NOTE ON THE REPRESENTATION OF THE VALUES OF POLYNOMIALS WITH REAL COEFFICIENTS FOR COMPLEX VALUES OF THE VARIABLE

BY

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§ 1.

In a previous paper [1] an extension was given of LILL's method, in which a polynomial with real coefficients was represented by polygonparts with angle φ. According to that, the value of the polynomial $f(z)$ can be determined graphically for each real value of z. For $\varphi = \pi/2$ we obtain the orthogones, which LILL used in resolving the real roots of a numerical equation. He also indicated [2] a method (without proof) to resolve graphically the complex roots of a numerical equation.

In the present note it is shown how this latter method can be extended to that of the polygonparts with angle φ, and how the values of a polynomial $f(z)$ for *complex* z can be represented.

§ 2.

We consider a polynomial of degree n

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n \qquad (a_0 \neq 0), \tag{1}$$

where a_i $(i = 0, 1, \dots, n)$ are real numbers.

We construct the polygonpart with angle φ of $f(z)$. As the directions of the a_i-sides are very important, it will be useful to give the rule for the construction of the polygonpart here again, but in another form. Each pair of consecutive sides include a fixed angle φ. The positive direction of the a_{i+1} side is obtained by a rotation of the positive a_i-direction in the point of intersection over an angle φ in the positive sense (that is counterclockwise). The positive a_0-direction is arbitrary. The a_{i+1}-side is drawn in the positive or negative direction according to the sign of the coefficient a_{i+1}.

(In fig. 1 the polynomial $f(z) = z^4 - 2z^3 - z^2 + 3z - 2$ is represented by the polygonpart $P_0P_1P_2P_3P_4P_5$).

Let the polygonpart with angle φ $P_0P_1P_2 \dots P_{n+1}$ (in fig. 1 with $n = 4$) represent the polynomial (1).

Without loss of generality we again may assume $a_0 > 0$.

If A_1 is an arbitrarily chosen point, then we consider $\overrightarrow{P_1A_1}$ as a complex vector with respect to P_1 as origin, and with the positive a_1-direction as positive axis of reals (shortly: *with respect to the a_1-side*).

In fig. 1 is $a_1 < 0$, so $\overrightarrow{P_1A_1} = \overline{P_1B_1} + i\,\overline{B_1A_1}$ [1]), and $\overrightarrow{A_1P_1} = -\overline{P_1B_1} - i\,\overline{B_1A_1}$.

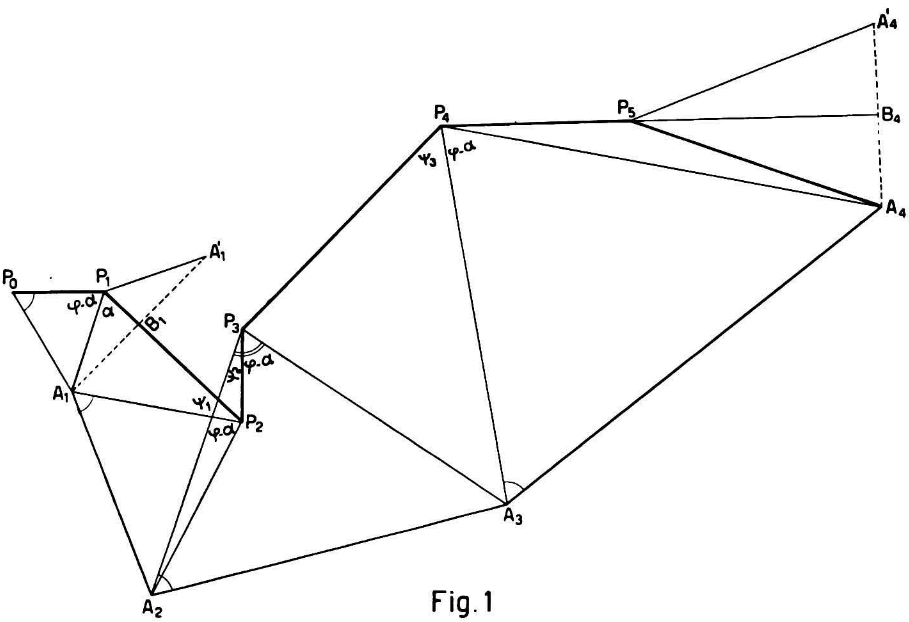

Fig. 1

Now we put

$$\overrightarrow{A_1P_1} = z\,a_0. \tag{2}$$

With each vector $\overrightarrow{A_1P_1}$ a complex number z corresponds, and conversely with each complex number z a vector $\overrightarrow{A_1P_1}$ with (2) can be found.

