Aerodynamics. — On the transmission of sound waves through a shock wave. By J. M. Burgers. (Mededeling No. 47 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

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1. Introduction. — The problem considered in the following pages is related to the one treated in a paper on the propagation of certain pressure disturbances in an ideal gas (1). As in that paper wave propagation in one dimension only is considered, while it is assumed that the gas carrying the waves obeys the ideal gas laws, internal friction, heat conductivity and heat radiation being neglected. It was shown in the paper mentioned that when two shock waves meet, or when a shock wave is met by an expansion wave, special frontiers will appear at which there is a change of the entropy of the gas. It will be the object of the present lines to show that a similar phenomenon occurs when a shock wave is met by sound waves.

In order to fix ideas it is convenient to take in view the case of a stationary shock wave, forming the boundary between two regions (comp. fig. 1), in one of which the gas moves with the supersonic velocity \( v_1 \) directed towards the shock wave, whereas in the other region the gas moves away from the shock wave with the subsonic velocity \( v_2 \). Indicating the pressure, density, temperature and the velocity of sound in the gas on one side of the shock wave resp. by \( p_1, \rho_1, T_1, c_1 \), and on the other side resp. by \( p_2, \rho_2, T_2, c_2 \), we have the following well known relations:

\[
\begin{align*}
\rho_1 v_1 &= \rho_2 v_2 \\
p_1 + \rho_1 v_1^2 &= p_2 + \rho_2 v_2^2 \\
k \frac{RT_1}{k-1} + \frac{v_1^2}{2} &= k \frac{RT_2}{k-1} + \frac{v_2^2}{2}
\end{align*}
\]

(1)

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Together with the formulae:

\[ \frac{p_1}{\varrho_1} = RT_1 = \frac{c_1^2}{k} \]
\[ \frac{p_2}{\varrho_2} = RT_2 = \frac{c_2^2}{k} \]  (2)

we have sufficient equations to calculate e.g. \( v_2 \), \( p_2 \), \( \varrho_2 \), \( T_2 \) and \( c_2 \) when \( v_1 \), \( p_1 \) and \( \varrho_1 \) have been given.

Next suppose that a sound wave is generated in the region 1 and is propagated relatively to the gas with the velocity \( c_1 \) towards the shock wave. Then a transmitted sound wave will appear in the region beyond the shock wave. It is impossible, however, that a reflected sound wave makes its appearance in region 1, as every reflected wave, although relatively to the gas it would travel away from the boundary with the velocity \( c_1 \), nevertheless is carried back into the boundary in consequence of the fact that the gas itself moves with a velocity exceeding \( c_1 \). Now at the boundary usually two conditions are observed, one referring to the continuity of mass transport, the other one referring to the pressures. In ordinary cases these two conditions suffice to determine the amplitudes of both the transmitted and the reflected waves. When no reflected waves are present, it seems to be impossible to satisfy both boundary conditions. This dilemma is solved by the circumstance that beyond the shock wave (i.e. in the region 2) not only a sound wave is propagated, but that at the same time in this region is produced a field of spatial inequalities in the entropy, which inequalities are transported by the moving gas with the velocity \( v_2 \) and which in a certain sense take the part of a second wave motion.

The physical cause of this fact is to be found in the circumstance that a shock wave introduces a change of the entropy of the gas which moves through the wave front. In the case of an undisturbed stationary shock wave this change of entropy is independent of the time. As soon, however, as sound waves are superimposed upon the shock wave, the change of entropy of the gas passing through it is no longer constant, and a periodic field of entropy changes makes its appearance.

Actually the situation is still more complicated. In reality there are three boundary conditions to be fulfilled, corresponding to equations (1) for the stationary shock wave, the third equation referring to the energy. Owing to the periodic changes of state of the gas (changes of velocity, of pressure and of density) produced on both sides of the shock wave, the position of the latter will not remain fixed, but will oscillate. The amplitude of the oscillations forms a third unknown in the problem, and it thus becomes possible to satisfy the three boundary conditions.

