**Mathematics.** — "*On a Function which Assumes any Value on a Non-Enumerable Set of Points in any Interval*". By Prof. J. WOLFF. (Communicated by Prof. R. WEITZENBÖCK).

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LEBESGUE has given an example of a function which assumes any value in any interval. This example is constructed by means of decimal developments. The points $x$ where the function assumes a given value $y$, form an *enumerable* set for any value of $y$ different from zero.

By the aid of a perfect set of points $P$ of which each pair of points has an irrational distance, (cf. These Proceedings **27** p. 95 and a communication of Prof. L. E. J. BROUWER in these Proceedings **27** p. 487) we shall construct a function which in any interval assumes any value on a *non-enumerable* set of points.

1. Any perfect set of points $P$ may be split into sets of points having the cardinal number $c$ which are the elements of a set of the cardinal number $c$. For the points of $P$ may be brought into (1,1) correspondence with the points of a plane $\pi$. To any straight line of a parallel pencil in $\pi$ there corresponds a sub-set of $P$ which has the cardinal number $c$. No pair of these sub-sets has any point in common.

2. Let $P$ be a linear perfect set of points of which any pair of points has an irrational distance. We call the rational numbers from minus infinite to plus infinite $r_1, r_2, \ldots$ Further we call $P_k$ the set which may be derived from $P$ by adding $r_k$ to every number of $P$. For $i \neq k$ $P_i$ and $P_k$ have no point in common. According to § 1 we split $P$ into sets of points $D(y)$ of which each has the cardinal number $c$ and which correspond one by one to the real numbers $y$. The set which is derived from $D(y)$ by adding $r_k$ to every number of $D(y)$, we call $D_k(y)$.

3. Now we define a function $f(x)$ in the following way:

$$f(x) = 0, \text{ if } x \text{ does not lie in any } P_k, \quad k = 1, 2, 3, \ldots$$
$$f(x) = y, \text{ if } x \text{ lies in } D_k(y), \quad k = 1, 2, 3, \ldots$$

Let $y$ be an arbitrary real number and let $I$ be an arbitrary interval. $D(y)$ contains a point of condensation $\xi(y)$, i.e.: in any interval containing $\xi(y)$, $D(y)$ is non-enumerable. $I$ contains a point $\xi_k(y)$ which is derived from $\xi(y)$ through addition of a rational number $r_k$, hence $D_k(y)$ is non-enumerable in $I$. And as $f(x) = y$ in $D_k(y)$, $f(x)$ assumes the value $y$ in a non-enumerable sub-set of $I$.

Hence in any interval our function assumes any value on a non-enumerable set of points.

4. We must remark that $f(x)$ has any rational number as period; and also that $f(x)$ only differs from zero on a set of the measure zero, to wit on the set which we find by combining $P_1, P_2, \ldots$, after excluding from each of these perfect sets of the measure zero the points where $f(x) = 0$, hence $D_k(0)$. Accordingly $\int_{-\infty}^{\infty} f(x)\,dx = 0$ in the sense of LEBESGUE: $f(x)$ is equivalent to the function which is identically zero. All $P_k$ are nowhere dense, hence the set of the points $x$ where $f(x) \neq 0$ belongs to the first category of BAIRE.

Let us finally remark that all this almost literally holds good if for the $x$-set and for the $y$-set we substitute spaces of any number of dimensions. Accordingly it is for instance possible to define three functions $u$, $v$ and $w$ of $x$ so that in the corresponding representation of the straight line $-\infty < x < \infty$ on the space $(u, v, w)$ the image of each $x$-interval covers the whole space a non-enumerable number of times.

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