We construct the points $A_2, A_3, \ldots, A_n$, such that the triangles $P_0P_1A_1$, $A_1P_2A_2$, $A_2P_3A_3$, $\ldots$, $A_{n-1}P_nA_n$ are conformable.

We shall prove that *the vector $\overrightarrow{A_nP_{n+1}}$ represents the value of the polynomial $f(z)$ with respect to the a_n-side.*

Proof. *With respect to the a_1-side:*

$$\arg\,^{2)}\, z = \arg \overrightarrow{A_1P_1} = \angle A_1P_1P_2 = \alpha, \ |\overrightarrow{A_1P_1}| = a_0|z|.$$

Hence

$$\overrightarrow{A_1P_2} = a_1 + a_0z.$$

$\arg \overrightarrow{A_1P_2} = \angle A_1P_2P_1 - \pi = \psi_1 - \pi.$

[1]) We remark that $\overrightarrow{XY} = -\overrightarrow{YX}$, and that $\overline{XY} = -\overline{YX}$.

[2]) With arg z we mean here the main argument of z.

With respect to the a_2-side:

$$\arg \overrightarrow{A_2P_2} = \psi_1 - \pi - \alpha = \arg z(a_1 + a_0 z),\ |\overrightarrow{A_2P_2}| = |z|\ |a_1 + a_0 z|.$$

Hence

$$\overrightarrow{A_2P_2} = z\,(a_1 + a_0 z),\ \text{and}\ \overrightarrow{A_3P_3} = a_2 + a_1 z + a_0 z^2.$$

$\arg \overrightarrow{A_2P_3} = \pi - \angle A_2P_3P_2 = \pi - \psi_2.$

With respect to the a_3-side:

$$\arg \overrightarrow{A_3P_3} = \pi - \psi_2 - \alpha = \arg z\,(a_2 + a_1 z + a_0 z^2),$$
$$|\overrightarrow{A_3P_3}| = |z|\ |a_2 + a_1 z + a_0 z^2|.$$

Hence

$$\overrightarrow{A_3P_3} = a_2 z + a_1 z^2 + a_0 z^3,\ \text{and}\ \overrightarrow{A_3P_4} = a_3 + a_2 z + a_1 z^2 + a_0 z^3.$$

Proceeding this process we find *with respect to the a_n-side:*

$$\arg \overrightarrow{A_nP_n} = \arg z\,(a_{n-1} + a_{n-1}\, z + \dots + a_0\, z^{n-1}),$$
$$|\overrightarrow{A_nP_n}| = |z\,(a_{n-1} + a_{n-2}\, z + \dots + a_0\, z^{n-1})|.$$

Hence

$$\overrightarrow{A_nP_n} = z\,(a_{n-1} + a_{n-2}\, z + \dots + a_0\, z^{n-1})$$

and

$$\overrightarrow{A_nP_{n+1}} = a_n + a_{n-1}\, z + a_{n-2}\, z^2 + \dots + a_0\, z^n = f(z).$$

In fig. 1 is $\overrightarrow{A_1P_1} = z = \frac{1}{2} - i$ and $\overrightarrow{A_4P_5} = f(\frac{1}{2} - i) = 2\frac{9}{16} - i$.

Remarks.

1. If the point A_1 (and so z) can be chosen such that $\overrightarrow{A_nP_{n+1}} = 0$, which means that the point A_n coincides with P_{n+1}, then z is found to be a root of the equation $f(z) = 0$.

2. It is obvious that with a point A_1', lying symmetrically to A_1 with respect to the a_1-side a point A_n' corresponds, symmetrically to A_n with respect to the a_n-side (in fig. 1 the point A_4'). So $\overrightarrow{A_n'P_{n+1}}$ represents the value $f(\bar{z})$, where $\bar{z}$ is the conjugate of z. This illustrates again the correctness of the wellknown theorem: If $f(z)$ is a polynomial with real coefficients, and z is a root of the equation $f(z) = 0$, then $\bar{z}$ is also a root.

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LITERATURE

1. MEULENBELD, B., Note on LILL's method of solution of numerical equations. Proc. Kon. Ned. Akad. v. Wetensch. **53**, 464 (1950).
2. LILL, M., Résolution graphique des équations algebriques, qui ont des racines imaginaires. Nouv. Ann. Math. **2**, 7 (1868).