Mathematics. — *On a problem in the theory of uniform distribution.* By P. ERDÖS and P. TURÁN. I. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 25, 1948.)

1. In a forthcoming paper [1]) we prove the following

Theorem I. If

$$f(z)=a_0+\ldots+a_n z^n=a_n \prod_{\nu=1}^{n}(z-z_\nu) \quad . \quad . \quad . \quad . \quad (1.1)$$

and

$$\max_{|z|=1}|f(z)|=M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

then for arbitrary fixed $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left|\sum_{\alpha \leqq \operatorname{arc} z_\nu \leqq \beta \bmod 2\pi}{}_{\nu} 1-\frac{\beta-\alpha}{2\pi}n\right|<16\sqrt{n \log \frac{M}{\sqrt{|a_0 a_n|}}}. \quad . \quad . \quad (1.3)$$

Here — and throughout the paper — the expression $\alpha \leqq \operatorname{arc} z \leqq \beta \bmod 2\pi$ means that the image of z on the complex plane lies in the angle formed by $\operatorname{arc} z=\alpha$ and $\operatorname{arc} z=\beta$.

The meaning of Theorem I is obviously that given a sequence of polynomials (the n-th of degree n) having the maximum modulus M_n on the unit circle and such that $\frac{M_n}{\sqrt{|a_0 a_n|}}$ increases "not too rapidly" (e.g. $\frac{M_n}{\sqrt{|a_0 a_n|}}<e^{\frac{n}{\log n}}$) then the roots of the n-th polynomial are uniformly distributed in the different angles even if the size of the angle tends to 0 with $1/n$ "not too rapidly".

2. It is natural to ask whether restricting only

$$\frac{1}{\sqrt{|a_0 a_n|}}\max_{|z|=\vartheta}|f(z)| \equiv \frac{M_\vartheta}{\sqrt{|a_0 a_n|}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

with fixed ϑ $(0<\vartheta<1)$ a similar equidistribution theorem can be deduced. It is easy to see that this is not the case. Indeed let

$$\left.\begin{aligned} \varphi_1(z)&=1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!} \\ f(z)&=z^n\varphi_1\left(\frac{\sqrt[n]{n!}}{z}\right)=1+\ldots+z^n. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.2)$$

[1]) Submitted to the Annals of Mathematics.

If z is on the circle $|z| = \frac{1}{2e}$ then from STIRLING's formula we have

$$w = \frac{\sqrt[n]{n!}}{|z|} > 2n$$

i.e. the terms $\frac{|w|^n}{n!}, \frac{|w|^{n-1}}{(n-1)!}, \ldots, \frac{|w|}{1}$, 1 decrease more rapidly then a geometric progression with quotient 1/2. Thus

$$\left|\varphi_1\left(\frac{\sqrt[n]{n!}}{z}\right)\right| < 2\frac{|w|^n}{n!} = \frac{2}{|z|^n}$$

$$\left|f\left(\frac{e^{i\vartheta}}{2e}\right)\right| < 2$$

i.e. for $|z| \leqq \frac{1}{2e}$

$$|f(z)| \leqq 2 \quad , \quad \frac{1}{\sqrt{|a_0 a_n|}} \max_{|z|=\frac{1}{2e}} |f(z)| \leqq 2.$$

On the other hand, as SZEGÖ [2]) showed, we have for the number N_1 resp. N_2 of roots of $\varphi_1(z)$ (i.e. also of $f(z)$) lying in the half plane $Rz \leqq 0$ resp. $\geqq 0$ the relations

$$\lim_{n\to\infty} \frac{N_1}{n} = \frac{1}{2} + \frac{1}{e\pi} \quad , \quad \lim_{n\to\infty} \frac{N_2}{n} = \frac{1}{2} - \frac{1}{e\pi},$$

i.e. the roots of $f(z)$ are not uniformly distributed in the different angles. Hence the polynomials (2.2) give the required counter-example for $\vartheta \leqq \frac{1}{2e}$. Choosing instead of the partial-sums of the exponential series the partial sums of certain MITTAG–LEFFLER functions [3]) we see that even if a sequence of polynomials (the n-th of degree n) divided by $\sqrt{|a_0 a_n|}$ remains uniformly bounded in n over a prescribed circle $|z| \leqq \vartheta$ with $0 < \vartheta < 1$, the roots of the n-th polynomial are not necessarily uniformly distributed in the different angles. No doubt, this fact throws a new light on the theorem stated in 1. and enhances its interest considerably.

