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CONTENTS

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CONTENTS

Pharmacology. — *On the Possibility of Testing Analeptics with the Aid of Fish.* H. A. DIJKERMAN, T. NIJZINK and P. E. VERKADE.

(Communicated at the meeting of April 24, 1948.)

During the last few years extensive investigations have been carried out at our laboratory as to the reactions of the goldfish, *Carassius auratus L.* (?), to a number of (local) anesthetics; only a small part of this work has as yet been published [1]). Starting from work of ADAMS *c.s.* [2]), we were able to show that the goldfish is eminently suitable as test animal for comparing the activity of at least certain types of (local) anesthetics. In our respective publications we emphasized the fact — which should be stipulated here once more — that in this test we undoubtedly are concerned with an action of the substances in question on the central nervous system.

During these investigations the question soon occurred to us whether with goldfish, with fish in general, a stimulating effect of substances like cardiazole, benzedrine and the like on the central nervous system, thus an antagonism between said substances and the anesthetics, exists and can be readily ascertained; at the time, as far as we are aware, such an antagonism was only known to exist with man and with a few warm-blooded animals. If so, attempts might be made, while availing ourselves of the results of our above-mentioned investigations, to work out a method for comparing the activity of analeptics using the goldfish or possibly some other kind of fish as test animal. A reliable and simple test for analeptics was, and in our opinion still is, needed.

In the literature we soon found KOCH's paper [3]), then of recent date, on the reaction of the stickleback (*Gasterosteus aculeatus L.*) to the combinations of ethyl urethan — cardiazole and ethyl urethan — lobeline. The 2–3 cm long and about one-year-old test animals were anesthesized by a 30 minutes' stay in a 0.6 % solution of ethyl urethan and subsequently transferred to water or to a highly dilute solution of the analeptic to be examined; KOCH then ascertained the time elapsing until the beginnings of awaking, for which fluttering of the lateral fins served as criterion. The concentrations examined were 5, 25 and 50 mg per litre in the case of cardiazole and 2.5, 5, 10 and 25 mg per litre in the case of lobeline. All the experiments were carried out at 18° C and the test

[1]) C. P. VAN DIJK, thesis Delft, 1946; P. H. WITJENS, thesis Delft, 1946; preliminary communication: P. E. VERKADE, C. P. VAN DIJK and P. H. WITJENS, Verslag Kon. Ned. Akad. v. Wetensch., Amsterdam, **55**, 5 (1946); P. H. WITJENS, C. P. VAN DIJK and P. E. VERKADE, Arch. intern. pharmacodynamie **74**, 178 (1947).

[2]) R. ADAMS, E. K. RIDEAL, W. B. BURNETT, R. L. JENKINS and E. E. DREGER, J. Am. Chem. Soc. **48**, 1758 (1926).

[3]) F. E. KOCH, Z. ges. exptl. Med. **108**, 695 (1941).

animals were previously kept at the same temperature for several days. The number of test animals was 100 in the control test and 50 in each of the tests with one of the analeptics. The numerical data thus obtained were elaborated statistically in the manner indicated by BURN [4]). Leaving aside the experiments with the lowest concentration of each of the two analeptics, the rather slight decreases observed in the average *duration of the anesthesia* if the fishes were introduced into a solution of an analeptic instead of into water are said to have been significant. KOCH thinks himself justified to the conclusion that his method is in principle suitable for ascertaining the "Weckwirkung" of central nervous system stimulants; his paper even contains a comparison of the activity of cardiazole, lobeline and an extract of *Lobelia inflata L.* on the basis of the rather unimpressive data obtained during the work in question.

During our investigations referred to above [1]) we found that the duration of the anesthesia of goldfish treated with a given solution of the anesthetics employed — 1-*n*-propoxy-2-amino-4-nitrobenzene and ethyl 4-aminobenzoate (anesthesine) — may fluctuate very considerably; for the latter anesthetic this had already been stated by ADAMS *c.s.* [2]). In our opinion one of the causes and perhaps even the main cause, of this phenomenon is the fact that the gill respiration, too, is affected by these anesthetics, such in degrees greatly varying from individual to individual; differences in the intensity of the gill respiration are bound to produce differences in the rate at which the anesthetic is removed from the organism.

It is obvious that the immediately preceding remarks give rise to serious objections to the test proposed by KOCH. This investigator also pointed out the pronounced scattering in the duration of the anesthesia of the sticklebacks used by him as test animals. This necessitated the use of large numbers of test animals, which, even though the test is simple, is to be considered a drawback. KOCH's paper does not contain information about the intensity of the gill respiration after a 30 minutes' stay of the fishes in the 0.6 % solution of ethyl urethan. It is, of course, conceivable that in this respect the reaction of the stickleback is quantitatively different from that of the goldfish, and also that the same applies to the effect of ethyl urethan as compared with that of the above-mentioned anesthetics employed by us. At any rate, for the test in question only such combinations of fish and anesthetic are suitable in which the gill respiration is not or only slightly affected by the anesthetic; indeed, differences in the intensity of the gill respiration are bound to produce differences in the rate of resorption of the analeptic by the fish.

On the other hand, it appears from the experience gained at our laboratory [1]) and by ADAMS *c.s.* [2]) that the *anesthesizing time, i.e.* the time elapsing between the moment the goldfish is introduced into a solution

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[4]) J. H. BURN, Biologische Bewertungsmethoden (Berlin, Jul. Springer, 1937), p. 22.

of an anesthetic and the moment it ceases to react to the strongest permissible pressure on the caudal fin and the dorsal fin, can be reproduced quite well within reasonable limits. In our opinion it was, therefore, very attractive to investigate whether the anesthesizing time can be affected by the type of analeptics in question. This can be done in two ways:

1°. By comparing the anesthesizing time pertaining to a chosen solution of an anesthetic and determined in the usual manner, with those anesthesizing times occurring in solutions which contain, besides the anesthetic in the same concentration, the analeptic to be examined.

2°. By comparing the former anesthesizing time with those anesthesizing times occurring in the case of fishes which have previously been treated during a given time with solutions of the analeptic to be examined.

We started by investigating whether any useful results could be obtained with the aid of the technique mentioned *sub* 1, this being of course experimentally the simplest one. Meanwhile, from the very beginning we were by no means blind to the fact that this technique presents the indisputable and serious drawback of the simultaneous use of two drugs. As test animals we used goldfishes and as anesthetics 1-*n*-propoxy-2-amino-4-nitrobenzene or ethyl 4-aminobenzoate (anesthesine), because a good deal of experience had been gained with these combinations of fish

TABLE 1. Anesthesizing times in minutes.

No.	1-*n*.propoxy-2-amino-4-nitrobenzene 7 mg/l									
	no analeptic	cardiazole mg/l					benzedrine-HCl mg/l			
		20	100	500	1000	0	5	20	100	200
1	10.5	13	12	12	10	14	9.5	12	9	12.5
2	9	14.5	15.5	14	11	13	8.5	11.5	8.5	13.5
3	9.5	13	13.5	15	12	13.5	10	8.5	8	9
4	11	16.5	13.5	10	9.5	20.5	12	8	10	10
5	10	14	12	13.5	10.5	11.5	9	9	8	7.5
6	13.5	12.5	9	10.5	12.5	14.5	13.5	11	7.5	10
7	9.5	12	13	11	8	14.5	11.5	15.5	9	11
8	9	12.5	12	16	7.5	24	14.5	11	8.5	11
9	11.5	10	16	13	8	13	10.5	9	9	11
10	18.5	16.5	11	15	12.5	18	11	9	9	10
11	15	11.5	13	11.5	8	11.5	11	10	8	8
12	26.5	10	14	10	10	10	8	8.5	10.5	12
13	13	10	12	11.5	10	15	14	8	9	9.5
14	12.5	10.5	14.5	12.5	9.5	11	10.5	14	10	10
15	11	13	12	13.5	10.5	13.5	9.5	9.5	9	8
16	15	10.5	11.5	9	7	10.5	8	8.5	7.5	8
17	19	11	14	10	11	10	9	9.5	9	7.5
18	14	10	14	10	11.5	9.5	8	7.5	10.5	8
19	10.5	14	13	13	14	12	13.5	8.5	8.5	8.5
20	13.5	10	17.5	10	9.5	12	7.5	10.5	7	6.5
Aver.:	13.1	12.3	13.2	12.1	10.1	13.6	10.2	10.0	8.8	9.6

and anesthetic in the course of the work already carried out at our laboratory and in particular also because the substances mentioned are already active in very low concentrations; we consider this to be an advantage especially over the ethyl urethan used by KOCH and afterwards also by other workers. The analeptics used by us were cardiazole and benzedrine hydrochloride. The tests were carried out at 20.0° C. The technique for the determination of the anesthesizing time was exactly that described elsewhere [1]); for the determination of the average anesthesizing time holding for a given solution use was invariably made of twenty fishes, which proved to be quite sufficient.

TABLE 2. Anesthesizing times in minutes.

No.	anesthesine 50 mg/l			
	no analeptic	cardiazole mg/l		
		20	100	500
1	10.5	13	17	6
2	8	13	12	8.5
3	11.5	11	8.5	10
4	7.5	12	13	9.5
5	13	14.5	14.5	8
6	15.5	12	19	6.5
7	12.5	13	12	8
8	8	10.5	15	8
9	8	10.5	13	11
10	11	17	15.5	12.5
11	9.5	8	20.5	17
12	11.5	10	19	14
13	10	9.5	16	15
14	12.5	25.5	17	10.5
15	17.5	10	17.5	10
16	12.5	11	16	6.5
17	9	11.5	12	8
18	14.5	12.5	10	10.5
19	10	12.5	11	12.5
20	11.5	10	13.5	13.5
Aver.:	11.2	12.4	14.6	10.3

Some results of our experiments are given in the tables 1 and 2, which as a whole speak for themselves. Only column 7 of Table 1 requires some explanation. The fishes which had been used for the experiments with a solution containing, besides the anesthetic, 1000 mg of cardiazole per litre were used again 17 days later for experiments with a solution of the anesthetic alone; the anesthesizing times then found are mentioned in the respective column. The average anesthesizing time (13.6 minutes) agreed completely with that originally found (13.1 minutes; column 2 of Table 1). We thus arrive at the conclusion that even the relatively very high concentration of cardiazole of 1000 mg per litre does not produce any lasting effect on the goldfish. If desired or necessary the goldfish may, therefore, be used repeatedly for experiments with cardiazole as here described, provided, of course, that sufficient rest periods are observed. According to our experience the same applies with respect to benzedrine hydrochloride.

It should be remarked here that the addition of cardiazole or benzedrine hydrochloride to the solutions of the anesthetics employed did not modify the pH to any appreciable extent. Moreover, as we soon hope to show in a separate paper, the results of experiments on goldfish with the abovementioned anesthetics are surprisingly little affected by the pH of the solution.

The data collected by us, only part of which has been given above, lead to the conclusion *that in none of the cases studied was the analeptic found to have any effect on the anesthesizing time*; none of the differences found between the average anesthesizing times in the absence and presence respectively of an analeptic was significant. Further work in this direction, *e.g.* with other kinds of fish or other anesthetics, appeared to us to be useless: we are convinced that the technique in question cannot lead to a comparison of the activities of analeptics.

In uttering this conviction we bear of course in mind the drawback — already referred to — of the simultaneous use of two drugs. We came across difficulties of this nature during some experiments with the combination of cocaine hydrochloride and cardiazole. Whereas these substances separately, in a concentration of 1000 and 500 mg per litre respectively, are tolerated quite well by the goldfish — as far as the former substance is concerned, naturally apart from the anesthesia caused by it, which was complete in about 15 minutes [5]) — the combination of the two substances, while maintaining the said concentrations, was found to have a fatal effect within a very short time.

The technique mentioned above *sub* 2, *i.e.* the subsequent treatment of fishes with a solution of an analeptic and of an anesthetic, has meanwhile been applied by URBAIN and BEAUVALLET [6]). In view of the experiments so far carried out by us in the manner in question, we are strongly inclined to doubt the accuracy of the results obtained by the said investigators. We wish to collect more numerical data and then intend to devote a separate paper to the technique in question.

Delft, Laboratory for Organic Chemistry
of the Technical University.

April 1948.

[5]) Cf. J. RÉGNIER, R. DAVID and R. SITRI, Compt. Rend. Soc. Biol. **129**, 476 (1938). These investigators used for anesthesizing experiments on the stickleback a solution of only 100 mg of cocaine hydrochloride per litre. In the case of the goldfish this anesthetic must be used in much higher concentrations, *e.g.* 1000 mg per litre.

[6]) G. URBAIN and M. BEAUVALLET, Compt. Rend. Soc. Biol. **139**, 576 (1945); **140**, 44 (1946); M. BEAUVALLET and G. URBAIN, ibid. **140**, 41 (1946).

Geophysics. — *On the large displacements commonly regarded as caused by* LOVE-*waves and similar dispersive surface-waves.* II. By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS JR.)

(Communicated at the meeting of April 24, 1948.)

§ 4. *Calculation of* $L^n(\varrho, z, u)$.

We change the path $0 \rightarrow {}^1/_2\pi$ of the ω integration (6) into ${}^1/_2\pi \rightarrow {}^1/_2\pi + i\infty$ and $i\infty \rightarrow 0$ (in $\omega = i\infty$ + real number the integrand vanishes). On the first traject α, β, s and the square root in the denominator are real; as $d\omega$ is imaginary the real part of the integral is equal to zero. The value on the second traject depends on the places of the two branch points of the integrand:
the first branch point (ω_n) is given by

$$\cos \omega_n = 1/\varrho \sqrt{u^2 \mathfrak{B}^2 - q_n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Passing this point from larger to smaller values of ω the radical

$$\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/\mathfrak{B}^2}$$

changes from positive real to positive imaginary. The point ω_n lies between 0 and ${}^1/_2\pi$ if $u < r_n/\mathfrak{B}$, where $r_n = (q_n^2 + \varrho^2)^{1/2}$; with increasing value of u the branch point moves towards $\omega = 0$ and reaches this point when $u = r_n/\mathfrak{B}$. At still larger values of u the branch point travels upwards along the imaginary axis.

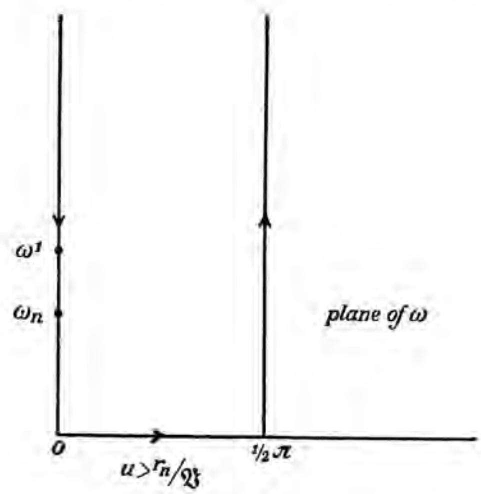

Fig. 3.

The second branch point (ω') is given by $\alpha = -i/\mathfrak{B}'$ (the branch points $+ i/\mathfrak{B}'$ and $\pm i/\mathfrak{B}$ do not occur in this part of the ω plane), or

$$u = (\varrho \cos \omega + q_n \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1})/\mathfrak{B}'.$$

Substituting this value of u back into (5) we obtain:

$$\alpha = \frac{-i\varrho \cos \omega (\varrho \cos \omega + i q \sqrt{1 - \mathfrak{B}'^2/\mathfrak{B}^2}) + q_n \sqrt{(\varrho \cos \omega \sqrt{1 - \mathfrak{B}'^2/\mathfrak{B}^2} + i q_n)^2}}{\mathfrak{B}' (\varrho^2 \cos^2 \omega + q_n^2)}$$

and this is, if $\mathfrak{B}' < \mathfrak{B}$, not equal to $-i/\mathfrak{B}'$. The point $\alpha = -i/\mathfrak{B}'$ is then represented by a point on blade B of the u-surface and has nothing to do with the ω-integration.

If however $\mathfrak{B}' > \mathfrak{B}$ we get

$$\alpha = \frac{-i \varrho \cos \omega (\varrho \cos \omega + q_n \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1}) + i q_n \sqrt{(\varrho \cos \omega \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1} - q_n)^2}}{\mathfrak{B}' (\varrho^2 \cos^2 \omega + q_n^2)}$$

which is equal to $-i/\mathfrak{B}'$ if

$$\varrho \cos \omega \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1} > q_n.$$

Using the angle ε of total reflection: $\varepsilon = arc \sin \mathfrak{B}/\mathfrak{B}'$ this inequality can be written as

$$\varrho > r_n \sin \varepsilon;$$

the branch point ω' is therefore of importance if the point (ϱ, z) lies outside the cone of total reflection $\varrho = r_n \sin \varepsilon$. In that case we have

$$\cos \omega' = (u \mathfrak{B}' - q_n \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1})/\varrho;$$

the point ω' travels with increasing value of u along the real ω-axis towards $\omega = 0$ and reaches this point when

$$u = (\varrho + q_n \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1})/\mathfrak{B}';$$

for larger values of u it moves along the imaginary axis.

First case: $\mathfrak{B} > \mathfrak{B}'$.

As long as $u < r_n/\mathfrak{B}$ no branch point exists on the imaginary ω-axis; α, and the square root in the denominator are imaginary and the real part of the integral on the traject $i\infty \to 0$ is then equal to zero. It follows:

$$L^n(\varrho, z, u) = 0 \quad \text{if} \quad u < r_n/\mathfrak{B} \quad \text{and} \quad \mathfrak{B} > \mathfrak{B}'$$

and

$$L^n(\varrho, z, u) = -2/\pi\, R \int_0^{\omega_n} \left(\frac{1-s}{1+e}\right)^n \frac{\alpha\, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/\mathfrak{B}^2}} \quad \text{if} \quad u > r_n/\mathfrak{N}.$$

The value of this function at $u \geqq r_n/\mathfrak{B}$ is very important as it determines to a large extent the motion existing in the points r_n immediately after the arrival of the wave C. For small values of ω_n equation (7) is approximately:

$$\varrho^2 (1-\omega_n^2) = u^2 \mathfrak{B}^2 - q_n^2,$$

or

$$\omega_n = i w_n \quad , \quad w_n = 1/\varrho \sqrt{u^2 \mathfrak{B}^2 - r_n^2}$$

On the small traject $\omega = 0$ to $\omega = \omega_n$ only the denominator of the integrand changes rapidly; the other factors remain in first approximation equal to their values at ω_n (or $\omega \approx 0$):

$$a \approx a_n = -\frac{i\varrho \cos \omega_n}{u_n \mathfrak{B}^2} \approx -i\varrho/r_n \mathfrak{B} = \frac{-i \sin \delta_n}{\mathfrak{B}} \quad , \quad \text{where } \sin \delta_n = \varrho/r_n$$

and

$$s \approx s_n = \frac{\mu' \sqrt{\mathfrak{B}^2/\mathfrak{B}'^2 - \sin^2 \delta_n}}{\mu \cos \delta_n}.$$

With $\omega = i\, w$ we obtain in first approximation:

$$L^n \approx \frac{2}{\pi r_n} \left(\frac{1-s}{1+s}\right)^n \int_0^{w_n} \frac{dw}{\sqrt{w_n^2 - w^2}}$$

and as the integral is, independent from w_n, equal to $\pi/2$ we obtain

$$L^n (\varrho, z, r_n/\mathfrak{B} + 0) = \frac{1}{r_n} \left(\frac{1-s_n}{1+s_n}\right)^n.$$

This result is not approximately but exactly true at $\omega_n = 0$ or $u = r_n/\mathfrak{B} + 0$; the function L^n changes discontinuously from zero to a finite value.

Second case: $\mathfrak{B}' > \mathfrak{B}$.

In this case two branch points are moving with increasing value of u along the real axis towards $\omega = 0$; the first one to reach this point is ω' as

$$(\varrho + q_n \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1})/\mathfrak{B}' < r_n/\mathfrak{B} \quad \text{if} \quad \varrho > r_n \sin \varepsilon.$$

(If $\varrho < r_n \sin \varepsilon$ the point ω' does not appear at all). At larger values of u ω' travels along the imaginary axis, followed by the branch point ω_n when $u > r_n/\mathfrak{B}$. On the traject $\omega = i\infty$ to $\omega = \omega'$ the integrand of L^n is real, $d\omega$ exepted; the real part of the integral is then zero. Between ω' and 0 the radical β' is imaginary and s is complex; hence

$$L^n (\varrho, z, u) = 0 \quad \text{if} \quad u < u' = (\varrho + q_n \sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1})/\mathfrak{B}' \quad \text{and}$$

$$L^n (\varrho, z, u) = -2/\pi\, R \int_0^{\omega'} \left(\frac{1-s}{1+s}\right)^n \frac{a\, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/\mathfrak{B}^2}} \quad \text{if } u > u'.$$

As the change from zero to a finite value marks the arrival of a disturbance at (ϱ, z) it is again of importance to calculate the value of L^n for values of $u \geq u'$. An approximation can be easily found by means of a development of the integrand; we obtain:

$$L^n(\varrho, z, u'+\delta) \approx 2\,\mu'/\mu\,\frac{n}{\varrho^2}\,\frac{\mathfrak{B}'\,\delta}{\sqrt{\operatorname{cotg}\varepsilon\,(\operatorname{cotg}\varepsilon - q_n/\varrho)^{1/2}}},\ \text{where } \delta \ll u'.$$

L^n changes therefore continuously from zero to a finite value at $u = u'$. But the arrival of the branch point ω_n at $\omega = 0$ causes a discontinuous change of L^n; in the same way as in the first case we find a discontinuous change equal to $1/r_n \cdot R\left(\frac{1-s_n}{1+s_n}\right)^n$. Hence

$$L^n(\varrho, z, r_n/\mathfrak{B}+0) = L^n(\varrho, z, r_n/\mathfrak{B}-0) + \frac{1}{r_n}\,R\left(\frac{1-s_n}{1+s_n}\right)^n.$$

§ 5. *The motion in the layer.*

Before proceeding with the evaluation of L^n for other values of u we use the obtained results to calculate the motion due to the wave C_1. This motion is determined by

$$C = \partial M/\partial\varrho \text{ and } M(\varrho, z, t) = \int_0^t F'(t-u)\cdot L(\varrho, z, u)\,du.$$

We start with the first case: $\mathfrak{B} > \mathfrak{B}'$.

a. If $u < r/\mathfrak{B}$ we found: $L = 0$; therefore $C = 0$ if $t < r/\mathfrak{B}$.

b. If $r/\mathfrak{B} < u < r_1/\mathfrak{B}$: $L = L^0 = 1/r$. Consequently the movement is, if $r/\mathfrak{B} < t < r_1/\mathfrak{B}$:

$$C = C^0 = \partial/\partial\varrho \int_{r/\mathfrak{B}}^t F'(t-u)\cdot 1/r\,du$$

or

$$C = C_0 = -\frac{\sin\psi}{r\mathfrak{B}}\,F'(t-r/\mathfrak{B}) - \frac{\sin\psi}{r^2}\,F(t-r/\mathfrak{B}) \text{ with } \psi = \operatorname{arc\,tg}\varrho/(f-z).$$

This motion which reaches the points r at $t = r/\mathfrak{B}$ is obviously caused by a disturbance which travels directly from the source at $r = 0$ to the point r.

c. If $r_1/\mathfrak{B} < u < r_2/\mathfrak{B}$ we have $L = L^0 + L^1$. The motion existing in ϱ, z at a time t between $r_1/\mathfrak{B}$ and $r_2/\mathfrak{B}$ is $C = C^0 + C^1$, where

$$C^1 = -\frac{\sin\psi_1}{\mathfrak{B}}\,F'(t-r/\mathfrak{B})\,L^1(\varrho, z, r_1/\mathfrak{B}+0) +$$

$$+\int_{r_1/\mathfrak{B}}^t F'(t-u)\cdot\partial/\partial\varrho\left\{2/\pi\,R\int_0^{\omega_1}\left(\frac{1-s}{1+s}\right)^1\frac{a\,d\omega}{\sqrt{u^2-(q_1^2+\varrho^2\cos^2\omega)/\mathfrak{B}^2}}\right\}du \quad (8)$$

with $\psi_1 = \operatorname{arc\,tg}\varrho/q_1$.

d. If $r_2/\mathfrak{B} < u < r_3/\mathfrak{B}$ we obtain the motion at the time t between $r_2/\mathfrak{B}$ and $r_3/\mathfrak{B}$:

$$C = C^0 + C^1 + C^2$$

where C^2 is obtained by changing the suffix and the exponent occurring in the expression of C^1 from 1 into 2.

The interpretation of these results is elementary: C^1 is caused by a disturbance which has been once reflected at $z = d$ and reaches ϱ, z while traveling in the $-z$ direction; if $q = 2d + f - z$ the disturbance moves from the source upwards to $z = 0$, is reflected at $z = 0$ and at $z = d$, and reaches then the point ϱ, z; if $q = 2d - f - z$ the motion is propagated downwards, reflected at $z = d$ and travels then to ϱ, z. At the reflection at $z = d$ the amplitude is multiplied by D_t.

At each moment $t = r_n/\mathfrak{B}$ a new motion starts at ϱ, z and each of these movements consists of a part which is proportional to $(D_t)^n/r_n$ and which exists during a time equal to the duration of the initial disturbance. The second part of these movements is expressed by the integrals (8); the second part of the direct wave C^0 is proportional $1/r^2$ and it is to be expected that the second part of the movements C^n are also proportional to $1/r_n^2$; the coefficient of reflection D_t is for every value of the variables smaller than 1; we have therefore:

$$2/\pi\, R \int_0^{\omega_n} \left(\frac{1-s}{1+s}\right)^n \frac{a\, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2\omega)/\mathfrak{B}^2}} < 2/\pi\, R \int_0^{\omega_n} P \frac{a\, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2\omega)/\mathfrak{B}^2}} = P/r_n$$

where P is a finite number (< 1). Differentiation $\partial/\partial\varrho$ yields: $P \sin\psi_n/r_n^2$ (a more complete treatment of this integral will be given in Chapter V where a similar integral will be investigated).

The second part of C^n is proportional to $1/r_n^2$ and can be neglected at large epicentral distances.

The other wave-systems existing in the layer (C_2 at $r < f$ and C_3, C_4 at $f < z < d$) are expressed by the same or similar functions as C_1; the movement at any point is of course equal to $C_1 + C_2$ or $C_3 + C_4$.

In the second case, where $\mathfrak{B}' > \mathfrak{B}$, the major part of the movement consist of the same series as in the previous case; the secondary parts, expressed by the ω-integrals, are also similar in both cases, but the order of succession in which they are observed is different. If $\mathfrak{B} > \mathfrak{B}'$ the major part is the fastest wave in the system C^n; the seismogram starts with the larger movement and shows at later times the minor disturbances, followed by the next large movement. If however $\mathfrak{B} < \mathfrak{B}'$ each wave C^n starts with a secondary motion as long as $r_n \sin\varepsilon < \varrho$; this is followed by the major part, and then the slow part of the secondary movement arrives. It is obvious that this is not very important as a seismologist usually observes only the surface $(1/\sqrt{\varrho})$ waves and the arrival of some $1/r$ waves. The only case in which the secondary motion is important is the following: the secondary wave of C^1 arrives at the time

$$t = \{\varrho + (2d - f - z)\sqrt{\mathfrak{B}'^2/\mathfrak{B}^2 - 1}\}/\mathfrak{B}'$$

and this time can be easily interpreted by assuming that the disturbance travels in the direction ε towards $z = d$, moves along the interface with velocity $\mathfrak{B}'$ and finally reaches ϱ, z, again traveling in the direction ε. At a certain distances ϱ this travel time is smaller then that of the direct wave and the seismogram starts with this wave. This small disturbance can be observed at not too large epicentral distances and is used in seismological prospecting ("geführte Welle" of O. v. SCHMIDT).

III. The large horizontal movements.

The large *SH* displacements observed in seismograms appear to be of the same order of magnitude as the Rayleigh-movement and the propagation of these displacements must therefore obey a $\varrho^{-1/2}$ law. The investigation of the previous chapter shows that the *SH* motion caused by a point source consists of a series of movements which are proportional to r_n^{-1} (or r_n^{-2}); their exists no wave with an amplitude proportional to $\varrho^{-1/2}$ and the Love-waves do not appear at all. The large *SH* movements are therefore not caused by a single $\varrho^{-1/2}$ (or surface) wave, but by the interference of a large number of small $1/r_n$ movements.

Consider the possibility of this occurrence:

the first condition to be fulfilled is of course that these interfering movements are to coincide; now the difference in travel time between the direct wave and the n^{th} reflected one is

$$\{\sqrt{\varrho^2 + (2nd - f - z)^2} - \sqrt{\varrho^2 + (f - z)^2}\}/\mathfrak{B}$$

and this is at large epicentral distances (at which the large *SH* movements occur) equal to

$$4n^2d^2/2\varrho\mathfrak{B}, \quad \text{provided} \quad 2nd \ll \varrho.$$

The time during which each seperate $1/r_n$ movement exists is equal to the duration τ of the initial disturbance; interference between this group of $n + 1$ waves is therefore possible if

$$4n^2d^2 \leqq 2\varrho\mathfrak{B}\tau \quad \text{or} \quad n \leqq \sqrt{2\varrho\mathfrak{B}\tau}/2d. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(as τ is in any case not very large the above inequality $2nd \ll \varrho$ is satisfied).

In the second place the coefficient of reflection has to be about equal to one, or $s_n \approx 0$. It is impossible to satisfy $s_n \approx 0$ if $\mathfrak{B} > \mathfrak{B}'$, but if $\mathfrak{B} < \mathfrak{B}'$ we obtain

$$\sin\delta_n = \mathfrak{B}/\mathfrak{B}'.$$

As $2\,nd \ll \varrho$ and $\sin \delta_n = \varrho/r_n$ this is only possible if $\mathfrak{V}/\mathfrak{V}' \approx 1$. Putting $\mathfrak{V}'^2/\mathfrak{V}^2 = 1 + \gamma$ where $\gamma \ll 1$ we get

$$|s_n| \approx \frac{\mu' \sqrt{\gamma - \cos^2 \delta_n}}{\mu \cos \delta_n} \quad \text{or} \quad |s_n| \approx \mu'/\mu \sqrt{\frac{\gamma \varrho^2}{4n^2 d^2} - 1}$$

using (1) this becomes

$$|s_n| \approx \mu'/\mu \sqrt{\frac{\gamma \varrho}{2 \mathfrak{V} \tau} - 1}$$

This is very small if $\gamma \approx 2\,\mathfrak{V}\tau/\varrho$, or

$$\mathfrak{V}'/\mathfrak{V} \approx 1 + \mathfrak{V}\tau/\varrho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When this condition is fulfilled a series of $\sqrt{2\varrho\mathfrak{V}\tau}/2d$ shocks which have all about the same amplitude-factor $1/\varrho$ interfere at the distance ϱ, thereby causing a movement which is proportional to

$$\frac{1}{d} \sqrt{\frac{\tau}{2\varrho\mathfrak{V}}}\, F'(t - r/\mathfrak{V}).$$

In a granitic layer ($\mathfrak{V} \approx 3{,}26$ km/sec) with a depth of 40 km this factor is equal to $1/40 \sqrt{\tau/\varrho} \cdot F'$; wether this is comparable with the amplitude-factor of the Rayleigh-movement depends on the mechanism at the hearth of the earthquake, especially on the ratio between the tangential and normal stresses and on the duration of the primary shock.

Equation (2) requires a continuous increase of $\mathfrak{V}$; such an increase of $\mathfrak{V}$ has been often observed and determined. GUTENBERG calculated a change of 0,001 z km/sec of the longitudinal velocity in the upper stratum of Southern California and obtained evidence of a similar increase of $\mathfrak{V}$.

The propagation of a disturbance in such a medium is not very different from that in a homogeneous layer:

A disturbance which starts from a point source in an almost horizontal, but downwards, direction is totally reflected at a depth d which varies with the direction of propagation; after this reflection this wave travels up and down between $z = 0$ and $z = d$ along slightly curved lines. The variation of the depths of the turning points of the first waves which reach a distant point ϱ as well as the curvature of their paths is very small.

The situation is therefore almost identical to that of the case dealt with above, if we suppose the source at the depth d (which does not alter the calculations).

Finally we investigate the differences between the interference phenomena at various points ϱ. According to the approximative theory (homogeneous layer) the interference starts at the arrival of the direct wave, at the time $t = r/\mathfrak{V}$. As $\varrho \gg d$ and the source is at the depth d we have $t = \varrho/\mathfrak{V} + d^2/2\varrho\mathfrak{V}$.

Now supposing that the large movements are due to surface waves traveling with a velocity $\mathfrak{V}_L$ along the free surface; then $t = \varrho/\mathfrak{V}_L$. It follows

$$1/\mathfrak{V}_L = 1/\mathfrak{V}\left(1+\frac{d^2}{2\varrho^2}\right);$$

consequently $\mathfrak{V}_L$ increases slowly with increasing value of ϱ, which is in accordance with the observations. In an inhomogeneous layer $\mathfrak{V}_L$ increases less slowly as the depth d at which the waves are totally reflected decreases with growing value of ϱ.

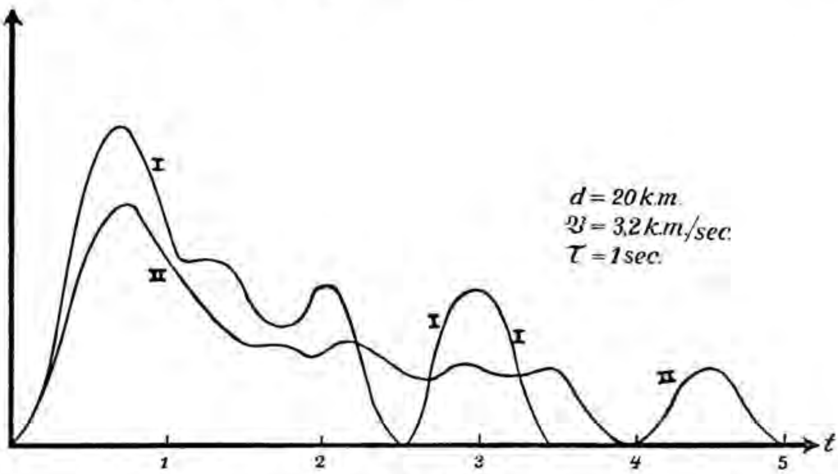

Fig. 4.
Curve I: $\varrho = 4000$ km; II: $\varrho = 8000$ km.
Primary disturbance: $\sin^2 \pi t$ $(0 < t < 1)$.

A second phenomenon due to interference is the following: the movement is at its maximum when the motion caused by the direct wave is past its maximum. The waves which arrive at later times interfere with the already decreasing movement, thereby delaying this decrease. As the difference in travel-time between two consecutive waves diminishes with growing value of ϱ the decrease of the large movements is slower at larger distances. The time during which the displacement is large increases therefore with the distance; this is perhaps the explanation of the well-known period-distance effect. However it must be remarked that any property connected with the periods of seismic waves can only be fully understood if the way in which these vibrations are caused is completely known. Assuming the time function $F(t)$ of the initial disturbance to be $F(t) = \sin^2 mt$, if $0 \leq t \leq \tau = \pi/m$ (and zero for other values of t) we have drawn the curves of Fig. 4, representing the amplitudes of the large SH movements at $\varrho = 4000$ and $\varrho = 8000$ k.m. (and $m = \pi$).

The propagation in an inhomogeneous medium has of course to be calculated and the above suggestions only indicate a possibility of interference. The above theory is exact if μ' is small in comparison with μ.

Mathematics. — *On the method of critical points.* By J. G. VAN DER CORPUT. First Communication [1]).

(Communicated at the meeting of May 29, 1948.)

Let me begin by thanking Prof. L. J. MORDELL most heartily for the kind words, with which he has welcomed and introduced me.

It is a very great honour to me to speak in your midst as lecturer for the year 1948 on the foundation to which is attached the famous name of Rouse Ball. Nomination as Rouse Ball lecturer is a high distinction for any mathematician, the more so because the invitation is from the university, justly celebrated all over the world, especially for mathematics which owe to it so many very important discoveries. I am referring not only to the eminent mathematicians now living and working at this university, but also to the dead, especially to HARDY, whose death was a great loss not only to Cambridge, not only to England but to the whole civilised world.

The invitation enables me to say something to this expert audience about a subject important not only for pure, but also for applied mathematics. The principle which underlies the theory I will treat in this lecture is so very simple, that it is quite obvious for every mathematician, but only during the last years has this idea taken a fixed form in my mind so that I can apply it now with success not only to the evaluation of integrals, but also to the solution of differential equations, integral equations and even functional equations.

The pleasure for me in treating this subject is the greater, because at the present you find almost nothing about it in the literature.

I could call my point of view in this lecture that of an asymptoticist. By what properties does the asymptoticist distinguish himself from the other mathematicians? Is it because he is less exact than his colleagues? Or that his results are only almost true? There was a time when all asymptotic expansions were forcibly expelled from mathematics, because it was thought that this pure territory would be in danger of being contaminated by those unworthy inhabitants. But that is long ago and the false accusation has been withdrawn. No, the difference is to be found elsewhere. In the last fifty years we have learned that the great numbers possess a regularity which the small numbers do not. The purpose of the asymptoticist is to study this regularity and in order not to be disturbed by the chaos reigning in the domain of the smaller numbers, he immediately excludes those from his investigations.

All considerations of an asymptoticist involve a variable, which I shall call ω. He studies quantities dependant on this variable, to be sure, but

1) Lecture given on April 27th 1948 as Rouse Ball lecturer at Cambridge (England).

he is only interested in the behaviour of those quantities for large values of ω. How they behave for lesser values of ω is quite indifferent to him.

From his point of view 3 and $3 + 10^{-\omega}$ are practically the same. In fact, if ω is greater than 1000, the two numbers are the same in the first thousand decimals. He calls these two numbers asymptotically equal one to another and he writes $3 \simeq 3 + 10^{-\omega}$. In general he calls two numbers α and β asymptotically equal, written $\alpha \simeq \beta$, if for every fixed positive integer h the product $\omega^h(\alpha - \beta)$ tends to zero, as ω tends to infinity. In other words: he requires that to each integer h correspond two positive numbers ω_h and c_h such that

$$|\alpha - \beta| \leqq \frac{c_h}{\omega^h} \quad \text{for } \omega \geqq \omega_h.$$

It is possible to prescribe moreover how ω_h and c_h may depend on h, and then we get an other notion of equality, but for simplicity I restrict myself here to the simplest case, in which ω_h and c_h may be arbitrary functions of h.

If α and β are asymptotically equal, I say that they belong to the same asymptotic class. The asymptoticist is not interested in the exact value of α. On the contrary, he is quite content if he knows to which class α belongs. If he has succeeded in finding this class, he says that the asymptotic character of α is uniquely determined. In short, he does not study numbers and functions, but only the asymptotic classes, formed by those numbers and functions. It appears that the study of those classes is in many respects less difficult than ordinary mathematics. In fact, in this branch of mathematics, which I call asymptotics, we have at our disposal some simple principles, which do not hold elsewhere. To one of those principles I want to draw your attention, namely to the principle of the critical points.

What are we doing in mathematics? At a certain moment we suppose some numbers, functions, sets and so on to be given, and to those quantities we apply some operation, for instance integration. In this manner we construct new quantities. For instance, if the functions x^{s-1} and $\varphi(x) = e^{-x}$ $(x \geqq 0)$ are known, we may construct under certain conditions the new function

$$\int_0^\infty x^{s-1} e^{-x} \varphi(x)\, dx.$$

This function can be extended analytically over the whole complex s-plane, with poles at the points $0, -1, -2, \ldots$. But what happens after an almost imperceptible change of the original function $\varphi(x)$, for instance if we put

$$\begin{aligned} \varphi(x) &= e^{-x} \qquad && (x \geqq 10^{-10}) \\ &= 0 && (0 \leqq x < 10^{-10}) \end{aligned}$$

Then we get a function with quite different properties, namely one without any poles. So we can write down in ordinary mathematics the following general rule: if we apply to one or more functions a certain operation, it is possible that the result obtains completely different properties, if the original functions are only slightly changed. In order to deduce the properties of the result, we therefore must have a complete knowledge of the original functions. And now the surprising fact in asymptotics: in many cases we almost need not know anything about the functions from which we start. In order to define uniquely the asymptotic behaviour of the result, it is in many cases sufficient to know how the original functions behave at certain points. These points are called the critical points.

Let us illustrate this by an simple example, by considering the integral

$$I_1 = \int_{-1}^{2} \frac{\cos \omega x^3}{1 + x + x^2}\, dx,$$

where ω denotes a large positive number. The graph of the integrand shows an enormous number of very rapid oscillations between the curves $y = \frac{1}{1 + x + x^2}$ and $y = \frac{-1}{1 + x + x^2}$. It is quite obvious that in this integral the waves neutralise each other partly. But only partly, of course not completely. First we have an irregularity at the origin, where the phase ωx^3 is stationary, that is, where its derivative vanishes. The length of the wave at the origin is much larger than at the other points; so that wave is not neutralised by the others. This fact was discovered already a century ago by STOKES who remarked that the order of magnitude of the integral depends in many cases on the behaviour of the integrand in the neighbourhood of the points at which the phase is stationary. The method of stationary phase so found which enables us to obtain for great values of ω a rough approximation of the integral, is not quite satisfactory, because there are still other irregularities, namely at the endpoints, where the waves suddenly break off. The waves in the vicinity of an endpoint furnish a contribution, which admittedly is not as great as that of the waves in the vicinity of the origin, but which is by no means negligible. Therefore, in order to study the asymptotic character of the integral, we must know the behaviour of the integrand not only at the origin, but also at the endpoints. But now there is this astonishing fact: that knowledge is not only necessary, but also sufficient. The other waves neutralise each other so perfectly that they have no influence whatever on the asymptotic character of the integral.

More precisely: consider the more general integral

$$I_2 = \int_a^b g(x)\, e^{i\omega f(x)}\, dx,$$

where a, b, $f(x)$ and $g(x)$ are independent of ω. I assume that $f(x)$ and $g(x)$ are infinitely often differentiable in the interval $a \leqq x \leqq b$, that the number of points between a and b at which the phase $\omega f(x)$ is stationary is at most finite and finally that to each of these points there corresponds at least one positive integer m such that $f^{(m)}(x) \neq 0$. Then the asymptotic character of the integral I_2 is completely determined, if the behaviour of the functions $f(x)$ and $g(x)$ is given in the vicinity of the critical points. These are the endpoints a and b and the points between a and b where the phase is stationary. It is perhaps not superfluous to add that for other kinds of integrals not only the endpoints and the points at which the phase is stationary may be critical, but also other points. For simplicity however I restrict myself in this lecture to the kind of integrals mentioned here.

The above assertion is almost obvious if we introduce neutralisators. A neutralisator $N_a(x)$ at the point a is a function of x, independent of ω, which is defined with all its derivates for every real x such that the two following conditions are satisfied:

1. At a the function takes the value 1, and all its derivatives vanish there.

2. There exists a positive number ε such that the neutralisator vanishes outside the interval $(a-\varepsilon,\ a+\varepsilon)$. I shall call this interval the active interval of the neutralisator.

If the critical points ξ lying between a and b are given, the integral I_2 may be written in the form

$$\left.\begin{aligned} I_2 = \int_a^b N_a(x) g(x) e^{i\omega f(x)} dx + \sum_\xi \int_a^b N_\xi(x) g(x) e^{i\omega f(x)} dx + \\ + \int_a^b N_b(x) g(x) e^{i\omega f(x)} dx + I_3, \end{aligned}\right\} \quad (1)$$

where the sum Σ is extended over all critical points ξ between a and b, and where

$$I_3 = \int_a^b H(x) e^{i\omega f(x)} dx; \quad H(x) = g(x) \{1 - N_a(x) - N_b(x) - \sum_\xi N_\xi(x)\}.$$

If the active intervals of the neutralisators are small enough, the last factor, and therefore also $H(x)$, vanishes with all its derivatives at each critical point.

The proof is established if we show that to each integer $q \geqq 0$ there corresponds a number c, independent of ω, such that

$$|I_3| \leqq c\,\omega^{-q}. \quad \ldots\ldots\ldots \quad (2)$$

In fact, I_3 is then asymptotically equal to 0 and I_2 is asymptotically equal to the sum on the right hand side of (1); as we may choose the active interval of each neutralisator as small as we like, this sum is completely determined by the behaviour of the functions $f(x)$ and $g(x)$ in the immediate vicinity of the critical points.

Now the proof of (2). We deduce this inequality for every function $H(x)$, which is independent of ω and which vanishes with all its derivatives at each critical point. Since the inequality is obvious for $q = 0$, I may suppose $q \geqq 1$ and assume that the inequality already has been proved with $q-1$ instead of q. Integration by parts gives

$$I_3 = -\frac{1}{i\omega}\int_a^b K(x)\, e^{i\omega f(x)}\, dx,$$

where

$$K(x) = \frac{d}{dx}\frac{H(x)}{f'(x)}$$

vanishes with each of its derivatives at every critical point. Therefore (2) is true with $q-1$ and $K(x)$ instead of q and $H(x)$. This establishes (2) and therefore the proof of the assertion.

The terms of the sum occurring on the right hand side of (1) are denoted respectively by

$$\int_a^{a+} g(x)\, e^{i\omega f(x)}\, dx; \int_{\xi-}^{\xi+} g(x)\, e^{i\omega f(x)}\, dx \text{ and } \int_{b-}^{b} g(x)\, e^{i\omega f(x)}\, dx,$$

so that we obtain

$$I_2 \cong \int_a^{a+} g(x)\, e^{i\omega f(x)}\, dx + \sum_{\xi}\int_{\xi-}^{\xi+} g(x)\, e^{i\omega f(x)}\, dx + \int_{b-}^{b} g(x)\, e^{i\omega f(x)}\, dx,$$

where the sum $\sum\limits_{\xi}$ is extended over the points ξ, lying between a and b, where the phase $\omega f(x)$ is stationary.

This theorem shows a great resemblance to the well known theorem of residues in the theory of complex functions, in which similarly the value of an integral is uniquely determined by the values which the integrand assumes at certain points (namely the singular points of the integrand). Therefore I call the above theorem the theorem of residues in asymptotics; the contribution of each critical point is called the residue at that point. As appears from the proof, the residue may be written in the form of the same integral, apart from the neutralisator which occurs as a supplementary factor of the integrand.

Just as in the theory of complex functions we have at our disposal very simple means which enable us to evaluate the residue at a critical point ξ. If m denotes the smallest positive integer such that the m^{th} derivative of f at ξ is not zero, then the residue at that point may be developed asymptotically in ascending powers of $\frac{1}{\omega^{\frac{1}{m}}}$ in the form

$$e^{i\omega f(\xi)}\left\{\frac{c_1}{\omega^{1/m}} + \frac{c_2}{\omega^{2/m}} + \ldots\right\},$$

where the coefficients $c_1, c_2, \ldots$ are independent of ω; the coefficient c_1 is uniquely determined by the values which $g(x)$ and $f^{(m)}(x)$ take at ξ, the coefficient c_2 by the values which $g(x)$, $g'(x)$, $f^{(m)}(x)$ and $f^{(m+1)}(x)$ take at ξ, and so on.

These simple remarks enable us to approximate very sharply to a great number of integrals. Let us return to the integral I_1, which possesses the critical points 0, -1 and 2. At either endpoint the phase is not stationary, so $m = 1$, and the residues at those points may be developed asymptotically in ascending powers of $\frac{1}{\omega}$. The residue of each endpoint is therefore at most of the same order of magnitude as $\frac{1}{\omega}$. In order to evaluate the residue at the origin, I compare the integral in question with the well known integral

$$\int_{-\infty}^{\infty} \cos \omega x^3 \, dx = \sqrt{3}\, \Gamma(\tfrac{4}{3})\, \omega^{-\frac{1}{3}};$$

which has only one critical point, namely the origin. The functions $g(x)$ and $f'''(x)$, occurring in the integral I_1 take at the origin the same values as the corresponding functions in the new integral. Consequently the asymptotic expansions of the two residues at the origin begin with the same term $\sqrt{3}\, \Gamma(\frac{4}{3})\, \omega^{-\frac{1}{3}}$, hence

$$I_1 = \sqrt{3}\, \Gamma(\tfrac{4}{3})\, \omega^{-\frac{1}{3}} + O(\omega^{-\frac{2}{3}}).$$

It is very easy to write down immediately the whole asymptotic expansion of this type of integral.

In our above example the preponderant term in this expansion is furnished, not by an endpoint, but by a point at which the phase is stationary. In general this is true, but not always. E.g. in the case of the integral

$$\int_{-1}^{2} \frac{x^2 \cos \omega x^3}{1 + x + x^2} \, dx$$

the preponderant term is furnished by the endpoints and not by the origin, so that here the original method of stationary phase would fail completely.

The method of critical points can be successfully applied to multiple integrals, for instance to a double integral of the form

$$I_3 = \iint_G g(x, y)\, e^{i \omega f(x, y)} \, dx \, dy,$$

where f and g and the two dimensional region G satisfy certain general conditions, for instance that they are independent of ω. In this case the critical points are first the points in the interior of G where the phase $\omega f(x, y)$ is stationary, that is where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$; further the vertices of the boundary of G; finally a boundary point is also critical, if the function $f(x, y)$, taken along the boundary curve, is stationary at that point.

In this manner we again obtain a theorem of residues, but now one which furnishes under very general conditions asymptotic expansions for double integrals. Just as with simple integrals, in general the preponderant term is furnished by a point with stationary phase, but nevertheless the contribution of the other critical points is by no means negligible. That the neglecting of this supplementary residues may have desastrous results, appears from geometrical optics.

Suppose a monochromatic point-source of light sends its rays through a lens placed obliquely in the bundle to a screen. According to geometrical optics we expect a circular image on the screen, but experiments [2]) show that, apart from this circle, there is also an asteroid. This is somewhat less clear, but still distinctly visible and has a very sharp boundary.

Geometrical optics does not give an explanation of this phenomenon, but the method of critical points furnishes an answer to this problem as follows.

On the screen we use a system of Cartesian coordinates ξ and η and in the plane of the lens coordinates x and y. At the point (ξ, η) on the screen we observe a vibration which may be characterised by the expression, $ce^{i\gamma t}$ where c defines the amplitude and the phase. If $c = 0$, then there is darkness at the point (ξ, η). Taking suitable units, we have under certain conditions

$$c = \iint e^{i\omega(\xi x + \eta y + x^2 - y^2)}\, dx\, dy,$$

where the integral is extended over the lens in the x–y-plane, which, for simplicity, we take to be the unit circle; ω is the reciprocal of the wave length of the light, and thus very large. According to the above theory c is approximately equal to the sum of the residues at the critical points.

The interior of the lens contains at most one critical point at which the phase is stationary. This point (x, y) is defined by

$$\xi + 2x = 0 \quad \text{and} \quad \eta - 2y = 0,$$

hence $x = -\frac{1}{2}\xi$, $y = \frac{1}{2}\eta$. We have to take into account only those points (ξ, η), for which $(-\frac{1}{2}\xi, \frac{1}{2}\eta)$ is a point of the lens, hence $\xi^2 + \eta^2 \leqq 4$. For the points (ξ, η) inside the circle $\xi^2 + \eta^2 \leqq 4$ the point of the lens with stationary phase furnishes a contribution, so that there is light at that point. In this way we find on the screen the circular image, predicted by geometrical optics. But the boundary also contains a critical point the contribution of which geometrical optics neglects. Putting $x = \cos t$ and $y = \sin t$, the phase $\xi x + \eta y + x^2 - y^2$ becomes $\xi \cos t + \eta \sin t + \cos 2t$ and this function is stationary if

$$-\frac{\xi}{\cos t} + \frac{\eta}{\sin t} = 4. \qquad (3)$$

Hence the contribution of the critical boundary point vanishes at every point (ξ, η), except when (ξ, η) lies on the lines given by (3). These lines cut off intercepts $-4 \cos t$ and $4 \sin t$ on the coordinate axes. These

[2]) NIENHUIS, Thesis (Groningen 1948).

lines envelop the asteroid $\xi^{\frac{2}{3}} + \eta^{\frac{2}{3}} = 4^{\frac{2}{3}}$. Hence it appears that the asteroid is the result of the action of the critical boundary point.

If the lens is not circular, different figures will be observed. Geometrical optics do not give an explication of these phenomena.

So far I just lifted a tip of the veil, but the method of critical points is much more powerful than I have sketched hitherto. For in many cases it enables us to determine the asymptotic character of the solutions of differential, integral and functional equations. I shall illustrate this by giving a very simple example. Consider the differential equation

$$\frac{dy}{dx} = (1 + x^2 + y^2)\, e^{i\omega x^3},$$

where ω denotes a large positive number. The problem is to establish the asymptotic character of the solution, which assumes a previously given value η at a certain point a, so that

$$y = \eta + \int_a^x (1 + u^2 + y^2(u))\, e^{i\omega u^3}\, du.$$

Fortunately the method of critical points may be applied to this integral, although the factor $1 + u^2 + y^2(u)$ depends now not only on u but also on ω.

First of all, taking a and x both positive or both negative and supposing for simplicity $a < x$, we obtain

$$y \simeq \eta + \int_a^{a+} + \int_{x-}^{x}$$

As I have said already the residue at x is equal to the corresponding integral taken from $x - \varepsilon$ to x (where $\varepsilon > 0$), provided that the integrand has a neutralisator at x as a supplementary factor. Integrating by parts, we find that the residue at x is

$$\frac{1}{3i\omega}\int_{x-}^{x} \frac{1 + u^2 + y^2(u)}{u^2}\, d\, e^{i\omega u^3} =$$

$$\frac{1}{3i\omega}\, \frac{1 + x^2 + y^2}{x^2}\, e^{i\omega x^3} - \frac{1}{3i\omega}\int_{x-}^{x} \left(\frac{1 + u^2 + y^2(u)}{u^2}\right)' e^{i\omega u^3}\, du;$$

in fact, the neutralisator used takes at x and $x - \varepsilon$ respectively the values 1 and 0.

The first term on the right hand side is of the same order of magnitude as $\frac{1}{\omega}$; the second is much smaller, viz. at most of the same order of magnitude as $\frac{1}{\omega^2}$. In this last term I may differentiate under the integral sign and then replace $y'(u)$ by $(1 + u^2 + y^2(u))\, e^{i\omega u^3}$. Further I may again integrate by parts, and so on. In this manner I find an asymptotic expansion for the residue at x. Apart from the sign we obtain an analogous expansion for the residue at a. Substituting these results we obtain

$$y \cong \Sigma\, c_{hkl}\, \omega^{-l}\, e^{i\omega(ha^3+kx^3)},$$

provided that $a < x$ and that a and x are both positive or both negative. The sum is extended over the integers $h \geqq 0$, $k \geqq 0$, $l \geqq h+k$;

$$c_{000} = \eta; \quad c_{101} = -\frac{1}{3i}\frac{1+a^2+\eta^2}{a^2}; \quad c_{011} = \frac{1}{3i}\frac{1+x^2+\eta^2}{x^2}.$$

If $a < 0 < x$ we have a supplementary residue, namely

$$\int\limits_{0-}^{0+} (1+u^2+y^2(u))\, e^{i\omega u^3}\, du.$$

Since the first and the second but not the third derivatives of the phase vanish at the origin, we obtain for this residue an asymptotic expansion of the form

$$\frac{c_1}{\omega^{\frac{1}{3}}} + \frac{c_2}{\omega^{\frac{2}{3}}} + \dots,$$

where for instance

$$c_1 = (1+y^2(0))\, e^{\frac{\pi i}{6}}\, \Gamma(\tfrac{1}{3}).$$

Thus we obtain also an asymptotic expansion for y, but one involving powers of $\frac{1}{\omega^{\frac{1}{3}}}$, therefore quite different from the development, found in the first case.

Ladies and Gentlemen, now I approach the end of this lecture, in which I have tried to give you some, perhaps vague, ideas of the importance of the method of critical points in the field of simple and multiple integrals, in that of ordinary and partial differential equations, and in that of integral and functional equations, but, before finishing, I would like to make a request. Under the auspices of the Mathematical Centre at Amsterdam a group of mathematicians has been formed to study the whole domain of asymptotic expansions, not only the theory, but also the applications in physics, astronomy, and so on. I estimate, that this investigation which has been started half a year ago, will take approximately five or six years. I should appreciate very much if any of you, who intends to take this up as a subject or has already worked on this theme, will inform the Mathematical Centre of his results. The study group at Amsterdam aims to give a general survey of the whole field, in which both theory and applications are to be dealt with, and will gratefully make use of your information. I shall regard it as a very fine outcome of this simple lecture, if cooperation will be established between English and Dutch mathematicians on a field, on which much still must and can be done.

Mathematics. — *On* MAHLER'*s partition problem.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of April 24, 1948.)

1. *Introduction.*

KURT MAHLER [1]) obtained a formula for the number $p(h)$ of partitions of the natural number h into powers of a given integer $r \geq 2$, i.e. the number of solutions of

$$h = h_0 + h_1 r + h_2 r^2 + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

in non-negative integers $h_0, h_1, h_2, \dots$ [2]). His result was

$$p(rh) = e^{O(1)} \sum_{n=0}^{\infty} r^{-\frac{1}{2}n(n-1)} h^n / n! \text{ }^{3)} \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

which leads to the explicit result

$$\left.\begin{aligned} \log p(rh) = \frac{1}{2 \log r} \left(\log \frac{h}{\log h} \right)^2 + \left(\frac{1}{2} + \frac{1}{\log r} + \frac{\log \log r}{\log r} \right) \log h - \\ - \left(1 + \frac{\log \log r}{\log r} \right) \log \log h + O(1). \end{aligned}\right\} \quad (1.3)$$

In the present paper we give a more precise analysis of the O-term in (1.3). It turns out to be of the form

$$\psi \left(\frac{\log h - \log \log h}{\log r} \right) + o(1) \quad . \quad . \quad . \quad . \quad . \quad (1.4)$$

where ψ is a certain periodic function with period 1; the $o(1)$ term can be further investigated.

The series on the right of (1.2) has a similar asymptotic behaviour, with a different periodic function however. It is a solution of the functional equation $F'(h) = F(hr^{-1})$. We shall develop asymptotic formulae for the solutions of that equation in a separate paper.

Our results on $p(rh)$ are found in the following way. Since

$$\sum_{h=0}^{\infty} p(h) x^h = \prod_{k=0}^{\infty} (1 - x^{r^k})^{-1} = f(x) \qquad (|x| < 1) \quad . \quad . \quad (1.5)$$

we have

$$p(h) = \frac{\varrho^{-h}}{2\pi} \int_0^{2\pi} f(\varrho e^{i\varphi}) e^{-hi\varphi} d\varphi. \qquad (0 < \varrho < 1). \quad . \quad . \quad (1.6)$$

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[1]) On a special functional equation, Journ. London Math. Soc. **15**, 115—123 (1940).

[2]) Of course only a finite number of the h_i are > 0.

[3]) Here $rh = h'$ must be an integer. In view of the generalisation to non-integral r we express (1.2) in terms of h instead of h'. See also (1.12).

The function $f(x)$, regular for $|x|<1$, can be calculated with great accuracy in the neighbourhood of the points $x = \exp(2\pi\nu i r^{-\mu})$ $(\nu, \mu = 0, 1, 2, \dots)$ by a formula derived in section 2. After that, evaluation of (1.6) leads to the announced results.

It must be noted that the most important contribution to (1.6) arises from the neighbourhood of the point $x = 1$. Much smaller conributions are given by the points $\exp(2\pi\nu i r^{-1})$ $(\nu = 1, \dots, r-1)$ and so on. These contributions are even much smaller than the errors we cannot avoid to make in the neighbourhood of $x = 1$. Therefore we restrict ourselves to a precise investigation of the neighbourhood of $x = 1$ with a relatively rough estimation of $f(x)$ on the remaining part of the circle $|x| = \varrho$.

In the sequel we shall use formula (1.16) instead of (1.6) because we want to generalise our considerations to the case that r is not an integer. First we develop the necessary formulae.

Henceforth r is an arbitrary number >1. Although not always stated explicitly, most functions in the sequel depend on r. Numbers depending on r only will be called constants.

Let $P(u)$ denote the number of solutions of

$$h_0 + h_1 r + h_2 r^2 + \dots \leqq u \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.7)$$

in non-negative integers $h_0, h_1, h_2, \dots$. The generating function is

$$F(s) = \prod_{k=0}^{\infty} (1 - e^{-sr^k})^{-1} = \int_{-\infty}^{\infty} e^{-su} \, dP(u), \quad (Re\, s > 0). \quad . \quad (1.8)$$

$F(s)$ reduces to $f(e^{-s})$ (see (1.5)) if r is an integer. The integral on the right of (1.8) is a STIELTJES integral. We notice that $P(u) = 0$ for $u < 0$, $P(0) = 1$. Furthermore we have

$$P(u) - P(u-1) = P(u/r) \qquad (-\infty < u < \infty). \quad . \quad . \quad . \quad (1.9)$$

Namely, $P(u) - P(u-1)$ denotes the number of solutions of

$$u - 1 < h_0 + h_1 r + h_2 r^2 + \dots \leqslant u, \quad . \quad . \quad . \quad (1.10)$$

which equals the number of solutions of

$$h_1 r + h_2 r^2 + \dots \qquad \leqq u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.11)$$

since for any solution $h_1, h_2, \dots$ of (1.11) just one non-negative integer h_0 satifying (1.10) can be found. Since (1.11) has $P(u/r)$ solutions we obtain (1.9).

If r and $u = h$ are integers (1.10) and (1.1) are equivalent, and so

$$P(h) = p(rh). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.12)$$

Moreover $P(y) = P([y])$ in that case, and

$$p(rh) = p(rh+1) = \dots = p(rh+r-1). \quad . \quad . \quad . \quad (1.13)$$

Again considering the general case we first notice that by a well-known inversion formula we have

$$\frac{P(u-0)+P(u+0)}{2}=\frac{1}{2\pi i}\lim_{A\to+\infty}\int_{a-iA}^{a+iA}F(s)\,e^{us}\frac{ds}{s}\quad (a>0).\quad (1.\,14)$$

Here the path of integration is the straight line.

It will be convenient to operate with $P_1(u)$ instead of $P(u)$, where

$$P_1(u)=\int_{u-1}^{u}P(v)\,dv=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}F(s)\,e^{us}(1-e^{-s})\frac{ds}{s^2}\quad (a>0).\quad (1.\,15)$$

The latter integral is absolutely convergent.

By (1.8) we have $(1-e^{-s})\,F(s)=F(rs)$. Consequently $P_1(u)$ can be written in the form

$$P_1(u)=\frac{r}{2\pi i}\int_{a-i\infty}^{a+i\infty}F(s)\,e^{us/r}\frac{ds}{s^2}\qquad (a>0).\quad .\quad .\quad .\quad (1.\,16)$$

From this formula we shall deduce the asymptotic behaviour of $P_1(u)$. The way back from $P_1(u)$ to $P(u)$ will be an easy one.

2. *An exact formula for $F(s)$.*

We shall derive a useful expression (formula (2.15) [4])) for the function (1.8), i.e.

$$F(s)=\prod_{k=0}^{\infty}(1-e^{-sr^k})^{-1}\qquad (Re\, s>0).\quad .\quad .\quad .\quad .\quad (2.\,1)$$

in the neighbourhood of $s=0$. For a moment we restrict ourselves to real and positive values of s. We have

$$\log F(s)=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}m^{-1}e^{-sr^km}.$$

From MELLIN's formula

$$e^{-w}=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(z)\,w^{-z}\,dz\qquad (a>0,\,w>0)$$

we now infer that

$$\log F(s)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\Gamma(z)\,m^{-1}\,(s r^k m)^{-z}\,dz.\quad .\quad .\quad (2.\,2)$$

[4]) The present author who found this formula in 1944 was informed after the war by Mr. MAHLER that a similar formula was communicated to him about 1923 by C. L. SIEGEL, who never published it. Therefore we present a full proof here.

the operation carried out being allowed in virtue of the convergence of

$$\int\limits_{a-i\infty}^{a+i\infty} \sum_0^\infty \sum_1^\infty |\Gamma(z)\, m^{-1} (s\, r^k m)^{-z}|\, dz.$$

Finally (2.2) leads to

$$\log F(s) = \frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} s^{-z} \frac{\Gamma(z)\,\zeta(1+z)}{1-r^{-z}}\, dz, \qquad (a > 0,\, s > 0) \quad (2.3)$$

where ζ denotes the RIEMANN ζ-function. Its functional equation gives

$$s^{-z}\, \Gamma(z)\, \zeta(1+z) = \frac{-\pi}{z \sin \frac{1}{2}\pi z} \cdot \left(\frac{2\pi}{s}\right)^z \zeta(-z), \quad . \quad . \quad . \quad (2.4)$$

and it is easily derived that for $0 < s < 2\pi r$ we have

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int\limits_{-n+\frac{1}{2}-i\infty}^{-n+\frac{1}{2}+i\infty} s^{-z} \frac{\Gamma(z)\,\zeta(1+z)}{1-r^{-z}}\, dz = 0$$

if n runs through the positive integers. It now follows from (2.3), (2.4) and from the estimation $\zeta(\sigma + it) = O(|t|)$ $(0 \leq \sigma \leq 1)$ that, for $0 < s < 2\pi r$, $\log F(s)$ equals the sum of the residues of

$$s^{-z}\, \Gamma(z)\zeta(1+z)(1-r^{-z})^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

We have to consider the roots $z = 2\pi ik/\log r$ of $1-r^{-z}$ $(k = 0, \pm 1, \pm 2, \ldots)$ and the poles $z = 0, -1, -2, \ldots$ of $\Gamma(z)$. The point $z = 0$ is the only pole of $\zeta(1+z)$ and thus it is a triple pole of (2.5).

First we evaluate the residue at $z = 0$. We have

$$z\,\zeta(z+1) = 1 + \gamma z + \gamma_2 z^2 + \ldots \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

where $\gamma = 0{,}5772157\ldots$ is EULER's constant, and $\gamma_2 = 0{,}0728158\ldots$. Furthermore

$$z\,\Gamma(z) = 1 - \gamma z + \tfrac{1}{2}\Gamma''(1)\, z^2 + \ldots, \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

where $\Gamma''(1) = \gamma^2 + \frac{1}{6}\pi^2$.

$$s^{-z} = 1 - z\log s + \tfrac{1}{2}(\log s)^2 z^2 - \ldots \quad . \quad . \quad . \quad . \quad . \quad (2.8)$$

$$\frac{z \log r}{1-r^{-z}} = 1 + \tfrac{1}{2} z \log r + \tfrac{1}{12} z^2 (\log r)^2 + \ldots \quad . \quad . \quad . \quad . \quad . \quad (2.9)$$

It follows from (2.6), (2.7), (2.8) and (2.9) that the residue of (2.5) at $z = 0$ equals

$$(\log s)^2/2\log r - \tfrac{1}{2}\log s + a_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.10)$$

where

$$a_0 = \{\gamma_2 - \tfrac{1}{2}\gamma^2 + \tfrac{1}{12}\pi^2 + \tfrac{1}{12}(\log r)^2\}/\log r. \quad . \quad . \quad . \quad (2.11)$$

The residues at $z = 2\pi ik/\log r$ ($k = \pm 1, \pm 2, \ldots$) are easily found to be

$$a_k\, s^{-2\pi ik/\log r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.12)$$

where

$$a_k = \Gamma\left(\frac{2\pi ik}{\log r}\right)\zeta\left(1 + \frac{2\pi ik}{\log r}\right)\Big/ \log r. \quad . \quad . \quad . \quad . \quad (2.13)$$

Finally we consider the poles $n = -1, -2, \ldots$ of $\Gamma(z)$. The residue of $\Gamma(z)$ at $z = -n$ is $(-1)^n/n!$; furthermore $\zeta(1-n) = (-1)^{n-1}\, n^{-1} B_n$, where the B_n are the BERNOULLI numbers defined by

$$y/(e^y - 1) = \sum_0^\infty y^n B_n/n! \quad ; \quad B_0 = 1, B_1 = -\tfrac{1}{2}, B_2 = \tfrac{1}{6}, B_3 = 0, \ldots$$

The residue of (2.5) at $z = -n$ thus amounts to

$$\beta_n s^n = \frac{B_n}{n \cdot n!} \cdot \frac{s^n}{r^n - 1} \qquad (n = 1, 2, 3, \ldots). \quad . \quad . \quad . \quad (2.14)$$

By taking the sum of the residues (2.10), (2.12), (2.14) we find

$$\log F(s) = \frac{(\log s)^2}{2\log r} - \tfrac{1}{2}\log s + \sum_{k=-\infty}^{\infty} a_k\, s^{-2\pi ik/\log r} + \sum_{n=1}^{\infty} \beta_n s^n. \quad (2.15)$$

The range of validity of (2.15), for which we took thus far $0 < s < 2\pi r$, can be continued over the semi-circle $Re\, s > 0$, $|s| < 2\pi r$. For, the last series on the right of (2.15) converges for $|s| < 2\pi r$, and the first one for $Re\, s > 0$. We have namely, for $k \to \pm\infty$,

$$a_k = O\left(e^{-\pi^2|k|/\log r}\,|k|^{-\frac{1}{2}}\log|k|\right) \quad . \quad . \quad . \quad . \quad . \quad (2.16)$$

since $|\Gamma(it)| \backsim \sqrt{2\pi}\, e^{-\frac{1}{2}\pi|t|}\,|t|^{-\frac{1}{2}}$ and $\zeta(1 + it) = O(\log|t|)$ for $t \to \pm\infty$.

3. *Preliminary estimations.*

In orther to deal with (1.16) we first derive some rough estimations for $F(\sigma + it)/F(\sigma)$. Here σ and t are real, $\sigma > 0$, and F is defined by (1.8) $=$ $=$ (2.1).

Lemma. We have [5])

$$|F(\sigma + it)/F(\sigma)| \leq 1 \qquad (\sigma > 0, -\infty < t < \infty) \quad . \quad . \quad . \quad . \quad (3.1)$$

$$|F(\sigma + it)/F(\sigma)| \leq \sigma^{c_1} \qquad (0 < \sigma \leq |t| \leq e^{-1}) \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

$$|F(\sigma + it)/F(\sigma)| \leq \sigma^{c_2 y^2} \qquad (0 \leq |t| \leq \sigma \leq 1, \sigma > 0, y = t/\sigma). \quad (3.3)$$

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[5]) The numbers $c_1, c_2, \ldots$ denote positive constants. If such a c occurs for the first time in a formula or statement we mean that it can be given a positive value such that the formula or statement is correct. At a second occurrence it keeps the value given to it the first time. The same applies to positive functions $c(\lambda)$ etc.

Proof. (3.1) immediately follows from

$$|1-e^{-\sigma r^k}| \leqslant |1-e^{-(\sigma+it)r^k}|.$$

More generally we have $(-\infty < t < \infty,\ \sigma > 0,\ K = 0, 1, 2, \ldots)$

$$\left|\frac{F(\sigma+it)}{F(\sigma)}\right| \leqslant \prod_{k=0}^{K} \left|\frac{1-e^{-(\sigma+it)r^k}}{1-e^{-\sigma r^k}}\right|^{-1}. \quad . \quad . \quad . \quad . \quad (3.4)$$

It is easily seen that

$$\left|\frac{1-e^{-(\sigma+it)r^k}}{1-e^{-\sigma r^k}}\right|^{-1} = \left\{1 + \frac{4\,e^{-\sigma r^k}\sin^2(\frac{1}{2}\,t\,r^k)}{(1-e^{-\sigma r^k})^2}\right\}^{-\frac{1}{2}}.$$

Assuming that

$$|t|\,r^K \leqslant 1 \quad , \quad \sigma r^K \leqslant 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

we have for $0 \leq k \leq K$

$$|\sin(\tfrac{1}{2}\,t\,r^k)| \geqslant \pi^{-1}\,|t|\,r^k,$$

$$\frac{(\sigma r^k)^2\,e^{-\sigma r^k}}{(1-e^{-\sigma r^k})^2} \geqslant \operatorname*{Min}_{e^{-1}\leq x\leq 1} \left(\frac{\log x^{-1}}{1-x}\right)^2 \cdot x > e^{-1}.$$

So if (3.5) holds, the right-hand-side of (3.4) is less than

$$\left\{1 + 4\left(\frac{t\,r^k}{\pi}\right)^2 \cdot (\sigma r^k)^{-2}\,e^{-1}\right\}^{-\frac{1}{2}(K+1)} \leqslant \left\{1 + \left(\frac{t}{4\sigma}\right)^2\right\}^{-\frac{1}{2}(K+1)}.$$

For K we may take

$$K = [\mathrm{Min}\,(\log\sigma^{-1}, \log|t|^{-1})/\log r]$$

so that for $0 < \sigma \leqslant 1, -1 \leqslant t \leqslant 1$ we obtain

$$\left|\frac{F(\sigma+it)}{F(\sigma)}\right| \leqslant \left\{1 + \left(\frac{t}{4\sigma}\right)^2\right\}^{\frac{\log \mathrm{Max}(\sigma, |t|)}{2\log r}} \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

We can now prove (3.2) and (3.3). First suppose $0 < \sigma \leq |t| \leq e^{-1}$. Putting $|t|\,\sigma^{-1} = y \geq 1$ we have, as $1 + 2^{-4}\,y^2 > (ey)^{c_3}$,

$$\log|F(\sigma+it)/F(\sigma)| \leqslant \frac{\log|t|}{2\log r} \cdot \log(1 + 2^{-4}\,y^2) \leqslant$$

$$\leqslant c_4 \log|t| \cdot \log e\,y = c_4 (\log\sigma + \log y)(1 + \log y).$$

The latter expression is a concave function of $\log y$, whose maximum in the interval $1 \leq y \leq e^{-1}\,\sigma^{-1}$ is attained either at $y = 1$ or at $y = e^{-1}\,\sigma^{-1}$. Both points give the value $c_4 \log \sigma$. This proves (3.2).

Second suppose $0 \leq |t| \leq \sigma \leq 1$, $\sigma > 0$. Since

$$1 + 2^{-4}\,y^2 \geqslant e^{c_5 y^2} \qquad (0 \leqslant y \leqslant 1)$$

we infer from (3.6)

$$|F(\sigma + it)/F(\sigma)| \leqslant e^{c_6 y^2 \log \sigma}$$

which proves (3.3).

4. *The asymptotic behaviour of $P_1(u)$.*

According to (2.15), a first approximation to $F(s)$ for s in the neighbourhood of the origin is $\exp\{(\log s)^2/2\log r\}$. If we had to evaluate the integral (cf. (1.16))

$$\int\limits_{a-i\infty}^{a+i\infty} e^{(\log s)^2/2\log r}\, e^{us/r}\, ds \qquad (a>0)$$

for large positive values of u we would choose an integration path passing through the saddle-point near the origin, i.e. the point $s=\sigma$, where σ satisfies

$$\frac{\log\sigma}{\sigma\log r} + u\,r^{-1} = 0, \qquad \sigma > 0. \quad \ldots \ldots \quad (4.1)$$

For $u>0$ the number σ is uniquely determined by (4.1). *Henceforth σ denotes this special function of u.* We now take $a=\sigma$ in (1.16) also:

$$P_1(u) = \frac{r}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} F(s)\, e^{us/r}\,\frac{ds}{s^2}. \quad \ldots \ldots \quad (4.2)$$

With the abbreviations

$$s = \sigma + it, \qquad t = y\sigma$$

$$\varkappa(u,y) = F(s)/F(\sigma)\cdot e^{iuy\sigma/r}(1+iy)^{-2} \quad \ldots \ldots \quad (4.3)$$

$$K(u) = \int\limits_{-\infty}^{\infty} \varkappa(u,y)\, dy. \quad \ldots \ldots \quad (4.4)$$

we have

$$P_1(u) = \frac{r}{2\pi\sigma} F(\sigma)\, e^{u\sigma/r} K(u) \quad \ldots \ldots \quad (4.5)$$

First we prove that for any number $\lambda>0$ we have

$$\left| K(u) - \int\limits_{-\lambda}^{\lambda} \varkappa(u,y)\, dy \right| \leqslant c_7\, \sigma^{c_8(\lambda)} \qquad (u > c_9) \quad \ldots \quad (4.6)$$

For $u\to\infty$ we have $\sigma\to 0$, and so c_9 can be chosen such that $u>c_9$ implies $\sigma<e^{-1}$. Now by (3.2), (3.3) and (4.3) we have for $u>c_9$

$$|\varkappa(u,y)| \leqslant \sigma^{c_{10}(\lambda)}(1+y^2)^{-1} \qquad (\lambda \leqslant |y| \leqslant e^{-1}\sigma^{-1})$$

with $c_{10}(\lambda) = \text{Min}\,(c_1, c_2\,\lambda^2)$ if $\lambda < 1$, $c_{10}(\lambda) = c_1$ if $\lambda \geq 1$. It follows, that, if $\lambda < e^{-1}\,\sigma^{-1}$,

$$\left| \int_{\lambda}^{e^{-1}\sigma^{-1}} \varkappa\,(u, y)\,dy \right| \leqslant \tfrac{1}{2}\,\pi\,\sigma^{c_{10}(\lambda)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

Furthermore we have, by (3.1), $|\varkappa(u, y)| \leq (1 + y^2)^{-1}$ for any u and y, and it follows that

$$\left| \int_{e^{-1}\sigma^{-1}}^{\infty} \varkappa\,(u, y)\,dy \right| \leqslant \int_{e^{-1}\sigma^{-1}}^{\infty} \frac{dy}{1+y^2} \leqslant e\,\sigma\,. \quad . \quad . \quad . \quad . \quad (4.8)$$

and also

$$\left| \int_{\lambda}^{\infty} \varkappa\,(u, y)\,dy \right| \leqslant e\,\sigma \text{ if } \lambda \geqslant e^{-1}\,\sigma^{-1}. \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

From (4.7), (4.8), (4.9) and the analogous inequalities for $y < 0$, (4.6) follows immediately ($c_7 = 2e$, $c_8\,(\lambda) = \text{Min}\ \{1, c_{10}\,(\lambda)\}$).

For a closer investigation of $\varkappa(u, y)$ for small values of $|y|$ we use (2.15). Introducing the abbreviations ($s = \sigma + i\,\sigma\,y$)

$$\sum_{-\infty}^{\infty} a_k\,s^{-2\pi i k/\log r} = g\,(s), \qquad \sum_{1}^{\infty} \beta_n\,s^n = \omega\,(s), \quad . \quad . \quad . \quad (4.10)$$

$$\left.\begin{aligned} \varrho\,(u, y) = \exp\,[\,\{\log^2(1 + iy) + 2\log\sigma \cdot \log\,(1 + iy)\}/2\log r - \\ - \tfrac{5}{2}\log\,(1 + iy) + i\,u\,y\,\sigma/r + g\,(s)] \end{aligned}\right\} \quad (4.11)$$

we have

$$\varkappa\,(u, y) = \varrho\,(u, y)\,e^{-g(\sigma) + \omega(s) - \omega(\sigma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.12)$$

If $\sigma < e^{-1}$, $|y| \leq 1$ we have

$$|\omega\,(s)| < c_{11}\,\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.13)$$

and by (2.16) and (4.10)

$$|g\,(\sigma + i\sigma y)| < c_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.14)$$

In virtue of (3.1) and (4.3) we have

$$|\varkappa(u, y)| \leqslant |\varkappa(u, 0)| = 1 \qquad (-\infty < y < \infty) \quad . \quad . \quad . \quad (4.15)$$

and it follows from (4.12), (4.13) and (4.14) that

$$\left| \int_{-\lambda}^{\lambda} \varkappa\,(u, y)\,dy - e^{-g(\sigma)} \int_{-\lambda}^{\lambda} \varrho\,(u, y)\,dy \right| < c_{13}\sigma \text{ for } 0 < \lambda < 1,\ u > c_9. \quad (4.16)$$

We transform $\int_{-\lambda}^{\lambda} \varrho(u, y)\, dy$ by introducing a new variable z by

$$\tfrac{1}{2} z^2 = \log(1 + iy) - iy. \qquad (4.17)$$

$$y = z + \mu_2 z^2 + \mu_3 z^3 + \ldots \qquad (|z| < c_{14}) \qquad (4.18)$$

On expressing u in terms of σ by (4.1) we obtain from (4.11)

$$\varrho(u, y) = \exp\left\{\frac{\log^2(1+iy) + z^2 \log \sigma}{2 \log r} - \tfrac{5}{2}\log(1+iy) + g(\sigma + i\sigma y)\right\} (4.19)$$

If we put, for a moment,

$$\log \sigma / \log r = v$$

the function $e^{g(s)}$ becomes an analytical function of the variables v and y in the range $|Im\, v| < \frac{1}{2}\pi/\log r$, $|y| < \cos\,(Im\, v \,\log r)$, since $g(s)$ is analytical for $Re\, s > 0$ (cf. (2.16)). Moreover it is a periodic function of v with period 1. It follows that $e^{g(s)}$ can be written in the form

$$e^{g(\sigma + i\sigma y)} = \sum_{n=0}^{\infty} \chi_n(v)\, y^n \qquad (4.20)$$

where the functions $\chi_n(v)$ are analytical in the strip $|Im\, v| < \frac{1}{2}\pi/\log r$, periodical mod 1, and satisfy

$$|\chi_n(v)| < c_{15}^{n+1} \qquad (-\infty < v < \infty,\ n = 0, 1, 2, \ldots).$$

If v is real (4.20) converges for $|y| < 1$.

We now easily deduce from (4.19) and (4.20) that $\varrho(u, y)\dfrac{dy}{dz}$ can be written in the form (σ real)

$$\varrho(u, y)\frac{dy}{dz} = e^{z^2 \log \sigma / 2 \log r} \sum_{n=0}^{\infty} \psi_n\left(\frac{\log \sigma}{\log r}\right) z^n \qquad (|z| < c_{16}). \quad (4.21)$$

Again, $\psi_n(v)$ is analytical for $|Im\, v| < \frac{1}{2}\pi/\log r$ and

$$|\psi_n(v)| < c_{17}^{n+1} \qquad (-\infty < v < \infty,\ n = 0, 1, 2, \ldots).$$

It is easily verified that

$$\psi_0(\log \sigma / \log r) = e^{g(\sigma)}. \qquad (4.22)$$

Now take $c_{18} < 1$ such that $-c_{18} \leq y \leq c_{18}$ implies $|z| < \frac{1}{2}c_{14}$, $c_{17}|z| < \frac{1}{2}$ and either $|\arg z| < \frac{1}{8}\pi$ or $|\arg -z| < \frac{1}{8}\pi$. Let ζ_1 and ζ_2 denote the values of z for $y = -c_{18}$ and $y = +c_{18}$, respectively, and put $Re\, \zeta_1^2 = Re\, \zeta_2^2 = c_{19}$. Now the integral

$$\int_{-c_{18}}^{c_{18}} \varrho(u, y)\, dy$$

can be expanded in a familiar way by means of (4.21): for any positive integer N we have $(u > c_9,\ \sigma < e^{-1})$

$$\left.\begin{aligned}&\left|\int_{-c_{18}}^{c_{18}} \varrho(u,y)\,dy - \sum_{n=0}^{2N+1} \psi_n\left(\frac{\log\sigma}{\log r}\right)\int_{-\infty}^{\infty} e^{-z^2\frac{\log\sigma^{-1}}{2\log r}} z^n\,dz\right| < \\ &< c_{20}(N)\,e^{-c_{18}\log\sigma^{-1}/2\log r} + \sum_{2N+2}^{\infty} c_{17}^{n+1}\left|\int_{\zeta_1}^{\zeta_2} e^{-z^2\frac{\log\sigma^{-1}}{2\log r}} z^n\,dz\right|.\end{aligned}\right\} \quad (4.23)$$

On taking for our integration path the broken line consisting of the segments $(\zeta_1, 0)$ and $(0, \zeta_2)$, we find because of

$$|c_{17}\,\zeta_{1,2}| < \tfrac{1}{2},\ |\arg -\zeta_1| < \tfrac{1}{8}\pi,\ |\arg \zeta_2| < \tfrac{1}{8}\pi$$

(put $|z| = t$ on both parts of the path) if $n \geq 2N+2$:

$$\left|\int_{\zeta_1}^{\zeta_2} e^{-z^2\log\sigma^{-1}/2\log r}\,z^n\,dz\right| <$$

$$< \int_{-|\zeta_1|}^{|\zeta_2|} (2c_{17})^{-n+2N+2}\,e^{-t^2\log\sigma^{-1}/4\log r}\,|t|^{2N+2}\,dt < \int_{-\infty}^{\infty}.$$

It now follows from (4.23) that $(u > c_9,\ \sigma < e^{-1})$

$$\left.\begin{aligned}&\left|\int_{-c_{18}}^{c_{18}} \varrho(u,y)\,dy - \right.\\ &-\left(\frac{2\pi\log r}{\log\sigma^{-1}}\right)^{\frac{1}{2}} \sum_{m=0}^{N}\left(\frac{\log r}{\log\sigma^{-1}}\right)^m \frac{(2m)!}{2^m m!}\,\psi_{2m}\left(\frac{\log\sigma}{\log r}\right)\left| < \frac{c_{21}(N)}{(\log\sigma^{-1})^{N+\frac{3}{2}}}.\right.\end{aligned}\right\} \quad (4.24)$$

On taking $\lambda = c_{18}$ in (4.6) and (4.16) we find (cf. (4.5), (4.1), (2.15), (4.13) and (4.14))

$$\left.\begin{aligned}P_1(u) = r\sqrt{\frac{\log r}{2\pi}}\,e^{\frac{(\log\sigma)^2}{2\log r} - \left(\frac{3}{2}+\frac{1}{\log r}\right)\log\sigma - \frac{1}{2}\log\log\sigma^{-1}} \times \\ \times\left\{\sum_{m=0}^{N}\frac{\varphi_m\left(\dfrac{\log\sigma}{\log r}\right)}{(\log\sigma^{-1})^m} + O\left(\frac{1}{(\log\sigma^{-1})^{N+1}}\right)\right\},\end{aligned}\right\} \quad . \quad . \quad (4.25)$$

where the functions

$$\varphi_m(v) = (\log r)^m \frac{(2m)!}{2^m m!}\,\psi_{2m}(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.26)$$

are periodic functions of v with period 1, analytical in $|Im\,v| < \frac{1}{2}\pi/\log r$. Especially

$$\varphi_0\left(\frac{\log\sigma}{\log r}\right) = e^{g(\sigma)} = \exp\left\{\sum_{-\infty}^{\infty} a_k\,e^{-2\pi i k\log\sigma/\log r}\right\}. \quad . \quad . \quad (4.27)$$

Formula (4.25) is our final result for the asymptotic behaviour of $P_1(u)$: σ is related to u by (4.1) and cannot be expressed explicitly in terms of u in a simple way.

5. *Final results concerning* $P(u)$.

The difference $P(u) - P_1(u)$ is relatively small. We have, by (1.15)

$$P(u-1) \leqslant P_1(u) \leqslant P(u), \qquad (5.1)$$

and on the other hand, by (1.9),

$$P(u) - P(u-1) = P(u/r) \leqslant P_1(u r^{-1} + 1). \qquad (5.2)$$

It follows that

$$0 \leqslant P(u) - P_1(u) \leqslant P_1(u r^{-1} + 1). \qquad (5.3)$$

In order to show that $P_1(ur^{-1} + 1)/P_1(u)$ is small we first give a first-order asymptotic expression for $P_1(u)$ explicitly in terms of u. It is readily derived from (4.1),

$$\sigma^{-1} = u r^{-1} \log r \cdot \log \sigma^{-1},$$

that, if $u \to \infty$,

$$\log \sigma^{-1} = \log u - \log\log u + \log\log r - \log r + O(\log\log u/\log u),$$

$$\log\, \log \sigma^{-1} = \log\log u + O(\log\log u/\log u),$$

$$\log^2 \sigma = (\log u - \log\log u)^2 - 2\log(r/\log r)\cdot(\log u - \log\log u) +$$
$$+ 2\log\log u + \log^2(r/\log r) + 2\log(r/\log r) + O\{(\log\log u)^2/\log u\},$$

and we obtain from (4.25)

$$\left.\begin{aligned} \log P_1(u) = {} & \frac{(\log u - \log\log u + \log\log r)^2}{2\log r} + \\ & + \left\{\frac{1}{2} + \frac{1}{\log r}\right\}\log u - \log\log u + \log\log r - \tfrac{1}{2}\log 2\pi + \\ & + \sum_{-\infty}^{\infty} a_k \exp\left\{2\pi i k\left(\frac{\log u - \log\log u + \log\log r}{\log r}\right)\right\} + \\ & \qquad\qquad + O\{(\log\log u)^2/\log u\}. \end{aligned}\right\} \qquad (5.4)$$

The a_n are defined by (2.11) and (2.13).

From (5.4) it is easily deduced that

$$P_1(u r^{-1} + 1)/P_1(u) = O\{\exp(-\log u + \log\log u)\} = O(u^{-\frac{1}{2}}).$$

Now (5.3) shows that (4.25) and (5.4) remain valid if $P_1(u)$ is replaced by $P(u)$.

If r and rh are integers we have $P(h) = p(rh)$; thus (5.4) proves (1.4). The more precise expansion (4.25) however cannot easily be expressed explicitly in terms of u (or rh).

March 1948.

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Mathematics. — *Über mehrwertige Aussagenkalküle und mehrwertige engere Prädikatenkalküle*. I. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of April 24, 1948.)

In einem neuerdings erschienenen Buche [1]) von S. A. KISS: *Transformations on lattices and structures of logic*, New York 1947, wird in inhaltlicher Weise eine 2^2-wertige Aussagenlogik eingeführt. Im folgenden wollen wir zeigen: 1° dass eine derartige Logik sich auch *formal* aufbauen lässt; 2° dass dieser Aufbau zu einem Aussagenlogik führt, welche *nicht nur eine 2^2-wertige Deutung, sondern sogar allgemein eine $2^n \times 2^m$-wertige Deutung* (n und m natürliche Zahlen) zulässt.

Eine Erweiterung des angewandten Verfahrens führt zu *Aussagenkalkülen mit $2^{n_1} \times 2^{n_2} \times \ldots \times 2^{n_m}$-wertigen Deutungen* ($n_1, n_2, \ldots, n_m$ natürliche Zahlen; m ganz und $\geqq 3$).

Völlig gleichartige Bemerkungen gelten für den engeren Prädikatenkalkül.

Erster Aufbau eines Aussagenkalküls $K^{(2)}$.

§ 1. Von den im folgenden durch grosse lateinische Buchstaben $A, B, C, \ldots$ [2]) angedeuteten *elementaren Aussagen* (*elementaren Kalkülformeln*) wird *nicht* angenommen, dass sie entweder wahr oder falsch sind. Wir lassen sowohl die Möglichkeit von endlich- wie die von abzählbar unendlich vielen elementaren Aussagen zu. Zu ihnen rechnen wir noch die durch die griechischen Buchstaben λ, ν, λ_2 und ν_2 dargestellten.

Als undefinierte *Grundverknüpfungen* nehmen wir zwei zweistellige Verknüpfungen $+$, $+_2$, und eine dritte einstellige Verknüpfung: *Komplement von*, angedeutet durch ein Akzent. Es gibt somit Kalkülformeln von der Art wie $X + Y$, $X +_2 Y$, X', Y'.

Einsetzungsregel E (erster Teil). Aus einer *Kalkülformel* erhält man wieder eine Kalkülformel, wenn man einen in ihr auftretenden grossen lateinischen Buchstaben durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln); dabei sind die grossen lateinischen Buchstaben, λ, ν, λ_2 und ν_2 als Kalkülformeln anzusehen, ferner mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R} + \mathfrak{S}$, $\mathfrak{R} +_2 \mathfrak{S}$, $\mathfrak{R}'$, $\mathfrak{S}'$ [3]).

Definition 1. $\mathfrak{A} \rightarrow \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$.

Definition 1ter. $\mathfrak{A} \rightarrow_3 \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' +_2 \mathfrak{B}$.

$\doteq \bar{\lambda}$, $\doteq \bar{\nu}$, $\doteq \bar{\lambda}_2$ und $\doteq \bar{\nu}_2$ sind *Modalitätszeichen*.

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1) Ir. TH. W. TE NUYL stellte mir freundlichst sein Exemplar zur Verfügung.

2) Daneben auch: $A_1, B_1, C_1, \ldots; A_2, B_2, \ldots; \ldots; A_n, \ldots$ (n eine natürliche Zahl).

3) Grosse deutsche Buchstaben werden immer Kalkülformeln andeuten.

Definition 2. Ein Ausdruck, bestehend aus einer Kalkülformel, gefolgt durch eines der Modalitätszeichen, und entstanden nach endlichmaliger Anwendung von Axiomen, Einsetzungsregel E, Schlussschema S oder (und) Verbindungsregel V, ist *ein Theorem*.

So sind alle Axiome als Theoreme zu betrachten.

Einsetzungsregel E (zweiter Teil). Ist $\mathfrak{A} \doteq \bar{\nu}\ [\mathfrak{A} \doteq \bar{\nu}_2]$ ein Theorem, und $\mathfrak{B}$ eine neue, aus $\mathfrak{A}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformel, so liefert auch $\mathfrak{B} \doteq \bar{\nu}\ [\mathfrak{B} \doteq \bar{\nu}_2]$ ein Theorem.

Erste Gruppe von Axiomen.

Axiom I. $[(X + X) \rightarrow X] \doteq \nu$.

Axiom II. $[X \rightarrow (X + Y)] \doteq \bar{\nu}$.

Axiom III. $[(X + Y) \rightarrow (Y + X)] \doteq \bar{\nu}$.

Axiom IV. $[(Y \rightarrow Z) \rightarrow \{(X + Y) \rightarrow (X + Z)\}] \doteq \bar{\nu}$.

Axiom V. $(\lambda \rightarrow X) \doteq \bar{\nu}$.

Axiom VI. $(X \rightarrow \nu) \doteq \bar{\nu}$.

Schlussschema S. Sind $\mathfrak{A} \doteq \bar{\nu}\ [\mathfrak{A} \doteq \bar{\nu}_2]$ und $(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \bar{\nu}\ [(\mathfrak{A} \rightarrow_3 \mathfrak{B}) \doteq \bar{\nu}_2]$ Theoreme, so ist auch

$$\mathfrak{B} \doteq \nu \qquad [\mathfrak{B} \doteq \bar{\nu}_2]$$

ein Theorem.

Zweite Gruppe von Axiomen.

Axiom I$^{\text{ter}}$. $[(X +_2 X) \rightarrow_3 X] \doteq \bar{\nu}_2$.

Axiom II$^{\text{ter}}$. $[X \rightarrow_3 (X +_2 Y)] \doteq \bar{\nu}_2$.

Axiom III$^{\text{ter}}$. $[(X +_2 Y) \rightarrow_3 (Y +_2 X)] \doteq \bar{\nu}_2$.

Axiom IV$^{\text{ter}}$. $[(Y \rightarrow_3 Z) \rightarrow_3 \{(X +_2 Y) \rightarrow_3 (X +_2 Z)\}] \doteq \nu_2$.

Axiom V$^{\text{ter}}$. $[\lambda_2 \rightarrow_3 X] \doteq \bar{\nu}_2$.

Axiom VI$^{\text{ter}}$. $[X \rightarrow_3 \nu_2] \doteq \nu_2$.

Verbindungsregel V. Sind

$$(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \nu \quad \text{und} \quad (\mathfrak{B} \rightarrow \mathfrak{A}) \doteq \bar{\nu}$$

Theoreme, so gilt dasselbe von

$$(\mathfrak{A} \rightarrow_3 \mathfrak{B}) \doteq \bar{\nu}_2 \quad \text{und} \quad (\mathfrak{B} \rightarrow_3 \mathfrak{A}) \doteq \bar{\nu}_2,$$

und umgekehrt.

Definition 3. Eine andere Schreibweise eines Theorems $\mathfrak{A} \doteq \bar{\nu}\ [\mathfrak{A}' \doteq \nu]$ ist $\mathfrak{A}' \doteq \bar{\lambda}\ [\mathfrak{A} \doteq \lambda]$; eine andere Schreibweise eines Theorems $\mathfrak{A} \doteq \nu_2\ [\mathfrak{A}' \doteq \nu_2]$ ist $\mathfrak{A}' \doteq \bar{\lambda}_2\ [\mathfrak{A} \doteq \lambda_2]$.

Die mit den Regeln E und S aus der ersten Gruppe von Axiomen ableitbaren Theoreme stehen dual gegenüber den mit diesen Regeln aus der zweiten Gruppe von Axiomen ableitbaren Theoremen.

Daneben haben wir noch das weiter unten folgende Dualitätsprinzip.

Definition 4. $\mathfrak{A} \cdot \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' + \mathfrak{B}')'$.

Definition 4$^{\text{bis}}$. $\mathfrak{A} \cdot_2 \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' +_2 \mathfrak{B}')'$.

Definition 1bis. $\mathfrak{A} \rightarrow_2 \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' \cdot \mathfrak{B}$.

Definition 1quater. $\mathfrak{A} \rightarrow_4 \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' \cdot_2 \mathfrak{B}$.

Dualitätsprinzip [4]. Ist $\mathfrak{A} \doteq \bar{\nu}$ $[\mathfrak{A} \doteq \bar{\nu}_2]$ ein Theorem, so auch $\mathfrak{B} \doteq \bar{\lambda}$ $[\mathfrak{B} \doteq \bar{\lambda}_2]$; dabei gehe Kalkülformel $\mathfrak{B}$ aus Kalkülformel $\mathfrak{A}$ dadurch hervor, dass: α) $+$ durch $\cdot$, und umgekehrt; $+_2$ durch $\cdot_2$, und umgekehrt, β) λ durch ν, und umgekehrt; λ_2 durch ν_2, und umgekehrt, γ) $\rightarrow$ durch $\rightarrow_2$, und umgekehrt; $\rightarrow_3$ durch $\rightarrow_4$, und umgekehrt, ersetzt werden; ev. vorkommende Akzenten sollen ungeändert bleiben. Ist, umgekehrt, bei denselben Kalkülformeln $\mathfrak{A}$ und $\mathfrak{B}$ $\mathfrak{B} \doteq \bar{\lambda}$ $[\mathfrak{B} \doteq \bar{\lambda}_2]$ ein Theorem, so auch $\mathfrak{A} \doteq \bar{\nu}$ $[\mathfrak{A} \doteq \bar{\nu}_2]$.

§ 2. Nach dem Vorigen wird es deutlich sein, *dass obiges System von Definitionen, Axiomen und Regeln dual steht gegenüber folgendem System, mit welchem es aequivalent ist* [5].

Die undefinierten Grundverknüpfungen werden diesmal angegeben durch $\cdot$, $\cdot_2$ und ein Akzent.

Einsetzungsregel E* (erster Teil). Aus einer Kalkülformel erhält man wieder eine Kalkülformel, wenn man einen in ihr auftretenden grossen lateinischen Buchstaben durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln); dabei sind die grossen lateinischen Buchstaben, λ, ν, λ_2 und ν_2 als Kalkülformeln anzusehen; ferner mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R} \cdot \mathfrak{S}$, $\mathfrak{R} \cdot_2 \mathfrak{S}$, $\mathfrak{R}'$, $\mathfrak{S}'$ [3]).

Statt der Definitionen 4 und 4bis brauchen wir hier die Definitionen 4* und 4*bis.

Definition 4*. $\mathfrak{A} + \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' \cdot \mathfrak{B}')'$.

Definition 4*bis. $\mathfrak{A} +_2 \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' \cdot_2 \mathfrak{B}')'$.

Die Definitionen 1—1quater lassen wir auch hier gelten.

Statt Definition 2 kommt die

Definition 2*. Ein Ausdruch, bestehend aus einer Kalkülformel, gefolgt durch eines der Modalitätszeichen (§ 1), und entstanden nach endlichmaliger Anwendung der Axiome und Regeln E*, S*, V* dieses Paragraphen, ist *ein Theorem*.

Einsetzungsregel E* (zweiter Teil). Ist $\mathfrak{A} \doteq \bar{\lambda}$ $[\mathfrak{A} \doteq \bar{\lambda}_2]$ ein Theorem, und $\mathfrak{B}$ eine neue, aus $\mathfrak{A}$ mittels der Einsetzungsregel E* (erster Teil) hervorgehende Kalkülformel, so liefert auch $\mathfrak{B} \doteq \bar{\lambda}$ $[\mathfrak{B} \doteq \bar{\lambda}_2]$ ein Theorem.

Erste Gruppe von Axiomen.

Axiom Ibis. $[(X \cdot X) \rightarrow_2 X] \doteq \bar{\lambda}$.

Axiom IIbis. $[X \rightarrow_2 (X \cdot Y)] \doteq \bar{\lambda}$.

Axiom IIIbis. $[(X \cdot Y) \rightarrow_2 (Y \cdot X)] \doteq \bar{\lambda}$.

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[4]) Zum Beweise vergleiche man J. RIDDER, Ueber den Aussagen- und den engeren Prädikatenkalkül I u. II. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam **49**, 1153—1164 (1946), und **50**, 24—30 (1947), insbes. 1153, 1154, 1156 und 26, 27 (Satz 42).

[5]) Vergleiche loc. cit. [4]), S. 27.

Axiom IVbis. $[(Y \rightarrow_2 Z) \rightarrow_2 \{(X \cdot Y) \rightarrow_2 (X \cdot Z)\}] \doteq \bar{\lambda}$.

Axiom Vbis. $(\nu \rightarrow_2 X) \doteq \bar{\lambda}$.

Axiom VIbis. $(X \rightarrow_2 \lambda) \doteq \bar{\lambda}$.

Schlussschema S*. Sind $\mathfrak{A} \doteq \bar{\lambda}$ $[\mathfrak{A} \doteq \bar{\lambda}_2]$ und $(\mathfrak{A} \rightarrow_2 \mathfrak{B}) \doteq \bar{\lambda}$ $[(\mathfrak{A} \rightarrow_4 \mathfrak{B}) \doteq \bar{\lambda}_2]$. Theoreme, so ist auch

$$\mathfrak{B} \doteq \bar{\lambda} \qquad [\mathfrak{B} \doteq \bar{\lambda}_2]$$

ein Theorem.

Zweite Gruppe von Axiomen.

Axiom Iquater. $[(X \cdot_2 X) \rightarrow_4 X] \doteq \bar{\lambda}_2$.

Axiom IIquater. $[X \rightarrow_4 (X \cdot_2 Y)] \doteq \bar{\lambda}_2$.

Axiom IIIquater. $[(X \cdot_2 Y) \rightarrow_4 (Y \cdot_2 X)] \doteq \bar{\lambda}_2$.

Axiom IVquater. $[(Y \rightarrow_4 Z) \rightarrow_4 \{(X \cdot_2 Y) \rightarrow_4 (X \cdot_2 Z)\}] \doteq \bar{\lambda}_2$.

Axiom Vquater. $(\nu_2 \rightarrow_4 X) \doteq \bar{\lambda}_2$.

Axiom VIquater. $(X \rightarrow_4 \lambda_2) \doteq \bar{\lambda}_2$.

Verbindungsregel V*. Sind

$$(\mathfrak{A} \rightarrow_2 \mathfrak{B}) \doteq \bar{\lambda} \quad \text{und} \quad (\mathfrak{B} \rightarrow_2 \mathfrak{A}) \doteq \bar{\lambda}$$

Theoreme, so gilt dasselbe von

$$(\mathfrak{A} \rightarrow_4 \mathfrak{B}) \doteq \bar{\lambda}_2 \quad \text{und} \quad (\mathfrak{B} \rightarrow_4 \mathfrak{A}) \doteq \bar{\lambda}_2,$$

und umgekehrt.

Definition 3*. Eine andere Schreibweise eines Theorems $\mathfrak{A} \doteq \bar{\lambda}$ $[\mathfrak{A}' \doteq \bar{\lambda}]$ ist $\mathfrak{A}' \doteq \bar{\nu}$ $[\mathfrak{A} \doteq \nu]$; eine andere Schreibweise eines Theorems $\mathfrak{A} \doteq \bar{\lambda}_2$ $[\mathfrak{A}' \doteq \bar{\lambda}_2]$ ist $\mathfrak{A}' \doteq \nu_2$ $[\mathfrak{A} \doteq \bar{\nu}_2]$.

§ 3. Ausgehend von den undefinierten Grundbegriffen $+$, $\cdot_2$ und $'$ oder von den undefinierten Grundbegriffen $\cdot$, $+_2$ und $'$ erhält man leicht zwei weitere, mit den beiden vorigen Systemen aequivalente Systeme von Definitionen, Axiomen und Regeln.

§ 4. **Definition 5.** $\mathfrak{A} \subset \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \bar{\nu}$.

Definition 5bis. $\mathfrak{A} \supset \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow_2 \mathfrak{B}) \doteq \bar{\lambda}$.

Definition 5ter. $\mathfrak{A} \subset_2 \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow_3 \mathfrak{B}) \doteq \bar{\nu}_2$.

Definition 5quater. $\mathfrak{A} \supset_2 \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow_4 \mathfrak{B}) \doteq \bar{\lambda}_2$.

Aus dem Axiomensystem des Par. 1 lassen sich die folgenden Sätze ableiten [6].

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[6]) Für die Beweise vergleiche man loc. cit. [4]), S. 1154—1158.

Satz 1. α) $\mathfrak{A} \subset \mathfrak{A}$ ist ein Theorem;

β) (Schlussschema) hat man als Theoreme $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{B} \subset \mathfrak{C}$, so gibt es auch ein Theorem $\mathfrak{A} \subset \mathfrak{C}$.

Satz 1ter. α) $\mathfrak{A} \subset_2 \mathfrak{A}$ ist ein Theorem;

β) (Schlussschema) hat man als Theoreme $\mathfrak{A} \subset_2 \mathfrak{B}$, $\mathfrak{B} \subset_2 \mathfrak{C}$, so gibt es auch ein Theorem $\mathfrak{A} \subset_2 \mathfrak{C}$.

Satz 2. α) $(\mathfrak{A} \cdot \mathfrak{B}) \subset \mathfrak{A}$ und $(\mathfrak{A} \cdot \mathfrak{B}) \subset \mathfrak{B}$ (sind Theoreme);

β) (Schlussschema) aus den Theoremen $\mathfrak{C} \subset \mathfrak{A}$ und $\mathfrak{C} \subset \mathfrak{B}$ folgt, dass sich auch schreiben lässt $\mathfrak{C} \subset (\mathfrak{A} \cdot \mathfrak{B})$.

Satz 2ter. α) $(\mathfrak{A} \cdot_2 \mathfrak{B}) \subset_2 \mathfrak{A}$ und $(\mathfrak{A} \cdot_2 \mathfrak{B}) \subset_2 \mathfrak{B}$ (sind Theoreme);

β) (Schlussschema) aus den Theoremen $\mathfrak{C} \subset_2 \mathfrak{A}$ und $\mathfrak{C} \subset_2 \mathfrak{B}$ folgt, dass sich auch schreiben lässt $\mathfrak{C} \subset_2 (\mathfrak{A} \cdot_2 \mathfrak{B})$.

Satz 3. α) $\mathfrak{A} \subset (\mathfrak{A} + \mathfrak{B})$ und $\mathfrak{B} \subset (\mathfrak{A} + \mathfrak{B})$;

β) (Schlussschema) aus $\mathfrak{A} \subset \mathfrak{C}$ und $\mathfrak{B} \subset \mathfrak{C}$ folgt, dass sich auch schreiben lässt $(\mathfrak{A} + \mathfrak{B}) \subset \mathfrak{C}$.

Satz 3ter. α) $\mathfrak{A} \subset_2 (\mathfrak{A} +_2 \mathfrak{B})$ und $\mathfrak{B} \subset_2 (\mathfrak{A} +_2 \mathfrak{B})$;

β) (Schlussschema) aus $\mathfrak{A} \subset_2 \mathfrak{C}$ und $\mathfrak{B} \subset_2 \mathfrak{C}$ folgt, dass sich auch schreiben lässt $(\mathfrak{A} +_2 \mathfrak{B}) \subset_2 \mathfrak{C}$.

Satz 4. $\lambda \subset \mathfrak{A}$.

Satz 4ter. $\lambda_2 \subset_2 \mathfrak{A}$.

Satz 5. $\mathfrak{A} \subset \nu$.

Satz 5ter. $\mathfrak{A} \subset_2 \nu_2$.

Definition 6. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzere Schreibweise von: $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$, oder — was wegen Verbindungsregel V, Definition 5 und Definition 5ter auf dasselbe hinauskommt — von: $\mathfrak{A} \subset_2 \mathfrak{B}$ und $\mathfrak{B} \subset_2 \mathfrak{A}$.

Satz 6. $[\mathfrak{A} \cdot (\mathfrak{B} + \mathfrak{C})] = [(\mathfrak{A} \cdot \mathfrak{B}) + (\mathfrak{A} \cdot \mathfrak{C})]$.

Satz 6ter. $[\mathfrak{A} \cdot_2 (\mathfrak{B} +_2 \mathfrak{C})] = [(\mathfrak{A} \cdot_2 \mathfrak{B}) +_2 (\mathfrak{A} \cdot_2 \mathfrak{C})]$.

Satz 7. α) $(\mathfrak{A} \cdot \mathfrak{A}') = \lambda$; β) $(\mathfrak{A} + \mathfrak{A}') = \nu$.

Satz 7ter. α) $(\mathfrak{A} \cdot_2 \mathfrak{A}') = \lambda_2$; β) $(\mathfrak{A} +_2 \mathfrak{A}') = \nu_2$.

Satz 8 (Satz 8ter). Lässt sich schreiben $\mathfrak{A} \doteq \bar{\nu}$, so auch $\mathfrak{A} = \nu$, und umgekehrt. [Lässt sich schreiben $\mathfrak{A} \doteq \bar{\nu}_2$, so auch $\mathfrak{A} = \nu_2$, und umgekehrt].

Satz 9 (Satz 9ter). Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$ [$\mathfrak{A} \subset_2 \mathfrak{B}$], und sind $\mathfrak{D}$ und $\mathfrak{E}$ neue, aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende, gleichgestaltete lateinische Buchstaben in beiden nicht oder in beiden an allen Stellen in gleicher Weise (d.h. durch gleichgestaltete Kalkülformeln) ersetzt sind, so hat man auch $\mathfrak{D} \subset \mathfrak{E}$ [$\mathfrak{D} \subset_2 \mathfrak{E}$].

Satz 10. Sind $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$ Theoreme, so gilt dasselbe von $\mathfrak{A} \subset_2 \mathfrak{B}$ und $\mathfrak{B} \subset_2 \mathfrak{A}$, und umgekehrt.

Zweiter Aufbau des Aussagenkalküls $K^{(2)}$.

§ 5. *Elementare Aussagen (elementare Kalkülformeln)* sollen wieder durch grosse lateinische Buchstaben [2]) und λ, ν, λ_2, ν_2 angedeutet werden.

Als undefinierte *Grundverknüpfungen* nehmen wir sechs zweistellige

Verknüpfungen $\subset$, $\subset_2$, $+$, $\cdot$, $+_2$, $\cdot_2$ und eine einstellige Verknüpfung: *Komplement von*, angedeutet durch ein Akzent.

Einsetzungsregel E_0 (erster Teil) habe den gleichen Wortlaut wie Einsetzungsregel E (erster Teil) mit dem Unterschiede, dass mit $\mathfrak{R}$ und $\mathfrak{S}$ (neben $\mathfrak{R} + \mathfrak{S}$, $\mathfrak{R} +_2 \mathfrak{S}$, $\mathfrak{R}'$, $\mathfrak{S}'$) auch $\mathfrak{R} \cdot \mathfrak{S}$ und $\mathfrak{R} \cdot_2 \mathfrak{S}$ als Formeln betrachtet werden sollen.

Den Inhalt der Sätze 9 und 9ter nehmen wir als zweiten Teil von Regel E_0 an. Also:

Einsetzungsregel E_0 (zweiter Teil). Lässt sich (auf Grund der nachfolgenden Axiome und Regel) schreiben $\mathfrak{A} \subset \mathfrak{B}$ [$\mathfrak{A} \subset_2 \mathfrak{B}$], mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, und sind $\mathfrak{D}$ und $\mathfrak{E}$ aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E_0 (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete lateinische Buchstaben in beiden nicht oder in beiden an allen Stellen durch gleichgestaltete Kalkülformeln ersetzt sind, so lässt sich auch schreiben $\mathfrak{D} \subset \mathfrak{E}$ [$\mathfrak{D} \subset_2 \mathfrak{E}$] [3]).

Definition 2_0. Ein Ausdruck bestehend aus zwei Kalkülformeln, getrennt durch $\subset$, $\supset$, $\subset_2$ oder $\supset_2$, und entstanden nach endlichmaliger Anwendung der hier folgenden Axiome, Einsetzungsregel E_0 oder (und) Verbindungsregel V_0 (nebst Definitionen), ist *ein Theorem*.

Definition 3_0. Eine andere Schreibweise eines Theorems $\mathfrak{A} \subset \mathfrak{B}$ ist $\mathfrak{B} \supset \mathfrak{A}$; eine andere Schreibweise eines Theorems $\mathfrak{A} \subset_2 \mathfrak{B}$ ist $\mathfrak{B} \supset_2 \mathfrak{A}$.

Erste und zweite Gruppe von Axiomen.

Axiom 1. α) $X \subset X$ (ist ein Theorem);

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{C}$, so ist auch $\mathfrak{A} \subset \mathfrak{C}$ ein Theorem.

Axiom 1ter. α) $X \subset_2 X$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset_2 \mathfrak{B}$ und $\mathfrak{B} \subset_2 \mathfrak{C}$, so ist auch $\mathfrak{A} \subset_2 \mathfrak{C}$ ein Theorem.

Axiom 2. α) $(X \cdot Y) \subset X$ und $(X \cdot Y) \subset Y$;

β) (Schlussschema) sind $\mathfrak{C} \subset \mathfrak{A}$ und $\mathfrak{C} \subset \mathfrak{B}$ Theoreme, so ist auch $\mathfrak{C} \subset (\mathfrak{A} \cdot \mathfrak{B})$ ein Theorem.

Axiom 2ter. α) $(X \cdot_2 Y) \subset_2 X$ und $(X \cdot_2 Y) \subset_2 Y$;

β) (Schlussschema) sind $\mathfrak{C} \subset_2 \mathfrak{A}$ und $\mathfrak{C} \subset_2 \mathfrak{B}$ Theoreme, so ist auch $\mathfrak{C} \subset_2 (\mathfrak{A} \cdot_2 \mathfrak{B})$ ein Theorem.

Axiom 3. α) $X \subset (X + Y)$ und $Y \subset (X + Y)$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset \mathfrak{C}$ und $\mathfrak{B} \subset \mathfrak{C}$, so auch $(\mathfrak{A} + \mathfrak{B}) \subset \mathfrak{C}$.

Axiom 3ter. α) $X \subset_2 (X +_2 Y)$ und $Y \subset_2 (X +_2 Y)$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset_2 \mathfrak{C}$ und $\mathfrak{B} \subset_2 \mathfrak{C}$, so auch $(\mathfrak{A} +_2 \mathfrak{B}) \subset_2 \mathfrak{C}$.

Axiom 4. $\lambda \subset X$.

Axiom 4ter. $\lambda_2 \subset_2 X$.

Axiom 5. $X \subset \nu$.

Axiom 5ter. $X \subset_2 \nu_2$.

Verbindungsregel V_0. Sind $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$ Theoreme, so gilt dasselbe von $\mathfrak{A} \subset_2 \mathfrak{B}$ und $\mathfrak{B} \subset_2 \mathfrak{A}$, und umgekehrt.

Definition 6_0. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzere Schreibweise von: $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$, oder — was wegen Verbindungsregel V_0 auf dasselbe hinauskommt — von: $\mathfrak{A} \subset_2 \mathfrak{B}$ und $\mathfrak{B} \subset_2 \mathfrak{A}$.

Axiom 6. $[X \cdot (Y + Z)] = [(X \cdot Y) + (X \cdot Z)]$.

Axiom 6ter. $[X \cdot_2 (Y +_2 Z)] = [(X \cdot_2 Y) +_2 (X \cdot_2 Z)]$.

Axiom 7. α) $(X \cdot X') = \lambda$; β) $(X + X') = \nu$.

Axiom 7ter. α) $(X \cdot_2 X') = \lambda_2$; β) $(X +_2 X') = \nu_2$.

Die aus den Axiomen 1, 2, 3, 4, 5, 6, 7 und den Regeln E_0 und V_0 ableitbaren Theoreme stehen dual gegenüber den aus den Axiomen 1ter, 2ter, 3ter, 4ter, 5ter, 6ter, 7ter und den Regeln E_0 und V_0 ableitbaren Theoremen.

Daneben gibt es das unten folgende Dualitätsprinzip [7]).

Dualitätsprinzip. Zu jedem mit den Axiomen 1—7, 1ter—7ter, Verbindungsregel V_0 und Einsetzungsregel E_0 ableitbaren Satz erhält man einen dualen, wenn: α) in jeder vorkommenden Kalkülformel $+$ durch $\cdot$, $+_2$ durch $\cdot_2$, λ durch ν, λ_2 durch ν_2, und umgekehrt, ersetzt werden; β) $\mathfrak{A} \subset \mathfrak{B}$ durch $\overline{\mathfrak{B}} \subset \overline{\mathfrak{A}}$, und $\mathfrak{A} \subset_2 \mathfrak{B}$ durch $\overline{\mathfrak{B}} \subset_2 \overline{\mathfrak{A}}$ ersetzt wird; dabei sollen $\overline{\mathfrak{A}}$ und $\overline{\mathfrak{B}}$ die gemäss α) aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ hervorgehenden Kalkülformeln sein.

Satz 11. α) $(\mathfrak{A} + \mathfrak{B}) = (\mathfrak{A}' \cdot \mathfrak{B}')'$, und $(\mathfrak{A} \cdot \mathfrak{B}) = (\mathfrak{A}' + \mathfrak{B}')'$;
β) $(\mathfrak{A} +_2 \mathfrak{B}) = (\mathfrak{A}' \cdot_2 \mathfrak{B}')'$, und $(\mathfrak{A} \cdot_2 \mathfrak{B}) = (\mathfrak{A}' +_2 \mathfrak{B}')'$.

Definition 1. $\mathfrak{A} \rightarrow \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$.

Definition 1ter. $\mathfrak{A} \rightarrow_3 \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' +_2 \mathfrak{B}$.

Satz 12. (Schlussschema) Lässt sich schreiben $\mathfrak{A} = \nu$ und $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$, so auch $\mathfrak{B} = \nu$.

Satz 12ter. (Schlussschema) Lässt sich schreiben $\mathfrak{A} = \nu_2$ und $(\mathfrak{A} \rightarrow_3 \mathfrak{B}) = \nu_2$, so auch $\mathfrak{B} = \nu_2$.

Satz 13. (Schlusschema) Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$, so auch $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$, und umgekehrt.

Satz 13ter. (Schlussschema) Lässt sich schreiben $\mathfrak{A} \subset_2 \mathfrak{B}$, so auch $(\mathfrak{A} \rightarrow_3 \mathfrak{B}) = \nu$, und umgekehrt.

Satz A. $[(X + X) \rightarrow X] = \nu$.

Satz Ater. $[(X +_2 X) \rightarrow_3 X] = \nu_2$.

Satz B. $[X \rightarrow (X + Y)] = \nu$.

Satz Bter. $[X \rightarrow_3 (X +_2 Y)] = \nu_2$.

Satz C. $[(X + Y) \rightarrow (Y + X)] = \nu$.

Satz Cter. $[(X +_2 Y) \rightarrow_3 (Y +_2 X)] = \nu_2$.

Satz D. $[(Y \rightarrow Z) \rightarrow \{(X + Y) \rightarrow (X + Z)\}] = \nu$.

Satz Dter. $[(Y \rightarrow_3 Z) \rightarrow_3 \{(X +_2 Y) \rightarrow_3 (X +_2 Z)\}] = \nu_2$.

Satz E. $(\lambda \rightarrow X) = \nu$.

Satz Eter. $(\lambda_2 \rightarrow_3 X) = \nu_2$.

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[7]) Für die Ableitung und die der weiteren Sätze dieses Par. vergl. loc. cit. [4]), S. 1160—1164.

Satz F. $(X \to \nu) = \nu$.

Satz Fter. $(X \to_3 \nu_2) = \nu_2$.

Satz 14 [Satz 14ter**].** Lässt sich schreiben $\mathfrak{A} = \nu$ $[\mathfrak{A} = \nu_2]$, und ist $\mathfrak{B}$ eine mittels der Einsetzungsregel E_0 (erster Teil) aus $\mathfrak{A}$ hervorgehende Kalkülformel, so darf man auch schreiben $\mathfrak{B} = \nu$ $[\mathfrak{B} = \nu_2]$.

Satz 15. α) Ist $\mathfrak{A} = \nu$ $[\mathfrak{A} = \lambda]$ ein Theorem, so auch $\mathfrak{A}' = \lambda$ $[\mathfrak{A}' = \nu]$.
β) Ist $\mathfrak{A} = \nu_2$ $[\mathfrak{A} = \lambda_2]$ ein Theorem, so auch $\mathfrak{A}' = \lambda_2$ $[\mathfrak{A}' = \nu_2]$.

Aus Verbindungsregel V_0 und den Sätzen 13, 13ter folgt:

Satz 16. Sind

$$(\mathfrak{A} \to \mathfrak{B}) = \nu \quad \text{und} \quad (\mathfrak{B} \to \mathfrak{A}) = \nu$$

Theoreme, so auch

$$(\mathfrak{A} \to_3 \mathfrak{B}) = \nu_2 \quad \text{und} \quad (\mathfrak{B} \to_3 \mathfrak{A}) = \nu_2,$$

und umgekehrt.

Definition 7_0 (Einführung der Modalitätszeichen). Die Schreibweisen $\mathfrak{A} \doteq \bar{\nu}$ und $\mathfrak{A} = \nu$ sollen einander ersetzen können; ebenso $\mathfrak{A} \doteq \bar{\lambda}$ und $\mathfrak{A} = \lambda$; ebenso $\mathfrak{A} \doteq \bar{\nu}$ und $\mathfrak{A} = \nu_2$; schliesslich auch $\mathfrak{A} \doteq \bar{\lambda}_2$ und $\mathfrak{A} = \lambda_2$.

§ 6. Aus den Sätzen 1—7, 1ter—7ter, 9, 9ter, 10 einerseits und den Sätzen 12, 12ter, A—F, Ater—Fter, 14, 14ter, 16 andererseits, nebst den Definitionen und den Sätzen 8, 8ter, 11, 13, 13ter und 15, folgt unmittelbar, *dass das Axiomensystem I—VI, I*ter*—VI*ter *mit der Einsetzungsregel E, Schlussschema S und Verbindungsregel V völlig gleichwertig ist mit dem Axiomensystem 1—7, 1*ter*—7*ter *mit der Einsetzungsregel E_0 und der Verbindungsregel V_0.* Beim ersten System sind $\doteq\bar{\lambda}$, $\doteq\bar{\lambda}_2$, $\cdot$, $\cdot_2$, $\to$, $\to_j$ $(j = 2, 3, 4)$ und $\subset$, $\subset_2$ $\supset$, $\supset_2$ abgeleitete Verknüpfungen, beim zweiten System sind es $\supset$, $\supset_2$ und $\doteq\bar{\nu}$, $\doteq\bar{\lambda}$, $\doteq\bar{\nu}_2$, $\doteq\bar{\lambda}_2$, $\to$, $\to_j$ $(j = 2, 3, 4)$.

§ 7. Es wird hiernach nicht schwer sein drei weitere Axiomensysteme zu formulieren, gleichwertig mit dem System von § 5, und in welchen die undefinierten Grundverknüpfungen dieselben sind wie in § 5 mit Ausnahme von $\subset$ und $\subset_2$, welche bzw. ersetzt sind durch $\supset$ und $\supset_2$; $\supset$ und $\subset_2$; $\subset$ und $\supset_2$.

Widerspruchsfreiheit und Deutungen der Axiomensysteme.

§ 8. *Die beiden einander gleichwertigen Axiomensysteme der Paragraphen 1 und 5 sind widerspruchsfrei* in dem Sinne, dass $X \doteq \bar{\nu}$ (oder $X = \nu$), $X \doteq \bar{\nu}_2$ (oder $X = \nu_2$), $X \doteq \bar{\lambda}$ (oder $X = \lambda$) und $X \doteq \bar{\lambda}_2$ (oder $X = \lambda_2$) in den Systemen nicht ableitbar sind.

Um diese Widerspruchsfreiheit einzusehen fügen wir den (nach Annahme endlich oder abzählbar unendlich vielen) elementaren Kalkülformeln $A, B, \ldots, A_1, \ldots, A_n, \ldots$ Paare arithmetischer Werte zu, welche nur (0,0) oder (1,0) oder (0,1) oder (1,1) sein sollen. $A + B$ $[\mathfrak{A} + \mathfrak{B}]$ ordnen wir das Wertepaar (0,0) zu, falls auch A [$\mathfrak{A}$] und B [$\mathfrak{B}$] das Paar (0,0) zugeordnet ist; die weiteren Zuordnungen sind aus den drei folgenden Tabellen ersichtlich.

$+$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(1,0)	(1,1)	(1,1)
(0,1)	(0,1)	(1,1)	(0,1)	(1,1)
(1,1)	(1,1)	(1,1)	(1,1)	(1,1)

$+_2$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,0)	(1,0)
(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
(0,1)	(0,0)	(1,0)	(0,1)	(1,1)
(1,1)	(1,0)	(1,0)	(1,1)	(1,1)

	/
(0,0)	(1,1)
(1,0)	(0,1)
(0,1)	(1,0)
(1,1)	(0,0)

Welche Wertepaare auch die Kalkülformeln X, Y, Z zugeordnet sind, in allen Fällen ist den in den Axiomen I—IV im Vordergliede vorkommenden Kalkülformeln dasselbe Wertepaar, und zwar (1,1) zugeordnet; daneben ist den in den Axiomen I^ter^—IV^ter^ im Vordergliede auftretenden Kalkülformeln immer das Wertepaar (1,0) zugeordnet.

λ sei immer das Wertepaar (0,0), ν das Wertepaar (1,1), λ_2 das Wertepaar (0,1) und ν_2 das Wertepaar (1,0) zugeordnet.

Den im Vordergliede von Axiom V und VI vorkommenden Kalkülformeln sind nun immer das Wertepaar (1,1), den im Vordergliede von Axiom V^ter^ und VI^ter^ vorkommenden Formeln immer das Wertepaar (1,0) zugeordnet.

Wendet man auf die in $\mathfrak{A} \dot{=} \bar{\nu}$ [$\mathfrak{A} \dot{=} \bar{\nu}_2$] vorkommende Kalkülformel $\mathfrak{A}$ die Einsetzungsregel E an, wobei sie in $\mathfrak{B}$ übergehen soll, und ist $\mathfrak{A}$ immer das Wertepaar (1,1) [das Wertepaar (1,0)] zugeordnet (wie auch die in ihr auftretenden, durch lateinische Buchstaben dargestellten, elementaren Aussagen bewertet sind), so ist klar, dass dasselbe von $\mathfrak{B}$ gilt.

Anwendung des Schlussschemas S führt von $\mathfrak{A} \dot{=} \bar{\nu}$ und $(\mathfrak{A} \to \mathfrak{B}) \dot{=} \bar{\nu}$, wobei $\mathfrak{A}$ und $\mathfrak{A} \to \mathfrak{B}$ immer (1,1) zugeordnet sein soll (wie auch die in ihr auftretenden, durch lateinischen Buchstaben dargestellten, elementaren Aussagen bewertet sind), zu $\mathfrak{B} \dot{=} \bar{\nu}$, wobei für $\mathfrak{B}$ dasselbe gilt; von $\mathfrak{A} \dot{=} \bar{\nu}_2$ und $(\mathfrak{A} \to_3 \mathfrak{B}) \dot{=} \bar{\nu}_2$, wobei $\mathfrak{A}$ und $\mathfrak{A} \to_3 \mathfrak{B}$ immer (1,0) zugeordnet sein soll, führt Schema S zu $\mathfrak{B} \dot{=} \bar{\nu}_2$, wobei für $\mathfrak{B}$ dasselbe gilt.

Verbindungsregel V führt von $(\mathfrak{A} \to \mathfrak{B}) \dot{=} \bar{\nu}$ und $(\mathfrak{B} \to \mathfrak{A}) \dot{=} \bar{\nu}$, bei welchen den Vordergliedern immer das Wertepaar (1,1) zugeordnet sein soll, d.h. bei welchen $\mathfrak{A}$ und $\mathfrak{B}$ immer *gleiche* (vielleicht wechselnde) Wertepaare zugeordnet sind, zu $(\mathfrak{A} \to_3 \mathfrak{B}) \dot{=} \bar{\nu}_2$ und $(\mathfrak{B} \to_3 \mathfrak{A}) \dot{=} \bar{\nu}_2$, be welchen den Vordergliedern immer das Wertepaar (1,0) zukommt. Auch die Umkehrung gilt.

In einem Theorem $\mathfrak{A} \dot{=} \bar{\nu}$ ist somit $\mathfrak{A}$ immer das Wertepaar (1,1), in einem Theorem $\mathfrak{A} \dot{=} \bar{\nu}_2$ ist $\mathfrak{A}$ immer das Wertepaar (1,0) zugeordnet, während wegen Definition 3 in Theoremen $\mathfrak{A} \dot{=} \bar{\lambda}$ oder $\mathfrak{A} \dot{=} \bar{\lambda}_2$ die immer zugeordneten Wertepaare (0,0) bzw. (0,1) sind.

$X \dot{=} \bar{\nu}$, $X \dot{=} \bar{\lambda}$, $X \dot{=} \bar{\nu}_2$ und $X \dot{=} \bar{\lambda}_2$ sind dadurch nicht ableitbar, denn

X kann willkürlich eines des Wertepaare (1,1), (1,0), (0,1), (0,0) zugeordnet werden. Daraus folgt unsere Behauptung [8]).

§ 9. Ein zweites Beweisverfahren, welches das vorige als Spezialfall umfasst, ist folgendes. Wir betrachten zwei Mengen M, N von endlich vielen (m bzw. n) diskreten Elementen (m und $n \geq 1$) [9]). Den durch grosse lateinische Buchstaben angedeuteten elementaren Kalkülformeln können als „Werte" die Paare von Teilmengen von M bzw. N zugeordnet werden (die Mengen M und N und ihre leeren Teilmengen mit eingeschlossen) [10]).

Beweisen wir diesmal die Widerspruchsfreiheit des Axiomensystems von § 5.

$A + B$ [$\mathfrak{A} + \mathfrak{B}$] ordnen wir den „Wert" $(T_1 + T_2, U_1 + U_2)$ zu, falls A [$\mathfrak{A}$] das Mengenpaar (T_1, U_1), B [$\mathfrak{B}$] das Mengenpaar (T_2, U_2) zugeordnet ist; $A \cdot B$ [$\mathfrak{A} \cdot \mathfrak{B}$] sei dann das Paar $(T_1 \cdot T_2, U_1 \cdot U_2)$ als „Wert" zugeordnet. Ausserdem seien dann $A +_2 B$ [$\mathfrak{A} +_2 \mathfrak{B}$] und $A \cdot_2 B$ [$\mathfrak{A} \cdot_2 \mathfrak{B}$] bzw. die Mengenpaare $(T_1 + T_2, U_1 \cdot U_2)$ und $(T_1 \cdot T_2, U_1 + U_2)$ zugeordnet. A' [$\mathfrak{A}'$] sei das Paar von Komplementär-

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[8]) *Die Verbindungsregel V ist von den Axiomen I—VI, I^ter^—VI^ter^, Einsetzungsregel E und Schlussschema S unabhängig.* Dies zeigt sich wie folgt. Den elementaren Kalkülformeln A, B, ..., A_1, ..., A_n, ... seien Paare arithmetischer Werte zugeordnet, welche nur (0,0), (0,1), (1,0) oder (1,1) sein sollen. Für die Grundverknüpfungen sollen wieder die im Texte gegebenen Zuordnungstabellen gelten. Für λ_2 seien als Zuordnungen die Wertepaare (0,0) und (0,1), für ν_2 die Wertepaare (1,0) und (1,1) möglich, während λ und ν nur (0,0) bzw. nur (1,1) zugeordnet sei. Wir denken uns nun bei der Ableitung von Theoremen die Verbindungsregel V nicht benutzt. Dann werden den in Theoremen $\mathfrak{A} \dot{-} \nu_2$ auftretenden Kalkülformeln $\mathfrak{A}$ immer eines der Wertepaare (1,0), (1,1), den in Theoremen $\mathfrak{A} \dot{-} \nu$ auftretenden Kalkülformeln $\mathfrak{A}$ immer das Wertepaar (1,1) zugeordnet sein.

Die Verbindungsregel V ist nun keine Folge der Axiome, Regel E und Schema S. Denn sonst müsste die Existenz von Theoremen $(\mathfrak{A} \rightarrow_3 \mathfrak{B}) \dot{-} \nu_2$ und $(\mathfrak{B} \rightarrow_3 \mathfrak{A}) \dot{-} \nu_2$ immer zur Folge haben, dass auch $(\mathfrak{A} \rightarrow \mathfrak{B}) \dot{-} \nu$ und $(\mathfrak{B} \rightarrow \mathfrak{A}) \dot{-} \nu$ Theoreme wären. Dies ist jedoch nicht der Fall. Denn $[\lambda_2 \rightarrow_3 (X +_2 X')'] \dot{-} \nu_2$ und $[(X +_2 X')' \rightarrow_3 \lambda_2] \dot{-} \nu_2$ sind (ohne Regel V) ableitbare Theoreme. Ordnen wir X (0,0), λ_2 (0,0) zu, so ist $(X +_2 X') + \lambda_2$ das Wertepaar (1,0) zugeordnet; es gibt somit kein Theorem der Form $[(X +_2 X')' \rightarrow \lambda_2] \dot{-} \nu$, was doch der Fall sein müsste, wenn Regel V ableitbar wäre.

Aus dem Axiomensystem des Par. 1 (also mit Einschluss der Verbindungsregel V) folgt das Theorem (Distributivitätssatz): *Immer ist* $[\varphi \circ (\psi \square \chi)] = [(\varphi \circ \psi) \square (\varphi \circ \chi)]$; *dabei stellt* $\circ$, *und ebenso* $\square$, *eine willkürliche der Grundverknüpfungen* $+$, $+_2$, $.$, $._2$ *dar, und stellt* φ, *ebenso wie* ψ *und* χ, *eine willkürliche der elementaren Kalkülformeln* λ, ν, λ_2, ν_2 *dar.* Zum Beweise beachte man die Theoreme: $\lambda_2 + \nu_2 = \nu$, und $\lambda +_2 \nu = \nu_2$, und die dualen Theoreme: $\nu_2 \,.\, \lambda_2 = \lambda$ bzw. $\nu \,._2\, \lambda = \lambda_2$. (Das erste dieser Theoreme folgt mit Einsetzungsregel E_0 aus Axiom 7, β).

[9]) Zum ersten Verfahren kommt man durch Betrachtung von Mengen M, N, deren jede aus einem einzelnen Element aufgebaut ist; die Paare von Teilmengen von M und N: $\{0,0\}$, $\{0, N\}$, $\{M, 0\}$ und $\{M, N\}$ seien bzw. mit (0,0), (0,1), (1,0) und (1,1) benannt.

[10]) Die Anzahl dieser Paare ist somit $2^m \times 2^n$.

mengen $(M - T_1, M - T_2)$ als „Wert" zugeordnet, falls A $[\mathfrak{A}]$ das Paar (T_1, T_2) zugeordnet ist.

λ sei als einzig möglicher „Wert" das Mengenpaar $(0,0)$, ν das Paar (M, N) zugeordnet; schliesslich seien diese einzig möglichen „Werte" für λ_2 und ν_2 bzw. $(0, N)$ und $(M, 0)$.

$\mathfrak{A} \subset \mathfrak{B}$ soll nur möglich sein, wenn für die $\mathfrak{A}$ und $\mathfrak{B}$ gemäss obigen Vorschriften zugeordneten Mengenpaare (T, U) bzw. (T_0, U_0) immer T eine Teilmenge von T_0, U eine Teilmenge von U_0 ist, wie auch den in $\mathfrak{A}$ und $\mathfrak{B}$ vorkommenden, durch grosse lateinische Buchstaben angedeuteten elementaren Kalkülformeln Paare von Teilmengen von M bzw. N zugeordnet sind.

$\mathfrak{A} \subset_2 \mathfrak{B}$ soll nur möglich sein, wenn für die $\mathfrak{A}$ und $\mathfrak{B}$ gemäss obigen Vorschriften zugeordneten Mengenpaare (T, U) bzw. (T_0, U_0) immer T eine Teilmenge von T_0, U_0 eine Teilmenge von U ist.

Es ist nun sofort klar, dass die Axiome 1—7, und 1ter—7ter sich in richtige Behauptungen über die Teilmengen von M und N übersetzen; dasselbe gilt für die Regel V_0.

Betrachten wir noch die Einsetzungsregel E_0. Ist $\mathfrak{A} \subset \mathfrak{B}$ $[\mathfrak{A} \subset_2 \mathfrak{B}]$ ein Theorem, so gilt für $\mathfrak{A}$ und $\mathfrak{B}$ gleichzeitig zugeordnete Mengenpaare (T_1, U_1) bzw. (T_2, U_2), dass T_1 Teilmenge von T_2, U_1 Teilmenge von U_2 ist [dass T_1 Teilmenge von T_2, U_2 Teilmenge von U_1 ist]; das bleibt so für die Kalkülformeln $\mathfrak{D}$ und $\mathfrak{E}$, falls $\mathfrak{D} \subset \mathfrak{E}$ $[\mathfrak{D} \subset_2 \mathfrak{E}]$ mittels Regel E_0 aus $\mathfrak{A} \subset \mathfrak{B}$ [aus $\mathfrak{A} \subset_2 \mathfrak{B}$] hervorgeht.

Aber daraus folgt, dass $X \doteq \bar{\nu}$ oder $X = \nu$ oder „$X \subset \nu$ und $\nu \subset X$" [dass $X \doteq \bar{\nu}_2$ oder $X = \nu_2$ oder „$X \subset \nu_2$ und $\nu_2 \subset X$"] nicht ableitbar ist. Denn jeder „Wert" von X, welcher nicht gleich dem Mengenpaare (M, N) ist [welcher nicht gleich dem Mengenpaare $(M, 0)$ ist], gibt eine unrichtige Behauptung über endliche Mengen.

Dass $X \doteq \bar{\lambda}$ und $X \doteq \bar{\lambda}_2$ nicht ableitbar sind, folgt daraus, dass dasselbe von $X = \lambda$ oder $X' = \nu$ und von $X = \lambda_2$ oder $X' = \nu_2$ gilt.

Dieses zweite Beweisverfahren zeigt, *dass die acht im Vorigen betrachteten aequivalenten Axiomensysteme einen $2^m \times 2^n$-wertigen Aussagenkalkül liefern (m und n natürliche Zahlen)*, welchen wir als *das Produkt eines 2^m-wertigen und eines 2^n-wertigen Aussagenkalküls* auffassen können; jeder dieser beiden letzten Kalküle lässt sich durch die loc. cit. [4]) gegebenen Axiomensysteme für den erweiterten RUSSELL–WHITEHEADschen Aussagenkalkül charakterisieren.

Mathematics. — *Recherche de la convergence négative dans les mathématiques intuitionistes.* By J. G. DIJKMAN. (Communicated by Prof. A. HEYTING.)

(Communicated at the meeting of April 24, 1948.)

§ 1. *Introduction.*

Dans les mathématiques intuitionistes on n'accepte pas l'axiome du principe tertii exclusi sans restriction, ce qui entraîne que l'on exige plus de l'idée d'existence dans les mathématiques intuitionistes que dans les mathématiques classiques.

Cette exigence d'existence plus précisée produit qu'il faut scinder plusieurs idées classiques en idées qui sont au point de vue des mathématiques classiques souvent équivalentes mais qui au point de vue des mathématiques intuitionistes ont une signification tout à fait différente.

Cela est aussi le cas avec l'idée de convergence.

Au lieu de la convergence classique M. le prof. L. E. J. BROUWER [1]) a introduit les classifications suivantes ayant des significations différentes dans les mathématiques intuitionistes:

convergence positive: La suite $\{a_n\}$ est convergente positivement s'il y a un nombre ,a, [2]) de sorte que pour chaque nombre $\varepsilon > 0$ nous puissions indiquer un entier $n(\varepsilon)$ avec la propriété $|a - a_n| < \varepsilon$ pour tous les $n > n(\varepsilon)$.

convergence négative: La suite $\{a_n\}$ est convergente négativement avec la limite ,a, si pour chaque nombre $\varepsilon > 0$, choisi arbitrairement, il est impossible qu'il y a une suite montante $\{n_i\}$ d'entiers avec la propriété $|a - a_{n_i}| > \varepsilon$ pour tous les i.

non-oscillant: La suite $\{a_n\}$ est non-oscillante si pour chaque nombre $\varepsilon > 0$ il est impossible qu'il y a deux suites montantes $\{n_i\}$ et $\{m_i\}$ d'entiers avec la propriété $|a_{n_i+m_i} - a_{n_i}| > \varepsilon$ pour tous les i.

Jusqu'aujourd'hui on n'a pas réussi à montrer qu'une suite convergente négativement n'a qu'une seule limite.

M. M. J. BELINFANTE [3]) a publié le théorème suivant:

Mettons que les suites $\{a_n\}$ et $\{b_n\}$ soient convergentes négativement avec la limite a resp. b, alors la suite $\{a_n + b_n\}$ est convergente négativement avec la limite $a + b$.

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1) Voir: L. E. J. BROUWER, Ueber die Bedeutung des Satzes vom ausgeschlossenen Dritten, Journal für die reine und angew. Mathematik, tome 154, page 1—7.

2) L'expression „il y a un nombre ,a", signifie que nous pouvons indiquer le nombre ,a, avec la précision désirée.

3) Voir: M. J. BELINFANTE, Zur Theorie der unendlichen Reihen: Sitzungsberichte der preuss. Akademie, Phys. mathem. Klasse 1929, page 639—660.

Le théorème indiqué ci-dessus non démontré suit très simplement du théorème publié par M. BELINFANTE (démonstration du théorème 8).

M. BELINFANTE donne du théorème la démonstration suivante:

De $|a_n + b_n - (a + b)| > \varepsilon$ (1) il suit que:
$|a - a_n| > \frac{1}{2}\varepsilon$ (2) ou $|b - b_n| > \frac{1}{2}\varepsilon$ (3)

Mettons que la suite d'indices $n_1 < n_2 < n_3 < \ldots$ est donnée pour quels indices l'inégalité (1) est vraie alors on peut en tirer les suites

$$p_1 < p_2 < p_3 < \ldots \text{ et } q_1 < q_2 < q_3 < \ldots$$

de sorte que chaque n_i appartient à une de ces deux suites tandis que les indices de la 1re resp. 2me suite satisfont à l'inégalité (2) resp. (3).

Ces suites ne peuvent pas être des suites montantes infiniment. Par conséquent la suite $n_1 < n_2 < n_3 < \ldots$ n'est pas une suite infinie. De cette façon M. BELINFANTE croit avoir indiqué une contradiction.

A propos de la remarque que cette démonstration, étant vraie, serait valable pour les deux théorèmes M. le Dr. A. HEYTING donna la critique suivante:

Pour une démonstration exacte il faut indiquer quelle série de ces deux est infinie.

Il est possible qu'il existe un problème principiellement irrésoluble.

Par suite il existe la possibilité que l'indication ci-dessus exigée est un problème principiellement sans résolution.

Alors dans ce cas il n'est pas permis de parler d'une contradiction.

Donc la démonstration indiquée n'est pas acceptable.

Nous avons la même difficulté pour la démonstration du théorème:

Le produit de deux suites convergentes négativement est convergent négativement avec la limite qui est égale au produit des limites.

En résumant nous avons donc cet état de choses:

On n'a pas réussi à montrer:

α) Une suite convergente négativement n'a qu'une seule limite.

β) La somme de deux suites convergentes négativement est convergente avec la limite égale à la somme des limites.

γ) Le produit de deux suites convergentes négativement est convergent négativement avec la limite égale au produit des limites.

Or on pourrait changer la définition de M. BROUWER en l'élargissant avec un de ces théorèmes.

Cependant en ajoutant le théorème α) nous rencontrons la même difficulté dans la démonstration du théorème β).

C'est pourquoi je donne une définition nouvelle dans ce qui suit, une définition qui est examinée dans les paragraphes suivants.

Cette idée nouvelle sera indiquée par: l'idée de convergence négative stricte.

§ 2. *Définition et „critère negatif de* CAUCHY".

Définition: La suite $\{a_n\}$ est dite convergente négativement au sens strict si l'on peut indiquer un nombre ,a, de sorte qu'il soit impossible pour chaque nombre $\varepsilon > 0$ qu'il n'y aurait pas une suite $\{n_i\}$ d'entiers consécutifs avec la propriété: $|a - a_{n_i}| < \varepsilon$ pour tous les i.

Ecrit par la notation symbolique:

$$(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a - a_n| < \varepsilon) \text{ [4]}.$$

Si nous écrivons la convergence positive de la suite $\{a_n\}$ par moyen de cette notation symbolique, nous obtenons:

$$(\forall \varepsilon)(\exists N)(n > N) \rightarrow |a - a_n| < \varepsilon).$$

Les exemples de suites convergentes négativement donnés par M. BROUWER peuvent être pris aussi comme des exemples de suites convergentes négativement au sens strict.

L'idée: convergence négative stricte est plus faible que la négation double de la convergence positive [5].

Le critère de CAUCHY est équivalent à la définition donnée pour la convergence positive. Ce critère de CAUCHY dit:

La suite $\{a_n\}$ est convergente positivement, si

$$(\forall \varepsilon)(\exists N)(n, m > N \rightarrow |a_n - a_m| < \varepsilon)$$

Pour une suite convergente négativement au sens strict il existe encore un critère qui est analogue au critère de CAUCHY. Nous indiquerons ce critère par: le critère négatif de CAUCHY. Il dit:

Théorème 1. Mettons que la suite $\{a_n\}$ est convergente négativement au sens strict avec la limite ,a, alors on a:

$$(\forall \varepsilon) \neg\neg (\exists N)(n, m > N \rightarrow |a_n - a_m| < \varepsilon).$$

Démonstration: La suite $\{a_n\}$ soit convergente négativement au sens strict avec la limite a.

Alors on a:

$$(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a - a_n| < \varepsilon) \quad \ldots \ldots \quad (1)$$

Mettons:

$$(\exists \eta) \neg (\exists N)(n, m > N \rightarrow |a_n - a_m| < \eta) \quad \ldots \ldots \quad (2)$$

Il est évident: $|a_n - a_m| \leq |a_n - a| + |a - a_m|$ et avec (2) il serait:

$$(\exists \eta) \neg (\exists N)(n, m > N \rightarrow |a_n - a| + |a - a_m| < \eta)$$

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[4]) Ici ε signifie un nombre quelconque; N et n sont des entiers. $(\forall \varepsilon)$ resp. $(\exists N)$ signifie: pour tous les $\varepsilon > 0$ resp. il existe un entier N. $\neg$ signifie le signe de négation. Alors: $(\forall \varepsilon) \neg\neg (\exists N)$ $(n > N \rightarrow |a - a_n| < \varepsilon)$ signifie: pour tous les $\varepsilon > 0$ il est impossible qu'il n'existât un entier N de sorte que de $n > N$ il suive: $|a - a_n| < \varepsilon$. Ici le signe d'implication: $\rightarrow$ est employé: $a \rightarrow b$ signifie: a entraîne b.

[5]) Voir: A. HEYTING: On weakened quantification, The Journal of Symbolir Logic, **11**, 119—121, Dec. 1946.

Alors nous aurions:

$$(\exists \eta) \neg (\exists N)(n > N \rightarrow |a - a_n| < \tfrac{1}{2}\eta)$$

contradictoire à (1).

Donc: si une suite est convergente négativement au sens strict elle satisfait au critère négatif de CAUCHY.

Si la suite $\{a_n\}$ satisfait au critère négatif de CAUCHY il n'en resulte pas que la suite est convergente négativement au sens strict, parce que nous pourrions dire cela seulement si le nombre ,a, qui se montrerait comme la limite, pouvait être calculé avec chaque degré de précision désiré.

Il est possible de montrer:

Théorème 2. Si la suite satisfait au critère négatif de CAUCHY alors la suite est non-oscillante.

Comme je restreindrai l'idée de non-oscillance je ne ferai pas la démonstration.

Inversement il n'est point permis de prétendre qu'il suit de la non-oscillance que la suite est convergente négativement au sens strict, comme il se montre des exemples, que M. BELINFANTE donne des suites convergentes négativement multiplement [6]).

§ 3. *Complétement de la définition de M.* BROUWER.

Nous pourrions compléter la définition de M. BROUWER ainsi:

La suite $\{a_n\}$ est convergent négativement avec la limite ,a, s'il est impossible pour chaque $\varepsilon > 0$, qu'il existe une suite montante $\{n_i\}$ d'entiers avec la propriété $|a - a_{n_i}| > \varepsilon$ pour tous les i pendant que:

$$(\forall \varepsilon) \neg\neg (\exists N)(n, m > N \rightarrow |a_n - a_m| < \varepsilon).$$

A côté de ce que M. BROUWER exige de la suite convergente négativement $\{a_n\}$ nous supposons que la suite satisfait au critère négatif de CAUCHY.

Nous prétendons:

Théorème 3. Mettons que la suite $\{a_n\}$ soit convergente négativement avec la limite ,a, d'apres cette définition, alors la suite $\{a_n\}$ est convergente négativement au sens strict.

Remarque: Les diverses transitions seront justifiées à la fin de la démonstration.

Démonstration: Mettons que nous pouvons indiquer un nombre $\varepsilon > 0$ de sorte que:

$$(\exists N)(n > N \rightarrow |a_n - a| < \eta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

alors pour ce nombre η nous aurions:

$$(\forall N) \neg\neg (\exists n)((n > N) \cdot (|a_n - a| > \tfrac{1}{2}\eta). \quad . \quad . \quad . \quad . \quad (2)$$

Nous avons: $(\forall \varepsilon) \neg\neg (\exists N)(n, m > N \rightarrow |a_n - a_m| < \varepsilon)$ („CAUCHY")

Choisissons $\varepsilon = \frac{1}{4}\eta$ alors pour cet ε on aurait:

$$\neg\neg (\exists N)(n, m > N \rightarrow |a_n - a_m| < \tfrac{1}{4}\eta).$$

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[6]) Voir: M. J. BELINFANTE, Ueber eine besondere Klasse von non-oszillierenden Reihen, Proc. Kon. Akad. v. Wetensch., Amsterdam, 33, 1170—1179 (1930).

Mettons que pour $\varepsilon = \frac{1}{4}\eta$ il serait:

$$(\exists N_1)(n, m > N_1 \rightarrow |a_n - a_m| < \tfrac{1}{4}\eta) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

De (2) et (3) il suit que:

$$\neg\neg(\exists N_2)(\nu > N_2 \rightarrow |a_\nu - a| > \tfrac{1}{5}\eta) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(4) est contradictoire à la définition de M. BROUWER. Cette contradiction ne peut pas naître de (3) alors:

$$\neg(\exists\eta)\neg(\exists N)(n > N \rightarrow |a_n - a| < \eta)$$

et par suite: $(\forall\eta)\neg\neg(\exists N)(n > N \rightarrow |a_n - a| < \eta)$ q.e.d.

Démonstration de la transition de (1) à (2).

Mettons que: $(\exists N)\neg(\exists\eta)((n > N)\cdot(|a_n - a| > \frac{1}{2}\eta))$

alors il serait: $(\exists N)(\forall n)(n > N \rightarrow |a_n - a| < \eta)$

et aussi: $(\exists N)(n > N \rightarrow |a_n| < \eta)$ contradictoire à (1).

Démonstration de la transition de (2) et (3) à (4).

Mettons:

$$\begin{cases} (\forall N)\neg\neg(\exists n)((n > N)\cdot(|a_n - a| > \tfrac{1}{2}\eta)). \quad . \quad . \quad . \quad (2) \\ (\forall N_1)(n, m > N_1 \rightarrow |a_n - a_m| < \tfrac{1}{4}\eta). \quad . \quad . \quad . \quad . \quad . \quad (3) \end{cases}$$

Nous montrons que pour cet η nous avons:

$$\neg\neg(\exists N_2)(\nu > N_2 \rightarrow |a_\nu - a| > \tfrac{1}{5}\eta) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Mettons en effet: $\neg(\exists N_2)(\nu > N_2 \rightarrow |a_\nu - a| > \frac{1}{5}\eta)$

alors nous avons:

$$(\forall N_2)\neg\neg(\exists\nu)((\nu > N_2)\cdot(|a_\nu - a| < \tfrac{9}{40}\eta))$$

et aussi:

$$\neg\neg(\exists\nu)((\nu > N_1)\cdot(|a_\nu - a| < \tfrac{9}{40}\eta))$$

de sorte qu'il suit de (3):

$$(\exists N_1)\neg\neg(\exists\nu)((\nu > N_1)\cdot(m > N_1 \rightarrow |a_\nu - a| + |a_\nu - a_m| < \tfrac{1}{2}\eta)).$$

Alors il est évident que:

$$(\exists N_1)\neg\neg(\exists\nu)((\nu > N_1)\cdot(m > N_1 \rightarrow |a - a_m| < \tfrac{1}{2}\eta)).$$

Nous pouvons en éliminer ν de sorte que nous obtenons:

$$(\exists N_1)\neg\neg(m > N_1 \rightarrow |a - a_m| < \tfrac{1}{2}\eta)$$

et il en résulte: $(\exists N_1)\neg(\exists m)((m > N_1)\cdot(|a - a_m| > \frac{1}{2}\eta))$

contradictoire à (2).

§ 4. *Des suites non-oscillantes.*

Nous voyons la difficulté rencontrée dans la démonstration de la propriété a) (§ 1) aussi dans la démonstration du théorème:

La somme de deux suites non-oscillantes est non-oscillante.

Pour cela nous restreindrons la définition.

Définition: La suite $\{a_n\}$ est appelée non-oscillante au sens strict, si:

$$(\forall \varepsilon) \neg\neg (\exists N) (n, m > N \rightarrow |a_n - a_m| < \varepsilon).$$

Cela signifie qu'une suite est non-oscillante au sens strict si elle satisfait au critère négatif de CAUCHY.

Il suit de cela aussi:

Théorème 4. Si la suite $\{a_n\}$ est convergente négativement au sens strict, la suite est aussi non-oscillante au sens strict.

L'idée nouvelle de la convergence négative est une précision de l'idée de convergence de M. BROUWER. Cela paraît de:

Théorème 5. Si une suite est convergente négativement au sens strict alors la suite est aussi convergente négativement suivant M. BROUWER.

Démonstration. Mettons que pour la suite $\{a_n\}$ un nombre a, soit donné tel que:

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a - a_n| < \varepsilon).$$

Alors il faut montrer que pour chaque $\varepsilon > 0$ il est impossible qu'il existe une suite montante $\{n_i\}$ d'entiers avec $|a - a_{n_i}| > \varepsilon$ pour tous les i. (Définition de M. BROUWER.)

Mettons qu'il existe un nombre $\varepsilon > 0$ pour lequel il y ait une suite $\{n_i\}$ avec $|a - a_{n_i}| > \varepsilon$ pour tous les i.

Alors il est impossible pour chaque entier N qu'il n'y ait pas un entier $n > N$ avec $|a - a_n| > \varepsilon$, et par conséquent:

$$(\exists \varepsilon) \neg (\exists N) (n > N \rightarrow |a - a_n| < \varepsilon),$$

contradictoire à la convergence négative au sens strict.

Par analogie on montre:

Théorème 6. Si une suite est non-oscillante au sens strict, elle est aussi non-oscillante d'après la définition de M. BROUWER.

§ 5. *Des théorèmes généraux de limites.*

Si la sutie $\{a_n\}$ est convergente négativement au sens strict avec la limite a, nous l'indiquons par: $\overline{\ }\mathrm{Lim.}\ a_n = a$.

Théorème 7. De $\overline{\ }\mathrm{Lim.}\ a_n = a$ et $\overline{\ }\mathrm{Lim.}\ b_n = b$ il suit que

$$\overline{\ }\mathrm{Lim.}\ (a_n + b_n) = a + b.$$

Démonstration:

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a_n - a| < \varepsilon) \quad \ldots \ldots \quad (1)$$

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |b_n - b| < \varepsilon) \quad \ldots \ldots \quad (2)$$

De (1) et (2) il suit [7])

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a_n - a| + |b_n - b| < 2\varepsilon) \quad . \quad . \quad . \quad (3)$$

Il est évident: $|a_n - a| + |b_n - b| \geqslant |a_n + b_n - (a + b)|$ . . . . (4)

De (3) et (4) il suit:

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a_n + b_n - (a + b)| < 2\varepsilon)$$

et alors: $^{-}\text{Lim.}\ (a_n + b_n) = a + b$.

Théorème 8. Une suite convergente négativement au sens strict n'a qu'une seule limite.

Démonstration: Il est évident que de $^{-}\text{Lim.}\ a_n = a$ il suit:

$$^{-}\text{Lim.}\ (-a_n) = -a.$$

Mettons: $^{-}\text{Lim.}\ a_n = a$ et aussi $^{-}\text{Lim.}\ a_n = b$.

Du théorème 7 il suit que $^{-}\text{Lim.}\ (a_n - a_n) = a - b$.

La suite $\{a_n - a_n\}$ ne consiste que de zéros et cette suite est convergente positivement avec la limite 0 et une suite convergente positivement n'a qu'une seule limite et par suite $a - b = 0$, alors $a = b$ [8]).

Théorème 9. De $^{-}\text{Lim.}\ a_n = a$ et $^{-}\text{Lim.}\ b_n = b$ il suit $^{-}\text{Lim.}\ a_n \cdot b_n = ab$.

Démonstration: De

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a_n - a| < \varepsilon) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

et

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |b_n - b| < \varepsilon) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

il suit [7]): $(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |(a_n - a)(b_n - b)| < \varepsilon^2)$

donc: $(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a_n b_n + a b - a b_n - b a_n| < \varepsilon^2)$

et alors

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |(a_n b_n - a b) + a (b - b_n) + b (a - a_n)| < \varepsilon^2) \quad . \quad (3)$$

De (1) il suit: $(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |b (a_n - a)| < \varepsilon b)$ . . . . (4)

et de (2) il suit: $(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a (b_n - b)| < \varepsilon a)$ . . . . (5)

De (3), (4) et (5) il suit:

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |(a_n b_n - a b) + a (b - b_n) + b (a - a_n)| + \\ + |b (a_n - a)| + |a (b_n - b)| < \varepsilon^2 + \varepsilon b + \varepsilon a)$$

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[7]) La transition de (1) et (2) à (3) est permis par la règle:

$$\{(\forall x) \neg\neg A(x)\} \wedge \{(\forall x) \neg\neg B(x)\} \rightarrow (\forall x) \neg\neg \{A(x) \wedge B(x)\}$$

Ces règles sont données par A. HEYTING: Die formalen Regeln der intuitionistischen Logik, Sitzungsber. der preuss. Akademie der Wissenschaften; Physikalisch-mathematische Klasse, (1930) page 42—56.

[8]) Je dois cette démonstration à M. A. HEYTING. La démonstration peut être donnée aussi comme suit: De

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a - a_n| < \varepsilon) \text{ et } (\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |b - a_n| > \varepsilon)$$

il en résulte;

$$(\forall \varepsilon) \neg\neg (\exists N) (n > N \rightarrow |a - b| < 2\varepsilon) \text{ alors } a = b.$$

et par suite:

$$(\forall\varepsilon)\neg\neg(\exists N)(n>N\rightarrow|a_nb_n-ab|<\varepsilon^2+\varepsilon a+\varepsilon b).$$

a et b sont supposés comme des nombres finis et alors $\varepsilon^2+\varepsilon a+\varepsilon b$. peut être choisi arbitrairement petit.

Théorème 10. Mettons que $^{\neg}$Lim. $a_n=a$ et $^{\neg}$Lim. $b_n=b$ et $|a_n|>|b_n|$ alors $|a|\geq|b|$. (C'est à dire: $\neg(|a|<|b|)$).

Démonstration: Mettons $|b|=|a|+\eta$ avec $\eta>0$.

$^{\neg}$Lim. $a_n=a$ donc:

$$(\forall\varepsilon)\neg\neg(\exists N)(n>N\rightarrow|a_n-a|<\tfrac{1}{3}\varepsilon). \quad . \quad . \quad . \quad . \quad (1)$$

$^{\neg}$Lim. $b_n=b$ donc:

$$(\forall\varepsilon)\neg\neg(\exists N)(n>N\rightarrow|b_n-b|<\tfrac{1}{3}\varepsilon). \quad . \quad . \quad . \quad . \quad (2)$$

De (1) il suit:

$$(\forall\varepsilon)\neg\neg(\exists N)(n>N\rightarrow|a_n|<|a|+\tfrac{1}{3}\varepsilon). \quad . \quad . \quad . \quad (3)$$

De (2) il suit:

$$(\forall\varepsilon)\neg\neg(\exists N)(n>N\rightarrow|b_n|>|b|-\tfrac{1}{3}\varepsilon=|a|+\eta-\tfrac{1}{3}\varepsilon). \quad . \quad (4)$$

Choisissons $\varepsilon<3\eta$. De (3) et (4) il suit: $\neg\neg(\exists N)(n>N)\rightarrow|a_n|<|b_n|)$ pendant que $|a_n|>|b_n|$ pour tous les n.

Remarque: L'exigence: $\neg(\exists N)(n>N\rightarrow|a_n|<|b_n|)$ est suffisante.

Théorème 11. De $^{\neg}$Lim. $b_n=b$ et $|b_n|>c>0$ il suit $^{\neg}$Lim. $\dfrac{1}{b_n}=\dfrac{1}{b}$.

Démonstration: Mettons:

$$(\exists\varepsilon)\neg(\exists N)\left(n>N\rightarrow\left|\frac{1}{b_n}-\frac{1}{b}\right|<\varepsilon\right)$$

alors il serait: $$(\exists\varepsilon)\neg(\exists N)\left(n>N\rightarrow\left|\frac{b_n-b}{b\,b_n}\right|<\varepsilon\right)$$

donc: $$(\exists\varepsilon)\neg(\exists N)(n>N\rightarrow|b_n-b|<\varepsilon|b|\,|b_n|)$$

et par suite: $$(\exists\varepsilon)\neg(\exists N)(n>N\rightarrow|b_n-b|<\varepsilon c^2)$$

et cela est impossoble parce que c est fini.

Théorème 12. Si la suite $\{a_n\}$ est convergente négativement au sens strict avec la limite ,a, et si la suite $\{b_n\}$ est donnée pour laquelle on a:

$$(\forall\varepsilon)\neg\neg(\exists N)(n>N\rightarrow|a_n-b_n|<\varepsilon)$$

alors aussi la suite $\{b_n\}$ est convergente négativement au sens strict avec la limite ,a.

Démonstration: $|b-a|\leq|b_n-a_n|+|a_n-a|$

alors: $$(\forall\varepsilon)\neg\neg(\exists M)(n>M\rightarrow|b_n-a|<2\varepsilon).$$

Théorème 13. Si les suites $\{a'_n\}$ et $\{a''_n\}$ sont convergentes négativement au sens strict avec la limite ,a, et si nous avons encore une suite $\{a_n\}$ pour laquelle est donnée

$$\neg\neg\,(\exists N)\,(n > N \rightarrow a_n' < a_n < a_n'')$$

alors la suite $\{a_n\}$ est convergente négativement au sens strict avec la limite a.

Démonstration: Mettons:

$$(\exists \varepsilon)\neg(\exists M)\,(n > M \rightarrow |a - a_n| < \varepsilon$$

alors nous aurions

$$(\exists \varepsilon)\neg(\exists M)\,(n > M \rightarrow |a - a_n'| + |a_n' - a_n| < \varepsilon).$$

Il est donné: $(\forall \varepsilon)\neg\neg(\exists N)\,(n > N \rightarrow |a - a_n'| < \tfrac{1}{2}\varepsilon)$

alors $(\exists \varepsilon)\neg(\exists N)\,(n > N \rightarrow |a_n' - a_n| < \tfrac{1}{2}\varepsilon)$ . . . . . (1)

Pourtant $(\forall \delta)\neg\neg(\exists N)\; n > N \rightarrow |a_n'' - a_n'| < \delta)$ . . . . . (2)

Maintenant nous avons:

$$\neg\neg(\exists N)\,(n > N \rightarrow a_n' < a_n < a_n'')$$

et par suite: $\neg\neg(\exists N)\,(n > N \rightarrow |a_n' - a_n| < |a_n' - a_n''|)$ . . . . . (3)

De (2) et (3) il suit

$$(\forall \delta)\neg\neg(\exists N)\,(n > N \rightarrow |a_n' - a_n| < \delta). \quad . \; . \; . \; . \; . \quad (4)$$

Si l'on choisit $\delta = \frac{1}{2}\varepsilon$ on aurait:

$\neg\neg(\exists N)\,(n > N \rightarrow |a_n' - a_n| < \frac{1}{2}\varepsilon)$ contradictoire à (1).

Du théorème 12 il suit:

Théorème 14. Chaque suite partielle de la suite convergente négativement au sens strict avec la limite ,a, est aussi convergente négativement au sens strict avec la limite ,a.

Théorème 15. Mettons que les nombres $a_1, a_2, a_3, \ldots$ sont positifs et mettons $A_n = \sum_{\nu=1}^{n} a_\nu$ pendant que A_n devient infini si n croît sans limite. Mettons encore $^{-}\mathrm{Lim.}\; x_n = x$.

Alors on a: $^{-}\mathrm{Lim.}\dfrac{\sum_{\nu=1}^{n} a_\nu x_\nu}{\sum_{\nu=1}^{n} a_\nu} = x$. „Théorème de limite de STOLZ-CAUCHY".

Démonstration: Il suffit que je donne la démonstration pour $x = 0$.

Alors il faut montrer: $(\forall \eta)\neg\neg(\exists M)\,(m > M \rightarrow \left|\dfrac{\sum_1^m a_n x_n}{\sum_1^m a_n}\right| < \eta)$ ou:

$$(\forall \eta)\neg\neg(\exists M)\,(m > M \rightarrow |\sum_1^m a_n x_n| < \eta \sum_1^m a_n).$$

Mettons: $(\exists\eta)\neg(\exists M)(m>M\rightarrow|\sum_1^m a_n x_n|<\eta\sum_1^m a_n)$ alors nous aurions:

$$(\exists\eta)\neg(\exists M)(m>M\rightarrow\sum_1^m a_n|x_n|<\eta\sum_1^m a_n) \text{ et par suite:}$$

$$(\exists\eta)\neg(\exists M)(m>M\rightarrow\sum_1^m a_n\{|x_n|-\eta\}<0). \quad (1)$$

Mettons que pour cet η: $(\exists N)(n>N\rightarrow|x_n|<\eta)$

Mettons encore: $$|x_n|+\delta_n=\eta \quad (2)$$

pour $n>N$, alors $\delta_n>0$ et de (2) il suit:

$$\sum_{N+1}^{N+\lambda} a_n|x_n|+\sum_{N+1}^{N+\lambda} a_n\delta_n=\eta\sum_{N+1}^{N+\lambda} a_n.$$

donc: $$\sum_{N+1}^{N+\lambda} a_n\{|x_n|-\eta\}=-\sum_{N+1}^{N+\lambda} a_n\delta_n$$

Pour $m=N+\lambda$ il est evident:

$$\left.\begin{aligned}\sum_1^m a_n\{|x_n|-\eta\}&=\sum_1^N a_n\{|x_n|-\eta\}+\\&+\sum_{N+1}^{N+\lambda} a_n\{|x_n|-\eta\}=\sum_1^N a_n\{|x_n|-\eta\}-\sum_{N+1}^{N+\lambda} a_n\delta_n,\end{aligned}\right\}. \quad (3)$$

alors pour cet η de (1) et (3) il suit:

$$\neg(\exists M)((M>N)\cdot(m<N\rightarrow\sum_1^N a_n\{|x_n|-\eta\}-\sum_{N+1}^m a_n\delta_n<0))$$

et donc:

$$\neg(\exists M)((M>N)\cdot(m>N\rightarrow\sum_{N+1}^m a_n\delta_n>\sum_1^N a_n\{|x_n|-\eta\})).$$

Maintenant nous avons:

$$(\forall\lambda)(\sum_{N+1}^{N+1} a_n\delta_n\leqslant\sum_1^N a_n\{|x_n|-\eta\}). \quad (4)$$

Mettons: $$(\exists N_1)(n>N_1\rightarrow|x_n|<\tfrac{1}{2}\eta). \quad (5)$$

alors pour cet entier N_1: $(n>N_1\rightarrow|x_n|+\delta'_n=\frac{1}{2}\eta)$ avec $\delta'_n>0$.
Il est évident:

$$\left.\begin{aligned}\sum_{N+1}^{N+\lambda} a_n\delta_n&=\sum_{N+1}^{N_1} a_n\delta_n+\sum_{N+1}^{N_1+\mu} a_n\{\eta-|x_n|\}=\sum_{N+1}^{N_1}+\\&+\sum_{N_1+1}^{N_1+\mu} a_n\{\tfrac{1}{2}\eta+\delta'_n\}\geqslant\sum_{N+1}^{N_1}+\tfrac{1}{2}\eta\sum_{N_1+1}^{N_1+\mu} a_n.\end{aligned}\right\}. \quad (6)$$

Il est donné que Σa_n devient infini si n croît sans limite, alors (1), (4) et (6) donnent une contradiction. Cette contradiction ne peut que naître de (1) et par suite:

$$\neg(\exists\eta)\neg(\exists M)\, m<M\rightarrow|\sum_1^m a_n x_n|<\eta\sum_1^m a_n)$$

et donc: $$(\forall\eta)\neg\neg(\exists M)(m>M\rightarrow|\sum_1^m a_n x_n|<\eta\sum_1^m a_n). \quad (7)$$

§ 6. *Convergence négative multiple.*

M. BELINFANTE a introduit l'idée de convergence multiple d'une suite. La définition de cette idée est:

Nous appelons la suite $\{a_n\}$ multiplement convergente négativement si nous pouvons indiquer p nombres $a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}, \ldots, a^{(p)}$ de sorte qu'il soit impossible pour chaque nombre $\varepsilon > 0$ qu'il existe p suites croissantes $n_1^{(r)} < n_2^{(r)} < n_3^{(r)} < \ldots$ d'entiers avec la propriété $|a^{(r)} - a_{n_i}^{(r)}| > \varepsilon$ pour tous les i et tous les r satisfaisant à $1 \leq r \leq p$.

Dans ce cas nous appelons la suite $\{a_n\}$ p-uplement convergente négativement avec la limite p-uple $a^{(1)}, a^{(2)}, a^{(3)}, \ldots, a^{(p)}$.

M. BELINFANTE a publié le théorème suivant [6]).

Si les suites $\{a_n\}$ resp. $\{b_n\}$ sont p-uplement resp. q-uplement convergente négativement, alors la suite $\{c_n\} = \{a_n + b_n\}$ est $p \cdot q$-uplement convergente négativement. Les $p \cdot q$-uples valeurs de la limite de la suite $\{c_n\}$ sont les sommes de chacune de ces p-uples valeurs de la limite de la suite $\{a_n\}$ et une q-uple valeur de la limite de la suite $\{b_n\}$.

Sur cette démonstration nous pouvons donner la même critique comme sur celle du théorème mentionnée à la page 1.

On rencontre aussi la même difficulté dans la démonstration du théorème: Mettons que les suites $\{a_n\}$ et $\{b_n\}$ sont p-uplement resp. q-uplement convergente négativement. Les $p \cdot q$-uples valeurs de la limite de la suite $\{c_n\}$ sont formées par les produits de chacune de ces p-uples valeurs de la limite de la suite $\{a_n\}$ avec une q-uple valeur de la limite de la suite $\{b_n\}$.

Alors il faut quitter les deux théorèmes parce qu'ils ne sont pas montrés. C'est pourquoi j'introduis une nouvelle idée de la convergence négative multiple, qui bien possède les deux propriétés.

J'indique cette idée dans ce qui suit par: l'idée de convergence négative multiple au sens strict.

Définition. La suite $\{a_n\}$ est p-uplement convergente négativement au sens strict si nous pouvons indiquer p nombres $a^{(1)}, a^{(2)}, \ldots, a^{(p)}$ de sorte que:

$$\neg\neg(\exists i)(\forall \varepsilon)\neg\neg(\exists N)(n > N \rightarrow |a_n - a^{(i)}| < \varepsilon).$$

En évitant la notation symbolique cette définition est donc:

Définition: La suite $\{a_n\}$ est p-uplement convergente négativement au sens strict si nous pouvons indiquer p nombres $a^{(1)}, a^{(2)}, \ldots, a^{(p)}$ de sorte qu'il est impossible qu'il n'existe pas un index i pour laquelle il est impossible pour chaque $\varepsilon > 0$ qu'il n'existe pas un entier N avec la propriété:

$$|a_n - a^{(i)}| < \varepsilon \text{ pour tous les } n > N.$$

Nous montrons:

Théorème: Si la suite a est p-uplement convergente au sens strict alors la suite $\{a_n\}$ est aussi p-uplement convergente négativement selon M. BELINFANTE.

Démonstration: Mettons que la suite $\{a_n\}$ soit p-uplement convergente négativement au sens strict avec les limites $a^{(1)}, a^{(2)}, \ldots, a^{(p)}$ et que pour $\varepsilon > 0$ p suites montantes $\{n_j^{(r)}\}$ $(1 \leq r \leq p)$ d'entiers sont données avec la propriété $|a_{n_j^{(r)}} - a^{(r)}| > \varepsilon$ pour tous les j et tous les r avec $1 \leq r \leq p$. Alors on a:

$$\neg (\exists i)(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a_n - a^{(i)}| < \varepsilon),$$

ce qui est contradictoire à la convergence négative multiple au sens strict, alors la suite est aussi p-uplement convergence négativement selon M. BELINFANTE.

Théorème: Mettons que la suite $\{a_n\}$ resp. $\{b_n\}$ soit p-uplement resp. q-uplement convergente négativement au sens strict avec les limites $a^{(1)}, a^{(2)}, \ldots, a^{(p)}$ resp. $b^{(1)}, b^{(2)}, b^{(3)}, \ldots, b^{(q)}$. Alors on a:

La suite $\{c_n\} = \{a_n + b_n\}$ est $p \cdot q$-uplement convergent négativement au sens strict avec les limites $c^{(k)} = a^{(i)} + b^{(j)}$ $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q;\ k = 1, 2, \ldots, pq)$.

Démonstration: Mettons:

$$\neg (\exists i,j)(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a_n + b_n - (a^{(i)} + b^{(j)}| < \varepsilon)$$

alors il en suit:

$$\neg (\exists i,j)(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a_n - a^{(i)}| + |b_n - b^{(j)}| < \varepsilon). \qquad (1)$$

Nous avons: $\left.\begin{array}{l} \neg\neg (\exists i)(\forall \varepsilon) \neg\neg (\exists N_1)(n > N_1 \rightarrow |a_n - a^{(i)}| < \frac{1}{2}\varepsilon) \\ \neg\neg (\exists j)(\forall \varepsilon) \neg\neg (\exists N_2)(n > N_2 \rightarrow |b_n - b^{(j)}| < \frac{1}{2}\varepsilon) \end{array}\right\}$ alors:

$$\neg\neg (\exists i,j)(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a_n - a^{(i)}| + |b_n - b^{(j)}| < \varepsilon)$$

contradictoire à (1), donc:

$$\neg\neg (\exists i,j)(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a_n + b_n - (a^{(i)} + b^{(j)})| < \varepsilon.$$

Les exemples de suites convergentes négativement multiplement donnés par M. BELINFANTE peuvent servir aussi comme des exemples de suites convergentes négativement multiplement au sens strict.

De la non-oscillance d'une suite ne suit point que la suite soit convergente négativement multiplement au sens strict. Cela paraît du théorème suivant:

Théorème. Une suite convergente négativement multiplement au sens strict est non-oscillante.

Démonstration: Mettons que la suite $\{a_n\}$ est p-uplement convergente négativement au sens strict avec les limites $a^{(1)}, a^{(2)}, \ldots, a^{(p)}$ alors on a:

$$\neg\neg (\exists i)(\forall \varepsilon) \neg\neg (\exists N)(n > N \rightarrow |a_n - a^{(i)}| < \varepsilon)$$

et aussi: $\quad (\forall \varepsilon) \neg\neg (\exists N)(n, m > N \rightarrow |a_n - a_m| < 2\varepsilon).$

alors la suite $\{a_n\}$ est non-oscillante.

Mathematics. — *On non-Archimedean normed spaces.* By I. S. COHEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of May 29, 1948.)

In a recent series of papers [1]) A. F. MONNA has developed systematically the properties of linear normed spaces over fields with non-Archimedean valuations. It is the purpose of the present paper to generalize these results and at the same time to provide simpler proofs for some of them. In § 1 we consider questions of finite dimensionality and local compactness. In § 2 we prove the HAHN–BANACH theorem for the case of a discrete valuation of the underlying field of scalars and show that it is not in general true for non-discrete valuations.

§ 1. *Topological properties of normed spaces.* Throughout this note K will denote a field complete with respect to a non-Archimedean valuation. We shall denote the value of an element a of K by $|a|$. If the valuation of K is discrete, we shall say that K is discrete. This does not, of course, mean that K has a discrete topology; in order that K have a discrete topology it is necessary and sufficient that the valuation be trivial.

Throughout this § (but not in the next one) we assume that the valuation of K is non-trivial.

By a (non-Archimedean) *normed space* over K we shall mean a linear space E over K such that to every $x \in E$ there is assigned a real number $\| x \|$ such that

$$\| x \| > 0 \text{ if } x \neq 0 \quad \ldots \quad (1)$$

$$\| x + y \| \leqq \max (\| x \|, \| y \|) \text{ for } x, y \in E \quad \ldots \quad (2)$$

$$\| a x \| = | a | \cdot \| x \| \text{ for } a \in K, x \in E. \quad \ldots \quad (3)$$

By means of the distance functions $d(x, y) = \| x - y \|$, E becomes a metric space. We say that E is *discrete* if the set of norms $\| x \|$ has no limit point but 0. This does not, of course, mean that E has a discrete topology. We observe that if E has a discrete topology, then the valuation of K is trivial (but not conversely). Also, if E is discrete in the sense just defined, then K is also discrete (but not conversely).

Throughout, E denotes a fixed normed space over K.

Lemma. Let F be a closed linear subspace of E, $y \in E$, y not in F. If the sequence $\{x_n + a_n y\}$ has a limit in E, where $x_n \in F$ and $a_n \in K$, then $\{x_n\}$ and $\{a_n\}$ have limits.

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[1]) Sur les espaces linéaires normés, I, II, III, IV. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, vol. 49 and V, id. vol. 51; cited as *M*. Further references can be found in this paper to some of Monna's previous work.

Proof. It is clearly sufficient to show that $\{a_n\}$ has a limit. Suppose, first, that $\lim_{n\to\infty} (x_n + a_n y) = 0$; we assert that $a_n \to 0$. For if not, then there exists a subsequence $\{a_{k_n}\}$ such that $|a_{k_n}| \geqq \varepsilon$, where ε is a fixed positive number. Then

$$\| a_{k_n}^{-1}(x_{k_n} + a_{k_n} y) \| = | a_{k_n} |^{-1} \cdot \| x_{k_n} + a_{k_n} y \| \leqq \varepsilon^{-1} \| x_{k_n} + a_{k_n} y \|,$$

which has the limit 0. This implies that $a_{k_n}^{-1} x_{k_n} + y \to 0$, and since F is closed, we arrive at the contradiction $y \in F$. Hence $a_n \to 0$.

In general, if $\lim_{n\to\infty} (x_n + a_n y)$ exists, then $(x_{n+1} - x_n) + (a_{n+1} - a_n) y \to 0$. By the previous result, $a_{n+1} - a_n \to 0$, hence $\{a_n\}$ is a fundamental sequence, which consequently has a limit since K is complete.

Theorem 1. If F is a closed linear subspace of E and $y_1, \ldots, y_m$ are any elements of E, then $F + Ky_1 + \ldots + Ky_m$ is closed. Any finite dimensional subspace of E is closed [2]).

Proof. The second statement follows from the first, with F the null subspace. By induction, it is sufficient to prove the first statement for $m = 1$. Hence we prove that $F + Ky$ (consisting of all $x + ay$, $x \in F$, $a \in K$) is closed; we may naturally assume that y is not in F. If, then, $\{x_n + a_n y\}$ $(x_n \in F, a_n \in K)$ has a limit in E, the lemma implies that $\{x_n\}$ has a limit x in F (since F is closed) and $\{a_n\}$ has a limit a. Thus $\lim_{n\to\infty} (x_n + a_n y) = x + ay \in F + Ky$, which is therefore closed.

A particular normed space over K can be obtained by taking the set K_m of all ordered m-tuples $A = (a_1, a_2, \ldots, a_m)$, $a_i \in K$. This becomes a linear space in the usual way, and it becomes a normed space if we define $\| A \| = \max (|a_1|, |a_2|, \ldots, |a_m|)$. Clearly K_m is complete; it is locally compact if K is.

Theorem 2. An m-dimensional normed space E is topologically isomorphic to K_m and is complete. It is locally compact if K is.

Proof. Let $\{x_1, \ldots, x_m\}$ constitute a basis for E. We get an isomorphism of K_m onto E by mapping $A = (a_1, \ldots, a_m)$ into $\sum_{i=1}^{m} a_i x_i$. This mapping is continuous, for if $M = \max (\| x_1 \|, \ldots, \| x_m \|)$, then $\| \sum_1^m a_i x_i \| \leqq M \| A \|$. To prove that the inverse mapping is continuous, we must show that if $\lim_{n\to\infty} \sum_{i=1}^{m} a_i^{(n)} x_i = 0$, where $a_i^{(n)} \in K$, then $\lim_{n\to\infty} a_i^{(n)} = 0$, $i = 1, \ldots, m$. Since by Theorem 1, $Kx_1 + \ldots + Kx_{m-1}$ is a closed linear subspace of E and since it does not contain x_m, it follows from the lemma that $\lim_{n\to\infty} a_m^{(n)} = 0$.

[2]) Cf. *M*, p. 649, Theorem 4. It is there assumed that E is discrete.

Then $\lim_{n\to\infty} \sum_{i=1}^{m-1} a_i^{(n)} x_i = 0$, and we repeat the above argument $m-1$ times. This proves the continuity of the inverse [3]).

Corollary. Two normed spaces of the same (finite) dimension over the same valued field are topologically isomorphic.

Theorem 3. The following three conditions on E are equivalent: [4])

(1) E is locally compact.

(2) If $C > 0$ and if $\varrho > 1$ is the value of some element of K, then there does not exist an infinite sequence $\{x_i\}$ such that

$$C \leqq \| x_i \| \leqq C\varrho^2, \quad C \leqq \| x_i - x_j \| \leqq C\varrho^2, \quad i \neq j.$$

(3) K is locally compact, and E is finite dimensional.

Proof. If E is locally compact, there exists an $\varepsilon > 0$ such that the sphere $\| x \| \leqq \varepsilon$ is compact. Since for each $a \neq 0$ in K the mapping $x \to ax$ is a homeomorphism of E on itself, it follows that the sphere $\| x \| \leqq | a | \varepsilon$ is compact. Since the valuation of K is non-trivial, $|a|$ can be arbitrarily large, and thus any closed bounded subset of E is compact. Hence no sequence satisfying the hypothesis of (2) can exist, since it would have no convergent subsequence. Thus (1) implies (2).