A slightly different problem is obtained when the original sound waves should have been produced in the region where the gas is moving with the subsonic velocity (i.e. in the region 2). In that case sound waves clearly cannot be transmitted into the region on the other side, as again every wave motion on that side inevitably will be carried into the boundary. Hence in this case we shall have a reflected sound wave only. At the same
time, however, there appears a field of periodic inequalities of entropy on the same side as where the reflected wave is formed, which field again is carried along by the moving gas with the velocity $v_2$. Also in this case the shock wave itself will oscillate. Again there are three unknowns, which can be determined in such a way that the three boundary conditions will be satisfied.

2. Equations for the sound waves and the entropy waves. — In deriving a set of equations which will enable us to describe the peculiar system of wave motion produced, we must keep in mind that in the domain where the gas moves with the supersonic velocity, the state of the gas will always be homogeneous, so that there pressure, density, temperature and entropy will have constant values. Hence in this region we have to do with ordinary sound waves, propagating themselves with the well known velocity of sound, deduced on the supposition that the changes of state of the gas are not only adiabatic, but at the same time isentropic.

In the region on the other side of the shock wave this latter supposition is untenable. In order to deduce an appropriate system of equations, we start from the equations of motion:

$$\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} &= -\rho \frac{\partial u}{\partial x}
\end{align*}$$

(3)

To these equations must be added the equation of energy, which expresses that energy is neither lost nor gained, and that the work which is performed by the pressure in changing the volume of an element, must find its equivalent in a change of the interior energy of the gas $U = RT/(k-1)$. This equation takes the form:

$$\frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} = p \left( \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right)$$

(4)

Together with the gas law:

$$\rho \dot{\theta} = RT$$

we have four equations, connecting the four variables $u, p, \rho, T$.

It is convenient to make a change of variables and to use as dependent variables the velocity of the gas $u$, the quantity $c$ defined as $\sqrt{kp/\rho}$, and the quantity $\theta = p/\rho^k$. The latter quantity is connected with the entropy $S$ per unit mass by means of the relation:

$$S = \frac{R}{k-1} \ln \theta + \text{constant}.$$
The temperature can be eliminated altogether, as \( T = c^2/kR \). The resulting equations obtain the form 2):

\[
\begin{align*}
\frac{\partial \Theta}{\partial t} + u \frac{\partial \Theta}{\partial x} &= 0 \\
\frac{\partial \Theta^2}{\partial t} + (u + c) \frac{\partial \Theta^2}{\partial x} \left( c + \frac{k-1}{2} u \right) &= c \left( \frac{\partial \Theta}{\partial t} + (u + c) \frac{\partial \Theta}{\partial x} \right) \ln \Theta.
\end{align*}
\]

In applying these equations to the case of sound waves we may consider the deviations of \( \Theta \), \( u \) and \( c \) from their normal constant values as small quantities of the first order. When all quantities of higher order are neglected, the factors \( u \) and \( c \) before a differential operator can be treated as constants. Writing \( \delta \Theta \), \( \delta c \), \( \delta u \) resp. for the deviations of \( \ln \Theta \), \( c \), \( u \) from their normal values, the equations reduce to:

\[
\begin{align*}
\frac{\partial \delta \Theta}{\partial t} + u \frac{\partial \delta \Theta}{\partial x} &= 0 \\
\frac{\partial \delta \Theta^2}{\partial t} + (u + c) \frac{\partial \delta \Theta^2}{\partial x} \left( c + \frac{k-1}{2} u \right) &= c \left( \frac{\partial \delta \Theta}{\partial t} + (u + c) \frac{\partial \delta \Theta}{\partial x} \right) \delta \Theta.
\end{align*}
\]

These equations are linear in the unknowns and can be used in describing the propagation of the combined system of sound and entropy waves of small amplitude.

