Mathematics. — On the uniform distribution modulo 1 of sequences \((f(n, \theta))\). By P. Erdős and J. F. Koksma. (Communicated by Prof. J. G. van der Corput.)

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I. Introduction. In a former paper \(^1\) we treated lacunary sequences. Now, using another method, we consider general sequences. For notation and definitions, see \(^1\). We prove

**Theorem 1.** Let \(f(1, \theta), f(2, \theta), \ldots\) be a sequence of real numbers, defined for each value of \(\theta\) of the segment \(a \leq \theta \leq \beta\), such that \(f(n, \theta)\) for \(n = 1, 2, \ldots\) as a function of \(\theta\), has a continuous derivative \(f'_{\theta}\) and such that the expression

\[
|f'_{\theta}(n_1, \theta) - f'_{\theta}(n_2, \theta)|
\]

for each couple of positive integers \(n_1 \neq n_2\) is either a non-decreasing or a non-increasing function of \(\theta\) on \(a \leq \theta \leq \beta\), the absolute value of which is \(\geq \delta\), where \(\delta\) denotes a positive number which does not depend on \(n_1, n_2, \) or \(\theta\). Then for almost all \(\theta\) the discrepancy \(D(N, \theta)\) of the sequence satisfies the inequality

\[
ND(N, \theta) = O(N^{1 \log^{1+\varepsilon} N}) \quad (\varepsilon > 0). \tag{1}
\]

Theorem 1 is a special case of the more general

**Theorem 2.** Let \(f(n, \theta)\) for \(n = 1, 2, \ldots\) denote a real continuous function of \(\theta\) on \(a \leq \theta \leq \beta\) and let

\[
\Phi(n_1, n_2, \theta) = f(n_1, \theta) - f(n_2, \theta) \quad \text{for} \quad n_1 \neq n_2
\]

have a continuous derivative \(\Phi'_{\theta}\) which is \(\neq 0\) and either non-decreasing or non-increasing on \(a \leq \theta \leq \beta\). Put

\[
A(M, N) = \frac{1}{N^2} \sum_{n_1 = M+2}^{M+N} \sum_{n_2 = M+1}^{M+n-1} \max \left( \frac{1}{|\Phi'_{\theta}(n_1, n_2, \alpha)|}, \frac{1}{|\Phi'_{\theta}(n_1, n_2, \beta)|} \right)
\]

and assume that for some constant \(\gamma \geq 1\)

\[
NA(M, N) \leq K_0 \log^\gamma N \quad \ldots \quad \ldots \quad \ldots \quad (2)
\]

for all couples of positive integers \(M, N\) where \(K_0\) is a positive constant. Then for almost all numbers \(\theta\) in \(a \leq \theta \leq \beta\) the discrepancy \(D(N, \theta)\) of the sequence \(\{f(1, \theta), f(2, \theta), \ldots\\}\) satisfies the inequality

\[
ND(N) = O(N^{1 \log^{1+\varepsilon} N}) \quad (\varepsilon > 0).
\]

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Remarks. 1. It is clear that the functions \( f(n, \theta) \) of Theorem 1 satisfy the assumptions of Theorem 2. For if one ranges the \( N \) numbers

\[ f(M + 1, \theta), f(M + 2, \theta), \ldots, f(M + N, \theta) \]

in order of magnitude, these numbers at each step increase with at least the amount \( \delta \) and we find

\[
\sum_{n=1}^{N} \left| f(n, \theta) - f(n_0, \theta) \right| < 2 \sum_{\mu=1}^{N} \frac{1}{\mu} < \frac{2}{\delta} \log 3N,
\]

hence

\[ NA(M, N) \leq \frac{2}{\delta} \log 3N = O(\log^\gamma N) \text{ for } \gamma = 1. \]

2. As Mr J. W. S. Cassels has shown us, he also proved Theorem 1. His very interesting method is different from ours. The proofs are completely independent from each other.