Suppose that (2) is satisfied. To see that (3) follows, let $x \neq 0$ be an element of E, $r = \| x \|$. Then there clearly exists no infinite sequence $\{a_i\}$ in K satisfying $Cr^{-1} \leqq |a_i| \leqq Cr^{-1}\varrho^2$, $Cr^{-1} \leqq |a_i - a_j| \leqq Cr^{-1}\varrho^2$ for $i \neq j$. The valuation of K must then be discrete, since otherwise we could find a sequence $\{a_i\}$ with distinct values between Cr^{-1} and $Cr^{-1}\varrho^2$, contradicting the previous statement. The residue field must be finite, for otherwise we have a sequence $\{b_i\}$ with $|b_i| = 1$, $|b_i - b_j| = 1$ for $i \neq j$; if $a \in K$, $Cr^{-1} \leqq |a| \leqq Cr^{-1}\varrho^2$, then $a_i = ab_i$ again gives a contradiction. It then follows that K is locally compact, for it is well known that a complete discrete valued field is locally compact if its residue field is finite.

To show that E is finite-dimensional, suppose it is not. Then we can find a sequence $\{x_i\}$ contradicting (2). For if $x_1, \ldots, x_n$ have been found, let F be the subspace of E which they generate; F is closed by Theorem 1. Since $E \neq F$, let y be an element not in F, and let d be the greatest lower bound of $\| y - x \|$ for $x \in F$. There exists an $x' \in F$ such that

$$d \leqq \| y - x' \| \leqq d\varrho;$$

replacing y by $y - x'$ we may assume that $d \leqq \| y \| \leqq d\varrho$, $\| y - x \| \geqq d$ for all $x \in F$. Let a in K be such that $Cd^{-1} \leqq |a| \leqq Cd^{-1}\varrho$. Placing $x_{x+1} = ay$ and continuing in this fashion, we violate (2). Hence E is finite-dimensional.

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[3]) MONNA proves (M, p. 651) that an m-dimensional space E is topological isomorphic to K_m under the additional hypothesis that E is complete. To prove the continuity of the inverse mapping he then uses a category argument.

[4]) Cf. M, pp. 646—651, Theorems 2, 3, 5, 6.

That (3) implies (1) follows from Theorem 2.

Corollary. A locally compact space is finite dimensional, complete, and discrete [5]).

The first two points follow from Theorems 3 and 2. If E were not discrete, we could find a sequence $\{x_i\}$ such that the $\| x_i \|$ are all distinct and lie between 1 and ϱ. No subsequence could converge, since it is clear (just as in the case of a non-Archimedean valued field) that in a convergent sequence $\{y_n\}$ which is not a null-sequence, the norms $\| y_n \|$ are ultimately constant. This contradicts local compactness.

§ 2. *The Hahn–Banach theorem.* In this § (as in the preceding one), K is a complete valued field, but we do not exclude the possibility that the valuation is trivial. By a linear functional on a normed space E we mean, as usual, a mapping f of E into K such that $f(ax+by) = af(x) + bf(y)$ for $x, y \in E$, $a, b \in K$; f is bounded if for some real number M,

$$| f(x) | \leq M \| x \|$$

for all $x \in E$, and the least such number is the bound of f.

Theorem 4. Let the valuation of K be discrete. Then any bounded linear functional on a subspace F of E can be extended to a linear functional on E with the same bound.

Proof. First consider the case where the valuation of K is trivial, so that $| a | = 1$ for $a \neq 0$ in K. If M is the bound of f, then the set E_0 of all $x \in E$ such that $\| x \| < M^{-1}$ is a subspace of E. If $F_0 = E_0 \cap F$, then $f(x) = 0$ for $x \in F_0$. Hence f defines a single-valued linear functional $\bar{f}$ on the projection space $F - F_0$, considered as a linear space (not normed) over K. But $F - F_0$ may be regarded as a subspace of $E - E_0$, and hence $\bar{f}$ can be extended to a linear functional $\bar{\Phi}$ on $E - E_0$. If, for $x \in E$, we define $\Phi(x) = \bar{\Phi}(\bar{x})$ where $\bar{x}$ is the projection of x, then Φ is a linear functional on E, and clearly $| \Phi(x) | \leq M \| x \|$.

Now suppose the valuation is not trivial and let ϱ be the least value greater than 1 of an element in K. Again let M denote the bound of f. By the usual well ordering procedure the theorem can be reduced to the case where $E = F + Ky$, $y \in E$. Since K is complete, f can be trivially extended to the closure of F, hence we may assume F is closed, $y \in F$.

Let $d \neq 0$ be the distance from y to F, let k be the integer satisfying

$$\varrho^{k-1} \leq Md < \varrho^k.$$

Then $M^{-1} \varrho^{k-1} \leq d < M^{-1} \varrho^k$, and hence there is an $x' \in F$ such that

[5]) As regards the finite dimensionality, see also IRVING KAPLANSKY, Topological Methods in Valuation Theory, Duke Mathematical Journal 14 (1947), 527—541, especially Theorem 7. Our proofs are simpler than KAPLANSKY's since we are dealing with a special case.

$\| y - x' \| < M^{-1} \varrho^k$. Replacing y by $y - x'$ we may assume:

$$\| y \| < M^{-1} \varrho^k, \qquad \| y - x \| \geq d \qquad \text{for all } x \in F.$$

If, now, $z \in E$, then $z = x + ay$, $x \in F$, $a \in K$, and we define

$$\Phi(z) = \Phi(x + ay) = f(x).$$

Clearly Φ is a linear functional which extends f, and we must show that $|\Phi(z)| \leq M \| z \|$.

Now if $\| x \| \neq \| ay \|$, then $\| x \| \leq \| x + ay \|$, hence

$$|\Phi(x + ay)| = |f(x)| \leq M \| x \| \leq M \| x + ay \|.$$

On the other hand, if $\| x \| = \| ay \|$, then

$$\| a^{-1} \| = \| y \| < M^{-1} \varrho^k,$$
$$|f(a^{-1}x)| \leq M \| a^{-1}x \| < \varrho^k.$$

Since the valuation of K is discrete, we have

$$|f(a^{-1}x)| \leq \varrho^{k-1}$$
$$|f(x)| \leq |a| \varrho^{k-1} \leq |a| Md \leq |a| M \| y + a^{-1}x \| = M \| x + ay \|.$$

Hence $|\Phi(x + ay)| \leq M \| x + ay \|$.

This theorem has been proved by MONNA (M, p. 685) for the case where E is discrete, and since this implies the discreteness of K, his result is contained in ours. Theorem 4 applies, moreover, whenever K is locally compact. We shall give an example to show that the theorem is not in general true without the assumption of discreteness, but first we prove:

Theorem 5. Let λ be any real number > 1, and let F be a subspace of E such that E is the closure of the subspace obtained by adjoining to F a denumerable set of elements. Then any linear functional f on F of bound M can be extended to a linear functional on E of bound not greater than λM.

Proof. This has been proved by MONNA (M, p. 687, Theorem 15a) in case $E = F + Ky$, $y \in E$. Now suppose E is the closure of $F + Ky_1 + + Ky_2 + \ldots$. Let $\{\lambda_i\}$ be a sequence of real numbers > 1 such that $\prod_1^\infty \lambda_i \leq \lambda$. By applying MONNA's result n times, we can extend f to a linear functional on $F + Ky_1 + \ldots + Ky_n$ of bound $\leq (\prod_1^\infty \lambda_i) M$, and the theorem then follows.

Corollary. If E is separable, then any bounded linear functional on a subspace can be extended to a bounded linear functional on E.

The counterexample referred to above is obtained as follows. Let E be the set of all power series $x = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \ldots$, where $\alpha_1, \alpha_2, \ldots$ is a set of rational numbers well ordered in the natural order, and $a_1, a_2, \ldots$ are taken from some field. We define $\| x \| = e^{-\alpha_1}$ if $a_1 \neq 0$. Defining addition and multiplication in the obvious way, we see that E is a field and $\| \dots \|$ is a valuation of E. Let K be the subfield consisting of all $\Sigma a_i t^{\alpha_i}$ in which $\{\alpha_i\}$ is a simple sequence which has ∞ as limit. We regard E as a normed space over K.

K is itself a subspace of E, and $f(x) = x$ (for $x \in K$) defines a linear functional on K of bound 1. We say that this cannot be extended to a linear functional Φ on E of bound 1. For if it could, consider $x = \Sigma a_i t^{\alpha_i}$ where $\{\alpha_i\}$ is a simple sequence having a finite limit. Then

$$\Phi(a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \dots) = c_1 t^{\gamma_1} + c_2 t^{\gamma_2} + \dots \in K.$$

Since $|\Phi(x)| \leqq \| x \|$, we have $\gamma_1 \geqq \alpha_1$. Now $\gamma_1 > \alpha_1$ is impossible, since

$$\Phi(a_2 t^{\alpha_2} + \dots) = -a_1 t^{\alpha_1} + c_1 t^{\gamma_1} + \dots, \text{ and } \alpha_1 < \alpha_2.$$

Hence $\alpha_1 = \gamma_1$, and the same reasoning shows that $c_1 = a_1$. Thus we have

$$\Phi(a_2 t^{\alpha_2} + \dots) = c_2 t^{\gamma_2} + \dots$$

Continuing, we obtain $\alpha_2 = \gamma_2$, $\alpha_3 = \gamma_3$, ..., which is impossible since $\gamma_i \to \infty$ and α_i is bounded.

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Mathematics. — *On induced group characters.* By T. A. SPRINGER. (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of May 29, 1948.)

In a recent paper [1]), R. BRAUER proved the following group theoretical statement, which has important consequences in analytic number theory [2]):

Any character of a finite group $\mathfrak{G}$ can be written as a linear combination with integral rational coefficients of characters induced by linear characters of subgroups of $\mathfrak{G}$.

The aim of the present note is to show that BRAUER's proof of this fact can be simplified a little.

1. We begin with the deduction of some properties of the characters induced by an arbitrary subgroup of $\mathfrak{G}$. These properties are not used explicitly in the proof of the main result but they can help to elucidate the idea of this proof.

Suppose that $\mathfrak{G}$ has order g, that $\mathfrak{H}$ is a proper subgroup of $\mathfrak{G}$ of order h, that g_A is the number of elements in the class $\mathfrak{K}_A$ of the element $A \in G$ and that h_B is the number of elements of the class $\mathfrak{k}_B$ of $B \in \mathfrak{H}$ in $\mathfrak{H}$.

Let $A_1, A_2, \dots, A_k$ and $B_1, B_2, \dots, B_\varkappa$ be complete systems of representatives of the classes of conjugate elements of $\mathfrak{G}$ and $\mathfrak{H}$, respectively. We put $g_{A_\lambda} = g_\lambda$ $(\lambda = 1, 2, \dots, k)$, $h_{B_\lambda} = h_\lambda$ $(\lambda = 1, 2, \dots, \varkappa)$.

Suppose, furthermore, that the classes $\mathfrak{K}_1, \mathfrak{K}_2, \dots, \mathfrak{K}_l$ of $\mathfrak{G}$ and no other classes contain elements of $\mathfrak{H}$. Then $\mathfrak{K}_\lambda \cap \mathfrak{H}$ splits up into a number of classes of $\mathfrak{H}$:

$$\mathfrak{K}_\lambda \cap \mathfrak{H} = \mathfrak{k}_\lambda^{(0)} + \mathfrak{k}_\lambda^{(1)} + \dots + \mathfrak{k}_\lambda^{(n_\lambda)} \qquad (\lambda = 1, 2, \dots, l).$$

We take a complete system of representatives $A_\lambda^{(0)} = A_\lambda$, $A_\lambda^{(1)}, \dots, A_\lambda^{(n_\lambda)}$ of these classes and denote by $h_\lambda^{(\mu)}$ the number of elements in $\mathfrak{k}_\lambda^{(\mu)}$. We may assume that the elements $A_\lambda^{(\mu)}$ occur amongst the elements B_λ.

Finally, we denote by $\varphi_1, \varphi_2, \dots, \varphi_k$ the irreducible characters of $\mathfrak{G}$, by $\omega_1, \omega_2, \dots, \omega_\varkappa$ those of $\mathfrak{H}$ and by $\omega_1^*, \omega_2^*, \dots, \omega_\varkappa^*$ the characters of $\mathfrak{G}$ induced by the latter ones. Then [3])

$$\left.\begin{aligned} \omega_\lambda^*(A) &= \textstyle\sum_{\varrho=1}^{k} c_{\varrho\lambda}\, \varphi_\varrho(A) \qquad (A \in \mathfrak{G}) \\ \varphi_\lambda(B) &= \textstyle\sum_{\varrho=1}^{\varkappa} c_{\lambda\varrho}\, \omega_\varrho(B) \qquad (B \in \mathfrak{H}) \end{aligned}\right\} \quad \dots\dots \quad (1)$$

where the $c_{\lambda\mu}$ are non negative integers.

[1]) R. BRAUER, On Artin's *L*-series with general group characters, Ann. of Math., **48**, 502—514 (1947).

[2]) Cf. Mathematical Problems (Princeton Bicentennial Conferences) p. 9.

[3]) By a theorem of FROBENIUS. For the properties of induced characters see e.g. A. SPEISER, Theorie der Gruppen von endlicher Ordnung, 3rd ed., § 64.

We will show first that, in a certain sense, the first relation (1) can be inverted. More precisely: we can find a linear combination $\Sigma_\varrho d_{\lambda\varrho}\omega_\varrho^*$ with rational $d_{\lambda\mu}$, such that

$\Sigma_\varrho d_{\lambda\varrho}\,\omega_\varrho^*(A) = \varphi_\lambda(A)$ if $\mathfrak{K}_A$ has elements in common with $\mathfrak{H}$,
$\Sigma_\varrho d_{\lambda\varrho}\,\omega_\varrho^*(A) = 0$ if $\mathfrak{K}_A$ has no elements in common with $\mathfrak{H}$.

If $\mathfrak{G} = \Sigma_{\varrho=1}^{i}\,\mathfrak{H}S_\varrho \quad \left(i = \frac{g}{h}\right)$, then [3])

$$\omega^*(A) = \sum_{\substack{\varrho=1\\ S_\varrho A S_\varrho^{-1}\in\mathfrak{H}}}^{i} \omega_\lambda(S_\varrho A S_\varrho^{-1}) = \frac{1}{h}\sum_{\substack{X\in\mathfrak{G}\\ XAX^{-1}\in\mathfrak{H}}} \omega_\lambda(XAX^{-1}) =$$
$$= \frac{1}{h}\cdot\frac{g}{g_A}\,\Sigma_{X\in\mathfrak{K}_A\cap\mathfrak{H}}\,\omega_\lambda(X).$$

It follows that

$$\omega_\lambda^*(A_\mu) = \frac{g}{g_\mu h}\,\{h_\mu^{(1)}\,\omega_\lambda(A_\mu^{(0)}) + \ldots + h_\mu^{(n_\mu)}\,\omega_\lambda(A_\mu^{(n_\mu)})\} \text{ if } 1 \leqslant \mu \leqslant l, \text{ and}$$
$$\omega_\lambda^*(A_\mu) = 0 \qquad \text{if } l < \mu \leqslant k.$$

This can be written in the form

$$\omega_\lambda^*(A_\mu) = \Sigma_{\varrho=1}^{\varkappa}\,\gamma_{\varrho\mu}\,\omega_\lambda(B_\varrho), \qquad (2)$$

where

$\gamma_{\lambda\mu} = 0$ if $l < \mu \leqslant k$;
$\gamma_{\lambda\mu} = 0$ if $1 \leqslant \mu \leqslant l$ and $B_\lambda = A_\alpha^{(\beta)}$ with $\alpha \neq \mu$;
$\gamma_{\lambda\mu} = \dfrac{g h_\mu^{(\nu)}}{g_\mu h}$ if $1 \leqslant \mu \leqslant l$ and $B_\lambda = A_\mu^{(\nu)}$.

We introduce a number of matrices: (λ is the row-index, μ the column-index; behind every matrix the number of rows and columns is indicated)

$$\Phi = (\varphi_\lambda(A_\mu))\,(k\times k) \quad , \quad \Phi_1 = (\varphi_\lambda(A_\mu))\,(k\times l);$$
$$\Omega^* = (\omega_\lambda^*(A_\mu))\,(\varkappa\times k) \quad , \quad \Omega_1^* = (\omega_\lambda^*(A_\mu))\,(\varkappa\times l);$$
$$\Omega = (\omega_\lambda(B_\mu))\,(\varkappa\times\varkappa) \quad , \quad \Omega_1 = (\omega_\lambda(A_\mu))\,(\varkappa\times l);$$
$$C = (c_{\lambda\mu})\,(k\times\varkappa);$$
$$\Gamma = (\gamma_{\lambda\mu})\,(\varkappa\times k);$$
$$G = \left(\frac{g_\lambda}{g}\,\delta_{\lambda\mu}\right)(k\times k) \quad , \quad G_1 = \left(\frac{g_\lambda}{g}\,\delta_{\lambda\mu}\right)(l\times l);$$
$$H = \left(\frac{h_\lambda}{h}\,\delta_{\lambda\mu}\right)(\varkappa\times\varkappa).$$

By (1) and (2) and by the orthogonality relations for the group characters we have

$$\left.\begin{aligned}
\Omega^{*} &= C'\Phi = \Omega\Gamma; \\
\Omega_1^{*} &= C'\Phi_1; \\
\Phi_1 &= C\Omega_1; \\
\Phi^{-1} &= G\overline{\Phi'} \quad , \quad \overline{\Phi_1'}\,\Phi_1 = G_1^{-1}; \\
\Omega^{-1} &= H\overline{\Omega'}.
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (A)$$

($\overline{\Phi'}, \ldots$ is obtained from $\Phi', \ldots$ by replacing all elements by their complex conjugates).

It follows from (A) that

$$C' = \Omega\Gamma\Phi^{-1} = \Omega(\Gamma G)\overline{\Phi'}.$$

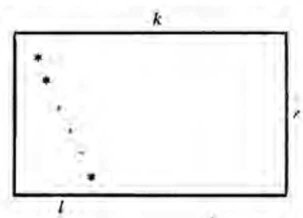
Therefore

$C = \overline{\Phi}(G\Gamma')\,\Omega' = \Phi\,(G\Gamma')\,\overline{\Omega'}$ (since the elements of C are real)

and

$$CC' = \Phi\,(G\Gamma')\,(\overline{\Omega'}\,\Omega)\,\Gamma\,\Phi^{-1} = \Phi\,(G\Gamma' H^{-1}\Gamma)\,\Phi^{-1}.$$

If the classes of $\mathfrak{H}$ are numbered suitably, Γ has a form like this



(zeros on the blank places).

In the columns $l+1, \ldots, k$ all elements are zero, in the columns $\mu\,(\mu = 1, \ldots, l)$ the element on the place of the class $\mathfrak{k}_\mu^{(\nu)}$ $(\nu = 0, 1, \ldots, n_\mu)$ is $\dfrac{gh_\mu^{(\nu)}}{g_\mu h}$, the elements on the other places are zero again. In ΓG, $H^{-1}\Gamma$ and $H^{-1}\Gamma G$ the zeros are on the same places as in Γ, the elements corresponding to $\dfrac{gh_\mu^{(\nu)}}{g_\mu h}$ are in these matrices $\dfrac{h_\mu^{(\nu)}}{h}$, $\dfrac{g}{g_\mu}$ and 1, respectively. It follows that $P = G\Gamma' H^{-1}\Gamma$ is a square $k \times k$-matrix of the form:

$$P = \begin{pmatrix} s_1 & & & & 0 \\ & \ddots & & & \\ & & s_l & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \text{ with}$$

$$s_\lambda = \frac{g}{g_\lambda} \sum_{\varrho=0}^{n_\lambda} \frac{h_\lambda^{(\varrho)}}{h} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Since $CC' = \Phi P \Phi^{-1}$, CC' has rank l. Then the same is true for C. [4]

[4]) For the inequality rank $C \geq l$ is obvious and the inequality rank $C \leq l$ follows easily if the minors of CC' are expressed in those of C.

Suppose $\Phi = (\Phi_1, \Phi_2)$. Then $CC' = (\Phi_1, \Phi_2) \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} G \begin{pmatrix} \overline{\Phi_1'} \\ \overline{\Phi_2'} \end{pmatrix} =$
$= (\Phi_1, \Phi_2) \begin{pmatrix} DG_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\Phi_1'} \\ \overline{\Phi_2'} \end{pmatrix} = \Phi_1 (DG_1) \overline{\Phi_1'}$.

Since $\overline{\Phi_1'}\, \Phi_1 = G_1^{-1}$, there is certainly a square $l \times l$-matrix $\triangle$ in Φ_1 such that det $(\triangle) \neq 0$. If the characters of $\mathfrak{G}$ are numbered suitably we may suppose that $\Phi_1 = \begin{pmatrix} \triangle \\ * \end{pmatrix}$. Then

$$CC' = \Phi_1 (DG_1) \overline{\Phi_1'} = \begin{pmatrix} \triangle \\ * \end{pmatrix} (DG_1) (\overline{\triangle'}, *) = \begin{pmatrix} \triangle (DG_1) \overline{\triangle'} & * \\ * & * \end{pmatrix}. \quad (4)$$

So there is certainly in the first l rows of C a square $l \times l$-matrix C_1 such that det $(C_1) \neq 0$. Again, one can number the characters of $\mathfrak{H}$ such that C_1 occurs also in the first l columns of C. Then C has the form

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

(C_1 is a $l \times l$-matrix, C_2 a $l \times (\varkappa - l)$-matrix etc.).

E_r denoting a unit matrix of degree r we have

$$\begin{pmatrix} C_1^{-1} & 0 \\ -C_3 C_1^{-1} & E_{k-l} \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} E_l & C_1^{-1} C_2 \\ 0 & -C_3 C_1^{-1} C_2 + C_4 \end{pmatrix}.$$

Since the last matrix has rank $\leqslant l$ we must have $C_4 = C_3 C_1^{-1} C_2$. It follows that

$$\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} (E_l, C_1^{-1} C_2) = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_3 C_1^{-1} C_2 \end{pmatrix} = C.$$

Therefore, if $M = \begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$ ($k \times l$-matrix) and $N = (E_l, C_1^{-1} C_2)$ ($l \times \varkappa$-matrix), we have $C = MN$.

Using (A) we obtain

$$\Phi_1 = C\,\Omega_1 = MN\,\Omega_1 = MR \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $R = N\,\Omega_1$ is a square $l \times l$-matrix, and

$$\left.\begin{aligned} \Omega_1^* &= C'\,\Phi_1 = N'\,M'\,\Phi_1, \\ N\,\Omega_1^* &= NN' \cdot M'\,\Omega_1 = NN' \cdot M'M \cdot R. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

By (5) we have $\triangle = C_1 R$, therefore det $(R) \neq 0$. Comparing

$$CC' = \begin{pmatrix} C_1 C_1' + C_2 C_2' & * \\ * & * \end{pmatrix}$$

with (4) we see that

$$C_1 C_1' + C_2 C_2' = \triangle (DG_1) \overline{\triangle'}.$$

It follows that $NN' = (E_l, C_1^{-1} C_2) \begin{pmatrix} E_l \\ C_2' C_1^{-1'} \end{pmatrix} = C_1^{-1} (C_1 C_1' + C_2 C_2') C_1^{-1'} =$
$= C_1^{-1} \triangle (DG_1) \overline{\triangle'} C_1^{-1'} = R(DG_1)\overline{R'}$. Moreover, by $G_1^{-1} = \overline{\Phi_1'}\, \Phi_1 =$
$= \overline{R'}\,(M'M)\,R$ we find $M'M = \overline{R'^{-1}}\, G_1^{-1} R^{-1}$.

Finally we get

$$NN' \cdot M'M = R(DG_1)\overline{R'} \cdot \overline{R'^{-1}}\, G_1^{-1} R^{-1} = R\,DR^{-1}$$

which proves that $\det\,(NN' \cdot M'M) \neq 0$. By (6) we have

$$R = (NN' \cdot M'M)^{-1} N \Omega_1^*$$

and from (5) we obtain $\Phi_1 = M(NN' \cdot M'M)^{-1} N\Omega_1^*$. Leaving matrix-notation (and using $\omega_\lambda^*(A_\mu) = 0$ if $l < \mu \leq k$) we find the enunciated result:

There exists a linear combination $\Sigma_\varrho d_{\lambda\varrho}\,\omega_\varrho^*$ such that $\Sigma_\varrho d_{\lambda\varrho}\,\omega_\varrho^*(A) = \varphi_\lambda(A)$ if $\mathfrak{K}_A$ has elements in common with $\mathfrak{H}$ and $=0$ if $\mathfrak{K}_A$ has no elements in common with $\mathfrak{H}$; moreover the $d_{\lambda\mu}$ are rational numbers such that $\det(D) \cdot d_{\lambda\mu} = (s_1 s_2 \ldots s_l) \cdot d_{\lambda\mu}$ *is an integer. Here s_λ is given by* (3).

2. We need some information about the numbers s_λ. In the first place it is easily seen that s_λ is an integer. For $\frac{g}{g_\lambda}$ is the order of the normalizer in $\mathfrak{G}$ of A_λ and $\frac{h}{h_\lambda^{(\varrho)}}$ is the order of the normalizer in $\mathfrak{H}$ of $A_\lambda^{(\varrho)}$. Since the latter group is the conjugate of a subgroup of the former group, the integer $\frac{h}{h_\lambda^{(\varrho)}}$ is a divisor of $\frac{g}{g_\lambda}$. Consequently $s_\lambda = \frac{g}{g_\lambda} \Sigma_{\varrho=0}^{n_\lambda} \frac{h_\lambda^{(\varrho)}}{h}$ is an integral number.

Secondly, suppose that $\mathfrak{H}$ is a q-Sylowgroup of $\mathfrak{G}$ of order q^α [5]. We will show that then $(s_\lambda, q) = 1$ $(\lambda = 1, 2, \ldots, l)$ [6].

We have $g = q^\alpha \widetilde{g}$, where $(\widetilde{g}, q) = 1$. Let $g_\lambda = q^{\alpha_\lambda} \widetilde{g}_\lambda ((\widetilde{g}_\lambda, q) = 1, \lambda = 1, \ldots, l)$. $\mathfrak{K}_\lambda$ can be split up into disjoint subsets, each of which contains all elements of $\mathfrak{G}$ that can be obtained from an element of $\mathfrak{G}$ by transformation with an element of $\mathfrak{H}$. Amongst these subsets occur the classes $\mathfrak{k}_\lambda^{(\nu)}$ $(\nu = 0, 1, \ldots, n_\lambda)$. We denote the other subsets by $\varkappa_\lambda^{(0)}, \varkappa_\lambda^{(1)}, \ldots, \varkappa_\lambda^{(m_\lambda)}$. Suppose that $\varkappa_\lambda^{(\nu)}$ contains $\widetilde{h}_\lambda^{(\nu)}$ elements and that $K_\lambda^{(\nu)}$ is a representative of $\varkappa_\lambda^{(\nu)}$. Then $K_\lambda^{(\nu)} \notin \mathfrak{H}$. Now $\widetilde{h}_\lambda^{(\nu)} = q^\alpha$: (the number of elements in $\mathfrak{H}$ commuting with $K_\lambda^{(\nu)}) = q^\alpha$: order $(\mathfrak{N}(K_\lambda^{(\nu)}) \cap \mathfrak{H})$. ($\mathfrak{N}(A)$ denotes the normalizer in $\mathfrak{G}$ of the element A). $\mathfrak{N}(K_\lambda^{(\nu)})$ has the order $q^{\alpha-\alpha_\lambda} . r$ $((r, q) = 1)$, since $K_\lambda^{(\nu)}$ is an element of $\mathfrak{K}_\lambda$. Therefore the order of $\mathfrak{N}(K_\lambda^{(\nu)}) \cap \mathfrak{H}$ does not exceed $q^{\alpha-\alpha_\lambda}$. If the order were $q^{\alpha-\alpha_\lambda}$ a Sylow-

[5]) For the properties of Sylowgroups to be used henceforth we refer to SPEISER, Ch. 5.

[6]) Cf. R. BRAUER, On the Cartan invariants of groups of finite order, Ann. of Math. **42**, 57 (1941).

group of $\mathfrak{N}(K_\lambda^{(\nu)})$ would be contained in $\mathfrak{H}$, but then $K_\lambda^{(\nu)}$ would be contained in $\mathfrak{H}$ (for $K_\lambda^{(\nu)}$ is contained in all Sylowgroups of $\mathfrak{N}(K_\lambda^{(\nu)})$ since these groups are conjugate [5])). This not being so, the order of $\mathfrak{N}(K_\lambda^{(\nu)}) \cap \mathfrak{H}$ is less than $q^{\alpha-\alpha_\lambda}$ and $\widetilde{h}_\lambda^{(\nu)} > q^{\alpha_\lambda}$.

Now

$$g_\lambda = q^{\alpha_\lambda} \widetilde{g}_\lambda = h_\lambda^{(0)} + h_\lambda^{(1)} + \ldots + h_\lambda^{(n_\lambda)} + \widetilde{h}_\lambda^{(0)} + \widetilde{h}_\lambda^{(1)} + \ldots + \widetilde{h}_\lambda^{(m_\lambda)}.$$

The numbers $\widetilde{h}_\lambda^{(\nu)}$ are divisible by $q^{\alpha_\lambda+1}$. Then the highest power of q dividing $h_\lambda^{(0)} + h_\lambda^{(1)} + \ldots + h_\lambda^{(h_\lambda)}$ must be q^{α_λ}. Now $(s_\lambda, q) = 1$ follows from

$$s_\lambda = \frac{g}{g_\lambda} \cdot \frac{h_\lambda^{(0)} + \ldots + h_\lambda^{(n_\lambda)}}{q^\alpha} = \frac{\widetilde{g}}{\widetilde{g}_\lambda} \frac{h_\lambda^{(0)} + \ldots + h_\lambda^{(n_\lambda)}}{q^{\alpha_\lambda}}.$$

3. We now proceed to the proof of BRAUER's theorem. From the previous notations we retain only those concerning $\mathfrak{G}$. Suppose that q is a prime divisor of g, that A is a q-regular element of $\mathfrak{G}$ of order a [7]) and that $\mathfrak{Q}$ is a q-Sylowgroup of the normalizer $\mathfrak{N}(A)$ of A in $\mathfrak{G}$. We denote by q^α the order of $\mathfrak{Q}$, by m the number of classes of $\mathfrak{Q}$ and by $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_m$ a complete system of representatives of these classes.

Let $\mathfrak{H}$ be the direct product of the cyclic group $\{A\}$ and of $\mathfrak{Q}$. Then an irreducible character $(\zeta_\lambda \vartheta_\mu)$ of $\mathfrak{H}$ is a product of an irreducible character ζ_λ of $\{A\}$ and an irreducible character ϑ_μ of $\mathfrak{Q}$:

$$(\zeta_\lambda \vartheta_\mu)(A^\nu Q) = \zeta_\lambda(A^\nu)\, \vartheta_\mu(Q) \qquad (Q \in \mathfrak{Q}).$$

If $(\zeta_\lambda \vartheta_\mu)^*$ is the character of $\mathfrak{G}$ induced by $(\zeta_\lambda \vartheta_\mu)$, we have

$$(\zeta_\lambda \vartheta_\mu)^* = \Sigma_{\varrho=1}^{k} r_{\lambda\mu\varrho} \varphi_\varrho,$$

where the $r_{\lambda\mu\varrho}$ are rational non negative integers.

Put

$$\psi_\lambda = \Sigma_{\varrho=1}^{a} \overline{\zeta_\varrho(A)}\, (\zeta_\varrho \vartheta_\lambda), \quad \psi_\lambda^* = \Sigma_{\varrho=1}^{a} \overline{(\zeta_\varrho(A)}\, (\zeta_\varrho \vartheta_\lambda)^* . \quad . \quad . \quad . \quad (7)$$

Then we have

$$\psi_\lambda(A^\nu Q) = \begin{cases} 0 & \text{if } \nu \neq 1 \\ a\, \vartheta_\lambda(Q) & \text{if } \nu = 1 \end{cases}$$

and

$$\psi_\lambda^* = \Sigma_{\varrho=1}^{k} w_{\varrho\lambda} \varphi_\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

if

$$w_{\lambda\mu} = \Sigma_{\varrho=1}^{a} \overline{\zeta_\varrho(A)}\, r_{\varrho\mu\lambda}.$$

[7]) This means that $(a, q) = 1$.

Suppose that $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_h$ are the classes of $\mathfrak{G}$ which contain elements with A as their q-regular factor [8]) and suppose that $A_1 = AQ_1$, $A_2 = AQ_2, \ldots, A_h = AQ_h$ are representatives of these classes. It may be assumed that $Q_1, Q_2, \ldots, Q_h$ are elements of $\mathfrak{Q}$. We note that (cf. (1))

$$\varphi_\lambda (AQ_\mu) = \Sigma_{\varrho, \tau}\, r_{\tau\varrho\lambda}\, \zeta_\tau (A)\, \vartheta_\varrho (Q_\mu) = \Sigma_{\varrho=1}^m\, \overline{w_{\lambda\varrho}}\, \vartheta_\varrho (Q_\mu) \quad (1 \leqslant \mu \leqslant l). \quad (9)$$

We now want to obtain a formula for $\psi_\lambda^* (AQ_\mu)$ similar to (2). We have

$$\psi_\lambda^* (AQ_\mu) = \frac{1}{aq^\alpha} \cdot \frac{g}{g_\mu} \cdot \Sigma_{X \in \mathfrak{K}_\mu \cap \mathfrak{H}}\, \psi_\lambda (X). \quad . \quad . \quad . \quad . \quad (10)$$

Since $\psi_\lambda (A^\nu Q) = 0$ if $\nu \neq 1$ $(Q \in \mathfrak{Q})$ we can restrict the summation to those X of $\mathfrak{K}_\mu \cap \mathfrak{H}$ which have the form AQ. $\mathfrak{K}_\mu \cap \mathfrak{H}$ splits up into a number of classes of $\mathfrak{H}$. Now a class of $\mathfrak{H}$ consists of the elements $A^\nu \times$ the elements of a class of $\mathfrak{Q}$. For $\psi_\lambda^* (AQ_\mu)$ we need the classes with $\nu = 1$ which are contained in $\mathfrak{K}_\mu$. Suppose that these classes are $\mathfrak{k}_\mu^{(0)}, \mathfrak{k}_\mu^{(1)}, \ldots, \mathfrak{k}_\mu^{(l_\mu)}$, that the number of elements in these classes is $q_\mu^{(0)}, q_\mu^{(1)}, \ldots, q_\mu^{(l_\mu)}$ and that $AQ_\mu^{(0)} = AQ_\mu$, $AQ_\mu^{(1)}, \ldots, AQ_\mu^{(l_\mu)}$ are representatives of these classes (the $Q_\mu^{(\nu)}$ are supposed to occur amongst the $\widetilde{Q}_\lambda$).

For some $X \in \mathfrak{G}$ we have $X (AQ_\mu) X^{-1} = AQ_\mu^{(\varrho)}$ $(\varrho = 0, 1, \ldots, l_\mu)$. Since the order of A is prime to q and since A and Q_μ, A and $Q_\mu^{(\varrho)}$ commute we infer, by raising both sides of this relation to a suitable power, that $XAX^{-1} = A$, $XQ_\mu X^{-1} = Q_\mu^{(\varrho)}$. It follows that X is an element of $\mathfrak{N}(A)$ and that the elements $Q_\mu^{(\varrho)}$ form a system of representatives for the classes of $\mathfrak{Q}$ belonging to the same class of $\mathfrak{N}(A)$ as Q_μ. Moreover, it is easily seen that $q_\mu^{(\varrho)}$ is the number of elements in the class of $Q_\mu^{(\varrho)}$ in $\mathfrak{Q}$.

Now we can write by (10)

$$\psi_\lambda^* (AQ_\mu) = \frac{1}{q^\alpha} \cdot \frac{g}{g_\mu} \cdot \Sigma_{\varrho=0}^{l_\mu}\, q_\mu^{(\varrho)}\, \vartheta_\lambda (Q_\mu^{(\varrho)}) \qquad \text{if } 1 \leqslant \mu \leqslant h$$

and we have $\psi_\lambda^* (A_\mu) = 0$ $\qquad$ if $h < \mu \leqslant k$.

Or

$$\psi_\lambda^* (A_\mu) = \Sigma_{\varrho=1}^k\, \beta_{\varrho\mu}\, \vartheta_\lambda (\widetilde{Q}_\varrho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where

$$\beta_{\lambda\mu} = 0 \qquad \text{if} \quad h < \mu \leqslant k$$

$$\beta_{\lambda\mu} = 0 \qquad \text{if} \quad \mu = 1, 2, \ldots, h \text{ and } \widetilde{Q}_\lambda = Q_\alpha^{(\beta)} \text{ with } \alpha \neq \mu;$$

$$\beta_{\lambda\mu} = \frac{g\, q_\mu^{(\nu)}}{g_\mu q^\alpha} \quad \text{if} \quad \mu = 1, 2, \ldots, h \text{ and } \widetilde{Q}_\lambda = Q_\mu^{(\nu)}.$$

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[8]) Any element $X \in \mathfrak{G}$ is a product of two commuting elements, one of which has an order which is a power of q whilst the other, the q-regular factor, has an order which is prime to q.

Again, we introduce some matrices:

$$\Phi_0 = (\varphi_\lambda (A\,Q_\mu))\ (k \times h);$$
$$\Psi^* = (\psi^*_\lambda\ (A_\mu))\ (m \times k),\ \Psi^*_0 = (\psi^*_\lambda\ (A\,Q_\mu))\ (m \times h);$$
$$\theta\ = (\vartheta_\lambda (\widetilde{Q}_\mu))\ (m \times m),\ \theta_0 = (\vartheta_\lambda (Q_\mu))\ (m \times h);$$
$$W = (w_{\lambda\mu})\ (k \times m);$$
$$B\ = (\beta_{\lambda\mu})\ (m \times k).$$

Then we have, by (8), (9) and (11)

$$\left.\begin{array}{l} \Psi\ = W'\,\Phi = \theta B \\ \Psi_0 = W'\,\Phi_0 \\ \Phi_0 = \overline{W}\,\theta_0 \end{array}\right\}.$$

Together with the orthogonality relations for the group characters of $\mathfrak{G}$ and $\mathfrak{Q}$ we have a system of equations which is completely equivalent to (A). In quite the same way [9]) as in **1** it is shown that one can find a matrix $P = (e_{\lambda\mu})$ such that $\Phi_0 = P\,\Psi^*_0$. The $e_{\lambda\mu}$ are algebraic numbers such that, if

$$\sigma_\lambda = \frac{g}{g_\lambda}\ \Sigma_{\varrho=0}^{l_\lambda}\ \frac{q_\lambda^{(\varrho)}}{q^\alpha} \qquad (\lambda = 1, \ldots, h)$$

the number $\sigma_1 . \sigma_2 \ldots \sigma_h . e_{\lambda\mu}$ is an algebraic integer.

Now $\frac{g}{g_\lambda}$ is the order of the normalizer in $\mathfrak{G}$ of $A Q_\lambda$ which is also the normalizer in $\mathfrak{N}(A)$ of Q_λ. Applying the results of **2** to $\mathfrak{N}(A)$ and its Sylowgroup $\mathfrak{Q}$ we find $(\sigma_\lambda, q) = 1$.

Now let Σ be a complete system of elements A representing the different classes of q-regular elements of $\mathfrak{G}$. Each of these elements A gives a system $\mathfrak{S}(A)$ of classes of $\mathfrak{G}$ containing elements with A as their q-regular factor and a class of $\mathfrak{G}$ belongs to exactly one system $\mathfrak{S}(A)$.

Applying what has been said above for these different A's we find by summation over these A's and by (7)

I) *There exists an integral rational number a_q, prime to q, such that for any φ_λ the character $a_q\,\varphi_\lambda$ is a linear combination with integral algebraic coefficients of characters $(\zeta_\lambda\,\vartheta_\mu)^*$ induced by characters of direct products $\{A\} \times \mathfrak{Q}$.*

This is the main point of the proof. After this we can proceed in the following way.

Suppose that $\{C\}$ is a cyclic subgroup of $\mathfrak{G}$ of order c and suppose that $\zeta_1, \zeta_2, \ldots, \zeta_c$ are its irreducible characters, inducing the characters $\zeta^*_1, \zeta^*_2, \ldots, \zeta^*_c$ of $\mathfrak{G}$. By

$$\Sigma_{\varrho=1}^{c}\ \overline{\zeta_\varrho\,(C)}\ \zeta_\varrho\,(C^\nu) = \begin{cases} 0 & \text{if } \nu \neq 1 \\ c & \text{if } \nu = 1 \end{cases}$$

[9]) The only (trivial) difference is that the matrix W, corresponding to C, has complex elements.

and by the formula for the induced characters from **1** we have

$$\xi_C^*(X) = \Sigma_{\varrho=1}^{c} \overline{\zeta_\varrho(C)}\, \zeta_\varrho^*(X) = \begin{cases} 0 & \text{if } X \notin \mathfrak{K}_C \\ \dfrac{g}{g_C} & \text{if } X \in \mathfrak{K}_C. \end{cases}$$

Therefore $\dfrac{g_C}{g}\,\xi_C^*(X)$ vanishes if $X \notin \mathfrak{K}_C$ and $\dfrac{g_C}{g}\,\xi_C^*(X) = 1$ if $X \in \mathfrak{K}_C$. Consequently:

II) *For any irreducible character* φ_λ, *the character* $g\,\varphi_\lambda$ *is a linear combination with integral algebraic coefficients of characters* ζ^* *induced by irreducible characters of cyclic subgroups of* $\mathfrak{G}$.

Now the greatest common divisor of the numbers a_q (q running through the prime divisors of g) and g clearly is 1. So we can find rational integers α_q and γ such that $\Sigma_{q|g}\,\alpha_q\, a_q + \gamma g = 1$.

By I and II $\Sigma_{q|g}\,\alpha_q\, a_q\, \varphi_\lambda + \gamma g \varphi_\lambda = \varphi_\lambda$ is a linear combination with integral algebraic coefficients of characters ω^* induced by characters $(\zeta_\lambda\, \vartheta_\mu)$, say $\varphi_\lambda = \Sigma_{\varrho=1}^{r}\, u_{\lambda\varrho}\, \omega_\varrho^*$. The ω_ϱ^* can be expressed in the φ_τ:

$$\omega_\varrho^* = \Sigma_{\tau=1}^{k}\, v_{\varrho\tau}\, \varphi_\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the $v_{\varrho\tau}$ are rational non negative integers).

Therefore

$$\Sigma_\varrho\, u_{\lambda\varrho}\, v_{\varrho\mu} = \delta_{\lambda\mu}.$$

It follows that the minors of degree k of $(v_{\lambda\mu})$ cannot have a common divisor $\neq 1$ [10]), consequently the k^{th} determinant divisor of the matrix $(v_{\lambda\mu})$ is 1. By (12) we get finally

III) *Any irreducible character of* φ_λ *of* $\mathfrak{G}$ *is a linear combination with integral rational coefficients of characters* $(\zeta_\lambda\, \vartheta_\mu)^*$.

Now it is not difficult to prove that a character $(\zeta_\lambda\, \vartheta_\mu)^*$ is induced by a linear character of a subgroup of $\mathfrak{G}$ [11]). Combining this with III we obtain BRAUER's theorem.

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[10]) If rational integers ξ_ϱ and integers η_ϱ of some algebraic numberfield satisfy $\Sigma_\varrho\, \xi_\varrho\, \eta_\varrho = 1$, then the numbers ξ_ϱ have the greatest common divisor 1.

[11]) This follows easily from the fact that BRAUER's theorem holds for a group whose order is a prime power (which can be proved simply). Cf. R. BRAUER, loc. cit. [1]), p. 512—514; see also SPEISER, § 63.

Physiology. — *Physiological investigations on bull semen.* By C. ROMIJN. (From the laboratory for Veterinary Physiology, State University, Utrecht). (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of May 29, 1948.)

Introduction.

Artificial insemination in domestic animals is a problem of great importance since many years. A considerable amount of work on problems of animal reproduction has been stimulated by the practice of artificial insemination. It is beyond doubt that increasing use of it will be made in many countries within a few years.

Starting with the work of Russian investigators, other scientists enlarged the field of research, especially the scientific work on the morphology and physiology of semen increased considerably. The evaluation of freshly drawn bull semen before use must be considered to be of prime importance. The secure maximam efficiency in conception rate, the methods of preliminary investigation of the semen should be modified to a maximal degree of accuracy.

Besides a morphological investigation of the semen with respect to the number of spermatozoa, their individual movement and mass movements (cloudy movements), the physico-chemical and physiological properties should be taken into consideration.

The author studied in the bull the aerobic metabolism of the semen as an indicator for the activity of the spermatozoa and the influence of the dilution fluid on it. The pH of freshly drawn bull semen has been measured and a possible relation between pH and semen quality discussed. The results are collected in the present article.

Methods.

The semen was collected with the artificial vagina in het clinic for Veterinary Obstetrics and Gynaecology and set at our disposal by the kindness of the director Prof. Dr. F. C. VAN DER KAAY. The samples of semen arrived in the physiological laboratory within 15 minutes after collection.

The oxygen consumption of the sperm was determined with a micro-respirometer of the Barcroft-type with modified respiration vessels (fig. 1).

Using the differential type of a respirometer, the thermobarometer can be avoided. To one of the respiration vessels 0.5 ccm. of semen and 1.5 ccm. of diluent are added and to the second 2 ccm. of diluent only. In the small cylindrical vessel on the bottom of the respiration chambers a small cylinder of filter paper was fixed, impregnated with 5 Mol. potassium hydroxide, to absorb the carbon dioxide. The capillary manometer was filled with petrol, coloured with a suitable dye (Sudan III). The total volume of each respiration vessel amounts about 16 ccm. The apparatus was calibrated by means of the methods described by DIXON (1934). A displacement of the manometerfluid of 10 mm. in one end of the manometer corresponds with an oxygen consumption of about 70 cmm.

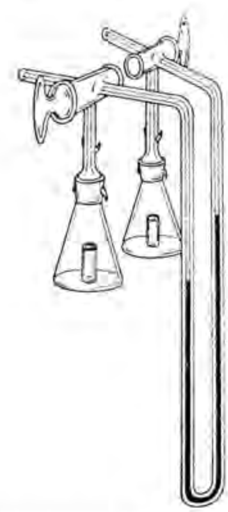

Fig. 1. Microrespirometer for the determination of the aerobic metabolism of semen.

The metabolism experiments were carried out in a waterbath at a temperature of 38.0° C., which remained constant within 0.1° C. Four respirometers were fixed on a framework and the whole shaken with a speed of 200 movements per minute and a maximal horizontal excursion of 3 cm. Manometer readings were made every 15 minutes and the oxygen consumption calculated per 10^9 sperm cells as a function of time.

The pH of the semen and of the diluents was measured with glass electrode with a "Arel"electrometer, an apparatus of Belgian origin and of the "Coleman"-type, using a saturated calomel electrode as reference electrode. A sample of 1 ccm. was sufficient and the apparatus was calibrated with buffersolutions, controled on pH with the hydrogen electrode, using an accurate potentiometer and a sensitive galvanometer.

Results.

In fig. 2 a typical curve is given. The oxygen consumption of 10^9 spermatozoa of the bull diluted with 0.85 % sodium chloride solution has been plotted against time (hours).

From fig. 2 the conclusion can be drawn that the oxygen consumption of bull sperm, diluted with three times the amount of physiological salt solution remains fairly constant for several hours. The amount of oxygen, consumed by 10^9 sperm cells during 2 hours at a temperature of 38° C. and a pH of 7.12 is about 250 cmm.

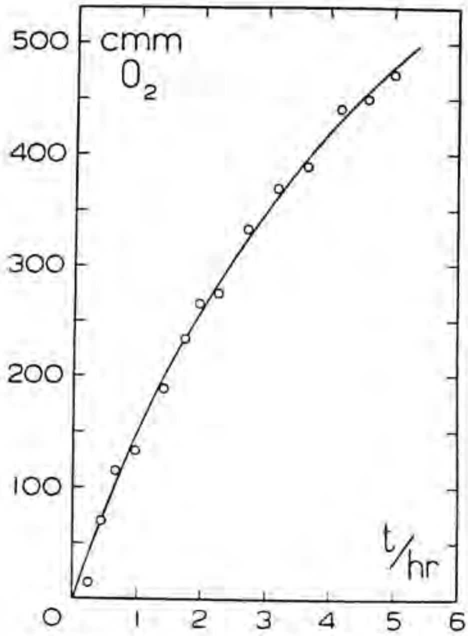

Fig. 2. Oxygen consumption of bull sperm, diluted with sodium chloride 0.85 %. Temp. 38.0° C. pH = 7.12.

In order to establish the influence of pH on the aerobic activity of bull semen, diluents containing phosphates in an isotonic concentration with a 0.85 % sodium chloride solution have been prepared. With the method described the oxygen consumption was measured at different pH. The results obtained with samples of semen of two bulls have been collected in fig. 3.

Overlooking fig. 3 the conclusion can be drawn that the aerobic activity of bull semen is greatly influenced by the pH of the dilution fluid, especially in the metabolism experiment of longer duration the influence is sharply pronounced. A pH of about 7 coincides with the optimal aerobic activity of the sperm; at this reaction of the medium the oxygen consumption is about proportional to the time (hours).

It seems therefore desirable for purposes of artificial insemination to prepare a dilution fluid of such a composition that firstly its concentration will be isotonic with a 0.85 % sodium chloride solution, secondly its pH after addition of a certain amount of bull semen will be about pH 7. Further details of the diluent will be mentioned below.

The activating influence of fresh egg yolk on bull semen is well-known; moreover this substance increases the time of survival after diluting the ejaculate. We measured the influence of egg yolk on the aerobic metabolism and the results of one experiment are collected in fig. 4.

From fig. 4 the activating influence of egg yolk on the aerobic metabolism is clearly demonstrated. Especially after 4 or 5 hours the amount of oxygen

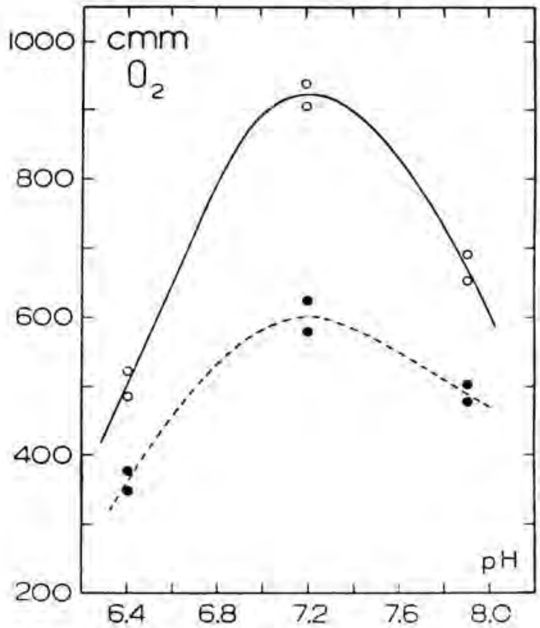

Fig. 3. pH and oxygen consumption of bull sperm.
0.5 ccm. of semen + 1.5 ccm. phosphate diluent.
Temp. 38° C.
(●) Oxygen consumption during two hours.
(○) ,, ,, ,, four ,,

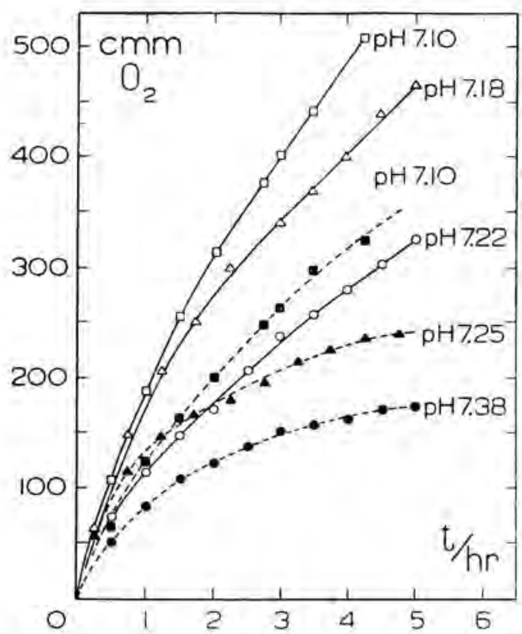

Fig. 4. Influence of egg yolk on the oxygen consumption of 10^9 spermatozoa.
(○, △, □) 0.5 ccm. semen + 1.5 ccm. egg yolk — phophate
(●, ▲, ■) 0.5 ccm. semen + 1.5 ccm. phosphate
Temp. 38° C. Number of spermatozoa 1027 × 10^6/ccm. of semen.

consumed in the presence of egg yolk makes about twice the amount without egg yolk. The influence of egg yolk on the oxygen consumption of bull sperm has been investigated on several samples of semen and the results partly collected in table I.

TABLE I.

Oxygen consumption of 10^9 spermatozoa during 4 hours at 38° C.
0.5 ccm. semen + 1.5 ccm. diluent.

Sample of bull semen	Number or spermatozoa $\times 10^6$ per 1 ccm.	pH of the semen	Oxygen consumption in cmm./4 hours	
			without egg yolk	with egg yolk
1	1132	6.60	228	378
2	1200	6.60	242	292
3	1358	6.41	74	221
4	906	6.49	154	320
5	808	6.48	153	219
6	1287	6.70	484	798
7	317	7.07	705	1313
8	604	6.79	496	748
9	1089	6.60	316	570
10	983	6.60	209	320
11	1348	6.67	284	538
12	900	6.90	184	215
13	845	6.89	232	420
14	314	7.19	260	435
15	802	6.70	256	375
16	665	6.66	224	424
17	580	7.00	326	495
18	952	—	383	623
19	1168	6.60	174	289
20	824	6.47	349	711
21	144	6.60	551	1580
22	723	6.49	188	346
23	1834	6.50	145	208
24	1364	6.60	241	403
25	1990	6.50	166	537
26	781	6.60	413	517
27	614	6.50	643	810
28	1880	6.63	283	359
29	1408	6.70	261	319
30	1623	6.32	285	475
31	891	6.48	252	375
32	977	6.23	325	394
33	828	6.78	244	381
34	692	6.77	197	375
35	1425	6.58	349	370
36	976	6.90	514	714
37	659	6.90	736	894
38	563	6.65	202	356
39	322	6.67	245	639

In nearly all the metabolism experiments carried out to investigate the influence of egg yolk on the oxygen consumption, the same result was obtained, though the increase in oxygen consumption is not the same in each sample of sperm in a quantitative respect. From a total number of about 200 ejaculates obtained from about 80 bulls in 180 cases a positive activation with egg yolk could be obtained, in 20 cases no activation or even an inhibition could be established. The exact mechanism of the egg yolk influence on the metabolism of bull sperm will be the subject of investigations in future.

According to TOSIC & WALTON (1946) the gradual decrease in oxygen uptake after addition of egg yolk would be caused by accumulation of hydrogen peroxide, formed by aerobic deamination of certain amino acids.

The results of our experiments confirm the evidence of a gradual inhibition of the oxygen consumption of the sperm egg yolk phosphate mixture (fig. 4) but it must be born in mind that the same decrease in oxygen intake can be detected in the sperm-phosphate mixture without egg yolk. Moreover even traces of H_2O_2 could not be detected in the sperm egg yolk phosphate medium after incubation with air during 4 hours at a temperature of 38° C. Catalase is fully absent from freshly drawn semen and egg yolk too does not contain this enzyme as we could establish with an accurate micromanometric technique. We suppose therefore that the inhibition of oxygen uptake after some hours at 38° C. must be caused not only by possible traces of H_2O_2 (TOSIC & WALTON) but accumulation of inhibiting metabolites of the sperm cells themselves will be of great importance.

The amount of egg yolk to be added to the dilution fluid of the sperm is matter of interest for the practice of artificial insemination. We investigated the influence of the egg yolk concentration on the aerobic metabolism of bull semen and the results have been collected in table II.

TABLE II.

Oxygen consumption of bull sperm (10^9 cells) at 38° C., diluted with phosphate buffer with varying amount of egg yolk.
0.5 ccm. sperm + 1.5 ccm. diluent. Number of spermatozoa in the ejaculate 6.31×10^9. pH = 7.28.

Egg yolk-phosphate	pH	Oxygen consumption (cmm.) after				
		1 hour	2 hrs	3 hrs	4 hrs	5 hrs
1 : 1	7.10	352	523	670	819	1005
1 : 2	7.21	384	530	648	780	910
1 : 3	7.21	270	461	579	680	824
1 : 4	7.12	457	575	610	780	932

Overlooking table II the conclusion is justified that to a certain degree the amount of egg yolk is of no influence on the activation of the sperm metabolism in a quantitative respect. We prefer therefore a dilution of

1 part of egg yolk with 3 parts of phosphate buffer. A concentration of 0.13 Mol. phosphate buffer was chosen, which is isotonic with a 0.85 % sodium chloride solution and 8 volumes of $Na_2HPO_4 . 2$ aq. 0.13 Mol. added to 2 volumes of KH_2PO_4 0.13 Mol. The pure salts after Sørensen have been used.

The activating properties of the egg contents are not limited to the yolk alone. The egg white stimulates the aerobic metabolism to the same degree (fig. 5).

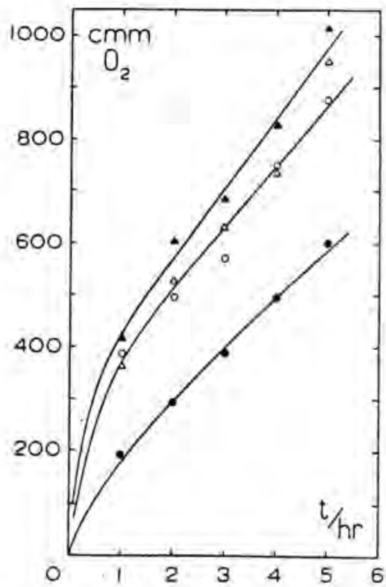

Fig. 5. Influence of yolk, white and total egg content on the oxygen consumption of bull semen.
0.5 ccm. semen + 1.5 ccm. of the diluent. Temp. 38° C.
(●) phosphate buffer pH 6.79
(○) yolk-phosphate pH 7.09
(▲) white-phosphate pH 7.45
(△) egg content-phosphate pH 7.30.

The pH of bull semen.

The pH of freshly drawn bull semen has been measured by WEBSTER (1932) on 7.0—7.5 by means of a somewhat primitive technique. According to DAVIES (1938) the pH of semen obtained with the artificial vagina has a pH lower than 7.0 and that collected after massage of the seminal vesiculae a pH higher than 7.0. HERMAN & SWANSON (1941) estimated the pH of the sperm of bulls with great fertility on 6.48, that of animals of inferior quality on 6.50. They could not establish a positive correlation between the pH of semen and fertility. ANDERSON (1942a, 1942b, 1944) however, who investigated this problem in East Africa found for the pH of the semen of bulls of good fertility values of pH 6.0 to pH 7.5. An average

TABLE III.

Sample of semen	Number of spermatozoa × 10^6 per 1 ccm.	Motility	pH	Decrease in pH after	
				1 hr	3 hrs
1	2800		5.88		
2	1412		6.30		
3	1168		6.60		
4	640	3	6.30		
5	1204	4	6.90		
6	644	4	7.22		
7	824	4	6.47		
8	568	4	6.89		
9	820	4	7.21		
10	528	4	6.85		
11	780	4	7.32		
12	781	4.	6.60		
13	614	4	6.50		
14	631	4	7.28		
15	892	4	7.01		
16	1353	4	6.95		
17	708	4	6.63		0.66
18	877	4	7.36	1.24	1.56
19	1287	4	6.70		
20	668	4	7.28	0.78	1.24
21	1020	4	6.71	0.21	0.37
22	756	4	7.07	0.82	0.87
23	317	4	7.07		
24	352	4	6.87	0.07	0.29
25	1132	4	6.60	0.52	0.70
26	902	4	6.52	0.53	0.46
27	1120	4	7.19	0.66	0.66
28	604	4	6.79		
29	1089	4	6.60		
30	659	4	7.07	0.35	0.77
31	1125	4	6.73	0.84	0.99
32	983	4	6.60	0.63	0.78
33	1348	4	6.67		
34	2144	4	6.70		
35	1160	4	7.02		
36	1880	4	6.63		
37	1408	4	6.70		
38	900	3	6.90	0.25	0.40
39	1200	4	6.60	0.70	0.77
40	845	4	6.89	0.66	0.89
41	1358	4	6.41	0.44	0.56
42	1990	4	6.50	0.76	1.01
43	906	4	6.49	0.48	0.67
44	808	4	6.48	0.39	0.44
45	1313	4	6.35	0.46	0.55
46	1147	4	6.60	0.72	0.73
47	1623	4	6.32	0.77	0.88
48	723	4	6.49	0.58	0.64
49	891	3	6.48	0.39	0.48
50	1296	4	6.20	0.21	0.30
51	1834	4	6.50	0.48	0.63
52	977	4	6.23	0.38	0.55
53	1364	4	6.60	0.88	0.95
54	475	4	6.59	0.26	0.49
55	684	3	6.70	0.30	0.37
56	828	4	6.78	0.50	0.95
57	1425	4	6.58		
58	692	4	6.77	0.47	0.73
59	976	4	6.90	1.02	1.25
60	991	3	6.60		

TABLE III. (*Continued*)

Sample of semen	Number of spermatozoa $\times 10^6$ per 1 ccm.	Motility	pH	Decrease in pH after	
				1 hr	3 hrs
61	659	4	6.90		
62	563	4	6.65	0.34	0.46
63	322	4	6.67	0.27	0.28
64	588	3	6.50	0.35	
65	350	3	6.60	0.39	0.46
66	802	4	6.70	0.48	0.60
67	665	3	6.66	0.36	0.34
68	580	4	7.00	0.53	0.84
69	613	3	6.80	0.61	0.93
70	1180	4	6.77	0.56	0.85
71	663	3	6.70	0.48	0.77
72	558	3	6.72		
73	1064	4	6.40	0.59	0.77
74	1130	4	6.39	0.29	0.31
75	661	4	6.33	0.40	0.40
76	700	4	6.67	0.53	0.69
77	364	3	6.79	0.44	0.28
78	472	3	6.48		
79	706	4	6.30	0.43	0.67
80	958	4	6.45	0.50	0.73
81	708	4	6.41	0.50	0.21
82	1020	4	6.30	0.52	0.72
83	637	4	6.60	0.22	0.36
84	1030	4	6.40	0.37	0.39
85	934	4	6.50		
86	1144	4	6.76	0.64	0.91
87	1128	4	6.40	0.37	0.45
88	623	4	6.20	0.26	0.45
89	1145	2	6.10	0.41	0.50
90	1050	4	6.63	0.91	0.91
91	684	2	6.72	0.39	0.33
92	926	2	6.25	0.30	
93	2278	3	6.51	0.41	0.41
94	618	2	6.61	0.32	0.43
95	445	2	6.89	0.51	0.96
96	1080	3	6.10	0.65	0.88
97	970	4	6.08	0.54	0.73
98	1170	4	6.00	0.47	0.71
99	885	4	6.15	0.37	0.50
100	1040	4	6.37	0.45	0.64
101	788	4	6.62	0.38	0.69
102	510	3	6.62	0.40	0.63
103	675	3	6.80	0.57	0.76
104	475	3	6.58	0.33	0.51
105	948	3	7.13	0.78	0.84
106	1534	4	6.32	0.25	0.48
107	939	4	7.00		
108	1035	4	6.89		
109	903	3	6.69		
110	412	4	6.63	0.03	0.05
111	904	4	6.40	0.57	0.64
112	1177	3	6.72	0.62	0.83
113	1757	4	6.57	0.48	0.68
114	1544	3	7.02	0.91	0.86
115	1383	3	6.50	0.37	0.61
116	1882	3	6.43	0.40	0.61
117	990	3	6.76	0.44	0.75
118	953	3	6.83	0.51	0.84
		Average	6.65	0.49	0.65

TABLE IV.

Sample of semen	Number of spermatozoa × 10^6 per 1 ccm.	Motility	pH	Decrease in pH after	
				1 hr	3 hrs
1	180	1	6.81		
2	44	1	7.78		
3	186	3	7.08		
4	72	2	7.89		
5	306	3	7.18	0.41	0.60
6	200	3	7.71	0.56	0.99
7	1.6	–	7.73	–0.38	–0.02
8	170	1	7.10		
9	592	2	7.35		
10	504	2	7.15		
11	117	–	8.24		
12	377	4	7.63		
13	96	2	7.03	–0.07	–0.28
14	456	3	6.70	0.40	0.79
15	144	2	6.60	–0.22	–0.30
16	156	–	7.05	0.15	0.07
17	314	2	7.19	0.32	0.48
18	808	3	6.60	0.09	0.10
19	439	1	6.59	0.14	0.16
20	506	2	6.70	0.22	0.22
21	744	2	6.78	–0.12	–0.32
22	445	–	7.22	0.00	–0.74
23	475	1	6.60	–0.07	–0.29
24	540	3	6.50	0.02	–0.01
25	441	4	6.70	0.31	0.25
26	439	2	7.35	0.68	0.87
27	208	1	6.98	0.38	0.38
28	271	1	6.98	0.00	–0.12
29	836	2	7.11	0.52	0.52
30	930	4	6.28	0.28	0.02
31	605		6.82		
32	589	4	6.71	0.00	0.02
33	339	4	7.11	0.07	0.00
34	210	1	7.69	0.31	0.39
35	546	3	6.40	0.00	–0.11
36	415	3	6.61	0.36	0.47
37	555	1	6.98	0.30	0.38
38	500	3	6.50	0.29	0.26
39	581	2	6.90	0.17	0.30
40	635	3	6.51		
41	745	2	7.01	0.22	0.36
42	450	2	7.00	0.20	0.44
43	697	4	6.63	0.22	0.29
44	1596	3	6.69	0.27	0.36
45	1788	1	6.78		
46	270	1	7.28		
47	204	2	7.80		
48	403	–	7.99	0.58	0.68
49	277	2	6.68	0.28	0.35
50	820	1	6.86	0.38	0.52
51	178	1	6.98		
52	526	2	7.40	0.10	0.21
53	449	3	6.90	0.12	0.51
54	115	–	7.84	0.39	0.55
55	1927	2	6.72	0.54	0.71
56	1035	2	6.81	0.31	0.39
57	1383	1	6.58	–0.15	0.00
58	1388	3	7.03	0.13	0.61
59	928	1	7.14	0.36	0.96
		Average	7.03	0.21	0.28

of pH 6.72 ± 0.02 could be established in samples of semen from 11 fertile bulls.

We investigated 177 samples of semen from different bulls of good and poorer fertility on pH, motility, number of spermatozoa and metabolism. 118 of these samples could be distinguished from the other 59 with respect to motility, number of cells and other morphological criteria and the results have been collected in tables III and IV. With respect to motility a classification into 4 groups has been proposed, indicated by the figures 1 to 4; the figure 4 corresponding with semen of highest sperm motility. The average pH value of 118 samples of good quality could be established on pH 6.65, that of 59 samples of poorer quality on pH 7.03. A conclusion about real difference in pH between good and bad samples can only be drawn after mathematical treatment of the results.

We calculated the standard deviation (s.d.) of the group of 118 samples on 0.29 and the coefficient of variation (C.V.) therefore on 4.4 %. The "standard error" can be calculated on 0.027 and the pH value of samples of good quality can be expressed by pH 6.65 ± 0.027.

The s.d. of the second group of sperm samples including the 59 ejaculates of poorer quality could be calculated on 0.44, the C.V. therefore on 6.3 % and the "standard error" on 0.057.

The average pH of bull sperm of bad quality can be expressed by pH 7.03 ± 0.057. From the values for the "standard error" mentioned above the conclusion after simple mathematical treatment can be drawn that the difference in pH between the two groups of sperm samples is a significant one; a value for △ of 6.03 could be calculated.

Besides morphological investigation the determination of the pH of an ejaculate will be of certain importance for the appraisal of bull semen. In general a pH higher than 7.03 indicates a sperm sample of poor quality,

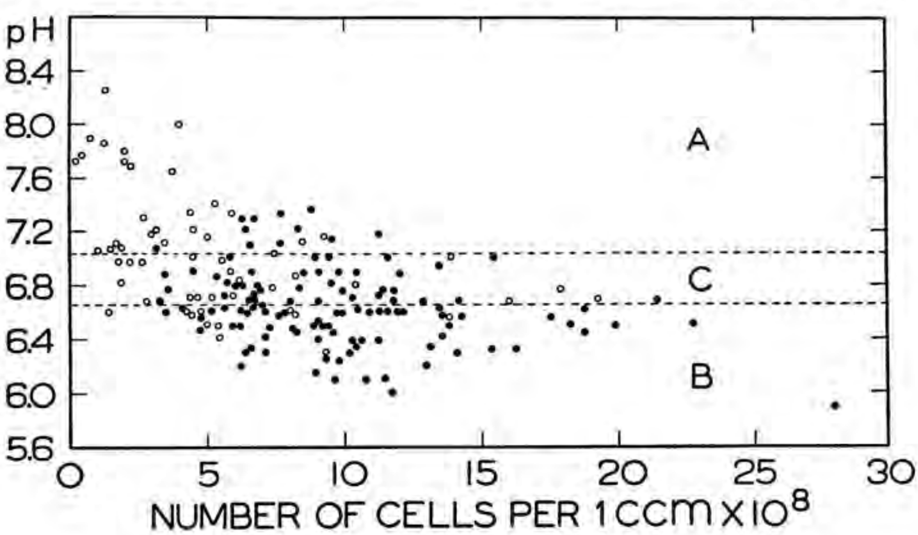

Fig. 6. For explanation see text.

a pH lower than 6.65 one of good quality. In fig. 6 the pH of all 177 investigated ejaculates has been plotted against the number of spermatozoa per 1 ccm. of semen.

A small number of spermatozoa per 1 ccm. of semen in general coincides with a low pH value and the reverse holds too. The dotted lines mark the zones A, B and C, enclosing the points of the sperm samples of poor (○), good (●) and doubtful quality. Especially the points between the two dotted lines (zone C) require our full attention. The pH of a freshly collected sample of bull sperm being lower than 7.03 and higher than 6.65, no direct conclusion about its quality can be obtained. About 50 per cent of the points within this pH range belong to the ejaculates of good quality, the remaining 50 per cent to those of poorer quality independant the number of spermatozoa. It will be necessary therefore to collect more data of such samples in order to obtain a correct idea about their quality.

We investigated the change in pH of ejaculates incubated during some hours at a temperature of 38° C. protected from air.

The results of these experiments showed that a significant difference exists in decrease in pH after one hour's incubation between samples of semen of good and poorer quality.

Storage of semen at low temperature (4° C.) during several hours did not change the acidity noteworthy:

30 min. after collection	pH 6.72	pH 6.72
1 hr ,, ,,	6.80	6.70
3 hrs ,, ,,	6.88	6.92
4 ,, ,, ,,	6.99	6.84
5 ,, ,, ,,	7.12	6.98
24 ,, ,, ,,	7.25	6.92
48 ,, ,, ,,	7.01	

A decrease in pH could not be observed, contrary a small increase in pH is very common, perhaps by evaluation of small amounts of ammonia.

Incubation at 38° C. protected from air will stimulate the accumulation of acid metabolites and the decrease in pH must be an indication for the activity of the sperm cells.

In ejaculates of good quality an average pH change of 0.49 after one hour's incubation could be measured, of 0.65 pH units after three hour's incubation at 38°C. In the samples of poor quality these figures amounted respectivally 0.21 and 0.28 pH units. Mathematical treatment of the results pointed out the significance of these differences.

It seems to us that measurement of the pH of the semen will be of certain importance for the appraisal of the ejaculate, in doubtful cases an incubation at 38° C. during one hour will be necessary to confirm the conclusion based on other characteristics.

In veterinary practice pH measurements cannot easily carried out with the pH electrometer. We developed a colorimetric technique for practical purpose, using the dye Brom-thymol blue as an indicator. The change in colour of this indicator falls between pH 6.00 to pH 7.6 from yellow to blue. 0.5 ccm. of semen is collected in a thoroughly cleaned pyrex tube and mixed with 1 ccm. of a dilution fluid. This diluent consists of 100 parts of a sodium chloride solution 0.85 per cent and 3.5 parts of the indicator-solution, prepared after the instructions given by CLARK (1923). Dilution of the semen in this way does not change the pH noteworthy and the decrease in pH during incubation is not altered too.

A yellow-green colour of the mixture indicates a sample of good quality, a bluish-green or blue colour one of bad quality. A distinct green colour needs the use of the incubation test. Colour change after one hour from green to yellow-green indicates a decrease in pH of sufficient quantity to reckon the semen to the ejaculates of better quality. When after one hour's incubation the green colour has not changed or a bluish hue may be visible, the activity of the semen is to low and it will be better not to use it for insemination purposes.

Summary.

A method for estimating the aerobic activity of bull sperm has been described and the influence of several factors on the oxygen consumption has been studied. At a pH of about 7.2 the aerobic activity is of maximal intensity, there is a rapid decrease towards the acid and alcaline side of the optimal pH. Fresh egg yolk, as well as egg white stimulate the oxygen consumption of the semen considerably in most of the samples investigated in this respect. Thits activation is relatively independant from the amount of egg yolk.

A significant difference could be established between the pH of samples of semen of good quality (average pH 6.65 $\pm$ 0.027) and that of samples of poorer quality (average pH 7.03 $\pm$ 0.057). Incubation of the sperm at a temperature of 38° C., protected from air, causes a decrease in pH of about 0.5 units in the ejaculates of better and of about 0.2 units in those of poorer quality.

A method for determination the pH of semen in veterinary practice has been developed.

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Aerodynamics. — *L'influence de la conduction thermique sur les anémomètres à fils chauds.* By R. BETCHOV. (Mededeling No. 55 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of May 29, 1948.)

Un anémomètre à fil chaud est le siège d'un phénomène de conduction thermique qui réduit sa sensibilité. La réponse du fil à une fluctuation du vent est modifiée en amplitude et en phase. L'étalonnage d'un fil avec un courant électrique pulsé peut également être faussé par la conduction. Il est possible de calculer les corrections après avoir mesuré le facteur traduisant l'effet de conduction thermique.

1. *Equation générale du fil chaud avec conduction vers les supports.* — Le fil chaud est toujours fixé entre deux supports dont nous supposons la température égale à celle de l'air ambiant et la conduction thermique infinie. Il est évident qu'une certaine quantité de chaleur est transmise par le fil à ces supports et que cette conduction refroidit les extrémités du fil. Sa température est donc fonction de la distance comptée à partir du milieu du fil, soit de la variable x, variant entre les limites $+l$ et $-l$ (la longueur totale du fil est $2l$).

L'équation d'énergie du fil doit être exprimée pour un petit segment dx et il nous faut introduire les grandeurs:

$\mathbf{R}$ et $\mathbf{R_0}$ = résistance du fil, par cm, respectivement à chaud et à froid;
$\varkappa$ = coefficient de conduction thermique, en watt/°C cm.

Nous utiliserons les symboles définis dans nos précédents articles [1]), soit:

R et R_0 = résistance totale du fil, à chaud et à froid,
a et b = coefficients de la formule de KING,
V = vitesse du vent,
I = courant électrique,
α = coefficient de variation thermique de R,
m = masse totale du fil,
s = chaleur spécifique, en calories/gr °C,
d = diamètre du fil.

Il faut remarquer que R_0, a, b, et m sont proportionnels à la longueur du fil.

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[1]) R. BETCHOV et E. KUYPER, Un amplificateur pour l'étude de la turbulence d'un écoulement d'air, Med. 50, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 1134—1141 (1947). — R. BETCHOV, L'inertie thermique des anémomètres à fil chaud, Med. 54, ibid. **51**, 224—233 (1948).

L'équation d'énergie s'écrit alors:

$$\mathbf{R}\,I^2 = \frac{a+b\sqrt{V}}{\alpha R_0}(\mathbf{R}-\mathbf{R_0}) + \frac{4{,}2\,m\,s}{\alpha R_0}\frac{\partial \mathbf{R}}{\partial t} - \frac{\pi\,\varkappa\,d^2}{4\alpha \mathbf{R_0}}\frac{\partial^2 \mathbf{R}}{\partial x^2} \quad . \quad . \quad (1)$$

Chal. fournie	Chal. transmise à l'air	Echauffement du fil	Chaleur de conduction.

2. *Solution stationaire.* — Si **R** ne dépend pas du temps, ainsi que V et I, l'intégration est facile. Comme conditions aux limites, nous prendrons: „Si $x = \pm l$, $\mathbf{R} = \mathbf{R_0}$". Nous poserons:

$$A = \frac{a+b\sqrt{V}}{\alpha R_0}\ ; \quad l^* = \sqrt{\frac{\pi\,\varkappa\,d^2}{4\,\alpha\,\mathbf{R_0}\,(A-I^2)}} \quad . \quad . \quad . \quad . \quad (2)$$

et l'équation (1) devient:

$$-\,l^{*2}\frac{d^2\mathbf{R}}{dx^2} + \mathbf{R} = \frac{A\,\mathbf{R_0}}{A-I^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

La solution est:

$$\mathbf{R} = \frac{A\,\mathbf{R_0}}{A-I^2}\left(1 - \frac{I^2}{A}\,\frac{\cosh x/l^*}{\cosh l/l^*}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

La résistance totale du fil est donnée par:

$$R = \int_{-l}^{+l} \mathbf{R}\,dx = \frac{A\,R_0}{A-I^2}\left(1 - \frac{I^2}{A}\,\frac{\mathrm{Tgh}\,\xi}{\xi}\right) \quad . \quad . \quad . \quad . \quad (5)$$

avec $\xi = l/l^*$. Nous désignerons l^* sous le nom de „longueur froide". Le rapport ξ traduit l'effet de conduction, il est en général supérieur à 2, ce qui permet de poser Tgh $\xi = 1$.

Pratiquement, les courbes donnant $R\,(I,\,V)$ ne permettent pas de déterminer ξ, elles accusent des écarts importants. En effet, la formule de King est approchée et les conditions idéales ne sont pas exactement remplies (densité de l'air uniforme, supports sans effets sur l'écoulement, fil rectiligne, conductivité de l'air indépendante de la température, etc.). La formule (5) nous donne l'importance relative de la conduction, mais la fonction $R\,(I,\,V)$ et ses dérivées partielles doivent être déduites de l'étalonnage statique du fil.

3. *Solution avec fluctuation du vent.* — Lorsque la vitesse du vent fluctue, la résistance par cm se compose d'un terme constant **R** et d'un terme **r** qui est fonction du temps. La fluctuation v de la vitesse V est petite (ordre de 1 %), et on peut écrire:

$$\mathbf{R}I^2 + \mathbf{r}\,I^2 = A(\mathbf{R}-\mathbf{R_0}) + A\,\mathbf{r} + u(\mathbf{R}-\mathbf{R_0}) + \frac{4{,}2\,ms}{\alpha R_0}\frac{\partial \mathbf{r}}{\partial t} - \frac{\pi\,\varkappa\,d^2}{4\alpha \mathbf{R_0}}\frac{\partial^2}{\partial x^2}(\mathbf{R}+\mathbf{r}) \quad (6)$$

avec:

$$u = \tfrac{1}{2}\,\frac{b\sqrt{V}}{\alpha R_0}\,\frac{v}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Ceci revient à négliger les termes d'ordre v^2. Nous supposons que le courant I reste exactement constant, ce que l'on peut obtenir avec le dispositif décrit dans notre deuxième article, p. 227—228. Les termes d'ordre zéro en u et $\mathbf{r}$ donnent l'équation (3) dont la solution est donnée par (4). En remplaçant u par $u\,\varepsilon^{j\omega t}$ et $\mathbf{r}$ par $\mathbf{r}\,\varepsilon^{j\omega t}$ et en utilisant (4), on obtient:

$$-l^{*2}\frac{d^2\mathbf{r}}{dx^2}+\mathbf{r}(1+j\,\omega/\omega^*)=-\frac{\mathbf{R_0}\,I^2\,u}{(A-I^2)^2}\left(1-\frac{\cosh x/l^*}{\cosh\xi}\right)\quad . \quad . \quad (8)$$

avec

$$\omega^*=\frac{a\,R_0}{4{,}2\,m\,s}\,(A-I^2).\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

La solution symétrique en x est:

$$\mathbf{r}=\frac{-\mathbf{R_0}\,I^2\,u}{(A-I^2)^2}\left(\frac{1}{1+j\,\omega/\omega^*}+j\,\frac{\omega^*}{\omega}\,\frac{\cosh x/l^*}{\cosh\xi}+C_0\cosh p\,x/l^*\right)\quad (10)$$

avec

$$p=\sqrt{1+j\,\omega/\omega^*}.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

La constante d'intégration C_0 est déterminée par la condition: $\mathbf{r}=0$ si $x=\pm\,l$, ce qui donne:

$$C_0=\frac{-1}{\cosh p\,\xi}\left(\frac{1}{1+j\,\omega/\omega^*}+j\,\frac{\omega^*}{\omega}\right).\quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

La résistance totale r est donnée, après intégration et élimination de C_0 avec (12), par:

$$\left.\begin{aligned} r=\int_{-l}^{+l}\mathbf{r}\,dx=&\frac{-R_0\,I^2\,u}{(A-I^2)^2\,(1+j\,\omega\,/\,\omega^*)}\\ &\left\{1-\frac{\operatorname{Tgh} p\,\xi}{p\,\xi}-\left(1-j\,\frac{\omega^*}{\omega}\right)\left(\frac{\operatorname{Tgh}\xi}{\xi}-\frac{\operatorname{Tgh} p\,\xi}{p\,\xi}\right)\right\}\end{aligned}\right\}.\quad (13)$$

La conduction se manifeste par la présence du terme ξ et son effet dépend du rapport ω/ω^*. Si $\xi>2$, on peut poser Tgh $\xi=1$, ainsi que Tgh $p\,\xi=1$; en effet, on peut décomposer p en parties réelles et imaginaires et écrire $p=\alpha+j\,\beta$ avec $\alpha\geqq 1$, ce qui donne:

$$\lim_{\xi\to\infty}(\operatorname{Tgh} p\,\xi)=\operatorname{Lim}\left(\frac{\operatorname{Tgh}\alpha\,\xi+j\operatorname{tg}\beta\,\xi}{1+j\operatorname{Tgh}\alpha\,\xi\operatorname{tg}\beta\,\xi}\right)=1\quad . \quad . \quad (14)$$

La résistance totale r est alors donnée par:

$$r=\frac{-R_0\,I^2\,u}{(A-I^2)^2\,(1+j\,\omega/\omega^*)}\left\{1-\frac{1}{\xi}\left[1+j\,\frac{\omega^*}{\omega}\,\frac{p-1}{p}\right]\right\}.\quad . \quad . \quad (15)$$

L'amplitude et la phase du terme entre [ ] sont représentées en fig. 1; l'amplitude varie entre la valeur 3/2 pour $\omega=0$ et la valeur 1 pour $\omega=\infty$ en décrivant à peu près un demi-cercle. Le déphasage est maximum aux environs de $\omega/\omega^* = 3/2$. Lors d'une mesure de turbulence, la tension alternative du fil rI est appliquée à un amplificateur qui corrige l'effet du terme $(1 + j\omega/\omega^*)$ et le carré moyen du signal de sortie est mesuré. Pour

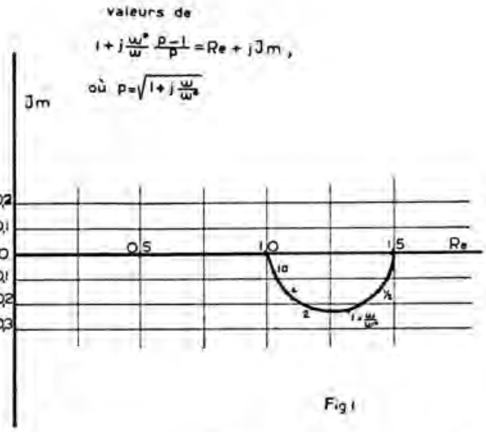

Fig. 1. Amplitudes et phases du terme de correction pour la conduction thermique, selon la formule (15).

déduire la fluctuation v de cette mesure à l'aide de la formule (15), il faudrait connaître les facteurs A, b/a et ξ contenus dans les formules (2), (7) et (15). Mais nous avons déjà signalé les écarts possibles. La difficulté peut être tournée en introduisant les dérivées partielles de la fonction $R(I, V)$; on peut les déduire immédiatement de l'étalonnage statique du fil. A partir des formules (5) et (15) on démontre facilement la relation:

$$r = \left(\frac{\partial R}{\partial V}\right)_I \cdot v \cdot \frac{1 - \frac{1}{\xi}\left[1 + j\frac{\omega^*}{\omega}\frac{p-1}{p}\right]}{\left(1 - \frac{3}{2\xi}\right)(1 + j\,\omega/\omega^*)} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Comme nous avons considéré r et v comme des fluctuations, il est naturel de constater que lorsque l'inertie thermique n'agit pas (ω/ω^* très petit), la conduction ne se manifeste pas explicitement. Elle agit sur la fonction $R(I, V)$ où son effet ne peut pas être séparé des autres perturbations. Avec ω/ω^* très grand, la conduction produit une tension plus élevée du fil, si l'on fait abstraction du terme $(1 + j\,\omega/\omega^*)$. Avec $\xi = 4$, cette majoration atteint 20 %.

La fluctuation de V se fait en général selon un spectre de composantes harmoniques et la correction de conduction, qui dépend de ω/ω^*, devrait se faire, en principe, pour chaque composante. Pratiquement, on observera le spectre de la turbulence et on fera une correction moyenne, basée sur une fréquence moyenne. Il serait possible de compenser cette erreur de conduction, en introduisant dans l'amplificateur un filtre donnant à peu près le diagramme inverse de la figure 1. On pourrait aussi introduire intentionellement un écart entre la fréquence propre du circuit self-résistance du préamplificateur décrit dans notre premier article et la fréquence propre du fil $\omega^*/2\pi$.

Cependant, avant d'appliquer rigoureusement la correction de conduction calculée, il faudrait être certain que les autres perturbations signalées n'agissent pas sur la réponse en fréquence de l'anémomètre. Les fils chauds que nous utilisons ne vérifient la formule (5) qu'à 5 ou 10 % près, et l'effet de conduction est de cet ordre. Nous considérons donc la formule (16) comme un exemple d'erreur possible, fonction de ω/ω^*. Elle nous indique la limite inférieure à la longueur des fils. Le choix de l'alliage utilisé doit également se faire de manière à ce que le rapport ξ reste aussi grand que possible (supérieur à 5, par exemple). On peut réduire la longueur froide du fil en élevant sa fréquence propre, mais cela oblige à travailler avec des fils de petit diamètre, plus fragiles.

4. *Solution avec courant pulsé.* — Nous avons décrit dans notre deuxième publication une méthode permettant de mesurer l'inertie thermique du fil (exprimée par la constante C), en faisant fluctuer I. Il convient donc d'examiner l'effet de conduction dans ce cas. En remplaçant dans l'équation (1), $\mathbf{R}$ par $\mathbf{R}+\mathbf{r}\,\varepsilon^{j\omega t}$ et I par $I+i\,\varepsilon^{j\omega t}$, après suppression des termes d'ordre 2 et introduction de la solution stationaire (4), on obtient une équation analogue à (8), soit:

$$-l^{*2}\frac{d^2\mathbf{r}}{dx^2}+\mathbf{r}\,(1+j\,\omega/\omega^*)=\frac{2\,A\,I\,i}{(A-I^2)^2}\left(1-\frac{I^2}{A}\,\frac{\cosh x/l^*}{\cosh\xi}\right). \quad (17)$$

La solution, compte tenu des conditions aux limites et intégrée de $-l$ à $+l$, donne:

$$r=\int_{-l}^{+l}\mathbf{r}\,dx=\frac{2\,A\,I R_0\,i}{(A-I^2)^2}\left\{\frac{1-(\operatorname{Tgh}p\,\xi)/p\,\xi}{1+j\,\omega/\omega^*}+j\,\frac{I^2\,\omega^*}{A\,\omega}\left(\frac{\operatorname{Tgh}\xi}{\xi}-\frac{\operatorname{Tgh}p\,\xi}{p\,\xi}\right)\right\}. \quad (18)$$

On remarque que le terme de conduction dépend de ξ, de ω/ω^* et du facteur I^2/A qui ne figure pas dans le terme correspondant lorsque V fluctue.

En utilisant le pont pour courants alternatifs décrit précédement, on observe la tension rI en fonction de ω, soit, avec approximation pour $\xi>2$:

$$e_{\tilde{\imath}}=r\,I=\frac{2\,A\,I^2\,R_0\,i}{(A-I^2)^2}\left\{\frac{1-1/p\,\xi}{1+j\,\omega/\omega^*}+j\,\frac{I^2}{A\,\xi}\,\frac{\omega^*}{\omega}\,\frac{p-1}{p}\right\}. \quad . \quad . \quad (19)$$

Lorsque $\omega \gg \omega^*$, on peut écrire:

$$e_{\tilde{\imath}}=\frac{2\,A\,I\,R_0\,i}{(A-I^2)^2}\,\frac{\omega^*}{j\,\omega}\left(1-\frac{I^2}{A\,\xi}\right). \quad . \quad . \quad . \quad . \quad . \quad (20)$$

La conduction réduit donc la tension de sortie, mais elle réduit dans le même rapport la résistance totale R, comme le démontre la formule (5). La relation utilisée dans notre précédente publication était:

$$e_{\tilde{\omega}} = \frac{C R I^2 i}{\omega/2\pi} \quad \text{avec} \quad C = a R_0/4{,}2\,\pi\, m s \quad . \quad . \quad . \quad . \quad (21)$$

[comp. les formules (11) et (15) de cette publication]; elle n'est donc pas modifiée par la conduction, à un degré d'approximation suffisant (Tgh $\xi = 1$). Mais l'effet de conduction n'est pas négligeable si ω est de l'ordre de ω^*. Une mesure de ω^* faite dans ces conditions, par exemple en observant un déphasage de 45 degrés, peut être affectée d'une erreur importante.

C'est à cet effet que nous attribuons une partie des erreurs observées par SIMMONS [2]) qui cherchait à mesurer ω^* en travaillant dans les conditions suivantes:

fil de Pt, $d = 2{,}5$ microns, $2l = 1{,}2$ mm,
$V = 2$ m/s, $T = 500$ °C.

Nous en déduisons, approximativement:

$$R/R_0 = 2{,}5, \quad I^2/A = 0{,}6 \quad \text{et} \quad 4 < \xi < 5$$

Avec $\xi = 4{,}5$, le déphasage se fait selon un arc de tangente 0,85, environ, au lieu de 1, et l'amplitude de la tension est environ 0,5 au lieu de 0,7, en prenant pour unité la tension sans conduction ni inertie.

Cela n'explique pas les erreurs de 100 % signalées pas SIMMONS, mais il est certain que le rôle de la conduction ne peut pas être négligé.

Avec un même genre de fil, nous avons obtenu une même constante C à différentes températures (entre 150 et 500 °C): nous avions:

fil de Pt, $d = 2$ microns, $2l = 0{,}65$ mm,
$R_0 = 30$ ohms; $C = 1{,}3 \cdot 10^6$.

5. *Mesure de la longueur froide.* — Nous avons plaçé un fil chaud dans un bulbe de verre et pompé l'air jusqu'à un vide de l'ordre de 10^{-7} mm Hg. La mesure de R en fonction de I donne une valeur approchée de ξ. En effet, en posant $A = 0$ dans l'équation (5) et en éliminant le facteur j qui apparaît, on obtient:

$$R = R_0 \frac{\operatorname{tg} \xi_0}{\xi_0} \quad \text{avec} \quad \xi_0 = [\xi(A=0)] = I \sqrt{\frac{4 l a R_0}{\pi \varkappa d^2}} \quad . \quad . \quad (22)$$

La comparaison entre la courbe $R(I)$ et la courbe $y = \operatorname{tg} x/x$ donne le facteur ξ_0/I. On peut alors déduire $\varkappa$ de la relation:

$$\varkappa = \frac{4 l a R_0}{\pi d^2} \frac{I^2}{\xi_0^2} = 2\pi c\, 4{,}2\, s\, \delta\, l^2 I^2/\xi_0^2 \quad . \quad . \quad . \quad . \quad (23)$$

[2]) L. F. G. SIMMONS, Note on Errors arising in Measurements of Turbulence, Aeron. Res. Council Techn. Rep. No. 1919, May 1939 (4042).

mais la relation utile pour la correction de conduction nous donne:

$$\xi = \xi_0 \sqrt{\frac{A - I^2}{I^2}} \simeq \xi_0 \sqrt{\frac{R_0}{R - R_0}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

La comparaison entre $R\,(I)$ et $y = \mathrm{tg}\, x/x$ ne peut se faire que lorsque R/R_0 est supérieur à 1,05 (erreurs de mesure) et la température inférieure à 700 degrés, environ (effet du rayonnement et variation de α et $\varkappa$ avec T).

6. *Correction pour un changement de la température ambiante.* — Dans tous nos calculs nous avons supposé que la température T_a de l'air restait constante; il peut arriver qu'elle varie de manière appréciable et cela introduit une correction spéciale. Si la température de l'air s'élève de t degrés, les grandeurs R_0 et α deviennent R_{01} et α_1, soit:

$$R = R_0\,(1 + \alpha\, T) = R_{01}\,(1 + \alpha_1\, T - \alpha_1\, t) \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

où

$$R_{01} = R_0\,(1 + \alpha\, t), \quad \alpha_1 = \alpha / 1 + \alpha\, t \,. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Le produit αR_0 reste constant, ainsi que la constante C. Si l'on néglige l'effet sur la conductivité de l'air, sa densité et sa chaleur spécifique, qui figurent dans les grandeurs a et b de la formule de KING, le terme A reste également constant. Pratiquement, les changements d'humidité et les dépôts d'impuretés sur le fil produisent d'appréciables différences, mais la température de l'air ambiant agit peu sur ces constantes du fil. Un effet de 1 % sur R_0 et α se produit, avec un fil de Pt pur, avec $t = 3$ °C environ; avec 10 et 20 % d'Ir, il faut que t atteigne 8 et 12 °C pour un même résultat.

7. *Calcul approché d'un fil chaud.* — A partir des constantes indiquées par les „International Critical Tables" et des formules résumées ci-dessous, nous avons tracé un système d'abaques.

Les figures 2 et 3 indiquent les valeurs de α, ϱ et $\alpha\varrho$ en fonction du taux d'Ir, ainsi que la relation entre R/R_0 et T. Pratiquement on ne dépasse pas $T = 700$ °C. (A cette température, un fil de 10 microns de diamètre perd en radiation le 5 % de l'énergie fournie, environ; cet effet du rayonnement devient plus important avec de plus forts diamètres).

On voit que $\alpha\varrho$ varie modérément avec le taux d'Ir, et nous avons calculé la figure 4 en supposant $\alpha\varrho = 3.10^{-8}$. Cette figure correspond à la formule de KING, soit:

$$R\,I^2 = A\,(R - R_0), \quad \text{où}:$$

$$A = \frac{a + b\sqrt{V}}{\alpha R_0} = \left(\frac{\varkappa' + \sqrt{2\pi\, s'\, \delta'\, \varkappa'\, d\, V}}{4/\pi\ \alpha\varrho} \right) d^2 \quad . \quad . \quad . \quad . \qquad (27)$$

$\varkappa'$ = conductivité de l'air = 0,23 watt/cm °C.
s' = chal. spéc. de l'air à vol. const. = 0,71 joule/gr °C.
δ' = densité de l'air = 1,29 mgr/cm³.

C'est ainsi qu'avec un fil de $d = 7$ microns, avec $V = 10$ m/sec, la

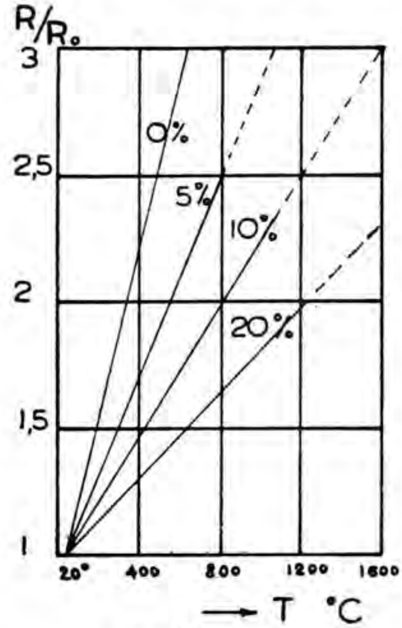

fig. 2

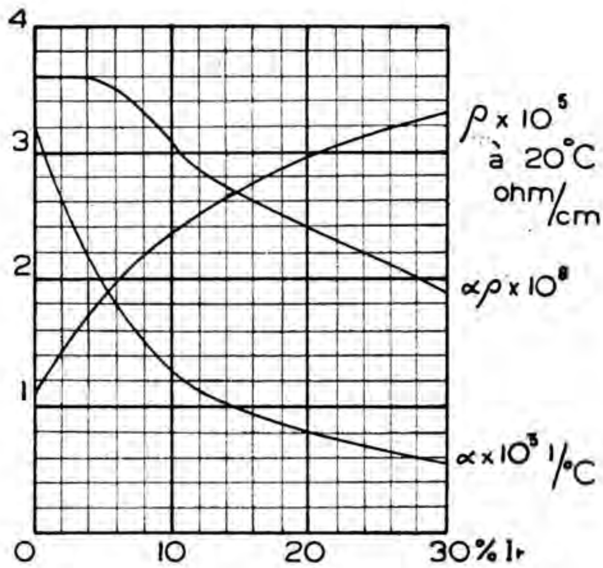

fig. 3

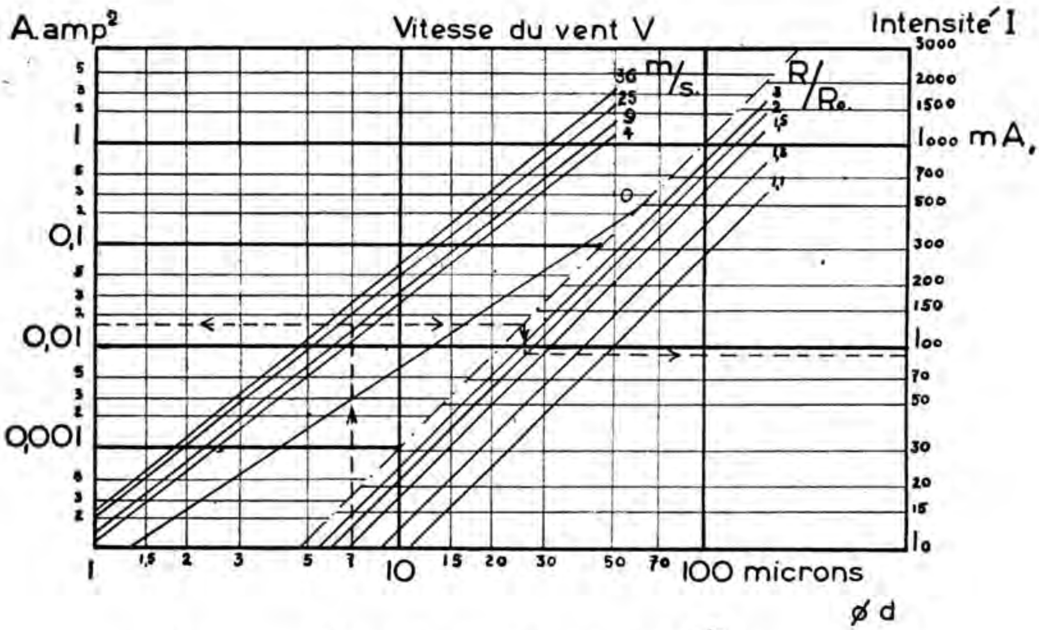

fig. 4 $RI^2 = A\,(R - R_0)\ ,\ A = \dfrac{a + b\sqrt{v}}{\alpha R_0}$

figure nous donne $A = 1{,}7 \cdot 10^{-2}$ amp²; avec $R/R_0 = 2$, le courant sera $I = 90$ mA. Ce graphique correspond à un vent perpendiculaire au fil, de température 20 °C. L'expérience confirme en première approximation les valeurs de A, selon KING; cependant A augmente légèrement avec la température du fil, probablement parce que l'air autour du fil est échauffé, et devient meilleur conducteur de la chaleur ($\varkappa'$ augmente avec la température). La droite $V = 0$ n'a pas de sens physique (rôle de la convection).

La figure 5 donne la résistance R_0 par mm, et la constante C en fonction du diamètre et du taux d'Ir, ou inversément. La constante C correspond à la formule:

$$C = \alpha R_0 / 4{,}2\,\pi\, m s = 16\,\alpha \varrho / \pi^3 s \delta d^4 \quad . \; . \; . \; . \; . \; . \quad (28)$$

où nous avons pris $s = 0{,}033$ cal/gr °C, ce qui suppose $T = 300$ °C; s augmente avec T de environ 7.10^{-6} cal/gr °C par degré, avec du Pt pur. Le fil de notre exemple donne, avec 10 % Ir, 6 ohms par mm à froid et $C = 2.10^4$.

Les points indiqués sur la figure 5 correspondent à nos mesures de C et R_0, avec divers échantillons.

La figure 6 indique la fréquence propre f^*, avec environ 10 % d'Ir, selon la formule:

$$f^* = \omega^*/2\pi = \frac{C R_0}{2R} A \quad . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

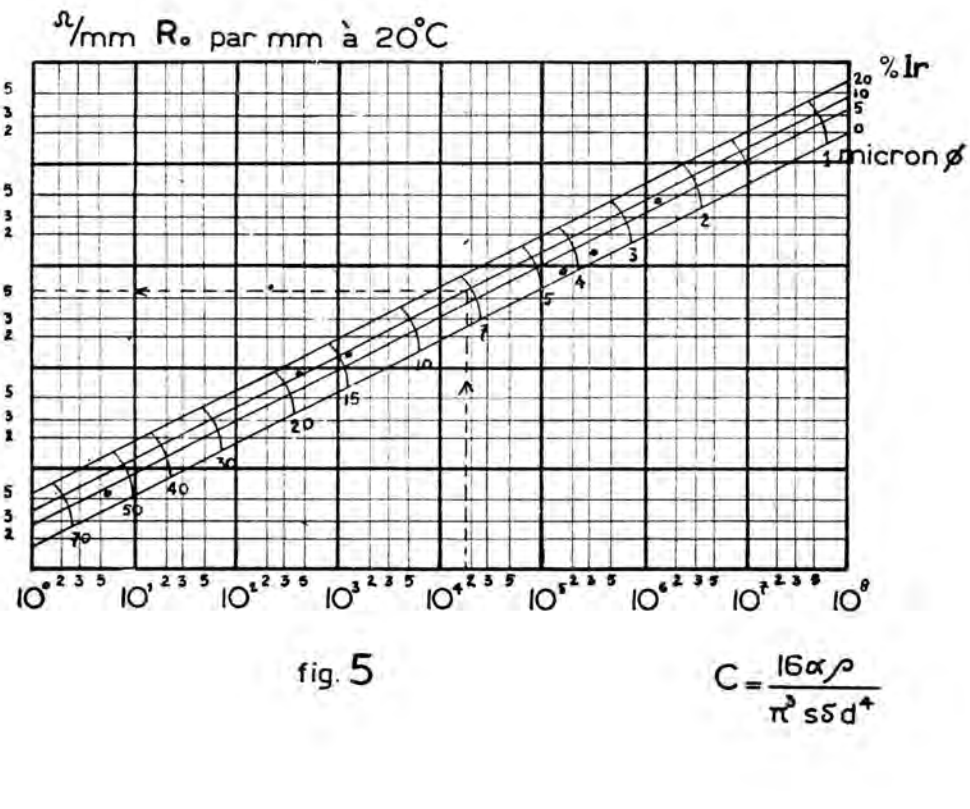

fig. 5

$$C = \frac{16\alpha\rho}{\pi^3 s\delta d^4}$$

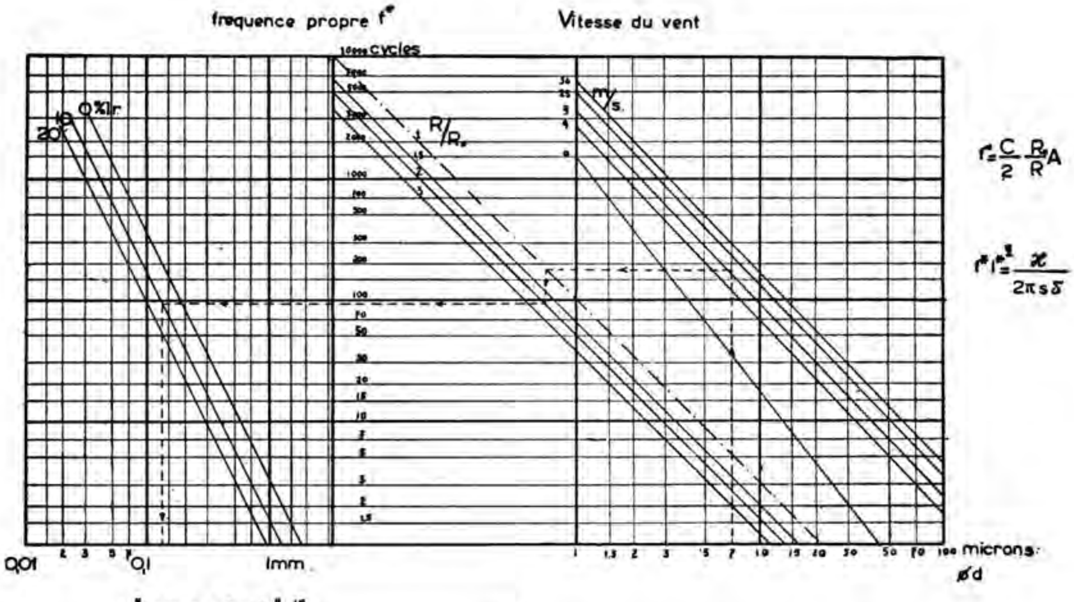

fig. 6

Notre exemple donne un fréquence de 90 périodes environ. La longueur froide est également indiquée sur la partie gauche de la figure 6. En combinant ω^* et l^* (voir formules (2) et (8)), on obtient:

$$f^* l^{*2} = \varkappa / 2\pi s \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

et le fil traité en exemple nous donne $l^* = 0{,}13$ mm.

On voit que l'Ir réduit l'effet de conduction, et cet avantage s'ajoute aux points déjà signalés : plus grande solidité, petit effet des variations de température de l'air ambiant. En revanche, le vieillissement du fil est important.

Le calcul des fils de tungstène (Wolfram) donne des résultats analogues, (le diamètre étant tenu constant) avec:

$$\left.\begin{aligned} A_w &= 1{,}13\, A_{Pt.pur} \\ R_{0w} &= 0{,}5 \;\; R_{0Pt} \\ C_w &= 0{,}95\, C_{Pt} \\ f^*_w &= 1{,}05\, f^*_{Pt} \\ l^*_w &= 1{,}5 \;\; l^*_{Pt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Astronomy. — *A working model of the solar atmosphere.* By C. DE JAGER.
(Communicated by Prof. M. MINNAERT.)

(Communicated at the meeting of April 24, 1948.)

Abstract. From the observed limb darkening of the sun, the temperature in the solar atmosphere may be directly determined as a function of the optical depth.

Such a relation, lately derived by BARBIER, is used in the present paper in order to construct an improved theoretical model of the outer layers. This model is extended to deeper layers, by considerations on the convection zone.

For exact calculations about Fraunhofer lines, a model of the solar atmosphere is of primary importance. It seems not well possible to derive such a model from theory only, because several sub-problems for such a calculation have not yet been solved in a satisfactory way: *a*) the blanketing effect; *b*) the ratio $\varkappa_\lambda/\bar{\varkappa}$; *c*) the convective layer; *d*) the chemical composition of the solar atmosphere and especially the ratio H/He.

This unsatisfactory state of things becomes apparent, when theory is compared to the observations of the limb darkening and the brightness of the disc.

CHANDRASEKHAR and MÜNCH, taking no account of the blanketing effect, find that the ratio between the theoretical and the observed values of $\varkappa_\lambda/\bar{\varkappa}$ should be of the order 1,42 [1]). BARBIER [2]) computes the blanketing effect bij MILNE's simple theory, and finds a good agreement if MULDER's number η is chosen equal to 0,068 (which is rather at the low side). The value of $\varkappa_{5010}/\bar{\varkappa}$, then found from his data, is compared to the theoretical value of CHANDRASEKHAR and MÜNCH in table I. The ratio

TABLE I.

τ_0	$\varkappa_0/\bar{\varkappa}$ (*B*)	$\varkappa_0/\bar{\varkappa}$ (*CH-M*)	Theory/"obs."
0.30	0.70	0.87	1.25
0.60	0.70	0.84	1.20
1.00	0.71	0.80	1.13
1.40	0.76	0.77	1.01
1.80	0.82	0.74	0.90

between theoretical and "observed" values is now of the order of 1,25 near the boundary of the atmosphere and decreases to 1 and even less at greater depths. Still another amount for the blanketing effect is deduced from the more elaborated theories of WOOLLEY and of MÜNCH, who predict a lower boundary temperature than BARBIER's 4900°.

[1]) Ap. J. **104**, 446 (1946).

[2]) Ann. d'Ap., **9**, 173 (1946).

Summarizing the effects of blanketing and of the departures from the theoretical values of $\varkappa_\lambda/\bar{\varkappa}$, we may assume that they both tend to decrease toward deeper layers. This is shown by table I and by all theories of the blanketing effect.

We therefore will adopt the following "working program" for the calculation of the solar atmosphere:

In the outermost layers, where both blanketing and deviations from the theoretical values of $\varkappa_\lambda/\bar{\varkappa}$ have a considerable but not well-known influence, BARBIER's *"observed" temperatures are adopted without a detailed theoretical explanation. From these data, we calculate the corresponding values of P and P_e. In the deeper layers, where convection plays a role, the atmosphere is calculated theoretically.*

Radiative part of the atmosphere. Let us provisionally assume that $N_H : N_{He} = 5$, in accordance with UNSÖLD's data for disturbed chromospheric regions and with M. SCHWARZSCHILD's data [3]) for the interior of the sun. A direct determination of this ratio in the normal solar spectrum is impossible.

The model of the outer radiative part of the solar atmosphere is calculated according to BARBIER's method which was modified in two respects:

a) P_e/P was not calculated according to the method proposed by BARBIER but was extracted from STRÖMGREN's more exact data [4]).

b) The influence of He was taken into account.

It is assumed throughout this paper that $\log H - \log Met = 3{,}8$.

For the mass absorption coefficient k_0 and for the pressure ratio we find [5]):

$$k_0 = \frac{\varkappa_0 \cdot N_{H^-}}{m_H \cdot N_H + m_{He} \cdot N_{He}} = \frac{\varkappa_0 \cdot N_{H^-}}{m_H \cdot N_H} \cdot \frac{1}{1{,}80}$$

$$P_e/P = \left(\frac{N_{H^+}}{N_H} + x_M\right)(1 + 0{,}2 + x_M)^{-1} \approx \frac{1}{1{,}20} \cdot (P_e/P)_{\text{STRÖMGREN}} \cdot$$

BARBIER's third approximation for P becomes:

$$P^2 = \frac{2\, g m_H}{k_0} \cdot 2{,}16 \int\limits_0^{\tau_0} \frac{d\,\tau_0}{\varphi(T) \cdot (P_e/P)_{\text{STRÖMGREN}}}$$

where $\varphi(T)$ has the same meaning as in BARBIER's paper. According to this scheme a model of the radiative solar atmosphere was now calculated (see table II).

There are some differences between this model and the models calculated

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[3]) Ap. J. **104**, 203 (1946).

[4]) Publ. København, **138** (1944).

[5]) In the subsequent formulae, $\varkappa_0$, k_0, τ_0 denote the absorption coefficients and the optical depth for $\lambda = 5010$ Å.

TABLE II. Model of the solar atmosphere.

τ_0	θ	log P	log P_e	γ	ξ (km/sec.)
0.01	1.019	4.13	0.27		
0.02	1.013	4.28	0.42		
0.03	1.008	4.38	0.50		
0.04	1.003	4.44	0.56		
0.06	0.993	4.54	0.65		
0.08	0.981	4.60	0.72		
0.10	0.973	4.65	0.78		
0.15	0.949	4.75	0.90		
0.20	0.933	4.81	1.00		
0.30	0.900	4.91	1.14		
0.40	0.873	4.97	1.28		
0.50	0.850	5.039	1.43		
0.60	0.830	5.056	1.54		
0.80	0.798	5.109	1.75		
1.00	0.772	5.141	1.94		
1.20	0.750	5.163	2.08		
1.40	0.733	5.180	2.24		
1.60	0.716	5.193	2.34		
1.80	0.702	5.205	2.46		
2.00	0.690	5.214	2.56		
2.50	0.664	5.231	2.76	0.08	1.12
3.00	0.634	5.244	2.99	0.09	1.26
4.00	0.614	5.263	3.14	0.11	1.46
5.00	0.602	5.281	3.25	0.11	1.52
7.00	0.588	5.311	3.35	0.11	1.53
9.00	0.581	5.333	3.41		

recently by STRÖMGREN [6]), MÜNCH [7]), and BARBIER [8]), partly due to the fact that the helium content has been taken into consideration and partly to the higher approximations used.

As compared to STRÖMGREN's model the differences are very small; STRÖMGREN's values of log P and log P_e at a certain value of $\bar{\tau}$ are almost identical with ours.

As compared to BARBIER's, the electron pressure remains almost the same, but the gas pressure has increased by a factor of about 1,5.

As compared to MÜNCH's model, the gas pressure as well as the electron pressure have in this work increased with a factor of about 1,25.

Boundary of the convection zone. The boundaries of this zone should be placed according to K. SCHWARZSCHILD's well-known condition at the depths were

$$\left(\frac{d\log T}{d\log P}\right)=\left(\frac{d\log T}{d\log P}\right)_{ad}.$$

[6]) Publ. København, **138** (1944).

[7]) Ap. J., **106**, 217 (1947).

[8]) L.c.

In our case this limit is found at $\tau_0 = 0{,}90$. However, WOOLLEY [9]) showed that for a mixture of gas and radiation SCHWARZSCHILD's condition should be replaced by

$$\left(L\frac{dT}{d\tau} + M\frac{dP}{d\tau} + \xi\frac{d\xi}{d\tau}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with

$$L = C_p = R\,(1+x)\{\tfrac{5}{2} + \tfrac{1}{2}\,x\,(1-x)\,(x/kT + \tfrac{5}{2})^2\}$$
$$M = -RTP^{-1}\,(1+x)\{1 + \tfrac{1}{2}\,x\,(1-x)\,(x/kT + \tfrac{5}{2})\}$$

ξ being a quantity, in general nearly equal to the ascending and descending velocity of the turbulent gases. It may be expected that at the boundary of the convection zone $\xi \approx 0$, so that equation (1) reduces to:

$$\left|\frac{dT}{d\tau}\right| = \left|\frac{M}{L}\frac{dP}{d\tau}\right| = \left|\frac{T}{P}\cdot\frac{\Gamma-1}{\Gamma}\frac{dP}{d\tau}\right| = \left|\frac{dT}{d\tau}\right|_{ad}$$

and the condition for convection to occur is:

$$\left|\frac{dT}{d\tau}\right| = \left|\frac{dT}{d\tau}\right|_{ad} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It should be noted that this condition is not identical with SCHWARZSCHILD's (it is however reduced to this when radiation is absent). This can be shown easily by calculating $\dfrac{dT}{d\tau}$ for identical sets of values of T, P and P_e in the cases of radiative and of adiabatic equilibrium. In general these calculations do not give the same results.

WOOLLEY's condition places the boundary of the convection zone at the depth where both members of (2) are equal. This turns out to be at $\tau_0 = 2{,}0$ for the model atmosphere, described by table II.

Convective part of the atmosphere.

For the structure of the convective layer below $\tau_0 = 2$, we take as a lead the analysis of BIERMANN [10]). Undoubtedly the uncertitudes are very great; but we must also consider that in the calculation of the escaping radiation the contribution of these deep layers plays only a minor role.

Next to the work of BIERMANN, we will make use of important relations, deduced by WOOLLEY [11]). Though his general conclusions concerning the convective layer are not confirmed by our work, his methods are very useful and will repeatedly be applied in the following.

From his discussion, BIERMANN concludes that probably the energy transport near the outer boundary of the convective layer is due mainly to radiation, while the convective transport becomes increasingly important at greater depths and is already predominant near $\tau_0 = 8$.

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[9]) M. N., **101**, 58 (1941).

[10]) Zs. für Ap., **21**, 320 (1942).

[11]) M. N., **101**, 52 (1941).

The gradual transition between the radiative and the convective transport cannot be expressed by making simply a connection between the $(\theta, \tau_0)_{rad}$ and the $(\theta, \tau)_{ad}$ curves; for each modification in the upper layers shifts the limiting curves towards which the deep layers converge. This transition however can be effectuated easily for the gradient $\frac{d \log T}{d \log P}$, from which the other physical parameters may be easily derived. The significance of such a transition is still better understood, if we start by estimating the ratio $\frac{\text{energy transport by convection}}{\text{total energy transport}}$, which is very fundamental.

This ratio is zero near the outer boundary, it increases with depth and tends to a limit for the central layers of the convection zone, which however is not unity, but smaller. For in each ascending gas current there will always be a temperature gradient, and the radiation will therefore always transport part of the energy. Generally this radiative contribution is neglected in the convection layer, but we wish to point out that its importance should be investigated; we shall put the convection transport at greater depths equal to 80—90 % of the total transport.

Model of the convection zone.

We assume the following model of the convection zone. Observations indicate that the solar convection is of the non-stationnary type: matter ascends in more or less regular bubbles (the granulae), these bubbles have a finite life-time; each bubble, when rising experiences resistance (SIEDENTOPF [12]). Outside the bubbles, matter is in downward motion (fig. 1).

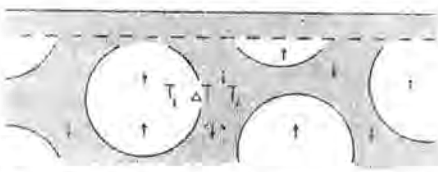

Fig. 1. Model of solar convection zone.

The ascending and descending currents have at each depth temperatures T_1 and T_2. We define the mean temperature T by

$$T_1 = T(1 + \gamma \cdot a_2)$$
$$T_2 = T(1 - \gamma \cdot a_1),$$

where γ is a small quantity.

a_1 and a_2 are the fractions of the sectional area at the depth considered, that is occupied by the upward resp. the downward moving matter. In a

[12]) A. N., **205**, 157 (1935).

similar way the densities ϱ_1 and ϱ_2 are defined. The relation between these quantities and the mean density ϱ is:

$$\varrho_1 = \varrho/(1 + a_2 \gamma Q)$$
$$\varrho_2 = \varrho/(1 - a_2 \gamma Q)$$

with $Q = 1 + \frac{T}{1+x} \cdot \left(\frac{\partial x}{\partial T}\right)_P$.

The pressure P has the same value in both columns. Henceforth we put $a_1 = a_2 = 0{,}5$; WOOLLEY showed [13]) that deviations from this value are of no practical importance. In any case: $T_1 - T_2\ (= \triangle T) = \gamma T$.

In order to perform an integration over the convective part of the solar atmosphere, four relations are required between the four variables T, P, γ (determining the temperature difference of both streams) and ξ (defining the velocity of the material).

The *first relation* to be used is SIEDENTOPF's well-known formula, derived from the equality between the upward accelaration and the frictional resistance:

$$\xi = 0{,}5 \sqrt{g \cdot l \triangle T/T}$$

The factor $\frac{1}{2}$ is introduced here to give account of the fact that in SIEDENTOPF's model the velocity is derived with respect to the surrounding matter, this matter being at rest in his model. This formula was transformed by WOOLLEY into

$$\xi_1 = g \cdot t \cdot \varrho \cdot \gamma \cdot Q/2\varrho_1. \qquad (3)$$

with ξ_1 = velocity in the ascending column. Henceforth we put for reasons of simplicity $\xi_1 = \xi_2 = \xi$.
g = acceleration of gravity.
t = mean life-time of the granulae $\approx 200^s$.
$Q = 1 + \frac{T}{1+x} \cdot \left(\frac{dx}{\partial T}\right)_P$.

Putting $\varrho/\varrho_1 = 1 + \gamma/2$, $t = 200^s$, $g = 2{,}75 \cdot 10^4$ and neglecting higher order terms of γ, (3) becomes

$$\xi = 1{,}38 \cdot 10^6 \cdot \gamma Q. \qquad (4)$$

Inserting numerical values, it is found that Q is near unity in the outer part of the convective layer.

The *second formula* to be used is the transport formula, analogous to WOOLLEY's formula (4, 6), also derived by TUOMINEN [14]) and others. It expresses that during the transport of the energy (which is partly radiative and partly convective) the total amount of transported energy remains constant:

$$\tfrac{1}{2}\, a c\, T_0^4 = \tfrac{4}{3} \frac{ac}{\varkappa\varrho} T^3 \cdot \frac{dT}{dh} + a_1 \cdot \varrho_1 \cdot \xi_1 \cdot c_v \cdot \gamma \cdot T. \qquad (5)$$

[13]) L.c. pag. 60.

[14]) Ann. d'Astroph. 8, 134 (1946).

(As to the factor c_v in the last member, compare BIERMANN [15])).
By substitution of (4) (putting $Q=1$), (5) may be reduced to

$$6{,}127 \cdot 10^{10} = \tfrac{4}{3}\, a\, c\, T^3 \frac{dT}{d\tau} + c_v \cdot T \cdot \gamma^2 \cdot \varrho \cdot 2{,}75 \cdot 10^6. \quad . \quad . \quad (6)$$

The *third formula* to be used for our purpose is WOOLLEY's formula (6, 4), which is strictly speaking only valid in the case of cylindrical flow. We assumed that this will practically be the case in the solar atmosphere. The formula is used by us in its simplified form, by putting $a_1 = a_2 = 0{,}5$; $Q = N = 1$:

$$g/\varkappa_0 - dP/d\tau_0 = \frac{\varrho_1^2 \xi^2}{4\,\varrho} \left(\frac{1}{T} \frac{dT}{d\tau_0} - \frac{1}{P} \frac{dP}{d\tau_0} \right). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Outside the convection zone the formula reduces to well-known formula for radiative equilibrium:

$$\frac{dP}{d\tau_0} = \frac{g}{\varkappa_0}$$

We will now give a detailed account of the consecutive steps by which we have derived a plausible description of the convective layer.

1. As we have explained on page 735, we begin by studying the ratio between the convective energy transport and the total energy transport. If all energy would be transported convectively, the radiation part of formula (6) being put equal to zero, γ^2 will acquire a maximum value

$$\gamma^2_{\max} = \frac{2{,}23 \cdot 10^4}{c_v \cdot \varrho \cdot T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If radiation and convection are both playing a role $\gamma^2/\gamma^2_{\max}$ will be equal to the ratio between the convective and the total energy transport. This should increase from 0 to neareby 1 between $\tau_0 = 2$ and $\tau_0 = 8$ (p. 735).

Just according to what function this transition occurs is not known. We will try several possible curves, which a priori seem equally well possible (fig. 2, curves I—IV).

2. The ratio $\gamma^2/\gamma^2_{\max}$ is introduced into equation (6), which by integration yields $T(\tau) = T'(\tau_0)$ (fig. 3).

3. We now make use of relation (7), which, by stepwise integration, must yield $P(\tau_0)$. In order to make the integration possible, all factors must be expressed as functions of P and τ_0.

$dT/d\tau_0$ has already been found as a function of τ_0.

$$\xi^2 = (1{,}38 \cdot 10^6 \cdot \gamma \cdot Q)^2 = (1{,}38 \cdot 10^6 \cdot Q)^2 \cdot \gamma^2/\gamma^2_{\max} \cdot \frac{2 \cdot 10^4}{c_v\, \varrho\, T} =$$

$$= (1{,}38 \cdot 10^6\, Q)^2 \cdot \gamma^2/\gamma^2_{\max} \cdot 2 \cdot 10^4 \cdot R/c_v\, P$$

$$\varrho_1^2/\varrho = \varrho/(1 + {}^1/_2 \cdot \gamma\, Q)^2 \approx \varrho\,(1 - \gamma \cdot Q) = P\,(1 - \gamma\, Q) \cdot / R \cdot T\,(\tau_0)$$

[15]) L.c. pag. 344 f.

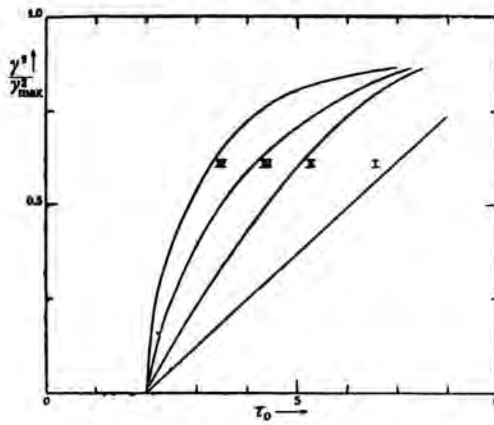

Fig. 2. The trial functions γ^2/γ^2_{max} (= H_{conv}/H_{total} as a function of τ_0.

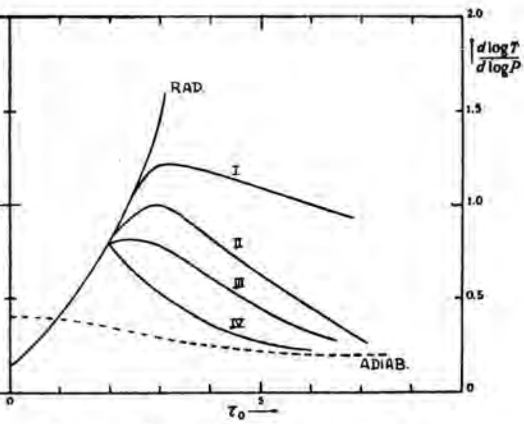

Fig. 4. The values of $(d \log T)/(d \log P)$ as a function of τ_0 for the models I—IV and for a model atmosphere in totally radiative equilibrium.

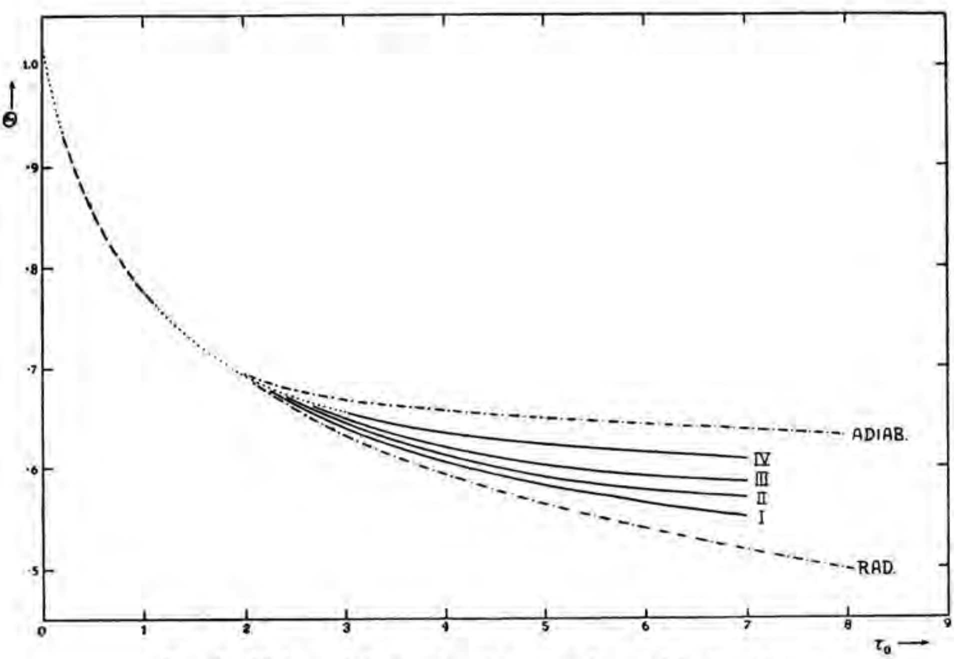

Fig. 3. Values of θ as a function of τ_0 for model atmospheres.

- – – – Temperature derived by BARBIER from observational data.
- ·········· Extrapolated part of BARBIER's curve.
- —·—·— Adiabatic and radiative limiting cases.
- ——— Model atmospheres I—IV.

$\varkappa_0$ is a function of P and T, computed in the usual way as the sum of the H and H^- absorption.

The integration of (7) is now carried out in steps, the consecutive values of $\varkappa_0$ being introduced at each stage.

4. From the functions T and P, the gradient $\dfrac{d \log T}{d \log P}$ is derived and drawn in fig. 4.

Critical considerations and conclusion.

We shall now try to compare the merits of the four functions which have been tried.

1. By inspection of fig. 4 it is seen at once that the gradient $\dfrac{d \log T}{d \log P}$ for curve I remains improbably great even in deep layers, and does not converge fast enough towards the adiabatic gradient. For this reason, curve I seems to be inacceptable.

2. A comparison with BARBIER's extrapolated formula would be in favour of curve IV. However, more exact results can only be obtained by computing directly the limb darkening from our T, τ_0-curves.

The results of these calculations which are performed for λ 5010 A are shown in table III. A comparison with the observed values (Abbot 1906—1908) shows that the best agreement between theory and observations occurs for our model II.

TABLE III. Limb darkening for model atmospheres. $\lambda = 5010$ A.

Model	$\sin \vartheta = 0.00$	0.55	0.75	0.875	0.95
Radiative	1.0000	0.8881	0.7700	0.6406	0.5068
I	1.0000	0.9035	0.7776	0.6467	0.5129
II	1.0000	0.8970	0.7823	0.6503	0.5171
III	1.0000	0.8994	0.7861	0.6532	0.5204
IV	1.0000	0.9152	0.7954	0.6604	0.5226
Obs. values	1.0000	0.8945	0.7773	0.6501	0.5171

3. We may check whether the observed brightness contrast between the granulation elements and the background is satisfactorily explained by our model. A measure for this contrast is obtained from the ratio

$$\frac{\Delta I}{I} = \frac{\int_0^\infty (e^{c_2/\lambda T_1} - 1)^{-1} \cdot e^{-\tau_0}\, d\tau_0 - \int_0^\infty (e^{c_2/\lambda T_2} - 1)^{-1} \cdot e^{-\tau_0}\, d\tau_0}{\int_0^\infty (e^{c_2/\lambda T} - 1)^{-1} \cdot e^{-\tau_0}\, d\tau_0}$$

For our four trial solutions, the following ratio's are found:

Functions	I	II	III	IV
$\dfrac{\Delta I}{I}$	0.11	0.15	0.20	0.24

Different values for this contrast have been found observationally, according to atmospheric and instrumental circumstances. The best modern values range between 15 % and 20 % (KEENAN) or 30 % and 40 % (WALDMEIER). From these, we conclude that there is a good general confirmation of our assumptions, and we find some evidence in favour of the curves III or IV.

4. Finally, our atmospheric model determines automatically the velocity of the ascending granules. This was derived directly from formula (4) and has been tabulated in table II for our model III. The results are in good general agreement with the turbulent velocities, derived from curves of growth or from individual profiles, for which 1 ... 1,7 km/sec is estimated.

Along these different lines of thought, we arrive at the conclusion, that our assumptions are in reasonable agreement with most observational facts. We shall assume as a working hypothesis our *model III*, which in several respects seems a good intermediary. The physical parameters for this model have been collected in table II.

WOOLLEY'*s objections against the existence of convection.*

WOOLLEY, from almost the same formulae as used in the present work, arrives at the conclusion that convection plays only an unimportant role in the solar atmosphere. His first argument is, that a convection, sufficient to modify the theoretical limb darkening into the observed darkening, would require improbably high velocities. His second argument is drawn from his formula (8,21), by which he computes that the convective energy transport is limited to a negligible fraction of the whole transport.

In his model however, atmospheric densities have been used (l.c. page 63) which are about a factor 10 smaller than these now generally assumed. His calculated velocities therefore are a factor 10 too great and his estimated maximum convectional transport is a factor 10.000 too small.

In order to show that his formulae are very well satisfied by the atmospheric data used by us, we compute the ratio $h = \frac{\text{volume}}{\text{area}}$ of the granulation elements by WOOLLEY's formula (5,55), assuming the high convection transport as given by our model III. For a depth $\tau_0 = 5$, we obtain as a result $h = 3000$ Km. This is rather greater than the tiniest granulae observed, but may be considered as of the right order of magnitude; the more so, because WOOLLEY's theory is based on the simplified assumption, that the upward and downward currents have each a constant temperature T_1 resp. T_2 over the entire height.

It generally appears that in considerations on the possibility of convection, the choice of an atmospheric model is of great importance.

The considerations exposed in this paper have only a preliminary character. The theoretical discussion may be improved by more precise calculations about the stream velocity and about the interaction between radiation and matter in the moving columns. Observations may bring important evidence by the photometry of the granulation at different distances from the centre and of the profiles of FRAUNHOFER lines with a high excitation potential. With that purpose, a rather extensive investigation of the hydrogen lines is being carried out by the writer.

In conclusion I should like to thank Professor M. MINNAERT for his advice, suggestions and encouragement during the course of this work.

Utrecht, Sterrewacht Sonnenborgh.

Chemotherapy. — *L'action inhibitrice des métaux sur la croissance du B. tuberculeux.* VI. *Soufre, sélénium et tellure.* By ONG SIAN GWAN. (Communicated by Prof. E. GORTER.)

(Communicated at the meeting of May 29, 1948.)

1. Les éléments du sixième groupe, sous-groupe *b*, du tableau périodique: oxygène, soufre, sélénium et tellure ne sont pas des métaux. Le tellure possède des propriétés physiques d'un métal, il peut être classé parmi les métalloïdes. L'oxygène est un élément indispensable à la croissance de B. tuberculeux. En effet, les recherches de NOVY et SOULE (1925), CORPER, LURIE et UYEI (1927) et de POTTER (1942) montrent que la suppression d'oxygène non seulement arrête la croissance de B. tuberculeux, mais elle le tue en même temps. En ce qui concerne l'influence de soufre, SAUTON (1912) a remarqué que la suppression de soufre dans son liquide synthétique fait diminuer le poids sec de récolte de 0,95 g à 0,04 g. RAULIN (1863, 1870) dans ses remarquables mémoires a constaté que le rapport des poids de récolte d'Ascophora migrans et d'Aspergillus niger avec et sans le concours de soufre varie de 2,0 à 15,4. L'addition de 1 g de soufre dans le milieu de culture peut développer 346 g de poids sec d'Aspergillus niger. Avec notre souche 1013 de B. tuberculeux nous avons obtenu une inhibition complète de croissance dans le liquide de SAUTON dépourvu de soufre. Le soufre ne peut pas être remplacé par le sélénium ou le tellure. Mais la présence simultanée de soufre et de sélénium ou de soufre et de tellure dans le liquide de SAUTON peut augmenter le poids de récolte de B. tuberculeux.

2. *Addition de soufre, sélénium ou tellure dans le milieu synthétique de* SAUTON.

Les éléments utilisés dans ces expériences sont les suivants: **1**. soufre pur cristallisé de POULENC, **2**. sélénium précipité de KAHLBAUM en poudre rouge foncé, **3**. sélénium puriss. en forme de perle de KAHLBAUM, il est utilisé seulement dans l'expérience *a*, **4**. tellurium in bacillus de KAHLBAUM dans les expériences *a*, *b*, *c* et *d* et **5**. tellure en poudre noir dans les autres expériences.

a. *Addition de 20 mg de S, Se ou Te par 100 cc de milieu synthétique de* SAUTON.

La souche utilisé, souche bovine Vallée, a subi 14 passges sur milieu de SAUTON, elle est désignée par V_{14}. Elle est agée de 14 jours et l'expérience a duré 30 jours. Les deux observations perdues (tableau 1) sont calculées à l'aide de la formule de ALLAN et WISHART (1930) et améliorée par YATES (1933).

TABLEAU 1.
Addition de 20 mg de soufre, sélénium ou tellure par 100 cc de milieu de SAUTON.
I. Tableau des observations.

Témoins mg	Soufre mg	Sélénium en poudre mg	Sélénium en perle mg	Tellure mg
396,5	244,0	325,0	694,0	262,0
696,5	757,0	716,0	730,0	158,5
191,5	844,0	707,5	665,0	685,5
358,0 [1]	728,5	638,5	303,5	296,0 [1]
Total 1642,5	2573,5	2387,0	2392,5	1402,0
Moyenne 410.6	643,4	596,8	598,1	350,5

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre rangées	197 515	3	65 838
Entre traitements	271 727	4	67 932
Résiduelle	533 607	10	53 361
Total	1 002 849	17	

[1]) Valeur calculée.

L'analyse de la variance montre qu'il n'existe pas de différences significatives entre les traitements considérés dans leur ensemble. Le coefficient de variation $C = s/\bar{\ } \times 100 = 44,4$ p. 100.

b. Addition de 20 mg de S, Se ou Te par 100 cc de milieu synthétique de SAUTON. *Méthode de carré latin.*

TABLEAU 2.
Addition de 20 mg de soufre, sélénium ou tellure par 100 cc de milieu de SAUTON.
Méthode du carré latin. Poids en mg.
I. Tableau des observations.

	Colonnes				Total
Rangées	T 458,0	S 439,5	Se 483,4	Te 470,0	1850,9
	S 368,5	Se 471,0	Te 499,5	T 435,0	1774,0
	Se 462,0	Te 429,5	T 405,5	S 479,0	1776,0
	Te 463,5	T 392,0	S 419,0	Se 483,8	1758,3
Total	1752,0	1732,0	1807,4	1867,8	7159,2
Moyenne	T 422,6	S 426,5	Se 475,1	Te 465,6	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre traitements	8 589	3	2 863
Résiduelle	11 790	12	983
Total	20 379	15	

T = témoin.

Cette méthode, due à R. A. FISHER, permet d'éliminer la différence de fertilité de milieu de culture dans deux directions normales: différences de fertilité entre rangées et entre colonnes.

Dans l'expérience considérée la souche utilisée est V_{11}, elle est agée de 23 jours et la durée de l'expérience est de 26 jours. L'analyse de la variance (tableau 2) montre le même résultat que celui de l'expérience précédente. Il n'existe par de différences significatives entre les traitements considérés dans leur ensemble Cependant en comparant les moyennes obtenues par Se et le témoin ou le tellure on constate que la différence est significative. En effet, une différence entre deux moyennes est significative si elle est supérieure à

$$\sqrt{\frac{983 \times 2}{4}} \times 2{,}179 = 48{,}3 \text{ mg}.$$

Le coefficient de variation est égal à $C = 7{,}0$ p. 100. La précision de cette expérience est $53361/983 = 54$ fois plus grande que celle de l'expérience précédente.

c. *Addition de 40 mg de S, Se ou Te par 100 cc de milieu synthétique de* SAUTON. *Méthode du carré latin.*

La souche utilisée V_{11} est agée de 30 jours, la durée de l'expérience est de 31 jours. On obtient le même résultat (tableau 3) que dans les deux expériences précédentes avec 20 mg d'élément. Il n'existe pas de différences significatives entre les traitements considérés dans leur ensemble. Le coefficient de variation est égal à $C = 4{,}1$ p. 100, il est extrêmement petit ce qui indique une grande précision.

TABLEAU 3.

Addition de 40 mg de soufre, sélénium ou tellure par 100 cc de milieu de SAUTON. Méthode du carré latin. Poids en mg.

I. Tableau des observations.

	Colonnes				Total
Rangées	T 641,1	S 732,0	Se 663,6	Te 619,9	2656,6
	S 651,0	T 630,7	Te 644,0	Se 612,3	2538,0
	Se 658,0	Te 618,8	S 649,0	T 614,2	2540,0
	Te 573,0	Se 657,0	T 641,8	S 598,1	2469,9
Total	2523,1	2638,5	2598,4	2444,5	10204,5
Moyenne	T 632,0	S 657,5	Se 647,7	Te 613,9	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre rangées	4 505	3	1 502
Entre colonnes	5 380	3	1 793
Entre traitements	4 367	3	1 456
Résiduelle	4 143	6	691
Total	18 395	15	

d. Addition de 80 mg de S, Se ou Te par 100 cc de milieu synthétique de SAUTON. *Méthode du carré latin.*

La souche utilisé V_{12} est agée de 28 jours et la durée de l'expérience est de 26 jours. Le tableau 4 et l'analyse de la variance montrent des différences significatives entre les traitements considérés dans leur ensemble. Une différence entre deux moyennes est significative si elle est supérieure à

$$\sqrt{\frac{3405 \times 2}{4}} \times 2{,}447 = 101{,}0 \text{ mg.}$$

TABLEAU 4.

Addition de 80 mg de soufre, sélénium ou tellure par 100 cc de milieu de SAUTON. Méthode du carré latin. Poids en mg.

I. Tableau des observations.

	Colonnes				Total
Rangées	T 66,4	S 104,4	Se 142,9	Te 110,0	423,7
	S 71,5	Te 433,6	T 116,0	Se 112,1	733,2
	Se 116,9	T 193,5	Te 144,1	S 79,3	533,8
	Te 449,4	Se 506,3	S 348,4	T 239,8	1543,9
Total	704,2	1237,8	751,4	541,2	3234,6
Moyenne	T 153,9	S 150,9	Se 219.6	Te 284,3	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre rangées	192 504	3	64 168
Entre colonnes	67 472	3	22 491
Entre traitements	47 998	3	15 999 *
Résiduelle	20 431	6	3 405
Total	328 405	15	

* probabilité $P = 0{,}05$.

On constate que la différence entre les moyennes de tellure et de témoin ou de soufre est significative. L'analyse de la variance montre également des différences systématiques entre les rangées et les colonnes, elles indiquent une variation de fertilité marquée dans deux directions normales. Le coefficient de variation est égal à $C = 28{,}9$ p. 100.

e. Comparaison des résultats obtenus par 20, 40 et 80 mg de S, Se ou Te.

L'analyse simultanée des trois carrés latins obtenus par 20, 40 et 80 mg d'élément permet de savoir s'il existe une différence significative de fertilité entre les carrés, si le résultat du traitement dépend du carré, c'est-à-dire s'il y a une interaction entre traitement et carré, s'il existe une différence significative entre les récoltes obtenues par les éléments séparés et enfin si la différence de récoltes obtenues par 20, 40 et 80 mg de même élément est significative.

TABLEAU 5.
Comparaison des cultures de B. tuberculeux obtenues avec 20, 40 ou 80 mg de soufre, sélénium ou tellure. Poids total en mg.
I. Tableau des observations.

Carré latin no.	Quantité d'élément utilisé en mg	Témoins	Soufre	Sélénium	Tellure	Total
1	20	1690,5	1706,0	1900,2	1862,5	7159,2
2	40	2527,8	2630,1	2590,9	2455,7	10204,5
3	80	615,7	603,6	878,2	1137,1	3234,6
Total		4834,0	4939,7	5369,3	5455,3	20598,3

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre carrés	1 526 163	2	763 082 ***
Entre traitements	23 782	3	7 927 **
Interaction: traitements-carrés	37 172	6	6 195 **
Entre rangées	197 009	6	32 835
Entre colonnes	72 852	6	12 142
Résiduelle	36 364	24	1 515
Total	1 893 342	47	

** probabilité $P = 0{,}01$. *** $P < 0{,}001$.

L'analyse de la variance représentée dans le tableau 5 montre des différences significatives entre les carrés ce qui indique une différence de fertilité entre les carrés. En effet, une différence entre deux poids totaux des carrés est significative si elle est supérieure à

$$\sqrt{1515 \times 16 \times 2} \times 2{,}064 = 454{,}5 \text{ mg}.$$

On constate que les trois carrés ont des fertilités différentes; en effet, les récoltes totales sont 3234,6 mg pour le carré avec 80 mg d'élément, 7159,2 mg avec 20 mg et 10 204,5 mg avec 40 mg d'élément.

Le calcul montre également une inter-action significative entre traitements et carrés. Le tableau des différences de récoltes obtenues par le soufre, le sélénium ou le tellure et le témoin (tableau 6) montre que l'augmentation maximum de récolte est obtenue avec le sélénium dans le carré 1, avec le soufre dans le carré 2 et avec le tellure dans le carré 3.

L'analyse de la variance montre aussi des différences significatives entre les traitements. Une différence significative entre deux traitements est égale à

$$\sqrt{1515 \times 12 \times 2} \times 2{,}064 = 393{,}6 \text{ mg}.$$

On en conclut que l'addition de soufre dans le milieu de SAUTON ne fait pas augmenter le poids de récolte. Par contre, l'addition de sélénium ou de tellure dans le milieu de SAUTON fait augmenter le poids de récolte. L'augmentation de récolte est significative non seulement par comparaison avec le témoin, mais elle l'est également par comparaison avec le soufre.

Enfin l'analyse de la variance montre que l'addition de 80 mg de tellure donne une augmentation de récolte plus grande que celle obtenue par 20 ou 40 mg de tellure. En effet, une différence entre deux valeurs quelconques dans le tableau 6 est significative si elle est supérieure à

$$\sqrt{1515 \times 8 \times 2} \times 2{,}064 = 321{,}3 \text{ mg.}$$

TABLEAU 6.

Différences de récoltes de B. tuberculeux cultivé avec S, Se ou Te et les témoins T. Poids en mg.

Quantité d'élément utilisé en mg	S—T	Se—T	Te—T	Total
20	15,5	**209,7**	172,0	397,2
40	**102,3**	63,1	—72,1	93,3
80	—12,1	262,5	**521,4**	771,8
Total	105,7	535,3	621,3	1262,3

L'examen des tableaux 5 et 6 montre que l'augmentation de récolte obtenue par addition de Se ou Te est plus élevée à mesure que la récolte des cultures témoins est plus petite. L'addition de S donne un effet contraire.

3. *Addition de soufre sélénium ou tellure dans le milieu synthétique de* SAUTON *dépourvu de soufre. Méthode du carré latin.*

L'expérience a pour but de savoir, si le sélénium peut remplacer le soufre dans le milieu de SAUTON. Pour cela le sulfate de magnésium dans la formule de SAUTON est remplacé par le citrate de magnésium. Le milieu de SAUTON contient 0,5 g de sulfate de magnésium par litre correspondant à 49,3 mg de magnésium et 65,0 mg de soufre. Le milieu utilisé dans l'expérience considérée contient seulement 3,4 mg de magnésium par litre sous forme de citrate de magnésium. Il n'est pas favorable à la croissance de B. tuberculeux, mais il a l'advantage de déceler une moindre action favorable sur la croissance de S ou Se ajouté au milieu. L'expérience est donc réalisée comme suit: **1**. flacons témoins contenant le milieu sans soufre, **2**. comme le témoin, et en outre 10 mg de soufre sublimé et lavé, **3**. comme le témoin, et en outre 50 mg de soufre sublimé et lavé, **4**. comme le témoin, et en outre 20 mg de sélénium en poudre.

La souche utilisée V_{13} est âgée de 31 jours et l'expérience a duré 55 jours. Deux cultures témoins (tableau 7) sont tombées dans le liquide au cours de l'expérience, de sorte que les poids obtenus et placés entre

TABLEAU 7.

Addition de soufre et de sélénium dans le milieu de SAUTON dépourvu de soufre. Méthode du carré latin. Poids en mg.

I. Tableau des observations.

	Colonnes				Total
Rangées	T (40,0)	50 S 59,7	Se 57,0	T (43,6)	200,3
	10 S 173,6	Se 53,0	50 S 226,4	10 S 69,2	522,2
	50 S 236,4	10 S 297,0	T 46,0	Se 53,9	633,3
	Se 53,5	T 47,7	10 S 241,8	50 S 331,6	674,6
Total	503,5	457,4	571,2	498,3	2030,4
Moyenne	T 44,3	10 S 195,4	50 S 213,5	Se 54,4	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre rangées	34 584	3	11 528
Entre traitements	97 113	3	32 371 **
Résiduelle	32 629	9	3 625
Total	164 326	15	

T, témoin sans soufre, 10 S ou 50 S, addition de 10 ou de 50 mg de S/100 cc., Se, addition de 20 mg de Se/100 cc.

parenthèses sont trop petits. L'analyse de la variance montre des différences significatives entre les traitements considérés dans leur ensemble. Une différence entre deux moyennes est significative si elle est supérieure à

$$\sqrt{\frac{3625 \times 2}{4}} \times 2{,}262 = 96{,}3 \text{ mg}.$$

On constate que l'addition de 10 et de 50 mg de soufre fait augmenter le poids de récolte. Il n'y a pas de différence entre les poids de récolte obtenus avec 10 et 50 mg de soufre. La différence de récoltes obtenues par le soufre et le sélénium est également significative. Le sélénium ne donne pas de différence avec la culture témoin, il ne peut donc pas remplacer le soufre dans le milieu de SAUTON. Cependant on a vu plus haut dans l'expérience *b* que la présence simultanée de sélénium et de soufre peut augmenter le poids de récolte. (tableaux 2 et 5)

4. *Addition de soufre, sélénium ou tellure dans le milieu synthétique de* SAUTON *dépourvu de soufre. Méthode du carré latin.*

Dans cette expérience le milieu de culture témoin contient la même quantité de magnésium comme dans la formule de SAUTON sous forme de nitrate de magnésium, le milieu ne contient pas de soufre. On fait de plus: **1.** comme le témoin, et en outre 20 mg de soufre sous forme de sulfate d'ammonium. Ce milieu contient donc du soufre et de l'azote en excès, **2.** comme le témoin, et en outre 20 mg de sélénium sous forme d'acide sélénieux H_2SeO_3. Ce milieu doit être neutralisé par l'ammoniaque avant la stérilisation. On constate après stérilisation un précipité fin et rouge de sélénium. 3. Comme le témoin, et en outre 30 mg de tellure en poudre noir. La culture de B. tuberculeux 1013 utilisée a été aimablement mise à notre disposition par M. L. E. DEN DOOREN DE JONG à Rotterdam; elle ést isolée d'une tuberculose rénale. Elle a subi quatre passages sur milieu de SAUTON, on la désigne par 1013 (4). L'âge de la culture est de 8 jours et l'expérience a duré 38 jours. Cette culture possède une remarquable propriété d'absorption des éléments ajoutés au milieu. La culture contenant du tellure se noircit complètement dès le lendemain et la voile flottante est entourée par du poudre de tellure. Trois jours après l'ensemencement la culture contenant du soufre en excès est colorée rouge-orangé, la culture témoin et la culture contenant du sélénium est d'une couleur jaune clair. Ce n'est qu'après cinq jours que la culture contenant du sélénium est devenu rouge. Les différentes couleurs observées ne changent pas pendant l'expérience; il n'y a pas de doute qu'elles sont dues à l'absorption de soufre, sélénium ou tellure par le B. tuberculeux.

La croissance de la culture témoin et de la culture contenant le sélénium ou le tellure sont complètement arrêtées pendant toute la durée de l'expérience. Par contre, la culture contenant du soufre augmente continu-

TABLEAU 8.

Addition de 20 mg de soufre ou de sélénium et de 30 mg de tellure par 100 cc de milieu de SAUTON dépourvu de soufre. Méthode du carré latin. Poids en mg. Souche de B. tuberculeux 1013 (4).

I. Tableau des observations.

	Colonnes				Total
Rangées	Se 0	S 305,2	Te 0	T 0	305,2
	T 0	Te 0	S 409,4	Se 0	409,4
	Te 0	T 0	Se 0	S 278,8	278,8
	S 284,0	Se 0	T 0	Te 0	284,0
Total	284,0	305,2	409,4	278,8	1277,4
Moyenne	T 0	S 319,4	Se 0	Te 0	

0, pas de croissance.

ellement. Le tableau 8 montre clairement le résultat obtenu, il montre que le sélénium ou le tellure ne peut pas remplacer le soufre dans le milieu de SAUTON, il montre de plus que la présence de soufre dans le milieu de SAUTON est indispensable à la croissance de B. tuberculeux, toute au moins pour la souche 1013 (4) utilisée.

5. *Addition de soufre, sélénium et tellure dans le milieu synthétique de* SAUTON. *Méthode du carré latin.*

La différence entre cette expérience et la précédente est que dans ce cas le milieu de culture témoin est le milieu complet de SAUTON. On fait de plus: 1. comme le témoin, et en outre 20 mg de soufre sous forme de sulfate d'ammonium. Ce milieu contient donc du soufre et de l'azote en excès, 2. comme le témoin, et en outre 20 mg de sélénium sous forme d'acide sélénieux. Le milieu doit être neutralisé avant la stérilisation; on constate après stérilisation un précipité rouge de sélénium, 3. comme le témoin, et en outre 30 mg de tellure en poudre noir. La souche utilisée 1013 (5) est âgée de 41 jours et la durée de l'expérience est de 32 jours.

On constate dès le lendemain un changement de couleur de B. tuberculeux introduit. La culture contenant le sélénium est rouge foncé, la culture contenant le tellure est complètement noire. Par contre, la culture témoin et celle contenant le soufre est d'une couleur normale jaune clair. Ce changement de couleur reste jusqu'à la fin de l'expérience. Cependant, on constate à la fin de l'expérience dans deux cultures contenant le sélénium quelques petits endroits d'un aspect normal jaune clair. Ces parties normales sont tournées vers le bas et elles sont plongées dans le liquide. On pourrait penser que l'absorption de sélénium aurait seulement lieu quand le B. tuberculeux est capable de respirer. Par contre, dans toutes les cultures contenant le tellure on observe des petites parties minces et normales au bord de la culture épaisse et noire, elles ne sont pas submergées dans le liquide.

TABLEAU 9.

Addition de 20 mg de soufre ou de sélénium et de 30 mg de tellure par 100 cc de milieu de SAUTON. Méthode du carré latin. Poids en mg. Souche de B. tuberculeux 1013 (5).

I. Tableau des observations.

	Colonnes				Total
Rangées	S 312,1	T 614,8	Se 3,1	Te 41,0	971,0
	T 315,3	Te 14,0	S 255,3	Se 5,6	590,2
	Te 60,2	Se 1,4	T 285,4	S 461,7	808,7
	Se 13,2	S 421,0 [1]	Te 19,2	T 281,8	735,2
Total	700,8	1051,2	563,0	790,1	3105,1
Moyenne	T 374,3	S 360,0	Se 5,8	Te 33,6	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Traitements	487 963	3	162 654 ***
Résiduelle	106 740	11	9 704
Total	594 703	14	

[1]) Valeur calculée.

Le résultat de l'expérience et l'analyse de la variance (tableau 9) montrent des différences significatives entre les traitements considérés dans leur ensemble. En effet, une différence entre deux moyennes est significative si elle est supérieure à 153,3 mg pour $P = 0{,}05$ et 309,1 mg pour $P = 0{,}001$. Le coefficient de variation est égal à $C = 50{,}8$ p. 100.

Il en résulte qu'il n'y a pas de différence entre les récoltes obtenues par le soufre et le témoin. L'addition de soufre dans le milieu de SAUTON ne fait pas augmenter la récolte, ce résultat correspond à celui obtenu dans les expériences *a*, *b*, *c* et *d* avec la souche VALLÉE. Le tellure et surtout le sélénium montrent une action inhibitrice considérable sur la croissance de la souche 1013 utilisée. Le poids moyen de la culture à sélénium est égal à 5,8 mg et celui de l'ensemencement au début de l'expérience est égal à 5,5 mg. On peut donc dire que pendant toute la durée de l'expérience la croissance est complètement arrêtée et que probablement le B. tuberculeux est tué. Si l'on compare les moyennes obtenues par le sélénium et le tellure à l'aide du test t, on obtient $t = 2{,}54$ et $n = 6$, cette valeur est égale au seuil de signification 0,05. L'action inhibitrice de sélénium est donc plus élevée que celle de tellure.

Nous avons essayé de traiter la tuberculose chez des souris après injection intraveineuse de cette souche sensible au sélénium et tellure par le sélénide de bismuth, Bi_2Se_3 et le telluride de bismuth, Bi_2Te_3. Le résultat in vivo correspond à celui obtenu in vitro, l'action de sélénide de bismuth est plus élevée que celle de telluride de bismuth.

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Zoology. — *The influence of higher concentrations of lithium chloride on maturation and first cleavages of the egg of Limnaea stagnalis.* II. By A. P. DE GROOT. (Zoological Laboratory, University of Utrecht.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of April 24, 1948.)

3. *Influence on cleavage.*

RAVEN and KLOMP (1946) showed that the lack of Ca^{++}-ions in the medium prevents a normal flattening of the blastomeres after the first cleavage, the result of which is a loosening of the vitelline membrane from the egg cortex. A normal cleavage was obtained by adding a small quantity of $CaCl_2$ to the medium.

As mentioned above, in certain concentrations of LiCl not only a first cleavage occurs, but even the blastomeres may flatten in a normal way, leading to the formation of a cleavage cavity in 0.1 and 0.05 LiCl. The presence of small quantities of $CaCl_2$ cannot be the cause of this. 1) The adhering capsule fluid was carefully removed from the egg surface by repeated washings. Eggs transferred to distilled water, after washing less intensively, developed abnormally. 2) If broken eggs were removed from the solution, normal cleavage still occurred. 3) The presence of some Ca as contamination in the LiCl employed cannot be excluded with certainty, but surely not in such a quantity as would be necessary to account for the effects. According to RAVEN and KLOMP (1946) an appreciable effect of $CaCl_2$ on cleavage can only be obtained with solutions of 0.005 % or higher. So, if Ca should be the cause of the normal flattening of the blastomeres in the LiCl solutions, a contamination of 10 % Ca would be necessary. This possibility may be left out of account. From the above considerations it may be concluded that besides $CaCl_2$ also LiCl is able to cause a flattening of the blastomeres after the first cleavage.

O. HUDIG (1946) found a normal cleavage of the *Limnaea* egg in KCl and in Na-citrate as well. In the former solution it was already observed by PASTEELS (1930) who treated the eggs of *Barnea candida.* PLOUGH (1927) observed a normal first cleavage of *Arbacia* eggs in seawater after precipitating the Ca with the aid of citrate. Thus in $CaCl_2$, LiCl, KCl and Na-citrate a normal first cleavage may occur.

RAVEN and MIGHORST (1946) found that a morula stage may be reached in distilled water after a temporary treatment of *Limnaea* eggs in $CaCl_2$. In the present investigation several egg-masses were treated in the same way with LiCl. It was observed that after a temporary treatment and subsequent transfer to distilled water, a cleavage might occur, whereas it was suppressed if the eggs remained in the employed concentration of LiCl;

after that development stopped, however. Probably, this discrepancy between the Li- and Ca-effect is due to a particular function of the Ca^{++}-ions in development. From the experiments of RAVEN and KLOMP (1946) it was concluded that Ca^{++}-ions affect the properties of the vitelline membrane. The investigation of O. HUDIG (1946) led to the conclusion that the absence of Ca^{++}-ions alters the egg cortex as well. In the present investigation it was noted that after transfer to LiCl the eggs stuck to the bottom of the containers. This occurred especially after removing the adhering capsule fluid by washing in distilled water. It is likely, therefore, that the outer layer of the egg, i.e. the vitelline membrane, is affected by LiCl.

As was already mentioned, in 0.05 % the first cleavage was completely normal. In higher concentrations, the cleavage is less normal or suppressed altogether, whereas in distilled water the blastomeres do never flatten. So there exists an optimal favourable concentration of LiCl with respect to segmentation.

4. *Inhibitory action.*

The observations described above show the possibility of stopping the processes of maturation and fertilization at various stages.

a. In 1.0 % LiCl, development may stop at the stage of the first maturation spindle even before the first maturation division. This was shown by the sectioned eggs. Since the *Limnaea* egg is deposited at a stage with the first maturation spindle in metaphase, the development must have been stopped nearly immediately after exposing the egg to this high concentration. The deviations of treated eggs, as compared with normal ones, are, apparently, of a degenerative nature only.

b. The inhibitory action of LiCl may bring the development to a standstill immediately after the first maturation division. This happened in several eggs in 0.5 and 0.4 %.

c. In 0.4 %, in some of the eggs a second maturation spindle was formed; however, a second polar body may not be extruded.

d. Other eggs in the same concentration were inhibited after the formation of the second polar body, showing karyomeres beneath the egg cortex or near the centre of the egg.

e. Finally, in 0.4 % a copulation of the pronuclei may occur. This was the most advanced stage, observed in this concentration. In spite of the large number of eggs, a first cleavage was never observed.

5. *Depolarization phenomena.*

A number of the above-mentioned abnormalities may certainly be considered as belonging to the group of phenomena called "depolarization" by DALCQ (1925).

a. The most fascinating manifestation of depolarization is certainly the displacement of the second maturation spindle from the animal pole, followed by a rotation of 180°, in such a way that the spindle places itself perpendicularly to the egg axis. RAVEN and MIGHORST (1946) observed this condition in a considerable number of eggs treated with 0.5 % $CaCl_2$. In the present investigation the same phenomenon occurred only in a few eggs. This may possibly be explained by the small percentage of eggs, which reached the stage of the second maturation spindle. Only in one out of three egg-masses, sectioned after treatment with 0.4 % LiCl, the rotation of the second naturation spindle was observed. This egg-mass had been transferred to the solution at a later stage than the other ones and showed the least disturbances. So a more frequent occurrence of a rotated maturation spindle may be expected in a somewhat lower concentration, or in the same solution if the treatment starts at a somewhat later stage of development.

b. If the position of the second maturation spindle is not affected and a second polar body is formed, a depolarization may become visible in an abnormal position of the karyomeres. Normally these are lying close beneath the egg cortex at the animal pole. In most of the eggs, sectioned after treatment with 0.4 % LiCl, the karyomeres were situated at a certain distance from the egg cortex, in some of them even in the centre. The attractive mechanisms, normally determining the position of the karyomères, seem to be weakened by the LiCl treatment.

c. According to DALCQ (1925) and PASTEELS (1930), the enlargement of polar bodies has to be considered as a depolarization phenomenon too. In the present investigation, polar bodies which easily could be recognized as giant polar bodies, occurred in concentrations of 0.2 up to 0.4 %, although in a very small number. The normal volume was surpassed several times by about 2.5 % of the observed polar bodies only.

d. Finally, the delayed migration of the sperm nucleus must be mentioned. Since the polarity of the egg must play a part in the normal displacement of the sperm nucleus to the animal pole, it is very likely that the inhibition of this phenomenon is caused by a disturbance of the attractive factors. Possibly, the accelerated migration observed in a number of eggs points in the same direction. We are inclined to classify both inhibition and acceleration as depolarization phenomena.

Possibly, more abnormalities are to be considered in the same way. As the polarity plays such an important part in maturation and segmentation, it is clear that damaging influences, although of a different nature, may cause deviations in all those processes which normally are determined by the polarity, i.e. depolarization. It is not likely that the rotation of the second maturation spindle has a particular place among these phenomena. The great variety of influences resulting in a rotation points in that direction. Not only $CaCl_2$ and LiCl, but also hypertonicity, hypotonicity, CO_2 and pure mechanical pressure (KING 1906) may cause this rotation. We are inclined to consider a depolarization phenomenon not as a special reaction to a special stimulus, but rather as an indication that a certain event is dependent on polar factors acting in a developing egg.

6. *Migration of the sperm nucleus.*

A number of eggs treated with 0.4 % LiCl showed a remarkable behaviour of the nuclear apparatus. About 30 min. before the second maturation division in the controls, the chromosomes had already developed into karyomeres, situated close to the egg cortex at the animal pole, in the remnants of the first maturation aster. At this moment the development of the sperm nucleus into a male pronucleus had already proceeded very far, and its migration to the animal pole had already started or was even completed. Normally the sperm nucleus retains its subcortical position and its original shape until about 20 min. after the completion of the second maturation division. Thus an acceleration, amounting to 1—1½ hours, had happened. Apparently, after the extrusion of the first polar body, a condition had been reached, which normally does not occur before the end of the second maturation division. The LiCl treatment seems to have activated at an early hour the directing factors acting upon the sperm nucleus. This effect of the LiCl treatment may be considered as belonging to a class of phenomena, called "mise à l'unisson" by BRACHET (1922): the sperm nucleus passes prematurely into a stage resembling that of the egg nucleus. An opposite effect was observed in other eggs, treated with the same solution. Here the moment of migration of the sperm nucleus was considerably delayed. In these cases, the eggs showed subsided egg-karyomeres. The factors responsible for the ascent of the sperm nucleus seem to be inhibited in this case. Possibly, this simultaneous effect on both egg karyomeres and sperm nucleus points to a relation between their positions. Moreover, it supports the view that the subsidence of egg kayomeres may be considered as a depolarization phenomenon.

7. *Cytoplasmic effects.*

The LiCl treatment does not only affect the nuclear processes, but also the cytoplasmic components. The structure of the cytoplasm is changed especially in higher concentrations, showing a more compact appearance. In concentrations more or less isotonic to the egg, this influence was not visible. Probably, it is the result of a withdrawal of water by hypertonicity of the medium.

The animal pole plasm, occurring normally one hour before the first cleavage, had been formed in none of the sectioned eggs. Only in a very few cases a doubtful indication of it was observed. Its formation is, apparently, suppressed by the LiCl treatment.

The subcortical plasm is influenced in a peculiar way; this effect is the more interesting owing to the concentration in which it is most pronounced. In a large number of sectioned eggs the subcortical plasm had not spread beneath the egg cortex in a normal way. Its distribution was most abnormal in eggs treated with 0.2 %. In a less degree, the abnormal situation occurred also in 0.4 and 0.5 % solutions, whereas in 1.0 % hardly any abnormality was visible. In the higher concentrations it had spread in a rather normal way, showing only unimportant accumulations at various places. In 0.2 %, however, a normal distribution was never observed. It is piled up at one or more places, even at the animal pole. With proceeding development of the egg the distribution becomes more regular and at the two-cell stage it is only slightly irregular.

Whereas nearly all the above-mentioned abnormalities can be ascribed to the hypertonicity of the solutions employed, it is not allowed to attribute the abnormalities of the subcortical and animal pole plasm to the same factor. The abnormal distribution of the subcortical plasm is most pronounced in an isotonic solution and, hence, cannot be due to hypertonicity. The suppression of the animal pole plasm happens in each concentration. So the specific properties of LiCl are to be adduced to explain these abnormalities.

In contrast with the abnormal behaviour of the animal pole plasm and subcortical plasm, the maturation divisions and the first cleavage did not show any irregularitie in about isotonic concentrations. From this we are forced to conclude that the nuclear cycle is disturbed especially by hypertonicity, whereas in more or less isotonic solutions of LiCl the cytoplasmic components of the egg are particularly affected. Further experiments will be needed to test this hypothesis.

The author is highly indebted to Prof. CHR. P. RAVEN for proposing and directing the investigation and for his most valuable criticism.

Summary.

1. Eggs of *Limnaea stagnalis* were treated shortly after oviposition with 12 different concentrations of LiCl, varying from 4.0 % to 0.05 % (osmotic pressure: 42.2—0.5 atm.).
2. The development does not proceed further than the second cleavage. It may be inhibited at various stages, dependent on concentration, stage of treatment, temperature and susceptibility of the eggs.
3. Above a certain concentration of the medium, all eggs orient with the animal pole downwards; probably, this is due to local differences in permeability of the egg cortex.
4. Abnormalities, considered as depolarization phenomena, were observed in concentrations between 0.5 and 0.2 %
5. In hypotonic solutions the first cleavage may be completely normal. From this it was concluded that LiCl may prevent the loosening of the vitelline membrane from the egg cortex.
6. The nuclear cycle of maturation and fertilization is disturbed in hypertonic solutions only, whereas the animal pole plasm and the

subcortical plasm show abnormalities in their distribution even in isotonic concentrations. Therefore, it is likely that LiCl exerts a specific influence on the cytoplasmic components of the *Limnaea* egg.

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CONTENTS

Geology. — *The root of the Alps.* By J. H. F. UMBGROVE.

(Communicated at the meeting of June 26, 1948.)

Introduction.

1. A prominent feature in ARGAND's tectonic synthesis of the Alps is the suggested northward overthrusting of the southern "hinterland" of the old Tethys geosyncline. In its most extreme expression this theory was formulated as overthrusting of Africa over Europe, the crystalline cores of the Austro-Alpine nappes being regarded as frontal overthrust parts of the hinterland. This theory has been generally accepted by Swiss geologists [1]). It is the leading theme in STAUB's "Der Bau der Alpen" and it was also propagated by COLLET in his well known book "The Structure of the Alps". So, for example, COLLET wrote: "the higher Prealps, that can be seen from Geneva, Lausanne and Berne, represent a small part of Africa resting on Europe or Eurasia".

2. However, recent investigations clearly show this idea to be untenable. In the first place the counterpart of the *Préalpes medianes* are known from French territory where they are called *zone briançonnaise et zone subbriançonnaise*. French geologists, however, always had good reasons for accepting a quite different view. In their opinion these masses originated from troughs along the external or convex side of the Pennine zone. Opinions changed so to speak at the SWISS frontier mainly under the influence of the tectonic interpretation of ARGAND and STAUB, but recently, TERCIER, was one of the first Swiss geologists to dissent. On account of detailed stratigraphic investigations in the *Préalpes medianes* — he presented strong arguments in favour of the opposite French theory. The sediments of the Prealps and the "klippes" are supposed originally to have accumulated somewhere in the northern part of the Pennine region.

3. Crystalline schists of the Ivrea zone (cf. fig. 7) were considered by ARGAND as the roots of the Prealps, which also were called lower East-Alpine nappes by STAUB. A quite different interpretation of the Ivrea zone was given by NOVARESE who considers this zone of schists as a mass of pre-Triassic rocks more or less comparable to the Aar-Gothard massif from a structural point of view. The northern boundary of the Ivrea zone is characterized as a strongly mylonitized zone of tectonic movements. His conclusion was adhered to by E. NIGGLI when this author attempted to give an explanation of the strongly positive anomalies of gravity found in the eastward continuation of the Ivrea zone near Lake Maggiore.

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[1]) One will find the same ideas in a recent paper by LOMBARD.

Apparently the Ivrea zone is not a zone of roots of nappes and E. HAUG was right when, as early as 1925, he tried to make clear that Switzerland never was covered by East-Alpine nappes. Results obtained in the Lombardic Alps have given conclusive evidence.

In this paper I hope to make clear that a dominating process in the formation of the Alps was progressive underthrusting towards the central belt from either side of the mountain-chain. This view was expressed by VENING MEINESZ in 1933 and it was also accepted by HESS in 1938. In order to examine the shape and dimensions of the resulting mountain-root geological as well as geophysical data will have to be taken into consideration.

Geological evidence.

4. The tectonic crush-zone along the northern boundary of the Ivrea strip can be followed eastward where it is generally called the Insubric line. The East-Alpine nappes occur north of the Insubric line whereas the Lombaric Alps are to be found south of the same line.

Recently de SITTER published a synthesis of the work of the Leyden School, carried out in the Lombardic Alps during the last decades. Here I want to stress only a few aspects of his results. In the first place so much seems to be established without doubt: the region south of the Insubric line did not give origin to overthrust sheets towards the north or northwest. Switzerland is not a tectonic "half window" in the sense of STAUB. *A fortiori* there is no question of overthrusting of a hinterland over the Alps. Whatever the interpretation of the East-Alpine nappes they originated from the Tethys geosyncline itself which apparently consisted of an intricate pattern of troughs and intervening ridges.

In the second place the results of the geological survey of the Lombardic Alps and the Orobic zone are in accordance with the interpretation of the Ivrea zone by NOVARESE. As a major feature the region south of the Insubric line is comparable to the northern Hercynian massifs like Aar-Gothard and Aiguilles Rouges-Mont Blanc. Both are blocks of the basement divided into numerous wedges showing differential movements and as a whole dipping towards the original geosyncline. Minor differences are due to their respective situation, the Ivrea-Insubric-Orobic region being at the concave, the "central massifs" at the convex side of the arc. According to LUGEON, GAGNEBIN and others down-sliding of nappes over rather great distances must have been a frequent process in the Helvetian Alps and the Prealps. Looking for similar features elsewhere we might expect to find them in the Lombardic Alps. Now, indeed, it seems to me hardly possible to show a more convincing example of down slided nappes than those shown in sections published by DOZY and DE SITTER.

5. Unrolling of the Alpine nappes reveals a shortening of the whole chain by a considerable amount. According to CADISCH an original width

of 630 km has been reduced to the present width of 150 km in post-Carboniferous times. Other estimates made by various authors vary between 200 km and more than 1000 km shortening. SONDER, however, who published a critical study on this subject agrees with 200 km as the minimum estimate allowable. Evidently the crystalline basement must have suffered the same shortening as the superstructure because autochthonous sediments are directly connected with the crystalline basement in the north (Aar massif) as well as in the South (Lugano-Lombardic region). The total mass of basement rocks incorporated in massifs and crystalline cores of nappes is far from sufficient to explain a crustal shortening of 200 kilometres even if the original crust was comparatively thin [2]). Hence the basement must have slid downward under the Alps so as to form a sialic mountain-root.

The question how the assymmetrical structure of the Alps originated above a symmetrical root was ably discussed by BUCHER. According to his opinion the asymmetry results from the arcuate shape of the geosyncline. Folds and thrustplanes will show overthrusting mainly towards the convex side of the arc because a movement meets less resistance towards the convex side than towards the concave side. Asymmetry is the rule, even in the rectilinear part of a geosyncline [3]).

Due to thermal processes in the root a migmatite front rises upward (cf. fig. 7). It is locally revealed at the surface by granite masses like the Bergell and Adamello massifs.

The situation of the young plutonic bodies along the Insubric zone or in its northern vicinity as well as their absence along the boundary of the belt of northern or so-called central massifs may be also due to the assymmetry. For the southern zone of basement rocks is much steeper than the central massifs and so facilitated the ascent of plutonic processes.

6. Considering the structural history of the Alps one notices an outward progression of tectonic action resulting in the addition of more and more structural elements. The complicated system of Mesozoic troughs and intervening ridges in the Pennine region was bounded on one side by the deeply subsiding Helvetian trough on the other side by the sedimentation troughs of the Southern Alps.

After the Oligocene paroxysm two new troughs came into existence, one further to the north (the Molasse trough) one to the south (the Lombardic trough). Probably the subsidence of these troughs kept pace with the rising movement of the folded belt in between them, due to a cause and effect relation. In the mean time denudation products from the Alps filled up the subsiding troughs on either side of the rising mountain-chain.

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[2]) This question is discussed at greater length in § 13.

[3]) "No wrinkle, however formed, would be expected to remain poised in perfect symmetry. As soon as it begins to lean, the bulk of further deformation is transferred to one side" (BUCHER, op. cit. p. 261, see also p. 483).

Due to a renewed compression towards the end of the Miocene Tortonian strata of the Lombardic trough were overrun by Triassic rocks of the Bergamask Alps, and the Helvetian nappes came to rest on Molasse deposits. Moreover differential movements along planes separating wedge-shaped parts of the basement caused the southern boundary zone of the Molasse trough including the frontal part of the Helvetian nappes to become tilted and adjusted into their present position.

Usually the crystalline wedges of basement rocks have been regarded as upthrust or even squeezed out masses due to pressure from the Pennine nappes. Though it is not denied that such a process may have played a role of some importance it should be granted that underthrusting towards the Alps might also be the main factor. This holds even for intricate situations such as those represented in the Windgälle and Jungfrau. Moreover the elevated position of the basement rocks is due to the same factor that was responsible for the great altitude of the Pennine nappes viz. a subsequent isostatic rise of the mountain-chain.

7. During the diastrophic phase of the Alps which ended in the Oligocene the region of the present Jura Mountains was affected by differential movements along basement blocks. The movements caused a roughly NE–SW pattern of faults and short anticlines as well as a number of small faults and intervening grabens in a longitudinal direction. However, the dominating pattern of longitudinal Jura folds originated with the Upper Miocene phase. Apparently renewed southward underthrusting of the basement was an important factor in the formation of these folds. Several other factors cooperated in the formation of the longitudinal folds and evidently these factors were not yet present in Oligocene times. Doubtless the high situated basement in the northern and western foreland

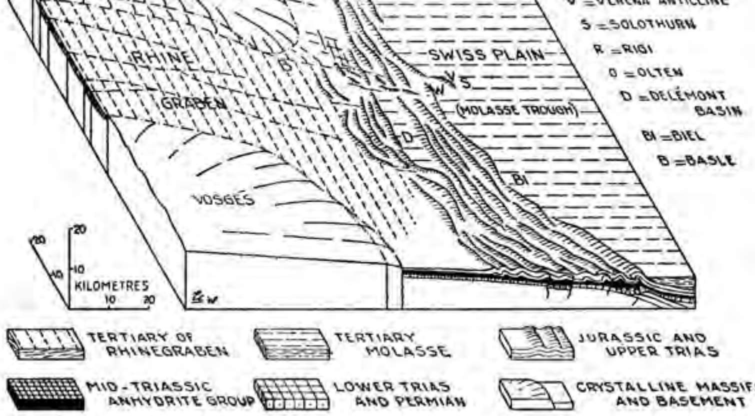

Fig. 1. Tectonogram of the Jura Mountains between the Rhine graben and the Swiss plain.

of the Juras was a factor of importance. The external shape of the Jura Mountains and the available space for the folds was controlled by it.

However, due to the general phenomenon of southward underthrusting a set of faults and intervening basement wedges originated on either side of the Molasse trough. The extern set of these tectonic elements probably caused the *décollement* of the Jura folds as they formed an obstacle against which the surface layers abutted. These layers became stripped off from the basement thereby gliding over the lubricating medium of the Mid-Triassic anhydrite group (fig. 1).

Other structural elements dating from Oligocene times (some even from earlier times) were active in the region of the present Jura Mountains. Several transverse folds reveal the influence and rejuvenation of an older pattern in a convincing way. Probably, however, several longitudinal elements of the basement had also a great influence in the arrangement of the Upper Miocene pattern of folds [4]).

Thus the Jura Mountains again reveal the progressive outward migration of tectonic activity in the basement [5]).

The Jura folds show a crustal shortening in the order of 10 kilometres, i.e. about 25 percent. However, it follows from the above given considerations that this does not imply a proportional amount of shortening for the whole Alps during the Upper Miocene phase of compression. For probably the greater part of the compression was taken up by the outer zones of the Alps and their foreland.

Geophysical evidence.

8. Uniting the results arrived at so far, a schematic and generalized section across the Alps ought to express a progressive underthrusting towards the mountain-chain from both the northern and southern foreland.

Evidently the two girdles of massif-like wedges were zones of very high friction and stress, especially their inner sides where they are bounded by the huge Pennine nappes and their roots. Probably the same strips were predestined to become major zones of movement during the subsequent process of restoration of isostatic equilibrium of the Alps. This suggestion finds a good confirmation by the distribution of seismic belts. In the western Alps ROTHÉ found two belts of seismic activity (fig. 2). The northern belt corresponds to the inner side of the zone of "Central Mas-

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4) The possible action of basement wedges was suggested by AUBERT. If sortlike structures exist their origin is probably due to the same phenomenon of Alpward underthrusting of the basement that caused similar though larger wedges along the intern margin of the Juras. If so they must have originated in the Upper Miocene. However, it seems to me very probable that longitudinal faults and graben-like structures of the type, which were called "pincées" by GLANGEAUD, and which date from Oligocene or even older times, were rejuvenated and had a great influence in the arrangement of the Upper Miocene pattern of folds.

5) A full discussion of the intricate problem of the origin of the Jura Mountains will be given in a separate paper.

sifs". The massifs themselves are practically aseismic. The conspicuous accumulation of epicentres between the massifs apparently means that similar tectonic elements are present at a lower level where they are buried in the intervening area of axial depression. The southern seismic belt corresponds to the boundary of the Pennine roots and the Ivrea zone which is characterized geologically as a steeply dipping zone of mylonitisation and greats faults (Ivrea, Tonale, Orobic and Insubric belts).

I cannot agree with OULIANOFF's criticism of ROTHÉ's interpretation. On the countrary, the two seismic zones can be followed eastward as far as the Eastern Alps [6]).

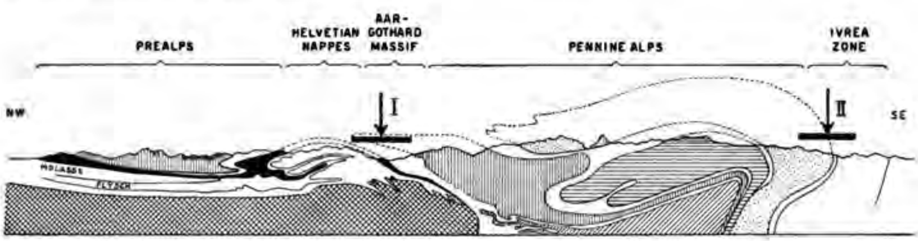

Fig. 2. Position of seismic belts (I and II) in the Alps (After ROTHÉ).

9. One of the principal conclusions arrived at so far leads us to deep reaching problems in a literal sense. For if underthrusting of the foreland towards the Alps took place from both the northern and the southern sides a sialic root of large dimensions must have been forced downward under the present mountain-chain.

Indeed, gravity anomalies found in the Alps clearly demonstrate the existence of a root of comparatively light material below the mountain-chain.

NIETHAMMER's map of Bouguer anomalies shows isanomale curves roughly parallel to the general trend of the mountain-chain. The anomalies gradually increase from zero along the northern margin of the Jura Mountains up to about — 150 in the Pennine Alps, whence they decrease again to zero when proceeding towards the southern margin of the Alps.

SALONEN's curves of Bouguer anomalies constructed at right angles to the trend of the Swiss mountains reveal an additional steepening below the Pennine Alps apart from the general increase of the anomalies towards the centre.

This feature is especially clear in profiles of the Eastern Alps constructed by HOLOPAINEN with the aid of a modified Bouguer reduction. The same phenomenon appears in all his curves of isostatic or Airy anomalies based on various assumptions of the thickness of the crust (T) and the degree of regional compensation (R). One of HOLOPAINEN's profiles is reproduced in fig. 3.

[6]) See HOLOPAINEN op. cit. 1947, p. 90 and also WANNER 1945.

The general conclusion deduced from studying these anomaly curves is: (1) the presence of a broad root of light material below the Alps and the adjacent regions gradually increasing from the northern and southern boundaries represented by the profiles towards the central belt of the Alps, (2) an additional root of light material below the central belt.

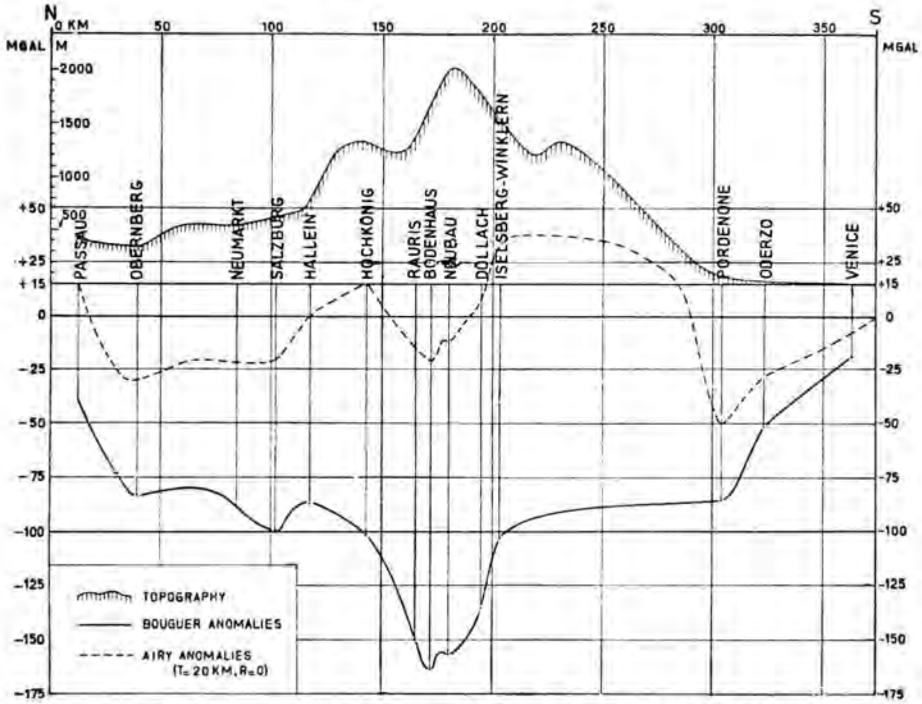

Fig. 3. Gravity profiles across the Eastern Alps (after HOLOPAINEN).

10. From a purely physical point of view numerous, not to say an infinite number of suppositions about the position of the light masses might be put forward.