3. So without imposing any further conditions on $\frac{M_\vartheta}{\sqrt{|a_0 a_n|}}$ (2. 1) can not lead to an equidistribution theorem similar to that of 1. However we

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[2]) G. SZEGÖ, Über eine Eigenschaft der Exponential-reihe. Sitzungsber. der Berliner Math. Ges. 50—64 (1924).

[3]) The distribution of roots of these partial-sums and even of the partial-sums of a general class of integral functions of finite positive order has been determined by P. ROSENBLOOM (to appear in the Transactions of Amer. Math. Soc.).

shall show that a simple additional condition can save the situation. We shall prove the following

Theorem II. If all the roots of a polynomial

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

are outside the open unit circle, and for a fixed $0 < \vartheta < 1$ we have for $|z| = \vartheta$ the inequality $|f(z)| \leqq M_\vartheta$ then writing (without loss of generality)

$$\frac{M_\vartheta}{\sqrt{|a_0 a_n|}} = e^{\frac{n}{g(n,\vartheta)}} \quad , \quad n \geqq g(n, \vartheta) \geqq 2 \quad . \quad . \quad . \quad . \quad (3.2)$$

we have for all $0 \leqq \alpha < \beta \leqq 2\pi$

$$\left| \sum_{\alpha \leqq \operatorname{arc} z_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta - \alpha}{2\pi} n \right| < C \log \frac{4}{\vartheta} \cdot \frac{n}{\log g(n, \vartheta)}, \quad . \quad (3.3)$$

where C denotes a *numerical* constant.

As we mentioned before while discussing Theorem I, if a given sequence of polynomials

$$f_n(z) = a_0^{(n)} + a_1^{(n)} z + \ldots + a_n^{(n)} z^n$$
$$n = 1, 2, \ldots$$

has the property that their absolute maxima $M^{(n)}$ on the unit circle satisfies

$$\frac{M^{(n)}}{\sqrt{|a_0^{(n)} a_n^{(n)}|}} = e^{o(n)}, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

then their roots are equidistributed in the different angles. Theorem II reveals the surprising fact that the much weaker condition concerning the absolute maxima $M_\vartheta^{(n)}$ on the circle $|z| = \vartheta$, $(0 < \vartheta < 1$ and fixed)

$$\frac{M_\vartheta^{(n)}}{\sqrt{|a_0^{(n)} a_n^{(n)}|}} = e^{o(n)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

can assure the equidistribution of the roots, if they are all $\geqq 1$ in absolute value.

In the case when $\frac{M_\vartheta}{\sqrt{|a_0 a_n|}}$ is "not too large", e.g. when

$$\frac{M_\vartheta}{\sqrt{|a_0 a_n|}} \leqq e^{\sqrt{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

the error term (3.3) is of order $n/\log n$. Curiously enough the same holds if $e^{\sqrt{n}}$ is replaced by n^{100} or even by a numerical constant say 10000. Though this error term is worse than that of Theorem I, DE BRUIJN [4])

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[4]) In a letter wherein he conjectured essentially our theorem II.