II. Some lemmas.

Lemma 1. Let \( f(n, \theta) \) for \( n = 1, 2, \ldots \) denote a real continuous function of \( \theta \) on \( \alpha \leq \theta \leq \beta \) and let

\[ \Phi(n_1, n_2, \theta) = f(n_1, \theta) - f(n_2, \theta) \text{ for } n_1 \neq n_2 \]

have a continuous derivative \( \Phi'_\theta \) which is \( \neq 0 \) and either non-decreasing or non-increasing on \( \alpha \leq \theta \leq \beta \). Finally put

\[
A_N = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{n_1=1}^{n-1} \max \left( \frac{1}{\Phi'_\theta(n_1, n_2, \alpha)}, \frac{1}{\Phi'_\theta(n_1, n_2, \beta)} \right).
\]

Then we have for \( N \geq 2, h > 0 \) (\( h \) not depending on \( n \) and \( \theta \))

\[
\int_{\alpha}^{\beta} \left| \sum_{n=1}^{N} e^{2\pi i \theta f(n, h)} \right| \theta \, d\theta \equiv (\beta - \alpha) N + \frac{A_N}{h} N^2.
\]

The proof of this lemma has been given by Koksma \( ^3 \).

Lemma 2. If \( u_1, u_2, \ldots \) is a real sequence and if \( D(N) \) denotes its discrepancy then for each integer \( m \geq 1 \), we have

\[
ND(N) \equiv K \left( \frac{N}{m+1} + \frac{m}{h} \sum_{n=1}^{N} \left| \sum_{h=1}^{N} e^{2\pi i n_1 h} \right| \right),
\]

where \( K \) denotes a numerical constant.

This lemma is an improvement proved by Erdős–Turán \( ^4 \) of the one-dimensional case of a theorem of van der Corput–Koksma \( ^5 \).

\( ^2 \) For litt. see \( ^1 \) and also \( ^5 \).


Lemma 3. If \( f(n, \theta) \) denotes the function of Lemma 1, and if \( D(N, \theta) \) denotes the discrepancy of the sequence \( f(1, \theta), f(2, \theta), \ldots \), then

\[
\int_a^\beta N^2 D^2(N, \theta) \, d\theta \equiv K_1 (N \log^2 N + A_N N^2 \log N).
\]

where \( K_1 \) depends on \( \beta - \alpha \) only.

Proof. Putting \( m = [\sqrt{N}] \), we have by Lemma 2

\[
N^2 D^2(N, \theta) \equiv K^2 \left( N + 2 \sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \sum_{n=1}^{\frac{N}{h}} e^{2\pi i h f(n, \theta)} \right) +
\]

\[
+ K^2 \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \sum_{n=1}^{N} e^{2\pi i h f(n, \theta)} \sum_{n=1}^{N} e^{2\pi i k f(n, \theta)}.
\]

Hence

\[
\int_a^\beta N^2 D^2(N, \theta) \, d\theta \equiv K^2 \left( N(\beta - \alpha) + 2 \sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \sum_{n=1}^{\frac{N}{h}} e^{2\pi i h f(n, \theta)} \left( \int_a^\beta \frac{1}{h} \, d\theta \right)^2 \sum_{n=1}^{N} e^{2\pi i k f(n, \theta)} \right) +
\]

\[
+ K^2 \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \sum_{n=1}^{N} e^{2\pi i h f(n, \theta)} \sum_{n=1}^{N} e^{2\pi i k f(n, \theta)} \left( \int_a^\beta \frac{1}{h} \, d\theta \right)^2 \left( \int_a^\beta \frac{1}{h} \, d\theta \right)^2.
\]

and by the CAUCHY–SCHWARZ inequality for integrals

\[
\equiv K^2 \left( N(\beta - \alpha) + 2 \sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \sum_{n=1}^{\frac{N}{h}} (\beta - \alpha)^2 N + \frac{\beta - \alpha}{h} A_N \cdot N^2 \right)^2 +
\]