3. Solution of equations (8)—(10) for the case of sound waves transmitted through a shock wave. — The form of the differential operators occurring in eqs. (8)—(10) induces us to look for solutions which are functions resp. of the following combinations of the independent variables:

\[
x - ut; 
(\quad x - (u + c)t; 
\quad x - (u - c)t.
\]

The first expression refers to phenomena which are simply carried along by the gas, moving with the velocity \( u \); the second and third expressions refer to phenomena which, apart from being carried along by the gas with the velocity \( u \), moreover show a propagation relatively to the gas with the velocity \( c \), either in the same direction as \( u \), or in the opposite direction.

In the case of the transmission of sound waves from the supersonic region into the subsonic region there clearly cannot appear waves moving with the velocity \( c \) against the motion of the gas (such waves might appear

2) These equations have been given on page 561 of the paper mentioned in footnote 1) (p. 273 above).
only in the case where obstructions would be present in the field at some distance downstream from the shock wave, which could produce a reflection travelling towards the shock wave). Hence we must discard the combination \( x-(u-c)t \), and restrict ourselves to expressions depending upon \( x-ut \) and \( x-(u+c)t \).

The following solution seems appropriate to this case:

\[
\delta \theta = \epsilon (x-ut) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)
\]

\[
\delta c + \frac{k-1}{2} \delta u = \frac{c}{2k} \epsilon (x-ut) + (k-1) \varphi (x-ut-ct) \quad \ldots \quad (12)
\]

\[
\delta c - \frac{k-1}{2} \delta u = \frac{c}{2k} \epsilon (x-ut) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)
\]

where \( \epsilon \) and \( \varphi \) provisionally are two arbitrary functions, afterwards to be fixed by means of the boundary conditions to be satisfied at the shock wave. From (12) and (13) we deduce:

\[
\delta c = \frac{c}{2k} \epsilon (x-ut) + \frac{k-1}{2} \varphi (x-ut-ct) \quad \ldots \quad (14)
\]

\[
\delta u = \varphi (x-ut-ct) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)
\]

The corresponding values of the deviations of the pressure, the density and the temperature are given by:

\[
\delta p = \rho c \varphi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)
\]

\[
\delta \rho = - \frac{\varrho}{k} \epsilon + \frac{\varrho}{c} \varphi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)
\]

\[
\delta T = \left( \frac{\epsilon}{k} + \frac{k-1}{c} \varphi \right) \quad \ldots \quad \ldots \quad (18)
\]

In the case of simple harmonic sound waves, the functions \( \epsilon \) and \( \varphi \) will become simple harmonic functions of their respective arguments, both having the same frequency with respect to the time as the original sound waves.

It will be seen from the expressions (15) and (16) that there will be a simple train of waves both in the velocity and in the pressure, propagated with the velocity \( c \) relatively to the moving gas. According to (11) there will also be a simple train of waves in the entropy field, carried along by the gas with the velocity \( u \) (and thus being stationary with respect to the gas). In the density and in the temperature on the other hand a double system of waves makes itself felt, which can give rise to a system of beats.

In the case of sound waves originating in the subsonic domain and originally travelling towards the shock wave, where a reflection takes place, the solution assumes the form:
\[ \delta \theta = \varepsilon (x - ut) \quad \ldots \ldots \ldots \ldots \ldots \quad (11a) \]
\[ \delta c + \frac{k-1}{2} \delta u = \frac{c}{2k} \varepsilon (x - ut) + (k-1) \varphi (x - ut - ct) \quad \ldots \ldots \ldots \ldots \ldots \quad (12a) \]
\[ \delta c - \frac{k-1}{2} \delta u = \frac{c}{2k} \varepsilon (x - ut) - (k-1) \varphi (x - ct + ct) \quad \ldots \ldots \ldots \ldots \ldots \quad (13a) \]