remarked that the order of the error term is best possible for every $0<\vartheta<1$; indeed

$$\vartheta=e^{-c} \quad , \quad c>0$$

and

$$l=\left[\frac{1}{c}\log n\right] \quad , \quad k=\left[\frac{cn}{\log n}\right],$$

the polynomial

$$f(z)=(1-z^{l})^{k}$$

satisfies

$$M_{\vartheta}\leqq(1+e^{-cl})^{k}<\exp(ke^{-cl})<\exp\left(\frac{cn}{\log n}e^{-c\left(\frac{1}{c}\log n-1\right)}\right)=$$

$$=\exp\left(\frac{cn}{\log n}\cdot\frac{1}{n}e^{c}\right)<2 \quad , \quad \frac{M_{\vartheta}}{\sqrt{|a_0 a_n|}}<2$$

for $n>n_0=n_0(\vartheta)$, and $f(z)$ has roots of multiplicity $>c\dfrac{n}{\log n}-1>$ $>\dfrac{c}{2}\cdot\dfrac{n}{\log n}$ for $n>n_1=n_1(\vartheta)$, which evidently shows that the error term in Theorem II is (with regard to n) the best possible. As a matter of fact essentially DE BRUIJN's example shows that Theorem II is for every admissible $g(n,\vartheta)$ essentially best possible. Indeed put

$$l_1=\left[\frac{1}{c}\log g(n,\vartheta)\right] \quad , \quad k_1=\left[\frac{cn}{\log g(n,\vartheta)}\right].$$

The polynomial

$$f_1(z)=(1-z^{l_1})^{k_1}$$

has a root of multiplicity

$$>\frac{cn}{\log g(n,\vartheta)}-1>\frac{c}{2}\cdot\frac{n}{\log g(n,\vartheta)}$$

though on the circle $|z|=e^{-c}=\vartheta$ we have

$$|f_1(z)|\leqq\left(1+e^{-c\left(\frac{1}{c}\log g(n,\vartheta)-1\right)}\right)^{\frac{cn}{\log g(n,\vartheta)}}<\exp\left(\frac{cn}{\log g(n,\vartheta)}\frac{e^{c}}{g(n,\vartheta)}\right)<$$

$$<\exp\left(\frac{n}{g(n,\vartheta)}\right),$$

if $g(n,\vartheta)$ is sufficiently large.

In our paper [1]) we have not dealt with the question whether the error term in Theorem I is best possible, with respect to n, but we can show by an example that it is essentially best possible; we do not discuss here the details.

For some further remarks about the relation of Theorem I and II see 10.

4. In 8 we shall see using a method due to I. SCHUR [5]), how the general case of Theorem II can be reduced to the case when all the roots of $f(z)$ lie on the unit circle. In 9. we easily deduce Theorem II thus specialised from the following

Theorem III. If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are real and

$$|s_k| \equiv \left| \sum_{\nu=1}^{n} e^{ki\varphi_\nu} \right| \leqq \psi(k) \qquad k = 1, 2, \ldots, m \quad . \quad . \quad . \quad (4.1)$$

$$m = m(n) \geqq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

then for arbitrary $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi}{}_\nu 1 - \frac{\beta - \alpha}{2\pi} n \right| < C \left(\frac{n}{m+1} + \sum_{k=1}^{m} \frac{\psi(k)}{k} \right)$$

with a numerical constant C.

5. Theorem III is obviously a "finite" form of the classical theorem of H. WEYL [6]) according to which if $\varphi_1, \varphi_2, \ldots$ is an infinite sequence of real numbers satisfying for every integer k the relation

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} e^{ki\varphi_\nu} = 0$$

then for every $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\substack{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi \\ \nu \leqq n}}{}_\nu 1$$

Another "finite" form of this theorem one can find on p. 101 of KOKSMA's well-known book [7]), where a sketch of the proof is also given. *Theorem* of VAN DER CORPUT and KOKSMA (in a slightly modified and restricted form). If there is a δ with $0 < \delta \leqq 1$ such that (4.1) holds for

$$k \leqq \left[\frac{K}{\delta} \log \frac{3}{\delta} \left(\log \log \frac{3}{\delta} \right)^2 \right] \equiv N_0,$$

K being a suitable numerical constant, then with the same K we have for all $0 \leqq \alpha < \beta \leqq 2\pi$

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi}{}_\nu 1 - \frac{\beta - \alpha}{2\pi} n \right| < K\delta n + 2K \sum_{1 \leqq k \leqq \frac{2}{\delta}} \frac{\psi(k)}{k} + $$
$$+ 2K \sum_{\frac{2}{\delta} < k \leqq N_0} \frac{\psi(k)}{k} e^{-\frac{k\delta}{89 \log^2 k\delta}}.$$

[5]) I. SCHUR, Sitzungsber. Berliner Akad. 403—428 (1933).