The question now arises what the most probable distribution is. The relatively sharp peaks of the curves furnish a means of limiting the number of suppositions considerably. For they make clear that most probably the disturbing masses occur in comparatively shallow levels. A comparison to the geological features at the surface furnishes a further means of arriving at an explanation, which in some cases seems highly probable. Thus, for example, one sharp downbending of the isostatic anomaly curves (fig. 3) corresponds exactly to the site of Molasse trough and a similar downbending corresponds to the Lombardic trough. Evidently the northern and southern strips of negative anomalies are due to the presence of the Molasse and Lombardic troughs. HOLOPAINEN thinks the negative anomaly is due either to the prism of young and light sediments with density of about 2.37 or to the non-equilibrium of the area or to a combination of both factors.

11. It is difficult to locate the disturbing mass which causes the negative bulge below the central belt. Tracing the structural history of the Alps we found that underthrusting towards the central belt was a dominating process. Hence the theory of a root of light material was postulated as a logical and necessary consequence. However, geological data cannot possibly furnish any evidence about the shape and dimensions of the root.

HOLOPAINEN's attempt at interpretation of the gravity data is based on several uncertain premisses.

His theory starts from two fundamental assumptions. One is a "normal anomaly" of + 15 milligal for the whole area of western Europe, as based on a formula accepted by HEISKANEN in 1938. The meaning of this positive anomaly is a mystery [7]). Moreover a different value may result

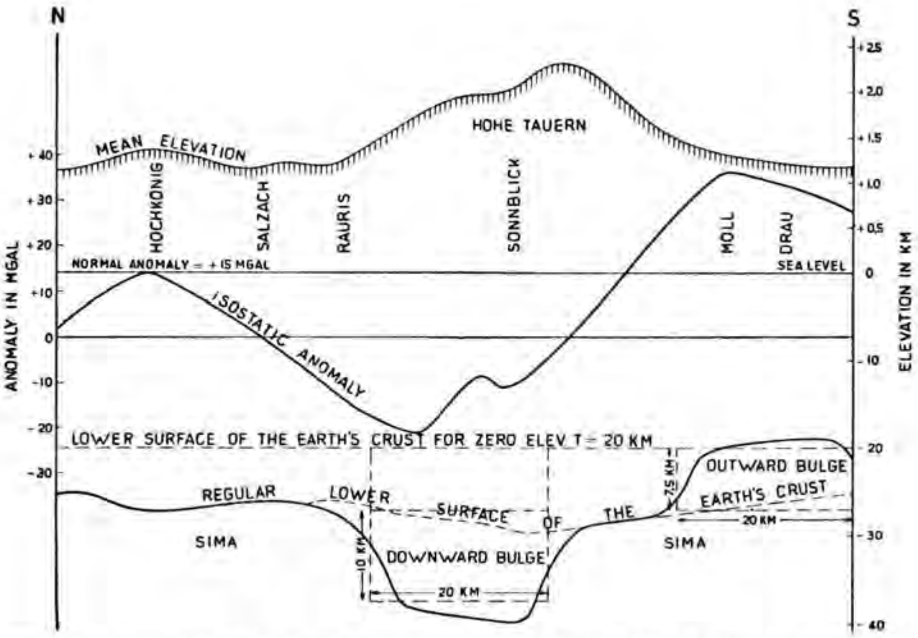

Fig. 4. Interpretation of isostatic anomalies in the Eastern Alps (after HOLOPAINEN).

if future investigations enable us to compute the "normal anomaly" on more numerous and more accurate data. The weight of this uncertainty will be clear if we realise that a second assumption is intimately connected with it, viz. the supposed thickness of 20 km of the earth's crust for zero elevation ($T = 20$). For the sake of simplicity HOLOPAINEN accepts the following model of the earth's crust. A crust 20 kilometres thick and with a density of 2.67 floating on a substratum with density 3.27. In his opinion these figures represent the most probable assumption. Of course the

[7]) The writer feels sincerely indebted to Professor VENING MEINESZ for his elucidating and stimulating discussion of the gravity anomalies of the Alps.

negative values partly depend on the accepted "normal anomaly". If the normal anomaly were to prove less than + 15 m.gal the result would be a corresponding increase of the thickness of the crust in HOLOPAINEN's model.

Moreover it seems more probable that the crust consists of several layers of varying density. For evident reasons the simplification introduced by HOLOPAINEN in his one layer model is another factor that affects the conclusions based on it. Fig. 4 clearly shows what his conclusions are. The lower surface of the crust would occur at a depth of 20 kilometres for zero elevation. An additional root of about 7,5 to 10 kilometres is drawn below the central Alps corresponding to the mean elevation of about 2 kilometres. Hence, a value of about 30 km would be probable for the thickness of the earth's crust under the mountain range. An additional root, called downward bulge, increases the thickness under the central belt by an amount of about 10 km over a north to south distance of about 20 km. According to HOLOPAINEN the downward bulge originated by a proces of downbuckling as suggested by VENING MEINESZ for the belt of strongly negative anomalies in the East Indies. It would be premature, however, to consider this model as a picture of the real situation.

12. Fig. 5 represent a crustal model built up of a granitic layer and intermediate layers above the substratum. The density of the intermediate layers is supposed to be greater than of the granitic layer and to be surpassed by the density of the substratum, though no special values will be introduced. The upper part of fig. 5, the geological profile, is based on STAUB's profile no. 4. which coincides with HOLOPAINEN's profiles II and II*a* (our fig. 3 and 4). Sedimentaries are marked by dots, crystalline cores of nappes are left white. Evidently the negative anomalies observed at the surface result from the combined influences of the sediments indicated by s and the roots r, r_1 and r_2. Probably the highly elevated pile of comparatively light sediments indicated by s in the geological section has a marked influence on the gravity curve. It is suggested that after its substraction from the total effect the resulting curve would coincide approximately with the dot-dash line. Hence the remaining negative must be due to the roots r, r_1 and r_2.

The downward bulge r is narrower than r_1 which in turn is narrower than r_2, but it is impossible to decide on the real magnitudes of the respective downward bulges, because data on the thicknesses of the crustal layers and their specific densities are still too few and of a too uncertain character. *Mutatis mutandis* the same holds good regarding the outward bulges a, a_1, a_2 and b, b_1, b_2 which correspond at the surface respectively to the zones of central massifs and southern basement rocks.

Possibly the positive belts are due to the higher level at which deeper and denser rocks became situated automatically when they had to follow the upward movement of the northern and southern belts of basement rocks to their present high situation. Of course, other factors may have been of additional importance. Intrusion of basic magma may be responsible for the relatively high positive anomalies in some areas, as suggested by E. NIGGLI for the region near Lake Maggiore.

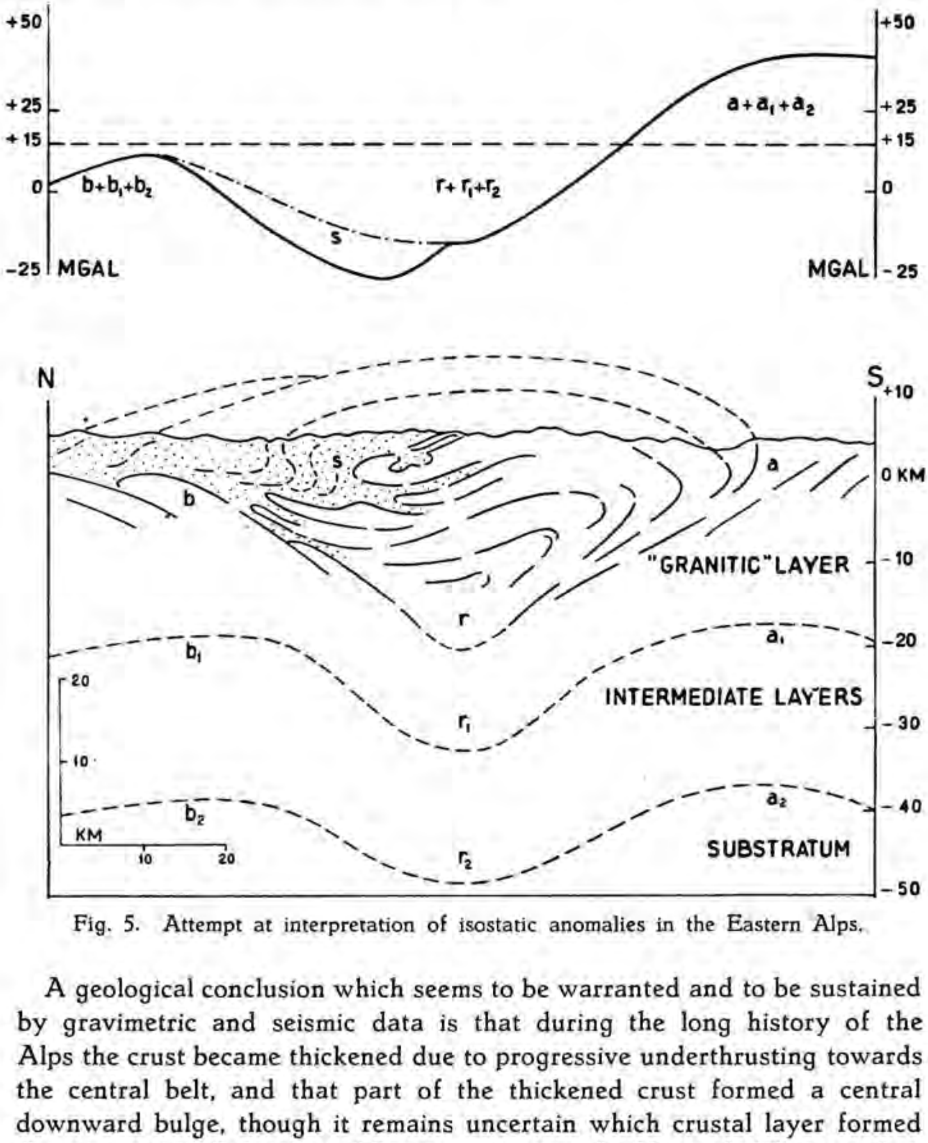

Fig. 5. Attempt at interpretation of isostatic anomalies in the Eastern Alps.

A geological conclusion which seems to be warranted and to be sustained by gravimetric and seismic data is that during the long history of the Alps the crust became thickened due to progressive underthrusting towards the central belt, and that part of the thickened crust formed a central downward bulge, though it remains uncertain which crustal layer formed the main part of the root. Moreover it seems highly probable in such a process that the phenomenon of underthrusting gradually migrated in a direction from the centre towards the "foreland" on both sides of the chain [8]).

[8]) In the East Indies a process of down-buckling of the crust was advocated by VENING MEINESZ in order to explain the occurrence of the belt of strongly negative

13. Possibly several problematic questions will be solved if more seismic data become available. What is known from seismic evidence about the thickenesses of the crustal layers in Europe was summarized by GUTENBERG in 1943 (Table I).

Though these data are too few for furnishing a final solution of several problems the seismological evidence is important in more than one respect. In the first place GUTENBERG's conclusion is in agreement with the geological and gravimetric evidence. "For Europe", he writes, "the maximum depth of the Mohorovičić layer is undoubtedly under the region of the Alps which means that the Alps have a root".

TABLE I.

Approximate thickness in km of crustal layers in Europe (after GUTENBERG).

Region	„granitic" layers (d_1)	intermediate layers (d_2)	Total
Northwestern Europe	30	10	40
Schwaebische Alb	25 — 30	20 — 25	50
Northern Alps and foreland	35	20 — 25	55 — 60
Tauern	35 — 40	20 — 25	55 — 65
Southern Carnic Alps	40	20 — 25	60 — 65
Yugoslavia	15	25	40

After this conclusion GUTENBERG continues [9]: "This root is mainly a result of a greater thickness of the uppermost (granitic) layer."

The total thickness of the upper layer under the central belt of the Alps consists of two parts, viz. (1) the crystalline cores of nappes and (2) a downward bulge. On the other hand the thickness of the intermediate layers under the Alps (20—25 km) is of the same order as the thickness of the intermediate layers under the foreland. Still, it is self-evident that during the process of down-buckling the intermediate layers must have formed an additional downward bulge similar to that of the upper layer. Therefore it appears logical to conclude that the root of the intermediate layers at any rate the bulk of it has disappeared by melting and spreading in the substratum [10]).

anomalies. The negative belt corresponds to a zone of strong diastrophism. Remarkably enough the negative belt is comparatively narrow, in spite of the fact that strong compression and possibly a corresponding rejuvenation of the sialic root occurred at several epochs. Apparently the processes involved in the formation of the root of the Alps were similar in so far as a central root originated and was perhaps rejuvenated eventually. But they were different in as much as the Alpine root grew ever broader during subsequent phases of diastrophism.

Perhaps this also explains why the negative anomalies of the central belt of the Alps are much smaller than those of the negative belt of the East Indies. (See also KUENEN, op. cit. 1936, pp. 202, 203.)

[9]) GUTENBERG, op. cit. 1943, p. 487.

[10]) See in this connection JEFFREYS, op. cit. 1929, pp. 295—296; and UMBGROVE, op. cit. 1947, pp. 85—86.

It is worth while to deduce the amount of basement rocks in the Alps before and after the crustal shortening of the Alpine cycle from a cross-section of 150 km of the present mountain-chain. Let us assume a crustal shortening of 200 km and a crustal thickness of 30 km at the beginning of the cycle in Triassic times.

From these minimum estimates follows that the original profile ought to show a total of crustal material in the order of 30×350 km^2 = = 10500 km^2. During the process of crustal shortening part of the crystalline basement became incorporated in massifs and the crystalline cores of nappes. In the profile represented by fig. 5 this amounts to 1500 km^2 above the zero line. This is a maximum estimate and it includes the part that disappeared by erosion. Allowing for a thickening of the crust under the whole area of the Alps in the order of 10 km below the zero line the total amount of crustal material now present, with the exception of the downward bulge, would amount to $1500 + (40 \times 150) = 7500$ km^2. Therefore a downward bulge with a profile in the order of $10500—7500 =$ = 3000 km^2 must have formed during the several epochs of compression of the Alpine cycle [11]). The figures chosen are unfavourable for finding a large downward bulge. Even if we would allow for still more unfavourable assumptions it seems inevitable to conclude that during the Alpine cycle crustal material formed a downward bulge which for the greater part has spread in the substratum. The only means of escaping this conclusion would be to start with a much thinner pre-Alpine crust and to allow for a much greater thickening of the crust under the whole area of the Alps. For the time being these seem very unprobable assumptions.

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[11]) As a consequence of the process of crustal shortening HOLMES is inclined to conclude that the continents must have grown progressively thicker and covered an ever smaller area in the course of geological history. As suggested by HOLMES this would involve a progressive increase of the rate of denudation and geosynclinal sedimentation as well as a progressive speeding up of orogenic processes. The theory of a remarkable acceleration of these phenomena, representing a genuine departure from the theory of uniformitarianism has been advocated by several authors. It is in good agreement with HOLMES' geological time curve based on the most probable ages of radioactive minerals and the maximum thickenesses of the geological systems. Still, however, the effect of the acceleration must not be overrated. For as far as can be ascertained the major cycles of diastrophism do not display a marked speeding up of their rhythm. As a matter of fact minor cycles plotted on the time scale show an increasing frequency but probably this phenomenon is due to the fact that unravelling the earth's structural history becomes ever more difficult the farther we try to penetrate into the past! Moreover one should not forget that after a phase of diastrophism, when the mountain belt regains isostatic equilibrium, part of the detritus is transported to the deep-sea and is forever lost from the continents. This amount should be taken into account when estimating the progressive thickening of the continents.

One of the most baffling problems of earth science is to find the motor of the deep seated processes which cause a rhythmic shortening of the earth's crust (and the earth's radius?). A vast increase of geological and geophysical data are needed before these problems can be attacked without entering into the realm of mere speculation.

14. In a far distant future one may expect the upper roots (r and r_1) to disappear due to the combined effect of continued denudation at the surface and isostatic rise of the root.

In this connection it is interesting to compare the seismic evidence found in the Sierra Nevada.

"All results available indicate that the root of the Sierra Nevada is due rather to an increase in the thickness of the deeper intermediate layers than in the thickness of the uppermost (granitic) layer" [12]). However, the Sierra is much older than the Alps.

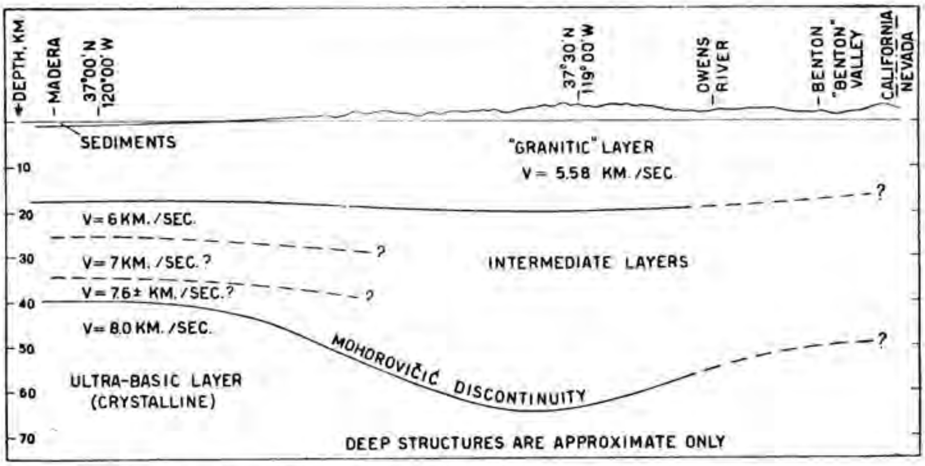

Fig. 6. Crustal layers in the Sierra Nevada (after GUTENBERG).

Fig. 6 shows the granitic layer without an appreciable root [13]) as contrasted to the large and broad root of the intermediate layers. The absence of a "granitic" root is what one might expect in a structure of great age, where there was ample time for an original root to disappear by the combined effect of isostatic rise and denudation. The presence of a root of the intermediate layers is an unexpected feature if it is true that it has already disappeared in the Alps.

If more seismic data confirm the marked difference between the Alps and the Sierra Nevada the cause of the different features must be sought in fundamental differences in the structural history of these mountain-chains.

For the time being, however, we must wait for more reliable data. As to the Alps GUTENBERG wrote: "More and better data on the velocities in the intermediate layers are necessary to find out how far an increase in the thickness of the intermediate layers contributes to the result."

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[12]) GUTENBERG, op. cit., p. 492.

[13]) "The results of a more detailed study now in progress, seem to exclude a root of the granitic layer extending below a depth of 30 km under the Sierra" (GUTENBERG, op. cit. 1945, p. 492).

Moreover, one may expect the boundaries between the crustal layers to be very irregular, especially under a region like the Alps. Therefore many more data are needed before one can construct a satisfactory picture of the thickness and distribution of crustal layers in the Alps.

A tentative and very schematic interpretation, taking into account the available geological and geophysical data is given in the tectonogram fig. 7.

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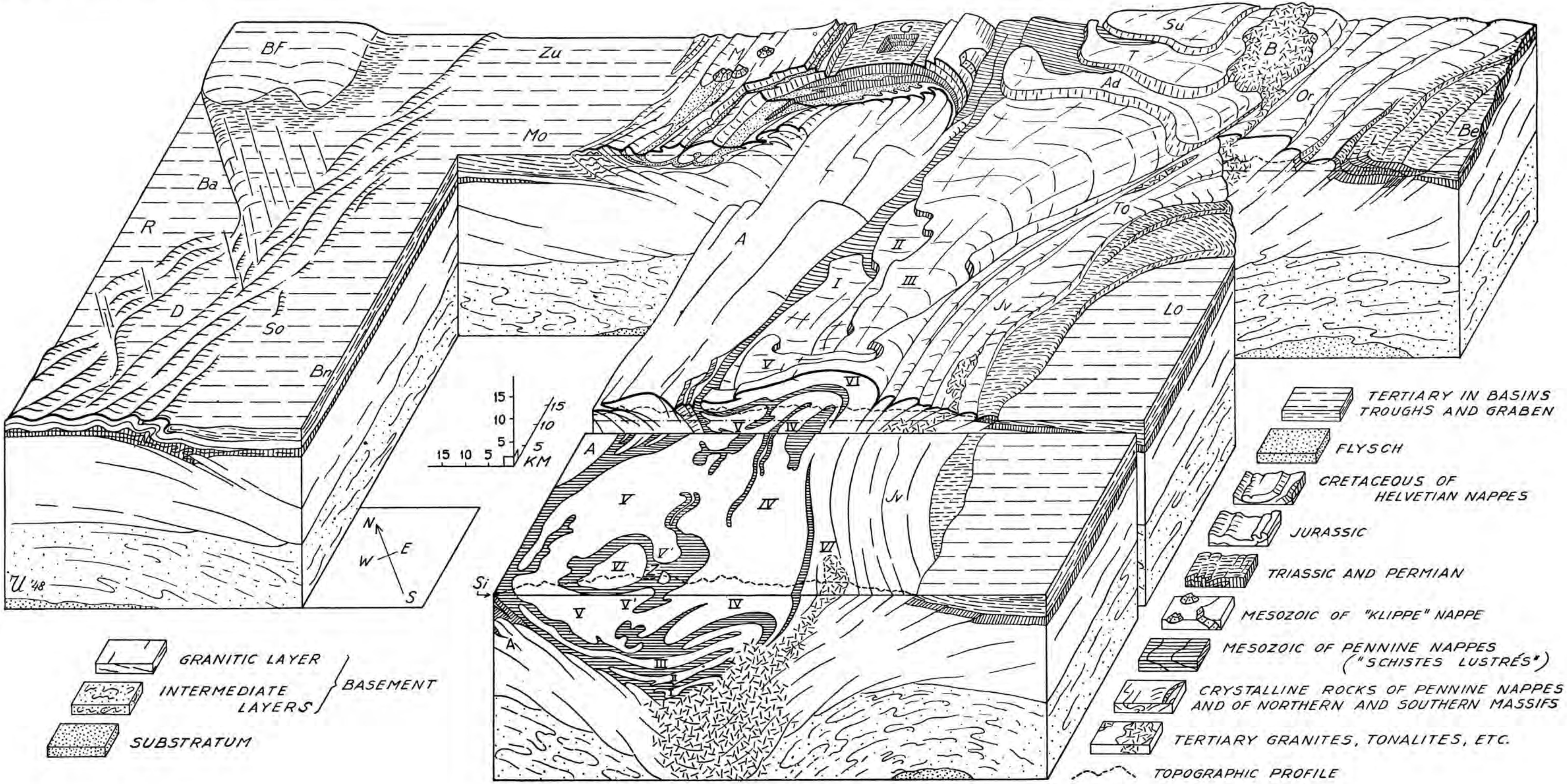
Fig. 7. Tectonogram of the Swiss Alps.

The left-hand part of the tectonogram shows the southern part of Rhine graben (R) and Black Forest massif (BF). South of them the Jura Mountains are represented schematically. The continuation of the section through the molasse trough (Mo) passes approximately along the famous "Axenstrasse". This part of the tectonodiagram, showing the structure of the so-called Helvetian Alps and remnants of the "Klippe" nappe (M) is largely based on a well-known blockdiagram by ARBENZ. The northern and southern massifs (A and Iv) as well as the Pennine nappes (I—VI, ad, T, and Su) are drawn in a very schematic manner, as is the whole blockdiagram. For the sake of clearness no "schistes lustrés" have been drawn between the Pennine nappes on the main block in the centre. They are shown, however, in the foreground block which is largely based on ARGAND and BEARTH. The profile on the right-hand part of the tectonogram, passing through the Orobic zone (Or) and The Lombardic Alps (L), is based on sections published by DOZY and DE SITTER. South of them follows the Lombardic trough (Lo).

A, Aar-Gothard massif; Ad, Adulla nappe; B, Bergell granite massif; Ba, Basle; Be, Bergamo; BF, Black Forest massif; Bn, Berne; D, Delemont basin; G, tectonic window of Glarus; L, Lombardic Alps; Lo, Lombardic trough; M, Mythen and Rotenfluh "Klippe"; Mo, Molasse trough; Or, Orobic (Insubric) zone; R, Rhine graben; So, Solothurn; Si, Sion; Su, Suretta nappe; T, Tambo nappe; To, Tonale zone; Z, Zermatt; Zu, Zurich.

VI	Dent Blanche nappe		
V'	St. Bernard nappe (backfold)	Michabel nappe	Pennine nappes
V	St. Bernard nappe		
IV	Monte Rosa nappe		
III	Monte Leone nappe	Simplon nappes	
II	Lebendun nappe		
I	Antigorio nappe		



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TERTIARY IN BASINS TROUGHS AND GRABEN
FLYSCH
CRETACEOUS OF HELVETIAN NAPPES
JURASSIC
TRIASSIC AND PERMIAN
MESOZOIC OF "KLIPPE" NAPPE
MESOZOIC OF PENNINE NAPPES ("SCHISTES LUSTRÉS")
CRYSTALLINE ROCKS OF PENNINE NAPPES AND OF NORTHERN AND SOUTHERN MASSIFS
TERTIARY GRANITES, TONALITES, ETC.
TOPOGRAPHIC PROFILE
GRANITIC LAYER
INTERMEDIATE LAYERS
BASEMENT
SUBSTRATUM

Fig. 7.

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Chemistry. — *Influence of temperature and catalysts on the bromination of naphthalene; the α-bromonaphthalene ⇄ β-bromonaphthalene equilibrium.* By J. P. WIBAUT and F. L. J. SIXMA (partly in collaboration with J. F. SUYVER and L. M. NIJLAND).

(Communicated at the meeting of June 26, 1948.)

§ 1. Since in 1835 LAURENT (1) described the bromination of naphthalene, this reaction has been repeatedly investigated in the liquid phase at temperatures below 100° C. According to the literature only one monobromonaphthalene is thus formed, namely α-bromonaphthalene, and in addition, dibromonaphthalene in proportion to the quantity of bromine used. In continuation of the investigations carried out in our laboratory into the substitution in the benzene nucleus at high temperatures, one of us (W.) and SUYVER (2) investigated the bromination of liquid naphthalene in the temperature range from 85—215° C. It was found that from the lowest temperature β-bromonaphthalene is formed in addition to α-bromonaphthalene. We determined the $\alpha : \beta$ ratio in the monobromonaphthalene mixture formed as a function of the reaction temperature and also found that this ratio is considerably influenced by the use of ferric chloride (or ferric bromide) as catalyst.

We shall first discuss these results, because the experiments on the reversible conversion α-bromonaphthalene ⇄ β-bromonaphthalene described in the present article links up with the work of WIBAUT and SUYVER.

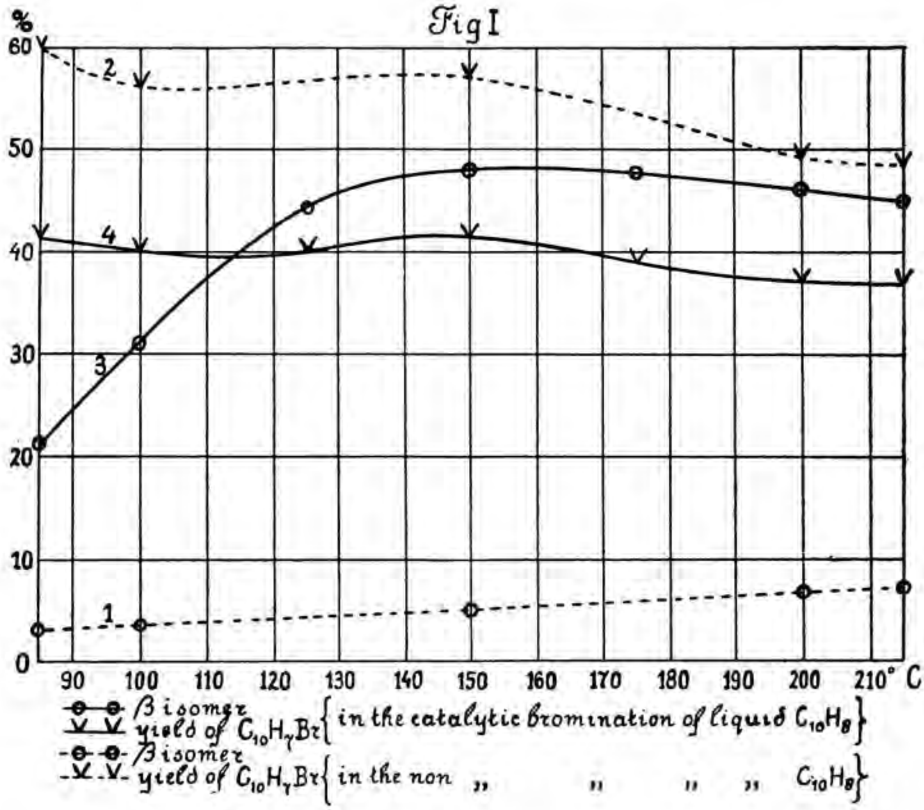
Curve 1 in fig. I shows the β-bromonaphthalene content in the monobromonaphthalene mixture formed in the non-catalytic bromination of liquid naphthalene with less than the theoretical quantity of bromine ($\frac{1}{2}$ mol of bromine : 1 mol of naphthalene).

Curve 2 represents the total yield of monobromonaphthalene calculated on bromine used. The decline in this curve at rising temperature is caused by the increasing dibromonaphthalene formation.

Curve 1 shows that the β-isomer content in the monobromonaphthalene mixture rises slowly with the temperature. The $\alpha : \beta$ ratio as a function of the absolute temperature can be represented by:

$$\ln \frac{C_\alpha}{C_\beta} = \frac{\varepsilon_\beta - \varepsilon_\alpha}{RT}$$

In this formula, which has been derived from a theory on reaction velocities developed by SCHEFFER (4), ε_α and ε_β represent the values of the energy of activation for bromination in the α- or β-position. The figures calculated with $\varepsilon_\beta - \varepsilon_\alpha = 2498$ cal/mol are in good agreement with the values found experimentally (3). The ratio in which the two isomeric bromonaphthalenes are formed in the temperature range from 85—215° C and in the absence of a catalyst is therefore determined by the difference in energy of activation required for substitution in the α- or β-position in the naphthalene molecule.



§ 2. Completely different results are obtained in the catalytic bromination of liquid naphthalene; 0.3 mol of bromine was added dropwise in these experiments to a mixture of 0.6 mol of molten naphthalene and 0.025 mol of pure ferric chloride. Curve 3 (fig. I) shows that under the catalytic influence of ferric chloride the quantity of β-bromonaphthalene formed is relatively far larger than in the non-catalytic reaction. That the total yields of monobromonaphthalene calculated on bromine (curve 4) are lower than those of curve 2 is due to the fact that in catalytic bromination more di- and tribromonaphthalenes are formed.

A large number of experiments carried out by SUYVER (5) showed that the figures given in fig. 3 for the β-bromonaphthalene content are reproducible under accurately defined conditions, such as for instance the quality of the ferric chloride used. Nevertheless, we found afterwards that the figures of curve 3 have no absolute value, because they are determined by the duration of the experiment, the rate at which the bromine is added and particularly by the activity of the catalyst. In the bromination of naphthalene at 150° C with ferric chloride as catalyst we even obtained 60 % of β-isomer in the monobromonaphthalene mixture. The maximum in curve 3 has no theoretical value. Curve 3 cannot be represented by a formula of SCHEFFER; the ratio of the isomers is not determined by the difference in energy of activation $\varepsilon_\beta - \varepsilon_\alpha$.

In SCHEFFER's theory it is assumed that the isomeric substitution products formed by simultaneous reactions are *not* converted into each other during these reactions. If, however, primarily formed α-bromonaphthalene should be converted into β-bromonaphthalene under the influence of ferric chloride, the results represented by curve 3 would become comprehensible.

From experiments carried out in conjunction with NIJLAND the following facts became clear. When a mixture consisting of 96 % of α-bromonaphthalene and 4 % of ferric chloride is heated for some hours at 150° C, various reactions develop, owing to which a little naphthalene and dibromonaphthalene is formed, 9 — 11 % of the α-bromonaphthalene originally present being converted into β-bromonaphthalene. This conversion $\alpha \rightleftarrows \beta$, however, proceeds too slowly and too incompletely to cause formation of 50 — 60 % of β-bromonaphthalene in the catalytic bromination of naphthalene at 150°.

Starting from the assumption that the action of ferric chloride towards α-bromonaphthalene cannot be compared with what takes place in the catalytic bromination of naphthalene, i.e. action of bromine in the presence of ferric chloride, the following experiments were made.

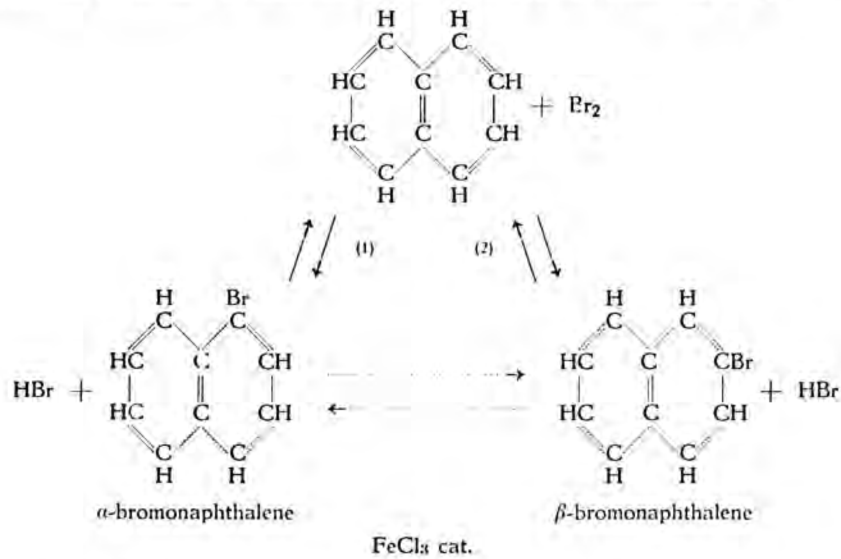
A mixture of 0.12 mol of α- or β-bromonaphthalene and 0.007 mol of ferric chloride was treated at 150° with less than the theoretical quantity of bromine; the quantity of bromine used varied from 0.04—0.06 mol. Only part of the monobromonaphthalene can therefore be further brominated. The reaction product contained a small quantity of naphthalene, monobromonaphthalene and dibromonaphthalene. The monobromonaphthalene mixture separated from it contained 40 % of α- and 60 % of β-isomer, it being immaterial whether α- or β-bromonaphthalene had been used as starting material.

When monobromonaphthalene is therefore brominated catalytically with less than the theoretical quantity of bromine reactions develop, as a result of which a 40 % α-bromonaphthalene $\rightleftarrows$ 60 % β-bromonaphthalene equilibrium is established. It was found that the equilibrium ratio is only slightly dependent on the temperature, because about the same values were found when the experiments were carried out at 150°, 200° or 250°. It is important that this conversion $\alpha \rightleftarrows \beta$ does not take place when the further bromination of α-bromonaphthalene is carried out without the addition of ferric chloride. During the *non-catalytic* bromination of liquid α-bromonaphthalene at 150°, 50 % of the starting product was converted into di- and tribromonaphthalenes; the monobromonaphthalene recovered consisted of the pure α-isomer.

§ 3. Then we carried out a systematic investigation into the reversible reactions which play a part in the catalytic bromination of naphthalene (6).

The fact that in catalytic bromination of bromonaphthalene a small quantity of naphthalene is formed, points towards the reversibility of the bromination reactions, since the naphthalene can only have been formed by debromination of bromonaphthalene. This reversibility was proved as follows: A current of gaseous hydrogen bromide was passed for 50 minutes through a mixture of 0.062 mol of monobromonaphthalene (α or β) and 0.002 mol of ferric chloride heated at 150°. During the experiment the reaction mixture was stirred vigorously, its temperature being kept constant within 1 — 2 degrees. After being worked up the monobromonaphthalene mixture contained about 60 % of β- and about 40 % of α-isomers, it being immaterial whether the starting product had been α- or β-bromonaphthalene. In addition, about 0.01 mol of naphthalene and about 0.004 mol of dibromonaphthalene had been formed. For checking purposes α-bromonaphthalene was heated with gaseous hydrogen bromide under pressure at 150° without adding ferric chloride. No reaction took place.

These experiments show that both the formation of α-bromonaphthalene and that of β-bromonaphthalene from naphthalene and bromine are reversible reactions, at least under the influence of ferric chloride:



α-bromonaphthalene β-bromonaphthalene

$FeCl_3$ cat.

If an equilibrium is established in the reactions 1 and 2, the catalytic debromination of α-bromonaphthalene or of β-bromonaphthalene will yield an α- and β-bromonaphthalene mixture, the $\alpha : \beta$ ratio in which is determined by the equilibrium constants of these reactions. If the equilibria of 1 and 2 are established rapidly enough we shall find that also in the *catalytic bromination of naphthalene* a mixture of α- and β-isomers is formed; the duration of the experiment, the temperature and the activity of the catalyst will determine whether the equilibrium belonging to a certain temperature will actually be established.

The experiments show that the equilibrium

$$C_{10}H_8 + Br_2 \rightleftarrows C_{10}H_7Br + HBr$$

is strongly in favour of bromonaphthalene. In spite of this one-sided equilibrium an *α-bromonaphthalene ⇄ β-bromonaphthalene equilibrium is established via the reversible reactions* 1 *and* 2; under the influence of ferric chloride the direct conversion $\alpha \rightleftarrows \beta$ develops so slowly that the $\alpha \rightleftarrows \beta$ equilibrium is not established in this way.

These debromination experiments were carried out at various temperatures. The $\alpha : \beta$ equilibrium ratio (table I) was determined at various temperatures from these results and from the experimental results on the bromination of monobromonaphthalene with less than the theoretical quantity of bromine (§ 2).

TABLE I. $\alpha : \beta$-bromonaphthalene equilibrium

Temperature °C	100	150	200	250
$^0/_0$ of β in equilibrium mixture of the monobromonaphthalenes	63.1	62.3	60.4	58.8

From the formula $$ln \frac{C_\alpha}{C_\beta} = \frac{\Delta Q}{RT} - \frac{\Sigma S_0}{R}$$

we calculate the heat of conversion $\alpha \rightleftarrows \beta : \Delta Q = -500$ calories/mol. In this calculation the variation of the entropy difference $S_\alpha - S_\beta$ with the temperature is ignored, which is permissible because the specific heats of α- and β-bromonaphthalene will not differ much. The error in the measurements of the equilibrium concentrations is ± 0.5 %, so that the error in ΔQ may be ± 130 calories/mol. The heat of conversion α-bromonaphthalene → β-bromonaphthalene is small as compared with the difference in energy of activation for α- and β-bromination in the liquid phase, because this $(\varepsilon_\beta - \varepsilon_\alpha)$ is about 2498 cal/mol. In comparing the figures of table I it appears that in the experiments on catalytic bromination of naphthalene, represented in curve 3 (fig. I), the $\alpha \rightleftarrows \beta$ equilibrium is not established. As mentioned above, however, it is possible to obtain the maximum β-isomer content (about 60 %) by catalytic bromination. The formation of large quantities of β-bromonaphthalene in the bromination of naphthalene with ferric chloride (or ferric bromide) as catalyst [1]) is therefore a result of the fact that the reactions 1 and 2 are reversible under the influence of the catalyst. It is useless to assume a "directing effect" of the catalyst, for small quantities of β-isomer are also formed in the non-catalytic bromination of naphthalene (§ 1). Similar debromination reactions take place when dibromonaphthalenes are heated at 150° with hydrogen bromide and ferric chloride as catalyst.

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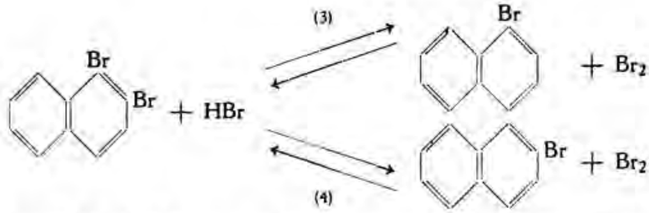
[1]) When using ferric bromide or ferrous bromide as catalysts, the results are quantitatively and qualitatively the same as those obtained with ferric chloride. For experimental reasons we used ferric chloride in the final experiments, because it is easier to obtain this substance in an anhydrous state.

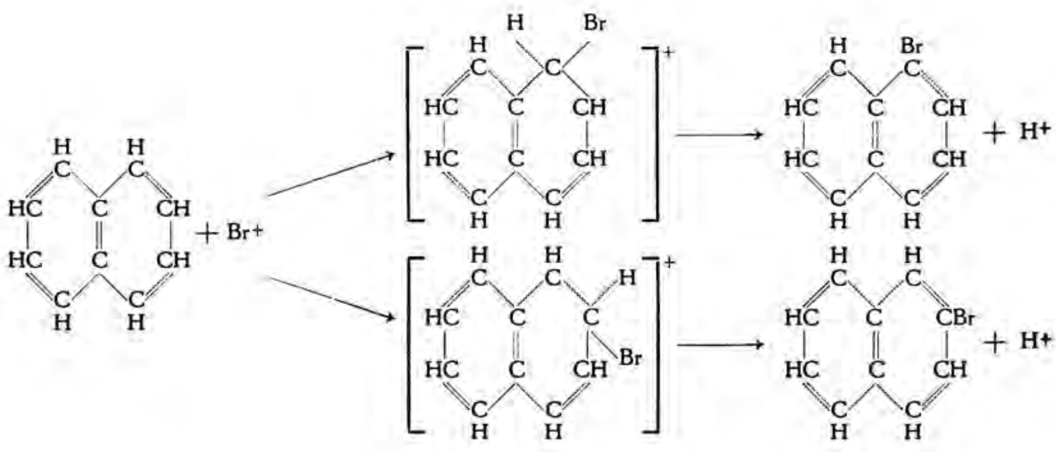
Apart from very little naphthalene, a mixture consisting of 56 % of β- and 44 % of α-bromonaphthalene is formed from 1 : 2-dibromonaphthalene under these conditions, which approach the equilibrium values $\alpha \rightleftarrows \beta$.



With 1 : 3- and 1 : 6-dibromonaphthalene a similar qualitative result was obtained. These observations afford an explanation for the experimentally established fact that in the further catalytic bromination of monobromonaphthalene with less than the theoretical quantity of bromine, the remaining monobromonaphthalene consists of a mixture of α- and β-bromonaphthalene, the equilibrium ratio $\alpha : \beta = 2 : 3$ being established under favourable conditions.

§ 4. Various investigations into the nuclear substitution of aromatic compounds at relatively low temperatures (liquid phase) have made it very likely that these reactions develop according to an ionogenic mechanism. For a bromination reaction it is therefore assumed that the bromine molecule is split into a positively and a negatively charged part, or what amounts to the same for the theory, that the bromine molecule is polarized. The positive part of the bromine molecule then forms a covalent bond with a carbon atom of the aromatic ring to be substituted. This requires two electrons to be available at this carbon atom. Under the influence of the positively charged bromine particle the naphthalene nucleus is polarized in such a way that two π-electrons are restricted to the reacting carbon atom (α or β). The above assumptions are expressed in the following scheme for naphthalene bromination:

$$Br_2 \rightarrow Br^+ + Br^-$$



The localization [2]) of two π-electrons to an α- or β-carbon atom of the naphthalene nucleus, in which the ground state of the aromatic nucleus is disturbed, requires energy. It may be expected that this energy will be different for the localization to an α- or to a β-carbon atom. If we assume that the energy required for the splitting of the bromine molecule is the same for the reaction with an α- or with a β-carbon atom, then the difference in energy of activation required for α- and β-substitution will be chiefly caused by the difference in energy required to restrict two π-electrons to the α- or β-carbon atom.

One of us (SIXMA) (6) has calculated the difference in energy of localization for the α- and the β-position by means of approximate methods elaborated in wave mechanics, i.e. by application of the "molecular orbital method". In this article only the result will be mentioned.

For the difference between the energy of a naphthalene nucleus, in which two π-electrons are restricted to an α-carbon atom and that of a naphthalene nucleus in which two π-electrons are restricted to a β-carbon atom we find:

$$\triangle E = E_\beta - E_\alpha = 3180 \text{ calories/mol.}$$

SUYVER and WIBAUT in their experiments calculated the difference in energy of activation for bromination in the β- and α-position as 2498 calories/mol for the reaction in the liquid phase and as 4215 calories/mol for the reaction in the gas phase. The agreement with the value found by application of the molecular orbital method is therefore satisfactory, considering that a number of approximations have been introduced into the theoretical derivation.

So far it has been assumed that the bromination of naphthalene proceeds as an electrophylic substitution. From the calculation it follows, however, that the same value is obtained for $\triangle E$, if one assumes substitution by bromine atoms (radical substitution) or by negatively charged bromine particles (nucleophylic substitution). The directing effect in the substitution in the naphthalene nucleus is therefore independent of the substitution mechanism when the temperature is not too high. In GOMBERG's reaction between naphthalene and the diazonium compound of methyl anthranilate, which reaction proceeds as a radical substitution, the substituent occupies the α-position, which is in agreement with the above conclusion. Also from a qualitative study by means of resonance structures, on the assumption that the molecule with the highest possible number of resonance structures has the greatest stability, it follows that substitution in the α-position requires least energy of activation.

The above therefore accounts for the fact that in the non-catalytic

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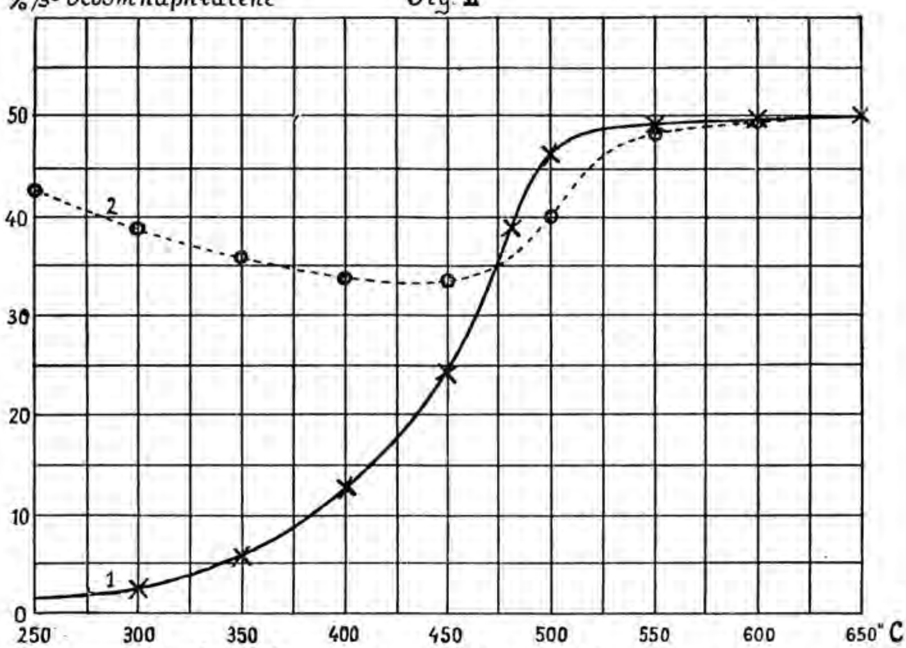
[2]) E. C. KOOYMAN and J. A. A. KETELAAR [Rec. trav. chim. **65**, 859 (1946)] have defined the concept localization of two π-electrons as the disturbance of the resonance by restriction of two of the π-electrons of the resonance system to two adjacent C-atoms reacting with ozone. In this case we extend the concept to restriction of two π-electrons to one carbon atom.

bromination of naphthalene below 300° chiefly α-substitution takes place. A catalyst such as ferric bromide (or ferric chloride) reduces the energy of the transition state of the reaction in such a way that the reaction becomes reversible. If the bromination develops according to an electrophylic mechanism it is easy to see that ferric bromide may promote the formation of positively and negatively charged bromine particles by the formation of a complex ion:

$$Br_2 \rightarrow Br^+ + Br^- ; \; FeBr_3 + Br^- \rightarrow [FeBr_4]^-$$
$$C_{10}H_8 + Br^+ \rightarrow C_{10}H_7Br + H^+$$
$$[FeBr_4]^- \rightarrow FeBr_3 + Br^-$$

§ 5. Let us now consider the formation of monobromonaphthalenes by direct bromination of naphthalene in the gas phase.

Fig. II



-×-× non catalytic bromination of $C_{10}H_8$ in the gaseous phase
-○-○- catalytic ” ” ” ” ” ” ” (FeBr₃ catalyst)

The results obtained by one of us (WIBAUT) in collaboration with SUYVER (2, 3) are represented in figure II.

Curve 1 shows the $\alpha : \beta$ ratio in the monobromonaphthalene mixture obtained by *non-catalytic bromination of naphthalene vapour*. In these experiments naphthalene vapour and bromine vapour were passed in the ratio of 1 mol of $C_{10}H_8$ to $\frac{1}{2}$ mol of Br_2 at a constant rate through a glass tube filled with glass wool or purified pumice and kept at constant temperature. The yields of monobromonaphthalene vary from 55 — 35 % (calculated on bromine) in the temperature range from 250 — 500°, but rapidly decrease above 550°, because the primarily formed bromonaphthalenes decompose at the elevated temperature. The lower part of curve 1 to approximately 300° is in agreement with a formula of SCHEFFER with $\varepsilon_\beta - \varepsilon_\alpha = 4215$ calories/mol. In this temperature range the $\alpha : \beta$ ratio is determined by the difference in energies of activation for α- and β-substitution, just as is the case in the non-catalytic bromination in the liquid phase.

According to an article published by SPEEKMAN (7) two cases must be distinguished in the bromination of gaseous bromobenzene, naphthalene, pyridine, etc. in the temperature range from 300—500°. At temperatures below 350° the reaction chiefly proceeds on the wall of the reaction vessel or at the surface of the pumice, glass wool or graphite with which the tube is filled (wall reaction); at 500° and higher the reaction proceeds between free molecules in the gas phase (gas reaction). In the transition zone from 350—400° the adsorption of the bromine and naphthalene molecules decreases, so that when the temperature rises the gas reaction will predominate over the wall reaction. Fact is that many qualitative observations on the bromination of gaseous aromatic compounds are in agreement with the assumption that the bromination at 300° chiefly develops as a wall reaction. From the experimentally established fact that at 500° and higher equal quantities of α- and β-bromonaphthalenes are formed SPEEKMAN concludes that in this temperature range the $\alpha : \beta$ ratio is not determined by the difference in energy of activation for α- and β-substitution, but is exclusively dependent on the probability of collision between the bromine molecule (or bromine atom, if one wants to assume atomic substitution) and an α- or β-position of the naphthalene molecule. As this probability is the same for both positions, equal quantities of α- and β-bromonaphthalene are formed. The question is now why from a certain temperature — in the case of naphthalene bromination from about 500° — the difference in energy of activation is no longer of importance. One of us (SIXMA) has treated this problem theoretically. Only the results of his studies are mentioned below. As the reaction temperature rises the energy of the molecule will increase. When the temperature is sufficiently high the average energy of the molecules may become of the same order of magnitude as the energy of activation of the reaction. Then the energy barrier is exceeded at each collision and the ratio of the substitution products is exclusively determined by *steric factors*.

The average energy of a system with three degrees of freedom is of the order of RT, the energy of activation of a halogenation reaction in the aromatic nucleus being of the order of 10,000—20,000 calories.

On the strength of these figures the monobromination of naphthalene would only yield equal quantities of α- and β-isomers at a temperature which might be roughly estimated at some thousands of KELVIN degrees. In SIXMA's (6) theory it is assumed that in naphthalene bromination the energy of the system is built up from 2n quadratic terms; the calculation shows that the transition temperature, i.e. the temperature at which equal quantities of α- and β-isomers begin to be formed is lower as n is larger.

This accounts in principal for the fact, that the transition temperature found experimentally, is much lower than might be expected according to the formula:

$$ln \frac{C_\alpha}{C_\beta} = \frac{\varepsilon_\beta - \varepsilon_\alpha}{R\,T};$$

it has not yet been possible to calculate this transition temperature. From the calculation it also follows that at a low temperature the reaction between bromine and naphthalene satisfies SCHEFFER's formula, also when the energy of the system must be described by $2n$ quadratic terms.

The catalytic bromination of naphthalene vapour shows quite a different development (curve 2 in fig. II). In these experiments carried out by SUYVER the reaction tube was filled with a catalyst consisting of *pumice impregnated with ferric bromide*. The presence of this catalyst not only increases the total rate of reaction at which monobromonaphthalene is formed, but it also highly promotes the β-substitution. The peculiar shape of this curve suggested that partial conversion $\alpha \rightleftarrows \beta$ takes place under the influence of ferric bromide. If, however, α- or β-bromonaphthalene vapour is passed slowly over the ferric bromide contact at 350° or at 500° no $\alpha \rightleftarrows \beta$ conversion takes place. In continuation of our experiments on the reversibility of the catalytic bromination of naphthalene in the liquid phase described in § 3, we have investigated the debromination reaction in the vapour phase.

We passed a mixture of gaseous α-bromonaphthalene and gaseous hydrogen bromide in the molecular ratio 1 : 15 over a ferric bromide pumice contact at 300° (0.06 mol of α-bromonaphthalene, 0.92 mol of hydrogen bromide in 300 minutes). The reaction product consisted of some naphthalene (about 0.001 mol) and the two monobromonaphthalenes in the ratio of 53.1 % of β- to 46.9 % of α-isomer. When the same experiment was made with β-bromonaphthalene as starting material, the monobromonaphthalene mixture obtained consisted of 60 % of β and 40 % of α.

As the ferric bromide is markedly volatile in the investigated temperature range and precipitates in cold parts of the apparatus, it is impossible to measure the heterogeneous gas equilibrium $\alpha \rightleftarrows \beta$ at various temperatures with accuracy. However, the experiments allow of the following conclusions. In the gas phase the bromination of naphthalene is reversible; also in case of a great excess of hydrogen bromide the equilibrium in the temperature range from 300 — 500° is still strongly in favour of the bromonaphthalenes.

As a result of the reversible reactions:

naphthalene + bromine

⇅ ⇅

HBr + α-bromonaphthalene β-bromonaphthalene + HBr a monobromonaphthalene mixture is formed, in which the $\alpha : \beta$ ratio may approach an equilibrium ratio. When we started in our experiments from the α- and β-bromonaphthalenes the same final condition was only roughly approached at 300° C. At 400° and 460° the rate of conversion $\alpha \rightleftarrows \beta$ is considerably lower than at 300°. This apparently paradoxal result must be accounted for as follows.

As the temperature rises the adsorption of bromonaphthalene and hydrogen bromide to the catalyst surface decreases, so that the ratio (adsorbed molecules) : (molecules in the gas phase) decreases. As we use flowing gases and the $\alpha \rightleftarrows \beta$ conversion only takes place at the catalyst surface, the number of α-molecules converted per unit of time into β-molecules (or conversely) will be smaller as the temperature is higher. Though the rate of reaction in the catalyst layer will be considerably increased when the temperature is raised, the above-mentioned decrease in adsorption, which has an opposite effect, apparently predominates. The curve 2 in fig. II does not represent equilibria, because they have not been established. At about 500° the curve of the catalytic bromination begins to approach that of the non-catalytic bromination and almost coincides with it at higher temperatures, because in this range the reaction proceeds almost exclusively in the gas phase and because the adsorption of the molecules to the catalyst layer has dropped to a very low value.

June, 1948.

Laboratory of Organic Chemistry of the University of Amsterdam.

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Mechanics. — *Non-linear relations between viscous stresses and instantaneous rate of deformation as a consequence of slow relaxation.* By J. M. BURGERS. (Mededeling No. 56 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of June 26, 1948.)

1. In connection with preparatory work for the International Rheological Congress to be held in September 1948 at Scheveningen (Holland) I had to study some papers by WEISSENBERG referring to the stresses called forward by deformation in visco-elastic materials, where laminar flow is accompanied by elastic deformations [1]). WEISSENBERG's treatment is rather of an abstract character and the reader is puzzled by the problem how a continuously progressing deformation as is found in laminar flow and a permanent elastic deformation can be present together, in particular as attention is drawn to the circumstance that the principal directions of these deformations may be different. However, a tangible picture can be obtained if we start from the idea that flow is possible in consequence of a relaxation phenomenon and bear in mind that, owing to the finite rate at which the re-arrangement of molecular structure takes place, every element of volume of a flowing medium will bear in its molecular pattern reminiscences of its past. In consequence of this circumstance the actual pattern during flow deviates from the equilibrium pattern. There is thus a state of physical deformation, which must be clearly distinguished from the progressively increasing deformation of the boundary surface of an element of volume as it occurs in flow.

In the ordinary theory of viscosity based on this idea it is supposed that the state of physical deformation is proportional to the instantaneous rate of deformation, as will be the case when the re-arrangement of the molecules takes place sufficiently quickly. On the other hand in liquids or fluids where this is not the case, "memory" will extend further into the past and the physical deformation will be determined by less simple relations.

The idea of a "memory" in matter has been introduced by BOLTZMANN, and was afterwards taken up by VOLTERRA and by VON KARMAN [2]). It

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[1]) Compare in particular: K. WEISSENBERG, La mécanique des corps déformables, Arch. Sciences phys. et natur. (Genève) (5) **17**, p. 1—105, 1935. — Of more recent publications may be mentioned: K. WEISSENBERG, A continuum theory of rheological phenomena, Nature **159**, p. 310, March 1, 1947.

[2]) See: V. VOLTERRA, Drei Vorlesungen über neuere Fortschritte der mathematischen Physik, Archiv d. Mathematik u. Physik (3) **22**, p. 97—182, 1914, in particular p. 155—171.
TH. VON KÁRMÁN, Das Gedächtnis der Materie, Die Naturwissenschaften **4**, p. 489, 1916.

is present also in an important model proposed by PRANDTL for the explanation of hysteresis and relaxation [3]. Although VOLTERRA has given some attention to the general case, the examples mainly considered refer to systems with displacements in a single direction, so that the problem of non coinciding spatial directions does not occur. In the following lines it will be shown how such a problem can arise for the elementary case of laminar motion, even if we keep to an extremely simplified qualitative treatment.

2. A few words on the theory of hydrodynamic viscosity may precede the consideration of the example. In hydrodynamical theory the components of the viscous stresses in a flowing liquid of constant density [4]) are assumed to be proportional to the corresponding components of the instantaneous rate of deformation. If the velocity components are denoted by u_i ($i = 1, 2, 3$, corresponding to the three axes of a system of rectangular coordinates x_1, x_2, x_3), the components of the instantaneous rate of deformation are given by [5]):

$$D_{ik} = \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \quad \ldots \ldots \ldots \ldots \quad (1)$$

and the components of the stress tensor have the values:

$$\tau_{ik} = \eta \, D_{ik} \quad \ldots \ldots \ldots \ldots \quad (2)$$

where η is the viscosity.

The theoretical explanation of this relationship starts from the idea that (*a*) every deformation of an element of volume of the liquid calls forward a change in the arrangement of the molecules, and that (*b*) when the element is left to itself a short but finite interval of time is needed before the normal statistical distribution of molecular distances and velocities has been restored. Whereas in the normal equilibrium state of an ordinary liquid the molecular field is statistically isotropic, there can be present consequently an anisotropy for a short period. The anisotropy of the arrangement gives rise to anisotropy of the field of inter-molecular forces, and as a result of this, to the appearance of stresses.

When deformations are changed or repeated periodically in intervals of time short compared with the time necessary for re-arrangement, the

[3]) L. PRANDTL, Ein Gedankenmodell zur kinetischen Theorie der festen Körper, Zeitschr. f. angew. Mathem. u. Mechanik 8, p. 85—106, 1928. Compare: First Report on Viscosity and Plasticity, Verhand. Kon. Nederl. Akademie v. Wetenschappen, (1) **15**, no. 3, p. 41—64 (1939).

[4]) To simplify incompressibility has been assumed so that the equation of continuity $\partial u_i / \partial x_i = 0$ (summation with respect to repeated indices) is satisfied by the components of the velocity. Also it is assumed that the stress components τ_{ik} refer to the deviatoric stresses only, so that $\tau_{ii} = 0$. No attention is given to the hydrostatic part of the stress (hydrostatic pressure).

[5]) A factor $\frac{1}{2}$ is sometimes inserted; in that case the factor η in eq. (2) must be replaced by 2η.

medium in general will show elastic behaviour, in first approximation with proportionality between stress and strain. On the other hand when the material is subjected to a process of progressive deformation, in particular when a stationary state of flow is present, the resultant effect of deformation and re-arrangement leads to the appearance of a statistically stationary anisotropic state of the molecular field, and thus to the appearance of a stationary system of stresses.

In the usual form of the theory the permanent state of physical deformation of the flowing medium, *i.e.* the state of anisotropy which is to be found in the pattern of molecular arrangement and of velocity distribution, is assumed to be equal to the product of the rate of deformation into a quantity having the dimensions of a time and called the "relaxation time" λ. When the stress is put equal to the product of the permanent physical deformation into the shear modulus G, the stress becomes equal to the rate of deformation multiplied by $G\lambda$. The product $G\lambda$ thefore represents the viscosity η, which can be calculated theoretically when it is possible to find G and λ [6]).

It is assumed in this mode of reasoning that the time of relaxation λ is very short compared with the time in which an appreciable geometrical deformation of the boundary surface of an element of volume takes place. The resulting physical deformation then will be slight, and as the equilibrium state itself is isotropic, the directions characterising the resulting anisotropy of the field will be the same as the directions characterising the tensor of the rate of deformation D_{ik}.

On the contrary when the relaxation time becomes large we must expect that the deviation of the molecular arrangement will bear reminiscences of a more remote past. The physical deformation will then no longer be simply proportional to the instantaneous rate of deformation, but will be determined by some integrated quantity. This brings the possibility that the tensor describing the physical deformation in general will not be parallel and proportional to the tensor of the instantaneous rate of deformation. At the same time the physical deformation may assume such a magnitude that in calculating the stresses account must be taken of stress-strain relationships for large deformations, which in general cannot be represented by linear formulae.

3. We now turn to the case of laminar flow. We assume the velocity u (parallel to the x-axis) to be given by $k\,y$; the component v being zero (we restrict ourselves to the x, y-plane). The tensor of the instantaneous rate of deformation is then given by:

$$D = \begin{Bmatrix} 0 & k \\ k & 0 \end{Bmatrix} \quad \ldots \ldots \ldots \ldots \quad (3)$$

[6]) The modern development of this theory is due to M. BORN and H. GREEN, and a particularly clear exposition is given by GREEN in a paper to be read before the International Rheological Congress, which will be published in the Proceedings of the Congress.

Now consider a point which at the instant $\bar{t}$ has the coordinates $\bar{x}, \bar{y}$; at the instant $t = \bar{t} - \lambda$ the coordinates have been:

$$x = \bar{x} - k\lambda\bar{y} \quad ; \quad y = \bar{y},$$

from which:

$$\bar{x} = x + k\lambda y \quad ; \quad \bar{y} = y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The physical state of deformation to be found at the instant $\bar{t}$ may be dependent on the whole series of preceding states, so that it should properly be calculated by means of an integral extending over the time from $t = -\infty$ until $t = \bar{t}$ [7]. However, the features necessary to explain WEISSENBERG's formulae come out already if we assume that the physical state of deformation is described by equations (4) in which λ will be considered as some given, finite quantity. The point of importance is that in this case we have to do with a *physical deformation of finite magnitude* and that account must be taken of this circumstance in calculating the stresses. Various methods have been proposed for dealing with finite deformations and the stresses accompanying them; in the present case it is convenient to make use of the system of formulae developed by WEISSENBERG for that purpose [8]. The transformation from the coordinates x, y to the coordinates $\bar{x}, \bar{y}$ is described with the aid of the matrix equation:

$$\|\bar{x}, \bar{y}\| = \|x, y\| \cdot \begin{Vmatrix} 1 & 0 \\ k\lambda & 1 \end{Vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Here $\|\bar{x}, \bar{y}\|$ and $\|x, y\|$ are matrices with a single row only, while the usual rule for matrix multiplication must be applied. We write ψ for the transformation matrix, $\widetilde{\psi}$ for its transposed form, so that:

$$\psi = \begin{Vmatrix} 1 & 0 \\ k\lambda & 1 \end{Vmatrix} \quad ; \quad \widetilde{\psi} = \begin{Vmatrix} 1 & k\lambda \\ 0 & 1 \end{Vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The matrix ψ is unsymmetrical and combines in itself a rotation with a deformation. The latter can be described by means of either of two symmetrical matrices ϑ_a, ϑ_p, determined by the equations [9]:

$$\vartheta_a^2 = \psi\,\widetilde{\psi} = \begin{Vmatrix} 1 & k\lambda \\ k\lambda & 1 + k^2\lambda^2 \end{Vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

$$\vartheta_p^2 = \widetilde{\psi}\,\psi = \begin{Vmatrix} 1 + k^2\lambda^2 & k\lambda \\ k\lambda & 1 \end{Vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

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[7]) See VOLTERRA, l.c. (footnote 2) above), p. 157 seq. — VOLTERRA, however, calculates the deformation at a given instant from the forces applied during the previous period, whereas in the text we are interested in the present state of physical deformation as it results from the geometrical deformation experienced by the material in the previous period.

[8]) K. WEISSENBERG, Arch. Sciences phys. et natur. (Genève) (5) **17**, p. 11 seq., 1935.

[9]) It is reminded that any power of a symmetrical matrix can be obtained as follows:

The matrices ϑ_a and ϑ_p differ in the order in which they must be combined with a matrix φ describing the rotation so as to obtain back the matrix ψ. This matrix φ (which is anti-symmetrical) is given by:

$$\varphi = \begin{Vmatrix} a & -\frac{1}{2}\,k\,\lambda\,a \\ \frac{1}{2}\,k\,\lambda\,a & a \end{Vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where a has been written for $(1 + \frac{1}{4}k^2\lambda^2)^{-1/2}$, and we have the equations:

$$\psi = \vartheta_a \cdot \varphi = \varphi \cdot \vartheta_p . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence ϑ_a represents a transformation which must precede the rotation, while ϑ_p is the transformation to be applied when the rotation is performed first.

Now the stress tensor is a function of ϑ_p [10]). For our purpose it is not so important which function is chosen, but the point to be observed is that the principal axes of the stress tensor are parallel to those of ϑ_p. It is not difficult to find the directions of the principal axes of ϑ_p and to calculate the corresponding eigenvalues. The first axis makes the angle Φ with the x-axis (counted anticlockwise from x to y) given by

$$tg\,\Phi = -\tfrac{1}{2}\,k\lambda + \sqrt{1 + \tfrac{1}{4}\,k^2\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which is smaller than 1, so that $\Phi < 45°$; the second axis is perpendicular to the first one. The eigenvalues are:

$$\tfrac{1}{2}\,k\lambda + \sqrt{1 + \tfrac{1}{4}\,k^2\lambda^2} \quad ; \quad -\tfrac{1}{2}\,k\lambda + \sqrt{1 + \tfrac{1}{4}\,k^2\lambda^2} \quad . \quad . \quad . \quad (11)$$

so that there is extension in the direction of the first axis and compression in that of the second axis.

Hence we see that when $k\lambda$ is treated as a finite quantity the principal direction of extension is *turned towards the x-axis*. We must expect the same for the principal tension stress. This is the result which is brought forward by WEISSENBERG; it can be interpreted by saying that a tension stress directed according to the x-axis is superposed on the usual system of shearing stresses τ_{yx} and τ_{xy}.

The principal directions of the given matrix are determined and its eigenvalues p_1, p_2, p_3 (for the threedimensional case) are calculated. The m-th power of the matrix then is a matrix with eigenvalues:

$$(p_1)^m \quad , \quad (p_2)^m \quad , \quad (p_3)^m ,$$

having the same orientation as the original matrix. Its expression with respect to any system of rectangular coordinates can be easily found in this way. By way of example we mention:

$$\vartheta_p = \begin{Vmatrix} (1 + \frac{1}{2}\,k^2\lambda^2)\,a & \frac{1}{2}\,k\,\lambda\,a \\ \frac{1}{2}\,k\,\lambda\,a & a \end{Vmatrix},$$

where $a = (1 + \frac{1}{4}k^2\lambda^2)^{-1/2}$.

[10]) Compare K. WEISSENBERG, l.c. p. 93, sub *b*). — It is confirmed by applying the equations given by R. S. RIVLIN, Large elastic deformations of isotropic materials, Philos. Trans. Roy. Soc. London **A 240**, p. 459, 1948, eqs. (3.9) and (3.10).

4. It will be possible to extend the formulae of the preceding section in such a way that they embrace the general case of a homogeneous field of deformation. On the other hand, when the velocity components are non linear functions of the coordinates, other features will come into the picture. It is doubtful whether a phenomenological treatment of such a case would be worthwhile, as it is to be expected that the actual physical relations and a proper analysis of the relaxation phenomenon and of the range of the inter-molecular forces (which range may be considerably extended in certain directions when the molecules are very long) will play an important part. We therefore leave aside the case of non-homogeneous fields.

Résumé.

Le but de cette note est de montrer comment certaines formules de WEISSENBERG ayant trait aux relations entre tensions et déformations dans un milieu „plasto-élastique" peuvent obtenir une illustration si on se base sur la théorie de la relaxation pour expliquer la possibilité d'un mouvement illimité d'un tel corps, et si on suppose que le temps de relaxation soit assez grand pour que l'état physique du corps en mouvement diffère beaucoup de l'état normal.

Resumo.

La jena artikolo celas montri kiamaniere kelkaj formuloj de WEISSENBERG pri la rilatoj inter tensioj kaj aliformiĝoj en medio plastik-elasta povas esti ilustrataj, kiam oni bazas sin sur la teorio de la malstreĉiĝo por klarigi la eblon de nelimita movado de tia korpo, supozante ke la tempo de malstreĉiĝo daŭru sufiĉe longe por ke la stato fizika de la korpo moviĝanta diferencu multe de la stato normala.

Mathematics. — *On the attraction between two perfectly conducting plates.* By H. B. G. CASIMIR.

(Communicated at the meeting of May 29, 1948.)

In a recent paper by POLDER and CASIMIR [1]) it is shown that the interaction between a perfectly conducting plate and an atom or molecule with a static polarizibility α is in the limit of large distances R given by

$$\delta E = -\frac{3}{8\pi}\hbar c \frac{\alpha}{R^4}$$

and that the interaction between two particles with static polarizibilities α_1 and α_2 is given in that limit by

$$\delta E = -\frac{23}{4\pi}\hbar c \frac{\alpha_1 \alpha_2}{R^7}.$$

These formulae are obtained by taking the usual VAN DER WAALS–LONDON forces as a starting point and correcting for retardation effects.

In a communication to the "Colloque sur la théorie de la liaison chimique" (Paris, 12—17 April, 1948) the present author was able to show that these expressions may also be derived through studying by means of classical electrodynamics the change of electromagnetic zero point energy. In this note we shall apply the same method to the interaction between two perfectly conducting plates.

Let us consider a cubic cavity of volume L^3 bounded by perfectly conducting walls and let a perfectly conducting square plate with side L be placed in this cavity parallel to the xy face and let us compare the situation in which this plate is at a small distance a from the xy face and the situation in which it is at a very large distance, say $L/2$. In both cases the expressions $\frac{1}{2}\Sigma\hbar\omega$ where the summation extends over all possible resonance frequencies of the cavities are divergent and devoid of physical meaning but the *difference* between these sums in the two situations, $\frac{1}{2}(\Sigma\hbar\omega)_I - \frac{1}{2}(\Sigma\hbar\omega)_{II}$, will be shown to have a well defined value and this value will be interpreted as the interaction between the plate and the xy face.

The possible vibrations of a cavity defined by

$$0 \leqslant x \leqslant L, \quad 0 \leqslant y \leqslant L, \quad 0 \leqslant z \leqslant a$$

have wave numbers

$$k_x = \frac{\pi}{L} n_x, \quad k_y = \frac{\pi}{L} n_y, \quad k_z = \frac{\pi}{a} n_z,$$

where n_x, n_y, n_z are positive integers;

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{\varkappa^2 + k_z^2}.$$

[1]) H. B. G. CASIMIR and D. POLDER, Phys. Rev., **73**, 360 (1948).

To every k_x, k_y, k_z correspond *two* standing waves unless one of the n_i is zero, when there is only one. For k_x, k_y this is without importance since for very large L we may regard k_x, k_y as continuous variables. Thus we find

$$\tfrac{1}{2}\Sigma\hbar\omega = \hbar c\frac{L^2}{\pi^2}\int_0^\infty\int_0^\infty\left[\tfrac{1}{2}\sqrt{k_x^2+k_y^2}+\sum_{n=1}^{\infty}\sqrt{n^2\frac{\pi^2}{a^2}+k_x^2+k_y^2}\right]dk_x\,dk_y$$

or, introducing polar coordinates in the $k_x\,k_y$ plane,

$$\tfrac{1}{2}\Sigma\hbar\omega = \hbar c\frac{L^2}{\pi^2}\cdot\frac{\pi}{2}\sum_{(0)1}^{\infty}\int_0^\infty\sqrt{\left(n^2\frac{\pi^2}{a^2}+\varkappa^2\right)}\,\varkappa\,d\varkappa,$$

where the notation $(0)1$ is meant to indicate that the term with $n=0$ has to be multiplied by $\frac{1}{2}$. For very large a also this last summation may be replaced by an integral and it is therefore easily seen that our interaction energy is given by

$$\delta E = \hbar c\frac{L^2}{\pi^2}\cdot\frac{\pi}{2}\left\{\sum_{(0)1}^{\infty}\int_0^\infty\sqrt{\left(n^2\frac{\pi^2}{a^2}+\varkappa^2\right)}\,\varkappa\,d\varkappa-\int_0^\infty\int_0^\infty\sqrt{(k_z^2+\varkappa^2)}\,\varkappa\,d\varkappa\left(\frac{a}{\pi}dk_z\right)\right\}.$$

In order to obtain a finite result it is necessary to multiply the integrands by a function $f(k/k_m)$ which is unity for $k \ll k_m$ but tends to zero sufficiently rapidly for $(k/k_m)\to\infty$, where k_m may be defined by $f(1)=\frac{1}{2}$. The physical meaning is obvious: for very short waves (X-rays e.g.) our plate is hardly an obstacle at all and therefore the zero point energy of these waves will not be influenced by the position of this plate.

Introducing the variable $u = a^2\varkappa^2/\pi^2$,

$$\delta E = L^2\hbar c\frac{\pi^2}{4a^3}\left\{\sum_{(0)1}^{\infty}\int_0^\infty\sqrt{n^2+u}\,f(\pi\sqrt{n^2+u}/a k_m)\,du\right.$$
$$\left.-\int_0^\infty\int_0^\infty\sqrt{n^2+u}\,f(\pi\sqrt{n^2+u}/a k_m)\,du\,dn\right\}.$$

We apply the EULER–MACLAURIN formula:

$$\sum_{(0)1}^{\infty}F(n)-\int_0^\infty F(n)\,dn = -\tfrac{1}{12}F'(0)+\tfrac{1}{24\times30}F'''(0)+\ldots$$

Introducing $w = u + n^2$ we have

$$F(n)=\int_{n^2}^{\infty}w^{1/2}f(w\pi/a k_m)\,dw,$$

whence

$$F'(n) = -2n^2 f(n^2\pi/a k_m)$$
$$F'(0)=0$$
$$F'''(0)=-4.$$

The higher derivatives will contain powers of (π/ak_m). Thus we find

$$\delta E/L^2 = -\hbar c \frac{\pi^2}{24 \times 30} \cdot \frac{1}{a^3},$$

a formula which holds as long as $ak_m \gg 1$. For the force per cm^2 we find

$$F = \hbar c \frac{\pi^2}{240} \frac{1}{a^4} = 0{,}013 \frac{1}{a_\mu^4} \text{ dyne/cm}^2$$

where a_μ is the distance measured in microns.

We are thus led to the following conclusions. There exists an attractive force between two metal plates which is independent of the material of the plates as long as the distance is so large that for wave lengths comparable with that distance the penetration depth is small compared with the distance. This force may be interpreted as a zero point pressure of electromagnetic waves.

Although the effect is small, an experimental confirmation seems not unfeasable and might be of a certain interest.

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Astronomy. — *The disc theory of the origin of the solar system.* By H. P. BERLAGE, Director Meteorological and Geophysical Service, Batavia.

(Communicated at the meeting of May 29, 1948.)

1. The difficulties raised against the encounter theory of the origin of the solar system, as well as the appearance of two other theories [1]) in recent years, may permit the author to review briefly the outcome of a series of papers published in these Proceedings between 1930 and 1940 [2]), including some corrections and extensions to which he found himself induced since. The more so, because he admits that in his odyssae through several attempts to attain a rational picture of the evolution of the planets, he was many times led astray, but is now in the position to formulate a rather concise theory as a working basis.

This theory is essentially monistic, considering the transformation of a nebula rotating round a heavy nucleus. Evidence converges towards this origin since careful analysis proves that also every dualistic conception of the solar system will lead us back to a primeval sun, surrounded by a very extensive gaseous envelope. SPITZER found the right expression, when writing [3]):

"Such an atmosphere is reminiscent of the Laplace nebular hypothesis, except that in this case there need be no lack of angular momentum. The validity of the encounter theory as an explanation of the origin of the solar system rests apparently on whether or not a non-uniformly rotating atmosphere could condense into solid bodies."

The author is convinced that this question should be answered in the affirmative.

2. Because the distribution of mass in the solar system is such that only one part in 700 is concentrated in the planets, ROCHE's model of a nucleus surrounded by a massless shell is applicable. Hence, the potential at any point in the atmosphere is Newtonian.

Let r denote the distance of a volume element from the axis of revolution, h its height above or below the equatorial plane, p and ϱ gaspressure and

[1]) H. ALFVÉN, On the cosmogony of the solar system, Stockholms Observatoriums Annaler, Band 14 No. 2, 1942; No. 5, 1943; No. 9, 1946. — C. F. V. WEIZSÄCKER, Ueber die Entstehung des Planetensystems, Zs. f. Ap. **22**, 319—355 (1943). Reviewed by G. GAMOV and J. A. HANEK, Ap. J., **101**, 249—254 (1945).

[2]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, 614 (1930); **33**, 719 (1930); **35**, 553 (1932); **37**, 221 (1934); **38**, 857 (1935); **43**, 532 (1940).

[3]) LYMAN SPITZER Jr., The Dissipation of Planetary Filaments, Ap. J. **90**, 674—688 (1939).

density, M the mass of the sun, γ the constant of gravitation, ω the angular velocity of an element of the nebula, then the two fundamental equations of equilibrium become

$$\gamma M h (r^2 + h^2)^{-\frac{3}{2}} + \frac{1}{\varrho} \frac{\partial p}{\partial h} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\gamma M r (r^2 + h^2)^{-\frac{3}{2}} + \frac{1}{\varrho} \frac{\partial p}{\partial r} = \omega^2 r . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In order to solve these equations we introduce the gas equation

$$p = f \varrho , \quad f = k \frac{T}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which T is the absolute temperature, μ the mean molecular weight of the gas and k a universal constant.

It is easily shown that, when the total mass of the planets was originally evenly distributed through a sphere of radius equal to, say, 500 astronomical units, this matter was opaque to solar radiation. Therefore, it is reasonable to assume that temperatures in the solar nebula will never have differed much from black body temperatures. This means a variation of T as the inverse square root of the distance from the sun.

It follows that

$$T = T_0 \left[\frac{r_0}{(r^2 + h^2)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

when r_0 is the radius of the sun and T_0 the temperature of the nebula where it is in touch with the solar atmosphere.

Let us further assume that the variation of μ can be represented adequately by a certain power of the solar distance.

If

$$\mu = \mu_0 \left[\frac{r_0}{(r^2 + h^2)^{\frac{1}{2}}} \right]^{\frac{1}{2} - s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$s = \frac{1}{2}$ means uniform chemical composition of the nebula.

We then have, when f_0 denotes the value of f at the surface of the sun

$$f = f_0 \left[\frac{r_0}{(r^2 + h^2)^{\frac{1}{2}}} \right]^{s} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Integrating (1) and (2), we obtain

$$\varrho = \varrho_e \left(1 + \frac{h^2}{r^2} \right)^{\frac{1}{2}s} \exp \left[- \frac{\gamma M}{r_0^s f_0 (1-s)} \left\{ \frac{1}{r^{1-s}} - \frac{1}{(r^2 + h^2)^{\frac{1}{2}(1-s)}} \right\} \right] \quad . \quad (7)$$

$$\frac{\gamma M}{r^2} + f_0 \left(\frac{r_0}{r} \right)^s \frac{d \log \varrho_e}{d r} - s f_0 \frac{r_0^s}{r^{s+1}} = \left(1 + \frac{h^2}{r^2} \right)^{\frac{1}{2}s} \omega^2 r \quad . \quad . \quad (8)$$

in which ϱ_e denotes the density of the gas in the equatorial plane. If it is given the two formulae define the structure of the nebula in steady motion.

Let us consider briefly 3 models which have special features, viz.:

$$\text{a)}\quad s=0\,,\quad \varrho=\varrho_e \exp\left[-\frac{\gamma M}{f_0}\left\{\frac{1}{r}-\frac{1}{(r^2+h^2)^{\frac{1}{2}}}\right\}\right] \quad \text{(7a)}$$

$$\frac{\gamma M}{r^2}+f_0\frac{d\log\varrho_e}{dr}=\omega^2 r \quad \text{(8a)}$$

$$\text{b)}\quad s=\tfrac{1}{2}\,,\quad \varrho=\varrho_e\left(1+\frac{h^2}{r^2}\right)^{\frac{1}{4}}\exp\left[-\frac{2\gamma M}{r_0^{\frac{1}{2}}f_0}\left\{\frac{1}{r^{\frac{1}{2}}}-\frac{1}{(r^2+h^2)^{\frac{1}{4}}}\right\}\right] \quad \text{(7b)}$$

$$\frac{\gamma M}{r^2}+f_0\left(\frac{r_0}{r}\right)^{\frac{1}{2}}\frac{d\log\varrho_e}{dr}-\tfrac{1}{2}f_0\frac{r_0^{\frac{1}{2}}}{r^{\frac{3}{2}}}=\left(1+\frac{h^2}{r^2}\right)^{\frac{1}{4}}\omega^2 r \quad \text{(8b)}$$

$$\text{c)}\quad s=1\,,\quad \varrho=\varrho_e\left(1+\frac{h^2}{r^2}\right)^{\frac{1}{2}}\exp\left[-\frac{\gamma M}{2r_0 f_0}\left(\frac{h}{r}\right)^2\right] \quad \text{(7c)}$$

$$\frac{\gamma M}{r^2}+f_0\frac{r_0}{r}\frac{d\log\varrho_e}{dr}-f_0\frac{r_0}{r^2}=\left(1+\frac{h^2}{r^2}\right)^{\frac{1}{2}}\omega^2 r \quad \text{(8c)}$$

a) $s=0$ means a decrease of the mean molecular weight of the gas with solar distance proportional to the decrease of temperature. The resulting structure is the simplest of the three, e.g. ϱ decreases monotonously with increasing h. The model is the only one showing the property that ω is a function of the distance r from the axis of rotation only.

b) $s=\frac{1}{2}$ means uniform chemical composition of the nebula. Many will claim the greatest a priori probability for this case. The equilibrium structure shows, however, one unreasonable feature. Lim $\varrho=\infty$ for lim $h=\infty$.

c) $s=1$ means an increase of the mean molecular weight of the gas with distance from the sun inversely proportional to the decrease of temperature. The infinite increase of μ with solar distance cannot be strictly true, of course, but the limiting value of ϱ is now zero for lim $h=\infty$. Another satisfactory feature of this structure is that lim $\omega=0$ for lim $h=\infty$. This too is valid for model b), but model c) is simpler. If ϱ_e is the same all over the equatorial plane, a meridional section through the nebula shows straight lines of equal density intersecting at the centre.

Now, the premises are such that clearly these 3 models encompass all reasonable possibilities. The real structure will have shown characteristics somewhere intermediate between the characteristics of the structures just described and probably not have differed much from model b). When the author proposes to proceed this investigation with model a), it is principally because the calculations in the next paragraphs are feasible only when the nebula rotates cylinderwise. Consequently all further steps taken are on surer ground. Yet, there might be more plausibility in assumption a) than seems to be the case.

For even if the chemical composition of the nebula was originally homogeneous, it is very improbable that this was maintained. The molecules did not move in free orbits. Gravitation was partly supported by gaspressure. Hence, the heavier molecules will have shown a tendency to gather towards the centre, the lighter ones towards the periphery.

Many will point out that the very high percentage of H and He present in the original nebula kept the mean molecular weight of the gas at some low value, say 4, all over. The actual constitution of the planets, however, is incompatible with such a large dosis of H and He and when the planets were not able to retain the H and He, it is extremely probable that even if H and He were once present in the nebula in the amount in which they are present in the sun now, they vanished from the nebula by evaporation into space before the planets were formed.

As to the probable existence of molecules in the nebula, we may mention that comets tails of the same low density consist of several molecular compounds of H, C, N, and O of which many show signs of disintegration only at small solar distances. We might even try to explain the general trend of the actual densities of the planets by a preponderance of molecules of heavier or less heavy weight where the planets condensed. There is at least no danger of losing any essential aspect of the development of the planets, when we go as far as the very simple assumption $f = f_0 =$ constant. Quantitatively model a) works out as follows.

We know that $Fe = 56$ and $SiO_2 = 60$ were predominant building materials of the nucleus and the outer shells of the earth. This would mean $\mu = 58$ at the solar distance where the earth was formed. The blackbody temperature at this same distance being 290°, we obtain $f_0 = 4 \times 10^8$ c.g.s. to base our further calculations upon.

3. For $h = \infty$ the density of the gas tends towards the finite value

$$\lim_{h=\infty} \varrho = \varrho_e \exp\left[-\frac{\gamma M}{r}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The solar nebula behaved morphologically like a vortex in the interstellar medium, as DESCARTES conceived it. As the interstellar medium established the external boundary conditions, the fact that (9) depends on r means, of course, that the motion of the nebula could not be strictly stationary. Independent internal boundary conditions are established by the sun at its surface. We have to consider the nebula as being in slow but gradual development, its structure being an equilibrium structure only in first approximation.

For $h \ll r$, (7a) reduces to

$$\varrho = \varrho_e \exp\left[-\frac{\gamma M}{2 f_0} \frac{h^2}{r^3}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence, with $\gamma = 6.67 \times 10^{-8}$ and $M = 2.00 \times 10^{33}$, we obtain roughly

$$\varrho = \varrho_e \times 10^{-\frac{5000}{r}\left(\frac{h}{r}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

when r and h are expressed in astronomical units.

In order to fix the ideas let us consider two instances. The density of the gas drops to a value 10^{-10} times the density in the equatorial plane at a vertical distance as small as $h = 0.1\ r$ for $r = 5$ astronomical units, that is as far out as Jupiter, whereas the drop is to one hundredth under the same conditions at $r = 25$ astronomical units, that is halfway Uranus and Neptune.

In other words, if these conditions of temperature and composition have prevailed in the solar nebula it has been a thin flat disc over most of its extent. The models b) and c) are for given total mass of the nebula less flat in the inner portions, but flatter in the outer portions, when compared with model a).

The theory of a gaseous envelope of the sun in steady motion and its flat disc-like shape was developed later independently by v. WEIZSÄCKER. He uses it at the base of an extremely interesting hypothesis of the origin of the planetary system, which is, however, fundamentally different from the hypothesis developed here.

4. The final structure of the disc is determined by viscosity. It must have acted first rapidly but roughly as turbulent viscosity, later, where turbulence dies out, slowly as the ordinary viscosity of laminar motion.

It was shown in Proc. **38** that the theorem of minimum loss of energy by viscosity requires, when squares and products of small quantities are neglected, that

$$2r\frac{d^2 \log \varrho_e}{dr^2} + \frac{d \log \varrho_e}{dr} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The solution of this equation is

$$\varrho_e = \varrho_0 \exp\left[-a r^{\frac{1}{2}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

when ϱ_0 and a are integration constants to be determined from observational data. These are the constant mass and the constant moment of momentum of the disc, or

$$m = 2.26 \times 10^{30}\ \text{gr}$$
$$\theta = 3.14 \times 10^{50}\ \text{c.g.s.}$$

In the way indicated in Proc. **35**, we obtain

$$a = 6.82 \times 10^{-7}\ \text{cm}^{-\frac{1}{2}}$$
$$\varrho_0 = 4.63 \times 10^{-9}\ \text{gr cm}^{-3}$$

We have now reached the point, where we can draw a picture of the nebula. Its meridional section is shown in Fig. 1. Lines of equal density suggest a toruslike structure. A 10^{-24} surface would indicate where the nebula merges into interstellar matter, since the latter is found to possess this order of density within the galactic system. A 10^{-22} line has been dotted, because even this hardly stands for the boundary surface. The 10^{-20} surface should be regarded as the most rational envelope of the original nebula. A density of 10^{-18} is reached in the earth's atmosphere at a height of 300 km, where we know the atmosphere is dense enough to cause it start visible phenomena like polar lights and the radiation of

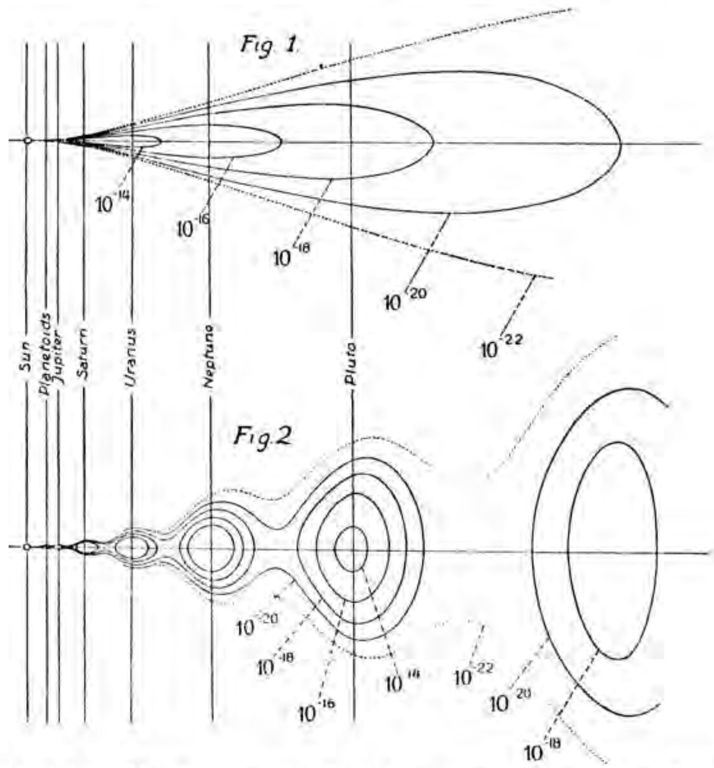

Fig. 1 and 2. Meridional section through gaseous disc generating planets.

meteorites. The surface 10^{-18} still envelopes Pluto's orbit. It might be argued that the matter between the envelopes 10^{-18} and 10^{-20} became the stock of our comets.

5. The motion described so far is steady. According to Proc. **35** it is also stable, where the following condition is satisfied,

$$\gamma M + f_0 \frac{d}{dr}\left(r^3 \frac{d \log \varrho_e}{dr}\right) > 0. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Applying this condition, we find that there is a critical radius

$$r_c = \left(\frac{4 \gamma M}{5 f_0 a}\right)^{\frac{2}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

such that the nebula will have been stable and in laminar motion up to a distance from the axis of rotation which is smaller than r_c, whereas the nebula will have been instable and in turbulent motion in its outer parts. Inserting the values of a and f_0 found before, we obtain

$$r_c = 5.31 \times 10^{15} \text{ cm}.$$

As this distance is far beyond Pluto's orbit, the solar nebula may be supposed to have settled down in laminar motion up to its outskirt.

As was shown in Proc. **35** a is independent of f_0. Hence, even if the molecular weight of the gas had been only one tenth of the weight which was assumed before, we find

$$r_c = 1.14 \times 10^{15}\ \mathrm{cm}$$

which proves that the nebula would still have been stable up to a distance beyond Pluto's orbit.

There seems to be no theoretical reason for large scale convective motions of the disc such as are at the basis of WEIZSÄCKER's theory of the origin of the planets and certainly even less so when WEIZSÄCKER's assumption that the particles move in free orbits would prove to be of vital significance. However, observational evidence favours the hypothesis that the motion of the disc actually was or at least remained turbulent down to a shorter distance from the sun than the theoretical limit. As a matter of fact, the motion of Pluto shows already signs of being less strictly regulated than the motion of the other planets, while the turbulent motion of the girdle of the nebula beyond Pluto might explain the retrograde motion of many comets.

6. So far we did develop the theory of the rotating solar nebula up to first order small quantities and found it steady within these limits. We know, however, that it cannot attain a strictly steady state and proceeds in continuous evolution. Hence, we are confronted with the problem whether the nebula will condense into planets or keep the disclike structure described so far. From Proc. **37** we know that steady state and KEPLER motion are only compatible, if the coefficient of viscosity would have adjusted itself throughout by adequate migration of molecules. The more probable method of compromise, however, is a dividing up of the nebula in concentric zones in each of which KEPLER's third law holds approximately whereas the radial gradient of the angular velocity shows discontinuities from zone to zone.

It is easily shown that this adjustment would result in the generation of circular zones in which the angular velocity is successively superior and inferior to the KEPLER velocity, which in the steady state means a fluctuation of the density ϱ_e with solar distance, that is a tendency towards concentric ring formation.

We may add that the formation of separate rings is the natural way in which the loss of energy by friction can decrease further and would eventually vanish when the rings rotate like rigid bodies. Thus fortified in our assumption that the planetary system once passed through the state of concentric gaseous rings, let us consider a configuration of the nebula which is infinitely near to the one described by (13), but differing from it by the addition in the exponent of an ondulating term. We than get

$$\varrho_e = \varrho_0 \exp\left[-a\, r^{\frac{1}{2}} + \varepsilon \sin f(r)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

the amplitude of the ondulation being an infinitesimal dimensionless quantity. As (13) was the solution of (12), the only condition which has to be satisfied by $f(r)$ is that the left side of (12) when multiplied by r, remains smaller than $\varkappa$ for every value of r, when $\varkappa$ denotes an infinitesimal dimensionless quantity.

Now, it is easily shown that among the functions expressing some power of r, the logarithmic function

$$f(r) = p \log r + q \quad \ldots \ldots \ldots \quad (17)$$

replacing the zero power is the only function which keeps the left side of (12) small for any value of r. This may be considered as an indication that whenever there is a tendency of the nebula to change from a disc into a series of concentric rings the radii r_n and r_{n-1} of two successive rings will satisfy the relation

$$\frac{r_n}{r_{n-1}} = \text{constant.} \quad \ldots \ldots \ldots \quad (18)$$

This, however, is the expression of BODE's rule, when we disregard the additional term [4]).

The mass of a planet m_p can be expressed by the formula

$$m_p = \text{constant} \times r^{\frac{5}{2}} \exp\left[-a\, r^{\frac{1}{2}}\right] \triangle r \quad \ldots \ldots \quad (19)$$

when $\triangle r$ denotes the difference between two successive radii, where minimum density occurs. As, however, following (18)

$$\triangle r = \text{constant} \times r_p \quad \ldots \ldots \ldots \quad (20)$$

when r_p is the planets distance from the sun, we find

$$m_p = \text{constant} \times r_p^{\frac{7}{2}} \exp\left[-a\, r_p^{\frac{1}{2}}\right]. \quad \ldots \ldots \quad (21)$$

This explains the general trend of the masses of the planets, which is increasing first and decreasing again, when we pass through the system from the centre towards the periphery.

Strictly, of course, when developing any arbitrary oscillation, we have to add in the exponent of (16) several terms

$$\varepsilon_1 \sin(p_1 \log r + q_1) + \varepsilon_2 \sin(p_2 \log r + q_2) + \ldots \quad \ldots \quad (22)$$

Observational data, however, favour the assumption that one of the partial terms is naturally stressed in the act of concentration and accounts for BODE's rule. The other terms account for the deviations of the distances of the planets from BODE's rule, as well as for the deviations of the masses of the planets from the general trend just mentioned.

The problem can also be treated as a problem of gravitational instability. In our nebula showing axial symmetry the tidal forces of the central body, which in other circumstances would tend to disintegrate local condens-

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[4]) The reader will have noticed that by erroneous reasoning the author was led in Proc. **43** to an alternative kind of density variation and to the following rule of distances

$$\sqrt{r_n} - \sqrt{r_{n-1}} = \text{constant.}$$

ations, are balanced by rotation. Therefore, its condensation into rings cannot differ much from what would happen, when a non-rotating disc of the same structure, but without central body, would be left to itself at a given moment. It is essentially the same process. The author even tried whether the variation of the intensity of the solar radiation with the sunspot cycle might have been the source of a pressure wave spreading through the disc lifting it over the threshold of gravitational instability and found that it would lead to the right dimensions.

7. Let us now prove that a variation of the course of the density from (13) to (16) may lead to a natural increase of the amplitude of the fluctuations introduced. In order to do this, we have to calculate the variations of mass δm, of angular momentum $\delta\theta$, of potential energy δV and of kinetic energy δU, involved in the density variation. Let the conditions

$$\delta m = 0\,, \quad \delta\theta = 0\,, \quad \delta(U+V) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

be satisfied. When these conditions are compatible with the condition

$$\delta U = -\delta V > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

we may assume that the disc follows this course and tends to change into a series of concentric rings.

In Proc. **43** the author gave affirmative evidence. There he assumed, however, that the nebula before becoming unstable, passes through a state of equilibrium in which $\delta U = 0$ and $\delta V = 0$ are satisfied up to first order small quantities automatically, confining himself to the calculation of second order small quantities. Apparently, this assumption is not allowed. Using the formulae developed in Proc. **43** the author was able to prove that condition (24) reads

$$\int_0^\infty r^{\frac{1}{4}} \exp\left[-a\, r^{\frac{1}{4}}\right] \sin\left(p \log r + q\right) dr < 0. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

It does not invalidate the conclusion reached in Proc. **43**. As (25) is independent from the other conditions and the left side an oscillating function, it can be satisfied.

The natural tendency leading in the direction of the formation of rings, we are tempted to follow the process of condensation through finite values of ε to its critical value. From the actual dimensions of the planetary system, we derive

$$p = 10.86$$

when 10 is the base of the logarithm in (25). Final condensation will occur when the density in the densiest part of every ring outgrows the minimum value stated in ROCHE's theorem, that is when

$$\varrho_0 \exp\left(-a\, r^{\frac{1}{4}} + \varepsilon\right] > 14.5\, M \left(\tfrac{4}{3}\pi r^3\right)^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

When ε_c is the oritical amplitude of the density wave and known values of a and ϱ_0 are inserted, we get

$$\varepsilon_c = (42.17 - 3\ {}^{10}\log r)\,({}^{10}\log e)^{-1} + 6.82 \times 10^{-7}\, r^{\frac{1}{4}}. \quad . \quad . \quad . \quad (27)$$

From this relation we obtain the following table

$r = \text{cm}$	ε_c
10^{12}	14.9
10^{13}	7.3
10^{14}	7.2
10^{15}	15.0

ε_c has a very flat minimum at $r = 7.74 \times 10^{13}$ cm, which is very near Jupiters mean distance from the sun. The lowest value of ε_c is 7.15. This proves that most probably condensation proceeded from Jupiter towards Mercury and from Jupiter towards Neptune, where ε_c reaches the values 10.6 and 10.3 respectively. The critical density ϱ_c in the equatorial plane which has to be surpassed is greatest for Mercury, where it reaches the value

$$\varrho_c = 3.50 \times 10^{-5} \text{ gr cm}^{-3}$$

In the nearest density throughs the density is then of the order 10^{-14} gr cm^{-3}. This shows how far the development towards individual rings has gone. It is reasonable to assume that the final condensation started with $\varepsilon = 8$. Fig. 2 shows how the meridional section through the disc looks like when the evolution has progressed up to this point.

We are still confronted with two questions. The first is whether in this stage kinetic energy can still grow at the cost of potential energy. We are allowed to answer it in the affirmative, since we will be able to show that there are cases in which the final system of condensed globes possesses more kinetic energy than the original nebula. The second question is, whether the motion of the gas could remain laminar up to the point of final condensation.

Introducing (16) in (14), we get the following condition of stability,

$$\frac{\gamma M}{f_0} - \tfrac{5}{4} a r^{\frac{3}{2}} + \varepsilon p r \{2 \cos (p \log r + q) - p \sin (p \log r + q)\} > 0. \quad (28)$$

The maximum value attained by the function between brackets is 11.25. When the other known values are inserted, we obtain

$$\varepsilon_c = 2.72 \times 10^{15} r^{-1} - 7.02 \times 10^{-9} r^{\frac{1}{2}} \,. \quad . \quad . \quad . \quad . \quad (29)$$

From this relation the following table can be derived

$r = \text{cm}$	ε_c
10^{12}	2720
10^{13}	272
10^{14}	27.2
10^{15}	2.5
5.31×10^{15}	0

As was already shown in 5., ε_c decreases to zero at a distance of

5.31×10^{15} cm from the sun. Beyond this distance the solar nebula would be in turbulent motion previous to any tendency of condensation. Both critical limits are equal at 3.02×10^{14} cm that is slightly beyond Uranus, where they reach the value 8.9. This proves that all the rings may have remained stable with the exception of the rings of Neptune and Pluto. They reached the limit of turbulent motion before reaching the limit of final condensation. According to our assumption this means that Neptune and Pluto were in danger to remain in the form of planetesimals. Neptune has evidently escaped this danger. Pluto perhaps not. Its excentric behaviour suggests that there may be other "plutoids".

Jupiter, when first starting to condense, may have drained his surroundings from matter before the other planets came into existence. This may be the reason, why there was almost no matter left for a planet in the place where we actually find the planetoids. The planetoids never gathered into one planet probably because matter in the ring was too dispersed.

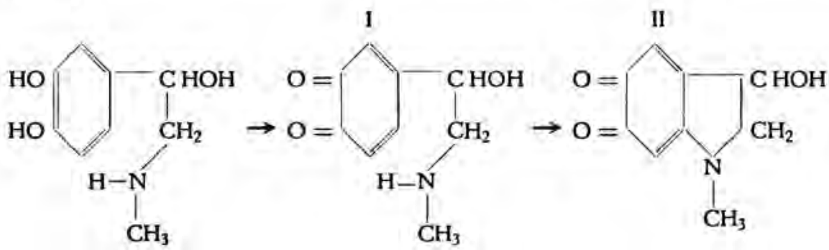
Saturn's ring remained in the planetesimal state as it revolves inside ROCHE's limit. On the other hand the low density of Saturn's satellite Mimas is only compatible with a planetesimal constitution like a comet's head, which proves that the final concentration of one globe from planetesimals is a possibility. Perhaps even most bodies in the solar system passed through the state of a cluster of meteorites. These considerations show that CHAMBERLIN and MOULTON's planetesimal theory may be essentially right, as the theories of DESCARTES, KANT and LAPLACE proved also essentially right.

What happens to the planetesimals, building up comet's heads, when they come too near to the sun and evaporate might be the opposite from the condensational process which once started in colder portions of the nebula.

Histology. — *The influence of adrenochrome on fibroblastic cells in vitro.* By P. J. GAILLARD and W. L. C. VEER [1]). (From the Lab. for exp. Histology and the Pathological Laboratory, State University of Leyden, Holland.)

(Communicated at the meeting of June 26, 1948.)

Adrenochrome is an oxidation product of adrenalin. The first step of this oxidation is a quinone (adrenalin-quinone I) formed by dehydrogenation of two H-atoms. (BALL and CHEN (1) proved the instability of this compound, which has a lifetime of less than one second.) During the next oxidation step adrenochrome (II) is formed, which is a dihydro-indole derivative.



Adrenochrome is made either enzymatically with pyrocatecholoxidase, tyrosinase, phenoloxidase, etc. (2, 3, 4) or chemically with silver oxide (5, 6, 7). It is a water soluble red substance, forming a monoxime (5, 8), a monosemicarbazone (7, 8), and a mono-p.bromophenylhydrazone (5).

GREEN and RICHTER (2) proved that also the cytochromes *a*, *b*, and *c* and the cytochrome-cytochromeoxidase system are able to transform adrenalin into adrenochrome. This makes it highly probable that also in the living organism adrenochrome might be formed. WAJZER (25, 26) even assumes adrenochrome (in protein linkage) to be a normal constituent of muscle.

Local anesthetics (9, 7) and vitamin P-like flavonoles and flavonones (10) can act as inhibitors of the enzymatic transformation process of adrenalin.

According to GREEN and RICHTER adrenochrome can act as a H-carrier in malic acid-, lactic acid- and β-hydroxybutyric acid dehydrogenase systems (see also POTTER (12)). At 20° C and at pH 7 the redox potential is abt. 0.044 V (4).

OVERBEEK and VEER (13) observed that adrenochrome is able to increase the O_2 consumption of mammalian erythrocytes yet not of a haemolysate in the same way as methylene blue. No appreciable increase

[1]) Present address: N.V. Organon, Oss.

of O_2 consumption was found in minced tissues or tissue slices under the influence of adrenochrome unless KCN was added.

MARQUARDT (14) described the influence of ascorbic acid on the transformation of adrenalin into adrenochrome and stressed the importance of the correlation between adrenalin, adrenochrome and ascorbic acid as products of the suprarenal gland. This correlation also seems to be of importance for the pigmentation occurring in ADDISON disease (29). Further, adrenochrome is able to form methaemoglobin from haemoglobin and to transform haemochromogen into verdohaemochromogen, which is an intermediate compound in the breakdown of haemoglobin to the bile pigments (5).

Adrenochrome does not possess a sympathomimetic activity (15, 16, 17) but it is able to decrease blood pressure in experimental hypertension in rats (18).

According to BACQ (16) and HEIRMAN (19) adrenoxin is formed on further oxidation of adrenochrome, PHILPOT (9) denying this process.

DEROUAUX and ROSKAM (20) described that adrenochrome, its monoxime and its monosemicarbazone can act haemostatically.

Capillary resistance increases after administration of adrenochrome, its oxime or iodo-adrenochrome (21).

The deamination of amino acids to imino acids by adrenochrome (22) belongs to the same group of phenomena as the well-known catalytic dehydrogenation of amino acids by quinones.

As to carbohydrate metabolism UTEWSKI (23) believes that adrenochrome is of importance in glycogenolysis. According to RANDALL (24) anaerobic glycolysis of rat brain homogenates is inhibited by adrenochrome. It arrests muscle glycolysis, prevents the formation of hexose diphosphate, and assists in the synthesis of glycogen from Cori-ester (WAJZER (25, 26)).

GREIF and STÖCKLEIN (27) claim that adrenochrome should be beneficial in the maintenance of diabetic patients. According to SNYDER et al. (7), however, adrenochrome and iodoadrenochrome are ineffective in lowering the blood sugar level of normal rabbits, potentiating the effect of insulin or neutralizing the effect of adrenalin (see also (28)).

Furthermore, adrenochrome has to be considered as a precursor of melanins (29).

In this respect COHEN (30) suggested that adrenochrome is first transformed into the 5 · 6-quinone of N-methyl-indoxyl (oxoadrenochrome) from which by means of polymerisation a melanin with an indirubinoid structure results.

As to the possible correlation between melanin-formation and haematopoetic processes VEER and GAILLARD (31) described that adrenochrome is able to increase the migration of bone marrow explants. However, the dose response curves were not of the same type as those obtained with anti-anaemic liver preparations.

Finally LETTRÉ's work (32) on the influence of adrenalin on cultivated fibroblastic explants from the heart anlage of a young chicken embryo must be mentioned. LETTRÉ described a metaphase stop under the influence of adrenalin and he suggested that an oxidation product of adrenalin should be responsible for this effect.

Adrenochrome being an important oxidation product of adrenalin might be of importance in this respect and therefore a number of experiments was performed in order to find out whether this view was correct.

Preliminary experiments with adrenochrome, colchicin and adrenalin.

The experiments were performed with a cell strain of the fibroblastic type derived from the anlage of the frontal bone of a 12 days old chicken embryo. The cultures were made according to the hanging drop method and the medium was composed of one drop of adult hen plasm, one drop of a 50 % press juice prepared from 8 days old chicken embryos and one drop of the test- or control fluid.

I. *Adrenochrome experiment* (27–2–1942).

Several concentrations were used, viz. 1/20, 1/60, 1/600, 1/6000, 1/60000, 1/600000, 1/6000000 and 1/∼ mg per culture. For each of these concentrations two cultures were taken and at the end of a 48 hours' growth period they were fixed with OsO_4 vapour and stained with dilute haematoxylin and eosin. The "migration" of the explants was estimated by means of projection according to EBELING. However, we did not use the planimeter values of the projection figures but the mean values of the radial migration expressed in millimeters. Table I gives the results.

TABLE I.

Concentration	migration
1/20 mg per culture	0.02 mm
1/160	1.03 „
1/600	1.52 „
1/6000	2.07 „
1/60000	1.87 „
1/600000	1.79 „
1/6000000	2.42 „
1/∼	2.34 „

So it appears that by adding 1/20 mg practically no migration occurred. In all other concentrations migration appeared to be possible. Yet, by reducing the amount of adrenochrome the values of the radial migration generally increased until the values obtained in the control-medium (1/∼ mg per culture).

Superficial examination of the mitotic figures led to the conclusion that they were absolutely normal both in appearance and in number, although in the growth zones of the explants cultivated with 1/60 mg of adrenochrome an increased number was likely to be observed.

II. *Colchicin experiment* (7–3–1942).

The same concentrations were used as in the adrenochrome experiment. The mean values of the radial migration measured after 48 hours are given in Table II.

TABLE II.

Concentration	migration
1/20 mg per culture	0.00 mm
1/60	0.05 „
1/600	0.24 „
1/6000	0.14 „
1/60000	0.12 „
1/600000	0.72 „
1/6000000	0.96 „
1/∼	1.18 „

It appears that apart from the concentrations 1/600000 and 1/6000000 the migration was very strongly suppressed. Hand in hand with this phenomenon a great number of typical C-mitoses could be observed.

This was not the case in the explants cultivated with 1/600000 and 1/6000000 mg of colchicin, where migration was not suppressed so severely and only very few mitotic figures could be detected (only part of them with the configuration of the well-known metaphase stop).

III. *Adrenalin experiment* (23–3–1942).

The following concentrations were used: 1 / 200, 1 / 6000, 1 / 60000, 1 / 600000, 1 / 6000000, 1 / 60000000 and 1/∼ mg per culture. The mean values of the radial migration are given in Table III.

TABLE III.

Concentration	migration
1/200 mg per culture	0.31 mm
1/6000	1.12 „
1/60000	1.55 „
1/600000	1.76 „
1/6000000	1.42 „
1/60000000	1.52 „
1/∼	1.31 „

From these figures it is obvious that only 1/200 mg of adrenalin severely suppressed radial migration but just in this concentration some typical metaphase stops could be observed, so substantiating LETTRÉ's results. Yet in all other concentrations the migration was not suppressed, nor did metaphase stops occur.

On the contrary most of the figures suggest a tendency to increase the value of radial migration and together with this phenomenon the mitotic cell divisions appeared fairly normal.

The conclusions from these preliminary experiments are rather simple. All three substances influenced the radial migration of the cultivated cells, though each in its own way. Colchicin suppressed the migration very severely (even by adding 1/60000 mg). Adrenochrome severely suppressed the migration only after the addition of 1/20 mg. However, a slight depressing effect could still be observed in the concentrations 1/60—1/600000 mg incl. Adrenalin suppressed the migration only when added to an amount of 1/200 mg, while the lower concentrations certainly did not act as inhibitors but even tended to increase the migration.

On behalf of the mitotic cell divisions a closer quantitative analysis of the explants used for these experiments seemed not to be justified, because especially after the addition of the higher doses of colchicin and adrenochrome the migration of cells was so strongly suppressed that practically no zones of outgrowth appeared. Therefore it was decided to repeat especially the experiments with colchicin and adrenochrome in such a way that a quantitative analysis became possible. For this purpose the explants were first cultured for about 15 hours without the addition of the substances to be tested. During this time a regular growth zone developed before the virtual beginning of the experiment, which moreover enabled us to judge the quality of the growth, so that only well-grown cultures could be chosen for the experiments. However, the most important advantage of this method is that the substances to be tested are able to intervene immediately in migration and mitosis of the cells. This was not the case in the earlier experiments, because after a complete renewal of the culture medium, invariably a lag period of about 4—6 hours occurs.

As a consequence, however, the substances could not be added in the same way as before yet according to the limits of the hanging drop technique, only by filling up the air space between the coagulum and the glass wall of the depression slide. As a rule, 15 drops had to be used for the purpose.

Migration (the radial migration of the cells per unit of time) and mitoses (the number of mitotic divisions in the growth zones specified according to the different phases) were judged at the end of another 9 hours' period after fixation in OsO_4 vapour and staining with dilute haematoxylin and eosin (according to the limits of this technique an observation time of 48 hours was not possible).

Final experiments with colchicin and adrenochrome.

Colchicin experiment (14–7–1942).

Twenty selected strain cultures were divided into two halves (1a, 1b; 2a, 2b etc.) and cultivated separately. After 15 hours of growth the air spaces of the depression slides were filled up with 15 drops of colchicin in the following concentrations:

	mg per ml		mg per ml
1a 15 drops of solution with	1/20000 [2])	11a 15 drops of solution with	1/5000000
1b	1/20000	11b	1/10000000
2a	1/20000	12a	1/10000000
2b	1/200000	12b	1/25000000
3a	1/20000	13a	1/10000000
3b	1/200000	13b	1/25000000
4a	1/200000	14a	1/25000000
4b	1/400000	14b	1/50000000
5a	1/200000	15a	1/25000000
5b	1/400000	15b	1/50000000
6a	1/400000	16a	1/50000000
6b	1/1000000	16b	1/100000000
7a	1/400000	17a	1/50000000
7b	1/1000000	17b	1/100000000
8a	1/1000000	18a	1/100000000
8b	1/5000000	18b	1/∼
9a	1/1000000	19a	1/100000000
9b	1/5000000	19b	1/∼
10a	1/5000000	20a	1/∼
10b	1/10000000	20b	1/∼

In this experiment radial migration appeared to be completely suppressed in the concentrations 1/20000—1/25000000 mg per ml. Even after the addition of 15 drops of solution with 1/50000000 or 1/100000000 mg per ml a pronounced decrease of radial migration as compared with the migration of the control cultures could be observed (abt. 50 % of the control values).

In the preliminary experiment migration was not influenced so severely. This difference might be due to the physiological properties of the explants.

Ordinarily after a lag period of 4—6 hours migration starts with maximum activity. This activity slowly decreases and at the end of some 3—5 days migration has fallen to zero.

These facts make it clear that suppression of migration is most difficult in the beginning of the growth period (preliminary experiment) and becomes more and more easy as the growth period proceeds (final experiment).

The mitotic cell divisions were counted with the help of an ocular counting grid. The mean numbers of mitotic cells per counting grid and per culture were estimated in all explants and the final values are summarized in Table IV.

Obviously in all concentrations an increase of the number of mitoses could be observed. For the concentration 1/100000000 mg/ml this increase is not important but in all other concentrations the increase varied between 500 % and 3500 % as compared with the control cultures. Moreover, nearly, though not absolutely, all mitotic figures showed the typical col-

[2]) Consequently the real amount per culture is approximately

$$\frac{15}{20} \times \frac{1}{20000} = \frac{1}{26667} \text{ mg etc.}$$

chicin character (whimsical shape of the cells, degeneration of the cytoplasm and clumping of the chromosomes in metaphase situation).

TABLE IV.

Concentration	mean number of mitoses per counting grid	mean number of mitoses per culture
1/20000 mg/ml	6.1	743
1/200000	5.3	587
1/400000	7.1	792
1/1000000	8.8	1106
1/5000000	8.7	1321
1/10000000	5.1	608
1/25000000	10.6	1498
1/50000000	1.7	268
1/100000000	0.5	73
1/∼	0.3	47

From a theoretical point of view it is interesting to note that suppression of radial migration and the occurrence of typical colchicin mitoses go hand in hand, which seems to indicate that also cytoplasmic processes are involved in the attack of colchicin on the cells.

Adrenochrome experiments (28–11–1942; 3–12–1942).

Both experiments were performed in an analogous way as the one with colchicin. Only were some higher concentrations added and some lower concentrations omitted. The adrenochrome solutions were added to the selected strain cultures after 18, resp. 17 hours of growth. The test or control fluid remained in the air space for 9 hours successively, after which the explants were fixed, stained and studied.

First experiment (28–11–1942).

Table V gives the mean values of the radial migration. M 1 is the mean value of the radial migration in mm during the 18 hours preceding the addition of adrenochrome. M 2 is the mean value of the radial migration in mm during the 9 hours after the addition of adrenochrome.

TABLE V.

Concentration	M 1	M 2
1/1000 mg/ml	0.62	0.28
1/5000	0.76	0.17
1/10000	0.69	0.24
1/20000	0.69	0.21
1/50000	0.66	0.14
1/100000	0.55	0.21
1/500000	0.55	0.24
1/1000000	0.59	0.21
1/10000000	0.58	0.14
1/25000000	0.52	0.14
1/∼	0.55	0.14

From this table it follows that none of the concentrations used led to a decrease of the migration. On the contrary, there is a tendency to increase the migration, especially in the higher concentrations.

Histologically the mitotic figures, though few in number, appeared to be quite normal. However, in this series a detailed quantitative analysis still could not be realized because of the fact that the values of M_1 varied too much, which indicated that the explants were not in an optimal condition for this kind of experiments. Therefore a second, analogous, experiment was carried out.

Second experiment (3–12–1942).

Table VI summarizes the results of measuring the radial migration before and after the addition of adrenochrome.

TABLE VI.

Concentration	M 1	M 2
1/50 mg/ml	0.59	0.12
1/100	0.52	0.07
1/1000	0.50	0.24
1/5000	0.55	0.21
1/10000	0.49	0.15
1/20000	0.47	0.19
1/50000	0.48	0.21
1/100000	0.59	0.16
1/500000	0.48	0.14
1/1000000	0.45	0.14
1/∼	0.47	0.09

The conclusion is that apart from the concentration 1/100 mg/ml no decrease of migration in relation to the controls (1/∼ mg/ml) occurred. On the contrary also in this experiment there appeared to be a marked tendency to increase migration in most of the concentrations used. This is in accordance with the results of the preceding experiment but, it must be mentioned here, not with those of the preliminary one of 27–2–1942. Possibly this difference can be explained either by the differences of the experimental procedure or by the characteristics of the adrenochrome itself. In the preliminary experiment the adrenochrome was added directly to the culture medium, that is to say at the very beginning of the growth period and before the usual lag period of about 4—6 hours. During this time, in which no signs of virtual growth can be observed, the relatively instable adrenochrome might have lost most of its characteristic activity so that it was no longer available in sufficient amounts for influencing the process of migration which started after that time. In both the final experiments the adrenochrome was added *during* the period of virtual growth and consequently should have been able to act immediately.

On the other hand the same facts as mentioned above (cp page 812) may have been of importance. Stimulation of migration is likely to be more difficult during a period of already maximum migration (preliminary experiment) than during a period of lessened migration activity (final experiment).

As to the mitotic cell divisions in all explants except those cultivated in 1/50 mg/ml, which partly liquefied the culture medium, the total number of mitoses in the growth zones was counted. The mean values per counting grid and per culture are mentioned in table VII.

TABLE VII.

Concentration	mean value of the number of mitoses per counting grid	mean value of the number of mitoses per culture
1/100 mg/ml	0.15	24.5
1/1000	0.34	84.5
1/5000	0.29	70.0
1/10000	0.44	75.5
1/20000	0.59	106.0
1/50000	0.63	119.0
1/100000	0.67	162.0
1/500000	0.57	113.5
1/1000000	0.69	68.0
1/∼	0.67	107.5

Obviously, apart from the concentration 1/100000 mg/ml not a single of the experimental cultures did show an increase of any importance of the number of mitoses per culture. Apart from the concentration 1/100000 mg/ml this result seems to indicate that the phenomenon of mitose stase does not occur unless such a stase should be masked by a strong decrease in the number of cells entering mitosis per unit of time.

Therefore and because of the deviating result in culturing with 1/100000 mg of adrenochrome per ml the numbers of the different mitotic stages were separately estimated and in table VIII the mean values of the quotient:

$$\frac{\text{Number of cells in metaphase}}{\text{Number of cells in pro-, ana- and telophase}} \text{ are given.}$$

As soon as a real metaphase stop should exist the value of the quotient as compared with the control values should have been forced to increase.

According to table VIII there are some deviations in indicating a relative increase of the number of metaphase stages. Again the concentration 1/100000 gives the highest deviation and so in this concentration a moderate metaphase stase seems likely to occur. On the other hand detailed microscopic observation led to the conclusion that the morphological aspects of the mitotic figures were quite normal. Moreover adrenochrome in contrast with colchicin appears to influence migration and mitosis independently and so it was concluded that adrenochrome does not act as a karyotoxic substance in the sense of colchicin.

However, the results (Table VII) do indicate that adrenochrome is able to suppress (in higher concentrations) the number of cells entering mitosis, and perhaps it is able, at least in one of the concentrations used (1/100000), to prolong the duration of the metaphase.

TABLE VIII.

Concentration	$\frac{\text{number of cells in metaphase}}{\text{number of cells in pro-, ana- and telophase}}$
1/100 mg/ml	0.89
1/1000	0.60
1/5000	1.08
1/10000	1.17
1/20000	0.99
1/50000	0.80
1/100000	1.21
1/500000	1.16
1/1000000	1.17
1/~	0.98

These conclusions seem to be obvious, although a stimulation, if any, of the number of cells entering mitosis per unit of time combined with a proportional shortening of the different stages of the mitotic process may have been overlooked. Such phenomena could have been analyzed only by means of cinemicrographic methods or by a fractionated photographic analysis, which we were not able to perform at the time the experiments were done.

Summary.

The effects of adrenochrome on the migration of the cells and on the process of mitotic cell division in explanted tissues of the fibroblastic type were studied. Apart from the highest concentrations used adrenochrome tended to *increase the radial* migration of the cultivated cells.

As to the process of mitotic cell division adrenochrome was *able to suppress the number of cells entering mitosis* in many of the concentrations used.

This phenomenon lessened with the decrease of the concentration added and showed no relations to the effects on the process of cell migration.

Microscopic examination revealed that the *aspects of the mitotic cells were absolutely normal* so that *adrenochrome certainly* does *not* act as a *karyotoxic substance* in the sense of colchicin. However, numerical evaluation of the pro-, meta-, ana- and telophase stages suggested that in one concentration adrenochrome was *able to prolong the duration of the metaphase* as compared with the duration of the other phases.

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Zoology. — *Specific characters in Millepora*. By H. BOSCHMA.

(Communicated at the meeting of May 29, 1948.)

In the course of time 56 different names have been used to indicate forms of the Hydrocoralline genus *Millepora*, which were considered as distinct species or varieties. Especially DUCHASSAING (1850) and DUCHASSAING & MICHELOTTI (1860, 1864) regarded specimens of a growth form slightly differing from that of previously described species as separate entities, they gave new specific names to 16 West Indian forms of *Millepora*.

Several authors commented upon the extreme variation of *Millepora* as a result of external influences. DANA (1848, p. 543) already remarked: "There is much difficulty in characterizing the Millepores, on account of the variations of form a species undergoes, and the absence of any good distinctions in the cells." It is indeed extremely difficult to find characters for specific distinction, and HICKSON (1898 *a*, *b*, 1899), who made elaborate studies concerning the species problem in *Millepora*, came to the conclusion that all the various forms are nothing else but results of the extreme possibilities for variation in the one species *Millepora alcicornis* L. HICKSON's studies were based on an extensive material of colonies of *Millepora* from various parts of the world. He tried to find characters for specific distinction in (1) the form of the corallum, (2) the size of the pores, (3) the degree of isolation of the cycles, (4) the presence or absence of ampullae, (5) the texture of the surface of the corallum, (6) the relative number of dactylopores and gastropores, (7) the anatomy of the soft parts. HICKSON failed to find distinct specific characters, and believed that the different growth forms were brought about by the various conditions of existence on the reefs only.

HICKSON's conclusions largely influenced the opinion of nearly all later investigators who published upon material belonging to *Millepora*. Almost invariably these authors remarked that the species problem in *Millepora* had been solved by HICKSON, and they identified their material as *Millepora alcicornis* L., though sometimes adding that the growth form corresponded with that of a certain form previously indicated with a separate name. Three authors expressing views different from those defended by HICKSON may be mentioned here. GRAVIER (1911) did not mention HICKSON's results and identified his material as *Millepora dichotoma* Forsk., whilst THIEL (1933) used the name *Millepora intricata* M. E. & H. for a colony of a delicately branched growth form which according to that author cannot easily be explained solely on account of influences of external conditions. CROSSLAND, who in a number of earlier papers had adopted HICKSON's views regarding the specific unity of all the forms of *Millepora*, in a recent publication (CROSSLAND, 1941) stated that in the Red Sea there occur three well defined species of the genus. The latter statement completely agrees with KLUNZINGER's (1879) results, who in an excellent manner described the specific characters of the three Red Sea forms, and gave the exact data of the synonymy of these three species, viz., *M. exaesa* Forsk., *M. dichotoma* Forsk., and *M. platyphylla* Hempr. & Ehrb.

In the last publication in which he gives his opinion on the variability of *Millepora*, HICKSON (1924, p. 145) writes: "The corallum assumes many variations of form. Sometimes it consists of thick massive plates, sometimes it is coarsely branched or becomes profusely ramified. These differences in form seem to be associated with differences of the immediate environment and cannot be used as characters for specific distinctions."

If this statement were right, colonies of *Millepora* living under the same conditions of existence always would show a highly similar growth form. On the reefs, however, often colonies of a strikingly dissimilar growth form may be found growing on the same spot, under exactly the same environmental factors. An instance of this kind is shown by the compound figured on the plate accompanying the present paper. The figure represents the region of contact of two colonies of *Millepora* growing side by side on the surf-swept edge of the reef of the island Edam in the Bay of Batavia. Each of the colonies with its broadest side was exposed to the full action of the waves, as they occupied the extreme seaward border of the reef. Consequently the external circumstances of the two colonies (depth of water, chemical and physical conditions of the water, exposure to wave action, lack of encumbrance by other colonies of corals) were exactly the same. In their growth form, however, the two colonies present striking differences. The one consists of vertically extending massive thick plates which among each other are combined into a more or less honeycombed mass, it is a typical representative of the species *Millepora platyphylla* Hempr. & Ehrb. The other colony in all essential characters corresponds with *Millepora murrayi* Quelch, it consists of a crowded mass of rather delicate branchlets. In their region of contact the heavier and apparently stronger colony has partly overgrown the more delicate specimen. The two colonies in their growth form are as unlike as possible, and these differences necessarily must be due to specific peculiarities, as the factors of the environment for each of the two colonies were exactly the same.

When corals of the genus *Millepora* are examined on the reefs, and attention is given to the conditions of the environment, it appears that, though really the colonies are variable to a considerable degree, this variation is restricted to certain limits. Colonies of a species which usually occurs as a decidedly ramified form under unfavourable conditions may remain rather compact and develop short stunted branches only. On the other hand colonies of a species which in its typical form grows out to thick and broad upstanding plates in an unsuitable locality may develop into a more or less branching form. But these deviations from the typical form never become so pronounced that there might be found a continuous series of growth forms connecting the one extreme with the other. There are a number of distinct species in the genus *Millepora*, each of which is characterized by definite peculiarities which remain obvious even if the specimens are living under unfavourable conditions. An exception form, however, the incrusting growths of *Millepora*, which by covering all kinds of objects with a thin layer of corallum assume the form of their substratum. These incrusting specimens as a rule do not show sufficient characters for a specific identification.

All more or less full grown colonies of *Millepora* can be arranged in groups showing characters of sufficient value to regard these groups as distinct species. The chief characters are those of the growth form, which, however, as a result of the pronounced variability, are not always easily to be defined.

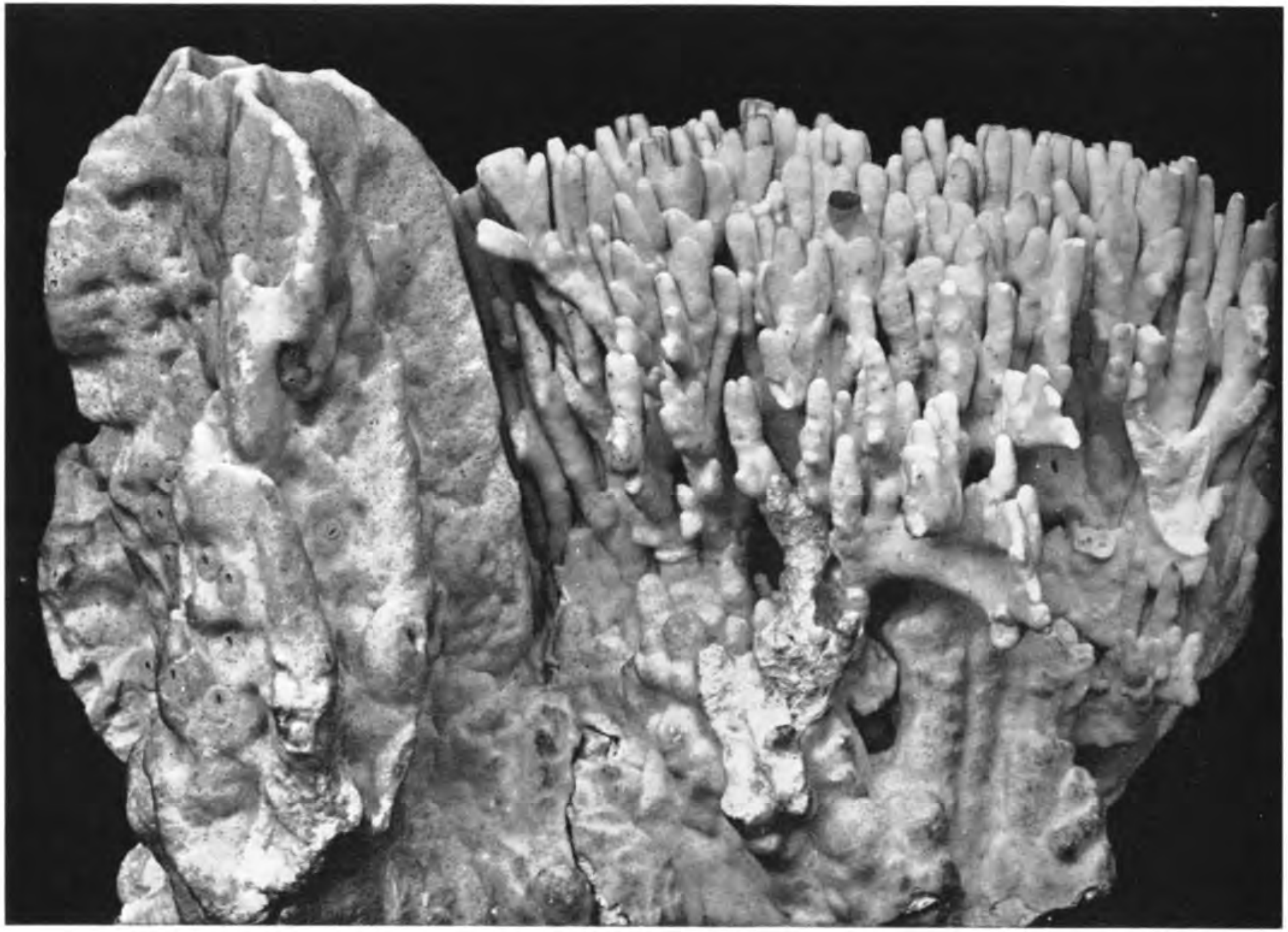
Besides the form of the corallum some of the other points in HICKSON's attempts to find specific characters may yield results of definite value, though often in a restricted sense.

If we take the extremes there is a difference in the size of the pores in the various species of *Millepora*. In *M. platyphylla* Hempr. & Ehrb. the pores invariably are large, whilst in *M. intricata* M. E. they always are small and often even difficult to detect. But in other species the size of the pores in different colonies may vary to a considerable degree, and even in parts of the same colony there may be a noticeable difference in the size of the pores.

The degree of isolation of the cycles in no manner can be used as a specific character. As HICKSON correctly observed the isolation of the cycles is highly different in various parts of one colony. It depends upon the rapidity of growth and the amount of light falling on the various parts of the corallum.

The presence or absence of ampullae cannot constitute a specific character. As a matter of fact ampullae form a part of the generative cycle of each colony, and they may be found, at least in certain parts of the year, in each specimen. On the other hand the shape of the ampullae, and especially that of the calcareous covering of these cavities, which is present before the medusae are set free, may be different in the various species, and for this reason may yield characters for a specific distinction. The available data as yet are too scanty for a decision in this respect.

The structure of the surface of the corallum in certain cases is of importance as a specific character. Many colonies of the West Indian species, *M. alcicornis* L., *M. squarrosa* Lamk., and *M. complanata* Lamk., have the surface profusely pitted with insignificant depressions in the centres of which there are the gastropores. But in other colonies of the same species this feature does not occur, so that then the surface of the corallum is even. It is interesting that these small depressions are found



The region of contact of two colonies of *Millepora, M. platyphylla* Hempr. & Ehrb. (left) and *M. murrayi* Quelch (right). The lower part of the colony of *M. murrayi* is covered with a thin layer of *M. platyphylla.* From the edge of the reef of the island Edam in the Bay of Batavia. Natural size.

in the three species only, they never occur in Indopacific species of the genus.

On the other hand many colonies of *M. platyphylla* Hempr. & Ehrb. have large parts of the surface covered with small warty excrescences on the tops of which the gastropores are found. Here again this character is not constant, as it may occur in a part of certain colonies only, and in other colonies of the same species it does not appear at all. But the occurrence of these warts is never observed in specimens belonging to other species, so that it, when occurring, is a means for identification of the specimen.

The relative number of dactylopores and gastropores does not furnish a character for specific distinction. This number to a certain degree is different in various parts of the same colony, it is highly dependent upon external conditions. Parts of vigorous growth generally show a larger amount of pores than regions of slow growth, and in various parts of one colony the relative number of dactylopores which surround each gastropore is rather variable.

The anatomy of the soft parts as yet has not given indications for specific distinctions. All the species examined in this respect have two kinds of nematocysts, larger and smaller. In the species *M. alcicornis* L., *M. complanata* Lamk., *M. platyphylla* Hempr. & Ehrb., *M. exaesa* Forsk., *M. tenella* Ortm., and *M. murrayi* Quelch these larger nematocysts have exactly the same shape, whilst their size is subject to slight variation only (length 25—30 μ). More detailed studies of the nematocysts in well preserved material perhaps may show that also here there are specific differences.

After having examined a fairly large amount of specimens from the reefs of islands in the Java Sea, and after careful studies on the material of the museums in Paris, Leiden, and Amsterdam, I came to the conclusion that at least nine species can be distinguished, each of which is characterized by distinct specific peculiarities. Moreover it appeared that almost all the forms described in previous literature could be identified as synonyms of the recognized species. A short summary of the species and their characters follows here.

Millepora alcicornis Linnaeus, 1758. Synonyms: *Millepora ramosa* Dana, 1848 (*Millepora alcicornis* var. *ramosa* Esper, 1790); *Millepora pumila* Dana, 1848; *Millepora moniliformis* Dana, 1848 (*Millepora alcicornis* var. *crustacea* Esper, 1790). Corallum of extremely variable form, consisting of usually flattened branches which may largely unite to form upstanding plates. The branches may, however, spread in all directions, or more or less solid masses may develop by pronounced fusion of the branches.

Millepora exaesa Forskål, 1775. Synonyms: *Millepora tuberculosa* Milne Edwards, 1857; *Millepora gonagra* Milne Edwards, 1860; *Millepora nodosa* Moseley, 1879 (*Millepora alcicornis* var. *nodosa* Esper, 1790). Corallum consisting of more or less rounded masses with a profusion of short and thick, knob-like branches.

Millepora dichotoma Forskål, 1775. Synonym: *Millepora reticularis* Milne Edwards, 1860. Corallum consisting of round branches which have a strong tendency to unite so as to form upstanding plates of a pronouncedly reticular character.

Millepora squarrosa Lamarck, 1816. Synonym: (?) *Millepora foliata* Milne Edwards, 1860. Corallum forming broad upstanding plates with thin edges, the surface covered with irregular ridges and tubercles, producing a frilled appearance of the colony.

Millepora complanata Lamarck, 1816. Synonym: *Millepora plicata* Dana, 1848 (*Millepora alcicornis* var. *plicata* Esper, 1790). Corallum consisting of flat upstanding plates with thin edges.

Millepora platyphylla Hemprich & Ehrenberg, 1834. Synonyms: *Millepora verrucosa* Milne Edwards, 1857; *Millepora incrassata* Milne Edwards, 1860 (*Millepora squarrosa* var. *incrassata* Dana, 1848); *Millepora ehrenbergi* Milne Edwards, 1860; *Millepora truncata* Vaughan, 1918 (*Millepora platyphylla* var. *truncata* Dana, 1848). Corallum forming thick upstanding plates with blunt edges. The plates have a tendency to unite in honeycombed masses.

Millepora intricata Milne Edwards, 1857. Corallum consisting of a complicated mass of thin branches spreading in all directions, and which are largely anastomosing among each other.

Millepora murrayi Quelch, 1884. Synonym: *Millepora confertissima* Quelch, 1886. Corallum branching in the shape of ogives with numerous vertical smaller branchlets, often forming dense masses.

Millepora tenella Ortmann, 1892. Synonym: *Millepora tortuosa* Dana, 1848 (the latter name is preoccupied by *Millepora tortuosa* Esper, 1790, which belongs to the coralline algae). Corallum rather spreadingly branching, each larger branch with numerous smaller branches in the same plane, these smaller branches rather distant from each other.

Of these nine species *M. alcicornis*, *M. squarrosa*, and *M. complanata* are restricted to the Atlantic region of America. *M. exaesa*, *M. dichotoma*, and *M. platyphylla* occur in the Red Sea and in other parts of the Indopacific region. *M. intricata*, *M. murrayi*, and *M. tenella* are living in the Indopacific region, but are not found in the Red Sea.

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Medicine. — *Determination of total base by exchange of ions.* By E. GORTER, A. VAN ROYEN (in collaboration with Miss A. KION, analyst).

(Communicated at the meeting of June 26, 1948.)

The methods accepted up to now for the determination of the total base (i.e. the sum of the metallic ions in blood serum), by electrolysis [1]), had certain disadvantages. The cations were separated by a mercury cathode, forming an amalgam, which could be broken down by a known amount of HCl. Besides the length of time needed for the analysis (about 2 hours), the chief difficulty was with the cellophane membranes, which would begin to leak at the most unexpected moments, giving too low a result.

We have succeeded in replacing this method by a much quicker one, in which use is made of a cation exchanger.

An ion exchanger may be defined as a solid, insoluble acid, of which the salts are also insoluble. There are many of these on the market. Most are synthetic resins, containing acid radicals. We use Dusarit, a sulphonated coal [2]).

Principle.

If a salt solution (e.g. NaCl) is passed through a filter bed of Dusarit, the $Na^{\cdot}$-ions are exchanged with the $H^{\cdot}$-ions, so that there remains as much HCl in the filtrate as there was NaCl passed through it.

When the Dusarit is saturated with $Na^{\cdot}$-ions, it can be regenerated by passage of an acid (e.g. HCl) over it, so that the $Na^{\cdot}$-ions that were combined with the Dusarit are replaced by the $H^{\cdot}$-ions.

The reaction may be represented:

$$DH + Na^{\cdot} \rightleftharpoons DNa + H^{\cdot}.$$

The same holds for other metallic ions. The amount of $H^{\cdot}$-ion found in the filtrate will be equivalent to the filtered cation and can be determined by titration.

Requirements.

1 Dusarit — Commercial Dusarit contains iron, calcium, sodium and other metals, and must be freed of them before use. This can be done by boiling for ten minutes in a glass

[1]) N. R. JOSEPH and W. C. STADIE, J. Biol. Chem. **125**, 795 (1938).

[2]) Dusarit is made by N.V. Activit, Amsterdam. — Fa. REINEVELD, Delft also have ion exchangers on the market.

	beaker, with continuous stirring. 100 grammes of Dusarit would take about 100 cc 20 % HCl. Pour out and repeat the process. Then wash off the HCl with distilled water. Dusarit is best kept under water, in a widemouthed stoppered bottle.
2 HCl 1n	for regeneration of Dusarit after use.
3 HCl 0,01n	for titration.
4 NaOH 0,1n	carbonate-free, titrated with oxalic acid (CO_2 free solution, with bromothymol blue as indicator, oxalic acid in the burette).
5 CO_2-free water	Distilled water, boiled for ten minutes, and cooled under air tight conditions.
6 Oxalic acid	for analysis.
7 Bromothymol blue	as indicator.

Procedure.

An apparatus is set up as shown in the illustration, with a layer of Dusarit about 10 cm high, care being taken that no air bubbles are left

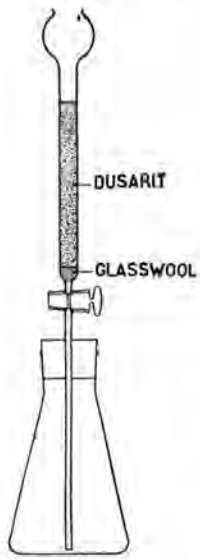

Fig. 1. Apparatus for estimation of total base.

in the Dusarit. For filling the apparatus it is best to mix the Dusarit with plenty of water. In all work with Dusarit, distilled water must be used: with tap water, the $Na^{\cdot}$- and $Ca^{\cdot\cdot}$-ions are taken up by the Dusarit, so that its capacity is lessened to an unknown extent. The Dusarit is supported on a layer of glass wool.

After filling the apparatus, and before every analysis, the Dusarit must be freed of traces of acid: 50 ml of water is enough for this. The water is allowed to run out until its level has sunk to the upper level of the Dusarit.

2 ml 0,1n NaOH is now added to the flask below, care being taken that the outlet of the funnel projects into the fluid.

Then 0,5 ml serum is added to the funnel: on opening the tap, it is drawn into the Dusarit layer. 50 ml CO_2 free water is added slowly, to wash out the acid formed, so that the Dusarit is not left dry. Total filtration time 3 minutes.

The excess NaOH is then titrated back, with freshly made 0,01 n HCl (diluted from 0,1 n HCl from the stock bottle), with B.T.B. as indicator. The acid had previously been titrated with 2 ml 0,1 n NaOH, with the same indicator.

Before or after the determination, a control experiment is done, with the same apparatus, in which only 50 ml CO_2 free water is passed through the Dusarit and received in 2 ml 0,1 n NaOH. The Dusarit soon gives off traces of acid, which must be taken into account when dealing with such small quantities.

The end-point of the titration is a blue-green colour (pH 7,4). There must not be too much shaking during the titration, so as not to dissolve a noticeable amount of CO_2 in the NaOH. If the proceeding is carried out quickly and carefully, a closed apparatus to prevent the entry of CO_2 is unnecessary.

After about ten analyses, the Dusarit becomes saturated with cations and must be regenerated. About 10 ml n HCl is slowly passed over the filter, and is then washed out with distilled water. The filter is then ready for the next estimation.

Calculation.

Suppose the concentration of HCl is y: in the control a ml HCl were used, and in the estimation itself b ml HCl, then (a—b) y m. equiv. NaOH were removed by the acid formed in the filter.

The serum, therefore, has 2000 (a—b) y m. equiv. cations per l.

Discussion.

In theory, one should titrate back the NaOH until the pH is that of the serum: this would be less than 7,4 in acidotic patients, and more in alkalotic. In practice, however, it makes no difference, as long as the same endpoint (7,4) is taken.

For the pH of serum, the formula of HASSELBALCH is used:

$$pH = 6{,}1 + \log \frac{NaHCO_3}{CO_2}$$

The ratio $NaHCO_3 : CO_2$ is $\pm 20 : 1$ at pH 7,4, while the $NaHCO_3$ concentration is $\pm$ 30 m. equiv. and the CO_2 concentration is therefore 1,5 m. equiv.

With a lowering of the pH to 7,1, this ratio must be $\pm$ 10 : i.e., there is 28,5 m. equiv. $NaHCO_3$ and $\pm$ 3 m. mol CO_2. Therefore, in titrating up to pH 7,1 instead of up to pH 7,4, there is a difference of 1,5 m. mol CO_2. In a total of $\pm$ 155 m. equiv. total base, this may be neglected, the accuracy not being greater than 2 m. equiv. (0,1 ml 0,01 n HCl). The electrolytic method has the same degree of accuracy.

Results.

A number of sera were examined by both methods, and the following figures were obtained:

Serum	m.equiv. NaCl	Resultant m.equiv. total base	
		Electrolysis	Ion exchange
—	100	97	97
—	100	97	96
—	100	97	97
Horse serum	—	177	176
„ „	—	179	178
Cow serum	—	172	173
„ „	—	172	172
„ „	—	168	169
„ „	—	168	170
Diluted serum	—	85	—
„ „	50	135	138
Cerebrospinal fluid	—	—	146
„ „	100	—	247

Geophysics. — *On the large displacements commonly regarded as caused by Love-waves and similar dispersive surface-waves.* III. By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS JR.)

(Communicated at the meeting of June 26, 1948.)

IV. The motion in an ocean caused by a subaqueous explosion.

§ 1. *The direct and the reflected wave.*

While in chapter II a purely transverse motion was examined we shall now consider the purely longitudinal movement propagated in a liquid layer. We again assume that the primary disturbance takes place at the time $t = 0$ in a point (at the depth f) of the layer and that the layer rests on a semi-infinite body.

The motion in the liquid (horizontal component U; vertical: W) satisfies:

$$U = \frac{\partial G}{\partial \varrho},\ W = \frac{\partial G}{\partial z} \text{ and } \sigma \frac{\partial^2 G}{\partial t^2} = \lambda \triangle G;$$

in the underlying medium the movement is partly transverse:

$$U' = \frac{\partial G'}{\partial \varrho} - \frac{\partial H'}{\partial z},\ W' = \frac{\partial G'}{\partial z} + \left(\frac{\partial H'}{\partial \varrho} + \frac{H'}{\varrho}\right), \quad \text{and}$$

$$\sigma' \frac{\partial^2 G'}{\partial t^2} = (\lambda' + 2\mu') \triangle G', \qquad \sigma' \frac{\partial^2}{\partial t^2}\left\{\frac{1}{\varrho}\frac{\partial(\varrho H')}{\partial \varrho}\right\} = \mu' \triangle \left\{\frac{1}{\varrho}\frac{\partial(\varrho H')}{\partial \varrho}\right\}.$$

We start with the movement which shows a discontinuity of W at the plane $z = f$:

$$W(z = f - 0) - W(z = f + 0) = E\, e^{\nu t} I_0(\nu \xi \varrho)$$

where E, ν and ξ are real constants. The motion is:

$$G = (A_1 e^{-\nu a z} + A_2 e^{\nu a z})\, e^{\nu t} I_0(\nu \xi \varrho), \text{ at } 0 < z < f.$$

$$G = (A_3 e^{-\nu a z} + A_4 e^{\nu a z})\, e^{\nu t} I_0(\nu \xi \varrho), \text{ ,, } f < z < d.$$

$$G' = A' e^{-\nu a' z + \nu t} I_0(\nu \xi \varrho) \text{ and } H' = B' e^{-\nu b' z + \nu t} I_1(\nu \xi \varrho) \text{ at } z > d,$$

with

$$a = \sqrt{\xi^2 + \frac{1}{V^2}},\ a' = \sqrt{\xi^2 + \frac{1}{V'^2}},\ b' = \sqrt{\xi^2 + \frac{1}{\mathfrak{B}'^2}}.$$

The amplitudes A_i, A' and B' are determined by the boundary conditions: at $z = 0$ the tension $= 0 : A_1 + A_2 = 0.$
at $z = d$ the normal component of the movement and the tension is continuous:

$$a A_3 e^{\nu a d} - a A_4 e^{-\nu a d} = - a' A' e^{-\nu a' d} + \xi B' e^{-\nu b' d}$$

$$\sigma(A_3 e^{\nu a d} + A_4 e^{-\nu a d}) = 2\mu'(c' A' e^{-\nu a' d} - b' \xi B' e^{-\nu b' d}), \text{ with } c' = \xi^2 + \frac{1}{2\mathfrak{B}'^2}.$$

and the tangential stress is equal to zero:

$$a'\,\xi\,A'\,e^{-\nu a'd} - c'\,B'\,e^{-\nu b'd} = 0.$$

Finally at $z = f$ the tension is continuous:

$$A_1\,e^{\nu af} + A_2\,e^{-\nu af} = A_3\,e^{\nu af} + A_4\,e^{-\nu af}$$

and W is discontinuous:

$$\nu a\,A_1\,e^{\nu af} - \nu a\,A_2\,e^{-\nu af} = \nu a\,A_3\,e^{\nu af} - \nu a\,A_4\,e^{-\nu af} + E.$$

The solution of these equations is

$$A_1 = -A_2 = \frac{E}{2a\nu}\,e^{-\nu af} + A_3 \qquad A' = \frac{E}{2a\nu}\cdot K_{ll}\,e^{-\nu(a-a')d}\,\frac{e^{\nu af}-e^{-\nu af}}{1+D_l\,e^{-2\nu ad}}$$

$$A_4 = \frac{E}{2a\nu}\,e^{\nu af} - \frac{E}{2a\nu}\,e^{-\nu af} - A_3 \qquad B' = \frac{E}{2a\nu}\cdot K_{lt}\,e^{-\nu(a-b')d}\,\frac{e^{\nu af}-e^{-\nu af}}{1+D_l\,e^{-2\nu ad}}$$

$$A_3 = \frac{E}{2a\nu}\cdot D_l\,e^{-2\nu ad}\,\frac{e^{\nu af}-e^{-\nu af}}{1+D_l\,e^{-2\nu ad}}$$

with

$$D_{ll} = \frac{4\sigma'\mathfrak{B}'^4\,a\,(c'^2 - a'b'\xi^2) - \sigma a'}{4\sigma'\mathfrak{B}'^4\,a\,(c'^2 - a'b'\xi^2) + \sigma a'},\quad K_{ll} = \frac{4\sigma\mathfrak{B}'^2\,a\,c'}{4\sigma'\mathfrak{B}'^4\,a\,(c'^2 - a'b'\xi^2) + \sigma a'},$$

$$K_{lt} = \frac{4\sigma\mathfrak{B}'^2\,a\,a'\,\xi}{4\sigma'\mathfrak{B}'^4\,a\,(c'^2 - a'b'\xi^2) + \sigma a'}$$

which are the coefficients of reflection (D) and refraction (K) at the interface when both media are infinite ($d = \infty$).

The Rayleigh function $c'^2 - a'b'\xi^2$ is positive for every real value of ξ and ν; the absolute value of D_l is therefore < 1 and the function $1 + D_l e^{-2\nu ad}$ can be developed into the uniformly convergent series $\sum_0^\infty (-D_l e^{-2\nu ad})^n$.

This movement with a discontinuity in all points of the plane $z = f$ can be changed into a movement which is only discontinuous in the point $z = f$, $\varrho = 0$ in the following way: the only parts of G and H which are entirely independent of the boundaries of the layer are

$$\frac{E}{2a\nu}\,I_0(\nu\xi\varrho)\cdot e^{\nu t - \nu a(f-z)},\ z < f \ \text{and}\ \frac{E}{2a\nu}\,I_0(\nu\xi\varrho)\cdot e^{\nu t-\nu a(z-f)},\ f < z < d.$$

These expressions represent therefore the motion which is directly propagated from the plane $z = f$ to the point ϱ, z. Applying the operator $\int_0^\infty 2\nu\xi^2 d\xi$ to these expressions the direct waves are transformed into

$$E\,e^{\nu t}\int_0^\infty \frac{\nu\xi}{\nu a}\,e^{-\nu a|f-z|}\,I_0(\nu\xi\varrho)\,d(\nu\xi),\ \text{ or: }\ E\,\frac{e^{\nu(t-r/V)}}{r}\ \text{ with }\ r^2 = \varrho^2 + (z-f)^2$$

which is a spherical wave originating at the point source $z = f$, $\varrho = 0$.

The movement in the layer caused by this wave is obtained by applying this operator to the above results:

$$G_1 = E\frac{e^{\nu(t-r/V)}}{r} + G_3(-z), \quad G_2 = -E\frac{e^{\nu(t-r_0/V)}}{r_0} - G_3(+z) \quad \text{if} \quad z < f,$$

$$G_4 = E\frac{e^{\nu(t-r/V)}}{r} - E\frac{e^{\nu(t-r_0/V)}}{r_0} - G_3(+z), \; f < z < d, \text{ with } r_0^2 = \varrho^2 + (z+f)^2$$

$$G_3(-z) = \nu E e^{\nu t} \sum_{n=1}^{\infty} \left\{ \int_0^\infty e^{-\nu a(2nd+f-z)} I_0(\nu \xi \varrho) \frac{(-D_l)^n}{a} d\xi - \right.$$
$$\left. - \int_0^\infty e^{-\nu a(2nd-f-z)} I_0(\nu \xi \varrho) \frac{(-D_l)^n}{a} d\xi \right\}.$$

Finally we change the time factor $e^{\nu t}$ into a function $F(t)$ by means of the inverse theorem; if $L(\varrho, z, u)$ is the Laplace function of $G_3 e^{-\nu t}$:

$$G_3 e^{-\nu t} = \nu \int_0^\infty e^{-\nu u} L(\varrho, z, u)\, du$$

then the movement caused by the disturbance $F(t)$ is determined by the following G-functions:

$$G_1 = E\frac{F(t-r/V)}{r} + G_3(-z), \quad G_2 = -E\frac{F(t-r_0/V)}{r_0} - G_3(+z), \quad (z<f)$$

$$G_4 = E\frac{F(t-r/V)}{r} - E\frac{F(t-r_0/V)}{r_0} - G_3(+z), \quad (f < z < d)$$

and

$$G_3 = \int_0^t \frac{\partial}{\partial t} F(t-u) \cdot L(\varrho, z, u)\, du.$$

The first parts of G_1 and G_4 constitute the disturbance propagated directly from $z = f$, $\varrho = 0$ to z, ϱ; the movement

$$\left.\begin{aligned} U &= -E \sin\psi \frac{F'(t-r/V)}{rV} - E\sin\psi \frac{F(t-r/V)}{r^2} \\ W &= -E\cos\psi \frac{F'(t-r/V)}{rV} - E\cos\psi \frac{F(t-r/V)}{r^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

where $\psi = \arcsin \varrho/r$ and $F' = \partial F/\partial t$, would also exist in an ocean which is infinite in every direction.

The first term of G_2 and the second one of G_4 yield:

$$\left.\begin{aligned} U_0 &= E \sin\psi_0 \frac{F'(t-r_0/V)}{r_0 V} + E\sin\psi_0 \frac{F(t'-r_0/V)}{r_0^2} \\ W_0 &= E\cos\psi_0 \frac{F'(t-r_0/V)}{r_0 V} + E\cos\psi_0 \frac{F(t-r_0/V)}{r_0^2} \end{aligned}\right\} \quad . \quad . \quad (2)$$

with $\psi_0 = \arcsin \varrho/r_0$;

this motion is independent of the underlying medium and would also occur in an ocean which is infinitely deep. The movement is due to the free surface $z = 0$ ("reflected wave").

§ 2. *The Laplace transformation of G_3.*

This transformation is entirely analogous to that of II § 3; the Laplace function $L_n(\varrho, z, u)$ of the n-th term of the G_3 series is equal to zero if $u < q_n/V$, where $q_n = 2nd \pm f \pm z$. For larger values of u we have

$$L_n(\varrho, z, u) = \frac{2}{\pi} R \int_0^{\pi/2} (-D_l)^n \frac{\xi \, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2}} \text{ (R means "real part of")}$$

with

$$\xi = \frac{-iu\varrho \cos \omega + q_n \sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2}}{q_n^2 + \varrho^2 \cos^2 \omega}$$

The velocity V in water being smaller than the transverse velocity $\mathfrak{V}'$ of the suboceanic rocks we have here the same situation as in case B of II § 3 ($\mathfrak{V} < \mathfrak{V}'$). Consequently the real part of the integral is zero if $u < u_1$, where

$$u_1 = \frac{\varrho}{V'} + q_n \sqrt{\frac{1}{V^2} - \frac{1}{V'^2}}.$$

At $u = u_1$ the branch point ω_1 at which $a' = 0$ reaches the imaginary ω axis provided

$$\varrho \sqrt{\frac{1}{V^2} - \frac{1}{V'^2}} > \frac{q_n}{V'} \; ; \quad \ldots \ldots \ldots \quad (3)$$

if ϱ and q_n satisfy this inequality the value of L_n is for each $u > u_1$:

$$L_n(\varrho, z, u) = \frac{2}{\pi} R \int_0^{\omega_1} (-D_l)^n \frac{\xi \, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2}}. \quad \ldots \quad (4)$$

With growing value of u the point ω_1 moves along the imaginary ω axis towards $i\infty$; at $u = u_2$

$$u_2 = \frac{\varrho}{\mathfrak{V}'} + q_n \sqrt{\frac{1}{V^2} - \frac{1}{\mathfrak{V}'^2}}$$

a second branch point ω_2 ($b' = 0$) appears on this axis, if

$$\varrho \sqrt{\frac{1}{V^2} - \frac{1}{\mathfrak{V}'^2}} > \frac{q_n}{\mathfrak{V}'}. \quad \ldots \ldots \ldots \quad (5)$$

At still larger values of u the branch point ω_0, determined by

$$u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2 = 0$$

travels also along this axis. This point reaches $\omega = 0$ at $u = u_0 = r_n/V$, where $r_n^2 = \varrho^2 + q_n^2$.

As we have seen in II § 3 the function L_n is discontinuous at $u = u_0$:

$$L_n(\varrho, z, u_0 + 0) = L_n(\varrho, z, u_0 - 0) + \frac{1}{r_n} R(-D_l)^n. \quad \ldots \quad (6)$$

Whereas ω_0 reaches the imaginary axis in any case, ω_1 and ω_2 do this only if ϱ, z fullfils the conditions (3) and (5); introducing the angles of total reflection

$$\varepsilon_1 = \arc\sin V/V' \text{ and } \varepsilon_2 = \arc\sin V/\mathfrak{B}'$$

these conditions become $\varrho > r_n \sin \varepsilon_1$, and $\varrho > r_n \sin \varepsilon_2$. If the point ϱ, z lies outside one or both cones of total reflection the traject of integration extends from $\omega = 0$ to $\omega = \omega_1$ (or ω_2). At values $u < u_0$ ξ and a' (possibly also b') are imaginary, but a is real; the absolute value of

$$D_l = \frac{4\,\sigma\,\mathfrak{B}'^4\,a\,(c'^2 - a'\,b'\,\xi^2) - \sigma\,a'}{4\,\sigma\,\mathfrak{B}'^4\,a\,(c'^2 - a'\,b'\,\xi^2) + \sigma\,a'}$$

is then equal to 1. Consequently the value of the integral varies at large epicentral distances ($\varrho \gg q_n$) as r_n^{-1}.

This is also true for the traject $\omega_0 \to \omega_1$ (or ω_2) when $u > u_0$ and the branch point ω_0 lies on the imaginary ω axis. On the traject $0 \to \omega_0$ however ξ is a complex quantity and the absolute value of D_l is not equal to 1; in order to determine the properties of the integral on this traject we proceed as follows:

Denoting the factors in the integrand which are a function of ξ^2 by $f(\xi^2)$ and the denominator by A^{-1} the Laplace function is

$$L_n = R \int_0^{\pi/2} f(\xi^2)\,A\,\xi\,d\omega, \text{ with } \xi = \frac{-i\,u\,\varrho\cos\omega + A^{-1}}{q_n^2 + \varrho^2\cos^2\omega}.$$

It is evident that

$$L_n = {}^1/_2 \int_0^{\pi} f(\xi^2)\,A\,\xi\,d\omega \text{ if } u > r_n/V \text{ (then } A \text{ is real).}$$

We now consider the integral

$$K_n = R \int_0^{\pi/2} f(\eta^2)\,A\,\eta\,d\omega, \text{ with } \eta = \frac{-i\,u\,\varrho\cos\omega - A^{-1}}{q_n^2 + \varrho^2\cos^2\omega}; \quad . \quad . \quad (7)$$

if $u > r_n/V$ we have again

$$K_n = {}^1/_2 \int_0^{\pi} f(\eta^2)\,A\,\eta\,d\omega.$$

Changing ω into $\omega' = \pi - \omega$ this becomes

$$K_n = {}^1/_2 \int_0^{\pi} f(\eta'^2)\,A\,\eta'\,d\omega', \text{ with } \eta' = \frac{+\,i\,u\,\varrho\cos\omega' - A^{-1}}{q_n^2 + \varrho^2\cos^2\omega'};$$

as $\eta' = -\xi$ it follows:

$$K_n = -\,{}^1/_2 \int_0^{\pi} f(\xi^2)\,A\,\xi\,d\omega = -\,L_n, \text{ or } L_n = \tfrac{1}{2}(L_n - K_n). \quad . \quad . \quad (8)$$

The value of K_n (ϱ, z, u) is calculated by an integration in the ω-plane along the contour $0 \to \tfrac{1}{2}\pi \to \tfrac{1}{2}\pi + i\infty \to i\infty \to 0$. The integrand has three branch points: $a' = 0$, $b' = 0$ and $A = 0$, and a pole at $\eta = \infty$.

The image of the branch point $\eta = -i/V'$ (the value $\eta = +i/V'$ is not represented in this part of the ω plane) is determined by substituting this value into

$$u = i\,\eta\,\varrho\cos\omega + q_n\,a.$$

Hence

$$u = \frac{\varrho \cos \omega}{V'} + q_n \sqrt{\frac{1}{V^2} - \frac{1}{V'^2}}.$$

Substituting this back into (7) we obtain

$$\eta = -\frac{i}{V'} \frac{\varrho^2 \cos^2 \omega + q_n \varrho \cos \omega \sqrt{V'^2/V^2 - 1} + q_n \sqrt{(\varrho \cos \omega \sqrt{V'^2/V^2 - 1} - q_n)^2}}{\varrho^2 \cos^2 \omega + q_n^2}$$

and this is equal to $-i/V'$ only if $\varrho \cos \omega \sqrt{V'^2/V^2 - 1} < q_n$ or $\varrho \cos \omega < q_n \, tg \, \varepsilon_1$. The branch point ω_1 ($a' = 0$) lies on the imaginary ω axis if ϱ, z lies inside the cone ε_1 of total reflection. The same applies to the branch point ω_2 ($b' = 0$). The third branch point ω_0 ($A = 0$) however is in any case represented by a point on this axis.

We have therefore three points on this axis ($\omega_1 > \omega_2 > \omega_0$) when $\varrho < r_n \sin \varepsilon_1$; two branch points ($\omega_2 > \omega_0$) when $r_n \sin \varepsilon_1 < \varrho < r_n \sin \varepsilon_2$ and only one (ω_0) when $r_n \sin \varepsilon_2 < \varrho$. The branch points ω_1 and ω_2 appear at the K_n integration when they do not occur at the L_n integration and reciprocally.

The pole of the first order $\eta = \infty$ is given by

$$\varrho^2 \cos^2 \omega + q_n^2 = 0, \text{ or } \cos \omega = \pm i \, q_n/\varrho.$$

The point $\cos \omega = + i \, q_n/\varrho$ is irrelevant to the K_n integration but $\omega = \arccos(-i q_n/\varrho)$ lies on the traject $\frac{1}{2}\pi \rightarrow \frac{1}{2}\pi + i\infty$. Integrating along a semi-circle enclosing this point we obtain the residue term of $K_n(\eta = \infty)$:

$$\frac{2 \cdot (-1)^n}{r_n}.$$

On the traject $\frac{1}{2}\pi \rightarrow \frac{1}{2}\pi + i\infty \rightarrow i\infty \rightarrow \omega_1$ (ω_2 or ω_0) the K_n integrand is real and as $d\omega$ is imaginary the real part of the integral is zero. On the traject $0 \rightarrow \omega_0$ A is real; the imaginary parts of ξ (in the function L_n) and η are the same but the real parts have opposite signs. As there exist no branch points on this traject it follows that the real parts of the K_n and the L_n integral are identical; in view of (8) it is evident that the traject $0 \rightarrow \omega_0$ disappears from the results. We obtain therefore:

$$\left.\begin{aligned}
L_n(\varrho, z, u) &= \frac{1}{\pi} R \int_{\omega_0}^{\omega_1} \{-D_l(\xi)\}^n \frac{\xi \, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2}} + \frac{(-1)^n}{r_n} \text{ if } \varrho > r_n \sin \varepsilon_2, \\
L_n(\varrho, z, u) &= \frac{1}{\pi} R \left\{ \int_{\omega_0}^{\omega_1} \{-D_l(\xi)\}^n \frac{\xi \, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2}} - \right. \\
&\left. - \int_{\omega_0}^{\omega_2} \{-D_l(\eta)\}^n \frac{\eta \, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2}} \right\} + \frac{(-1)^n}{r_n}, \text{ if } r_n \sin \varepsilon_2 > \varrho > r_n \sin \varepsilon_1. \\
L_n(\varrho, z, u) &= -\frac{1}{\pi} R \int_{\omega_0}^{\omega_1} \{-D_l(\eta)\}^n \frac{\eta \, d\omega}{\sqrt{u^2 - (q_n^2 + \varrho^2 \cos^2 \omega)/V^2}} + \frac{(-1)^n}{r_n} \text{ if } r_n \sin \varepsilon_1 > \varrho.
\end{aligned}\right\} \quad (9)$$

These expressions are only valid if $u > r_n/V$.

As in all these integrals $\xi(\eta)$ and a' (sometimes also b') are imaginary but a is real the absolute value of D_l is 1; at large epicentral distances the value of L_n diminishes as ι_n^{-1}.

§ 3. *The Stoneley function* $S(\varrho, z, u)$.

In the case $n = 1$ the K_n-contour encloses a second pole of the first order, namely the value of ω where the denominator of D_l is equal to zero or:

$$4\,\sigma\,\mathfrak{B}'^4\,a\,(c'^2 - a'\,b'\,\eta^2) + \sigma\,a' = 0.$$

Substituting $\eta = \pm i/C$ this equation becomes the Stoneley equation for the two media water and suboceanic rock:

$$\left(1 - \frac{C^2}{2\mathfrak{B}'^2}\right)^2 - \sqrt{\left(1 - \frac{C^2}{\mathfrak{B}'^2}\right)\left(1 - \frac{C^2}{V'^2}\right)} + \frac{\sigma}{4\sigma'}\frac{C^4}{\mathfrak{B}'^4}\sqrt{\frac{1 - C^2/V'^2}{1 - C^2/V^2}} = 0.$$

For very small values of C the left hand side is

$$\approx \tfrac{1}{2}\,C^2\left(\frac{1}{V'^2} - \frac{1}{\mathfrak{B}'^2}\right) < 0$$

while it is positive real for values of $C \lessapprox V(< \mathfrak{B}')$. The equation has therefore always a root $C < V < \mathfrak{B}'$ (see also the note at the end of this chapter).

The value of u at $\eta_0 = -i/C$ is ($n = 1$):

$$u = \frac{1}{C}\left(\varrho\cos\omega + i\,q_1\sqrt{1 - C^2/V^2}\right);$$

hence by substitution into (7):

$$\eta = -\frac{i}{C}\,\frac{\varrho^2\cos^2\omega + i\,q_1\,\varrho\cos\omega\sqrt{1 - C^2/V^2} - i\,q_1\sqrt{(\varrho\cos\omega\sqrt{1 - C^2/V^2} + i\,q_1)^2}}{\varrho^2\cos^2\omega + q_1^2}$$

which is equal to $-i/C$. The value η_0 is therefore represented by a point in the ω plane enclosed by the K_n-contour. Consequently we have if $n = 1$ to add the contribution of the pole η_0 to the results of § 2; this correction term is

$$S(\varrho, z, u) = \frac{1}{\pi}\,R\cdot 2\pi i\left\{-\eta\,A\,\frac{c'^2 - a'\,b'\,\eta^2 - \sigma\,a'/4\sigma'\mathfrak{B}'^4 a}{\partial/\partial\omega\,(c'^2 - a'\,b'\,\eta^2 + \sigma\,a'/4\sigma'\mathfrak{B}'^4 a)}\right\}_{\eta=\eta_0}.$$

As $\partial/\partial\omega = \partial\eta/\partial\omega \cdot \partial/\partial\eta$ and $\partial\eta/\partial\omega = -i\,\eta\,a\,A\,\varrho\sin\omega$ this becomes

$$S(\varrho, z, u) = 2\,R\left\{\frac{c'^2 - a'\,b'\,\eta^2 - \sigma\,a'/4\sigma'\mathfrak{B}'^4 a}{\partial/\partial\eta\,(c'^2 - a'\,b'\,\eta^2 + \sigma\,a'/4\sigma'\mathfrak{B}'^4 a)}\cdot\frac{1}{a\,\varrho\sin\omega}\right\}_{\eta=\eta_0}$$

or

$$S = s\,R\left(\frac{i}{\varrho\sin\omega}\right)_{\eta_0},\ \text{where } s \text{ is the constant:}$$

$$S = \frac{\sigma}{\sigma'} \frac{C^4}{\mathfrak{B}'^4} \frac{1-\dfrac{C^2}{V'^2}}{1-\dfrac{C^2}{V^2}} \cdot$$

$$\frac{\sqrt{1-\dfrac{C^2}{\mathfrak{B}'^2}}}{4-3\left(\dfrac{C^2}{V'^2}+\dfrac{C^2}{\mathfrak{B}'^2}\right)+\dfrac{C^4}{V'^2\mathfrak{B}'^2}-4\left(1-\dfrac{C^2}{2\mathfrak{B}'^2}\right)\sqrt{\left(1-\dfrac{C^2}{V'^2}\right)\left(1-\dfrac{C^2}{\mathfrak{B}'^2}\right)}+\dfrac{\sigma}{4\sigma'}\dfrac{C^4}{\mathfrak{B}'^4}\dfrac{\dfrac{C^2}{V^2}-\dfrac{C^2}{V'^2}}{1-\dfrac{C^2}{V^2}}\sqrt{\dfrac{1-\dfrac{C^2}{\mathfrak{B}'^2}}{1-\dfrac{C^2}{V^2}}}}.$$

Again

$$(\varrho \sin \omega)^2 = (r_1^2 - u^2 C^2 - q_1^2 C^2/V^2) + 2iu\, C q_1 \sqrt{1-C^2/V^2}$$

or $\qquad (\varrho \sin \omega)^2 = B e^{i\varphi}$ where

$$B = \{(r_1^2 - u^2 C^2 - q_1^2 C^2/V^2)^2 + 4u^2 C^2 q_1^2 (1 - C^2/V^2)\}^{1/2}$$

and

$$Q = \text{arc tan}\, \frac{2u C q_1 \sqrt{1-C^2/V^2}}{r_1^2 - u^2 C^2 - q_1^2 C^2/V^2}.$$

It follows

$$S(\varrho, z, u) = s B^{-1/2} \sin {}^1/_2\, \varphi.$$

At large distances ϱ ($\gg q_1$) this function is proportional to r_1^{-1} (and therefore of the same order as the rest of L_1) provided u is not $\approx r_1/C$. If $u \approx r_1/C$ S is proportional to $r_1^{-1/2}$ so that the Stoneley function then determines a major part of the disturbance at ϱ, z.

(To be continued.)

Mathematics. — *Über mehrwertige Aussagenkalküle und mehrwertige engere Prädikatenkalküle.* II. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of June 26, 1948.)

Aussagenkalküle $K^{(p)}$.

§ 10. Aussagenkalküle, welche als Produkte von 3, 4, ... erweiterten RUSSELL–WHITEHEADschen Aussagenkalkülen zu betrachten sind, lassen sich nach dem Vorigen leicht bilden. Es wird genügen den Fall *eines Produktes von drei „Faktoren"* zu betrachten; *dieser Kalkül* $K^{(3)}$ *wird dann* $2^m \times 2^n \times 2^p$*-wertig sein* ($m$, n, p willkürliche natürliche Zahlen).

Erstes Axiomensystem für $\mathbf{K}^{(3)}$.

Elementare Kalkülformeln: $A, B, \ldots, A_1, \ldots, A_n, \ldots$; ausserdem λ, ν, λ_2, ν_2, λ_3, ν_3, λ_4, ν_4.

Undefinierte Grundverknüpfungen: $+$, $+_2$, $+_3$, $+_4$ und $'$.

Einsetzungsregel E (erster Teil). Aus einer *Kalkülformel* erhält man wieder eine Kalkülformel, wenn man einen in ihr auftretenden grossen lateinischen Buchstaben durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln); dabei sind die grossen lateinischen Buchstaben, λ, ν, λ_2, ν_2, λ_3, ν_3, λ_4, ν_4 als Kalkülformeln anzusehen, ferner mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R} + \mathfrak{S}$, $\mathfrak{R} +_2 \mathfrak{S}$, $\mathfrak{R} +_3 \mathfrak{S}$, $\mathfrak{R} +_4 \mathfrak{S}$, $\mathfrak{R}'$ und $\mathfrak{S}'$ [1]).

Definition 1. $\mathfrak{A} \to \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$; $\mathfrak{A} \to_{2j-1} \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' +_j \mathfrak{B}$ ($j = 2$, 3 oder 4).

$\doteq \bar{\lambda}$, $\doteq \bar{\nu}$, $\doteq \bar{\lambda}_j$, $\doteq \bar{\nu}_j$ ($j = 2$, 3 oder 4) sind *Modalitätszeichen*.

Definition 2. Ein Ausdruck, bestehend aus einer Kalkülformel, gefolgt durch eines der Modalitätszeichen, und entstanden nach endlichmaliger Anwendung von Axiomen, Einsetzungsregel E, Schlussschema S oder (und) Verbindungsregel V, ist *ein Theorem*.

Einsetzungsregel E (zweiter Teil). Ist $\mathfrak{A} \doteq \bar{\nu}$ [$\mathfrak{A} \doteq \bar{\nu}_2$ oder $\mathfrak{A} \doteq \bar{\nu}_3$ oder $\mathfrak{A} \doteq \bar{\nu}_4$] ein Theorem, und $\mathfrak{B}$ eine neue, aus $\mathfrak{A}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformel, so liefert auch $\mathfrak{B} \doteq \bar{\nu}$ [$\mathfrak{B} \doteq \bar{\nu}_2$ bzw. $\mathfrak{B} \doteq \bar{\nu}_3$ bzw. $\mathfrak{B} \doteq \bar{\nu}_4$] ein Theorem.

Erste Gruppe von Axiomen.

Axiom I. $[(X + X) \to X] \doteq \bar{\nu}$.

Axiom II. $[X \to (X + Y)] \doteq \bar{\nu}$.

Axiom III. $[(X + Y) \to (Y + X)] \doteq \bar{\nu}$.

Axiom IV. $[(Y \to Z) \to \{(X + Y) \to (X + Z)\}] \doteq \bar{\nu}$.

Axiom V. $(\lambda \to X) \doteq \bar{\nu}$.

Axiom VI. $(X \to \nu) \doteq \bar{\nu}$.

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[1]) Grosse deutsche Buchstaben werden immer Kalkülformeln andeuten.

Zweite bis vierte Gruppe von Axiomen ($j = 2$, 3 oder 4).

Axiom I(2j—1). $[(X +_j X) \rightarrow_{2j-1} X] \doteq \nu_j$.
Axiom II(2j—1). $[X \rightarrow_{2j-1} (X +_j Y)] \doteq \nu_j$.
Axiom III(2j—1). $[(X +_j Y) \rightarrow_{2j-1} (Y +_j X)] \doteq \nu_j$.
Axiom IV(2j—1).

$$[(Y \rightarrow_{2j-1} Z] \rightarrow_{2j-1} \{(X +_j Y) \rightarrow_{2j-1} (X +_j Z)\}] \doteq \nu_j.$$

Axiom V(2j—1). $[\lambda_j \rightarrow_{2j-1} X] \doteq \nu_j$.
Axiom VI(2j—1). $[X \rightarrow_{2j-1} \nu_j] \doteq \nu_j$.

Schlussschema S. Sind $\mathfrak{A} \doteq \nu$ [$\mathfrak{A} \doteq \nu_2$ oder $\mathfrak{A} \doteq \nu_3$ oder $\mathfrak{A} \doteq \nu_4$] und $(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \nu$ [$(\mathfrak{A} \rightarrow_3 \mathfrak{B}) \doteq \nu_2$ bzw. $(\mathfrak{A} \rightarrow_5 \mathfrak{B}) \doteq \nu_3$ bzw. $(\mathfrak{A} \rightarrow_7 \mathfrak{B}) \doteq \nu_4$] Theoreme, so ist auch $\mathfrak{B} \doteq \nu$ [$\mathfrak{B} \doteq \nu_2$ bzw. $\mathfrak{B} \doteq \nu_3$ bzw. $\mathfrak{B} \doteq \nu_4$] ein Theorem.

Verbindungsregel V. Ist entweder

$$(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \nu \text{ und } (\mathfrak{B} \rightarrow \mathfrak{A}) \doteq \nu,$$

oder

$$(\mathfrak{A} \rightarrow_3 \mathfrak{B}) \doteq \nu_2 \text{ und } (\mathfrak{B} \rightarrow_3 \mathfrak{A}) \doteq \nu_2,$$

oder

$$(\mathfrak{A} \rightarrow_5 \mathfrak{B}) \doteq \nu_3 \text{ und } (\mathfrak{B} \rightarrow_5 \mathfrak{A}) \doteq \nu_3,$$

oder

$$(\mathfrak{A} \rightarrow_7 \mathfrak{B}) \doteq \nu_4 \text{ und } (\mathfrak{B} \rightarrow_7 \mathfrak{A}) \doteq \nu_4$$

ein Paar von Theoremen, so gilt dasselbe von den drei übrigen Paaren.

Definition 3. Eine andere Schreibweise eines Theorems $\mathfrak{A} \doteq \nu$ [$\mathfrak{A} \doteq \nu_j$ mit $j = 2$, 3 oder 4] ist $\mathfrak{A}' \doteq \lambda$ [$\mathfrak{A}' \doteq \lambda_j$ bzw. mit $j = 2$, 3 oder 4]; eine andere Schreibweise eines Theorems $\mathfrak{A} \doteq \lambda$ [$\mathfrak{A} \doteq \lambda_j$ mit $j = 2$, 3 oder 4] ist $\mathfrak{A}' \doteq \nu$ [$\mathfrak{A}' \doteq \nu_j$ bzw. mit $j = 2$, 3 oder 4].

Definition 4. $\mathfrak{A} \cdot \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' + \mathfrak{B}')'$; $\mathfrak{A} \cdot_j \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' +_j \mathfrak{B}')'$ ($j = 2$, 3 oder 4).

Definition 1bis. $\mathfrak{A} \rightarrow_2 \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' \cdot \mathfrak{B}$; $\mathfrak{A} \rightarrow_{2j} \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' \cdot_j \mathfrak{B}$ ($j = 2$, 3 oder 4).

Dualitätsprinzip. Ist $\mathfrak{A} \doteq \nu$ [$\mathfrak{A} \doteq \nu_j$ ($j = 2$, 3 oder 4)] ein Theorem, so auch $\mathfrak{B} \doteq \lambda$ [$\mathfrak{B} \doteq \lambda_j$ (bzw. $j = 2$, 3 oder 4)]; dabei gehe Kalkülformel $\mathfrak{B}$ aus Kalkülformel $\mathfrak{A}$ dadurch hervor, dass: α) $+$ durch $\cdot$, und umgekehrt; $+_j$ durch $\cdot_j$, und umgekehrt ($j = 2$, 3 oder 4), β) λ durch ν, und umgekehrt; λ_j durch ν_j, und umgekehrt ($j = 2$, 3 oder 4), γ) $\rightarrow$ durch $\rightarrow_2$, und umgekehrt; $\rightarrow_{2j-1}$ durch $\rightarrow_{2j}$, und umgekehrt ($j = 2$, 3 oder 4), ersetzt werden; ev. vorkommende Akzenten sollen ungeändert bleiben. Ist, umgekehrt, bei denselben Kalkülformeln $\mathfrak{A}$ und $\mathfrak{B}$ $\mathfrak{B} \doteq \lambda$ [$\mathfrak{B} \doteq \lambda_j$ ($j = 2$, 3 oder 4)] ein Theorem, so auch $\mathfrak{A} \doteq \nu$ [$\mathfrak{A} \doteq \nu_j$ (bzw. $j = 2$, 3 oder 4)].

§ 11 [2]). **Definition 5.** $\mathfrak{A} \subset \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \bar{\nu}$; $\mathfrak{A} \supset \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow_2 \mathfrak{B}) \doteq \bar{\lambda}$. Ebenso sind $\mathfrak{A} \subset_j \mathfrak{B}$ und $\mathfrak{A} \supset_j \mathfrak{B}$ andere Schreibweisen von $(\mathfrak{A} \rightarrow_{2j-1} \mathfrak{B}) \doteq \bar{\nu}_j$ bzw. $(\mathfrak{A} \rightarrow_{2j} \mathfrak{B}) \doteq \bar{\lambda}_j$ ($j = 2$, 3 oder 4).

Aus dem Axiomensystem des Par. 10 lassen sich die folgenden Sätze ableiten.

Satz 1. α) $\mathfrak{A} \subset \mathfrak{A}$ [$\mathfrak{A} \subset_j \mathfrak{A}$, mit $j = 2$, 3 oder 4] ist ein Theorem;

β) (Schlussschema) hat man als Theoreme $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{B} \subset \mathfrak{C}$ [$\mathfrak{A} \subset_j \mathfrak{B}$, $\mathfrak{B} \subset_j \mathfrak{C}$, mit $j = 2$, 3 oder 4], so gibt es auch ein Theorem $\mathfrak{A} \subset \mathfrak{C}$ [$\mathfrak{A} \subset_j \mathfrak{C}$, bzw. mit $j = 2$, 3 oder 4].

Satz 2. α) $(\mathfrak{A} \cdot \mathfrak{B}) \subset \mathfrak{A}$ und $(\mathfrak{A} \cdot \mathfrak{B}) \subset \mathfrak{B}$ [$(\mathfrak{A} \cdot_j \mathfrak{B}) \subset_j \mathfrak{A}$ und $(\mathfrak{A} \cdot_j \mathfrak{B}) \subset_j \mathfrak{B}$, mit $j = 2$, 3 oder 4] (sind Theoreme);

β) (Schlussschema) aus den Theoremen $\mathfrak{C} \subset \mathfrak{A}$ und $\mathfrak{C} \subset \mathfrak{B}$ [$\mathfrak{C} \subset_j \mathfrak{A}$ und $\mathfrak{C} \subset_j \mathfrak{B}$, mit $j = 2$, 3 oder 4] folgt, dass sich auch schreiben lässt $\mathfrak{C} \subset (\mathfrak{A} \cdot \mathfrak{B})$ [$\mathfrak{C} \subset_j (\mathfrak{A} \cdot_j \mathfrak{B})$, bzw. mit $j = 2$, 3 oder 4].

Satz 3. α) $\mathfrak{A} \subset (\mathfrak{A} + \mathfrak{B})$ und $\mathfrak{B} \subset (\mathfrak{A} + \mathfrak{B})$ [$\mathfrak{A} \subset_j (\mathfrak{A} +_j \mathfrak{B})$ und $\mathfrak{B} \subset_j (\mathfrak{A} +_j \mathfrak{B})$, mit $j = 2$, 3 oder 4];

β) (Schlussschema) aus $\mathfrak{A} \subset \mathfrak{C}$ und $\mathfrak{B} \subset \mathfrak{C}$ [$\mathfrak{A} \subset_j \mathfrak{C}$ und $\mathfrak{B} \subset_j \mathfrak{C}$, mit $j = 2$, 3 oder 4] folgt das Theorem $(\mathfrak{A} + \mathfrak{B}) \subset \mathfrak{C}$ [das Theorem $(\mathfrak{A} +_j \mathfrak{B}) \subset_j \mathfrak{C}$, bzw. mit $j = 2$, 3 oder 4].

Satz 4. $\lambda \subset \mathfrak{A}$, und $\lambda_j \subset_j \mathfrak{A}$, mit $j = 2$, 3 oder 4.

