Mathematics. — *On a problem in the theory of uniform distribution.* By P. ERDÖS and P. TURÁN. II. (Communicated by Prof. J. G. VAN DER CORPUT.)

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9. After this first reduction of our problem we transform it in the following way. Let for the polynomial $\psi(z)$ defined in (8.4)

$$M_\vartheta^* = \max_{|z|=\vartheta} |\psi(z)| = \max_{|z|=\vartheta} |\psi_1(z)| = \max_{|z|=\vartheta} \left| \prod_{\nu=1}^{n} (1 - z e^{-i\varphi_\nu}) \right| . \quad (9.1)$$

Then

$$\max_{|z|=\vartheta} \log |\psi_1(z)| = \log M_\vartheta^* . \quad \ldots \ldots \quad (9.2)$$

Since $\log \psi_1(z)$ is regular for $|z| = \vartheta$ we have by the classical theorem of HADAMARD–BOREL–CARATHEODORY that for $|z| \leqq \frac{\vartheta}{2}$, since $\log \psi_1(0) = 0$,

$$|\log \psi_1(z)| \leqq 2 \log M_\vartheta^*$$

i.e.

$$\left| \sum_{\nu=1}^{n} \log (1 - z e^{-i\varphi_\nu}) \right| \leqq 2 \log M_\vartheta^*$$

or denoting $\sum_{\nu=1}^{n} e^{-ki\varphi_\nu}$ by s_k,

$$\left| \sum_{k=1}^{\infty} \frac{z^k}{k} s_{-k} \right| \leqq 2 \log M_\vartheta^* , \qquad |z| \leqq \frac{\vartheta}{2} . \quad \ldots \ldots \quad (9.3)$$

Then CAUCHY's estimation gives

$$\left.\begin{aligned} |s_k| = |s_{-k}| &\leqq k \frac{2 \log M_\vartheta^*}{\left(\frac{\vartheta}{2}\right)^k} = \\ &= 2k \left(\frac{2}{\vartheta}\right)^k \log M_\vartheta^* < \left(\frac{4}{\vartheta}\right)^k \log M_\vartheta^* = \left(\frac{4}{\vartheta}\right)^k \frac{n}{g^*(n, \vartheta)} . \end{aligned}\right\} . \quad (9.4)$$

Owing to the trivial inequality $|s_k| \leqq n$ the inequality (9.4) is restrictive only for those k's for which

$$k \leqq \frac{\log g^*(n, \vartheta)}{\log \frac{4}{\vartheta}} . \quad \ldots \ldots \quad (9.5)$$

Hence the proof of Theorem II is reduced to the question whether or not the inequalities (9.4) with the restriction (9.5) involve equidistribution of the φ_ν's mod 2π. In other words we reduced the proof of Theorem II to the proof of Theorem III with

$$\psi(k) = \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^k, \quad m = \left[\frac{\log g^*(n,\vartheta)}{2\log\frac{4}{\vartheta}}\right]. \quad . \quad . \quad (9.6)$$

Since

$$\sum_{\nu=1}^{m}\frac{\psi(k)}{k} < \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^{\frac{m}{2}}\left(1+\log\frac{m}{2}\right) + \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^m\cdot\frac{2}{m}\cdot\frac{m}{2} <$$

$$< \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^m\left\{1+2\left(\frac{\vartheta}{4}\right)^{\frac{m}{2}}\log\frac{m}{2}\right\} \leqq$$

$$\leqq \frac{2n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^m \leqq \frac{2n}{\sqrt{g^*(n,\vartheta)}} < \frac{3n}{\log g^*(n,\vartheta)}$$

and

$$\frac{n}{m+1} < 2\log\frac{4}{\vartheta}\cdot\frac{n}{\log g^*(n,\vartheta)}$$

we obtain — anticipating Theorem III — that for every $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left|\sum_{\alpha\leqq\varphi_\nu\leqq\beta \bmod 2\pi} 1 - \frac{\beta-\alpha}{2\pi}n\right| < 5C\log\frac{4}{\vartheta}\cdot\frac{n}{\log g^*(n,\vartheta)}$$

i.e. Theorem II will be proved.