[6]) H. WEYL, Über die Gleichverteilung von Zahlen mod Eins. Math. Ann., **77**, 313—352 (1916).

[7]) J. F. KOKSMA, Diophantische Approximationen. Ergebn. der Math. und ihrer Grenzgebiete (1936).

Choosing $\delta = \frac{1}{m+1}$ we see that in order to obtain the same error term as in (4. 2) we have to restrict more of the s_k's, i.e. Theorem III is sharper than the theorem of VAN DER CORPUT–KOKSMA. As a matter of fact one can not deduce Theorem II from it (whereas it can be deduced from Theorem III). This improvement was obtained roughly speaking by using "DUNHAM JACKSON means" of the FOURIER series of the periodic *discontinuous* function $f_2(x)$ defined by

$$\begin{aligned} f_2(x) &= 1 \quad \text{for} \quad 0 \leqq x < a \\ f_2(x) &= 0 \quad \text{for} \quad a \leqq x < 2\pi \end{aligned}$$

instead of the partial sums of the *continuous* function $f_3(x)$ defined by

$$\begin{aligned} f_3(x) &= 1 \quad \text{for} \quad \eta \leqq x \leqq a - \eta \\ f_3(x) &= 0 \quad \text{for} \quad a \leqq x \leqq 2\pi, \end{aligned} \qquad 0 < \eta < \frac{a}{2}$$

and linear in the remaining intervals.

6. We consider Theorem III in the special case

$$|s_k| \leqq k^\lambda \quad , \quad \lambda > 1 \text{ and fixed}$$

for all $k \leqq n^{1/\lambda}$.

Then the error term in (4. 2), if $m \leqq n^{1/\lambda}$, is

$$< C\left(\frac{n}{m} + m^\lambda\right)$$

i.e. choosing $m = [n^{1/(\lambda+1)}]$ we obtain the following

Corollary. If $\varphi_1, \ldots, \varphi_n$ are real, $\lambda \geqq 1$ and

$$\left.\begin{aligned} &\left|\sum_{\nu=1}^{n} e^{ki\varphi_\nu}\right| \leqq k^\lambda \\ &1 \leqq k \leqq n^{\frac{1}{\lambda+1}} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6.\ 1)$$

then for all $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left|\sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta - \alpha}{2\pi} n\right| < C n^{\frac{\lambda}{\lambda+1}} \quad \ldots\ldots \quad (6.\ 2)$$

with a numerical constant C.

The interesting question whether the estimation (6. 2) is best possible or not, remains open.

7. L. KALMÁR [8]) made the remarkable discovery that if the roots of the

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[8]) L. KALMÁR, Az interpolatióról (hungarian). Matematikai és Fizikai Lapok 1926, p. 120—149. The expression $T_n(z)$ in (7.3) denotes the classical CEBICSEF-polynomial

$$T_n(z) = \left(\frac{z + \sqrt{z^2 - 1}}{2}\right)^n + \left(\frac{z - \sqrt{z^2 - 1}}{2}\right)^n.$$

He actually proved a more general theorem when the roots of polynomials $\omega_n(z)$ lie on a prescribed closed JORDAN-curve.

