Mathematics. — *"The Complete System of Invariants of Two Covariant Antisymmetrical Tensors of the Second Order and an arbitrary Number of Vectors"*. By G. F. C. GRISS. (Communicated by Prof. R. WEITZENBÖCK).

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1. We start from a system of 2 antisymmetrical tensors p_{ik} and q_{ik} and r covariant vectors ${}_1a_i, \ldots, {}_ra_i\,(i, k = 1, \ldots, n)$ and we shall determine its complete system of invariants by the aid of the symbolical method [1].

We write symbolically

$$p_{ik} = p_i\, p_k = -p_k\, p_i \qquad (i, k = 1, \ldots, n),$$

where $p_1 \ldots p_n$ are complex symbols [1], and we introduce aequivalent complex symbols

$$(p_i)_1 = (p_i)_2 = (p_i)_3 = \ldots .$$

The same with q_{ik}.

According to the first principal theorem of the symbolical method any whole rational invariant of the given tensors is a sum of products of bracket-factors of the following form:

$$(p_1^2 \ldots p_\mu^2\, p_{\mu+1} \ldots p_\nu\; q_1^2 \ldots q_\lambda^2\, q_{\lambda+1} \ldots q_\varkappa\; {}_1a \ldots {}_\sigma a) \quad \mu+\nu+\lambda+\varkappa+\sigma = n. \quad (1)$$

For $\mu = \nu$ and $\lambda = \varkappa$ the bracket-factor is itself an invariant and we shall prove the

THEOREM: *The (smallest) complete system of invariants of the antisymmetrical tensors p_{ik} and q_{ik} and the vectors ${}_1a_i, \ldots, {}_ra_i$ are formed by the following brackets*

$$(p_1^2 \ldots p_\mu^2\, q_1^2 \ldots q_\lambda^2\; {}_1a \ldots {}_\rho a) \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

If the number of vectors is $\geqq n$, there are of course invariants without p and q, e.g. $({}_1a \ldots {}_na)$. If n is even there are invariants without a's, to wit

$$(p_1^2 \ldots p_\mu^2\, q_1^2 \ldots q_\lambda^2).$$

2. PROOF. We consider an invariant I consisting of a product of brackets and containing in particular (1) where $\mu \neq \nu$ and $\lambda \neq \varkappa$. Find the other bracket containing the symbol $p_{\mu+1}$ and transform by the aid of an identity of the symbolical method so that this 2nd symbol gets into (1). We find:

[1]) Cf. R. WEITZENBÖCK. Invariantentheorie. Groningen (1923). Especially Abschnitt III.

$$\left.\begin{aligned}
&(2\mu+2)(p_1^2\dots p_\mu^2 p_{\mu+1}\dots p_\nu\; q_1^2\dots q_\lambda^2 q_{\lambda+1}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)(p_{\mu+1}\dots)=\\
&=(-1)^{\nu-\mu}2\lambda(p_1^2\dots p_{\mu+1}^2 p_{\mu+2}\dots p_\nu q_1^2\dots q_{\lambda-1}^2 q_\lambda\; q_{\lambda+1}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)(q_\lambda\dots)+\\
&+\sum_\pi(-1)^\pi(p_1^2\dots p_{\mu+1}^2 p_{\mu+2}\dots p_{\mu+\pi-1}p_{\mu+\pi+1}\dots p_\nu\; q_1^2\dots q_\lambda^2 q_{\lambda+1}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)(p_{\mu+\pi}\dots)+\\
&+\sum_\tau(-1)^{\nu-\mu+\tau}(p_1^2\dots p_{\mu+1}^2 p_{\mu+2}\dots p_\nu\; q_1^2\dots q_\lambda^2 q_{\lambda+1}\dots q_{\lambda+\tau-1}q_{\lambda+\tau+1}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)(q_{\lambda+\tau}\dots)+\\
&+\sum_\rho(-1)^{\nu-\mu+\varkappa-\lambda+\rho}(p_1^2\dots p_{\mu+1}^2 p_{\mu+2}\dots p_\nu q_1^2\dots q_\lambda^2 q_{\lambda+1}\dots q_\varkappa\; {}_1a\dots{}_{\rho-1}a\; {}_{\rho+1}a\dots{}_\sigma a)({}_\rho a\dots).
\end{aligned}\right\}\quad(3)$$

