Physics.—The Determination of the Earth’s plasticity from the post-glacial uplift of Scandinavia: Isostatic adjustment. By F. A. Vening Meinesz.

(Communicated at the meeting of September 25, 1937.)

In “Physics”, August 1935, N. A. Haskell gives a new and noteworthy treatment of the problem to determine the Earth’s viscosity from the post-glacial uplift of Scandinavia, attributing this uplift, as it is generally done, to the readjustment of the isostatic equilibrium after the disappearance of the ice-load. He finds a value for the kinematic viscosity $\nu$ of the order of $3 \times 10^{21}$ c.g.s. units, which corresponds to about $10^{22}$ c.g.s. units for the coefficient of viscosity $\eta$, $\eta$ being equal to $\rho \times \nu$ and putting the density $\rho$ at 3.3. In 1934 the writer$^1$), by a rough deduction, derived a value for $\eta$ of $4 \times 10^{22}$, also from data about these phenomena. Another deduction from these data was made in 1934 by R. W. Van Bemmelen and H. P. Berlage$^2$), who find a value for $\eta$ of $1.3 \times 10^{20}$, but this figure is based on a special hypothesis about the structure and dynamics of the Earth’s crust and so it cannot be directly compared to the other two values.

The paper of Mr. Haskell induced the writer to take the subject up again to see where the difference between the results for $\eta$, although not great considering the uncertainties of the subject, find their origin. One point of difference strikes at once. Mr. Haskell uses the data about the uplift of Scandinavia after the disappearance of the ice-load, as they are given by F. Nansen in “The Earth’s Crust, its surface forms, and isostatic adjustment”, Oslo 1928, while the writer has used the present rate of uplift as it is given by levelling, in combination with the gravity figures; the negative anomalies indicate the amount of deviation from isostatic equilibrium and the rising is supposed to be exclusively caused by isostatic readjustment. It seems worth while to combine these three sources of evidence together and to see what results of it; this is the object of this paper.

By means of the following considerations it is possible to divide the

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problem in two parts which can be treated separately. We begin by assuming that everywhere the rate of readjustment of the isostatic equilibrium is proportional to the deviation from isostasy; this will in most cases be a sufficient approximation. Let \( \zeta \) be the downward deviation of the Earth's surface from the position of isostatic equilibrium and let \( \dot{\zeta} = w_0 \) be the velocity of the Earth's surface in the same direction. Our assumption then says

\[
w_0 = \dot{\zeta} = -K\zeta.
\]

which, integrated, gives

\[
\zeta = \zeta(0) e^{-Kt}.
\]

\( \zeta(0) \) = value of \( \zeta \) for \( t = 0 \).

The first part of our problem is to determine \( K \) from one of the formula's (1) or (2). From the value of \( K \) we can derive a quantity which is valuable for investigations about isostasy and correlated subjects, viz. the time \( T \) in which half of the isostatic anomaly disappears by readjustment. We find

\[
T = \frac{\log 2}{K \log e} = \frac{0.69315}{K}.
\]

\( T \) might be called the halving period for isostatic adjustment.

The second part of our problem is to derive data about the Earth's plasticity from the value of \( K \). We shall come back to this problem afterwards and the result of the investigation will show — as also Haskell's investigation shows — that \( K \) not only depends on the plasticity of the Earth but also on the horizontal dimensions of the phenomenon; it is, at least approximately, proportional to them. So \( T \) is, with the same approximation, inversely proportional to the horizontal dimensions.

For the determination of \( K \) we have first the data used by Haskell. From the curves about the uplift of Scandinavia given by Nansen he uses the following figures for the uplift \( \zeta \) and for the rate of uplift \( w_0 \) both of Angermanland, which is considered to be about the centre of the rising area. According to (1) this gives the values for \( K \) as noted in the third column

<table>
<thead>
<tr>
<th>Year B.C.</th>
<th>( \zeta ) (meters)</th>
<th>( w_0 ) (cm/year)</th>
<th>( K ) (sec(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>147</td>
<td>3.9</td>
<td>( 8.4 \times 10^{-12} )</td>
</tr>
<tr>
<td>4000</td>
<td>118</td>
<td>2.7</td>
<td>7.3</td>
</tr>
<tr>
<td>3000</td>
<td>94</td>
<td>2.2</td>
<td>7.4</td>
</tr>
<tr>
<td>2000</td>
<td>74</td>
<td>1.8</td>
<td>7.7</td>
</tr>
</tbody>
</table>

Mean value for \( K = 7.7 \times 10^{-12} \) . . . (4)
These data do well agree with each other but there is one objection to them. The figures for \( \zeta \) are based on the assumption that they represent the total uplift till equilibrium is reached, while the figures are given by Nansen as the total upheaval up to the present time. As it is generally considered probable that the equilibrium has not yet been attained, we have to add the amount which is still lacking.