polynomials

$$\omega_n(z) = (z - x_1^{(n)})(z - x_2^{(n)}) \ldots (z - x_n^{(n)}) \quad . \quad . \quad . \quad . \quad (7.1)$$

satisfy for all $n = 1, 2, \ldots$ the inequalities

$$1 \geqq x_1^{(n)} \geqq x_2^{(n)} \geqq \ldots \geqq x_n^{(n)} \geqq -1 \quad . \quad . \quad . \quad . \quad . \quad (7.2)$$

and if the polynomials $\omega_n(z)$ have the asymptotic representation (with the obvious meaning of the n-th root)

$$\lim_{n \to \infty} \sqrt[n]{\frac{\omega_n(z)}{T_n(z)}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.3)$$

on the complex $z = x + i \,.\, y$-plane, cut along the segment $-1 \leqq x \leqq 1$, then the roots $x_\nu^{(n)}$ are uniformly distributed *in* FEJÉR's *sense* [9]) in $[-1, +1]$ i.e. writing

$$x_\nu^{(n)} = \cos \vartheta_\nu^{(n)}, \ \nu = 1, 2, \ldots, n, \ n = 1, 2, \ldots, 0 \leqq \vartheta_1^{(n)} \leqq \vartheta_2^{(n)} \leqq \ldots \leqq \vartheta_n^{(n)} \leqq \pi \quad (7.4)$$

we have for every $0 \leqq \alpha < \beta \leqq 2\pi$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\alpha \leqq \vartheta_\nu^{(n)} \leqq \beta} 1 = \frac{\beta - \alpha}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.5)$$

It is easy to see from (7.3) that the polynomial

$$2^n w^n \omega_n \left(\frac{w + \frac{1}{w}}{2} \right) = F_n(w)$$

has all its roots on the unit-circle and for $|w| < 1$

$$\lim_{n \to \infty} \sqrt[n]{F_n(w)} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.6)$$

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[9]) L. FEJÉR, Interpolation und konforme Abbildung. Gött. Nachr. 1918, p. 319—331. Generally if l is a given JORDAN-curve and the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}$ $(n = 1, 2, \ldots)$ are on l, he calls the points $z_\nu^{(n)}$ uniformly distributed over l if mapping conformally the outside of l onto the outside of $|w| =$ and continuously on the boundary, the maps

$$w_1^{(n)} = e^{i\varphi_1^{(n)}}, \ldots, w_n^{(n)} = e^{i\varphi_n^{(n)}}$$

are uniformly distributed over the unit-circle in WEYL's sense i.e. for every $0 \leqq \alpha < \beta \leqq 2\pi$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\alpha \leqq \varphi_\nu^{(n)} \leqq \beta} 1 = \frac{\beta - \alpha}{2\pi}.$$

In the case when l degenerates into a doubly-covered segment $-1 \leqq x \leqq +1$ then the mapping function is $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$ and we get the definition (7.4)—(7.5). This definition of equidistribution fits in with various function-theoretical problems even in the case of segments.

It is natural to ask for the "finite" analogon of KALMÁR's theorem. Making the following weaker assumption that the roots of the polynomial

$$f(z)=a_0+\ldots+z^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.7)$$

lie in $|z|\geqq 1$ and on a fixed circle $|z|=\vartheta$, $0<\vartheta<1$ we have

$$|f(z)|\leqq 1+\varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.8)$$

we may expect that the error term in the distribution of arc z_ν is much smaller if ε is "small". But the example of DE BRUIJN

$$f_4(z)=(1-z^{l_2})^{k_2}$$

$$l_2=[\omega \log n], \qquad k_2=\left[\frac{1}{\omega}\cdot\frac{n}{\log n}\right]$$

with sufficiently large $\omega>\omega_0(\vartheta)$ shows as before that even under the assumptions (7. 7) and (7. 8) we can not get a better error term than $0\left(\frac{n}{\log n}\right)$.

As was conjectured by DE BRUIJN, we can prove that the error term is $0\left(\frac{\log n}{n}\right)$, only if we assume that the sequence $f_n(z)$ of polynomials (7. 7) satisfies $\lim\limits_{n\to\infty}\max\limits_{|z|=\vartheta}|f_n(z)|=1$ for every positive $\vartheta<1$.