Satz 5. $\mathfrak{A} \subset \nu$, und $\mathfrak{A} \subset_j \nu_j$, mit $j = 2$, 3 oder 4.

Definition 6. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzre Schreibweise von: $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$, oder — was wegen Verbindungsregel V und Definition 5 auf dasselbe hinauskommt — von: $\mathfrak{A} \subset_j \mathfrak{B}$ und $\mathfrak{B} \subset_j \mathfrak{A}$ ($j = 2$ oder 3 oder 4).

Satz 6. $[\mathfrak{A} \cdot (\mathfrak{B} + \mathfrak{C})] = [\mathfrak{A} \cdot \mathfrak{B}) + (\mathfrak{A} \cdot \mathfrak{C})]$;
$[\mathfrak{A} \cdot_j (\mathfrak{B} +_j \mathfrak{C})] = [(\mathfrak{A} \cdot_j \mathfrak{B}) +_j (\mathfrak{A} \cdot_j \mathfrak{B})]$ ($j = 2$, 3 oder 4).

Satz 7. α) $(\mathfrak{A} \cdot \mathfrak{A}') = \lambda$;
$(\mathfrak{A} \cdot_j \mathfrak{A}') = \lambda_j$ ($j = 2$, 3 oder 4).
β) $(\mathfrak{A} + \mathfrak{A}') = \nu$;
$(\mathfrak{A} +_j \mathfrak{A}') = \nu_j$ ($j = 2$, 3 oder 4).

Satz 8. Lässt sich schreiben $\mathfrak{A} \doteq \bar{\nu}$, so auch $\mathfrak{A} = \nu$, und umgekehrt. Lässt sich schreiben $\mathfrak{A} \doteq \bar{\nu}_j$, so auch $\mathfrak{A} = \nu_j$, und umgekehrt ($j = 2$, 3 oder 4).

Satz 9. Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$ [$\mathfrak{A} \subset_j \mathfrak{B}$, mit $j = 2$, 3 oder 4], und sind $\mathfrak{D}$ und $\mathfrak{E}$ neue, aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende, gleichgestaltete lateinische Buchstaben in beiden nicht oder

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[2]) Vergleiche Teil I, § 4.

in beiden an allen Stellen in gleicher Weise (d.h. durch gleichgestaltete Kalkülformeln) ersetzt sind, so hat man auch $\mathfrak{D} \subset \mathfrak{E}$ [$\mathfrak{D} \subset_j \mathfrak{E}$ bzw. mit $j = 2, 3$ oder 4].

Satz 10. Sind $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$ Theoreme, so gilt dasselbe von $\mathfrak{A} \subset_j \mathfrak{B}$ und $\mathfrak{B} \subset_j \mathfrak{A}$ ($j = 2$ oder 3 oder 4), und umgekehrt.

§ 12. *Widerspruchsfreiheit und $2^m \times 2^n \times 2^p$-Wertigkeit des Axiomensystems von Par. 10* (m, n und p natürliche Zahlen). Wir betrachten drei Mengen M, N, P von endlich vielen (m bzw. n bzw. p) diskreten Elementen. Den durch grosse lateinische Buchstaben angedeuteten Kalkülformeln können als „Werte" die Tripel von Teilmengen von M bzw. N bzw. P hinzugefügt werden (die Mengen M, N, P und ihre leeren Teilmengen mit eingeschlossen)[3]).

$A + B$, $A +_2 B$, $A +_3 B$ und $A +_4 B$ ordnen wir als „Wert" bzw. $(T_1 + T_2, U_1 + U_2, V_1 + V_2)$, $(T_1 + T_2, U_1 + U_2, V_1 \cdot V_2)$, $(T_1 + T_2, U_1 \cdot U_2, V_1 + V_2)$ und $(T_1 \cdot T_2, U_1 + U_2, V_1 + V_2)$ zu, falls A das Tripel (T_1, U_1, V_1), B das Tripel (T_2, U_2, V_2) zugeordnet ist; eine analoge Verabredung wird gemacht bei Kalkülformeln wie $\mathfrak{A}$ und $\mathfrak{B}$.

A' [$\mathfrak{A}'$] wird als „Wert" $(M—T, N—U, P—V)$ zugeordnet, falls (T, U, V) ein „Wert" von A[$\mathfrak{A}$] ist.

λ sei als einziger „Wert" $(0, 0, 0)$ zugeordnet, ν das Tripel (M, N, P); λ_2 $(0, 0, P)$ und ν_2 $(M, N, 0)$; λ_3 $(0, N, 0)$ und ν_3 $(M, 0, P)$; schliesslich λ_4 $(M, 0, 0)$ und ν_4 $(0, N, P)$.

Welche „Werte" X, Y und Z auch zugeordnet sind, in allen Fällen ist den in den Axiomen I—VI im Vorderglied vorkommenden Kalkülformeln der „Wert" (M, N, P) zugeordnet; für die Axiome $\mathrm{I}^{(3)}$—$\mathrm{VI}^{(3)}$ ist dieser gemeinsame „Wert" $(M, N, 0)$, für die Axiome $\mathrm{I}^{(5)}$—$\mathrm{VI}^{(5)}$ $(M, 0, P)$, und für die Axiome $\mathrm{I}^{(7)}$—$\mathrm{VI}^{(7)}$ $(0, N, P)$.

Ausserdem führt Einsetzungsregel E von Theoremen $\mathfrak{A} \doteq \bar{\nu}$ **oder** $\doteq \bar{\nu}_j$ $(j = 2, 3, 4)$, bei welchen $\mathfrak{A}$ immer (M, N, P) bzw. $(M, N, 0)$ bzw. $(M, 0, P)$ bzw. $(0, N, P)$ als „Wert" zugeordnet ist, zu einem Theorem $\mathfrak{B} \doteq \bar{\nu}$ **bzw.** $\doteq \bar{\nu}_j$ $(j = 2, 3, 4)$, bei welchem für $\mathfrak{B}$ dasselbe gilt.

Eine analoge Eigenschaft zeigt das Schlussschema S.

Betrachten wir schliesslich die Verbindungsregel V, und sei z.B. bekannt, dass $(\mathfrak{A} \rightarrow_3 \mathfrak{B}) \doteq \bar{\nu}_2$ und $(\mathfrak{B} \rightarrow_3 \mathfrak{A}) \doteq \bar{\nu}_2$ Theoreme sind; dann können $\mathfrak{A} \rightarrow_3 \mathfrak{B}$ und $\mathfrak{B} \rightarrow_3 \mathfrak{A}$ nur den „Wert" $(M, N, 0)$ haben. Daraus folgt, dass $\mathfrak{A}$ und $\mathfrak{B}$ immer g l e i c h e „Werte" zugeordnet sein müssen, wie auch die in ihnen vorkommenden, durch lateinische Buchstaben dargestellten elementaren Kalkülformeln bewertet sind. Somit sind $\mathfrak{A} \rightarrow \mathfrak{B}$ und $\mathfrak{B} \rightarrow \mathfrak{A}$ immer mit (M, N, P), $\mathfrak{A} \rightarrow_5 \mathfrak{B}$ und $\mathfrak{B} \rightarrow_5 \mathfrak{A}$ mit $(M, 0, P)$, und $\mathfrak{A} \rightarrow_7 \mathfrak{B}$ und $\mathfrak{B} \rightarrow_7 \mathfrak{A}$ mit $(0, N, P)$ bewertet.

$X \doteq \bar{\nu}$ ist somit nicht ableitbar; denn X lässt sich ein von (M, N, P)

[3]) Die Anzahl dieser „Werte" ist somit $2^m \times 2^n \times 2^p$.

verschiedener „Wert" hinzufügen. Ebenso sind nicht ableitbar $X \doteq \bar{\lambda}$, $X \doteq \bar{\nu}_j$ und $X \doteq \bar{\lambda}_j$ $(j = 2, 3$ oder $4)$.

Aus dem Vorigen folgt die Widerspruchtsfreiheit und die $2^m \times 2^n \times 2^p$-Wertigkeit des Axiomensystems von § 10.

Es treten hier 2^3 Modalitätszeichen auf. Natürlich ist es z.B. möglich nur Theoreme zu betrachten, in welchen $+_4$, $\cdot_4$, $\rightarrow_7$, $\rightarrow_8$, $\doteq \bar{\nu}_4$, $\doteq \bar{\lambda}_4$ nicht auftreten. Dann hat man *einen Teilkalkül, welcher die Axiome* $\mathrm{I}^{(7)}$—$\mathrm{VI}^{(7)}$ *nicht benutzt, und in welchem nur 6 Modalitätszeichen vorkommen:* in Einsetzungsregel *E*, Schlussschema *S* und Verbindungsregel *V* können dabei die Teile, welche sich auf Theoreme mit $\doteq \bar{\nu}_4$ beziehen, fortbleiben.

§ 13 [4]). Es wird nicht schwierig sein *die Gleichwertigkeit des Axiomensystems von* § 10 *mit dem hier folgenden* zu beweisen.

Zweites Axiomensystem für $K^{(3)}$.

Elementare Kalkülformeln: $A, B, \ldots, A_1, \ldots, A_n, \ldots;$

$$\lambda, \nu, \lambda_2, \nu_2, \lambda_3, \nu_3, \lambda_4, \nu_4.$$

Undefinierte Grundverknüpfungen: $\subset, \subset_2, \subset_3, \subset_4;$
$+, \cdot; +_2, \cdot_2; +_3, \cdot_3; +_4, \cdot_4;$ das Akzent $'$.

Einsetzungsregel E_0 (erster Teil) habe den gleichen Wortlaut wie Einsetzungsregel E (erster Teil), in § 10, mit dem Unterschiede, dass mit $\mathfrak{R}$ und $\mathfrak{S}$ (neben $\mathfrak{R} + \mathfrak{S}, \ldots, \mathfrak{R}', \mathfrak{S}'$) auch $\mathfrak{R} \cdot \mathfrak{S}$ und $\mathfrak{R} \cdot_j \mathfrak{S}$ $(j = 2, 3$ oder $4)$ als Formeln betrachtet werden sollen.

Einsetzungsregel E_0 (zweiter Teil). Lässt sich (auf Grund der nachfolgenden Axiome und Regel) schreiben $\mathfrak{A} \subset \mathfrak{B}$ [$\mathfrak{A} \subset_j \mathfrak{B}$ mit $j = 2, 3$ oder 4], mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, und sind $\mathfrak{D}$ und $\mathfrak{E}$ aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E_0 (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete lateinische Buchstaben in beiden nicht oder in beiden an allen Stellen durch gleichgestaltete Kalkülformeln ersetzt sind, so lässt sich auch schreiben $\mathfrak{D} \subset \mathfrak{E}$ [$\mathfrak{D} \subset_j \mathfrak{E}$ bzw. mit $j = 2, 3$ oder 4].

Definition 2_0. Ein Ausdruck bestehend aus zwei Kalkülformeln, getrennt durch $\subset$, $\supset$, $\subset_j$ oder $\supset_j$ $(j = 2, 3$ oder $4)$, und entstanden nach endlichmaliger Anwendung der hier folgenden Axiome, Einsetzungsregel E_0 oder (und) Verbindungsregel V_0 (nebst Definitionen) ist *ein Theorem*.

Definition 3_0. Eine andere Schreibweise eines Theorems $\mathfrak{A} \subset \mathfrak{B}$ ist $\mathfrak{B} \supset \mathfrak{A}$; eine andere Schreibweise eines Theorems $\mathfrak{A} \subset_j \mathfrak{B}$ ist $\mathfrak{B} \supset_j \mathfrak{A}$ $(j = 2, 3$ oder $4)$.

Verbindungsregel V_0. Sind $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$ Theoreme, so gilt dasselbe von $\mathfrak{A} \subset_j \mathfrak{B}$ und $\mathfrak{B} \subset_j \mathfrak{A}$ $(j = 2, 3$ oder $4)$, und umgekehrt.

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4) Vergleiche Teil I, §§ 5 u. 6. Man benutze die Sätze von § 11.

Definition 6_0. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzere Schreibweise von: $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$, oder — was wegen Verbindungsregel V_0 auf dasselbe hinauskommt — von: $\mathfrak{A} \subset_j \mathfrak{B}$ und $\mathfrak{B} \subset_j \mathfrak{A}$ ($j = 2$, 3 oder 4).

Erste Gruppe von Axiomen.

Axiom 1. α) $X \subset X$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{C}$, so ist auch $\mathfrak{A} \subset \mathfrak{C}$ ein Theorem.

Axiom 2. α) $(X \cdot Y) \subset X$ und $(X \cdot Y) \subset Y$;

β) Schlussschema) lässt sich schreiben $\mathfrak{C} \subset \mathfrak{A}$ und $\mathfrak{C} \subset \mathfrak{B}$, so auch $\mathfrak{C} \subset (\mathfrak{A} \cdot \mathfrak{B})$.

Axiom 3. α) $X \subset (X + Y)$ und $Y \subset (X + Y)$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset \mathfrak{C}$ und $\mathfrak{B} \subset \mathfrak{C}$, so auch $(\mathfrak{A} + \mathfrak{B}) \subset \mathfrak{C}$.

Axiom 4. $\lambda \subset X$.

Axiom 5. $X \subset \nu$.

Axiom 6. $[X \cdot (Y + Z)] = [X \cdot Y) + (X \cdot Z)]$.

Axiom 7. α) $(X \cdot X') = \lambda$; β) $(X + X') = \nu$.

Zweite bis vierte Gruppe von Axiomen ($j = 2$, 3 oder 4).

Axiom 1(2j—1). α) $X \subset_j X$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset_j \mathfrak{B}$ und $\mathfrak{B} \subset_j \mathfrak{C}$, so auch $\mathfrak{A} \subset_j \mathfrak{C}$.

Axiom 2(2j—1). α) $(X \cdot_j Y) \subset_j X$ und $(X \cdot_j Y) \subset_j Y$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{C} \subset_j \mathfrak{A}$ und $\mathfrak{C} \subset_j \mathfrak{B}$, so auch $\mathfrak{C} \subset_j (A \cdot_j \mathfrak{B})$.

Axiom 3(2j—1). α) $X \subset_j (X +_j Y)$ und $Y \subset_j (X +_j Y)$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset_j \mathfrak{C}$ und $\mathfrak{B} \subset_j \mathfrak{C}$, so auch $(\mathfrak{A} +_j \mathfrak{B}) \subset_j \mathfrak{C}$.

Axiom 4(2j—1). $\lambda_j \subset_j X$.

Axiom 5(2j—1). $X \subset_j \nu_j$.

Axiom 6(2j—1). $[X \cdot_j (Y +_j Z)] = [(X \cdot_j Y) +_j (X \cdot_j Z)]$.

Axiom 7(2j—1). α) $(X \cdot_j X') = \lambda_j$; β) $(X +_j X') = \nu_j$.

Definition 1 (wie in § 10). $\mathfrak{A} \rightarrow \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$; $\mathfrak{A} \rightarrow_{2j-1} \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' +_j \mathfrak{B}$ ($j = 2$, 3 oder 4).

Definition 7_0 (Einführung der Modalitätszeichen). Die Schreibweisen $\mathfrak{A} \doteq \bar{\nu}$ und $\mathfrak{A} = \nu$ sollen einander ersetzen können; ebenso $\mathfrak{A} \doteq \bar{\lambda}$ und $\mathfrak{A} = \lambda$; ebenso $\mathfrak{A} \doteq \bar{\nu}_j$ und $\mathfrak{A} = \nu_j$ ($j = 2$, 3 oder 4); schliesslich auch $\mathfrak{A} \doteq \bar{\lambda}_j$ und $\mathfrak{A} = \lambda_j$ ($j = 2$, 3 oder 4).

Die engeren Prädikatenkalküle $P^{(k)}$.

§ 14. Es wird genügen *zwei aequivalente, widerspruchsfreie Axiomen-*

systeme für den engeren Prädikatenkalkül $P^{(2)}$ zu geben; auch *dieser Kalkül ist* $2^m \times 2^n$*-wertig.*

Erstes Axiomensystem.

Undefinierte *Grundverknüpfungen:* $+, +_2, \Pi, \Pi_2, \Sigma, \Sigma_2, '$.

Elementare Aussagen (elementare Kalkülformeln): $A, B, \ldots, A_1, \ldots, A_n, \ldots; \lambda, \nu, \lambda_2, \nu_2$.

Gegenstandsvariable: $x, y, z, u, v, w, x_1, \ldots, w_1, x_2, \ldots, w_2, \ldots, x_n, \ldots$

Prädikatenvariable: $F(\cdot), F(\cdot,\cdot), F(\cdot,\cdot,\cdot), \ldots, G(\cdot), G(\cdot,\cdot), \ldots, F_1(\cdot), F_1(\cdot,\cdot), \ldots, G_1(\cdot), G_1(\cdot,\cdot), \ldots, F_n(\cdot), \ldots$

Einsetzungsregel E (erster Teil). Als *Kalkülformeln* sind anzusehen:

1° die durch grosse lateinische Buchstaben dargestellten elementaren Kalkülformeln;

2° die elementaren Kalkülformeln $\lambda, \nu, \lambda_2, \nu_2$;

3° Prädikatenvariable, bei denen die Leerstellen durch Gegenstandsvariable ausgefüllt sind;

4° wenn $\mathfrak{A}(x)$ eine Formel ist, in der (unter anderen oder allein) x als freie Variable auftritt, auch $\Pi_{(x)}\mathfrak{A}(x), \Pi_{2(x)}\mathfrak{A}(x), \Sigma_{(x)}\mathfrak{A}(x)$ und $\Sigma_{2(x)}\mathfrak{A}(x)$ (dasselbe entsprechend bei anderen freien Variablen);

5° mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R} + \mathfrak{S}, \mathfrak{R} +_2 \mathfrak{S}, \mathfrak{R}', \mathfrak{S}'$ [1]); dabei sollen $\mathfrak{R}$ und $\mathfrak{S}$ nicht eine gleiche Gegenstandsvariable enthalten, welche in der einen Formel gebunden, in der anderen frei vorkommt [5]).

Aus einer Kalkülformel erhält man wieder eine Kalkülformel, wenn man eine in ihr auftretende, eine elementare Aussage darstellende, grosse lateinische Buchstabe durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln), oder wenn man eine freie Gegenstandsvariable durch eine andere ersetzt, oder wenn man eine Prädikatenvariable mit n Leerstellen durch eine Formel, die mindestens n freie Gegenstandsvariable enthält, ersetzt [6]).

Definition 1. $\mathfrak{A} \rightarrow \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$.

Definition 1ter. $\mathfrak{A} \rightarrow_3 \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' +_2 \mathfrak{B}$.

Definition 2. Ein Ausdruck, bestehend aus einer Kalkülformel, gefolgt durch eines der *Modalitätszeichen* $\doteq \bar{\nu}, \doteq \bar{\lambda}, \doteq \bar{\nu}_2, \doteq \bar{\lambda}_2$, und entstanden nach endlichmaliger Anwendung von Axiomen, Einsetzungsregel E, Umbenennungsregel U, Schlussschema S, Schema Σ, Schema Π oder (und) Verbindungsregel V, ist *ein Theorem.*

Einsetzungsregel E (zweiter Teil). Ist $\mathfrak{A} \doteq \bar{\nu}$ $[\mathfrak{A} \doteq \bar{\nu}_2]$ ein Theorem, und $\mathfrak{B}$ eine neue, aus $\mathfrak{A}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformel, so ist auch $\mathfrak{B} \doteq \bar{\nu}$ $[\mathfrak{B} \doteq \bar{\nu}_2]$ ein Theorem.

Umbenennungsregel U. In den Theoremen (siehe Definition 2) sind

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[5]) In einer Formel soll dieselbe Variable n i c h t gleichzeitig in freier und in gebundener Form vorkommen.

[6]) Eine nähere Formulierung bei HILBERT–ACKERMANN, Grundzüge der theoretischen Logik, 2e Aufl. 1938, S. 56 u. 57.

Umbenennungen für die gebundenen Gegenstandsvariablen erlaubt gemäss HILBERT–ACKERMANN, loc. cit. 5), S. 57, δ).

Erste Gruppe von Axiomen: die Axiome I—VI sind dieselben wie in Teil I, § 1;

Axiom VII. $[\Pi_{(x)} F(x) \to F(y)] \doteq \bar{\nu}$;

Axiom VIII. $[F(y) \to \Sigma_{(x)} F(x)] \doteq \bar{\nu}$.

Schlussschema S. Sind $\mathfrak{A} \doteq \bar{\nu}\, [\mathfrak{A} \doteq \bar{\nu}_2]$ und $(\mathfrak{A} \to \mathfrak{B}) \doteq \bar{\nu}\, [(\mathfrak{A} \to_3 \mathfrak{B}) \doteq \bar{\nu}_2]$ Theoreme, so ist auch

$$\mathfrak{B} \doteq \bar{\nu} \qquad [\mathfrak{B} \doteq \bar{\nu}_2]$$

ein Theorem.

Schlussschema Σ. Ist $\{\mathfrak{B}(x) \to \mathfrak{A}\} \doteq \bar{\nu}\, [\{\mathfrak{B}(x) \to_3 \mathfrak{A}\} \doteq \bar{\nu}_2]$ **ein** Theorem, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, so ist auch

$$\{\Sigma_{(x)} \mathfrak{B}(x) \to \mathfrak{A}\} \doteq \bar{\nu}\, [\{\Sigma_{2(x)} \mathfrak{B}(x) \to_3 \mathfrak{A}\} \doteq \bar{\nu}_2]$$

ein Theorem.

Schlussschema Π. Ist $\{\mathfrak{A} \to \mathfrak{B}(x)\} \doteq \bar{\nu}\, [\{\mathfrak{A} \to_3 \mathfrak{B}(x)\} \doteq \bar{\nu}_2]$, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, ein Theorem, so auch

$$\{\mathfrak{A} \to \Pi_{(x)} \mathfrak{B}(x)\} \doteq \bar{\nu}\, [\{\mathfrak{A} \to_3 \Pi_{2(x)} \mathfrak{B}(x)\} \doteq \bar{\nu}_2].$$

Zweite Gruppe von Axiomen: die Axiome I^{ter}—VI^{ter} sind dieselben wie Teil I, § 1;

Axiom VII^{ter}. $[\Pi_{2(x)} F(x) \to_3 F(y)] \doteq \bar{\nu}_2$;

Axiom VIII^{ter}. $[F(y) \to_3 \Sigma_{2(x)} F(x)] \doteq \bar{\nu}_2$.

Verbindungsregel V. Sind

$$(\mathfrak{A} \to \mathfrak{B}) \doteq \bar{\nu} \quad \text{und} \quad (\mathfrak{B} \to \mathfrak{A}) \doteq \bar{\nu}$$

Theoreme, so gilt dasselbe von

$$(\mathfrak{A} \to_3 \mathfrak{B}) \doteq \bar{\nu}_2 \qquad \text{und} \qquad (\mathfrak{B} \to_3 \mathfrak{A}) \doteq \bar{\nu}_2,$$

und umgekehrt.

Definition 3, 4, 4bis, 1bis und 1^{quater} sollen den gleichen Wortlaut wie in I, § 1 haben.

Dualitätsprinzip. Dies hat den gleichen Wortlaut wie in I, § 1; nur soll zu den dort angegebenen Vertauschungen hinzugefügt werden: δ) jedes Allzeichen $\Pi_{(\cdot)}$ soll durch ein Seinszeichen $\Sigma_{(\cdot)}$, und umgekehrt, ersetzt werden; jedes Allzeichen $\Pi_{2(\cdot)}$ soll durch ein Seinszeichen $\Sigma_{2(\cdot)}$, und umgekehrt, ersetzt werden.

Definition 5, 5bis, 5ter u. 5^{quater} sollen den gleichen Wortlaut wie in I, § 4 haben.

§ 15. Zweites Axiomensystem für $P^{(2)}$.

Undefinierte *Grundverknüpfungen:* $\subset$, $\subset_2$, $+$, $\cdot$, $+_2$, $\cdot_2$, Π, Π_2, Σ, Σ_2, $'$.

Elementare Aussagen, Gegenstandsvariable und *Prädikatenvariable* wie in § 14.

Einsetzungsregel E_0 (erster Teil) habe den gleichen Wortlaut wie Einsetzungsregel E (erster Teil) in § 14; nur füge man unter 5° $\mathfrak{R} \cdot \mathfrak{S}$ und $\mathfrak{R} \cdot_2 \mathfrak{S}$ zu.

Einsetzungsregel E_0 (zweiter Teil). Lässt sich (auf Grund der Axiome und Regeln dieses Par.) schreiben $\mathfrak{A} \subset \mathfrak{B}$ [$\mathfrak{A} \subset_2 \mathfrak{B}$], mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, welche nicht eine gleiche Gegenstandsvariable enthalten, gebunden in der einen und frei in der anderen Formel, und sind $\mathfrak{D}$ und $\mathfrak{E}$ aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E_0 (erster Teil) hervorgehende Kalkülformeln, wobei 1° sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete grosse lateinische Buchstaben in beiden nicht oder in beiden an allen Stellen durch gleichgestaltete Kalkülformeln ersetzt sind, 2° analoge Bedingungen für freie Gegenstandsvariable und für Prädikatenvariable erfüllt sind, 3° in $\mathfrak{D}$ und $\mathfrak{E}$ nicht eine gleiche Gegenstandsvariable gebunden in der einen und frei in der anderen vorkommt, so lässt sich auch schreiben $\mathfrak{D} \subset \mathfrak{E}$ [$\mathfrak{D} \subset_2 \mathfrak{E}$].

Umbenennungsregel U_0. Lässt sich (auf Grund der Axiome und Regeln dieses Par.) schreiben $\mathfrak{A} \subset \mathfrak{B}$ [$\mathfrak{A} \subset_2 \mathfrak{B}$], mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, welche nicht eine gleiche Gegenstandsvariable enthalten, gebunden in der einen und frei in der anderen Formel, und sind $\mathfrak{D}$ und $\mathfrak{E}$ aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ entstanden durch Umbenennung einer gleichen gebundenen Gegenstandsvariablen [7]) gemäss HILBERT–ACKERMANN, loc. cit. 5), S. 57, δ), wobei in $\mathfrak{D}$ und $\mathfrak{E}$ nicht eine gleiche Gegenstandsvariable gebunden in der einen und frei in der anderen vorkommt, so lässt sich auch schreiben $\mathfrak{D} \subset \mathfrak{E}$ [$\mathfrak{D} \subset_2 \mathfrak{E}$].

Definition 2_0. Ein Ausdruck bestehend aus zwei Kalkülformeln, gefolgt durch $\subset$, $\supset$, $\subset_2$ oder $\supset_2$, und entstanden nach endlichmaliger Anwendung der Axiome und Regeln (nebst Definitionen) dieses Par., ist *ein Theorem.*

Definition 3_0. Eine andere Schreibweise eines Theorems $\mathfrak{A} \subset \mathfrak{B}$ ist $\mathfrak{B} \supset \mathfrak{A}$; eine andere Schreibweise eines Theorems $\mathfrak{A} \subset_2 \mathfrak{B}$ ist $\mathfrak{B} \supset_2 \mathfrak{A}$.

Verbindungsregel V_0. Sind $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$ Theoreme, so gilt dasselbe von $\mathfrak{A} \subset_2 \mathfrak{B}$ und $\mathfrak{B} \subset_2 \mathfrak{A}$, und umgekehrt.

Definition 6_0. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzere Schreibweise von: $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$, oder — was wegen Verbindungsregel V_0 auf dasselbe hinauskommt — von: $\mathfrak{A} \subset_2 \mathfrak{B}$ und $\mathfrak{B} \subset_2 \mathfrak{A}$.

Erste und zweite Gruppe von Axiomen: die Axiome 1—7, 1ter—7ter sind dieselben wie in I, § 5 [8]);

Axiom 8. $\Pi_{(x)} F(x) \subset F(y)$.

Axiom 8ter. $\Pi_{2(x)} F(x) \subset_2 F(y)$.

Axiom 9. $F(y) \subset \Sigma_{(x)} F(x)$.

Axiom 9ter. $F(y) \subset_2 \Sigma_{2(x)} F(x)$.

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[7]) Kommt die betrachtete Variable nur $\mathfrak{A}$ oder nur in $\mathfrak{B}$ vor, so findet die Umbenennung natürlich nur in einer von beiden Formeln statt.

[8]) Dabei soll für jedes in den Schlussschematen auftretende $\mathfrak{M} \subset \mathfrak{N}$ oder $\mathfrak{M} \subset_2 \mathfrak{N}$ gefordert werden, dass keine Gegenstandsvariable gleichzeitig in $\mathfrak{M}$ frei und in $\mathfrak{N}$ gebunden, oder umgekehrt, vorkommt; dasselbe für $\mathfrak{A} + \mathfrak{B}$, $\mathfrak{A} \cdot \mathfrak{B}$, $\mathfrak{A} +_2 \mathfrak{B}$ und $\mathfrak{A} \cdot_2 \mathfrak{B}$.

Schlussschema Σ_0. Ist $\mathfrak{B}(x) \subset \mathfrak{A}$ [ist $\mathfrak{B}(x) \subset_2 \mathfrak{A}$] ein Theorem, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, so ist auch

$$\Sigma_{(x)} \mathfrak{B}(x) \subset \mathfrak{A} \qquad [\Sigma_{2(x)} \mathfrak{B}(x) \subset_2 \mathfrak{A}]$$

ein Theorem.

Schlussschema Π_0. Ist $\mathfrak{A} \subset \mathfrak{B}(x)$ [ist $\mathfrak{A} \subset_2 \mathfrak{B}(x)$] ein Theorem, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, so ist auch

$$\mathfrak{A} \subset \Pi_{(x)} \mathfrak{B}(x) \qquad [\mathfrak{A} \subset_2 \Pi_{2(x)} \mathfrak{B}(x)]$$

ein Theorem.

Definition 1—1quater. $\mathfrak{A} \rightarrow \mathfrak{B}$, $\mathfrak{A} \rightarrow_2 \mathfrak{B}$, $\mathfrak{A} \rightarrow_3 \mathfrak{B}$ und $\mathfrak{A} \rightarrow_4 \mathfrak{B}$ sind andere Schreibweisen bzw. von $\mathfrak{A}' + \mathfrak{B}$, $\mathfrak{A}' \cdot \mathfrak{B}$, $\mathfrak{A}' +_2 \mathfrak{B}$ und $\mathfrak{A}' \cdot_2 \mathfrak{B}$.

Definition 7_0 (Einführung der Modalitätszeichen). Die Schreibweisen $\mathfrak{A} \doteq \bar{\nu}$ und $\mathfrak{A} = \nu$ sollen einander ersetzen können; ebenso $\mathfrak{A} \doteq \bar{\lambda}$ und $\mathfrak{A} = \lambda$; ebenso $\mathfrak{A} \doteq \bar{\nu}_2$ und $\mathfrak{A} = \nu_2$; schliesslich auch $\mathfrak{A} \doteq \bar{\lambda}_2$ und $\mathfrak{A} = \lambda_2$.

Mathematics. — *On sums of powers of linear forms III.* By R. A. RANKIN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 26, 1948.)

1. In this paper I obtain improved estimates of the minimum value of the sum of the βth powers of n *real* homogeneous forms in n variables, when $1 \leq \beta \leq 2$ and n is large.

It is convenient to regard $x_1, x_2, \ldots, x_n$ as Cartesian coordinates of a point P in Euclidean n-dimensional space E_n. The point O for which each x_k vanishes is called the origin, and P is called a lattice point if each coordinate is an integer.

For $n \geq 2$, let

$$L_j(P) = L_j = \sum_{k=1}^{n} a_{jk} x_k \qquad (j = 1, 2, \ldots, n) \quad . \quad . \quad . \quad . \quad (1)$$

be n real linear forms in the n real variables $x_1, x_2, \ldots, x_n$ with determinant $D = \| a_{jk} \| \neq 0$. Write

$$g_\beta(P) = g_\beta(x_1, x_2, \ldots, x_n) = \left\{ \sum_{j=1}^{n} |L_j|^\beta \right\}^{1/\beta} \quad . \quad . \quad . \quad . \quad (2)$$

and put

$$\alpha\beta = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since $1 \leq \beta \leq 2$, it follows that $\frac{1}{2} \leq \alpha \leq 1$.

As in **(10)**, we define $M(g_\beta)$ to be the lower bound of $g_\beta(P)$ for all lattice points P other than O, and write M_β for the least upper bound of $M(g_\beta)$ for all sets of forms L_j with a fixed determinant D.

MINKOWSKI **(7)** showed that

$$M(g_\beta) \leqslant M_\beta \leqslant 2 I_\beta^{-1/n} = n^\alpha |D|^{1/n} / A_1'(\alpha), \quad . \quad . \quad . \quad . \quad (4)$$

where

$$I_\beta = \frac{2^n \Gamma^n(1+\alpha)}{|D| \, \Gamma(1+n\alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is the content of the region $g_\beta(P) \leq 1$. For large n

$$A_1'(\alpha) \backsim A_1(\alpha) = (e/\alpha)^\alpha \, \Gamma(1+\alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The estimate (4) has been improved by several writers since MINKOWSKI. We mention only those improvements which have been made for $\beta < 2$ and large n. For the case $\beta = 2$, see **(9)**, for example; for other results when $1 \leq \beta \leq 2$, see the work of HUA **(6)**, MULLENDER **(8)** and the author **(10)**.

By using his results for $\beta = 2$, in conjunction with SCHWARZ's inequality, BLICHFELDT (1) showed, in 1914, that

$$M_1 \leqslant n^{\frac{1}{2}} M_2 \leqslant (2n)^{\frac{1}{2}} (1 + \tfrac{1}{2} n)^{1/n} I_2^{-1/n} = n \, |D|^{1/n} / A_2', \quad . \quad . \quad . \quad (7)$$

where

$$A_2' \backsim A_2 = (\pi e)^{\frac{1}{2}} > A_1 (1) = e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Later, however, he improved this estimate considerably, obtaining (2)

$$M_1 \leqslant n \, |D|^{1/n} / A_3', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$A_3' \backsim A_3 = (\tfrac{3}{2} \pi e)^{\frac{1}{2}} = 3.57905 \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Recently HLAWKA [1]) (5) has generalised BLICHFELDT's result to values of β which satisfy $1 \leq \beta \leq 2$. (Both BLICHFELDT and HLAWKA consider, in addition, linear forms whose coefficients may be complex.) For real linear forms his results take the form

$$M_\beta \leqslant n^\alpha \, |D|^{1/n} / A_4' (\alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where, for large n,

$$A_4' (\alpha) \backsim A_4 (\alpha) = (\tfrac{3}{2})^{\alpha - \frac{1}{2}} (\pi e)^{\frac{1}{2}} = (\tfrac{2}{3})^{1-\alpha} A_3. \quad . \quad . \quad . \quad . \quad (12)$$

In the present paper, I shall show that

$$M_\beta \leqslant n^\alpha \, |D|^{1/n} / A_5' (\alpha, \delta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$A_5' (\alpha, \delta) = 2^{-\alpha} n^\delta \left(\frac{2-\delta}{1-\delta} \right)^{\alpha - \delta} \left\{ \frac{(1 + n\delta)(\alpha + \gamma - 2)}{(1-\delta) I_\gamma |D|} \right\}^{-1/n} \quad . \quad . \quad . \quad (14)$$

Here δ is any number satisfying the inequalities [2])

$$\tfrac{1}{2} \leqslant \delta \leqslant \alpha \leqslant 1, \;\; \delta \leqslant \tfrac{1}{3} (1 + \alpha) < 1, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and $\gamma \delta = 1$. For large n,

$$A_5' (\alpha, \delta) \backsim A_5 (\alpha, \delta) = 2^{1-\alpha} \left(\frac{2-\delta}{1-\delta} \right)^{\alpha - \delta} \left(\frac{e}{\delta} \right)^\delta \Gamma (1 + \delta). \quad . \quad . \quad (16)$$

In particular, for $\delta = \frac{1}{2}$, we have

$$A_5 (\alpha, \tfrac{1}{2}) = A_4 (\alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

However, for

$$\alpha > \alpha_0 = \frac{1}{2} + \frac{3}{4} \left\{ \log \frac{3}{2} - \frac{\Gamma'}{\Gamma} \left(\frac{3}{2} \right) \right\} = \frac{1}{4} (3C + 3 \log 6 - 4) = 0.77673 \ldots, \quad (18)$$

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[1]) I am indebted to Professor HLAWKA for sending me the proof sheets of his paper.

[2]) The inequality $\delta \leq \frac{1}{3} (1 + \alpha)$ is more restrictive than is absolutely necessary, but is sufficient for our purpose; it could be replaced by the weaker, but more complicated, inequality $\delta^2 - \delta (3 - \alpha) + 1 \geq 0$.

where C is EULER's constant 0.57721..., it is possible to improve (11) by choosing δ suitably. For

$$\left[\frac{\partial}{\partial \delta} \log A_5(a, \delta)\right]_{\delta=\frac{1}{2}} = \tfrac{4}{3}(a - a_0),$$

so that, if $a > a_0$, $\frac{1}{2} < \frac{1}{3}(1 + a)$ and therefore

$$A_5(a, \delta) > A_5(a, \tfrac{1}{2})$$

for some $\delta > \frac{1}{2}$.

In particular, for $a = \beta = 1$, the function $A_5(1, \delta)$ attains a maximum in the neighbourhood of $\delta = \delta_1 = 0.6455$, and

$$A_5(1, \delta_1) = 3.65931 \ldots > A_3 = A_4(1) = 3.57905 \ldots \quad . \quad (19)$$

The following table gives values of $A_5(a, \delta)$ for $a = 0.8, 0.85, 0.9, 0.95$ and 1.0. The corresponding values of δ, given in the second column, have been chosen near the maxima of $A_5(a, \delta)$ for the a in question. In the table, and throughout the paper, the numerical values given to the functions $A_r(a, \delta)$ $(1 \leq r \leq 5)$ are lower bounds, and are not necessarily the closest values to within 5 places of decimals.

a	δ	$A_5(a, \delta)$	$A_5(a, \frac{1}{2}) = A_4(a)$
0.80	0.51	3.30085	3.30027
0.85	0.54	3.37434	3.36786
0.90	0.57	3.45658	3.43683
0.95	0.60	3.55028	3.50722
1.00	0.6455	3.65931	3.57905

In conclusion, it is worth remarking that it follows from (13), (16) and (19) that a lattice point P, other than the origin, exists such that the product of the homogeneous forms $L_1, L_2, \ldots, L_n$ satisfies the inequality

$$|L_1 L_2 \ldots L_n|^{1/n} < |D|^{1/n}/3.65931 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

for sufficiently large n. It has been shown by DAVENPORT (3) that, for some lattice point $P \neq O$,

$$|L_1 L_2 \ldots L_n|^{1/n} \leqslant M_2 (n \varkappa^{1-1/n})^{-\frac{1}{2}} \leqslant |D|^{\frac{1}{n}} \{(\varkappa \pi e)^{\frac{1}{2}} + o(1)\}^{-1} \quad (21)$$

where $\varkappa \doteqdot 1.47$. Since $(\varkappa \pi e)^{\frac{1}{2}} < (\frac{3}{2} \pi e)^{\frac{1}{2}} = A_3 < 3.65931$, (20) represents an improvement on the second inequality of (21).

2. As in the two previous papers in this series, it is convenient to work in terms of packing constants rather than in terms of the minima M_β. We recapitulate briefly the ideas there introduced.

Let

$$\varrho'(g_\beta) = \{\tfrac{1}{2} M(g_\beta)\}^n I_\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Then, if each of the forms is multiplied by the same constant factor t, we obtain n new forms with a minimum $t\,M(g)$ and $t^{-n}I_\beta$ in place of I_β. It follows that $\varrho'(g_\beta)$ remains unaltered, and we can therefore simplify by choosing t so that

$$M(g_\beta) = 2.$$

We also write

$$\varrho'_\beta = (\tfrac{1}{2}\,M_\beta)^n\,I_\beta,$$

so that ϱ'_β is the least upper bound of $\varrho'(g_\beta)$ for all sets of real forms L_j.

For any point P_j of E_n we denote by $K_\beta(P_j)$ the body consisting of all points such that

$$g_\beta(P-P_j) < 1.$$

If $P_1, P_2, \ldots$ is a configuration of points (not necessarily lattice points) in E_n such that

$$g_\beta(P_i-P_j) \geqslant 2 \qquad (i \neq j) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

we say that the points $P_1, P_2, \ldots$ form a g_β-packing [3]) in E_n. This is true, in particular, if $P_1, P_2, \ldots$ are lattice points since we have taken $M(g_\beta) = 2$, but for the remainder of the paper we shall consider the more general case when the P_j need not be lattice points.

3. We prove here a number of lemmas.

Define

$$S_q(m) = S_q = \sum_{i=1}^{m} |m-2i+1|^q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Then we have

Lemma 1. *If* $q \geq 1$,

$$S_q \leqslant \frac{m^{q+1}}{q+1}.$$

We first show that

$$2\,u^q \leqslant \int_{u-1}^{u+1} x^q\,dx \qquad (u \geqslant 1).$$

We have

$$\int_{u-1}^{u+1} x^q\,dx - 2\,u^q = \int_{u-1}^{u+1} (x^q-u^q)\,dx = \int_0^1 [\{(u+h)^q-u^q\}-\{u^q-(u-h)^q\}]\,dh.$$

Now $(u+h)^q - u^q$ is a non-decreasing function of u for fixed $h>0$ since its derivative $q\{(u+h)^{q-1} - u^{q-1}\} \geq 0$; the integrand of the last integral is therefore not negative. Accordingly

$$S_q(m) = 2 \sum_{1 \leq i \leq \frac{1}{2}m} (m-2i+1)^q \leqslant \int_0^m x^q\,dx = \frac{m^{q+1}}{q+1}.$$

---

[3]) Since the bodies $K_\beta(P_1)$, $K_\beta(P_2)$, ... are convex, it can be shown that they do not overlap when their centres form a g_β-packing. This justifies the use of the word 'packing'. We do not, however, make any use of this property in the present paper, the inequality (23) being all that we require.

Lemma 2. *For $\mu \geq 1$ and $m \geq 1$,*

$$\left(1-\frac{1}{m}\right)^{\mu} \geqslant 1-\frac{\mu}{m}.$$

For

$$\frac{\mu}{m} \geqslant \mu \int_{1-1/m}^{1} u^{\mu-1}\, du = 1-\left(1-\frac{1}{m}\right)^{\mu}.$$

Lemma 3. *If $b_1 \geq b_2 \geq \ldots \geq b_m$, then*

$$\sum_{i<j} |b_i - b_j| \leqslant \sum_{i=1}^{m} |m-2i+1|\,|b_i|.$$

This lemma is due to BLICHFELDT (**2**). See also HLAWKA (**5**), Hilfssatz 3. We have

$$\sum_{i<j} |b_i - b_j| = \sum_{i=1}^{m} (m-2i+1)\, b_i \leqslant \sum_{i=1}^{m} |m-2i+1|\,|b_i|.$$

Lemma 4. *If $b_1, b_2, \ldots, b_m$ $(m \geq 2)$ are any numbers, real or complex, and $1 \leq \gamma \leq 2$, then*

$$\sum_{1 \leq i<j \leq m} |b_i - b_j|^{\gamma} \leqslant m^{\gamma-1}(m-1)^{2-\gamma} \sum_{j=1}^{m} |b_j|^{\gamma}.$$

This follows from the inequalities

$$\sum_{i<j} |b_i - b_j|^2 = m \sum_{j=1}^{m} |b_j|^2 - \left|\sum_{j=1}^{m} b_j\right|^2 \leqslant m \sum_{j=1}^{m} |b_j|^2,$$

$$\sum_{i<j} |b_i - b_j| \leqslant \sum_{i<j} \{|b_i| + |b_j|\} = (m-1) \sum_{j=1}^{m} |b_j|,$$

and MARCEL RIESZ's convexity theorem applied to the forms

$$X_{ij} = b_i - b_j.$$

See Theorem 296 of HARDY, LITTLEWOOD and PÓLYA (**4**), and Lemma 1 of (**10**).

In the following three lemmas we assume that β and γ satisfy the inequalities

$$1 \leqslant \beta \leqslant \gamma \leqslant 2 \quad , \quad \gamma > 1, \qquad \qquad (25)$$

and we put

$$q = \frac{\gamma}{\gamma-1}, \qquad \qquad (26)$$

$$\mu = q(\alpha+\gamma-2), \qquad \qquad (27)$$

where α is defined by (3). It is easily verified that $q > 1$ and that $\mu > 0$.

Lemma 5. *If $b_1, b_2, \ldots, b_m$ $(m \geq 2)$ are real numbers, then*

$$\sum_{1 \leq i<j \leq m} |b_i - b_j|^{\beta} \leqslant S_q^{1-\beta/\gamma}\, m^{\beta-1} (m-1)^{(\beta-1)(2-\gamma)/(\gamma-1)} \left\{\sum_{j=1}^{m} |b_j|^{\gamma}\right\}^{\beta/\gamma}.$$

When $\beta = \gamma$ the result follows immediately from Lemma 4, so that we shall suppose that $1 \leq \beta < \gamma \leq 2$.

We then have, by HÖLDER's inequality,

$$\sum_{i<j} |b_i - b_j|^\beta \leqslant \left\{ \sum_{i<j} |b_i - b_j| \right\}^{(\gamma-\beta)/(\gamma-1)} \left\{ \sum_{i=j} |b_i - b_j|^\gamma \right\}^{(\beta-1)/(\gamma-1)} . \quad (28)$$

But, by Lemma 3 and HÖLDER's inequality ($\gamma > 1$),

$$\sum_{i<j} |b_i - b_j| \leqslant S_q^{1/q} \left(\sum_{j=1}^{m} |b_j|^\gamma \right)^{1/\gamma} \quad \dots \dots \quad (29)$$

The restriction $b_1 \geq b_2 \geq \dots \geq b_m$ of Lemma 3 is clearly unnecessary since both sides are symmetric functions of all the b_j. The lemma now follows from (28), (29) and Lemma 4.

Lemma 6. *If $P_1, P_2, \dots, P_m$ form a g_β-packing in E_n, then, for any point P,*

$$\sum_{j=1}^{m} g_\gamma^\gamma (P - P_j) \geqslant 2^{(1-\alpha)\gamma} m^{(2\alpha-1)\gamma} (m-1)^\mu (n S_q)^{1-\alpha\gamma}.$$

The result is trivial when $m = 1$ and we shall therefore assume that $m \geq 2$. Write $L_{jk} = L_k (P - P_j)$. Then we have, by (23) and Lemma 5,

$$\tfrac{1}{2} m (m-1) 2^\beta \leqslant \sum_{1 \leq i < j \leq m} g_\beta^\beta (P_i - P_j) = \sum_{k=1}^{n} \sum_{i<j} |L_{ik} - L_{jk}|^\beta$$

$$\leqslant S_q^{1-\beta/\gamma} m^{\beta-1} (m-1)^{(\beta-1)(2-\gamma)/(\gamma-1)} \sum_{k=1}^{n} \left\{ \sum_{j=1}^{m} |L_{jk}|^\gamma \right\}^{\beta/\gamma} .$$

The lemma now follows, since, by HÖLDER's inequality,

$$\sum_{k=1}^{n} \left\{ \sum_{j=1}^{m} |L_{jk}|^\gamma \right\}^{\beta/\gamma} \leqslant \left\{ \sum_{k=1}^{n} \sum_{j=1}^{m} |L_{jk}|^\gamma \right\}^{\beta/\gamma} n^{1-\beta/\gamma}.$$

Lemma 7. *If $P_1, P_2, \dots, P_m$ form a g_β-packing in E_n, then, for any point P,*

$$\sum_{j=1}^{m} g_\gamma^\gamma (P - P_j) \geqslant \lambda^\gamma m \left(\frac{m-1}{m} \right)^\mu$$

where

$$\lambda = 2^{1-\alpha} \left\{ \frac{2\gamma - 1}{n(\gamma-1)} \right\}^{\alpha - 1/\gamma} \quad \dots \dots \quad (30)$$

The lemma follows immediately from Lemmas 1 and 6.

We now put

$$\delta = 1/\gamma \quad \dots \dots \quad (31)$$

and introduce the further restriction

$$\delta \leqslant \tfrac{1}{2} (1 + \alpha). \quad \dots \dots \quad (32)$$

This, together with the inequalities (25), is equivalent to (15) which will be assumed to hold for the remainder of the paper. It follows that

$$(1-a)\,\delta \leqslant (2-3\,\delta)\,\delta = (1-\delta)^2-(1-2\,\delta)^2 \leqslant (1-\delta)^2.$$

Therefore, by (27) and (31),

$$\mu = \frac{(1-\delta)-(1-a)\,\delta}{\delta\,(1-\delta)} \geqslant \frac{(1-\delta)-(1-\delta)^2}{\delta\,(1-\delta)} = 1.$$

Thus μ now satisfies the restriction imposed in Lemma 2.

4. Let C_n be an n-dimensional hypercube in E_n of edge L, and let $N(L)$ be the maximum number of bodies $K_\beta(P_j)$ $(j=1, 2, \ldots)$ which can be placed in C_n so that (i) their centres P_j form a g_β-packing, and (ii) every point of each $K_\beta(P_j)$ lies inside C_n. We can then define the packing constant $\varrho(g_\beta)$ to be

$$\varrho\,(g_\beta) = \lim_{L\to\infty} \frac{I_\beta\, N(L)}{L^n}. \quad \ldots \ldots \quad (33)$$

If b' is the radius of the smallest hypersphere containing $K_\beta(O)$ with centre at O, the argument given in § 5 of (9) (with b' in place of b) shows that this limit exists. Let ϱ_β be the least upper bound of $\varrho(g_\beta)$ for all sets of forms $L_1, L_2, \ldots, L_n$.

We can also define a regular packing constant as follows. Let $N'(L)$ be the maximum number of bodies $K_\beta(P_j)$ which can be placed in any C_n subject to the conditions (i) and (ii) above, and such that, in addition, the centres P_j are lattice points. The regular packing constant is then defined to be

$$\lim_{L\to\infty} \frac{I_\beta\, N'(L)}{L^n},$$

and it is obvious that this limit is precisely I_β, i.e. the number $\varrho'(g_\beta)$ defined by (22), since $M(g)=2$. Clearly

$$\varrho'\,(g_\beta) \leqslant \varrho\,(g_\beta) \quad , \quad \varrho'_\beta \leqslant \varrho_\beta. \quad \ldots \ldots \quad (34)$$

We now consider any configuration of bodies $K_\beta(P_j)$ in C_n which satisfies the conditions (i) and (ii) above. We replace each $K_\beta(P_j)$ by a new body $K^*_\gamma(P_j)$ consisting of all points P such that

$$g_\gamma\,(P-P_j) < \lambda,$$

where γ, the reciprocal of δ, is any number satisfying (15). With each point P of $K^*_\gamma(P_j)$ we associate a density function $\sigma(u_j)$, where $u_j = g_\gamma\,(P-P_j)$ and

$$\sigma\,(u) = \frac{1}{\mu}\left\{1-\left(\frac{u}{\lambda}\right)^\gamma\right\}. \quad \ldots \ldots \quad (35)$$

the numbers μ and λ being defined by (27) and (30).

Since we have shown in § 3 that $\mu \geq 1$, it follows from Lemmas 2 and 7 that, if P belongs to exactly m bodies $K_\gamma^*(P_j)$, then

$$\sum_{j=1}^{m} \sigma(u_j) = \frac{m}{\mu} - \frac{1}{\mu\lambda^\gamma} \sum_{j=1}^{m} g_\gamma^\gamma (P-P_j)$$

$$\leqslant \frac{m}{\mu} \left\{ 1 - \left(\frac{m-1}{m} \right)^\mu \right\} \leqslant 1.$$

Thus the total density at each point of C_n cannot exceed unity, and we conclude that

$$(L + 2b^*)^n > N(L)\, \mathfrak{M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

where b^* is the radius of the smallest hypersphere with centre O which contains $K_\gamma^*(O)$, and $\mathfrak{M}$ is the 'mass' of $K_\gamma^*(O)$. Clearly

$$\mathfrak{M} = I_\gamma \int_0^\lambda \sigma(u)\, du^n.$$

If we now divide each side of (36) by $I_\beta N(L)$ and let $L \to \infty$, we deduce that

$$\frac{1}{\varrho_\beta'} \geqslant \frac{1}{\varrho_\beta} \geqslant \frac{I_\gamma}{I_\beta} \int_0^\lambda \sigma(u)\, du^n = \frac{I_\gamma}{\mu I_\beta} \, \frac{\lambda^n}{1 + n\delta}.$$

Since

$$\varrho_\beta' = (\tfrac{1}{2} M_\beta)^n I_\beta,$$

it follows that

$$M_\beta \leqslant 2\lambda^{-1} \left\{ \frac{I_\gamma}{\mu(1 + n\delta)} \right\}^{1/n}$$

which is (13) and (16) follows on using STIRLING's theorem.

I wish to express my thanks to Professor L. J. MORDELL for his helpful criticism of this paper.

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Mathematics. — *On Two Problems of* MAHLER. By J. W. S. CASSELS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 29, 1948.)

In this note, I use the language, and, as far as possible, the notation of the fundamental papers of MAHLER [**1**, **2**]. In particular X and $|X|$ denote a point in n-dimensional space and its distance from the origin, O, respectively, Λ is a lattice with a point at the origin, K is a star-body, and $d(\Lambda)$ and $\triangle(K)$ have the meanings assigned [**1** § 1 and § 3]. MAHLER discusses unbounded star-bodies K with a group Γ of automorphisms (homogeneous linear substitutions) Ω and the properties

(i) *There exists at least one K-admissible Λ (i.e. K is of "finite type").*

(ii) *There exists a constant c depending only on K and Γ such that if $X \in K$, there is some $\Omega \in \Gamma$ (which may depend on X) such that*

$$|\Omega X| \leqslant c.$$

(iii) *Let $X \neq O$ be any point, not necessarily $X \in K$, and let m be any number however large. Then there exists an $\Omega \in \Gamma$ such that* [1])

$$|\Omega X| \geqslant m.$$

An unbounded star-body K satisfying (i) and (ii) is said by MAHLER to be automorphic. If K satisfies (iii) as well, we shall call it "specially automorphic". MAHLER puts forward the two problems [2])

Problem α. Is every specially automorphic star-body also boundedly reducible? [**2** Problem 10, p. 629: Definition C, p. 524]

Problem β. Does every critical lattice of an automorphic star-body K have at least one point on the boundary of K? [**1** Problem D, p. 177].

We answer both questions by the following

Theorem. *There exists a specially automorphic star-body K with a critical lattice Λ having no points on the boundary of K.*

The theorem answers α in the negative, since if $H \subset K$ is a *bounded* star-body it cannot have Λ as a critical lattice, as every critical lattice of H has a point on the boundary of H [**1** Theorem 11, p. 162]. It answers β in the negative, since a specially automorphic K is *a fortiori* automorphic.

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[1]) In MAHLER's notation [**1**, § 18] condition (iii) is expressed as

$$\Sigma_\Gamma = J_\Gamma = \{0\}$$

[2]) Mr. ROGERS informs me that he and Prof. DAVENPORT have also found the answer to problem α. They use the star domain $-1 \leq xy \leq k$, where $k > 0$ is irrational, but do not determine a critical lattice.

We actually will construct K in two dimensions (i.e. K is a "star domain"), but, before doing so, enunciate some properties of the star domains

$$J_r : |xy| \leqslant r,$$

where r is a parameter.

Lemma 1.

$$\triangle(J_r) = 5^{1/2} r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If Λ is J_r-admissible but $d(\Lambda) < 2^{2/3} r$, then $\Lambda = \Lambda_{a,b}$ for some pair of real numbers a, b satisfying

$$r \leqslant |a\, b| < 2^{3/2} 5^{-1/2} r,$$

where $\Lambda_{a,b}$ is the lattice with basis

$$P = (a, b), \qquad P_1 = (\varepsilon\, a, -\varepsilon^{-1}\, b)$$

and

$$\varepsilon = \tfrac{1}{2}(1 + 5^{1/2})$$

Conversely, J_r admits all $\Lambda_{a,b}$ for which $|a\, b| \geq r$ and then

$$d(\Lambda_{a,b}) = 5^{1/2} |a\, b|$$

This is, effectively, the first of the well-known MARKOFF chain of theorems [3 Chapter VII. For the lattice-theoretic interpretation, 4 Chapter XXIV].

Lemma 2. *The points*

$$\pm P_n = (\varepsilon^n\, a, (-\varepsilon)^{-n}\, b)$$

where n is an integer (positive, negative, or zero), are all points of $\Lambda_{a,b}$. They are the only points of $\Lambda_{a,b}$ other than O for which $|x\, y| < 2|a\, b|$ (i.e. which are inner points of $J_{2|ab|}$).

For we introduce ξ, η co-ordinates by

$$x = a(\xi + \varepsilon\, \eta), \qquad y = b(\xi - \varepsilon^{-1}\, \eta) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

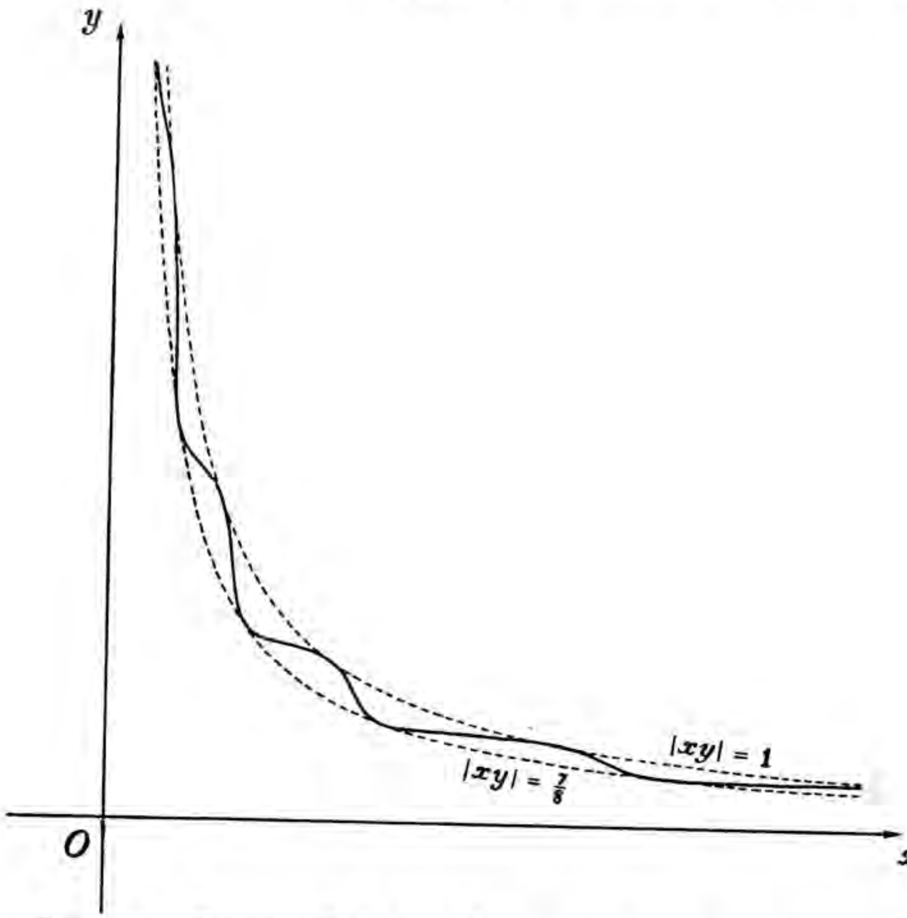
so that the points of $\Lambda_{a,b}$ are precisely those with rational integer ξ, η co-ordinates. Further $\xi + \varepsilon\, \eta$ runs through all the integers of the algebraic number-field $R(5^{1/2})$ as (ξ, η) runs through all pairs of rational integers, and $\xi - \varepsilon^{-1}\, \eta$ is the conjugate of $\xi + \varepsilon\, \eta$. But then $|x\, y| = |a\, b \text{ Norm } (\xi + \varepsilon\, \eta)| \geq 2|a\, b|$, except when $\xi = \eta = 0$ or when $\xi + \varepsilon\, \eta$ is a unit of $R(5^{1/2})$ i.e. when $\xi + \varepsilon\, \eta = \pm \varepsilon^n$, for some integer n. The result now follows from (2).

We define the domain [3]) K by

$$K: |xy| \leqslant 1 + 2^{-4}(\sin 2\pi t - 1);\ t = (\log|x| - \log|y|)/2\log 2.$$

K is a star domain since, in polar, (ϱ, φ), co-ordinates the definition of K clearly gives a non-zero upper bound for ϱ in terms of φ which is unique and continuous except that it tends to ∞ when φ tends to an integer multiple of $\pi/2$. Further

$$J_{\frac{7}{8}} \subset K \subset J_1 \quad \ldots \ldots \ldots \ldots \quad (3)$$



Lemma 3. *K is specially automorphic.*

For (i) holds since, by (3), any J_1-admissible lattice is K-admissible, and (ii) and (iii) hold since K has the automorphisms

$$\Omega^n : x' = 2^n x,\ y' = 2^{-n} y,$$

which clearly have the required properties.

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[3]) The portion of K in each quadrant is similar to that in the first. The diagram (for which I am indebted to Mr. C. S. DAVIS) is not to scale, the distance between the dotted hyperbolae having been exaggerated for the sake of clarity [by putting 2^{-3} for 2^{-4} in the definition of K].

Lemma 4.

$$\triangle(K) = 5^{1/2}.$$

It follows from (1) and (3) that $\triangle(K) \leq \triangle(J_1) = 5^{1/2}$ and so it is enough to show that $d(\Lambda) \geq 5^{1/2}$ for every K-admissible lattice Λ. Suppose not, and let $d(\Lambda) = (1-\eta)5^{1/2}$ for some K-admissible Λ, where $\eta > 0$ is a constant. Since Λ is certainly $J_{\frac{1}{4}}$-admissible and

$$(1-\eta)\, 5^{1/2} < 5^{1/2} < \tfrac{7}{8}\, 2^{3/2},$$

there are some constants a, b such that $|a\,b| = 1-\eta$ and $\Lambda = \Lambda_{a,b}$, by lemma 1. But then, by lemma 2, $P_n \in \Lambda = \Lambda_{a,b}$ and hence, as Λ is K-admissible,

$$|x_n\, y_n| \geqslant 1 + 2^{-4}(\sin 2\,\pi\, t_n - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$(x_n, y_n) = (\varepsilon^n\, a, (-\varepsilon)^{-n}\, b),$$

$$t_n = (\log|x_n| - \log|y_n|)/2\log 2 = t_0 + n\,\theta \text{ (say)}$$

and

$$\theta = \log\varepsilon/\log 2.$$

By a well-known theorem of KRONECKER [4, Chapter XXIII], there is an integer N such that

$$\sin 2\,\pi\, t_N = \sin 2\,\pi\,(t_0 + N\,\theta) > 1 - 2^4\,\eta$$

since θ is clearly irrational. As $|x_n\, y_n| = |a\,b| = 1-\eta$ this contradicts (4).

Lemma 5. *The lattice $\Lambda = \Lambda_{1,1}$ is a critical lattice of K, but it has no points on the boundary of K.*

$\Lambda_{1,1}$ is K-admissible, since it is J_1-admissible, and hence critical, since $d(\Lambda_{1,1}) = \triangle(K) = 5^{1/2}$. By lemma 2, the only points of $\Lambda_{1,1}$ which might possibly lie on the boundary of K are the

$$\pm P_n = \pm(x_n, y_n) = \pm(\varepsilon^n, (-\varepsilon)^{-n})$$

But then $t_n = t_0 + n\,\theta = n\,\theta$, and so

$$1 = |x_n\, y_n| > 1 + 2^{-4}(\sin 2\,\pi\, t_n - 1) = 1 + 2^{-4}(\sin 2\,\pi\, n\,\theta - 1)$$

since θ is irrational. This concludes the proof of the lemma, and also of the theorem since $\Lambda = \Lambda_{1,1}$ has the required properties.

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Mathematics. — *On the inversion of k-dimensional Fourier–Stieltjes-integrals.* By D. VAN DANTZIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 26, 1948.)

1. Let $V(X)$ be an absolutely additive set-function defined on all B-measurable subsets X of a k-dimensional Euclidean space R. The value of V on the k-dimensional interval (shortly: bar)

$$a_\lambda < x_\lambda \leqq a_\lambda + h_\lambda \qquad \lambda = 1, \ldots, k$$

where x_λ are Cartesian coordinates in R, will be denoted by $V(a, h)$; the Lebesgue–Stieltjes-integral of a bounded B-measurable function

$$f(x) = f(x_1, \ldots, x_k)$$

with respect to V by

$$\int f(x)\, V(x, dx)$$

the domain of integration, whenever it is not mentioned, being R.

The characteristic function $\varphi_V(t) = \varphi_V(t_1, \ldots, t_k)$ is defined as

$$\varphi_V(t) = \int e^{i \Sigma x t}\, V(x, dx)$$

where Σxt stands for $\sum_1^k {}_\lambda\, x_\lambda t_\lambda$. It exists and is continuous for every $t \in R$.

The absolute-value-function of $V(X)$ is denoted by $|V|(X)$.

V being absolutely additive, $|V|$ also is; $|V|(R) = C_1$ is finite and $|\varphi_V(t)| \leqq C_1$ for every $t \in R$.

The purpose of this paper is to give a formula expressing $V(X)$ by means of $\varphi_V(t)$. It contains as a special case such a formula due to P. LÉVY for $k = 1$ [1]) and to E. K. HAVILLAND for arbitrary k [2]).

2. If $H(X)$ is another absolutely additive set-function, then

$$\int H(t, dt)\, \varphi_V(t) = \int H(t, dt) \int e^{i\Sigma xt}\, V(x, dx) =$$
$$= \int V(x, dx) \int e^{i\Sigma xt}\, H(t, dt) = \int V(x, dx)\, \varphi_H(t),$$

the integrals being interchangeable because they converge absolutely.

If $I(t; A)$ for every $t \in R$ is a (not necessarily absolutely) additive function of A defined for a certain family of B-measurable sets A, and for every A is a bounded B-measurable function of t, then

$$W(A) \overset{\mathrm{df}}{=} \int H(t, dt)\, I(t; A)\, \varphi_V(t) = \int V(x, dx) \int H(t, dt)\, e^{i\Sigma xt}\, I(t; A)$$

also is an additive function of A, defined on the same family.

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[1]) P. LÉVY, Calcul des Probabilités, Paris 1925.

[2]) E. K. HAVILLAND. On the inversion formula for Fourier-Stieltjes transforms in more than one dimension, Amer. Journ. Math. **57**, 94—100 and 382—388 (1935).

C.f. for [1]) and [2]): H. CRAMÈR, Random Variables and Probability Distributions, Cambridge 1937.

In particular we take

$$I(t;A) \overset{\mathrm{df}}{=} \int_A e^{-i\Sigma x t}\, dx$$

(where dx stands for $dx_1 \ldots dx_k$) which is additive on the family F of all sets A consisting of a finite number of bars in R. On a bar $A = (a, h)$ (open, closed or partially closed) the value of $I(t; A)$ is

$$e^{-i\Sigma a t} \prod_\lambda \frac{e^{-it_\lambda h_\lambda} - 1}{-it_\lambda}.$$

Then, for every $x \in R$:

$$\int H(t, dt)\, e^{i\Sigma x t} \int_A e^{-i\Sigma y t}\, dy = \int_A dy \int H(t, dt)\, e^{i\Sigma(x-y)t} =$$
$$= \int_A dy\, \varphi_H(x-y), \text{ hence } W(A) = \int V(x, dx) \int_A dy\, \varphi_H(x-y).$$

All these results remain true if V and H are complex set-functions, the real and imaginary part of which are absolutely additive.

3. Now let $X\lambda$, λ being a real number, be the set of all points $x\lambda = (x_1\lambda, \ldots, x_k\lambda)$ with $x \in X$. Then $I(t; A\delta^{-1}) = \delta^{-k} I(t\delta^{-1}; A)$. We put:

$$W_\delta(A) \overset{\mathrm{df}}{=} \int H(t, dt)\, \varphi_V(t\delta^{-1})\, I(t; A\delta^{-1}) =$$
$$= \delta^{-k} \int V(x, dx) \int_A dy \int H(t, dt)\, e^{i\Sigma(x-y)t\delta^{-1}} =$$
$$= \delta^{-k} \int V(x, dx) \int_A dy\, \varphi_H\left(\frac{x-y}{\delta}\right) = \int V(x, dx) \int_{B(x)} dz\, \varphi_H(z) = \int V(x, dx)\, G(B(x)),$$

where $B(x)$ is the set of all z with $(x - z\delta) \in A$, and $G(B) \overset{\mathrm{df}}{=} \int_B dz\, \varphi_H(z)$ exists and is additive on the family of all sets B consisting of a finite number of bars.

4. We shall prove the following *Theorem*:

If 1°. A is a bar in R; $A' = \overline{A} \cap \overline{R - A}$ its boundary [3]),

2°. $V(X)$ is absolutely additive in R; $|V|(X) \leq C_1$,

3°. $|V|(A') = 0$,

4°. $H(X)$ is absolutely additive in R;

$$\varphi_H(x) \overset{\mathrm{df}}{=} \int e^{i\Sigma x t}\, H(t, dt),$$

5°. $G(B) \overset{\mathrm{df}}{=} \int_B dx\, \varphi_H(x)$ is bounded on the family of all bars in R; $|G(B)| \leq C_2$,

6°. A number $a = a(\varepsilon) > 0$ exists for every $\varepsilon > 0$, such that $|G(B)| \leq \varepsilon$ for every bar B, contained in the set R_a of all points $x \in R$ with $\|x\| \overset{\mathrm{df}}{=} \operatorname{Max}_\lambda |x_\lambda| \geq a$,

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[3]) $\overline{A}$ being the closure of A and $A \cap B$ the intersection of A and B.

7°. $G(R) = 1$,[4])

8°. $W_\delta(A) \overset{\mathrm{df}}{=} \delta^{-k} \int H(t, dt)\ \varphi_V(t\delta^{-1}) I(t\delta^{-1}; A)$, where

$$I(t; A) \overset{\mathrm{df}}{=} \int_A e^{-i\Sigma xt}\, dx,$$

then

$$\lim_{\delta \to 0} W_\delta(A) = V(A).$$

5. *Proof:* Let ε be a positive number and U a neighbourhood of the boundary A' of the bar A. As $|V|(A') = 0$, U can be chosen so small that $|V|(U) \leqq \frac{\varepsilon}{4} \operatorname{Min}(\frac{1}{C_2}, 1)$. Putting $(R-U) \cap (R-A) = F_1$ and $(R-U) \cap A = F_2$, the closed sets F_1 and F_2 have positive distances ϱ_1 and ϱ_2 [5]) from A and $R-A$ respectively. We take

$$\delta \leqq \left\{ a\left(\frac{\varepsilon}{8kC_1}\right)\right\}^{-1} \cdot \operatorname{Min}(\varrho_1, \varrho_2)$$

and intend to prove then: $|W_\delta(A) - V(A)| \leqq \varepsilon$.

Therefore we consider $W_\delta(A)$ as the sum of three parts I_1, I_2 and I_3, obtained by extending the integral (cf. 3) $\int V(x, dx) G(B(x))$ over F_1, F_2 and U respectively.

It will then be sufficient to prove:

1°. $|I_1| \leqq \frac{\varepsilon}{4}$

2°. $|I_2 - V(F_2)| \leqq \frac{\varepsilon}{4}$

3°. $|I_3| \leqq \frac{\varepsilon}{4}$

4°. $|V(A) - V(F_2)| \leqq \frac{\varepsilon}{4}$.

The latter inequality is trivial as

$$|V(A) - V(F_2)| = |\int_{A-F_2} V(x, dx)| \leqq |V|(A-F_2) \leqq |V|(U) \leqq \frac{\varepsilon}{4}.$$

The third inequality follows from

$$|I_3| = |\int_U V(x, dx)\, G(B(x))| \leqq$$

$$\leqq \int_U |V(x, dx)|\, |G(B(x))| \leqq C_2 \cdot |V|(U) \leqq \frac{\varepsilon}{4}.$$

In order to prove the first one we remark that in it $B(x)$, being the set of

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[4]) $G(R) \overset{\mathrm{df}}{=} \lim G(B)$ when B expands towards infinity in all directions independently.

[5]) Distances in the sense of $\| x - y \|$, c.f. condition 6° of the theorem.

all $z = \frac{x-y}{\delta}$ with $y \in A$, whereas $x \in F_1$, is a bar contained in R_a, where $a = a\left(\frac{\varepsilon}{8kC_1}\right)$.

In fact $\| x-y \| \geq \varrho_1 \geqq a\delta$, hence $\| z \| \geq a$.

By condition 6° we then have $| G(B(x)| \leq \frac{\varepsilon}{8kC_1}$, hence

$$|I_1| = |\int_{F_1} V(x, dx)\, G(B(x))| \leqq \int_{F_1} |V(x, dx)|\, |G(B(x))| \leqq$$
$$\leqq |V|(F_1) \cdot \frac{\varepsilon}{8kC_1} \leqq \frac{\varepsilon}{8} < \frac{\varepsilon}{4}$$

by condition 2°.

Finally the proof of the second inequality follows from the fact that $G(B)$ is additive with $G(R) = 1$:

$$|\int_{F_2} V(x, dx)\, G(B(x)) - V(F_2)| = |\int_{F_2} V(x, dx)\,\{1 - G(R-B(x))\} - V(F_2)| =$$
$$= |\int_{F_2} V(x, dx)\, G(R-B(x))| \leqq \int_{F_2} |V(x, dx)|\, |G(R-B(x))|.$$

Like before, if $z = \frac{x-y}{\delta} \in R - B(x)$, then $y \in R - A$, whereas $x \in F_2$. Hence $\| x-y \| \geq \varrho(R-A, F_2) = \varrho_2 \geq a\delta$ and $\| z \| \geq a$, so that $R - B(x) \subset R_a$. It then follows from condition 6°, that

$$|G(R-B(x))| \leqq 2k \cdot \frac{\varepsilon}{8kC_1} = \frac{\varepsilon}{4C_1}.$$

This leads immediately to the second inequality, so that the theorem is proved.

6. *Corollary*. From the proof of the theorem it follows that the inequality $| W_\delta(A) - V(A)| \leq \varepsilon$ holds for all $\delta \leq \delta_0$ and all bars A, such that the neighbourhood U of A' consisting of all points having a distance $\varrho \leq a\left(\frac{\varepsilon}{8kC_1}\right)\delta_0$ from A' satisfies the inequality $|V|(U) \leq \frac{\varepsilon}{4}$. Min $(C_2^{-1}, 1)$. In particular this is the case for all bars, having their boundary sufficiently far away. In fact taking for U_1 a "neighbourhood of infinity" (i.e. a domain such that $R - U_1$ is compact) so small that $|V|(U_1) \leq \frac{\varepsilon}{4}$. Min $(C_2^{-1}, 1)$ and for U the set of all points having a distance $\varrho > a\left(\frac{\varepsilon}{8kC_1}\right)\delta_0$ from $R - U_1$, every bar having its boundary in U will satisfy the condition. More in particular this is the case for every bar $A\lambda$, where A is any bar not containing O on its boundary and all sufficiently large λ.

Remark. The theorem and its corollary remain true if the term "bar" is defined as a set belonging to a class F of B-measurable sets, such that

1°. $I(t; A)$ exists on each of them,

2°. if A is a "bar" then the set $x + A\lambda$ of all $z \in R$ with $\frac{z-x}{\lambda} \in A$, for every $x \in R$ and every λ belongs to F,

3°. the intersection of two "bars" is a "bar",

4°. the complement of every "bar" A is the sum of at most K "bars" without common points, K being a number independent of A. The factor $8k$ occurring in $a\left(\frac{\varepsilon}{8kC_1}\right)$ has then to be replaced by K.

Here condition 4° can be omitted if to condition 6° of the theorem the following one is added:

6*. a number $a^* = a^*(\varepsilon)$ exists for every $\varepsilon > 0$, such that

$$|G(R_{a^*} - B)| \leqq \varepsilon$$

for every bar B with $R - B \subset R_{a^*}$.

Then the inequalities remain valid, if $a\left(\frac{\varepsilon}{8kC_1}\right)$ everywhere is replaced by $\operatorname{Max}\left(a\left(\frac{\varepsilon}{4C_1}\right), a^*\left(\frac{\varepsilon}{4C_1}\right)\right)$.

7. We shall now express the conditions 4°—7° of the theorem in terms of sufficient conditions for the auxiliary function $H(X)$.

A bar B will be called here *closed alee*, if it consists of those points t of its (ordinary) closure $\overline{B}$, for which a point $a \in \overline{B}$ exists, such that for every λ $0 \leqq \frac{a_\lambda}{t_\lambda} < 1$ if $t_\lambda \neq 0$ and $a_\lambda = 0$ if $t_\lambda = 0$.

A function $h(t)$ will be called ***continuous from the lee-side*** if

$$\lim_{n \to \infty} h(t^{(n)}) = h(t),$$

whenever $\lim_{n \to \infty} t_\lambda^{(n)} = t_\lambda$ with $|t_\lambda^{(n)}| \geqq |t_\lambda|$ for every λ.

If the sum $\Sigma (-1)^{p_1 + \dots + p_k} h(a_1^{(p_1)}, \dots, a_k^{(p_k)})$, the summation being extended over all combinations of p_λ equal to 0 or 1 ($\lambda = 1, \dots, k$), is defined to be the value $\psi(B)$ of a set-function ψ on the bar closed alee, whose closure is the set $a_\lambda^{(0)} \leqq t_\lambda \leqq a_\lambda^{(1)}$ ($\lambda = 1, \dots, k$), then, if h is of bounded variation, continuous from the lee-side and vanishing everywhere at infinity (i.e. $\lim h(t) = 0$ if at least one of the t_λ tends towards infinity), ψ is absolutely additive.

Under these conditions $\int_\Omega dh(t) = \psi(\Omega)$ and $\int_\Omega |dh(t)| = |\psi|(\Omega)$ for any B-measurable set Ω.

The conditions 4°—7° may now be replaced by the following set of conditions:

A. $H(X) = \int_X h(t)dt$, where $h(t)$ is a function of bounded variation, its total variation being $\leqq C_3$, vanishing everywhere at infinity.

B. $h(t)$ is continuous from the lee-side [6]).

C. $h(t)$ is symmetrical in every coördinate t_λ:
$h(t^{(1)}) = h(t^{(2)})$ if $|t_\lambda^{(1)}| = |t_\lambda^{(2)}|$ for every λ.

D. $h(t)$ is absolutely integrable over R: $\int_R |h(t)|\, dt = C_4$.

E. $h(0) = (2\pi)^{-k}$.

In the proof we use the following notations:

$$E = E(t) = E(t_1, \ldots, t_k) = E_1(t_1) \ldots E_k(t_k)$$

is a product of k one-dimensional continuous functions of bounded variation, each depending on one of the coordinates of R only. Then in a Stieltjes-integral we have: $dE = dE_1 \ldots dE_k$.

We write: $E_{\lambda_1 \ldots \lambda_m} \overset{\mathrm{df}}{=} E_{\lambda_1} \ldots E_{\lambda_m}$ $(m \leq k)$ and $h(t^*, t_{\lambda_1}, \ldots, t_{\lambda_m})$ for $h(t)$ if we want to draw the attention especially towards $t_{\lambda_1}, \ldots, t_{\lambda_m}$, t^* signifying the remaining t_λ.

Further, when B is the bar $a_\lambda^{(0)} < t_\lambda \leq a_\lambda^{(1)}$, we define:

$$[hE_{\lambda_1 \ldots \lambda_m}] = \Sigma (-1)^{p_1 + \cdots + p_m} h(t^*, a_{\lambda_1}^{(p_1)}, \ldots, a_{\lambda_m}^{(p_m)}) \cdot E_{\lambda_1}(a_{\lambda_1}^{(p_1)}) \ldots E_{\lambda_m}(a_{\lambda_m}^{(p_m)})$$

and $[h] = h$, the summation being extended over all combinations of p_λ equal to 0 or 1 $(\lambda = 1, \ldots, m)$. In these and the following formulae we shall always suppose all λ_j to be different from one another.

In the proof of the following lemma we shall use the formula:

$$\int_{t_0}^{t_1} dh(t) \int_B f(x)\, d_x\, g(x, t) = \int_B f(x)\, d_x \left\{ \int_{t_0}^{t_1} g(x, t)\, dh(t) \right\}$$

where x stands for $(x_1, \ldots, x_k)$ [7]) and the formula for integration by parts of the one-dimensional Stieltjes-integral which we may use because E_λ is supposed to be continuous for all λ [8]).

8. *Lemma:* [9]) If $h(t)$ is a function of bounded variation and

$$E = E(t) = E_1(t_1) \ldots E_k(t_k)$$

a product of k one-dimensional functions of bounded variation, each depending on one coordinate only, then

$$\int_B h(t)\, dE(t) = \sum_0^k {}^{\nu} (-1)^\nu \sum_1^{k} {}^{*} \int_{a_{\lambda_1}^{(0)}}^{a_{\lambda_1}^{(1)}} \cdots \int_{a_{\lambda_\nu}^{(0)}}^{a_{\lambda_\nu}^{(1)}} E_{\lambda_1 \ldots \lambda_\nu}\, d_{\lambda_1 \ldots \lambda_\nu} [hE_{\lambda_{\nu+1} \ldots \lambda_k}]$$

with $\sum_1^k {}^{*} = \sum_1^k {}_{\lambda_1, \ldots, \lambda_\nu}$ **with** $1 < \lambda_1 < \ldots < \lambda_\nu \equiv k$ [10]).

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6) Consequently the k-dimensional integral $\int_\Omega |dh(t)| = |\psi|(\Omega) = 0$ if Ω is the set of all coordinate-hyperplanes (i.e. the set of all t with $\prod_\lambda t_\lambda = 0$).

7) L. C. YOUNG, The theory of integration, 1927, p. 40.

8) S. SAKS, Theory of the integral, 1937, p. 102.

9) I owe the lemma and its proof to Mr. J. HEMELRIJK.

10) The summation Σ^* will always be extended over those λ's which form the index of the factor $E_{\lambda_1 \ldots \lambda_\nu}$ under the integral-sign.

Proof: For one dimension we have, according to the formula of integration by parts:

$$\int_{a^{(0)}}^{a^{(1)}} h(t_1)\, dE(t_1) = [h(t_1)\, E(t_1)]_{a^{(0)}}^{a^{(1)}} - \int_{a^{(0)}}^{a^{(1)}} E(t_1)\, dh(t_1)$$

in accordance with the lemma.

This formula holds also true, if instead of $h(t_1)$ and $dh(t_1)$ we write $h(t^*, t_1)$ and $d_1 h(t^*, t_1)$ respectively, because $h(t_1, \ldots, t_k)$ being of bounded variation, $h(t^*, t_1)$ is of bounded variation with respect to t_1 [11]).

Supposing the lemma to be valid for $k = m$ we have for $k = m + 1$ (dE being equal to $dE_{m+1} \cdot dE_{1 \ldots m}$):

$$\int_B h(t)\, dE(t) = \sum_0^m{}^{\nu} (-1)^{\nu} \sum_1^m{}^{*} \int_{a^{(0)}_{m+1}}^{a^{(1)}_{m+1}} dE_{m+1} \int_{a^{(0)}_{\lambda_1}}^{a^{(1)}_{\lambda_1}} \cdots \int_{a^{(0)}_{\lambda_\nu}}^{a^{(1)}_{\lambda_\nu}} E_{\lambda_1 \ldots \lambda_\nu}\, d_{\lambda_1 \ldots \lambda_\nu} [h E_{\lambda_{\nu+1} \ldots \lambda_m}]$$

which, according to the formula of L. C. YOUNG, mentioned above is:

$$= \sum_0^m{}^{\nu} (-1)^{\nu} \sum{}^{*} \int \cdots \int E_{\lambda_1 \ldots \lambda_\nu}\, d_{\lambda_1 \ldots \lambda_\nu} \{ \int [h E_{\lambda_{\nu+1} \ldots \lambda_m}]\, dE_{m+1} \}$$

where the limits of integration are the same as in the previous equation. Integrating the expression between the curved brackets by parts, we get:

$$= \sum_0^m{}^{\nu} (-1)^{\nu} \sum{}^{*} \int \cdots \int E_{\lambda_1 \ldots \lambda_\nu}\, d_{\lambda_1 \ldots \lambda_\nu} [h\, E_{\lambda_{\nu+1} \ldots \lambda_m, m+1}] +$$

$$+ \sum_1^{m+1}{}^{\mu} (-1)^{\mu} \sum{}^{*} \int \cdots \int E_{\lambda_1 \ldots \lambda_{\mu-1}, m+1}\, d_{\lambda_1 \ldots \lambda_{\mu-1}, m+1} [h E_{\lambda_\mu \ldots \lambda_m}] =$$

$$= \sum_0^{m+1}{}^{\sigma} (-1)^{\sigma} \sum{}^{*} \int \cdots \int E_{\lambda_1 \ldots \lambda_\sigma}\, d_{\lambda_1 \ldots \lambda_\sigma} [h E_{\lambda_{\sigma+1} \ldots \lambda_{m+1}}]$$

which proves the lemma.

9. We now prove the statement made in point 7. Putting

$$E = \prod_{\lambda=1}^{k} \frac{e^{i z_\lambda t_\lambda} - 1}{i z_\lambda}$$

and B_s being a bar symmetrical with respect to O:

$$-b < t_\lambda \leq b$$

we get, because of the symmetry of $h(t)$ (condition C):

$$\int_{B_s} h(t)\, dE(t) = \int_{B_s} h(t) \prod_{\lambda=1}^{k} e^{i z_\lambda t_\lambda}\, dt_\lambda =$$

$$= \sum_0^k{}^{\nu} (-1)^{\nu} \sum{}^{*} 2^{k-\nu} \prod_{\mu=\nu+1}^{k} \frac{\sin b z_{\lambda_\mu}}{z_{\lambda_\mu}} \int_{-b}^{+b} \cdots \int_{-b}^{+b} \prod_{\sigma=1}^{\nu} \frac{e^{i z_{\lambda_\sigma} t_{\lambda_\sigma}} - 1}{i z_{\lambda_\sigma}} \cdot$$

$$\cdot\, d_{\lambda_1 \ldots \lambda_\nu} h(t^*, t_{\lambda_{\nu+1}} = b, \ldots, t_{\lambda_k} = b).$$

[11]) Cf. E. W. HOBSON, Theory of functions I (3rd ed. 1927) p. 343--347; the definition according to HARDY and KRAUSE has been used here.

All terms of this sum, except the last one ($\nu = k$), tend to zero when $b \to \infty$, because each of them is absolutely

$$\leqq 2^k \prod_\lambda |z_\lambda^{-1}| \, |\psi| \, (R - B_s)$$

and $|\psi| \, (R - B_s) \to 0$ for $b \to \infty$.

From this we may conclude, in accordance with the definition of $\int\limits_R$ as given in the footnote of point 4:

$$\varphi_H(z) = \int\limits_R h(t) \prod_\lambda e^{iz_\lambda t_\lambda} \, dt_\lambda = \int\limits_R \prod_\lambda \frac{e^{iz_\lambda t_\lambda} - 1}{iz_\lambda} \, dh(t)$$

for $h(t)$ is absolutely integrable over R (condition D) and

$$\int\limits_R |dh(t)| = |\psi| (R) \leq C_3$$

is finite (condition A).

We shall now proceed to prove that condition 6° of point 4 follows from conditions A ... E.

If B is a bar $a_\lambda < z_\lambda \leq b_\lambda$ ($\lambda = 1, \ldots, k$) with (for instance) $0 < a \leq a_k$, we have to prove, that, by choosing a sufficiently large,

$$|G(B)| = \Big|\int\limits_B dz \, \varphi_H(z)\Big|$$

becomes $\leq \varepsilon$.

$$G(B) = \int\limits_B dz \, \varphi_H(z) = \int\limits_B dz \int\limits_R \prod_\lambda \frac{e^{iz_\lambda t_\lambda} - 1}{iz_\lambda} \, dh(t) =$$

$$= \int\limits_R dh(t) \prod_\lambda \int\limits_{a_\lambda}^{b_\lambda} \frac{e^{iz_\lambda t_\lambda} - 1}{iz_\lambda} \, dz_\lambda = \int\limits_R dh(t) \prod_\lambda \int\limits_{a_\lambda}^{b_\lambda} \frac{\sin z_\lambda t_\lambda}{z_\lambda} \, dz_\lambda$$

because the integrals, being absolutely convergent, may be interchanged, whereas the imaginary parts vanish because of the symmetry of $h(t)$.

Now for all λ we have:

$$\left| \int\limits_{a_\lambda}^{p_\lambda} \frac{\sin z_\lambda t_\lambda}{z_\lambda} \, dz_\lambda \right| \leqq \int\limits_{-\pi}^{+\pi} \frac{\sin x}{x} \, dx = p\pi$$

p being a numerical constant > 1 and < 2, whereas for $\lambda = k$:

$$\left| \int\limits_{a_k}^{b_k} \frac{\sin z_k t_k}{z_k} \, dz_k \right| \leqq \frac{2}{a \, |t_k|} \quad \text{in consequence of } 0 < a \leqq a_k.$$

Dividing R into two parts R' and R'' with $|t_k| > \eta$ and $|t_k| \leqq \eta$ respectively, we can (according to condition B, cf. the footnote) choose η so small, that

$$\int\limits_{t_k=-\eta}^{t_k=+\eta} \int\limits_{-\infty}^{+\infty} \ldots \int |dh(t)| \leqq \frac{\varepsilon}{2 (p\pi)^k}. \text{ Taking } a \geqq \frac{4 (p\pi)^{k-1}}{\varepsilon \cdot \eta} C_3$$

we get $|G(B)| < \varepsilon$.

Condition 6° being proved, it is easy to see, that 5° also is fulfilled with $C_2 = (p\pi)^k C_3$. Finally condition 4° and 7° follow from the absolute integrability of $h(t)$ and from condition E respectively.

10. *Remark:* Taking in particular $H(X) = \int_x h(t)dt$ with

$$h(t) = (2\pi)^{-k} \prod_\lambda \bar{\imath}(1-t_\lambda)\,\bar{\imath}(1+t_\lambda), \text{ where } \bar{\imath}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leqq 0 \end{cases}$$

we find:

$$\varphi_H(z) = \prod_\lambda \frac{\sin z_\lambda}{z_\lambda} \text{ and } G(z, h) = \prod_\lambda \left\{ \frac{1}{\pi} \int_{z_\lambda}^{z_\lambda + h_\lambda} \frac{\sin u_\lambda}{u_\lambda} du_\lambda \right\}.$$

Conditions B, C and E are clearly satisfied; so is A with $C_3 = \pi^{-k}$. Moreover we may take $\eta(\varepsilon) = 1$, hence

$$a(\varepsilon) = 4(p\pi)^{k-1}\varepsilon^{-1} C_3 = 4p^{k-1}(\pi\varepsilon)^{-1} \text{ and } C_2 = p^k.$$

Hence, replacing t by $t\delta$, the theorem yields the *corollary*:

$$V(A) = \lim_{\delta \to 0} (2\pi)^{-k} \int_{-\delta^{-1}}^{+\delta^{-1}} \cdots \int dt_1 \ldots dt_k \, \varphi_V(t) I(t; A)$$

which is equivalent with LÉVY's theorem for $k = 1$ and with HAVILLAND's theorem for arbitrary k.

According to the corollary of point 6 we have:

$$\left| V(A) - (2\pi)^{-k} \int_{-\delta^{-1}}^{+\delta^{-1}} \cdots \int dt_1 \ldots dt_k \, \varphi_V(t) I(t; A) \right| \leqq \varepsilon$$

for any bar A (in the sense of point 1) and any $\delta > 0$ for which the inequality $|V|(U) < \frac{\varepsilon}{4p^k}$ holds, where U is the set of all points having a distance $\leqq \frac{24\,pk^{k-1}\,C_1}{\pi\varepsilon}$ from A'.

11. *Remark:* Taking as a second special case $h(t) = (2\pi)^{-k} e^{-\frac{1}{2}Q(t)}$ where $Q(t)$ is a positive definite homogeneous quadratic form with determinant $\triangle$, we get:

$$\varphi_H(x) = (2\pi)^{-\frac{1}{2}k} \triangle^{-\frac{1}{2}} e^{-\frac{1}{2}q(x)},$$

where $q(x)$ is the reciprocal quadratic form of $Q(t)$; the conditions 4° — 7° are easily shown to be satisfied with $C_2 = 1$, whereas $a(\varepsilon)$ can be taken as $\sqrt{2b(\varepsilon)}$, where

$$\left\{ \Gamma\left(\frac{k}{2}\right) \right\}^{-1} \int_{b(\varepsilon)}^{\infty} e^{-u} u^{\frac{1}{2}k-1} du = \varepsilon.$$

We obtain:

$$V(A) = \lim_{\delta \to 0} W_\delta(A) =$$

$$= \lim_{\delta \to 0} (2\pi)^{-k} \int \underset{-\infty}{\overset{+\infty}{\dots}} \int dt_1 \dots dt_k \, e^{-\frac{1}{2}\delta^2 Q(t)} \varphi_V(t) \int \underset{A}{\dots} \int e^{-i\Sigma xt} \, dx_1 \dots dx_k =$$

$$= \lim_{\delta \to 0} (2\pi)^{-k} \int \underset{A}{\dots} \int dx_1 \dots dx_k \int \underset{-\infty}{\overset{+\infty}{\dots}} \int dt_1 \dots dt_k \, \varphi_V(t) \, e^{-\frac{1}{2}\delta^2 Q(t) - i\Sigma tx}.$$

As in this example $G(B)$ is not only additive, but absolutely additive, so that the conditions 5° and 6° can be satisfied for arbitrary B-measurable sets B, this latter equality holds for *arbitrary* domains A not containing any point of discontinuity of V on their boundary. By going through the argument of points 5 and 6 with the distance of points defined by $\varrho(x, y) = \sqrt{q(x-y)}$ instead of $\| x-y \|$, by remarking that $0 \leqq G(B) \leq \varepsilon$ for any set B having a distance $\sqrt{q(x)} \geq a(\varepsilon)$ from 0, and that $0 \leq G(B) \leq 1$ for every set B, and by taking the integrals 3° and 4° together, we find that

$$\left| V(A) - \int \underset{A}{\dots} \int dx_1 \dots dx_k \int \underset{-\infty}{\overset{+\infty}{\dots}} \int dt_1 \dots dt_k \, \varphi_V(t) \, e^{-\frac{1}{2}\delta^2 Q(t) - i\Sigma tx} \right| \leqq \varepsilon$$

for any δ and any domain A such that the open set U, consisting of all points having a distance $\sqrt{q(x-y)} < \delta a\left(\frac{\varepsilon}{3C_1}\right)$ from A', has a $|V|(U) \leq \frac{\varepsilon}{3}$

$a(\varepsilon) = \sqrt{2b(\varepsilon)}$, $b(\varepsilon)$ being defined by the tail of the $\Gamma\left(\frac{k}{2}\right)$-funciton like above.

Mathematics. — *Uitbreiding van enige identiteiten.* I. By J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of May 29, 1948.)

In de verslagen der Ned. Akad. van Wetensch., Afd. Natuurkunde, Vol. LII, N⁰. 4, 1943, gaf ik „Eenige identiteiten", die volgden uit formules, voorkomende in mijn artikel, geplaatst in de Proceedings, vol. XLV nos. 9 en 10, 1942, getiteld „Extension d'une série des fonctions de BESSEL, due à LOMMEL, et quelques séries des fonctions de BESSEL analogues", in het volgende aangeduid met I.

Deze identiteiten betroffen andere uitdrukkingen voor

$$\sum_{p=0}^{s} \frac{(\pm 1)^p (2s-2p+\nu)^k}{p!\,\Gamma(2s-p+\nu+1)} = (\pm 1)^s \sum_{p=0}^{s} \frac{(\pm 1)^p (2p+\nu)^k}{(s-p)!\,\Gamma(s+p+\nu+1)} \quad (k \text{ geheel} > 0)$$

in de volgende gevallen:

$(-1)^p$, $\nu=1$, k oneven; $(-1)^p$, $\nu=2$, k even; $(+1)^p$, ν willekeurig (in het bijzonder $\nu=1$ en $\nu=2$), k oneven; $(+1)^p$, $\nu=1$ en $\nu=2$, k even.

Het is echter ook mogelijk andere uitdrukkingen daarvoor aan te geven in de algemene gevallen:

$(+1)^p$, ν willekeurig, k even; $(-1)^p$, ν willekeurig, k oneven en k even.

1. Om daartoe te komen gaan we allereerst uit van de recurrente betrekkingen (25) en (25′) van I, n.l.

$$S_{\nu,2k+2}(x) = \nu^{2k+1}\frac{x}{2} I_{\nu-1}(x) - xS_{\nu+1,0}(x) - x\sum_{p_1=0}^{k-1}\binom{2k+1}{2p_1+1} S_{\nu+1,2p_1+2}(x). \quad (1)$$

$$\left.\begin{aligned} S_{\nu,2k+2}(x) &= (\nu-2)^{2k+1}\frac{x}{2} I_{\nu-1}(x) + xS_{\nu-1,0}(x) + \\ &+ x\sum_{p_1=0}^{k-1}\binom{2k+1}{2p_1+2} S_{\nu-1,2p_1+2}(x), \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

waarin $I_\mu(x)$ de BESSEL'sche functie voorstelt en $S_{\nu,k}(x) = \sum_{n=0}^{\infty} (-1)^n (\nu+2n)^k I_{\nu+2n}(x)$ is.

Vervangen we in (1) $S_{\nu+1,2p_1+2}(x)$ door de uitdrukking, die uit (1) volgt door daarin ν en k te vervangen door $\nu+1$ resp p_1 (en gelijktijdig p_1 door p_2), waardoor $S_{\nu,2k+2}(x)$ wordt uitgedrukt in $S_{\nu+2,2p_2+2}(x)$, en substitueren we hiervoor weer de uitdrukking, die uit (1) volgt na vervanging van ν door $\nu+2$, k door p_2 (en gelijktijdig p_1 door p_3), dan vinden we, zo voortgaande, ten slotte de algemene formule (ν willekeurig):

$$\left.\begin{array}{l} S_{\nu,2k+2}(x)=\dfrac{x}{2}\sum\limits_{r=0}^{k}(-x)^r\sum\limits_{p_1=0}^{k-r}\binom{2k+1}{2p_1+2r}\sum\limits_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-2}\cdots \\ \cdots\sum\limits_{p_r=0}^{p_{r-1}}\binom{2p_{r-1}+3}{2p_r+2}\{(\nu+r)^{2p_r+1}I_{\nu+r-1}(x)-2S_{\nu+r+1,0}(x)\}. \end{array}\right\} \quad (3)$$

Op overeenkomstige wijze volgt uit (2):

$$\left.\begin{array}{l} S_{\nu,2k+2}(x)=\dfrac{x}{2}\sum\limits_{r=0}^{k}x^r\sum\limits_{p_1=0}^{k-r}\binom{2k+1}{2p_1+2r}\sum\limits_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-2}\cdots \\ \cdots\sum\limits_{p_r=0}^{p_{r-1}}\binom{2p_{r-1}+3}{2p_r+2}\{(\nu-r-2)^{2p_r+1}I_{\nu-r-1}(x)+2S_{\nu-r-1,0}(x)\}. \end{array}\right\} \quad (4)$$

Hierin is, zo $r=0$ is, voor

$$\sum_{p_r=0}^{p_{r-1}}\binom{2p_{r-1}+3}{2p_r+2}(\nu+r)^{2p_r+1} \text{ resp. } (\nu-r-2)^{2p_r+1}$$

te nemen ν^{2k+1} resp. $(\nu-2)^{2k+1}$. De afleiding dezer formules werd in I achterwege gelaten, omdat hierdoor $S_{\nu,2k+2}(x)$, ν willekeurig, wordt uitgedrukt in oneindige reeksen $S_{\mu,0}(x)$, die slechts door een eindige vorm zijn voor te stellen, en dus ook $S_{\nu,2k+2}(x)$, indien μ geheel is, hetgeen in I als doel werd gesteld.

Met het oog op de algemene identiteiten, waarom het nu gaat, zijn deze formules wel van belang.

Door nl. in (1) en (2) te substitueren:

$$S_{\nu,2k+2}(x)=\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}\sum_{p=0}^{s}\frac{(2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)},$$

$$\left(\frac{x}{2}\right)^{r+1}\{(\nu+r)^{2p_r+1}I_{\nu+r-1}(x)-2S_{\nu+r-1,0}(x)\}=$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m+2r+\nu}}{m!\,\Gamma(m+r+\nu)}(\nu+r)^{2p_r+1}-2\sum_{m=0}^{\infty}(-1)^m\left(\frac{x}{2}\right)^{2m+2r+\nu+2}\sum_{p=0}^{m}\frac{1}{(m-p)!\,\Gamma(m+p+r+\nu+2)}=$$

$$=(-1)^r\sum_{s=r}^{\infty}\frac{(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s+\nu)}(\nu+r)^{2p_r+1}+(-1)^r\,2\sum_{s=r+1}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}\sum_{p=0}^{s-r-1}\frac{1}{(s-r-p-1)!\,\Gamma(s+p+\nu+1)},$$

$$\left(\frac{x}{2}\right)^{r+1}\{(\nu-r-2)^{2p_r+1}I_{\nu-r-1}(x)+2\,S_{\nu-r-1,0}(x)\}=$$

$$=\sum_{s=0}^{\infty}\frac{(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-r+\nu)}(\nu-r-2)^{2p_r+1}+2\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}\sum_{p=0}^{s}\frac{1}{(s-p)!\,\Gamma(s-r+p+\nu)}$$

en daarna aan elkaar gelijk te stellen de coëfficienten van $\left(\frac{x}{2}\right)^{2s+\nu}$ in beide leden, worden we gevoerd tot de algemene identiteiten (ν willekeurig):

$$\left.\begin{aligned}
&\sum_{p=0}^{s}\frac{(2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)}= \\
&=\sum_{r=0}^{k}2^r\sum_{p_1=0}^{k-r}\binom{2k+1}{2p_1+2r}\sum_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-2}\cdots \\
&\cdots\sum_{p_r=0}^{p_{r-1}}\binom{2p_{r-1}+3}{2p_r+2}\left\{\frac{(\nu+r)^{2p_r+1}}{(s-r)!\,\Gamma(s+\nu)}+2\sum_{p=0}^{s-r-1}\frac{1}{(s-r-p-1)!\,\Gamma(s+p+\nu+1)}\right\}= \\
&=\sum_{r=0}^{k}2^r\sum_{p_1=0}^{k-r}\binom{2k+1}{2p_1+2r}\sum_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-2}\cdots \\
&\cdots\sum_{p_r=0}^{p_{r-1}}\binom{2p_{r-1}+3}{2p_r+2}\left\{\frac{(\nu-r-2)^{2p_r+1}}{s!\,\Gamma(s-r+\nu)}+2\sum_{p=0}^{s}\frac{1}{(s-p)!\,\Gamma(s-r+p+\nu)}\right\}
\end{aligned}\right\}. \quad (5)$$

Hierin is, zo $r=0$ is, voor $\sum_{p_r=0}^{p_{r-1}}\binom{2p_{r-1}+3}{2p_r+2}(\nu+r)^{2p_r+1}$ resp. $(\nu-r-2)^{2p_r+1}$ te nemen ν^{2k+1} resp. $(\nu-2)^{2k+1}$.

Door in (1) voor $S_{\nu+1,2p_1+2}(x)$ te substitueren de uitdrukking, die uit (2) volgt door vervanging van ν en k door $\nu+1$ resp. p_1 (en gelijktijdig p_1 door p_2), evenzo in (2) voor $S_{\nu-1,p_1+2}(x)$ te substitueren de uitdrukking, die uit (1) volgt door vervanging van ν en k door $\nu-1$ resp. p_1 (en gelijktijdig p_1 door p_2), worden verkregen de recurrente betrekkingen, reeds in I onder (26) en (26′) vermeld:

$$\begin{aligned}
S_{\nu,2k+2}(x) &= \nu^{2k+1}\frac{x}{2}I_{\nu-1}(x)-xS_{\nu+1,0}(x)- \\
&-\frac{x^2}{2}I_\nu(x)\sum_{p_1=0}^{k-1}\binom{2k+1}{2p_1+2}(\nu-1)^{2p_1+1}-x^2S_{\nu,0}(x)\sum_{p_1=0}^{k-1}\binom{2k+1}{2p_1+2}- \\
&\qquad -x^2\sum_{p_1=0}^{k-2}\binom{2k+1}{2p_1+4}\sum_{p_2=0}^{p_1}\binom{2p_1+3}{2p_2+2}S_{\nu,2p_2+2}(x)
\end{aligned}$$

en

$$\begin{aligned}
S_{\nu,2k+2}(x) &= (\nu-2)^{2k+1}\frac{x}{2}I_{\nu-1}(x)+xS_{\nu-1,0}(x)+ \\
&+\frac{x^2}{2}I_{\nu-2}(x)\sum_{p_1=0}^{k-1}\binom{2k+1}{2p_1+2}(\nu-1)^{2p_1+1}-x^2S_{\nu,0}(x)\sum_{p_1=0}^{k-1}\binom{2k+1}{2p_1+2}- \\
&\qquad -x^2\sum_{p_1=0}^{k-2}\binom{2k+1}{2p_1+4}\sum_{p_2=0}^{p_1}\binom{2p_1+3}{2p_2+2}I_{\nu,2p_2+2}(x).
\end{aligned}$$

Deze voeren ieder afzonderlijk, op gelijke wijze als in het vorige geval, tot algemene formules (ν willekeurig), waardoor $S_{\nu,2k+2}(x)$ wordt uitgedrukt in $S_{\nu+1,0}(x)$ en $S_{\nu,0}(x)$ resp. $S_{\nu-1,0}(x)$ en $S_{\nu,0}(x)$, hetgeen in I weer achterwege werd gelaten. We vinden dan:

$$\left.\begin{aligned}
S_{\nu,2k+2}(x) &= \frac{x}{2} \sum_{r=0}^{\leq \frac{k}{2}} (-x^2)^r \sum_{p_1=0}^{k-2r} \binom{2k+1}{2p_1+4r} \sum_{p_2=0}^{p_1} \binom{2p_1+4r-1}{2p_2+4r-2} \cdots \\
&\quad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+3}{2p_{2r}+2} \{ \nu^{2p_{2r}+1} I_{\nu-1}(x) - 2S_{\nu+1,0}(x) \} - \\
&- 2\left(\frac{x}{2}\right)^2 \sum_{r=0}^{\leq \frac{k-1}{2}} (-x^2)^r \sum_{p_1=0}^{k-2r-1} \binom{2k+1}{2p_1+4r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+4r+1}{2p_2+4r} \cdots \\
&\quad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+3}{2p_{2r+1}+2} \{ (\nu-1)^{2p_{2r+1}+1} I_\nu(x) + 2S_{\nu,0}(x) \} = \\
&= \frac{x}{2} \sum_{r=0}^{\leq \frac{k}{2}} (-x^2)^r \sum_{p_1=0}^{k-2r} \binom{2k+1}{2p_1+4r} \sum_{p_2=0}^{p_1} \binom{2p_1+4r-1}{2p_2+4r-2} \cdots \\
&\quad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+3}{2p_{2r}+2} \{ (\nu-2)^{2p_{2r}+1} I_{\nu-1}(x) + 2S_{\nu-1,0}(x) \} + \\
&+ 2\left(\frac{x}{2}\right)^2 \sum_{r=0}^{\leq \frac{k-1}{2}} (-x^2)^r \sum_{p_1=0}^{k-2r-1} \binom{2k+1}{2p_1+4r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+4r+1}{2p_2+4r} \cdots \\
&\quad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+3}{2p_{2r+1}+2} \{ (\nu-1)^{2p_{2r+1}+1} I_{\nu-2}(x) - 2S_{\nu,0}(x) \}.
\end{aligned}\right| . \qquad (6)$$

Hierin is, zo $r = 0$ is, voor

$$\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+3}{2p_{2r}+2} \nu^{2p_{2r}+1} \text{ resp. } (\nu-2)^{2p_{2r}+1}$$

te nemen ν^{2k+1} res. $(\nu-2)^{2k+1}$.

Door in (6) te substitueren:

$$S_{\nu,2k+2}(x) = \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s} \frac{(2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{ \nu^{2p_{2r}+1} I_{\nu-1}(x) - 2S_{\nu+1,0}(x) \} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\dfrac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)} \nu^{2p_{2r+1}+1} +$$

$$+ (-1)^r\, 2 \sum_{s=r+1}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s-r-1} \frac{1}{(s-r-p-1)!\,\Gamma(s-r+p+\nu+1)}.$$

$$\left(\frac{x}{2}\right)^{2r+2}\{(\nu-1)^{2p_{2r+1}+1} I_\nu(x) + 2S_{\nu,0}(x)\} =$$

$$= (-1)^{r+1} \sum_{s=r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r-1)!\,\Gamma(s-r+\nu)} (\nu-1)^{2p_{2r+1}+1} +$$

$$+ (-1)^{r+1}\, 2 \sum_{s=r+1}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s-r-1} \frac{1}{(s-r-p-1)!\,\Gamma(s-r+p+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+1}\{\nu-2)^{2p_{2r}+1} I_{\nu-1}(x) + 2S_{\nu-1,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)} (\nu-2)^{2p_{2r}+1} +$$

$$+ (-1)^r\, 2 \sum_{s=r}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s-r} \frac{1}{(s-r-p)!\,\Gamma(s-r+p+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+2}\{(\nu-1)^{2p_{2r+1}+1} I_{\nu-2}(x) - 2S_{\nu,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu+1)} (\nu-1)^{2p_{2r+1}+1} -$$

$$-(-1)^r\, 2 \sum_{s=r+1}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s-r-1} \frac{1}{(s-r-p-1)!\,\Gamma(s-r+p+\nu)},$$

en daarna aan elkaar gelijk te stellen de coefficienten van $\left(\frac{x}{2}\right)^{2s+\nu}$ in beide leden, worden we gevoerd tot de algemene identiteiten (ν willekeurig):

$$\left.\begin{aligned}
&\sum_{p=0}^{s} \frac{(2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)} = \\
&= \sum_{r=0}^{\leq \frac{k}{2}} 2^{2r} \sum_{p_1=0}^{k-2r} \binom{2k+1}{2p_1+4r} \sum_{p_2=0}^{p_1} \binom{2p_1+4r-1}{2p_2+4r-2} \cdots \\
&\quad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+3}{2p_{2r}+2} \left\{ \frac{\nu^{2p_{2r}+1}}{(s-r)!\,\Gamma(s-r+\nu)} + 2\sum_{p=0}^{s-r-1} \frac{1}{(s-r-p-1)!\,\Gamma(s-r+p+\nu+1)} \right\} + \\
&+ 2 \sum_{r=0}^{\leq \frac{k-1}{2}} 2^{2r} \sum_{p_1=0}^{k-2r-1} \binom{2k+1}{2p_1+4r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+4r+1}{2p_2+4r} \cdots \\
&\quad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+3}{2p_{2r+1}+2} \left\{ \frac{(\nu-1)^{2p_{2r+1}+1}}{(s-r-1)!\,\Gamma(s-r+\nu)} + 2\sum_{p=0}^{s-r-1} \frac{1}{(s-r-p-1)!\,\Gamma(s-r+p+\nu)} \right\} =
\end{aligned}\right\} . \quad (7)$$

(Zie volgende pagina)

$$\left.\begin{aligned}
&= \sum_{r=0}^{\leqq \frac{k}{2}} 2^{2r} \sum_{p_1=0}^{k-2r} \binom{2k+1}{2p_1+4r} \sum_{p_2=0}^{p_1} \binom{2p_1+4r-1}{2p_2+4r-2} \cdots \\
&\cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+3}{2p_{2r}+2} \left\{ \frac{(\nu-2)^{2p_{2r}+1}}{(s-r)!\,\Gamma(s-r+\nu)} + 2 \sum_{p=0}^{s-r} \frac{1}{(s-r+p)!\,\Gamma(s-r+p+\nu)} \right\} + \\
&+ 2 \sum_{r=0}^{\leqq \frac{k-1}{2}} 2^{2r} \sum_{p_1=0}^{k-2r-1} \binom{2k+1}{2p_1+4r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+4r+1}{2p_2+4r} \cdots \\
&\cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+3}{2p_{2r+1}+2} \left\{ \frac{(\nu-1)^{2p_{2r+1}+1}}{(s-r)!\,\Gamma(s-r+\nu-1)} + 2 \sum_{p=0}^{s-r-1} \frac{1}{(s-r-p-1)!\,\Gamma(s-r+p+\nu)} \right\}
\end{aligned}\right\} . \quad (7)$$

Hierin is, zo $r=0$ is, voor $\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+3}{2p_{2r}+2} \nu^{2p_{2r}+1}$ resp. $(\nu-2)^{2p_{2r}+1}$ te nemen ν^{2k+1} resp. $(\nu-2)^{2k+1}$.

Uit (5) en (7) volgt voor $k=0, \nu$ willekeurig:

$$\sum_{p=0}^{s} \frac{(2p+\nu)^2}{(s-p)!\,\Gamma(s+p+\nu+1)} = \frac{\nu}{s!\,\Gamma(s+\nu)} + 2 \sum_{p=0}^{s-1} \frac{1}{(s-p-1)!\,\Gamma(s+p+\nu+1)}. \quad (8)$$

In verband met de indentiteiten:

$$\sum_{p=0}^{n} \frac{1}{(n-p)!\,(n+p)!} = \frac{2^{2n-1}}{(2n)!} + \frac{1}{2(n!)^2}, \quad \sum_{p=0}^{n} \frac{1}{(n-p)!\,(n+p+1)!} = \frac{2^{2n}}{(2n+1)!},$$

$$\sum_{p=0}^{n} \frac{1}{(n-p)!\,(n+p+2)!} = \frac{2^{2n+1}}{(2n+2)!} - \frac{1}{2\{(n+1)!\}^2},$$

welke o.a. af te leiden zijn uit:

$$S_{0,0}(x) = \sum_{n=0}^{\infty} (-1)^n I_{2n}(x) = \tfrac{1}{2}\{I_0(x) + \cos x\}, \; S_{1,0}(x) = \sum_{n=0}^{\infty} (-1)^n I_{2n+1}(x) = \tfrac{1}{2} \sin x,$$

$$S_{2,0}(x) = \sum_{n=0}^{\infty} (-1)^n I_{2n+2}(x) = \tfrac{1}{2}\{I_0(x) - \cos x\}$$

volgt uit (7) voor $\nu = 1$:

$$\sum_{p=0}^{s} \frac{(2p+1)^{2k+2}}{(s-p)!\,(s+p+1)!} =$$

$$= 2^{2s} \sum_{r=0}^{k} \frac{1}{(2s-r)!} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-2} \cdots \sum_{p_r=0}^{p_{r-1}} \binom{2p_{r-1}+3}{2p_r+2},$$

evenzo uit de tweede van (7) voor $\nu = 2$:

$$\sum_{p=0}^{s} \frac{(2p+2)^{2k+2}}{(s-p)!\,(s+p+2)!} =$$

$$= 2^{2s+1} \sum_{r=0}^{k} \frac{1}{(2s-r+1)!} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \sum_{p_r=0}^{p_{r-1}} \binom{2p_{r-1}+3}{2p_r+2},$$

welke bijzondere gevallen voorkomen als (9) en (10) in het aangehaalde artikel „Eenige identiteiten".

Mathematics. — *Lattice points in non-convex regions.* I. By P. MULLENDER.
(Communicated by Prof. A. HEYTING.)

(Communicated at the meeting of May 29, 1948.)

Introduction. We consider an n-dimensional space of points $(z_1, \ldots, z_n)$ with real coordinates and suppose Λ to be an n-dimensional lattice in that space with determinant $\triangle \neq 0$. We are interested in those lattices, of which at least one point other than the origin O is contained in a given region R. And we try to find constants A, such that any lattice Λ with $|\triangle| \leq A$ has a point other than O contained in R. In this paper we only consider non-convex regions R, which are closed and symmetrical about O. We use the following theorem, which is a corollary of a theorem of BLICHFELDT [1]):

Theorem 1. *Let K be any bounded region of volume V_k and let R be a closed region containing all the points $P_1 - P_2$ with P_1 and P_2 in K, then R contains a point other than O of any lattice with determinant $\triangle \neq 0$, for which $|\triangle| \leq V_k$.*

So our problem is, to construct a region K, with the largest possible volume V_k, such that all points $P_1 - P_2$, with P_1 and P_2 in K, are points of R.

It is very easy to solve this problem, if R is convex (and symmetrical about O). The best possible region K is then the set of points we obtain by multiplying the coordinates of every point of R by $\frac{1}{2}$. But, if R is a non-convex region, such a general solution is not known. In 1945, however, MORDELL [2]) stated a few theorems concerning non-convex regions, which possessed a certain amount of generality. He gave in fact a generalisation of BLICHFELDT's application of theorem 1 to the problem of the simultaneous approximation of $n-1$ real numbers by rational fractions with the same denominator.

In this paper we give a further development of BLICHFELDT's method, in some cases leading to better results.

I. A two-dimensional construction of the region K.

1. The Problem. We suppose that p and q are positive integers and $p + q = n$. Let

$$M \equiv M(u) \equiv M(u_1, \ldots, u_p) \text{ and } N \equiv N(v) \equiv N(v_1, \ldots, v_q)$$

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1) H. F. BLICHFELDT, A New Principle in the Geometry of Numbers with some Applications, Trans. Am. Math. Soc., **15** 227—235 (1914).

2) L. J. MORDELL, Lattice points in some n-dimensional non-convex regions, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 773—792 (1946).

be defined for all real values of the variables $u_1, \dots, u_p, v_1, \dots, v_q$. We suppose that these functions have the following properties:

$$M(u) > 0, \text{ if } (u) \neq (0), \; N(v) > 0, \text{ if } (v) \neq (0); \quad . \quad . \quad . \quad . \quad (1)$$

$$M(tu) = |t|\, M(u), \; N(tv) = |t|\, N(v), \text{ for any } t; \quad . \quad . \quad . \quad . \quad (2)$$

$$M(u') + M(u'') \geqq M(u' + u''), \; N(v') + N(v'') \geqq N(v' + v'') \quad . \quad (3)$$

Let $f(x)$ be a single valued, positive, steadily decreasing, differentiable function of x, defined for $0 < x < a$, with steadily increasing derivative. Put $f(0) = \lim_{x \to +0} f(x)$ and $f(a) = \lim_{x \to a-0} f(x)$, where $f(0)$ and a may be infinite. We suppose that $f(a) = 0$.

Throughout part I of this paper we suppose R to be defined by the inequalities

$$x \leqq a, \qquad y \leqq f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$x = M(z_1, \dots, z_p) \text{ and } y = N(z_{p+1}, \dots, z_n).$$

Clearly R is closed and symmetrical about O. If a and $f(0)$ both are finite, then R is also bounded, since by (1) and (2) all the coordinates z are bounded.

In the same way we may define a bounded region K by

$$x \leqq a, \qquad y \leqq \varphi(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where a is finite and $\varphi(x)$ is a certain bounded, single valued, positive function of x, defined for $0 \leq x \leq a$. And we have achieved our purpose, i.e. this region K satisfies our condition, if

$$F(x_1, x_2) \equiv f(x_1 + x_2) - \varphi(x_1) - \varphi(x_2) \geqq 0, \quad . \quad . \quad . \quad . \quad (6)$$

for all x_1 and x_2 of the interval $0 \leq x \leq a$. For suppose

$$P_1 = (z'_1, \dots, z'_n) \text{ and } P_2 = (z''_1, \dots, z''_n)$$

are points of K, then

$$\begin{aligned}
N(z'_{p+1} - z''_{p+1}, \dots, z'_n - z''_n) &\leqq \\
&\leqq N(z'_{p+1}, \dots, z'_n) + N(z''_{p+1}, \dots, z''_n) && \text{by (2) and (3)} \\
&\leqq \varphi(M(z'_1, \dots, z'_p)) + \varphi(M(z''_1, \dots, z''_p)) && \text{by hypothesis} \\
&\leqq f(M(z'_1, \dots, z'_p) + M(z''_1, \dots, z''_p)) && \text{by (6),} \\
&\leqq f(M(z'_1 - z''_1, \dots, z'_p - z''_p)), && \text{by (2) and (3),}
\end{aligned}$$

as $f(x)$ is decreasing, and this means, that $P_1 - P_2$ is contained in R.

Hence our problem is reduced to the two-dimensional one of finding a function $\varphi(x)$, such that, for a given $f(x)$, the inequality (6) is satisfied.

2. The Solution. First we state the solution of MORDELL.

Take $0 < \xi < a$ and suppose that

$$f(\xi) + \xi f'(\xi) > 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This means, that the tangent of the curve $y = f(x)$ at the point $(\xi, f(\xi))$ makes an intercept on the x-axis, greater than 2ξ. MORDELL shows, that this condition does not involve any loss of generally, as it always can be satisfied, if necessary by interchanging the coordinates x and y, except in the case $x \cdot f(x) =$ constant.

Now we can state MORDELL's result in the following form:

Theorem 2. *The inequality*

$$F(x_1, x_2) \equiv f(x_1 + x_2) - \varphi(x_1) - \varphi(x_2) \geqq 0 \quad . \quad . \quad . \quad . \quad (6)$$

is satisfied by the function $\varphi(x)$, *defined by*

$$\varphi(x) = \tfrac{1}{2} f(\xi) - \tfrac{1}{2} \xi f'(\xi) + x \,.\, f'(\xi), \text{ for } 0 \leqq x \leqq \xi, \quad . \quad . \quad (8)$$

$$\varphi(x) = -\tfrac{1}{2} f(\xi) + \tfrac{1}{2} \xi f'(\xi) + f(x), \text{ for } \xi \leqq x \leqq a, \quad . \quad . \quad (9)$$

with a *following from*

$$f(a) = \tfrac{1}{2} f(\xi) - \tfrac{1}{2} \xi f'(\xi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

for all x_1 *and* x_2 *of the interval* $0 \leq x \leq a$.

MORDELL was led to this result by geometrical considerations. The theorems we want to prove in this paper have also been found in a geometrical way. But, as the description of the geometrical ideas leading to the proofs would be very complicated, we shall only give the analytical demonstration.

Let $g(x)$ be a single valued, differentiable function of x, defined for $0 \leq x \leq \frac{1}{2}\xi$, with $g(\frac{1}{2}\xi) = \frac{1}{2}\xi$. We assume that $g'(x) < 0$. Hence $\frac{1}{2}\xi < g(0) = \beta$ say. Let $\beta < a - \frac{1}{2}\xi$.

If we put

$$\bar{x} = g(x),$$

then x is also a single valued, differentiable function of $\bar{x}$,

$$x = h(\bar{x}),$$

defined for $\frac{1}{2}\xi \leq \bar{x} \leq \beta$ and

$$h'(\bar{x}) \equiv h'(g(x)) \equiv \frac{1}{g'(x)} \equiv \frac{1}{g'(h(\bar{x}))}.$$

We now define

$$\varphi(x) = \tfrac{1}{2} f(\xi) - \int_x^{\frac{1}{2}\xi} f'(t + g(t))\, dt, \text{ for } 0 \leqq x \leqq \tfrac{1}{2}\xi, \quad . \quad . \quad . \quad (11)$$

$$\varphi(x) = \tfrac{1}{2} f(\xi) + \int_{\frac{1}{2}\xi}^{x} f'(h(t) + t)\, dt, \text{ for } \tfrac{1}{2}\xi \leqq x \leqq \beta, \quad . \quad . \quad (12)$$

$$\varphi(x) = f(x) - \varphi(0), \qquad \text{for } \beta \leqq x \leqq a \quad . \quad . \quad (13)$$

Clearly (11) and (12) indicate the same value for $\varphi(\frac{1}{2}\xi)$, namely $\varphi(\frac{1}{2}\xi) = \frac{1}{2}f(\xi)$. We must, however, prove that (12) and (13) also indicate the same value for $\varphi(\beta)$.

For $0 \leqq x \leqq \frac{1}{2}\xi$

$$F(x, g(x)) = f(x + g(x)) - f(\xi) + \int_x^{\frac{1}{2}\xi} f'(t + g(t))\, dt - \int_{\frac{1}{2}\xi}^{g(x)} f'(h(t) + t)\, dt,$$

by (6), (11) and (12). But

$$\int_{\frac{1}{2}\xi}^{g(x)} f'(h(t) + t)\, dt = - \int_x^{\frac{1}{2}\xi} f'(\tau + g(\tau))\, g'(\tau)\, d\tau,$$

by the substitution $\tau = h(t)$, since $t = g(\tau)$ and $dt = g'(\tau) d\tau$. Again replacing τ by t, we get

$$\int_x^{\frac{1}{2}\xi} f'(t + g(t))\, dt - \int_{\frac{1}{2}\xi}^{g(x)} f'(h(t) + t)\, dt = \int_x^{\frac{1}{2}\xi} f'(t + g(t)) \cdot (1 + g'(t))\, dt =$$

$$= \int_{x+g(x)}^{\xi} f'(s)\, ds = f(\xi) - f(x + g(x)).$$

Hence

$$F(x, g(x)) = f(x + g(x)) - \varphi(x) - \varphi(g(x)) = 0, \quad . \quad . \quad . \quad (14)$$

for $0 \leqq x \leqq \frac{1}{2}\xi$.

In particular

$$F(0, \beta) = f(\beta) - \varphi(0) - \varphi(\beta) = 0.$$

Hence we have, by (11) and (12),

$$\varphi(\beta) = f(\beta) - \varphi(0),$$

and this is the same value as is indicated by (13). It follows that $\varphi(x)$ is a continuous function of x.

Further, by (11), (12) and (13),

$$\varphi'(x) = f'(x + g(x)), \text{ for } 0 \leqq x \leqq \tfrac{1}{2}\xi, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\varphi'(x) = f'(h(x) + x), \text{ for } \tfrac{1}{2}\xi \leqq x \leqq \beta, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\varphi'(x) = f'(x), \qquad \text{for } \beta \leqq x \leqq a \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Since $f'(\frac{1}{2}\xi + g(\frac{1}{2}\xi)) = f'(\xi) = f'(h(\frac{1}{2}\xi) + \frac{1}{2}\xi)$ and $f'(h(\beta) + \beta) = f'(\beta)$, $\varphi'(x)$ is also continuous and we can say that $\varphi(x)$ is differentiable for all $x \geqq 0$.

By (15), (16) and (17), $\varphi'(x) < 0$ for all $x \geqq 0$ and so $\varphi(x)$ is steadily decreasing. Now $\varphi(\frac{1}{2}\xi) = \frac{1}{2} f(\xi) > 0$ and, for sufficiently large $x \geqq \beta$, $\varphi(x) = f(x) - \varphi(0) < 0$, since $\lim\limits_{x \to \alpha - 0} f(x) = 0$. Hence there exists exactly one number α, defined by

$$\varphi(\alpha) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and clearly $\frac{1}{2}\xi < \alpha < a$.

We now want to use this function $\varphi(x)$, but only in the interval $0 \leqq x \leqq \alpha$, to define the region K. That we do indeed find a suitable region K by doing so, follows from:

Theorem 3. *The function $\varphi(x)$, defined by (11), (12) and (13), satisfies the inequality (6) for all x_1 and x_2 of the interval $0 \leqq x \leqq a$ and hence also of the interval $0 \leqq x \leqq \alpha$.*

By taking $g(x) \equiv \xi - x$ we obtain theorem 2 as a corollary of this theorem.

R e m a r k. Our function $\varphi(x)$ can also be replaced by a more general one. But since we have not been able to find any improvement of the results by the use of this generalisation, we have restricted ourselves to the given solution.

In order to apply theorem 3 we only need to calculate V_k.

Writing

$$\iint \dots \int_{M \leqq 1} du_1 \dots du_p = P \text{ and } \iint \dots \int_{N \leqq 1} dv_1 \dots dv_q = Q, \quad . \quad . \quad (19)$$

where $M = M(u_1, \dots, u_p)$ and $N = N(v_1, \dots, v_q)$ are defined as in § 1, we have, by (2), for any positive c and d,

$$\iint \dots \int_{M \leqq c} du_1 \dots du_p = P c^p \text{ and } \iint \dots \int_{N \leqq d} dv_1 \dots dv_q = Q d^q.$$

Hence, for any continuous, single valued, positive function $\varphi(x)$, defined for $0 \leq x \leq x_0$, say,

$$J(x) \equiv \iint \dots \int_{\substack{N \leqq \varphi(M) \\ M \leqq x}} du_1 \dots du_p dv_1 \dots dv_q = Q \iint \dots \int_{M \leqq x} (\varphi(M))^q du_1 \dots du_p =$$

$$= Q.\, P.\, p. \int_0^x (\varphi(x))^q x^{p-1} dx,$$

or

$$J(x) = p P Q \int_0^x (\varphi(x))^q x^{p-1} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now defining $\varphi(x)$ by (11), (12) and (13) and taking $x_0 = \alpha$, we have

$$V_k = J(\alpha) = pPQ \int_0^\alpha (\varphi(x))^q x^{p-1} dx \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence, by theorems 1 and 3:

Theorem 4. *If R is given by (4), then R contains a point other than O of any lattice Λ with determinant $\triangle \neq 0$, for which*

$$|\triangle| \leqq pPQ \int_0^\alpha (\varphi(x))^q x^{p-1} dx,$$

where P and Q are defined by (19), $\varphi(x)$ by (11), (12) and (13) and α by (18).

3. The Proof. For the proof of theorem 3 six cases are to be considered, the remaining cases arise by mere interchange of x_1 and x_2.

(I) $$0 \leqq x_1 \leqq \tfrac{1}{2}\xi, \qquad 0 \leqq x_2 \leqq \tfrac{1}{2}\xi.$$

Then, by (15),

$$F'_{x_1}(x_1, x_2) = f'(x_1 + x_2) - \varphi'(x_1) = f'(x_1 + x_2) - f'(x_1 + g(x_1)).$$

Since $g(x_1) \geq \frac{1}{2}\xi \geq x_2$ and $f'(x)$ is increasing, it follows that

$$F'_{x_1}(x_1, x_2) \leq 0.$$

Similarly $F'_{x_2}(x_1, x_2) \leq 0$. Hence, by (14),

$$F(x_1, x_2) \geqq F(\tfrac{1}{2}\xi, \tfrac{1}{2}\xi) = 0.$$

(II) $$0 \leqq x_1 \leqq \tfrac{1}{2}\xi, \qquad \tfrac{1}{2}\xi \leqq x_2 \leqq \beta.$$

Then again

$$F'_{x_1}(x_1, x_2) = f'(x_1 + x_2) - f'(x_1 + g(x_1)).$$

But in this case

$$F'_{x_1}(x_1, x_2) \gtreqless 0, \text{ if } x_2 \gtreqless g(x_1), \text{ i.e. if } x_1 \gtreqless h(x_2).$$

Hence $F(x_1, x_2)$ attains its least value if $x_1 = h(x_2)$. And so, by (14),

$$F(x_1, x_2) \geqq F(h(x_2), x_2) = F(h(x_2), g(h(x_2))) = 0.$$

(III) $$0 \leqq x_1 \leqq \tfrac{1}{2}\xi, \qquad \beta \leqq x_2 \leqq a.$$

As before,

$$F'_{x_1}(x_1, x_2) = f'(x_1 + x_2) - f'(x_1 + g(x_1)).$$

This implies here $F'_{x_1}(x_1, x_2) \geq 0$, since $g(x_1) \leq \beta \leq x_2$. Hence, by (13),

$$F(x_1, x_2) \geqq F(0, x_2) = f(x_2) - \varphi(0) - \varphi(x_2) = 0.$$

(IV) $$\tfrac{1}{2}\xi \leqq x_1 \leqq \beta, \qquad \tfrac{1}{2}\xi \leqq x_2 \leqq \beta.$$

By (16),

$$F'_{x_1}(x_1, x_2) = f'(x_1 + x_2) - f'(h(x_1) + x_1).$$

Since $h(x_1) \leqq \frac{1}{2}\xi \leqq x_2$, $F'_{x_1}(x_1, x_2) \geqq 0$ and similarly $F'_{x_2}(x_1, x_2) \geqq 0$. Hence, by (14),

$$F(x_1, x_2) \geqq F(\tfrac{1}{2}\xi, \tfrac{1}{2}\xi) = 0.$$

(V) $$\tfrac{1}{2}\xi \leqq x_1 \leqq \beta, \qquad \beta \leqq x_2 \leqq a.$$

Again

$$F'_{x_1}(x_1, x_2) = f'(x_1 + x_2) - f'(h(x_1) + x_1)$$

and $F'_{x_1}(x_1, x_2) > 0$, since also in this case $h(x_1) \leqq \frac{1}{2}\xi < x_2$. Further, by (17),

$$F'_{x_2}(x_1, x_2) = f'(x_1 + x_2) - f'(x_2) > 0.$$

Hence, by (IV),

$$F(x_1, x_2) \geqq F(\tfrac{1}{2}\xi, \beta) \geqq 0.$$

(VI) $$\beta \leqq x_1 \leqq a, \qquad \beta \leqq x_2 \leqq a.$$

Then

$$F'_{x_1}(x_1, x_2) = f'(x_1 + x_2) - f'(x_1) > 0, \; F'_{x_2}(x_1, x_2) = f'(x_1 + x_2) - f'(x_2) > 0$$

and hence, again by (IV),

$$F(x_1, x_2) \geqq F(\beta, \beta) \geqq 0.$$

4. The Application. By choosing various functions for $M(u_1, \ldots, u_p)$, $N(v_1, \ldots, v_q)$, $g(x)$ and $f(x)$, many applications of our theorems may be given. If we take $g(x) \equiv \xi - x$, we obtain the same results as MORDELL. But now we can improve those results by defining $g(x)$ in a more general way [3]). We restrict ourselves to the very important case, given by

$$f(x) \equiv \frac{1}{x^{p/q}}. \quad \ldots \ldots \ldots \quad (22)$$

In the expression in theorem 4, P and Q are not affected by the way of defining $g(x)$. Therefore we are only interested in the numerical value we obtain for

$$J \equiv p \int_0^\alpha (\varphi(x))^q x^{p-1} dx. \quad \ldots \ldots \ldots \quad (23)$$

We shall show that the value we find for J on taking $g(x) \equiv \xi - x$ is less than the values we obtain defining $g(x)$ otherwise.

We start however by demonstrating, that, if $f(x)$ is given by (22), we can prove more than is stated in theorem 4.

Since $a = \infty$ we can write (4) in the form

$$(M(z_1, \ldots, z_p))^p \cdot (N(z_1, \ldots, z_q))^q \leqq 1. \quad \ldots \ldots \quad (24)$$

For $p < q$, MORDELL's condition is satisfied, since then

$$f(\xi) + \xi f'(\xi) = \frac{1}{\xi^{p/q}} - \frac{p}{q\,\xi^{p/q}} = \frac{q-p}{q\,\xi^{p/q}} > 0. \quad \ldots \quad (25)$$

First let $\xi = 1$. Then we have to define $g(x) \equiv g_1(x)$ as a differentiable function of x in the interval $0 \leqq x \leqq \frac{1}{2}$, with $g'_1(x) < 0$, $g_1(\frac{1}{2}) = \frac{1}{2}$ and $g_1(0) = \beta_1 > \frac{1}{2}$. Putting further $\bar{x} = g_1(x)$, we have $h_1(\bar{x}) = x$ for $\frac{1}{2} \leqq \bar{x} \leqq \beta_1$.

Next we define for arbitrary ξ

$$g_\xi(x) = \xi g_1(x/\xi) \text{ and } h_\xi(\bar{x}) = \xi h_1(\bar{x}/\xi),$$

where $0 \leqq x \leqq \frac{1}{2}\xi$ and $\frac{1}{2}\xi \leqq \bar{x} \leqq \beta_\xi = \xi \beta_1$.

Then

$$\varphi_\xi(x) = \frac{1}{2\,\xi^{p/q}} + \frac{p}{q\,\xi^{p/q}} \int_{x/\xi}^{\frac{1}{2}} \frac{dt}{(t + g_1(t))^{1+p/q}}, \text{ for } 0 \leqq x \leqq \tfrac{1}{2}\xi. \quad (26)$$

$$\varphi_\xi(x) = \frac{1}{2\,\xi^{p/q}} - \frac{p}{q\,\xi^{p/q}} \int_{\frac{1}{2}}^{x/\xi} \frac{dt}{(h_1(t) + t)^{1+p/q}}, \text{ for } \tfrac{1}{2}\xi \leqq x \leqq \xi\beta_1. \quad (27)$$

$$\varphi_\xi(x) = \frac{1}{x^{p/q}} - \varphi_\xi(0), \quad \text{for } \xi\beta_1 \leqq x \quad \ldots \ldots \quad (28)$$

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[3]) KOKSMA and MEULENBELD, Proc. Ned. Akad. v. Wetensch., Amsterdam, **44** (1941), **45** (1942), have also applied the method of BLICHFELDT to several approximation problems. Their results are included in those given in my Amsterdam dissertation (1945) of which a summary can be found in the same journal **50**, 173—185 (1947). All these results however can be improved in the same way and to the same extend as MORDELL's results by the method given here.

From this it follows immediately, that

$$\varphi_\xi(x) = \frac{1}{\xi^{p/q}} \varphi_1(x/\xi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Defining α_ξ by $\varphi_\xi(\alpha_\xi) = 0$, we have $\alpha_\xi = \xi\, \alpha_1$.

Now, by (21)

$$V_k = p\,P\,Q \int_0^{\alpha_\xi} (\varphi_\xi(x))^q\, x^{p-1}\, dx,$$

or

$$V_k = p\,P\,Q \int_0^{\xi\alpha_1} (\varphi_1(x/\xi))^q\, (x/\xi)^{p-1} \frac{dx}{\xi} = p\,P\,Q \int_0^{\alpha_1} (\varphi_1(x))^p\, x^{p-1}\, dx.$$

That means, V_k is independent of the choice of ξ.

Now, by choosing ξ arbitrarily small, we obtain an arbitrarily small value of $\alpha_\xi = \xi\, \alpha_1$, e.g. $\alpha_\xi < \frac{1}{2}\varepsilon$. Then, if K is defined by (5), with $\varphi(x) \equiv \varphi_\xi(x)$ and $\alpha = \alpha_\xi$, we know that $M \equiv M(z_1, \ldots, z_p) < \frac{1}{2}\varepsilon$ for any point $P = (z_1, \ldots, z_n)$ of K. Hence $M < \varepsilon$ for any point $P_1 - P_2$, with P_1 and P_2 contained in K. In this case therefore we still can apply theorem 1 to K and R, if we define R as the set of points $(z_1, \ldots, z_n)$ not only satisfying (24), but also satisfying the inequality $M < \varepsilon$.

Hence there are points other than O of any lattice Λ, with determinant $\triangle \neq 0$, for which $|\triangle| \leq V_k$, satisfying not only (24), but also $M < \varepsilon$ with arbitrarily small ε.

Similarly, by taking ξ arbitrarily large and consequently

$$\varphi_\xi(0) = \frac{1}{\xi^{p/q}} \varphi_1(0)$$

arbitrarily small, we can prove that there are also points of Λ other than O, satisfying both (24) and $N < \varepsilon$.

Hence

Theorem 5. *Given any positive ε, there are an infinity of points satisfying (24), with $M < \varepsilon$ and also an infinity of points satisfying (24), with $N < \varepsilon$ of any lattice Λ with determinant $\triangle \neq 0$, for which*

$$|\triangle| \leqq p\,P\,Q \int_0^{\alpha_1} (\varphi_1(x))^q\, x^{p-1}\, dx,$$

Where P and Q are defined by (19), $\varphi_1(x)$ by (26), (27) and (28), with $\xi = 1$ and α_1 by $\varphi_1(\alpha_1) = 0$.

Let c be a constant, satisfying

$$c > \tfrac{1}{2}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Then we define $g_1(x) \equiv g(x)$ by

$$g(x) \equiv \frac{2c - (2c+1)\,x}{2c-1}, \text{ for } 0 \leqq x \leqq \tfrac{1}{2}, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and we have for $h_1(x) \equiv h(x)$

$$h(x) \equiv \frac{2c-(2c-1)\,x}{2c+1},\ \text{for}\ \tfrac{1}{2} \leqq x \leqq \frac{2c}{2c-1} \quad . \quad . \quad . \quad (32)$$

Clearly $g'(x) < 0$ and $h'(x) < 0$.

It is easy to verify that we find for $\varphi_1(x) = \varphi(x)$:

$$\varphi(x) \equiv c - \left(\frac{2c-1}{2}\right)^{1+p/q} \cdot \frac{1}{(c-x)^{p/q}},\ \text{for}\ 0 \leqq x \leqq \tfrac{1}{2}. \quad . \quad . \quad (33)$$

$$\varphi(x) \equiv -c + \left(\frac{2c+1}{2}\right)^{1+p/q} \cdot \frac{1}{(c+x)^{p/q}},\ \text{for}\ \tfrac{1}{2} \leqq x \leqq \frac{2c}{2c-1}. \quad (34)$$

$$\varphi(x) \equiv -c + c\left(\frac{2c-1}{2c}\right)^{1+p/q} + \frac{1}{x^{p/q}},\ \text{for}\ \frac{2c}{2c-1} \leqq x \quad . \quad . \quad (35)$$

Now the third equation for $\varphi(x)$ has only a meaning for our purpose, if $a \geq \dfrac{2c}{2c-1}$, i.e. if

$$\varphi\left(\frac{2c}{2c-1}\right) = -c + \frac{2c+1}{2}\left(\frac{2c-1}{2c}\right)^{p/q} \geqq 0,$$

or

$$\left(1-\frac{1}{2c}\right)^{p} \cdot \left(1+\frac{1}{2c}\right)^{q} \geqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Provided that this inequality is satisfied, it follows from (18), that

$$a = \frac{1}{\left\{c - c\left(\dfrac{2c-1}{2c}\right)^{1+p/q}\right\}^{q/p}}.$$

We take $p = 1$ and $q = n-1$ and then we find for J in (23), after some transformation of the integrals,

$$J = J_1 + J_2 + J_3,$$

where

$$J_1 = \frac{2c-1}{2} \int_1^{\frac{2c}{2c-1}} \left\{c - \frac{2c-1}{2} \cdot \frac{1}{x^{\frac{1}{n-1}}}\right\}^{n-1} dx,$$

$$J_2 = \frac{2c+1}{2} \int_1^{\frac{2c}{2c-1}} \left\{-c + \frac{2c+1}{2} \cdot \frac{1}{x^{\frac{1}{n-1}}}\right\}^{n-1} cx,$$

$$J_3 = (n-1) \int_{c\left(\frac{2c}{2c-1}\right)^{\frac{1}{n-1}} - \frac{2c-1}{2}}^{1} (1-x)^{n-1} \frac{dx}{x}.$$

Hence, for $n \to \infty$,

$$J_1 \sim \frac{1}{2^n}\left\{\left(\frac{2c}{2c-1}\right)^{2c-1} - \frac{2c-1}{2c}\right\},$$

$$J_2 \sim \frac{1}{2^n}\left\{\frac{2c+1}{2c} - \frac{2c+1}{2c}\cdot\left(\frac{2c-1}{2c}\right)^{2c}\right\},$$

$$J_3 \sim \frac{2}{2^n}\left(\frac{2c-1}{2c}\right)^{2c},$$

and so

$$J \sim \frac{1}{2^n}\left\{\left(\frac{2c}{2c-1}\right)^{2c-1} + \left(\frac{2c-1}{2c}\right)^{2c+1} + \frac{1}{c}\right\}.$$

On letting c tend to infinity we obtain the results of BLICHFELDT and MORDELL [4]). Then clearly

$$J \sim \frac{1}{2^n}\left(e + \frac{1}{e}\right) = \frac{3{,}086}{2^n}.$$

For $c = 1$ however, we have

$$J \sim \frac{1}{2^n}\,(2 + \tfrac{1}{8} + 1) = \frac{3{,}125}{2}.$$

Also for small values of n we obtain an improvement of MORDELL's results when choosing a suitable value of c. Take e.g. $p = 1$, $q = 2$. Then (36) is satisfied for $c \geqq \frac{1}{4}(\sqrt{5}+1) = 0{,}81$. In this case MORDELL's method gives

$$J = \frac{2{,}270}{8},$$

whereas for $c = 2\frac{1}{2}$,

$$J = \frac{2{,}283}{8}.$$

Our results give e.g. an improvement of the results of BLICHFELDT concerning the approximation of irrationals by rationals.

Take

$$M(z_1) = |z_1|, \quad N(z_2, \ldots, z_n) = \max(|z_2|, \ldots, |z_n|).$$

Then clearly $P = 2$ and $Q = 2^{n-1}$. Hence by (21) and (23)

$$V_k = 2^n J.$$

Now, by theorem 5, there are an infinity of points $(z_1, \ldots, z_n)$ of any lattice Λ, of which the determinant $\triangle$ satisfies

$$0 < |\triangle| \leqq 2^n J, \quad \ldots \ldots \ldots \quad (37)$$

[4]) Also of KOKSMA and MEULENBELD and of my own, cf. [3]).

such that

$$M(z_1) \cdot (N(z_2, \ldots, z_n))^{n-1} \leqq 1 \text{ and } N(z_2, \ldots, z_n) < \varepsilon \,. \quad . \quad (38)$$

We define Λ as follows:

$$z_1 = x, \quad z_{k+1} = -C(a_k x - y_k). \quad (k = 1, \ldots, n-1)$$

with positive C, arbitrary real $a_1, \ldots, a_{n-1}$ and integers $x, y_1, \ldots, y_{n-1}$. Then

$$\Delta = C^{n-1}$$

and we can replace (37) by

$$C \leqq (2^n J)^{\frac{1}{n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

In place of (38) we write now

$$|x|^{\frac{1}{n-1}} \cdot \max_{1 \leqq k \leqq n-1} |a_k x - y_k| \leqq \frac{1}{C}, \qquad \max_{1 \leqq k \leqq n-1} |a_k x - y_k| < \varepsilon \quad . \quad (40)$$

Taking $\varepsilon < 1$, there can be only one solution of these inequalities with $x = 0$, namely $x = y_1 = \ldots = y_{n-1} = 0$. Hence, if C satisfies (39), there are an infinity of solutions of the system of inequalities

$$|a_1 x - y_1| \leqq \frac{1}{C\,|x|^{\frac{1}{n-1}}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$|a_{n-1} x - y_{n-1}| \leqq \frac{1}{C\,|x|^{\frac{1}{n-1}}}.$$

Any improvement of the value we find for J involves an improvement of the value we may substitute for C.

Mathematics. — *Local connectedness and quasiorder*. By J. DE GROOT. (Communicated by Prof. A. HEYTING.)

(Communicated at the meeting of May 29, 1948.)

1. We assume that all our sets are separable and metric.

Definition of quasiorder. A point p of a set M is of quasiorder q in M if p is contained in arbitrarily small neighbourhoods $U(p)$ with boundaries $\mathfrak{R}(U)$ consisting of exactly q components, and if p is not contained in arbitrarily small neighbourhoods with boundaries consisting of less than q components (q being finite, or $\aleph_0$, or $\aleph$). If p is contained in arbitrarily small neighbourhoods with boundaries consisting of a finite number of components and the quasiorder is not finite in p, we put the quasiorder equal to ω.

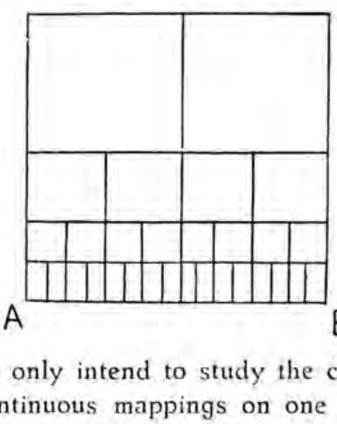
Therefore q may be

$$0, 1, 2, 3, \ldots, n, \ldots, \omega, \aleph_0, \aleph. \qquad (1)$$

It is worth noting that the neighbourhoods in consideration need not necessarily be open. Any set $U \supset O \supset p$ with open O is a neighbourhood U of p, and $\mathfrak{R}(U) = \overline{U} \cdot \overline{(M-U)}$.

Taking this into consideration one sees for instance that the rational locally connected curve, as indicated in the figure, is of quasiorder 1 in any point of AB. The quasiorder of a set M is the smallest number in (1), greater than or equal to the quasiorder of any point of M.

The quasiorder is obviously a topological invariant.



In the following we only intend to study the connection between local connectedness and continuous mappings on one side and quasiorder on the other side, and more especially for continua. The following observations are, moreover, just as true for generalised continua (microcompact connected sets) as for continua. Finally, in 4., the sets of quasiorder $\leq \aleph_0$ will be classed more precisely and we shall find new continuous invariants for continua.

Local connectedness may be expressed as a boundary-property in the following way:

2. **Theorem I.** *A continuum M is locally connected if and only if the quasiorder of M is $\leqq \omega$.*

Proof. Be the quasiorder of $M \leqq \omega$. Then a point p of M has an arbitrarily small neighbourhood U with a boundary $\mathfrak{R}(U)$ consisting of a finite number — say n — of components. If we can prove that $\overline{U}$ has a finite number of components we shall have shown the local connectedness of M in p, since the component of $\overline{U}$ containing p is, in this case, a connected neighbourhood of p. — Now, any component C of $\overline{U}$ intersects $\mathfrak{R}(U)$: for, if not, any point $r \in \mathfrak{R}(U)$ and C may be separated in $\overline{U}$:

$$\overline{U} = A_1 + A_2 \qquad r \in A_1, C \in A_2 \qquad (A_1, A_2 \text{ disjoint, closed})$$

The compact set $\mathfrak{R}(U)$ is covered by a finite number of sets A_1. From this it follows that the intersection

$$A = \Pi A_2$$

of the corresponding sets A_2 is open and closed in M and contains C, in contradiction to the connectedness of M.

Conversely, be M a locally connected continuum, $U(p)$ a neighbourhood of $p \in M$ and $V(p)$ a region with

$$p \in \overline{V}(p) \subset U(p).$$

The compact boundary $\mathfrak{R}(V)$ is covered by a finite number of regions R_i $(i = 1, 2, \ldots, s)$. We may suppose

$$\overline{R}_i \cdot p = 0, \qquad \overline{R}_i \subset U(p).$$

Be D as subset of $\sum_{i=1}^{s} R_i$ with

$$\overline{D} = \overline{\sum_{i=1}^{s} R_i - D} = \sum_{i=1}^{s} \overline{R}_i.$$

The set

$$U' = (V(p) + \sum_{i=1}^{s} R_i) - D$$

is a neighbourhood of p, contained in $U(p)$ and has a boundary

$$\mathfrak{R}(U') = \sum_{i=1}^{s} \overline{R}_i$$

such that $\mathfrak{R}(U')$ has a finite number of components, q.e.d.

3. The quasiorder therefore makes it possible to give a topologically invariant classification of the continua, where the important class of locally connected continua appears — according to theorem I — as one definite class. On the one side the locally connected continua are classed yet further

according to their quasiorder 1, 2, ..., n, ..., ω; on the other hand the not-locally-connected continua are divided into two groups, corresponding with the quasiorders $\aleph_0$ and $\aleph$ respectively. We can easily show by way of examples that there must exist continua of every quasiorder (quasiorder 0 excepted), while conversely also any continuum (more generally: any compact set whatever) has apparently exactly one quasiorder. Since every locally connected continuum may be mapped continuously only on any locally connected continuum, the notion "being of quasiorder $\leq \omega$" is a continuous invariant. We shall prove that the property "being of quasiorder $\leq \aleph_0$" is also a continuous invariant. It is, however, not true — as may be shown by way of examples and as follows from 4. — that any set of quasiorder $\aleph_0$ may be mapped continuously on any other set of order $\aleph_0$.

Theorem II. *The notions "being a continuum of quasiorder $\leq \omega$" and "being a continuum of quasiorder $\leq \aleph_0$ "are continuous invariants* [1]).

Proof. First we contend

a) In a continuum M any point of quasiorder $\leq \aleph_0$ has arbitrarily small neighbourhoods which are the sum of a countable number of components (the reverse is not true). For, if $U(p)$ is a neighbourhood of p in M with $\mathfrak{R}(U)$ consisting of a countable number of components, any component of $\overline{U(p)}$ intersects $\mathfrak{R}(U)$; otherwise one might conclude (like in the beginning of the proof of Theorem I) that M is not connected. Therefore $\overline{U(p)}$ is the required neighbourhood.

b) In a continuum M of quasiorder $\leq \aleph_0$ the quasiorder of any compact subset $D \subset M$ in M is $\leq \aleph_0$ (the definition of the quasiorder of D in respect to M is obvious). Indeed; the compact boundary $\mathfrak{R}(V)$ of an arbitrarily small open neighbourhood $V = V(D)$ of D in M is covered by a finite number of sets R_i $(i = 1, 2, \ldots, s)$, each R_i being an arbitrarily small compact neighbourhood of a point of $\mathfrak{R}(V)$ and the sum of a countable number of components, which is possible according to *a*).

$S = \sum_{i=1}^{s} R_i$ is therefore a compact set consisting of countably many components:

$$S = \sum_k C_k,$$

and we may assume $S \cdot V = 0$.

Take in every component C_j which consists of more points than one (a non degenerate component) a subset D_j with $\overline{D_j} = \overline{C_j - D_j} = C_j$.

[1]) It is trivial that "being of quasiorder $\leq \aleph$" is a continuous invariant. The notion "being a continuum of quasiorder $\leq n$" is not a continuous invariant. — *The theorem is not true for compact sets instead of continua;* e.g. it is possible to map the discontinuum of Cantor — a set of quasiorder 0 — continuously on any compact set whatever, and so also on a compact set of quasiorder $\aleph$.

The set

$$V' = (\overline{V} + S) - \sum_j D_j$$

is a neighbourhood of V; its boundary $\Re(V')$ is a (proper or improper) subset of S and contains all nondegenerate components C_j and an unknown number of degenerate components. Therefore $\Re(V')$ is the sum of countably many components, q.e.d.

If M has a quasiorder $\leq \aleph_0$ and $M' = f(M)$ is a continuous mapping, we must prove that the quasiorder of M' is $\leq \aleph_0$. For an open neighbourhood $U' = U'(p')$ of p' in M' one may determine an open neighbourhood $U = U(D)$ in M of the compact set $D = f^{-1}(p)$ with

$$f(U) \subset U'.$$

According to b), there is a neighbourhood $O = O(D) \subset U$ of D in M with $\Re(O)$ being the sum of countably many components. Since $D = f^{-1}(p)$, f is continuous and M compact, one may find a compact neighbourhood $N = N(p)$ — chosen sufficiently small — such that

$$f^{-1}(N) \subset O.$$

In particular $$R = f^{-1}(\Re(N)) \subset O.$$

R and D are disjoint compact subsets of M. Therefore there is a compact neighbourhood $O' = O'(D) \subset O$ [2]) not intersecting R, such that $\Re(O')$ is the sum of a countable number of components (again according to b)).

Therefore

$$R \subset O - O' \subset \overline{O - O'}$$

while $\Re(O - O')$ consists of a countable number of components. But then $\overline{O - O'}$ also has a countable number of components, since M is a continuum (proof analogous to that of a)).

The set

$$V'' = N + f(\overline{O - O'})$$

is a neighbourhood of p in M' contained in U'. $f(\overline{O - O'})$ is a compact set which is the sum of a countable number of components, and which does not contain p':

$$p \not\subset f(\overline{O - O'}) = \sum_k C'_k.$$

Take in every nondegenerate C'_j a subset D'_j with

$$\overline{D'_j} = \overline{C'_j - D'_j} = C'_j.$$

Then

$$V''' = V'' - \sum_j D'_j$$

is the required neighbourhood of p' contained in U' while its boundary $\Re(V''')$ is the sum of a countable number of components.

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[2]) We may even assume $0' \subset \overline{0} - \Re(0)$.

4. Just like the classification according to the order of a point, known in the theory of curves, may be further refined — for sets of order $\leqq \aleph_0$ — according to "genus" and "typus" (conf. K. MENGER, Kurventheorie, p. 293), likewise one may give a further classification of the sets quasiorder $\leqq \aleph_0$ according to "quasigenus" and "quasitypus". While these classifications naturally give topological invariants, the classification according to quasigenus has, moreover, the advantage — for continua — that it gives continuous invariants (and not countably many at that), contrary to the class, according to "genus", which does not give continuous invariants (which follows immediately from the fact that a segment may be mapped continuously on a square).

Definition of quasigenus. The derived sets of a compact countable set A may be well-ordered:

$$A = A_0 \supset A_1 \supset A_2 \supset \ldots A_\omega \subset A_{\omega+1} \supset \ldots A_{\omega.2} \supset \ldots A_\alpha = 0.$$

Here α is an ordinal number of the first or second class, A_β the intersection of all sets A_γ with $\gamma < \beta$, if β is a limit-number; and A_β the derived set of $A_{\beta-1}$ if β has a predecessor $\beta - 1$.

M is of quasigenus $\leq \alpha$ (α being an ordinal number of the first or second class), if any point p of M is contained in arbitrarily small neighbourhoods with boundaries $\mathfrak{R}$ such that for the component-space [3]) $A = C\,(\mathfrak{R})$ of $\mathfrak{R}$, $A_\alpha = 0$. The smallest number α for which this is true is the quasigenus of M [4]).

A set of quasiorder $\leq \omega$ for instance is of quasigenus 0 or 1 (0 in case the set is 0-dimensional). According to Theorem I the locally connected continua are identical with the continua of quasigenus 1. — One may construct continua of any quasigenus α. To this end one takes a compact countable set A with $A_\alpha = 0$, $A_\beta \neq 0$, $\beta < \alpha$ on the segment $0 \leq x \leq 1$, $y = 0$. Each point of A is connected by a straight line with the point $(0,1)$. The so-formed continuum is apparently of quasigenus α.

Now we may generalize Theorem II in the following way:

Theorem III. *The property "being a continuum of quasigenus $\leqq \alpha$" is a continuous invariant, where α is an arbitrary ordinal number of the first or second class.*

Proof. We shall not give an extensive proof, because it is running along the same lines as the previous proof: that is, if one only takes into

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[3]) We assume the existence of boundaries $\mathfrak{R}$ which are the sum of a countable number of components: $\mathfrak{R} = \Sigma\, C$. By identifying the components C to one point, the component-space $A = C(\mathfrak{R})$ comes into being; since M is compact, A is a countable compact space, being the upper semi-continuous decomposition-space of $\mathfrak{R}$.

[4]) It is easy to prove, that for any compact set M of quasiorder $\leq \aleph_0$, the quasigenus in one definite number d.

account that the following is true: The simple fact used in the previous proof, that a space which is the sum of a countable number of components keeps this property when it is mapped continuously, must now be altered to: If $\mathfrak{R}$ is a compact space with a countable component-space $A = C(\mathfrak{R})$, for which $A_\alpha = 0$, then for any continuous mappong $\mathfrak{R}' = f(\mathfrak{R})$, with $A' = C(\mathfrak{R}')$:

$$A'_\alpha = 0.$$

This is, however, true. For, the continuous mapping $\mathfrak{R}' = f(\mathfrak{R})$ induces a continuous mapping of the component-space A on the component-space A'. Both are compact countable spaces. For these are all continuous invariants known (see Proc. Ned. Akad. v. Wetensch., Amsterdam, **48** (1945), p. 246, Theorem V); especially $A_\alpha = 0$ leads to $A'_\alpha = 0$.

Amsterdam, 1948.

Mathematics. — *A note on Mathieu functions.* By C. J. BOUWKAMP. (Communicated by Prof. BALTH. VAN DER POL.)

(Communicated at the meeting of June 26, 1948.)

To the best of my knowledge, there have been given four different proofs (see ref. 1–4) of the following theorem:

The differential equation of Mathieu (see ref. 5)

$$y'' + (a - 2q \cos 2z)\, y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

cannot have two linearly independent solutions periodic in z of period 2π unless $q = 0$, $a = n^2$, where n is an integer.

The main purpose of this note is to communicate of this theorem a fifth proof which may be of some interest on account of its simplicity. Henceforth, any solution of (A) that is periodic in z of period 2π will be termed Mathieu function. The parameters a and q are not restricted to real values; the case $q = 0$ is trivial.

As is well known (see ref. 5), there exist four different types of Mathieu function, each of wich possesses its characteristic Fourier-series expansion, *viz.*

(*i*) functions that are even in z, of (least) period π,

$$\mathrm{ce}_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(q) \cos 2rz\, ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(*ii*) functions that are even in z, of (least) period 2π,

$$\mathrm{ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(q) \cos (2r+1)\, z\, ; \quad . \quad . \quad . \quad . \quad (2)$$

(*iii*) functions that are odd in z, of (least) period 2π,

$$\mathrm{se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(q) \sin (2r+1)\, z\, ; \quad . \quad . \quad . \quad . \quad (3)$$

(*iv*) functions that are odd in z, of (least) period π,

$$\mathrm{se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(q) \sin (2r+2)\, z. \quad . \quad . \quad . \quad . \quad (4)$$

The relation that is required between a and q, in order that Mathieu's differential equation shall admit of solutions of one of the types (1)–(4), is most simply expressed in terms of a transcendental equation involving an infinite continued fraction (see ref. 5).

These eigenvalue-equations are, in order,

$$0 = -\frac{a}{2} - \frac{q^2}{|4-a} - \frac{q^2}{|16-a} - \frac{q^2}{|36-a} - \ldots, \quad . \quad . \quad . \quad \text{(I)}$$

$$0 = q + 1 - a - \frac{q^2}{|9-a} - \frac{q^2}{|25-a} - \frac{q^2}{|49-a} - \ldots, \quad . \quad . \quad \text{(II)}$$

$$0 = -q + 1 - a - \frac{q^2}{|9-a} - \frac{q^2}{|25-a} - \frac{q^2}{|49-a} - \ldots, \quad . \quad \text{(III)}$$

$$0 = 4 - a - \frac{q^2}{|16-a} - \frac{q^2}{|36-a} - \frac{q^2}{|64-a} - \ldots, \quad . \quad . \quad \text{(IV)}$$

each of which has an infinite number of roots $a_r(q)$, where q is regarded as the independent variable.

All this has substantially been known since HEINE (ref. 6), well before the publication of INCE's paper referred to (ref. 1).

Let us now turn to the new proof in question. The theorem of INCE will have been proved when any pair out of equations (I)–(IV) is shown to have no roots in common ($q \neq 0$). Now, it is quite obvious that (I) and (IV) have no common root. Indeed, those values of a that satisfy (IV), for some fixed q, make the right-hand member of (I) infinite. Further, (II) and (III) cannot have any root in common because the right-hand side of (II) equals $2q \neq 0$ for those values of a that satisfy (III).

There does not seem to exist a direct proof that the remaining pairs of equations, *viz.* (I, II), (I, III), (II, IV), and (III, IV), behave in the same manner. However, the non-existence of common roots in these cases can be proved from the very start, that is, from the differential equation itself.

Obviously, any point z_0 of the finite plane of complex z is an ordinary point of Mathieu's equation. That is to say, the values of $y(z_0)$ and $y'(z_0)$ may be chosen at will; the relevant solution of Mathieu's equation then is unique, for any fixed a and q. This implies that the general solution of Mathieu's equation cannot vanish at some $z = z_0$. Neither can its derivative. In particular, the functions (1) and (2) cannot be coexistent solutions; if they were, the general solution would be an even function of z, thus having a vanishing derivative at $z = 0$. Similarly, the odd functions (3) and (4) cannot form a fundamental system of solutions. Further, the combinations (2,4) and (1,3) are impossible because then the general solution, respectively its derivative, would vanish at $z = \pi/2$. This completes the proof.

In concluding, I give some numerical information in connection with the eigenvalues of $\mathrm{ce}_0\,(z, q)$ and $\mathrm{ce}_2\,(z, q)$ for *purely imaginary* q ($= \pm is$, $s > 0$). As has been observed by MULHOLLAND and GOLDSTEIN (ref. 7), the function $a(q)$ may show branch points on the imaginary axis of q. The pair of these singularities nearest the origin, in case (I), is at $q = \pm i s_0$ where $s_0 \approx 1.468$ (Ref. 5 and 7). For this critical value of q the functions

$ce_0(z, q)$ and $ce_2(z, q)$ are no longer distinct; the relevant eigenvalue corresponds to a double root of (I). When $0 \leq s \leq s_0$ the eigenvalues of ce_0 and ce_2 are distinct and real; when $s > s_0$ they are conjugate-complex. It has been suggested by MC LACHLAN (ref. 5) that the critical eigenvalue $a_0 = a(i\,s_0)$ is equal to 2.

If this was true, s_0 would be the positive root of

$$1 = \frac{s^2|}{|2} + \frac{s^2|}{|14} + \frac{s^2|}{|34} + \frac{s^2|}{|62} + \dots \quad (5)$$

Since I do not understand why this should be so, and, moreover, since careful extrapolation of the numerical results of MULHOLLAND and GOLDSTEIN indicate that the critical eigenvalue slightly exceeds 2, I have taken the trouble to carry out some calculations in the critical range. It is thereby advisable to regard a as the independent variable instead of q. See table I.

TABLE I.

ce_0		ce_2	
a	s	a	s
2.0	1.467344...	2.20	1.466506...
2.05	1.468497...	2.15	1.468084...
2.075	1.468734...	2.10	1.468745...
2.080	1.468754...	2.095	1.468761...
2.085	1.468766...	2.090	1.468768...
$a_0 = 2.088\dots$		$s_0 = 1.468768\dots$	

Since $s = 1.46876852\dots$ when $a = 2.088$, we have, rounding off,

$$s_0 = 1.468769,$$

in which the error in the last decimal does not exceed half a unit. This should be compared with the root of (5), $\sigma = 1.467344\dots$.

June, 1948.

Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands.

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Geology. — *Phenomena of mineralisation at the Mezzel creek near Bommerig in the Geul Valley (South Limburg, Holland).* By P. DE WIJKERSLOOTH. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of May 29, 1948.)

We know that the river Geul flows through the rich ore area of Moresnet and Bleyberg (see fig. 1) of eastern Belgium. The lead-zinc ores of Sippenaecken (on the right bank of the Geul) occur in the immediate vicinity of the dutch border and were at the end of the nineteenth century exploited.

The question can be put forward as to the existence of ore-deposits in the watershed of the Geul in South-Limburg, also for the reason that further to the north in the coal region of South-Limburg considerable veins of lead-zinc ores have been found.

In a former paper ("Sur la région métallifère de Moresnet-Bleyberg-Stolberg-Limburg néerlandais" see bibliogr.) we dealt with this problem

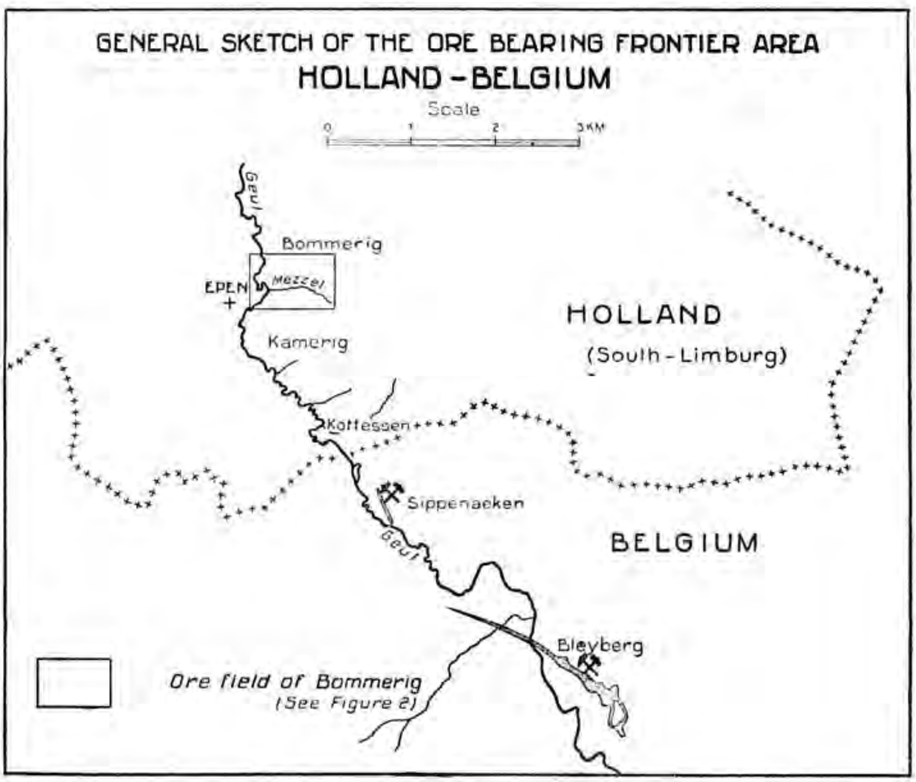

Fig. 1.

and stressed the fact that the ore-deposits of eastern Belgium continue into Holland and that the souh-eastern part of South-Limburg is therefore part of this important ore zone.

On this basis the Geul-Valley between Epen and the Belgian border, and particularily the right side of the valley, was thoroughly investigated for ore deposits (see fig. 1). The "Cavando Acquiro"company, formed especially for this purpose, had a series of geophysical (geoelectrical) measurements made, resulting in the locating of nine faults suspected of carrying ore. These (faults) showed remarkable parallelism and had a general NW—SE strike. There thus exists a fault area having the same orientation as that in eastern Belgium and here we get the impression that we are dealing with the northern prolongation of the mineralized fault-complex of the belgian region.

The most northerly of the nine fault zones lies in the neighbourhood of the Mezzel creek at Bommerig. It was found to be particularly promising and therefore it was decided to make here several bore holes. The results obtained here will be dealt with further on in this paper.

The fault zone of Bommerig comes to the surface in the upper part of the Mezzel creek. It has a NW—SE strike and maintains this strike at the point where the creek swings in western direction.

From geophysical data the trend of the mineralisation was plotted running parallel to the above mentioned outcrop of the fault zone for a distance of 80—100 metres to the SW (see fig. 2).

Hence it was concluded that the fault has a SW dip and is mineralized in depth.

Both conclusions were confirmed by the drilling results. The Bommerig fault zone dips indeed at 35° to the SW (see fig. 3) and carries really ore. It is a pleasure to record that the geophysical survey and the results obtained from the borings agreed so harmoniously.

The mineralisation of the Bommerig fault zone is of two different kinds. Higher up in this zone iron ores predominate while in the deeper parts they are less evident and give place to weak but still unmistakeable lead-, zinc- and copper-mineralisations.

The iron mineralisation is especially characteristic. It has taken place mostly in the Aachen argillaceous sand series of the lower Senonian age and is present in iron carbonate form (Sphaerosiderite), closely associated with the kaolinisation of the Aachen clay. Both products of mineralisation have always been regarded in the Moresnet area as marking an upper-level zone of the lead-zinc formations and have played here an important role in the discovery of underlying lead-zinc ores. Several authors have stressed the importance of these minerals for revealing the presence of the lead and zinc ores (DELANOUË was the first to mention this, see bibliog.).

The sphaerosiderite occurs mostly in the form of small rounded grains of 0,5—1 mm in diameter, which mostly are lying in kaolinised clay. The borings yielded large quantities of this mineral which came up with the

ORE FIELD OF BOMMERIG

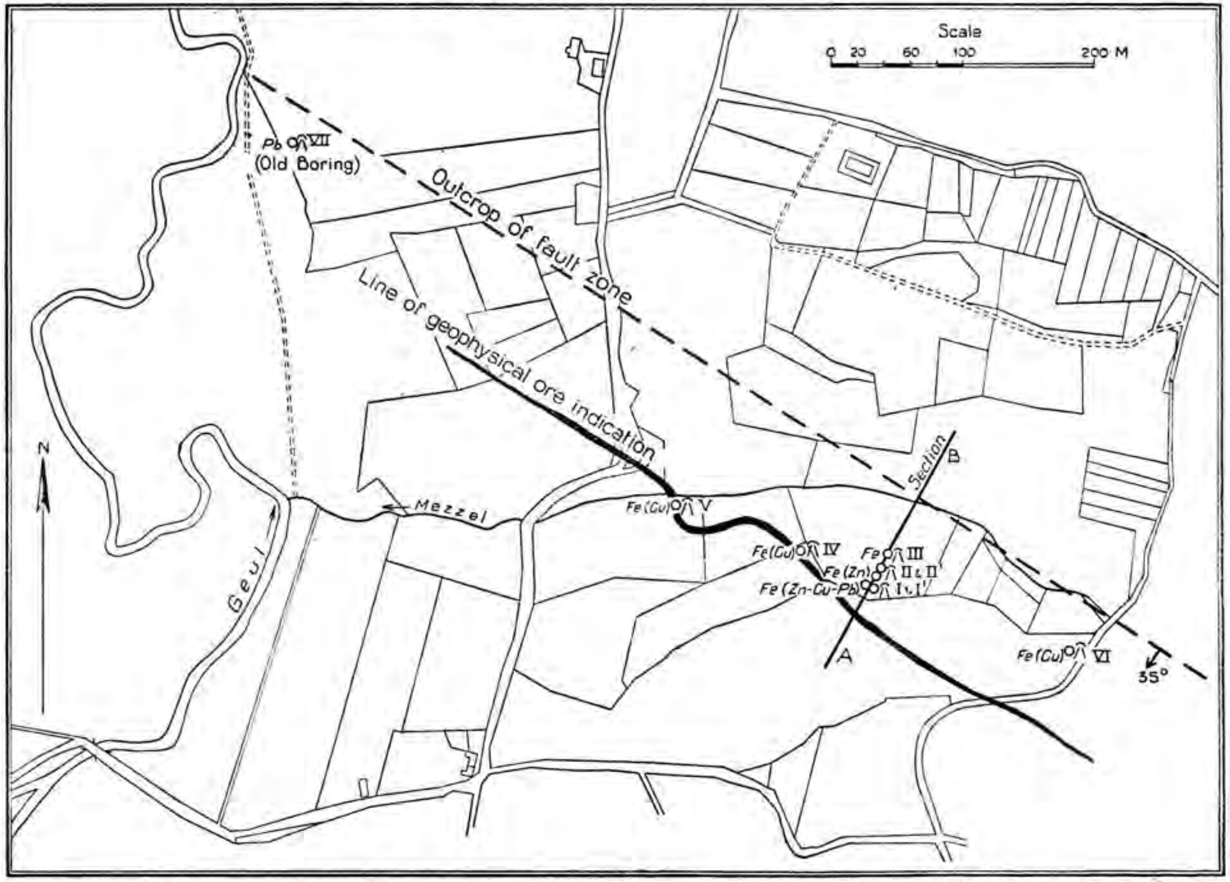

Fig. 2.

Boring Nº I	(Cav. Acq.)	Depth	71,05 m	(Dip 45°)
Boring Nº I'	(Cav. Acq.)	Depth	115,20 m	(Dip 45°)
Boring Nº II	(Cav. Acq.)	Depth	72,80 m	(Vertical)
Boring Nº II'	(Cav. Acq.)	Depth	96,80 m	(Vertical)
Boring Nº III	(Cav. Acq.)	Depth	81,75 m	(Vertical)
Boring Nº IV	(Cav. Acq.)	Depth	400,27 m	(Vertical)
Boring Nº V	(Cav. Acq.)	Depth	155,13 m	(Vertical)
Boring Nº VI	(Cav. Acq.)	Depth	83,57 m	(Vertical)
Boring Nº VII	(Bergw. Verg)	Depth	56,20 m	(Vertical)

drilling fluid. Apart of this, larger masses of sphaerosiderite and limonite have been found, being recovered from cores. The thickness of the sphaerosiderite mineralisation is generally from 2—5 metres (see fig. 3).

The chemical composition of the sphaerosiderite has been kindly calculated for us by Dr. W. VAN TONGEREN. It is as follows:

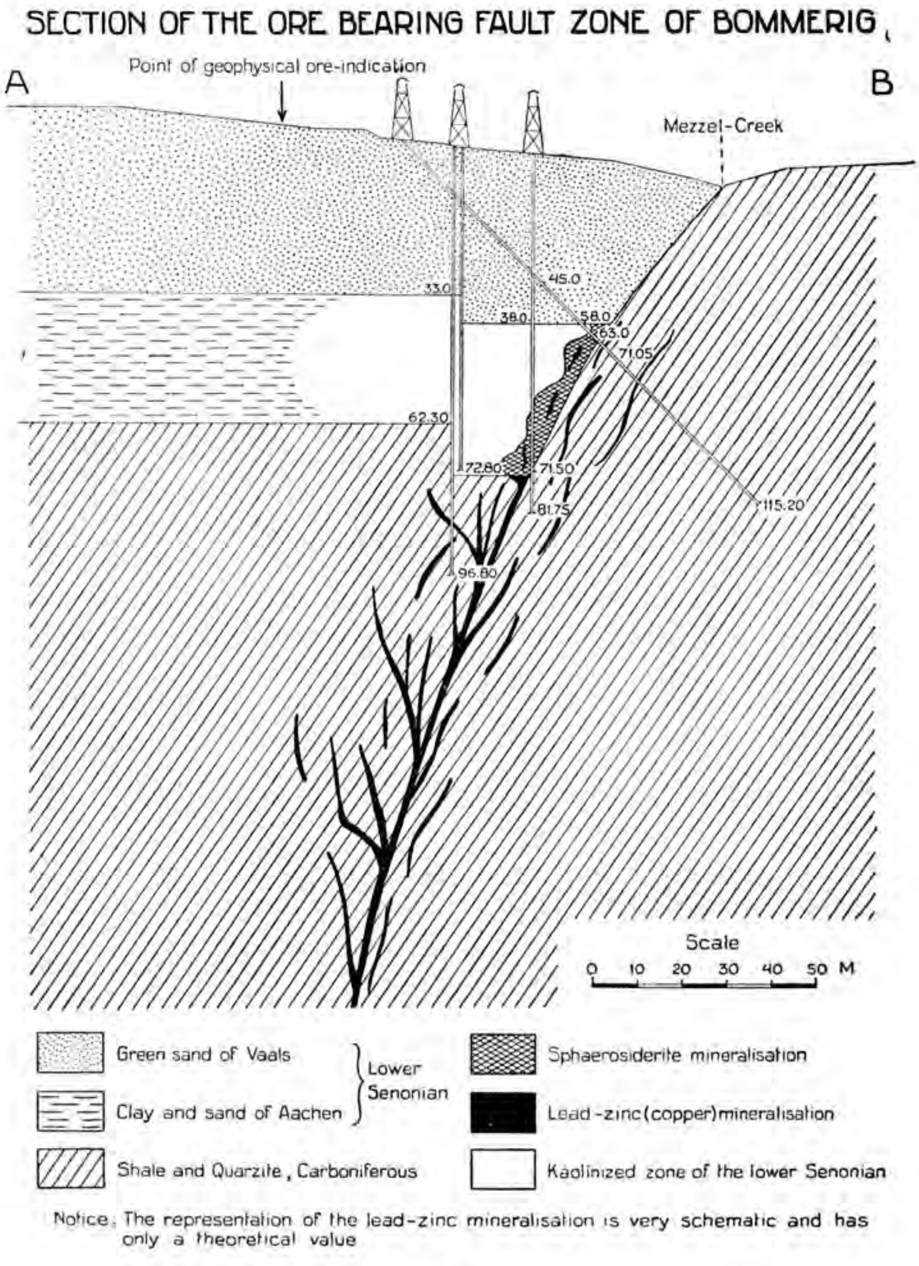

Fig. 3.

Fe_2O_3	.... 2,20 %	CaO	.... 2,78 %
FeO	.... 48,05 %	MgO	.... 0,22 %
MnO	.... 2,48 %	SiO_2	.... 6,20 %
CO_2	.... 33,24 %	Al_2O_3	.... 4,44 %
H_2O	.... 0,53 %	P_2O_5	.... 0,23 %
		TiO_2	.... 0,19 %

The striking manganese and phosphorus content is characteristic. In this relation we must point out that the sphaerosiderite of eastern Belgium also contains both these elements and the manganese often occurs in still larger quantities.

The deeper carboniferous beds, which are overlain by the Lower Senonian are traversed by numerous small fissures which are frequently filled with blende, galena and chalcopyrite, the blende being of a reddish variety. The galena has often a three dimensional trellis structure.

Although larger accumulations of these minerals have been found nowhere, the occurrence in the fissures is widespread. A marked „Erzdurchtrümmerung" of the carboniferous rock is evident.

The carbonate and sulphide mineralisations, also found separate generally, are nevertheless connected by transition types. We get sphaerosideritic ores with traces of blende, galena and chalcopyrite and also sulphides accompanied by sphaerosiderite. This shows clearly their genetic connection.

It would have been very interesting if the borings would have reached the carboniferous limestone in which usually the more important ore concentrations occur, as indicated by the Moresnet ore deposits. It is a pity that none of the borings reached this limestone formation and it remains, therefore, an insolved problem as to what extent the intersection of the Bommerig fault zone with the carboniferous limestone is mineralized. Considering the sphaerosiderite deposits of the lower Senonian and the sulphide impregnation of the carboniferous shale series, it may be assumed that there will be a possibility of finding real ore concentrations here.

In concluding this short report we must point out that for the first time the presence of mineralized Lower Senonian has been carried out. The importance of this lies in the fact that the upper age-level of the lead-zinc mineralization can now be fixed more exactly than before.

As a result of our investigations it is found that the lead-zinc mineralization must have continued into lower senonian time.

On the other hand the geological study of the lead-zinc deposits of eastern Belgium and Aachen has shown us that the calamine and "schalenblende" ores of these areas have been formed on the plane of the senonian transgression.

It has therefore to be accepted that the main mineralization took place at the end of cenomanian time while the final stage was carried on into the lower Senonian.

The investigation of the Bommerig ore zone did not only supply data of local importance but also facts of general significance with respect to our knowledge of the history of the lead-zinc formation.

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Zoology. — *The influence of thiourea on the development of Limnaea stagnalis L.* By F. H. SOBELS. (From the Zoological Laboratory, Dept. of General Zoology and Dept. of Endocrinology, University of Utrecht.) (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of May 29, 1948.)

Introduction.

The influence of thyroxine on the development of Invertebrates has been studied by several authors. O. V. HYKES (1930, 1931) examined the influence of this compound on the development of *Physa* and *Paracentrotus*. P. WEISS (1928) stimulated metamorphosis of Ascidia larvae by means of thyroidextract; and B. CHATZILLO (1936) describes the accelerating influence of thyroid hormone on the later stages of development of *Limnaea stagnalis*. The question whether there is an accelerating or an inhibiting influence of this Vertebrate hormone on the development of eggs and larvae of Invertebrates provoked an extensive discussion.

G. BEVELANDER (1946) investigated the influence of thiourea on the eggs of *Arbacia punctulata*. Fertilized eggs, exposed to a concentration of 1 % thiourea do not cleave; at a concentration of 0.5 % a specific inhibition of gastrulation occurs. This would be due to a selective action of thiourea on the blastula stage. According to BEVELANDER a specific physiological influence of this compound, already known during several years for its influence on the Vertebrate organism, on the eggs of the sea urchin exists.

The purpose of my investigation was to determine the influence of thiourea on the cleavage and later development of eggs of the pond snail, *Limnaea stagnalis*.

Methods.

Egg-masses were obtained in the usual way by stimulation of the snails with *Hydrocharis* (RAVEN and BRETSCHNEIDER 1942); the eggs were separated from the mucus and cultured in little glass dishes. Thiourea was applied in concentrations of 0.5, 0.75, 1 and 2 % (0.07, 0.10, 0.13 and 0.26 molar, respectively) dissolved in distilled water. For the treatment of decapsulated eggs, however, the thiourea was dissolved in a 0.04 % $CaCl_2$ solution, in order to prevent abnormal cleavage due to a lack of Ca^{++}-ions (HUDIG 1946). If the eggs were to be observed for several days, they were cultured in petri-dishes without water on a bottom of 2 % agar; in some experiments, thiourea was dissolved in the agar in the concentration desired.

Decapsulation was performed by pricking the egg capsules on a dry glass plate under a binocular microscope; after this manipulation the eggs were pipetted into the thiourea solutions.