We have written I as a sum of invariants where (1) is replaced by a bracket-factor which contains the square of one more symbol p. By continuing this we split I into invariants each of which contains a bracket-factor in which all the symbols p are quadratic. Finally we can obtain that the symbols p are quadratic in all the brackets. In this case all the brackets have the form:

$$(p_1^2\dots p_\mu^2\; q_1^2\dots q_\lambda^2 q_{\lambda+1}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)\quad\dots\dots\quad(4)$$

We apply a similar transformation as (3):

$$\left.\begin{aligned}
&(2\lambda+2)(p_1^2\dots p_\mu^2\; q_1^2\dots q_\lambda^2\; q_{\lambda+1}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)(q_{\lambda+1}\, p^2\dots q^2\dots q_\alpha\dots a\dots)=\\
&=-2\mu(p_1 p_2^2\dots p_\mu^2 q_1^2\dots q_\lambda^2 q_{\lambda+1}^2 q_{\lambda+2}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)(p_1\, p^2\dots q^2\dots q_\alpha\dots a\dots)+
\end{aligned}\right\}\quad(5)$$

+ terms which contain one symbol q more quadratically and which consist again of brackets of the type (4). We transform the first term once more by conveying p_1 out of the first bracket-factor into the second. In this way we split our invariant I' into invariants with brackets of the type (4) which contain either a symbol more quadratically or a bracket-factor that contains a symbol q more and a symbol p less quadratically than (4). If we continue this we get, apart from the invariants which contain a symbol more quadratically, invariants of which the 1st bracket-factor has the form

$$(q_1^2\dots q_\lambda^2\; q_{\lambda+1}\dots q_\varkappa\; {}_1a\dots{}_\sigma a)\quad\dots\dots\quad(6)$$

The last of these transformations leads to exclusively invariants that contain one symbol more quadratically, and of which the brackets have kept the type (4).

If we continue this long enough we obtain a whole rational function of brackets in which all the symbols are quadratic.

The form of the invariants shows directly that the complete system of invariants (2) cannot be replaced by a smaller one.

3. We shall now treat the case that, besides ${}_1a_i,\dots,{}_ra_i$, p_{ik} and q_{ik} an arbitrary number of contravariant vectors ${}_1u^i\dots{}_su^i$ are given. In this case we have, besides (1), symbolical factors of the following types:

$$({}_{i_1}u' \ldots {}_{i_n}u'), \qquad (7)$$

$$({}_l a\ {}_h u'), \qquad (8)$$

$$(p\ {}_h u') \qquad (9)$$

and

$$(q\ {}_h u'), \qquad (10)$$

where $i_1 \ldots i_n$ are n numbers out of $1, \ldots, s$ and $l = 1, \ldots, r$, $h = 1, \ldots, s$.

(7) and (8) are invariants; with (9) and (10) we form directly the invariants

$$(p\ {}_h u')(p\ {}_k u') \qquad (11)$$

and

$$(q\ {}_h u')(q\ {}_k u') \qquad (12)$$

where

$$h, k = 1, \ldots, s, \quad h \neq k.$$

If now we choose again an invariant which is a product of symbolical factors, we may at once disregard factors of the types (7) and (8) and we need only consider the types (1), (9) and (10). Now the factors of type (1) can contain a symbol p of which the aequivalent symbol is contained in $(p\ {}_h u')$. Disregarding such symbols p and q, which we shall indicate resp. by π and ϱ, we can, according to the proof given above, reduce all the brackets of the type (1) to the more simple type

$$(p_1^2 \ldots p_\mu^2\ q_1^2 \ldots q_\nu^2\ \pi_1 \ldots \pi_\lambda\ \varrho_1 \ldots \varrho_\varkappa\ {}_1 a \ldots {}_\sigma a) \qquad (13)$$

In this case, however, the invariant that contains this factor, may be split into the product of 2 invariants of which one has the type

$$\left.\begin{array}{l}(p_1^2 \ldots p_\mu^2\ q_1^2 \ldots q_\nu^2\ \pi_1 \ldots \pi_\lambda\ \varrho_1 \ldots \varrho_\varkappa\ {}_1 a \ldots {}_\sigma a)(\pi_1\ {}_1 u') \ldots \\ \qquad \ldots (\pi_\lambda\ {}_\lambda u')(\varrho_1\ {}_{\lambda+1} u') \ldots (\varrho_\varkappa\ {}_{\lambda+\varkappa} u').\end{array}\right\} \qquad (14)$$

We can continue in an analogous way with the other factor. There remains, therefore, only the invariant (14) to investigate. We transform all the $\varrho_\varkappa$ in the 1st factor. This fails only when at a given moment that first factor does not contain any π or p; in this case we have

$$(q_1^2 \ldots q_\nu^2\ \varrho_1 \ldots \varrho_\lambda\ {}_1 a \ldots {}_\sigma a)(\varrho_1\ {}_1 u') \ldots (\varrho_\lambda\ {}_\lambda u') \qquad (15)$$

multiplied by invariants of the type (11). (15), however, gives invariants of the types (2) and (12).