We might perhaps deduce this amount from the above figures for the rate of uplift by making use of the assumption that this speed must be proportional to \( \zeta \) but the result would be questionable as neither this assumption nor the figures for the uplift are accurate enough for allowing such a deduction. A better base seems to be the isostatic anomaly which is still existing and which may be assumed to be caused by the fact that the equilibrium has not yet been reached.

The mean figure of this negative anomaly for the central part of the depressed area may be estimated at about — 25 milligal 1).

Putting the density \( \varrho \) of the plastic layer under the crust in which the phenomenon takes place at 3.3, we may compute the amount \( \zeta \) which the crust has to rise before the equilibrium is established, from the formula

\[
A_n = -2\pi k^2 \varrho \zeta \quad (k^2 = \text{NEWTON's constant})
\]

and we find

\[
\zeta = 180 \text{ meters.}
\]

Adopting this figure for the depression at the present time, we find for the depressions at the periods under consideration

<table>
<thead>
<tr>
<th>Year</th>
<th>( \zeta ) meters</th>
<th>( K ) ( 10^{12} \text{ sec}^{-1} )</th>
<th>( w_0 ) cm/year</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000 B.C.</td>
<td>327</td>
<td>2.74</td>
<td>2.84</td>
</tr>
<tr>
<td>4000</td>
<td>298</td>
<td>2.71</td>
<td>2.59</td>
</tr>
<tr>
<td>3000</td>
<td>274</td>
<td>2.72</td>
<td>2.38</td>
</tr>
<tr>
<td>2000</td>
<td>254</td>
<td>2.80</td>
<td>2.21</td>
</tr>
</tbody>
</table>

Mean value of \( K = 2.76 \times 10^{-12} \text{ sec}^{-1} \) . . . (5)

Combining each figure of \( \zeta \) with the value of 180 meters at the present time, we can compute \( K \) from formula (2) and we find the four values of the second column of the above table; the mean value is 2.76 \( \times \) \( 10^{-12} \) sec\(^{-1}\). These values are in good harmony with each other. Using this mean value for \( K \) we can by means of formula (1) compute the rates of uplift and we find the figures of the third column of the

1) See e.g. Map of "Kossmat", "Geologische Erläuterungen zur Frage der isostatischen Reduktionsmethoden", giving the anomalies after isostatic reduction by HEISKANEN according to the HAYFORD–BOWIE system.
table. These figures slightly deviate from the values given by Nansen, but the writer does not know whether the differences are inside the uncertainties of these figures; the first figure, for 5000 B.C., shows by far the greatest discrepancy.

There is still one piece of information which we have not yet taken into account, i.e. the rate of uplift in the present time as it is provided by the levelling of Scandinavia. For the central part of the depressed area this rate may be put at 1.1 cm/year. By means of the above value of $K$ we can deduce this rate also from the present value of $\zeta$ of 180 meters; by applying formula (1) we find 1.57 cm/year. This also seems to be in fair agreement.

There appears, however, to be a tendency for the rates of uplift in the first part of the total time-interval of 6900 years that is considered, to be greater than the computed values and those in the second part to be smaller. We can easily find possible causes for it. In the first place it may be caused by the depressed area gradually becoming smaller. As $K$ decreases with a decrease of the horizontal dimensions, the rate of uplift would decrease more than the formula for constant $K$ indicates.

In the second place the assumption may be wrong that there have been no other processes going on in this part of the Earth's crust besides the isostatic readjustment. Part of the vertical movements or part of the gravity anomalies might have been caused by other phenomena. Considering these possibilities it is indeed remarkable that the agreement is as good as it is.