The eggs were fixed in Bouin, sectioned at 7.5 μ and stained with iron haematoxylin-saffranin or azan. To distinguish the nineteen stages until the 4th cleavage, RAVEN's normal table (1946) has been used.

Experiments.

1. *Inhibition and delay of development.*

Eggs of different egg-masses (Tu 1—5) were treated with a 1 % thiourea solution, immediately after laying. A 3 hours' treatment causes already an inhibition of development. After exposure to the thiourea solution during 4—5 hours inhibition of 1st cleavage occurs. When the eggs are transferred to distilled water, recovery is still possible after a 5 hours' treatment. A longer exposure of eggs in their capsules to a 1 % thiourea solution gives no recovery; mostly an abnormal 3rd cleavage occurs.

The eggs of different batches show great variations in susceptibility, but as a rule all eggs of one batch are arrested in the same stage of development.

By studying eggs which after a 3—4 hours' treatment were transferred to distilled water and, after thorough washing, were laid out on an agar bottom, it becomes evident that this relatively short treatment causes a marked delay of development at later stages.

When treatment is not beginning at the uncleaved stage, but somewhat later, the inhibition of development is less pronounced.

In experiment Tu-9 eggs were exposed during 15 hours to a 0.5 % thiourea solution. During the first 3 days, no difference in rate of development with the controls could be observed. After 6 days, however, the treated eggs showed a marked delay in development, which was still more pronounced after 9 days.

A 3—4 hours' treatment with a 2 % thiourea solution causes also an inhibition and delay of development.

2. *Arrest of development.*

Uncleaved eggs were placed on agar, containing 1 % of thiourea. Development was arrested at the 4-cell stage. A similar phenomenon could be observed after 10 hours' treatment with a 1 % solution. Most eggs were arrested in the 8-cell stage, some eggs, however, did not develop beyond the uncleaved stage. Exposed to a 2 % solution during 6 hours a few eggs were arrested in the 2-cell stage, most eggs did not cleave.

Decapsulated eggs in a 1 % solution (Tu 33, 34) remain uncleaved or are arrested in the 4-cell stage.

Eggs of Tu-38, decapsulated immediately after laying and transferred to a 0.75 % solution remain uncleaved, but live for several hours. We shall refer to this special case later.

Batches Tu 40—48 showed that the eggs are more susceptible to the treatment at the 2-cell stage than at later stages of development. The eggs can still be arrested at the 4-cell stage by a treatment beginning during the first cleavage (RAVEN stage 5 at the latest), and at the 8-cell stage by a treatment beginning at a late 2-cell stage (RAVEN stage 8); when the treatment is begun still later, development proceeds beyond these stages and mortality is rather low.

At a high temperature (33° C) arrest in the uncleaved or 4-cell stage is accelerated. Batches placed all night in a 1 % solution at a temperature of streaming tap water showed the same abnormalities of development as those treated at room temperature during a much shorter time. Hence the susceptibility of the eggs seems to be dependent on temperature.

An inspection of the eggs fixed after arrest at the 4-cell stage reveals that the nuclei are not always in the same phase. In some cases they have formed monasters with disorderly arranged chromosomes, in other batches development has stopped at the prophase stage or at an anaphase stage of division. Eggs arrested either in the 4- or 8-cell stage preserved the capacity to form a wide cleavage cavity; in this case the nuclei are always in the interphase- or prophase stage.

3. *Abnormalities of cleavage.*

A 1 % Tu solution applied during 5—6 hours produces interesting abnormalities of cleavage (fig. 1). These abnormalities occur especially

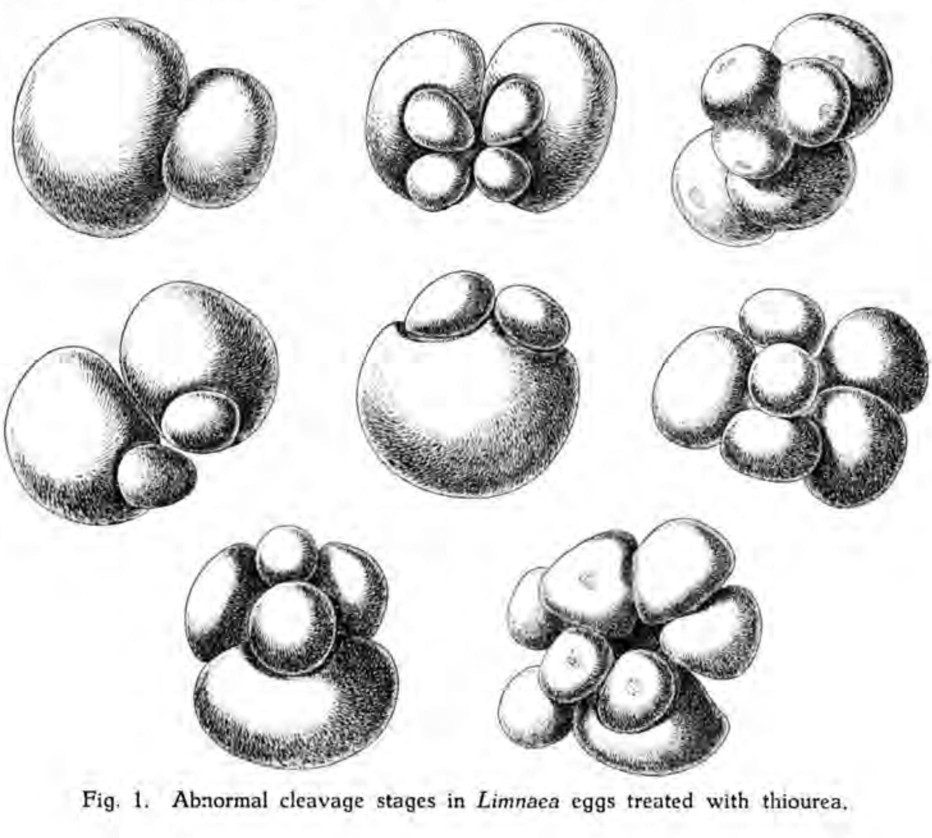

Fig. 1. Abnormal cleavage stages in *Limnaea* eggs treated with thiourea.

after the 3rd cleavage, but also at earlier stages. Abnormal 3-, 5- and 6-cell stages occur. The blastomeres show great divergences in size. A very unequal distribution of yolk among the blastomeres may occur. In many cases the synchronism of cell divisions is entirely or partially disturbed.

A treatment with a 2 % thiourea solution yields many 2-cell stages with blastomeres of different size. In some cases there is only an apparent cleavage by formation of giant polar bodies.

Fig. 2 shows cases where the blastomeres at the 2nd cleavage are not

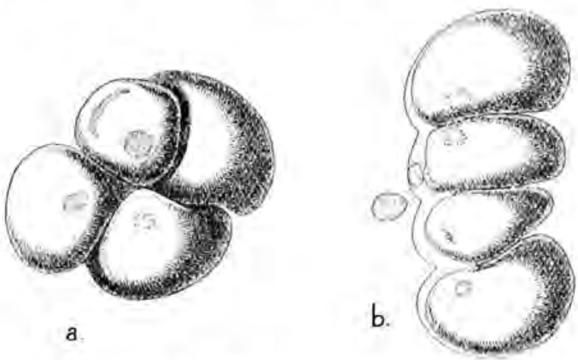

Fig. 2. Abnormal 4-cell stages.

formed in a normal way; there is a disturbance in direction of cleavages. This phenomenon can be observed especially in those cases where the 2nd cleavage is delayed in one of the blastomeres; when, after some time, cleavage begins in this blastomere too, it divides at right angles to the other one. This abnormality has been observed in several batches treated with 0.75 % Tu after decapsulation.

4. *Mechanisms of abnormal cleavage.*

In the abnormal cleavage the following points are of interest:

a. Cell division may be arrested. This may lead to uncleaved eggs with several groups of clotted chromosomes (fig. 3). At the second cleavage

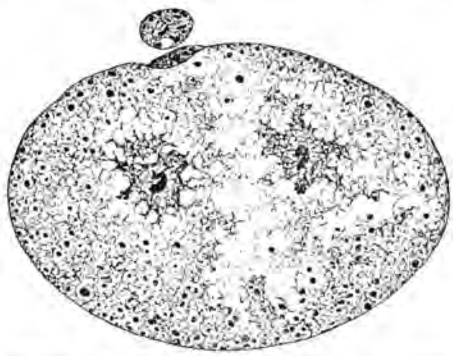

Fig. 3. Arrest of cell division. Beginning pycnosis of nuclei.

one of the cells may fail to divide, which leads to a 3-cell stage. As a rule these cells with arrested division contain 2 nuclei; apparently mitosis has taken place, but has not been followed by cell division. In most cases, no further mitosis takes place. Also at later cleavages, arrest of cell division may occur in one or more of the blastomeres.

b. The nuclei show various types of degeneration.

1^0. *Fragmentation of chrosmosomes.* In some cases cleavage spindles with disorderly arranged and fragmented chromosomes occur; sometimes

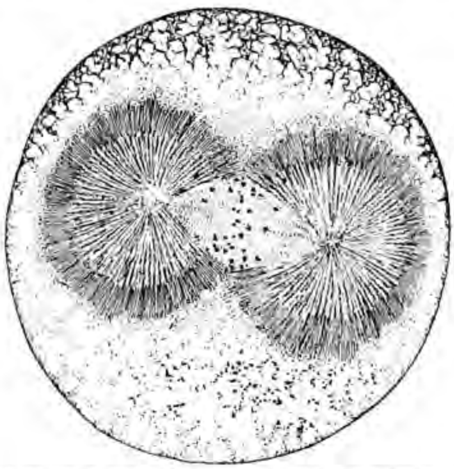

Fig. 4. First cleavage splindle with fragmented and disorderly arranged chromosomes. Spindle and asters abnormally enlarged.

the latter are widely scattered throughout the cytoplasm (fig. 4). In other cases the number of chromosomes formed in the spindles is evidently reduced.

2^0. Often *pycnosis* occurs; this is clearly demonstrated in the case of fig. 3.

3^0. *Monasters* are also found, especially in eggs arrested at the 4-cell stage. They contain irregular or clotted masses of chromosomes in their central part.

c. In many cases anachronisms of cleavage occur. At the 2nd cleavage one of the blastomeres may divide much later than the other, so that a temporary 3-cell stage is intercalated between the 2- and 4-cell stages. The mitosis in this half is also delayed as compared with the other one.

d. Most interesting are deviations in the position of cleavage spindles, which have been observed often in these eggs. In one batch which had been decapsulated and put into a 0.75 % Tu solution immediately after oviposition, no cleavage occurred (Tu 38); the eggs were fixed after 8 hours, when the controls were in the 8-cell stage. The maturation divisions had taken place normally; 2 polar bodies were present in all eggs. The animal pole plasm had been formed under the egg cortex of the animal side. The cleavage spindle had been formed, but at this moment the development had taken an abnormal course. The asters of the cleavage spindle had grown to enormous sizes (fig. 4); the spindle itself had also enlarged and elongated, so that in many eggs the asters were pressed against the egg cortex at opposite points of the surface (fig. 5). The

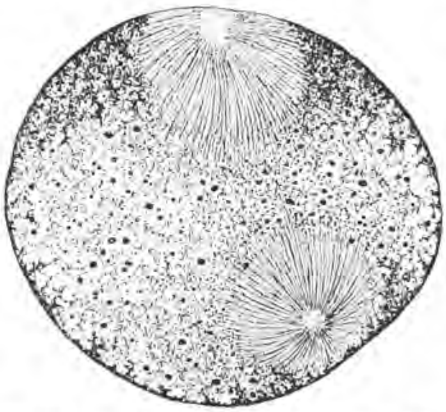

Fig. 5. Enlarged and rotated cleavage spindle in egg with inhibition of first cleavage.

chromosomes had lost their regular arrangement, they were lying scattered over the whole cytoplasm. Most interesting is, however, the fact that in many of these eggs the spindle had rotated into the direction of the egg axis; one of the asters is situated at the animal pole, pushing aside the

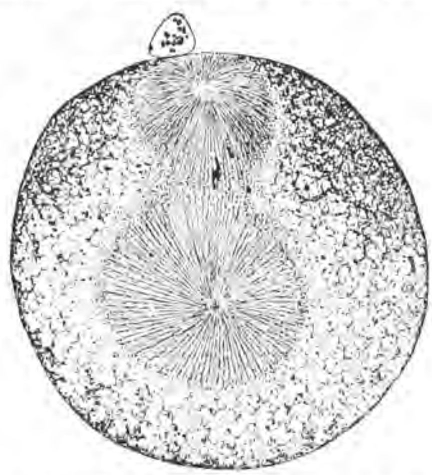

Fig. 6. Enlarged cleavage spindle, rotated into the direction of the egg axis, in egg with inhibition of first cleavage.

animal pole plasm, the other is lying at or near the vegetative pole (fig. 5, 6). Among 17 eggs, in 4 cases this had occurred; in 4 other eggs the spindles had an oblique position intermediate between that described above and its normal position perpendicular to the egg axis.

In the embryos in which cleavage is delayed in one of the blastomeres of the 2-cell stage, the spindle of this cell rotates in such a way that it comes to lie parallel to the egg axis. In this way, 4-cell stages with a tetraëdic position of the cells are formed (fig. 2*a*).

In other cases at the second cleavage one of the spindles may place itself perpendicularly to the plane of first cleavage which leads to a *T* shaped 4-cell stage (fig. 7). Sometimes both cleavage spindles take this position which gives rise to a cleavage stage with 4 cells in a line (fig. 2*b*).

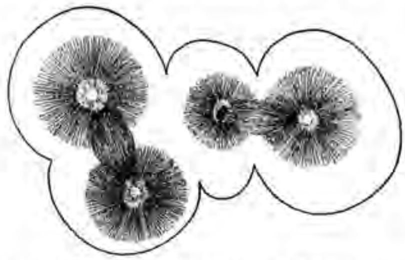

Fig. 7. Deviation of cleavage spindle in 2nd cleavage of *Limnaea* egg treated with thiourea.

Discussion.

Eggs in their capsules subjected to a thiourea solution of 1 % show an inhibition of development at the first cleavage stages. Until 5 hours of treatment this inhibition is reversible; but during further development the influence of thiourea shows itself by a delay of development. A treatment begun in the uncleaved stage causes the most pronounced inhibition and delay, possibly the influence on the nuclear apparatus is much stronger in this case.

A solution of 0.5 % applied for a long time shows itself to be poisonous too. Probably a higher temperature increases the susceptibility of the eggs for the influence of thiourea.

The inhibition of development seems to be in accordance with the facts found by BEVELANDER (1946) in *Arbacia*. In my experiments, however, there was no specific arrest of development in the gastrula stage.

My experiments show that an arrest of development at the 4-cell stage can still be brought about by a treatment beginning during the first cleavage, whereas development can be arrested at the 8-cell stage, when treatment begins at the utmost at a late 2-cell stage. When treatment begins still later even a relatively longer thiourea treatment causes less injury.

In WOKER's experiments (1943, 1944), *Tubifex* eggs subjected to colchicine show an irreversible arrest of cleavage in the 2-cell stage, as colchicine is acting directly on the cell nucleus in the interphase stage immediately after first cleavage. WOKER speaks, therefore, of a "critical phase" of greatest susceptibility. In the thiourea treatment in *Limnaea* the arrest of development does not occur immediately, but only after a certain lapse of time. This may indicate that the "critical concentration" of thiourea within the eggs is only gradually reached by the endosmosis of the agent into the cells.

Considering the eggs with abnormal cleavage, it is remarkable that many of these eggs show the tendency to continue the normal sequence of spiral cleavage, but by the influence of thiourea certain divisions are skipped. As soon, however, as a deviation of the direction of cleavage spindles takes place a total aberration of the normal type of cleavage occurs.

These spindle rotations may be explained by weakening of the factors, governing the position of the spindles in normal development, by the action of thiourea. However, many of these cases call forth another explanation. In most cases, the spindle rotations occur in cells in which cell division has been delayed for a considerable time. In the eggs described above (p. 905) where the first cleavage spindle had rotated into a vertical position, meanwhile the 3rd cleavage had taken place in the controls. It might be supposed that the conditions governing the succession of spindle positions in normal development change relatively independent of the other developmental processes, so that in these eggs, which have skipped two cleavages, the spindles are forced into a position corresponding to that of 3rd cleavage.

This explanation can also be applied to those cases described above, where a delay of 2nd cleavage in one of the blastomeres causes the spindle of this cell to rotate into the position of a 3rd cleavage spindle.

HÖRSTADIUS (1928) reports on comparable phenomena in eggs of *Paracentrotus lividus* caused by shaking or exposure to diluted seawater. In these cases the formation of mitotic spindles can be delayed, but the spindles are turning synchronously with those of the control eggs. In this sea urchin a cytoplasmic factor, determining the formation of micromeres and located at the vegetative side of the egg, plays a part. We do not know, however, which is the determining factor for spindle rotation preceding the 3rd cleavage in the *Limnaea* egg. Similar observations have been made by CONKLIN (1938) in *Crepidula* eggs treated with low temperatures; after returning the eggs to 20°, some divisions may be skipped, but the subsequent cleavages (e.g. formation of micromeres) nevertheless take place in the normal direction.

RAVEN and MIGHORST (1946) report on a deviation of the second maturation division in eggs of *Limnaea* exposed to a 0.5 % $CaCl_2$ solution. In some egg masses a depolarization occurs. The second maturation spindle looses its contact with the animal pole, becomes centrally located and assumes a position at right angles to the polar axis.

Our hypothesis seems to give a possible explanation of these facts as there is a tendency of the second maturation spindle to assume the position of the first cleavage spindle in normal eggs.

PASTEELS (1931) displaces the 1st cleavage spindle in eggs of *Barnea candida* by ultraviolet radiation. There is no alteration, however, of the position of the spindle to the egg axis or of the spiral cleavage type.

PASTEELS (1930) reports on some cases of orientation of the first cleavage spindle in the direction of the egg axis. According to his interpretation, the spermatozoon has penetrated in these cases at the lateral side of the egg and the male pronucleus has been formed in an abnormal place. As soon as the spindle figure moves to the egg cortex, as occurs in all eggs of Lamellibranchiates, regulation takes place and the axis of mitosis places itself at right angles to the egg axis. PASTEELS explains these phenomena by the action of a cytoplasmic factor, which determines the position of the spindle.

Possibly, treatment with thiourea solution influences cytoplasmic factors responsible for cell division. It seems probable that these factors are located in the cortical or subcortical plasm.

On the other hand, the phenomena of nuclear degeneration seem to point to a direct action of the thiourea solution on the nuclear apparatus.

Summary.

1. Eggs of *Limnaea stagnalis* were treated at several stages with solutions of thiourea varying between 0.5 and 2 %.
2. Treatment of eggs within the capsules with a 1 % thiourea solution immediately after oviposition causes an inhibition of development. When the duration of the treatment is 5 hours or less, recovery occurs, but at later stages development is delayed as compared with the controls.
3. A longer treatment with thiourea solutions varying between 0.75 and 2 % causes an arrest of development either in the uncleaved or in the 4- or 8-cell stages. With respect to this arrest, the stage in which treatment begins is of great importance.
4. A longer treatment also causes many abnormalities of cleavage. The synchronism of cleavage divisions may be entirely or partially disturbed.
5. Cell divisions may be arrested; the nuclei show various types of degeneration, e.g. fragmentation of chromosomes, pycnosis and monasters; anachronisms of cleavage also occur.
6. In many cases, deviations from the normal direction of spindles occur, especially in delayed cleavages. They point to a relative independence of the factors governing the direction of the spindles in spiral cleavage.

I am highly indebted to Prof. RAVEN for his valuable assistance during this investigation and his critical advice in the presentation of the results.

I wish to thank Mr. LEVER and Mr. BRETSCHNEIDER for their encouragement and assistance in various respects.

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Medicine (Chemotherapy). — *L'action inhibitrice des métaux sur la croissance du B. tuberculeux.* VII. *Manganèse et rhénium.* By ONG SIAN GWAN. (Communicated by Prof. E. GORTER.)

(Communicated at the meeting of May 29, 1948.)

1. Le manganèse et le rhénium appartiennent au groupe VII, sous-groupe *a*, du tableau périodique. Le rhénium n'a aucune action inhibitrice sur la croissance de B. tuberculeux, le manganèse par contre montre une forte action inhibitrice même en presence de sérum. Ceci explique le bon résultat obtenu par WALBUM depuis 1924 dans le traitement de la tuberculose expérimentale par le chlorure de manganèse. WALBUM (1926, a) a également montré que l'injection préalable de chlorure de manganese chez un cobaye tuberculeux protège l'animal contre un choc tuberculinique.

HELMS (1925) et HELMS et FREDERIKSEN (1927) ont traité 115 tuberculeux pulmonaires par des injections intraveineuses de chlorure de manganèse. Ils ont constaté l'amélioration dans 77,3 p. 100, la disparition des crachats dans 46 p. 100, la disparition de B. tuberculeux dans les crachats de 56,5 p. 100, la disparition de la fièvre dans 70 p. 100 et l'augmentation de poids dans 87 p. 100 des cas avec une moyenne de 2.72 kg et un maximum de 10,5 kg. Un résultat comparable a été obtenu par LUNDE (1926) chez 58 tuberculeux.

Par contre, HÖEG LARSEN et TÖRNING (1927) n'ont obtenu aucun résultat chez 31 malades traités par des injections de chlorure de manganèse.

2. *Action inhibitrice de manganèse et de rhénium sur la croissance de B. tuberculeux.*

Les métaux utilisés dans toutes les expériences sont les suivants: **1.** manganum metallicum fusum MERCK, **2.** rhénium pursiss. en poudre de HEYL. Dans certaines expériences on utilise le sulfate de manganèse de ANALAR, analytical reagent.

a. Expérience réalisée avec 5 mg de Mn ou de Re et de 4,06 mg de sulfate de manganèse par 100 cc de milieu de SAUTON. *Méthode du carré latin.*

La concentration de Mn dans le milieu contenant le sulfate de manganèse est égale à 1 : 100 000.

La souche de B. tuberculeux utilisé 1030 (2) a été aimablement mise à notre disposition par M. L. E. DEN DOOREN DE JONG. La culture est âgée de 57 jours et la durée de l'expérience est de 33 jours. Le tableau 1 et l'analyse de la variance montrent qu'il n'y a pas de différences significatives entre les traitements considérés dans leur ensemble. Le coefficient de variation est égal à $C = 11{,}8$ p. 100.

b. Expérience réalisée avec 10 mg de Mn ou de Re et 8,12 mg de *sulfate de manganèse par 100 cc de milieu de* SAUTON. *Méthode du carré latin.*

La concentration de Mn dans le milieu contenant le sulfate de manganèse est égale à 1 : 50 000. La souche utilisée 1030 (3) est âgée de 33 jours

TABLEAU 1.

Action inhibitrice de 5 mg de Mn ou de Re et de 4,06 mg de $MnSO_4 . 4 H_2O$/100 cc, concentration de Mn 1 : 100 000. Poids en mg. Méthode du carré latin.

I. Tableau des observations.

	Colonnes				Total
Rangées	Mn 615,4	T 713,9	Mn 524,7	Re 720,0	2574,0
	T 685,7	$MnSO_4$ 747,6	Re 691,2	Mn 869,8	2994,3
	Re 737,6	Mn 768,4	T 679,4	$MnSO_4$ 521,1	2706,5
	$MnSO_4$ 727,2	Re 729,9	Mn 754,0	T 624,4	2835,5
Total	2765,9	2959,8	2649,3	2735,3	11110,3
Moyenne	T 675,9	Mn 751,9	$MnSO_4$ 630,2	Re 719,7	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre traitements	33 669	3	11 223
Entre rangées	24 205	3	8 068
Résiduelle	60 270	9	6 697
Total	118 144	15	

T, témoin.

TABLEAU 2.

Action inhibitrice de 10 mg de Mn ou de Re et de 8,12 mg de $MnSO_4 . 4 H_2O$/100 cc, concentration de Mn 1 : 50 000. Poids en mg. Méthode du carré latin.

I. Tableau des observations.

	Colonnes				Total
Rangées	Mn 10,7	$MnSO_4$ 566,5	Re 507,8	T 689,6	1774,6
	Re 532,6	Mn 751,4	T 719,1	$MnSO_4$ 576,6	2579,7
	T 528,0	Re 626,2	$MnSO_4$ 589,8	Mn 628,4	2372,4
	$MnSO_4$ 598,6	T 666,6	Mn 14,6	Re 631,3	1911,1
Total	1669,9	2610,7	1831,3	2525,9	8637,8
Moyenne	T 650,8	Mn 351,3	$MnSO_4$ 582,9	Re 574,5	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre traitements	203 941	3	67 980
Entre colonnes	171 640	3	57 213
Résiduelle	328 569	9	36 508
Total	704 150	15	

et l'expérience a duré 67 jours. L'analyse de la variance (tableau 2) montre qu'il n'y a pas de différences significatives entre les traitements considérés dans leur ensemble. Une différence significative entre deux moyennes est égale à 305,6 mg. On constate que la différence entre les moyennes obtenues par le témoin et le manganèse est significative. Le coefficient de variation est égal à $C = 35,4$ p. 100.

c. Expérience réalisée avec 10 mg de Mn ou de Re par 100 cc de milieu de SAUTON.

Cette expérience est comparable à la précédente, mais la souche utilisée 1030(1) est plus âgée: 51 jours et la durée de l'expérience est de 57 jours. De plus le poids moyen de B. tuberculeux utilisé pour l'ensemencement est plus petit, il est dans ce cas 1,4 mg de poids sec au lieu de 3,8 mg dans l'expérience précédente. On constate maintenant une différence très significative entre les moyennes de Mn et de témoin (tableau 3).

TABLEAU 3.

Action inhibitrice de 10 mg de Mn ou de Re par 100 cc de milieu de SAUTON. Poids en mg.

	Colonnes			Total
Rangées	Re 0,5	Mn 40,0	T 5,5	46,0
	Mn 0	Re 637,2	T 657,4	1294,6
	T 0	Mn 0	Re 14,4	14,4
	Re 753,5	T 703,8	Mn 0	1457,3
Total	754,0	1381,0	677,3	2812,3
Moyenne	T 341,7	Mn 10,0	Re 351,4	

d. Expérience réalisée avec 10 mg de Mn et 8,12 mg de sulfate de manganèse par 100 cc de milieu de SAUTON.

Dans le milieu contenant le sulfate de manganèse la concentration de Mn est égale à 1 : 50 000. On utilisé la souche bovine VALLÉE après 15 passages sur millieu de SAUTON: V_{15}, l'âge de la culture est de 68 jours et la durée de l'expérience est de 30 jours. Le poids sec moyen de B. tuberculeux transplanté est 4,8 mg.

L'analyse de la variance (tableau 4) montre qu'il existe des différences significatives entre les traitements considérés dans leur ensemble. Une différence significative entre deux moyennes pour le seuil de signification $P = 0,01$ est égale à $\Delta = 545,9$ mg et pour $P = 0,001$, $\Delta = 771,7$ mg. L'action inhibitrice de sulfate de manganèse et surtout de manganèse métallique sur la croissance sont donc très marquées.

e. Expérience réalisée avec 20 mg de Mn ou de Re par 100 cc de milieu de SAUTON.

La souche utilisée V_{12} est âgée de 13 jours et la durée de l'expérience est de 25 jours. Le tableau 5 et l'analyse de la variance montrent des différences très significatives entre les traitements considérés dans leur ensemble. L'action inhibitrice obtenu par 20 mg de Mn est très élevée,

TABLEAU 4.

Action inhibitrice de 10 mg de Mn ou de 8,12 de $MnSO_4 . 4 H_2O$ par 100 cc de milieu de SAUTON. Poids en mg.

I. Tableau des observations.

	Colonnes								Total
Rangées	T	904,8	A	504,6	T	997,2	Mn	67,6	2474,2
	Mn	0	T	1034,3	Mn	276,6	$MnSO_4$	67,0	1377,9
	$MnSO_4$	0,3	A	853,0	$MnSO_4$	347,6	T	990,8	2191,7
	T	1058,4	Mn	635,2	A	1035,6	$MnSO_4$	758,4	3487,6
	A	1132,4	$MnSO_4$	1153,7	Mn	14,0	A	1069,0	3369,1
Total		3095,9		4180,8		2671,0		2952,8	12900,5
Moyenne	T	997,1	Mn	198,7	$MnSO_4$	465,4	A	918,9	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre rangées	764 037	4	191 009
Entre traitements	2 152 321	3	717 440 ***
Résiduelle	766 481	12	63 873
Total	3 682 839	19	

*** probabilité < 0,001.

A, substance organique, non déterminée.

TABLEAU 5.

Action inhibitrice de 20 mg de Mn ou de Re par 100 cc de milieu de SAUTON. Poids en mg.

I. Tableau des observations.

	Témoins	Manganèse	Rhénium	Total
Rangées	707,3	9,8	726,0	1443,1
	730,5	1,7	723,5	1455,7
	726,0	12,8	788,5	1527,3
	757,0	43,6 [1]	781,4 [1]	1582,0
Total	2920,8	67,9	3019,4	6008,1
Moyenne	730,2	17,0	754,9	500,7

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre rangées	4 208	3	1 403
Entre traitements	1 404 802	2	702 401 ***
Résiduelle	1 637	4	409
Total	1 410 647	9	

[1]) Valeur calculée.

elle est plus élevée que celle obtenu par 10 mg de Mn en utilisant la même souche VALLÉE (expérience *d*). Ce résultat inattendu a été constaté plusieurs fois avec d'autres éléments.

3. *Action inhibitrice de manganèse en présence de sérum sanguin sur la croissance de B. tuberculeux.*

a. *Expérience réalisée avec 10 mg de Mn ou 40,6 mg de sulfate de manganèse par 100 cc de milieu de* SAUTON.

La concentration de Mn dans le milieu avec sulfate de manganèse est égale à 1 : 10 000. On ajoute dans les flacons contenent le manganèse et le sulfate de manganèse 5 cc de sérum de chèvre non chauffé, les flacons témoins ne contiennent pas de sérum. La souche utilisée, V_{13} est âgée de 51 jours et la durée de l'expérience est de 68 jours.

TABLEAU 6.

Action inhibitrice de 10 mg de Mn ou de 40,6 mg de $MnSO_4 . 4 H_2O$ en présence de 5 cc de sérum. Concentration de Mn 1 : 10 000. Poids en mg.

	Colonnes			
Rangées	Mn 0 $MnSO_4$ 0 T 1049,8	T 948,3 Mn 0 $MnSO_4$ 0	Mn 0 T (226,7) $MnSO_4$ 0	$MnSO_4$ 0 T 711,1 Mn 0
Total	1049,8	948,3	(226,7)	711,1
Moyenne	T 733,8	Mn 0	$MnSO_4$ 0	

0, pas de croissance.

Dans tous les flacons contenant Mn + sérum ou $MnSO_4$ + sérum on ne constate pas de croissance de B. tuberculeux pendant toute la durée de l'expérience (tableau 6). Si l'on compare cette expérience avec les expériences précédentes *d* et *e* on serait en droit de conclure que l'action

TABLEAU 7.

Action inhibitrice de 10 mg de Mn et de 10 mg de Mn en présence de 5 cc de sérum. Poids en mg. Méthode du carré latin.

I. Tableau des observations.

	Colonnes				Total
Rangées	Mn+sér 4,1	Mn 497,8	T 967,0	Sér 1227,6	2696,5
	T 860,6	Sér 1210,4	Mn+sér 727,2	Mn 525,0	3323,2
	Sér 1253,2	T 998,7	Mn 94,6	Mn+sér 834,2	3180,7
	Mn 737,4	Mn+sér 36,7	Sér 1377,8	T 857,0	3008,9
Total	2855,3	2743,6	3166,6	3443,8	12209,3
Moyenne	T 920,8	Sér 1267,3	Mn 463,7	Mn+sér 400,6	

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre traitements	2 000 509	3	666 836 **
Résiduelle	833 499	12	69 458
Total	2 834 008	15	

Sér = sérum.

** probabilité 0,01.

de Mn en présence de sérum est plus élevée que sans sérum. Pour résoudre cette question l'expérience suivante a été réalisée.

b. Expérience réalisée avec 10 mg de Mn par 100 cc de milieu de SAUTON *avec ou sans sérum.*

L'expérience est réalisée comme suit: **1.** témoin contenant le milieu de SAUTON, **2.** comme le témoin, et en outre 5 cc de sérum de chèvre non chauffé, **3.** comme le témoin, et en outre 10 mg de Mn, **4.** comme le témoin, et en outre 10 mg de Mn et 5 cc de sérum de chèvre non chauffé.

Le tableau 7 montre le résultat obtenu, l'analyse de la variance montre des différences significatives entre les traitements considérés dans leur ensemble. Une différence entre deux moyennes est significative si elle est plus grande que 406,1 mg. On constate qu'il n'y a pas de différence significative entre l'action inhibitrice de Mn avec et sans sérum. Le poids moyen de la culture avec sérum est plus élevé que celui sans sérum, mais la différence n'est pas significative. On constate cependant une croissance appréciable dans les cultures avec sérum deux jours après l'ensemencement, celle-ci ne se produit pas dans les cultures témoins.

4. *Addition de 15 mg de manganèse au cours de la croissance de B. tuberculeux.*

L'addition de Mn au cours de la croissance permet de savoir si la croissance est diminuée ou si elle est complètement arretée. Dans le dernier cas le B. tuberculeux pourrait être tué.

Pour savoir si la croissance est arretée on compare à la fin de l'expérience le poids moyen des cultures avec Mn avec celui des cultures témoins au moment de l'addition de Mn. Enfin, pour savoir si la croissance est diminuée

TABLEAU 8.

Addition de 15 mg de Mn au cours de la croissance de B. tuberculeux. Poids en mg.

I. Tableau des observations.

Mn_{15} 671,1	T_{31} 585,5	Mn_{31} 658,4	T_{40} 668,6	T_{8} 4,7	Mn_{8} 604,4	T_{15} 57,6
T_{15} 116,1	Mn_{15} 845,6	T_{31} 633,4	Mn_{31} 677,4	T_{40} 702,0	T_{8} 2,7	Mn_{8} 680,6
T_{40} 672,0	T_{8} 1,7	Mn_{8} 687,7	T_{15} 34,1	Mn_{15} 474,9	T_{31} 589,1	Mn_{31} 731,0
Mn_{8} 846,6	T_{15} 58,0	Mn_{15} 701,0	T_{31} 631,0	Mn_{31} 599,7	T_{40} 626,4	T_{8} 32,4
Moyenne	T_{8} 10,4	T_{15} 66,4	T_{31} 609,8	T_{40} 667,3		
	Mn_{8} 704,8	Mn_{15} 673,2	Mn_{31} 666,6			

II. Tableau d'analyse de la variance

Source de variation	Somme des carrés	Degrés de liberté	Variance moyenne
Entre rangées	18 678	3	6 226
Entre traitements	3 952	3	1 317
Résiduelle	93 917	10	9 392
Total	116 547	16	

T_i, poids de culture témoin au bout de *i* jours.

Mn_i, poids de culture au bout de 40 jours et après addition de Mn au bout de *i* jours.

on compare à la fin de l'expérience le poids moyen des cultures avec Mn avec celui des cultures témoins de même âge.

L'expérience est réalisée avec la souche V_{12}, âgée de 42 jours, la durée de l'expérience est de 40 jours.

La figure 1 et le tableau 8 montrent le résultat obtenu. On constate que

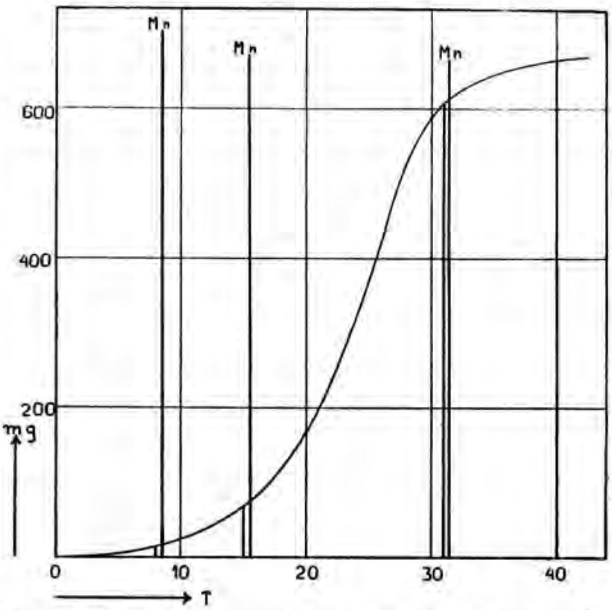

Fig. 1. Addition de 15 mg de manganèse au cours de la croissance de B. tuberculeux. Poids en mg. T, temps en jours. Mn, poids moyen des cultures de B. tuberculeux de 40 jours apres addition de Mn au bout de 8, 15 et 31 jours.

l'addition de Mn au bout de 8, 15 et 31 jours n'a aucun influence sur la croissance de B. tuberculeux. Les poids obtenus par le manganèse ont été comparés dans l'analyse de la variance avec le poids témoin de 40 jours.

5. *Cuture de B. tuberculeux ayant été en contact avec le manganèse.* L'expérience a pour but de savoir si le B. tuberculeux qui a été en contact avec le manganèse pendant un temps déterminé et ensuite transplanté dans un milieu normal montre une croissance diminuée ou arrêtée. Pour cela on compare le poids moyen obtenu avec le poids moyen des cultures témoins dans les mêmes conditions.

L'expérience est exécutée comme suit: on prend deux flacons contenant le milieu de SAUTON et deux autres flacons contenant le même milieu et en outre 20 mg de Mn. Les quatre flacons sont ensemencés avec la souche V_{12} de 42 jours. Au bout de 8, 15 et 31 jours on fait de chaque culture deux cultures dans un milieu de SAUTON. On obtient ainsi quatre cultures de B. tuberculeux ayant été en contact avec Mn et quatre cultures de B. tuberculeux témoin. On place les cultures à l'étuve et on détermine le poids sec après 31—32 jours.

Le tableau 9 montre le résultat obtenu. Malheureusement la plupart des cultures témoins sont tombées dans le liquide au cours de l'expérience de sorte que les poids obtenus entre parenthèses sont trop petits. Néanmoins le résultat obtenu est très net. On constate qu'après huit jours de contact

TABLEAU 9.

Culture de B. tuberculeux ayant été en contact avec le Mn: 20 mg de Mn par 100 cc de milieu de SAUTON. Poids en mg.

Durée de contact avec Mn en jours	Durée de l'expérience en jours	Témoins	Manganèse
8	32	743,1 (58,1) (109,5) (139,5)	3,8 477,3 (53,2) 62,4
Total Moyenne		1050,2 262,6	596,7 149,2
15	31	(99,5) (110,4) (47,0) 759,2	0 1,4 0,4 0,1
Total Moyenne		1016,1 254,0	1,9 0,5
31	31	573,6 542,3 (61,5) (29,0)	0,7 2,7 1,9 1,0
Total Moyenne		1206,4 301,6	6,3 1,6

Les poids entre parenthèses représentent des cultures tombées dans le liquide pendant l'expérience, ils sont donc trop petits.

avec le manganèse la croissance est diminuée et qu'après 15 et 31 jours de contact la croissance est complètement arrêtée et le B. tuberculeux est problablement tué.

Nous remercions vivement MM. E. VAN DER LAAN et J. VAN DEN DULK de ses précieux conseils dans l'analyse de la variance.

Résumé. 1. Le rhénium n'a aucune action inhibitrice sur la croissance de B. tuberculeux.

2. Le manganèse et le sulfate de manganèse ont une action inhibitrice marquée meme en présence de sérum sanguin.

3. L'addition de manganèse au cours de la croissance n'a aucune influence inhibitrice.

4. Par contre, la culture de B. tuberculeux ayant été en contact avec le manganèse montre une croissance diminuée. Si le contact avec le

manganèse est prolongé la croissance est complètement arrêtée et le B. tuberculeux est probablement tué.

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Medicine (In fectious diseases). — *Some observations on the rabbit-pox virus.* By F. WENSINCK. (From the "Instituut voor praeventieve Geneeskunde", Leiden.) (Communicated by Prof. H. W. JULIUS.)

(Communicated at the meeting of May 29, 1948.)

The infection of rabbits with neuro-vaccine may lead to the outbreak of a rabbit-pox epidemic (LEVADITI and NICOLAU, 1923; LEVADITI and SANCHIS–BAYARRI, 1927). Undoubtedly, the great power of resistance of the vaccine-virus plays an important role. These epidemics exhibit a strongly divergent aspect; their character varies from a suddenly exploding infectious disease with a rich symptomatology and high mortality (NICOLAU and KOPCIOWSKA, 1929; GREENE, 1933, 1934, 1935; HU et al., 1936; PEARCE et al., 1933, 1936; ROSAHN et al., 1936) to those latent virus-infections, which either lead to immunity (NICOLAU and KOPCIOWSKA, 1929 b; DURAN REYNALS, 1931) or become manifest only after some intervention.

These latent virus-infections are especially insidious when rabbits are intracerebrally injected, when virus-stocks have to be preserved (e.g. herpes-virus) and when an attempt is made to prove the existence of some kind of virus in the diseased material (LEVADITI, LEPINE and SCHOEN, 1931).

Under certain circumstances — one is easily inclined to forget that neuro-vaccine has been used in one's laboratory — the affinity of herpes- as well as of rabbit-pox-virus to the skin and nervous system can make it difficult to distinguish between the two viruses. Above all this can be said of the meningoencephalitis herpetica and rabbit-pox meningitis, which, becoming manifest after intracerebral injections, show a close resemblance, considered from a clinical point of view.

The purpose of this article is to emphasize histopathological similarities and differences and to indicate how cutaneous inoculation of the cerebral substance on the guinea pig is able to prevent a wrong diagnosis.

General data.

In the period between September 1942 and September 1943, 55 rabbits were intracerebrally injected with liquor-samples from 52 patients (suffering from multiple sclerosis, neuritis retrobulbaris and a number of other organic nervous diseases). As a rule 0.2 cc of each liquor was used per rabbit. After an average of 16—17 days, a great number of rabbits died as a result of this liquor-injection. Neither the origin of the liquor-samples, nor those anomalies which can be ascertained during the routine-examination (albumen, cells, etc.) showed any relation to these cases. For a shorter or longer period, about half of the animals that died showed perceptible symptoms of meningo-encephalitis. When other rabbits were given intracerebral injections with the cerebral emulsions of the dead rabbits, this led to the appearance of a meningoencephalitis, whereby the time between injection and death decreased to 4—5 days. Bacteriologically the rabbit-cerebra were sterile.

However, intracerebral injection of sterilized liquids (physiological NaCl-solution and glycerine) and cerebral emulsions from new rabbits, appeared to have the same results, while control experiments with some of our liquor samples, made by dr. J. H. BEKKER (State Public Health Laboratories, Utrecht) proved to be negative. When, besides, it became evident, that after close examination of a number of rabbits, tongue-pox could be traced, we felt sufficiently justified in considering a latent infection with neuro-vaccine as proved.

This was also demonstrated by the fact that it was possible to prove the existence of the virus in the nasal secretion (also in the case of spontaneously infected rabbits) by guinea pig inoculation, even after filtration through a Seitz E K filter-pad. Spontaneous infection caused almost negligible symptoms of sickness. In a solitary instance, a rabbit suffered from a purulent rhinitis.

As, in the past, rabbit-pox epidemics — the virulent type as well as that of latent infection — have already been the subject of lengthly considerations, only two details will be discussed here.

I. *Meningitis,* caused by the rabbit-pox virus.

a. From a clinical point-of-view, it is difficult to distinguish rabbit-pox meningitis from herpesmeningoencephalitis. In both cases, paralysis, ophisthotonus, insults, gnashing of teeth, excessive salivation and various stages of agitation are manifest in various combinations. In the case of *herpes-meningo-encephalitis, attacks* are repeatedly noticed, whereby the following phenomena are of importance. Turning its head backwards, the animal rises onto its hind legs. With its forelegs, it makes convulsive movements of a very slight amplitudo (cf. LEVADITI, 1926). In the case of rabbit-pox meningitis on the contrary, we never noticed these attacks, at least not in their extreme form. The clinical phenomena can be explained by damage suffered by the central nervous system. It should, however, be observed that excessive salivation also occurs with animals without encephalitis. Next to a central cause excessive salivation can also have a local one; repeatedly, we noticed it with rabbits suffering from tongue-pox without encephalitic-symptoms.

b. Before discussing the histopathology of rabbit-pox meningitis, we propose to give a short description of the structure of the cerebral membranes of the rabbit. From publications by GOLMANN (1931) and SSOLOWJEW and ARIEL (1933) we are able to acquire a good idea of the various nucleal forms in the leptomeninx. The authors lay especial stress on existing differences in the fibrous structure of pia and arachnoidea.

When the cerebra are removed from the cranial cavity, the dura is not taken away. In the microscopic preparations, we only have to deal with arachnoidea and pia. On the subdural and subarachnoideal side, the arachnoidea is coated with cover cells. The trabeculae arachnoideales connecting the arachnoidea-membrane with the pia and the vessels, running through the subarachnoideal space are also coated with these cover-cells. The pia is a layer of cover-cells and fibrocytes, grown together with the cerebral surface. In leptomeninx parts, cut tangentially and in so-called plane preparations (fig. 1) the differences between cover-cells and fibrocytes are clear: the cover-cells have a big, round, bladdery nucleus, poor in chromatine, whilst the fibrocytes show a protracted, more or less irregular and chromatine-rich nucleus.

In the arachnoideal trabeculae too (fig. 2) the cover-cells are clearly visible, as well as in the pia (fig. 3). Finally, fig. 4 shows that the cover-cells, which coat the entire subarachnoideal space, form a serried whole, like, for example, the pleura-endothelium.

With most rabbits, which had been intracerebrally injected with rabbit-pox-virus (about a hundred) a meningitis with — locally — many disintegrated leucocytes was found. Generally, the inflammation remained restricted to the membranes; we often found a superficial perivascular cell-infiltration under the inflamed membranes, seldom, however, in the deeper layers of the cortex. In most cases, symptoms of inflammation were found locally; in a few cases only, were they to be considered as diffuse. There was a preference for the meninges of the convexity which penetrate together with a sulcus and the meninges coating the hyppocampus (fig. 5 and 6).

In most cases, we discovered meningitis, varying from moderate to pronounced. Often a pronounced meningitis was also of an extensive character, although in various cerebra important changes were found only locally.

Character of the changes (fig. 7). Meningitis caused by the rabbit-pox virus is characterized by an infiltration of leucocytes and lymphocytes into the meninges and into the wall of the superficial cortical blood-vessels. In some cases an infiltrate consists almost exclusively of disintegrated leucocytes and we were struck by the fact that, generally, few leucocytes were found intact.

In connection with the transmission of the virus and the shortening of the time between injection and death, we found that the number of rabbits with histological changes in the cerebral membranes had increased. Neither the character of the inflammation symptoms nor the intensity changed, however.

c. The histopathological picture, which we described, shows a resemblance with the "pustule meningée", i.e. meningitis in the case of rabbit-pox, as described by NICOLAU and KOPCIOWSKA (1931). These authors point out that intracerebral injection with neuro-vaccine results in a completely identical situation, which is very conceivable, considering the close congeniality between neuro-vaccine and rabbit-pox virus.

d. From the above-mentioned data it can be concluded that the histopathological picture of the rabbit-pox meningitis may show a great similarity with herpes-meningoencephalitis (cf. LEVADITI 1926). Also in the case of herpesmeningoencephalitis, lymphocytes, plasmacells and polynuclear cells are found in the membranes, but generally the cerebral-tissue itself is more strongly affected than in the case of rabbit-pox meningitis. This is proved by the strong perivascular cell-infiltration and the penetration of leucocytes into deeper layers of the cortex and into the white matter. When, however, in the case of pronounced rabbit-pox meningitis, the perivascular infiltration in the cortex layers becomes intensive, or if during a case of slight meningoencephalitis herpetica the encephalitis is histologically little manifest, then, the making of a diagnosis becomes more or less arbitrary. A comparison between figs 5/6 and 8/9 makes it clear that in cases where the differences are not so evident, it may be extremely difficult to make a distinction. In such a case, it might be advisable to attempt to use other means in order to judge the nature of the isolated virus. It appeared to us that the following method is of value.

II. When the rabbit-pox virus is rubbed into skin-scarifications of the guinea pig a dermatitis arises, which develops as follows. Two days after the inoculation, the skin turns red and begins to swell. On the third day, yellow-white coloured papula appear on this erytheme, which develop into pox at the end of the third or the beginning of the fourth day. Then, in

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Fig. 1. Plane preparation of the arachnoidea-membrane (subarachnoideal side); haematoxyline-eosine.
D: Large chromatine-poor *nuclei* of cover cells.
OD: Smaller, chromatine-richer nuclei of immature cover cells.
F: Nuclei of fibrocytes.

Fig. 2. Trabeculae arachnoideales, plane preparation; haemotoxyline; *D*: cover cells.

Fig. 3. Cerebral coupe with pia-section, which has been torn loose, now lying flat; cresylviolet. A large number of nuclei of the cover cell type.

Fig. 4. Plane preparation of arachnoidea-membrane; silver impregnation (Ranvier), coloured afterwards with haemotoxyline; drawing.

Fig. 5. Rabbit-pox meningitis; cresylviolet; photo, giving a general view of the subject. Meningitis of great extensiveness on the convexity; vigorous basal meningitis.

Fig. 6. Rabbit-pox meningitis of moderate intensity with perivascular cell-infiltration in the superficial cortex layers; cresylviolet.

Fig. 7. Rabbit-pox meningitis; cresylviolet.
D: Large-sized nuclei of cover-cells. On several places in the preparation, pycnotic nuclei of leucocytes are visible.

Fig. 8. Herpes-meningoencephalitis; cresylviolet; photo, giving a general view of the subject. Pronounced inflammation around vessels, as far as within the white matter.

Fig. 9. Herpes-meningoencephalitis; cresylviolet.
Perivascular cell-infiltration around vessels in the whole cortex with disturbance of the cortex-structure.

F. WENSINCK: *Some observations on the rabbit-pox virus.*

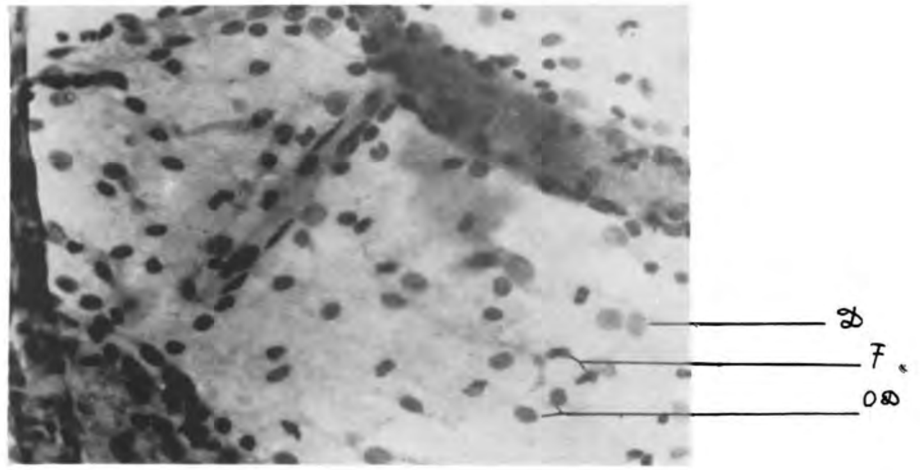
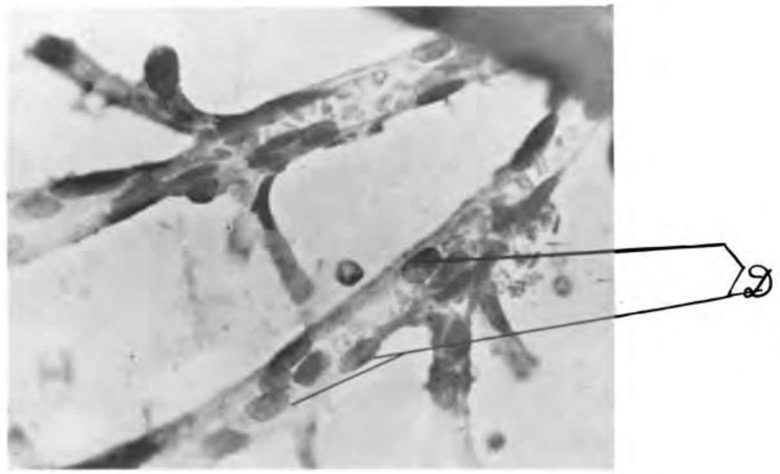


Fig. 1.



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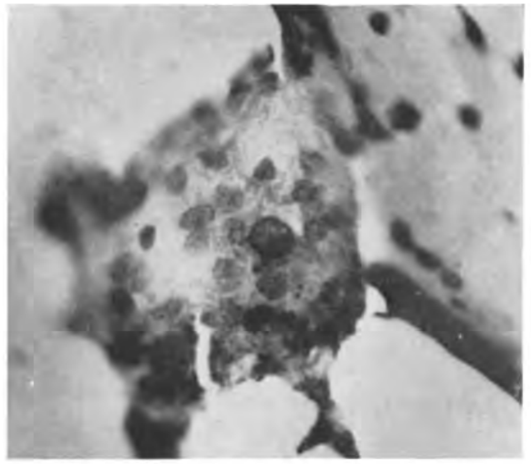

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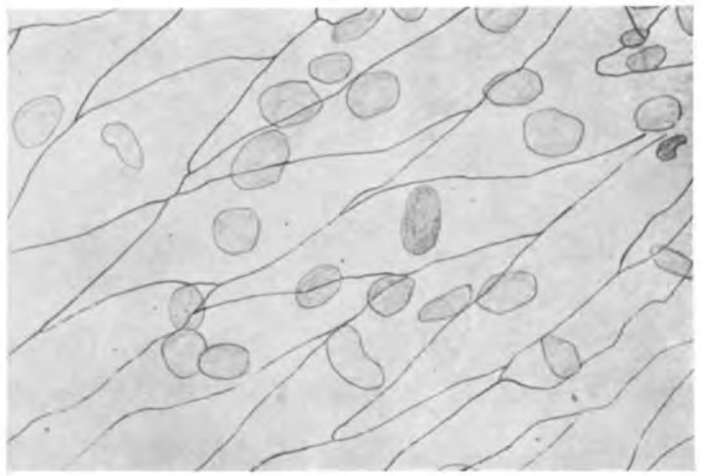

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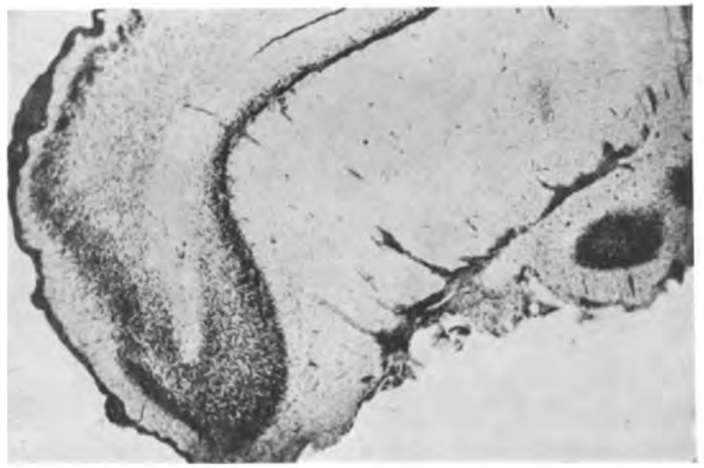

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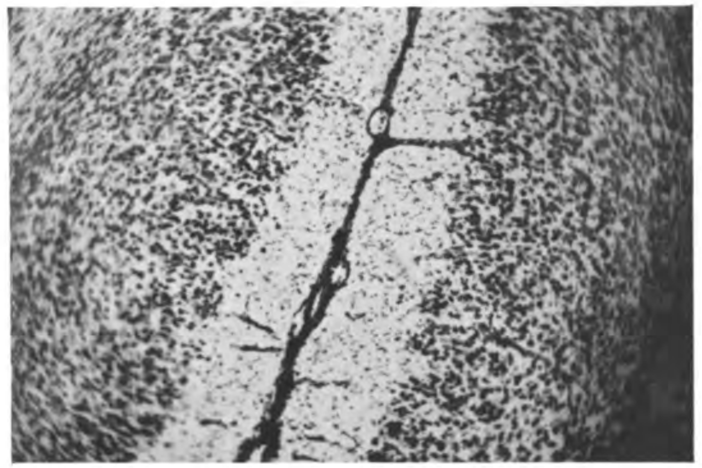

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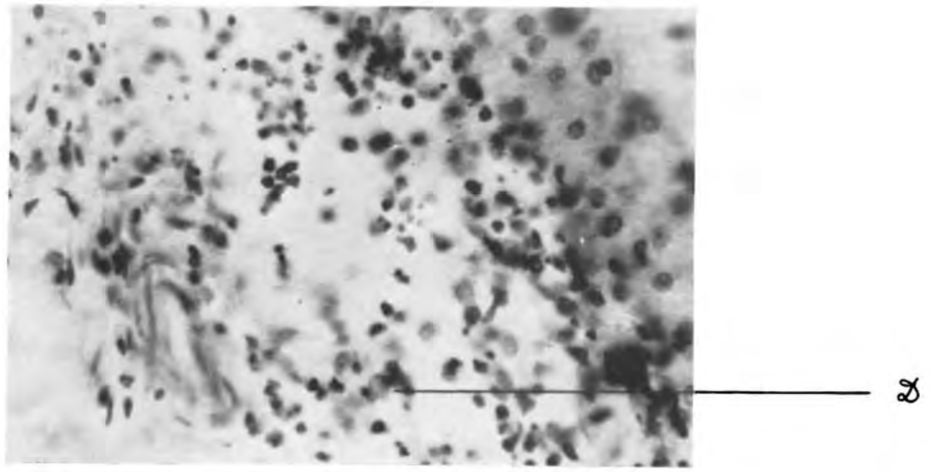

Fig. 7.

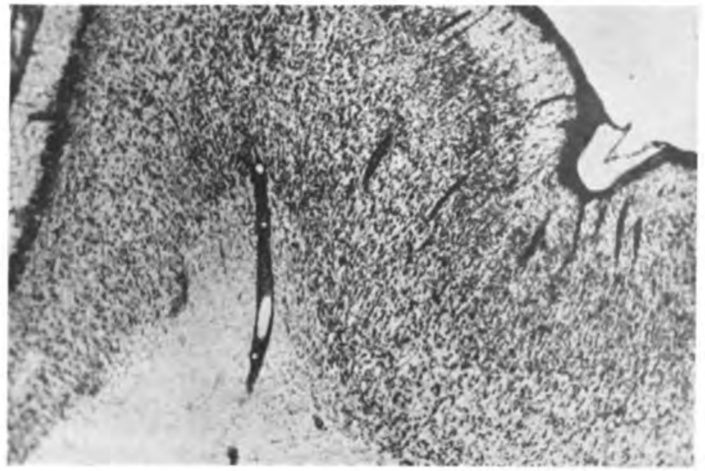

Fig. 8.

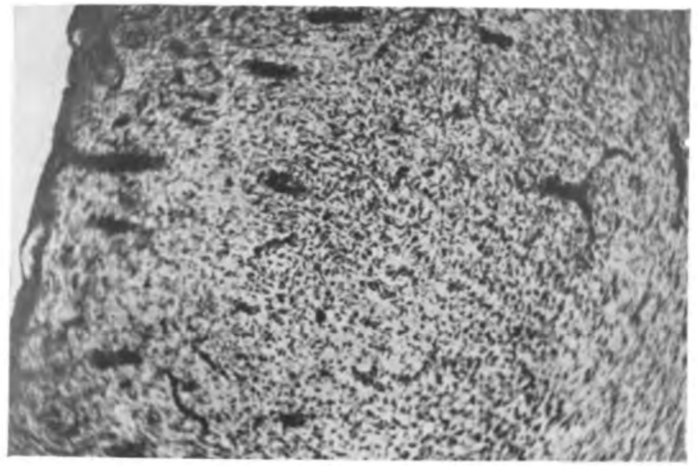

Fig. 9.

the centre of the papula, there is a depression on which a small scab forms. On the sixth day the pox begin to dry up and two weeks after the inoculation, the skin is normal again. Pox never appear before the third day. To get an impression of the regularity of the process, following the cutaneous inoculation, 21 guinea pigs were inoculated with rabbit-pox virus of seventeen strains (brain substances from 2nd to 8th transmission). The result was as follows:

12 out of 21 animals showed pronounced pox formation.
3 out of 21 animals showed moderate pox formation.
5 out of 21 animals showed only slight pox formation.
1 out of 21 animals did not show pox formation at all.

Consequently, the guinea pig is very sensitive to cutaneous inoculation with the virus.

Although the animals were often kept in the same cage with infected rabbits we never saw a spontaneous encephalitis in guinea pigs, nor rhinitis or dermatitis (cf. PEARCE et al., 1936 *b*).

The phenomena, which can be observed after a second cutaneous inoculation of the guinea pig, are an indication of immunity or allergy. The allergic reaction is extremely characteristic and distinguishes itself by the appearance of miniature pox. These have the size of a pin's head, are mostly small in number and appear as early as the 2nd day after inoculation. Besides, these miniature pox disappear again on the third day.

Approximately 17 days after the first cutaneous inoculation with rabbit-pox virus, the 21 guinea pigs, mentioned above were again inoculated. After the 2nd inoculation, the animals, which had strongly reacted upon the first inoculation (12/21) either showed no symptoms (10/12) or some miniature pox. Animals, which had reacted less strongly upon the first inoculation (3/21) showed no symptoms (2/3) or miniature pox only (1/3). Animals, which had shown only slight pox formation after the first inoculation (5/21), all reacted with the formation of miniature pox after the second one. The animal, which had not reacted upon the first inoculation, showed skin-pox after the 2nd inoculation; these skin-pox appeared after three days.

These characteristic skin-phenomena could also be generated by means of cutaneous inoculation with other material than cerebral substance (nasal secretion and saliva) of rabbits, spontaneously or artificially infected with rabbit-pox virus.

From the above-mentioned data it may be concluded that, in the case of a cutaneous inoculation, the guinea pig's sensitiveness to the rabbit-pox virus is such that, by means of this experimental animal, the virus can be reliably demonstrated.

Regarding the changes of the guinea pigs skin after a herpes-inoculation there is, however, no unanimity (cf. LEVADITI, 1926; HAUDUROY, 1929). If cutaneous symptoms were generated, they consisted of vesicles (BEDSON and CRAWFORD, 1927); intra- and subcutaneous herpes-inoculation on the sole of the foot led regularly to the formation of vesicles. So, if changes occur, they are, in any case, of another character than the rabbit-pox dermatitis.

In our own experiments, ten guinea pigs showed no skin-changes whatever, when we used herpes-virus from the first three transmissions of two herpes-strains. These herpes-strains had been isolated from the eruptions of two female patients, suffering from herpes labialis. The isolation had taken place by means of the intracerebral injection of a rabbit. The herpes-strains had been cultivated until the third passage. The six rabbits, which had been injected, all died. Clinically, five and histopathologically, four of these animals showed meningoencephalitis. Of five animals, nasal scrapings and cerebral emulsions were rubbed into skin-scarifications of two guinea pigs each.

These cutaneous inoculations caused no reactions in the guinea pigs.

Summary.

1. When rabbits were intracerebrally injected with liquor cerebrospinalis of patients, suffering from organic nervous diseases, this led to symptoms, which after closer examination, turned out to be caused by an infection with rabbit-pox virus.

2. This infection led only to unobtrusive spontaneous symptoms, such as rhinitis. Intracerebral injections of liquor and sterilized liquids made this latent infection manifest.

3. The clinical phenomena, as well as the histopathological picture of the central nervous system in the case of rabbit-pox meningitis, are, in several respects, similar to a herpesmeningoencephalitis. Sometimes, the histological pictures of both diseases are difficult to distinguish from each other.

4. The rabbit-pox virus can simply be demonstrated by means of cutaneous inoculation on the guinea pig, whereby a characteristic dermatitis arises. When the guinea pig is re-inoculated it turns out to be immune or to react with an allergic reaction, whereby miniature pox appear, which disappear soon afterwards.

5. Cutaneous inoculation of the guinea pig with herpes virus did not lead to any inflammation of the skin.

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CONTENTS

Physics. — *About Mountain-formation on the Earth.* By F. A. VENING MEINESZ.

(Communicated at the meeting of September 25, 1948.)

If we study the way in which mountains and mountain-ranges have formed on the Earth's surface we shall find much that is unknown or conjectural. We can no doubt state that there are several different types between which we can distinguish. In many cases we shall find that these types merge into each other.

In the first place we can distinguish between mountain-formations of a linear kind where one horizontal dimension is much smaller than the other and those of which the horizontal dimensions in the two directions have the same order of magnitude. Besides the isolated volcano the latter type as far as the continents are concerned mainly exists of areas which for some reason or another have risen to some elevation above sea-level and which have since been dissected by erosion. The resulting topography depends on the time during which this erosion has operated and the conditions to which it was subject; it is hardly necessary to go into details here as we know that geomorphologists have made extensive studies of these processes.

Sub-oceanic topography of this kind has been less intensively investigated [1]) and so a few remarks may be made. It is clear that no erosion can alter the submarine topography but sedimentation may perhaps do so, although in many cases probably all features will be covered by approximately the same layer and so the topography must in its general lines remain the same. It might be possible that steeper slopes may thus be made somewhat more gradual but the writer has no evidence in this sense. He had the opportunity to make soundings over several deep troughs and several submarine mountains of probably volcanic origin and no sign could be found of a softening of the slopes. As an instance he may mention the northern slope of the Romanche Trough (near the equator between Africa and S. America) which is unusually steep; the grade is about 60°. This may, however, be caused by its being a young formation and so we can not conclude much for the topography in general. Leaving aside again the case of isolated volcanoes [2]) and considering now banks or plateaus of the non-linear kind we may try to make out their way of originating. We have only two series of data which can give us a hold on this problem, detailed

[1]) We shall not deal here with the topography of the continental shelves and the submarine canyons about which much research has been made.

[2]) An important study of this subject is given by H. H. HESS in "Drowned ancient islands of the Pacific Basin", Am. J. O. Sc. 244, Nov. 1946, pp. 772—791.

soundings and gravity results, and in general this will still leave us a wide field of conjecture. If we find a bank which in general shows local isostatic compensation as e.g. two gravity profiles running up to the Bromley Plateau (S. Atlantic, east of S. America at about 31° S) appear to indicate, the most obvious supposition is that it is caused by a local thickening of the sialic layer which seems a not unlikely assumption considering the fact that the continents themselves are no doubt much larger features of this same kind. But it might of course be also possible that some unknown cause in the substratum has brought about a rising of the crust in this area as we no doubt find them in so many continental areas where epeirogenetic movements are not uncommon, especially near or over old tectonic areas of former periods. The causes of such vertical movements, rising as well as sinking, are still unknown; in some cases perhaps convection-currents in the substratum might be supposed to be responsible for them but this is no more than a hypothesis.

A third possibility is no doubt that the topography is caused by plateau extrusions but in that case a purely local isostatic compensation seems hardly likely.

We shall not further enlarge here on the case of topography of the non-linear kind; in this paper we shall especially consider the mountain-formation on the earth's surface of the more or less linear type. We can distinguish here between two cases with a third case in a more or less intermediate position.

In the first place we have the well-known geosyncline belts which give rise to folded mountain-ranges; we shall presently come back to them more in detail. They generally occur in the continents or in areas bordering on them; the last occurs in the Antilles, in the Southern Antilles and along the eastern and south-eastern border of Asia and the eastern border of Australia and New Zealand. Although geologists have often surmised connections of these belts through the oceans, as e.g. between the Antilles and the Mediterranean and Alpine geosynclinal belt there is no proof that these connections do exist. In the North Atlantic in particular no evidence in this sense can be found and the topography as well as the gravity-field show quite different features. In general these geosynclinal belts are characterized by strong folding and overthrusting of the surface layers and follow slowly curving tracks; in the belt themselves the individual tectonic axis are also curved as e.g. in the Mediterranean and Alpine belt. No evidence of such belts has as yet been found in the central parts of the Atlantic, the Indian Ocean or the Pacific.

In the second place we have linear topography along more or less straight lines which keep their direction over greater distances than the former group and where no large overthrusts are found in the surface layers of the crust. As we shall see these belts of topography lack many of the properties of the geosynclinal belts as e.g. the presence of deep earthquakes. Usually a second direction is also prevailing in the topography which makes an angle of 60°—90° with the first. The topography makes the impression to be for a great part of volcanic origin and we find especially great outcrops in areas where a belt in the first direction crosses one in the second. We find this type of topography in the North and South Atlantic and in the central Pacific east of the andesite line. It is not certain whether it is also found in the Indian Ocean and perhaps also in the continents.

In connection with the properties described it is obvious to attribute this type of topography to a set of fault-planes in the crust along which shearing has mainly taken place in a horizontal sense; as it is always the case with shear it can occur in two directions enclosing an angle of 60°—90°.

For the geosynclinal belts we shall adhere to the usual view for which much evidence is present that in this case the blocks of the Earth's crust on both sides of the belt are pressed together and that the crust in the belt gives way and allows a shortening of the distance of the order of several tens of kilometers.

A third case of an intermediary character can be distinguished if in geosynclinal belts straight tectonic axis occur making a small angle with the direction of the relative movement of the two crustal blocks on both sides of the axis. In that case the relative movement of these blocks is mainly shearing along a fault-plane coinciding with the tectonic axis while, if any, only a small component of the movement is present at right angles to the fault-plane causing a slight overriding of one of the blocks over the other which thus is somewhat pressed downwards. An example of this may probably be found on the west-coasts of Sumatra and of California and Mexico, where slight ocean-deeps are found at the foot of the continental slope while fault-planes with strong relative movements are known parallel to the coast. In the case of Sumatra this belt is forming part of the great Indonesian arc which is curved over the greatest part of its course. For this curved part the crust may be supposed to bulge downwards under the effect of the great compression in the way the strong negative gravity anomalies over these belts have led us to suppose. Where the belt runs over Sumatra in a direction nearly coinciding with the relative movement of the two crustal blocks it is straight and the gravity field no longer points to a more or less symmetrical crustal down-bulge in the tectonic belt but to an overriding of the Sumatra block over the ocean-block. We thus see that the facts seem to indicate that also in the geosynclinal belts there are parts of the tectonic axis where the relative movement of the crustal blocks on both sides has the character of a mainly horizontal shearing movement along a fault-plane. These parts are not accompanied by deep earthquakes. This case thus seems to approach the second one although it forms part of a belt of the first-mentioned kind.

We shall now examine our cases somewhat more in detail. In the first place we shall look at the geosyncline belts. As a typical example of such a belt in the present period we may mention the eastern part of the Indonesian archipelago where we find the outer arc around the Banda Sea characterized by folding and overthrusting of the surface layers, by shallow-focus earthquakes obviously caused by the crust's deformation in this belt and by strong negative anomalies which may probably be interpreted as the effect of the downward bulge of the crust in the denser substratum which is brought about by the crust's compression and which thus brings lighter matter in the place of heavier masses.

Inside this arc we have the second one consisting of volcanoes, and under or near this arc we find the intermediate shocks. The topography is more or less locally compensated. Inside this arc we have the deep Banda basin and the deep earthquake foci.

This general distribution is typical for orogenic island arcs. We usually find the shallow shocks in the zone of folding and the intermediate and deep shocks on the inside of the curve and mostly on the continental side, the deep shocks at larger distances than the intermediate ones and both at distances from the tectonic zone equal in order of magnitude to their depths.

We find great differences in elevation in the tectonic zone. South of Java e.g. it lies at depths of 1300 m to more than 2000 m while in Timor it comes up to elevations above sea-level of more than 2900 m. These last elevations may perhaps be explained by a beginning of the readjustment of the isostatic balance by some local disengaging of the crust from its surroundings. The rising thus taking place must have an irregular character depending on the amount of loosening in the belt. Probably the rising is usually accompanied by a sinking of the neighbouring belts which have been pushed upwards by the central belt but which sink back as this belt detaches itself.

A second possibility of explaining differences in elevation in the tectonic belt may be found in a thinner surface layer being squeezed out and folded when the downbulge of the main crust takes place. This may well be dependent on the thickness of the sedimentary layer; as this layer is no doubt less resistant than the deeper crustal layers, it can well be understood that a great part of the layer is not engulfed in the downward movement. This line of explanation has e.g. been advanced by HESS [3]) for making clear that the southern part of the tectonic belt in the Antilles comes here and there above sea-level, as e.g. in Barbados and Tobago, while the northern part which is indicated by the belt of negative anomalies is found at great depth; for a great part it does not even show a ridge. In the southern part the neighbouring South American continent can account for a thicker sedimental layer than in the north. UMBGROVE adopts the same line of reasoning for explaining the absence of a tectonic second arc

[3]) HESS, H. H., Recent advances in interpretation of gravity anomalies and island-arc structure (Adv. Rep. Comm. o. Cont. a. Oceanic Str., 1939).

in the Marianas-Bonin arc where only the volcanic arc has been developed; he assumes a much thinner sialic layer here than nearer to the Asiatic continent.

The inner arc is always volcanic and the volcanoes are often reaching above sea-level. Its volcanic character gives to this arc likewise an irregular length-profile. It is possible that in this arc the surface layers also show some folding as e.g. in the south of Java but this is always less intensive than in the tectonic arc itself.

We have already mentioned that in the straight parts of these arcs as e.g. in the Sumatra area, the relative movement of the crustal blocks may probably be assumed to be mainly shear in the sense of the tectonic axis while only a small component at right angles to this axis is present; this last component may be supposed to cause an overriding of the Sumatra block over the Indian Ocean block and no longer a large down-bulge of the crust; this view is in harmony with the smaller size of the negative anomalies in the tectonic belt here than in the other parts of the arc and in the asymmetric character of the anomaly-curve. This straight part is not accompanied by deep shocks.

The deep earthquakes only occur in the tectonic belts surrounding the Pacific.

It is not yet quite certain whether in the present period also tectonic belts are present on the Earth's surface where early stages of the orogenetic cycle occur as e.g. the first syncline stage when folding has not yet taken place on a large scale or takes place for the first time. Later stages than the one described above can no doubt be found as e.g. in the Alps where the folding appears to be nearly finished and where the tectonic belt has risen to so great a height that only slight remnants of the negative anomalies are left. Earthquakes are rare here although not yet quite absent and volcanoes have become extinct.

As it is well known many geologists think that these great tectonic deformations of the Earth's crust take place in cycles of some 50—70 million years which more or less periodically occur over the whole globe and which are separated by periods of rest of some 100—150 years. The writer may refer here to UMBGROVE's book, "the Pulse of the Earth", which deals especially with this subject, to KUENEN's study [4]) and to GRIGGS's [5]) and his own papers [6]) which try to account for this periodicity by assuming great scale convection-currents between the crust and the core. We shall presently come back to this hypothesis.

[4]) KUENEN, PH. H., Major geological cycles; Proc. Ned. Akad. v. Wetensch., Amsterdam, 44 (1941).

[5]) GRIGGS, DAVID, A theory of mountain-building, Am. J. o. Sc. 237, Sept. 1939, pp. 611—650.

[6]) VENING MEINESZ, F. A., Major tectonic phenomena and the hypothesis of convection-currents in the Earth; J. o. the Geol. Soc. o. London, 103, 3, 1947, pp. 191—207.

VENING MEINESZ, F. A., Gravity Expeditions at sea, Vol. IV, Ch. II, 1948, Waltman, Delft.

It seems difficult to understand the curved track of the tectonic axis in the orogenetic belts and up to now no clear explanation has been given. It seems strange that in this way the length of the deformed belt becomes greater than is necessary and that thus the deformation energy should appear to be larger than needed. We may perhaps come to some deeper understanding by first considering the parts of the tectonic belt where the main phenomenon is shear. In the Indonesian archipelago these are the Sumatra and Philippine parts.

Examining the problems of mechanics involved in these parts where the shear occurs along a plane enclosing only a small angle with the direction of the relative movement of the crustal blocks on both sides, it is difficult to come to another conclusion than that the fault-plane must already have been present before the tectonic phenomenon took place; it is otherwise hard to explain how a plane in this direction could come into existence. Accepting its presence beforehand it is clear that only little energy is needed for causing shear in this sense and so we can understand that the tectonic belt will as far as possible follow these old lines. It is thus perhaps possible to account for the whole arc as of course the two lines on the sides — in the East Indies the belts west of Sumatra and east of the Philippines — have to be connected by a curved line for becoming a continuous belt.

The presence of old lines on the Earth's surface is more and more widely admitted and as these lines are straight, it seems likely that they really represent fault-planes through the crust. Since a long time geologists as DAUBRÉE, SEDERHOLM, SONDER, a.o. have written about such old lines which they usually thought to have the directions SE-NW and SW-NE and they have in general been supposed to be very old, having been reopened from time to time when new forces occurred in the crust. The writer himself has advanced a hypothesis for explaining their presence [7]); he showed that a shift of the poles over some 70° along the meridian of 90° E could bring about a net of shear-planes in the crust which would well agree with the existing directions.

In general we may state that the presence of a pre-existing net of fault-planes in the crust would make it much more easy to understand the twisted and complicated courses which the tectonic axis have taken in the orogenetic belts. Particularly marked instances of such complicated courses may be found in the Mediterranean-Alpine belt, in the West Indies and in the East Indies.

A last point which has to be broached in connection with the orogenetic belts on Earth is the question about the cause of these deformations of the crust. It has often been supposed that this cause has to be looked for in currents in the substratum which because of the very high viscosity must exert great stresses on the lower boundary of the crust. A plausible

[7]) VENING MEINESZ, F. A., Shear Patterns of the Earth's crust; Transactions Amer. Geophys. Union, 28, 1, 1947, pp. 1—61.

explanation of such currents could perhaps be found in the cooling of the Earth which must bring about a cause of instability in the plastic layer below the crust and if this layer is homogeneous over a sufficient thickness this must give rise to convection-currents. As most seismologists do not accept a density discontinuity in the mantle of the Earth between the rigid crust and the core, it seems at least possible that this entire layer would be subject to such currents which thus would assume continental dimensions in a horizontal sense. Smaller currents going less deep would of course also be possible. As the writer has put forward elsewhere [8]) he thinks that both types are present. The smaller type could perhaps account for the deep earthquakes and for the sinking down of the deep basins in the eastern part of the East Indian archipelago and in other island-arc areas while the larger type could explain the great tectonic phenomena in the orogenetic geosyncline belts of the Earth.

In both cases, however, the hypothesis gains in plausibility if we assume a small strength to be present in the convection-layer which has to be overcome before the current can set in; the current is then supposed to make only about a half-turn, bringing the layer of lower temperature down and the layer of higher temperature on top and thus restoring the stability. GRIGGS [9]) has already pointed this out for the larger type of convection-current in 1939 in his important paper on mountain-formation in which he explains that thus the large periods of rest can be accounted for between the periods of great tectonic activity as mentioned before while the writer has shown the same to be the case for the great time-lag of some 15.000.000 years between the last folding-period in the outer Banda arc and the sinking down of the Banda basin.

A further asset of the hypothesis that large scale convection-currents are responsible for the great tectonic phenomena can be found in the possibility thus to explain the regressions of the oceans during the first part of the period of tectonic activity and the long period of transgression during the period of rest; the difference in temperature in the rising and sinking columns during the time that the current is going can account for these surface movements.

It appears to the writer that the number of phenomena which can thus be understood is sufficient for at least considering the supposition of convection-currents as a serious working hypothesis.

Much less can be said about the second kind of topography on Earth, the straight-line type of mountain-formation of which we find examples in the central parts of the Pacific and in the North and South Atlantic, possibly likewise in the Indian Ocean, in Africa and Europe and perhaps

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8) VENING MEINESZ, F. A., Major tectonic phenomena and the hypothesis of convectioncurrents in the Earth, J. o. the Geol. Soc. of London, 103, 3, 1947, pp. 191—207.

9) GRIGGS, DAVID, A theory of mountain-building. Am. J. o. Sc. 237, Sept. 1939, pp. 611—650.

also in other continents. We do not know much about the period in which this topography came into being nor whether the history of these mountain-formations shows the same more or less periodic character as the geosyncline belts. The important discovery by HESS of many flattened submarine cones in the Pacific of which the upper flat surface is found at variable depths up to more than 2000 m [10]) may perhaps shed light on this difficult problem. It led him to the interpretation that each cone has been a volcano subjected to marine planation and sinking down below sea-level because of the slow sedimentation in the ocean and the resulting sinking of the ocean-floor with the volcanoes resting on it. If this ingenious hypothesis can be accepted — and the writer thinks that there can hardly be any doubt about it — we might come to an estimate of the age of the volcanoes and if the observed material increases sufficiently this might enable us to get an idea whether there have been periods of great volcanic activity separated by intervals of small action.

Another question which at least for the ocean-areas can not yet be answered is whether part of the topography along these straight lines is non-volcanic. It makes the impression that the general features of the Mid-Atlantic Ridge in the North Atlantic have local isostatic compensation while most of the submarine volcanoes are regionally compensated but the gravity material leading to these conclusions is yet insufficient for making sure about them. If they prove to be true it appears probable that part of the Mid-Atlantic Ridge in its general lines has been brought about by other phenomena than volcanism. We might perhaps suppose that the shear along the fault-planes in the crust not only causes volcanoes but also a pressing up of matter along the plane.

Such a mechanism has already been supposed for the west-coasts of Sumatra and California and Mexico and we might no doubt surmise it also for other straight-line parts of the mountain-systems in the continents as we may perhaps find in the three ranges around Bohemia, the Thüringer Wald, great parts of the Apennines and the Dalmation mountains and other ranges which may possibly be reckoned to this type of mountain-formation. Besides their straight courses we may perhaps find a reason for doing so in the fact that they all seem to belong to the system of two directions of topography usually admitted to be present in the European continent. They are certainly in agreement with the before-mentioned net of shear-planes derived by the writer from the hypothesis of a shift of the poles.

Among the few data we seem to have about the volcanic topography in the oceans which we have attributed to shear along a system of straight fault-planes we found a certain predominating of volcanic activity in the places where faults in the two prevailing directions cross each other. We

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[10]) H. H. HESS, Drowned ancient islands of the Pacific basin, Am. J. o. Sc., 244, Nov. 1946, pp. 772—791.

need hardly say that this appears to be in good harmony with our supposition of two sets of fault-planes along which shear occurs. If at the same time movement takes place in both directions, the points of crossing must in particular show a lockering of the Earth's crust and so this must be favorable for the rising of magma.

Before leaving our subject we may examine one problem more which presents itself. We have already drawn attention to the fact that in the central parts of the oceans we seem mainly to find the straight-lined type of topography; this appears at least to be true for the Pacific and the Atlantic and there is not much reason for a contrary opinion in the Indian Ocean although the scarcity of soundings there does not yet allow a good conclusion in this case. The question arises why no geosynclines seem to occur in those areas.

An answer to this question may perhaps be looked for in two directions. In the first place the crust under the oceans has another constitution than under the continents; it has no doubt a thinner granite layer and perhaps this is even absent over large areas while the sedimentary layer must likewise be much thinner and of a different type. It might be possible that this different constitution would lead to other physical constants and to a different behaviour with regard to the stresses working in the crust. The deviation in thermal conditions caused by the low temperature near the ocean floor and probably also by a smaller content of radio-active matter in the crust may also contribute to the bringing about of another reaction of the crust to the stresses. This might have the effect of making the crust more resistant against down-buckling which process must involve a good deal of plastic deformation in the crust and of gliding of one layer on another. The result might, therefore, be that the crust gives way to shear before the stresses have reached the values needed for down-buckling.

In the second place we might try to find an explanation by assuming that the conditions below the oceanic crust prevent convection-currents to come into being and this could account for the absence of sufficient stresses in the crust for causing down-buckling. We have already mentioned that probably the subcrustal matter has a certain strength which must be overcome before a convection-current can set in and this involves the need of sufficient temperature differences in a horizontal sense for bringing about differences of pressure large enough for starting the movement. Now it could well be understood that such differences of temperature in a horizontal sense are present on the border of the continents where the oceanic and continental temperature distributions occur side by side, or even perhaps in the middle of the continents where the constitution of the crust and, therefore, of the thermal and radio-active properties may change sufficiently from place to place for causing temperature differences, but that under the oceanic crust the conditions are too regular for such effects. This could thus explain the absence of convection-currents below the oceans and, as a consequence of this, the absence of all tectonic activity in the crust.

The above sketch of the great features of mountain-formation on the Earth's surface is only a first tentative attempt to arrive at some deeper understanding of these great problems. Much more research will no doubt be necessary before a well-founded treatment can be given.

Summary.

After a short discussion of the non-linear topography on Earth a survey is given of the linear mountain-formations among which three groups are distinguished, firstly the geosyncline belts characterized by folding and overthrusting, by shallow as well as deep earthquakes and by a curved track of the axis, in the second place the straight-lined type mostly accompanied by strong volcanism where two directions predominate over large areas of the crust probably representing fault-planes through the whole crust along which shear occurs and where deep shocks are absent, and thirdly the intermediate case that in geosyncline belts straight fault-planes are found along which mainly shear takes place, possibly coupled with some overriding of one block over the other (e.g. Sumatra, California, Mexico, etc.).

In the central part of the Pacific, in the Atlantic and possibly also in the Indian Ocean it is probable that only the second type occurs. A possible cause might be the absence of convection-currents or the different constitution of the crust in those areas.

A more detailed discussion is given of the three types of mountain-formation as well as of the way they can have come into being. For the compression in the crust which leads to the phenomena in geosyncline belts, convection-currents are supposed to be responsible, but the writer thinks that the course of the belts can not be understood without admitting pre-existing fault-planes. The presence of a net of such fault-planes over the whole Earth might also explain the second type of mountain-formation. The net is probably very old but the faults must since continually have been rejuvenated. The writer thinks that his hypothesis about the net having originally been brought about by a shift of the poles, gives a satisfactory explanation.

Zoology. — *Some Rhizocephalan Parasites of Maiid Crabs.* By H. Boschma.

(Communicated at the meeting of September 25, 1948.)

Sacculina spectabilis nov. spec.

p. p. *Sacculina pilosa* Van Kampen & Boschma, 1925.
p. p. *Sacculina rotundata* Boschma, 1931.

South of Salawati (1°42'.5 S, 130°47'.5 E), 32 m, Siboga Expedition Sta. 164, 1 specimen on *Paramithrax* (*Chlorinoides*) *longispinosus* (De Haan) (holotype).

Diagnosis. Male genital organs in the posterior part of the body, outside the visceral mass, completely separated. Testes more or less globular, rather abruptly passing into the comparatively wide vasa deferentia. Colleteric glands with a small number of canals. External cuticle of the mantle with groups of hyaline spines consisting of a kind of chitin which is different from that of the main layers of this cuticle. The spines take their origin from the inner surface of distinctly cup-shaped excrescences which are implanted on or in the main layers of the external cuticle. The height of the excrescences as a whole from the base of the common part to the tips of the spines may amount to 175 μ. The individual spines have a length of up to about 90 μ. Retinacula unknown, probably not occurring.

There is not a figure of the external shape of the specimen, as previously it was regarded as forming a part of the species at that time identified as *Sacculina pilosa* Kossmann (Van Kampen & Boschma, 1925). In the cited paper we remarked that the parasite had a more or less unsymmetrical shape, its dimensions were $10.5 \times 7.5 \times 5$ mm. The mantle opening, on the anterior margin, was slightly protruding above its surroundings. The mantle showed a slight wrinkling.

The data given below are taken from the series of longitudinal sections. These distinctly show the configuration of the male organs; the colleteric glands, however, seem to be underdeveloped, as also the whole visceral mass, on account of the presence of a large Bopyrid parasite occupying the greater part of the mantle cavity.

The chief particulars of the male organs are shown in fig. 1. The vasa deferentia (fig. 1*a*) are rather wide, they possess a well developed system of ridges penetrating into the lumen. In a section from a more dorsal region the narrow canals with their chitinous walls which connect the vasa deferentia with the testes are represented, penetrating the thick walls of the testes (fig. 1*b*). In a section from a still more dorsal region the testes are shown in their largest size (fig. 1*c*), here the left is still devoid of a cavity, whilst the right has a fairly large lumen. In a previous paper (BOSCHMA, 1931, fig. 11*c*) a section is represented which corresponds with that of fig. 1*b* in the present paper.

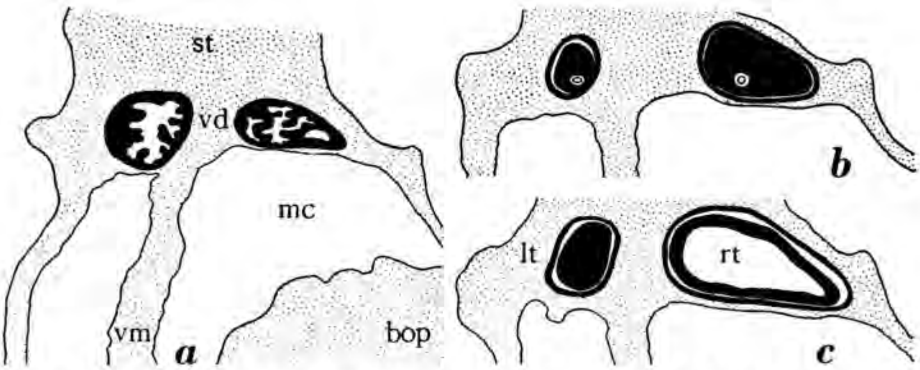

Fig. 1. *Sacculina spectabilis* nov. spec. Longitudinal sections of the posterior part of the body; *a* through the vasa deferentia, each following section from a more dorsal region. *bop*, Bopyrid parasite in mantle cavity; *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *st*, stalk; *vd*, vasa deferentia; *vm*, visceral mass. × 36.

The colleteric glands are very indistinct, they seem to possess a small number of canals only. But whether this is a specific character or brought about on account of the presence of the Bopyrid parasite must remain undecided.

The excrescences of the external cuticle of the mantle are of a peculiar shape, different from those of other species with excrescences consisting of compounds of spines. They were figured in two previous papers (VAN KAMPEN & BOSCHMA, 1925, fig. 7, and BOSCHMA, 1931, fig. 3*a*, *b*), whilst two of these excrescences (or, more precisely, sections of them) are represented in fig. 2 of the present paper. The figure distinctly shows how the numerous spines are implanted on the inner surface of cup-shaped large excrescences. As a rule the spines are longest in the central parts of the cups (here they may reach a length of about 90 μ), whilst towards the margins they gradually become shorter. The height of the whole excrescences may amount to 175 μ. The largest excrescences have a diameter of about 140 μ.

Retinacula have not been found on the internal cuticle of the mantle, in all probability they do not occur in the species.

There are a number of species of the genus in which the excrescences of the external cuticle are composed of groups of spines which are united on a common basal part. Of these species *Sacculina rotundata* Miers and *Sacculina carpiliae* Guérin–Ganivet have excrescences of a similar size as those of *Sacculina spectabilis*. The latter species, however, is at once to be distinguished from the other two by the peculiar shape of the basal parts. In the two other species all the spines of one excrescence as a rule are of approximately the same length, whilst in *Sacculina spectabilis* the central spines are decidedly longer than those on the margin of the cup-shaped basal part. These differences are striking enough to regard *Sacculina spectabilis* as a well defined distinct species.

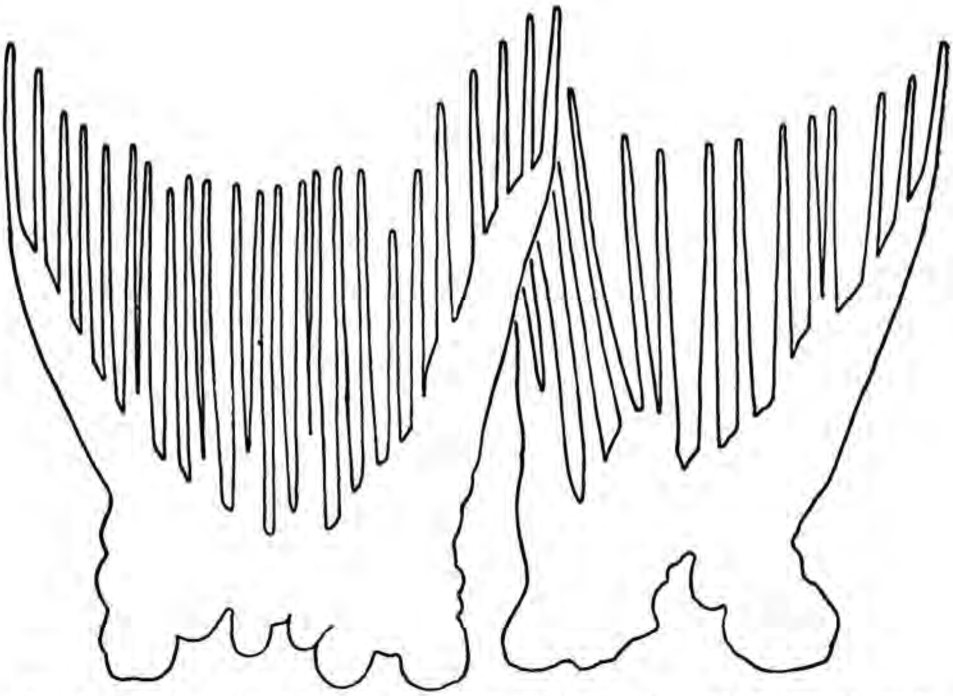

Fig. 2. *Sacculina spectabilis* nov. spec. Sections of excrescences of the external cuticle of the mantle. × 530.

Sacculina synaptothrix nov. spec.

p. p. *Sacculina pilosa* Boschma, 1928.
p. p. *Sacculina rotundata* Boschma, 1931.

Halmaheira, BERNSTEIN leg. (collection Leiden Museum), 1 specimen on *Tiarinia gracilis* Dana (holotype).

Obi latoe, shore or reef, April 23—27, 1930, Snellius Expedition, 2 specimens on *Tylocarcinus styx* (Herbst) (paratypes).

Diagnosis. Male genital organs in the posterior part of the body, outside the visceral mass, completely separated. Testes more or less globular, rather abruptly passing into the comparatively wide vasa deferentia. Colleteric glands with a comparatively small number of canals (14—22 in a longitudinal section). External cuticle of the mantle with groups of hyaline spines consisting of a kind of chitin which is different from that of the main layers of this cuticle. In groups of 2 to 6 the spines are united on comparatively thick branches, which in their turn are united on a common basal part. The length of the individual spines may amount to 60 μ, that of the branches including the spines to 70 μ, the whole excrescences, including the basal part, may reach a height of 120 μ. Retinacula unknown, probably not occurring.

The three specimens, one a parasite of *Tiarinia gracilis*, the two others of *Tylocarcinus styx*, in all their characters sufficiently agree to be regarded as belonging to the same species.

The external shape of the specimen on *Tiarinia* is strikingly similar to that of one of the specimens on *Tylocarcinus*, both have the same nearly

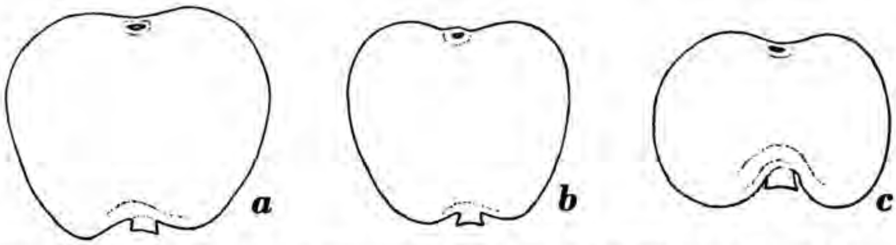

Fig. 3. *Sacculina synaptothrix* nov. spec. *a*, specimen on *Tiarinia gracilis* Dana; *b* and *c*, specimens on *Tylocarcinus styx* (Herbst). Left side. The larger diameter in mm is: *a*, 6; *b*, 6; *c*, 5½.

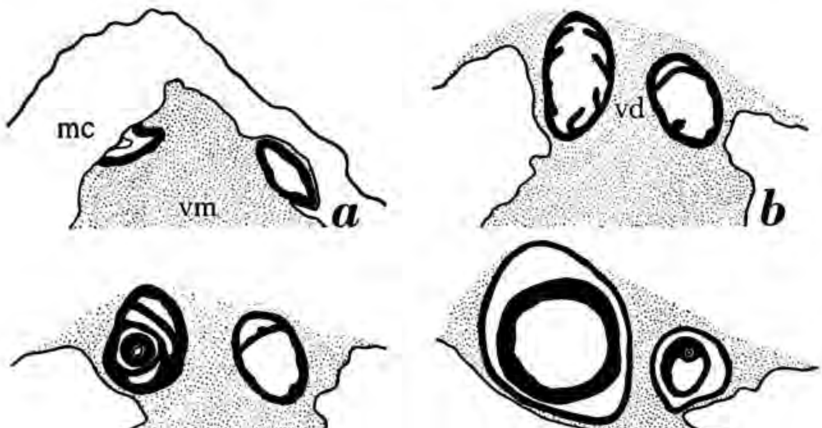

Fig. 4. *Sacculina synaptothrix* nov. spec., specimen on *Tiarinia gracilis* Dana. Longitudinal sections of the posterior part of the body; a through the region of the male genital openings, each following section from a more dorsal region. *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *vd*, vasa deferentia; *vm*, visceral mass. × 45.

circular shape with slight concavities at the anterior and the posterior margins (fig. 3*a*, *b*). The external shape of the second specimen on *Tylocarcinus* is slightly different from that of the other two as it is more or less panduriform (fig. 3*c*). In all the specimens the mantle opening lies at the anterior margin of the left side (the surface regarding the thorax of the host, cf. BOSCHMA, 1948). Its surroundings do not noticeably protrude above the surface of the mantle. With the exception of a few little pronounced grooves in the posterior region the mantle to the naked eye presents a smooth surface.

Longitudinal sections have been made of all the three specimens, these show that they agree in all the salient points of their internal anatomy.

In the specimen on *Tiarinia* the male genital openings are found on the posterior extremity of the visceral mass ventrally from the region of attachment of the stalk (fig. 4*a*). Towards the dorsal region the vasa deferentia grow out into rather voluminous sacs which possess a few ridges on their internal surface (fig. 4*b*, *c*). Farther dorsally a narrow tube with a well developed chitinous wall forms the connection between the vas deferens and the testis (left in fig. 4*c*, right in fig. 4*d*). In a more dorsal region the testis of each side gradually obtains its greatest dimension. In this specimen the left testis grows to a somewhat larger size than the right

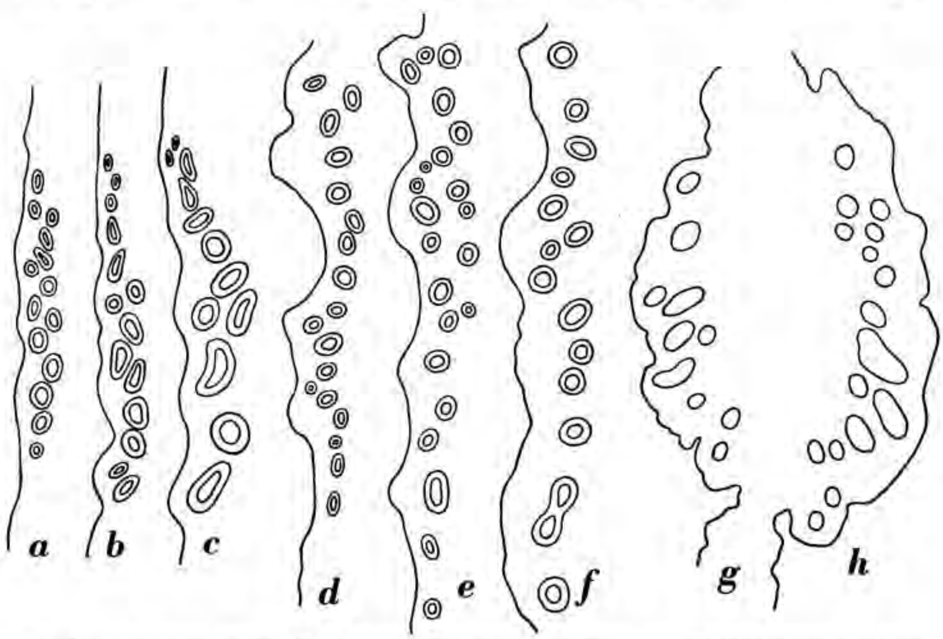

Fig. 5. *Sacculina synaptothrix* nov. spec. *a*—*c*, specimen on *Tiarinia gracilis* Dana, longitudinal sections of one of the colleteric glands; *a* from a rather peripheral region, each following section from a more central part. *d*—*f*, corresponding sections of one of the colleteric glands of one of the specimens on *Tylocarcinus styx* (Herbst). *g*, longitudinal section of the left colleteric gland of the other specimen on *Tylocarcinus styx* (Herbst). *h*, longitudinal section of the right colleteric gland of the same specimen. *a*—*f*, × 80; *g*, *h*, × 135.

(fig. 4*d*, *e*), the cavity of the latter remains much smaller than that of the former. The extreme dorsal part of the two testes is represented in the section of fig. 4*f*.

The male organs of the two specimens on *Tylocarcinus* mutually are of an entirely similar structure, moreover they so completely correspond with those of the specimen on *Tiarinia* that only the chief individual peculiarities may be mentioned here.

In one of the specimens on *Tylocarcinus* the vasa deferentia show a greater amount of ridges than in the specimen on *Tiarinia*. In this specimen the two testes are well developed and have approximately the same size (fig. 7).

In the other specimen on *Tylocarcinus* the ridges of the vasa deferentia are inconspicuous. Here the two testes are of decidedly different size, the right being much larger and more fully developed, presenting a much wider cavity than the left (fig. 8).

The colleteric glands of the three specimens are strongly alike in shape and in the number and distribution of canals. The colleteric glands of the specimen on *Tiarinia* (fig. 5*a*—*c*) are rather flat organs, which show maximally 14 canals in a longitudinal section of the region of their most

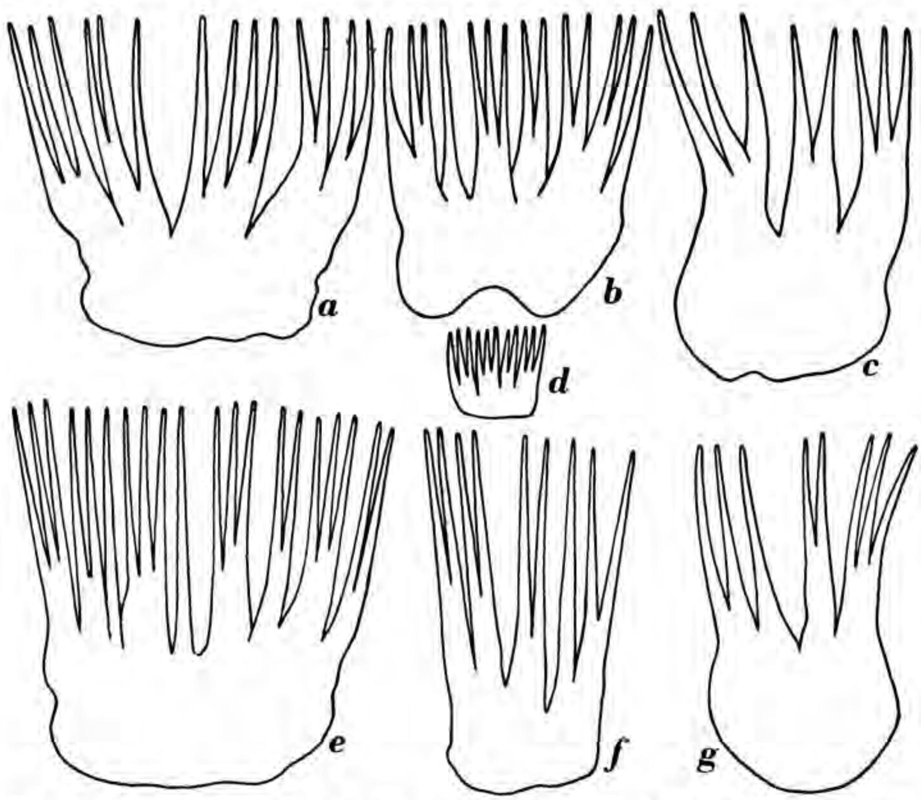

Fig. 6. *Sacculina synaptothrix* nov. spec., specimen on *Tiarinia gracilis* Dana, excrescences from various parts of the external cuticle. × 530.

strongly branched part. These canals are arranged in one or two rows more or less parallel to the surface of the visceral mass.

In the first specimen on *Tylocarcinus* (fig. 5*d*—*f*) the colleteric glands are strikingly similar to those of the specimen on *Tiarinia*, they have approximately the same shape and show a system of canals of about the same arrangement. The maximum number of canals found in a longitudinal section in this specimen is 22.

In the second specimen on *Tylocarcinus* (fig. 5*g*, *h*) the colleteric glands are slightly more protruding over the surface of the visceral mass than those of the other specimens; their system of canals is not as distinct as in the former specimens, as the canals are not noticeably lined with chitin. The arrangement of the canals, however, is quite similar to that found in the other specimens. The largest number of canals counted in one longitudinal section of this specimen is 16.

In the specimen on *Tiarinia* the shape and size of the excrescences of the external cuticle of the mantle varies to some degree in different parts of the mantle. They invariably consist of groups of spines which are united

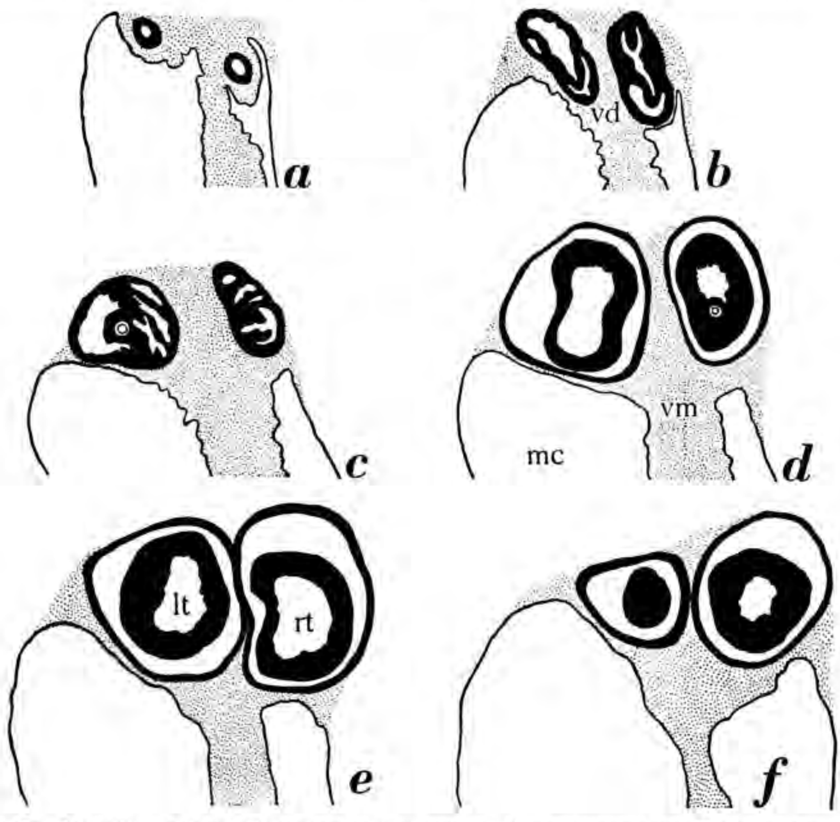

Fig. 7. *Sacculina synaptothrix* nov. spec., one of the specimens on *Tylocarcinus styx* (Herbst), longitudinal sections of the posterior part of the body; *a* from the ventral region of the male organs, each following section from a more dorsal part. *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *vd*, vasa deferentia; *vm*, visceral mass. × 45.

on thicker branches, the latter again being united on a common basal part. The free spines as a rule are not longer than 45 μ (exceptionally to 65 μ), the branches including the spines generally have a length of about 60 μ, whilst the complete excrescences when well developed have a length of

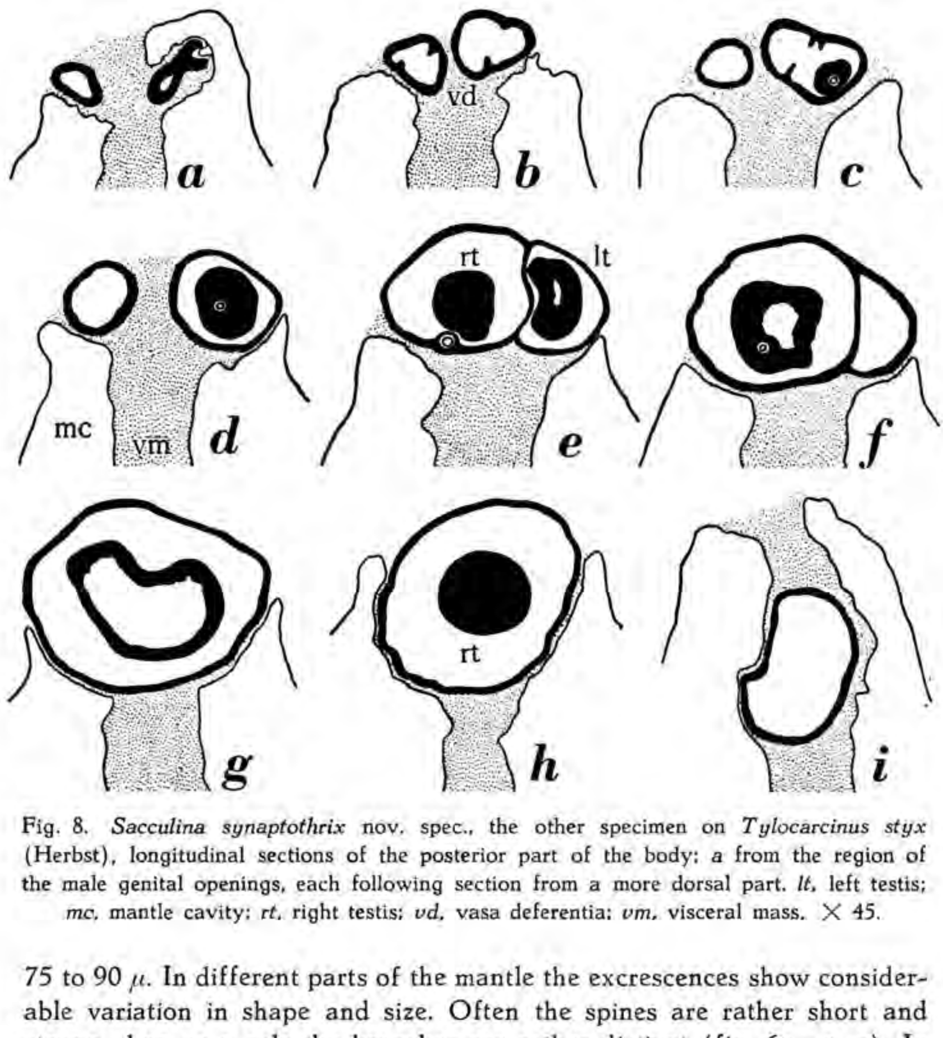

Fig. 8. *Sacculina synaptothrix* nov. spec., the other specimen on *Tylocarcinus styx* (Herbst), longitudinal sections of the posterior part of the body: *a* from the region of the male genital openings, each following section from a more dorsal part. *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *vd*, vasa deferentia; *vm*, visceral mass. × 45.

75 to 90 μ. In different parts of the mantle the excrescences show considerable variation in shape and size. Often the spines are rather short and strong, then as a rule the branches are rather distinct (fig. 6*a*—*c*, *g*). In other excrescences the spines are more slender and the branches are more crowded (fig. 6e, *f*). In the vicinity of the stalk the excrescences are much smaller, though of the same general character as elsewhere (fig. 6*d*).

The excrescences of the specimens on *Tylocarcinus* are of a somewhat different type from those of the former specimen, though in general they correspond in all important details. Here again small groups of spines are united on rather pronounced branches which in their turn are united on a well developed common basal part. The spines as a rule have a length of 20 to 30 μ, exceptionally to 40 μ, the branches (including the spines) have a length of 45 to 60 μ, and in both specimens the maximal length of the excrescences is about 120 μ, though as a rule they remain shorter. Excrescences of various size and shape of one of specimens on *Tylocarcinus* are shown in fig. 9*a*, *b*, of the other specimen from the same crab in fig. 9*c*—*e*.

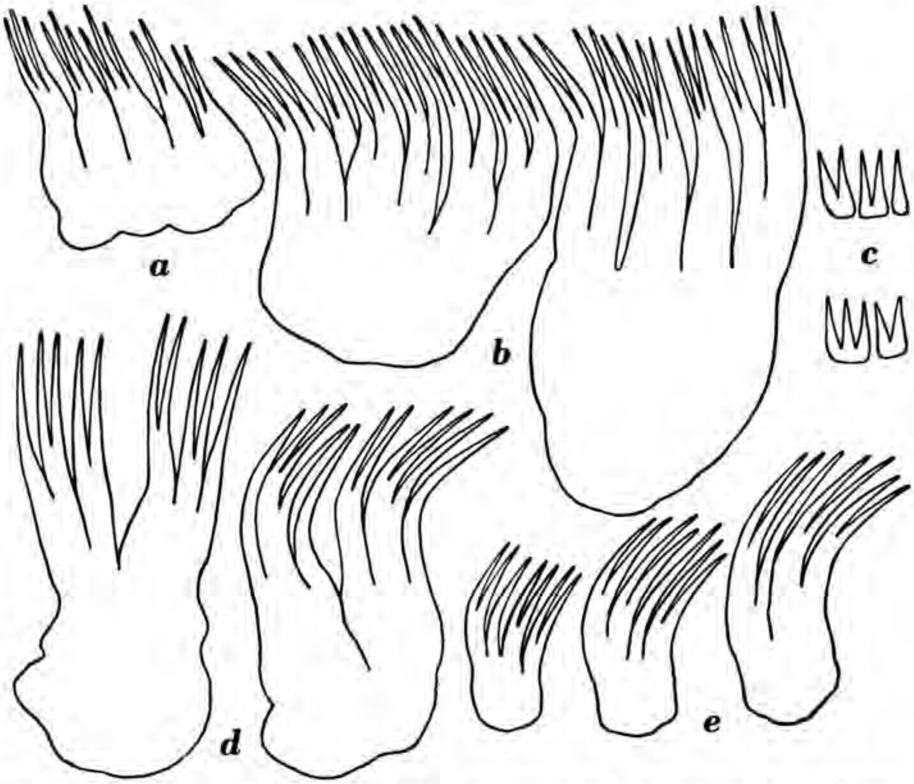

Fig. 9. *Sacculina synaptothrix* nov. spec., specimens on *Tylocarcinus styx* (Herbst), excrescences of the external cuticle. *a*, *b*, from one of the specimens; *c*—*e*, from the other specimen. × 530.

The very small excrescences (fig. 9c) are from the vicinity of the stalk, where they do not grow out to the full size.

Retinacula were not found on the parts of the internal cuticle examined for this purpose, so that in all probability these organs do not occur in the species.

The most striking specific character of *Sacculina synaptothrix* is the peculiar structure of the excrescences of the external cuticle, different from those in other species especially on account of the presence of distinct branches.

Sacculina glabra V. K. & B.

Sacculina glabra Van Kampen & Boschma, 1925; Boschma, 1931; Boschma, 1937.

Beo, Talaud Islands, shore or reef, June 14—21, 1930, Snellius Expedition, 1 specimen on *Tiarinia gracilis* Dana.

The type specimen of *Sacculina glabra* was a parasite of the Maiid crab *Hyastenus subinermis* Zehntner; the present specimen, a parasite of the Maiid crab *Tiarinia gracilis* Dana, completely corresponds with the former in all important details. The external shape of the two specimens only is somewhat different. In the type specimen the mantle opening is found at the extremity of a rather narrow tube, in the present specimen the mantle opening and its surrounding tube are rather wide (fig. 10*a*). Moreover, in the type specimen the mantle shows a system of well marked grooves, whilst in the present specimen the surface of the mantle is smooth. The dimensions of the specimen on *Tiarinia* are 5 × 4 × 2½ mm.

From the type specimen a series of transverse sections was made (cf. BOSCHMA, 1937, fig. 38), from the specimen on *Tiarinia* a series of longitudinal sections. Therefore in the present specimen the male organs could be examined in transverse sections. A comparison with the longitudinal sections of the male organs in the type specimen shows that these organs in both specimens exactly agree in every detail. Fig. 11*a* shows

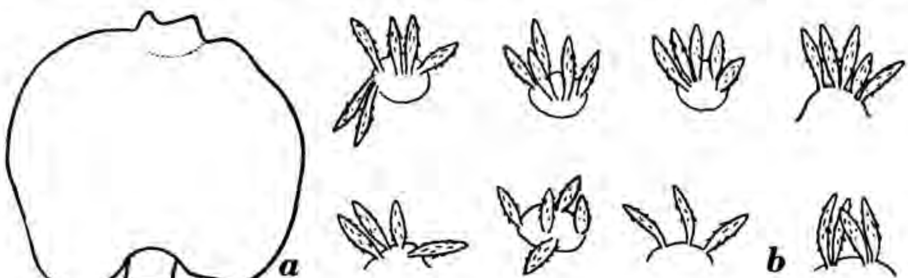

Fig. 10. *Sacculina glabra* V. K. & B., specimen on *Tiarinia gracilis* Dana. *a*, left side of the specimen (greater diameter 5 mm). *b*, retinacula. *b*, × 530.

the ventral part of the two vasa deferentia, which here form comparatively narrow canals. Towards a more dorsal region the male organs considerably increase in size (fig. 11*b*) and still more dorsally their cavities become much narrower on account of irregular rather thick ridges developing on their inner walls (fig. 11*c*). In a still more dorsal region the two male organs again become canals with more or less circular contours and more or less circular cavities (fig. 11*d*).

The colleteric glands of the two specimens also have the same structure and a corresponding number of canals. A direct comparison is not possible as the one specimen was sectioned transversely and the other longitudinally. In the transverse sections of the colleteric glands of the type specimen the maximum number of canals is 23, in the longitudinal sections of the colleteric glands of the specimen dealt with here the maximum number of canals is 32 (fig. 11*e*, *f*). These numbers are of the same order. In the specimen dealt with here the canals of the colleteric glands have well developed chitinous walls, whilst in the type specimen hardly any chitin was present in these glands. These differences, however, depend upon different stages in the life history of the animals.

The external cuticle of the mantle is very thin, in the specimen dealt with here its thickness in various parts of the mantle varies from 3 to 12 μ. This is in accordance with that in the type specimen (about 7—8 μ).

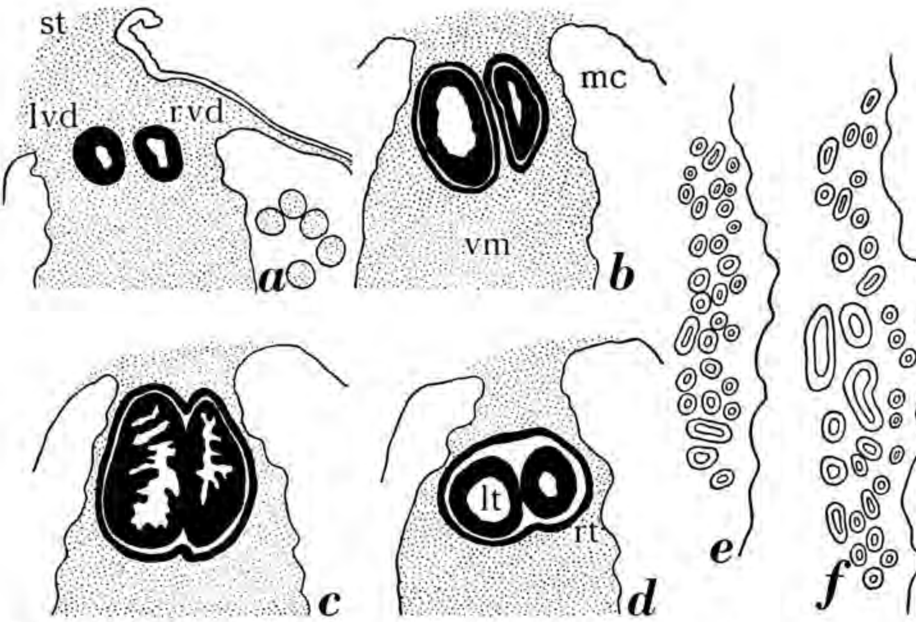

Fig. 11. *Sacculina glabra* V. K. & B., specimen on *Tiarinia gracilis* Dana. *a—d*, longitudinal sections of the posterior part of the body; *a* from the ventral region of the vasa deferentia, each following section from a more dorsal part. *e*, *f*, longitudinal sections of one of the colleteric glands, *e* from a more peripheral region than *f*.
a—d, × 45; *e*, *f*, × 80.

In the type specimen no retinacula were found. They occur, however, on the internal cuticle of the mantle of the specimen on *Tiarinia*. The retinacula are more or less evenly distributed over the surface of the internal cuticle, they consist of a basal part and 3 to 6 spindles of rather slender shape (fig. 10*b*). The length of these spindles is about 15 μ, they possess distinct barbs.

Sacculina glabra on account of the peculiar structure of the male organs is a well characterized species, different from all others which have an external cuticle devoid of excrescences.

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Neurology. — *Traumatic microcephaly after spontaneous delivery at full term.* By B. BROUWER and C. DE LANGE. (From the Neurological Clinic of the Wilhelmina Hospital and the Dutch Central Institute for Brain Research at Amsterdam.)

(Communicated at the meeting of September 25, 1948.)

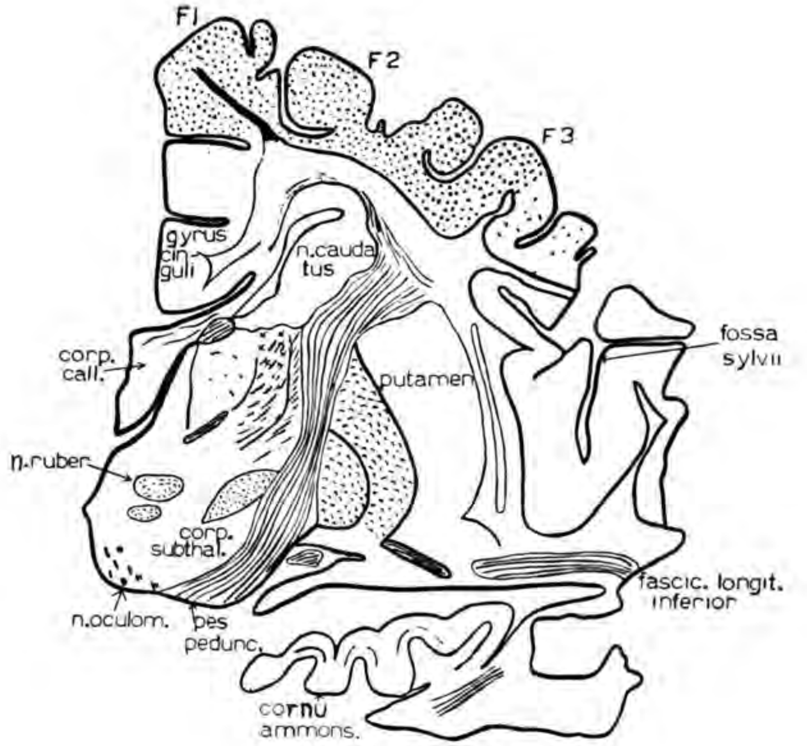
The investigations of SCHWARTZ [13]) and the Frankfurt school have shown that damage to the brain may occur in the mature newborn child after spontaneous delivery. The pathological examination of such cases has taught that haemorrhages in various parts of the central nervous system are frequent, but that also other destructive processes (encephalomalacy, formation of cavities, etc.) may be present, especially in the areas of the brain from which the venous blood flows into the vena magna Galeni. Therefore in such cases the destruction is seen in the white matter of the telencephalon (centrum semi-ovale) and in the striate bodies with their surroundings, whereas the cortex escapes damage. The birth injury is seldom confined to the pallium. In the course of a year the Central Institute for Brain Research received the brain of four children with brain injuries, although these children were fullborn and have been delivered without artificial help. One of these is described here because the examination in serial sections yielded unexpected results.

Case history.

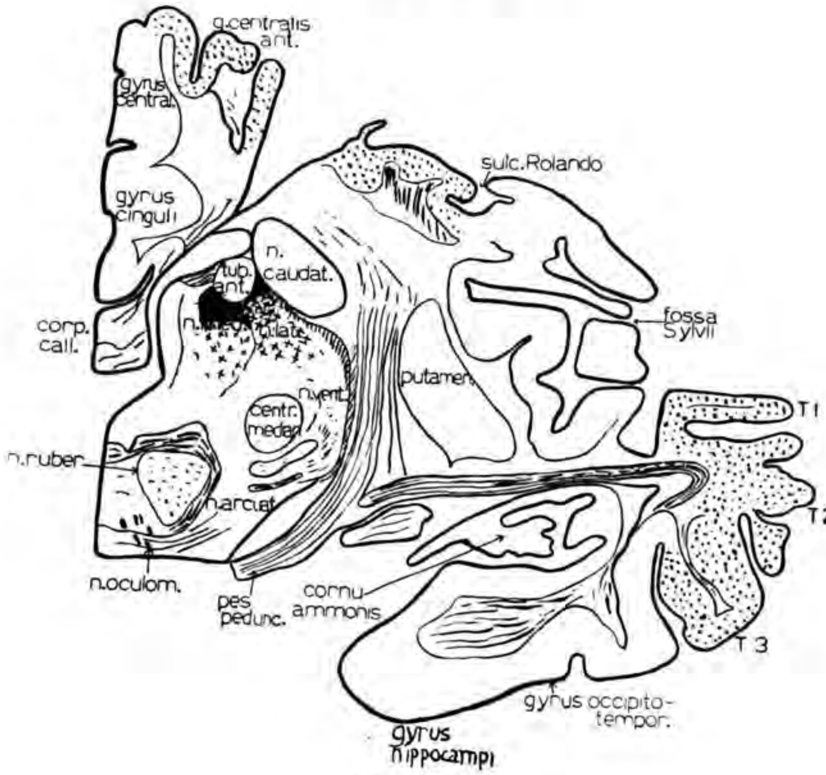
The girl was spontaneously born on the 29th of November 1945 in a normal position. She was a mature child and the birth-weight was 3530 gram. For some days the child was very cyanotic, had convulsions and breathing was irregular. She was treated in the "Onze Lieve Vrouwe Gasthuis", at Amsterdam, with oxygen and several injections. Röntgenograms of the skull revealed that the anterior fontanel was present, but rather small. The coronal sutures were almost invisible, but the occipital sutures (lambda) were wide. After a short time the child's condition improved and she could be sent home. During the first weeks the parents did not see any abnormality, but at the age of two months convulsions appeared in the right arm and later on contractions in the muscles of the eye-balls and of other parts of the body were observed. Since these attacks increased in number the patient was admitted again to the same hospital. She was now 5 months old, but was microcephalic and idiotic. The circumference of the head was 39 cm. She suffered from frequent epileptic fits, but was not paralysed. The kneejerks were lively. The fundus oculi showed no alterations. The internal organs and he blood picture were normal (Wassermann-test negative). Repeated lumbar punctures did not yield any liquor. The Röntgenological examination revealed the microcephaly but the base of the skull was normal. The suturae were small, but distinctly present. The anterior fontanel was 1 cm^2, the posterior fontanel was closed.

On the third of June 1946, the child was transferred to the Neurological Clinic of the University for neurosurgical treatment. Here also clonic contractions appeared frequently, mainly on the right side.The microcephaly was especially pronounced in fronto-parietal

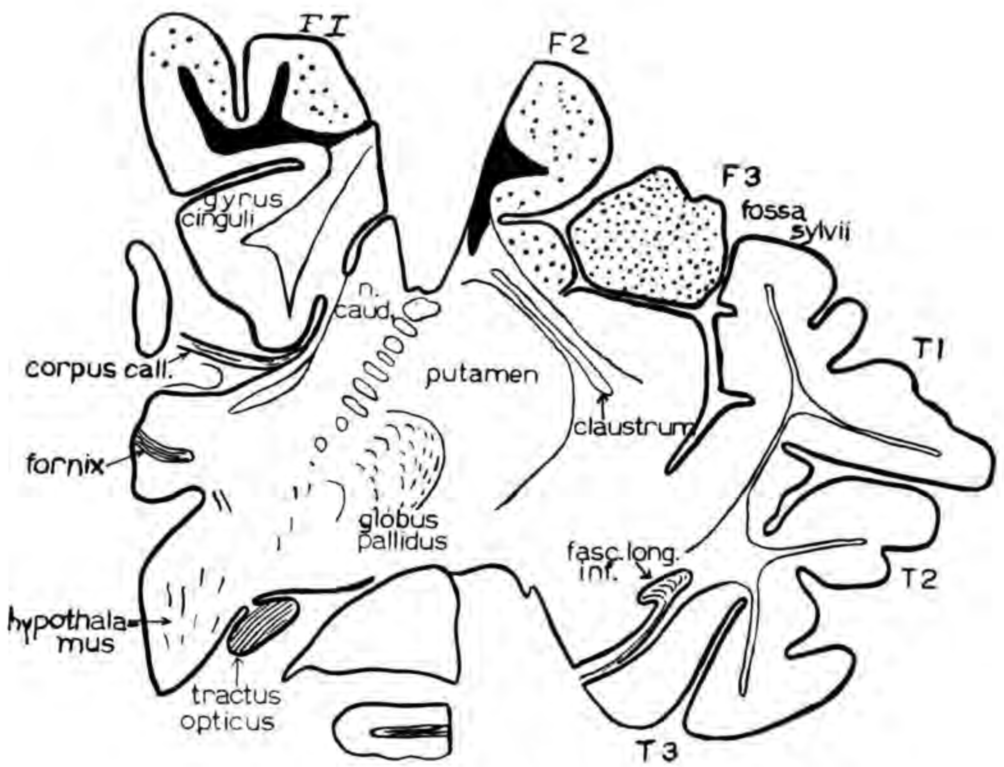
B. BROUWER and C. DE LANGE: *Traumatic microcephaly after spontaneous delivery at full term.*



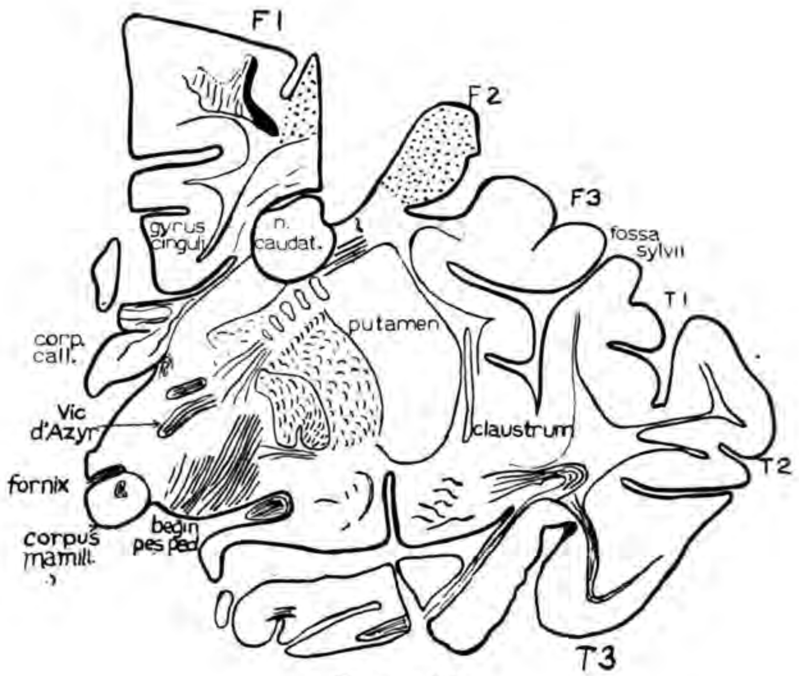
Drawing IV.



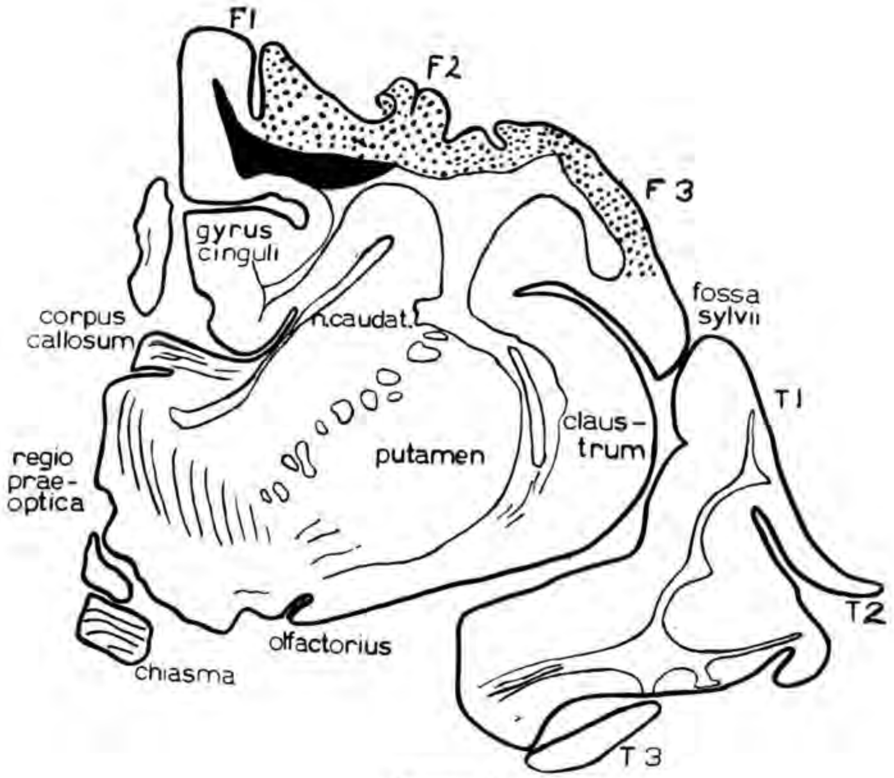
Drawing V.



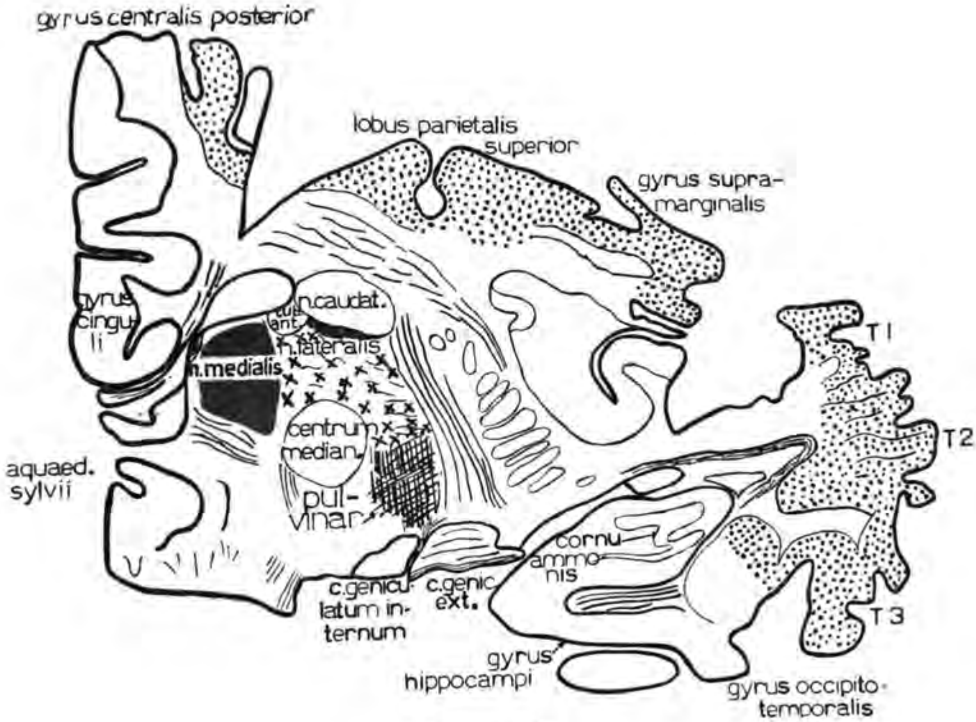
Drawing II.



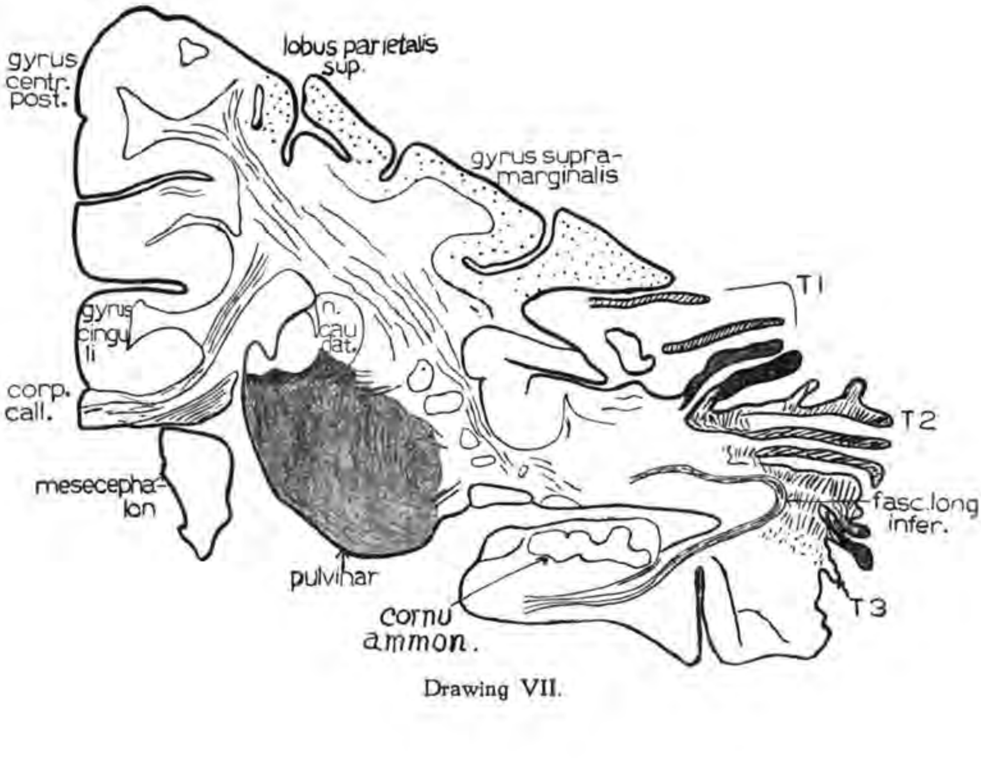
Drawing III.



Drawing I.



Drawing VI.



Drawing VII.

direction. The child was idiotic, did not react to light but slightly to acoustic stimuli. There was no spontaneous nystagmus, but transient strabismus convergens was seen. There were no signs of meningitis. The arm reflexes were normal, the knee reflexes lively. The tonus in the left extremities was increased. A foot clonus was seen at that side. The Babinski sign was present on both sides, just as in normal children of the same age.

On the proposal of Prof. BIEMOND, the neurosurgeon Dr NOORDENBOS, performed a trepanation on the right frontal side of the skull. During the operation several epileptic attacks occurred. The child died some hours after the operation, at the age of about 6 months.

Anatomical investigation.

The post-mortem examination was made in the Institute of the pathologist, Prof. Dr H. T. DEELMAN. The chief results were as follows. The frontal part of the skull was too small and too narrow. There was a subdural haematoma in the parietal region of both sides, which was a little larger on the left. The cerebrum was poorly developed, especially in its frontal part (microcephaly). There was polygyria but no real congenital microgyria. Many gyri were much too small and irregularly formed. The post-mortem examination of the internal organs did not show any changes, especially no signs of congenital malformation. Prof. Dr H. T. DEELMAN and Prof. Dr A. BIEMOND gave the brain to the Dutch Central Institute for Brain Research for further examination.

Description of the macroscopical conditions.

The brain is registered number B^1 No. 11. The weight of the cerebrum in toto is 345 gram. The weight of the telencephalon, with the optic thalamus and metathalamus is 234 gram, that of the cerebellum and the other part of the rhombencephalon is 104 gram. The cerebellum is large in comparison with the remaining part of the brain.

There is no leptomeningitis. The bulbi olfactorii, the optic nerves and the chiasma show no alterations. The arteria basilaris and the arteriae vertebrales are normal. Both hemispheres of the telencephalon are nearly symmetrically reduced in size. A determination of the various sulci and gyri is hardly possible. On both sides, however, the fossa Sylvii may be identified and probably on the left side also the sulcus Rolando. Figure 1 shows the extension of the microgyria and polygyria seen from the dorsal side. The microgyria is of a rough type, varying very much from place to place and asymmetrically distributed over the pallium. Figure 2 shows that the pseudo-microgyria is very pronounced in the occipital lobes. Here is a clear distinction between the medial part, which seems normal and the lateral side, which is seriously damaged. The gyri fusiformes and occipitotemporales show only some anomalies in the configuration of the smaller gyri but are otherwise normal. The left uncus is also very well developed, in the right uncus some microgyric spots are visible. In the basal part of the frontal lobes is also microgyria, but the cortex is not so greatly altered as in the lateral parts of the occipital lobes.

After cutting the brain in a frontal plane a marked reduction of the centrum semi-ovale is seen, being especially pronounced in the frontal half of the brain. At the level of the optic thalamus the sulci and gyri of the dorsal part of the pallium seem to be implanted in the striate bodies. Here the centrum semi-ovale and the cortex show a slightly yellow colour. In the lateral part of the hemispheres there are many "pori". The ventricles are rather small. The corpus callosum is smaller than in a normal brain of the same age. The striate bodies seem to be too large, the optic thalamus too small. The capsula interna and the pes pedunculus are well formed, and show the normal white colour.

Technique.

Pieces from various parts of the cerebral cortex were removed for staining after NISSL, BIELSCHOWSKY, HOLZER, SUDAN, and with haematoxylin-eosin and frozen sections for staining after SUDAN III and with haematoxylin. Further, pieces of the cerebellar cortex, the striate bodies and the optic thalamus were stained with haematoxylin-eosin and after

NISSL. Some sections of the medulla oblongata at the level of the maximum of the inferior olives were stained after WEIGERT–PAL, VAN GIESON, NISSL and with haematoxylin-eosin. Serial sections were made through the left half of the brain from the level of the septum pellucidum down to the midbrain and one in 5 were alternately stained after WEIGERT–PAL and VAN GIESON (thickness 30 μ). Serial sections were also made of the right half of the brain at the same levels as the other half, and stained after NISSL, and here and there with haematoxylin-eosin (1 out of 5 sections, thickness 25 μ). Furthermore serial sections were made of the right occipital lobe and stained after NISSL, some with haematoxylin-eosin (1 out of 10, thickness 25 μ).

Description of the microscopical conditions.

The *dura mater* is thickened by increase of connective tissue. The pia arachnoidea contains many cells and blood-vessels in the parts of the pallium where the underlying cortex is disturbed. In some sections the tunica media of a bloodvessel is much enlarged. Most of the cells in the pia arachnoidea are small and round, but there are also larger cells, showing a red protoplasm in the Sudan sections. Further there are cells resembling rod-cells or cells of the connective tissue. Perivascular infiltrations are not present. There is no real leptomeningitis.

Paraffin sections of the *cerebral cortex* do not show any alterations in the right uncus, the gyrus centralis posterior, the gyrus hippocampi and the insula. The other parts of the cortex are damaged. We describe here the lateral part of the occipital lobe, in which the alterations are very intensive. The cyto-architectonic structure is totally disturbed (Fig. 3). Many ganglion cells and fibres are degenerated. In the white substance of the gyri many cells contain fat. In the cortex itself fat cells are also present filling up the walls of the bloodvessels or forming conglomerations in which rests of bloodlumina are visible. In several places these conglomerations are changed into "pori", when the cells, filled up with lipoid substance, have been transported to the venous sinus. Now and then in these pori fat cells are seen and some of the content of these pori is coloured red with eosin. In the oldest pori membranes of glia fibres are seen. There is further much gliosis between these conglomerations and the pori.

These pori are chiefly found in the deeper layers of the cortex reaching here and there the surface or the white substance ("Rindenblasen-porencephalie" of SCHWARTZ). Besides these destructive processes with formation of small cavities there are also parts of the occipital lobe in which fibrillar glia and astrocytes are present.

In several sections of this lobe many portions of cerebral tissue are apparently isolated from the adjacent parts of the cortex (Fig. 4). They are surrounded by a small layer of connective tissue originating from the wall of a bloodvessel. In various places such isolated pieces are lying together giving the impression of a "cactus formation" (Fig. 5). In these isolated pieces chiefly neuroglia and bloodvessels are seen. Here and there such a piece of cerebral tissue is also seen in the lumen of a bloodvessel. In several places of the cerebral cortex calcification (or pseudocalcification) is visible.

Changes of similar intensity are seen in some other parts of the cerebral cortex, but in most of the sections the pathological process is less pronounced. Usually a fibrillar gliosis is seen in the three upper layers of the cortex, here and there with growth of blood-vessels. As a rule several ganglion cells in the deeper layers are still unimpaired, for example in the left motor area. The three frontal gyri are all damaged, the lesion being most pronounced in the basal part of the left gyrus frontalis I, in which all the above described alterations are visible with extensive formation of pori and in which several parts are calcified (pseudo calcification ?). The sections of the left uncus show the same picture, with many isolated pieces of cerebral tissue as described in the occipital lobe. The sections of the right gyrus temporalis I, stained after VAN GIESON show destruction of the cortical layers with an intensive increase of neuroglia. The left gyrus temporalis I contains formation of pori, but is otherwise relatively intact. The right inferior parietal lobe has lost its architectonic structure and many ganglion cells have disappeared, while the pia arachnoidea contains many cells in this region.

The paraffin sections of the striate body and the cerebellum do not show any alterations. The sections of the medulla oblongata through the level of the maximal development of the inferior olives may be regarded as normal. The pyramidal tracts for instance are well formed, showing myelinisation in accordance with the age of the child.

Description of the serial sections.

a. *Series through the corpora striata and the optic thalamus with the surrounding pallium.*

The left half of the cerebrum is alternatively stained after WEIGERT–PAL and VAN GIESON, the right after NISSL, and here and there with haematoxylin-eosin. These series reach from the level of the septum pellucidum down to the caudal part of the pulvinar. We have made drawings of the various levels and half are reproduced (in I—VII), in which the left side of the brain is pictured. There are some differences in the extension of the pathological process between the left and the right side but this is not essential to an understanding of the alterations.

The cerebrum is much too small, but there are no arguments for the assumption of a general or local congenital disturbance in development. There are for instance no heterotopias, so frequently seen in microcephaly. Neither any rests of an encephalitis are found. The primary destruction is seen only in the pallium, not in the deeper structures of the brain. We mentioned above that part of both hemispheres showed a yellow colour, yet on the microscopical examination no rests of haemorrhagies are found. The ependym of the lateral and third ventricles and of the aquaeductus Sylvii is not altered.

Drawing I shows that the cortex of all the frontal gyri is primarily changed. The number of myelinated fibres is small. This similar destruction is seen in the white substance of the first and second frontal gyrus reaching the border of the centrum semi-ovale. The latter is much too small and insufficiently myelinated with exception of the fibres proceeding to the capsula interna. In NISSL sections stripes of microglia cells are visible, which seem to connect the various smaller gyri. The cells of the gyrus cinguli are strikingly normal and this is true of both sides. There is only a reduction of myelinated fibres caused by loss of associative connections. As this remains so throughout the whole series no further mention will be made. The temporal lobe is not yet altered at this level. The corpus callosum is too thin, is poorly myelinated and shows secondary degeneration of several fibres. The chiasma is normally stained, as is also true of the optic nerves seen in more orally situated sections. The septum pellucidum contains many cells and its myelinisation is sufficient. The regio praeoptica, the caudate nucleus with the putamen, the claustrum, the capsula interna and the olfactory region may be regarded as normal. In the sections following the level of drawing I, the hypothalamus appears, having normal cells and fibres. The nucleus supra-opticus, the nucleus paraventricularis and the nucleus mamillo-infundibularis are clearly visible. The temporal lobes and the commissura anterior are very well myelinated.

At the level of drawing II the globus pallidus appears. The striate bodies are strongly developed throughout the whole series; the myelinisation is normal and there is no loss of cells in the putamen, the caudate nucleus and the globus pallidus. Some portions of the frontal cortex are removed for detailed examination. Several ganglion cells in the three frontal gyri are normal, although the architectonic structure of the cortex is disturbed, and the myelinisation is poor. The fibrae arcuatae (U-fibres), so often intact in pathological processes of the telencephalon, are visible only at some spots. The myelinisation in the three temporal lobes is sufficient, but not so as in more orally situated sections. A number of the fibres appears to have degenerated. It must be noted that at the right side the uncus shows microgyria with destruction of cells, as described above. The regio innominata and the nucleus amygdalae are normal. At the same level the cells of the tuber cinereum are seen. The nucleus hypothalamicus dorsomedialis and the nucleus ventro-medialis show no alterations. The fornix descendens is very well myelinated. This is also true of the optic tract, which is normal up to the level of the external geniculate body. The commissures of v. GUDDEN and MEYNERT are myelinated. Drawing II shows also the fibres of the fasciculus longitudinalis inferior.

At the level of drawing III the third frontal gyrus may be regarded as normal. The taenia thalami is markedly myelinated and remains so throughout the whole series. The hypothalamus is very well developed. The mamillary bodies have a normal capsula and the Nissl sections show that the cells must be regarded as normal. The fornix descendens terminates in these ganglia. The ansa lenticularis is well myelinated. The beginning of the cornu Ammonis is visible. In drawing IV the neothalamus appears in the sections. The temporal gyri have not been reproduced in this drawing because they were cut off in this section but in the preceding and following preparations it is clear that the cortex of this part of the neopallium is preserved (Fig. 6). The cornu Ammonis is normal and remains as such in the remainder of this series. This is also true of the nucleus ruber and the corpus subthalamicum. The capsula interna and the pes pedunculi are darkly stained in the WEIGERT–PAL preparations and no secondary degeneration of fibres has been seen. The cells in the reticular area bordering the capsula interna may be regarded as normal, but there is a slight loss of cells in the nucleus latero-dorsalis, not in the nucleus ventralis anterior or in the oral part of the nucleus medialis thalami.

The neothalamus is seen more fully in drawing V, in which also the nucleus anterior thalami (tuberculum anterius) is visible. In the sections between the levels of the drawings IV and V the first temporal lobe reveals damage and the centralis anterior appears. At the level of the drawing V the medial part of the gyrus centralis anterior is unchanged, but the remaining lobi are altered. A part of the gyrus centralis posterior is already seen but shows no changes. The three gyri of the temporal lobe are now altered on both sides, but the gyrus occipito-temporalis and the gyrus hippocampi are unchanged. In the neothalamus the nucleus medialis shows retrograde degeneration of many cells, while the fibrillar glia is increased, especially in the lateral part (Figures 7, 8). The various cell groups in the medial part of the neothalamus on the border of the third ventricle are normal. The cells of the anterior nucleus are very well developed throughout the whole series, but in the nucleus latero-dorsalis a retrograde degeneration of several cells is seen although less intensive than in the medial nucleus. We did not find any changes in the nucleus ventralis, the centrum medianum (nucleus LUYSII) nor the nucleus arcuatus of FLECHSIG. In contrast with the cellular changes in the neothalamus the various fibre-tracts do not show any alterations in the preparations stained after WEIGERT–PAL.

At the level reproduced in drawing VI the gyrus centralis posterior is seen at its maximum. The medial part shows no change, the dorso- and lateral parts, however, are damaged. Also the lobus parietalis inferior (gyrus supra-marginalis) and the three temporal gyri are primarily altered (Figure 9). The centrum semi-ovale is broader than in more orally situated levels but remains smaller than in normal sections and the number of myelinated fibres is subnormal. The oral part of the pulvinar appears here and shows retrograde degeneration of ganglion cells with a secondary growth of neuroglia. The medial nucleus also is still seriously altered, whereas all the cell groups on the border of the third ventricle are well developed. Also in the nucleus latero-dorsalis the condition is abnormal. Especially at the ventral border of the anterior nucleus many cells are lost. The centrum medianum has a normal appearance. This is also true of the ganglion habenulae, the corpus geniculatum mediale and the grey substance surrounding the aquaeductus Sylvii.

In the sections caudal to this level the alterations in the three temporal gyri (drawing VII) are increasing in intensity (Fig. 10). In the white substance of the temporal lobe the fasciculus longitudinalis inferior remains well myelinated. The gyrus supra-marginalis is seriously affected. This is also true of the gyrus angularis, which appears a little more caudally in the series. At this level the neothalamus is only represented by the pulvinar, in which many cells are lost, while others show retrograde degeneration. The increase of the fibrillar neuroglia is intense at this level. The large ganglion cells, which were always visible near the border of the third ventricle, have now disappeared. The mesencephalon does not show any changes.

b. *Series through the right occipital lobe.*

The NISSL-sections show a distinct difference between the medial and the lateral part of the occipital lobe. The architectonics of the regio calcarina is unchanged, and the cells do not show any symptoms of degeneration. The cortex of the lateral part, however, is intensively altered, as described above. The white substance of the occipital lobe is too small, but there are no primary lesions. The left occipital lobe macroscopically was in the same condition as the right one, so that an examination in serial sections seemed superfluous.

Summarising the pathological facts, a destructive process in the neopallium is found owing to many ganglion cells and nerve fibres having disappeared and the architectonics being lost. The intensity of this process varies in the different parts of the pallium and its extension is seen in the seven drawings. From these it is clear that chiefly the dorsal and lateral parts of the hemispheres are affected. The archipallium and the palaeopallium are normal. Furthermore, the centrum semi-ovale is reduced in size, especially in the frontal half of the cerebrum and the myelinisation is insufficient. The corpus callosum is too small and shows secondary degeneration of fibres. Of the deeper situated parts of the brain it may be stated that the corpora striata, the hypo-epi- and subthalamus and the capsula interna are unchanged. In the neothalamus the nucleus medialis, nucleus dorso-lateralis and the pulvinar show retrograde degeneration of cells, with secondary increase of glia in many places. The other thalamic nuclei may be regarded as normal.

Discussion.

With regard to the clinical aspect of our observation the facts once more may be stressed that the child was full-born and that no artificial help was necessary at birth. During the first days the child was very ill, but then a period of several months followed during which the parents did not see any abnormal symptoms. Thereafter epileptic seizures and other cerebral symptoms appeared. Such a course with a prolonged latent period before the appearance of more serious symptoms is frequently seen after birth traumata. In our case the clinical syndrome was dominated by the microcephaly with idiocy and epileptic convulsions. Spastic pareses, so often seen after lesions of the brain at birth, were not present here.

From our description it will be clear that several pathological changes must be regarded as being the direct consequence of the traumatic action with asphyxia at birth. This trauma must have been rather severe, as is proved by the existence of the subdural haematoma in both parietal regions, evoking a yellow staining of the underlying cortex. The polygyria and microgyria were not caused by a congenital malformation, for on microscopical examination, destruction and shrinking of the various layers of the cortex were found with secondary increase of neuroglia. In actual congenital microgyria, caused by a pathological development of the various cellular layers of the cortex, such shrinking is not seen, whereas the surface of the brain in such cases shows locally small ridges, giving an aspect not unlike a cauliflower (NIEUWENHUYSE [11])).

The question arises whether the microcephaly may also be regarded as the result of the trauma at birth. The conception of GIACOMINI, according to which a microcephalia vera must be distinguished from a pseudomicrocephalia, is generally accepted. To the first group cases belong in which a general malformation of the brain has arisen in an early period of the embryonic life, caused by hereditary or by exogenous factors (intoxication, syphilis, disturbances in nutrition, X-rays, etc.). These factors may also be active in the cases belonging to the pseudomicrocephaly, but in these a localised process in the brain is found, which caused loss of cerebral tissue in the foetal period, such as encephalitis, meningitis, porencephaly, hydrocephalus etc. (V. MONAKOW [10]), GREENFIELD and WOLFSOHN [3]) a.o.). In both groups of congenital microcephaly the usual symptoms of malformation, especially heterotopia, are found on microscopical examination.

LITTLE regarded several cases of microcephaly as the result of the traumatic influence at birth, and his opinion is confirmed by anatomical investigations (ANTON [1]) et alii). Recently KUHN [4]) has described the results of a clinical-anatomical investigation of such a case of microcephaly, in which severe traumatic alterations in the brain were found. In his case the circumference of the skull at the age of 5 months was almost the same as at birth (increasing from 37 cm to 38 cm). The formative stimulus for the growth of the skull was lacking here, because the brain remained too small.

The fact that in our case the systematic examination of the brain in serial sections did not show any symptom of malformation, is a strong argument for the conception that here the microcephaly was of traumatic origin. The existence of the subdural haematoma in both parietal regions proves that the brain must have been compressed in a lateral rather than in a dorso-ventral direction. This would account for the escape of several basal parts of the brain (e.g. the cornu Ammonis). The centrum semi-ovale did not show any traumatic alterations, but it was much too small and poorly myelinated. It seems likely that this also was caused by the compression at birth.

From our description of the pathological alterations in the cortex of the neopallium it is clear that the damage was not limited to a compression of parietal regions, but extended much further over the hemispheres. Similar findings are reported in the literature. SCHWARTZ [13]) already mentioned that often more recent changes may be found amongst the older lesions of the cortex, indicating that the pathological process had not come to a standstill ("akute Spätfolge"). This is confirmed by WOHLWILL [15]), who has pointed out that in these cerebra many disturbances in the circulation arise owing to traumatic alterations in the innervation of the walls of the blood-vessels, causing insufficient nourishment of the brain-tissue, resulting in degeneration. According to this author the normal and pathological stimuli of the extra-uterine life have an additional destructive influence on such processes, where once the neuro-vascular apparatus has been damaged.

In regard to the traumatic alterations in the cortex cerebri we would call special attention to those cortical alterations referred to as the "cactus formation" (figure no. 5). They are composed of pieces brain-tissue, lying apparently isolated in the pia arachnoidea and grouped together in a formation resembling a cactus. They must be definitely distinguished from the so-called hollow gyri of SCHWARTZ, in which all the tissue of a gyrus has disappeared with the exception of the lamina zonalis. We may further draw attention to the fact, that repeatedly isolated pieces of brain-tissue are found in the lumen of blood-vessels, leading to disturbances in the circulation and further destruction of the cortex. Finally, we may again stress the fact that in various parts of the cortex, especially in the deeper layers, ganglion cells are still intact.

In his investigations on malformations of the brain, ANTON [1]) found that in microcephaly the projection fibres are frequently less involved in the process than the associative elements. Such a contrast is also plain in our observation. The capsula interna and the pes pedunculi are not too small and are well myelinated, whereas the corpus callosum is arrested in its development and shows symptoms of a secondary degeneration of fibre tracts. The fasciculus longitudinalis inferior, with the corpus geniculatum externum, are developed in accordance with the age of the child. This is not difficult to understand, because the striate area in the occipital lobe was unimpaired. The relations between the internal geniculate body and the cortex are more difficult to explain. This ganglion did not show any changes, whereas the posterior part of the first temporal gyrus, in which the acoustic radiation ends, had clearly been damaged. In this part of the neopallium not all the fibres and cells were degenerated, so that we can only conclude that the association systems had suffered more intensively than the projection fibres.

The difference between association and projection systems is also seen in the nuclei of the neothalamus. In our case the following nuclei did not show any retrograde degeneration: the nucleus anterior (tuberculum anterius), the nucleus ventralis, the nucleus ventralis anterior, the centrum medianum of LUYS, the nucleus arcuatus of FLECHSIG and the medially situated cell-groups near the border of the third ventricle. Retrograde degeneration of the ganglion cells and reactive increase of neuroglia were found in the nucleus medialis thalami, the nucleus latero-dorsalis and the pulvinar. Recent investigations have rendered it very doubtful as to whether the centrum medianum (the nucleus medialis b. of VON MONAKOW) has direct connections with the neopallium. WALKER [14]), for instance, did not find any changes in this area after hemidecortication in the macaque monkey. The nucleus arcuatus of FLECHSIG, on the other hand, is connected with the neopallium, i.e. with the operculum and the insula (DROOGLEEVER FORTUYN [2])). The latter part of the cortex was normal in our case, but the operculum revealed alterations which evidently were not sufficient to cause a reaction in the cells of the arcuate nucleus. It is generally accepted that the cell groups, lying medially in the neothalamus in the immediate neighbourhood of the third ventricle are not projected to the pallium. Although there is still some difference of opinion regarding the connections of the nucleus anterior with the gyri of the frontal brain, it is now certain according to experiments in animals that the chief projection of this nucleus is to the gyrus cinguli (LE GROS CLARK [5]), DROOGLEEVER FORTUYN [2])). This gyrus is unchanged in both hemispheres, hence we understand the normal appearance of the nucleus anterior. There is a general agreement of opinion that the nucleus ventralis, which may be divided into several smaller cell-groups, is chiefly projected to the gyrus centralis posterior, partly also to the anterior. It is very probable that this latter projection originates in the nucleus ventralis anterior, in which cerebellar systems from the dentate nucleus terminate. Since the gyrus centralis anterior is damaged and partly also the gyrus centralis posterior an explanation is required why the central nuclei of the neothalamus remained intact.

In regard to the projection of the medial nucleus on the pallium it may be concluded that the original conception of V. MONAKOW's school, namely that this nucleus is connected with the frontal area of the brain, has been confirmed by recent clinical-anatomical and experimental investigations. The nucleus medialis increases in size together with these frontal regions. From the results of the experimental-anatomical investigations it is probable that the various subdivisions of this nucleus have a separate projection to the cortex. In our case several parts of the nucleus medialis showed groups of normal ganglion cells within the areas of degeneration. It was tempting to make some deductions respecting a more precise localisation. However, the alterations of the frontal cortex were too diffuse to permit of a definite conclusion.

The alterations in the nucleus latero-dorsalis were less intensive than in the nucleus medialis, although the degeneration of ganglion cells and increase of neuroglia were marked in several places. The projection of this nucleus is still a matter of controversy among the various investigators. Nevertheless, the original conception of VON MONAKOW [9]), namely that the nucleus lateralis is chiefly connected with the parietal lobe would seem to be confirmed in recent experimental-anatomical studies. The great alteration of this cortical area explains the degeneration of the nucleus latero-dorsalis. This is also true of the pulvinar. The earlier experimental-anatomical studies of MINKOWSKI [7, 8]) and others have demonstrated, that the pulvinar is projected towards the posterior part of the parietal and temporal gyri. POLJAK [12]) came to the conclusion that in monkeys the chief area of projection is found in the gyri forming the walls of the posterior part of the sulcus fossae Sylvii ("the posterior Sylvian receptive region"). LE GROS CLARK and NORTHFIELD [6]) confirmed this, but proved that in the macaque monkey the pulvinar has also connections with the cortex of the occipital lobe, bordering the striate area (field 18 and 19 of BRODMANN). WALKER [14]), after his experiments on the chimpanzee, found that the lateral part of the pulvinar is projected to the gyrus parietalis superior, the medial part to the gyrus supra-marginalis and the ventral part to the temporo-occipital region. As to the relations in man the classic conception of VON MONAKOW of the gyrus angularis being the chief end-station of the fibre-systems originating in the pulvinar perhaps is still the best. In our case not only the gyrus angularis was impaired by the pathological process, but also the gyrus supra-marginalis and the cortex, bordering the striate area, so that the serious alterations in the pulvinar are not surprising.

The nucleus medialis, the nucleus latero-dorsalis and the pulvinar are projected to cortical regions, which FLECHSIG termed "association areas". His division of the cerebral cortex into "projection" and "association" areas, in which the former are connected with lower parts of the central nervous system by ascending and descending systems, whereas the latter are not, can not be any longer accepted. The association areas of FLECHSIG have also many connections with extra-cortical portions of the central nervous system e.g. with the neothalamus. Although FLECHSIG's conception has been disproved, it yet seems to us that a division of projective and associative parts of the brain is still of value. Several of the so-called association areas are connected with those nuclei of the neothalamus in which no direct afferent sensory fibres from lower parts of the brain terminate. The investigations of LE GROS CLARK [5]) and others have shown that all sensory fibres of the medial fillet, the spino-thalamic tract and the trigeminal fillet end exclusively in the nucleus ventralis and the nucleus arcuatus of FLECHSIG. These experiments refuted the earlier conception that trigeminal fibres proceed to the medial nucleus. The cerebellar fibres terminate in the nucleus ventralis anterior, not in the latero-dorsal nucleus. The optic fibres proceed to the external geniculate body, passing only the pulvinar; the acoustic fibres end in the internal geniculate body. The medial and the latero-dorsal nucleus together with the pulvinar may be regarded as "associative" thalamic nuclei. They are connected with the "association" areas of the cortex on one hand, on the other by small fibre systems with the remaining thalamic nuclei. These associative thalamic nuclei increase in size in the ascending scale of mammals and are most strongly developed in man. In these nuclei stimuli from lower parts of the central nervous system may be brought to a higher functional level (LE GROS CLARK [5])). In the brain described in these pages almost exclusively associative systems have been damaged by the trauma at birth, while the projective systems remained unimpaired. This explains why the clinical syndrome was dominated by idioty and epileptic convulsions.

Summary.

In this article a report is given on the clinical-anatomical investigation of the brain of a child, 6 months of age, damaged at birth, although the child was born spontaneously and à terme. During the first days following birth the child showed severe symptoms of asphyxia and epileptic convulsions. After that a period of two months followed during which the child did not show any pathological symptoms. Suddenly the epileptic seizures reappeared and it gradually became clear that the skull was not growing and that the child did not develop mentally. At the post mortem examination a subdural haematoma was found in both parietal regions of the brain. Furthermore microcephaly, microgyria and polygyria were present. Histological examination of the brain failed to show any symptoms of foetal malformations or encephalitis. All the pathological alterations could be derived from the traumatic accident at birth. The authors have accepted the opinion of WOHLWILL [15]) that in such cases the pathological process progresses even after the traumatic influence has ceased. Disturbances in the neuro-vascular mechanism lead to further destruction or degeneration of the brain-tissue. The alterations in the cortex, characterised by resorption and displacement of brain-tissue, degeneration of cells and fibres and increase of neuroglia, were chiefly localised in the lateral parts of the hemispheres. The distribution of cortical alterations was presented in drawings. Owing to traumatic influence the growth of the telencephalon was arrested (microcephaly), while the centrum semi-ovale remained too small and poorly myelinated. Examination in serial sections of the corpus striatum, the epi- and hypothalamus, the subthalamus and the capsula interna with the pes pedunculi showed normal relations. In the optic thalamus retrograde degeneration of ganglion cells was found only in the nucleus

medialis, the nucleus latero-dorsalis and the pulvinar. These nuclei were regarded as associative thalamic nuclei through which — in cooperation with the neopallium — a higher functional level is reached. The birth injury impaired almost exclusively the associative system of the cortex and caused a retrograde degeneration in the cells of the "associative" nuclei of the neothalamus.

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B. BROUWER and C. DE LANGE: *Traumatic microcephaly after spontaneous delivery at full term.*

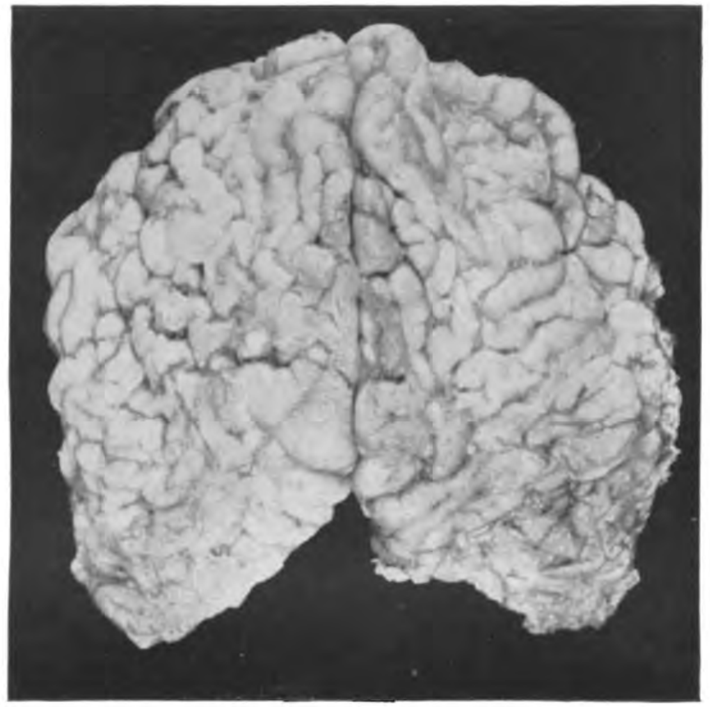

Fig. 1.
Microcephalia and Polygyria.

Fig. 2.
Microcephalia and Polygyria.

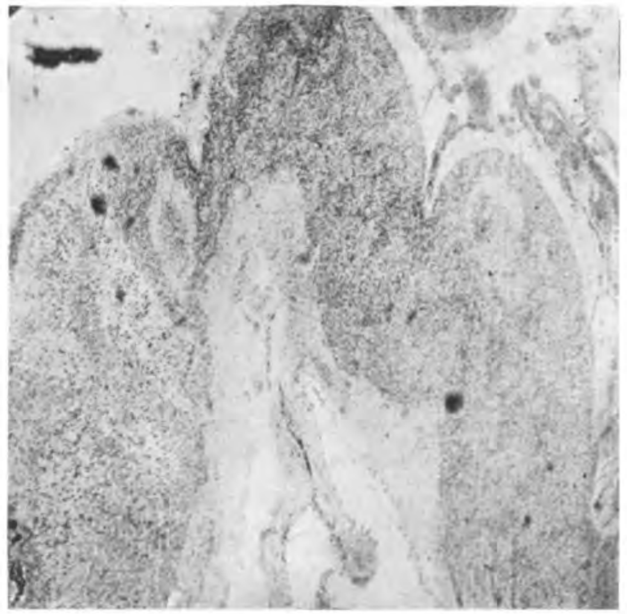

Fig. 3.
Cyto-architecture seriously damaged in the right occipital lobe. (NISSL-preparation.)

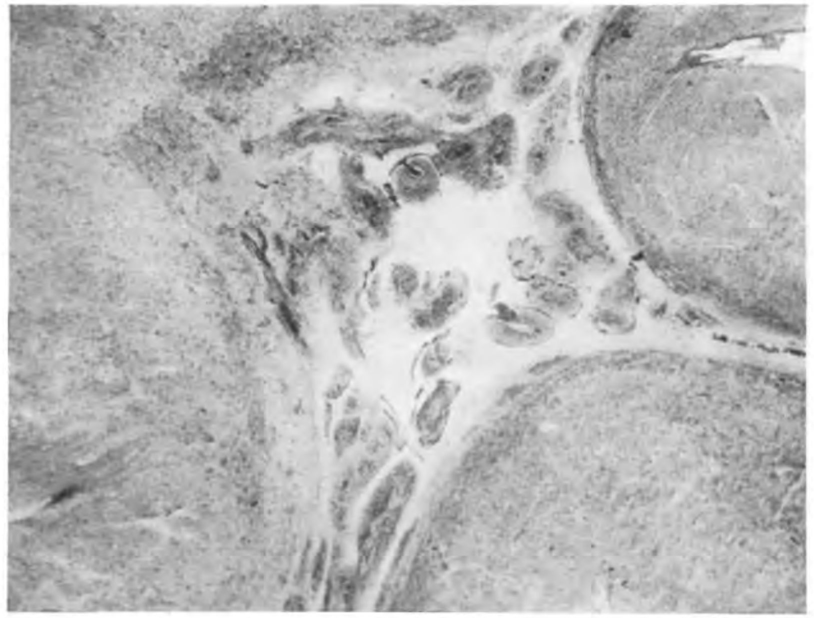

Fig. 4.
Isolated portions of cerebral tissue. (NISSL-preparation.)

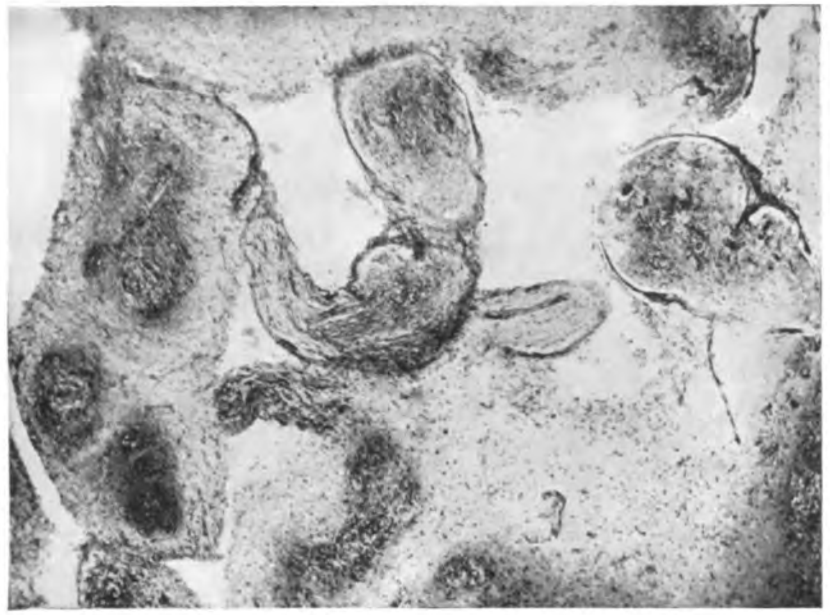

Fig. 5.
"Cactus" formation in the cortex. (NISSL-preparation.)

Fig. 6.
Section at the level of the beginning of the neothalamus, stained after WEIGERT-PAL.

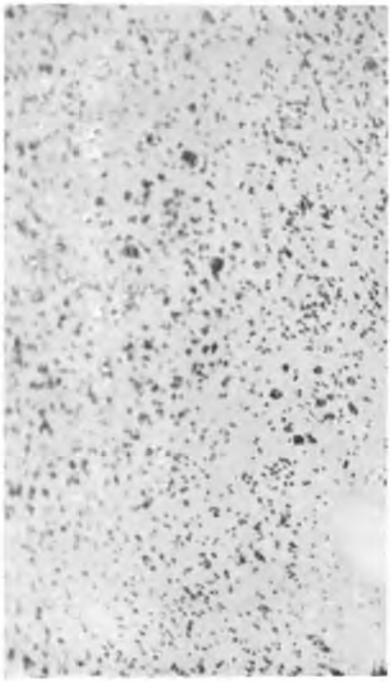

Fig. 7.
Degeneration in the nucleus medialis thalami. (NISSL-preparation.)

Fig. 8.
Increase of glia in the nucleus medialis thalami. (V. GIESON-preparation.)

Fig. 9.
Primary lesion of the gyrus supra-marginalis and of the temporal lobe. (V. GIESON-preparation.)

Mathematics. — *Essentieel-negatieve eigenschappen.* By L. E. J. BROUWER.

(Communicated at the meeting of September 25, 1948.)

Met het oog op de aan een eventueele affirmatieve, respectievelijk negatielooze wiskunde [1]) toe te kennen draagwijdte is wellicht de publicatie van nut van een sinds 1927 in mijn colleges en voordrachten nu en dan aangevoerd eenvoudig en aanschouwelijk voorbeeld van realisatie eener enkelvoudig-negatieve eigenschap (d.w.z. absurditeit eener constructieve eigenschap) op een wijze die zeer zeker geen hoop op vervorming tot corollarium eener constructieve eigenschap rechtvaardigt. Het bestaat in twee reëele getallen, die verschillend zijn zonder dat een van beide als grooter of kleiner dan het andere, laat staan als van het andere verwijderd, kan worden aangewezen.

Zij α een wiskundige assertie, die *niet kan worden getoetst,* d.w.z. dat geen methode bekend is, om hetzij haar absurditeit, hetzij de absurditeit harer absurditeit af te leiden [2]).

Dan kan het scheppende subject in verband met deze assertie α naar het volgende voorschrift een onbepaald voortschrijdende sequentie van rationale getallen $a_1, a_2, a_3, \ldots$ creëeren: Zoolang bij de keuze der a_n aan het scheppende subject nòch de juistheid, nòch de absurditeit van α is gebleken, wordt iedere a_n gelijk aan 0 gekozen. Zoodra echter tusschen de keuze van a_{r-1} en die van a_r aan het scheppende subject de juistheid van α is gebleken, wordt zoowel a_r, als voor iedere natuurlijk getal ν ook $a_{r+\nu}$, gelijk aan 2^{-r} gekozen. En zoodra tusschen de keuze van a_{s-1} en die van a_s aan het scheppende subject de absurditeit van α is gebleken, wordt zoowel a_s, als voor ieder natuurlijk getal ν ook $a_{s+\nu}$, gelijk aan -2^{-s} gekozen.

Deze onbepaald voortschrijdende sequentie $a_1, a_2, a_3, \ldots$ is positief convergent, bepaalt dus een reëel getal ϱ.

Gold voor dit reëele getal ϱ de relatie $\varrho > 0$, dan zou ϱ niet < 0 kunnen zijn, dus zou vaststaan dat nimmer de absurditeit van α kan blijken, dus ware de absurditeit der absurditeit van α bekend, dus ware α getoetst, hetgeen niet het geval is. *De relatie* $\varrho > 0$ *geldt dus niet.*

Gold verder voor het reëele getal ϱ de relatie $\varrho < 0$, dan zou ϱ niet > 0

[1]) Vgl. G. F. C. GRISS, Negatielooze intuïtionistische wiskunde, Verslagen dezer Akademie, **53**, p. 261 (1944); Negationless intuitionistic mathematics, deze Proceedings, **49**, p. 1127 (1946); D. VAN DANTZIG, On the principles of intuitionistic and affirmative mathematics, deze Proceedings, **50**, p. 918 en p. 1092 (1947).

[2]) Bijvoorbeeld de assertie, dat een viertal natuurlijke getallen $n > 2$, a, b en c bestaat, waarvoor de betrekking $a^n + b^n = c^n$ geldt, of de assertie, dat in de decimale ontwikkeling van π tien op elkaar volgende cijfers voorkomen, die een sequentie 0123456789 vormen.

kunnen zijn, dus zou vaststaan dat nimmer de juistheid van α kan blijken, dus ware de absurditeit van α bekend, dus ware wederom α getoetst, hetgeen niet het geval is. *De relatie* $\varrho < 0$ *geldt dus evenmin.*

Onderstellen we ten slotte de geldigheid der relatie $\varrho = 0$. In dat geval zou ϱ nòch < 0, nòch > 0 kunnen zijn, dus nòch de absurditeit, nòch de juistheid van α ooit kunnen blijken, dus ware zoowel de absurditeit der absurditeit, als de absurditeit van α bekend, een contradictie. *De relatie* $\varrho = 0$ *is dus contradictoor. D.w.z. de reëele getallen* ϱ *en* 0 *zijn verschillend.*

Zoodat voor de reëele getallen ϱ en 0 de enkelvoudig-negatieve eigenschap $\varrho \neq 0$ bestaat zonder aanwezigheid van een der eigenschappen $\varrho > 0$ en $\varrho < 0$, laat staan van een der constructieve eigenschappen $\varrho \circ\!> 0$ en $\varrho <\!\circ\, 0$. Voor reëele getallen is dus de relatie $\neq$ een essentieel-negatieve relatie.

Op analoge wijze ontstaat, indien aan het einde der derde alinea -2^{-s} vervangen wordt door 2^{-s}, voor de reëele getallen ϱ en 0 de enkelvoudig-negatieve eigenschap $\varrho > 0$ zonder aanwezigheid der constructieve eigenschap $\varrho \circ\!> 0$. Ook de relatie $>$ der virtueele ordening van het continuum is derhalve een essentieel-negatieve relatie.

Astronomy. — *Types of satellite systems and the disc theory of the origin of the planetary system.* By H. P. BERLAGE, Director Meteorological and Geophysical Service, Batavia.

(Communicated at the meeting of May 29, 1948.)

1. There is no doubt that the satellite systems have grown in the same way as the primary system, only on a smaller scale. According to (10) *) conformal systems are born when the gravitational potential of the central body is of the same order of magnitude at similar distances and this condition appears to be satisfied in most of the cases. Moreover there is no discrepancy between the distribution of the masses and the distribution of the angular momenta in the satellite systems, such as there is in the main system, with the only exception of the system Earth-Moon. It will have to be considered specially. Consequently much can be learned from an analysis of the satellite systems. Space does not allow to go into much detail. However, some general information is easily obtained and very useful.

Let us imagine the case that only one satellite would grow from the nebula revolving round a primary. Formulae developed in Proc. **35** need only be slightly revised in order to prove that the nebula is characterized in first approximation by

$$\left.\begin{aligned} m &= 2\,k\,\varrho_0\,6!\,a^{-7} \\ \theta &= 2\,k\,(\gamma M)^{\frac{1}{2}}\,\varrho_0\,7!\,a^{-8} \\ U &= k\,\gamma\,M\,\varrho_0\,4!\,a^{-5} \end{aligned}\right\}\; k = (2\pi)^{\frac{1}{2}} \left(\frac{f_0}{\gamma M}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (30)$$

When the satellite evolving from this nebula would circulate at a distance r from its primary, 3 conditions have to be satisfied, viz.:

$$m = 2\,k\,\varrho_0\,6!\,a^{-7} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$m\,r^{\frac{1}{2}} = 2\,k\,\varrho_0\,7!\,a^{-8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$m\,r^{-1} > 2\,k\,\varrho_0\,4!\,a^{-5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

However, from (31) and (32) follows

$$a\,r^{\frac{1}{2}} = 7 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and

$$m\,r^{-1} = \tfrac{30}{49} \cdot 2\,k\,\varrho_0\,4!\,a^{-5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Condition (33) not being satisfied, we conclude that *the evolution of one satellite from the nebula surrounding a primary is highly improbable.*

*) Preceding Article.

It is easy to confirm this from general theory. We know that condition (25) *) has to be satisfied. Now, let us assume

$$\sin(p \log r + q) = \cos\left(p \log \frac{a^2 r}{49}\right). \quad . \quad . \quad . \quad . \quad (36)$$

This is a fluctuation of the density distribution reaching a maximum where

$$r = \frac{49}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which is the distance where the only satellite would circulate. The smaller the value of p, the lesser the masses of other satellites. For $p = 0$ we would indeed get only one satellite. However, this is the most improbable case to occur, because this changes

$$\int_0^\infty r^{\frac{7}{2}} \exp\left[-a r^{\frac{1}{2}}\right] \cos\left(p \log \frac{a^2 r}{49}\right) dr \quad . \quad . \quad . \quad . \quad . \quad (38)$$

into

$$\int_0^\infty r^{\frac{7}{2}} \exp\left[-a r^{\frac{1}{2}}\right] dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

which is the highest positive value reached by (38).

The first part of the integrand,

$$r^{\frac{7}{2}} \exp\left[-a r^{\frac{1}{2}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

has its maximum too at (37) and therefore the condition $(38) < 0$ is not likely to be satisfied by any value of p. On the contrary, most likely to be satisfied is — $(38) < 0$ due to an ondulation in the density distribution showing maxima, where the former showed minima. In this case, when p is small two satellites are generated with masses of importance, the other ones being negligible. Evidently *even numbers have preference above uneven numbers.*

Now, let us assume that a system of two satellites is born whose masses are equal and whose distances from the primare are r_1 and r_2. Then the following conditions have to be satisfied,

$$2m = 2k \varrho_0 6!\, a^{-7} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$m\left(r_1^{\frac{1}{2}} + r_2^{\frac{1}{2}}\right) = 2k \varrho_0 7!\, a^{-8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$m\left(\frac{1}{r_1} + \frac{1}{r_2}\right) > 2k \varrho_0 4!\, a^{-5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Treating the inequality as an equality, the solution of the equations is

$$r_1 = \left(\frac{7 + 2\sqrt{2}}{a}\right)^2, \quad r_2 = \left(\frac{7 - 2\sqrt{2}}{a}\right)^2. \quad . \quad . \quad . \quad . \quad (44)$$

This shows that there are real solutions in cases of two and more satellites. The systems Earth-Moon and Neptune-Triton, however, require special consideration.