from which:
\[ \delta c = \frac{c}{2k} \varepsilon + \frac{k-1}{2} \varphi - \frac{k-1}{2} \Phi \quad \ldots \ldots \ldots \ldots \ldots \quad (14a) \]
\[ \delta u = \varphi + \Phi \quad \ldots \ldots \ldots \ldots \ldots \quad (15a) \]

and further:
\[ \delta p = \varphi c (\varphi - \Phi) \quad \ldots \ldots \ldots \ldots \ldots \quad (16a) \]
\[ \delta \varrho = \frac{\varrho}{k} \varepsilon + \frac{\varrho}{c} (\varphi - \Phi) \quad \ldots \ldots \ldots \ldots \ldots \quad (17a) \]
\[ \delta T = T \left\{ \frac{\varepsilon}{k} + \frac{k-1}{c} (\varphi - \Phi) \right\} \quad \ldots \ldots \ldots \ldots \ldots \quad (18a) \]

The function \( \Phi (x - ut + ct) \) introduced here describes the original sound waves, travelling relatively to the gas with the velocity \( c \) in the opposite direction of the flow velocity \( u \); hence \( \Phi \) is to be considered as a given function.

As equations (11)—(18a) all refer to the domain where the flow is subsonic, the flow velocity \( u \) introduced here corresponds to \( v_2 \) in fig. 1, while \( c \) corresponds to \( c_2 \); similarly for \( \varrho \) must be read \( \varrho_2 \) and for \( T : T_2 \).

It is convenient to add to these results the corresponding expressions for the sound waves which in the first case are present in the supersonic region (in the second case sound waves are absent from this region). They have the form:
\[ \delta \theta = 0 \quad \ldots \ldots \ldots \ldots \ldots \quad (11b) \]
\[ \delta c + \frac{k-1}{2} \delta u = (k-1) \cdot F (x - ut - ct) \quad \ldots \ldots \ldots \ldots \ldots \quad (12b) \]
\[ \delta c - \frac{k-1}{2} \delta u = 0 \quad \ldots \ldots \ldots \ldots \ldots \quad (13b) \]

from which:
\[ \delta c = \frac{k-1}{2} F \quad \ldots \ldots \ldots \ldots \ldots \quad (14b) \]
\[ \delta u = F \quad \ldots \ldots \ldots \ldots \ldots \quad (15b) \]
\[ \delta p = \varrho c F \quad \ldots \ldots \ldots \ldots \ldots \quad (16b) \]
\[ \delta \varrho = \frac{\varrho}{c} F \quad \ldots \ldots \ldots \ldots \ldots \quad (17b) \]
\[ \delta T = \frac{k-1}{c} TF \quad \ldots \ldots \ldots \ldots \ldots \quad (18b) \]
Here \( F \) represents a given function, while now \( u, c, \varrho, T \) resp. correspond to \( v_1, c_1, \varrho_1, T_1 \).

4. The motion of the shock wave front itself and the boundary conditions. — We write \( \xi \) for the velocity of the shock wave front, which velocity in the case of a shock wave disturbed by sound waves will be a small quantity of the first order. The conditions to be fulfilled at the shock wave can be deduced from eqs. (1) by subtracting the amount \( \xi \) from all velocities, simultaneously paying regard to the changes of the various quantities, described by \( \delta u, \delta \varrho \) etc. In this way the following set of equations is obtained 3):

\[
(q_1 + \delta q_1)(v_1 + \delta u_1 - \xi) = (q_2 + \delta q_2)(v_2 + \delta u_2 - \xi)
\]
\[
(p_1 + \delta p_1)(v_1 + \delta u_1 - \xi)^2 = (p_2 + \delta p_2)(v_2 + \delta u_2 - \xi)^2
\]
\[
\frac{kR}{k-1}(T_1 + \delta T_1) + \frac{(v_1 + \delta u_1 - \xi)^2}{2} = \frac{kR}{k-1}(T_2 + \delta T_2) + \frac{(v_2 + \delta u_2 - \xi)^2}{2}.
\]