10. Before turning to the proof of Theorem III we sketch the corresponding reasoning for Theorem I. In this case — as we remarked in [1]) — the general case can be reduced to the case when all the roots lie on the unit circle. Then in (9.2) M_ϑ^* is replaced by $\max_{|z|=1}|\psi(z)| = M$. Applying the theorem of HADAMARD–BOREL–CARATHEODORY to the interior circle $|z| = \varrho$, where we determine ϱ suitable later, we obtain

$$|\log\psi_1(z)| \leqq \frac{2\log M}{1-\varrho}, \qquad |z| \leqq \varrho$$

resp.

$$\left|\sum_{k=1}^{\infty}\frac{z^k}{k}s_{-k}\right| < \frac{2\log M}{1-\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.1)$$

CAUCHY's estimation gives

$$|s_k| = |s_{-k}| \leqq k\,\frac{2\log M}{1-\varrho}\cdot\frac{1}{\varrho^k}.$$

Choosing $\varrho = 1 - \frac{1}{k+1}$

$$\left.\begin{array}{l} |s_k| \leqq 20\, k^2 \log M \\ k = 1, 2, \ldots \end{array}\right\} \quad \ldots\ldots \quad (10.2)$$

This gives according to the corollary of Theorem III an error term $0(n^{2/3})$ only. Hence Theorem I seems to be much deeper. This seems to justify the use of more difficult analytical tools in the proof of [1]).

11. For the proof of theorem III we need some simple auxiliary considerations.

Let

$$R = \int_0^{2\pi} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{2} t}{\sin \frac{t}{2}} \right)^4 dt. \quad \ldots\ldots \quad (11.1)$$

Obviously

$$\left.\begin{array}{l} R > 4 \int_0^{2\pi} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{2} t}{t} \right)^4 dt = 4 \left(\frac{\left[\frac{m}{2}\right]+1}{2} \right)^3 \int_0^{\pi\left(\left[\frac{m}{2}\right]+1\right)} \left(\frac{\sin y}{y} \right)^4 dy > \\ > \frac{1}{2} \left(\frac{m}{2} \right)^3 \int_0^{\frac{\pi}{2}} \left(\frac{\sin y}{y} \right)^4 dy > c_1 m^3 \end{array}\right\} \quad (11.2)$$

where c_1 and later $c_2, \ldots$ denote numerical constants. Further

$$\left.\begin{array}{l} R = 2 \int_0^{\pi} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{2} t}{\sin \frac{t}{2}} \right)^4 dt \leqq 2\pi^4 \int_0^{\pi} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{2} t}{t} \right)^4 dt < \\ < \frac{\pi^4}{4} \left(\left[\frac{m}{2}\right]+1 \right)^3 \int_0^{\infty} \left(\frac{\sin y}{y} \right)^4 dy < c_2 m^3. \end{array}\right\} \quad (11.3)$$

12. Let a be a parameter subjected only to the restriction

$$\pi \geqq a \geqq \frac{10}{m+1} \quad \ldots\ldots \quad (12.1)$$

and let

$$\pi_m(x, a) = \frac{1}{R}\int_{-x}^{a-x} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{2} t}{\sin \frac{t}{2}} \right)^4 dt. \quad . \quad . \quad . \quad (12.2)$$

We have also

$$\pi_m(x, a) = \frac{1}{R}\int_{0}^{a} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{t} (t-x)}{\sin \frac{t-x}{2}} \right)^4 dt \quad . \quad . \quad (12.3)$$

and since the integrand of (12. 2) is an even function of t

$$\pi_m(x, a) = \frac{1}{R}\int_{x-a}^{x} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{2} t}{\sin \frac{t}{2}} \right)^4 dt. \quad . \quad . \quad . \quad . \quad (12.4)$$

Since generally

$$\left(\frac{\sin \frac{k+1}{2} y}{\sin \frac{y}{2}} \right)^2 = (k+1) + k \cdot 2\cos y + (k-1)\, 2\cos 2y + \ldots 1 \cdot 2 \cos k y,$$

we obtain at once that the integrand of (12. 2), i.e. $\pi_m(x, a)$ itself, is a trigonometric polynomial of order $\leqq m$

$$\pi_m(x, a) = a_0(a) + \sum_{1 \leqq \nu \leqq m} (a_\nu(a) \cos \nu x + b_\nu(a) \sin \nu x). \quad . \quad (12.5)$$

13. We need some information about the coefficients in (12. 5). Evidently, using (12. 3),

$$\left. \begin{aligned} a_0(a) &= \frac{1}{2\pi}\int_0^{2\pi} \pi_m(x, a)\, dx = \frac{1}{2\pi R}\int_0^a dt \int_0^{2\pi} \left(\frac{\sin \frac{\left[\frac{m}{2}\right]+1}{2} (t-x)}{\sin \frac{t-x}{2}} \right)^4 dx = \\ &= \frac{1}{2\pi R}\int_0^a dt \cdot R = \frac{a}{2\pi}. \end{aligned} \right\} \quad (13.1)$$