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*) Preceding Article.

With two equal masses

$$\frac{r_1}{r_2} = 6.53.$$

This high quotient shows us that few satellites with greatly diverging orbits are more likely to evolve than a numerous sequence. The same can be inferred from general theory. Evidently (38) tends to 0 when p increases. Hence the smallest p-values satisfying the conditions for actual development will have preference. The 4 big planets, the 4 big satellites of Jupiter and the 4 satellites of Uranus suggest that 4 is the most preferential number.

2. The system Earth-Moon consists of a primary with only one satellite, which is, as we have seen, the most improbable case to occur. However, it is not difficult to show, why a body like our moon could actually develop. According to the elementary theory of 1, the following condition would have to be satisfied in cases of masses m which are negligible compared with M

$$\gamma M m r^{-1} > 2 k \gamma M 4! a^{-5}. \quad \dots \dots \quad (45)$$

In cases of masses m which are not negligible compared with M we have to add to the left side the potential energy lost in the condensation of the satellite from almost infinitely dispersed matter.

When the density of the satellite is assumed uniform, we get the corrected condition

$$\frac{\gamma M m}{r} + \frac{3}{5}\frac{\gamma m^2}{r_s} > 2 k \gamma M 4! a^{-5} \quad \dots \dots \quad (46)$$

or

$$\frac{m}{r}\left(1 + \frac{3}{5}\frac{m r}{M r_s}\right) > 2 k 4! a^{-5} \quad \dots \dots \quad (47)$$

when r_s denotes the radius of the satellite.

(33) and (35) show that this condition is satisfied, when

$$\frac{3}{5}\frac{m r}{M r_s} > \frac{19}{49} \quad \dots \dots \quad (48)$$

or, say, when

$$\frac{3}{5}\frac{m r}{M r_s} > \frac{2}{5} \quad \dots \dots \quad (49)$$

hence, when

$$r > \frac{2}{3}\frac{M}{m} r_s. \quad \dots \dots \quad (50)$$

In the case of earth and moon

$$\frac{M}{m} = 81, \quad r_s = \tfrac{1}{4} r_e$$

when r_e is the radius of the earth, while (50) changes into

$$r > {}^{83}/_{6}\, r_e \; . \qquad (51)$$

This conditions proves that the moon can have developed, but only at an initial distance more than 14 times the earth's radius. Its actual distance is 60 times the earth's radius. This makes its growth possible with the restriction that G. H. DARWIN's theory of the moon's recession from the earth would have to count with the new limit to its original distance from the earth. It is far wider than ROCHE's limit.

3. Neptune's system is the second example of a planet with only one satellite. Therefore it looks like more than a fortuitous coincidence that both examples of a planet with one satellite show an exceptionally high proportion between the mass of the satellite and the mass of the planet. Triton's mass is estimated by NICHOLSON at 5 times the mass of our moon [5]) or the 300th part of Neptune's mass. Yet, in the case of Neptune the proportion between the masses of satellite and planet is too low to account for the generation of only one satellite at Triton's actual distance.

When we assume that Triton presents the same mean density as Neptune then it is easily shown that the condition (50) limiting the distance at which a solitary satellite could condense, turns into

$$r > 30\, r_n$$

when r_n denotes the radius of Neptune. The actual distance of Triton from Neptune is 13.3 times the planet's radius. This would imply that we are here investigating the case of a satellite which has approached its parent planet in course of time. It is the reverse of what has happened in the system Earth-Moon. But presisely this is what should have happened in the system Neptune-Triton by the same influence of tidal friction which eliminated our Moon from the Earth. The reason is that the Moon's revolution and the Earth's rotation are in the same sense, whereas Triton's revolution and Neptune's rotation are in opposite sense.

That by this same process Neptune's rotation has been slowed down would explain why its present rotation period is 15h.8, whereas the rotation periods of the other big planets are 9h.9 for Jupiter, 10h.2 for Saturn and 10h.7 for Uranus [6]).

[5]) Publ. Astr. Soc. Pacific. **43**, 261 (1931).

[6]) K. HIMPEL, Erdgeschichte und Kosmogonie, p. 122.

Geophysics. — *On the large displacements commonly regarded as caused by Love-waves and similar dispersive surface-waves.* IV. By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS JR.)

(Communicated at the meeting of June 26, 1948.)

§ 4. *The waves reflected at the interface.*

The movement in the ocean caused by the subjacent medium starts at a time $t = t_1$ where

$$t_1 = \frac{\varrho}{V'} + \frac{q_1 \cos \varepsilon}{V}, \quad q_1 = 2d - f \pm z$$

as $L(\varrho, z, u) \equiv 0$ if $u < t_1$. The disturbance which reaches ϱ, z at $t = t_1$ has travelled along the surface of cone ε from the source to the bottom, with velocity V, has then been transmitted in every direction along the bottom and reaches ϱ, z while travelling upwards in the direction ε. This movement is obtained by substituting (4) into

$$U = \frac{\partial G}{\partial \varrho}, \quad W = \frac{\partial G}{\partial z} \text{ and } G = \int_0^t \frac{\partial}{\partial t} F(t-u) \cdot L(\varrho, z, u)\, du.$$

We get:

$$\left.\begin{aligned} U_1 &= -\frac{2}{\pi} \int_{t_1}^{t} F'(t-u) \cdot \frac{\partial}{\partial \varrho} R \int_0^{iw_1} \frac{\xi D_l}{\sqrt{u^2 - (q_1^2 + \varrho^2 \cos^2 \omega)/V^2}}\, d\omega \cdot du \\ W_1 &= -\frac{2}{\pi} \int_{t_1}^{t} F'(t-u) \cdot \frac{\partial}{\partial z} R \int_0^{iw_1} \frac{\xi D_l}{\sqrt{u^2 - (q_1^2 + \varrho^2 \cos^2 \omega)/V^2}}\, d\omega \cdot du \end{aligned}\right\} t_1 < t < r_1/V \quad . \qquad (10)$$

with $\qquad \cosh w_1 = \dfrac{u V'}{\varrho} - \dfrac{q_1}{\varrho} \sqrt{\dfrac{V'^2}{V^2} - 1}$ and $r_1^2 = \varrho^2 + q_1^2$.

At the time $t = r_1/V$ a new disturbance reaches ϱ, z: the Laplace function is discontinuous at $u = r_1/V$ which entails a new term of $U_1 (W_1)$. For instance:

$$\begin{aligned} U &= \frac{\partial}{\partial \varrho} \int_{t_1}^{t} F'(t-u) \cdot L_n(\varrho, z, u)\, du \\ &= \frac{\partial}{\partial \varrho} \int_{t_1}^{r_1/V} F'(t-u) \cdot L_n(\varrho, z, u)\, du + \frac{\partial}{\partial \varrho} \int_{r_1/V}^{t} F'(t-u) \cdot L_n(\varrho, z, u)\, du \\ &= \int_{t_1}^{t} F'(t-u) \cdot \frac{\partial}{\partial \varrho} L_n(\varrho, z, u)\, du + \\ &\qquad + F'(t - r_1/V) \frac{\partial}{\partial \varrho} \{ L(\varrho, z, u = r_1/V + 0) - L(\varrho, z, u = r_1/V - 0) \}. \end{aligned}$$

Therefore with (6):

$$\left.\begin{aligned} U_1 &= -E\sin\psi_1\cdot\frac{D_1}{r_1V}F'(t-r_1/V)+E\int_{t_1}^{t}F'(t-u)\cdot\frac{\partial}{\partial\varrho}L_1(\varrho,z,u)\,du \\ W_1 &= -E\cos\psi_1\cdot\frac{D_1}{r_1V}F'(t-r_1/V)+E\int_{t_1}^{t}F'(t-u)\cdot\frac{\partial}{\partial z}L_1(\varrho,z,u)\,du \end{aligned}\right\}\ t>r_1/V. \quad (11$$

D_1 = the real part of the value of D_l at $\xi = -i\sin\psi_1/V$, while $\sin\psi_1 = \varrho/r_1$

L_1 = expression (9) with $n = 1$ together with the Stoneley function. The movement U_1, W_1 will be called the first reflected wave.

At the time $t = t_2$, where

$$t_2 = \frac{\varrho}{V'} + \frac{q_2\cos\varepsilon}{V},\quad q_2 = 2\times 2d - f \pm z$$

the second reflected wave appears at ϱ, z:

$$U_2 = \frac{2}{\pi}\int_{t_2}^{t}F'(t-u)\cdot\frac{\partial}{\partial\varrho}R\int_0^{iw_2}\frac{\xi(-D_l)^2}{\sqrt{u^2-(q_2^2+\varrho^2\cos^2\omega)/V^2}}\,d\omega\cdot du,\ \text{if } t_2<t<r_2/V \quad (12)$$

(and a similar expression for W_2), with

$$\cosh w_2 = \frac{u\,V'}{\varrho} - \frac{q_2}{\varrho}\sqrt{\frac{V'^2}{V^2}-1},\quad r_2^2 = \varrho^2 + q_2^2$$

At $t = r_2/V$ this movement changes discontinuously into

$$U_2 = E\sin\psi_2\cdot\frac{(-D_2)^2}{r_2V}F'(t-r_2/V)+E\int_{t_2}^{t}F'(t-u)\cdot\frac{\partial}{\partial\varrho}L_2(\varrho,z,u)\,du,\ \text{if } t>r_2/V \quad (13)$$

$(-D_2)^2$ = the real part of $(-D_l)^2$ in which $\xi = -i\sin\psi_2/V$ and $\sin\psi_2 = \varrho/r_2$.

It will be clear that at a time t between t_{n-1} and t_n the movement due to the reflected waves is

$$U = \sum_{m=1}^{n-1}U_m \text{ and } W = \sum_{m=1}^{n-1}W_m$$

where U_m is obtained by changing the exponent and the suffix 2 of (13) into m, while W_m is found by changing in U_m $\partial/\partial\varrho$ into $\partial/\partial z$.

At not too small epicentral distances ($\varrho \gg q_n$) the movement consists of a part which is proportional to r_n^{-1} and a part expressed by the integrals in (11) or (13). With the exception of the Stoneley function, contained in L_1, these integrals are (if $\varrho \gg q_n$) proportional to r_n^{-2}; these disturbances will certainly not be observed. This S function is also proportional to r_1^{-2} provided u is not about equal to r_1/C.

The observed motion consists therefore of a series of shocks $F'(t-r_n/V)$ with amplitudes diminishing with increasing n (as the coefficient of reflection $|D_l| < 1$ and r_n increases as $2nd$). Moreover at a time $t \approx r_1/C$ the "Stoneley wave" arrives; this movement is given by

$$U_s\,(\text{or } W_s) = s\int_{r_1/V}^{t}F'(t-u)\cdot\frac{\partial}{\partial\varrho}\left(\text{or }\frac{\partial}{\partial z}\right)\{B^{-1/2}\sin\ {}^1/_2\,\varphi\}\,du.$$

After some reductions we obtain

$$\left.\begin{aligned}\frac{\partial}{\partial\varrho}\{B^{-1/2}\sin\tfrac{1}{2}\varphi\} &= -B^{-3/2}\cdot\varrho\sin\tfrac{3}{2}\varphi \text{ and}\\ \frac{\partial}{\partial z}\{B^{-1/2}\sin\tfrac{1}{2}\varphi\} &= \pm B^{-3/2}\sqrt{1-C^2/V^2}\,B_1\cos(\tfrac{3}{2}\varphi+\varphi_1)\end{aligned}\right\}. \quad (14)$$

where

$$B_1 = \sqrt{u^2 C^2 + q_1^2(1-C^2/V^2)} \text{ and } \sin\varphi_1 = \frac{q_1(1-C^2/V^2)}{B_1}.$$

In the second expression (14) we have to take the $+(-)$ sign if

$$q_1 = 2d \pm f + z \; (q_1 = 2d \pm f - z).$$

This becomes at $u = \frac{r_1}{C} + \tau$, with $\tau \ll \frac{r_1}{C}$, and $r_1 \gg q_1$:

$$B = 2q_1 r_1\sqrt{1-C^2/V^2}\sqrt{\left(\frac{C\tau}{q_1\sqrt{1-C^2/V^2}}\right)^2+1}, \quad B_1 = r_1$$

$$\sin\varphi = \left\{1+\left(\frac{C\tau}{q_1\sqrt{1-C^2/V^2}}\right)^2\right\}^{-1/2} \quad \text{and} \quad \varphi_1 = 0;$$

hence:

$$\left.\begin{aligned}U_s &= \frac{-s}{2\sqrt{2}(1-C^2/V^2)^{3/4}}\,\frac{E}{q_1^{1/2}\varrho^{1/2}}\int F'(t-u)\cdot\sin\tfrac{3}{2}\varphi\sin^{3/2}\varphi\cdot du\\ W_s &= \frac{\pm s}{2\sqrt{2}(1-C^2/V^2)^{1/4}}\,\frac{E}{q_1^{1/2}\varrho^{1/2}}\int F'(t-u)\cdot\cos\tfrac{3}{2}\varphi\sin^{3/2}\varphi\cdot du\end{aligned}\right\}. \quad (15)$$

with

$$\varphi = \text{arc cotg}\,\frac{C\tau}{q_1\sqrt{1-C^2/V^2}}.$$

The integration (15) has to be carried out in the interval

$$\frac{r_1}{C} - \tau_1 < u < \frac{r_1}{C} + \tau_1$$

where τ_1 is a value of τ at which $\varphi \ll \frac{1}{2}\pi$. The result being of course proportional to $\varrho^{-1/2}$ the possibility exists that the Stoneley movement is observed as a major disturbance.

We finish this paragraph with some remarks relating to the interpretation of the obtained results. The indication "n-th reflected wave" implies the common interpretation of the disturbance which arrives at $t = r_n/V$, namely a wave which has travelled n times up and down between $z = 0$ and $z = d$. It is perhaps not superfluous to emphasize that this interpretation is nothing more than a convenient way to describe this movement and that it gives not an exact description of the path travelled by this disturbance. There is only one disturbance at ϱ, z, caused by the subjacent medium, of which the way along which it has been propagated, is known:

the beginning of the first reflected wave. This disturbance has travelled in the shortest possible time (t_1) from the hearth towards the interface and back to the point ϱ, z. At any time $t > t_1$ disturbances which have travelled in every possible way, with velocities V, V' or $\mathfrak{V}'$ and with a travel time equal to t, reach ϱ, z; for instance a disturbance propagated vertically downwards and afterwards travelling horizontally at a depth $z > d$ through the underlying medium, while reaching ϱ, z (in the ocean) in some oblique direction may very well contribute to say the second reflected wave ($t = r_2/V$).

The Stoneley (or Rayleigh) movement is therefore not a phenomenon which happens only at the first reflection ($n = 1$) at $z = d$ — this would be very remarkable. Properly speaking there exists no first reflection at all; the movement is only a part of the disturbance caused by the second medium.

§ 5. *Surface waves in the ocean.*

In seismology a surface wave is usually understood as a motion which decreases with increasing epicentral distance as $\varrho^{-1/2}$, changes exponentially with the depth and is large in comparison to the r^{-1} shocks. Such a movement however is never observed during an earthquake. CAGNIARD proved in 1939 that the Rayleigh (or Stoneley) wave although proportional to $\varrho^{-1/2}$ decreases not exponentially with z but as $q^{-3/2}$; this author examined the other properties of this movement very thoroughly especially with regard to the appearance of vibrations caused by a single impulse (the method of § 4 is essentially the same as that used by CAGNIARD).

Moreover it is by no means true that a $\varrho^{-1/2}$ wave forms in any circumstances the major part of the disturbance; to demonstrate this we use some numerical values:

$$V' = 3\,V,\ V' = \mathfrak{V}'\sqrt{3}\ (\text{or } \lambda' = \mu') \text{ and } \sigma' = 3\,\sigma.$$

The Stoneley velocity is then $C = 0{,}9984$ V; it follows $s = 0{,}2231$ and $\sqrt{1 - C^2/V^2} = 0{,}0553$. The motion is:

$$U_s = \frac{6{,}065}{q^{3/2}\varrho^{1/2}} E \int F'(t-u) \cdot \sin \tfrac{3}{2}\varphi \sin^{3/2}\varphi \cdot du$$

$$W_s = \pm \frac{0{,}3354}{q^{3/2}\varrho^{1/2}} E \int F'(t-u) \cdot \cos \tfrac{3}{2}\varphi \sin^{3/2}\varphi \cdot du$$

$$\text{with } u = \frac{\varrho}{C} + 0{,}0553\,\frac{q_1}{C}\,\text{cotg}\,\varphi.$$

The functions $\sin \frac{3}{2}\varphi \sin^{3/2}\varphi$ and $\cos \frac{3}{2}\varphi \sin^{3/2}\varphi$ differ sensibly from zero in the interval $|\text{cotg}\,\varphi| \lessapprox 4$; we have therefore to integrate for values of $u = \frac{\varrho}{C} \pm \tau_1$ where $\tau_1 \approx 0{,}25\frac{q_1}{C}$. As $q_1 = 2d - f - z$ this becomes for the movement of the free surface, if we suppose that $f \ll d$:

$$\tau_1 \approx \frac{d}{2V}.$$

The surface wave caused by a disturbance with duration τ_0 is observed at ϱ, 0 during the time

$$\text{beginning at } \frac{\varrho}{C} - \frac{d}{2V} \text{ and ending at } \frac{\varrho}{C} + \frac{d}{2V} + \tau_0.$$

Now suppose that $\tau_0 \ll d/V$; the integration has then only to be carried out in the small interval τ_0 and the result decreases with decreasing τ_0. The Stoneley wave caused by a very short explosion is therefore not large in comparison with r^{-1} shocks and will if τ_0 is small enough, not be observed.

In the opposite case where $\tau_0 \gg d/V$ the Stoneley movement caused by a single impulse — for instance $F(t) = \sin^2 \pi t/\tau_0$ — is considerably larger than the r^{-1} shocks. In this case $F'(t-u)$ changes slowly in the interval $u = \varrho/C \pm \tau$ and we obtain in first approximation

$$\int F'(t-u) \cdot \sin \tfrac{3}{2} \varphi \sin^{3/2} \varphi \cdot du \approx F'(t-\varrho/C) \int \sin \tfrac{3}{2} \varphi \sin^{3/2} \varphi \, du.$$

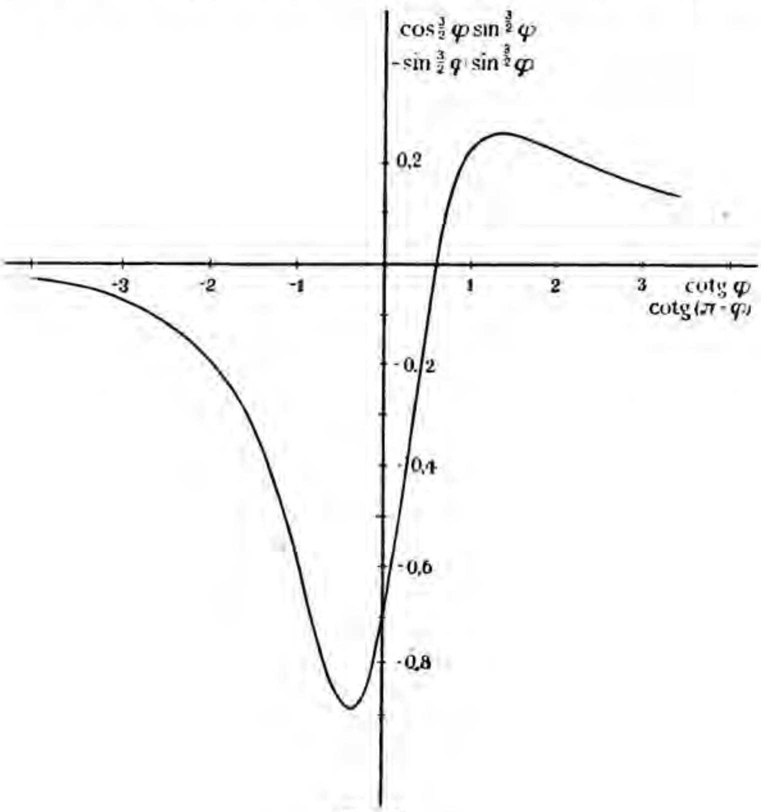

Fig. 5.

An elementary calculation, based on the graphic representation of $\sin \frac{3}{2} \varphi \sin^{3/2} \varphi$ (Fig. 5, which is identical to Fig. 54 in CAGNIARD's work), shows that the integral is about equal to $\dfrac{1}{10}\dfrac{d}{V}$. (This is also valid for the function $\cos\frac{3}{2}\varphi \sin^{5/2}\varphi$, which changes into $\sin\frac{3}{2}\varphi \sin^{5/2}\varphi$ by changing φ into $\pi - \varphi$.) The motion is therefore about:

$$U_s \approx \frac{0{,}2\,E}{\varrho^{1/2}\,d^{1/2}\,V}\,F'(t-\varrho/C) \text{ and } W_s \approx \frac{0{,}01\,E}{\varrho^{1/2}\,d^{1/2}\,V}\,F'(t-\varrho/C).$$

Comparing this with the direct wave, which is at large values of ϱ:

$$U \approx \frac{E}{\varrho V}\,F'(t-\varrho/C) \text{ and } W \approx 0$$

we get

$$\frac{U_s}{U} \approx \sqrt{\frac{\varrho}{25\,d}}.$$

Assuming the ocean to be 4 km deep this ratio is equal to 3 at 900 km and equal to 10 at 10^4 km distance from the hearth.

For values of $\tau_0 \approx d/V$ the Stoneley movement can be calculated by numerical integration; its maximum value is of the same order as in the case $\tau_0 \gg d/V$. A more detailed investigation lies outside the scope of the present paper.

Apart from this major movement other surface waves can be formed by interference of several reflected waves. As the coefficient of reflection at large values of ϱ is about equal to 1 (contrary to D_t in the case of purely transverse waves, which is ≈ -1) a large movement is built up by these waves if they reach the point $\varrho, 0$ with a time difference $< \tau_0$. The difference in travel time between the direct and the n-th reflected wave is

$$t_n - \frac{r}{V} \approx \frac{2\,n^2\,d^2}{\varrho V}.$$

Consequently a group of

$$n = \sqrt{\frac{\varrho\,\tau_0\,V}{2\,d^2}}$$

waves with amplitudes proportional to ϱ^{-1} interferes, resulting in a $\varrho^{-1/2}$ movement. Supposing $d = 4$ km and $V = 1{,}5$ km/sec. the number of interfering shocks at a distance of 4000 km is about $13{,}7\ \sqrt{\tau_0}$; the maximum value is therefore about $10\ \sqrt{\tau_0}$ times larger then the movement caused by a single ϱ^{-1} shock.

The occurrence of both kinds of surface waves appears only to be possible if the duration of the primary impulse is not to small ($\approx$ 1 sec.). The observations of the seismic waves generated by the very short subaqueous explosions at Bikini confirm this result as no surface waves were recorded. Observations at distances of 100—350 km of the Helgoland explosion (April 18, 1947) are also in accordance with this theory; only small waves were registered.

Note on the theory of CAGNIARD.

CAGNIARD states (page 233) that a Rayleigh (Stoneley) wave cannot appear in a solid body covered by a liquid; in view of the fact that a large part of the earth is covered by the oceans this statement is rather important. Moreover regarding the terrestrial atmosphere as a liquid ($\mu = 0$) this also applies to the propagation of seismic waves on the continents; consequently there exists, according to CAGNIARD, no Rayleigh wave at all during any seismic disturbance. However as σ/σ' is in this case very small ($\approx 5 \cdot 10^{-4}$) CAGNIARD obtains two complex roots of the equation

$$a\,(c'^2 - a'\,b'\,\xi^2) + \frac{\sigma\,a'}{4\,\sigma'\,\mathfrak{V}'^4} = 0. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

by an approximative calculation; these roots appear to be

$$\xi_0 = \frac{\pm i - 7{,}5 \cdot 10^{-6}}{\mathfrak{V}'_R}$$

where $\mathfrak{V}'_R$ = the ordinary Rayleigh velocity ($= 0{,}9194\,\mathfrak{V}'$ if $V' = \mathfrak{V}'\sqrt{3}$).

These two roots determine a movement which is at not too large epicentral distances almost identical to a Rayleigh wave (CAGNIARD § 101).

As these results are in contradiction to the theory developed in this chapter we re-examine equation (16).

1. For large (absolute) values of ξ the left-hand side of (16) is

$$L \approx \frac{\xi^3}{i}\left(\frac{1}{\mathfrak{V}'^2} - \frac{1}{V'^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

hence L is at $\xi = +\,i\,\infty$ negative imaginary.

The smallest of the three velocities V, V' and $\mathfrak{V}'$ is either V or $\mathfrak{V}'$:

if $\mathfrak{V}' > V$ L is positive imaginary at $\xi = i/V$, ($a = 0$)

and if $\mathfrak{V}' < V$ L is positive imaginary at $\xi = i/\mathfrak{V}'_R$, ($c'^2 - a'b'\xi^2 = 0$).

It follows that (16) admits two purely imaginary roots $\pm\, i/C$, where $C < V$ when $V < \mathfrak{V}'$ and $C < \mathfrak{V}'_R$ when $\mathfrak{V}' < V'$.

2. Equation (16) yields no other roots than these two; this may be shown by a method which is also used by CAGNIARD (§ 19): suppose that ξ describes a circle in the positive direction with a very large radius and its centre at $\xi = 0$, and also but in the opposite direction a curve infinitely close to the line connecting the branch points of L. The number of times that the image of L turns around the origin in the positive direction is equal to the number of roots of $L = 0$.

When ξ describes the circle L travels according to (17) three times in the same direction along this circle. The number of roots is certainly even as L contains only ξ^2; three roots being impossible the equation has only the two roots already obtained. (This can be confirmed by the investigation of the curve described by L when ξ travels along the branch points; it appears that the image of L turns once around $\xi = 0$ in the negative direction.)

3. With decreasing value of σ (λ remaining constant) V and C increase; when V becomes larger then $\mathfrak{V}'$ the Stoneley velocity C approaches the Rayleigh velocity $\mathfrak{V}'_R$ and is equal to $\mathfrak{V}'_R$ if $V = \infty$ ($\sigma = 0$). Using the numerical values $V = 0{,}340$ km/sec., $V' = \mathfrak{V}' \sqrt{3}$, $\mathfrak{V}' = 10$ V and $\sigma/\sigma' = 5 \cdot 10^{-4}$ we obtain $\lambda/\lambda' = 5 \cdot 10^{-6}$; therefore $V = \mathfrak{V}'$ if $\sigma/\sigma' = 5 \cdot 10^{-6}$. At smaller values of σ/σ' C approaches $\mathfrak{V}'_R$ as a limit.

The two values ξ_0 calculated by CAGNIARD are no roots of (16): at $\xi = \xi_0$ L is very small in absolute value but not equal to zero. The roots of $L = 0$ are always purely imaginary and are about equal to $\pm i/\mathfrak{V}'_R$ when σ/σ' is much smaller then the existing ratio of $5 \cdot 10^{-4}$.

4. The result is that, because of equation (16), a Rayleigh wave ($C \approx \mathfrak{V}'_R$) cannot occur in a disturbance propagated through a continental layer. It may be difficult to ascertain wether this is in agreement with the seismic observations as it is not impossible that this wave coincides with surface waves caused by superposition. The theoretical investigation of this occurrence involves the examination of the propagation of a disturbance in an inhomogeneous body (chapter III).

However it is far from certain that (16) is applicable to the system: solid body covered by air. The appearance of the term with σ/σ' in (16) is due to the second boundary equation at $z = d$ (§ 1) which requires the continuity of the normal tension. This condition has certainly to be fulfilled if the two media are welded together or — as it happens in the interior of the earth — are firmly pressed together. This is also true at the bottom of the sea, but it is to be doubted if this condition exists at the upper surface of the earth during a seismic disturbance. The contact between a gas and a solid body is different from that between the ocean and sub-oceanic rock under a pressure of some hundreds of atmospheres. It is perhaps more reasonable to suppose that the upper surface of the earth is completely free in which case the ordinary Rayleigh equation obtains.

(To be continued.)

L. CAGNIARD: Réflexion et réfraction des ondes séismiques progressives. (Paris, 1939.)

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Physics. — *Superquantization.* I. By H. J. GROENEWOLD. (Koninklijk Nederlands Meteorologisch Instituut te De Bilt.) (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of September 25, 1948.)

1. Introduction.

1.1 *Quantization processes.* We begin with shortly recalling the general situation in the current theories of quantization.

1.11 *Historical situation.* 1.111 *Classical theory.* In classical theory, which is the starting point, one has particles (e.g. electrons) and fields (e.g. the electromagnetic field). If necessary we shall distinguish classical concepts by a prefix c and e.g. write c-particles and c-fields. c-particles and c-fields may interact with each other: c-particles generate c-fields; c-fields act upon c-particles. In the usual dualistic theory c-particles and c-fields are quantized separately in the following steps.

1.112 *Particle quantization.* The classical quantities belonging to c-particles are replaced by quantum operators. The latter act on wave functions, which represent the states of the c-particles. If there is a fixed number n of c-particles, the wave functions depend on n complete sets of coordinates and are called n-particle wave functions. The classical equations of motion are replaced by wave equations (SCHRÖDINGER representation) or by operator equations (HEISENBERG representation). The wave functions can be considered to describe quantum fields (q-fields) related to the c-particles.

1.113 *Field quantization.* c-fields are quantized in a somewhat analogous way. The field quantities are replaced by field operators. The field equations are replaced by operator equations. The quantized c-fields show discrete properties, which can be interpreted from a particle point of view. They can be considered to describe quantum particles (q-particles) related to the c-fields (e.g. photons related to the electromagnetic field). The field operators can be expressed in terms of creation and annihilation operators, which describe creation (emission) and annihilation (absorption) of q-particles.

1.114 *Superquantization.* The 1-particle wave functions of ordinarily quantized c-particles can in their turn be superquantized by a process, which is almost entirely analogous with the quantization of c-fields. The 1-particle wave functions are replaced by wave operators, the wave equations by superquantized wave equations. Also this form of field quantization leads to the aspect of q-particles. The wave operators can again be expressed in terms of creation and annihilation operators, which describe creation and annihilation of q-particles.

1.12 *Logical situation.* 1.121 *Particle quantization.* Historically ordinary particle quantization makes the progressive step from the less correct classical theory towards the relatively more correct ordinary quantum theory. As soon as the latter has entirely been established, the quantization process has performed its pioneering duty. This historical aspect of the relation between classical and quantum theory has to be sharply distinguished from logical problems on the inner structure of ordinary quantum theory and its relations to observation. In this paper we shall not make such problems.

1.122 *Superquantization.* With regard to 1-particle wave functions the process of superquantization, which introduces the aspect of many q-particles, undoubtedly makes a progressive step. It is, however, perhaps not always clearly apprehended that superquantization does not lead one step beyond the complete ordinary quantum theory of c-particles with many-particle wave functions. In fact superquantization can be entirely understood within the scope of the latter theory. This paper only contains news for those, who are still surprised by this point of view.

We shall assume the reader to be familiar with the usual theories of superquantization. As it has been said before, these theories introduce wave operators and creation and annihilation operators, which describe creation and annihilation of q-particles. In our line of reasoning we shall in the ordinary quantum theory with many-particle wave functions of c-particles introduce creation and annihilation operators, which act on the many-particle wave functions and which describe creation and annihilation of c-particles, and wave operators. This introduction is not a matter of a new assumption, which leads beyond the ordinary quantum theory, it is merely a matter of definitions within this theory. Our aim will be to chose these definitions in such a way, that our creation and annihilation operators and wave operators become isomorphic to those of superquantization. In distinction with the latter process we shall denote our process as "superquantization".

If we succeed to establish the isomorphy, the two descriptions are entirely equivalent. The one does not go beyond the other. A further consequence of the isomorphy between the creation and annihilation operators of c-particles and those of q-particles is that there is no logical distinction between c-particles and q-particles. Therefore the two concepts can be identified and the prefix can be omitted.

In introducing the wave operators we shall not immediately make the right choice, which gives the required isomorphy with superquantization. We shall begin with a preliminary choice, first in statical, thereafter in dynamical representation. Later on comparison with the present theories of superquantization will show how the choice still has to be modified. Historically the introduction of wave operators and every step in our derivation is entirely guided by superquantization. On logical grounds, however, the latter would deserve to be put between the humiliating quotation marks, which we have reserved for our process, just as well.

1.123 *Field quantization.* It is beyond doubt that historically field quantization is a progressive step. As soon as the quantization has been established we may with lack of historical reverence come to a logically more coherent picture by the following reinterpretation. Though originally we had no c-particles, the q-particles related to the c-field are identified with c-particles and the prefix is omitted. The creation and annihilation operators act on many-particle wave functions. The latter describe the q-field related to the particles. From this point of view the situation after field quantization is entirely similar to the situation after particle quantization. c-fields must then be considered as similar to 1-particle wave functions (appropriately normalized), field quantization as similar to superquantization.

1.124 *Particles and fields.* It may be useful to state precisely what we now mean with particles and fields. Our fields are described by ordinary wave functions depending on complete sets of coordinates. Every complete set represents one particle (for indiscernible particles there is no individual one-to-one correspondence, only an assembly of n sets corresponds to an assembly of n particles).

1.2 *Source particles and carrier particles.* 1.21 *Dualistic theory.* In classical theory there are particles and fields (in the sense of 1.111, not of 1.124). Though they may interact with each other, their equations of motion are treated separately. Therefore the theory is dualistic. After quantization, which is also done separately, we get in both cases wave functions related to particles. That does not mean, however, that from our point of view quantum theory is unitary. For one obtains different kinds of particles. They may interact with each other, but their wave equations are still treated separately. Therefore the theory is still dualistic.

1.211 *Interaction.* In classical theory a field generated at a certain place and time by one particle may at another place and time act on another particle (e.g. an electromagnetic field generated by and acting on electrons). In this way by intervention of the field the particles indirectly interact with each other at different places and times. After quantization we get two kinds of particles. A particle of the first kind in a certain region of space-time emits a particle of the second kind, which in another region of space-time is absorbed by another particle of the first kind (e.g. a photon emitted and absorbed by electrons). In this way by intervention of the particles of the second kind the particles of the first kind interact with each other in different regions of space-time. Because of this behaviour we shall call particles of the first kind source particles and particles of the second kind carrier particles. If particles of a certain kind interact with particles of various other kinds, it may occur that in some interactions they behave as source particles and in other interactions as carrier particles (e.g.

charged mesons interacting with photons act as source particles, interacting with nucleons they act as carrier particles). Therefore the classification is not always unique.

1.212 *Observability.* The ordinary particle operators in general represent observables (particle observables). For some kinds of particles also the wave operators (if they are of commutator type) represent observables (field observables), for other kinds (if they are of anticommutator type) they do not. Whereas particles of both kinds can in classical theory be approximated by means of c-particles, only particles of the first kind can in classical theory be approximated by means of c-fields. The latter approximation will in general be used if they act as carrier particles, the former if they act as source particles.

1.22 *Unitary theories.* There are two ideals of unitary theories. They are opposite to each other, though a priori they need not exclude each other.

1.221 *Unitary f-theory.* Classical unitary field theory considers particles as singularities of the fields. The equations of motion and all other properties of the particles are entirely determined by those of the fields. Also the quantization of particles is determined by the quantization of fields. A complete theory has not been given. It might be difficult to fit such a theory in our scheme in which particle quantization and not field quantization is treated as the important step.

1.222 *Unitary p-theory.* Classical unitary particle theory starts with nothing else than particles, which have retarded and/or advanced interaction at a distance. The description of this interaction may be extremely complicated. Not before all equations of motion have been established, fields may be introduced in order to simplify the description. The equations of motion and all other properties of the fields are entirely determined by those of the particles and/or by the way in which the fields are introduced. Also the quantization of fields is entirely determined by the quantization of interacting particles. Or: quantum unitary p-theory starts with nothing else than source particles, which have retarded and/or advanced interaction at a distance. Not before all wave equations have been established, carrier particles may be introduced. The wave equations and all other properties of the carrier particles are entirely determined by those of the source particles and/or the way in which the carrier particles are introduced. Also here a complete theory has not been given. Such a theory would readily fit into our scheme and greatly simplify its principles.

In a unitary p-theory field observations could be entirely reduced to particle observations.

1.3 *Plus and minus troubles.* It is obvious that our whole picture badly suffers from oversimplification. Among all more or less concealed complications there is one group of difficulties, which during our derivations will become too apparent to be entirely ignored. They are about positive and negative states and positive and negative particles. We shall have to deal with them in some extent, even though we shall not bother about the malignant divergencies with which they are connected.

1.31 *Positive and negative states.* 1.311 *Energy states.* In all cases we shall meet positive and negative energy states. There are two kinds of difficulties with negative states:

d_1 how to distinguish them properly from positive states;

d_2 how to get rid of them or to give them a proper physical meaning.

1.312 *Density states.* For some kinds of particles (those for which the wave equations contain second order time derivatives) one gets also saddled with negative density states. They entail the same two kinds of difficulties as negative energy states. The negative density states combine with the negative energy states in such a way, that it sometimes nearly looks as if the difficulties cancel.

1.32 *Positive and negative particles.* The creation or annihilation of pairs of opposite particles should be treated with due regard to the interaction between the two particles. If one tries to do so, one gets entangled with divergencies. These and other difficulties have not been solved, they have only been circumnavigated by means of an ingeneous trick. This trick (the hole theory) is to neglect the interaction and to put on a par creation or annihilation of a particle in a given energy state and annihilation or creation of an opposite particle in the opposite energy state. This trick has either openly or tacitly been performed in nearly all current theories. If we want to establish isomorphy with these theories, we are also bound to perform the trick. That truly makes a step beyond the original ordinary theory. It should be emphasized, however, that this step is entirely distinct from processes of quantization. And further it should not be forgotten that it is a tentative capriole rather than a firm step, which leads out of the difficulties.

1.4 *Typical processes and queries.* The following processes P_1, P_2 and P_3 and queries Q_1, Q_2 and Q_3 are loosely in parallel with each other and with the foregoing sections 1.1, 1.2 and 1.3.

1.41 *Processes.* Creation and annihilation operators serve to describe the elementary processes of creation and annihilation of particles. There are some typical compound processes, which can be described by pairs of such operators and which deserve our special attention:

P_1 transition of a particle from one state to another state;

P_2 birth and death of a carrier particle;

P_3 birth or death of a pair of opposite particles.

P_1 can be conceived so that a particle is annihilated in one state and a particle of the same kind is created in another state. In P_2 the carrier particle is first created somewhere and later annihilated elsewhere. The trick of the hole theory tries to reduce the pair creation or annihilation in P_3 to a transition of the type P_1 between a positive and a negative energy state.

1.42 *Queries*. Some typical problems in which we are interested are:

Q_1 does superquantization lead beyond ordinary quantum theory of particles?

Q_2 how are the relations between the properties of carrier particles and those of source particles; can in particular the former be derived from the latter?

Q_3 how shall negative states and pair processes be dealt with?

We shall be concerned with Q_1 and in the meantime leave Q_2 aside. We cannot do so with Q_3, which interferes so strongly with Q_1, that it has to be considered to some extent.

1.5 *Present theories*. There are two types of present theories based on superquantization [1]) with which we shall try to establish isomorphy. Of the one type [2]) DIRAC's hole theory [3]) is typical for half-odd spin and PAULI–WEISSKOPF's theory [4]) for integer spin. The other type is DIRAC's 1942 theory [5]) [6]). We shall find that the latter type is entirely equivalent with the ordinary quantum theory of particles. The theories of the first type can only be obtained after performing the trick of hole theory. From this point of view they are all hole theories. We shall not discuss the trick. That entirely belongs to the problems of Q_3.

It needs hardly to be repeated that in all this there is nothing new. This paper does not claim any originality.

2. "Superquantization".

In this section we introduce in the ordinary quantum theory of particles preliminary wave operators, which already strongly resemble those of the present theories.

2.1 *Wave functions*. Before doing so we have first to deal with the wave functions.

2.11 *Notation*. Our notation will appear rather cumbersome. In fact a careful detailed notation is essential for a rigorous discussion. The usual notations are not sufficient for this special purpose.

2.12 *Wave equations*. A system of n particles can in ordinary quantum theory be described by a many-times theory, in which there is a separate time coordinate for each particle, and in some cases also by a single-time theory, in which there is only one time coordinate for the whole system. Let (x_k) stand for the complete set of all individual variables of the kth particle except for the individual time t_k or the common time t. $(x_1t_1, \dots x_nt_n \mid \Psi$ or $(x_1, \dots x_n; t \mid \Psi$ will stand for the wave function, $\Psi^\dagger \mid x_nt_n, \dots x_1t_1)$ or $\Psi^\dagger \mid t; x_n, \dots x_1)$ for its HERMITIAN adjoint. Before a Ψ an operator in the (x_k) and t_k or t will act to the right, after a $\Psi^\dagger$ it will act to the left (between a $\Psi^\dagger$ and a Ψ it does in general not matter which way).

For a system of identical particles the wave equations have the form

$$\mathbf{K}\{x_k t_k\}(x_1 t_1, \dots x_n t_n \mid \Psi = 0 \qquad (k = 1, \dots n)\,. \quad . \quad . \quad . \quad (1)$$

in many-times theory, whereas in single-time theory there is a single wave equation

$$\mathbf{K}\{x_1, \ldots x_n; t\}(x_1, \ldots x_n; t)\,\Psi = 0 \quad \ldots \ldots \quad (1')$$

The HERMITIAN operators $\mathbf{K}$ may depend on and act on the variables mentioned in curled brackets and also on variables of particles of different kinds with which those of the kind considered interact. $\mathbf{K}\{x_1, \ldots x_n; t\}$ is symmetrical in the (x_k).

Even in intrinsical relativistic invariant theories the explicit invariance may be more or less obscured by the particular part of the time coordinates. A striking case is a single-time theory derived from a relativistic invariant many-times theory.

Where in the following no special reference is made to expressions for single-time theory, they will (as far as they exist) be supposed to be similar to those of many-times theory with all t_k $(k = 1, \ldots n)$ replaced by t.

2.13 *Permutations.* The permutations $\mathbf{P}$ of all sets $(x_1 t_1), \ldots (x_n t_n)$ or $(x_1), \ldots (x_n)$ form the symmetrical group S_n. Every even or odd permutation $\mathbf{P}_{even}$ or $\mathbf{P}_{odd}$ can be regarded as the product of respectively an even or odd number of interchanges of pairs. In a 1-dimensional representation of S_n all pair interchanges must have the same representative, either -1 (F–D representation) or $+1$ (B–E representation). The representative $\delta_\mathbf{P}$ of a permutation $\mathbf{P}$ is then given by

	F–D	B–E
$\mathbf{P}_{even}$	$+1$	$+1$
$\mathbf{P}_{odd}$	-1	$+1$

. . . . (2)

2.14 *Statistics.* The equations (1) permute if we commute all $\mathbf{K}$ with a $\mathbf{P}$. In (1′) $\mathbf{K}$ commutes with all $\mathbf{P}$. Therefore all $\mathbf{P}$ are integrals of motion for (1) as well as for (1′).

Because identical particles are indiscernible, the wave functions Ψ must provide a 1-dimensional representation of S_n

$$\mathbf{P}\,\Psi = \delta_\mathbf{P}\,\Psi \quad \ldots \ldots \ldots \quad (3)$$

If this is a F–D representation, all Ψ are antisymmetrical (FERMI–DIRAC statistics), if it is a B–E representation, all Ψ are symmetrical (BOSE–EINSTEIN statistics).

For a given representation the symmetry operator $\mathbf{S}_n$ will be defined by

$$\mathbf{S}_n = \frac{1}{n!} \sum_\mathbf{P} \delta_\mathbf{P}\,\mathbf{P}, \quad \ldots \ldots \ldots \quad (4)$$

where the sum is over all $n!$ permutations. $\mathbf{S}_n$ is HERMITIAN and idempotent $(\mathbf{S}_n^2 = \mathbf{S}_n)$, the eigenvalues are 1 and 0. The eigenfunctions belonging to 1 are of the required symmetry (either antisymmetrical or symmetrical), those belonging to 0 must be rejected. $\mathbf{S}_n$ is a projection operator selecting the (anti-)symmetrical component of Ψ. As soon as all Ψ have the required symmetry, $\mathbf{S}_n$ can be replaced by the eigenvalue 1.

2.15 *Density operator.* For particles of a given kind we have HERMITIAN density operators $\varrho\{x_k t_k\}$ (commuting for different k). In a relativistic theory they are the time components of the 4-velocity operators. In some cases (e.g. if (1) is a second order differential equation in the time coordinates i.e. for integral spin) the density operators are indefinite. Then we have to distinguish between positive and negative density states. That is the first difficulty d_1. We try to form operators $\eta\{x_k t_k\}$ (also commuting for different k), which satisfy the conditions

- c_1 η is HERMITIAN $\eta = \eta^\dagger$ and unitary $\eta^2 = 1$;
- c_2 η commutes with ϱ;
- c_3 $\eta\varrho$ (which is HERMITIAN by c_1 and c_2) is positive definite;
- c_4 η is uniquely determined and in a relativistic invariant theory it is invariant.

Because of c_1 the eigenvalues of η are $+1$ and -1. c_2 and c_3 can be met by taking for η the operator with the same eigenstates and the same sign of eigenvalues as ϱ. c_4 is liable to give trouble. In fact it may be too stringent. If η can be found, the eigenstates with positive eigenvalue will determine the positive density states, those with negative eigenvalue the negative density states. If ϱ is positive definite, then $\eta = 1$ and all states are positive.

2.16 *Inner product.* The density operators determine the metric in the HILBERT space of wave functions. In case of an indefinite density operator this metric would also become indefinite. That is the second difficulty d_2. We pass it off in a rather primitive way. If η has been found, we make use of c_3 and take as the inner product of $\Psi^\dagger$ and Ψ'

$$\left.\begin{aligned}\int\ldots\int(dx_1)\ldots(dx_n)\,\Psi^\dagger\,|\,x_n t_n,\ldots x_1 t_1)\,\eta\,\{x_1 t_1\}\,\varrho\,\{x_1 t_1\}\ldots\eta\,\{x_n t_n\}\,\varrho\,\{x_n t_n\}\\ (x_1 t_1,\ldots x_n t_n\,|\,\Psi'.\end{aligned}\right\} \quad (5)$$

Thus by intercalating the factors η we have obtained a positive definite metric. $\int(dx)$ means integration over all continuous variables and summation over all discrete variables of the set (x). If the wave functions are spinor quantities we get an inner spinor product, if they are 4-tensor quantities we get an inner 4-tensor product. The density operators are subjected to the condition that if Ψ and Ψ' satisfy the wave equations (1) or (1′), the inner product (5) or (5′) has to be an integral of motion. In a relativistic theory more generally the 4-divergence of the 4-velocity has to vanish.

It cannot be said that in 2.15 and 2.16 the difficulties d_1 and d_2 with negative density states have been solved in an elegant or even in a satisfactory way.

2.17 *(x) and (μ) representation.* We shall have to consider transformations between various representations.

Choose a complete system of orthonormal individual wave functions $(xt\,|\,\psi\,|\,\mu)$

$$\left.\begin{aligned}&\int(dx)(\mu\,|\psi^{\dagger}|\,xt)\,\eta\{xt\}\,\varrho\{xt\}\,(xt\,|\,\psi\,|\,\mu')=\delta_{\mu\mu'};\\&\int(dx')\,\{\sum_{(\mu)}(xt\,|\,\psi\,|\,\mu)\,(\mu\,|\psi^{\dagger}|\,x't)\}\,\eta\{x't\}\,\varrho\{x't\}(x't\,|\,\psi=(xt\,|\,\psi.\\&\int(dx)\,\psi^{\dagger}\,|xt)\,\eta\{xt\}\,\varrho\{xt\}\,\{\sum_{(\mu)}(xt\,|\,\psi\,|\,\mu)(\mu\,|\,\psi^{\dagger}\,|\,x'\,t)\}=\psi^{\dagger}\,|x't).\end{aligned}\right\}\;. \tag{6}$$

$\sum\limits_{(\mu)}$ means summation over all discrete parameters and integration over all continuous parameters of the set (μ). The first equation, which is concerned with the inner product of the wave functions, expresses the orthonormality. The second and third equation are HERMITIAN adjoint to each other. They are concerned with the outer product of the wave functions and express the completeness.

For the product representation we need the complete system of orthonormal (anti-)symmetrized products

$$(x_1t_1,\ldots x_nt_n\,|\,\Psi\,|\,\mu_n,\ldots\mu_1)=\left(\frac{n!}{\prod\limits_{(\nu)} z_\nu!}\right)^{1/2}\mathbf{S}_n\,(x_1t_1\,|\,\psi\,|\,\mu_1)\ldots(x_nt_n\,|\,\psi\,|\,\mu_n)\,. \tag{7}$$

z_ν is the number of (μ_k), which are equal to (ν), i.e. the occupation number of the state (ν). For FERMI–DIRAC statistics all terms with $z_\nu > 1$ cancel (PAULI's exclusion principle), so that always $z_\nu! = 1$. In order that the wave functions (7) shall be complete in single-time theory, it is necessary that $\eta\{x_kt\}\,\varrho\{x_kt\}$ commutes with $(x_lt\,|\,\psi\,|\,\mu)$ for $l \neq k$, so that it may not contain differentiation with respect to t. In those cases (e.g. if (1) is a second order differential equation in the time coordinates i.e. in case of integral spin) for which this condition cannot be satisfied a description in single-time theory is excluded.

(7) may serve as the kernel of the transformation between (x) and (μ) representation. The two representations are not entirely similar, because they stand on a different footing with symmetry. In the following it is in general sufficient to remember that the wave functions can be linearly expressed in the products (7). A throughout performance of the transformation would, however, be rather illustrative.

2.2 *Wave operators; statical representation.* Now we shall introduce creation and annihilation operators and preliminary wave operators, first in a statical representation.

2.21 *Creation and annihilation operators.* We define the operators $(\mu\,|\,\mathbf{a}^{\dagger}\,|\,xt\}$ and $\{xt\,|\,\mathbf{a}\,|\,\mu)$ by requiring that for arbitrary n the operators $(\mu\,|\,\mathbf{a}^{\dagger}\,|\,xt\}z_\mu^{-1/2}$ and $z_\mu^{-1/2}\{xt\,|\,\mathbf{a}\,|\,\mu)$ respectively take a factor $(x_nt_n\,|\,\psi\,|\,\mu)$ out of $(x_1t_1,\ldots x_nt_n\,|\,\Psi\,|\,\mu_n,\ldots\mu_1)$ or insert a factor $(x_nt_n\,|\,\psi\,|\,\mu)$ into $(x_1t_1,\ldots x_{n-1}t_{n-1}\,|\,\Psi\,|\,\mu_{n-1},\ldots\mu_1)$ and further restore the (anti-)symmetry and normalization. At this stage the separation of the factor $z_\mu^{-1/2}$ is still rather artificial. It is obvious that this choice has already been made with the purpose of later establishing the required isomorphy. $\mathbf{a}^\dagger$ and $\mathbf{a}$ are now given by

$$\left.\begin{aligned}&(\mu|\mathbf{a}^\dagger|xt\}(x_1t_1,\ldots x_nt_n|\Psi=\\ &\qquad=n^{1/2}\int(dx_n)\,(\mu|\psi^\dagger|x_nt_n)\,\eta\{x_nt_n\}\,\varrho\{x_nt_n\}\,\mathbf{S}_n(x_1t_1,\ldots x_nt_n|\Psi,\\ &\{xt|\mathbf{a}|\mu)\,(x_1t_1,\ldots x_{n-1}t_{n-1}|\Psi=n^{1/2}\,\mathbf{S}_n(x_nt_n|\psi|\mu)(x_1t_1,\ldots x_{n-1}t_{n-1}|\Psi.\end{aligned}\right\}. \quad (8)$$

Acting to the left they give

$$\left.\begin{aligned}&\Psi^\dagger|x_{n-1}t_{n-1},\ldots x_1t_1)(\mu|\mathbf{a}^\dagger|xt\}=\Psi^\dagger|x_{n-1}t_{n-1},\ldots x_1t_1)(\mu|\psi^\dagger|x_nt_n)\,\mathbf{S}_n\,n^{1/2},\\ &\Psi^\dagger|x_nt_n,\ldots x_1t_1)\,\{xt|\mathbf{a}|\mu)=\\ &\qquad=\int dx_n)\,\Psi^\dagger|x_nt_n,\ldots x_1t_1)\,\mathbf{S}_n\,\eta\{x_nt_n\}\,\varrho\{x_nt_n\}\,(x_nt_n|\psi|\mu)\,n^{1/2}.\end{aligned}\right\}. \quad (8^\dagger)$$

So they are HERMITIAN adjoint to each other. If acting to the right, $\{xt\,|\,\mathbf{a}\,|\,\mu)$ will be denoted as the creation operator, $(\mu\,|\,\mathbf{a}^\dagger\,|\,xt\}$ as the annihilation operator belonging to the state (μ). If acting to the left, they reverse their roles. They satisfy the commutation relations

$$\left.\begin{aligned}&[(\mu|\mathbf{a}^\dagger|xt\}\,,\,\{xt|\mathbf{a}|\mu')]^\pm=\delta_{\mu\mu'}\,\mathbf{S};\\ &[\{xt|\mathbf{a}|\mu)\,,\,\{xt|\mathbf{a}|\mu')]^\pm=0\quad,\quad[(\mu|\mathbf{a}^\dagger|xt\}\,,\,(\mu'|\mathbf{a}^\dagger|xt\}]^\pm=0\end{aligned}\right\}. \quad (9)$$

(upper sign for F–D statistics, lower sign for B–E statistics).

2.22 *Transition operators.* Incidentally it may be mentioned that the products $\{xt\,|\,\mathbf{a}\,|\,\mu')\,(\mu\,|\,\mathbf{a}^\dagger\,|\,xt\}$ can serve as transition operators between the states (μ) and (μ') (process P_1). The products with $\mu=\mu'$ can be considered as projection operators of the states (μ).

2.23 *Wave operators.* Now we define the preliminary statical wave operators by

$$\left.\begin{aligned}&(ys|\boldsymbol{\psi}|xt\}=\sum_{(\mu)}(ys|\psi|\mu)\,(\mu|\mathbf{a}^\dagger|xt\}\,,\\ &\{xt|\boldsymbol{\psi}^\dagger|ys)=\sum_{(\mu)}\{xt|\mathbf{a}|\mu)\,(\mu|\psi^\dagger|ys).\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad (10)$$

They are HERMITIAN adjoint to each other. In many-times theory s will always be kept equal to t_n, if the annihilation operator acts on a wave function with n sets (x_kt_k) (or the creation operator on a wave function with $n-1$ sets). In single-time theory s will be kept equal to t. (10) can somehow be considered as a transformation between (μ) and (y) representation. Meanwhile the operation is always on the (xt).

As a consequence of (10) the wave operators satisfy the commutation relations

$$\left.\begin{aligned}&[(ys|\boldsymbol{\psi}|xt\}\,,\,\{xt|\boldsymbol{\psi}^\dagger|y's)]^\pm=\sum_{(\mu)}(ys|\psi|\mu)\,(\mu|\psi^\dagger|y's)\,\mathbf{S};\\ &[\{xt|\boldsymbol{\psi}^\dagger|ys)\,,\,\{xt|\boldsymbol{\psi}^\dagger|y's)]^\pm=0\quad,\quad[(ys|\boldsymbol{\psi}|xt\}\,,\,(y's|\boldsymbol{\psi}|xt\}]^\pm=0.\end{aligned}\right\}. \quad (11)$$

The sum in the first relation has the property described in the second and third equation of (6).

2.24 *Substitution operators.* The wave operators, which operate on the (xt), often occur combined with an operation in the (ys) in such a way that the total resulting operation has a simple meaning. We represent the combination by the substitution operators $\{ys\,|\,\psi\,|\,xt\}$ and $\{xt\,|\,\psi^\dagger\,|\,ys\}$ (mind the backets, which distinguish them from the wave operators and which indicate operation on the (xt) as well as on the (ys)). They are defined by

$$\left.\begin{aligned}&\{ys|\psi|xt\}\,(x_1t_1,\dots x_nt_n|\,\Psi=(ys|\psi|xt\}\,(x_1t_1,\dots x_nt_n|\,\Psi,\\&\{xt|\psi^\dagger|ys\}\,'x_1t_1,\dots x_{n-1}t_{n-1};ys|=\\&\qquad\qquad=\int(dy)\,\{xt|\psi^\dagger|ys)\,\eta\,\{ys\}\,\varrho\,\{ys\}\,(x_1t_1,\dots x_{n-1}t_{n-1};yx|\,\Psi.\end{aligned}\right\}. \quad (12)$$

If they act to the left we get

$$\left.\begin{aligned}&\Psi^\dagger|ys;x_{n-1}t_{n-1},\dots x_1t_1)\,\{ys|\psi|xt\}=\\&\qquad\qquad=\int(dy)\,\Psi^\dagger|ys;x_{n-1}t_{n-1},\dots x_1t_1)\,\eta\,\{ys\}\,\varrho\,\{ys\{\,(ys|\psi|xt\},\\&\Psi^\dagger|x_nt_n,\dots x_1t_1)\{xt|\psi^\dagger|ys\{=\Psi^\dagger|x_nt_n,\dots x_1t_1)\,\{xt|\psi^\dagger|ys).\end{aligned}\right\}. \quad (12^\dagger)$$

which are the HERMITIAN adjoint relations.

It may seem as if the operators introduced in 2.21, 2.23 and 2.24 have by little and little become more and more complicated. In fact, however, with this chain of definitions we have nearly gone round. The meaning of the substitution operators is clearly illustrated by the result of (12) and $(12^\dagger)$

$$\left.\begin{aligned}&\{ys|\psi|xt\}\,(x_1t_1,\dots x_nt_n|\,\Psi=(n^{1/2}\,\mathbf{S}_n\,(x_1t_1,\dots x_nt_n|\,\Psi)_{(x_nt_n)\to(ys)},\\&\}xt|\psi^\dagger|ys\{\,(x_1t_1,\dots x_{n-1}t_{n-1};ys|\,\Psi=\\&\qquad\qquad=\mathbf{S}_n\,n^{1/2}((x_1t_1,\dots x_{n-1}t_{n-1};ys|\,\Psi)_{(ys)\to(x_nt_n)}\end{aligned}\right\}. \quad (13)$$

and

$$\left.\begin{aligned}&\Psi^\dagger|x_1t_1,\dots x_{n-1}t_{n-1};ys)\,\{ys|\psi|xt\}=\\&\qquad\qquad=(\Psi^\dagger|x_1t_1,\dots x_{n-1}t_{n-1};ys))_{(ys)\to(x_nt_n)}\,n^{1/2}\,\mathbf{S}_n,\\&\Psi^\dagger|x_1t_1,\dots x_nt_n)\{xt|\psi^\dagger|ys\}=(\Psi^\dagger(x_1t_1,\dots x_nt_n)\,\mathbf{S}_n\,n^{1/2})_{(x_nt_n)\to(ys)},\end{aligned}\right\}. \quad (13^\dagger)$$

with $s=t_n$ in all substitutions. This shows that:

S_s An annihilation substitution operator (anti-)symmetrizes the wave function in all sets (x_kt_k), multiplies it by the square root of the number of these sets and replaces the last set by a new set (ys) with the same value of time. A creation substitution operator replaces a set (ys) by an additional set (x_kt_k) with the same value of time, multiplies the wave function by the square root of the number of all sets (x_kt_k) and (anti-)symmetrizes the result.

2.25 *Particle operators.* As particle operators we take the ordinary HERMITIAN operators acting on the (x_kt_k) and symmetrical in these sets, i.e. commuting with all permutation operators $\mathbf{P}$. For a given number n of particles such an operator can be written as a sum of homogeneous particle operators $\mathbf{R}^{(m)}$, each with a different value of m ($m\leq n$; in practice $m=0,1,2$), of the form

$$\mathbf{R}^{(m)}\{x_1t_1,\dots x_nt_n\}=\sum_{k_1\dots k_m}\mathbf{R}\{x_{k_1}t_{k_1},\dots x_{k_m}t_{k_m}\}\quad(k_1,\dots k_m\text{ all different}). \quad (14)$$

$\mathbf{R}\{x_1t_1, \dots x_mt_m\}$ needs not to be symmetrical, but it can be taken so and then it is uniquely determined by $\mathbf{R}^{(m)}$. If it operates on a function of the required symmetry type, $\mathbf{R}^{(m)}$ can be written

$$\mathbf{R}^{(m)}\{x_1t_1, \dots x_nt_n\} = n \dots (n-m+1)\,\mathbf{S}_n\,\mathbf{R}\{x_nt_n, \dots x_{n-m+1}t_{n-m+1}\}\,\mathbf{S}_n. \quad (15)$$

According to S_s this is equivalent with

$$\left.\begin{array}{l}\mathbf{R}^{(m)}\{x_1t_1, \dots\} = \{xt|\psi^\dagger|y_1s_1\}\dots \\ \quad \dots\{xt|\psi^\dagger|y_ms_m\}\mathbf{R}\{y_1s_1, \dots y_ms_m\}\{y_ms_m|\psi|xt\}\dots\{y_1s_1|\psi|xt\}.\end{array}\right\} \quad (16)$$

This form is rather trivial. With the help of (12) it can be written

$$\left.\begin{array}{l}\mathbf{R}^{(m)}\{x_1t_1, \dots\} = \\ \quad = \int\dots\int(dy_1)\dots(dy_m)\{xt|\psi^\dagger|y_1s_1)\dots\{xt|\psi^\dagger|y_ms_m)\,\eta\{y_1s_1\}\,\varrho\{y_1s_1\}\dots \\ \quad \dots\eta\{y_ms_m\}\,\varrho\{y_ms_m\}\,\mathbf{R}\{y_1s_1, \dots y_ms_m\}(y_ms_m|\psi|xt\}\dots(y_1s_1|\psi|xt\},\end{array}\right\} \quad (17)$$

if it acts to the right and with the help of $(12^\dagger)$

$$\left.\begin{array}{l}\mathbf{R}^{(m)}\{x_1t_1, \dots\} = \\ \quad = \int\dots\int(dy_1)\dots(dy_m)\{xt|\psi^\dagger|y_1s_1)\dots\{xt|\psi^\dagger|y_ms_m)\,\mathbf{R}\{y_1s_1, \dots y_ms_m\} \\ \quad \eta\{y_ms_m\}\,\varrho\{y_ms_m\}\dots\eta\{y_1s_1\}\,\varrho\{y_1s_1\}(y_ms_m|\psi|xt\}\dots(y_1s_1|\psi|xt\},\end{array}\right\} \quad (17^\dagger)$$

if it acts to the left. In case of noncommutability of $\mathbf{R}$ with $\eta\varrho$ the distinction between (17) and $(17^\dagger)$ should well be observed. The expressions in the right hand members of (16) and (17), $(17^\dagger)$ do not explicitly contain the number n of particles.

In many-times theory the important homogeneous particle operators are of the type $\mathbf{R}^{(1)}\{x_1t_1, \dots\}$, formed from the individual operators $\mathbf{R}\{x_kt_k\}$.

2.26 *Observable particle quantities.* All observable particle quantities can be built up from expressions of the type

$$\int\dots\int(dx_1)\dots(dx_n)\,\Psi^\dagger|x_nt_n, \dots x_1t_1)\,\mathbf{R}^{(m)}\{x_1t_1, \dots x_nt_n\}(x_1t_1, \dots x_nt_n|\Psi'. \quad (18)$$

$\mathbf{R}^{(m)}$ may contain ϱ and η and the determination of its correct form may be a thorny matter. But that belongs to problem Q_3 and for problem Q_1 we need not care about it. In (18) one can if one likes express $\mathbf{R}^{(m)}$ according to (16) or (17), $(17^\dagger)$.

2.3 *Wave operators; dynamical representation.* Now we turn from the statical to the dynamical representation.

2.31 *Dynamical representations.* If we consider the time dependence of (18), the part of the motion of the system is entirely contained in the wave functions Ψ (SCHRÖDINGER representation). This dynamical time dependence can be transfered to the operators $\mathbf{R}$ (HEISENBERG representation). In the latter case the wave functions Ψ are replaced by their "initial" values Ψ' and the operators $\mathbf{R}$ by operators $\mathbf{R}'$, which contain, besides the explicit time dependence of $\mathbf{R}$, also the motion of the system.

Also in HEISENBERG representation the particle operators can be expressed according to (16) or (17), $(17^\dagger)$, in which the $\mathbf{R}'\{xt\}$ and $\mathbf{R}'\{ys\}$ then appear with a dash. If one is willing to use these expressions throughout, it is also possible to transfer the dynamical time dependence to the substitution and wave operators ψ. In that case we get substitution and wave operators $\boldsymbol{\psi}'$ containing the motion of the system and in (16) and (17), ($17^\dagger$) the $\mathbf{R}'\{xt\}$ appear with a dash, the $\mathbf{R}\{ys\}$ without.

The direct transfer of the dynamical time dependence from the Ψ to the $\mathbf{R}$ can hardly be described in such general terms as we are using at the moment. But if we succeed in transfering it from the Ψ to the $\boldsymbol{\psi}$, the transfer from the $\boldsymbol{\psi}$ to the $\mathbf{R}$ can immediately be found from (16) or (17), ($17^\dagger$) (with undashed $\mathbf{R}\{ys\}$ and dashed $\boldsymbol{\psi}'$ and $\mathbf{R}'\{xt\}$).

The various dynamical representations are compared in outline in the following scheme. The explicit time dependence of the undashed operators $\mathbf{R}$ has been left out of account. The suffixes 0 are explained further on. The underlined quantities contain the motion of the particles. The initial conditions are always contained in Ψ.

elementary	e_1 Schrödinger	e_2 Heisenberg	
	$\underline{(xt \vert \Psi}$	$(xt_0 \vert \Psi$	
	$\mathbf{R}\{x\}$	$\underline{\mathbf{R}'\{xt\}}$	
"superquantized"	s_1	s_2 "Schrödinger"	s_3 "Heisenberg"
	$\underline{(xt \vert \Psi}$	$(xt_0 \vert \Psi'$	$(xt_0 \vert \Psi'$
	$(ys \vert \boldsymbol{\psi} \vert xt\}$	$\underline{(ys \vert \boldsymbol{\psi}' \vert xt_0\}}$	$(ys_0 \vert \boldsymbol{\psi} \vert xt_0\}$
	$\mathbf{R}\{y\}$	$\mathbf{R}\{y\}$	$\underline{\mathbf{R}'\{ys\}}$

2.32 *"Superquantized wave equations"*. In transfering the dynamical time dependence from the Ψ to the $\boldsymbol{\psi}$ (representation s_2), we replace the wave functions $(x_1t_1, \dots x_nt_n \mid \Psi$ by their value $(x_1t_{10}, \dots x_nt_{n0} \mid \Psi'$ at a fixed set of initial times t_{k0} $(k = 1, \dots n)$. The statical wave operators $(ys \mid \boldsymbol{\psi} \mid xt\}$ and $\{xt \mid \boldsymbol{\psi}^\dagger \mid ys)$ are replaced by the dynamical wave operators $(ys \mid \boldsymbol{\psi}' \mid xt_0\}$ and $\{xt_0 \mid \boldsymbol{\psi}'^\dagger \mid ys)$. The values of s are not fixed to the t_0, but keep step with the t in the same way as before. These substitutions should not alter the initial value and the time dependence of (18), in which (16) or (17), ($17^\dagger$) has been inserted for $\mathbf{R}^{(m)}\{xt\}$. At $s = t_0$ the dashed operators have to be equal to the undashed ones at $t = t_0$. Further also at $s = t_0$ they have to satisfy (11). Finally they have to satisfy the "superquantized wave equations"

$$\left.\begin{aligned} &\mathbf{K}\{ys\}\,(ys \mid \boldsymbol{\psi}' \mid xt_0\} = 0, \\ &\{xt_0 \mid \boldsymbol{\psi}'^\dagger \mid ys)\,\mathbf{K}\{ys\} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in many-times theory or

$$\left.\begin{aligned} &(\mathbf{K}\{x_1, \dots y; t\} - \mathbf{K}\{x_1, \dots; t\})(ys \mid \boldsymbol{\psi}' \mid xt_0\} = 0, \\ &\{xt_0 \mid \boldsymbol{\psi}'^\dagger \mid ys)(\mathbf{K}\{x_1, \dots, y; t\} - \mathbf{K}\{x_1, \dots; t\}) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (19')$$

in single-time theory. (With the help of $\mathbf{K}^{(1)}\ \{x_1t_1, \dots\}$ (19) can also be written similar to (19')).

The $\boldsymbol{\psi}$ were independent of the dynamics of the system. They are relatively simple universal operators. The $\boldsymbol{\psi}'$ contain (for given initial conditions) the entire motion of the system. They will be frightfully complicated.

(To be continued.)

Chemistry. — *On the crystal structure of strychnine sulfate and selenate. II. [010] projection and structure formula.* By C. BOKHOVEN, J. C. SCHOONE and J. M. BIJVOET.

(Communicated at the meeting of September 25, 1948.)

Fig. 1 gives the Fourier synthesis of the electron distribution in the [010] projection of strychnine sulfate [1]).

The Fourier coefficients were deduced from the photometrically measured intensities of Weissenberg diagrams. Their signs were derived from the comparison of the structure factors of the isomorphous sulfate and selenate. Full data will be published in due time in the Acta Crystallographica.

The atomic configuration revealed by the Fourier map affords good agreement between calculated and observed intensities, which was still improved by small adjustments of the atomic parameters in a somewhat systematic way.

With this projection and the known interatomic distances only one model is compatible; its bonds are designed in fig. 1 [2]).

This conclusion was made easier by reference to a rough [001] projection. The structure of the strychnine molecule is seen in fig. 2 and again in fig. 3 in a projection suitable to act as an ordinary structure formula. It would not be easy to differentiate in this frame between *C*, *N* and *O* atoms by means of X-rays.

We think our deduction of the strychnine structure convincing already at this stage of the analysis. The definite proof however has to be supplied by a further study of the — non symmetrical — [001] projection, which is in progress now.

The above result was reached before we were aware of the fact that quite recently organic chemistry itself succeeded in making the right choice between the models in question [3]).

So our X-ray result constitutes an independent confirmation of this choice.

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[1]) C. BOKHOVEN, J. C. SCHOONE and J. M. BIJVOET, These Proceedings, L, 825 (1947).

[2]) Our hearty thanks are due to Mr D. M. W. DEN BOER for cooperation in this part of the investigation.

[3]) R. ROBINSON, Nature **162**, 177 (1948).

C. BOKHOVEN, J. C. SCHOONE and J. M. BIJVOET: *On the crystal structure of strychnine sulfate and selenate. II. [010] projection and structure formula.*

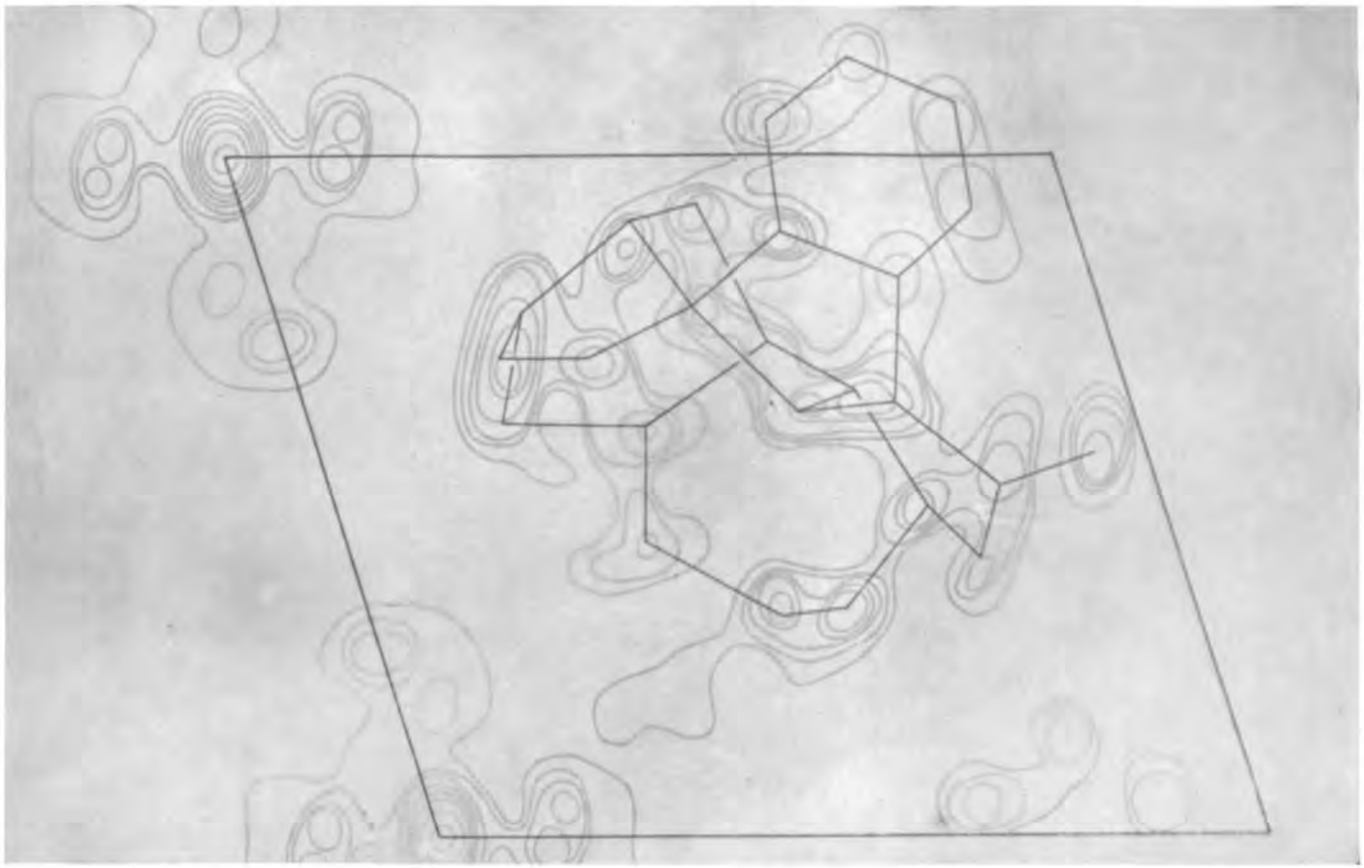

Fig. 1. Electron-density map of strychnine sulfate $(C_{21}H_{22}N_2O_2)_2 . H_2SO_4 5$ aq in [010] propection. In the corners of the cell SO_4 groups with the water-molecules around it *). In the middle the strychnine molecule. This projection may be compared with the model of fig. 2 or with the projections of the latter in a slightly different orientation, fig. 3.

*) Four water molecules are located each on top of an O-atom of the SO_4 group. The vague maximum between the SO_4 group and the strychnine molecule corresponds in total density to about half a water molecule. It may possibly be attributed to the fifth molecule in statistical distribution over the twofold position about the axis of symmetry.

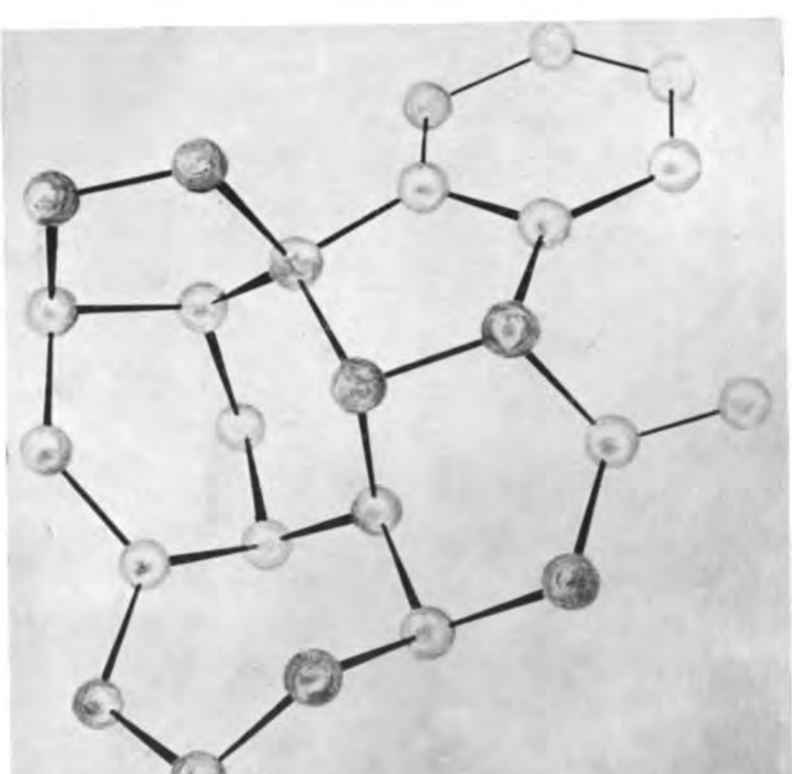

Fig. 2. Model of the strychnine molecule.

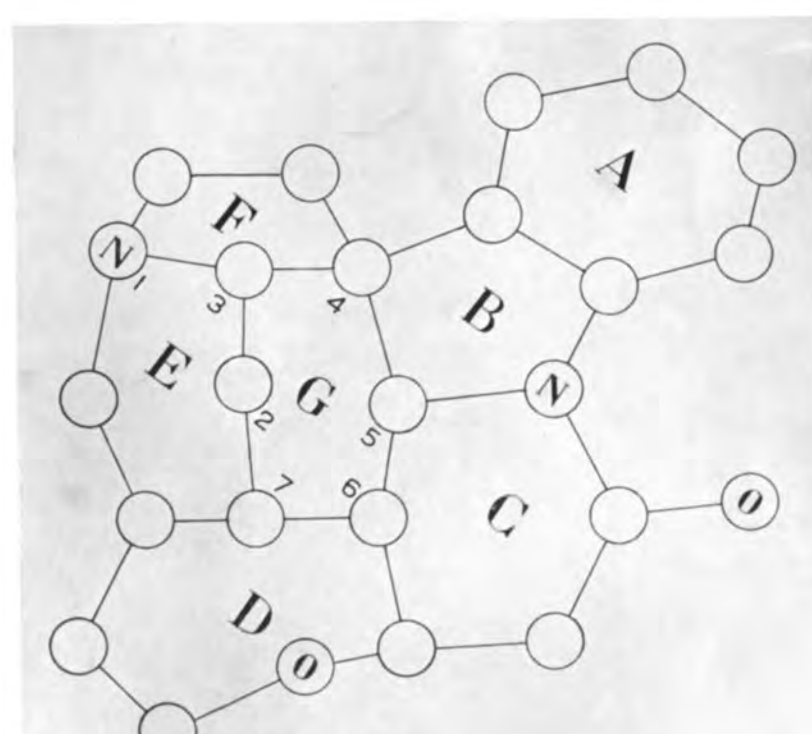

Fig. 3. Projection of the strychnine molecule in a direction slightly different from that of fig. 1 or 2.

Mathematics. — *Über mehrwertige Aussagenkalküle und mehrwertige engere Prädikatenkalküle.* III. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 25, 1948.)

Der Aussagenkalkül *KL*.

§ 16. Nach GÖDEL [1]) lässt sich das durch das BECKERsche Zusatzaxiom $\mathbf{N}p \rightarrow \mathbf{NN}p$ und die hier folgenden Axiome V und VI ergänzte LEWISsche System of Strict Implication durch folgendes Axiomensystem charakterisieren.

Einsetzungsregel E (erster Teil). Aus einer *Kalkülformel* erhält man wieder eine Kalkülformel, wenn man einen in ihr auftretenden grossen lateinischen Buchstaben durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln); dabei sind die grossen lateinischen Buchstaben, λ und ν als Kalkülformeln anzusehen, ferner mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R} + \mathfrak{S}$, $\mathfrak{R}'$, $\mathfrak{S}'$ und $\mathbf{N}\mathfrak{R}$, $\mathbf{N}\mathfrak{S}$.

Definition 1. $\mathfrak{A} \rightarrow \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$.

$\doteq \bar{\nu}$, $\doteq \bar{\lambda}$ sind *Modalitätszeichen*.

Definition 2. Ein Ausdruck, bestehend aus einer Kalkülformel gefolgt durch $\doteq \bar{\nu}$, oder $\doteq \bar{\lambda}$, und entstanden nach endlichmaliger Anwendung von Axiomen, Einsetzungsregel *E*, Schlussschema *S* oder (und) Ableitungsregel *A*, ist *ein Theorem*.

Einsetzungsregel E (zweiter Teil). Ist $\mathfrak{A} \doteq \bar{\nu}$ ein Theorem, und $\mathfrak{B}$ eine aus $\mathfrak{A}$ mittels der Einsetzungsregel *E* (erster Teil) hervorgehende Kalkülformel, so liefert auch $\mathfrak{B} \doteq \bar{\nu}$ ein Theorem.

Axiom I. $[(X + X) \rightarrow X] \doteq \bar{\nu}$.

Axiom II. $[X \rightarrow (X + Y)] \doteq \bar{\nu}$.

Axiom III. $[(X + Y) \rightarrow (Y + X)] \doteq \bar{\nu}$.

Axiom IV. $[(Y \rightarrow Z) \rightarrow \{(X + Y) \rightarrow (X + Z)\}] \doteq \bar{\nu}$.

Schlussschema S. Sind

$$\mathfrak{A} \doteq \bar{\nu} \text{ und } \quad [\mathfrak{A} \rightarrow \mathfrak{B}] \doteq \bar{\nu}$$

Theoreme, so ist auch

$$\mathfrak{B} \doteq \bar{\nu}$$

ein Theorem.

Definition 3. Eine andere Schreibweise eines Theorems $\mathfrak{A} \doteq \bar{\nu}$ ist $\mathfrak{A}' \doteq \bar{\lambda}$; eine andere Schreibweise eines Theorems $\mathfrak{A}' \doteq \bar{\nu}$ ist $\mathfrak{A} \doteq \bar{\lambda}$.

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[1]) Siehe Ergebnisse eines mathem. Kolloquiums herausgegeben von Karl Menger, Heft 4 (1933), Leipzig u. Berlin, S. 39. Die Axiome V und VI fehlen an der zitierten Stelle.

Definition 4. $\mathfrak{A} \cdot \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' + \mathfrak{B}')'$.

Definition 5. $\mathfrak{A} \rightarrow_2 \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' \cdot \mathfrak{B}$.

Axiom V. $[\lambda \rightarrow X] \doteq \bar{\nu}$.

Axiom VI. $[X \rightarrow \nu] \doteq \bar{\nu}$.

Dualitätsprinzip $\mathbf{D}_1$ [2]. Ist $\mathfrak{A} \doteq \bar{\nu}$ ein Theorem, so auch $\mathfrak{B} \doteq \bar{\lambda}$; dabei gehe Kalkülformel $\mathfrak{B}$ aus Kalkülformel $\mathfrak{A}$ dadurch hervor, dass: α) $+$ durch $\cdot$, und umgekehrt, β) λ durch ν, und umgekehrt, γ) $\rightarrow$ durch $\rightarrow_2$, und umgekehrt, ersetzt werden; ev. vorkommende Akzenten und **N** sollen ungeändert bleiben. Ist, umgekehrt, bei denselben Kalkülformeln $\mathfrak{A}$ und $\mathfrak{B}$ $\mathfrak{B} \doteq \bar{\lambda}$ ein Theorem, so auch $\mathfrak{A} \doteq \bar{\nu}$.

Axiom VII. $[(\mathbf{N}X) \rightarrow X] \doteq \bar{\nu}$.

Axiom VIII. $[\{\mathbf{N}(X \rightarrow Y)\} \rightarrow \{(\mathbf{N}X) \rightarrow (\mathbf{N}Y)\}] \doteq \bar{\nu}$.

Axiom IX. $[(\mathbf{N}X) \rightarrow \{\mathbf{N}(\mathbf{N}X)\}] \doteq \bar{\nu}$.

Ableitungsregel A. Ist $\mathfrak{A} \doteq \bar{\nu}$ ein Theorem, so auch $(\mathbf{N}\mathfrak{A}) \doteq \bar{\nu}$.

Definition 6. $\mathbf{M}\mathfrak{A}$ ist eine andere Schreibweise von $[\mathbf{N}\mathfrak{A}']'$.

Satz 1 (Schlussschema). Ist $\mathfrak{A} \doteq \bar{\lambda}$ ein Theorem, so auch $(\mathbf{M}\mathfrak{A}) \doteq \bar{\lambda}$.

Beweis. Eine andere Schreibweise von $\mathfrak{A} \doteq \bar{\lambda}$ ist $\mathfrak{A}' \doteq \bar{\nu}$. Also ist, nach Ableitungsregel A, auch $[\mathbf{N}\mathfrak{A}'] \doteq \bar{\nu}$ ein Theorem oder, anders geschrieben, $[N\mathfrak{A}']' \doteq \bar{\lambda}$, d.h. $(\mathbf{M}\mathfrak{A}) \doteq \bar{\lambda}$.

Satz 2. $[(\mathbf{M}X) \rightarrow_2 X] \doteq \bar{\lambda}$ (ist ein Theorem).

Beweis. Axiom VII und Einsetzungsregel E liefern:

$$[(\mathbf{N}X') \rightarrow X'] \doteq \bar{\nu},$$

oder

$$[(\mathbf{N}X')' + X'] \doteq \bar{\nu},$$

oder

$$(\mathbf{M}X + X') \doteq \bar{\nu},$$

oder

$$[(\mathbf{M}X)' \cdot X] \doteq \bar{\lambda},$$

oder

$$[(\mathbf{M}X) \rightarrow_2 X] \doteq \bar{\lambda}.$$

Satz 3. $[\{\mathbf{M}(X \rightarrow_2 Y)\} \rightarrow_2 \{(\mathbf{M}X) \rightarrow_2 (\mathbf{M}Y)\}] \doteq \bar{\lambda}$.

Beweis. Axiom VIII und Einsetzungsregel E liefern:

$$[\{\mathbf{N}(X' \rightarrow Y')\} \rightarrow \{(\mathbf{N}X') \rightarrow (\mathbf{N}Y')\}] \doteq \bar{\nu},$$

oder

$$[\{\mathbf{N}(X + Y')\}' + \{(\mathbf{N}X')' + (\mathbf{N}Y')\}] \doteq \bar{\nu},$$

oder

$$[\{\mathbf{N}(X' \cdot Y)'\}' + \{(\mathbf{N}X')' + (\mathbf{N}Y')\}] \doteq \bar{\nu},$$

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[2]) Siehe J. RIDDER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 1153—1164 (1946), u. **50**, 24—30 (1947), insbes. S. 1153, 1154, 1156 u. S. 26, 27 (Satz 42).

oder

$$[\mathbf{M}(X' \cdot Y) + \{(\mathbf{M}X) + (\mathbf{M}Y)'\}] \doteq \bar{\nu},$$

oder

$$[\{\mathbf{M}(X' \cdot Y)\}' \cdot \{(\mathbf{M}X)' \cdot (\mathbf{M}Y)\}] \doteq \bar{\lambda},$$

oder

$$[\{\mathbf{M}(X \rightarrow_2 Y)\} \rightarrow_2 \{(\mathbf{M}X) \rightarrow_2 (\mathbf{M}Y)\}] \doteq \bar{\lambda}.$$

Satz 4. $[(\mathbf{M}X) \rightarrow_2 \{\mathbf{M}(\mathbf{M}X)\}] \doteq \bar{\lambda}.$

Beweis. Axiom IX und Einsetzungsregel E liefern:

$$[(\mathbf{N}X') \rightarrow \{\mathbf{N}(\mathbf{N}X')\}] \doteq \bar{\nu},$$

oder

$$[(\mathbf{N}X')' + \{\mathbf{N}(\mathbf{N}X')\}] \doteq \bar{\nu},$$

oder

$$[(\mathbf{N}X') \cdot \{\mathbf{N}(\mathbf{N}X')\}'] \doteq \bar{\lambda},$$

oder

$$\{(\mathbf{N}X') \cdot [\mathbf{M}\{(\mathbf{N}X')'\}]\} \doteq \bar{\lambda},$$

oder

$$[(\mathbf{M}X)' \cdot \{\mathbf{M}(\mathbf{M}X)\}] \doteq \bar{\lambda},$$

oder

$$[(\mathbf{M}X) \rightarrow_2 \{\mathbf{M}(\mathbf{M}X)\}] \doteq \bar{\lambda}.$$

Aus den Axiomen VII, VIII, IX, Ableitungsregel A einerseits und den Sätzen 1—4 andererseits folgt, dass obiges Dualitätsprinzip sich erweitern lässt zu folgendem

Prinzip $\mathbf{D}_2$. Ist $\mathfrak{A} \doteq \bar{\nu}$ ein Theorem, so auch $\mathfrak{B} \doteq \bar{\lambda}$; dabei gehe $\mathfrak{B}$ aus $\mathfrak{A}$ dadurch hervor, dass: α) $+$ durch $\cdot$, und umgekehrt, β) λ durch ν, und umgekehrt, γ) $\rightarrow$ durch $\rightarrow_2$, und umgekehrt, δ) $\mathbf{N}$ durch $\mathbf{M}$, und umgekehrt, ersetzt werden; ev. vorkommende Akzenten sollen ungeändert bleiben. Ist, umgekehrt, bei denselben Kalkülformeln $\mathfrak{A}$ und $\mathfrak{B}$ $\mathfrak{B} \doteq \bar{\lambda}$ ein Theorem, so auch $\mathfrak{A} \doteq \bar{\nu}$.

§ 17. *Das Axiomensystem des Par. 16 ist widerspruchsfrei* in dem Sinne, dass $X \doteq \bar{\nu}$ und $X \doteq \bar{\lambda}$ nicht aus ihm ableitbar sind.

Beweis. Wir betrachten eine Menge M von n diskreten Elementen ($n \geqq 1$).

Den durch grosse lateinische Buchstaben angedeuteten elementaren Kalkülformeln sollen als „Werte" die Teilmengen von M zugeordnet werden können (die leere Menge 0 und die Menge M selbst mit einbegriffen).

$A + B$ [$\mathfrak{A} + \mathfrak{B}$] ordnen wir den „Wert" $T_1 + T_2$ zu, falls A [$\mathfrak{A}$] die Teilmenge T_1, B [$\mathfrak{B}$] die Teilmenge T_2 zugeordnet ist; $A \cdot B$ [$\mathfrak{A} \cdot \mathfrak{B}$] sei dann $T_1 \cdot T_2$ als „Wert" zugeordnet. $A'[\mathfrak{A}']$ sei die Komplementärmenge $M—T_1$ als „Wert" zugeordnet, falls $A[\mathfrak{A}]$ T_1 zugeordnet ist.

Ist $A[\mathfrak{A}]$ M zugeordnet, so sei auch $\mathbf{N}A[\mathbf{N}\mathfrak{A}]$ die Menge M als „Wert" zugeordnet; ist $A[\mathfrak{A}]$ die leere Menge oder eine echte nicht leere Teilmenge von M als „Wert" zugeordnet, so sei der zugehörige „Wert" von $\mathbf{N}A[\mathbf{N}\mathfrak{A}]$ die leere Menge.

λ sei als einzig möglicher „Wert" die leere Menge 0, ν als einzig möglicher „Wert" die Menge M zugeordnet.

Welche „Werte" auch den Kalkülformeln X, Y, Z zugeordnet sind, in allen Fällen ist den in den Axiomen I—IX im Voerderglied vorkommenden Kalkülformeln die Menge M zugeordnet.

Ausserdem führen Einsetzungsregel E, Schlussschema S und Ableitungsregel A immer von Theoremen $\mathfrak{A} \doteq \bar{\nu}$ mit der im letzten Absatz genannten Eigenschaft zu Theoremen $\mathfrak{B} \doteq \bar{\nu}$ mit derselben Eigenschaft. Schliesslich führt Anwendung von Definition 3 von Theoremen $\mathfrak{A} \doteq \bar{\nu}$, bei welchen $\mathfrak{A}$ nur den „Wert" M haben kann, zu Theoremen $\mathfrak{B} \doteq \bar{\lambda}$, bei welchen $\mathfrak{B}$ nur den „Wert" 0 haben kann, und umgekehrt.

Damit ist die Nicht-Ableitbarkeit von $X \doteq \bar{\nu}$ und die von $X \doteq \bar{\lambda}$ bewiesen.

Das Beweisverfahren zeigt ausserdem, *dass der Kalkül KL ein 2^n-wertiger Kalkül ist* ($n \geqq 1$).

§ 18. Das Axiomensystem des Par. 16 steht dual gegenüber folgendem System; die Systeme sind gleichwertig [3]).

Definition 2_0. Ein Ausdruck bestehend aus einer Kalkülformel, gefolgt durch $\doteq \bar{\nu}$ oder $\doteq \bar{\lambda}$, und entstanden nach endlichmaliger Anwendung von Axiomen (dieses Par.), Einsetzungsregel E_0, Schlussschema S_0 oder (und) Ableitungsregel A_0 ist *ein Theorem*.

Einsetzungsregel $\mathbf{E}_0$, aus Regel E hervorgehend durch Änderung von $\mathfrak{R} + \mathfrak{S}$ in $\mathfrak{R} \cdot \mathfrak{S}$, und von $\mathbf{N}\mathfrak{R}$ in $\mathbf{M}\mathfrak{R}$, von $\mathbf{N}\mathfrak{S}$ in $\mathbf{M}\mathfrak{S}$, schliesslich von $\mathfrak{A} \doteq \bar{\nu}$ in $\mathfrak{A} \doteq \bar{\lambda}$, von $\mathfrak{B} \doteq \bar{\nu}$ in $\mathfrak{B} \doteq \bar{\lambda}$.

Schlussschema $\mathbf{S}_0$. Sind

$$\mathfrak{A} \doteq \bar{\lambda} \quad \text{und} \quad [\mathfrak{A} \rightarrow_2 \mathfrak{B}] \doteq \bar{\lambda}$$

Theoreme, so auch

$$\mathfrak{B} \doteq \bar{\lambda}.$$

Axiom $\mathbf{I}_0$. $[(X \cdot X) \rightarrow_2 X] \doteq \bar{\lambda}$.

Axiom $\mathbf{II}_0$. $[X \rightarrow_2 (X \cdot Y)] \doteq \bar{\lambda}$.

Axiom $\mathbf{III}_0$. $[(X \cdot Y) \rightarrow_2 (Y \cdot X)] \doteq \bar{\lambda}$.

Axiom $\mathbf{IV}_0$. $[(Y \rightarrow_2 Z) \rightarrow_2 \{(X \cdot Y) \rightarrow_2 (X \cdot Z)\}] \doteq \bar{\lambda}$.

Axiom $\mathbf{V}_0$. $[\nu \rightarrow_2 X] \doteq \bar{\lambda}$.

---

[3]) Vergl. § 16 und RIDDER, loc. cit. 2), S. 26 u. 27 (Par. 10).

Axiom VI$_0$. $[X \rightarrow_2 \lambda] \doteq \bar{\lambda}$.

Axiom VII$_0$. $[(\mathbf{M}X) \rightarrow_2 X] \doteq \bar{\lambda}$.

Axiom VIII$_0$. $[\{\mathbf{M}(X \rightarrow_2 Y)\} \rightarrow_2 \{(\mathbf{M}X) \rightarrow_2 (\mathbf{M}Y)\}] \doteq \bar{\lambda}$.

Axiom IX$_0$. $[(\mathbf{M}X) \rightarrow_2 \{\mathbf{M}(\mathbf{M}X)\}] \doteq \bar{\lambda}$.

Ableitungsregel A$_0$. Ist $\mathfrak{A} \doteq \bar{\lambda}$ ein Theorem, so auch $(\mathbf{M}\mathfrak{A}) \doteq \bar{\lambda}$.

Die Definitionen 1, 3 und 5 bleiben erhalten.

Definition 4$_0$. $\mathfrak{A} + \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' \cdot \mathfrak{B}')'$.

Definition 6$_0$. $\mathbf{N}\mathfrak{A}$ ist eine andere Schreibweise von $[\mathbf{M}\mathfrak{A}']'$.

Die Aussagenkalküle $KL^{(p)}$ $(p \geqq 2)$.

§ 19. Die in dem erweiterten RUSSELL–WHITEHEADschen Aussagenkalkül K gültige Dualität [4]) ermöglichte in Teil I und Teil II [5]) die Einführung von Aussagenkalkülen $K^{(p)}$ $(p \geqq 2)$, welche als „Produkte" von p erweiterten RUSSELL–WHITEHEADschen Kalkülen K aufzufassen sind.

Ebenso ermöglicht die Dualität von §§ 16 und 18 die Einführung von Aussagenkalkülen $KL^{(p)}$ $(p \geqq 2)$, welche als „Produkte" von p erweiterten LEWISschen Kalkülen KL (§ 16) aufzufassen sind.

Die näheren Ausführungen dieses Gedankens liegen auf der Hand [6]).

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[4]) Siehe RIDDER, loc. cit. 2), S. 26, 27 (Satz 42).

[5]) Siehe insbes. Teil I dieser Arbeit, §§ 8, 9 und Teil II, § 12.

[6]) Jeder Kalkül $KL^{(p)}$ enthält wieder eine von den Axiomen und den übrigen Schlussregeln unabhängige Verbindungsregel.

Mathematics. — *Uitbreiding van enige identiteiten*. II. By J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of May 29, 1948.)

2. Voorts gaan we uit van de betrekkingen (41) en (41′), voorkomende in I, nl.

$$S'_{\nu,2k+1}(x)=\nu^{2k}\frac{x}{2}I_{\nu-1}(x)+xS'_{\nu+1,0}(x)+x\sum_{p_1=0}^{k-1}\binom{2k}{2p_1+2}S'_{\nu+1,2p_1+2}(x). \quad (9)$$

en

$$S'_{\nu,2k+2}(x)=-(\nu-2)^{2k+1}\frac{x}{2}I_{\nu-1}(x)+x\sum_{p_1=0}^{k}\binom{2k+1}{2p_1+1}S'_{\nu-1,2p_1+1}(x), \quad (10)$$

waarin $S'_{\nu,k}(x)=\sum_{n=0}^{\infty}(\nu+2n)^k I_{\nu+2n}(x)$ is.

We kunnen hieraan nog toevoegen:

$$\left.\begin{aligned}S'_{\nu,2k+1}(x)=-(\nu-2)^{2k}\frac{x}{2}I_{\nu-1}(x)+xS'_{\nu-1,0}(x)+\\ +x\sum_{p_1=0}^{k-1}\binom{2k}{2p_1+2}S'_{\nu-1,2p_1+2}(x)\end{aligned}\right\}. \quad (11)$$

en

$$S'_{\nu,2k+2}(x)=\nu^{2k+1}\frac{x}{2}I_{\nu-1}(x)+x\sum_{p_1=0}^{k}\binom{2k+1}{2p_1+1}S'_{\nu+1,2p_1+1}(x), \quad (12)$$

welke formules op overeenkomstige wijze zijn afgeleid als is aangegeven in I § 5 bij de afleiding van soortgelijke formules voor $S_{\nu,2k+2}(x)$.

Door nu met elkaar te combineren de betrekkingen (9) en (12), (11) en (10), (9) en (10), (11) en (12), op de wijze als in het voorgaande is uitgevoerd, vindt men de volgende algemene formules (ν willekeurig):

$$\left.\begin{aligned}S'_{\nu,2k+1}(x)=\frac{x}{2}\sum_{r=0}^{k}x^{2r}\sum_{p_1=0}^{k-r}\binom{2k}{2p_1+2r}\sum_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-1}\cdots\\ \cdots\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+1}{2p_{2r}+1}\{(\nu+2r)^{2p_{2r}}I_{\nu+2r-1}(x)+2S'_{\nu+2r+1,0}(x)\}+\\ +2\left(\frac{x}{2}\right)^2\sum_{r=0}^{k}x^{2r}I_{\nu+2r}(x)\sum_{p_1=0}^{k-r-1}\binom{2k}{2p_1+2r+2}\sum_{p_2=0}^{p_1}\binom{2p_1+2r+1}{2p_2+2r+1}\cdots\\ \cdots\sum_{p_{2r+1}=0}^{p_{2r}}\binom{2p_{2r}+2}{2p_{2r+1}+2}(\nu+2r+1)^{2p_{2r+1}+1}=\\ =\frac{x}{2}\sum_{r=0}^{k}x^{2r}\sum_{p_1=0}^{k-r}\binom{2k}{2p_1+2r}\sum_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-1}\cdots\\ \cdots\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+1}{2p_{2r}+1}\{-(\nu-2r-2)^{2p_{2r}}I_{\nu-2r-1}(x)+2S'_{\nu-2r-1,0}(x)\}-\end{aligned}\right\}. \quad (13)$$

(Zie volgende pagina)

$$
\left.
\begin{aligned}
&-2\left(\frac{x}{2}\right)^2 \sum_{r=0}^{k-1} x^{2r} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-2r-3)^{2p_{2r+1}+1} = \\
&\quad = \frac{x}{2} \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \{ \nu^{2p_{2r}} I_{\nu-1}(x) + 2\, S'_{\nu+1,0}(x) \} - \\
&-2\left(\frac{x}{2}\right)^2 I_\nu(x) \sum_{r=0}^{k-1} x^{2r} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}+1} = \\
&\quad = \frac{x}{2} \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \{ -(\nu-2)^{2p_{2r}} I_{\nu-1}(x) + 2\, S'_{\nu-1,0}(x) \} + \\
&+2\left(\frac{x}{2}\right)^2 I_{\nu-2}(x) \sum_{r=0}^{k-1} x^{2r} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}+1} .
\end{aligned}
\right\} \quad (13)
$$

Hierin is, zo $r=0$ is, voor $\sum\limits_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} (\nu+r)^{2p_{2r}}$ resp. $(\nu-2r-2)^{2p_{2r}}$ resp. $\nu^{2p_{2r}}$ resp. $(\nu-2)^{2p_{2r}}$ te nemen ν^{2k} resp. $(\nu-2)^{2k}$.

Op gelijke wijze komt men door combinatie van de betrekkingen (12) en (9), (10) en (11), (12) en (11), (10) en (9) tot de volgende algemene formules (ν willekeurig):

$$
\left.
\begin{aligned}
S'_{\nu,2k+2}(x) = {} & 2\left(\frac{x}{2}\right)^2 \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \{(\nu+2r+1)^{2p_{2r+1}} I_{\nu+2r}(x) + 2 S'_{\nu+2r+2,0}(x)\} + \\
&+ \frac{x}{2} \sum_{r=0}^{k} x^{2r} I_{\nu+2r-1}(x) \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu+2r)^{2p_{2r}+1} =
\end{aligned}
\right\} \quad (14)
$$

(Zie volgende pagina)

$$\left.\begin{aligned}
&=2\left(\frac{x}{2}\right)^2\sum_{r=0}^{k}x^{2r}\sum_{p_1=0}^{k-r}\binom{2k+1}{2p_1+2r+1}\sum_{p_2=0}^{p_1}\binom{2p_1+2r}{2p_2+2r}\cdots\\
&\quad\cdots\sum_{p_{2r+1}=0}^{p_{2r}}\binom{p_{2r}+1}{2p_{2r+1}+1}\{-(\nu-2r-3)^{2p_{2r+1}}I_{\nu-2r-2}(x)+2S'_{\nu-2r-2,0}(x)\}-\\
&-\frac{x}{2}\sum_{r=0}^{k-1}x^{2r}I_{\nu-2r-1}(x)\sum_{p_1=0}^{k-r-1}{}'\binom{2k+1}{2p_1+2r}\sum_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-1}\cdots\\
&\qquad\cdots\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+2}{2p_{2r}+2}(\nu-2r-2)^{2p_{2r}+1}=\\
&=\frac{x}{2}\sum_{r=0}^{k}x^{2r}\sum_{p_1=0}^{k-r}\binom{2k+1}{2p_1+2r+1}\sum_{p_2=0}^{p_1}\binom{2p_1+2r}{2p_2+2r}\cdots\\
&\quad\cdots\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+2}{2p_{2r}+2}[\nu^{2p_{2r}+1}I_{\nu-1}(x)+x\sum_{p_{2r+1}=0}^{p_{2r}}\binom{2p_{2r}+1}{2p_{2r+1}+1}\\
&\qquad\{-(\nu-1)^{2p_{2r+1}}I_\nu(x)+2S'_{\nu,0}(x)\}]=\\
&=\frac{x}{2}\sum_{r=0}^{k}x^{2r}\sum_{p_1=0}^{k-r}\binom{2k+1}{2p_1+2r+1}\sum_{p_2=0}^{p_1}\binom{2p_1+2r}{2p_2+2r}\cdots\\
&\quad\cdots\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+2}{2p_{2r}+2}[-(\nu-2)^{2p_{2r}+1}I_{\nu-1}(x)+x\sum_{p_{2r+1}=0}^{p_{2r}}\binom{2p_{2r}+1}{2p_{2r+1}+1}\\
&\qquad\{(\nu-1)^{2p_{2r+1}}I_{\nu-2}(x)+2S'_{\nu,0}(x)\}]
\end{aligned}\right\}\quad(14)$$

Hierin is, zo $r=0$ is, voor

$$\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+2}{2p_{2r}+2}(\nu+2r)^{2p_{2r}+1}$$

resp. $(\nu-2r-2)^{2p_{2r}+1}$ resp. $\nu^{2p_{2r}+1}$ resp. $(\nu-2)^{2p_{2r}+1}$ te nemen ν^{2k+1} resp. $(\nu-2)^{2k+1}$.

In verband met de identiteit, geldig voor alle ν, waarvan de juistheid o.a. door volledige inductie kan worden aangetoond:

$$\sum_{p=0}^{s}\frac{(-1)^p}{(s-p)!\,\Gamma(s+p+\nu+1)}=\frac{1}{(2s+\nu)\,s!\,\Gamma(s+\nu)},$$

kan men voor $S'_{\nu,0}(x)$ schrijven:

$$S'_{\nu,0}(x)=\sum_{n=0}^{\infty}I_{\nu+2n}(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m+2n+\nu}}{m!\,\Gamma(m+2n+\nu+1)}=$$

$$=\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}\sum_{p=0}^{s}\frac{(-1)^p}{(s-p)!\,\Gamma(s+p+\nu+1)}=$$

$$=\sum_{s=0}^{\infty}\frac{(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}}{(2s+\nu)\,s!\,\Gamma(s+\nu)}.$$

Door nu in (13) te substitueren:

$$S'_{\nu,2k+1}(x) = \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+1}}{(s-p)!\,\Gamma(s+p+\nu+1)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{(\nu+2r)^2 p_{2r}\, I_{\nu+2r-1}(x) + 2S'_{\nu+2r+1,0}(x)\} =$$

$$= \sum_{s=2r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r)!\,\Gamma(s+\nu)} \left\{(\nu+2r)^2 p_{2r} - \frac{2(s-2r)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu+2r}(x) = -\sum_{s=2r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r-1)!\,\Gamma(s+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{-(\nu-2r-2)^2 p_{2r}\, I_{\nu-2r-1}(x) + 2S'_{\nu-2r-1,0}(x)\} =$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu)} \left\{-(\nu-2r-2)^2 p_{2r} + \frac{2(s-2r+\nu-1)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu-2r-2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu-1)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{\nu^2 p_{2r}\, I_{\nu-1}(x) + 2S'_{\nu+1,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)} \left\{\nu^2 p_{2r} - \frac{2(s-r)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu}(x) = -(-1)^r \sum_{s=r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r-1)!\,\Gamma(s-r+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{-(\nu-2)^2 p_{2r}\, I_{\nu-1}(x) + 2S'_{\nu-1,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)} \left\{-(\nu-2)^2 p_{2r} + \frac{2(s-r+\nu-1)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu-2}(x) = (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu-1)},$$

vindt men na gelijkstelling der coëfficienten van $\left(\frac{x}{2}\right)^{2s+\nu}$ in beide leden de algemene indentiteiten (ν willekeurig):

$$\sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+1}}{(s-p)!\,\Gamma(s+p+\nu+1)} =$$

$$\left.\begin{aligned}
&= \frac{1}{\Gamma(s+\nu)} \sum_{r=0}^{k} \frac{2^{2r}}{(s-2r)!} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ (\nu+2r)^{2p_{2r}} - \frac{2(s-2r)}{2s-2r+\nu-1} \right\} - \\
&- \frac{1}{\Gamma(s+\nu)} \sum_{r=3}^{k-1} \frac{2^{2r+1}}{(s-2r-1)!} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu+2r+1)^{2p_{2r+1}+1} =
\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}
&= \frac{1}{s!} \sum_{r=0}^{k} \frac{2^{2r}}{\Gamma(s-2r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ -(\nu-2r-2)^{2p_{2r}} + \frac{2(s-2r+\nu-1)}{2s-2r+\nu-1} \right\} - \\
&- \frac{1}{s!} \sum_{\nu=0}^{k-1} \frac{2^{2r+1}}{\Gamma(s-2r+\nu-1)} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-2r-3)^{2p_{2r+1}+1} =
\end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned}
&= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ \nu^{2p_{2r}} - \frac{2(s-r)}{2s-2r+\nu-1} \right\} + \\
&+ \sum_{r=0}^{k-1} \frac{(-1)^r\, 2^{2r+1}}{(s-r-1)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}+1} =
\end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned}
&= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ -(\nu-2)^{2p_{2r}} + \frac{2(s-r+\nu-1)}{2s-2r+\nu-1} \right\} + \\
&+ \sum_{r=0}^{k-1} \frac{(-1)^r\, 2^{2r+1}}{(s-r)!\,\Gamma(s-r+\nu-1)} \sum_{p_1=0}^{k-\nu-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}+1}.
\end{aligned}\right\} \quad (18)$$

Hierin is, zo $r=0$ is, voor $\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} (\nu+2r)^{2p_{2r}}$ resp. $(\nu-2r-2)^{2p_{2r}}$ resp. $\nu^{2p_{2r}}$ resp. $(\nu-2)^{2p_{2r}}$ te nemen ν^{2k} resp. $(\nu-2)^{2k}$.

Door in (14) te substitueren:

$$S'_{\nu,2k+2}(x) = \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{(\nu+2r+1)^{2p_{2r+1}} I_{\nu+2r}(x) + 2S'_{\nu+2r+2,0}(x)\} =$$

$$= \sum_{s=2r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r-1)!\,\Gamma(s+\nu)} \left\{-(\nu+2r+1)^{2p_{2r+1}} + \frac{2(s-2r-1)}{2s-2r+\nu-2}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+1} I_{r+2r-1}(x) = \sum_{s=2r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r)!\,\Gamma(s+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{-(\nu-2r-3)^{2p_{2r+1}} I_{\nu-2r-2}(x) + 2S'_{\nu-2r-2,0}(x)\} =$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu-1)} \left\{-(\nu-2r-3)^{2p_{2r+1}} + \frac{2(s-2r+\nu-2)}{2s-2r+\nu-2}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+1} I_{\nu-2r-1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{-(\nu-1)^{2p_{2r+1}} I_\nu(x) + 2S'_{\nu,0}(x)\} =$$

$$= (-1)^r \sum_{s=r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r-1)!\,\Gamma(s-r+\nu)} \left\{(\nu-1)^{2p_{2r+1}} - \frac{2(s-r+\nu-1)}{2s-2r+\nu-2}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+1} I_{\nu-1}(x) = (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{(\nu-1)^{2p_{2r+1}} I_{\nu-2}(x) + 2S'_{\nu,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1^{\,s} \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu-1)} \left\{(\nu-1)^{2p_{2r+1}} - \frac{2(s-r)}{2s-2r+\nu-2}\right\},$$

vindt men na gelijkstelling der coëfficienten van $\left(\frac{x}{2}\right)^{2s+\nu}$ in beide leden de volgende algemene identiteiten (ν willekeurig):

$$\sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)} =$$

$$\left.\begin{aligned}
&= \frac{1}{\Gamma(s+\nu)} \sum_{r=0}^{k} \frac{2^{2r+1}}{(s-2r-1)!} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\quad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \left\{ -(\nu+2r+1)^{2p_{2r+1}} + \frac{2(s-2r-1)}{2s-2r+\nu-2} \right\} + \\
&+ \frac{1}{\Gamma(s+\nu)} \sum_{r=0}^{k} \frac{2^{2r}}{(s-2r)!} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu+2r)^{2p_{2r}+1} =
\end{aligned}\right\} . \quad (19)$$

$$\left.\begin{aligned}
&= \frac{1}{s!} \sum_{r=0}^{k} \frac{2^{2r+1}}{\Gamma(s-2r+\nu-1)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\quad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \left\{ -(\nu-2r-3)^{2p_{2r+1}} + \frac{2(s-2r+\nu-2)}{2s-2r+\nu-2} \right\} - \\
&\quad - \frac{1}{s!} \sum_{r=0}^{k} \frac{2^{2r}}{\Gamma(s-2r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu-2r-2)^{2p_{2r}+1} =
\end{aligned}\right\} . \quad (20)$$

$$\left.\begin{aligned}
&= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} \left[\nu^{2p_{2r}+1} + 2(s-r) \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \left\{ (\nu-1)^{2p_{2r+1}} - \frac{2(s-r+\nu-1)}{2s-2r+\nu-2} \right\} \right] =
\end{aligned}\right\} . \quad (21$$

$$\left.\begin{aligned}
&= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} \left[-(\nu-2)^{2p_{2r}+1} + 2(s-r+\nu-1) \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \right. \\
&\qquad\qquad \left. \left\{ (\nu-1)^{2p_{2r+1}} - \frac{2(s-r)}{2s-2r+r-2} \right\} \right] .
\end{aligned}\right\} . \quad (22)$$

Hierin is, zo $r=0$ is, voor $\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu+2r)^{2p_{2r}+1}$ resp. $(\nu-2r-2)^{2p_{2r}+1}$ resp. $\nu^{2p_{2r}+1}$ resp. $(\nu-2)^{2p_{2r}+1}$ te nemen ν^{2k+1} resp. $(\nu-2)^{2k+1}$.

Daar in (17) voor $r=s$ geldt: $\nu^{2p_{2r}} - \dfrac{2\,(s-r)}{2s-2r+\nu-1} = \nu^{2p_{2s}}$, terwijl deze uitdrukking $=0$ is voor $r \neq s$ en $\nu=1$, evenzo in (18) voor $r=s$ geldt: $-(\nu-2)^{2p_{2r}} + \dfrac{2\,(s-r+\nu-1)}{2\,s-2r+\nu-1} = 2-(\nu-2)^{2p_{2r}}$, terwijl deze uitdrukking $=0$ is voor $r \neq s$ en $\nu=1$, volgt uit (17) zowel als (18) voor $\nu=1$:

$$\left.\begin{aligned}\sum_{p=0}^{s} \frac{(-1)^p\,(2p+1)^{2k+1}}{(s-p)\,!\,(s+p+1)\,!} = (-1)^s\,2^{2s} \sum_{p_1=0}^{k-s} \binom{2\,k}{2p_1+2s} \sum_{p_2=0}^{p_1} \binom{2p_1+2s-1}{2p_2+2s-1} \cdots \\ \cdots \sum_{p_{2s}=0}^{p_{2s-1}} \binom{2p_{2s-1}+1}{2p_{2s}+1}\end{aligned}\right\} . \quad (23)$$

Zo geldt in (22) voor $r=s$: $(\nu-1)^{2p_{2r+1}} - \dfrac{2\,(s-r)}{2s-2r+\nu-2} = (\nu-1)^{2p_{2s+1}}$, terwijl deze uitdrukking $=0$ is voor $r \neq s$ en $\nu=2$. Derhalve volgt uit (22) voor $\nu=2$:

$$\left.\begin{aligned}\sum_{p=0}^{s} \frac{(-1)^p(2p+2)^{2k+2}}{(s-p)\,!\,(s+p+2)\,!} = (-1)^s\,2^{2s+1} \sum_{p_1=0}^{k-s} \binom{2k+1}{2p_1+2s+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2s}{2p_2+2s} \cdots \\ \cdots \sum_{p_{2s+1}=0}^{p_{2s}} \binom{2p_{2s}+1}{2p_{2s+1}+1}\end{aligned}\right\} . \quad (24)$$

Deze beide identiteiten stemmen overeen met die, welke vermeld zijn in I onder (47) en (48).

Substitueren wij in (15) t/m (18), evenzo in (19) t/m (20) $k=0$, dan vinden we (ν willekeurig):

$$\sum_{p=0}^{s} \frac{(-1)^p\,(2p+\nu)}{(s-p)\,!\,\Gamma(s+p+\nu+1)} = \frac{\nu-1}{(2s+\nu+2)\,s\,!\,\Gamma(s+\nu)} \quad . \quad . \quad . \quad (25)$$

en

$$\sum_{p=0}^{s} \frac{(-1)^p\,(2p+\nu)^2}{(s-p)\,!\,\Gamma(s+p+\nu+1)} = \frac{\nu\,(\nu-2)}{(2s+\nu-2)\,s\,!\,\Gamma(s+\nu)} \quad . \quad . \quad . \quad (26)$$

Onderstellen we $k>0$, dan volgt uit (18) voor $\nu=2$:

$$\left.\begin{aligned}\sum_{p=0}^{s} \frac{(-1)^p\,(2p+2)^{2k+1}}{(s-p)\,!\,(s+p+2)\,!} = \frac{2}{(2s+1)\,(s\,!)^2} + \\ +2 \sum_{r=0}^{k-1} \frac{(-1)^r\,2^{2r}}{\{(s-r)\,!\}^2} \sum_{p_1=0}^{k-r-1} \binom{2\,k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+2} \cdots \\ \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} \left[1+2(s-r)\left\{2p_{2r+1}+1 - \frac{2\,(s-r)}{2s-2r-1} \sum_{p_{2r+2}=0}^{p_{2r+1}} \binom{2p_{2r+1}+1}{2p_{2r+2}+1}\right\}\right]\end{aligned}\right\} . \quad (27)$$

Evenzo volgt, zo $k > 0$ is, uit (21) en (22) voor $\nu = 1$:

$$\left.\begin{array}{l}
\displaystyle\sum_{p=0}^{s} \frac{(-1)^p (2p+1)^{2k-2}}{(s-p)!\,(s+p+1)!} = -\frac{1+4sk+4s^2 \displaystyle\sum_{p_1=0}^{k-1} \binom{2k+1}{2p_1+3}}{(2s-1)\,(s!)^2} - \\
\displaystyle -4 \sum_{r=0}^{k-1} \frac{(-1)^r\, 2^{2r}}{\{(s-r-1)!\}^2} \sum_{p_1=0}^{k-r-1} \binom{2k+1}{2p_1+2r+3} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+2}{2p_2+2r+2} \cdots \\
\displaystyle \cdots \sum_{p_{2r+2}}^{p_{2r+1}} \binom{2p_{2r+1}+2}{2p_{2r+2}+2} \left[1+2(s-r-1) \left\{ 2p_{2r+2}+1 - \frac{2(s-r-1)}{2s-2r-3} \sum_{p_{2r+3}=0}^{p_{2r+2}} \binom{2p_{2r+2}+1}{2p_{2r+3}+1} \right\} \right]
\end{array}\right\} . \quad (28)$$

In zekere zin zijn (27) en (28) tegenhangers van (23) en (24).

Mathematics. — *Modern operational calculus based on the two-sided Laplace integral.* I. By BALTH. VAN DER POL and H. BREMMER.

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1. *Introduction.*

The operational calculus, as often used by technicians, goes back to OLIVER HEAVISIDE, who introduced his heuristic methods with very great practical success. Although his approach is far from being mathematically rigorous, HEAVISIDE himself already drew attention to the fact that his methods and procedures could be derived from the Laplace transform [1]).

HEAVISIDE's operational methods were mainly meant as a tool for investigating linear electrical systems to which at the time $t=0$ suddenly an electromotive force was applied, the system being originally at rest. Therefore the transform $f^*(p)$ of the time function $h(t)$, as used by HEAVISIDE and most of his followers, is the following:

$$f^*(p) = p\int_0^\infty e^{-pt}\, h(t)\, dt, \quad \ldots \ldots \ldots \quad (1)$$

with 0 as lower limit of integration.

All the work by BROMWICH, CARSON, VAN DER POL, NIESSEN, WAGNER, HUMBERT, MCLACHLAN and many others in this field is based on the *one-sided* Laplace transform. However, already before 1940 we worked out an operational calculus, well suited for practical applications, which is based *ab initio* on the *two-sided* Laplace transform

$$f(p) = p\int_{-\infty}^\infty e^{-pt}\, h(t)\, dt \quad \ldots \ldots \ldots \quad (2)$$

with $-\infty$ as lower limit of integration instead of 0. Henceforth, the integral relation (2) is shortly written as follows:

$$f(p) \doteqdot h(t), \qquad \alpha < \mathrm{Re}\, p < \beta,$$

where the strip of convergence of the integral (2), viz., $\alpha < \mathrm{Re}\, p < \beta$, has to be specified explicitly. When, moreover, an 'original' $h(x, y)$ of two variables x and y is transformed as

$$f(p, q) = pq\int_{-\infty}^\infty e^{-px}\int_{-\infty}^\infty e^{-qy}\, h(x, y)\, dx\, dy,$$

[1]) O. HEAVISIDE, Electromagnetic Theory, London, Benn Brothers, 1922, Vol. III, p. 236.

we write for short

$$f(p,q) \mathrel{\ddot{\overline{\overline{..}}}} h(x,y). \qquad \text{etc.}^{2)}$$

Although the Laplace transform as such has long been known, there is certainly room for an operational calculus based on this Laplace transform (particularly the two-sided) because the operational (or symbolic) methods often lead in an extremely short way to a solution of complicated problems, once the rules and theorems of this calculus have been mastered. This is true not only for many technical problems, but also for large parts of the analysis, e.g., linear differential equations (both with constant and variable coefficients), difference equations, partial differential equations, integral equations, potential theory, number theory, etc. The situation here is analogous to that of the theory of linear equations or vector analysis where complicated calculations can often be reduced to simple procedures, owing to the introduction of determinants, matrices and concepts such as gradients, curl-vectors.

It is just this new symbolism of the two-sided Laplace transform which shows its great heuristic value, and many new results have been obtained during the eight years of its application [3]).

In practical applications of the operational calculus as expounded below, it is very seldom necessary to refer to the inversion integral corresponding to (2), viz.

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \, \frac{f(p)}{p} \, dp. \qquad (\alpha < c < \beta) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Moreover, an explicit use of the Laplace integral (2) as such is only rarely needed since the available rules usually enable us to find the solution of our problem right away. However, the Laplace transform being the rigorous mathematical basis, every step in the process of an operational

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[2]) We originally introduced the symbol $\doteqdot$ for a one-sided Laplace transform. Some authors use with the same meaning the symbol $\supset$, which, however, might be confused with a similar symbol used in the theory of sets and which has already a different mathematical meaning. We shall therefore adhere to the definition of $\mathrel{\dot{\overline{\overline{.}}}}$ as given above and which therefore represents the two-sided Laplace transform. The upper dot is always towards the original so that we can write either

$$f(p) \mathrel{\underset{\cdot}{\overline{\overline{\cdot}}}} h(t)$$

or

$$h(t) \mathrel{\overset{\cdot}{\overline{\overline{\cdot}}}} f(p),$$

both being short-hand notations for (2).

[3]) An extensive volume on the operational calculus based on the two-sided Laplace transform is now in course of publication at the Cambridge University Press. Several new results were already published during and after the war in the 'Wiskundige Opgaven', Groningen, Noordhoff; 1943, numbers 33, 34, 35, 36, 38; 1946, numbers 77, 120.

solution of a problem can fully be interpreted in terms of these transforms; thus a completely rigorous control of all the steps is always possible.

It is the purpose of the present paper to expound the general lines and also the advantages of this new form of the operational calculus and to point out the improvements and gains with respect to the older form based on the one-sided Laplace integral.

2. *The strip of convergence.*

In the consideration of one-sided Laplace integrals, the indication of the strip of convergence is usually omitted. This will cause no misunderstanding since, in that case, the strip is always a domain of the p-plane reaching to infinity at the right whereas the left boundary $\mathrm{Re}\, p = a$ is determined by the condition of convergence of the one-sided Laplace integral. In the case of two-sided Laplace integrals, however, there may exist several strips for which one and the same image function $f(p)$ corresponds to different originals $h(t)$. Consequently, the specification of the strip is absolutely necessary. In special cases this strip may cover the total p-plane, e.g., when the original is non-vanishing in a finite interval of t only. Less trivial examples showing strips coinciding with the total p-plane, are:

$$e^{-\alpha t^2} \doteq \sqrt{\frac{\pi}{\alpha}}\, p\, e^{\frac{p^2}{4\alpha}}, \qquad -\infty < \mathrm{Re}\, p < \infty,$$

$$\frac{1}{2}\left(\frac{d^2}{dt^2} - \frac{1}{4}\right)\left\{e^{\frac{t}{2}}\, \theta_3(0, e^{2t})\right\} \doteq p\, \xi\, (p + \tfrac{1}{2}), \qquad -\infty < \mathrm{Re}\, p < \infty,$$

where

$$\theta_3(0, x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x},$$

and

$$\xi(p) = (p-1)\, \pi^{-\frac{p}{2}}\, \Pi\left(\frac{p}{2}\right) \zeta(p)$$

is, as usual, defined so that the functional equation for Riemann's ζ-function is equivalent to the observation that $\xi(p + \frac{1}{2})$ is an even function of p.

3. *The unit function.*

The discontinuous function $U(t)$ defined by

$$U(t) = \begin{cases} 1 & (t > 0) \\ \frac{1}{2} & (t = 0) \\ 0 & (t < 0) \end{cases}$$

is called the 'unit function'. It was already considered by CAUCHY, who named it 'coefficient limitateur' or 'restricteur' [4]). For our purpose, this

[4]) Enzyklopaedie Math. Wiss. II, 1, 2, p. 1324.

function is particularly important because in using it we can consider the one-sided operational calculus as a special case of the two-sided calculus. In fact, originals vanishing for $t < 0$ may be written as

$$h^*(t) = h(t)\, U(t),$$

while the corresponding two-sided Laplace transform

$$p \int_{-\infty}^{\infty} e^{-pt}\, h(t)\, U(t)\, dt = p \int_{0}^{\infty} e^{-pt}\, h(t)\, dt,$$

is automatically reduced to a one-sided transform. When using therefore the two-sided calculus, it is not allowed to omit the factor $U(t)$ in one-sided originals (which are therefore zero for $t < 0$). This is clear since a given original $h(t)$ may have an image without more, as well as another image after it has been replaced by zero for $t < 0$. An illustrative example is the following

$$\frac{1}{e^t + 1} \doteqdot -\frac{\pi p}{\sin(\pi p)}, \qquad -1 < \mathrm{Re}\, p < 0, \tag{4a}$$

$$\frac{U(t)}{e^t + 1} \doteqdot \frac{p}{2} \left\{ \psi\left(\frac{p}{2}\right) - \psi\left(\frac{p-1}{2}\right) \right\}, \qquad -1 < \mathrm{Re} p < \infty \tag{4b}$$

(ψ = logarithmic derivative of GAUSS's Π-function).

The unit function also plays a role in many other questions occurring in the operational calculus. In this respect we mention the notation

$$h(t)\ U(t-a)$$

for an arbitrary function $h(t)$ which is made to vanish for $t < a$ (compare the example of section 9).

4. *The shift rule.*

In the two-sided calculus most of the elementary 'rules' have a slightly simpler form than in the older one-sided calculus. E.g., in the differentiation rule (stating that a differentiation of the original corresponds to a multiplication by p of the image) the restriction $h(0) = 0$ of the one-sided calculus can be dropped (see section 7).

A simplification also occurs in the case of the 'shift rule'. In the one-sided calculus, which only concerns positive arguments of the original, this rule reads:

Given $$h(t) \doteqdot f(p),$$

then we have

(1) if $\lambda > 0$,

$$e^{\lambda p} f(p) \doteqdot h(t + \lambda);$$

(2) if $\lambda < 0$,

$$e^{\lambda p} f(p) \doteqdot \begin{cases} h(t+\lambda) & (t > -\lambda) \\ 0 & (0 < t < -\lambda). \end{cases} \tag{5a}$$

In the two-sided calculus, however, the distinction between positive and negative values of λ disappears. The final formulation there simply amounts to:

Given $$h(t) \doteqdot f(p), \qquad \alpha < \operatorname{Re} p < \beta,$$

then we have

$$h(t+\lambda) \doteqdot e^{\lambda p} f(p), \qquad \alpha < \operatorname{Re} p < \beta. \tag{5b}$$

The one-sided rule (5a) is obtained as a special case of the two-sided rule (5*b*) by substituting in the latter $h(t)U(t)$ for $h(t)$.

We give here two applications of the general shift rule (5*b*) illustrating, moreover, the usefulness of the unit function.

(*A*) The construction of the image of 'step functions', i.e., of functions $h(t)$ that are constant between two consecutive integer values of t. As an example we consider the one-sided function that increases by unity at each of the points $t = \log n$ (n integer). This function may be represented by

$$\sum_{n=1}^{\infty} U(t-\log n) = \sum_{n=1}^{\infty} U(e^t - n) = \sum_{n=1}^{[e^t]} 1 = [e^t]$$

($[x]$ = greatest integer not greater than x).

Starting from the fundamental relation

$$U(t) \doteqdot 1, \qquad 0 < \operatorname{Re} p < \infty,$$

which follows at once from (2), the shift rule yields

$$U(t-\log n) \doteqdot e^{-p \log n} = \frac{1}{n^p}, \qquad 0 < \operatorname{Re} p < \infty,$$

so that we have

$$[e^t] = \sum_{n=1}^{\infty} U(t-\log n) \doteqdot \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The image here found is the Dirichlet series for the ζ-function; since this series only converges for $\operatorname{Re} p > 1$, we obtain at once the operational relation

$$[e^t] \doteqdot \zeta(p), \qquad 1 < \operatorname{Re} p < \infty.$$

This transform can be made the basis of a large part of modern arithmetic [5].

(*B*) A function given originally in the interval $0 < t < 1$ can easily be continued periodically outside this interval. In these cases we can start from the relation

$$h(t)\{U(t) - U(t-1)\} \doteqdot f(p), \qquad -\infty < \operatorname{Re} p < \infty.$$

[5]) See BALTH. VAN DER POL, Application of the Operational or Symbolic Calculus to the Theory of Prime Numbers, Phil. Mag. **26**. 925 (1938).

expressing that the given function, which is zero for $t < 0$ and $t > 1$, will in general have some image $f(p)$. The periodic function in question is representable by

$$h(t-[t])\,U(t) = h(t)\,\{U(t)-U(t-1)\} + \\ + h(t-1)\,\{U(t-1)-U(t-2)\} + \\ + h(t-2)\,\{U(t-2)-U(t-3)\} + \ldots$$

The shift rule at once leads to the corresponding image, viz.

$$h(t-[t])\,U(t) \doteqdot f(p) + e^{-p} f(p) + e^{-2p} f(p) + \ldots$$

The summation of this geometric series is possible for all p having positive real part. Thus we arrive at the final result

$$h(t-[t])\,U(t) \doteqdot \frac{f(p)}{1-e^{-p}}, \qquad 0 < \operatorname{Re} p < \infty.$$

5. *The rule for the composition product.*

In the one-sided calculus this rule reads:

Given $\qquad h_1(t) \doteqdot f_1(p) \quad ; \quad h_2(t) \doteqdot f_2(p).$

then it follows that

$$\int_0^t h_1(\tau)\,h_2(t-\tau)\,d\tau \doteqdot \frac{1}{p}\,f_1(p)\,f_2(p). \quad \ldots \ldots \quad (6a)$$

The corresponding rule in the two-sided calculus is simpler insofar as the composition integral (sometimes the term 'convolution' is used) [6]) has constant limits of integration. In fact, the complete rule now becomes:

Given

$$h_1(t) \doteqdot f_1(p), \qquad \alpha_1 < \operatorname{Re} p < \beta_1,$$
$$h_2(t) \doteqdot f_2(p), \qquad \alpha_2 < \operatorname{Re} p < \beta_2,$$

then it follows that

$$\left.\begin{array}{l} \displaystyle\int_{-\infty}^{\infty} h_1(\tau)\,h_2(t-\tau)\,d\tau \doteqdot \frac{1}{p}\,f_1(p)\,f_2(p), \\ \max(\alpha_1, \alpha_2) < \operatorname{Re} p < \min(\beta_1, \beta_2). \end{array}\right\} \quad \ldots \ldots \quad (6b)$$

It has to be stressed that an image of a composition product exists only when the two initial strips of convergence overlap. The existence of the corresponding common strip is guaranteed in the case of one-sided originals

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[6]) We prefer the term 'composition product' (as given by VOLTERRA) instead of 'convolution' because the fundamental principle can be extended to more (and even to an infinite number of) dimensions, in which case the 'folding' idea is lost.

(since each strip extends to infinity in the right part of the p-plane). In the two-sided calculus, however, the condition of overlapping is not self-evident; it is expressed analytically by

$$\max\ (\alpha_1, \alpha_2) < \operatorname{Re} p < \min\ (\beta_1, \beta_2).$$

In this connection we remark that, if a common strip is lacking, the rule (6b) is still applicable after a transformation, of one or both of the primary relations, with the aid of the 'attenuation rule'.

The latter states, in both the one-sided and the two-sided calculus:

Given $\qquad h(t) \doteqdot f(p), \qquad \alpha < \operatorname{Re} p < \beta,$

then it follows that

$$e^{-at}\, h(t) \doteqdot \frac{p}{(p+a)} f(p+a), \qquad \alpha - \operatorname{Re} a < \operatorname{Re} p < \beta - \operatorname{Re} a.$$

An example may illustrate the possibility of such an indirect application of the composition-product rule. We start from the relation

$$e^{-e^{-t}} \doteqdot \Pi(p), \qquad 0 < \operatorname{Re} p < \infty, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

which is easily verified by reducing its Laplace integral to Euler's second integral for the Π-function. According to the rule concerning the transformations of t into $-t$, we have also:

$$e^{-e^{t}} \doteqdot -\Pi(-p), \qquad -\infty < \operatorname{Re} p < 0. \quad . \quad . \quad . \quad . \qquad (8)$$

The strips of convergence of (7) and (8) are not overlapping but adjacent. In order to construct a composition product, we replace (7) by the following relation, obtained with the aid of the attenuation rule,

$$e^{-t}\, e^{-e^{-t}} \doteqdot \frac{p}{(p+1)}\, \Pi(p+1) = p\, \Pi(p), \qquad -1 < \operatorname{Re} p < \infty. \quad . \qquad (9)$$

Now, the relations (8) and (9) have a common strip, viz. $-1 < \operatorname{Re} p < 0$. The composition-product rule can now be used and leads to

$$\int_{-\infty}^{\infty} e^{-\tau}\, e^{-e^{-\tau}} \cdot e^{-e^{t-\tau}}\, d\tau \doteqdot -\Pi(p)\, \Pi(-p), \qquad -1 < \operatorname{Re} p < 0. \quad (10)$$

The substitution $e^{-\tau} = s$ transforms the integral into the original of the relation (4a), so that each of the two functions

$$-\Pi(p)\, \Pi(-p) \quad \text{and} \quad -\frac{\pi p}{\sin(\pi p)}$$

are found as image of $\dfrac{1}{e^t+1}$ (for $-1 < \operatorname{Re} p < 0$). By virtue of the uniqueness of the Laplace integral, we thus have demonstrated the relation

$$\Pi(p)\, \Pi(-p) = \frac{\pi p}{\sin(\pi p)}$$

for $-1 < \mathrm{Re}\, p < 0$. The validity of this formula for other values of p then follows from the principle of analytic continuation.

We conclude our considerations on the composition-product rule with two remarks:

(1) The one-sided form (6*a*) is obtained as a special case of the two-sided form (6*b*) by a substitution of $h_1(t)\, U(t)$ and $h_2(t)\, U(t)$ for $h_1(t)$ and $h_2(t)$ respectively, which substitution automatically introduces the limits of integration of (6*a*);

(2) By identifying $h_2(t) \risingdotseq f_2(p)$ with the operational relations

$$\begin{aligned} U(t) &\risingdotseq 1, & 0 &< \mathrm{Re}\, p < \infty, \\ -U(-t) &\risingdotseq 1, & -\infty &< \mathrm{Re}\, p < 0, \end{aligned}$$

respectively, we get the new *integration rule*:

Given $$h(t) \risingdotseq f(p), \qquad \alpha < \mathrm{Re}\, p < \beta,$$

then it follows that

$$\int_{-\infty}^{t} h(\tau)\, d\tau \risingdotseq \frac{1}{p} f(p), \qquad 0 < \mathrm{Re}\, p < \beta,$$

$$\int_{\infty}^{t} h(\tau)\, d\tau \risingdotseq \frac{1}{p} f(p), \qquad \alpha < \mathrm{Re}\, p < 0.$$

6. *General advantages of the two-sided calculus.*

Some very striking advantages are:

(1) A simple formulation of the general operational rules (compare the two preceding sections).

(2) The possibility of treating functions whose two-sided image is simpler than their one-sided image or whose one-sided image even is lacking. In this respect we refer to the examples (4) and (7). Another typical two-sided original is

$$\frac{1}{e^{e^{-t}} - 1} \risingdotseq \Pi(p)\, \zeta(p), \qquad 1 < \mathrm{Re}\, p < \infty,$$

while many other relations of this kind are dealt with in section 10.

(3) The possibility of considering images having no original at all in the one-sided calculus. The increase of the number of available image functions follows, e.g., from the theorem that in the one-sided calculus there do not exist images $f(p)$ having equidistant zeros on a line parallel to the real p-axis. Such a restriction does not occur in the two-sided calculus. An example showing such a two-sided Laplace transform with an infinity of zeros on the real p-axis itself, is given by

$$2^{-\frac{p}{2}} \sin\left(\frac{\pi}{4} p\right) \Pi(p) \risingdotseq e^{-e^{-t}} \sin(e^{-t}), \qquad -1 < \mathrm{Re}\, p < \infty.$$

Botany. — *Some remarks on Nummulites javanus Verb. and Nummulites perforatus de Montf.* By TJ. VAN ANDEL. (Communicated by Prof. PH. H. KUENEN.)

(Communicated at the meeting of September 25, 1948.)

Abstract.

Specimina of *Nummulites perforatus* de Montf. from Timor have been studied in order to solve the problem of the systematic position of *N. javanus* Verb. This species is found to belong to *N. perforatus*. The opinion of DOORNINK, who considers it to be partly N. perforatus and partly N. gizehensis Forsk. is rejected. Several other theories on this point are discussed and rejected. *N. perforatus* from Timor occurs in two varieties, 1 and 2, which are identical with γ and δ of Verbeek (var. 1) and β Verb. (var. 2). The existence of the name *N. javanus* Verb. is not justified.
The megalospheric form belonging to *N. perforatus* is *N. bagelensis* var. Ia Verb.

1808 *Eogon perforatus* Denys de Montfort: Conchyliologie systématique, t. I, p. 166—167.
1826 *Nummulina perforata* Montf. d'ORBIGNY: Tableau méthodique de la classe des Céphalopodes. Ann. des Sc. Nat. VII, p. 296. Forma A.
1840 *Nummulites obtusa* J. de C. SOWERBY: Systematic list of organic remains of Cutch. Transact. Geol. Soc. of London (2)V, p. 329. Forma B.
1853 *Nummulites perforatus* d'Orb. d'ARCHIAC et HAIME: p. 115—120.
1881 *Nummulites perforatus* d'Orb. DE LA HARPE: VIII, p. 130—140.
1896 *Nummulites javanus*, VERBEEK et FENNEMA: p. 1096 Forma B.
1896 *Nummulites bagelensis* Ia VERBEEK et FENNEMA: p. 1101, Forma A.
1912 *Nummulites laevigatus* pars, DOUVILLÉ: p. 261 Forma B.
1915 *Nummulites bagelensis* II, var. megaspherica RUTTEN: in WATERSCHOOT VAN DER GRACHT: p. 53 Forma A.
1915 *Nummulites Vredenburgi* Prever pars. DOLLFUSS: p. 15.
1926 *Nummulites obtusus* Sowerby, NUTTALL: p. 138 Forma B.
1929 *Camerina obtusa* Sowerby GERTH: p. 592—593 Forma B.
1929 *Camerina gizehensis* Forsk GERTH: ibid Forma B.
1932 *Camerina perforata* de Montf. DOORNINK: p. 6 Forma B.
1932 *Camerina gizehensis* Forsk. DOORNINK: ibid Forma B.
1932 *Camerina bagelensis* Ia Verb. DOORNINK: ibid Forma B.
1934 *Camerina javana* Verb. CAUDRI: p. 63—64 Forma B.
1934 *Camerina perforata* de Montf. HENRICI: p. 21—25 Forma A. et B.

Other synonyms see BOUSSAC (1911)

Since in 1896 VERBEEK and FENNEMA based their description of *Nummulites javanus* (p. 1096) on javanese specimina, this species has been the subject of much discussion. Soon it appeared to be identical with previously described European species in so many respects, that its independance was almost universally denied. Only CAUDRI in 1934 (p. 64) still maintained the name *Camerina javana* (Verb.). The question to which species it thus belonged led to a long and complicated discussion. GERTH (1929, p. 592) believes it to be partly *N. obtusus* Sowerby, partly *N. gizehensis* Forsk. DOUVILLÉ (1912, p. 261) believes one of VERBEEK's varieties can be identified as *N. laevigatus* Bruguière. DOLFUSS considers *N. javanus* as a transitional form between *N. Vredenburgi* Prever of the *laevigata*-group and the group of *perforata* de Montf. (1915, p. 15). DOORNINK spreads the varieties of *N. javanus* over *N. perforatus* de Montf. and *N. gizehensis* Forsk (1932, p. 6). And HENRICI rejects all these ideas and reckons *N. javanus* as a whole to be part of *N. perforatus* (1934, p. 25).

Forma A also shares in the nomenclatory confusion, although the question has drawn attention in a lesser degree. Mention of this form is very scarce. VERBEEK and FENNEMA described also a group of forms together under the name of *N. bagelensis*, divided in two groups, each with a megalospheric (Ia, IIc) and a microspheric (Ib, IId) form. Already the authors thought it possible that one of those types belongs to *N. javanus* as its A-form. RUTTEN (1914, p. 53—55) found *N. javanus* and *N. bagelensis* together in the same rock without any other Nummulitidae, but did not conclude that they belong together. And DOORNINK adds *N. bagelensis* (Ia) to that part of *N. javanus* that he reckons to be *N. gizehensis* (1932, p. 10). HENRICI, although rejecting the conclusion of DOORNINK concerning *N. gizehensis*, adds *N. bagelensis* Ia to *N. javanus*, under the name of *N. perforatus*.

With the help of extensive material collected by D. TAPPENBECK in the Mollo region on Dutch Timor and used stratigraphically in his thesis (1939) the present author has tried to solve the problem of the systematic place of *N. javanus* Verb., deciding in favour of the opinion of HENRICI (1934) in spite of later objections by CAUDRI (1934).

The material appeared on examination to consist of two closely related microspheric and one megalospheric form. Both microspheric types, being identical in all important characters, belong without doubt to the same species, forming two varieties (1 and 2) of it.

D e s c r i p t i o n.

Forma B, var. 1 (fig. 1, 2)

Shape: disc lenticular or flat, often saddle-shaped or with undulating border. Edge sharp. One side often flatter than the other.

Surface: with strongly curved or meandriform raised lines, joining in one or more points. Often not very clear or lacking. Surface in that case smooth, some structure visible only after etching with HCl. Granulations visible on the border, after etching also on the whole shell, numerous, irregularly distributed.

Septal filaments visible after grinding down part of the shell, sometimes more or less meandriform, branching in the direction of the border, sometimes reticulate, branching and anastomosing in elongated and irregular meshes; nearer and more parallel to the median layer of the shell simply curved and furcated, radiating in whirling shapes from

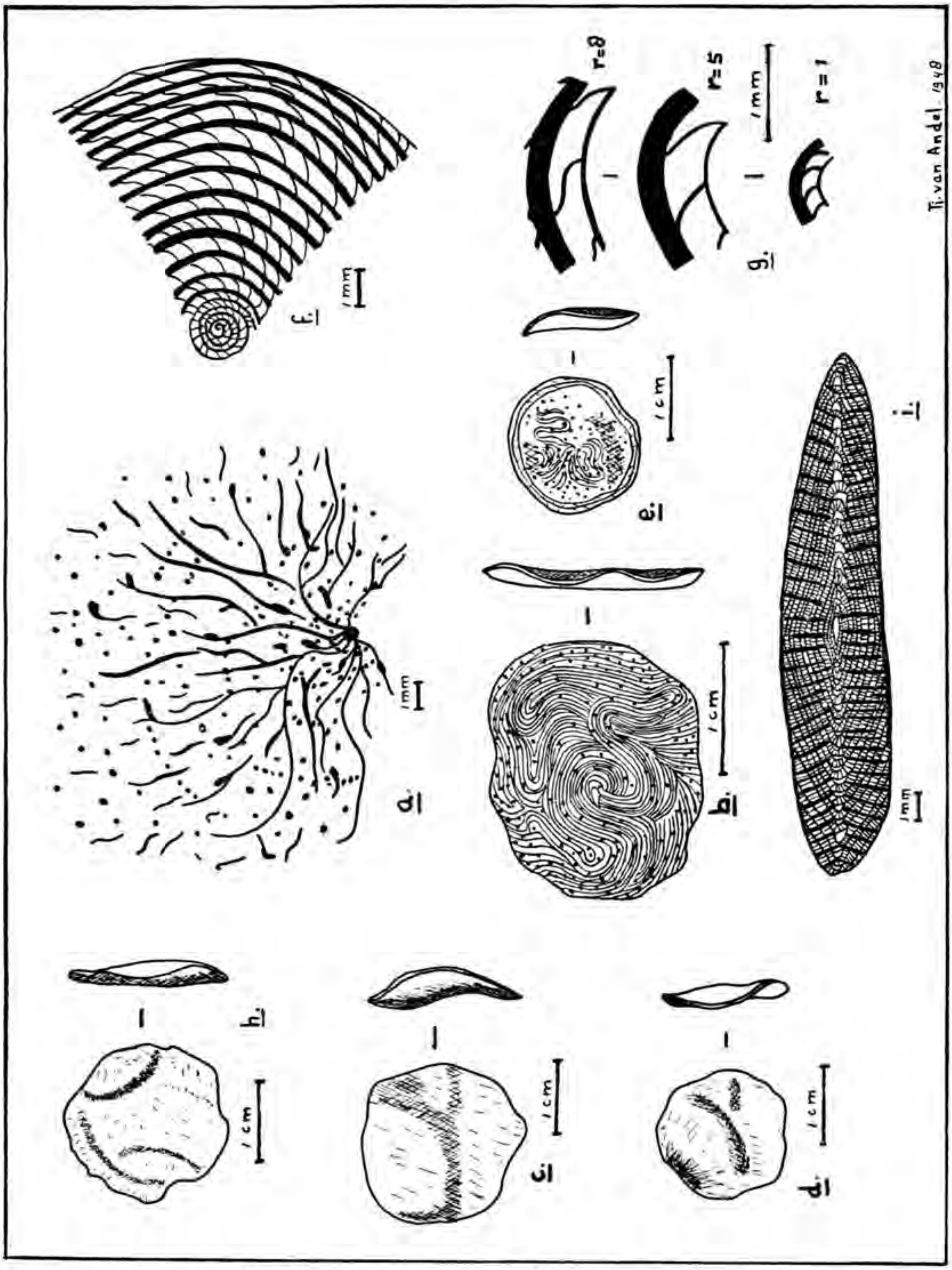

Fig. 1. Nummulites perforatus de Montf. Forma *B*, var. 1.

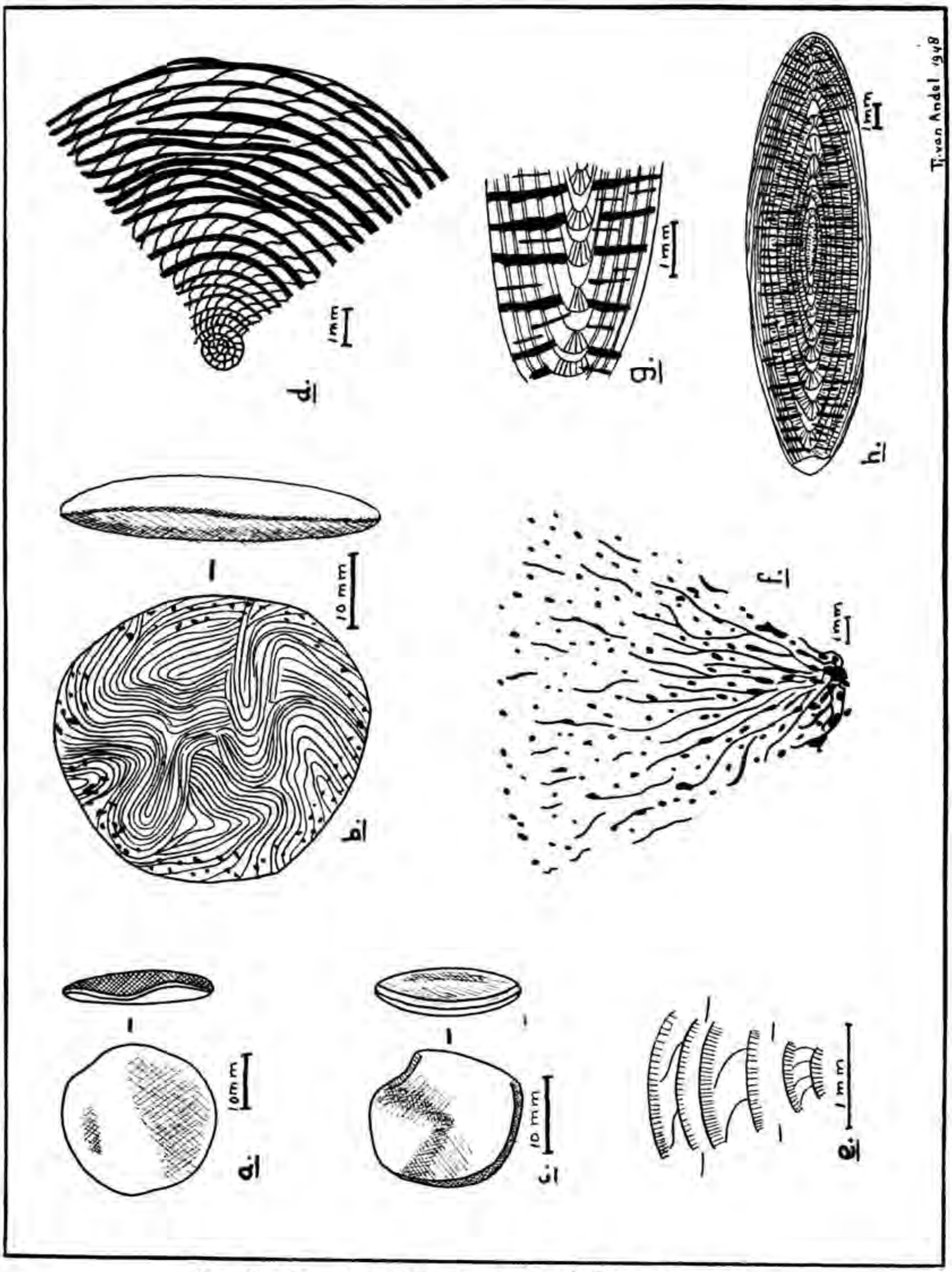

Fig. 2. Nummulites perforatus de Montf. Forma *B*, var. 2.

a central point. Sections near the median layer always show this type, even when meandriform near the surface.

Pillars numerous, irregular, mostly flattened, concentrated in the centre, mainly placed between the septal filaments but also for some part upon them. Mostly placed on the outside of each whorl.

Whorls in the centre evenly and narowly wound. Gradually widening. Sometimes on the periphery narrower again. Regularly increasing chamberheight or a constant height from a certain distance from the centre are also often found. Whorls on the outer part of the spiral irregular, no longer circular. The undulation of the whorls, sometimes described, is caused by sectioning an undulating median layer. Sometimes extra whorls between the normal ones occur, mostly shorter than one winding. Distance between two whorls very variable; even in one whorl. Number of whorls $1\frac{1}{2} - 2 \times$ mm radius.

Septa thin and strongly curved, falcate or *S*-shaped; in the central part mostly perpendicular to the preceding whorl, more oblique up to 45° in the outer part; meeting the next whorl under 45°. Always widely spaced.

Chambers near the centre higher than long or as high as long, on the periphery always two or more times as long as high (height measured parallel to the radius of the shell, length measured tangentionally. Central chamber invisible. Chamberform very characteristic, falcate with sharp corners.

Spiral wall with variable thickness, always less than the chamberheight. In an *axial section* spiral walls very thick; touching each other everywhere, except in the median layer. Pillars numerous, number increasing in the direction of the median layer, often reaching the surface even in the central part of the shell, cylindrically, generally placed on the outer side of each whorl.

Forma B, var. 2 (fig. 3, 4)

Only small difference with the above described specimina. The differences are as follows. Disc flat or lenticular, often rather globular. Thickness much greater, as compared to the diameter, than in var. 1. Edge obtuse or rounded. Granulations even after etching only on the border part. Number of whorls slightly greater, number of chambers per whorl less. Average diameter greater, whorls more irregular, loose extra whorls often occurring, chamberheight very variable. In an axial section spiral walls mostly not touching, pillars in the central part not reaching the surface. Further particularities in the figures.

Both types are thus very similar, especially concerning the important features of septa and shape of the chambers. The most important and constant difference lies in the shape of the shell. Other differences in diameter, number of whorls and chambers are only statistically discernable.

In the table below some important numerical data are given.

A comparison of the specimina described with older descriptions shows the great resemblance to *Nummulites javanus* Verb. (VERBEEK and FENNEMA 1896, p. 1096) Dimensions, shape, surface structure, septal filaments, distribution of pillars, form, and number of chambers and whorls are completely identical with their description and figures (Pl. III 45—47, IV 56—68, V 69—73, VII 94). VERBEEK distinguishes four varieties, two, α and β, with obtuse edge and great thickness in relation to their diameter, the others thin and with sharp edge. Our var. I contains specimina of Verbeeks γ and δ together with many transitions. We were not able to find any useful limit between those two varieties. Var. I from Timor is thus identical with γ and δ Verbeek.

TABLE I.

	Var. I			Var. II		
	Number of specim.	Aver.	Min. max.	Number of specim.	Aver.	Min. max.
Diameter	22	18 mm	9 – 25	14	20,3	15 – 30
Thickness	21	4	2,5 – 7	11	6,6	5 – 9
Number of whorls	21	14	7 – 20	13	19	11 – 31
Whorls at $r = 5$ mm	14	10	6 – 13	7	11	9 – 12
$r = 10$ mm	9	15 – 16	12 – 20	7	18	15 – 20
Whorls from centr. – 1 mm	13	5	4 – 6	6	5	5 – 6
1 – 3 mm	13	4 – 5	3 – 9	7	4	3 – 5
3 – 10 mm	7	8	6 – 17	5	9	6 – 11
Septa per 1/4 whorl $r = 5$	13	11 – 12	8 – 13	7	11	5 – 14
$r = 8$	10	15	12 – 16	5	12	8 – 16
Chamberheight × length $r = 1$	13	0,25 × 0,25	0,15 – 0,5 × 0,15 – 0,3	7	0,28 × 0,21	0,2 – 0,35 × 0,15 – 0,38
$r = 5$	13	0,45 × 0,60	0,30 – 0,5 × 0,50 – 0,8	7	0,46 × 0,64	0,35 – 0,55 × 0,45 – 0,85
$r = 8$	10	0,40 × 0,90	0,30 – 0,6 × 0,50 – 1,2	6	0,57 × 1,00	0,50 – 0,75 × 0,80 – 1,15
Thickness spiralblade	13	0,35	0,10 – 0,30	7	0,30	0,15 – 0,50

Var. 2 from Timor shows a resemblance to var. β Verb. in some points (number of whorls, shape, edge, dimensions). A few specimina however possess the greater dimensions of var. α Verb. (33 mm). Pure specimina of this variety are not found, so I cannot decide upon the problem of its autonomy. DOORNINK (1932, p. 5) too noticed the vagueness of VERBEEK's varieties.

The following table contains a comparison between VERBEEK's data and the dimensions of the Timor Nummulitidae. It is clear that even the largest specimina of var. 2 differ still much with var. α Verb.

TABLE II.

	Nummulites javanus Verb.				Spec. from Timor			
					1		2	
	α	β	γ	δ	*a*	*b*	*a*	*c*
Border	obtuse	obtuse	sharp	sharp	sharp	sharp	obtuse	obtuse
Diameter	22	21	21	16.5	18	16	20	30
Thickness	7	7 – 9	4 – 5	4.5	4	4	6.5	9
Whorls at $r = 5$ mm	14 – 15	11	12 – 13	10 – 12	10	11	11	31
$r = 10$ mm	25 – 28	21	21 – 22	—	15 – 16	—	18	19
Septa to 1/4 whorl $r = 5$ mm	12 – 16	14 – 15	9 – 10	10 – 12	11	11	11	14
$r = 10$ mm	30 – 36	25	16 – 20	—	17	—	14	18

Column *a* of the Timor specimina contains the average values, column *c* the largest values, column *b* the smallest individuals. The conclusion seems justified that the Timor Nummulitidae are identical with *N. javanus* Verb. Furthermore we can distinguish two varieties 1 ($= \gamma$ and δ Verbeek) and 2 ($= \beta$ Verbeek). The position of α Verb. remains obscure.

We have to consider now the theory of DOORNINK (1932, p. 6) who wants to spread *N. javanus* over *N. perforatus* de Montf. and *N. gizehensis* Forsk. Close inspection of his arguments shows this view to be untenable as appears already from the strong resemblance between the two varieties. This opinion is also expressed by HENRICI (1934, p. 25).

Descriptions of both species by DE LA HARPE (1880—1881, VIII, p. 115, 1883, p. 32—49); BOUSSAC (1911 p. 74) and ROZLOZSNIK (1926 p. 170, 220, 1929, p. 43, 47) show many points of resemblance. Both possess a sharp or obtuse edge, dimensions are quite identical. DOORNINK (1932, p. 6) considers the possession of one flat and one convex side to be characteristic for *N. gizehensis*, but *N. perforatus* also often shows this feature. The septal filaments of *N. gizehensis* are mostly more meandriform, but this is also found sometimes with *N. perforatus*. Both the specimina illustrated by VERBEEK (1896, Pl. III, 49, 51, 54; Pl. IV 58, 63; Pl. V 71) and those from Timor are within the range of variation of *N. perforatus*. Granulations in both *perforatus* and *gizehensis* lie mainly between and not (as DOORNINK (p. 6) mentions for *gizehensis*) upon the filaments. Studying the figures of BOUSSAC, DE LA HARPE and ROZLOZSNIK there can be no doubt on this point. DOORNINK himself gives no figures of the septal filaments of his specimina.

The most important difference between both species is the shape of the chambers, a feature not considered by DOORNINK. In the chambers of *N. gizehensis* the height is always more than the length, at best they are equal. Septa are straight or only slightly curved and approximatively perpendicular to the spiral wall. The chambers thus show a typical arcade form (DE LA HARPE 1880—1881, p. 115, 1883, p. 32). Septa always close together. *N. perforatus* on the other hand has chambers many times as long as high, and strongly curved or falcate septa, which meet the spiral wall at an angle of about 45°. Chambers thus low and falcate. Both the Timor material and the figures of VERBEEK (pl. III, IV, V) and DOORNINK (pl. II) show this perforata chambertype. This important characteristic (BOUSSAC 1911, p. 8) enables us to join without any doubt our specimina, together with DOORNINK's and VERBEEK's material, to *N. perforatus*.

BOUSSAC records the occurrence of two main types of *N. perforatus*, corresponding with our varieties 1 and 2. Considering this complete resemblance to an European species the maintaining of *N. javanus* as a species of its own, as CAUDRI (1934, p. 64) does, loses its importance. In India specimina of *N. perforatus* are found (NUTTALL 1926) under the name *Camerina obtusa* de Sow, a synonymy for the microspheric form of *N. perforatus*), which bridge the gap between Europe and the East-Indian Archipelago. Remains the opinion of DOUVILLÉ, who combines part of *N. javanus* with *N. laevigatus* Bruguière, basing his opinion upon the occurrence of pillars on the septal filaments (1912, p. 261). This opinion lacks other arguments and the fine and regular meshes of the filamental net of *laevigata* with all pillars on the points of junction, compared with the coarse net of *perforata*, strongly opposes it. The idea of DOLFUSS (1915, p. 15) that *N. javanus* would form a transition between *N. perforatus* and *N. laevigatus*, must be rejected on the same arguments.

The unquestionable relation between *N. laevigatus* and *N. perforatus* leads BOUSSAC to the idea, that the latter is the result of further evolution of *N. laevigatus* (1911, p. 75). Because of its different chamberform lies *N. gizehensis*. outside this group. ABRARD (1928, p. 89) supposes *perforatus* to originate from *gizehensis* in the same way as this is the case with *N. Brogniarti* and *N. laevigatus*. But the only argument, the very slight resemblance between *gizehensis* and *Brogniarti*, is of very little value.

Less complicated is the case of the related megalospheric form, also found in great quantities in the Timor material. Except these A. and the above described B. forms only very few Nummulitidae are found so the relation seems fairly well established.

D e s c r i p t i o n (fig. 3)

Shape globular to thick lenticular or double conical, edge sharp.

Surface smooth. *Septal filaments* visible after some grinding, *S*-shaped, sometimes branching and curved as forma *B*, sometimes relatively straight and radial, in centre joining in a whirl.

Pillars irregular, not numerous, concentrated in the central part, mostly on the outside of the spiral wall, and placed between the septal filaments.

Whorls very regular, chamber height only slightly increasing from centre to border. Chamberlength regularly increasing, chambers higher than long.

Central chamber egg-shaped and fairly large. *Septa* thin, falcate, perpendicular to the preceding spiral wall. In an axial section a few cylindrical pillars are visible, mostly in

TABLE III.

	Timor			*N. gizeh.* Forma *A.* after DOORNINK		
	number	aver.	max. and min.	number	aver.	max. and min.
Diameter	15	4,2	3—6,8	4	4,3	3,7—4,5
Thickness	14	2,2	1,7—2,8	2	2,1	2,0—2,3
Whorls	13	5	4—6	4	5	4—6
Chambers i/th 1st whorl	15	3	3	2	3	3
3rd whorl	7	7	6—8	2	7	7
5th whorl	5	10	9—10	1	9	9
Chamberheight × length 1st whorl	5	0,28×0,30	0,35—0,15 × 0,35—0,55	2	0,20×0,33	0,15—0,24 × 0,3—0
Chamberheight × length 3rd whorl	6	0,32×0,45	0,25—0,38 × 0.35—0,52	2	0,40×0,40	0,38—0,45
5th whorl	7	0,25×0,60	0,22—0,30 × 0,50—0,75	1	0,30—0,53	—
Central chamber	12	0,70×0,60	0,55—0,90 × 0,50—0,85	3	0,65×0,60	0,60—0,75 × 0,60

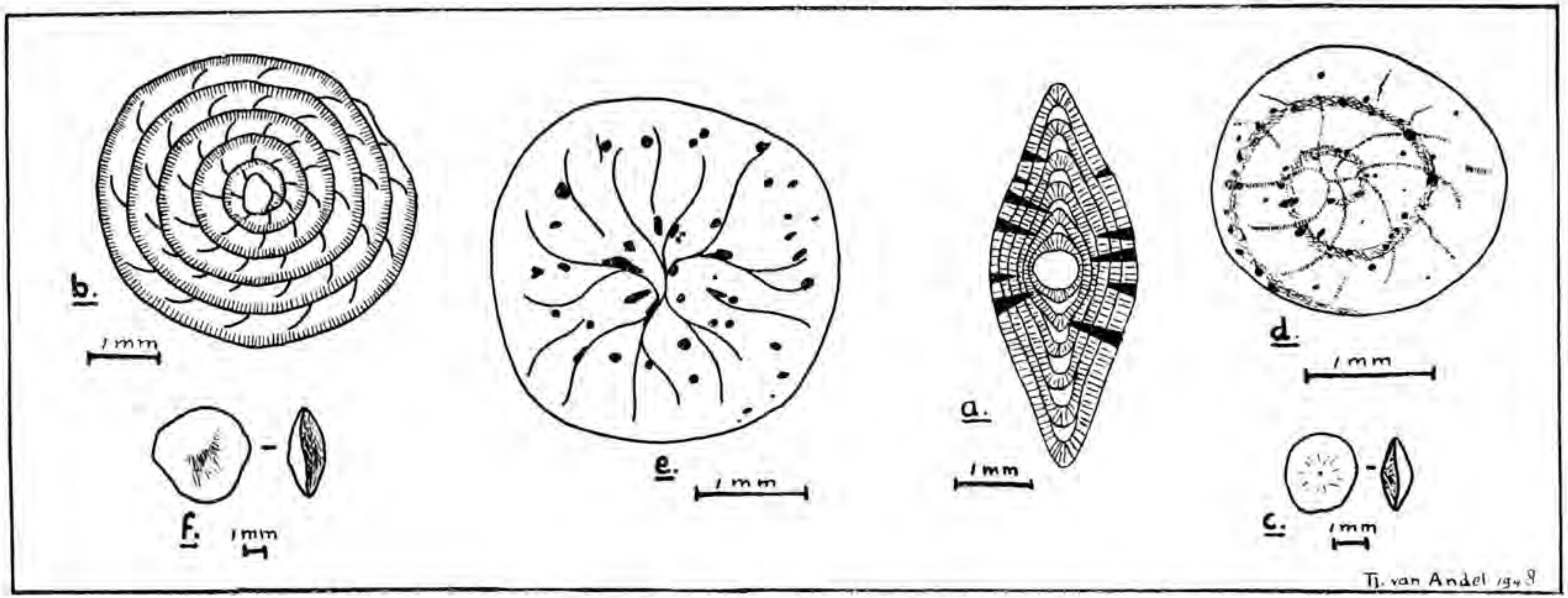

Fig. 3. Nummulites perforatus de Montf. Forma *A*.

the central part and reaching the surface. Spiral walls thick, touching each other, structure therefore compact. No central column.

The shape of chambers and septal filaments is completely identical with forma B. of *N. perforatus*. This and the occurring together of both forms is in favour of combining the two. I consider it logical to give both forms of one species the same name. The name *N. obtusus* Sowerby, given to the B. form, is the youngest and must be rejected in favour of *N. perforatus*, originally given to the A. form, for both.

If we compare this forma A. from Timor with *N. bagelensis* VERB. (VERBEEK 1896, p. 110) the resemblance to *N. bagelensis* Ia in form and dimensions of chambers and septal filaments is very clear. They are without doubt identical. HENRICI too describes the same form from Timor and declares it to be *N. bagelensis* Verb. Ia and the A. form of *N. perforatus*. On Celebes RUTTEN (1914, p. 54) found a variety of *N. bagelensis* together with *N. javanus* var. γ. He describes it as a variety apart, called megasferica, part of *N. bagelensis* II Verb. because of its dimensions (2,5—3,5 mm). The difference with the normal *N. bagelensis* IIc is a smaller number of whorls (3—4 against 4—6 for IIc), no central column and a central chamber with the same dimensions as var. Ia Verb. (0,50—0,80 mm against IIc 0,10—0,30 mm). All those differences however bring it to var. Ia Verb., so the A-form RUTTEN found to *N. javanus* γ is the same as the one found at Timor.

Only in the work of DOORNINK we find any further observations concerning this A-form. By the courtesy of Prof. H. A. BROUWER, director of the Amsterdam Geological Institute I was able to study the original thin sections made by DOORNINK. This and the figures in his book (Pl. II, 1—2) leads to the conclusion, that this form cannot belong to *N. gizehensis* owing to the great difference in chamberform, which is of the *perforatus* type. The A-form of *N. gizehensis* (*N. curvispirus*, cf. also ROZLOZSNIK 1929, p. 220, and DE LA HARPE 1883, p. 32) shows the arcade form, as is definitely stated by the authors. In other features (form, dimensions, number of whorls) the two species resemble each other much, thus explaining the incorrect determination.

Amsterdam, March 1948.

Geological Institute.

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Paleontology. — *Pleistocene Vertebrates from Celebes.* I. *Celebochoerus heekereni nov. gen. nov. spec.* By D. A. HOOIJER. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of September 25, 1948.)

Some months ago, through the courtesy of Prof. Dr A. J. BERNET KEMPERS, Head of the Archaeological Survey of the Dutch East Indies, I received a small collection of fossil vertebrate remains that were recently found by Mr. H. R. VAN HEEKEREN, prehistorian to the said Survey, at Desa Beru and at Sompoh (12 km N. of Beru) near Tjabengè (Sopeng district), between the Walanae river and the Singkang depression, about 100 km N.E. of Macassar in S. Celebes. The exact stratigraphical position of the specimens is uncertain since they were not found in situ but consist of surface finds. They were, however, associated with stone flakes identical to the uppermost Middle, or Upper Pleistocene Sangiran culture of Java (DE TERRA, 1943, p. 456). Notes on the matrix still adhering to some of the specimens will be given in connection with the descriptions of these fossils.

The specimens collected by Mr. VAN HEEKEREN are the first Pleistocene vertebrates ever found in the island of Celebes, and, therefore, they are of paramount importance in relation to problems of paleozoogeography. The recent fauna of Celebes, with its peculiar intermixture of Oriental (at least three fourths of the total) and Australian elements as well as several endemic genera, has required much racking of the brains on the part of zoogeographers who at various times classed Celebes with the Oriental region, the Australian region, or with a transition zone between the two (MAYR, 1944). Not less than four land bridges past which the various species reached the island were demanded by P. and F. SARASIN (1901), viz., the Java bridge, the Philippine bridge, the Moluccas bridge, and the Flores bridge. However, only the second and third of these may have formed a continental connection and thus may have been used by terrestrial animals (STRESEMANN, 1939).

The Pleistocene fauna of Celebes, as far as at present known from Mr. VAN HEEKEREN's collection, consists of a peculiar new giant suid that forms the object of the present paper, and a fossil Babirusa, a small archidiskodont elephant, a pigmy buffalo apparently representing the Anoa, and a giant tortoise the descriptions of which are forthcoming.

The Pleistocene fossils add at least one genus of "Oriental" (western for Celebes) origin, viz., *Archidiskodon*, and also one apparently endemic genus (*Celebochoerus*) to the fauna of Celebes. It is important to note that none of the fossil species found in Celebes (with the possible exception of the tortoise) is closely related to or identical with any member of the Pleistocene fauna of Java.

Celebochoerus nov. gen.

Diagnosis: A giant suid with upper canines subtriangular in cross section and only slightly constricted at the pulp cavity. The anterior surface is at right angles to the upper and is but slightly narrower than the latter. Curvature of the canines like in *Sus*: the broad upper surface, with a median longitudinal groove, is situated at the inner curve. The lower canines, however, are small relative to the upper, as in *Phacochoerus*. Upper and anterior surface of upper C completely coated with an enamel layer that has less developed or is absent posteriorly.

Genotype: *Celebochoerus heekereni* nov. spec.

Celebochoerus heekereni nov. spec.

Diagnosis: The specific diagnosis is the same as the generic diagnosis presented above.

Holotype: The base of a left upper canine figured in the present paper (pl. I, figs. 1—3, textfig. 1*a*).

Paratype: Portion of right upper canine figured pl. I, figs 4—6 and textfig. 1*b*.

Locality: Desa Beru, Tjabengè (Sopeng district), about 100 km N.E. of Macassar, S. Celebes.

Age: Pleistocene.

A giant pig is represented in the collection by two portions of upper C, one of the left and one of the right side. In the left specimen (pl. I, fig. 3) the pulp cavity is shown at one end. The greatest length of this specimen is 62 mm. At the distal end the tooth has broken off obliquely but presented an almost flat surface that has been polished by me to be able to study the cross section more in detail.

The tooth is subtriangular in cross section with the angles rounded, and thus presents three surfaces for description, the anterior, the upper and the posterior according to the position of the upper canine in the maxillary. The anterior surface stands about at right angles to the upper surface and is the narrowest of the three; the posterior surface then, of course, is the widest. The tooth is but slightly curved, with the upper surface situated at the inner curve, and is not twisted.

The anterior surface is almost flat except for a shallow longitudinal groove at one-third of its width from the anterior upper angle. The anterior surface passes into the upper and the posterior surface by well-rounded angles; the angle between the anterior and the posterior surface is more gradually rounded off than the anterior upper angle.

The upper surface of the canine has a wavy outline due to the presence of a shallow longitudinal median groove.This groove occupies the middle third of the upper surface but is at the most hardly two mm in depth, being least marked at the proximal end of the tooth. At the distal end it is somewhat more distinctly marked off posteriorly than anteriorly.

The posterior surface passes gradually into the upper and the anterior surface of the tooth; below the rounded posterior upper angle it is flattened for about one-third of its width but then presents a distinct longitudinal groove which is nearer to the lower than to the upper margin. This posterior longitudinal groove is almost as deep as the upper but is narrower. Like the upper groove it becomes more distinct when passing outward. It is separated from the flat portion above it by a marked ridge. Its lower margin is finely grooved longitudinally.

After the distal surface of the tooth fragment had been polished it showed the tooth to be built up from concentrical layers of dentine. The anterior and upper surfaces are coated with an enamel layer slightly over one mm in thickness, that cannot be traced along the posterior surface. The dimensions of the tooth in cross section are: 33 mm horizontally, and 31 mm vertically when taken in the middle of the fragment. The tooth is a little constricted at the pulp cavity.

The second specimen of upper C (pl. I, figs. 4—6, textfig. 1*b*) agrees with the holotype in every important character and differs only in the

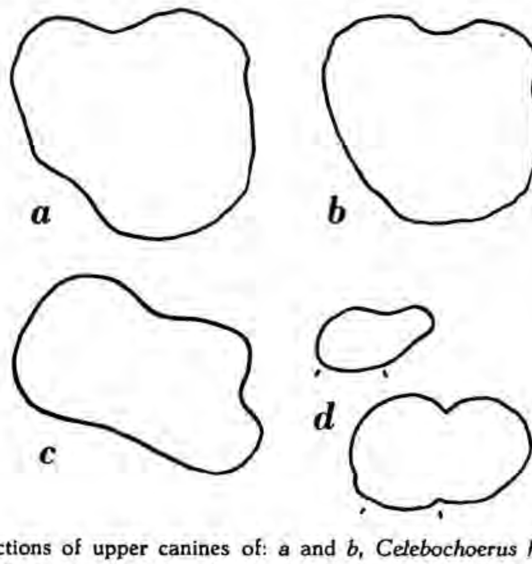

Fig. 1. Cross sections of upper canines of: a and *b*, *Celebochoerus heekereni* nov. gen. nov. spec.; *a*, left canine (holotype); *b*, right canine (paratype); c, *Phacochoerus aethiopicus* (Pallas), Leiden Museum, cat. ost. n; *d*. *Sus cristatus vittatus* Boie, Leiden Museum, reg. no. 872, at pulp cavity and at alveolar margin. In all figures the anterior surface is to the right, and the upper surface above. Natural size.

configuration of certain details that can safely be regarded as individual. The fragment belongs to a right upper canine and is 70 mm in length. There is an extensive, transversally concave, wear facet caused by the abrasion of the lower canine, that extends almost all over the anterior surface leaving only a narrow proximal strip of that surface intact. Added to that the lower surface of the tooth, viz., that where the anterior and

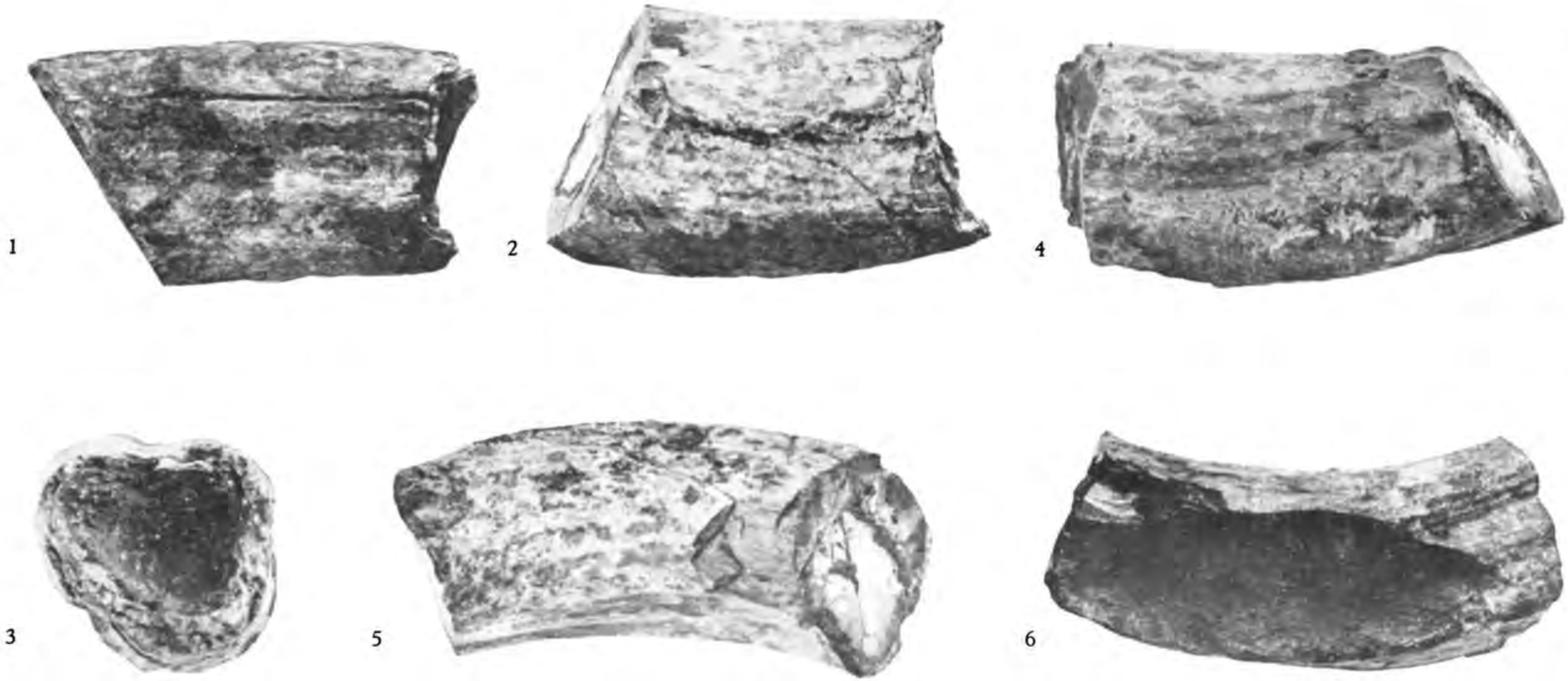

Fig. 1—6. *Celebochoerus heekereni* nov. gen. nov. spec.; figs. 1—3, left upper canine (holotype); fig. 1, upper view; fig. 2, posterior view; fig. 3, pulp cavity; figs. 4—6, right upper canine (paratype); fig. 4, upper view; fig. 5, posterior view; fig. 6, anterior view. All figures natural size.

the posterior surface pass into one another, is superficially damaged. Like the foregoing specimen it is slightly wider (horizontal diameter 28 mm) than high (vertical diameter 27 mm) at the narrowest proximal part where the cross section was taken; it increases distinctly in diameters toward the distal or outer end that is unfortunately incomplete because of wear and damage. The present tooth fragment is more curved and smaller than that described above, and is slightly twisted too, to the effect that the groove in the upper surface is nearer to the anterior margin at the distal than at the proximal end.

As is evident from the inspection of the cross sections of the fossil specimens presented in fig. 1*a* and *b* the median upper groove is more marked off on either side in the paratype (*b*) than it is in the holotype (*a*), but it is equally shallow. The anterior longitudinal groove, too, has more distinctly developed in the paratype, but it is found in the same position (at one-third of the width) on the anterior surface that is at right angles to the upper surface. In addition, the anterior surface of the paratype is faintly striated longitudinally. Instead of one well-marked longitudinal groove in the lower half of the posterior surface as found in the holotype we find a few striae on the posterior surface of the paratype which, therefore, is more gradually rounded from side to side than the posterior surface of the holotype.

The polished proximal surface of the paratype shows the enamel layer at the upper and anterior surface to be thicker than that in the holotype, even exceeding 2 mm in thickness posteriorly of the upper longitudinal groove. In the paratype the enamel coating is also shown on the posterior surface, though barely half a mm in thickness.

The present fossil Celebean canines are clearly distinct from recent male upper canines of *Sus* and *Potamochoerus* [1]) in being much less dorsoventrally compressed and in having a continuous enamel layer along the upper and anterior surfaces. The male upper C in *Sus scrofa* L., *Sus cristatus* Wagner, *Sus verrucosus* Müller et Schlegel, *Sus barbatus* Müller, *Sus celebensis* Müller, *Potamochoerus porcus* (L.) and *Potamochoerus larvatus* (Cuvier) is subquadrangular to sub-oval in cross section, the horizontal diameter often considerably exceeding the vertical. There is a median longitudinal groove in the upper surface, but it may have so weakly developed as to be practically absent. Apart from narrow longitudinal enamel ridges along the anterior upper and posterior upper edges of the tooth and one on the posterior surface, if any, the lower surface exclusively is coated with enamel that shows longitudinal striae of variable development. This longitudinal strip of enamel is often confined to the posterior half of the lower surface. Added to that the wear facet on the anterior

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[1]) The male upper C of the babirusa now inhabiting Celebes is slender and sub-oval in cross section without enamel coating; this form needs no further consideration here.

surface of the male upper C in *Sus* and *Potamochoerus* is never concave transversely as it is in the paratype of *Celebochoerus*. Wear facets of a shape similar to that in the latter occur only in forms in which the upper canine is much broader than the lower, viz., in *Phacochoerus*.

We may thus safely infer from the transversely concave wear facet that the Celebes upper canines belong to a suid in which the lower canines are decidedly narrower than the upper.

The *Phacochoerus* upper canine, however, is totally different from that of *Celebochoerus* (as well as from the upper C in *Sus*) in being compressed in a different way; the anterior upper and the posterior lower surfaces are the widest and the narrow posterior surface is situated at the inner curve instead of the broad upper surface as is the case in *Sus*, *Potamochoerus* and *Celebochoerus*. As can be seen from the cross section presented in fig. 1*c* wide and shallow longitudinal grooves occur on the broad antero-superior and postero-inferior surfaces, while a third groove is found on the narrow anterior surface that forms an obtuse angle with the antero-superiór surface so as to stand about vertically and that is worn away by the antagonist. The *Phacochoerus* canine has no coating of enamel except at the tip. Pleistocene upper canines of *Phacochoerus* from N. Africa which Prof. ARAMBOURG has kindly shown to me agree with the recent in every detail of their structure.

From the characters of the enamel capping found on unworn tips of the upper C of *Phacochoerus*, STEHLIN (1899, p. 278, fig. VI) concludes that the upper canine of *Phacochoerus* can theoretically be derived from that of *Sus* by the expansion of the posterior surface of the latter. The homologue of the lower surface of the *Sus* canine that is coated with enamel, according to STEHLIN is found in the rounded antero-inferior edge of the *Phacochoerus* canine. Along this edge a strip of enamel has developed too, though only 5 or 6 mm in width and not extending beyond 6 cm from the tip. Near the tip, like in *Sus*, an enamel ridge occurs along the anterior upper edge of the *Phacochoerus* canine, and the anterior surfaces of the canines in both genera consequently are regarded as homologous by STEHLIN. At the posterior upper edge of the tip of the *Phacochoerus* canine, however, two enamel ridges have developed. The anterior of the two is regarded by STEHLIN as the equivalent of the posterior upper enamel ridge, and the posterior is regarded as representing the ridge that may be found on the posterior surface of the *Sus* canine. Consequently, the whole of the surface of the *Phacochoerus* upper canine occupied by the posterior lower longitudinal groove and a good deal of the convexity between the latter and the anterior upper longitudinal groove belong to the posterior surface in the sense of *Sus*. The posterior surface thus appears to have extra-ordinarily developed in *Phacochoerus* as compared to its extent in *Sus*.