3) The equations also can be deduced from equations (3) and (4) of the text above by making a change to coordinates \( t_1, x_1 \) in such a way that \( t_1 = t : x_1 = x - \psi(t) \), where \( \psi(t) \) provisionally is an arbitrary function of \( t \), which is independent of \( x \). We write: \( \psi'(t) = \xi \). The equations then assume the form (with a change of order):

\[
\frac{\partial \varrho}{\partial t_1} + (u - \xi) \frac{\partial \varrho}{\partial x_1} = -\varrho \frac{\partial u}{\partial x_1}
\]
\[
\frac{\partial u}{\partial t_1} + (u - \xi) \frac{\partial u}{\partial x_1} = -\frac{1}{\varrho} \frac{\partial p}{\partial x_1}
\]
\[
\frac{\partial U}{\partial t_1} + (u - \xi) \frac{\partial U}{\partial x_1} = \frac{p}{\varrho^2} \left( \frac{\partial \varrho}{\partial t_1} + (u - \xi) \frac{\partial \varrho}{\partial x_1} \right).
\]

We multiply the second and third one by \( \varepsilon \); moreover the third equation is transformed by introducing the expression for \( U \), eliminating the derivatives of \( \varrho \) bij means of the first one and adding to it the second one multiplied by \( \varepsilon(u - \xi) \). In this way we arrive at the following system:

\[
\frac{\partial \varrho}{\partial t_1} + (u - \xi) \frac{\partial \varrho}{\partial x_1} + \varepsilon \frac{\partial u}{\partial x_1} = 0. \quad (I)
\]
\[
\varepsilon \frac{\partial u}{\partial t_1} + \varepsilon(u - \xi) \frac{\partial u}{\partial x_1} + \frac{\partial p}{\partial x_1} = 0. \quad (II)
\]
\[
\frac{R \varrho}{k-1} \frac{\partial T}{\partial t_1} + \varepsilon(u - \xi) \frac{\partial u}{\partial t_1} + \frac{R}{k-1} \varepsilon(u - \xi) \frac{\partial T}{\partial x_1} + \frac{p}{\varrho} \frac{\partial u}{\partial x_1} + \varepsilon(u - \xi) \frac{\partial p}{\partial x_1} + \varepsilon(u - \xi)^2 \frac{\partial u}{\partial x_1} = 0 \quad (III)
\]
When we restrict to the case of sound waves originating in the region 1 (the supersonic region), and for $\delta u_1, \delta \varrho_1, \ldots \delta u_2, \ldots$ introduce the appropriate expressions obtained from eqs. (14b)—(18b) and from eqs. (14)—(18) respectively, neglecting quantities of higher order of magnitude than the first and striking out all quantities of zero order on the ground that they already satisfy eqs. (1), the following results are obtained:

$$\frac{\varrho_1 v_1}{c_1} F + \varrho_1 (F-\xi) = -\frac{\varrho_2 v_2}{k} \varepsilon + \frac{\varrho_2 v_2}{c_2} \varphi + \varrho_2 (\varphi-\xi) \quad . \quad (19)$$

$$\frac{\varrho_1 v_1}{c_1} F + \frac{\varrho_1 v_1^2}{c_1} F + 2 \varrho_1 v_1 (F-\xi) =$$

$$= \frac{\varrho_2 v_2}{c_2} \varphi - \frac{\varrho_2 v_2}{k} \varepsilon + \frac{\varrho_2 v_2}{c_2} \varphi + 2 \varrho_2 (\varphi-\xi) \quad . \quad (20)$$

$$\frac{kRT}{c_1} F + \varrho_1 (F-\xi) = \frac{RT}{k-1} \varepsilon + \frac{kRT}{c_2} \varphi + \varrho_2 (\varphi-\xi) \quad . \quad (21)$$

In these equations we must take the values of the functions $F, \varphi$ and $\varepsilon$ at the point $x = 0$; hence $F, \varphi$ and $\varepsilon$, like $\xi$, in these equations are functions of the time $t$ only. In the case of simple harmonic functions they all can be taken as having the same time factor, e.g. $\sin \nu (t - t_0)$; as this factor drops out, the equations in reality determine the ratio of the amplitudes of the waves.