Taking the mean of the two values for the present rate of uplift in the central area, we may adopt for this rate the value of 1.3 cm/year. Combining this with the value of $\zeta$ of 180 meters, formula (1) gives

$$K = 2.3 \times 10^{-12} \text{ sec}^{-1}. \ldots \ldots \ldots \ldots (6)$$

which we shall finally adopt.

This value is 0.3 times the value (4) derived from the figures taken by Haskell and so the resulting value for $\eta$, which is inversely proportional to it, will be 10/3 times as great. This gives

$$\eta = 3 \times 10^{22} \text{ c. g. s. units.} \ldots \ldots \ldots \ldots (7)$$

The main cause of the difference from Haskell's figure, which is about the discrepancy mentioned in the beginning of this paper, is caused by Haskell's assumption that equilibrium has already been attained, while the above value has been found by assuming that part of the depression is still unadjusted.

From this value of $K$ we deduce by means of formula (3)

$$T = 9600 \text{ years.}$$
So it appears that in Scandinavia the period in which half of the depression disappears is about 10000 years.

The second problem is to derive from \(K\) data about the plasticity of the Earth. For doing this we have to make assumptions about the properties of the Earth’s layers. The classic assumption is to suppose that the crust is not plastic and that it is floating on a plastic layer of constant density which is assumed to have the properties of a viscous fluid with a coefficient of viscosity in the same way as \textit{Newtonian} liquids show. We shall adhere to this assumption here, as it was also done by \textsc{Haskell} in his paper and in the same way as he did, we shall neglect the elastic forces working in the crust. We thus confine our problem to the movements of the plastic substratum.

We shall further make some simplifying assumptions for making the solution more easy. It cannot in this way compete with the merits of the elegant solution of \textsc{Haskell}, which is certainly better adapted to the conditions of the problem, but the solution will thus become relatively simple and so it may have its uses. It will make it, moreover, easy to introduce afterwards a complication of the basic assumptions by supposing the viscosity to decrease with greater depth and this may be a better approximation of the conditions in the Earth.

Our first assumption is that the problem is two-dimensional. Taking the \(X\) axis in horizontal direction and the \(Z\) axis vertically downwards, the equations for the substratum are

\[
\begin{align*}
\eta \nabla^2 u - \frac{\partial p}{\partial x} &= 0, & \sigma_x &= -p - 2 \eta \frac{\partial w}{\partial z} \\
\eta \nabla^2 w - \frac{\partial p}{\partial z} + \tau g &= 0, & \sigma_z &= -p + 2 \eta \frac{\partial w}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, & \tau_y &= \eta \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)
\end{align*}
\]

in which \(\sigma_x\) and \(\sigma_z\) = the normal stress components, \\
\(\tau_y\) = the shearing stress component, \\
\(p = -\frac{1}{\eta} (\sigma_x + \sigma_z)\) \\
\(u\) and \(w\) = the components of the speed, \\
\(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\).

Our second assumption is that \(u\) and \(w\) are functions of \(x\) multiplied by

\[e^{i\pi \frac{x}{L}}\]
or, in other words, that \( u \) and \( w \) are periodic in the direction of the \( X \) axis with a half wave-length \( L \). The equations now become

\[
-\eta \frac{\pi^2}{L^2} u + \eta \frac{\partial^2 u}{\partial z^2} - i \frac{\pi}{L} p = 0 \quad (9A)
\]

\[
\alpha_x = -p - 2\eta \frac{\partial w}{\partial z} \quad (9D)
\]

\[
-\eta \frac{\pi^2}{L^2} w + \eta \frac{\partial^2 w}{\partial z^2} - \frac{\partial p}{\partial z} + \varrho g = 0 \quad (9B)
\]

\[
\alpha_z = -p + 2\eta \frac{\partial w}{\partial z} \quad (9E)
\]

\[
u = i \frac{L}{\pi} \frac{\partial w}{\partial z} \quad . \quad . \quad . \quad . \quad (9C)
\]

\[
\tau_y = \eta \left( \frac{\partial u}{\partial z} + i \frac{\pi}{L} w \right) \quad (9F)
\]

Introduction (9 C) in (9 A) and eliminating \( p \) from (9 A) and (9 B) we get the equation for \( w \)

\[
\frac{\partial^4 w}{\partial z^4} - 2 \frac{\pi^2}{L^2} \frac{\partial^2 w}{\partial z^2} + \frac{\pi^4}{L^4} w - \pi^2 \frac{\varrho g}{\eta L^2} = 0 \quad . \quad . \quad . \quad (10)
\]

of which the general solution is the sum of four terms, powers of \( e \), the exponents being the roots of a biquadratic equation, yielding four roots that are two and two equal. From this solution the formula's for \( u \), \( p \), \( \sigma_x \), \( \sigma_z \) and \( \tau_y \) are easily derived by means of the formula's 9 C, D, E and F.