8. As mentioned in 4. we start with the following remark of I. SCHUR. Let $z=re^{i\varphi}$ be a fixed point on the complex-plane, and $\zeta=\varrho e^{i\gamma}$ move along the line arc $\zeta=\gamma$ (γ fixed). Then

$$|z-\zeta|^2=r^2+\varrho^2-2r\varrho\cos(\varphi-a)$$

$$\frac{|z-\zeta|^2}{|\zeta|}=\frac{r^2}{\varrho}+\varrho-2r\cos(\varphi-a).$$

If ϱ moves from $+\infty$ to $\varrho=r$ the expression on the right decreases monotonically; hence if $\zeta=\zeta_0=\varrho_0 e^{i\gamma_0}$, $\varrho_0>1$ and $|z|=r\leqq 1$ then

$$\frac{|z-\zeta_0|^2}{|\zeta_0|}\geqq|z-e^{i\gamma_0}|^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.1)$$

This is the remark we need.

Let

$$f(z)=a_0+\ldots+a_n z^n=a_n\prod_{\nu=1}^{n}(z-z_\nu)=a_n\prod_{\nu=1}^{n}(z-\varrho_\nu e^{i\varphi_\nu}) \quad (8.2)$$

be the polynomial of Theorem II; let

$$|z_\nu|=\varrho_\nu\geqq 1, \qquad \nu=1,2,\ldots,n \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.3)$$

and the ϑ of this Theorem be fixed. Let z be on the circumference of the circle $|z| = \vartheta$ and ζ_0 be any of the roots z_ν. Then the remark (8. 1) gives

$$\frac{|z-z_\nu|^2}{|z_\nu|} \geqq |z-e^{i\varphi_\nu}|^2, \qquad \nu = 1, 2, \ldots, n.$$

Multiplying all these inequalities we obtain on the whole $|z| = \vartheta$ the inequality

$$|\psi(z)|^2 \equiv \left| \prod_{\nu=1}^{n} (z-e^{i\varphi_\nu}) \right|^2 \leqq \frac{|f(z)|^2}{|a_0 a_n|} \quad . \quad . \quad . \quad . \quad (8.4)$$

i.e. a fortiori

$$M_\vartheta^* = \max_{|z|=\vartheta} |\psi(z)| \leqq \max_{|z|=\vartheta} \frac{|f(z)|}{\sqrt{|a_0 a_n|}} = \frac{M_\vartheta}{\sqrt{|a_0 a_n|}}. \quad . \quad . \quad (8.5)$$

The distribution of the roots of $\psi(z)$ in the different angles is identical with that of $f(z)$. Now assume that Theorem II is proved in the case when all the roots lie on the unit circle; then

$$\psi(z) = 1 + b_1 z + \ldots + b_n z^n, \qquad |b_n| = 1 \quad . \quad . \quad . \quad (8.6)$$

and with

$$\max_{|z|=\vartheta} |\psi(z)| = M_\vartheta^* = e^{\frac{n}{g^*(n,\vartheta)}}$$

we have for all $0 \leqq \alpha < \beta \leqq 2\pi$

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi} L - \frac{\beta-\alpha}{2\pi} n \right| < C \log \frac{4}{\vartheta} \cdot \frac{n}{\log g^*(n,\vartheta)}. \quad . \quad (8.7)$$

Then using (8. 5) we have

$$e^{\frac{n}{g^*(n,\vartheta)}} = M_\vartheta^* \leqq \frac{M_\vartheta}{\sqrt{|a_0 a_n|}} = e^{\frac{n}{g(n,\vartheta)}}, \text{ i. e. } \log g(n,\vartheta) \leqq \log g^*(n,\vartheta)$$

i.e. from (8. 7) a fortiori

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta-\alpha}{2\pi} n \right| < C \log \frac{4}{\vartheta} \cdot \frac{n}{\log g(n,\vartheta)}.$$

Hence Theorem II will indeed be entirely established once we prove it in the special case when all the roots lie on the unit circle.