By making use of the first line of eqs. (1) in order to eliminate $\varrho_1$ and

$$\varrho (u-\xi) = \text{constant}. \quad \ldots \quad (IV)$$

Introducing this result into the second equation we obtain upon integration:

$$\varrho (u-\xi)^2 + p = \text{constant}. \quad \ldots \quad (V)$$

The third one gives:

$$\varrho (u-\xi) \frac{RT}{k-1} + (u-\xi) p + \frac{\varrho (u-\xi)^3}{2} = \text{constant}.$$  

or, after division by $\varrho (u-\xi)$ and making use of $p/\varrho = RT$:

$$\frac{kRT}{k-1} + \frac{(u-\xi)^2}{2} = \text{constant}. \quad \ldots \quad (VI)$$
and of eqs. (2) in order to eliminate the temperatures, we find, after some reductions:

\[ F \left( \frac{c_1 + v_1}{v_1} \right) \left\{ c_1 (c_1 + v_1) + \left( k + \frac{v_1}{c_1} \right) v_1 (v_1 - v_2) \right\} = \left\{ \right\} \]

\[ = \varphi \frac{c_2 + v_2}{v_2} \left\{ c_2 (c_2 + v_2) + (k - 1) v_2 (v_1 - v_2) \right\} \]

\[ \epsilon \frac{v_2}{k} = - \frac{F (c_1 + v_1)^2}{c_1 v_1} + \varphi \frac{(c_2 + v_2)^2}{c_2 v_2} \ldots \ldots \ldots \]

\[ (v_1 - v_2) \xi = F \left( \frac{c_1 + v_1}{c_1} \right) \frac{(c_1 + v_1)(v_1 - v_2)}{c_2 v_2} - \varphi \frac{(c_2 + v_2) v_1}{v_2} \ldots \ldots \ldots \]

The first one of these formulae determines the ratio of the transmitted wave \( \varphi \) to the original wave \( F \). Having found this ratio the values of \( \epsilon \) and \( \xi \) can be obtained by means of the other two equations.

In the case of waves originating in the subsonic domain, which are reflected at the shock wave, a similar set of expressions is obtained. In this case the equations corresponding resp. with (19)—(21) are found by substituting zero for \( F \), and \( \varphi + \Phi \) for \( \varphi \) in terms referring to \( \partial u \), while \( \varphi - \Phi \) must be substituted for \( \varphi \) in terms referring to \( \partial p, \partial q, \delta T \). The final equations take the form:

\[ \Phi \left( c_2 - v_2 \right) \left\{ c_2 (c_2 - v_2) + (k - 1) v_2 (v_1 - v_2) \right\} = \left\{ \right\} \]

\[ = \varphi \left( c_2 + v_2 \right) \left\{ c_2 (c_2 + v_2) + (k - 1) v_2 (v_1 - v_2) \right\} \]

\[ \epsilon \frac{v_2}{k} = - \Phi \frac{(c_2 - v_2)^2}{c_2 v_2} + \varphi \frac{(c_2 + v_2)^2}{c_2 v_2} \ldots \ldots \ldots \]

\[ (v_1 - v_2) \xi = \Phi \frac{(c_2 - v_2) v_1}{v_2} - \varphi \frac{(c_2 + v_2) v_1}{v_2} \ldots \ldots \ldots \]

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4) In order to obtain this equation in the form given above use has been made of the relation:

\[ \frac{c_1^2}{v_1} + k v_1 = \frac{c_2^2}{v_2} + k v_2. \]

which follows from the second one of eqs. (1). The final result is not wholly symmetrical. It will be observed, however, that when \( c_1 = c_2, v_1 = v_2 \), the equation gives \( F = \varphi \), as the unsymmetrical terms then reduce to zero.