Introducing the conditions that \( u \) and \( w \) disappear for \( z = \infty \) and that \( \tau_y = 0 \) for \( z = 0 \), we find the solution in the shape

\[
\begin{align*}
u &= \frac{A}{2\eta} z e^{-\frac{\pi z}{L}} \cos \pi \frac{x}{L} \\
w &= -\frac{A}{2\eta} \left( z + \frac{L}{\pi} \right) e^{-\frac{\pi z}{L}} \sin \pi \frac{x}{L} \\
\sigma_z &= -\varrho g z + \pi \frac{A}{L} \left( z + \frac{L}{\pi} \right) e^{-\frac{\pi z}{L}} \sin \pi \frac{x}{L} \\
p &= \varrho g z - A e^{-\frac{\pi z}{L}} \sin \pi \frac{x}{L}
\end{align*}
\]

For \( z = 0 \) we have

\[
\begin{align*}
w_0 &= -\frac{AL}{2\pi \eta} \sin \pi \frac{x}{L} \\
\sigma_{x_0} &= A \sin \pi \frac{x}{L}
\end{align*}
\]

We may put \( \sigma_{x_0} \) equal to the deviation from the equilibrium surface \( \zeta \) multiplied with \( \varrho g \) and we thus have

\[
A = \varrho g \zeta_0 \quad . \quad . \quad . \quad . \quad . \quad (13)
\]

(\( \zeta_0 \) = maximum value of \( \zeta \), i.e. for \( x = \frac{1}{2} L \)).
The maximum value of \( w_0 \) (also for \( x = \frac{1}{2} L \)) becomes

\[
   w_0 = -\frac{\varrho g L}{2 \pi \eta} \zeta_0 . \quad \quad \quad \quad \quad \quad \quad (14)
\]

and so we see that the rate of uplift is proportional to \( \zeta \) as well as to \( L \). So the assumption made in the beginning of this paper comes true. For the ratio \( K \) we find

\[
   K = \frac{\varrho g L}{2 \pi \eta} . \quad \quad \quad \quad \quad \quad \quad (15)
\]

The deviation of the equilibrium surface is given by

\[
   \zeta = \zeta_0 \sin \pi \frac{x}{L} . \quad \quad \quad \quad \quad \quad \quad (16)
\]

It is represented by a sine-curve. For the part between \( x = 0 \) and \( x = L \) it is a fair representation of the Scandinavian conditions.

We can use (15) for the computation of \( \eta \). We shall put for \( K \) the value (6) which we have finally adopted and for \( L \) the value of 1400 km. This value is derived from the data about the former and the present uplift 1). It is also in good agreement with fig. 3 of HASKELL’s paper, where the depression is somewhat broader but circular while in our case it is assumed two-dimensional. For \( \varrho \) we introduce the density of the substratum i.e. 3.3. We thus get

\[
   \eta = 3 \times 10^{22} \text{ gram cm}^{-1} \text{ sec}^{-1} . \quad \quad \quad \quad \quad \quad \quad (7)
\]

which is in perfect agreement with the value (7) which we had derived directly from HASKELL’s solution. The resulting kinematic viscosity \( \nu = \eta / \varrho \) is

\[
   \nu = 9 \times 10^{21} \text{ cm}^2 \text{ sec}^{-1} .
\]