Consequently there remains:

$$(p_1^2 \ldots p_\mu^2\ q_1^2 \ldots q_\nu^2\ \pi_1 \ldots \pi_\lambda\ {}_1 a \ldots {}_\sigma a)(\pi_1\ {}_1 u') \ldots (\pi_\lambda\ {}_\lambda u') \qquad (16)$$

We remark that $\lambda \leqq n$ and we may disregard those invariants (16) for which $2\nu \leqq \lambda$; in this case we transform all the q's out of the former factor into the latter.

THEOREM: *The smallest complete system of invariants of the antisymmetrical tensors p_{ik} and q_{ik} and an arbitrary number of covariant and contravariant vectors is*

$$({}_l a\ {}_k u'),\quad ({}_{i_1}u' \dots {}_{i_n}u'),\quad (p\ {}_h u')(p\ {}_k u'),\quad (q\ {}_h u')(q\ {}_k u'),$$

$$(p_1^2 \dots p_\mu^2\ q_1^2 \dots q_\lambda^2\ {}_1 a \dots {}_\sigma a),$$

$$(p_1^2 \dots p_\mu^2\ q_1^2 \dots q_\nu^2\ \pi_1 \dots \pi_\lambda\ {}_1 a \dots {}_\rho a)(\pi_1\ {}_{i_1}u') \dots (\pi_\lambda\ {}_{i_\lambda}u') \quad (2\nu > \lambda > 0;\ i_\alpha \neq i_\beta).$$

It remains still to show that this system is the smallest. For this we need only prove that an invariant of the latter type cannot be reduced to the other invariants by the aid of the symbolical identities. We can only transform one or another π in the 1st factor, through which, disregarding invariants with one square more, there arises an invariant with a factor $(\varrho u')$. We can only continue this with another π until all the symbols π are conveyed to the 1st factor. There remains, however, an invariant which does not belong to the indicated system, because, according to $2\nu > \lambda$, the first factor contains at least one symbol q.

4. We shall discuss a few special cases.

1. THEOREM: *The (smallest) complete system of invariants of an antisymmetrical tensor p_{ik} and an arbitrary number of vectors is*

$$({}_l a\ {}_k u') \qquad ({}_{i_1}u' \dots {}_{i_n}u') \qquad (p\ {}_h u')(p\ {}_k u')$$

$$(p_1^2 \dots p_\mu^2\ {}_1 a \dots {}_\sigma a).$$

2. *If 2 covariant vectors a_i and b_i and no contravariant ones are given, we have for $n = 2k$ the complete system*

$$\left.\begin{array}{ll} (p_1^2 \dots p_i^2\ q_1^2 \dots q_{k-i}^2) & (i = 0, \dots, k) \\ (ab\,p_1^2 \dots p_i^2 q_1^2 \dots q_{k-i-1}^2) & (i = 0, \dots, k-1); \end{array}\right\} \quad . \quad . \quad . \quad (17)$$

for $n = 2k + 1$

$$\left.\begin{array}{ll} (a p_1^2 \dots p_i^2\ q_1^2 \dots q_{k-i}^2) & (i = 0, \dots, k) \\ (b p_1^2 \dots p_i^2\ q_1^2 \dots q_{k-i}^2) & (i = 0, \dots, k). \end{array}\right\} \quad . \quad . \quad . \quad (18)$$

In both cases we have $n+1$ relative, hence n absolute invariants $I_1, \dots, I_n$. Accordingly all algebraic absolute invariants may be expressed in these n absolute invariants. As on the other hand, the given tensors and vectors have $2 \times \frac{1}{2} n(n-1) + 2n = n^2 + n$ independent components, we must get at least n independent absolute invariants through elimination of the coefficients of the transformation. Hence $I_1, \dots, I_n$ are n independent absolute invariants.