It needs no comment that the above derivation of the *Phacochoerus* canine from that of *Sus* is purely hypothetical. Intermediate stages are unknown, and STEHLIN (l.c., pp. 279, 285) accepts from this peculiar canine development *Phacochoerus* to have arisen independently from *Sus* at least since the lower Oligocene.

The subtriangular shape of the *Celebochoerus* canines in cross section can be derived from the *Sus* canine as well. There can be no doubt that the median upper longitudinal groove in the *Celebochoerus* canine is the homologue of that in *Sus*; in both genera this groove even occurs in the same position, at the inner curve of the tooth. In *Sus* the longitudinal upper groove is quite shallow at the pulp cavity (fig. 1*d* upper figure), while it becomes deeper (and even sharply pinched in in cross section) when passing outward (fig. 1*d* lower figure). The two cross sections just referred to were taken from one and the same specimen, a left upper C of *Sus cristatus vittatus* Boie; the smaller was taken at the pulp cavity while the larger was taken at the alveolar margin. The drawings are orientated in the same way and the extent of the enamel layer at the posterior moiety of the lower surface is indicated by small stripes. The enamel covering is seen to be about of the same width in both sections. However, in the smaller section a shallow longitudinal groove is found antero-superiorly of the enamel-coated area, while this groove has disappeared in the larger section. In the latter the anterior surface is gradually rounded from the anterior margin of the lower enamel strip to the sharply defined upper groove. In the larger section we find another longitudinal groove, in the lower part of the posterior surface, that is not represented in the section at the pulp cavity, the anterior groove flattening out about where the posterior groove begins when passing outward along the tooth. In these two grooves we may very well have the homologues of the anterior and the posterior longitudinal grooves in our Celebes canines.

At any rate the derivation of the *Celebochoerus* canines from that of *Sus* requires less imagination than that of *Phacochoerus* from *Sus*. The only steps involved in this transformation are the inferior protrusion of the lower surface especially toward the anterior side so as to make the anterior surface flattened at about right angles to the upper surface, and the extension of the anterior and posterior grooves, which are virtually present in the *Sus* canine, along the whole of the length of the canine. Then, of course, there is the question of the difference in size. To a certain extent the first step is required for the change of a *Sus* canine into a *Phacochoerus* canine too, as is evident from the anterior inferior protrusion of the latter (fig. 1*c*) that is completely absent in the *Sus* canine (fig. 1*d*). I must emphasize that the derivation presented above is by no means considered to follow the probable phylogenetic course.

STEHLIN (1899—1900) deals at length with the upper canines of the recent and fossil Suidae. Most of the forms can be passed in silence because of their being too small. The only form that needs to be considered here is the male upper canine of *Listriodon*, first described by VON MEYER (1846, p. 467). Though in size this tooth comes very near the upper canine of

Celebochoerus, it differs from the latter in the more marked constriction of the pulp cavity and in the distribution of the enamel which forms a wide lower strip and two ridges on the upper surface only, like in *Sus*. Unlike *Phacochoerus*, *Listriodon* has a marked sexual difference in size of the canines, those of the females having weakly developed (STEHLIN, 1899, p. 283). It is impossible as yet to make out whether the Celebes canines belong to males or to females. The smaller of the two might be a female canine and in that case the sexual difference in *Celebochoerus* is very small, as it is in *Phacochoerus* too.

The upper canines of *Omochoerus heseloni* (Leakey) from the Lower Pleistocene of Omo, N. of Lake Rudolph in Abyssinia described by ARAMBOURG (1948, p. 341/42, fig. 35, pl. XX, figs. 1, 3, 3*a*) resemble those of *Celebochoerus* not only in size but also in the shape of the cross section (l.c., fig. 36 *D*) which is only more elongated horizontally. Through the courtesy of Prof. ARAMBOURG I could examine the specimens. One measures 37 mm horizontally and 29 mm vertically, the other 35 mm horizontally and 27 mm vertically. The lower surface is coated with heavily ridged enamel. The enamel coating becomes thinner toward and on the anterior surface and is absent on the upper surface. On the posterior surface, enamel has not developed except along the posterior upper edge. The *Omochoerus* upper canine thus agrees with that of *Sus* in the absence of enamel along the upper surface, which on the contrary has a thick enamel coating in the upper canine of *Celebochoerus*.

Among the remains figured under the name *Sus giganteus* in the "Fauna Antiqua Sivalensis" by FALCONER and CAUTLEY (1847, pl. 69 (except fig. 5), pl. 70, figs. 4—8 and pl. 71 figs. 12—19) there is a fragment of an upper canine (l.c., pl. 71, figs. 19, 19*a* and 19*b*). The fragment is only 60 mm long, the greater transverse diameter is given as 29 mm, the lesser as 20 mm (FALCONER, 1868 I, p. 512). From the figured cross section I measure 30 and 25 mm respectively. The difference between the two diameters thus is greater than that in our specimens. The cross section of the Siwalik specimen moreover is not subtriangular but rather elliptical. The latter is curved in the same way as in *Sus*; a shallow longitudinal groove occurs on the broad surface at the inner curve. The Siwalik canine agrees with that of *Celebochoerus* in the extent of the enamel covering that is indicated in the figure as a continuous layer all around the canine.

Very few words are bestowed on the upper canine by subsequent workers on fossil Siwalik Suidae, most attention being paid to the lower C, the premolars and the molars. The upper canine figured by FALCONER and CAUTLEY is recorded by LYDEKKER (1885, p. 269) but the specific identity is said to be uncertain. STEHLIN (1899, p. 266) is of the same opinion.

In PILGRIM's work on the fossil Suidae of India one upper C only is figured (PILGRIM, 1926, pl. XII, figs. 2*a* and 2*b*). The specimen originates from the Chinji zone and is tentatively referred to *Listriodon pentapotamiae* (Falconer) (l.c., p. 32); it differs from the *Celebochoerus* canines in being smaller and much more curved. COLBERT (1935, p. 233, fig. 108) figures an upper C from the Lower Siwaliks assigned to *L. pentapotamiae* that seems to differ from our specimens in the same points as does that of PILGRIM's figure. An isolated upper C from the Lower Siwaliks, the locality of which is unknown, is left specifically unidentified (COLBERT, 1935, p. 263). The upper C figured by FALCONER and CAUTLEY, ignored in the descriptions of *Sus giganteus* by LYDEKKER (1884) and PILGRIM (1926) is listed without reserve under *Sivachoerus giganteus* by COLBERT (1935, p. 228).

Male upper canines of *Sus lydekkeri* Zdansky have been described from the Pleistocene of Chou Kou Tien in N. China by ZDANSKY (1928, p. 92, pl. X, fig. 5) and YOUNG (1932, p. 8). They agree with the upper canine of recent *Sus* in the restriction of enamel to two ridges on the upper and one strip on the lower surface respectively, and differ from the latter only in their superior dimensions. ZDANSKY gives the maximal horizontal diameter of the upper C as 32 mm (YOUNG gives 27 mm), and the vertical as about 20 mm (YOUNG 22 mm) [2]). Of *Listriodon gigas* Pearson (1928, p. 7, fig. 1) from the Ping Fan district of Kansu in China an upper C is figured in situ in a portion of a left maxillary. No reference, however, is made to this tooth in the text except the statement that the ventral enamel band is very wide, which is sufficient for our present purpose.

The upper canines of *Sus brachygnathus* Dubois and *Sus macrognathus* Dubois from the Pleistocene of Java, the only of the Greater Sunda Islands in which fossil pigs have been found thus far, have the enamel restricted in the typical *Sus* fashion, and added to that they are much smaller than the Celebes canines. Large pigs such as *Celebochoerus* have not been found to occur in the Pleistocene of Java.

Here our survey of the fossil Suidae comes to an end. One important point has at least become certain, viz., that the shape of the upper canine in the Celebes form is as distinct from that in recent and fossil *Sus* as, e.g., that of *Phacochoerus*. We may thus assume that *Celebochoerus* probably also splitted off at an early date from the main stock of the Suidae. Future finds of premolars and molars of this fossil form will enable us to determine its affinities more exactly than is at present possible.

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Zoology. — *On the thickness of the layer of blubber in Antarctic Blue and Fin Whales.* I. By E. J. SLIJPER (Institute of Veterinary Anatomy, State University, Utrecht). (Communicated by Prof. G. KREDIET.) [1])

(Communicated at the meeting of September 25, 1948.)

1. *Introduction. Material.*

According to the International Convention for the Regulation of Whaling (Washington 1946; Schedule art. 8*a*), the Antarctic whaling season is to be closed when 16.000 Blue Whale Units shall have been caught by all expeditions together. Consequently the total amount of oil that can be obtained during a certain season from the Antarctic whaling grounds, will be largely influenced by the fatness of the whales. Therefore it will be of great importance to have a thorough knowledge about the different factors determining this fatness. Almost the only available data about this subject, however, are those about the production of whale oil in barrels per Blue Whale Unit that can be found in the International Whaling Statistics, in certain numbers of the Hvalrådetsskrifter and in the Norsk Hvalfangsttidende. It has already been pointed out by MACKINTOSH (1942; p. 208) that these data can only give a very rough impression about the fatness of whales, because they are usually determined on the score of the production of a whole week or even of a whole month and because the oil from the different species of whales is mixed in the factory-tanks without separate measurement. The principal facts that are shown by these data are the increase of the yield of oil during the progress of the season and the fact that there is a fairly large variability in the data according to the different seasons and the different areas of the Antarctic.

MACKINTOSH and WHEELER (1929), however, have already shown that there is still an other way to get an impression about the fatness of whales, i.e. by measuring the thickness of the layer of blubber (see also HEYERDAHL, 1932). Unfortunately these authors have only made their researches on this subject on the land stations at South Georgia. Measurements that were afterwards taken by members of the Discovery staff on board of floating factories, have not yet been published.

During the first and second Antarctic expedition of the Dutch floating factory "Willem Barendsz" some data about the thickness of the layer of blubber in Blue and Fin Whales (*Balaenoptera musculus* L. and *Balaenoptera physalus* L.) could be collected by the author (season 1946—1947) and by Mr W. G. BRAAMS, biological student, who sailed in 1947—

[1]) This publication was made on the basis of the researchwork achieved by the team for researchwork on whales, of the organisation T. N. O.

1948 as junior whaling inspector. Grateful acknowledgement is made to Mr BRAAMS for the conscientious way in which he has collected the material. In table 1 a review is given of the number of whales caught during these two expeditions and of the number of animals from which blubber-measurements were taken.

TABLE 1.

General review of the catch of the f.f. "Willem Barendsz" during the season 1946—1947 and 1947—1948 with data about the number of whales from which blubber-measurements were taken.

	Number of whales caught		Number of whales from which measurements were taken	
	1946—1947	1947—1948	1946—1947	1947—1948
Blue Whale				
♂ Immature < 74′	43	61	7	17
Mature > 74′	150	117	11	43
Total ♂	193	178	18	60
♀ Immature < 77′	56	94	9	22
Mature > 77′	152	59	22	23
Total ♀	208	153	31	45
Total Blue	401	331	49	105
Fin Whale				
♂ Immature < 63′	17	37	3	15
Mature > 63′	160	397	8	55
Total ♂	177	434	11	70
♀ Immature < 65′	16	52	1	11
Mature > 65′	156	340	20	76
Total ♀	172	392	21	87
Total Fin	349	826	32	157
Total Baleen Whales	750	1157	11	262

MACKINTOSH and WHEELER (1929) took their measurements of the layer of blubber only at one single point of the body, viz. at the lateral side midway between the dorsal fin and the anus. During the first season of "Willem Barendsz" measurements were taken at a comparatively large number of different points, during the second season this number was reduced to five, viz.: 1. dorso-median line at the level of the flipper; 2. dorso-median line just cranially of the dorsal fin; 3. lateral side midway between the dorsal fin and the anus; 4. ventro-median line just cranially of the anus; 5. ventro-median line just cranially of the umbilicus (see fig. 13—16). In fig. 7—12 the thickness of the blubber in the different parts of the body has been plotted against the time at which the whales were caught. Since the thickness of the blubber varies with the length and general condition of the animals, different symbols have been used for the different groups of size, for the sexes, for pregnant whales etc. The material of the first season was so small that no reliable average thicknesses for the different months could be calculated (see table 5). This, however, could be done for the second season and consequently from table 5, fig. 7—12 and 13—16 a general impression about the changes in the thickness of the blubber during the progress of the season could be obtained. This general impression is represented by the curves of fig. 17—18. In spite of the fact that the material collected by Mr BRAAMS during the second season was much larger and completer than that of the first season, the total amount of data is still comparatively small. Moreover it will be shown that in both seasons the expedition operated under more or less abnormal conditions. But exactly because these conditions permitted us to collect data from localities not frequently visited in normal seasons, I believe that it may be useful to publish and to discuss our results, so that they may be compared with those of future researches.

2. *The thickness of the layer of blubber in the different parts of the body.*

In Whales no distinction can be made between the corium and the subcutis. Under the epidermis there is only one single layer of fibrous connective tissue containing a large amount of fat cells: the layer of blubber. The thickness of this layer, however, is very variable in the different parts of the body. The data collected from a great number of whales of different size and condition and caught at different times, showed that broadly outlined the mutual relation in the thickness of the blubber is always the same. In fig. 1 this mutual relation is shown if the

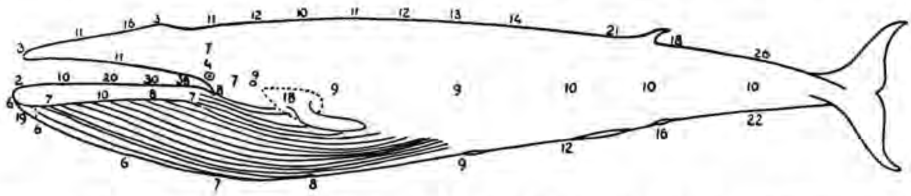

Fig. 1. Schematic drawing of a Blue or Fin Whale indicating the mutual thickness of the layer of blubber at different points of the body, if this thickness at a point midway between the dorsal fin and the anus is fixed at 10 cm.

thickness at the lateral side of the body midway between the dorsal fin and the anus is fixed at 10 cm. It then appears that after calculating, approximately the same relative measurements that are given in fig. 1, can be found in any Blue or Fin Whale and in any part of the season. Almost exactly the same distribution of the blubber-thickness over the body was found in some foetal Blue and Fin Whales. Only at the dorsal side of the trunk the relative thickness was less than in the adult animals.

From these facts the conclusion may be drawn that the differences in thickness of the blubber in the various parts of the body of whales do not depend on the metabolism of the animals but on demands made by the general shape of the body, the locomotion or other mechanical factors. The data about the thickness of the skin in mammals given by SCHUMACHER (1931) show that it is highly probable that the various stresses appearing in the different parts of the skin when the animal is moving, may be held responsible for the differences in thickness. On the other hand there is not the slightest accordance in the distribution of the thickness of blubber over the body in the big Whales and that of the corium of a neonatus of the Manatee (MATTHES, 1929). So there is a great possibility that the differences in the thickness of blubber play an important part in the formation of the general body-outlines, i.e. in the stream-line of the body.

Fig. 1 shows that at the tip of the snout on the dorsal part of the head, the layer of blubber is very thin. It increases gradually in the caudal direction and just before the blowhole there is a very distinct dorso-median hump which is only caused by a marked thickening of the blubber. Laterally of the blowhole the blubber is very thin. In the rostral part of the head the lateral parts of the blubber have the same thickness as the median, but caudally of the blowhole the blubber is much thicker in the dorso-median line than at the lateral parts. It is especially very thin round about the eye, but increases slightly in thickness towards the angle of the mouth and towards the external ear-opening. Fig. 2 shows that the lower jaw

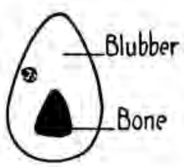

Fig. 2. Very schematic cross-section through the middle of the lower jaw of a Blue or Fin Whale.

possesses a dorsal crest consisting entirely of blubber. The height of this crest increases from the tip of the jaw up to the coronoid process and then decreases towards the angle of the mouth. The blubber of the ventral side of the jaw has its greatest thickness in the middle. Just a little caudally and ventrally of the tip of the lower jaw there is a mighty thickening of the blubber in the ventro-median line. On the ventro-lateral and ventral side of the head and in the cranial part of the trunk (up to the umbilicus), i.e. in the region of the longitudinal grooves, the blubber shows a moderate thickness.

About the trunk in general it may be said that the layer of blubber is thickest in the dorso-median line and that it increases a little along its whole surface in the caudal direction. This increase in thickness, however, is very marked in the dorsal and ventral parts of the body between the level of the genital aperture and that of the anus. In the lateral parts the thickness here remains the same. A very characteristic region, where the layer of blubber is almost twice as thick as in the surrounding parts, is always found around the basis of the flipper. Its shape shows a comparatively great individual variability, but mostly it has the shape of a pointed oval with the rounded part close behind the basis of the flipper and tapering into a cranio-dorsal direction towards the external ear-opening.

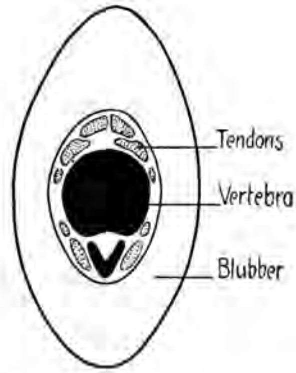

Fig. 3. Very schematic cross-section through the middle of the tail of a Blue or Fin Whale.

A very distinct thickening is also found just in front of the dorsal fin in the dorso-median line.

Fig. 1 and 3 show that the thickness of the lateral parts of the blubber of the tail does not differ from that of the lateral parts of the trunk. In the dorsal and ventral regions, however, the thickness increases rapidly caudally of the anus and the dorsal fin, so that a very high dorsal and ventral crest of the tail is formed which consists entirely of blubber. These crests most surely play an important part in the moulding of the streamline of the body. The crests show almost the same height up to the insertion of the flukes and then decrease rapidly on the dorso- and ventromedian side of the tail-fin. The dorsal crest is about $^1/_4$ or $^1/_5$ higher than the ventral one.

HEYERDAHL (1932; pl. 3, fig. 1) has made some very interesting observations on the fat-percentage in the different parts of the blubber of a Fin Whale. According to his analyses there is a general increase in fat-content of the dorsal and ventral parts of the blubber in a cranio-caudal direction. Moreover the percentage of fat in the dorsal blubber was much higher than in the ventral parts of the same level of the body. Although the number of available data is still very small, the conclusion might be drawn that in adult whales there is a distinct correlation between the thickness of the layer of blubber and its percentage of fat. The thicker the blubber, the higher its fat-content.

3. *The thickness of the layer of blubber according to species, size and general condition of the whales.*

Fig. 4 shows that in the foetus the absolute thickness of the blubber increases with increasing length of the body. Exactly the same, however, may be said about the relative thickness of the blubber, i.e. the thickness of the blubber in $^0/_{00}$ of the body-length. In foetuses of the Blue Whale of

240—660 cm of body-length the relative thickness of the blubber at the lateral side on the level of the umbilicus increased from about 5,0 ‰ up

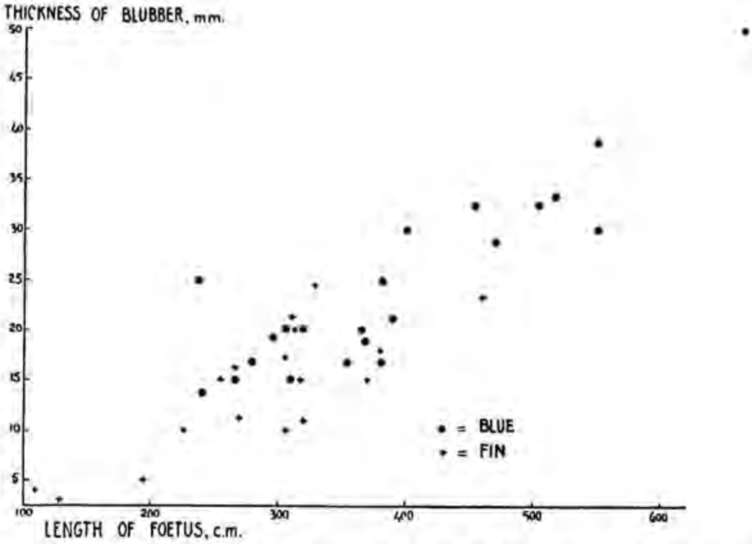

Fig. 4. Diagram indicating the increase in thickness of the *layer of blubber* in *Blue* and *Fin Whales* before birth, with increasing length of the *foetusses.*

to 7,5 ‰. In Fin Whale foetuses from 200—460 cm there was an increase of about 2,5 up to about 6,0 ‰.

The adult Blue Whales caught during the first season showed an average relative blubber-thickness of 5,3 and the adult Fin Whales of 4,6 ‰. These data are in perfect accordance with the opinion of GRAY (1928, 1930) that the relative thickness of the blubber in calves would be greater than in adult animals. Thus during the period of the most rapid growth, i.e. the period between birth and sexual maturity, an apparent decrease in relative thickness of the blubber may be observed. This phenomenon may be partly connected with the fact that with increasing content of the body the surface and consequently the loss of heat decreases. For the greater part, however, it may be connected with the fact that the layer of blubber of a foetus contains almost no fat (according to HEYERDAHL, 1932, p. 79: 5,35 % in a Blue Whale foetus of 366 cm). Although there are no data available about the fat-percentage of the blubber in calves, it is highly probable that this fat-percentage gradually increases from birth up to adolescence (according to HEYERDAHL, 1932, p. 79: 54—67 % in young Blue Whales of 56 and 58 feet length). This means that the relative thickness of the blubber may decrease with increasing fat-percentage, i.e. increasing capacity to prevent loss of heat.

In the adult or nearly adult animals, i.e. in the animals which are generally caught by the catchers, just the reverse may be observed. Although there is a considerable variability, the data of "Willem Barendsz" confirm those of MACKINTOSH and WHEELER (1929) and HEYERDAHL (1932). They show that with increasing body-size there is not only an increase in the absolute (fig. 7—9, 10—12) but also in the relative thickness of the layer of blubber (fig. 17). Unfortunately no definite conclusions can be drawn from the tables of HEYERDAHL (1932) with regard to the relation between the body-length and the fat-percentage. It is highly possible that the difference in blubber-thickness between big and small animals, may be explained by the fact that in small animals a greater percentage of the food is used for growth and basal metabolism, so that a smaller part is available for the production of store-fat. The data shown in fig. 7—12 confirm also the results of MACKINTOSH and WHEELER (1929) and HEYERDAHL (1932) that pregnant females have a thicker layer of blubber than non-pregnant females or males of the same length. HEYERDAHL (1932; p. 54, table 8*a*, 8*b*) has shown that in pregnant females also the percentage of fat in the layer of blubber is much higher than in the other animals. The thickness of the blubber and its percentage of fat are smallest in lactating whales. According to HAVINGA (1933) the Common Seal (*Phoca vitulina* L.) shows the same facts with regard to pregnant and lactating animals.

TABLE 2.

Average thickness of the layer of blubber at a point on the flank midway between the dorsal fin and the anus in ‰ of the length of the animal.
The data bear upon the entire whaling season (resp. 17-12-'46—7-4-'47 and 8-12-'47—31-3-'48)

	All animals			Adult ♂ and adult non-pregnant ♀		
	Blue	Fin	Difference in %	Blue	Fin	Difference in %
S. Georgia 1925—1927 [1]	3.5	3.0	14	3.7	3.4	8
W. B. 1946—1947	5.5	4.7	14	5.3	4.6	13
W. B. 1947—1948	5.3	4.5	15	5.1	4.5	12

[1]) According to MACKINTOSH and WHEELER (1929).

From the data collected in table 2 the conclusion may be drawn that there is a difference of about 14 % between the thickness of the blubber in Blue and in Fin Whales. Table 5 shows that, broadly outlined, this difference can be found in any part of the season, since the increase in thickness is the same in both species (see sub 5). According to the International Whaling Statistics and the paper of MACKINTOSH (1942) the difference in length between Blue and Fin Whales is about 14 % at sexual maturity, 11—14 % (resp. South Georgia and Antarctic pelagic) at the moment when the animals have attained their average length and 15 % at their maximum size. Consequently the conclusion may be drawn that the difference in blubber-thickness between the two species of whales entirely depends on their normal difference in size and not on other factors as for example their metabolism. From the paper of HEYERDAHL (1932) no definite relation appears between the fat-percentage of the blubber in Blue and Fin Whales.

In calculating Blue Whale Units, two Fin Whales are reduced to one Blue Whale. Thus it might be of certain importance to know whether this is in accordance with a difference in length of about 14 % or not. LAURIE (1933) designed a curve for the relation between length and weight in Blue Whales. The weights of Fin Whales that have been found by ZENKOVIC (1937) and QUIRING (1943) later on, fall quite within the range of this curve. According to this relation a Fin Whale of moderate size would have about $^3/_5$ of the weight of an average Blue Whale. Taking into consideration that the relative thickness of the blubber decreases with decreasing length and weight, it seems to be highly probable that the output of the average Blue Whale indeed may be twice that of the average Fin Whale. I am, however, fully aware of the fact that for the present only a very rough impression about the relation Blue-Fin can be obtained, since the production of oil from a whale does not depend only on its size or on the amount of fat in the blubber, but also on the fat-percentage of the meat, the bone and the internal organs. Researches made on board of "Willem Barendsz" by FELTMANN, SLIJPER and VERVOORT (1948) have shown that the fat-percentage of meat and bone was almost the same in Blue and Fin Whales with the exception of the meat of the tail which was distinctly fatter in the Blue Whale.

4. *The thickness of the layer of blubber according to the different seasons and the different whaling grounds.*

The data of table 2 show that the average thickness of the blubber measured at South Georgia by MACKINTOSH and WHEELER (1929) in the seasons 1925—'26 and 1926—'27 is much smaller than the average thickness during the two expeditions of the f.f. "Willem Barendsz", the blubber of the second season being quite a little bit thinner than that of the first. Now we may ask whether this difference may be ascribed to local or to seasonal influences or probably to both. It may be supposed that the curves for the output in the different seasons and on the different whaling grounds of the Antarctic also may give a certain impression about the local and seasonal variations in the thickness of the blubber. The curves of fig. 5 show that at South Georgia as well as in Antarctic pelagic whaling there are large seasonal variations in the yield of oil. Apart from minor variations there is a distinct conformity between the ups and downs in the two curves, indicating that for the whole Antarctic there must be good and bad seasons, i.e. seasons in which the whales are generally fat or lean. Apparently this will be connected with general feeding-conditions caused by differences of the temperature of the sea-water or other factors affecting the development of the krill. Besides it is quite evident that the output at S. Georgia is nearly always smaller than the yield of factory-

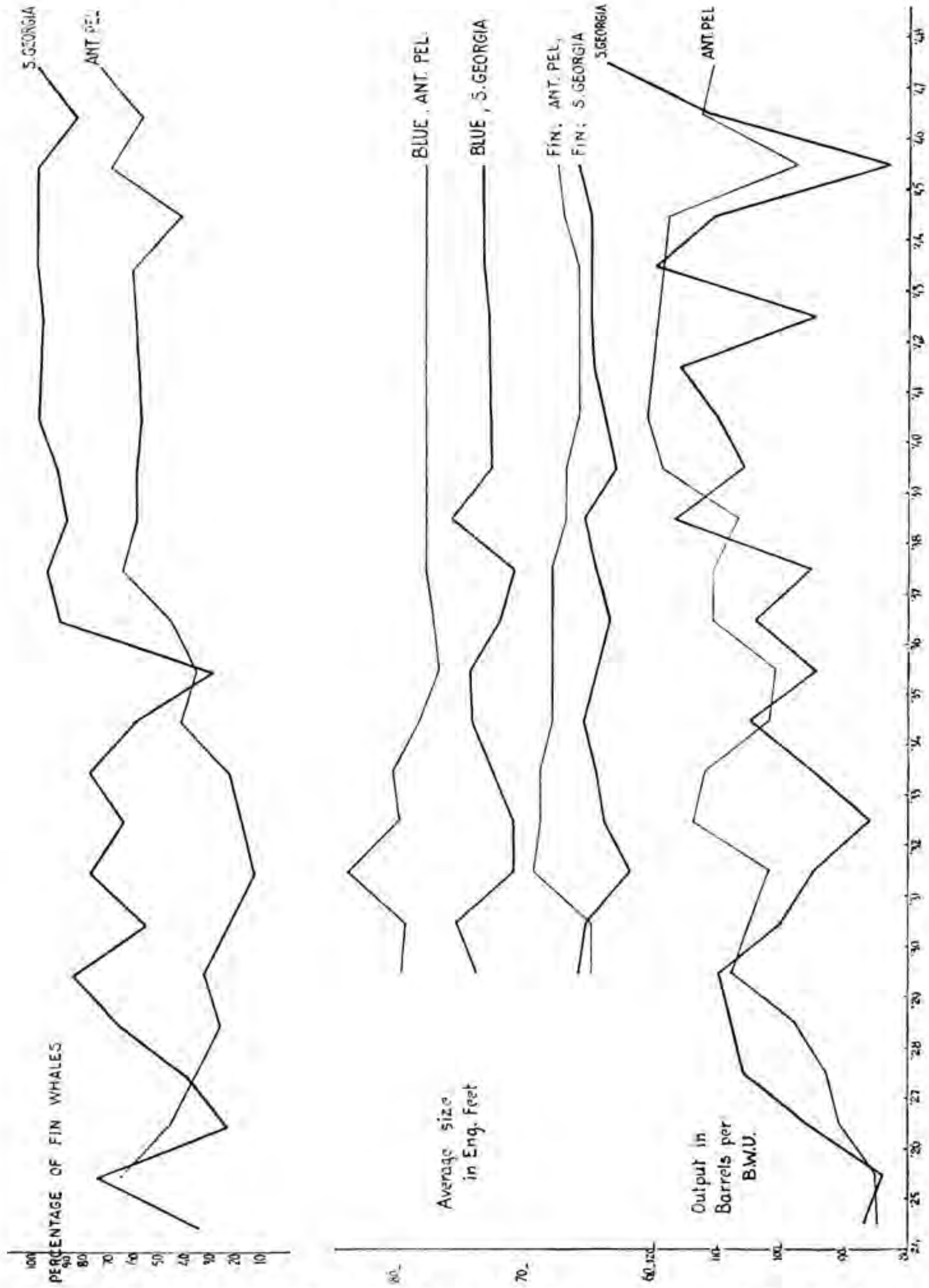

Fig. 5. Diagram indicating the seasonal variations in the *percentage* of *Fin Whales*, in the *average size* of whales and in the *output of oil* in barrels per Blue Whale Unit from 1924/25—1947/48 at *South Georgia* and in *Antarctic pelagic whaling* (whaling carried on by floating factories). 1 Barrel = 1/6 ton (1016 kg). Blue Whale Units are calculated on the basis 1 Blue Whale = 2 Fin Whales = 2½ Humpback Whales = 6 Sei Whales.

ships, indicating that the whales caught at S. Georgia are generally leaner than on the ordinary Antarctic pelagic whaling grounds.

This fact may be ascribed to three different causes: In the first place the condition of the whales at S. Georgia may be worse than in other areas of the Antarctic, in consequence of the more northern situation of the island. For the percentage of lean whales just arriving from subtropical waters may be larger. In the second place it may be caused by the fact that the percentage of immature and small animals in the catch is larger at S. Georgia than in Antarctic pelagic whaling (fig 5). The absolute and relative thickness of the blubber in these small animals is less than in their bigger relatives (p. 1039), but the B.W.U. are calculated independently of the size of the whales. And in the third place at S. Georgia a greater percentage of Fin Whales are caught (fig. 5). Now it might be possible that one Blue Whale would not exactly represent two Fin Whales.

MACKINTOSH (1942; p. 208, 250) suggests that on an average one Fin Whale might represent a larger amount of oil than ½ Blue Whale. Then the increase of the output of oil per B.W.U. during the progress of the season may at least partly be caused by the increasing percentage of Fin Whales in the catch. Fig. 20 and 25 show that for the season 1947—1948 there is a very marked resemblance indeed between the curve for the output and that for the percentage of Fin Whales. There was, however, not the slightest correlation during the season 1946—1947 (fig. 18 and 20) and from some parts of these curves just the opposite conclusion might be drawn. To make a perfectly reliable calculation will be very difficult indeed, since the difference in fatness between the two species varies considerably with the locality where and with the part of the season in which the whales are caught. It will be shown later on that, broadly outlined, during the second part of the season 1946—1947 the catch of "Willem Barendsz" consisted of comparatively big and consequently comparatively fat Blue and Fin Whales. During the second part of the season 1947—1948, however, the average size of the Blue Whales was much smaller and consequently comparatively lean Blue Whales, but comparatively fat Fin Whales were caught. It is quite possible that one fat Blue Whale represents a little more oil than two fat Fin Whales, but one lean Blue Whale a little less. It may be expected that a large number of concise data about the thickness of the blubber collected during many different seasons and on different whaling grounds, may give us in future a better insight in this question. Moreover fig. 5 shows that there is no correlation between the curve for the percentage of Fin Whales and that for the output of oil per B.W.U. neither at S. Georgia nor in Antarctic pelagic Whaling. It is true that this fact gives not a perfectly reliable indication that the correlation does not exist, since the general seasonal influences and the percentage of Fin Whales might reinforce or weaken each other. I believe, however, that at present there is no reason to suppose that the relation one Blue = two Fin differs much from the real relation in the yield of oil that can be obtained from these two species (see also p. 1040).

Fig. 5 shows that, according to S. Georgia, there is a distinct mutual correlation between the curves indicating the average length of the Blue and the Fin Whales as well as a distinct correlation between these curves and the curve for the output of oil. In this respect it should, however, be taken into consideration that the catch of the seasons 1940—1945 was highly influenced by war- and post-war circumstances. Although less obvious, the curve for the yield of oil in Antarctic pelagic whaling shows the same correlation. The fact that this correlation is less distinct may be explained by the fact that the catch indicated as "Antarctic pelagic" is made by a great number of very different floating factories in which the whales are processed in different ways. So the differences in the manner of processing may influence the output of oil to a greater degree than at the land-stations of South Georgia. This is clearly shown by the data of table 3.

TABLE 3.

Survey of the variability in the output of oil per B. W. U. according to the different factory-ships and land-stations in Antarctic pelagic whaling and at South Georgia. The data are taken from International Whaling Statistics and from PAULSEN (1947, 1948).

	Output of oil in barrels per B. W. U. in the season:				
	1936—1937	1937—1938	1945—1946	1946—1947	1947—1948
Antarctic pelagic					
Number of factory-ships	30	31	9	15	17
Output	92—125	93—133	89—110	78—133[1])	94—131
South Georgia					
Number of land-stations	2	2	3	3	3
Output	101—107	94—98	80—84	110—114	126—130

[1]) 101—133 if the two Japanese factory-ships are excluded.

Now the conclusion may be drawn that probably the output of oil is chiefly influenced by general seasonal factors (feeding etc.) and by the average length of the whales that are caught. The data about the thickness of the blubber at S. Georgia were collected by MACKINTOSH and WHEELER (1929) in the seasons 1925—1926 and 1926—1927. Fig. 5 shows that in both seasons, but especially in the first, the output of oil at S. Georgia was very small; they were bad seasons.

Table 4 and fig. 5 show that according to the output of oil during the last two Antarctic seasons average values were obtained with the exception of the last season at S. Georgia, which is characterised by a very high output. In 1946—1947 the output of "Willem Barendsz" was much higher than the average of all factory-ships but in 1947—1948 it was just a little

TABLE 4.

Some data about the catch in the Antarctic whaling seasons 1946—1947 and 1947—1948. From PAULSEN (1947, 1948)

	1946—1947		1947—1948	
	Percentage Fin Whales	Output per B.W.U.	Percentage Fin Whales	Output per B.W.U.
Antarctic pelagic	59	113	74	110
South Georgia	84	112	98	128
"Willem Barendsz"	47	133	71	105

below this average. A comparison with the percentage of Fin Whales does not give the impression that this factor may give an explanation of the facts. A comparison of fig. 22, 23 and 26, however, shows that the percentage of immature Whales caught by "Willem Barendsz" in 1946—1947 was much below and in 1947—1948 much above the average. This fact bears particularly on the Blue Whales, but the percentage of immature Fin Whales in the second season was still 25 % higher than in the first. It will be quite evident that a large number of immature Whales lessens

TABLE 5.

Average thickness of the layer of blubber in $^0/_{00}$ of the length of the animals in the different groups of size and the different months of the season.
In italics: number of animals from which measurements were taken.

		Blue Whales						
		Males			Females			
		<74'	74'—80'	>80'	<77'	77'—80'	81'—85'	>85'
Dec.	1946—1947	*2* 3.9	*2* 5.7	*1* 6.2	*3* 5.0	*1* 6.8		
	1947—1948	*9* 4.7	*11* 5.0	*1* 4.6	*12* 4.0	*2* 5.0	*5* 6.3	*2* 5.0
Jan.	1946—1947		*1* 5.8	*2* 4.2	*2* 4.6	*1* 3.1	*5* 5.4	*2* 4.6
	1947—1948	*3* 3.3	*11* 4.5	*1* 4.9	*5* 5.3	*1* 5.3	*1* 4.4	*1* 5.3
Febr.	1946—1947	*2* 5.4	*1* 4.0		*1* 6.1	*1* 4.3		*2* 5.6
	1947—1948	*2* 4.8	*3* 5.4	*1* 5.9	*5* 6.4		*1* 6.4	*3* 5.9
March	1946—1947		*1* 6.4		*1* 5.7	*1* 9.1	*1* 7.2	*1* 5.3
	1947—1948	*2* 5.6	*8* 5.1	*6* 5.1	*2* 5.8	*2* 6.4	*2* 5.3	*2* 6.6
		Fin Whales						
		Males			Females			
		<63'	63'—70'	>70'	<65'	65'—70'	>70'	
Dec.	1946—1947	*3* 4.3			*1* 3.0		*5* 4.2	
	1947—1948	*4* 4.3	*11* 4.3		*1* 3.7	*8* 3.9	*7* 4.8	
Jan.	1946—1947	*1* 5.0			*1* 4.2	*1* 4.0	*3* 4.5	
	1947—1948	*3* 4.5	*16* 4.4		*3* 3.4	*10* 4.6	*10* 4.3	
Febr.	1946—1947	*1* 7.2	*2* 4.6				*2* 4.4	
	1947—1948	*6* 4.8	*15* 4.6	*1* 5.8	*5* 5.1	*13* 5.5	*10* 5.8	
March	1946—1947		*1* 5.2			*1* 4.8	*2* 5.6	
	1947—1948	*2* 5.4	*13* 4.7		*1* 4.5	*7* 6.0	*10* 6.2	

the average length of the animals to a marked degree. Although no reliable conclusions may be drawn from the scanty data in table 5, it might appear that the differences in the average thickness of the blubber between the two seasons, do not depend on differences in any special group of whales or any special part of the season. The differences in the average thickness of the blubber appears to be directly correlated with the very marked differences in the average size of the animals between the two seasons (table 6).

Summarizing it may be said that the differences in the average relative thickness of the blubber as they have been discussed above, may be caused by general seasonal variations (the data of MACKINTOSH and WHEELER were collected in a bad season), as well as by variations depending on the locality where the whales are caught. These local variations chiefly depend on the average length of the whales, so for example on the number of very big or very small animals in the catch. There is not only a very marked difference between the average length at S. Georgia and in Antarctic pelagic whaling, but also between the different areas of the pelagic whaling grounds (BERGERSEN, LIE and RUUD, 1939; p. 24).

(To be continued.)

KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

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CONTENTS

Geology. — *Origin of the Jura Mountains.* By J. H. F. UMBGROVE.

(Communicated at the meeting of October 30, 1948.)

1. The oldest rocks in the Jura Mountains belong to one special stratum of the Mid-Triassic. Even in tunnels and in the core of deeply cut anticlines no older rocks appear to view. This layer of the so-called anhydrite group contains beds of rock-salt. Apparently, as suggested by BUXTORF since 1908, the anhydrite group played the rôle of a lubricant over which the overlying strata could glide and — detached from the underlying basement — were compressed into folds. Hence the Jura Mountains are considered as a surface phenomenon, a superficial type of folding which Swiss geologists have named *décollement.*

The arcuate chain of the Juras is framed by the basement blocks of Black Forest, Vosges, and Plateau Central. And in between the two last named the high situation of the basement is revealed by a few smaller outcrops of crystalline rocks (fig. 1).

Apparently the site and extern boundary of the Jura Mountains is in some way related to the surrounding "frame work" of high situated basement rocks the movement of the surface layers having been hampered by and becoming adapted to the updomed basement of the surroundings. In the depressed area of the Rhine graben the frontal chains of the Jura Mountains protrude farther northward than in the adjacent sectors which are opposite the higher situated blocks of Black Forest and Vosges.

On the other hand the whole arrangement of the Juras, as well as their time of origin clearly show a relation to the Alps. Two structural trends occur in the Jura Mountains. In the region between Basle and the Swiss plain [1]) the dominant pattern consists of longitudinal folds with E–W trends, the second feature is formed by several short anticlines with N.N.E. to N.E. trend. Thus the rectangular shape of the Delémont basin is due to a framework of folds consisting of these two elements. Another example is to be seen even as far south as the Verena anticline, a short and faulted anticline protruding from Molasse strata between Solothurn and the Weissenstein. The latter is one of the highest and southernmost longitudinal anticlines of this part of the Jura Mountains. To these examples many others enumerated by VONDERSCHMITT could be added. Moreover numerous faults run in the same direction. Now this direction is also displayed by the faults of the Rhine graben. Moreover they are of the same age, viz. they originated in Oligocene times, though some were rejuvenated in Miocene times. There even existed a connection between the Rhine graben and the Molasse trough across the present area of the Jura Mountains in Oligocene (Stampian) times, the so-called Rauracian graben (fig. 1).

[1]) See fig. 1 and also the tectonogram in UMBGROVE, op. cit. 1948, p. 764, fig. 1.

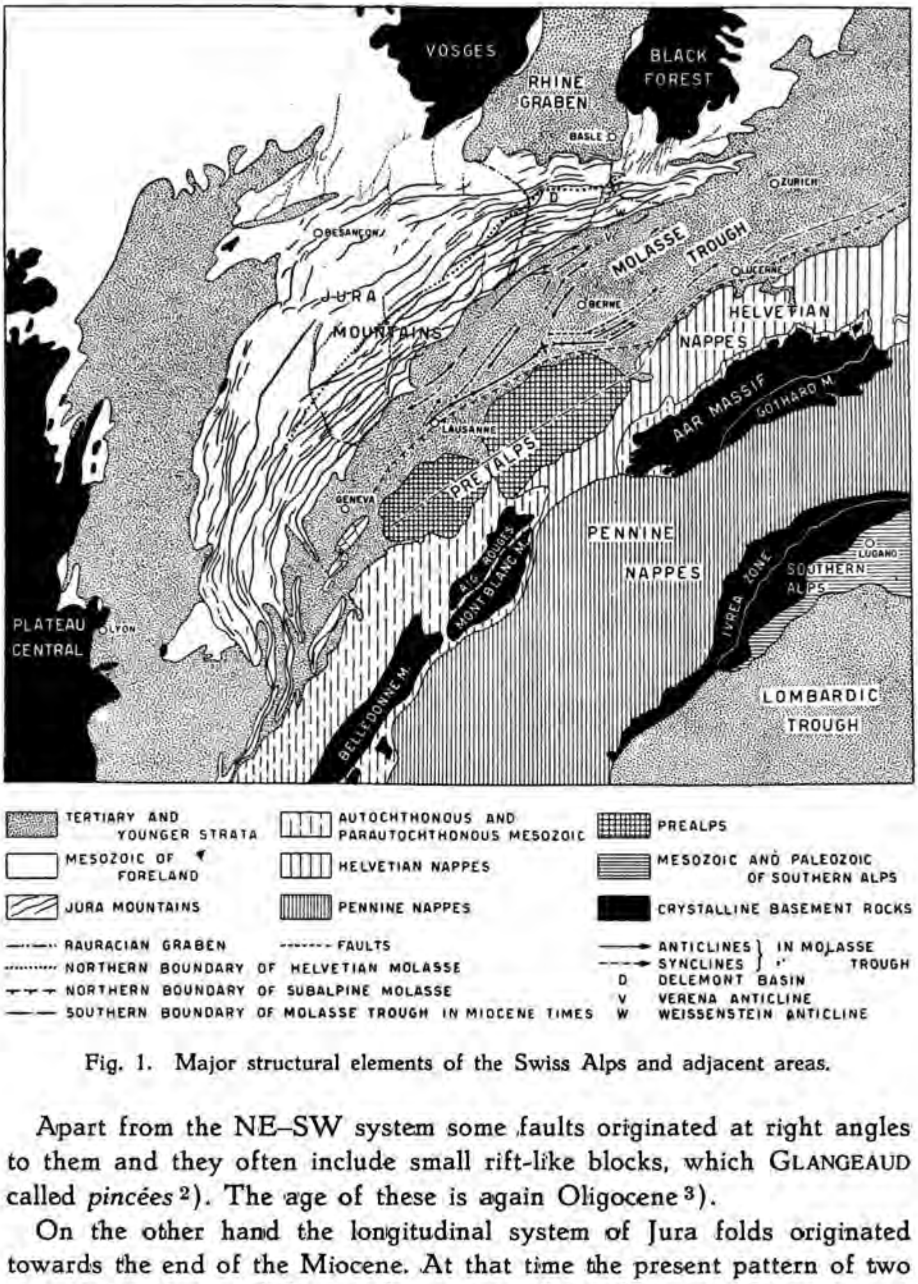

Fig. 1. Major structural elements of the Swiss Alps and adjacent areas.

Apart from the NE–SW system some faults originated at right angles to them and they often include small rift-like blocks, which GLANGEAUD called *pincées* [2]). The age of these is again Oligocene [3]).

On the other hand the longitudinal system of Jura folds originated towards the end of the Miocene. At that time the present pattern of two intersecting structural elements became completed. However, the transverse folds are quite different from the folds of the dominating longitudinal system. They are less numerous, they can be followed over a short distance only, and some of them show the characteristic arrangement which is called *en*

[2]) GLANGEAUD, 1944, p. 27.

[3]) HEIM, 1919, p. 571, Taf. XXI; BUXTORF, 1934, p. 529, fig. 3, and GLANGEAUD, 1944, p. 24, 25 and fig. 2, 7 and 8.

échelon. After a discussion of several theories VONDERSCHMITT concludes these features can be best explained in the following manner. A system of short *en échelon* folds originated in the strata overlying the fault zone when adjacent blocks with their basement at different depth were subjected to differential movements.

Probably similar movements caused the Vosges to be moved relatively in a southern direction as compared to the Black Forest block which underwent a relative movement northward. However, the same final positions would result if all the blocks moved southward but over different distances.

The longitudinal Jura chains originated towards the end of the Miocene [4]). So they did not yet exist when the sediments of the Molasse trough accumulated. As a matter of fact Molasse sediments spread over large areas of the present Jura Mountains (fig. 3) and still occur in several synclines including the Delémont basin. With the exception of the area of the Rauracian graben their thickness is not appreciable if compared to the thickness of the deposits in the Molasse trough. Aquitanian strata of the Molasse trough overlain by Burdigalian still have a thickness of 400 m as far north as the southernmost Jura chain near Büttenberg (which is south of the Rauracian graben between Solothurn and Biel [5]). However, in the syncline of Tavannes-Court, which is about 8 km northward in the Jura Mountains, Aquitanian strata are lacking and the Burdigalian rests directly on Oligocene (Stampian) sediments. Hence the southern boundary of the Jura chains coincides approximately with the original outer boundary of the Molasse trough. A theory on the origin of the Juras ought to explain this coincidence.

2. ALBERT HEIM compared the Jura Mountains to a table cloth that being pushed to one side became rumpled into ridges and valleys. In his opinion the push came from the south. When the Alps were being compressed they are supposed to have pushed the Jura Mountains northward in front of them. Gliding over the lubricant layer of the Mid-Triassic the upper strata were pushed into folds over the unfolded basement consisting of Lower Triassic, Permian, and crystalline rocks. One wonders how this process took place, separated as the Alps are from the Weissenstein by a stretch of 20 miles of a low and rather flat country, the so-called Swiss plain which is the surface of the Molasse trough (fig. 2).

Undoubtedly the tectonics of the Molasse trough will prove more intricate the more detailed mapping, deep borings, and geophysical

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[4]) According to BUXTORF (1938, p. 381) the Upper Miocene folding probably can be devided into a pre-Pontian, and a post-Pontian phase. However, he remarked: "vielleicht gibt die künftige Prüfung die Notwendigkeit ihrer Verlegung in etwas frühere Abschnitte des Obermiocän". For the sake of arguement we will indicate the time of *décollement* as: Upper Miocene.

[5]) BAUMBERGER, p. 75.

investigations will be carried out. As a matter of fact a map of the western part of the trough published recently by KOPP shows several additional anticlines and synclines when compared to BAUMBERGER's map of 1934 (fig. 1).

Still, however, the fact remains that the moderate undulations of the Molasse strata form a strong contrast to the structure of the Jura Mountains. If a push from the Alps was transferred through the prism of Molasse sediments so as to cause the folding of the Juras why then was only the southern part of the trough strongly influenced by this pushing action, whereas the northern part remained nearly undisturbed? [6])

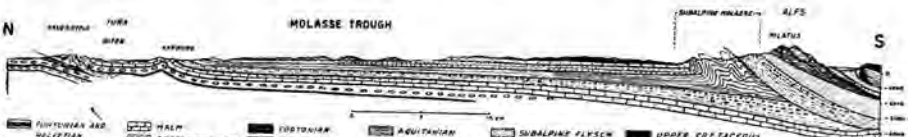

Fig. 2. Section across the Molasse trough (from BAUMBERGER).

This is still less comprehensible if downsliding was a main factor in the *mise en place* of the Helvetian nappes as is at present believed by some geologists.

Moreover the disturbed structure of the subalpine Molasse as well as the frontal parts of the Helvetian nappes are due to tectonic action of the basement according to several authors. In the central massifs the basement of both the Helvetian nappes and the Swiss plain is exposed. These massifs, consisting of a great number of wedges of crystalline rocks, were subjected to differential movements and the frontal parts of some of them reach as far as the Windgälle, Jungfrau, Breithorn and Muthorn, where they overthrust Mesozoic and Tertiary rocks of the High Calcareous Alps. It can hardly be denied that movements of similar wedges in the basement were responsible for the tectonics of the subalpine Molasse. This theory was offered by GÜNZLER (fig. 3), it was expressed by BERSIER [7]) in a section across the Molasse trough near Geneva, (fig. 4 and 5) and again by HABICHT in his generalized section across the subalpine Molasse near Ricken—Speer.

Generally, the wedge-shaped elements have been considered as a result of northward overthrusting. More probably, however, the main factor responsible for their origin was a process of progressive underthrusting of the foreland towards the Alps, as was explained at some length by the present author in a previous paper.

[6]) Cf. BAUMBERGER, op. cit. p. 73, 74; BAILEY, op. cit. p. 33.

[7]) According to KOPP the Molasse in BERSIER's section is much too thick. Instead of 5000 metres he thinks about 3000 metres would be a more probable figure and more in accordance with seismic data.

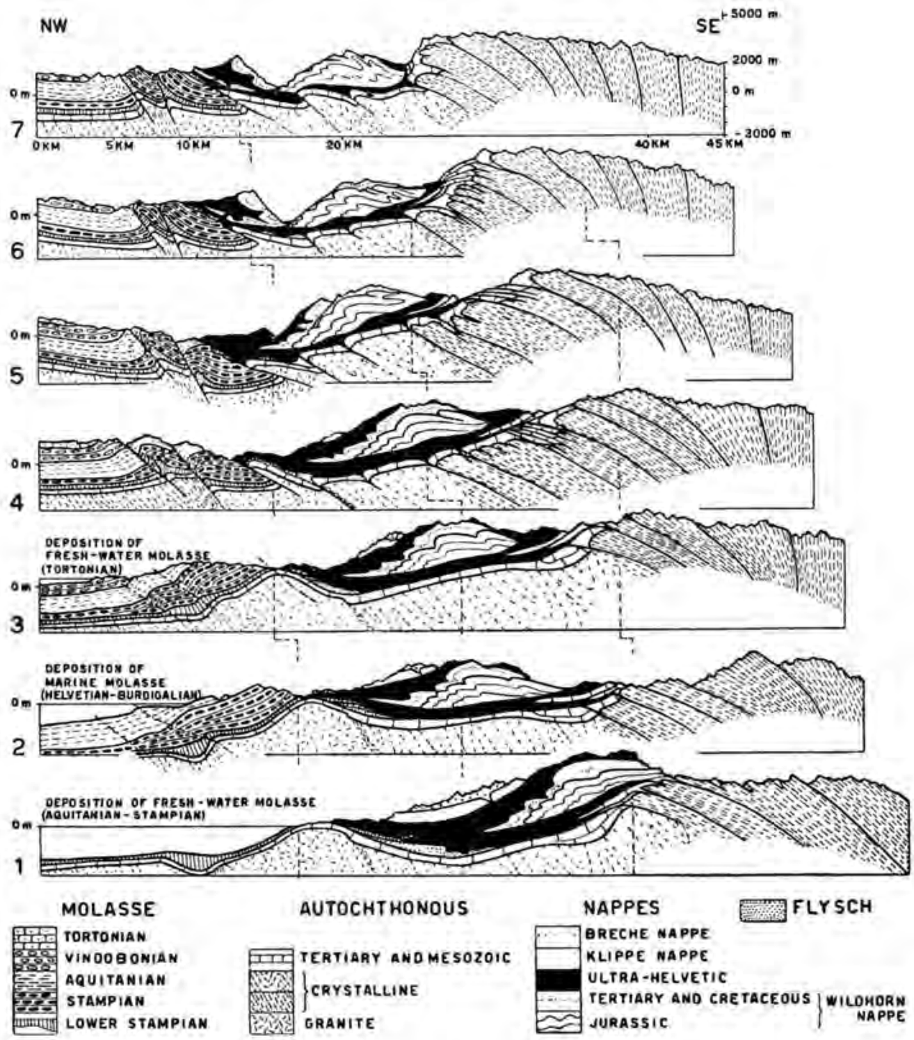

Fig. 3. Differential movements along basement wedges affecting the Central Massifs and the southern part of the Molasse trough (adapted from GÜNZLER).

3. BERSIER's profile (fig. 4) expresses a theory which is different from HEIM's inasmuch as a tangential push from the south is supposed to be transmitted underneath the Molasse trough towards its northwestern border where it caused the folding of the Jura Moutains. With respect to this representation the question again arises: why did the Juras become folded while an intervening pile of Molasse strata remained nearly undisturbed?

How far the typical sequence of Triassic and Jurassic layers extend beyond the Jura Mountains in S. and S.E. direction is unknown. The Trias of the Alps is developed in a quite different manner. However, to postulate that the lubricating layer coincides exactly with the present

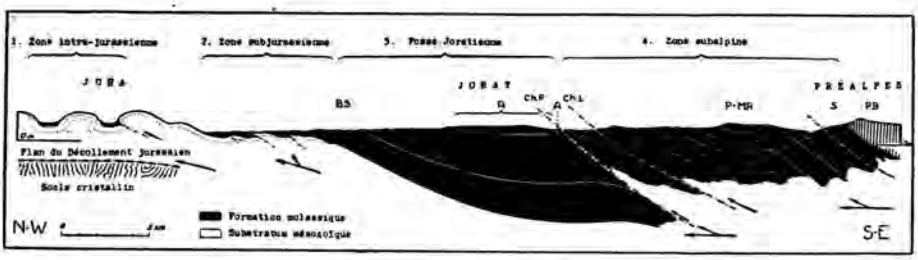

Fig. 4. Section across the western part of the Molasse trough (after BERSIER), compare fig. 5.

boundary of the Jura Mountains would be a too unsatisfactory theory. Besides it is not in agreement with known facts.

In the same manner the thickness and weight of the Molasse sediments cannot possibly furnish a full explanation because the Molasse trough extends much farther eastward than the easternmost Jura chains. Probably the processes involved were of a more complicated nature.

4. LUGEON went still further and considered the Jura Mountains as a nappe which slid into its present position under the influence of gravity. According to one suggestion a vertical uplift of the "central massifs" (Aar—Gothard, Aiguilles Rouges—M. Blanc) caused a downsliding of the plastic layers in N. and NW directions. The impulse is supposed to be transferred below the Molasse trough. The masses sliding and accumulating under the influence of gravity are supposed to have caused the *décollement* of the Jura Mountains and their adaptation to the surrounding framework of basement rocks. However, not only does the basement below the Jura Mountains dip towards the S and SE, i.e. in the opposite direction of the supposed gliding of the Jura strata, but one might reasonably expect an enormous gap either between the Juras and the Molasse trough or between the latter and the High Calcareous Alps.

Therefore LUGEON thinks the pressing out of plastic layers below the weight of sediments of the Molasse trough in an outward direction must have caused the folding of the Juras. Even this suggestion does not explain the shortening of the upper structure as displayed by the Jura Mountains. Moreover instead of folding in the Upper Miocene one might expect a gradual process keeping pace with the accumulation of Molasse sediments and causing the out-squeezing of the plastic layers along the outer boundary of the Molasse trough. According to LUGEON the extension of the Jura Mountains corresponds to the occurence of the series of Central Massifs. The latter disappear below the Helvetian nappes and possibly continue eastward in deeper realms. The point of disappearance would find its counterpart in the easternmost Jura anticline.

However, a line connecting the westernmost outcrops of the central massifs with those of the Vosges is far from parallel with a line which connects the easternmost outcrop of the central massifs with the eastern boundary of the Black Forest massif. This fact can easily be seen on a geological map and is clearly illustrated by OULIANOFF who also suggests an original connection between Vosges—Black Forest and central massifs, which was broken up subsequently. On the other hand there is a close relation between the extent and arrangement of the Jura anticlines and the situation of basement rocks along their outer boundary.

5. In short, in the simile of the table cloth being crumpled and forced over the table by some sort of pushing activety from the south, the following main problems remain unsolved: (1) what was the pushing element causing the *décollement*, (2) why does the inner boundary of the Jura Mountains coincide with the outer boundary of the Molasse trough, (3) why did the time of the *décollement* coincide with the Upper Miocene phase of diastrophism whereas such a phenomenon did not occur at the time of the much stronger Oligocene movements, (4) why did the greater part of the Molasse trough remain nearly undisturbed?

Theoretically there is a second possible way of crumpling a table cloth. Imagine we keep the cloth fixed at a certain point and then pull the table southward instead of pushing its cover northward. The result would be the same. Though an inconvenient manner of demonstration the simile is probably more relevant to what actually occurred when the Jura Mountains originated.

Our present knowledge of the structure of the Alps leads to the conclusion that a process of major importance was progressive underthrusting of both the northern and southern "forelands" towards the central belt of the Alps. The same process probably played an important part in the formation of the Jura Mountains.

Due to the process of the Miocene *décollement* the superstructure suffered a compression of about 6 to 10 kilometres or 25 percent. If we accept a process of southward underthrusting of the basement, one question still remains unanswered, viz. why did not the Jura strata slip under the Molasse trough, or more concisely what was the obstacle that kept the "table cloth" fixed at the present inner margin of the Jura Mountains?

The Oligocene movements in the Jura Mountains were contemporaneous with a phase of strong diastrophism in the Alps. However, only faults and some accompanying transverse folds came into being at that time. The dominating pattern of much stronger developed longitudinal folds originated during the more recent though not so strong phases of compression in the Upper Miocene. There must, therefore, be a reason not only why the longitudinal pattern originated in the Upper Miocene but also why it did not come into being with the much stronger Oligocene phase of diastrophism. Apparently the factor — or factors — which caused the upper layers to become stripped off from their basement so as to result in the Miocene *décollement* was — or were — not yet present during the Oligocene diastrophism. After the Oligocene diastrophism the Molasse trough came into being. Therefore it seems reasonable to consider the possible rôle played by this new element. Moreover the Rhine graben came into being in the Oligocene, though probably a first downward movement started already in the Eocene. However, the rising movement of the Rhine Shield which caused the formation of Vosges and Black Forest in the shape of high situated blocks took place in more recent times and had reached an appreciable amount in the Upper Miocene.

Hence the influence of the updoming basement of the foreland is a second factor which needs further consideration.

6. During the formation of the Molasse trough the basement under the trough subsided by an amount of about 3000 metres. In this movement the marginal flexures were the week strips predestined to become zones of tectonic disturbances during the subsequent epoch of diastrophism. Thus the tectonic disturbances along the inner boundary of the trough have to be considered as a necessary consequence of the general southward underthrusting of the basement during the Upper Miocene phase. Proceeding northward the next zone of weakness is the outer margin of the Molasse trough. Here, however, the situation was different inasmuch as the bottom of the trough slopes in the same direction as the underthrusting basement. Probably, therefore, the faults and crystalline basement wedges which originated along the outer margin of the trough were of minor importance if compared to those along the inner margin. A wedge-shaped element of the basement along the outer boundary of the Molasse trough was already drawn by STAUB [8]). BERSIER's profile (fig. 4) can be easily completed in a similar way (fig. 5).

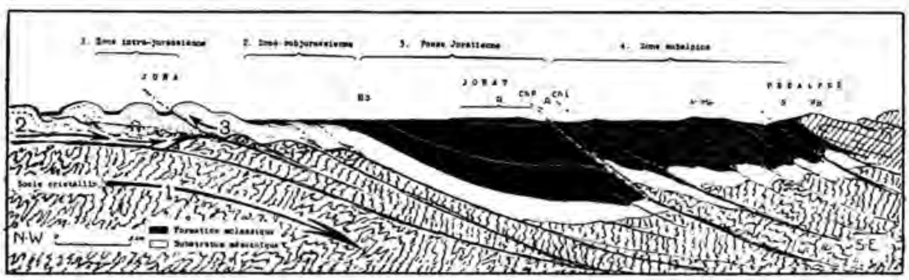

Fig. 5. The same section as fig. 4 with addition of basement. The arrow shows supposed direction of underthrusting.

Possibly the prism of Molasse sediments, if of a certain thickness and rigidity, formed in itself another and additional obstacle. Let us accept,

[8]) R. STAUB, op. cit. profiles 16—21. Wedges in the floor of the trough originated only where the trough is very narrow; see STAUB profile 21 (below the Salève fold).

for a moment, one or more obstacles along the outer boundary of the trough which prevented the upper layers of the Juras from being dragged along when southward underthrusting of the basement occurred. Even then the problem is not yet solved. For the Molasse trough extends much farther eastward than the Jura chains. Therefore the problem is only solved if there is a plausible reason for supposing the basement wedges not to have originated along the outer margin of the trough eastward of the present Jura Mountains.

7. I believe there is indeed a plausible reason for the origin of basement wedges along the inner side of the present Jura Mountains and their absence more eastward viz. the origin and extension of the other "new" element. Its influence was already mentioned in § 1. The shape of the Jura arc is adapted to the surrounding outcrops of basement rocks. Its eastward end corresponds exactly to the eastern boundary of the Black Forest (fig. 1), and the northernmost Jura folds protrude farther northward in the Rhine graben as contrasted to the adjacent sectors opposite Vosges and Black Forest. Hence it seems clear that without the presence of the highly situated basement rocks the Jura arc would not have formed.

It must be realised that only in some special areas the basement rocks have been elevated so high as to form large horst-like blocks like Vosges and Black Forest. However, a general tilting movement took place in a much wider surrounding area. Even in the Rhine graben the renewed movements along faults that occurred in Aquitanian times were accompanied by elevation according to VONDERSCHMITT [9]). So the surface of the basement under the Jura strata became gradually sloping upwards in an outward direction. This holds good also for the western part of the Juras. For even in this region, where the Juras are separated from the Plateau Central by the Saone-Rhône depression, outcrops of basement rocks are found along the outer margin of the Jura arc (fig. 1). Though the outer boundary of the Juras has the general appearance of a gently curved arc, it actually consists of two arcs (fig. 1). According to LUGEON a southwestern arc called *arc Lédonien* reaches as far as Salins from where a second arc, the so-called *arc bisontin* can be traced eastwards. Possibly this phenomenon is due to different amounts of tilt of the basement under the two respective arcs, the surface under the Lédonian arc dipping at a slightly greater angle if compared to the basement under the "arc bisontin".

Apparently subsidence of the Molasse trough was not sufficient to predestine the origin of basement wedges along its outer boundary. Only when, at the same time, the outer boundary of the trough became the inner boundary of an area with opposite movement —i.e. of an upward tilting of the present area of the Juras — the hinge line between the two opposed movements was weakened to such a degree that basement wedges

[9]) VONDERSCHMITT, 1942, p. 94.

and consequently a *décollement* could originate. Tentatively the situation is represented by blocks A—D of fig. 6. The tilted basement is shown by blocks A, B and C.

Block D represents a schematic profile east of the Black Forest. There the basement has not been tilted in an outward direction. Consequently no basement wedges originated and no *décollement* took place.

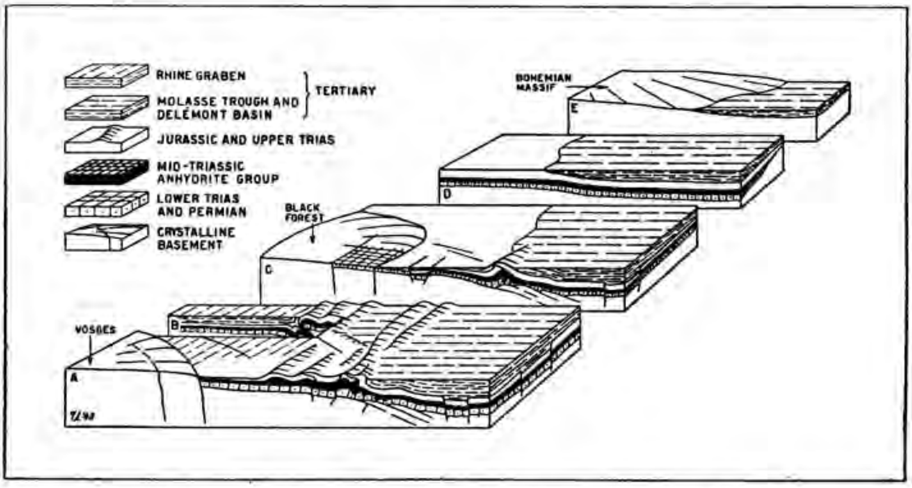

Fig. 6. Schematic representation of different sectors between Vosges and Bohemian Massif.

Still further eastward again there is a high situated block of basement rocks, the Bohemian massif (fig. 6, block E). There, however, neither Jura strata nor a lubricating layer are present. Moreover the prism of Molasse sediments is thinner and rests directly on the granites and gneisses of the Bohemian massif [10]).

Moreover, it is the upward slope of the basement that — combined with four other factors — was of deciding importance in controlling the outer boundary and shape of the Jura arc as well as the character of the folding.

The four other factors were: (1) the lubricating layer, (2) a certain thickness [11]) of the upper structure, (3) the limited space in transverse direction due to the obstacles along the outer boundary of the Molasse trough, and (4) Alpward underthrusting of the basement.

In the Rhine graben the basement is at a lower level than on the adjacent blocks. It is for this reason that *ceteris paribus* the outermost folds of the

[10]) See KOBER, op. cit. pp. 159 and 161; STAUB op. cit. profiles 1—3 (compare also profile 23). On STAUB's profiles no wedge-shaped obstacle of basement rocks is drawn along the northern margin of the trough.

[11]) The relation between the size of a fold and the thickness of the strata participating in the folding was discussed by DE SITTER (op. cit. 1929, with several examples from the Jura Mountains).

longitudinal system occur farther northward than on the higher blocks on either side (where similar conditions controlling the outer boundary of the arc are to be found more southward, see fig. 6, blocks A, B, C).

In short, the fact that a structure like the Jura Mountains is an exceptional feature means that its formation is due to the accidental presence and interaction of several factors. If we accept a southward movement of the basement the evident reason why the layers above the lubricant layer became folded within the limited space of the Juras is the presence of the upward slope of the basement rocks in an outward direction. In this process, however, one or more other factors were responsible for the coincidence of the inner margin of the Juras with the outer margin of part of the Molasse trough.

8. It seems improbable that the basement under the Juras is so smooth as was originally supposed. On the contrary structural elements of the basement dating from Oligocene times (some even from earlier times) presumably had a great influence in the development and arrangement of the Upper Miocene folds. Regarding the transverse elements this was already mentioned in § 1. Several transverse folds reveal the influence of an older pattern of the substratum in a convincing way. According to LINIGER the anticlines wich surround the Delémont basin existed even in pre-Stampian times, but they became rejuvenated subsequently, once in the Oligocene, and once again in the Miocene. Some of the Oligocene transverse faults became rejuvenated during the Miocene phases of movement. Among them is the well-known set of transcurrent faults. Probably, however, several longitudinal elements of the basement also had a great influence in the arrangement of the Upper Miocene pattern of folds.

The possible influence of basement wedges was suggested by AUBERT whose opinion is clearly represented in fig. 7. If such structures exist their

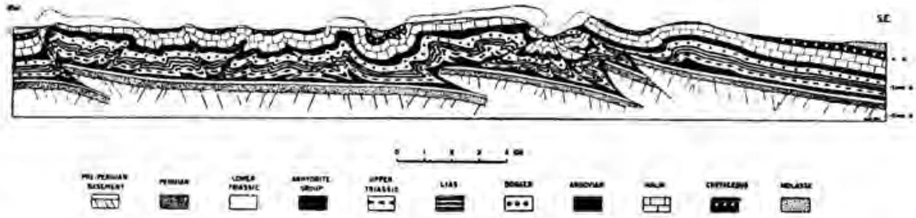

Fig. 7. Wedge-shaped structures in the basement of the Jura Mountains (after AUBERT).

origin is probably due to the same phenomenon of Alpward underthrusting of the basement that caused similar though larger wedges along the inner margin of the Juras. If so they must also have originated in the Upper Miocene. Another possibility is that longitudinal faults and relatively small graben-like structures (the *pincées* of GLANGEAUD, cf. § 1) dating from Oligocene or even older times had a great influence in the arrangement of the Upper Miocene pattern of the Juras. This idea is substantiated in the schematic and tentative blockdiagrams of fig. 6 and 8. Most probably both features played a rôle of importance.

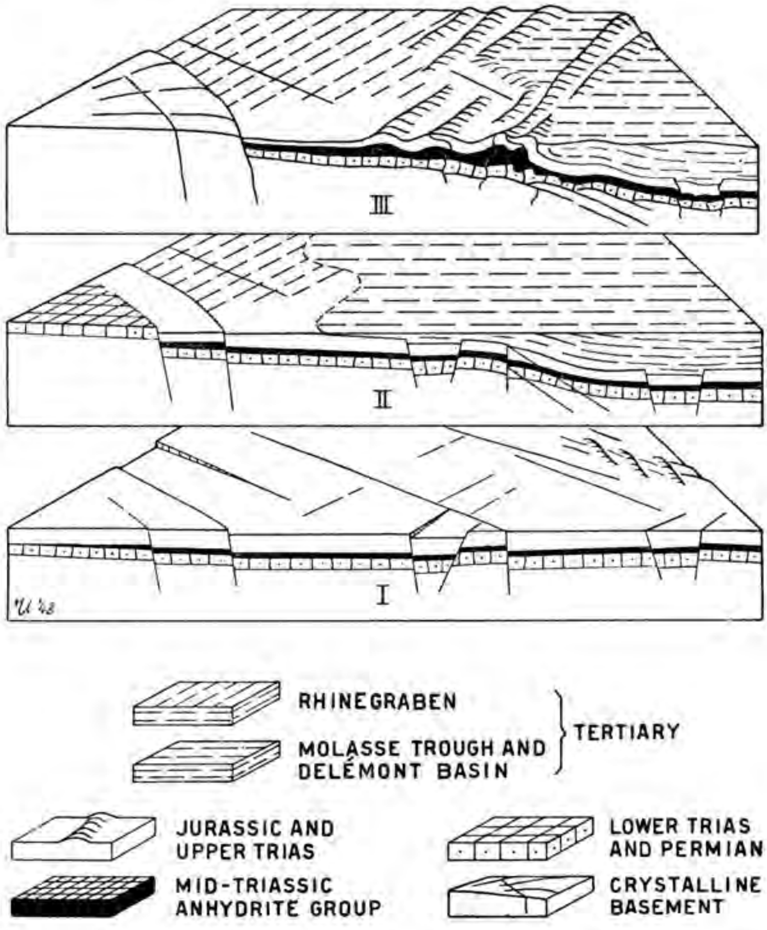

Fig. 8. The suggested sequence of events that gave origin to the Jura Mountains.

9. Finally, the suggested sequence of events that gave origin to the Jura Mountains is represented schematically by fig. 8 whereas a tectonogram showing the area under discussion in its coherence with the structural elements of the Alps, was published in my previous paper on *the root of the Alps.*

Block I of fig. 8 represents the situation after the Oligocene diastrophism. At a later stage the Molasse trough came into being (block II). The basement became gradually tilted outward and in part highly elevated to form the high situated blocks of the foreland (Black Forest, Vosges, Plateau Central) as well as the smaller outcrops in between them. During the

Upper Miocene phases of compression the inner margin of the trough became strongly influenced by movements along faults — parallel to the margin of the trough — which were due to a southward underthrusting of the basement and which caused the subalpine Molasse to become tilted and warped (fig 3). Due to the same general process of underthrusting another set of faults and intervening basement wedges originated along the next zone of weakness, viz. along the outer boundary of the trough (fig. 8, block III).

However, they originated only where the flexure became weakened by the upward tilting of the basement in the adjacent surrounding area. Without this additional weakening of the flexure no basement wedges and therefore no Jura Mountains would have originated (cf. fig. 6, A—D).

The wedges formed an obstacle which prevented the strata above the lubricating layer of the Mid-Triassic from being dragged along southward and downward so as to partake in the movement of the basement underthrusting towards the Alps. Possibly the prism of Molasse sediments formed an additional factor. The superstructure gliding over the lubricating layer became stripped off and rumpled into folds which in their general arrangement had to become adapted to the high situated basement. Possibly the connection between the lubricating layer below the Juras and its continuation under the Molasse trough was already broken or squeezed off during the subsidence of the trough and the additional tilting of the surrounding foreland. If so, this was a third reason for the coincidence of the outer margin of the Molasse trough and the inner boundary of the *décollement*. How far the sub-Jurassic layers extend under the Molasse trough is unknown. The manner in which they are drawn in fig. 6 is entirely arbitrary. The same holds good for the extension of the Jura-layers. The wedges are also drawn very schematically in the blockdiagram. Undoubtedly the situation is much more complicated, and possibly it would be better to imagine zones of imbrication instead of wedges which are a too simplistic representation. Among the Oligocene structural elements that became rejuvenated during the Upper Miocene phases is a set of transcurrent faults.

Though the major problematic points enumerated in § 5, seem largely cleared up by the suggested sequence of events, several questions need further examination. A better understanding of the complicated mechanism involved and a quantitative estimate of the rôle played by the interacting factors are wanted before the problem can be considered as solved. It may be expected that a thorough seismic investigation of the Juras and adjacent strips, combined with a gravimetric survey, will prove of great importance in this respect and, possibly, it will reveal also several unknown and unexpected features, and new problems.

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Aerodynamics. — *L'analyse spectrale de la turbulence.* By R. BETCHOV. (Mededeling No. 58a uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of October 30, 1948.)

Nous donnons ici la théorie et la description d'un appareil permettant d'étudier la composition spectrale de la turbulence d'un écoulement aérodynamique, et en général de tout phénomène similaire, traduit par un signal électrique.

I. *Spectre de Fourier de la turbulence.*

La fluctuation $v(t)$ de la vitesse V d'un courant d'air peut être convertie, à l'aide d'un anémomètre à fil chaud, en une tension électrique qui lui est proportionnelle, si l'on a pris soin de compenser l'inertie thermique du fil et si l'on néglige quelques autres défauts du fil chaud. Cette fonction v ne présente pas de discontinuités. Nous supposerons que l'observation se fait entre les instants $-T$ et $+T$, et que v est nulle en dehors de cet intervalle (fig. 1). Cela correspond donc au cas pratique et nous assure que les

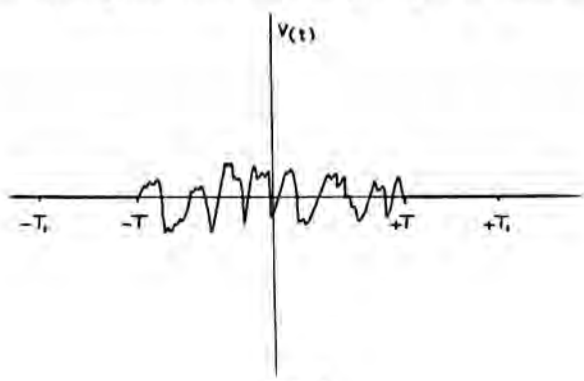

Fig. 1.

intégrales suivantes existent:

$$\int_{-\infty}^{+\infty} v(t)\, dt \quad \text{et} \quad \int_{-\infty}^{+\infty} v^2(t)\, dt. \quad \ldots \ldots \quad (1)$$

La transformée $\varphi(s)$ est alors définie par les équations:

$$v(t) = \int_{-\infty}^{+\infty} \varphi(s)\, e^{jst}\, ds. \quad \ldots \ldots \quad (2)$$

$$\varphi(s) = \frac{1}{2\pi} \int_{-T}^{+T} v(t)\, e^{-jst}\, dt. \quad \ldots \ldots \quad (3)$$

La fonction v est toujours réelle, et en désignant par une (*) les grandeurs conjugées, nous avons:

$$v = v^* = \int_{-\infty}^{+\infty} \varphi^*(s)\, e^{-jst}\, ds \quad \text{et} \quad \varphi^*(s) = \varphi(-s). \quad . \quad . \quad . \quad (4)$$

Pour mesurer la turbulence, définie par la formule:

$$\tau = \frac{\sqrt{\overline{(v^2)}}}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

on utilise un thermo-couple et un galvanomètre. Ce dispositif donne la valeur moyenne de v^2, mais l'indication du galvanomètre peut encore fluctuer si les variations de v^2 ne sont pas toutes plus rapides que la fréquence propre du système, correspondant à une période de 10 à 20 secondes.

Le carré de v est donné par:

$$v^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi^*(r)\, \varphi(s)\, e^{j(s-r)t}\, dr\, ds. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Pour obtenir le carré moyen, calculons l'intégrale:

$$\int_{-T_1}^{+T_1} v^2\, dt = 2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi^*(r)\, \varphi(s)\, \frac{\sin(s-r)\, T_1}{(s-r)}\, dr\, ds \quad . \quad . \quad . \quad . \quad (7)$$

prise entre les limites $+T_1$ et $-T_1$ où $T_1 > T$ (fig. 1).

Lorsque T_1 tend vers l'infini, on a:

$$\int_{-T_1}^{+T_1} v^2\, dt = 2\pi \int_{-\infty}^{+\infty} \varphi^*(s)\, \varphi(s)\, ds = 4\pi \int_0^{\infty} \varphi(s)\, \varphi(-s)\, ds \quad . \quad . \quad . \quad (8)$$

en vertu de la relation:

$$\lim_{\lambda \to \infty} \int_{-\infty}^{+\infty} f(x)\, \frac{\sin \lambda (x-\xi)}{(x-\xi)}\, dx = \pi f(\xi). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Comme nous prenons v nulle si t est extérieur au domaine $\pm T$, le carré moyen est donné par:

$$\overline{v^2} = \frac{1}{2T}\int_{-T}^{+T} v^2(t)\, dt = \frac{1}{2T}\int_{-T_1}^{+T_1} v^2\, dt = \frac{2\pi}{T}\int_0^{\infty} \varphi(s)\, \varphi(-s)\, ds. \quad . \quad (10)$$

Ceci implique que l'amplitude de φ est proportionnelle à $\sqrt{T}$, ce qui est normal. Nous définissons encore la fonction ψ:

$$\psi(s) = \frac{2\pi}{T}\, s\, \varphi(s)\, \varphi(-s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

telle que:

$$\overline{(v^2)} = \int_0^\infty \psi(s)\, d(\mathrm{Log}\, s). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Pour mesurer ψ, il faut transmettre le signal v à travers un filtre passe-bande. Désignons par $A(s; s_0)$ la fonction complexe donnant la réponse du filtre, accordé sur la fréquence s_0 et attaqué par la fréquence s. La tension de sortie sera alors:

$$w(t) = \int_{-\infty}^{+\infty} A(s)\, \varphi(s)\, e^{jst}\, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Le filtre aura évidemment la propriété $A^*(s) = A(-s)$, puisque les seuls éléments physiques sensibles à la fréquence sont des impédances imaginaires. La fonction $w(t)$ donnant le signal de sortie du filtre est continue et nulle en dehors des limites $\pm T$, sauf un petit signal amorti après $t = +T$. En appliquant w au thermo-couple, on obtient une indication du galvanomètre proportionnelle à:

$$\overline{(w^2)} = \frac{1}{2T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(s)\, A^*(r)\, \varphi(s)\, \varphi^*(r) \left\{ \int_{-T_1}^{+T_1} e^{j(s-r)t}\, dt \right\} dr\, ds. \quad . \quad (14)$$

Le raisonnement employé ci-dessus nous donne:

$$\overline{(w^2)} = \frac{2\pi}{T} \int_0^\infty \varphi(s)\, \varphi(-s)\, A(s)\, A(-s)\, ds. \quad . \quad . \quad . \quad . \quad (15)$$

Si la fonction A est différente de zéro dans une bande étroite autour de s_0, et si $\varphi(s)\,\varphi(-s)$ ne varie pas trop dans cet intervalle, on peut écrire:

$$\overline{w^2} = \frac{2\pi}{T}\, \varphi(s_0)\, \varphi(-s_0)\, \lambda(s_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

où la grandeur $\lambda(s_0)$ représentant la largeur de la bande, est définie par:

$$\lambda(s_0) = \int_0^\infty A(s)\, A(-s)\, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Si λ est proportionnelle à s_0, on a w^2 proportionnelle à ψ, et le spectre ψ peut être observé en faisant varier s_0.

Nous avons observé que, si l'on travaille dans une région du spectre où ψ varie rapidement en fonction de s, la fonction w présente des battements irréguliers, d'autant plus prononcés que la bande du filtre est plus étroite. Ces variations en amplitude sont assez lentes pour agir sur le galvanomètre.

Il est alors délicat de considérer l'indication moyenne de cet instrument comme mesure de ψ. En effet, son dispositif d'amortissement ne fonctionne pas nécessairement selon une loi linéaire, et il peut se produire une sorte de détection. Avec différentes sélectivités, et compte tenu de l'effet sur la largeur de bande, nous avons observé un effet de cet ordre.

Ces „fluctuations de fluctuations" imposent une limite à la sélectivité du filtre.

On pourrait appliquer la tension fournie par le thermo-couple (quelques millivolts) à une grille de lampe, à travers un filtre passe-bas de très grande constante de temps et avoir ainsi un système strictement linéaire.

II. *Le dispositif de filtrage.*

Nous utilisons un amplificateur à contre-réaction, ne comportant aucune self-induction.

Prenons un signal d'entrée „monochromatique", que nous représenterons par $v\,e^{j\omega t}$. Il est appliqué à l'une des grilles d'un premier étage constitué par deux pentodes égales, dont grilles et écrans sont fortement couplés (voir fig. 2). La grille de la seconde lampe reçoit la tension de réaction que nous définirons plus tard (voir formule 21). La plaque de cette seconde lampe transmet le signal $g(u-v)e^{j\omega t}$ à la grille d'une triode. Le facteur d'amplification g vaut environ 80.

Le couplage doit être direct, pour éviter l'auto-oscillation de l'appareil. Cette triode n'amplifie pas, mais elle fournit, avec des basses impédances de sortie, une tension de cathode $g(v-u)$ et une tension d'anode $g\,k(u-v)$, où k désigne le rapport des résistances d'anode et de cathode, rapport voisin de 2 et où nous avons supprimé les termes $e^{j\omega t}$. Ces tensions sont appliquées au filtre schématisé en fig. 3. La tension de sortie du filtre, soit U, est donnée par:

$$U = g\,(v-u)\,F(\omega) \quad \ldots\ldots\ldots \quad (18)$$

avec:

$$F(\omega) = \frac{(C_2/C_1 + R_1/R_2 - k) + j\,(\omega/\omega_1 - \omega_2/\omega)}{(1 + C_2/C_1 + R_1/R_2) + j\,(\omega/\omega_1 - \omega_2/\omega)} \quad \ldots \quad (19)$$

Valeurs des éléments de la figure 2

1	1 M Ω log.	12	5 K Ω
2	300 K Ω	13	50 K Ω
3	500 K Ω	14	800 Ω
4	1200 Ω	15	100 K Ω
5	4.700 Ω	16	250 K Ω
6	1 K Ω	20	1 microfarad
7,	7′, 7″, réglage	21	sept capacités, de 0,7 ; 0.22 ; 0,07 ; 0,022 jusqu'à 0,0007 muf
8	25 K Ω	22	2 muf, faible, capacité avec la masse
9	50 K Ω, lin. bobiné	23	2 muf.
10	100 K Ω log. avec interrupteur		
11	1 M Ω		

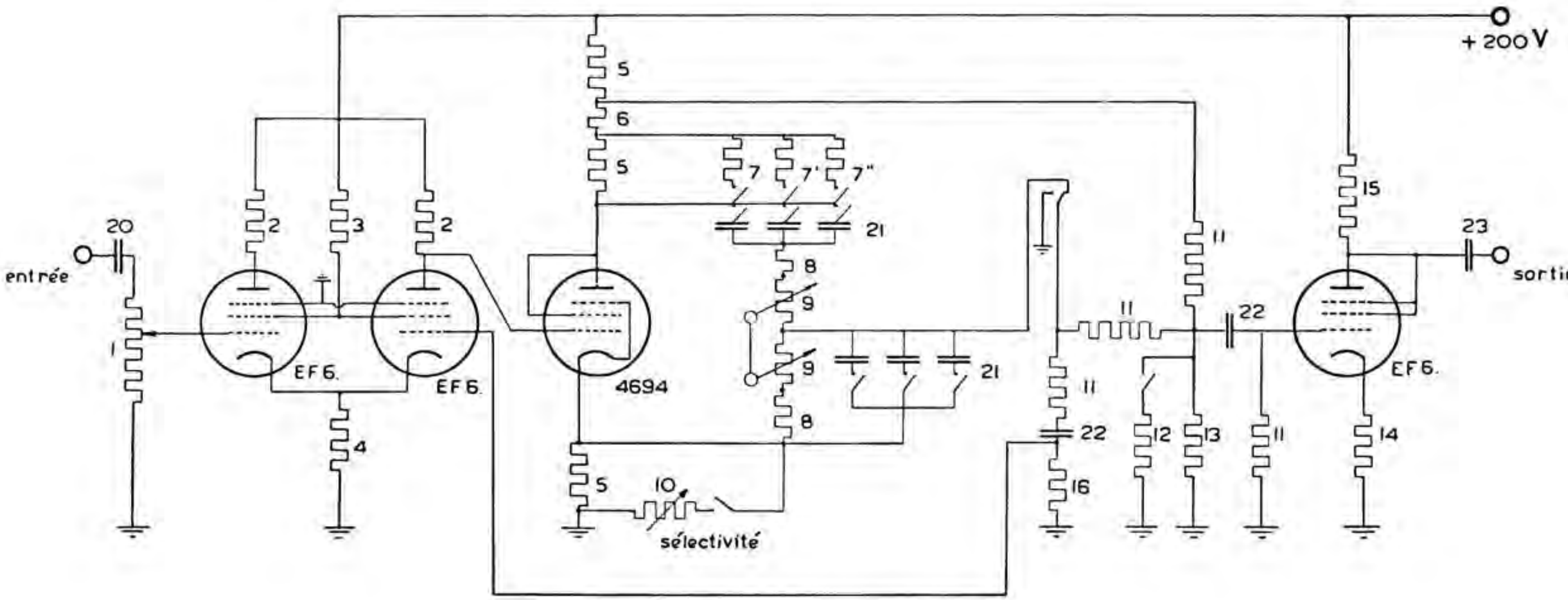

Fig. 2. Schéma du spectromètre.
Le nombre des paires de condensateurs et des résistances de réglage a été réduit de 7 à 3, pour simplifier le schéma.

où:

$$\omega_1 = 1/R_1 C_2 \qquad \omega_2 = 1/R_2 C_1.$$

Il faut régler k de façon à rendre le terme $m = (C_2/C_1 + R_1/R_2 - k)$ suffisamment petit. La fonction $F(\omega)$ est alors voisine de zéro si $\omega = \omega_0$, avec:

$$\omega_0 = \sqrt{\omega_1 \omega_2}. \quad \ldots \ldots \ldots \quad (20)$$

D'autre part, F tend vers 1 si ω va vers zéro ou vers l'infini. Pratiquement nous avons $\omega_1 = \omega_2 = \omega_0$ et F est invariante à la transformation de ω/ω_0 en ω_0/ω.

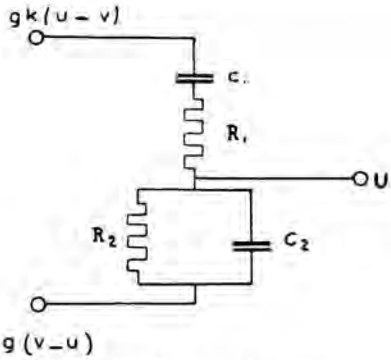

Fig. 3. Principe du filtre.

Les résistances R_1 et R_2 sont constituées chacunes par un élément fixe de 25 Kohms et par un potentiomètre linéaire et bobiné, de 50 Kohms. Ces potentiomètres sont montés sur un même axe et permettent de faire varier R_1 et R_2 dans un rapport de 1 à 3. En ajustant l'un des éléments d'environ 25 Kohms, on peut obtenir un rapport R_1/R_2 constant et voisin de l'unité, dans tout le domaine des potentiomètres, sauf quelques écarts, de l'ordre de 1 %. En utilisant différentes paires de condensateurs C_1 et C_2, on peut faire varier $\omega_0/2\pi$ de 3 à 10.000 cycles. Nous utilisons sept paires de condensateurs, dont trois seulement sont représentées sur le schéma (fig. 2).

Le rapport C_2/C_1 dépend de la paire enclanchée, mais reste voisin de 1; il faut donc ajuster le rapport k de manière à ce que le terme m reste constant, aux petites variations de R_1/R_2 près, quelle que soit la valeur de ω_0. Un jeu de résistances de réglage, allant de 10 à 80 Kohms, nous permet de travailler avec $m = 0{,}03$. La valeur minimum de F est alors $0{,}01 \pm 0{,}003$.

La tension U est appliquée à un atténuateur qui fournit la tension $u = \gamma U$, avec γ voisin de 0,2. On a alors:

$$u = \gamma g (v - u) F(\omega) = v \frac{\gamma g F}{1 + \gamma g F} \quad \ldots \ldots \quad (21)$$

d'où l'on déduit:

$$v - u = \frac{v}{1 + \gamma g F}. \quad \ldots \ldots \ldots \quad (22)$$

Le rapport entre le signal $(v - u)$ à F minimum et à $F = 1$ n'est pas assez élevé. Pour améliorer la réponse, nous faisons la somme de la tension U et de la tension $g(u - v)$ fournie par une fraction de la résistance d'anode. Cette somme est amplifiée une fois et donne le signal de sortie w.

Compte tenu du changement de signe, et à une constante multiplicative près, on a:

$$w = g\,(v - u) - U = v\,\frac{1 - F}{1/g + \gamma F}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Nous écrirons:

$$w = \frac{w_{\max}}{1 + jMz}$$

avec

$$z = \omega/\omega_1 - \omega_2/\omega \quad ; \quad M = \frac{1 + \gamma g}{3 + m\gamma g} \quad ; \quad w_{\max} = vg\,\frac{3 - m}{3 + \gamma g m}. \quad (24)$$

Pratiquement on atteint facilement $w/v = 0{,}1$ avec F très voisin de 1, et $w/v = 55$ à F minimum. Le réglage soigné permet d'atteindre un contraste plus élevé, mais l'appareil se derégle lentement et cela n'est pas nécessaire. D'autre part les formules calculées ci-dessus négligent différents facteurs, par exemple le rôle de l'impédance du filtre dans le circuit de la triode.

Les défauts de linéarité des potentiomètres agissent sur w lorsque $\omega = \omega_0$, et produisent d'importantes variations de w_{max}. En augmentant la sélectivité, on accentue encore cet effet.

Le facteur γ a été déterminé sur la base de ces considérations et des „fluctuations de fluctuations" décrites ci-dessus.

Nous avons également la possibilité de transmettre à travers l'appareil le signal d'entrée non filtré, en supprimant le feed-back et en réduisant l'amplification (résistance No. 12). On pourrait également obtenir un filtre passe-haut ou passe-bas.

Selon la définition usuelle, la sélectivité est mesurée par le rapport $2\,(\omega_s/\omega_0 - 1)$ avec ω_s telle que l'amplitude de w tombe de w_{max} à $0{,}707\ w_{max}$. A partir de (24), on déduit:

$$2\,(\omega_s/\omega_0 - 1) \simeq 1/M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Nous avons environ $1/M = 0{,}2$, ce qui donne autour de $\omega_0/2\pi = 100$ périodes, une bande large de 20 cycles. Mais il nous faut une définition plus satisfaisante, en accord avec (17). Traçons, avec un signal d'entrée „monochromatique" la courbe de w^2 en fonction de ω, et également en fonction de Log ω. Une telle courbe est donnée en fig. 4, pour $\omega_0/2\pi = 850$ périodes. La surface de la première courbe est donnée par:

$$\int_0^{\infty} w^2(\omega)\,d\omega = \frac{\omega_0}{2}\,w_{\max}^2 \int_{-\infty}^{+\infty} \frac{dx}{1 + M^2\left(x - \dfrac{1}{x}\right)^2}. \quad . \quad . \quad . \quad . \quad (26)$$

L'intégration dans le plan complexe, autour des pôles $+\sqrt{1-1/4\,M^2}+$ $+\,j/2\,M$ et $-\sqrt{1-1/4\,M^2}+j/2\,M$ donne:

$$\int_0^\infty w^2\,d\omega = \frac{\pi}{2}\,\frac{\omega_0}{M}\,w^2_{max} \quad \ldots\ldots\ldots \quad (27)$$

La surface de la seconde courbe correspond à:

$$\left.\begin{aligned} \int_0^\infty w^2\,d\,(\mathrm{Log}\,\omega) &= w^2_{max}\int_0^\infty \frac{dx}{x\left(1+M^2\left(x-\frac{1}{x}\right)^2\right)} \\ &= \frac{2\,w^2_{max}}{\sqrt{4\,M^2-1}}\,\mathrm{arc\ tang}\,\sqrt{4\,M^2-1} \simeq \frac{\pi}{2}\,\frac{1}{M}\,w^2_{max} \end{aligned}\right\} \ldots \quad (28)$$

Avec M de l'ordre de 5, on voit que la seule différence entre les résultats selon (27) et (28) réside dans la présence du facteur ω_0. Les écarts des potentiomètres font varier w_{max} et M de $\pm$ 6 %, ce qui fait varier la surface des courbes de $\pm$ 6 %.

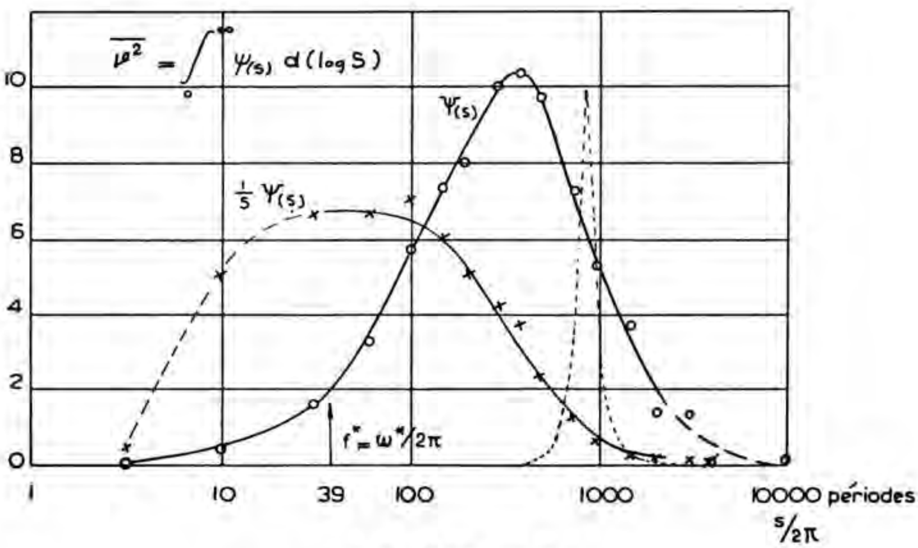

Fig. 4. Spectres ψ et φ.
Vent de 10 m/sec., turbulence de 0,8 %. La caractéristique de l'appareil est tracée autour de 850 périodes.

L'appareil travaille avec une bande passante proportionnelle à la fréquence d'accord, ce qui justifie l'hypothèse faite pour le calcul de w^2. Si le signal d'entrée est turbulent, la fonction w^2 donnée par le thermo-couple correspond à la fonction ψ.

Nous pouvons augmenter la sélectivité, en réduisant la résistance de cathode de la triode. Cela revient à élever la valeur de k et la bande devient très étroite, jusqu'à ce que l'oscillation de l'appareil soit déclanchée. Mais la valeur de w_{max} varie selon la position des potentiomètres, et la sélectivité variable ne peut être utilisée que pour des observations qualitatives.

III. *Résultats de mesures et remarques diverses.*

Nous donnons ici quelques résultats d'expériences et signalons quelques problèmes particuliers. Nous utilisons les symboles définis dans nos publications antérieures que nous désignons par I, II et III [1]).

Le fil de Pt. utilisé contenait 10 % d'Ir., diamètre de 15 microns, longueur de 1,98 mm. Il était soudé à un support ne vibrant pas, avec une tension de l'ordre de 3 grammes (la limite de ce matériel est atteinte à $4 \cdot 16^6$ gr/cm^2, soit ici avec une tension de 7 gr.). L'étalonnage statique, soit le relevé des courbes $I(R, V)$, nous donne I_0. La résistance de la ligne reliant le fil chaud au pont de WHEATSTONE doit être compensée.

Selon la formule de KING, la grandeur:

$$A = \frac{R I^2}{R - R_0} = (\varkappa' + \sqrt{2\pi \varkappa' s' \delta' V d}) \frac{d^2}{4/\pi \, a \varrho} \quad . \quad . \quad . \quad (29)$$

ne devrait pas dépendre de la température T. L'expérience montre que le facteur de $\sqrt{V}$ reste constant, mais que le terme additif en $\varkappa'$ augmente avec T. Tout se passe comme si la formule de KING était appliquable avec $\varkappa'$ et δ' fonctions de T, à pression constante. En première approximation, $\varkappa'\delta'$ resterait constant. Cependant il conviendrait de donner une solution exacte de ce problème, en tenant compte de la modification de l'écoulement aérodynamique produite par l'échauffement de l'air et des variations des coefficients avec T.

Le fil est ensuite étalonné dynamiquement, selon la méthode décrite en II, et nous avons à plusieurs reprises obtenus $C = 800$. La fréquence propre f^* a été évaluée à 39 cycles et le préamplificateur ajusté à cette valeur.

La mesure de la turbulence peut alors avoir lieu, et avec une vitesse de vent et une résistance du fil donnée, on lit au galvanomètre du thermocouple une indication σ. Immédiatement après, nous appliquons à l'amplificateur une tension de fréquence connue (1000 cycles) et mesurons l'amplitude $e(\sigma)$ donnant la déflection σ (mesure en volts efficaces !). Cette tension est équivalente à la turbulence, et le bruit de fond est éliminé, si l'on suppose qu'il n'a aucune corrélation avec la turbulence. En première

[1]) I. R. BETCHOV et E. KUYPER, Un amplificateur pour l'étude de la turbulence... (Meded. 50), Proc. Acad. Sciences Amsterdam, **50**, 1134—1141 (1947);

II. R. BETCHOV, L'inertie thermique des anémomètres... (Meded. 54), ibid. **51**, 224—233 (1948);

III. R. BETCHOV, L'influence de la conduction thermique... (Meded. 55) ibid. **51**, 721—730 (1948).

approximation, la turbulence est donnée par les formules (11) et (14) de I:

$$\tau = \frac{4 \cdot 1000 \cdot e(\sigma)}{C\,RI\,(I^2 - I_0^2)}. \quad \ldots \ldots \ldots \quad (30)$$

Notre tunnel, à circuit ouvert, ne comporte qu'une grille de filtrage (mailles carrées, côté de 3 cm, profondeur de 18 cm). A cette grille succède une contraction de 3 à 1, donnant une section de travail de 40 × 40 cm. Aux basses vitesses (5 m/sec) la turbulence est de 0,6 %; elle s'élève avec la vitesse et atteint 1 % à 25 m/sec. Les vibrations mécaniques du tunnel sont liées à cette augmentation.

Le bruit du tunnel ne contribue pas directement à cet effet; si nous supposons que les fluctuations acoustiques de la pression atteignent au maximum 100 dynes/cm^2, soit 10^{-4} at., les fluctuations de la densité de l'air δ' (voir formule 29) produisent une turbulence fictive de 0,01 %, que l'on peut négliger.

Le spectre de la turbulence, soit la fonction $\psi(\omega)$, est donné par le spectromètre, après déduction du spectre du bruit de fond des amplificateurs et correction de conduction thermique. Ce spectre de fond n'est sensible que de 1000 à 6000 périodes. La figure 4 donne le spectre ψ avec $V = 10$ m/sec, ainsi que la courbe calculée ψ/ω, proportionnelle à la norme de $\varphi(\omega)$. On voit qu'il serait préférable d'avoir des appareils amplifiant au delà de 3000 cycles.

La partie principale du spectre est au dessus de la fréquence propre du fil, soit 39 cycles, ce qui garantit une reproduction correcte. En effet, avec un fil plus mince, l'erreur sur f^* et l'erreur de compensation auraient des conséquences plus graves. Pour placer f^* au delà du spectre, soit à 1000 cycles au moins, il faudrait un diamètre de 1 à 2 microns.

La correction de longueur froide a été calculée selon la formule (15) de III, et les figures 1 et 6 de III.

En prenant $l^* = 0{,}2$ mm, soit $\xi = 5$, on trouve que l'indication du spectromètre doit être majorée de 30 % entre 3 et 10 cycles, de 16 % à 40 cycles, 8 % à 100 cycles et 0 % au delà de 400 cycles. La surface de la courbe est ainsi augmentée de 3 %, ce qui correspond à une majoration de la turbulence de 1,5 %. De plus, tout le signal a été diminué selon $(1 - 1/\xi)$, et notre expression (30) de la turbulence doit être multipliée par 1,26, ce qui nous a donné $\tau = 0{,}70$ %.

La correction selon la constante h du circuit de chauffage (formules (3) et (9) de I), donne 1 %, et il nous faudrait encore tenir compte d'un dernier effet: l'auto-corrélation. Nous ne voulons pas entrer dans le détail de cette correction, que nous évaluons à 10 %, et qui est due au fait que plus le fil est long, plus il y a de chance qu'un supplément de vitesse du vent en un point soit compensé par une diminution en quelque autre point du fil. Cet effet est lié au degré de corrélation de la turbulence.

La turbulence vraie atteint ainsi 0,8 % environ.

Aerodynamics. — *Spectral analysis of an irregular function.* By J. M. BURGERS. (Mededeling No. 58*b* uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of October 30, 1948.)

1. The object of the following lines is to give a few comments on the concept of the spectrum of an irregularly oscillating function of the time, which is treated by R. BETCHOV in the accompanying article [1]). BETCHOV has represented the function $v(t)$ considered by him, by means of a Fourier integral. In order to be able to do this, it was necessary to suppose that the function $v(t)$ existed only in an interval of finite duration, $-T < t < +T$, outside of which it was replaced by zero. Although this is sufficient for many purposes, there is something artificial in the procedure, as actually the function $v(t)$ may represent some natural phenomenon continually going on and exhibiting always the same statistical character (one may think *e.g.* of the oscillations of the atmospheric pressure). The Fourier integral, however, cannot be applied for an infinite interval of time in such a case, as the functions considered here do not satisfy the condition of having "limited total fluctuation" over an infinite interval.

It is possible to define the spectrum of such a function in a different way, by establishing a relation between the spectrum and the "coefficient of correlation" of the function $v(t)$. This has been shown by TAYLOR [2]). For the demonstration of his formula TAYLOR still refers to the Fourier integral of $v(t)$. We will show that a relation, very similar to that given by TAYLOR, can be deduced by investigating what is obtained when an electric signal, proportional to $v(t)$, is passed through a "filtering circuit". In this way we keep as closely as possible to the experimental method applied for spectral analysis, without making use of any supposition concerning the possibility of Fourier analysis of the function $v(t)$.

2. Many types of filtering circuits can be constructed, but in order to keep the mathematical deductions as simple as possible, the following example has been taken, in which the transformed signal $w(t)$ is connected with the signal $v(t)$ by the differential equation:

$$\frac{d^2 w}{dt^2} + 2 p \omega \frac{dw}{dt} + \omega^2 w = 2 \sqrt{p}\, \omega \frac{dv}{dt} \quad . \; . \; . \; . \; . \quad (1)$$

Here ω and p are adjustable constants; p must be smaller than 1 and

[1]) R. BETCHOV, L'analyse spectrale de la turbulence, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam. **51**, 1063—1072 (1948).

[2]) G. I. TAYLOR, The spectrum of turbulence, Proc. Royal Society London, A **164**, 476 (1938).

actually will be taken very small. The factor before dv/dt on the right hand side is unimportant, as the signal coming out of the filtering circuit can be amplified; the factor $2\sqrt{p}\,\omega$ introduced in (1) has been chosen for convenience.

That the differential equation (1) defines a filter, is seen if v is replaced by e^{ist}. The value of w then becomes:

$$w = \frac{2\sqrt{p}}{2p + i(s/\omega - \omega/s)} e^{ist}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This gives a maximum for $s = \omega$; the sharpness of the maximum increases when p is taken smaller and smaller, its actual width being proportional to $p\,\omega$ [3]).

When $v(t)$ is an irregular function, always presenting the same statistical character, we can use the integral of (1):

$$w = 2\sqrt{\frac{p}{1-p^2}} \int_0^\infty d\tau \; v\left(t - \frac{\tau}{\omega}\right) e^{-p\tau} \cos(\tau\sqrt{1-p^2} + \varepsilon) \quad . \quad . \quad (3)$$

(where ε is defined by $\sin \varepsilon = p$) for the calculation of mean values referring to w, from mean values referring to v.

The most important quantities in this respect are $\overline{w(t)\ w(t+\eta)}$ and $\overline{v(t)\ v(t+\eta)}$, where η is a fixed interval of time. In both cases the mean values are taken with respect to t, over a period of sufficient duration in order that mean values can be treated as invariant with respect to a displacement of this period forward or backward along the time-scale. We suppose that the second quantity is given, in the form:

$$\overline{v(t)\, v(t+\eta)} = A^2 R(\eta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where A^2 is the mean value of $v(t)^2$, while $R(\eta)$ represents the *coefficient of correlation* of the function $v(t)$.

Knowledge of the value of A^2 and of the function $R(\eta)$ will embody all that we assume to be given concerning the statistical character of the function $v(t)$. The function $R(\eta)$ is a measure for the rapidity of change with time of the function $v(t)$ on one hand, and for the degree of regularity or irregularity of $v(t)$ on the other hand. We have $R(0) = 1$, which at the same time is the maximum value $R(\eta)$ can attain. In all practical cases it is possible to find a value θ, such that:

$$R(\eta) = 0 \text{ for } \eta > \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

It is not to be excluded that within the range $0 < \eta < \theta$ the coefficient of correlation $R(\eta)$ may pass through negative values.

The function $R(\eta)$ is symmetrical, so that

$$R(\eta) = R(-\eta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

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[3]) As dr. BETCHOV has told me, the differential equation for the filtering circuit described by him can be brought into a form similar to (1).

3. Starting from the integral (2) the following relation can be deduced:

$$\left.\begin{aligned}&\overline{w(t)\,w(t+\eta)}=\\&=\frac{4p}{1-p^2}\int_0^\infty d\tau_1\int_0^\infty d\tau_2\,\overline{v\left(t-\frac{\tau_1}{\omega}\right)v\left(t+\eta-\frac{\tau_2}{\omega}\right)}\,e^{-p(\tau_1+\tau_2)}\\&\qquad\cos(\tau_1\sqrt{1-p^2}+\varepsilon)\cos(\tau_2\sqrt{1-p^2}+\varepsilon).\end{aligned}\right\}. \quad (6)$$

This equation can be transformed by introducing $\sigma=\tau_2+\tau_1$ and $\delta=\tau_2-\tau_1$ as new variables. We will pass over the details and mention the result [4]):

$$\left.\begin{aligned}&\overline{w(t)\,w(t+\eta)}=\\&=\frac{A^2}{\cos\varepsilon}\int_0^\infty d\delta\left\{R\left(\eta+\frac{\delta}{\omega}\right)+R\left(\eta-\frac{\delta}{\omega}\right)\right\}e^{-p\delta}\cos(\delta\sqrt{1-p^2}+\varepsilon).\end{aligned}\right\} \quad (7)$$

The mean square of w is obtained from this formula if we take $\eta=0$. Making use of (5*b*) and assuming p to be so small that p^2 can be neglected in comparison with 1, we find:

$$\overline{w^2}=2\,A^2\int_0^\infty d\delta\,R\left(\frac{\delta}{\omega}\right)e^{-p\delta}\cos(\delta+\varepsilon). \quad (8)$$

We introduce the functions:

$$I_1(\omega)=\omega\int_0^\infty d\delta_1\,R(\delta_1)\,e^{-p\omega\delta_1}\cos\omega\,\delta_1 \quad (9a)$$

$$I_2(\omega)=\omega\int_0^\infty d\delta_1\,R(\delta_1)\,e^{-p\omega\delta_1}\sin\omega\,\delta_1 \quad (9b)$$

These functions are generalised Fourier transforms of the correlation coefficient $R(\eta)$. When p is so small that the quantity $p\,\omega\,\theta$ will be small compared with unity for an important range of values of ω, the exponential factor in (9*a*) and (9*b*) is of little importance and we obtain ordinary Fourier transforms.

Equation (8) now gives:

$$\overline{w^2}=2\,A^2\,\{I_1(\omega)-p\,I_2(\omega)\}. \quad (10)$$

Hence the mean amplitude of the signal coming out of the filtering circuit, for a given value of ω, is determined by the Fourier transform of the correlation function for that value of ω.

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[4]) The calculations necessary for the reduction of the right hand side of (6) are of similar nature as various calculations given by dr. TCHEN CHAN-MOU in: "Mean value and correlation problems connected with the motion of small particles" (thesis Delft, 1947; Meded. no. 51 Labor. v. Aero- en Hydrodynamica der T. H.; see pp. 88/89 and 99/100). — TCHEN has also given a definition of a function presenting always the same statistical character (l.c. p. 52).

4. In the second place we take $\eta > \theta$, which allows us to discard the term $R(\eta + \delta/\omega)$ in (7). We then find, again with $\cos \varepsilon \cong 1$:

$$\overline{w(t)\, w(t+\eta)} = 2\, A^2\, e^{-p\omega\eta} \{ I_1^* \cos(\omega\eta + \varepsilon) + I_2^* \sin(\omega\eta + \varepsilon) \} \quad . \quad (11)$$

where

$$I_1^*(\omega) = \tfrac{1}{2}\, \omega \int_{-\infty}^{+\infty} d\delta_1\, R(\delta_1)\, e^{p\omega\delta_1} \cos \omega\delta_1 \quad . \quad . \quad . \quad . \quad (12a)$$

$$I_2^*(\omega) = \tfrac{1}{2}\, \omega \int_{-\infty}^{+\infty} d\delta_1\, R(\delta_1)\, e^{p\omega\delta_1} \sin \omega\delta_1 . \quad . \quad . \quad . \quad . \quad (12b)$$

In those cases where $p\,\omega\,\theta$ is small compared with unity, I_1^* will be practically equal to I_1, while I_2^* will be of the order of p.

From (11) it appears that the signal $w(t)$ transmitted by the filter exhibits a correlation extending over an interval of time which is no longer directly dependent on θ (the maximum duration for which correlation is found in the original function $v(t)$), and which actually can be much greater than θ. The presence of the harmonic functions of η in the right hand member of (11) proves that the transmitted signal is nearly a harmonic oscillation of frequency ω. The exponential factor $e^{-p\omega\eta}$, however, indicates that the correlation does not extend over an indefinite period, but becomes gradually less when η is increased. This is exactly what we must expect, when the filter does not give a pure harmonic oscillation, but a band of frequencies of breadth proportional to $p\,\omega$.

We can interprete this result by saying that the filter *produces* an approximately harmonic oscillation out of the irregular function $v(t)$, in such a way that the amplitude of this oscillation depends on the Fourier transform of $R(\eta)$. This is in accordance with TAYLOR's result, but we have obtained it in a form adapted to the experimental method for spectral analysis. In particular it was not necessary to assume that $v(t)$ itself was composed of harmonic components; nor did we require any knowledge of phase relations for such components, if they were present — this in contrast with the Fourier integral, where the phase of the transformed function is of great importance.

The result obtained can also be considered as an illustration of the controversy, once famous in the theory of optics, whether the colours of the spectrum can be said to be originally present in white light, or whether they are produced by the spectroscope.

(To be continued.)

Zoology. — *The influence of an electric field on the eggs of Limnaea stagnalis L.* By CHR. P. RAVEN. (Zoological Laboratory, University of Utrecht.)

(Communicated at the meeting of October 30, 1948.)

Previous experiments have shown that a polar gradient field plays an important part in the development of the eggs of *Limnaea* (e.g. RAVEN 1947). The experiments of DE GROOT (1948) indicate that in this gradient field two groups of factors are involved, governing the evolution of the nuclei and the distribution of the cytoplasmic substances, respectively. Nothing is known, however, on the nature of these factors. Therefore, an attempt was made to get some insight into this problem of the physical nature of the gradient field.

It seemed likely that electric forces might play some part in the movements of the constituents of the cell. As a matter of fact, such forces have often been put forward for the explanation of the process of mitosis. It has been shown experimentally that various constituents of the cell have different electric charges (e.g. PENTIMALLI 1909, MC CLENDON 1910, HARDY 1913, MEIER 1921, ZEIDLER 1925, BOTTA 1932, VON LEHOTZKY 1936, CHURNEY and KLEIN 1937).

The presumption that electric forces might be responsible for some of the movements of cell constituents obtained further support from observations of RUITER and BUNGENBERG DE JONG (1947) on the behaviour of coacervate drops in an electric field. In these drops, streaming movements were observed very similar to the shift of cytoplasmic substances during the early development of *Limnaea*.

A direct measure of the electric potential at different points of the *Limnaea* egg did not seem possible, on account of its minute size. Therefore, an attempt was made to study the electric properties of the egg indirectly, by exposing it to a strong electric field. It was supposed that such an external field would interfere in some way or other with the presumed intracellular electric fields within the egg, thereby causing a deviation of some of the constituents of the cell, which might permit some conclusion on the existence and action of these intracellular fields.

First, a problem had to be solved. The eggs had to be submitted to the action of the electric field while lying in water. Though, by using distilled water, the electric current can be made very small, still some electrolysis, with formation of acids and bases at the electrodes, will take place. As I intended to use rather long exposure times, precautions had to be taken against the actions on the eggs of these products formed at the electrodes.

In literature, some dispositions have been described for the exposure of cells and tissues to electric fields while preventing the injurious action of ions (e.g. PENTIMALLI 1909, GUILLIERMOND and CHOUCROUN 1936). However, none of them appeared to be adapted to our present purpose. Therefore, a new apparatus was constructed preventing with certainty any influence on the eggs of the substances formed at the electrodes [1]).

Methods.

A rectangular cuvette (length 80 mm, width 27 mm, height 19 mm) is divided by 3 transverse sills into 4 compartments (fig. 1). The middle sill S_1 has a breadth of 18 mm, the other sills S_2 and S_3 are 3,5 mm in

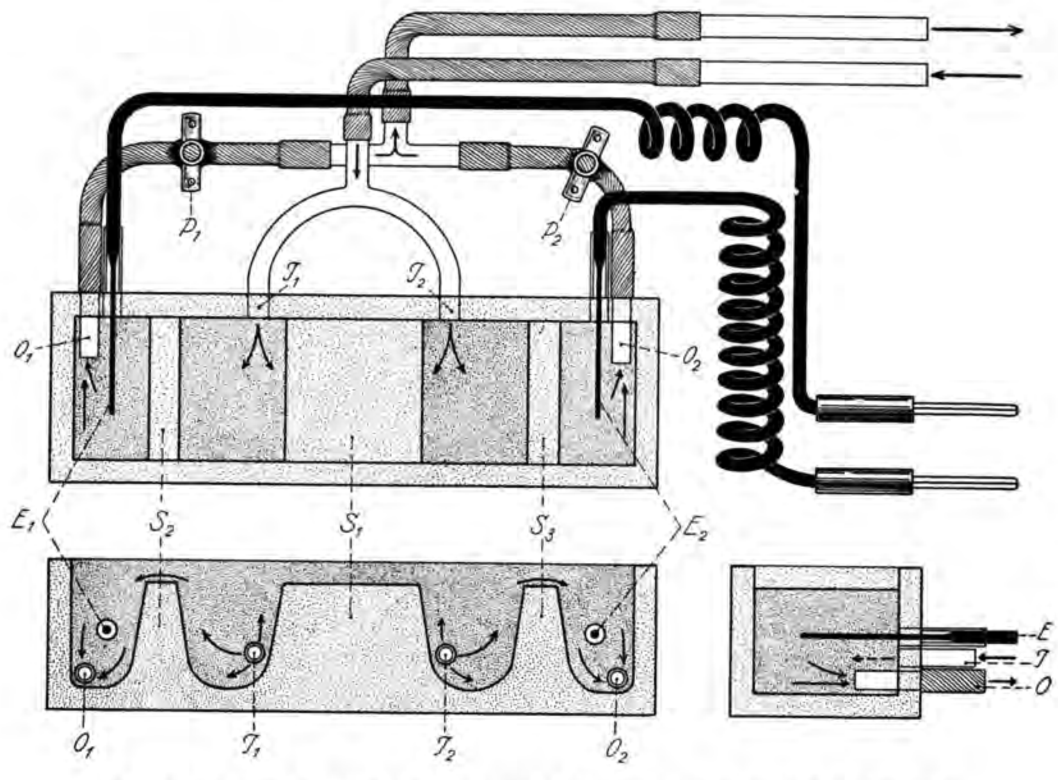

Fig. 1. Electrophoresis c^ll with removal of products formed at the electrodes. Description cf. text.

breadth. They reach from the bottom till 3 mm under the top. The cuvette with its sills is made of plastics ("Kallodent" 333, made by I.C.I.) and has been cast, as a piece, in a plaster of Paris mould. The platinum electrodes E_1 and E_2 are inserted into the outer compartments by bores into one of their side-walls. Nearer the bottom of these compartments,

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[1]) The author is indebted to prof. H. G. BUNGENBERG DE JONG for his valuable advice in the construction of the apparatus.

small glass tubes (O_1, O_2) pierce the walls, forming a pair of outlets. The inlets of the water-supply I_1 and I_2 are in the inner compartments bordering on the middle sill S_1.

The water-supply is fed by two bottles, which are connected by rubber tubing to a T-piece; these conduits can be closed by means of pressing-screws. In this way, the composition of the medium flowing through the cuvette can be changed e.g. in the vital staining experiments. The main is divided by another T-piece leading to the inlets I_1 and I_2.

The outlets O_1 and O_2 are connected by rubber tubing to a T-piece which communicates with the escape-pipe. This consists of a glass tube bent at right angles, which can be slid up and down through a bore of a cork stopper (fig. 2).

Fig. 2. Water discharge with adjustable level of outflow.

Before the beginning of an experiment, the cuvette is filled with distilled water from the water-supply. The eggs are put on top of the middle sill S_1 and brought into focus under a binocular microscope. Then the cuvette is covered with a large cover-slip, care being taken that no air-bubbles are enclosed; as the water against the plastic brim forms a convex surface, this can be accomplished without any difficulty. Then the water-supply and -discharge are set into action and, by shifting the levels of store-bottles and escape-pipe, adjusted in such a way that neither any leakage occurs nor air-bubbles are drawn in under the margin of the coverslip. By focussing on small particles in the water, brought in accidentally with the eggs, the direction of water current over S_1 is determined; by lightly turning the pressing-screws P_1 and P_2 (fig. 1), the pressures at both sides of S_1 are equalized so that no water current over S_1 remains. Now, a regular current of water passes from the inlets I to the outlets O over the sills S_2 and S_3. When the electric current is switched in, the products of electrolysis formed at the electrodes (except small gas bubbles accumulating beneath the cover slip) are carried off immediately.

The efficacy of this disposition has been tested by means of indicator solutions. When the electric current passes through a diluted neutralred solution, yellow clouds are formed at the kathode; they remain confined to the kathode compartment, however, and do not surpass the sill which borders the latter. With a cresolred solution (yellow), purple streaks arise at the kathode and are carried away through the outlet; in the same way, in bromothymolblue solution (yellow) blue clouds are formed at the kathode and immediately carried away. We may be sure, therefore, that no acids and bases formed at the electrodes reach the eggs.

Fig. 3 shows the circuit. The direct current of the town-net (220 V) passes through a potentiometer P, from which a voltage varying between 0 and 200 V can be tapped off. The micro-ampèremeter measures the current passing through the cuvette.

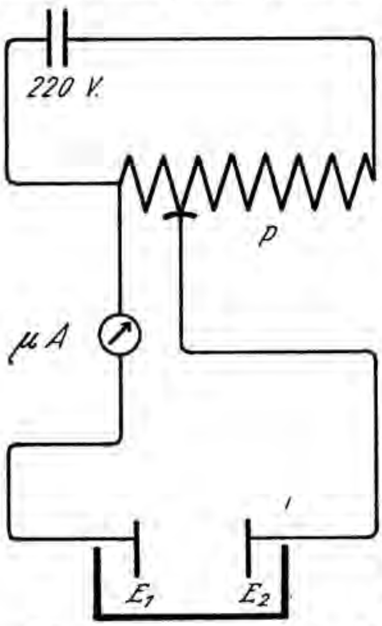

Fig. 3. Electric circuit for electrophoresis experiments.

The eggs are situated in the narrow space between the surface of sill S_1 and the cover-slip, 3 mm in height, 18 mm in breadth and about 20 mm in width. The lines of force of the electric field are passing from one side to the other through this space; the field will be practically homogeneous in this region. The resistance wil be highest at the level of the sills, whereas the deep troughs between them will offer only little resistance to the current. Therefore, for the computation of the potential gradient these troughs have been neglected, the distance of the electrodes being taken as the sum of the breadths of the sills, i.e. 25 mm. Hence, with a potential difference between the electrodes of 75 Volt, the field strength is 30 Volt/cm.

The electric current has only been small; when distilled water is used, as in most of our experiments, with a potential difference of 200 Volt the current strength does not exceed 50—60 μAmp. (current density = 85—100 μAmp./cm^2).

Results.

1. *Egg capsules.*

In a first series of experiments, whole egg capsules, freed as much as possible from the outer jelly, have been studied.

As soon as the circuit is switched in, the egg capsules begin to move in the direction of the anode. In most cases, they are sticking to the bottom by rests of the jelly so that only a twitching movement occurs; sometimes, however, they move on in the direction of the anode and drop into the trough which borders the sill at this side. This movement is rather quick; though no precise measurements have been made, the estimated electrophoretic mobility (μ/sec/Volt/cm) lies in the neighbourhood of 10.

These observations seem to prove that the egg capsules have a negative charge. However, another explanation has to be considered. Under the influence of the potential difference an electro-osmotic movement of the water will occur, and it might be supposed that the resulting water current accounts for the movement of the egg capsules (cf. ABRAMSON 1934). As a matter of fact, the presence of electro-osmotic currents is indicated by a displacement of small particles, directed to the kathode, right along the bottom of the container. It may be assumed that a similar current exists at the under side of the cover-slip. As the electrophoresis cell is open at both ends, the existence of a counter-current through the middle of the space containing the eggs does not seem very likely in these circumstances, however. Anyhow, such a counter-current would be far too weak to account for the observed anodal movement of the egg capsules. We may conclude, therefore, that these capsules have a negative charge.

The eggs within the capsules do not exhibit, as a rule, any visible movement when the circuit is switched in. However, after the lapse of some minutes they are found in most of the capsules in the half facing the anode. Hence, their electrophoretic movement is very slow; it has to be considered, however, that they are moving through the rather viscous egg capsule fluid. The eggs show no orientation with respect to the electric field, their animal poles pointing in various directions.

After about 10 minutes (at 50—80 V/cm), in the egg capsule fluid a distinct stratification appears. At the anodal side a dense yellow-coloured substance is accumulated, whereas the kathodal side is occupied by a very clear transparent fluid. Both substances are separated by a sharp boundary. The egg cell is lying in the yellow part. When the circuit is broken, this stratification disappears within some minutes, the components of the egg capsule fluid mingle again into a homogeneous liquid. When the egg capsules are left in the field (80 V/cm) for 20—30 minutes, however, at the boundary between the dense and the transparent part of the egg capsule fluid a kind of flocculation occurs giving rise to the formation of a turbid zone; after breaking the circuit, this zone remains visible for hours, but on the next day the egg capsule fluid has got a homogeneous appearance again. In some cases, egg capsules have been observed to burst and collapse in the electric field.

As to the development of the eggs exposed to the electric field, the following observations have been made. In some cases, eggs have been observed to cleave in a normal way and simultaneously with untreated controls, after being exposed to a field of 70 Volt/cm for 40 minutes. 3 Eggs exposed immediately after the extrusion of the 1st polar body to a field of 50 V/cm for 15 minutes, then another 5 minutes to 70 V/cm, afterwards cultured on agar bottom in the usual way, developed to normal hippo-stage embryos. 16 Eggs treated for 55 minutes (50 V/cm for the first 20 minutes, then increased to 80 V/cm), beginning immediately after laying and ending 20 minutes after the extrusion of the 1st polar body, yielded 5 normal hippo-stage embryos. In another case, of 9 eggs treated between first and second polar body formation with 80 Volt/cm for 45 minutes, 4 developed to normal trochophores. As a matter of fact, mortality in the treated cultures is somewhat higher than in the controls; since a partial flocculation of the egg capsule fluid had taken place in these capsules at the end of exposure, this is not astonishing. The fact, however, that a large percentage of the treated eggs showed a normal development proves that even so strong an electric field does not seriously interfere with the developmental processes in the uncleaved egg when this is surrounded by the egg capsule.

2. *Decapsulated eggs.*

Decapsulated eggs, studied in distilled water, as a rule show no reaction when the circuit is switched in: no displacement or rotation of the eggs takes place. In some cases, however, the eggs showed a slight movement towards the kathode. As these eggs are resting upon the bottom of the electrophoresis cell, it seems probable that this displacement is caused by the electro-osmotic flow of water along the bottom. No orientation of the eggs in the electric field takes place, the animal poles pointing in all directions, but mostly upwards.

Whereas the eggs, when surrounded by the egg capsules, can endure even an exposure to 80 Volt/cm for a longer time without immediate detrimental effects, decapsulated eggs in distilled water soon cytolyse under the influence of stronger electric fields. In one case, of 11 eggs belonging to one batch, in a field of 70 Volt/cm 9 cytolysed within 5 minutes; in another, 4 out of 11 eggs cytolysed within 10 minutes in 80 Volt/cm. The incidence of cytolysis is dependent on the strength of the field; on an average, after 45 minute exposure 15 % of the eggs have cytolysed in 30 Volt/cm, 27 % in 40 Volt/cm, 44 % in 50 Volt/cm. However, different batches show great differences in their resistance against cytolysis.

Cytolysis proceeds very rapidly; therefore, it is not easy to study its course. As far as could be made out, as a rule it begins with an outflow of protoplasm at the side of the egg facing the anode; at first, the vitelline membrane remains intact, the outflowing cytoplasm spreads under the membrane to the other side of the egg. After some seconds, the membrane bursts, mostly at the anodal side, and the whole contents of the egg are dispersed, forming a cloud in the water. Immediately afterwards, in most cases this "cloud" of protoplasm begins to move to the anode rather quickly, and eventually drops into the trough at this side. The whole process, as a rule, does not take longer than 20 seconds.

Not always cytolysis proceeds in this way. Occasionally, the vitelline membrane remains intact, but is distended considerably by its cytolysed contents; these "swollen" cytolysed eggs do not move to the anode as do those which have burst. In some cases, at the moment of cytolysis the eggs show an indistinct stratification, the anodal side being more orange, whereas the kathodal side is light yellow; furthermore, the pole of the egg facing the anode often shows a conical or somewhat nipple-shaped evagination.

According to VON LEHOTZKY (1936), when an electric current passes through the cells of the onion, their protoplasm shows an acid reaction at the anodal and an alkaline reaction at the kathodal side. I have investigated by means of vital staining experiments if the same holds true in the case of *Limnaea* eggs. Strongly diluted solutions of neutralred, nile blue hydrochloride, cresolred and bromothymolblue have been used; they were flowing through the electric cuvette in the ordinary way.

With neutralred, the intact eggs did not stain perceptibly during the experiment in the weak solutions used. However, at the moment of cytolysis they became deep red at the side facing the anode, whereas the rest of the egg remained yellow.

With nile blue hydrochloride, the same phenomena were observed: The eggs did not stain while they were intact. As soon as cytolysis occurred, however, the anodal side became deep blue, whereas the rest of the egg remained yellow. This was even the case in extremely diluted solutions which had no visible colour.

In cresolred, neither intact nor cytolysed eggs showed a distinct differential colour; in bromothymolblue intact eggs remained colourless, whereas the "swollen" cytolysed eggs showed a light greenish colour all over.

We may conclude from these observations that intact eggs are relatively impermeable to the dyes used. As soon as the egg cortex is destroyed at the anodal side with beginning cytolysis, the outflowing cytoplasm stains heavily with neutralred and nile blue hydrochloride.

These observations gave no definite answer as regards the occurrence of pH differences in the egg. Therefore, in some experiments another procedure was followed. Egg capsules were placed, immediately after laying, in somewhat stronger solutions of neutralred or nile blue hydrochloride. After some hours, the heavily-stained eggs were decapsulated and exposed to the electric field.

Eggs treated in this way with neutralred are deep red except the area surrounding the animal pole which is orange brown. In distilled water, the eggs give off a part of the absorbed dye substance, which forms reddish clouds in the water; at the same time, they become paler, more brownish red, with yellow-brown animal pole area; a red band of granules remains in the neighbourhood of the equator. No colour differences related to the direction of the electric field, even in a field of 60 V/cm, could be observed. The eggs cytolyse in the usual way; the vitelline membrane bursts, perhaps, less often at the anodal side than in unstained eggs. No regular colour differences in cytolysing or cytolysed eggs have been observed.

After staining in nile blue hydrochloride, similar observations have been made. The eggs are deep blue. Cytolysis occurs in the usual way, but also in this case the point at which the membrane bursts shows no clear relation to the direction of the field. No colour differences in intact or cytolysing eggs can be observed; only after cytolysis is complete, the anodal side of the eggs shows, perhaps, a somewhat heavier staining.

We must conclude, therefore, that our observations do not point to the occurrence of pH differences in the cytoplasm of eggs under the influence of the electric field.

In another experiment, VON LEHOTZKY (1936) treated the onion cells after exposure to the electric field with a mixture of eosin and methylen blue, and observed that the original kathodal side of the cells stained red, the anodal side blue. This observation proves, according to him, that both sides of the cell have got an opposite electric charge.

In my experiments, methylen blue gave the same results as neutralred and nile blue hydrochloride: intact eggs remained colourless, in cytolysing eggs the outflowing cytoplasm at the anodal side stained immediately. In eosin intact eggs remained colourless, cytolysed eggs stained uniformly red after some minutes.

In many instances it has been observed that the decapsulated eggs in a field of 40—50 V/cm pursued their development synchronously with the controls; both the extrusion of first and second polar body, and first and second cleavage have been observed to occur in the electric field. However, in some cases where the eggs had been exposed to the field for about 2 hours, cleavage took place with some delay and only in part of the eggs. No abnormalities in the situation of polar bodies or the direction of cleavages have been observed.

In order to study with greater accuracy the influence of the field on the structure of the eggs, in a number of cases the eggs have been fixed at the end of the experiment. In order to obtain a maximum effect, the treatment was continued until part of the eggs began to cytolyse. At this moment they were sucked up in a pipette, either immediately after or without previous breaking of the circuit, and transferred to Bouin's fluid; the whole procedure took no more than 2—3 seconds. In this way, is was hoped to preserve eventual distortions in the structure of the egg caused by the electric field. At the same time, it would permit to study more accurately the course of cytolysis. In order to fix a greater number of the eggs in the act of cytolysing, in some instances they were subjected to a field of 30 V/cm for at least half an hour, then the field was increased to 40—50 V/cm. Shortly afterwards, many of the eggs began to cytolyse simultaneously.

Table I summarizes these experiments.

TABLE I.

Exp.	Treatment	Stage at fixation	Number of eggs		
			Total	Intact	Cytolysing
RB I	80 V/cm, 20 min.	1st. matur. spindle	4	2	2
RK I	30 V/cm, 30 min.; then 40 V/cm	Formation of 1st polar body	13	10	3
RC I	40 V/cm, 30 min.	2d matur. spindle	5	5	—
RB II	40 V/cm, 60 min.	2d pol. b. just formed	15	15	—
RK II	30 V/cm, 35 min.; then 50 V/cm	Pronuclei	12	10	2
RJ I	30 V/cm, 80 min.; then 40 V/cm, 5 min.	1st cleavage spindle	3	3	—
RK III	30 V/cm, 35 min.; then 40 V/cm, 5 min.	1st cleavage spindle	10	9	1
RJ II	30 V/cm, 35 min.	Late 2 cell stage	7	5	2
			69	59	10

The study of the intact eggs reveals that they show an entirely normal structure. They have pursued their development synchronously with the controls, and resemble, in every respect, normal eggs of the same stage of development. Even the intact eggs of RB I, which had been exposed to a field of 80 V/cm, show nothing abnormal. The only exception to this rule form two eggs of RB II. In one of these eggs, which had been fixed just after the extrusion of the 2d polar body, the first polar body is still connected with the egg; in the other, the first polar body has not been extruded, but remained as a conical projection of the egg surface, whereas the second maturation spindle, which is in early telophase, has remained "submerged" in the interior of the egg. In 4 other eggs of the same batch, there is an indication that the second polar body has been displaced slightly to one side, so that the 2d maturation spindle is somewhat distorted; this is not very conspicuous, however. In no other case any effect of the electric field on the situation and structure of chromosomes, spindles, pronuclei or cytoplasmic substances has been detected.

The eggs fixed during the course of cytolysis offer many interesting aspects. Fortunately, various phases of the process, from its very first beginning to the final stage of complete cytolysis, are present in my material. This permits us to obtain a survey of the events which follow one another quite rapidly once the process has begun. These changes are most conspicuous in sectioned eggs, which after fixation in Bouin's fluid have been stained with azan; the following description refers to such eggs.

The first indications of beginning cytolysis have been found in 2 eggs of RK II. In both, at two diametrically opposite points of the egg surface, there is a small area where the cytoplasm exhibits distinct signs of disintegration, which are not quite identical on both sides of the egg, however. On one side, probably that of the kathode (K), there is some confluence of cytoplasmic vacuoles; the surface layer of the egg has remained intact, but the egg surface is somewhat irregular. On the opposite side (A), on the contrary, in a restricted area the protoplasmic structure has been entirely destroyed; a liquefied space has been formed, which is filled, in the sections, with a fine coagulum. The egg cortex is also involved in this process of liquefaction, but the vitelline membrane has remained intact and stands out clearly and sharply (fig. 4*a*). At the same time, a remarkable change in the colourability of the eggs has occurred. Whereas in the normal eggs the red colour of azocarmine is predominant, in those with beginning cytolysis the colour in the central part of the egg has changed to bluish. Moreover, the vacuoles in this part are enlarged and the cytoplasmic meshes between them are narrower: evidently, this central part of the egg is strongly hydrated.

Further stages of the process of cytolysis have been found in RK I. One of them corresponds in the main to those just described, but the irregular contour of the egg at the presumed kathodal side is somewhat more conspicuous. The other two cytolysing eggs of this batch represent a further phase. They are distinguished from normal eggs already at low magnifications by their blue colour, which has now spread over the whole egg except only a narrow zone on one side. On the opposite side (A), where in the previous stage the egg cortex had been destroyed by liquefaction, now the outflowing of the cytoplasm under the vitelline membrane has begun; a comparison with our previous observations on the process of cytolysis makes it clear that this is, probably, the anodal side. In one of the eggs, this outflowing cytoplasm is still restricted to a small part of the circumference of the egg (fig. 4*b*); in the other one, it has flown beneath the vitelline membrane till the opposite side of the egg (fig. 4*c*). It contains both β- and γ-granules of the proteid yolk. The hydrated zone, which occupied the centre of the egg in the previous stage, now has been displaced toward the kathodal half; by further swelling and confluence of the vacuoles a clear fluid space has been formed; the original cytoplasmic meshes have been reduced to an irregular network of fine threads. On the kathodal side a rather narrow superficial layer (K) has been formed which differs from the rest of the egg by its denser appearance and its red violet colour; it covers about half of the circumference of the egg. In the middle of this region, the contour of the egg is unsharp and irregular, the vitelline membrane is not clearly visible (fig. 4*b*). The

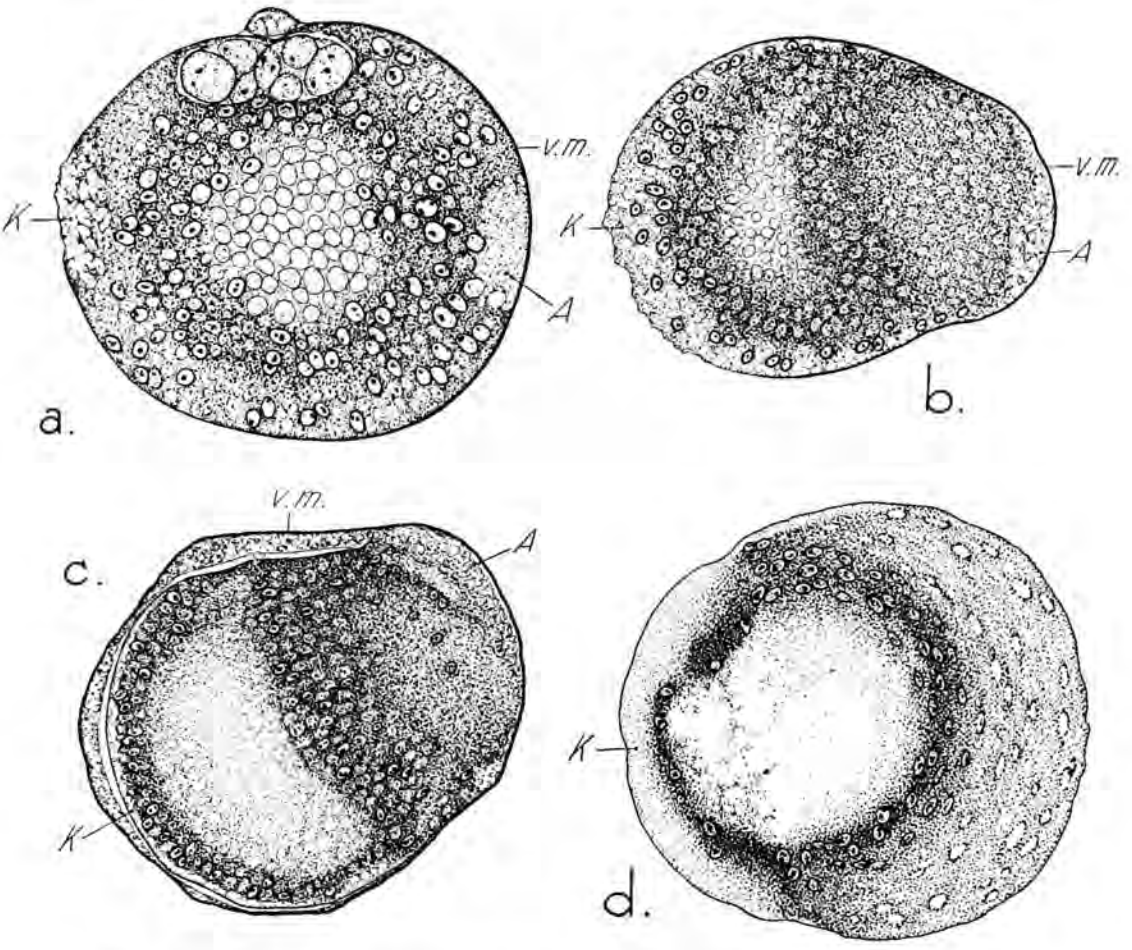

Fig. 4. Cytolysis of *Limnaea* eggs in electric field. *a*, Disintegration at anodal and kathodal pole. Hydration of egg centre. *b*. Beginning outflow of cytoplasm at anodal side. c. Outflowing cytoplasm spreading beneath vitelline membrane (v. m.). *d*. Stratification of cytolysed egg.

maturation spindle with its chromosomes is still clearly visible, but in one of the eggs, in which it is situated near the disintegrating region it is somewhat distorted.

Finally, an egg of RB I represents an advanced stage of cytolysis. The egg is now clearly stratified, and the opposite sides are distinguished by striking colour differences (fig. 4*d*). On one side, the dense superficial layer of the previous stage has condensed still more and now has a deep red colour in azan-stained sections (*K*). It is separated by a narrow blue layer from the hydrated area, which has enlarged, but is still situated nearer the kathodal side of the egg; the fine cytoplasmic network pervading it is still more reduced. The rest of the egg is blue and contains many vacuolar spaces; its structure is quite different from that of a normal egg. A somewhat denser layer surrounds the hydrated area. By the extension of the latter, the maturation spindle has been pushed aside and distorted.

Discussion.

1. The electrophoretic movements observed in the electric field show clearly that the egg capsules of *Limnaea* in distilled water have a negative electric charge. Presumably, the same holds true of the egg cells in the capsules, though their displacement is less conspicuous and might be caused by the electrophoresis of the substances of the egg capsule fluid. When the egg cells are decapsulated and transferred to distilled water, no electrophoretic movement or, at the utmost, a weak displacement towards the kathode takes place. Cytolysed eggs, however, immediately get a strong negative charge and move towards the anode. As a matter of fact, this refers only to those eggs bursting in the process of cytolysis; when the vitelline membrane remains intact, no such electrophoretic movement has been observed.

Neither in the egg capsule fluid nor in distilled water do the eggs orient with respect to the direction of the electric field; the animal pole may point in all directions.

2. Both in the capsules and in distilled water the eggs in an electric field pursue their development in a normal way synchronously with the controls. Eggs exposed in the capsules to a field of 80 V/cm for as long as 45 minutes may develop to normal trochophore or hippo-stage embryos. With decapsulated eggs the corresponding experiment could not be made, in consequence of the technical difficulties of rearing the eggs once they are out of their capsules. However, the study of eggs fixed after treatment has shown that such eggs exhibit no abnormalities in structure as long as cytolysis has not yet begun. No displacements of egg components, even in a field of 80 V/cm, have been found, with the exception of one batch, in which some eggs showed disturbances in the extrusion of polar bodies, or small lateral displacements of the 2nd polar body at the time of its extrusion. In another batch (RK I), in which the extrusion of the 1st polar body took place at the moment of fixation, no such displacements have been observed, however. Hence, we may say that in general the structure

of *Limnaea* eggs is not disturbed by the influence of an electric field, as long as they are intact. In this respect, our results correspond to those of GUILLIERMOND and CHOUCROUN (1936) who found that a field of the same order of magnitude did not influence plant cells as long as they were in the living state. To explain this fact, the authors mention two possibilities: either could the cell be surrounded by a protecting conductive layer, or its material could become polarized, so that the electric charges thereby produced at its surface keep in equilibrium the action of the external field. The latter explanation seems more probable.

3. Contrary to the eggs in the capsules, those in distilled water cytolyse in stronger electric fields. Great differences in susceptibility between different batches and individual eggs exist, however. Cytolysis does not occur at once, but after a certain latency period; this period is the shorter, on an average, the stronger the field is. During this latency period no visible changes of the egg can be observed. However, from the observation that, after a prolonged stay in a weak field, many eggs begin to cytolyse immediately after the intensity of the field has been moderately increased, we may conclude that during this latency period certain changes are going on in the eggs preparing the way for cytolysis. The vital staining experiments seem to show, however, that these changes do not consist in the appearance of pH differences in the eggs.

Once initiated, cytolysis takes a rapid and characteristic course. The first changes observed are a hydration in the centre of the egg, combined with a disintegration of cytoplasmic structure both at the anodal and kathodal pole of the egg. On the side of the anode, this leads to a total destruction of a circumscribed part of the egg cortex. At the same time, the colourability of the cytoplasm with azan (after Bouin fixation) changes from red to blue, beginning in the hydrated centre and spreading soon over most of the cytoplasm, with the exception of a narrow peripheral zone at the side of the kathode. At a certain moment, the cytoplasm begins to flow out through the gap at the anodal pole, spreading beneath the vitelline membrane, which may remain intact but in most cases soon bursts. The hydrated zone exhibits a further swelling and is displaced somewhat to the kathodal side. The differences in colourability of the egg substances intensify; the end of the process shows a total stratification of the egg. The whole process, from the beginning outflow of cytoplasm to a full stratification and disintegration of the egg, takes only a few seconds.

4. We may conclude from these observations that the egg substances resist eventual displacing forces due to the electric field as long as the egg cortex is intact. Once the integrity of the cortex is broken, these forces have free scope and lead to a rapid stratification of egg substances. The same has been observed by GUILLIERMOND and CHOUCROUN (1936) for plant cells; as soon as the cells are killed by a prolonged action of the field, electrophoresis begins and the elements of the cell are displaced according to the charge they have at that moment.

Summary.

1. The influence of an electric field on eggs of *Limnaea stagnalis* has been studied by means of an electric cell, in which the products formed at the electrodes were prevented to reach the eggs.

2. The egg capsules have a negative electric charge. The same holds true, perhaps, of the egg cells in the capsules. Decapsulated eggs in distilled water have no charge at all or only a weak positive charge. Immediately after cytolysis, however, they get a strong negative charge.

3. Neither in the egg capsule fluid nor in distilled water do the eggs orient with respect to the direction of the electric field.

4. Both in the capsules and in distilled water the eggs in an electric field may pursue their development in a normal way synchronously with the controls. Eggs treated in the capsules may develop to normal embryos. In general, no displacements of egg components occur, even in a field of 80 V/cm, as long as the eggs remain intact.

5. Decapsulated eggs in distilled water cytolyse in stronger electric fields. Cytolysis takes a rapid and characteristic course and leads in a short time to a total stratification of the egg contents and a disintegration of the egg.

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Physics. — "*Superquantization.*" II. By H. J. GROENEWOLD. (Koninklijk Nederlands Meteorologisch Instituut te De Bilt.) (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of September 25, 1948.)

2.33 *Dynamical wave operators.* We can write down an explicit expression for the preliminary dynamical wave operators if we have a complete system of orthonormal solutions (satisfying (6)) of

$$\mathbf{K}\{xt\}(xt|\psi=0 \tag{20}$$

in many-times theory or of

$$(\mathbf{K}\{x_1,\dots x_{n-1},x;t\}-\mathbf{K}\{x_1,\dots x_{n-1};t\})(xt|\psi_n=0 \tag{20'}$$

in single-time theory. In single-time theory the solutions $(xt\,|\,\psi_n\,|\,\mu)$ will in general depend on the variables $(x_1,\dots x_{n-1})$. In many cases of single-time theory and in all interesting cases of many-times theory the operator $\mathbf{K}$ and therefore also the solutions $(xt\,|\,\psi_n\,|\,\mu)$ (the suffix n has to be dropped in many-times theory) contain creation and annihilation operators of particles of another kind (in particular of carrier particles) with which those of the considered kind interact. All this makes them extremely complicated.

The dynamical wave operators are then given by

$$\left.\begin{aligned}(ys\,|\,\psi'\,|\,xt_0\}(x_1t_{10},\dots x_nt_{n_0}\,|\,\Psi' &= \sum_{(\mu)}(ys\,|\,\psi_n\,|\,\mu)\int(dx_n)(\mu\,|\,\psi_n'\,|\,xt_0)\,n^{1/2}\,\mathrm{S}_n\\ &\qquad \eta\{x_nt_{n_0}\}\,\varrho\{x_nt_{n_0}\}(x_1t_{10},\dots x_nt_{n_0}\,|\,\Psi',\\ \{xt_0\,|\,\psi'^{\dagger}\,|\,ys)(x_1t_{10},\dots x_{n-1}t_{n-10};ys\,|\,\Psi' &= \sum_{(\mu)}\mathrm{S}_n\,n^{1/2}(x_nt_{n_0}\,|\,\psi_n'\,|\,\mu)(\mu\,|\,\psi_n^{\dagger}\,|\,ys)\\ &\qquad (x_1t_{10},\dots x_{n-1}t_{n-10};ys\,|\,\Psi',\end{aligned}\right\} \tag{21}$$

if operating to the right and by the HERMITIAN adjoint relations ($21^{\dagger}$), if operating to the left and with $s=t_n$ everywhere. The ψ' appear as a generalization of the ψ. If we replace the t_{k0} by t_k, we get the undashed ψ again.

The dynamical wave operators satisfy the commutation relations

$$[(ys\,|\,\psi'\,|\,xt_0\},\{xt_0\,|\,\psi'^{\dagger}\,|\,y's')]^{\pm}=\sum_{(\mu)}(ys\,|\,\psi_n\,|\,\mu)(\mu\,|\,\psi_n^{\dagger}\,|\,y's')\,\mathrm{S}_n,\quad\text{etc.,} \tag{22}$$

similar to (11), only s and s' may now have different values. The sum satisfies the wave equations similar to (20) or (20′) with $\mathbf{K}\{ys\}$ (at the left) as well as with $\mathbf{K}\{y's'\}$ (at the right). For $s=s'$ it has the properties described in (6).

2.34 *Dynamical substitution operators.* If we form the dynamical substitution operators similar to (12), ($12^{\dagger}$), their meaning can still readily be demonstrated.

We shall say that with respect to the kth set of variables a wave function is up to date if the time coordinate has the value t_k and that it is at the beginning if the value of the time coordinate is t_{k0}. In the representation s_1 the Ψ are up to date in all sets (xt) and (ys). In s_3 the Ψ' are at the beginning in all sets (xt_0) and (ys_0). In s_2, with which we are dealing for the moment, the Ψ' are at the beginning in the sets (xt_0) and up to date in the sets (ys).

If we compare the dynamical substitution operators with the statical ones, particularly observing the time dependence, we see that:

S_d The dynamical substitution operators have the same properties as the statical substitution operators as summed up in S_s, but in addition a dynamical annihilation substitution operator brings the wave function up to date in the set (ys), which replaces the last set $(x_k t_{k0})$, and a dynamical creation substitution operator puts the wave function back to the beginning in the new set $(x_k t_{k0})$, which replaces the set (ys).

2.35 *Dynamical particle operators.* If in (16) or (17), (17†) the $\boldsymbol{\psi}$ are replaced by $\boldsymbol{\psi}'$, we find the dynamical homogeneous particle operators $\mathbf{R}'^{(m)}\{x_1t_1, \ldots\}$, which contain the entire motion of the system. Operators of the type $\mathbf{R}'\{x_1t_1, \ldots;\ y_1s_1, \ldots\}$ will be dynamical in the sets (xt), not in the sets (ys).

If in many-times theory $\mathbf{R}^{(1)}\{x_1t_1, \ldots\}$ is formed from the individual operators $\mathbf{R}\{x_kt_k\}$, we can define the dynamical individual operators $\mathbf{R}'\{xt\}$ by $\mathbf{R}'^{(1)}\{x_1t_1, \ldots; xt\} - \mathbf{R}'^{(1)}\{x_1t_1, \ldots\}$.

The dynamical $\mathbf{K}'\{xt\}$ in many-times theory or $\mathbf{K}'\{x_1, \ldots; t\}$ in single-time theory vanish according to the "superquantized wave equation" (19) or (19').

If after having formed the dynamical particle operators one forgets everything about wave operators, one is left with the elementary HEISENBERG representation e_2.

3. Special cases.

Before facing the general formalism developed so far with present theories, we first derive the explicit expression for the right hand member of (22) (in which we omit $\mathbf{S}_n$) in some special cases of typical kinds of particles. Successively we consider particles of spin ½, 0 and 1.

3.1 *Spin ½.* The 1-particle wave functions are spinors. In many-times theory the operators $\mathbf{K}\{xt\}$ in (1) read

$$\mathbf{K}\{xt\} = \left(\frac{\hbar}{i}\frac{\partial}{\partial t} - e\,\varphi(x)\right) + \vec{\alpha}\left(\frac{\hbar c}{i}\frac{\vec{\partial}}{\partial x} + e\,\vec{a}(x)\right) + \beta\, m\, c^2. \quad (23)$$

The 4-velocity operator is $(1, \vec{\alpha})$, so the density operator $\varrho = 1$. This is positive definite, therefore $\eta = 1$.

For free particles (zero external field $(\varphi, \vec{a})$ or zero charge e) a complete system of orthonormal solutions (satisfying (6)) of (20) is given by

$$(xt\,|\,\psi\,|\,\xi_\pm r) = b\,|\,\xi_\pm r)\, e^{-\frac{i}{\hbar c}(\pm(\xi^2+m^2c^4)^{1/2}ct - \vec{\xi}\vec{x})}/(h\,c)^{3/2} \quad . \quad . \quad . \quad (24)$$

with two spinors $b\,|\,\vec{\xi}_\pm\, r)$ $(r = 1, 2)$, for which

$$\left.\begin{aligned} &(r\,\vec{\xi}_\pm\,|\,b\cdot b\,|\,\vec{\xi}_\pm\, s) = \delta_{rs}, \\ &\sum_r b\,|\,\vec{\xi}_\pm\, r)\,(r\,\vec{\xi}_\pm\,|\,b = (\pm(\xi^2+m^2c^4)^{1/2} + \vec{\alpha}\,\vec{\xi} + \beta\, m\, c^2)/\pm 2\,(\xi^2+m^2c^4)^{1/2}. \end{aligned}\right\} \quad (25)$$

This gives for the right hand member of (22)

$$\left.\begin{aligned} &\sum_r \sum_\pm \int (d\vec{\xi})\,(ys\,|\,\psi\,|\,\vec{\xi}_\pm r)(r\,\vec{\xi}_\pm\,|\,\psi^\dagger\,|\,y'\,s') = \\ &\qquad = \left(-\frac{\hbar}{i}\frac{\partial}{\partial s} + \vec{\alpha}\,\frac{\hbar c}{i}\frac{\vec{\partial}}{\partial y} + \beta\, m\, c^2\right) D_a(y-y', s-s'). \end{aligned}\right\} \quad (26)$$

3.2 *D-fuctions.* The functions D_a above in 3.1 and D_s below in 3.3 and 3.4 are given by

$$\left.\begin{aligned} D_{\substack{a\\s}}(x,t) &= \int \frac{(d\vec{\xi})}{(h\,c)^3}\, \frac{e^{-\frac{i}{\hbar}(\xi^2+m^2c^4)^{1/2}t} \mp e^{\frac{i}{\hbar}(\xi^2+m^2c^4)^{1/2}t}}{2\,(\xi^2+m^2c^4)^{1/2}}\, e^{\frac{i}{\hbar c}\vec{\xi}\vec{x}} = \\ &= \frac{1}{4\pi\hbar c x}\frac{\partial}{\partial x} F_{\substack{a\\s}0}\left(\frac{mc}{\hbar}\,|\,c^2t^2 - x^2\,|^{1/2}\right) = \\ &= \frac{m}{4\pi\hbar^2} F_{\substack{a\\s}1}\left(\frac{mc}{\hbar}\,|\,c^2t^2-x^2\,|^{1/2}\right)\Big/ |\,c^2t^2-x^2\,|^{1/2}. \end{aligned}\right\} \quad (27)$$

Inside and outside the lightcone F_a and F_s stand for various kinds of BESSEL functions as indicated in fig. 1.

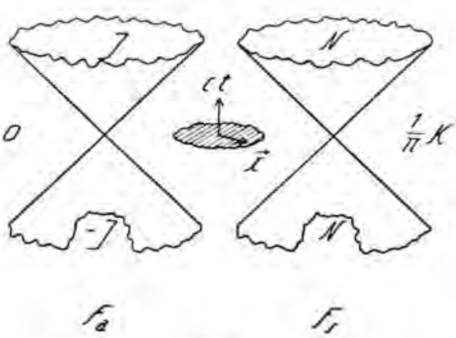

Fig. 1.
(Instead of $\pm J$ read $\pm iJ$.)

For $m = 0$ (27) degenerates into

$$D_a = -\frac{i}{h\,c}\,\delta(c^2t^2 - x^2) \quad , \quad D_s = -\frac{1}{\pi\, h\, c}\,\frac{1}{c^2t^2-x^2}. \quad . \quad . \quad (27^*)$$

3.3 *Spin* 0. The 1-particle wave functions are scalars or pseudoscalars. We consider the scalar case. The wave equations can be derived from

$$\left.\begin{aligned}\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\alpha_k}}+e\,\varphi_{\alpha_k}(x_k)\right)\Psi=m\,c^2{}_{\alpha_k}\Psi,\\ \left(\frac{\hbar c}{i}\frac{\partial}{\partial x_{k\alpha_k}}+e\,\varphi^{\alpha_k}(x_k)\right){}_{\alpha_k}\Psi=-\,m\,c^2\,\Psi.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (28)$$

This gives for the operator $\mathbf{K}\{xt\}$

$$\mathbf{K}\{xt\}=\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_\alpha}+e\,\varphi^\alpha(x)\right)\left(\frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha}+e\,\varphi_\alpha(x)\right)+m^2c^4. \quad . \quad (29)$$

In 4-dimensional time-space the metric is indefinite

$$(-g_{00}=g_{11}=g_{22}=g_{33}=1),$$

but since the wave functions are scalars this has no direct consequence for the metric in HILBERT space. Meanwhile the 4-velocity operator

$$\left(\frac{\hbar c}{i}\frac{\delta}{\delta x^\alpha}-\frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha}-2\,e\,\varphi_\alpha(x)\right)$$

gives the density operator

$$\left(\frac{\hbar}{i}\frac{\delta}{\delta t}-\frac{\hbar}{i}\frac{\partial}{\partial t}-2\,e\,\varphi(x)\right)$$

which happens to be indefinite as well (∂ is meant to operate to the right, δ to the left). So we have to determine the operator η.

The conditions c_1, c_2, c_3 of 2.15 can readily be satisfied, but c_4 is somewhat knotty. In an external field the components of the 4-velocity operator commute neither with each other, nor with $\mathbf{K}\{xt\}$. Therefore there are no simultaneous eigenstates. If we brush aside this difficulty, we might say that, if acting on a solution of the wave equation, the 4-velocity operator behaves time-like with positive and negative eigenvalues of the time component separated by a gap (of $2mc^2$). If η acts on a solution of the wave equation (and that is all we need), c_4 is satisfied as long as there is such a distinct and invariant separation between positive and negative solutions. This holds exactly in zero external field, but is liable to break down in "hard" fields (hard enough for pair creation and annihilation).

For free particles a complete system of orthonormal solutions (satisfying (6)) of (20) is given by

$$(xt\,|\,\psi\,|\,\vec{\xi}_\pm)=e^{-\frac{i}{\hbar c}\left(\pm(\xi^2+m^2c^4)^{1/2}c\,t-\vec{\xi}\,\vec{x}\right)}/(\xi^2+m^2c^4)^{1/4}(2\,h^3c^3)^{1/2}. \quad . \quad (30)$$

This gives for the right hand member of (22)

$$\sum_\pm\int(d\vec{\xi})(ys\,|\,\psi\,|\,\vec{\xi}_\pm)(\vec{\xi}_\pm\,|\,\psi^\dagger\,|\,y's')=D_s(y-y',s-s')\ . \quad . \quad . \quad (31)$$

3.4 *Spin* 1. 3.41 $m \neq 0$. The 1-particle wave functions are vectors or pseudovectors. We consider the vector case. The wave equations can be derived from

$$\left.\begin{aligned}\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\alpha_k}}+e\varphi_{\alpha_k}(x_k)\right)_{\dots\beta_k\dots}\Psi-\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\beta_k}}+e\varphi_{\beta_k}(x_k)\right)_{\dots\alpha_k\dots}\Psi\\ =mc^2{}_{\dots(\alpha_k\beta_k)\dots}\Psi,\\ \left(\frac{\hbar c}{i}\frac{\partial}{\partial x_{k\alpha_k}}+e\varphi^{\alpha_k}(x_k)\right)_{\dots(\alpha_k\beta_k)\dots}\Psi=-mc^2{}_{\dots\beta_k\dots}\Psi.\end{aligned}\right\} \quad (32)$$

This gives for the operator $\mathbf{K}\{xt\}$ (operating on ${}_{\dots\alpha\dots}\Psi$ or $\Psi^{\dagger\dots\beta\dots}$)

$$\mathbf{K}^\alpha_\beta\{xt\}=\left(\frac{\hbar c}{i}\frac{\partial}{\partial x^\gamma}+e\varphi^\gamma(x)\right)\left(\frac{\hbar c}{i}\frac{\partial}{\partial x^\delta}+e\varphi_\delta(x)\right)(\delta^\delta_\gamma\delta^\alpha_\beta-\delta^\alpha_\gamma\delta^\delta_\beta)+m^2c^4 \quad (33)$$

or the adjoint representation. (It should be observed that for $\gamma \neq \delta$ the factor operators cannot be commuted).

The 4-velocity operator (acting between $\Psi^{\dagger\dots\beta\dots}$ and ${}^{\dots\alpha\dots}\Psi$ is

$$\left(\frac{\hbar c}{i}\frac{\delta}{\delta x_\delta}-\frac{\hbar c}{i}\frac{\partial}{\partial x_\delta}-2e\varphi^\delta(x)\right)(g_{\gamma\delta}\,g_{\beta\alpha}-g_{\gamma\alpha}\,g_{\beta\delta}-g_{\gamma\beta}\,g_{\alpha\delta}) \quad . \quad (34)$$

The density operator ($\gamma = 0$) is indefinite like for zero spin. The indefinite g's even threaten to lead to further difficulties. In fact they do not for free particles (as we shall see), but they are liable to do so in "hard" fields.

For free particles the wave equations reduce to two sets

$$\left.\begin{aligned}\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\beta_k}}\frac{\hbar c}{i}\frac{\partial}{\partial x_{k\beta_k}}+m_k^2c^4\right)_{\dots\alpha_k\dots}\Psi=0,\\ \frac{\hbar c}{i}\frac{\partial}{\partial x_{k\alpha_k}}{}_{\dots\alpha_k\dots}\Psi=0.\end{aligned}\right\} \quad . \quad . \quad . \quad (35)$$

The second set of equations can be regarded as supplementary conditions to the first set. Owing to them the second and third term in the last factor of (34) can be dropped. That makes the density operator equal to

$$\left(\frac{\hbar}{i}\frac{\delta}{\delta t}-\frac{\hbar}{i}\frac{\partial}{\partial t}\right)g^{\beta\alpha}.$$

A complete system of orthonormal solutions (satisfying (6)) of (35) is given by

$$(xt|_\alpha\psi|\vec{\xi}_\pm r)={}_\alpha b|\vec{\xi}_\pm r)\,e^{-\frac{i}{\hbar c}(\pm(\xi^2+m^2c^4)^{1/2}ct-\vec{\xi}\vec{x})}/(\xi^2+m^2c^4)^{1/4}(2h^3c^3)^{1/2} \quad . \quad (36)$$

with 3 4-vectors ${}_\alpha b|\xi_\pm r)$ $(r = 1, 2, 3)$ satisfying the supplementary conditions

$$\xi^\alpha_\pm\,{}_\alpha b|\vec{\xi}_\pm r)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

(where $\xi^0_\pm = \pm(\xi^2 + m^2c^4)^{1/2}$) and for which

$$\left.\begin{aligned}(r\vec{\xi}_\pm|b^{\dagger\alpha}\cdot{}_\alpha b|\vec{\xi}_\pm s)=\delta_{rs},\\ \sum_r{}_\alpha b|\vec{\xi}_\pm r)(r\vec{\xi}_\pm|b^\dagger_\beta=g_{\alpha\beta}+\frac{\xi_\alpha\xi_\beta}{m^2c^4}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Because of (37) the b's are space-like, so that with regard to them $g_{\alpha\beta}$ behaves positive definite. That is why there are (for free particles) no further difficulties with the indefinite metric than for zero spin. The right hand member of (22) becomes

$$\left.\begin{aligned}&\sum_r \sum_{\pm} \int (d\vec{\xi})\,(ys\,|\,_\alpha\psi\,|\,\vec{\xi}_\pm r)\,(r\vec{\xi}_\pm\,|\,\psi_\beta\,|\,y's')=\\&\qquad=\left(g_{\alpha\beta}+\frac{\hbar c}{i}\frac{\partial}{\partial y^\alpha}\frac{\hbar c}{i}\frac{\partial}{\partial y^\beta}\Big/m^2c^4\right)D_s(y-y',s-s').\end{aligned}\right\} \quad (39)$$

3.42 $m=0$. Zero restmass forms a singular case for which the foregoing treatment breaks down after (37). ξ^α is now a zero-vector and not all b's satisfying the supplementary conditions are space-like. This gives difficulties with the indefinite $g_{\alpha\beta}$. Invariant relations similar to (38) cannot be formed. Instead of helping any longer, the supplementary conditions (37) only stand in the path. Now the supplementary conditions in (35) apply to the wave functions Ψ. They may be imposed on the complete system of individual reference functions ψ and on the wave operators ψ if such is possible and useful, but they need not if it is a nuisance or impossible. So we only impose them on the Ψ. If we like we can write them as

$$\frac{\partial}{\partial y_\beta}\,(ys\,|\,_\beta\boldsymbol{\psi}^\alpha\,|\,xt\}\,(x_1 t_1,\dots|_{\alpha_1\dots}\,\Psi=0 \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in s_1 representation or as

$$\frac{\partial}{\partial y_\beta}\,(ys\,|\,_\beta\boldsymbol{\psi}'^\alpha\,|\,xt_0\}\,(x_1 t_{10},\dots|_{\alpha_1\dots}\,\Psi'=0 \quad . \quad . \quad . \quad . \quad (41)$$

in s_2 representation. With the supplementary conditions imposed on the $\boldsymbol{\psi}$, already the operators to the left of Ψ in (40) and (41) would themselves have been identically zero.

As we suppose the zero mass particles to be uncharged, the density operator is $\left(\frac{\hbar}{i}\frac{\delta}{\delta t}-\frac{\hbar}{i}\frac{\partial}{\partial t}\right)g^{\alpha\beta}$. The positive and negative states of the first factor are the same as for zero spin. In order to distinguish between positive and negative states of the second factor we choose an arbitrary time-like 4-vector j^α. As positive vectors we take those orthogonal to j^α, as negative vectors those parallel to j^α. The operator η is then the product of the corresponding operator for zero spin (which because of $e=0$ exactly satisfies c_4) and the 4-tensor $(g_{\alpha\beta}+2j_\alpha j_\beta)$. The latter factor does not satisfy c_4 because it depends on the choice of j^α.

As we drop (37), we now get in (36) (with $m=0$) 4 4-vectors $_\alpha b\,|\,\vec{\xi}_\pm\, r)$ $(r=1,2,3,4)$ for which

$$\left.\begin{aligned}&(r\vec{\xi}_\pm\,|\,b^{\dagger\alpha}\cdot{}_\alpha b\,|\,\vec{\xi}_\pm s)=\delta_{rs},\\&\sum_r {}_\alpha b\,|\,\vec{\xi}_\pm r)\,(r\vec{\xi}_\pm\,|\,b^\dagger_\beta=g_{\alpha\beta}+2j_\alpha j_\beta.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

3 b's are space-like, 1 is time-like. The right hand member of (22) becomes

$$\sum_{r}\sum_{\pm}\int (d\vec{\xi})\,(ys|{}_{\alpha}\psi|\vec{\xi}_{\pm}r)\,(r\vec{\xi}_{\pm}|\psi^{\dagger}_{\beta}|y's') = (g_{\alpha\beta} + 2j_{\alpha}j_{\beta})\,D_s(y-y', s-s'). \quad (43)$$

4. Present theories.

Now we compare the results of our primitive form of "superquantization" with the starting point of the present theories.

4.1 *Notation.* In the present theories the functions on which the wave operators act are in general hardly taken into consideration. Consequently the variables on which they depend are usually not explicitly mentioned even in the wave operators. In our notation that would mean that in the expressions for the wave operators not the (xt), but only the (ys) are written down. This incomplete notation, which is quite sufficient for every-day use, is perhaps one of the main factors, which make that the meaning of the wave operators is not always clearly understood.

4.2 *Commutation relations.* If our wave operators shall be isomorphic to those of the present theories, they have to satisfy the same commutation relations.

4.21 *Field quantization.* In the present theories the field operators (more precisely the sum of creation and annihilation field operators, which is HERMITIAN) resulting from quantization of classical fields represent field observables. We have not considered this kind of observations. As soon as the desired isomorphy has been established, our field operators can be interpreted in the same way. The question how far field measurements can be interpreted by particle measurements belongs to problem Q_2.

For the moment we are only interested in the consequences with regard to the commutation relations. Because there can be no signals between two world points with a space-like connection, field observations in two such points cannot affect each other. Therefore the corresponding field operators in two such points must commute with each other.

Incidentally this also indicates that carrier particles obey B–E statistics. The problem whether that can be explained again belongs to Q_2.

4.22 *Superquantization.* More generally all wave operators of the present theories obey PAULI's postulate [7]) that in world points with a space-like connection they commute or anti-commute. In other words their (anti-)-commutators vanish outside the light cone.

4.23 *Discrepancies.* The commutation relations (31), (39) and (43) of the wave operators as we have preliminary defined them contain D_s, which according to fig. 1 does not vanish outside the light cone. Therefore our preliminary wave operators cannot be isomorphic with those of the present theories. We must try to modify the preliminary definition (10) of the wave operators in such a way, that they fit into the recognized commutation relations, without spoiling those properties, which are already all right. Now D_s in (31), (39) and (43) has to be replaced by D_a and moreover

$(g_{\alpha\beta} + j_\alpha j_\beta)$ in (43) by $g_{\alpha\beta}$. More generally the right hand member of (22) has to be replaced by

$$\sum_{(\mu)} \eta\{ys\}(ys|\psi_n|\mu)(\mu|\psi_n^\dagger|y's')\,\mathrm{S}_n = \sum_{(\mu)} (ys|\psi_n|\mu)(\mu|\psi_n^\dagger|y's')\eta\{y's'\}\,\mathrm{S}_n. \quad (44)$$

This modified expression satisfies the same wave equation as the original one. For $s = s'$ it has also the properties described in (6), which now only should be read in a different way.

4.3 *Modified wave operators.* We can make the modification in two different ways, which establish the required isomorphy with two different types of present theories: DIRAC's 1942 theory and the current hole theories.

4.31 DIRAC's 1942 *theory*. One way to obtain the modification (44) is to define the modified wave operators $(ys|\psi_D|xt\}$ and $\{xt|\psi_D^\dagger|ys)$ by

$$\left.\begin{aligned}(ys|\psi_D|xt\} &= (ys|\psi|xt\},\\ \{xt|\psi_D^\dagger|ys) &= \{xt|\psi^\dagger|ys)\eta\{ys\},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10\,D)$$

if they are operating to the right and

$$\left.\begin{aligned}(ys|\psi_D|xt\} &= \eta\{ys\}(ys|\psi|xt\},\\ \{xt|\psi_D^\dagger|ys) &= \{xt|\psi^\dagger|ys),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10\,D^\dagger)$$

if they are operating to the left. They are HERMITIAN adjoint to each other. In s_2 representation they satisfy the "superquantized wave equations" (19) or (19′). Their commutation relations yield the required form (44). If (12), (12†) and (17), (17†) are written with the D-modified wave operators, the factors η are swallowed up by the creation wave operators; the resulting expressions remain unaltered. (If we let also the factors ϱ be swallowed up, we get a description with canonical conjugates).

If the substitution operators are correspondingly modified, we see that:

S_D The D-modified substitution operators are almost identical with the preliminary ones. The D-modified creation substitution operators only give an extra factor -1 wherever a negative density function in (ys) is replaced by the same function in $(x_k t_k)$.

The D-modified wave operators $(ys|\psi_D|xt\}$ and $\{xt|\psi_D^\dagger|ys)$ are now isomorphic with the fields $U(y)$ and $U^*(y)$ (in PAULI's notation [6])) of DIRAC's 1942 theory. A further discussion of the latter theory belongs to problem Q_3.

4.32 *Current theories.* 4.321 *Positive and negative states.* Up to now we could completely avoid to speak about positive and negative energy states and positive and negative particles. They are, however, so narrowly interwoven with already the wave operators of the current theories, that we have to deal with them in some extent. A complete discussion would lead into problem Q_3.

4.3211 *Energy states.* For all spin values the energy-momentum operator of a particle is $\left(\frac{\hbar c}{i}\frac{\delta}{\delta x^\alpha}-\frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha}\right)/2$. The kinetic energy-momentum operator is therefore $\left(\frac{\hbar c}{i}\frac{\delta}{\delta x^\alpha}-\frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha}-2e\,\varphi_\alpha(x)\right)/2$. The kinetic energy operator $(\alpha=0)$ is indefinite in exactly the same way as the density operator for zero spin. We distinguish between positive and negative energy states in exactly the same way as between positive and negative density states for zero spin by means of an operator ζ, which is identical with η of the latter case and therefore also makes the same difficulties.

4.3212 *Charge conjugated states.* We consider two kinds of particles, which only differ in the sign of their charge e (for $e=0$ the two kinds are identical). They can also be considered as particles of one kind with a charge operator $\mathbf{e}$ with eigenvalues $\pm e$. To each particle state corresponds another state (the charge conjugated state) with the opposite charge and energy-momentum vector (the charge conjugated state of ψ is e.g. the complex conjugate $\bar{\psi}$ in case of integer spin; in case of spin $\frac{1}{2}$ it is $\bar{\varrho}_1\bar{\psi}$ in a representation in which $\vec{\alpha}$ and ϱ_3 have real matrixelements). Then also the kinetic energy-momentum is opposite. If one of the states is a positive energy state, the other is a negative energy state. The charge conjugate of $\psi\,|\,\mu)$ will be written as $\psi\,|\,\tilde{\mu})$.

The connection between creation of a particle in one state and annihilation of a particle in the charge conjugated state is one of the fundamental topics of problem Q_3.

4.322 *Hole theories.* We now turn to the current theories, but leave aside for a moment the photon case (spin 1, $m=0$), which we shall deal with later on.

4.3221 *H-revision.* In order to obtain isomorphy with the current wave operators we have to replace in (10) the creation and annihilation operators of negative energy states respectively by the annihilation and creation operators of the charge conjugated states. This H-revision is a part of the hole trick. The other part is the omission of the interaction between the two opposite particles during a process of pair creation or annihilation, but that entirely belongs to Q_3. The H-revised wave operators become

$$\left.\begin{aligned}(ys|\psi_H|xt\}&=\sum_{(\mu)}(1+\zeta\{ys\})/2\;\;(ys|\psi|\mu)(\mu|\mathbf{a}^\dagger|xt\}\\&\quad+\sum_{(\mu)}(1-\zeta\{ys\})/2\;\;(ys|\psi|\mu)\{xt|\mathbf{a}|\tilde{\mu}),\\\{xt|\psi_H^\dagger|ys)&=\sum_{(\mu)}\{xt|\mathbf{a}|\mu)(\mu^\dagger|ys)\;\;(1+\zeta\{ys\})/2\\&\quad+\sum_{(\mu)}(\tilde{\mu}|\mathbf{a}^\dagger|xt\}(\mu|\psi^\dagger|ys)\;\;(1-\zeta\{ys\})/2.\end{aligned}\right\}\quad.\quad(10H)$$

They are HERMITIAN adjoint to each other. In s_2 representation they satisfy the "hyperquantized wave equations" (19) or (19′). In the same representation they satisfy the commutation relations

$$\left.\begin{aligned}[(ys|\psi'_H|xt_0\},\{xt_0|\psi'^{\dagger}_H|y's')]^{\pm} &= \sum_{(\mu)} \begin{matrix}1\\ \zeta\{ys\}\end{matrix} (ys|\psi_n|\mu)(\mu|\psi_n^{\dagger}|y's')\,\mathrm{S}_n \\ &= \sum_{(\mu)} (ys|\psi_n|\mu)(\mu|\psi_n^{\dagger}|y's') \begin{matrix}1\\ \zeta\{y's'\}\end{matrix} \mathrm{S}_n, \text{ etc.}\end{aligned}\right\} . \quad (45)$$

The upper factor refers to F–D statistics, the lower factor to B–E statistics.

4.3222 *Spin and statistics.* The right hand part of (45) is equal to the required form (44) for half-odd spin ($\eta = 1$) only in case of F–D statistics, for integer spin ($\eta = \zeta$) only in case of B–E statistics. The theoretical derivation [7] of this connection between spin and statistics, which is due to PAULI and BELINFANTE, is based on

b_1 PAULI's postulate (cf. 4.22);

b_2 the H-revision (hole trick).

b_1 is not satisfied by our preliminary wave operators defined by (10). b_2 has neither been performed in our preliminary picture nor in its D-modified form, which is equivalent to DIRAC's 1942 theory. The necessity of b_1 and b_2 has not been unshakably established. b_1 and b_2 are sufficient but not necessary conditions for the connection between spin and statistics. The latter connection is the only point, which is directly backed by experimental evidence.

4.3223 *H-revised operators.* In order to make (17), (17†) isomorphic with the corresponding expressions in the current theories, they have to be written with the H-revised wave operators and the factors η have to be dropped. If the same is done with (12), (12†), the relation (16) remains unaltered. But the substitution operators and the particle operators are essentially changed.

4.32231 *Substitution operators.* The former are revised in such a way that:

S_H The H-revised creation/annihilation substitution operators differ from the preliminary ones in so far as, when taking out/inserting a negative energy function in (ys), the latter insert/take out the corresponding function in (xt) or (xt_0), but the former take out/insert the charge conjugated function in (xt) or (xt_0); further the H-revised creation operator gives an extra factor -1 if it takes out a negative density function in (ys).

4.32232 *Particle operators.* The particle operators are essentially changed by the H-revision. That makes that the whole theory is essentially changed. This change is not a matter of quantization, it is only a result of the hole trick (which moreover omits the interaction between the two opposite particles during a process of pair creation or annihilation). The discussion of the change belongs to problem Q_3.

4.3224 *Photons.* In the photon case there is a slight complication, because both factors of the density operator $\left(\frac{\hbar}{i}\frac{\delta}{\delta t}-\frac{\hbar}{i}\frac{\partial}{\partial t}\right) g_{\alpha\beta}$ in 3.42 are indefinite. One way to meet with these difficulties is to perform the interchange of creation and annihilation operatore of "charge" conjugated states for those negative energy functions, which are positive vectors, and for those positive energy functions, which are negative vectors. There are other ways. None of these ways has been followed in the current theories, for the current theories do not ask for an explicit realization of the wave operators. In fact they need not and that is also why the theory is ultimately independent of the choice of j^a in 3.42. As a starting point of quantum electrodynamics one can take the commutation relations and they do not depend on j^a. Then the usual course is to choose a time axis and to perform a transformation, which eliminates the scalar and longitudinal field operators, so that only the transverse ones are left. Because they are space-like, there are no further difficulties with the indefinite metrical tensor. (In our way of reasoning we might say that j^a is chosen in the direction of the time axis). Though this representation depends on the choice of the time axis, the processes which it describes do not. So the theory is ultimately (in its observable consequences) invariant and independent of j^a.

4.4 *Half-odd and integer spin.* One remark might be added about the characteristic difference between half-odd and integer spin in problem Q_1. The differential operator $\mathbf{K}\{xt\}$ is (in particular in the time coordinate) of 1st order for half-odd spin and of 2nd order for integer spin. The density operator is a zero order differential operator, which is even equal to 1, in the first case. It is a 1st order differential operator in the time coordinate and even indefinite in the second case. The different density operators can be regarded as characteristic for the difference between the two cases. The 2nd order wave equations for integer spin can be reduced to 1st order equations of more complicated wave functions as (28) and (32). This gives for various purposes a simpler description indeed. But it does not reduce the density operator to 1. It is this density operator, which can be hold responsible for many of the complications in case of integral spin.

5. Conclusion.

5.1 *Plus and minus troubles.* Not all difficulties mentioned in 1.3 could be shifted to problem Q_3 and none of them has been solved. Those of 1.311 appeared already in our preliminary picture in case of integral spin, those of 1.312 and 1.32 only in its H-revised form.

5.11 *Distinction of negative states* (d_1). We have postulated an operator η, which is $+1$ for positive and -1 for negative density states and an operator ζ, which is $+1$ for positive and -1 for negative energy states.

These operators have only been determined in case of free particles and their existence in presence of hard external fields has not even been warranted.

5.12 *Elimination of negative states* (d_2). In the ordinary quantum theory of particles we have multiplied the original indefinite density operator by η. In this way we have provisionally brought about an artificially definite metric in HILBERT space.

In the H-revision the negative energy states have been attended by means of the hole trick.

5.2 *"Superquantization"*. 5.21 *Wave operators.* By introducing creation and annihilation operators and wave operators we have brought the ordinary quantum theory of particles in "superquantized" form. The wave operators form an indispensable tool for describing interaction processes in which particles are created or annihilated (e.g. P_2 and P_3).

The wave operators have first been defined preliminary by (10).

5.22 *Present theories.* 5.221 DIRAC'*s* 1942 *theory.* In order to obtain DIRAC's 1942 theory we had to redefine the wave operators according to the D-modification:

D the operators $\eta\,\{ys\}$ are taken up in the creation wave operators.

Then the original indefinite density operator is left unaccompanied by η. That makes the artificially definite metric look indefinite.

Though DIRAC's 1942 theory is the youngest of the present theories, its starting point is the most primitive. The discussion of its consequences belongs to Q_3.

5.222 *Hole theories.* In order to obtain the older current theories we had to apply the H-revision:

H_1 creation/annihilation operators of negative energy states are replaced by annihilation/creation operators of the charge conjugated states;

H_2 the operators $\eta\{ys\}$ are replaced by 1.

Again the original indefinite density operator is left unaccompanied by η. Contrary to the D-modification, the H-revision makes a real change in the theory. This change is not a quantization process.

5.23 *Quantization processes.* Thus our picture of the quantization processes is:

PQ Starting from classical particle theory, ordinary particle quantization is the first and only step of quantization. The ordinary quantum theory of particles is equivalent to DIRAC's 1942 theory. If afterwards the hole trick is performed we get the hole theories.

FQ Starting from classical field theory, field quantization is the first and only step of quantization. The quantized field theory is equivalent to ordinary quantum theory of particles.

Therefore all present theories are equivalent with ordinary quantum theory of particles in which in some cases (hole theories) the hole trick has been performed. None of them contains a second step of quantization, which goes beyond the first step.

Summary.

It often appears that one is not always clearly conscious of the relations between ordinary quantization of classical particle theory, quantization of classical field theory and superquantization of ordinary quantum theory of particles. In this paper the situation has been looked at from a perhaps unorthodox point of view. All present quantum theories can without a further process of quantization be derived from ordinary quantum theory of particles. The latter is already equivalent with DIRAC's 1942 theory. The older current theories can be obtained by performing a trick, which is not a matter of quantization and which is characteristic for hole theories.

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Zoology. — *Note on some Crustacea Decapoda Natantia from Surinam.* By L. B. HOLTHUIS. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of October 30, 1948.)