We may now introduce the assumption that \( \eta \) is not constant but varying with the depth \( z \). We find in this case for the general equations of motion for the three dimensional case

\[
   \begin{align*}
   \eta \nabla^2 u & - \frac{\partial p}{\partial x} + X + \frac{\partial \eta}{\partial z} \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial z} \frac{\partial w}{\partial x} = 0 \\
   \eta \nabla^2 v & - \frac{\partial p}{\partial y} + Y + \frac{\partial \eta}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial \eta}{\partial z} \frac{\partial w}{\partial y} = 0 \\
   \eta \nabla^2 w & - \frac{\partial p}{\partial z} + Z + 2 \frac{\partial \eta}{\partial z} \frac{\partial w}{\partial z} = 0 
   \end{align*}
\]

1) See e.g. maps on pages 90 and 112 of “Isostasie und Schweremessung”, by Dr. A. BORN, Berlin 1923.
in which
\[ p = -\frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \]

The equations for the stress-components are not altered:
\[\sigma_x = -p + 2\eta \frac{\partial u}{\partial x} \quad \tau_x = \eta \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \text{ etc.} \]

Applying this to the two-dimensional case and assuming the way \( \eta \) decreases with the depth \( z \) to be given by the formula
\[ \eta = \eta_0 e^{-0.693 \frac{z}{D}}, \ldots \ldots \ldots \ldots \ldots (18) \]

\((D = \text{depth at which } \eta = \frac{1}{2} \eta_0)\)

we can follow the same way as we have done for the case of constant \( \eta \).

The solution is more complicated, the four roots of the equation for the exponents of the \( e \) terms are all different now and the final formula’s are long but in principle there is no difficulty. We shall only give here the result for the relation between \( K, \eta_0 \) and \( D \) incase the ratio \( a = 0.22 \frac{L}{D} \) is smaller than 1.

\[ (1 - \frac{3}{2} a + a^2 - \frac{1}{2} a^3 \ldots) \eta_0 = \frac{\varrho g L}{2\pi K} \ldots \ldots \ldots \ldots (19) \]

Adopting the same values of \( \varrho, L \) and \( K \) as for the former case we find for a few values of the depth \( D \)

<table>
<thead>
<tr>
<th>( D )</th>
<th>( \eta_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \infty )</td>
<td>( 3 \times 10^{22} \text{ gram cm}^{-1} \text{ sec}^{-1} )</td>
</tr>
<tr>
<td>1500 km</td>
<td>( 4 \times 10^{22} )</td>
</tr>
<tr>
<td>1000 ,,</td>
<td>( 5 \times 10^{22} )</td>
</tr>
<tr>
<td>700 ,,</td>
<td>( 6 \times 10^{22} )</td>
</tr>
<tr>
<td>154 ,,</td>
<td>( 26 \times 10^{22} )</td>
</tr>
</tbody>
</table>

The last value \((a = 2)\) has been found by applying the general formula’s.

So we see that incase \( \eta \) sinks to half its value at 700 km depth, the value for \( \eta_0 \) directly below the crust is twice as much as for constant \( \eta \) and that for \( D \) about 150 km it rises to \( 8\frac{1}{2} \) times as much.

The quantities \( T \) and \( K \) still merit a further consideration. We found that the halving period \( T \) for isostatic adjustment is about 10000 years for Scandinavia and in the second place that, at least approximately, it
may in general be expected to be inversely proportional to the diameter of the phenomenon. This allows the more general statement, which only for the second property more or less depends on assumptions about the way in which the isostatic adjustment takes place, that, if $T$ is the halving period of adjustment in thousands of years and $L$ the diameter in thousands of kilometers, we may expect for the two-dimensional case (one dimension assumed infinite)

$$ TL = 14 $$

for the circular case

$$ TL = 20. $$

In the same way we may derive the following general statement from the present rate of uplift of Scandinavia of 1.3 cm per year and from the present depression there of 180 meters. If $w$ is the rate of uplift in cm per year, $\zeta$ the depression in kilometers and $L$ again the diameter in thousands of kilometers, we may expect for the two-dimensional case

$$ w = 5 \zeta L \text{ cm/year} $$

for the circular case

$$ w = 3\frac{1}{2} \zeta L \text{ cm/year}. $$

In the same way we find that, if $dA$ is the disappearance per year of the isostatic anomaly $A$ we may expect for the two-dimensional case

$$ dA = 5 \times 10^{-5} A L \text{ per year} $$

for the circular case

$$ dA = 3\frac{1}{2} \times 10^{-5} A L \text{ per year}. $$