Further using (12. 2)

$$a_\nu(a)=\frac{1}{\pi}\int_0^{2\pi}\pi_m(x,a)\cos\nu x\,dx=-\frac{1}{\pi\nu}\int_0^{2\pi}\frac{\partial\pi_m}{\partial x}\sin\nu x\,dx=$$

$$=-\frac{1}{\pi\nu R}\int_0^{2\pi}\sin\nu x\left\{\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}(a-x)}{\sin\frac{a-x}{2}}\right)^4+\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}x}{\sin\frac{x}{2}}\right)^4\right\}dx$$

i.e.

$$\left.\begin{array}{l}|a_\nu(a)|\leqq\dfrac{1}{\pi\nu R}\displaystyle\int_0^{2\pi}\left\{\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}(a-x)}{\sin\frac{a-x}{2}}\right)^4+\right.\\ \left.+\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}x}{\sin\frac{x}{2}}\right)^4\right\}dx=\dfrac{2}{\pi\nu},\qquad 1\leqq\nu\leqq m;\end{array}\right\} \quad . \quad (13.2)$$

similarly

$$\left.\begin{array}{l}|b_\nu(a)|\leqq\dfrac{2}{\pi\nu}\\ 1\leqq\nu\leqq m.\end{array}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (13.3)$$

14. We need also some information about the shape of the graph of $\pi_m(x,a)$. The definition of R and representation (12. 2) give immediately for every real x

$$0\leqq\pi_m(x,a)\leqq 1. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (14.1)$$

We consider $\pi_m(x,a)$ in the interval

$$\frac{2}{m+1}\leqq x\leqq a-\frac{2}{m+1} \quad . \ . \ . \ . \ . \ . \quad (14.2)$$

(this has a meaning owing to (12. 1)). Using the representation (12. 2) and the estimation (11. 3) we have in the range (14. 2)

$$\left.\begin{array}{l}\pi_m(x,a)>\dfrac{1}{c_2m^3}\displaystyle\int_0^{\frac{1}{1+\left[\frac{m}{2}\right]}}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{\frac{t}{2}}\right)^4dt=\\ =\dfrac{1}{c_2m^3}\left(\dfrac{\left[\frac{m}{2}\right]+1}{2}\right)^3\displaystyle\int_0^{\frac{1}{2}}\left(\frac{\sin y}{y}\right)^4dy>c_3.\end{array}\right\} \quad . \ . \quad (14.3)$$

Further, for $a < x \leqq \frac{3}{2}\pi$ we have, using the estimation (11.2) and the representation (12.4)

$$\left.\begin{aligned}\pi_m(x, a) &< \frac{1}{c_1 m^3}\int_{x-a}^{\frac{3}{2}\pi}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{\sin\frac{t}{2}}\right)^4 dt < \frac{\pi^4}{c_1 m^3}\int_{x-a}^{\frac{3}{2}\pi}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{t}\right)^4 dt < \\ &< \frac{\pi^4}{c_1 m^3}\left(\frac{\left[\frac{m}{2}\right]+1}{2}\right)^3\int_{\frac{\left[\frac{m}{2}\right]+1}{2}(x-a)}^{\infty}\left(\frac{\sin y}{y}\right)^4 dy < c_4\int_{\frac{m}{4}(x-a)}^{\infty}\frac{dy}{y^4} < \frac{c_5}{(m(x-a))^3}.\end{aligned}\right\} \quad (14.4)$$

Finally, for $\frac{3}{2}\pi \leqq x \leqq 2\pi$ we have from the estimation (11.2) and the representation (12.4), since $x - a \geqq \frac{3}{2}\pi - \pi = \frac{\pi}{2}$

$$\left.\begin{aligned}\pi_m(x, a) &< \int_{\frac{\pi}{2}}^{x}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{\sin\frac{t}{2}}\right)^4 dt < c_6\int_{\frac{\pi}{2}}^{x}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{2\pi - t}\right)^4 dt = \\ &= c_6\int_{2\pi-x}^{\frac{3\pi}{2}}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}y}{y}\right)^4 dy < \frac{c_7}{(m(2\pi - x))^3}\end{aligned}\right\} \quad (14.5)$$

15. Now we are going to prove the following

Lemma. Assuming (4.1) and (4.2) we have for the number N of the φ_ν's lying in an arbitrary interval of length $\frac{10}{m+1}$ with $m > 20$ the inequality

$$N < c_8\left(\frac{n}{m+1} + \sum_{\nu=1}^{m}\frac{\psi(\nu)}{\nu}\right). \quad . \quad . \quad . \quad . \quad . \quad (15.1)$$

Proof. Without loss of generality we may suppose that our interval is

$$\frac{2}{m+1} \leqq x \leqq \frac{12}{m+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.2)$$

We consider the polynomial $\pi_m(x, \gamma)$ where

$$\gamma = \frac{14}{m+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.3)$$