\[
+ \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \left( (\beta - \alpha) N + \frac{1}{h} A_N \cdot N^2 \right)^2 \left( (\beta - \alpha) N + \frac{1}{k} A_N \cdot N^2 \right)^2
\]

by Lemma 1. Hence by the CAUCHY–SCHWARZ inequality for sums

\[
\int_a^\beta N^2 D^2(N, \theta) \, d\theta \equiv K^2 \left( N(\beta - \alpha) + 2 \sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \sum_{n=1}^{\frac{N}{h}} \frac{\beta - \alpha}{h} \sqrt{N} + \sum_{h=1}^{[\sqrt{N}]} \frac{\sqrt{N} \beta - \alpha}{h} \sqrt{A_N} \cdot N \right) +
\]

\[
+ \left( \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \left( (\beta - \alpha) N + \frac{1}{h} A_N \cdot N^2 \right)^2 \right) \left( (\beta - \alpha) N + \frac{1}{k} A_N \cdot N^2 \right)^2
\]

\[
\equiv K_2 (N + N \log N + \sqrt{A_N} N^2 + N \log^2 N + A_N N^2 \log N),
\]

where \( K_2 \) only depends on \( K \) and \( \beta - \alpha \).

Now if

\[
\sqrt{A_N} N^2 \log N > A_N N^2 \log N,
\]

we should have

\[
1 > \sqrt{A_N} N \log N.
\]
hence

\[ A_N N^2 \log N < \sqrt{A_N} N^1 < N. \]

Therefore

\[ \int_{\alpha}^{\beta} N^2 D^2(N, \theta) d\theta \leq K_1 (N \log^2 N + A_N N^2 \log N). \]

Q.e.d.

**Lemma 4.** Let \( F(M. N) = F(M, N, \theta) \) denote a function of \( \theta \) on a segment \( \alpha \leq \theta \leq \beta \) for each couple of positive integers \( M \) and \( N \), such that

\[ |F(M, N)| \leq |F(M, N_1)| + |F(M + N_1, N - N_1)| \quad \ldots \quad (3) \]

for each triple \( M, N \) and \( N_1 \leq N \) and such that \( F \) belongs to the class \( L^2 \) over the segment. Let further

\[ \int_{\alpha}^{\beta} |F(M, N, \theta)|^2 d\theta \leq K_3 N \log^a N \]

\( K_3 > 0 \) and \( a, \) being real constants. Then for almost all \( \theta \) in \( \alpha \leq \theta \leq \beta \) we have

\[ |F(0, N, \theta)| = O\left( N^{1+\frac{a+3+\epsilon}{2}} \right) \quad (\epsilon > 0). \]

This lemma is a special case of a theorem of Gál–Koksma, the proof of which will appear before long \(^6\).

III. We now prove Theorem 2.

Let \( M \) denote an arbitrary integer \( \geq 1 \) and consider the functions

\[ f(M + 1, \theta), f(M + 2, \theta), \ldots; \quad \ldots \quad (4) \]

these functions satisfy the assumptions of Lemma 1 and the corresponding number \( A_N \) is exactly identical with the number \( A(M, N) \) which we have defined in Theorem 2. Denoting the discrepancy of the sequence (4) by \( D(M, N, \theta) \), we have by Lemma 3, applied to the sequence (4),

\[ \int_{\alpha}^{\beta} N^2 D^2(M, N, \theta) d\theta \leq K_1 (N \log^2 N + A(M, N) N^2 \log N) \leq K_4 N \log^{1+\gamma} N \]

because of (2). Now it is easily seen from the definition of \( D(N) \), that if we put

\[ F(M, N, \theta) = N D(M, N, \theta), \]

the relation (3) is satisfied. Hence Theorem 2 follows immediately from Lemma 4 with \( a = 1 + \gamma \).