3. *If 3 covariant vectors a_i, b_i and x_i are given, we have for $n = 2k$ a complete system, which, besides of* (17), *consists of*

$$\left.\begin{array}{ll}(xap_1^2\dots p_i^2\, q_1^2\dots q_{k-i-1}^2) & (i=0,\dots,k-1)\\ (xbp_1^2\dots p_i^2\, q_1^2\dots q_{k-i-1}^2) & (i=0,\dots,k-1).\end{array}\right\}\quad\dots\quad(19)$$

For $n=2k+1$ we get besides (18)

$$\left.\begin{array}{ll}(xp_1^2\dots p_i^2\, q_1^2\dots q_{k-i}^2) & (i=0,\dots,k)\\ (xabp_1^2\dots p_i^2\, q_1^2\dots q_{k-i-1}^2) & (i=0,\dots,k-1).\end{array}\right\}\quad\dots\quad(20)$$

In the same way as above we find in both cases $2n$ independent absolute invariants. We find in particular n independent absolute invariants $(x\,{}_h\alpha')$, which we can also consider as covariants of the system a_i, b_i, p_{ik} and q_{ik}. Consequently the n contravariant vectors ${}_h\alpha^i$ are linearly independent.

We can also prove the latter by direct calculation of the determinant $D=|\,{}_h\alpha^i\,|$ on the supposition that for $n=2k$

$$I=(abq_1^2\dots q_{k-1}^2)\neq 0 \text{ and } I'=(p_1^2\dots p_k^2)\neq 0,$$

whereas all the other invariants op (17) are zero; in this case we find

$$D=c\,I^k\,I'^{k-1},\quad\text{where}\quad c\neq 0.$$

For $n=2k+1$ we calculate D in the same way by assuming

$$I=(a\,p_1^2\dots p_k^2)\neq 0\quad\text{en}\quad I'=(b\,q_1^2\dots q_k^2)\neq 0,$$

whereas all the other invariants of (18) are zero; in this case

$$D=c'\,I^k\,I'^k,\quad\text{where}\quad c'\neq 0.$$

4. *For 2 covariant vectors a_i and b_i and one contravariant vector u we find, besides* (17) *and* (18), *for $n=2k$ and $n=2k+1$ resp.*

$$\left.\begin{array}{lll}(a\,u') & (p\,u')\,(a\,p\,p_1^2\dots p_i^2\, q_1^2\dots q_{k-i-1}^2) & (i=0,\dots,k-2)\\ (b\,u') & (p\,u')\,(b\,p\,p_1^2\dots p_i^2\, q_1^2\dots q_{k-i-1}^2) & (i=0,\dots k-2)\end{array}\right\}\quad\dots\quad(21)$$

and

$$\left.\begin{array}{lll}(a\,u') & (p\,u')\,(p\,p_1^2\dots p_i^2\, q_1^2\dots q_{k-1}^2) & (i=0\dots,k-1)\\ (b\,u') & (p\,u')\,(a\,b\,p\,p_1^2\dots p_i^2\, q_1^2\dots q_{k-i-1}^2) & (i=0,\dots k-2)\end{array}\right\}\quad\dots\quad(22)$$

We find again $2n$ independent absolute invariants among which in particular n independent absolute invariants of the form $({}_h\beta\,u')$, which may be considered as contravariants of the system a_i , b_i , p_{ik} and q_{ik}.

Accordingly the covariant vectors ${}_h\beta_i$, defined in this way, are linearly independent. The minors of the determinant of these covariant vectors divided by the determinant give n linear independent contravariant vectors that depend linearly on the contravariant vectors ${}_h\alpha^i$ mentioned under 3.

5. Finally for $n = 4$ [1]) we have the

THEOREM: *The (smallest) complete system of invariants of 2 antisymmetrical tensors p_{ik} and q_{ik} and an arbitrary number of vectors is formed by*

$$({}_l a \; {}_h u') \qquad ({}_{i_1}u' \; {}_{i_2}u' \; {}_{i_3}u' \; {}_{i_4}u') \qquad (p_1^2 \; p_2^2) \qquad (p^2 \; q^2) \qquad (q_1^2 \; q_2^2)$$

$$(p^2 \; {}_l a \; {}_m a) \qquad (q^2 \; {}_l a \; {}_m a) \qquad (p \; {}_h u')\,(p \; {}_k u') \qquad (q \; {}_h u')\,(q \; {}_k u')$$

$$(q^2 \; p \; {}_l a)\,(p \; {}_h u').$$

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[1]) Cf. F. MERTENS. Invariante Gebilde von Nullsystemen. Wiener Berichte. Band XCVII (1888).