Replacing x in

$$\pi_m(x, \gamma) = a_0(\gamma) + \sum_{\nu=1}^{m} (a_\nu(\gamma) \cos \nu x + b_\nu(\gamma) \sin \nu x)$$

by $\varphi_1, \varphi_2, \ldots, \varphi_n$ and summing we obtain owing to (13.1)

$$\sum_{l=1}^{n} \pi_m(\varphi_l, \gamma) = \frac{\gamma}{2\pi} n + \sum_{\nu=1}^{m} a_\nu(\gamma) \sum_{l=1}^{n} \cos \nu \varphi_l + \sum_{\nu=1}^{m} b_\nu(\gamma) \sum_{l=1}^{n} \sin \nu \varphi_l.$$

In the interval (15.2) condition (14.2) is satisfied i.e. from (14.3) and the non-negativity of $\pi_m(x, \gamma)$ we have

$$c_3 N \leqq \sum_{l=1}^{n} \pi_m(\varphi_l, \gamma) \leqq \frac{7}{\pi} \cdot \frac{n}{m+1} + \sum_{\nu=1}^{m} (|a_\nu(\gamma)| + |b_\nu(\gamma)|) \left| \sum_{l=1}^{n} e^{\nu i \varphi_l} \right|.$$

Applying (4.1), (13.2) and (13.3) we obtain further

$$c_3 N < \frac{7}{\pi} \cdot \frac{n}{m+1} + \frac{4}{\pi} \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu}. \qquad \text{Q. e. d.}$$

16. Now we turn to the proof of theorem III. Let d be given satisfying

$$\frac{10}{m+1} \leqq d \leqq \pi \quad \ldots\ldots\ldots \quad (16.1)$$

and consider the polynomial $\pi_m(x, d)$. Replacing x in

$$\pi_m(x, d) = \frac{d}{2\pi} + \sum_{\nu=1}^{m} (a_\nu(d) \cos \nu x + b_\nu(d) \sin \nu x)$$

by $\varphi_1, \varphi_2, \ldots, \varphi_n$ and summing we obtain

$$\sum_{j=1}^{n} \pi_m(\varphi_j, d) = \frac{d}{2\pi} n + \sum_{\nu=1}^{m} a_\nu(d) \sum_{j=1}^{n} \cos \nu \varphi_j + \sum_{\nu=1}^{m} b_\nu(d) \sum_{j=1}^{n} \sin \nu \varphi_j.$$

Arguing as before we obtain

$$\sum_{j=1}^{n} \pi_m(\varphi_j, d) > \frac{d}{2\pi} n - \frac{4}{\pi} \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu}. \quad \ldots\ldots \quad (16.2)$$

Now denoting the number of φ_ν's in $0 \leqq x \leqq d$ by $N(d)$, the contribution of these φ_ν's is, owing to $\pi_m(x, d) \leqq 1$,

$$\leqq N(d). \quad \ldots\ldots\ldots \quad (16.3)$$

To obtain an upper estimation for the contribution of the other φ_ν's we construct successive contiguous intervals of length $\frac{10}{m+1}$ each starting from $x = d$ and covering the interval $d \leqq x \leqq \frac{3}{2}\pi$. The contributions of the φ_ν in the interval

$$D_k \ : \ d + k \frac{10}{m+1} \leqq x \leqq d + (k+1) \frac{10}{m+1}$$

is owing to the lemma

$$< c_8 \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right) \max_{x \in D_k} \pi_m(x, d)$$

which is, by (14. 4)

$$< c_8 \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right) \max_{x \in D_k} \frac{c_5}{(m(x-d))^3} <$$

$$< c_9 \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right) \cdot \frac{1}{k^3}.$$

Hence summing over k

$$\sum_{d \leqq \varphi_j \leqq \frac{3}{2}\pi} \pi_m(\varphi_j, d) < c_{10} \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right). \quad . \quad . \quad (16.4)$$

The contribution of the φ_ν lying in the remaining interval $\frac{3\pi}{2} < x \leqq 2\pi$ we can estimate similarly. Combining (16. 2), (16. 3) and (16. 4) we obtain

$$N(d) > \frac{d}{2\pi} \cdot n - c_{11} \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right). \quad . \quad . \quad . \quad (16.5)$$

Obviously the same estimation holds for the number $N(c, c+d)$ of the φ_ν's for which $c \leqq \varphi_\nu \leqq c + d \bmod 2\pi$. The restriction $d \leqq \pi$ is obviously unnecessary, if we replace c_{11} by $2c_{11} = c_{12}$.

To obtain the upper estimation of $N(c, c+d) - \frac{d}{2\pi} n$ we have, due to

$$N(c_1 c + d) = n - N(0, c) - N(c + d, 2\pi) \qquad \text{Q. e. d.}$$

merely to apply (16. 5) twice.

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