Recently the Rijksmuseum van Natuurlijke Historie at Leiden received a small, but extremely interesting collection of Decapod Crustacea from Dutch Guiana, which was collected and donated by Dr. D. C. GEIJSKES of the Agricultural Experiment Station at Paramaribo. The present paper is an enumeration of the Natantia of this collection. Some specimens belonging to the genus *Macrobrachium* were too young to be identified with certainty, and several among them probably belong to undescribed species. These specimens are not included in the present list, just like a new species of *Palaemon,* which will be described in the near future by Dr. WALDO L. SCHMITT, head curator of the department of Zoology of the United States National Museum at Washington, D.C.

Dr. GEIJSKES kindly provided me with detailed descriptions of the various localities in which the specimens were collected, for which I wish to tender him my best thanks.

The shrimpfauna of Surinam is so poorly known, that it certainly is worth while to pay some attention to it. This is the more so as some of the species are of economic importance. JOHNSON & LINDNER (1934) for instance state that in Surinam "Fresh shrimp are taken from the river and dried shrimp are imported in large quantities from the United States". Five species of shrimps were obtained by Dr. GEIJSKES from shrimp traps, which were placed by the population in the mouth of the Surinam River in the outward flowing water during low tide. The specimens contained in these traps for the larger part belong to *Penaeus aztecus* Ives (the grooved shrimp), while also *Xiphopenaeus krøyeri* (Heller) (the sea bob) and the new species of *Palaemon* are represented in considerable numbers. *Xiphopenaeus krøyeri, Penaeus aztecus* and the two other members of the *Penaeus brasiliensis* group are of some economic importance in the southern United States. In the latter region, however, *Penaeus setiferus* (Linnaeus), a species not represented in the present collection, is the most important species of shrimp from a commercial point of view.

Penaeus aztecus Ives, 1891

Mouth of Surinam River, near Resolutie. In shrimp traps. Bottom mud, water muddy brown, salinity 15890 mg/l Cl, temperature of the water 27° C, December 22, 1942, 9 h. a. m.—110 specimens.

BURKENROAD (1939) divided the species, which up to that time was known as *Penaeus brasiliensis* Latr., into three distinct species: *Penaeus* *brasiliensis* Latr., *Penaeus aztecus* Ives, and *Penaeus duorarum* Burkenroad. These three species only can be separated on the shape of the thelycum and the petasma in adult specimens. Juvenile specimens are very difficult to distinguish. In the present collection the large specimens showed the characteristics said by BURKENROAD to be typical for *Penaeus aztecus* Ives. It is not certain, however, that all specimens belong to that species as a large part of the material is immature.

The species is known from the Atlantic coast of America from New Jersey to Uruguay.

Xiphopenaeus krøyeri (Heller, 1862)

Mouth of Surinam River, near Resolutie. In shrimp traps. Bottom mud, water muddy brown, salinity 15890 mg/l Cl, temperature of the water 27° C, December 22, 1942, 9 h. a.m.—16 specimens.

The specimens generally are larger than those of *Penaeus aztecus* from the same capture.

Xiphopenaeus krøyeri occurs along the Atlantic coast of America from South Carolina to Brazil. Like the previous species the present form is of some economic importance in the Southern U.S.A.

Acetes americanus Ortmann, 1893

Mouth of Surinam River, near Resolutie. In shrimp traps. Bottom mud, water muddy brown, salinity 15890 mg/l Cl., temperature of the water 27° C, December 22, 1942, 9 h. a.m.—5 specimens.

BURKENROAD (1934) considers the species of *Acetes* from the Atlantic coast of America, which have one tooth behind the tip of the rostrum, to belong to one species. According to this author *Acetes brasiliensis* of HANSEN is identical with ORTMANN's *Acetes americanus*, while *Acetes carolinae* Hansen at most may be considered a subspecies of *A. americanus*. BURKENROAD himself described two new subspecies: *A. americanus louisianensis*, and *A. americanus limonensis*. He thus considers *Acetes america-*

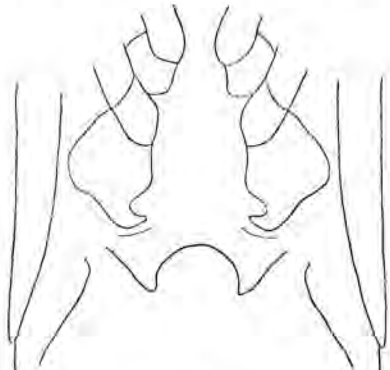

Fig. 1. *Acetes americanus* Ortmann, female specimen. Genital sternite in ventral view. × 20.

nus to consist of four subspecies: *A. americanus carolinae* Hansen from North Carolina, *A. americanus louisianensis* Burkenroad from Louisiana, *A. americanus limonensis* Burkenroad from Panama and *A. americanus americanus* Ortmann from Brazil. The subspecies *louisianensis* and *limonensis* are intermediate between the subspecies *carolinae* and *americanus*. The present specimens from Surinam, which all are females, nicely fit in the picture, by being intermediate between *A. americanus americanus* and *A. americanus limonensis*. The females of the four subspecies namely differ in the width of the concavity in the middle of the posterior margin of the genital sternite. In the Carolina specimens this concavity is deeper than broad, while it becomes gradually shallower in the more southern forms, being shallowest in *Acetes americanus americanus*. In the specimens from Surinam, the concavity is shallower than in those from Panama, but deeper than in those from Brazil. The specimens from Surinam have the same rights as those from Panama and as those from Louisiana to be considered to belong to a distinct subspecies. But considering the gradual transition of the character in material from the various regions from North of South, it seems to be not very useful to coin subspecific names for all the forms of the intermediate regions. We do better in my opinion to consider *Acetes americanus* Ortmann to be a large variable species with two extreme forms *A. americanus americanus* from the southern part of its range of distribution, which has the emargination of the genital sternite of the female very shallow and *A. americanus carolinae* Hansen from the northern part of the range of distribution with this emargination very deep.

Hippolysmata (Exhippolysmata) oplophoroides nov. spec.

Mouth of Surinam River, near Resolutie. In shrimp traps. Bottom mud, water muddy brown, salinity 15890 mg/l Cl., temperature of the water 27° C, December 22, 1942, 9 h. a.m.—2 specimens.

Description. The rostrum is long, slender, and directed somewhat upwards. It reaches with about half its length beyond the scaphocerite. In the basal part of the upper margin 9 or 10 teeth are placed close together, forming an elevated basal crest. One tooth is placed some distance behind the crest. Three or four teeth of the crest are placed behind, the others in front of the posterior limit of the orbit. The rest of the upper margin bears 5 or 6 widely separated teeth. The lower margin is provided with 10 to 13 teeth. The carapace is coarsely pitted and is provided with an antennal and a pterygostomian spine.

The abdomen, just like the carapace, is coarsely and shallowly pitted. The dorsal parts of all abdominal segments are evenly rounded, except that of the third segment, which bears a dorsal carina ending in a strong posteriorly directed spine, which overreaches the posterior margin of the third segment. This feature, together with that of the rostrum give the species a superficial resemblance with species of *Oplophorus*, for which reason the trivial name *oplophoroides* is given to the present form. The pleurae of the second to fourth segments are produced in a rather narrow, but rounded top. The pleurae of the fifth and sixth segments end in a distinct sharp tooth. The posterolateral angle of the sixth segment is very slenderly spiniform. The sixth segment is less than twice as long as the fifth, and less than half as long as the telson. The telson is elongate triangular and gradually tapers to a slender point. The dorsal surface bears two pairs of spinules: the anterior pair lies at one third of the length of the telson from the anterior margin, and the posterior pair lies closer to the anterior pair that to the tip of the telson. No spinules are seen on the lateral margin or near the top of the telson.

The eyes are well developed. No ocellus is present.

The basal segment of the antennular peduncle has the stylocerite rather broad and pointed, it reaches beyond the middle of the basal segment. No spines are present on any of the three peduncular joints. The second joint is somewhat longer than the third and together they are shorter than the first. The upper flagellum is simple and has about 20 to 25 basal joints thickened.

The scaphocerite is almost thrice as long as broad. The outer margin is slightly concave and ends in a strong final tooth, which fails to attain the end of the lamella. A slightly ventrally directed outer spine is present near the base of the scaphocerite.

The mandible lacks the incisor process and the palp; the molar process is well developed, like in BALSS's figure of the mandible in *H. hastatoides*. The maxillula has the inner lacinia very small, while the upper lacinia is large and high; the palp is indistinctly bilobed. The maxilla and the first maxillipede do not differ from those of *Hippolysmata hastatoides* as figured by BALSS (1925). The second maxillipede, however, possesses a well developed podobranch, which is partly concealed by the large epipod. The third maxillipede is slender, it reaches about to the end of the scaphocerite. The last joint is about 1.5 times as long as the penultimate and provided with 4 or 5 strong spines near the top. The antepenultimate joint is about twice as long as the penultimate. The exopod fails to reach the end of the antepenultimate joint. At the base of the third maxillipede 2 small but distinct gills are present. In my opinion these gills are the pleurobranch and the arthrobranch of the third maxillipede. BALSS (1925) states that *Hippolysmata hastatoides* is provided with a pleurobranch on the third and with an arthrobranch on the second maxillipede. The branchial formula in my specimens, which in most respects agrees with that of *Hippolysmata hastatoides*, is the following

	maxillipedes			pereiopods				
	I	II	III	I	II	III	IV	V
pleurobranchs	—	—	1	1	1	1	1	1
arthrobranchs	—	—	1	—	—	—	—	—
podobranchs	—	1	—	—	—	—	—	—
exopods	1	1	1	—	—	—	—	—
epipods	1	1	1	1	1	1	1	—

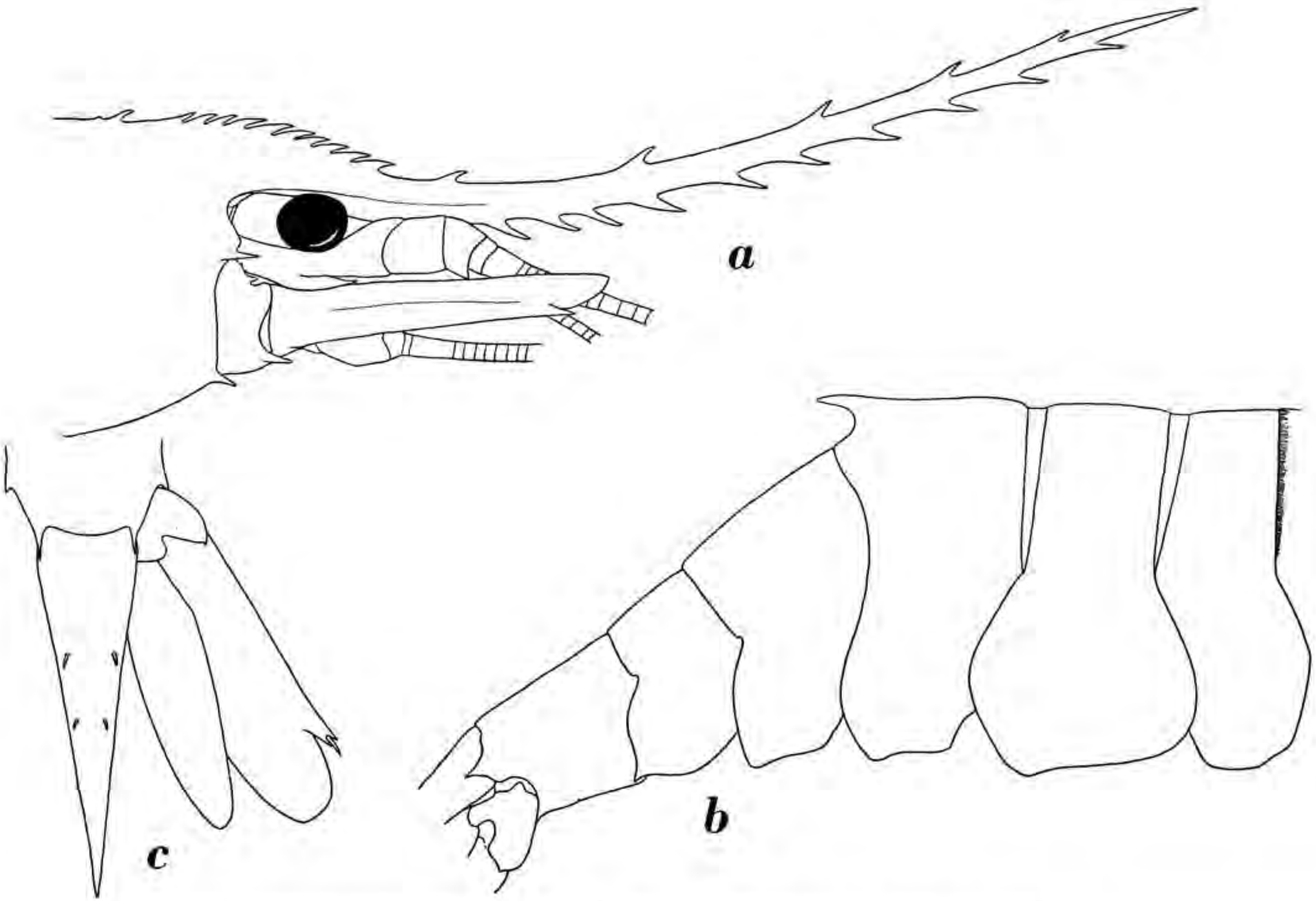

Fig. 2. *Hippolysmata* (*Exhippolysmata*) *oplophoroides* nov. spec. *a*. anterior part of the body in lateral view; *b*. abdomen in lateral view; c. telson and uropod in dorsal view. *a*—*c*, × 7.

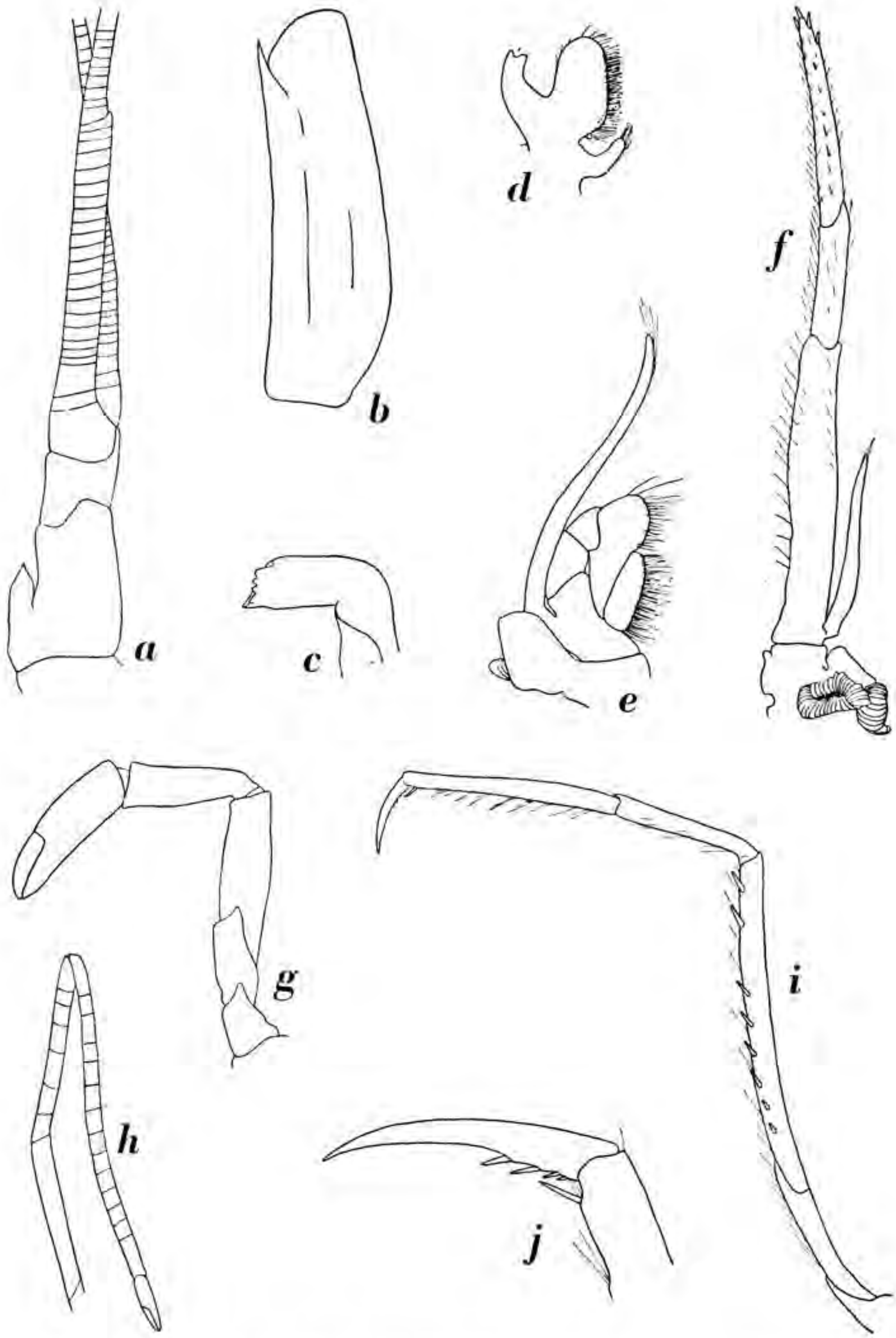

Fig. 3. *Hippolysmata* (*Exhippolysmata*) *oplophoroides* nov. spec. *a.* antennula; *b.* scaphocerite; *c.* mandible; *d.* maxillula; *e.* second maxillipede; *f.* third maxillipede; *g.* first pereiopod; *h.* second pereiopod; *i.* third pereiopod; *j.* dactylus of third pereiopod. *a, b, f—i,* × 8; *c—e,* × 12; *j,* × 28.

The epipods on the first four periopods are very small, but distinct. The first pereiopods are equal, they reach somewhat beyond the end of the antennal peduncle. The fingers are short and blunt. The fixed finger ends in a dark coloured sharp point, which fits between the two points in which the dactylus ends. The tips of the dactylus too have a darker colour than the rest of that finger. The outer surface of the closed fingers is convex, the inner surface is strongly concave. The fingers measure 5/8 of the length of the palm. The carpus is slightly shorter than the chela and is 3/4 of the length of the merus. The ischium reaches in its posterior part with an elongate process beyond the base of the merus. The second legs are slender, the left and right are slightly unequal in size, but equal in shape. They almost reach to the end of the third maxillipede. The chela is small and slender. The carpus is about 5 times as long as the chela. It is divided into 13 to 15 joints. The first and the last of these are longer than the other joints. The merus is 4/7 of the length of the carpus and consists of 7 or 8 joints. The ischium is slightly shorter than the merus and is not subdivided. The last three legs are slender and similar in shape. The third leg reaches with the dactylus beyond the scaphocerite. The dactylus is simple and slender, it possesses about four small spines in the proximal part of the posterior margin. The propodus is slender, being about thrice as long as the dactylus. The posterior margin of the propodus bears some hairs, while a pair of spines is present near the base of the dactylus. The carpus is 2/3 of the length of the propodus and 3/7 of the length of the merus. The merus is provided near the posterior margin with 6 to 8 strong movable spines. The ischium is less than half as long as the merus.

The pleopods in my two specimens, which both are females, are normal in shape. The endopod of the first pleopod ends in a narrowly elongated tip.

The uropods are elongate. The outer margin of the exopod ends in two distinct teeth, between which a slender movable spine is present.

Size. The two female specimens (both of which non-ovigerous) are 47 and 51 mm long.

Remarks. Up till now three species of the present subgenus have been described: *Hippolysmata ensirostris* Kemp (with the var. *punctata* Kemp), *Hippolysmata tugelae* (Stebbing) and *Hippolysmata hastatoides* (Balss). The former two species probably are identical (vid. HOLTHUIS, 1947, p. 74). *Hippolysmata ensirostris* is known from India and the Malay Archipelago, *H. tugelae* from the Cape region, while *H. hastatoides* is known from the west coast of Africa from Cameroon to Angola. *Hippolysmata oplophoroides* may be recognized at once from these species by possessing a distinct spine at the third abdominal segment. From *Hippolysmata hastatoides* the present species moreover differs in the dentition of the rostrum, by having the ultimate half of the rostrum provided with teeth on the dorsal margin and by possessing more ventral rostral teeth. Furthermore the telson in *H. oplophoroides* is more slender than in *H.*

hastatoides, while, if BALSS's (1925) figure 74 is correct, also the uropods in the new species are different by having the outer margin of the exopod ending in two teeth and a movable spine (BALSS figures only 1 tooth there).

Euryrhynchus wrzesniowskii Miers, 1877

Zanderij I, a locality about 40 km S. of Paramaribo along the railroad from Paramaribo into the interior. Savanna region, Troelinde creek, a forest creek with brownish acid water (pH 4.5), temperature 24°.5 C., Januari 14, 1943. — 7 specimens.

Sectie Q, a locality on the railroad from Paramaribo to the interior, about 70 km S. of that town. Savanna region, small forest creek, with shingle bottom and clear water (pH 5.4), temperature 23° C, February 6, 1942. — 7 specimens and June 7, 1944. — 1 specimen.

This is the third record of this curious Crustacean in literature. The species was described by MIERS (1877) from Cayenne, French Guiana. Then it was reported upon again in 1935 by GORDON, who described and figured specimens from the Upper Cuyuni and Mazaruni River basins, British Guiana. GORDON's specimens were found in a swamp, which "occupied a hollow, without outlet, on rather high ground in the forest, and therefore in the full shade of the canopy. The bottom of the swamp was covered by a very thick layer of dead leaves... The water was very yellow and highly deoxygenated. There was much H_2S among the leaves. The plankton was slight except for *Euryrhynchus*, which was present in considerable numbers in the water and among the upper layer of the leaves, and small fishes (*Rivulus urophthalmus* and *Pyrrhulina filamentosa*) swimming chiefly near the surface of the water." Though the records of the species from British and French Guiana made it highly probable that the species should occur also in Dutch Guiana, the find of it in the latter region is very interesting and the detailed description of the habitat by Dr. GEIJSKES is a welcome addition to GORDON's description of the curious environments in which the species lives.

Macrobrachium jelskii (Miers, 1877)

Nannikreek, Nickerie River basin, W. Surinam, Near Dam van Wouw. Nannikreek is a swamp creek with rather acid water (pH about 5), which is used for irrigating the rice fields, February 12, 1942. — 4 juveniles.

The specimens, though rather young, agree quite well with the descriptions given in literature and with adult material at my disposal. The species is known from Venezuela, Trinidad, Dutch and French Guiana.

Macrobrachium brasiliense (Heller, 1862)

Right Coppename River, line 3, From a well in the Emma Mountains. Altitude 150 m. This well is situated at the line between granite and diabase, the water is clear, with a temperature 23° C, pH. 6, October 30, 1943. — 1 specimen.

Brownsberg, Saramacca River basin, about 120 km S. of Paramaribo. In mountain creek, altitude 400 m, water clear, pH 6.2, temperature 22.5° C, September 16, 1938. — 8 specimens.

As several of the specimens are adult males, the identity of the material could be made fully certain. *Macrobrachium brasiliense* is known from British Guiana, Colombia (Orinoco River basin), and from the upper Amazon basin in W. Brazil, E. Ecuador and N.E. Peru.

Macrobrachium surinamicum nov. spec.

Mouth of Surinam River, Juli 5, 1944. — 6 specimens.

As this species will be described more extensively in a future publication, here only the most important characters are given:

The rostrum is about straight, with 13 to 16 dorsal and 4 to 6 ventral teeth, which are regularly divided over the rostral margins. Three or four rostral teeth are placed behind the orbit. The carapace in adult males is smooth, just like the abdomen.

The telson has the posterior margin distinct and provided with two pairs of spines, the inner of which overreaches the tip of the telson.

In the adult male the second pereiopods are equal in shape, but unequal in size. The joints are spinulate. The fingers bear one or two teeth in the proximal part of their cutting edges, while distally of these large teeth the edges bear about 12 distinct blunt teeth, which are smaller than the proximal teeth, and which diminish in size anteriorly. No velvety pubescence is present on the chela (except for a small row of pubescence close along the cutting edges), but a layer of short velvety hairs is present on the lower surface of the carpus and merus.

The specimens of this species seen by me are up to 55 mm long.

Type: The holotype of this species is a specimen from Plantation "Geyersvlijt" near Paramaribo, Surinam, July, 1911, W. C. VAN HEURN coll. The specimen is preserved in the Leiden Museum.

The species is readily distinguished from allied species by the shape of the rostrum and the second legs: especially by the large number of rostral teeth placed behind the orbit, by the large number of ventral rostral teeth, and by the dentition of the cutting edges of the second legs. I have seen material of *Macrobrachium surinamicum* from Colombia, British and Dutch Guiana.

Macrobrachium? olfersii (Wiegmann, 1836)

Wilhelmina Mountains, Zandkreek, Lijn I, Central Surinam. Creek with clear water and a sandy bottom with some rocks, pH 6.1, temperature of the water 23° C, August 18 and 19 and September 2, 1943. — 5 specimens.

Poeloegoedoe Falls, Marowijne River, E. Surinam. Broad river with rapids. The shrimps were collected between Podostemonaceae of the genus *Mourera*, pH 6.1, temperature of the water 30° C, August 31, 1939. — 2 incomplete specimens.

As all the specimens available are small and some of them are moreover incomplete by missing several of the legs, it is impossible to state with certainty to which species they belong. They show most resemblance to *Macrobrachium olfersii* (Wiegmann), a species occurring in fresh water of the continent of Central and South America from S. Mexico to S. Brazil, while the species moreover probably is introduced in Florida.

Palaemonetes carteri Gordon, 1935

Zanderij I, a locality about 40 km S. of Paramaribo along the railroad from that town into the interior. Savanna region, Troelinde creek, a forest creek with brownish acid water with pH 4.5 and temperature 24.5° C, January 14, 1943. — 1 specimen.

Sectie Q, a locality on the railroad from Paramaribo to the interior, about 70 km S. of that town. Savanna region, small forest creek, with shingle bottom and clear water (pH 5.4), temperature 23° C, June 7, 1947. — 9 specimens.

Kabelstation, a locality likewise situated on the railroad from Paramaribo to the interior, still farther inland and close near the Suriname River. Savanna region. Shrimps found in a pool of brownish fresh water in an excavation in the kaolin-like clayish soil, which excavation was made during the building of the railroad, September 23, 1938. — 6 specimens.

The specimens entirely agree with the description given by GORDON (1935) of specimens which originated from the Mazaruni and Upper Cuyuni River basins in British Guiana. It is curious that a large part of my specimens were found in company of *Euryrhynchus wrzesniowskii*, while all specimens of *Palaemonetes carteri* recorded by GORDON came from different localities as her *Euryrhynchus* specimens. The present record of the species is the second in literature.

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Zoology. — *On the thickness of the layer of blubber in Antarctic Blue and Fin Whales.* II. By E. J. SLIJPER (Institute of Veterinary Anatomy, State University, Utrecht). (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of September 25, 1948.)

5. *The increase in thickness of the layer of blubber during the Antarctic whaling season.*

To get an impression about the increase in thickness of the blubber during the season, in a great number of diagrams the absolute thickness of the blubber was plotted against the time at which the whales were caught. Separate diagrams were made for the different species and the different points of the body where the measurements were taken, and different symbols were used for the different size-groups and sexes, for pregnant and lactating whales (fig. 6). Some of these diagrams are given

Fig. 6. Explanation of symbols used in fig. 7—12.

	Blue Whales		*Fin Whales*
		Females	
∘	< 77′ (23.5 m)		< 65′ (19.8 m) sexually immature
•	77′ — 80′ (23.5—24.4 m)		< 65′ (19.8 m) sexually mature
●	81′ — 85′ (24.7—25.9 m)		65′ — 70′ (19.8—21.3 m)
⬤	> 85′ (25.9 m)		> 70′ (21.3 m)
✦		pregnant	
✱		lactating	
		Males	
▴	< 74′ (22.6 m)		< 63′ (19.2 m) sexually immature
▴	74′ — 80′ (22.6—24.4 m)		< 63′ (19.2 m) sexually mature
▲	81′ — 85′ (24.7—25.9 m)		63′ — 70′ (19.2—21.3 m)
▲	> 85′ (25.9 m)		> 70′ (21.3 m)

in fig. 7—9 (first season) and 10—12 (second season). Moreover for the second season the average thickness of the blubber was calculated per month, per species-, sex- and size-group, and so a great number of curves, represented in fig. 13—16 was obtained.

These diagrams and curves show that there is a fairly great individual variability, so that in future a great number of measurements must be made. They show also a large variability due to sex, length and condition of the whales which have already been discussed sub 3. Besides there is

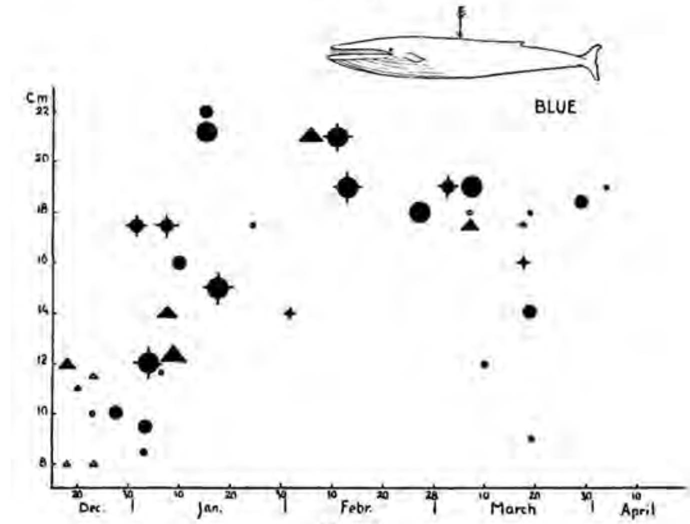

Fig. 7.

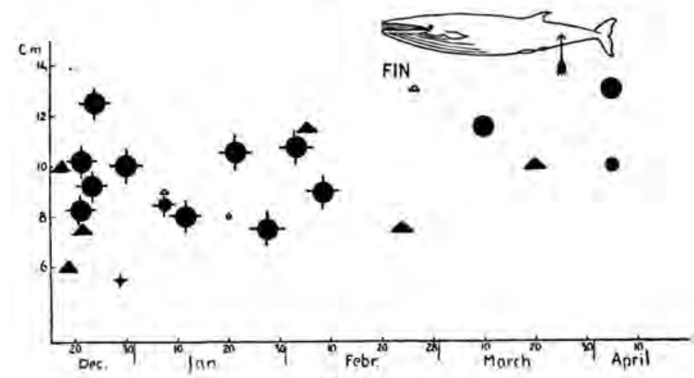

Fig. 8.

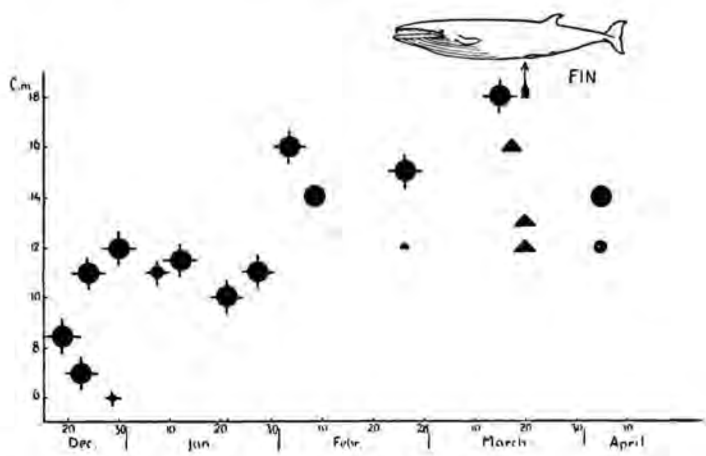

Fig. 9.

Fig. 7—9. Diagrams indicating the increase in absolute thickness of the *layer of blubber* in *Whales* measured at the point indicated by the arrow during the season *1946—1947.* For explanation of symbols see fig. 6.

a very large variability in the course of the curves according to the points of the body where the measurements were taken, although no distinct type of curve appears to refer to any special point of the body. The conclusion, however, may be drawn that the point where the blubber was measured by MACKINTOSH and WHEELER (1929; lateral side midway between the dorsal fin and the anus) is not a very favourable one, since the layer of blubber at this point is comparatively thin and consequently the increase is also comparatively small. The courses of all separate curves and diagrams of a certain season can be summarized in one single resultant, represented in fig. 17 (1946—1947) and 18 (1947—1948). Then it appears that the curves for the measurements taken at the dorso-median line just cranially of the dorsal fin and at the ventro-median line just cranially of the anus, show the greatest resemblance with the resultant-curve. Thus if in future it will not be possible to take measurements at several different points of the body, preference should be given to these two points, the more so as the layer of blubber is fairly thick here.

The curves of MACKINTOSH and WHEELER (1929) have been made according to the relative thickness of the blubber. This, however, may not be considered an objection to compare them with our data, since it is shown in fig. 17 that there is a perfect correlation between the curve for the relative and for the absolute thickness of the blubber. Although the curves of MACKINTOSH and WHEELER (1929) show a marked variability as to sex, size and condition of the animals, it is also possible to construct a resultant, which has been given in fig. 19. Grateful acknowledgement is made to Dr N. A. MACKINTOSH (London) for his letter of July 29th, 1947 in which he communicates the results of a large number of blubber-measurements of Fin Whales also taken at South Georgia. Dr MACKINTOSH writes that "it now seems that the average blubber-thickness at South Georgia generally tends to fall off a little from about October to December and increases from about January to April" (see fig. 19). From all those curves and diagrams the conclusion may be drawn that the increase in thickness of the layer of blubber takes place in about the same way in Blue and Fin Whales, as well as in whales of different sex and size.

It is very striking that the two curves of the blubber-thickness of the whales caught by "Willem Barendsz" (fig. 18, 20) show a perfect correlation with the curve for the increase in the output of oil per B.W.U. during the season. For S. Georgia there are no data available about the increase of the output during the season. Now it is a matter of fact that the output of oil in whales quite certainly is not determined by the thickness of the blubber alone. Apart from more or less thorough methods in processing the carcases on board of the factory-ships (p. 1124), the yield of oil depends on the absolute size of the whales (in calculating B.W.U. the size is not taken into account), the thickness of the blubber, the percentage of fat in the blubber (this, however, increases probably proportionally with

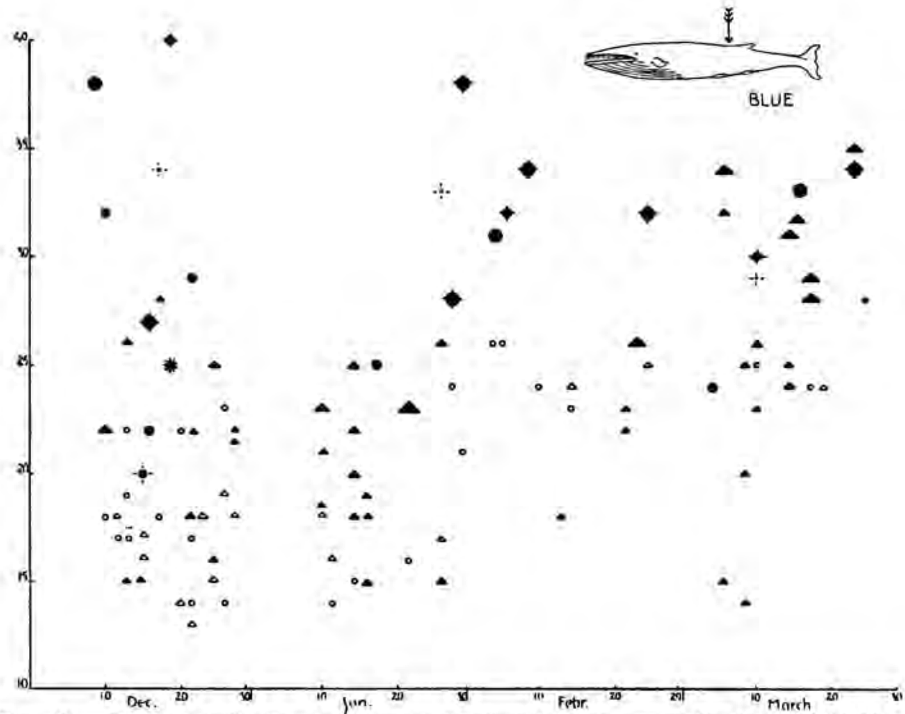

Fig. 10. Diagram indicating the increase in absolute thickness of the *layer of blubber* in *Blue Whales* measured at the dorso-median line just cranial of the dorsal fin during the season *1947—1948*. For explanation of symbols see fig. 6.

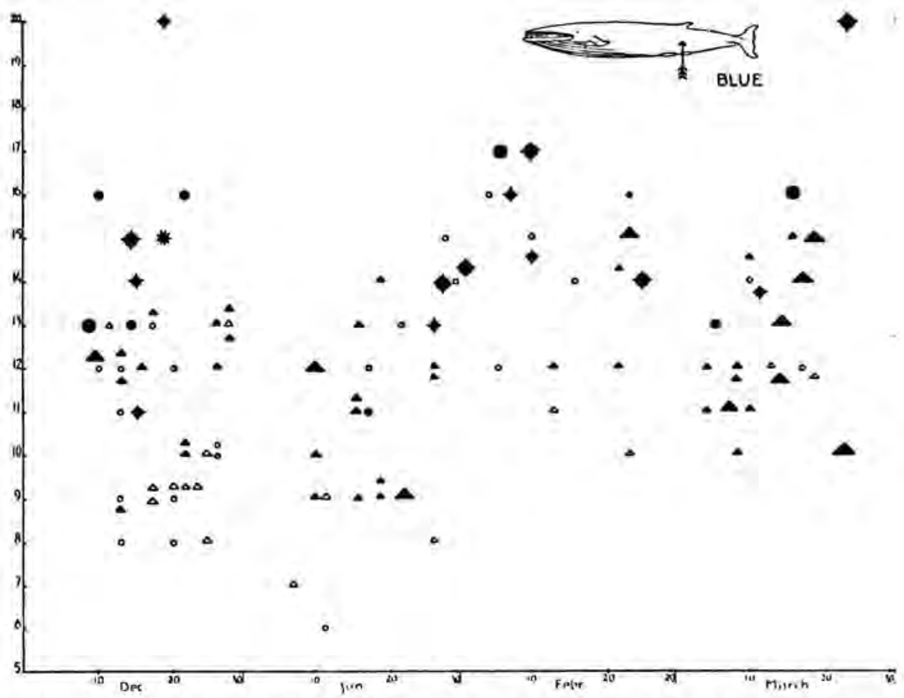

Fig. 11. Diagram indicating the increase in absolute thickness of the *layer of blubber* in *Blue Whales* measured at the lateral side midway between the dorsal fin and the anus during the season *1947—1948*. For explanation of symbols see fig. 6.

increasing blubber-thickness; see page 1037), as well as on the percentage of fat in the meat, the bones and the internal organs of the whales.

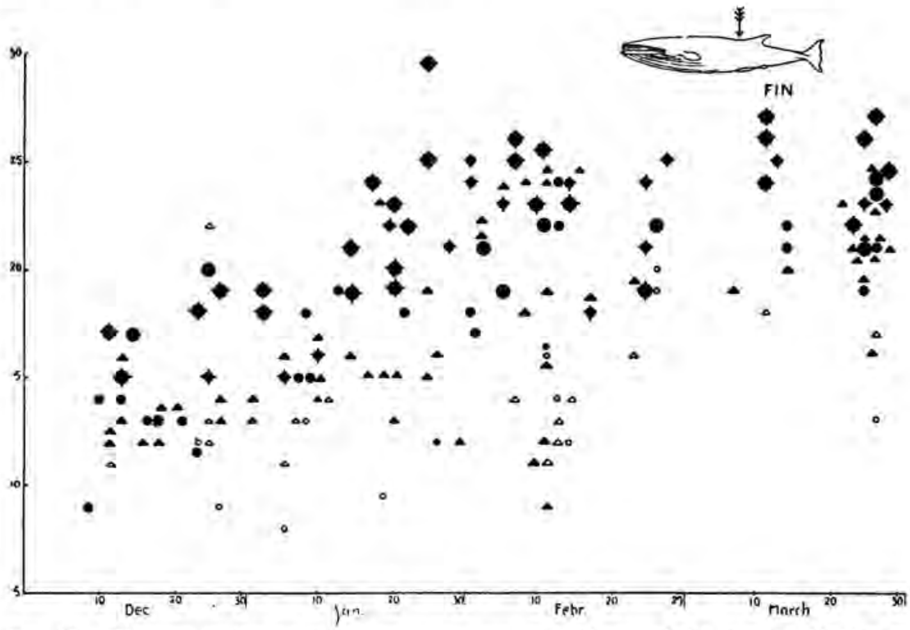

Fig. 12. Diagram indicating the increase in absolute thickness of the *layer of blubber* in *Fin Whales* measured at the dorso-median line just cranial of the dorsal fin during the season *1947—1948*. For explanation of symbols see fig. 6.

Unfortunately no reliable data are known about this subject and also the researches made on board "Willem Barendsz" by FELTMANN, SLIJPER and VERVOORT (1948) did not permit to draw any conclusions about the increase of the fat-percentage of meat and bone during the season. HEYERDAHL (1932; p. 94) supposed that the deposit of fat would take place at first in the internal organs, then in the meat and the internal layer of the blubber and finally in the outer layer of the blubber. For the present I cannot agree with this opinion since, according to a superficial impression I got during the first season of "Willem Barendsz", the fat of the internal organs did not increase to a marked degree before the last part of the season.

It would be very tempting to consider the correlation between the curves for the blubber-thickness and for the output of oil as an argument that the internal fat (meat, bone etc.) increases proportionally with the fat in the blubber and that consequently the thickness of the blubber might be considered a reliable indication for the oil-production of a certain animal. It is, however, highly probable that other factors also play an important part in modelling the shape of the output-curve. For example it might be supposed that an increase in thickness of the blubber or an increase in fat-percentage of the organs would cause a rise of the curve but that at the same time a

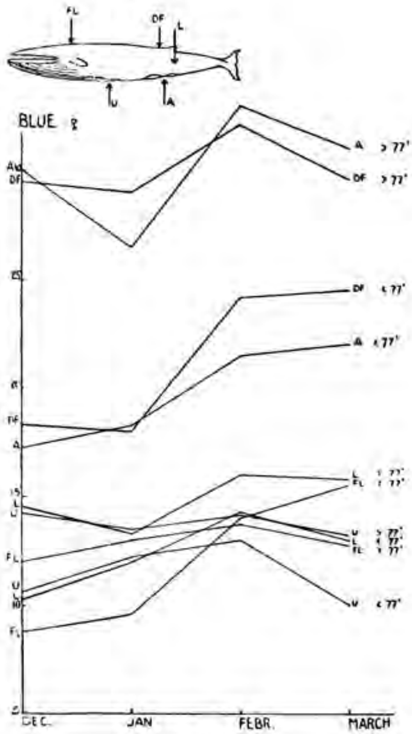

Fig. 13.

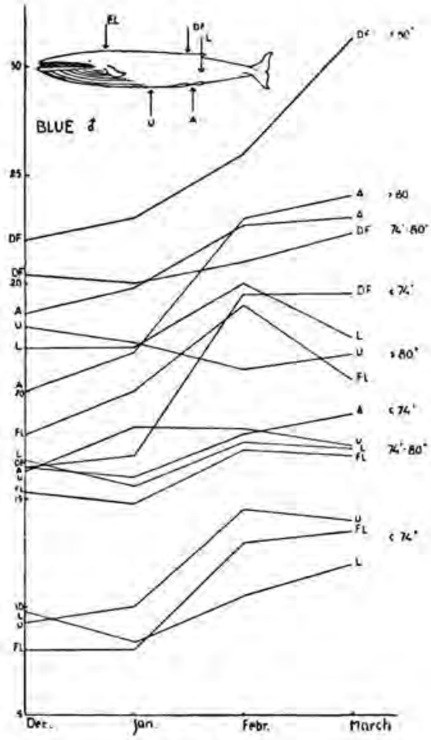

Fig. 14.

Fig. 13—14. Curves indicating the average absolute thickness of the *layer of blubber* in *Blue Whales* during the season *1947—1948*, calculated per month for the different groups of size and the different points of the animals.

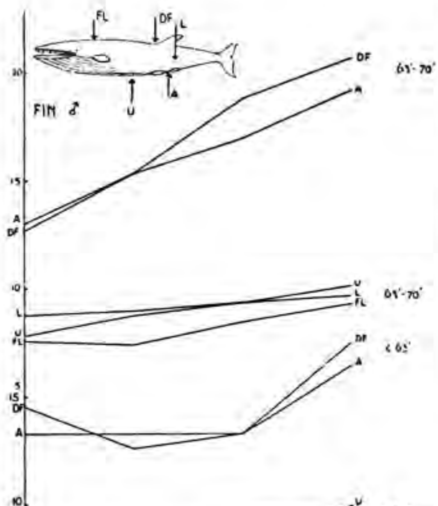

Fig. 15.

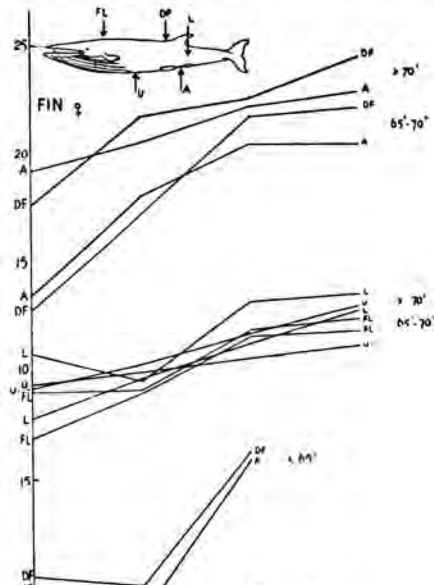

Fig. 16.

Fig. 15—16. Curves indicating the average absolute thickness of the *layer of blubber* in *Fin Whales* during the season *1947—1948* calculated per month for the different groups of size and the different points of the animals.

decrease of the average size of the animals or a decrease of the number of pregnant females would cause a decline. It will be shown that the data collected on the two expeditions of "Willem Barendsz" are able to give

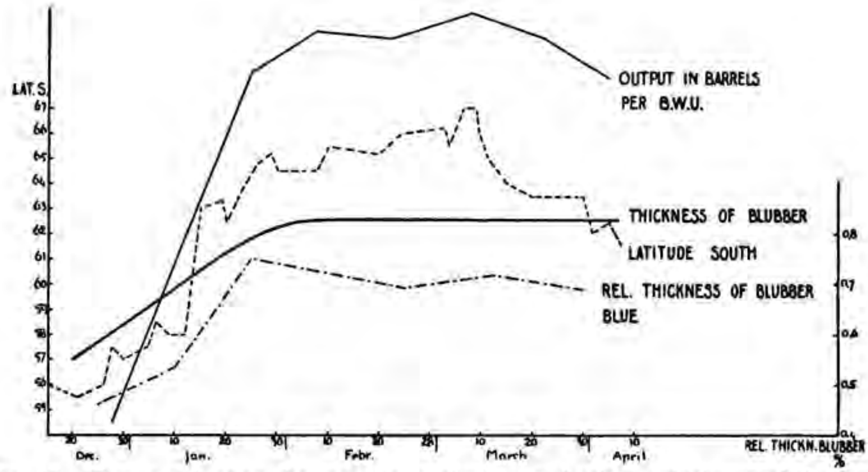

Fig. 17. Curves indicating the increase in absolute and relative thickness of the *layer of blubber*, the *output in barrels per B. W. U.* (see fig. 5) and the *latitude South* at which whaling took place. All curves bear on f.f. *"Willem Barendsz"* during the season *1946—1947*.

a better insight in this question. This requires, however, at first a discussion about the circumstances under which the expeditions were operating during the two seasons.

It appears from fig. 18 and 20 that the shape of the output-curves for

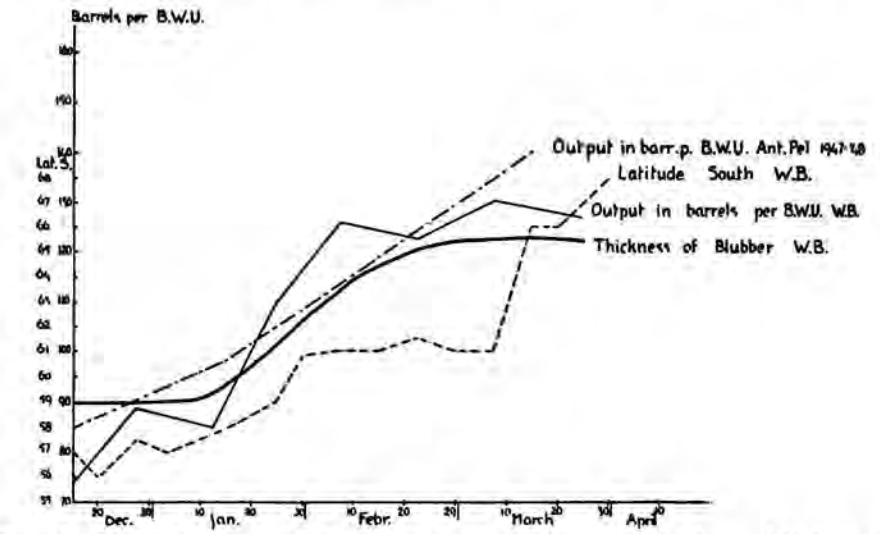

Fig. 18. Curves indicating the increase in absolute thickness of the *layer of blubber*, the *output in barrels per B. W. U.* (see fig. 5) and the *latitude South* at which whaling took place. The curves bear on f.f. *"Willem Barendsz"* during the season *1947—1948* and on the average figures for *Antarctic pelagic* whaling during the same season published by PAULSEN (1948).

both seasons differs markedly from the shape of the curves that are normally obtained in Antarctic pelagic whaling. The shape of the curve for the Norvegian expeditions shows that for the season 1946—1947 this difference probably does not depend on general seasonal influences, but that it must have been caused by the very abnormal conditions under which whaling took place on "Willem Barendsz" during its first season. In the first period of this season (17-12-'46—13-1-'47) the expedition operated between 7° and 2° E. and 56°—58° S., during the second period (14-1-'47—7-4-'47) between 7° and 26° W. and 63°—67° S. This means that during the first period whaling was carried on in an area far north of the boundary of the pack-ice, whereas most factory-ships start the season at the boundary of the pack-ice or even in the outer zone of the ice.

TABLE 6.

Some data about the composition of the catch of "Willem Barendsz" compared with average figures for Antarctic pelagic whaling and for South Georgia.

	"Willem Barendsz"			Antarctic pelagic 1934—1939 whole season [1])	South Georgia 1934—1939 whole season [1])
	Season 1946—1947		Season 1947—1948 whole season		
	First period 17-12—13-1	Whole season			
Average length of animals in Eng. feet					
Blue	77.8	78.3	75.8	78.1	73.4
Fin	67.5	68.6	67.8	67.6	64.8
Immature animals in % of total number of animals of the group					
Blue ♂ (<74')	35.0	22.3	40.0	26.2	
♀ (<77')	37.4	26.9	57.6	35.2	
Fin ♂ (<63')	10.1	9.6	10.2	16.4	
♀ (<65')	13.3	9 2	15.2	19.1	
Pregnant females in % of total number of adult females					
Blue	25.2	45.4	63.8	66.2	
Fin	57.2	40.0	53.0	79.6	

[1]) From International Whaling Statistics XVI and MACKINTOSH (1942; p. 277).

Now the data collected in table 6 and fig. 21—24 show that during the first period there was a higher percentage of Fin Whales in the catch than may be considered as normal for Antarctic pelagic whaling. The percentage more resembled that of S. Georgia, an island which is situated at about 54° 30′ S. The length of the Blue Whales was under the average of the other part of the season and also under the average for factory-ships. The average length of the Fin Whales was normal. The percentage of immature Blue Whales was higher than normal, that of the Fin Whales showed no marked differences with the average in other seasons. During the whole season 1946—1947 the percentage of pregnant females was lower than normal. The curve for the Fin Whales, however, has a normal

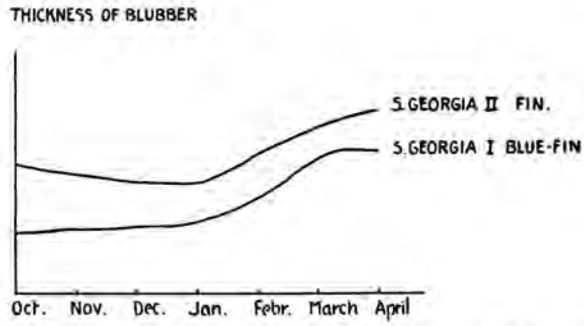

Fig. 19. Curves indicating the increase in relative thickness of the *layer of blubber* during the Antarctic whaling season at *South Georgia*. I = average curve for Blue and Fin Whales during the seasons 1925—1926 and 1926—1927 according to MACKINTOSH and WHEELER (1929); II = average curve for a larger number of Fin Whales according to a letter received from Dr MACKINTOSH.

shape, whereas that for the Blue Whales shows abnormally low percentages in the first period. Summarizing it may be said that during the first period the composition of the catch showed much more resemblance with the figures from S. Georgia (see also MACKINTOSH and WHEELER, 1929; table p. 456) than with those of Antarctic pelagic whaling. The

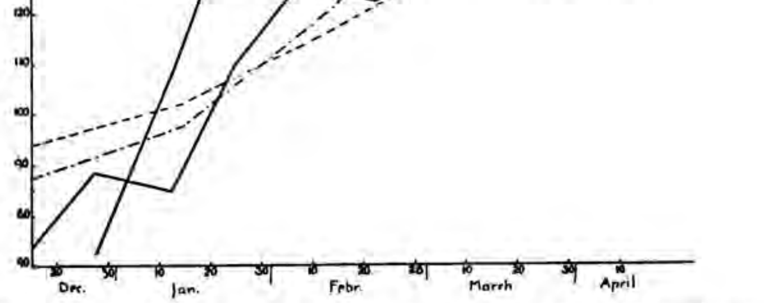

Fig. 20. Curves indicating the increase in *output of oil* per Blue Whale Unit during the Antarctic whaling season. The curves bear on the two seasons of f.f. "*Willem Barendsz*", on the average of *Antarctic pelagic whaling* from 1934—1939 (according to BERGERSEN LIE and RUUD, 1939) and on the results of the *Norvegian expeditions* during the season 1946—'47 (according to PAULSEN, 1947).

sudden rise of the output-curve in January shows a distinct correlation with the curve for latitude South (fig. 17) and the possibility may not be

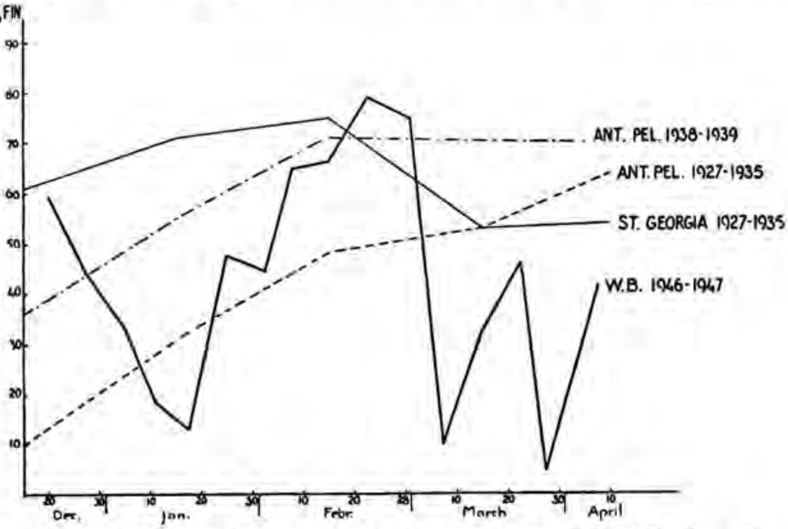

Fig. 21. Curves indicating the variations in the *percentage of Fin Whales* in the catch of *"Willem Barendsz"* during the season *1946—1947*. The other curves bear on average figures for *Antarctic pelagic whaling* and whaling at *S. Georgia* in different seasons (according to MACKINTOSH, 1942).

excluded that the curve for the thickness of blubber would also have shown a more sudden rise if more data had been available.

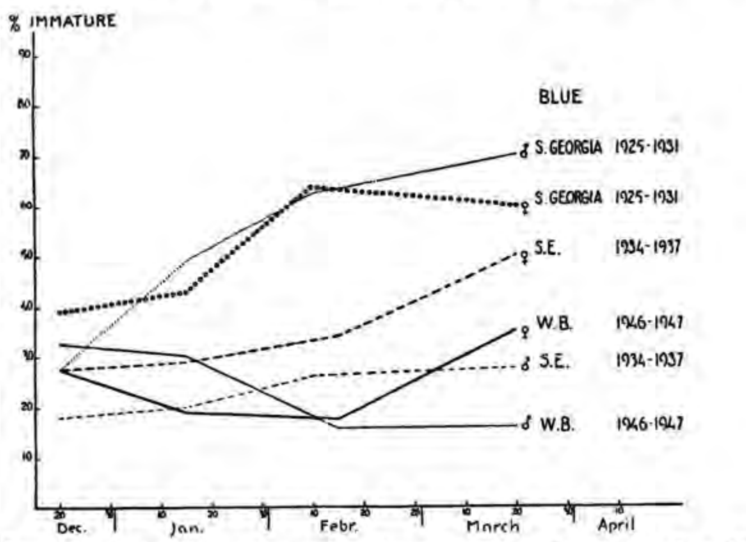

Fig. 22. Curves indicating the variations in the *percentage of immature Blue Whales* (♂ < 74'; ♀ < 77') in the catch of f.f. *"Willem Barendsz"* during the season *1946—1947*. The curves indicate average data calculated per month. They represent percentages of the total number of male or female Blue Whales that have been caught. The other curves bear on average data from *South Georgia* (season *1925—'31*) and on data from f.f. *"Southern Empress"* collected in area IV during the seasons 1934—'37 (MACKINTOSH, 1942).

In the season 1947—1948 the expedition operated from 8-12-'47 until 30-3-'48 in area II and III (from 17° E.—36° W.). Up to 10-3-'48 whaling was carried on between 55° and 61° S. and during the last part of the season between 66° and 68° S. In the Western parts of the Antarctic 1947—1948 seems not to have been a quite normal season, since the continuous north wind prevented the drifting of the pack-ice, at least at different localities. Up to the beginning of March "Willem Barendsz" could not get into the pack-ice. Now it is a well-known fact that Blue Whales and especially the adult animals live mostly in the outer zone of

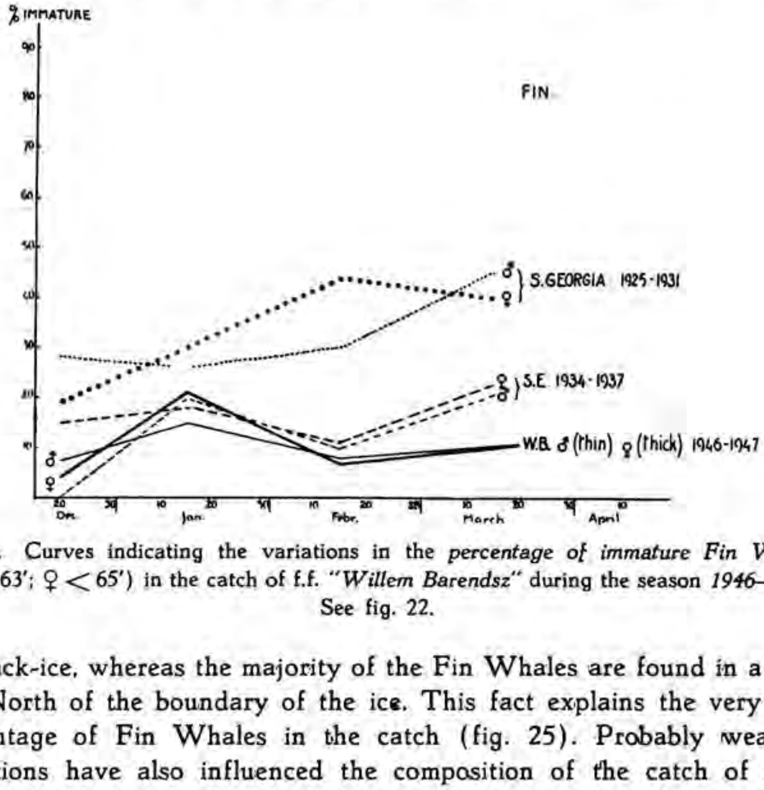

Fig. 23. Curves indicating the variations in the *percentage of immature Fin Whales* (♂ < 63'; ♀ < 65') in the catch of f.f. *"Willem Barendsz"* during the season *1946—1947*. See fig. 22.

the pack-ice, whereas the majority of the Fin Whales are found in a zone just North of the boundary of the ice. This fact explains the very high percentage of Fin Whales in the catch (fig. 25). Probably weather-conditions have also influenced the composition of the catch of other expeditions. For according to the survey of PAULSEN (1948) a very high percentage of Fin Whales has been found in the catch of factory-ships operating in area II, III and IV (77 %, see also fig. 5). The three expeditions that operated in area V, however, showed a Fin Whale percentage of 50 %. As far as is known at present they could already get into the ice at the beginning of the season, but this seems to have been also possible in area IV. The average output per B.W.U. of all factory-ships, as well as the quite normal shape of the output-curve given in fig. 18, show that weather-conditions have not influenced the results of Antarctic whaling on the whole in the same way as the results of "Willem Barendsz". Apparently the Fin Whales were comparatively fat, a supposition that is supported by the very high output at South Georgia.

(To be continued.)

Mathematics. — *Modern operational calculus based on the two-sided Laplace integral.* II. By BALTH. VAN DER POL and H. BREMMER.

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7. *The impulse or delta function.*

For a convenient use of the operational calculus it is advisable to introduce Dirac's impulse function $\delta(x)$. This improper 'function' may be defined by the three properties

$$(1) \quad \delta(x) = 0 \quad \text{if} \quad x \neq 0, \qquad (11a)$$

$$(2) \quad \delta(0) \to \infty, \qquad (11b)$$

$$(3) \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \qquad (11c)$$

Its many physical applications are well known; further, its importance for technical problems is evident from the use of pulse voltages, e.g., in radar technique.

The properties of the impulse function may be understood very simply by considering this 'function' as the derivative of the unit function:

$$\delta(x) = \frac{d}{dx} U(x). \qquad (12)$$

From this definition, (11*a*) and (11*b*) are evident at once; moreover the substitution of (12) in (11*c*) leads to an interpretation of (11*c*) as the Stieltjes integral:

$$\int_{-\infty}^{\infty} d\, U(x) = 1.$$

Further, an application to (12) of the differentiation rule suggests the operational relation:

$$\delta(t) \doteqdot p. \qquad -\infty < \operatorname{Re} p < \infty. \qquad (13)$$

Its validity may be justified in two different ways:

(*a*) with the aid of the corresponding Laplace integral. In fact, by considering again a Stieltjes integral, the Laplace transform of (13) appears to be:

$$p \int_{-\infty}^{\infty} e^{-pt}\, \delta(t)\, dt = p \int_{-\infty}^{\infty} e^{-pt}\, d\, U(t) = p;$$

(*b*) by working out the inversion integral (3). If we consider the latter as a Cesàro limit of the first order, with respect to the part above the real axis and the part below the real axis of the path of integration $\operatorname{Re} p = c$, we find for the original of $f(p) = p$:

$$\frac{1}{2\pi i} \lim_{\lambda \to \infty} \left\{ \int_{c-i\lambda}^{c} e^{pt} \left(1 - \frac{p}{c-i\lambda}\right) dp + \int_{c}^{c+i\lambda} e^{pt} \left(1 - \frac{p}{c+i\lambda}\right) dp \right\} =$$

$$= \lim_{\lambda \to \infty} \frac{e^{ct}}{\pi t^2} \frac{2\lambda \sin^2\left(\frac{\lambda t}{2}\right) + \sin(\lambda t) - \lambda t}{c^2 + \lambda^2} =$$

$$= \lim_{\lambda \to \infty} \frac{2 e^{ct}}{\pi \lambda t^2} \sin^2\left(\frac{\lambda t}{2}\right).$$

As a matter of fact this limit is identical with the δ-function because it behaves in exactly the same way and has the same properties as (11). Further the limit in question exists for $-\infty < c < \infty$ and is independent of c; this is in accordance with the strip of convergence $-\infty < \operatorname{Re} p < \infty$ of (13).

The usefulness of the impulse function is based particularly on the 'sifting integral':

$$\int_{-\infty}^{\infty} h(\tau)\, \delta(t-\tau)\, d\tau = h(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

From an operational point of view this property is evident at once by considering (14) as a composition product of the two functions $h(t) \doteqdot f(p)$ and $\delta(t) \doteqdot p$, because its image becomes

$$\frac{1}{p} \cdot f(p) \cdot p = f(p).$$

Also the introduction of derivatives of the impulse function itself often appears to be convenient; the corresponding operational relation is

$$\delta^{(n-1)}(t) \doteqdot p^n, \qquad -\infty < \operatorname{Re} p < \infty.$$

From the very large number of succesful operational applications of the impulse function we only mention the following:

(*a*) One may drop the restriction $h(0) = 0$ when differentiating one-sided originals $h(t)\, U(t)$. In fact, such a differentiation can be performed as follows:

$$\left.\begin{aligned} \frac{d}{dt}\{h(t)\, U(t)\} &= h'(t)\, U(t) + h(t)\, \delta(t) = \\ &= h'(t)\, U(t) + h(0)\, \delta(t). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

The identity

$$h(t)\,\delta(t) = h(0)\,\delta(t)$$

here used, is easily verified. The differentiation mentioned may be illustrated by the example of

$$\cos t\; U(t) \doteqdot \frac{p^2}{p^2+1}, \qquad 0 < \mathrm{Re}\, p < \infty.$$

We formally obtain

$$\frac{d}{dt}\{\cos t\; U(t)\} \doteqdot \frac{p^3}{p^2+1}, \qquad 0 < \mathrm{Re}\, p < \infty,$$

or

$$-\sin t\; U(t) + \delta(t) \doteqdot -\frac{p}{p^2+1} + p, \qquad 0 < \mathrm{Re}\, p < \infty.$$

The first term of the latter relation corresponds to a well-known image; the second term is fully in accordance with (13). Differentiations of this type are particularly important for the consideration of differential equations with boundary conditions (see next section).

(*b*) *Image functions* can be constructed for originals being linear combinations of impulse functions. Thus in the most general case we are led to the following operational relations containing an infinite series of impulse functions:

$$\sum_{n=0}^{\infty} a_n\, \delta(t-t_n) \doteqdot p \sum_{n=0}^{\infty} a_n\, e^{-t_n p}.$$

When t is understood as being the time, this relation links a summation over impulses, occurring at arbitrary times t_n with arbitrary amplitudes a_n, to a general Dirichlet series. The originals of similar relations may describe, e.g., the electric current corresponding to the so-called pulse modulation of electrotechnics.

(*c*) Impulse functions may also occur in the image functions. Since each image is an analytic function inside the strip of convergence, the operational relations corresponding to the images in question necessarily have in the p-plane a strip degenerated into a line. Examples are:

$$\frac{1}{2\pi} \doteqdot p\,\delta(ip), \qquad \mathrm{Re}\, p = 0,$$

$$\frac{1}{2\pi}\sin t \doteqdot \delta(p^2+1), \qquad \mathrm{Re}\, p = 0.$$

8. *Linear differential equations (constant coefficients) with boundary conditions.*

In order to illustrate the operational treatment of such equations, we consider a very simple special case. Let it be required to solve the equation of the second order with constant coefficients

$$a_0 \frac{d^2 h}{dx^2} + a_1 \frac{dh}{dx} + a_2 h = \varphi(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

while the initial values $h(0)$ and $h'(0)$ of $h(x)$ and $h'(x)$ at $x = 0$ are given. With a view to the boundary conditions we derive, with the aid of the procedure indicated in (15), a new equation for the one-sided function

$$h^*(x) = h(x)\ U(x).$$

We find consecutively:

$$\frac{dh^*}{dx} = \frac{dh}{dx} U(x) + h(0)\ \delta(x),$$

$$\frac{d^2 h^*}{dx^2} = \frac{d^2 h}{dx^2} U(x) + h'(0)\ \delta(x) + h(0)\ \delta'(x),$$

so that the equation for h^* becomes

$$\left. \begin{array}{l} a_0 \dfrac{d^2 h^*}{dx^2} + a_1 \dfrac{dh^*}{dx} + a_2 h^* = \varphi(x)\ U(x) + \\ + \{a_0 h'(0) + a_1\ h(0)\}\ \delta(x) + a_0\ h(0)\ \delta'(x), \end{array} \right\} \quad . \quad . \quad . \quad (17)$$

where use is made of (16).

Next we identify the independent variable x with the operational variable t while assuming, for some strip of convergence $\mathrm{Re}\, p > a$,

$$h^*(t) = h(t)\ U(t) \doteqdot f^*(p).$$

Thereupon we determine the image of (17), this determination being equivalent to a multiplication throughout by the operator

$$\int_{-\infty}^{\infty} e^{-pt} \ldots dt.$$

This operational transposition of (17), performed with the aid of the relations (13) and (14) for the impulse function and its derivatives, reduces the differential equation (16) to a simple algebraic equation, viz.

$$(a_0 p^2 + a_1 p + a_2) f^*(p) = \Phi^*(p) + \{a_0 h'(0) + a_1 h(0)\}\ p + a_0\ h(0)\ p^2,$$

$\Phi^*(p)$ being the image of $\varphi(x)\ U(x)$. The solution of this algebraic equation can be written as follows:

$$f^*(p) = f_1^*(p) + f_2^*(p),$$

in which

$$f_1^*(p) = \frac{\Phi^*(p)}{a_0 p^2 + a_1 p + a_2},$$

$$f_2^*(p) = \frac{\{a_0 h'(0) + a_1 h(0)\}\ p + a_0 h(0)\ p^2}{a_0 p^2 + a_1 p + a_2}.$$

Our problem is solved when the originals $h_1(t)$ and $h_2(t)$ of $f_1^*(p)$ and $f_2^*(p)$, respectively, are found. Evidently $h_1(t)$ is independent of the initial values $h(0)$ and $h'(0)$, but corresponds to the fictitious boundary conditions $h(0) = h'(0) = 0$. Therefore, $h_1(t)$ represents the complete solution if the system were at rest for $t < 0$ so that $h(t)$ and $h'(t)$ vanish for $t = 0$. On the other hand, $h_2(t)$ is independent of $\varphi(t)$; it therefore represents the complete solution if the system were no more affected by external forces for $t > 0$, the boundary conditions at $t = 0$ then accounting for the effect of such forces in the past $(t < 0)$. In electrotechnical terms, $h_1(t)$ corresponds to a switch-on phenomenon, $h_2(t)$ to a switch-off phenomenon.

We emphasize the fact that in the above analysis we assumed that $h(t)\, U(t)$ and $\varphi(t)\, U(t)$ do have an operational image. In the applied method the operational treatment was preceded by the derivation of a new differential equation (17) for the one-sided function $h(t)\, U(t)$, an equation that, in general, is inhomogeneous even if the original equation (16) happens to be homogeneous, the boundary conditions appearing through the delta functions in the right-hand member. It will be clear that the method described is applicable to all kinds of linear differential equations with constant coefficients.

9. *Linear differential equations with variable coefficients.*

The procedure of the foregoing section may also be applied to equations with variable coefficients. Since, in general, the latter are transformed by an operational transposition into another differential equation, the method is efficient only if the process leads to a lowering of the order of the equation, or to the reduction to a simpler equation of the same order.

As an example we discuss the equation

$$(1 - x^2)\, h''(x) - 2\, x\, h'(x) + \nu\, (\nu + 1)\, h(x) = 0 \quad . \quad . \quad . \quad (18)$$

for the Legendre functions of the first and second kinds, $P_\nu(x)$ and $Q_\nu(x)$. As to the functions $P_\nu(t)$, an image is to be expected only if these functions are cut-off at some point $t = a$, and replaced by zero for $t < a$. When deriving the new equation for

$$h^*(t) = P_\nu(t)\; U(t - a),$$

we need successively:

$$\left.\begin{aligned}
\frac{dh^*(t)}{dt} &= P'_\nu(t)\; U(t-a) + P_\nu(a)\; \delta(t-a), \\
t\, \frac{dh^*(t)}{dt} &= t\, P'_\nu(t)\; U(t-a) + a\; P_\nu(a)\; \delta(t-a), \\
\frac{d^2 h^*(t)}{dt^2} &= P''_\nu(t)\; U(t-a) + P'_\nu(a)\; \delta(t-a) + P_\nu(a)\; \delta'(t-a), \\
t^2\, \frac{d^2 h^*(t)}{dt^2} &= t^2 P''_\nu(t)\; U(t-a) + a^2\; P'_\nu(a)\; \delta(t-a) + P_\nu(a)\; t^2\, \delta'(t-a).
\end{aligned}\right\} \quad (19)$$

The last term of (19) can further be simplified with the aid of the properties:

$$\left.\begin{aligned} x\,\delta(x) &= -\,\delta'(x), \\ x^2\,\delta(x) &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which are easy to verify by deriving the images of the left and right members, using for the left members the operational relation

$$t^n\,h(t) \doteqdot p\left(-\frac{d}{dp}\right)^n \frac{f(p)}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This demonstration of (20) is based on the uniqueness of the original, the image being given. Such uniqueness exists for 'almost all' arguments (i.e., apart from a set of arguments of zero measure) when the originals are restricted to L-functions, i.e., functions of which the absolute value is integrable over every finite interval [1]).

An application of (20) to the last term of (19) now yields

$$\begin{aligned} t^2\,\delta'(t-a) &= a^2\,\delta'(t-a) + 2\,a\,(t-a)\,\delta'(t-a) + (t-a)^2\,\delta'(t-a) = \\ &= a^2\,\delta'(t-a) - 2\,a\,\delta(t-a). \end{aligned}$$

With the aid of the above formulae we arrive at the following differential equation for $h^*(t)$:

$$\left.\begin{aligned} &\left\{(1-t^2)\frac{d^2}{dt^2} - 2\,t\,\frac{d}{dt} + \nu(\nu+1)\right\} h^*(t) = \\ &= (1-a^2)\,\{P'_\nu(a)\,\delta(t-a) + P_\nu(a)\,\delta'(t-a)\}. \end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

Obviously the equation for h^* is homogeneous again, similar to that for h, if $a = \pm 1$. However the validity of (22) is questionable when $a = -1$, since $P_\nu(z)$ is in general singular for $z = -1$. Therefore we limit ourselves to $a = +1$, in which case the image $f^*(p)$ of $h^*(t) = P_\nu(t)\,U(t-1)$ is to be found from an operational transposition of the homogeneous equation

$$\left\{(1-t^2)\frac{d^2}{dt^2} - 2\,t\,\frac{d}{dt} + \nu(\nu+1)\right\} h^*(t) = 0.$$

With the aid of (21) we thus find

$$p^2\,\frac{d^2 f^*}{dp^2} - \{\nu(\nu+1) + p^2\}\,f^* = 0,$$

which equation can be reduced to that of the Bessel functions. The required solution of the latter is determined by the condition of finiteness for $p \to \infty$ (this condition has to be satisfied according to a Tauber theorem for

[1]) Compare D. V. WIDDER, The Laplace transform, Princeton, 1941, p. 244.

operational relations); therefore $f^*(p)$ will be proportional to the modified Hankel function

$$\sqrt{p}\, K_{\nu+1/2}(p).$$

When accounting moreover for the property $P_\nu(1) = 1$, we finally arrive at the operational relation:

$$P_\nu(t)\, U(t-1) \doteqdot \sqrt{\frac{2p}{\pi}}\, K_{\nu+1/2}(p), \qquad 0 < \operatorname{Re} p < \infty. \quad . \quad . \quad (23)$$

The simple form of this result is a consequence of the choice of $a = +1$ for the point of cutting off the complete function $P_\nu(t)$. When taking, e.g., $a = 0$, the equation (22) will no longer be homogeneous, and the corresponding image becomes more intricate. For integer values n of ν it may be seen that

$$P_n(t)\, U(t) \doteqdot \left[\left[\sqrt{\frac{2p}{\pi}}\, K_{n+1/2}(p)\right]\right] =$$
$$= \left[\left[p^{n+1}\left(-\frac{1}{p}\frac{d}{dp}\right)^n \left(\frac{e^{-p}}{p}\right)\right]\right], \qquad 0 < \operatorname{Re} p < \infty,$$

in which the symbol [[ ]] indicates the omission of positive powers of p when developing the expression between double brackets into a Laurent series.

10. *Originals with arguments of exponential character.*

Operational relations with originals of this type appear to be very fertile for a simple derivation of many new relations between known functions. Most of these relations exist only as a two-sided Laplace transform and they can often be found with the aid of the following rule. We start with the essentially one-sided relation

$$h(t)\, U(t) \doteqdot f(p), \qquad 0 < \operatorname{Re} p < \infty. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Next we derive, with the substitution $w = e^t$ in the corresponding Laplace integral, the relation

$$h(e^t) \doteqdot p \int_0^\infty \frac{h(w)}{w^{p+1}}\, dw.$$

In the right member we write

$$\frac{1}{w^{p+1}} = \frac{1}{\Pi(p)} \int_0^\infty e^{-ws} s^p\, ds,$$

so as to obtain a repeated integral. Upon reversing the order of integration, the w-integral is recognized as a Laplace integral for the original relation (24), but for the argument s instead of p. The evaluation of the repeated integral thus leads to the following image of $h(e^t)$, expressed as an integral over the first image $f(s)$:

$$h(e^t) \doteqdot \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} f(s)\, ds.$$

Moreover, the substitution $s = e^{-\tau}$ shows that this image may also be interpreted as the definition integral of a new, third, relation. As a consequence, we obtain a triple of corresponding operational relations, viz.

$$\left\{\begin{aligned} & h(t)\, U(t) \doteqdot f(p), \qquad 0 < \mathrm{Re}\, p < \infty, \quad \ldots\ldots \quad (25a) \\ & h(e^t) \doteqdot \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} f(s)\, ds \equiv f_1(p), \quad \ldots\ldots \quad (25b) \\ & f(e^{-t}) \doteqdot \Pi(p)\, f_1(p), \quad \ldots\ldots \quad (25c) \end{aligned}\right.$$

in which the strips of convergence of the last two relations cannot be given generally and therefore should be determined in each particular case. We emphasize that, in this derivation, we arrive at the *two-sided* original of (25*b*) while starting from the *one-sided* original of (25*a*); this is inevitable since the transformation $t \to e^{t'}$ automatically leads to the range of $-\infty$ to $+\infty$ for t' when $0 < t < \infty$.

We give two examples of the triples described.

(*a*) Substituting for (25*a*) a well-known operational relation for the Bessel functions, we thus obtain the triple

$$\left\{\begin{aligned} & t^{\nu/2} J_\nu(2\sqrt{t})\, U(t) \doteqdot \frac{1}{p^\nu} e^{-\frac{1}{p}}, \qquad 0 < \mathrm{Re}\, p < \infty, \\ & \qquad\qquad (\mathrm{Re}\, \nu > -1) \,.\;.\;. \quad (26a) \\ & e^{\frac{\nu t}{2}} J_\nu(2 e^{\frac{t}{2}}) \doteqdot \frac{\Gamma(\nu - p)}{\Gamma(p)}, \qquad \tfrac{1}{2}\mathrm{Re}\, \nu - \tfrac{3}{4} < \mathrm{Re}\, p < \mathrm{Re}\, \nu, \\ & \qquad\qquad (\mathrm{Re}\, \nu > -\tfrac{3}{2}) \,.\;.\;. \quad (26b) \\ & e^{\nu t} e^{-e^t} \doteqdot p\, \Gamma(\nu - p), \qquad -\infty < \mathrm{Re}\, p < \mathrm{Re}\, \nu; \;.\;.\;.\; \quad (26c) \end{aligned}\right.$$

(*b*) Starting from the above relation (23) for the Legendre functions, we get, after some reductions, the triple

$$\left\{\begin{aligned} & P_\nu(t)\, U(t-1) \doteqdot \sqrt{\frac{2p}{\pi}}\, K_{\nu+\frac{1}{2}}(p), \qquad 0 < \mathrm{Re}\, p < \infty, \quad \ldots\ldots \quad (27a) \\ & P_\nu(e^t)\, U(t) \doteqdot \frac{2^{p-1}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{p-\nu}{2}\right) \Gamma\left(\frac{p+\nu+1}{2}\right)}{\Gamma(p)}, \\ & \qquad\qquad |\mathrm{Re}\, \nu + \tfrac{1}{2}| - \tfrac{1}{2} < \mathrm{Re}\, p < \infty, \quad (27b) \\ & e^{-\frac{t}{2}} K_{\nu+\frac{1}{2}}(e^{-t}) \doteqdot p\, 2^{p-\frac{3}{2}}\, \Gamma\left(\frac{p-\nu}{2}\right) \Gamma\left(\frac{p+\nu+1}{2}\right). \\ & \qquad\qquad |\mathrm{Re}\, \nu + \tfrac{1}{2}| - \tfrac{1}{2} < \mathrm{Re}\, p < \infty. \quad (27c) \end{aligned}\right.$$

Many other exponential relations can thus be derived from (25). Particularly the general hypergeometric functions can with much success be treated in this way. As an illustration we give two relations for the special hypergeometric function with three parameters α, β and γ. We have

$$\left.\begin{aligned} &\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,F(\alpha,\beta;\gamma;-e^{-t}) \doteqdot \frac{\Gamma(p+1)\,\Gamma(\alpha-p)\,\Gamma(\beta-p)}{\Gamma(\gamma-p)},\\ &\qquad\qquad 0<\operatorname{Re}p<\min(\operatorname{Re}\alpha,\operatorname{Re}\beta),\end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} &\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,F(\alpha,\beta;\gamma;1-e^{-t}) \doteqdot\\ &\qquad \Gamma(p+1)\,\Gamma(\alpha-p)\,\Gamma(\beta-p)\,\Gamma(p+\gamma-\alpha-\beta),\\ &\qquad\qquad 0<\operatorname{Re}p<\min(\operatorname{Re}\alpha,\operatorname{Re}\beta),\\ &\qquad\qquad\qquad \operatorname{Re}(\gamma-\alpha-\beta)>0.\end{aligned}\right\} \quad . \quad . \quad \tag{29}$$

Other relations of exponential character may also be derived without using (25). Thus the operational relation for the one-sided original, but of exponential type,

$$\left.\begin{aligned} &(1-e^{-t})^{\gamma-1}\,F(\alpha,\beta;\gamma;1-e^{-t})\,U(t) \doteqdot \Gamma(\gamma)\,\frac{\Gamma(p+1)\,\Gamma(p-\alpha-\beta+\gamma)}{\Gamma(p-\alpha+\gamma)\,\Gamma(p-\beta+\gamma)},\\ &\qquad\qquad 0<\operatorname{Re}p<\infty, \qquad (\operatorname{Re}\gamma>0)\end{aligned}\right\} \tag{30}$$

can be verified by transposing operationally term-by-term the series defining the hypergeometric function in the left member. The gamma functions, occurring in relations of this type, often lead to great simplifications in the case of integer values of the parameters; thus we get from (27*b*), for integral order $\nu = n$, the elegant expression:

$$P_n(e^t)\,U(t) \doteqdot \frac{(p+1)(p+3)\dots(p+2n-1)}{(p-2)(p-4)\dots(p-2n)}, \qquad 2n<\operatorname{Re}p<\infty.$$

Applications of relations with an original of exponential type often yield identities the verification of which is not at all simple without the use of the corresponding images. We give the following examples.

(*A*) The inversion integral (3), applied to (29), leads to

$$\left.\begin{aligned} &\Gamma(b+c)\,\Gamma(b+d)\,e^{-at}f(a+c,a+d;a+b+c+d;1-e^{-t}) =\\ &\qquad = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{pt}\,\Gamma(p+a)\,\Gamma(p+b)\,\Gamma(c-p)\,\Gamma(d-p)\,dp,\\ &\qquad\qquad -\operatorname{Re}a<c_0<\min(\operatorname{Re}c,\operatorname{Re}d),\end{aligned}\right\} \tag{31}$$

where

$$f(\alpha,\beta;\gamma;x) \equiv \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,F(\alpha,\beta;\gamma;x),$$

and which can be considered as a generalization of a formula of Barnes, the latter being the special case for $t = 0$;

(B) The following elementary identity, containing nothing else but the duplication formula of the Γ-function,

$$\frac{\pi}{4^{\alpha+\beta-\gamma-1}} \frac{\Gamma(2p+1)\,\Gamma(2\alpha-2p-1)\,\Gamma(2\beta-2p-1)}{\Gamma(2\gamma-2p-1)} =$$

$$= \frac{1}{p} \cdot \frac{\Gamma(p+1)\,\Gamma(\alpha-p)\,\Gamma(\beta-p)}{\Gamma(\gamma-p)} \quad \frac{p}{(p+\frac{1}{2})} \frac{\Gamma(p+\frac{3}{2})\,\Gamma(\alpha-p-\frac{1}{2})\,\Gamma(\beta-p-\frac{1}{2})}{\Gamma(\gamma-p-\frac{1}{2})},$$

can be transposed operationally with the aid of (28), it being considered as the image of a composition product. After some reductions we thus find the following, apparently new, integral relation for the hypergeometric function:

$$\left.\begin{aligned} &\int_0^\infty f(\alpha,\beta;\gamma;-xs)\, f\left(\alpha,\beta;\gamma;-\frac{x}{s}\right)\frac{ds}{\sqrt{s}} = \\ &= \frac{\pi}{4^{\alpha+\beta-\gamma-1}} \frac{f(2\alpha-1,2\beta-1;2\gamma-1;-x)}{\sqrt{x}}, \\ &\qquad \min(\operatorname{Re}\alpha, \operatorname{Re}\beta) > \tfrac{1}{2};\ x>0; \end{aligned}\right\} \quad . \ . \ . \ . \ (32)$$

(C) An integral equation of the type

$$g(x) = \int_{-\infty}^{\infty} K\left(\frac{\xi}{x}\right) h(\xi)\, d\xi,$$

where K is an arbitrary kernel, may be reduced, by the substitutions $x = e^{-t}$, $\xi = e^{-\tau}$, to another integral equation of which the kernel depends only on the difference $t - \tau$; next, it may be solved with the aid of a composition product. An application of (29) thus leads to the following pair of corresponding integrals:

$$\left\{\begin{aligned} g(x) &= \int_0^x F\left(\alpha,\beta;\gamma;1-\frac{\xi}{x}\right)\left(1-\frac{\xi}{x}\right)^{\gamma-1} h(\xi)\, d\xi, \\ h(x) &= \frac{\sin(\gamma\pi)}{\pi}\, x^{\alpha-1} \int_0^x F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; 1-\frac{\xi}{x}\right)\frac{\xi^{\gamma-\alpha}}{(x-\xi)^\gamma}\, g'(\xi)\, d\xi, \\ &\qquad \max(0,\alpha-1) < \gamma < 1; \end{aligned}\right\} (33)$$

(D) New identities can often easily be derived by a differentiation of an exponential relation with respect to one of its parameters. Thus a differentiation with respect to ν of (27b), after some reductions, leads to the following interesting expression for the derivative of a spherical harmonic with respect to its order:

$$\frac{\partial}{\partial\nu} P_\nu(x) = \int_{\frac{1}{x}}^{1} P_\nu(xu)\, \frac{(u^\nu - u^{-\nu-1})}{(u^2-1)}\, du. \qquad (x>1)$$

In the derivation of this formula use was made of the following operational relation:

$$p\,\{\psi(p+a)-\psi(p+b)\} \doteqdot U(t)\,\frac{(e^{-bt}-e^{-at})}{(e^{t}-1)},$$
$$-1-\min(a,b)<\operatorname{Re} p<\infty,$$

which is of some interest already by itself.

11. *Operational identities.*

By this term we mean identities, usually of the integral type, that contain one or more operational originals and images. Examples of these identities relating to a single operational relation

$$h(t) \doteqdot f(p), \qquad \alpha<\operatorname{Re} p<\beta,$$

are

$$\int_a^b \frac{f(s)}{s}\,ds=\int_{-\infty}^{\infty}\frac{(e^{-as}-e^{-bs})}{s}\,h(s)\,ds, \qquad (a,b)\subset(\alpha,\beta),$$

and

$$\frac{1}{\Pi(\nu)}\int_0^{\infty} f(s)\,s^{\nu-1}\,ds=\int_0^{\infty}\frac{h(s)}{s^{\nu+1}}\,ds \qquad (\alpha\leqslant 0;\ \operatorname{Re}\nu>-1).$$

These identities are easily verified, assuming the convergence of the integral, by replacing $f(s)$ by the corresponding Laplace integral.

Another identity, using two operational relations with one-sided originals, viz:

$$h_1(t)\ U(t) \doteqdot f_1(p), \qquad 0<\operatorname{Re} p<\infty,$$
$$h_2(t)\ U(t) \doteqdot f_2(p), \qquad 0<\operatorname{Re} p<\infty,$$

is given by the so-called exchange identity:

$$\int_0^{\infty} h_1(a\,s)\,\frac{f_2(s)}{s}\,ds=\int_0^{\infty} h_2(a\,s)\,\frac{f_1(s)}{s}\,ds. \qquad (a>0) \quad . \quad . \quad (34)$$

An application of this expression to the relation

$$J_\nu(t)\ U(t) \doteqdot \frac{p}{\sqrt{p^2+1}}\,e^{-\nu\,\operatorname{arc\,sinh} p}, \qquad 0<\operatorname{Re} p<\infty,\ (\operatorname{Re}\nu>-1)$$

e.g., leads immediately to the following expression symmetrical in μ and ν:

$$\int_0^{\infty} J_\mu(\sinh u)\,e^{-\nu u}\,du=\int_0^{\infty} J_\nu(\sinh u)\,e^{-\mu u}\,du,$$
$$\operatorname{Re}(\mu,\nu)>0.$$

12. *Generating functions.*

A final remark will show the ease with which discontinuous functions may be studied operationally. Let a generating function

$$\chi(x, a) = \sum_{n=-\infty}^{\infty} \varphi_n(a)\, x^n \qquad (a < |x| < b) \quad . \quad . \quad . \quad . \quad (35)$$

be given. The corresponding operational relation,

$$\varphi_{[t]}(a) \doteqdot (1 - e^{-p})\, \chi(e^{-p}, a), \qquad -\log b < \operatorname{Re} p < -\log a, \quad . \quad (36)$$

then follows at once. It can be verified by splitting the Laplace integral into intervals of unit length. We note that the annular region of convergence of the original Laurent series has thus been transformed, by the substitution $x = e^{-p}$, into a strip of convergence between two parallel lines.

An application of (36) to the generating function for the Bessel functions of integral orders, viz.

$$e^{\frac{\alpha}{2}\left(x - \frac{1}{x}\right)} = \sum_{n=-\infty}^{\infty} J_n(\alpha)\, x^n,$$

thus yields the operational relation

$$J_{[t]}(\alpha) \doteqdot (1 - e^{-p})\, e^{-\alpha \sinh p}, \qquad -\infty < \operatorname{Re} p < \infty. \quad . \quad . \quad . \quad (37)$$

The significance of such relations may be illustrated by the inversion integral of (37). This integral, taken along the imaginary p-axis ($c = 0$), can be reduced, by substituting $p = i\omega$, to the following real expression:

$$J_{[\nu]}(\alpha) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \frac{\omega}{2}}{\omega} \cos \{(\nu - \tfrac{1}{2})\,\omega - \alpha \sin \omega\}\, d\omega,$$

where in the left-hand side we obtain a Bessel function of integral order $[\nu]$, whereas in the integral a continuous ν occurs. When $\nu = n$ (integer) the left-hand side should be replaced by $\frac{1}{2}J_{n-1}(\alpha) + \frac{1}{2}J_n(\alpha)$.

In the special case of a Taylor series the parameter a of (35) vanishes, so that the abscissa, $-\log a$, of the right boundary of the strip of convergence of (36) tends to infinity. Again, an example is given by

$$P_{[t]}(\cos \vartheta)\, U(t) \doteqdot \frac{\sqrt{2}\sinh \frac{p}{2}}{\sqrt{\cosh p - \cos \vartheta}}, \qquad 0 < \operatorname{Re} p < \infty,$$

which operational relation can be obtained from the generating function for the polynomials of Legendre.

Although in this paper many derivations have been given in outline only, it is hoped that this short exposé of a new operational calculus based on the two-sided Laplace integral, may show its great fertility in both abstract and applied mathematics.

Mathematics. — *On Differentiable Linesystems of one Dual Variable.* I. By N. H. KUIPER (Princeton N.Y.) (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 30, 1948.)

CONTENTS.

1. *Introduction.*

Lines, combined with one of their two directions, in a Euclidean three-dimensional space, can be represented by unitvectors with three components over the ring of dual numbers. STUDY [1], BLASCHKE [3] [1]).

Dual numbers are pairs of real numbers with two laws of composition: $A + B \equiv (a + \varepsilon\bar{a}) + (b + \varepsilon\bar{b}) = (a + b) + \varepsilon(\bar{a} + \bar{b})$; $A \cdot B = ab + \varepsilon(\bar{a}b + a\bar{b})$; $(\varepsilon^2 = 0)$. A dual element will be denoted by a capital; the real part by the same letter small, the other part by the same small letter with a bar. Vectors will be denoted by German letters. A unit dual vector is for example $\mathfrak{A} = \mathfrak{a} + \varepsilon\bar{\mathfrak{a}} \equiv (A_1, A_2, A_3) = (a_1 + \varepsilon\bar{a}_1, a_2 + \varepsilon\bar{a}_2, a_3 + \varepsilon\bar{a}_3)$; $\mathfrak{A}^2 = \mathfrak{a}^2 + 2\varepsilon\, \mathfrak{a}\bar{\mathfrak{a}} = 1$. $\mathfrak{a}$ represents the direction of a line. $\bar{\mathfrak{a}}$ is the moment of a univector in the directed line, with respect to the origin of the coordinate system.

Distance, angle and *orientation* of two directed lines $\mathfrak{A}$ and $\mathfrak{B}$ $(\mathfrak{A}^2 = \mathfrak{B}^2 = 1)$ are determined by the scalar product

$$\cos \Phi = \mathfrak{A}\mathfrak{B}, \ \Phi = \varphi + \varepsilon\bar{\varphi} = \text{angle} + \varepsilon \text{ distance}, \ 0 \leq \varphi \leq \pi \qquad (1)$$

The orientation (positive or negative) is the sign of φ. $\varphi\bar{\varphi}/|\varphi\bar{\varphi}|$ determines the orientation of $\mathfrak{A}$ with respect to $\mathfrak{B}$.

Cos Φ is an example of a differentiable function of a dual variable, which can be defined in analogy to a differentiable function of a

1) Bibliography is found at the end of this paper.

complex variable. A differentiable function of a dual variable $X = x + \varepsilon \bar{x}$ has the form

$$F(X) = F(x) + \varepsilon \bar{x} F'(x), \; F'(x) = dF(x)/dx \quad . \quad . \quad . \quad (2)$$

e.g. $\cos X = \cos x - \varepsilon \bar{x} \sin x$.
$F(X)$ in (2) is the unique differentiable continuation of $F(x)$
Formels for differentiation and integration are

$$\left. \begin{array}{l} dF(X)/dX \equiv F'(X) = F'(x) + \varepsilon \bar{x} F''(x) \\ \qquad \text{(in particular for } X = x + \varepsilon \cdot 0 : F'(X) = F'(x)) \\ \int_A^X F(Y)\, dY = \int_a^x F(y)\, dy + \varepsilon (\bar{x} \cdot F(x) - \bar{a} \cdot F(a)) \end{array} \right\} . \quad (3)$$

The important properties of vectoranalysis are valid for the vectorspace over the ring of dual numbers. Moreover, identities in real variables induce identities in dual variables, obtained by differentiable continuation (2). The Euclidean threedimensional linegeometry, expressed with the help of dual unitvectors, is therefore closely analogous to the spherical geometry, expressed with the help of real unitvectors. Properties of elementary spherical geometry can be carried over to linegeometry by some simple translationrules. For example the theorem on the perpendiculars in a triangle and the theorem of Desargues. KUIPER [7] Ch. 1.

2. *The triangle-inequality in the linegeometry* [2]).

A property, which is not expressible in the form of identities in vectors, but which also admits an analogy in linegeometry, is the triangle-inequality.

The dual numbers are *ordered* as follows

$$A = a + \varepsilon \bar{a} > B = b + \varepsilon \bar{b} \leftrightarrows \left\{ \begin{array}{ll} a > b & \text{in case } a \neq b \\ \bar{a} > \bar{b} & \text{in case } a = b \end{array} \right\} . \quad . \quad (4)$$

We define the norm of a dual number A by $|A| = |a| + \varepsilon |\bar{a}|$ (5)

Now it is a matter of simple geometrical considerations, to show that the dual angles between three directed lines $(0 \leq \Phi_{12}, \Phi_{23}, \Phi_{31} \leq \pi)$ obey

$$|\Phi_{13}| \leqq |\Phi_{12}| + |\Phi_{23}| . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Necessary and sufficient conditions for equality in (6) are

a) The three lines are in parallel planes. No two lines are parallel (same direction). If $\mathfrak{A}$ is the line which lies in the middle plane, and if the directions of the lines are represented on a unitsphere, then lies the direction of $\mathfrak{A}$ on the geodesic arc $\leq \pi$ between the other directions (on the unitsphere).

Or b) Two, not three, lines are parallel. The plane of these lines contains the common perpendicular of the three lines. The third line intersects the plane of the other two in a point not between these lines.

Or c) The three lines are parallel and are in one plane.

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[2]) R. DEBEVER proposed the problem of a metric in the space of threedimensional lines to me.

3. *D-systems* [3]). *Introduction continued.*

The differential geometry of sphere-curves can also be carried over by analogy into the geometry of lines in Euclidean threespace. The theory of reguli (ruled surfaces) was developped with this analogy in mind (BLASCHKE [3]). Those properties of reguli which are analogous to properties of spherical curves, can be expressed by the same formulas (compare for example BIRAN [6] and SABAN [9]).

However, reguli are not the linesystems with the closest analogy to spherical curves. *A better analogy is obtained from D-systems* [3]), *differentiable linesystems of one dual parameter.* A D-system in general is a very particular linecongruence (two real parameters). That is why D-systems have not had much attention. We want to study D-systems because of their simple character due to the mentioned "close analogy", and because of their relation to reguli. Every regulus determines by differentiable continuation ((2)) a unique D-system in which it is contained. *The differential invariants and the properties, which the regulus has in analogy to the sphere-curve, are the differential invariants and the properties of the related D-system.*

A complete system of differential invariants of a regulus can be obtained in two parts: 1. the invariants of a complete system of the D-system determined by the regulus. Properties analogous to properties of sphere-curves are expressed in *these* invariants. 2. the invariants which determine the position of the regulus in the related D-system (parameter of distribution, § 6).

It is our aim to study D-systems in relation with their reguli, and to show some properties of reguli to be properties of the related D-systems.

4. *The first invariant orthogonal system at a line of a D-system.*

A regulus is determined by a dual unitvector, function of one real variable: $\mathfrak{A} = \mathfrak{A}(t)$, $\mathfrak{A}^2 = 1$. The unique dual differentiable continuation of this function is given by (2):

$$\mathfrak{A}(T) = \mathfrak{A}\,(t) + \varepsilon \bar{t}\, \mathfrak{A}'(t),\ \mathfrak{A}^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$\mathfrak{A}(T)$ is the D-system "related to" the regulus $\mathfrak{A}(t)$.

> In this paper all real functions of real variables will be assumed to be analytic, though the greater part of the paper is also valid under weaker differentiability conditions.

W. BLASCHKE determined an invariant orthogonal system at a line of a regulus. We obtain the first invariant orthogonal system at a line $(T = 0)$ of a D-system $\mathfrak{A}(T)$ in formally the same way. The three mutually orthogonal and intersecting lines $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ of this system obey

$$d\mathfrak{A}/dT = \dot{\mathfrak{A}} = P\mathfrak{A}_1,\ \mathfrak{A}_2 = \mathfrak{A} \times \mathfrak{A}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

[3]) STUDY [1] p. 305 called these systems *"Synektische Strahlensysteme"*.

When $P^2 = \dot{\mathfrak{A}}^2$ is not a zerodivisor of the ring of dual numbers, then has (8) two trivially related solutions $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{A}, \mathfrak{A}_1^0 = -\mathfrak{A}_1, \mathfrak{A}_2^0 = -\mathfrak{A}_2$ (N is zerodivisor, if $M \neq 0$ exists, such that $N \cdot M = 0$). Usually we choose the solution for which $P > 0 \cdot P = + \sqrt{\dot{\mathfrak{A}}^2}$. We can decompose any dual vector with respect to the system $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ and we find in particular with the help of

$$\mathfrak{A}^2 = 1, \mathfrak{A}\dot{\mathfrak{A}} = 0, \mathfrak{A}\mathfrak{A}_1 = 0, \dot{\mathfrak{A}}\mathfrak{A}_1 + \mathfrak{A}\dot{\mathfrak{A}}_1 = 0, \text{ etc.}$$

$$\left.\begin{array}{lll} \dot{\mathfrak{A}} = & P\mathfrak{A}_1 & P^2 = \dot{\mathfrak{A}}^2 \\ \dot{\mathfrak{A}}_1 = -P\mathfrak{A} & & +Q\mathfrak{A}_2;\ P^2Q = (\mathfrak{A}\,\dot{\mathfrak{A}}\,\ddot{\mathfrak{A}}) \\ \dot{\mathfrak{A}}_2 = & -Q\mathfrak{A}_1 & \end{array}\right\} \quad . \quad . \quad . \quad (9)$$

From (9) and (3) follows (compare BLASCHKE [3] or KUIPER [7]).

Theorem 1. *The Blaschke-system at a line of a regulus (not cylinder) coincides with the first invariant orthogonal system of the related D-system at that line:*

$$T = t + \varepsilon \cdot 0 \quad ; \quad \dot{\mathfrak{A}} = \frac{d\mathfrak{A}(T)}{dT} = \frac{d\mathfrak{A}(t)}{dt}.$$

If $\dot{\mathfrak{A}}^2$ *is a zerodivisor* we get, because

$$\dot{\mathfrak{A}}(T) = \dot{\mathfrak{A}}(t) + \varepsilon\bar{t}\ddot{\mathfrak{A}}(t) = \dot{\mathfrak{a}}(t) + \varepsilon[\dot{\bar{\mathfrak{a}}}(t) + \bar{t}\ddot{\mathfrak{a}}(t)] \quad ((3)).$$
$$\dot{\mathfrak{a}}(t) = 0.$$

The direction $\mathfrak{a}(t)$ of lines in a first order neighbourhood of the considered line is constant.

From $\mathfrak{A}(T) = \mathfrak{A}(t) + \varepsilon\bar{t}\dot{\mathfrak{A}}(t) = \mathfrak{A}(t) + \varepsilon\bar{t}\dot{\mathfrak{a}}(t)$ ((2))
we also conclude that the D-system, function of the real variables t, $\bar{t}$, does not essentially depend on the variable $\bar{t}$ at the considered line ($T = 0$):

$$\left[\frac{d}{d\bar{t}}\mathfrak{A}(t,\bar{t})\right]_{T=t+\varepsilon\bar{t}=0} = 0.$$

The D-system is *degenerated* to a *cylinder*. If this is the case at all lines of the D-system, it is a cylinder in the usual sense.

A D-system is *completely degenerated* when $\dot{\mathfrak{A}}(T) = 0$.

The first invariant orthogonal system is not determined ((8)) at a line where a D-system is degenerated, but not completely. Only the directions of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are then uniquely defined. We will restrict to D-systems for which it is possible to choose $\mathfrak{A}_1(T)$ as *differentiable* function of T such that (8) holds. With this choice also (9) holds.

From (7) and the remarks above we conclude to

Theorem 2. *The D-system related to a regulus, which is not a cylinder,* is *a congruence* (linesystem depending essentially on two real parameters). *The D-system related to a cylinder is the cylinder itself.*

The geometrical construction of the D-system related to a regulus (not cylinder) is easily obtained, once we know the geometrical meaning of $\mathfrak{A}_2(T)$. $\mathfrak{A}_2(T)$ is the common perpendicular of $\mathfrak{A}(T)$ and any line of the D-system in the first-order-neighbourhood of $\mathfrak{A}(T)$:

$$\mathfrak{A}_2(T)\,\mathfrak{A}(T) = 0,\ \mathfrak{A}_2(T)\,[\mathfrak{A}(T) + \dot{\mathfrak{A}}(T)\,\triangle T] = 0 \quad ((9))$$

The lines $\mathfrak{A}(t + \varepsilon \bar{t})$ (t fixed, $\bar{t}$ variable) belong to the first-order-neighbourhood of $\mathfrak{A}(t)$

$$\mathfrak{A}(t + \varepsilon \bar{t}) = \mathfrak{A}(t) + \dot{\mathfrak{A}}(t) \cdot \varepsilon \bar{t}$$

Hence these lines $\mathfrak{A}(T)$, *which have the same direction* $\mathfrak{a}(t)$ *as* $\mathfrak{A}(t)$, *intersect* $\mathfrak{A}_2(t)$ *perpendicular*. From these properties follows, given $\mathfrak{A}(t)$, the construction of $\mathfrak{A}(T)$.

> The third axis $\mathfrak{A}_2(t)$ of the Blaschke-system at a line of a regulus (not cylinder) meets the rule $\mathfrak{A}(t)$ perpendicular and in the strictionpoint, and is perpendicular to the asymtotic tangentplane of the rule (BLASCHKE [3]).

Notice that $\mathfrak{A}_2(T)$ is common perpendicular of $\mathfrak{A}(T)$ and a line in the first-order-neighbourhood of $\mathfrak{A}(T)$ in *any* regulus in the D-system containing the line $\mathfrak{A}(T)$. Hence all these reguli have at $\mathfrak{A}(T)$ the same strictionpoint. The reguli of a *general* congruence have at a line a variable strictionpoint.

5. *The invariant dual parameter.*

The dual angle $\triangle \Psi$ between two nearby lines of a D-system

$\mathfrak{A}(T)$ and $\mathfrak{A}(T + \triangle T) = \mathfrak{A} + \dot{\mathfrak{A}} \cdot \triangle T + \ldots = \mathfrak{A} + P\,\mathfrak{A}_1 \cdot \triangle T + \ldots$

is, *modulo* $(\triangle T)^2$, determined by

$$\begin{cases} \mathfrak{A}\,(T + \triangle T) \cdot \mathfrak{A}\,(T) = \cos \triangle \Psi = 1 \\ \mathfrak{A}\,(T + \triangle T) \cdot \mathfrak{A}_1(T) = \sin \triangle \Psi = \triangle \Psi = P \cdot \triangle T \\ \mathfrak{A}\,(T + \triangle T) \cdot \mathfrak{A}_2(T) = 0 \end{cases}$$

The infinitesimal dual angle between two infinitesimally near lines is therefore in the usual notation

$$d\,\Psi = P\,dT. \quad \ldots\ldots\ldots \quad (10)$$

$S = \int P\,dT = S(T)$, defined but for a constant, is called *the dual arclength of the D-system*. The equation $S = S(T)$ can be solved for T, if the D-system is nondegenerate [4]) ($dS/dT = P$ is not a zerodivisor). Non-

[4]) Proof: $S = S(T) = s + \varepsilon \bar{s} = \int p(t)\,dt + \varepsilon\,[\int \bar{p}(t)\,dt + \bar{t}\,p(t)] + K \quad ((3))$.

$$\therefore \left| \frac{\partial\,(s, \bar{s})}{\partial\,(t, \bar{t})} \right| = (p\,(t))^2 \neq 0.$$

Hence t, $\bar{t}$ can be solved as functions of s, $\bar{s}$; and dT/dS exists because $dT/dS = P^{-1}$.

degenerate D-systems can therefore be represented with the help of thiş *invariant dual parameter S.* The equations (9) simplify with the invariant parameter to

$$\left.\begin{array}{lll} \dot{\mathfrak{A}} = \mathfrak{A}_1 & & \\ \dot{\mathfrak{A}}_1 = -\mathfrak{A} & + \operatorname{cotg} \Phi \cdot \mathfrak{A}_2 & \operatorname{cotg} \Phi = (\mathfrak{A}\, \dot{\mathfrak{A}}\, \ddot{\mathfrak{A}}) \\ \dot{\mathfrak{A}}_2 = -\operatorname{cotg} \Phi \cdot \mathfrak{A}_1 & & \end{array}\right\} \quad (11)$$

6. *The reguli in a D-system.*

A regulus, contained in a given nondegenerate D-system $\mathfrak{A} = \mathfrak{A}(T)$, resp. $\mathfrak{A} = \mathfrak{A}(S)$, is defined by a function

$$T = T(u) = t(u) + \varepsilon \bar{t}(u), \text{ resp. } S = S(u) = s(u) + \varepsilon \bar{s}(u).$$

The dual angle between two infinitely near lines of the regulus is

$$d\Psi = d\psi + \varepsilon d\bar{\psi} = P \frac{dT}{du} du = (p + \varepsilon \bar{p})(t_u + \varepsilon \bar{t}_u) du \quad \left(t_u = \frac{dt}{du}\right)$$

$$= [p t_u + \varepsilon (\bar{p} t_u + p \bar{t}_u)] du$$

$$= \frac{dS}{du} du = (s_u + \varepsilon \bar{s}_u) du$$

The *parameter of distribution of the regulus* is (BLASCHKE [3])

$$\delta = \frac{d\bar{\psi}}{d\psi} = \frac{\bar{p} t_u + p \bar{t}_u}{p t_u} = \frac{\bar{s}_u}{s_u} = \frac{d\bar{s}}{ds}. \quad \dots\dots \quad (12)$$

The *cylinders* in the D-system $\mathfrak{A}(T)$, resp. $\mathfrak{A}(S)$, obey $t_u = s_u = 0$. The rules of these reguli are the lines $\mathfrak{A}(t + \varepsilon \bar{t})$ (t fixed) mutually parallel, and contained in the plane $\mathfrak{A}(t) - \mathfrak{A}_2(t)$. The other peculiar reguli in the D-system are the *torsi* generated by the *tangents to a curve.* They can also be developed over the plane and have a parameter of distribution $\delta = 0$:

$$\bar{p} t_u + p \bar{t}_u = \bar{s}_u = 0 \quad , \quad \bar{s} \text{ constant.}$$

The reguli in the D-system with a constant parameter of distribution are given by linear equations in s and $\bar{s}$: $a \cdot s + b \cdot \bar{s} + c = 0$.

A regulus, not a cylinder, contained in a given D-system $\mathfrak{A}(S)$, can also be defined by $\bar{s} = \bar{s}(s)$. This function again is determined by $\delta = \delta(s)$ and $\bar{s}(0)$ ((12)):

$$\bar{s} = \int_0^s \delta(s)\, ds + \bar{s}(0).$$

Theorem 3. *A regulus, not cylinder, in a given D-system with invariant parameter,* $\mathfrak{A}(S)$, *is determined by one rule* ($\mathfrak{A}(S)$, $S = 0$ $(+\varepsilon \bar{s}(0))$) *and the parameter of distribution* $\delta(s)$, *function of the invariant parameter of direction* s (s is the arclength on a unitsphere, on which the directions of the rules of the regulus may be represented).

7. *Geometrical classification of non-degenerate D-systems.*

$\mathfrak{A}(S)$ be a given non-degenerate D-system with invariant parameter S. Then is $\mathfrak{A}(s) = \mathfrak{A}(s + \varepsilon \cdot 0)$ a *torsus*, i.e. a regulus generated by tangents to a curve (§ 6).

$$\mathfrak{A}_2(S) = \mathfrak{A}_2(s) + \varepsilon \bar{s}\, \dot{\mathfrak{A}}_2(s) = \mathfrak{A}_2(s) - \varepsilon \bar{s} \text{ cotg } \Phi \cdot \mathfrak{A}_1(s) \quad ((11))$$

(s fixed, $\bar{s}$ variable) are lines that intersect the line $\mathfrak{A}(s)$, and are perpendicular to the constant (is asymtotic!) tangent plane along the rule $\mathfrak{A}(s)$ of the torsus. $\mathfrak{A}_2(S)$ (S variable) consists of the normals to this torsus.

In case $\mathfrak{A}_2(S)$ is not degenerated, the relation between $\mathfrak{A}(S)$ and $\mathfrak{A}_2(S)$ is symmetric ((9) (11)), and hence also $\mathfrak{A}(S)$ consists of the normals to a torsus.

When $\mathfrak{A}_2(S)$ is degenerated, but not completely, then is $\mathfrak{A}_2(S)$ a cylinder. $\mathfrak{A}(S)$ intersects $\mathfrak{A}_2(S)$, and also the lines in first order neighbourhood of $\mathfrak{A}_2(S)$, perpendicular ((11)). $\mathfrak{A}(S)$ consists of the perpendicular tangents of a cylinder.

When $\mathfrak{A}_2(S)$ is completely degenerated, consists $\mathfrak{A}(S)$ of the perpendiculars to the line $\mathfrak{A}_2 = \mathfrak{A}_2(S)$.

Theorem 4. *A non-degenerate D-system consists of* a) *the normals to a torsus,* b) *the perpendicular tangents of a cylinder, or* c) *the normals to a line. A degenerate D-system is a cylinder. A completely-degenerated D-system is a line.*

(Compare STUDY [1] p. 305).

If a regulus has a related D-system on which a) of theorem 4 is applicable, then a torsus as mentioned can be constructed as follows: $(\mathfrak{A}\, \mathfrak{A}_1\, \mathfrak{A}_2\, X_u)$ be the invariant (Blaschke-)system of axes at the line $t = u$ of the regulus $\mathfrak{A}(t) \cdot x(t)$ be an orthogonal trajectory of the rules of $\mathfrak{A}(t) \cdot \mathfrak{B}(u)$ is the line parallel to $\mathfrak{A}_2(u)$ and passing through the point $x(u)$. Then is $\mathfrak{B}(t)$ the required torsus.

8. *The momentanous axes of a D-system.*

In the theory of D-systems we can very well use the terminology of cinematics. We call the dual parameter in $\mathfrak{A}(T)$: time. The moments of time are ordered by (4). $(\mathfrak{A}\, \mathfrak{A}_1\, \mathfrak{A}_2) = (\mathfrak{A}\, \mathfrak{A}_1\, \mathfrak{A}_2)\,(T)$ is a Motion. The momentanous axis of this Motion and hence of the D-system is defined by

$$\mathfrak{S} = \Lambda\, \mathfrak{A} + \Lambda_1\, \mathfrak{A}_1 + \Lambda_2\, \mathfrak{A}_2\,,\ \dot{\mathfrak{S}} = \dot{\Lambda} = \dot{\Lambda}_1 = \dot{\Lambda}_2 = 0\,,\ \mathfrak{S}^2 = 1, \Lambda_2 > 0. \quad (13)$$

With (9) follows

$$\Lambda_1 P = \Lambda_1 Q = \Lambda P - \Lambda_2 Q = 0.$$

These equations have a unique solution if $P^2 + Q^2 \neq 0$ ($P^2 + Q^2$ is never a zerodivisor different from zero; $p > 0$)

$$\mathfrak{S} = \frac{Q\, \mathfrak{A} + P\, \mathfrak{A}_2}{V\,(P^2 + Q^2)} = \cos \Phi \cdot \mathfrak{A} + \sin \Phi \cdot \mathfrak{A}_2. \quad . \quad . \quad . \quad (13)$$

Φ, also in (11), is the dual angle between a line of the D-system and its momentanous axis: $\mathfrak{S}\,\mathfrak{A} = \cos \Phi$.

The system of momentanous axes of a D-system is another D-system, eventually degenerated. The dual arclength of $\mathfrak{S}(T)$ is obtained from (13)

$$\dot{\mathfrak{S}} = \dot{\Phi}\,(-\sin \Phi \cdot \mathfrak{A} + \cos \Phi \cdot \mathfrak{A}_2) = \dot{\Phi} \cdot \mathfrak{S}_1.$$

$$\text{Dual arclength of } \mathfrak{S}(T) = \int \dot{\Phi}\, dT = \Phi (+ \text{ a dual constant})! \quad . \quad (14)$$

In words

Theorem 5. *The dual angle Φ between $\mathfrak{A}(T)$ and its momentanous axis $\mathfrak{S}(T)$ is dual arclength of the D-system of momentanous axes $\mathfrak{S}(T)$. $\mathfrak{A}(T)$ is a "developed D-system", developed from the "enveloped D-system" $\mathfrak{S}(T)$.*

It is easily seen that if $\mathfrak{A}(T)$ and $\mathfrak{B}(T)$ are developed from the same non-degenerate D-system $\mathfrak{S}(T)$, then is (compare KUIPER [7] ch. 3)

$$\mathfrak{S}_2 = -\mathfrak{A}_1 = -\mathfrak{B}_1$$

$$\mathfrak{B}(T) = \cos K \cdot \mathfrak{A}(T) + \sin K \cdot \mathfrak{A}_2(T)$$

$$\Phi_b(T) - \Phi_a(T) = K \text{ (dual constant).}$$

The dual angle between $\mathfrak{A}(T)$ and $\mathfrak{B}(T)$ is constant: K.

(13) and (14) are also valid when $\mathfrak{A}(T)$ is degenerated but $q \neq 0$:

$$\mathfrak{S} = \frac{Q\mathfrak{A} + \varepsilon \bar{p} \mathfrak{A}_2}{Q} = \mathfrak{A} + \varepsilon \frac{\bar{p}}{q} \mathfrak{A}_2$$

$$\sin \Phi = \sin \varphi + \varepsilon \bar{\varphi} \cos \varphi = \varepsilon \bar{p}/q$$

$$\Phi = 0 + \varepsilon \bar{p}/q, \quad \varphi = 0, \quad \bar{\varphi} = \bar{p}/q.$$

The momentanous axis at a line where a D-system is a cylinder, is parallel to that line. In particular:

Theorem 6. *The enveloped D-system of a cylinder is another cylinder.* $\bar{p}/q$ is the radius of curvature of an orthogonal trajectory of the rules of the first cylinder.

To get the definition and computation of *the dual velocity of a Motion* $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T)$, we consider a line perpendicular to the momentanous axis $\mathfrak{S}(T)$, namely $\mathfrak{A}_1(T)$. We consider the dual angle $\Delta\chi$ between $\mathfrak{A}_1(T)$ and $\mathfrak{A}_1(T+\Delta T)$. *Modulo* $(\Delta T)^2$ we have (see (9))

$$\mathfrak{A}_1(T+\Delta T) \cdot \mathfrak{S} = [\mathfrak{A}_1 + (-P\mathfrak{A} + Q\mathfrak{A}_2)\Delta T] \cdot \frac{Q\mathfrak{A} + P\mathfrak{A}_2}{\sqrt{(P^2+Q^2)}} = 0$$

$$\mathfrak{A}_1(T+\Delta T) \cdot \mathfrak{A}_1 = 1 = \cos \Delta\chi$$

$$\mathfrak{A}_1(T+\Delta T) \cdot (\mathfrak{S} \times \mathfrak{A}_1) = \mathfrak{A}_1(T+\Delta T) \cdot \frac{-P\mathfrak{A} + Q\mathfrak{A}_2}{\sqrt{(P^2+Q^2)}} = \sqrt{P^2+Q^2}\,\Delta T$$

$$= \sin \Delta\chi = \Delta\chi.$$

The dual velocity is

$$\frac{d\chi}{dT} = V(P^2 + Q^2) > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The dual velocity is positive with respect to the momentanous axis $\mathfrak{S}$. It has the orientation of the rotation over the smallest angle which sends $\mathfrak{A}_1$ into $\mathfrak{S} \times \mathfrak{A}_1$; $\mathfrak{A}_1 \times (\mathfrak{S} \times \mathfrak{A}_1) = \mathfrak{S}$.

From (13) we get an important interpretation of the dual motion of the first invariant orthogonal system of a D-system $\mathfrak{A}(T)$ with non-degenerate enveloped D-system $\mathfrak{S}(T) = \mathfrak{S}(\Phi(T))$. $\mathfrak{S}(\Phi)$ be therefore considered as a D-system in an *immobile* space. The system of lines

$$\mathfrak{R}(\chi) = \cos\chi \cdot \mathfrak{A} + \sin\chi \cdot \mathfrak{A}_2, \ \chi \text{ variable},$$

is a "geodesic D-system" (§ 11) in a *mobile* space attached to the orthogonal system $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T)$.

The dual motion $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T)$ *is then obtained by "developing the mobile D-system* $\mathfrak{R}$ *along the immobile D-system* $\mathfrak{S}$*", i.e. such that for any* Φ, $\mathfrak{S}(\Phi)$, $\mathfrak{S}_1(\Phi)$ *coincides with* $\mathfrak{R}(\Phi)$, $\mathfrak{R}_1(\Phi)$.

Remark. A *general* dual motion, represented by

$$\mathfrak{A}(T),\ \mathfrak{B}(T),\ \mathfrak{C}(T),\ \mathfrak{A}^2 = \mathfrak{B}^2 = \mathfrak{C}^2 = 1,\ \mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{C} = \mathfrak{C}\mathfrak{A} = 0,$$

is obtained by development of a D-system $\mathfrak{S}^*(\Phi)$ in a mobile space (here $\mathfrak{S}^*(\Phi)$ need not be geodesic!) along another D-system $\mathfrak{S}(\Phi)$ in an immobile space.

The results of this § can easily be specialised to reguli; namely by restriction of the dual parameters to values $T = T(u)$, $S = S(u)$, $\Phi = \Phi(u)$. $\mathfrak{A}(T(u))$ for example be a regulus in the D-system $\mathfrak{A}(T)$ with non-degenerate enveloped D-system $\mathfrak{S}(T)$. The motion of the invariant orthogonal system of Blaschke $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T(u))$ (theorem 1), can be obtained by gliding development of the regulus

$$\mathfrak{R}(\Phi(u)) = \cos\Phi(u) \cdot \mathfrak{A} + \sin\Phi(u) \cdot \mathfrak{A}_2$$

in the mobile space attached to $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T(u))$, along the regulus $\mathfrak{S}(\Phi(u))$ in the immobile space, such that at the moments u, $(\mathfrak{S}, \mathfrak{S}_1)(\Phi(u))$ coincides with $(\mathfrak{R}, \mathfrak{R}_1)(\Phi(u))$. Notice that the reguli $\mathfrak{S}(\Phi(u))$ and $\mathfrak{R}(\Phi(u))$ have the same parameter of distribution

$$\delta = \frac{d\bar{\varphi}}{d\varphi} = \frac{d\bar{\varphi}}{du} \Big/ \frac{d\varphi}{du}.$$

The "enveloped regulus" of a regulus and the dual velocity of the invariant orthogonal system of Blaschke of a regulus were studied by BIRAN [6 b, c]. Analogues of the formulas and constructions of Euler–Savary were studied by DISTELI [2], GARNIER [5], VAN HAASTEREN [8], BIRAN [6 d]. Compare § 13.

(To be continued.)

Mathematics. — *On a problem in the theory of uniform distribution.* By P. ERDÖS and P. TURÁN. I. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 25, 1948.)

1. In a forthcoming paper [1]) we prove the following

Theorem I. If

$$f(z) = a_0 + \ldots + a_n z^n = a_n \prod_{\nu=1}^{n} (z - z_\nu) \quad . \quad . \quad . \quad . \quad (1.1)$$

and

$$\max_{|z|=1} |f(z)| = M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

then for arbitrary fixed $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left| \sum_{\alpha \leqq \operatorname{arc} z_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta - \alpha}{2\pi} n \right| < 16 \sqrt{n \log \frac{M}{\sqrt{|a_0 a_n|}}}. \quad . \quad . \quad (1.3)$$

Here — and throughout the paper — the expression $\alpha \leqq \operatorname{arc} z \leqq \beta \bmod 2\pi$ means that the image of z on the complex plane lies in the angle formed by $\operatorname{arc} z = \alpha$ and $\operatorname{arc} z = \beta$.

The meaning of Theorem I is obviously that given a sequence of polynomials (the n-th of degree n) having the maximum modulus M_n on the unit circle and such that $\frac{M_n}{\sqrt{|a_0 a_n|}}$ increases "not too rapidly" (e.g. $\frac{M_n}{\sqrt{|a_0 a_n|}} < e^{\frac{n}{\log n}}$) then the roots of the n-th polynomial are uniformly distributed in the different angles even if the size of the angle tends to 0 with $1/n$ "not too rapidly".

2. It is natural to ask whether restricting only

$$\frac{1}{\sqrt{|a_0 a_n|}} \max_{|z|=\vartheta} |f(z)| \equiv \frac{M_\vartheta}{\sqrt{|a_0 a_n|}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

with fixed ϑ $(0 < \vartheta < 1)$ a similar equidistribution theorem can be deduced. It is easy to see that this is not the case. Indeed let

$$\left.\begin{aligned} \varphi_1(z) &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} \\ f(z) &= z^n \varphi_1\left(\frac{\sqrt[n]{n!}}{z}\right) = 1 + \ldots + z^n. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.2)$$

[1]) Submitted to the Annals of Mathematics.

If z is on the circle $|z| = \frac{1}{2e}$ then from STIRLING's formula we have

$$w = \frac{\sqrt[n]{n!}}{|z|} > 2n$$

i.e. the terms $\frac{|w|^n}{n!}, \frac{|w|^{n-1}}{(n-1)!}, \ldots, \frac{|w|}{1}$, 1 decrease more rapidly then a geometric progression with quotient 1/2. Thus

$$\left|\varphi_1\left(\frac{\sqrt[n]{n!}}{z}\right)\right| < 2\frac{|w|^n}{n!} = \frac{2}{|z|^n}$$

$$\left|f\left(\frac{e^{i\vartheta}}{2e}\right)\right| < 2$$

i.e. for $|z| \leqq \frac{1}{2e}$

$$|f(z)| \leqq 2 \quad , \quad \frac{1}{\sqrt{|a_0 a_n|}} \max_{|z|=\frac{1}{2e}} |f(z)| \leqq 2.$$

On the other hand, as SZEGÖ [2]) showed, we have for the number N_1 resp. N_2 of roots of $\varphi_1(z)$ (i.e. also of $f(z)$) lying in the half plane $Rz \leqq 0$ resp. $\geqq 0$ the relations

$$\lim_{n\to\infty} \frac{N_1}{n} = \frac{1}{2} + \frac{1}{e\pi} \quad , \quad \lim_{n\to\infty} \frac{N_2}{n} = \frac{1}{2} - \frac{1}{e\pi},$$

i.e. the roots of $f(z)$ are not uniformly distributed in the different angles. Hence the polynomials (2.2) give the required counter-example for $\vartheta \leqq \frac{1}{2e}$. Choosing instead of the partial-sums of the exponential series the partial sums of certain MITTAG-LEFFLER functions [3]) we see that even if a sequence of polynomials (the n-th of degree n) divided by $\sqrt{|a_0 a_n|}$ remains uniformly bounded in n over a prescribed circle $|z| \leqq \vartheta$ with $0 < \vartheta < 1$, the roots of the n-th polynomial are not necessarily uniformly distributed in the different angles. No doubt, this fact throws a new light on the theorem stated in 1. and enhances its interest considerably.

3. So without imposing any further conditions on $\frac{M_\vartheta}{\sqrt{|a_0 a_n|}}$ (2.1) can not lead to an equidistribution theorem similar to that of 1. However we

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[2]) G. SZEGÖ, Über eine Eigenschaft der Exponential-reihe. Sitzungsber. der Berliner Math. Ges. 50—64 (1924).

[3]) The distribution of roots of these partial-sums and even of the partial-sums of a general class of integral functions of finite positive order has been determined by P. ROSENBLOOM (to appear in the Transactions of Amer. Math. Soc.).

shall show that a simple additional condition can save the situation. We shall prove the following

Theorem II. If all the roots of a polynomial

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

are outside the open unit circle, and for a fixed $0 < \vartheta < 1$ we have for $|z| = \vartheta$ the inequality $|f(z)| \leq M_\vartheta$ then writing (without loss of generality)

$$\frac{M_\vartheta}{\sqrt{|a_0 a_n|}} = e^{\frac{n}{g(n,\vartheta)}} \quad , \quad n \geqq g(n, \vartheta) \geqq 2 \quad . \quad . \quad . \quad . \quad (3.2)$$

we have for all $0 \leq \alpha < \beta \leq 2\pi$

$$\left| \sum_{\alpha \leqq \operatorname{arc} z_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta - \alpha}{2\pi} n \right| < C \log \frac{4}{\vartheta} \cdot \frac{n}{\log g(n, \vartheta)}, \quad . \quad (3.3)$$

where C denotes a *numerical* constant.

As we mentioned before while discussing Theorem I, if a given sequence of polynomials

$$f_n(z) = a_0^{(n)} + a_1^{(n)} z + \ldots + a_n^{(n)} z^n$$
$$n = 1, 2, \ldots$$

has the property that their absolute maxima $M^{(n)}$ on the unit circle satisfies

$$\frac{M^{(n)}}{\sqrt{|a_0^{(n)} a_n^{(n)}|}} = e^{o(n)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

then their roots are equidistributed in the different angles. Theorem II reveals the surprising fact that the much weaker condition concerning the absolute maxima $M_\vartheta^{(n)}$ on the circle $|z| = \vartheta$, $(0 < \vartheta < 1$ and fixed$)$

$$\frac{M_\vartheta^{(n)}}{\sqrt{|a_0^{(n)} a_n^{(n)}|}} = e^{o(n)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

can assure the equidistribution of the roots, if they are all $\geqq 1$ in absolute value.

In the case when $\dfrac{M_\vartheta}{\sqrt{|a_0 a_n|}}$ is "not too large", e.g. when

$$\frac{M_\vartheta}{\sqrt{|a_0 a_n|}} \leqq e^{\sqrt{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

the error term (3.3) is of order $n/\log n$. Curiously enough the same holds if $e^{\sqrt{n}}$ is replaced by n^{100} or even by a numerical constant say 10000. Though this error term is worse than that of Theorem I, DE BRUIJN [4])

[4]) In a letter wherein he conjectured essentially our theorem II.

remarked that the order of the error term is best possible for every $0 < \vartheta < 1$; indeed

$$\vartheta = e^{-c} \quad , \quad c > 0$$

and

$$l = \left[\frac{1}{c}\log n\right] \quad , \quad k = \left[\frac{cn}{\log n}\right],$$

the polynomial

$$f(z) = (1 - z^l)^k$$

satisfies

$$M_\vartheta \leqq (1 + e^{-cl})^k < \exp(k e^{-cl}) < \exp\left(\frac{cn}{\log n}\, e^{-c\left(\frac{1}{c}\log n - 1\right)}\right) =$$
$$= \exp\left(\frac{cn}{\log n}\cdot\frac{1}{n}\, e^c\right) < 2 \quad , \quad \frac{M_\vartheta}{\sqrt{|a_0 a_n|}} < 2$$

for $n > n_0 = n_0(\vartheta)$, and $f(z)$ has roots of multiplicity $> c\dfrac{n}{\log n} - 1 >$ $> \dfrac{c}{2}\cdot\dfrac{n}{\log n}$ for $n > n_1 = n_1(\vartheta)$, which evidently shows that the error term in Theorem II is (with regard to n) the best possible. As a matter of fact essentially DE BRUIJN's example shows that Theorem II is for every admissible $g(n, \vartheta)$ essentially best possible. Indeed put

$$l_1 = \left[\frac{1}{c}\log g(n,\vartheta)\right] \quad , \quad k_1 = \left[\frac{cn}{\log g(n,\vartheta)}\right].$$

The polynomial

$$f_1(z) = (1 - z^{l_1})^{k_1}$$

has a root of multiplicity

$$> \frac{cn}{\log g(n,\vartheta)} - 1 > \frac{c}{2}\cdot\frac{n}{\log g(n,\vartheta)}$$

though on the circle $|z| = e^{-c} = \vartheta$ we have

$$|f_1(z)| \leqq \left(1 + e^{-c\left(\frac{1}{c}\log g(n,\vartheta) - 1\right)}\right)^{\frac{cn}{\log g(n,\vartheta)}} < \exp\left(\frac{cn}{\log g(n,\vartheta)}\,\frac{e^c}{g(n,\vartheta)}\right) <$$
$$< \exp\left(\frac{n}{g(n,\vartheta)}\right),$$

if $g(n, \vartheta)$ is sufficiently large.

In our paper [1]) we have not dealt with the question whether the error term in Theorem I is best possible, with respect to n, but we can show by an example that it is essentially best possible; we do not discuss here the details.

For some further remarks about the relation of Theorem I and II see 10.

4. In 8 we shall see using a method due to I. SCHUR [5]), how the general case of Theorem II can be reduced to the case when all the roots of $f(z)$ lie on the unit circle. In 9, we easily deduce Theorem II thus specialised from the following

Theorem III. If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are real and

$$|s_k| \equiv \left| \sum_{\nu=1}^{n} e^{ki\varphi_\nu} \right| \leqq \psi(k) \qquad k = 1, 2, \ldots, m \quad . \quad . \quad . \quad (4.1)$$

$$m = m(n) \geqq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

then for arbitrary $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta - \alpha}{2\pi} n \right| < C \left(\frac{n}{m+1} + \sum_{k=1}^{m} \frac{\psi(k)}{k} \right)$$

with a numerical constant C.

5. Theorem III is obviously a "finite" form of the classical theorem of H. WEYL [6]) according to which if $\varphi_1, \varphi_2, \ldots$ is an infinite sequence of real numbers satisfying for every integer k the relation

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} e^{ki\varphi_\nu} = 0$$

then for every $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\substack{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi \\ \nu \leqq n}} 1$$

Another "finite" form of this theorem one can find on p. 101 of KOKSMA's well-known book [7]), where a sketch of the proof is also given. *Theorem* of VAN DER CORPUT and KOKSMA (in a slightly modified and restricted form). If there is a δ with $0 < \delta \leqq 1$ such that (4.1) holds for

$$k \leqq \left[\frac{K}{\delta} \log \frac{3}{\delta} \left(\log \log \frac{3}{\delta} \right)^2 \right] \equiv N_0.$$

K being a suitable numerical constant, then with the same K we have for all $0 \leqq \alpha < \beta \leqq 2\pi$

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta - \alpha}{2\pi} n \right| < K\delta n + 2K \sum_{1 \leqq k \leqq \frac{2}{\delta}} \frac{\psi(k)}{k} +$$

$$+ 2K \sum_{\frac{2}{\delta} < k \leqq N_0} \frac{\psi(k)}{k} e^{-\frac{k\delta}{89 \log^2 k\delta}}.$$

[5]) I. SCHUR, Sitzungsber. Berliner Akad. 403—428 (1933).

[6]) H. WEYL, Über die Gleichverteilung von Zahlen mod Eins. Math. Ann., **77**, 313—352 (1916).

[7]) J. F. KOKSMA, Diophantische Approximationen. Ergebn. der Math. und ihrer Grenzgebiete (1936).

Choosing $\delta = \frac{1}{m+1}$ we see that in order to obtain the same error term as in (4.2) we have to restrict more of the s_k's, i.e. Theorem III is sharper than the theorem of VAN DER CORPUT–KOKSMA. As a matter of fact one can not deduce Theorem II from it (whereas it can be deduced from Theorem III). This improvement was obtained roughly speaking by using "DUNHAM JACKSON means" of the FOURIER series of the periodic *discontinuous* function $f_2(x)$ defined by

$$\begin{aligned} f_2(x) &= 1 \quad \text{for} \quad 0 \leqq x < a \\ f_2(x) &= 0 \quad \text{for} \quad a \leqq x < 2\pi \end{aligned}$$

instead of the partial sums of the *continuous* function $f_3(x)$ defined by

$$\begin{aligned} f_3(x) &= 1 \quad \text{for} \quad \eta \leqq x \leqq a - \eta \\ f_3(x) &= 0 \quad \text{for} \quad a \leqq x \leqq 2\pi, \end{aligned} \qquad 0 < \eta < \frac{a}{2}$$

and linear in the remaining intervals.

6. We consider Theorem III in the special case

$$|s_k| \leqq k^\lambda \quad , \quad \lambda > 1 \text{ and fixed}$$

for all $k \leqq n^{1/\lambda}$.

Then the error term in (4.2), if $m \leqq n^{1/\lambda}$, is

$$< C\left(\frac{n}{m} + m^\lambda\right)$$

i.e. choosing $m = [n^{1/(\lambda+1)}]$ we obtain the following

Corollary. If $\varphi_1, \ldots, \varphi_n$ are real, $\lambda \geqq 1$ and

$$\left.\begin{aligned} \left| \sum_{\nu=1}^{n} e^{ki\varphi_\nu} \right| &\leqq k^\lambda \\ 1 \leqq k &\leqq n^{\frac{1}{\lambda+1}} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (6.1)$$

then for all $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi}^{\nu} 1 - \frac{\beta - \alpha}{2\pi} n \right| < C n^{\frac{\lambda}{\lambda+1}} \quad \ldots \ldots \quad (6.2)$$

with a numerical constant C.

The interesting question whether the estimation (6.2) is best possible or not, remains open.

7. L. KALMÁR [8]) made the remarkable discovery that if the roots of the

[8]) L. KALMÁR, Az interpolatióról (hungarian). Matematikai és Fizikai Lapok 1926, p. 120—149. The expression $T_n(z)$ in (7.3) denotes the classical CEBICSEF-polynomial

$$T_n(z) = \left(\frac{z + \sqrt{z^2-1}}{2}\right)^n + \left(\frac{z - \sqrt{z^2-1}}{2}\right)^n.$$

He actually proved a more general theorem when the roots of polynomials $\omega_n(z)$ lie on a prescribed closed JORDAN-curve.

polynomials

$$\omega_n(z) = (z - x_1^{(n)})(z - x_2^{(n)}) \ldots (z - x_n^{(n)}) \quad . \quad . \quad . \quad . \quad (7.1)$$

satisfy for all $n = 1, 2, \ldots$ the inequalities

$$1 \geqq x_1^{(n)} \geqq x_2^{(n)} \geqq \ldots \geqq x_n^{(n)} \geqq -1 \quad . \quad . \quad . \quad . \quad . \quad (7.2)$$

and if the polynomials $\omega_n(z)$ have the asymptotic representation (with the obvious meaning of the n-th root)

$$\lim_{n \to \infty} \sqrt[n]{\frac{\omega_n(z)}{T_n(z)}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.3)$$

on the complex $z = x + i \, . \, y$-plane, cut along the segment $-1 \leq x \leq 1$, then the roots $x_\nu^{(n)}$ are uniformly distributed *in* FEJÉR's *sense* [9]) in $[-1, +1]$ i.e. writing

$$x_\nu^{(n)} = \cos \vartheta_\nu^{(n)}, \ \nu = 1, 2, \ldots, n, \ n = 1, 2, \ldots, 0 \geqq \vartheta_1^{(n)} \geqq \vartheta_2^{(n)} \geqq \ldots \geqq \vartheta_n^{(n)} \geqq \pi \quad (7.4)$$

we have for every $0 \leq \alpha < \beta \leq 2\pi$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\alpha \leqq \vartheta_\nu^{(n)} \leqq \beta} 1 = \frac{\beta - \alpha}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.5)$$

It is easy to see from (7. 3) that the polynomial

$$2^n w^n \omega_n \left(\frac{w + \dfrac{1}{w}}{2} \right) = F_n(w)$$

has all its roots on the unit-circle and for $|w| < 1$

$$\lim_{n \to \infty} \sqrt[n]{F_n(w)} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.6)$$

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[9]) L. FEJÉR. Interpolation und konforme Abbildung. Gött. Nachr. 1918, p. 319—331. Generally if l is a given JORDAN-curve and the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}$ $(n = 1, 2, \ldots)$ are on l, he calls the points $z_\nu^{(n)}$ uniformly distributed over l if mapping conformally the outside of l onto the outside of $|w| =$ and continuously on the boundary, the maps

$$w_1^{(n)} = e^{i\varphi_1^{(n)}}, \ldots, w_n^{(n)} = e^{i\varphi_n^{(n)}}$$

are uniformly distributed over the unit-circle in WEYL's sense i.e. for every $0 \leqq \alpha < \beta \leqq 2\pi$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\alpha \leqq \varphi_\nu^{(n)} \leqq \beta} 1 = \frac{\beta - \alpha}{2\pi}.$$

In the case when l degenerates into a doubly-covered segment $-1 \leqq x \leqq +1$ then the mapping function is $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$ and we get the definition (7.4)—(7.5). This definition of equidistribution fits in with various function-theoretical problems even in the case of segments.

It is natural to ask for the "finite" analogon of KALMÁR's theorem. Making the following weaker assumption that the roots of the polynomial

$$f(z) = a_0 + \ldots + z^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.7)$$

lie in $|z| \geqq 1$ and on a fixed circle $|z| = \vartheta$, $0 < \vartheta < 1$ we have

$$|f(z)| \equiv 1 + \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.8)$$

we may expect that the error term in the distribution of arc z_ν is much smaller if ε is "small". But the example of DE BRUIJN

$$f_1(z) = (1 - z^{l_2})^{k_2}$$

$$l_2 = [\omega \log n], \qquad k_2 = \left[\frac{1}{\omega} \cdot \frac{n}{\log n}\right]$$

with sufficiently large $\omega > \omega_0(\vartheta)$ shows as before that even under the assumptions (7.7) and (7.8) we can not get a better error term than $O\left(\frac{n}{\log n}\right)$.

As was conjectured by DE BRUIJN, we can prove that the error term is $O\left(\frac{\log n}{n}\right)$, only if we assume that the sequence $f_n(z)$ of polynomials (7.7) satisfies $\lim\limits_{n \to \infty} \max\limits_{|z| = \vartheta} |f_n(z)| = 1$ for every positive $\vartheta < 1$.

8. As mentioned in 4. we start with the following remark of I. SCHUR. Let $z = re^{i\varphi}$ be a fixed point on the complex-plane, and $\zeta = \varrho e^{i\gamma}$ move along the line arc $\zeta = \gamma$ (γ fixed). Then

$$|z - \zeta|^2 = r^2 + \varrho^2 - 2 r \varrho \cos(\varphi - \alpha)$$

$$\frac{|z - \zeta|^2}{|\zeta|} = \frac{r^2}{\varrho} + \varrho - 2 r \cos(\varphi - \alpha).$$

If ϱ moves from $+\infty$ to $\varrho = r$ the expression on the right decreases monotonically; hence if $\zeta = \zeta_0 = \varrho_0 e^{i\gamma_0}$, $\varrho_0 > 1$ and $|z| = r \leqq 1$ then

$$\frac{|z - \zeta_0|^2}{|\zeta_0|} \geqq |z - e^{i\gamma_0}|^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.1)$$

This is the remark we need.

Let

$$f(z) = a_0 + \ldots + a_n z^n = a_n \prod_{\nu=1}^{n} (z - z_\nu) = a_n \prod_{\nu=1}^{n} (z - \varrho_\nu e^{i\varphi_\nu}) \quad (8.2)$$

be the polynomial of Theorem II; let

$$|z_\nu| = \varrho_\nu \geqq 1, \qquad \nu = 1, 2, \ldots, n \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.3)$$

and the ϑ of this Theorem be fixed. Let z be on the circumference of the circle $|z| = \vartheta$ and ζ_0 be any of the roots z_ν. Then the remark (8. 1) gives

$$\frac{|z-z_\nu|^2}{|z_\nu|} \geqq |z-e^{i\varphi_\nu}|^2, \qquad \nu = 1, 2, \ldots, n.$$

Multiplying all these inequalities we obtain on the whole $|z| = \vartheta$ the inequality

$$|\psi(z)|^2 \equiv \left| \prod_{\nu=1}^{n} (z-e^{i\varphi_\nu}) \right|^2 \leqq \frac{|f(z)|^2}{|a_0 a_n|} \quad . \quad . \quad . \quad . \quad (8.4)$$

i.e. a fortiori

$$M_\vartheta^* = \max_{|z|=\vartheta} |\psi(z)| \leqq \max_{|z|=\vartheta} \frac{|f(z)|}{\sqrt{|a_0 a_n|}} = \frac{M_\vartheta}{\sqrt{|a_0 a_n|}}. \quad . \quad . \quad . \quad (8.5)$$

The distribution of the roots of $\psi(z)$ in the different angles is identical with that of $f(z)$. Now assume that Theorem II is proved in the case when all the roots lie on the unit circle; then

$$\psi(z) = 1 + b_1 z + \ldots + b_n z^n, \qquad |b_n| = 1 \quad . \quad . \quad . \quad (8.6)$$

and with

$$\max_{|z|=\vartheta} |\psi(z)| = M_\vartheta^* = e^{\frac{n}{g^*(n,\vartheta)}}$$

we have for all $0 \leqq \alpha < \beta \leqq 2\pi$

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi} L - \frac{\beta-\alpha}{2\pi} n \right| < C \log \frac{4}{\vartheta} \cdot \frac{n}{\log g^*(n,\vartheta)}. \quad . \quad (8.7)$$

Then using (8. 5) we have

$$e^{\frac{n}{g^*(n,\vartheta)}} = M_\vartheta^* \leqq \frac{M_\vartheta}{\sqrt{|a_0 a_n|}} = e^{\frac{n}{g(n,\vartheta)}}, \text{ i. e. } \log g(n,\vartheta) \leqq \log g^*(n,\vartheta)$$

i.e. from (8. 7) a fortiori

$$\left| \sum_{\alpha \leqq \varphi_\nu \leqq \beta \bmod 2\pi} 1 - \frac{\beta-\alpha}{2\pi} n \right| < C \log \frac{4}{\vartheta} \cdot \frac{n}{\log g(n,\vartheta)}.$$

Hence Theorem II will indeed be entirely established once we prove it in the special case when all the roots lie on the unit circle.

Mathematics. — *On the representation of* 1, 2, ..., *N by differences*. By P. ERDÖS and I. S. GÁL. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 30, 1948.)

L. RÉDEI and A. RÉNYI called the set of integers $a_1, a_2, \ldots, a_{k(n)}$ in their paper [1]) a difference-basis with respect to n if every positive integer ν; $0 < \nu \leq n$ can be represented in the form $\nu = a_i - a_j$. Let $n^* = \min k(n)$ denote the minimal value of $k(n)$ for a given n. L. RÉDEI and A. RÉNYI proved, that

$$1^*) \qquad \lim_{n\to\infty} \frac{n^*}{\sqrt{n}} \text{ exists,}$$

$$2^*) \qquad \lim \frac{n^*}{\sqrt{n}} = \inf \frac{n^*}{\sqrt{n}} \text{ (inf denotes the greatest lower bound)}$$

$$3^*) \qquad \sqrt{2+\frac{4}{3\pi}} \leqslant \lim_{n\to\infty} \frac{n^*}{\sqrt{n}} \leqslant \sqrt{\frac{8}{3}} \text{ holds.}$$

Somewhat earlier A. BRAUER [2]) considered the similar problem of a difference-basis $a_1 < a_2 < \ldots < a_{l(n)}$ with respect to n, the elements of which satisfy the inequality $0 \leq a_i \leq n$; $i = 1, 2, \ldots, n$. In what follows difference-bases of A. BRAUER's type shall be called "restricted difference-basis with respect to n".

L. RÉDEI proposed the following question: Let $n_0 = \min l(n)$ denote the minimal number of terms of a restricted difference-basis with respect to n, minimum being meant for fixed n. Does the set of numbers $\frac{n_0}{\sqrt{n}}$ converge to a limit? Further if the limit exists, how can it be estimated from above? In this note we prove the following results:

Theorem: *If* $n_0 = \min l(n)$ *for fixed* n, *where* $l(n)$ *denotes the number of terms of a restricted difference-basis with respect to* n, *then*

$$1^0) \qquad \lim_{n\to\infty} \frac{n_0}{\sqrt{n}} \text{ exists,}$$

$$2^0) \qquad \lim \frac{n_0}{\sqrt{n}} = \inf \frac{n_0}{\sqrt{n}}.$$

$$3^0) \qquad \sqrt{2+\frac{4}{3\pi}} \leqslant \lim \frac{n_0}{\sqrt{n}} \leqslant \sqrt{\frac{8}{3}} \text{ holds.}$$

[1]) L. RÉDEI and A. RÉNYI, On the representation of 1, 2, ..., N by differences. Recueil Mathématique, T. 61, 1948.

[2]) A. BRAUER, A problem of additive number-theory and its application in electrical engineering, Journ. of the Elisha Mitchell Scientific Society, Vol. 61, pp. 55—66.

Proof: Obviously if we can prove 1°), then the inequality

$$\sqrt{2+\frac{4}{3\pi}} \leqslant \lim \frac{n_0}{\sqrt{n}}$$

follows at once from 3*). Similarly it can be seen from 2°) that

$$\lim \frac{n^*}{\sqrt{n}} \leqslant \lim \frac{n_0}{\sqrt{n}} \leqslant \sqrt{\frac{8}{3}}.$$

Namely the numbers 0, 1, 4, 6 form a restricted difference-basis with respect to $n = 6$, therefore

$$\inf \frac{n^*}{\sqrt{n}} \leqslant \inf \frac{n_0}{\sqrt{n}} \leqslant \frac{4}{\sqrt{6}} = \sqrt{\frac{8}{3}}.$$

Consequently it is sufficient to prove the statements 1°) and 2°). The following proof of these results contains a new proof of 1*) and 2*) too, only the restriction $0 \leq a_i \leq n$; $i = 1, 2, \ldots, n$ must be omitted [3]).

I. Consider a fixed value of n and denote

$$a_1 < a_2 < \ldots < n_0 \; ; \; (0 \leqslant a_i \leqslant n \; ; \; i = 1, 2, \ldots n_0) \quad \ldots \quad (1)$$

the (restricted) difference-basis with respect to n, having a minimal number of terms. Further let us have $N \geq 7\,(n+1)$ and choose the prime p such that

$$M = N - (n+1)(p^2+p+1) \geqslant 0. \quad \ldots\ldots \quad (2)$$

Later we shall determine the exact value of the prime p.

J. SINGER [4]) has proved that there exist $p+1$ integers b_k; $k = 1, 2, \ldots, p+1$ such that the differences $b_k - b_l$ represent a complete system of residues modulo $p^2 + p + 1$. We can choose these residues $b_1, b_2, \ldots, b_{p+1}$ in such a manner that

$$0 \leqslant b_1 < b_2 < \ldots < b_{p+1} < p^2 + p + 1 = m \quad \ldots\ldots \quad (3)$$

Hence if $0 \leq \nu \leq m - 1$ (ν integer) there exist two residues b_k and b_l such that either $\nu = b_k - b_l$ or $\nu - m = b_k - b_l$.

Now let us consider the integers

$$a_i m + b_k \quad ; \quad i = 1, 2, \ldots, \bar{n} \quad ; \quad k = 1, 2, \ldots, p+1 \quad \ldots \quad (4)$$

(If $0 \leq a_i \leq n$; $i = 1, 2, \ldots, n$ then according to (2) we have $0 \leq a_i m + b_k < m\,n + m \leq \mathrm{N}$). Every ν; $0 \leq \nu \leq m\,n$ is the difference of two numbers $a_i m + b_k$ and $a_j m + b_l$. In fact put $\nu = \nu_1 m + \nu_2$, $0 \leq \nu_1 \leq n-1$, $0 \leq \nu_2 \leq m - 1$. If ν_2 has a representation $\nu_2 = b_k - b_l$ then a_i and a_j shall be choosen so that $\nu_1 = a_i - a_j$. Consequently we obtain a

[3]) Our proof is similar to that of RÉDEI and RÉNYI.

[4]) J. SINGER, Trans. Amer. Math. Soc. 1938, T. 43, pp. 377—385, and VIJAYARAGHAVAN–S. CHOWLA, Proc. Nat. Acad. Sci. India, Sect. A. T. 15, 1945, p. 194.

representation $\nu = (a_i m + b_k) - (a_j m + b_l)$. If however $\nu_2 - m$ can be represented in the form $b_k - b_l$ then $\nu = (\nu_1 + 1)m + (\nu_2 - m)$ where $\nu_1 + 1 \leq n$. Consequently there exists a pair a_i, a_j with the property $\nu_1 + 1 = a_i - a_j$. Thus $\nu = (a_i m + b_k) - (a_j m + b_l)$. Taking all these facts into account, it follows that the set of the integers $a_i m + b_k$ in (4) is a restricted difference-basis with respect to mn.

Finally we consider the integers

$$0, 1, 2, \ldots, [\sqrt{M}], N, N-[\sqrt{M}], N-2[\sqrt{M}], \ldots, N-([\sqrt{M}]+1)[\sqrt{M}]. \quad (5)$$

(Every one of these numbers satisfies the condition $0 \leq \nu \leq N$.) Obviously we can represent every satisfying $N - [\sqrt{M}]([\sqrt{M}] + 2) \leq \nu \leq N$ as the difference of two members of the set (5). Taking into account the inequality $[\sqrt{M}] > \sqrt{M} - 1$ we obtain from (2) that

$$N-[\sqrt{M}]([\sqrt{M}]+2) < N-(\sqrt{M}-1)(\sqrt{M}+1) = N-M+1 = nm+1$$

and thus $N - [\sqrt{M}]([\sqrt{M}] + 2) \leq mn$. Consequently every ν satisfying $mn \leq \nu \leq N$ is the difference of two members of the set (5).

Therefore every ν; $0 \leq \nu \leq N$ is the difference of two integers of the sets (4) and (5) respectively. That is to say, the union of the sets (4) and (5) gives a restricted difference-basis of N. The sets (4) and (5) having $\bar{n}(p+1)$ and $2[\sqrt{M}] + 2$ terms respectively, we obtain

$$N_0 \leqslant n_0(p+1) + 2[\sqrt{M}] + 2. \quad \ldots \ldots \quad (6)$$

II. Hitherto we have for p and N only the restrictions $N \geq 7(n+1)$ and the inequality (2). Now we shall determine the exact value of the prime p. An immediate consequence of the prime number theorem is the following fact: If $\delta > 0$ and $x \geq x(\delta)$, there exists a prime such that $x \leq p < (1+\delta)x$. Therefore $x^2 \leq p^2 < (1+\delta)^2 x^2$ and thus

$$x^2 + x + 1 \leq p^2 + p + 1 < (1+\delta)^2 (x^2 + x + 1).$$

Let us denote

$$(1+\delta)^2 (x^2 + x + 1) = \frac{N}{n+1} \text{ and } (1+\delta)^{-2} = 1 - \frac{\varepsilon^2}{36}.$$

Consequently if $\varepsilon > 0$ is an arbitrary small fixed number, there exists a p such that

$$\left(1 - \frac{\varepsilon^2}{36}\right) \frac{N}{n+1} \leqslant p^2 + p + 1 < \frac{N}{n+1}$$

if only $N \geq N_1(\varepsilon, n)$. Thus $0 < M = N - (n+1)(p^2 + p + 1) \leq \frac{\varepsilon^2}{36} N$ that is to say we can choose p such a manner that

$$2[\sqrt{M}] \leqslant 2\sqrt{M} < \frac{\varepsilon}{3}\sqrt{N}, \quad \ldots \ldots \quad (7)$$

if only $N \geq N_1(\varepsilon, n)$. According to (2) we have $N \geq mn = n(p^2 + p + 1) > np^2$ i.e.

$$p < \frac{\sqrt{N}}{\sqrt{n}} \quad \ldots \ldots \ldots \ldots \quad (8)$$

if $N \geq 7\,(n+1)$. Taking into account that $\bar{n} \leq n$ and the fact that n and $\varepsilon > 0$ are fixed, we have $n_0 < \frac{\varepsilon}{3}\sqrt{N}$ if only $N \geq N_2(\varepsilon, n)$.

Consequently according to (6), (7) and (8) it follows

$$N_0 < \frac{n_0}{\sqrt{n}}\sqrt{N} + \frac{\varepsilon}{3}\sqrt{N} + 2 + n_0 < \sqrt{N}\left(\frac{n_0}{\sqrt{n}} + \varepsilon\right)$$

i.e.

$$\frac{N_0}{\sqrt{N}} < \frac{n_0}{\sqrt{n}} + \varepsilon \quad \ldots \ldots \ldots \ldots \quad (9)$$

for arbitrary small, fixed $\varepsilon > 0$, if only $N \geq N_3\,(\varepsilon, n)$.

III. From the inequality (9) we have at once the estimate

$$\overline{\lim}\,\frac{N_0}{\sqrt{N}} \leqslant \frac{n_0}{\sqrt{n}} + \varepsilon$$

for arbitrary positive ε. Thus it follows

$$\overline{\lim}\,\frac{N_0}{\sqrt{N}} \leqslant \frac{n_0}{\sqrt{n}}$$

and since the integer n is arbitrary we have

$$\lim \frac{N_0}{\sqrt{N}} \leqslant \inf \frac{n_0}{\sqrt{n}} \leqslant \underline{\lim}\,\frac{N_0}{\sqrt{n}}.$$

Therefore

$$\overline{\lim}\,\frac{N_0}{\sqrt{N}} = \underline{\lim}\,\frac{N_0}{\sqrt{N}} = \lim \frac{n_0}{\sqrt{n}} = \inf \frac{n_0}{\sqrt{n}}.$$

Thus 1°) and 2°) is proved, the proof of 1*) and 2*) is clearly the same except that the condition $0 \leq a_i \leq n$ has to be omitted.

Astronomy. — *Solar centre-to-limb variations of the profiles of H_α—H_ζ.* By C. DE JAGER. (Communicated by Prof. M. MINNAERT.)

(Communicated at the meeting of October 30, 1948.)

Resumo. En ĉi tiu noto ni komunikas la rezultojn de fotografaj mezuradoj de la intensec-profiloj pri la linioj H_α, H_β, H_γ, H_δ kai H_ζ de la sunspektro, en diversaj distancoj de la centro de la sundisko. Ni aldonis koncizan komparon de ĉi tiu materialo kun tiu, kiun publikigis aliaj observantoj.

Abstract. In this note we communicate the results of photographic determinations of the intensity profiles for the lines H_α, H_β, H_γ, H_δ and H_ζ of the solar spectrum, at different distances from the centre of the disc. A short comparison of the material with that of other observers is given.

The hydrogen lines are in many respects very interesting lines in the solar spectrum. Their investigation may yield important information on a number of problems concerning the structure of the solar atmosphere and the process of Fraunhofer-line formation. Some of these problems will be outlined.

1. The high excitation potential of the Balmer lines and still more of the Paschen lines makes the wings of these lines extremely useful for an investigation of the deep photospheric layers. Their usefulness in this respect is only equalled by the oxygen lines at $\lambda\lambda$ 7774 and 8446 [1]. Moreover, the strength of the absorption in the wings of these lines as compared to the continuous absorption is neither dependent on the abundance of hydrogen in the solar atmosphere nor on the electron pressure; it is only a function of the temperature. Therefore the investigation of the wings is a new method to find the temperature distribution in the solar atmosphere.

2. It may be that in this case the simple process of line formation is somewhat complicated by the reemission mechanism. Considering the role of non-coherent scattering, it seems advisable to describe the formation of the strong Fraunhofer lines by the "absorption mechanism". Still there remains doubt, whether in the uppermost layers of the solar atmosphere the scattering mechanism does not become increasingly important, in comparison to the absorption. And finally it is probable that in these high atmospheric levels fluorescence and non-compensated cycles play also a role in the formation of the line profiles.

3. In the high levels, which are especially important for the formation of the cores of the Balmer lines, deviations from thermodynamic equilibrium

[1]) These lines have been investigated at Utrecht by the present writer.

may play a role. Miss ROSA [2]) showed indeed that the centre-to-limb variations of Balmer lines could only be explained by assuming that in the highest layers super-excitation of the second level of the hydrogen atom plays an important role.

4. It is further well-known that HOUTGAST [3]) could only explain the centre-to-limb variations of strong Fraunhofer lines by introducing an extinction factor in the transport formula; this can be explained physically by non-coherent scattering. For the Balmer lines, similar processes may be involved.

5. REDMAN [4]) could only explain his eclipse observations by the assumption that the solar surface is not smooth but rough. By this "roughening effect", observations at the extreme limb correspond to a heliocentric angle ϑ considerably smaller than 90°. It seems that for the explanation of the centre-to-limb variations of weak Fraunhofer lines and for the explanation of the low excitation temperatures found in the solar atmosphere this "roughening" can be introduced succesfully [5]).

As a contribution towards the solution of some of these problems at least, the present investigation had been started. Provisionally, only the direct observational results will be communicated, while the theoretical consequences will be developed in a future paper.

Observational technique.

During the months January to June 1947, the Balmer lines H_3—H_{16} and the Paschen lines P_5—P_6 were investigated by photographic photometry. The results here published refer to the lines H_3—H_8, which are the best suited for a precise investigation.

The observations were made with the solar grating spectrograph of Sonnenborgh Observatory, Utrecht. The solar image had a diameter of 67 mm. Most observations were made in the second order spectrum (dispersion 2 Å/mm) with the exception of the H_α- and some of the H_β-spectra, which were made in the first order (4 Å/mm). In all spectral regions except for H_β the intensity of stray light in the spectrograph was reduced by placing colour filters in front of the slit. For H_β no suitable filters were available.

The intensity of stray light from the central parts of the solar image, which might influence the spectra of the limb, was investigated by measuring several times the intensity of the sky-light outside the solar image at different distances from the limb. The effect of this light on the limb spectra was computed and it was shown that stray light had no appreciable effect.

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[2]) Z. Ap. **24**, 28 (1947).

[3]) Diss. Utrecht 1942.

[4]) M. N., **103**, 173 (1943).

[5]) D. S. EVANS, M. N., **107**, 433 (1947).
J. C. PECKER, Ann. d'Ap. not yet published.

Other corrections were made for Rowland ghosts while also the intensities at distances of 6,5 and 13 Å from the centre of the line were corrected if necessary (At these distances the first and second ghosts are formed).

A third correction was applied for the influence of the apparatus. A simple apparatus function (given in the mathematical form of a Voigt profile and determined by its two Voigt parameters) was determined for each of the 48 plates in order to take account of the variability of the instrumental curve due to turbulence inside the spectrograph tube. It appeared afterwards that the influence of the apparatus function was relatively small, and that it was not necessary to take into account the variations of this function from plate to plate. From all observations for one line a mean could be determined.

A complete set of observations consisted of three plates I, II, III. The spectra numbered from I, 1 to I, 8 in table I are found on the first plate; the spectra II, 1 to II, 8 are found on the second plate, each number corresponding to another distance from the centre; on the third plate were taken four spectra III, 1—4 at points near the northern limb, and four spectra III, 5—8 at the corresponding points near the southern limb. The values of cos ϑ, corresponding with these spectra are also given in Table I.

Atmospheric scintillation and driving errors of the clockwork may intro-

TABLE I.

Points on the Solar Disc investigated.

Designation of spectrum	r/R	cos ϑ
I,1	·000	1·000
I,2	·442	·897
I,3	·605	·796
I,4	·722	·692
I,5	·817	·577
I,6	·886	·464
I,7	·928	·373
I,8	·964	·266
II,1	·322	·947
II,2	·528	·849
II,3	·671	·742
II,4	·770	·638
II,5	·844	·537
II,6	·910	·415
II,7	·950	·312
II,8	·978	·209
III,1; III,5	·975	·222
III,2; III,6	·983	·184
III,3; III,7	·987	·161
III,4; III,8	·993	·123

TABLE II.

Hα Residual Intensities.

$\Delta\lambda$(Å) / $\cos\vartheta$	0	·2	·4	·6	·8	1·0	1·25	1·5	2·0	3	4	5	6	7	8	9	10	12	14	16	18	20
1·000	172	204	279	429	575	657	703	733	768	823	862	892	910	922	941	949	955	969	976	977	984	(982)
·947	(167)	(213)	(274)	435	(575)	660	711	736	773	822	868	891	907	922	927	946	957	969	978	983		
·897	195	257	331	453	599	(719)	747	774	805	849	882	929	935	944	947	952	962	973	978	986	(989)	
·849	203	226	302	(445)	586	677	727	760	793	843	874	900	920	933	943	951	955	966	972	977	980	(981)
·796	209	258	328	450	593	681	732	766	800	849	892	909	922	933	943	950	961	974	983	985	(989)	
·742	203	235	322	470	610	699	768	818	820	860	903	919	930	939	946	953	962	971	978	984	984	(991)
·692	201	224	312	480	640	725	763	786	820	873	899	922	936	943	946	955	964	969	980	988	(991)	(991)
·638	197	225	303	455	633	740	785	808	831	880	899	920	930	937	946	955	967	978	980	986	(987)	(989)
·577	218	241	323	467	657	745	793	814	850	893	921	939	947	951	956	964	973	978	987	(984)		
·537	222	241	337	506	678	768	812	823	870	903	924	938	946	953	956	964	966	975	978	981	(986)	
·464	215	247	330	513	661	775	827	849	879	909	935	946	953	958	965	966	974	977	985			
·415	197	226	305	485	692	798	841	868	889	918	942	956	962	966	970	974	978	980	982	983		
·373	227	243	341	524	724	839	875	905	915	950	959	967	975	977	978	977	979	(989)				
·312	193	224	321	492	695	805	862	888	910	937	951	956	966	966	969	974	981	981	(985)			
·266	203	230	323	518	690	847	882	907	914	936	950	961	967	976	980	979	981	987	(992)			
·222	(185)	(201)	(281)	518	676	(823)	881	899	921	954	961	970	979	981	987	(989)	(991)	(992)				
·209	215	236	327	523	728	868	906	915	936	955	964	969	972	975	978	978	980	981	(982)			
·184	(194)	(224)	(300)	(505)	(722)	(856)	(905)	(917)	(933)	958	964	973	978	981	984	985	986	(988)	(991)			
·161	(206)	(242)	(333)	526	722	886	926	926	944	965	973	974	977	(982)	(984)	986	(987)	(991)				
·123	(224)	(266)	(349)	(573)	712	868	926	929	950	976	979	985	990	992	(991)	989	(992)					

TABLE II (continued).

H_β *Residual Intensities.*

$\cos\vartheta$ \ $\Delta\lambda$(Å)	·0	·2	·4	·6	·8	1·0	1·25	1·50	2	3	4	5	6	7	8	9	10	12	14
1·000	176	239	403	562	612	647	685	718	765	824	864	890	914	925	937	955	964	(975)	(975)
·947	(179)	230	406	563	620	665	709	739	776	826	868	893	913	928	941	(945)	(945)		
.897	180	253	416	538	610	644	676	713	773	827	867	890	908	916	926	937	946	966	
·849	(172)	231	440	599	656	675	714	750	797	867	894	910	927	936	937	(945)	(945)		
796	193	247	420	583	629	669	708	718	781	839	886	911	929	943	946	955	(966)		
·742	177	223	447	608	669	700	725	760	773	842	866	897	924	936	942	946			
·692	187	256	455	636	684	701	729	771	817	862	888	909	929	934	939	946	948	966	(980)
·638	173	246	435	640	688	716	744	770	797	855	876	892	910	923	933	942	950		
·577	196	262	445	603	709	733	761	792	838	886	923	945	956	961	(966)	(964)	(976)		
·537	200	227	472	655	693	736	764	783	(811)	858	881	901	919	929	935	940	(940)		
·464	183	248	(425)	612	670	713	737	762	843	853	880	898	923	933	943	955	(949)	(969)	
·415	195	276	469	681	720	766	802	833	858	899	920	927	943	945	948	953			
·373	189	255	464	678	751	788	817	841	880	908	935	946	961	970	971	972	977		
·312	222	296	516	751	814	843	867	888	906	912	930	942	954	954	(955)				
·266	189	250	470	680	762	815	836	853	875	896	914	935	952	954	962	962	(965)	(973)	
·222	(179)	(240)	(475)	(735)	804	851	872	892	913	(922)	(950)								
·209	202	272	(503)	745	817	850	871	885	912	930	943	953	(961)						
·184	(207)	(236)	(470)	(754)	(867)	(875)	(910)	(906)	(918)	(940)									
·161	(216)	(272)	(499)	(778)	843	877	898	912	922	(930)									
·123	(212)	(298)	(483)	(781)	853	880	897	915	924	(933)	(954)								

TABLE II (continued).

H_γ Residual Intensities.

$\Delta\lambda$(Å) / cos ϑ	·0	·2	·4	·6	·8	1·0	1·25	1·50	2·	3	4	5	6	7	8	9
1·000	185	259	400	508	577	(627)	(686)	(732)	(780)	(858)	897	929	960	(979)	(989)	
·947	(183)	272	415	499	559	(612)	(652)	(722)	(779)	(852)	887	934	963	974	(982)	(989)
·897	191	261	(434)	552	615	(663)	(740)	(769)	(828)	(863)	919	951	970	978	(982)	(987)
·849	(178)	262	425	535	581	(632)	(692)	(743)	(796)	(863)	918	951	968	976	987	989
·796	189	285	449	572	627	(711)	(743)	(774)	(830)	(894)	918	947	969	976	(965)	(984)
·742	181	282	450	562	614	(661)	(707)	(764)	(826)	(893)	928	950	963	976	982	
·692	207	295	481	595	653	(699)	(745)	(786)	(841)	(912)	934	960	971	980	983	(987)
·638	194	273	472	588	(609)	(702)	(748)	(800)	(847)	(912)	936	964	978	984		
·577	215	303	495	632	694	(745)	(796)	(826)	(887)	(928)	960	984	987	(988)		
·537	185	283	508	626	681	(720)	(775)	(823)	(875)	(928)	963	972	984			
·464	235	323	538	660	(734)	(780)	(832)	(863)	(921)	(953)	960	987	993			
·415	195	(309)	548	665	(723)	(774)	(824)	(862)	(911)	(955)	965	977	990	(998)		
·373	242	(352)	607	752	(793)	(829)	(873)	(906)	(937)	(953)	970	981	(902)			
·312	214	356	580	711	(771)	(807)	(854)	(886)	(928)	(951)	970	985				
·266	250	381	644	778	(821)	(848)	(890)	(921)	(943)	(973)	985					
·222	(210)	324	624	769	(806)	(870)	(906)	(937)	(977)							
·209	197	329	614	748	(821)	(846)	(882)	(912)	(968)	(967)	977	(992)				
·184	(250)	352	675	838	(855)	(893)	(925)	(943)	(970)							
·161	(207)	(333)	666	837	(850)	(891)	(918)	(949)	(978)							
·123	(224)	(337)	661	825	867	(912)	(937)	(979)	(981)							

TABLE II (continued)

H_δ Residual Intensities.

$\Delta\lambda$(Å) / cos ϑ	·0	·2	·4	·6	·8	1·0	1·25	1·50	2	3	4	5	6	7	8	9	10
1·000	226	305	447	(561)	(615)	(654)	(696)	718	773	833	871	(892)	(912)	(931)	950	(960)	
·947	(188)	242	413	(541)	(592)	(643)	(712)	734	780	(874)	(923)	(947)	(970)	(971)	(976)		(980)
·897	231	327	481	(597)	(642)	(686)	(723)	760	807	878	906	(933)	(952)	(969)	(967)	976	
·849	198	281	431	(548)	(619)	(662)	(707)	743	792	878	958	(931)	951	961	(963)	(972)	
·796	(229)	(320)	(477)	(590)	(654)	(690)	(728)	(769)	(828)	(875)	(916)	(931)	(944)	(956)	(968)		
·742	201	294	462	(590)	(646)	(688)	(738)	750	804	878	917	938	951	(965)	(959)	(973)	
·692	(257)	328	511	(647)	(703)	(745)	(780)	807	847	900	922	(931)	(947)	(966)	(969)	(980)	
·638	208	318	471	(593)	(662)	(718)	(770)	783	837	900	937	955	963	(974)	(986)		
·577	245	(388)	566	(672)	(725)	(769)	(800)	832	869	921	(943)	(958)	(970)	(976)	(979)	(984)	
·537	253	355	527	(629)	(698)	(727)	(788)	807	848	886	919	944	(965)	(974)	(980)		
·464	(277)	(386)	(558)	(687)	(740)	(783)	(815)	(842)	(899)	(931)	(952)	(966)	(978)	(982)	(985)		
·415	264	366	544	(696)	(742)	(774)	(795)	833	874	921	943	(966)	(970)	(981)	(987)		
·373	288	407	605	(732)	(779)	(810)	(844)	862	896	952	(971)						
·312	(293)	419	615	(746)	(808)	(844)	(862)	895	919	(947)	(957)	(962)	(974)	(988)			
·266	(308)	437	679	(777)	(835)	(874)	(895)	917	932	967							
·222	316	461	642	(807)	(840)	(878)	(897)	917	944	961	(979)						
·209	299	438	655	(775)	(827)	(853)	(871)	(890)	906	(939)	(955)	(962)	(978)	(978)			
·184	308	469	694	(838)	(879)	(895)	(922)	938	955	964	978						
·161	338	479	700	(855)	(899)	(899)	(919)	924	936	966	978	(988)	(955)				
·123	345	484	695	(837)	(869)	(885)	(904)	922	944	964	(984)						

TABLE II (continued)

H_ζ Residual Intensities.

$\Delta\lambda$ (Å) / $\cos\vartheta$	·0	·2	·4	·6	·8	1·0	1·25	1·50	2	3	4	5	6	7
1·000	263	334	500	567	606	629	784	719	779	853	922			
·947	283	326	469	530	569	605	628	674	727	828	894			
·897	334	343	495	553	596	647	676	726	779	867	941			
·849	288	393	517	573	619	644	680	715	765	833	896			
·796	268	369	497	555	592	622	669	761	775	864	943	980		
·742	263	393	535	577	612	652	677	705	774	860	930			
·692	303	407	533	578	613	635	667	711	773	882	938	980		
·638	273	393	533	593	632	666	690	720	774	838	873	917		966
·577	367	509	598	627	654	674	696	737	768	853	925			
·537	304	398	610	637	660	691	721	754	786	871	939	967		
·464		485	643	682	702	728	759	795	844	948	990			
·415	323	467	674	702	718	753	775	812	821	904	951			
·374	312	486	707	728	752	773	789	811	869	942	975			
·312	351	472	683	695	720	748	763	797	848	918	951			
·266	353	453	685	724	738	763	785	805	864	932	978			
·222	418	485	758	807	823	844	863	878	899	938	960	983		
·209	400	483	704	766	779	799	820	844	867	911	962			
·184	438	516	765	806	822	836	858	881	895	926	970	967		983
·161	465	580	794	832	843	864	877	893	915	939	969			
·123	503	617	839	871	881	900	912	932	940	967	987			

All observations are made in the *red wing* of the line.

The central intensities of these profiles are uncertain.

duce deviations up to 0,1 or 0,3 mm in the position of the solar image. In the extreme limb spectra this may introduce considerable errors (e.g. an error of ± 0.3 mm in the observations at cos $\vartheta = 0{,}123$ gives values of cos ϑ of 0,070 or 0,148 respectively).

The results are given in Table II. It should be noted that all residual intensities in tables II and III are multiplied by a factor 1000. Each point is the mean of the violet and the red wing of the line and is also the mean of several plates. No points are given in Table II unless 3 measurements at least were available to take the mean value. If the mean value was formed from less than six points, the number is put between brackets. This is also the case when the differences in residual intensities between the both wings of the line exceeded 2 % without any obvious reason, or when only one wing was available for measurement.

Discussion of the results and comparison with observations of other observers.

In the literature data are found from D. S. EVANS [6]) who observed H_α at five distances from the centre of the solar disc. Moreover Professor P.

TABLE III.

Residual Intensities in H_δ from sprectra taken by J. HOUTGAST at Utrecht.

sin ϑ	$\Delta\lambda$(Å) / cos ϑ	·000	·102	·204	·307	·410	·512	·75	1·01	1·45
·00	1·00	204	244	295	391	457	530	602	636	703
·60	·80	216	267	344	388	496	558	622	674	734
·80	·60	285	338	425	500	585	649	734	771	796
·90	·436	281	348	427	524	622	680	746	794	844
·95	·312	285	368	506	614	668	708	790	833	868
·98	·199	342	378	500	624	732	772	832	869	898
·99	·135	327	383	622	734	774	807	850	887	911

sin ϑ	$\Delta\lambda$(Å) / cos ϑ	2·17	2·89	4·34	5·78	7·23	8·66	10·1	11·5	13·0
·00	1·00	769	830	872	916	933	940	959	981	991
·60	·80	807	850	910			967	969	974	983
·80	·60	871	902	936	953	962	971	979	994	994
·90	·436	904	925	942	958	964	977	987	994	
·95	·312	891	926	931	953	977	984	998		
·98	·199	929	949	955	975	973	977	986	991	
·99	·135	912	947	966	988	992				

[6]) M. N. **99**, 156 (1939).

TEN BRUGGENCATE, Göttingen, was so kind as to send us his observations, made in 1947, about at the same time as our Utrecht observations. The writer is greatly indebted to Professor TEN BRUGGENCATE for the permission to see his data before publication. These observations were made at seven distances to the sun's centre and for H_α, H_β, H_γ, and H_δ. Finally unpublished observations of H_δ, also for seven distances from the centre of the sun, made by HOUTGAST at Utrecht and reduced by the writer, were available for comparison. We kindly thank Dr. J. HOUTGAST, who generously gave us permission to make use of this observational material. These results are summarised in Table III.

When the Utrecht observations are plotted as a function of cos ϑ, the points are found to scatter by about 1 %. A comparison with the other authors shows that differences up to 10 % occur. However, it does not appear that the observations of one author are systematically too high or too low.

The Utrecht observations of H_α are in general intermediate between the results of EVANS and of TEN BRUGGENCATE. A rather striking difference is found for the H_δ line, which according to our measurements has an appreciably broader core for some values of cos ϑ than according to TEN BRUGGENCATE. For the half width of H_δ for cos $\vartheta = 1{,}0$ one finds for example: TEN BRUGGENCATE 1,32 Å; HOUTGAST 1,40 Å; this note 1,50 Å, while the Utrecht photometric atlas shows a half width of 1,65 Å.

A detailed technical discussion would be necessary in order to elucidate such discrepancies. However, it seems clear that their origin is to be explained at least partly by the uncertain extrapolation of the line profile at places where blended lines interfere.

The observational results in any case appear sufficiently well established that theoretical conclusions may be drawn in a reliable way, except for very special problems.

I am much indepted to Prof. MINNAERT, on whose instigation this work was carried out. In the reductions I have been materially aided by Mr. J. A. A. M. DAMEN STERCK.

Utrecht, Sterrewacht Sonnenborgh.

Paleontology. — *Pleistocene Vertebrates from Celebes.* II. *Testudo margae nov. spec.*. By D. A. HOOIJER. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of October 30, 1948.)

The remains of a gigantic land-tortoise dealt with in the present paper form part of a collection of Pleistocene Vertebrates made by Mr. H. R. VAN HEEKEREN, prehistorian to the Archaeological Survey of the Dutch East Indies, at Desa Beru and at Sompoh, near Tjabengè (Sopeng district), about 100 km N.E. of Macassar in S. Celebes. I am greatly indebted to Prof. Dr. A. J. BERNET KEMPERS, Head of the Archaeological Survey of the Dutch East Indies, for entrusting this valuable material to me, and to Miss D. M. A. BATE, Dr. W. E. SWINTON, and Mr. J. C. BATTERSBY, of the British Museum (Natural History) who kindly placed at my disposal the collections of recent and extinct giant land-tortoises in the Museums at Tring and London.

The Pleistocene fauna associated with the gigantic tortoise of S. Celebes contains a peculiar suid (*Celebochoerus heekereni* Hooijer), a pigmy archidiskodont elephant, a babirusa, and an anoa as noticed already in an earlier paper (HOOIJER, 1948). Stone flakes found with these fossils are identical with those of the uppermost Middle, or Upper Pleistocene Sangiran culture of Java.

Land-tortoises as large as the Pleistocene Celebean form are confined at the present day to the Galâpagos Islands on the Equator in the Eastern Pacific, and to some islands in the Western Indian Ocean. These animals are now practically exterminated by Man who found that they were good to eat. Fossil evidence shows that these giant forms once were much more widely distributed; remains of gigantic land-tortoises have been found in Tertiary and Pleistocene deposits in the Holarctic as well as the Ethiopian and Oriental Regions. How these terrestrial animals got to the oceanic islands is a moot question; it is claimed that land-tortoises are drowned within a few hours (GADOW, 1909, p. 373), but it is also stated (SIMPSON, 1943, p. 420) that they float and can survive long periods in salt water. The existence of a *Testudo* species in Patagonia before the late Tertiary land-bridge existed (SIMPSON, 1942) adds to the evidence that the distribution of *Testudo* is not dependent on and not in the main a result of land connections. Our species thus might well have reached the island of Celebes overseas from the Asiatic continent and not by any land connection.

The fossil which forms the object of the following description was embedded in a matrix consisting chiefly of calcite grains of irregular form and containing grains of quartz and also some alkaline felspar.

Testudo margae nov. spec.

Diagnosis: A gigantic species (carapace exceeding one metre in straight length). Shaft of scapula and proscapular process more compressed antero-posteriorly, coracoid facet more elongated and deviating from the vertical axis of the scapula at a more acute angle than in the Galápagos, Seychelles and Aldabra-Madagascar species of *Testudo*. Distinguished from the Mascarene tortoises by its less slenderly built scapula, and by the coracoid not being ankylosed to the scapula.

Holotype: The right scapula described and figured in the present paper.

Locality: Desa Beru, Tjabengè (Sopeng district), about 100 km N.E. of Macassar, S. Celebes.

Age: Pleistocene.

Name: I have named this species in honour to my wife.

Our evidence of the existence of a gigantic land-tortoise in the Pleistocene of S. Celebes is a scapula, of the right side. The bone consists of two pieces which fortunately fit nicely (pl. I). The proscapular process (pl. I fig. 1, lower side at the right) has broken off at its base. The coracoid was not ankylosed to the scapula; the articulating surface for it on the scapula, however, is entire. The bone shows some longitudinal fissures, one on the anterior and one on the posterior side, that are filled with matrix. Near the proximal end there are some more fissures dividing the glenoid cavity into four parts which remained, however, almost in their natural position; the fissures are only a few mm wide. Otherwise the scapula is perfect.

The shaft of the scapula is much compressed antero-posteriorly in its middle portion and widens both proximally and distally. The distal knob of the scapula (above in the figures) that reaches the carapace, is distended chiefly in its antero-posterior diameter (pl. I fig. 2 and fig. 4). It is rounded anteriorly and depressed postero-distally. At the middle of its height the scapula presents an elongated O in cross section, the lateral surface (pl. I fig. 4, and to the left, above the lip of the glenoid cavity, in pl. I fig. 1) is not wider than the medial surface (pl. I fig. 2). The posterior surface is convex transversely throughout its length, most markedly so distally, whilst the anterior surface is flattened in its lower and middle portions, becoming strongly convex above. The shaft increases in transverse diameters only in its proximal fourth, below the fracture. The medial surface (to the right in pl. I fig. 1) is concave longitudinally especially in this portion, where it passes gradually into the upper surface of the proscapular process, the inwardly and anteriorly directed projection of the scapula, only the base of which is preserved in the present specimen. The lateral surface is very slightly convex longitudinally except above the lip of the glenoid cavity which projects slightly too far because of the outward displacement of the lateral marginal fragment.

The glenoid cavity projects well over the posterior, lateral, and especially

D. A. HOOIJER: *Pleistocene Vertebrates from Celebes.* II. *Testudo margae nov. spec.*

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3

2

1

Testudo margae nov. spec., right scapula (holotype), from the Pleistocene of Desa Beru, Tjabengè (Sopeng district), about 100 km N.E. of Macassar, S. Celebes. Fig. 1, anterior view; fig. 2, medial view; fig. 3, posterior view; fig. 4, lateral view.
All figures one-half natural size.

over the anterior border of the scapular shaft; it is distinctly concave in transverse direction (pl. I fig. 1 and fig. 3) and but slightly so in antero-posterior direction. It is elongated transversely and truncated at the medial side, where it passes abruptly into a very oblique facet descending toward the anterior side. This facet is the articulating surface for the coracoid, which bone would have made up a further part of the glenoid cavity. The coracoid surface meets the vertical axis of the scapula at an angle of 45° (pl. I fig. 2 and fig. 4), and is roughened and subtriangular in shape. Its lateral margin, forming the suture with the glenoid cavity, is straight; its posterior margin that projects even more than the glenoid cavity, is convex, but the medial margin, below the base of the proscapular process, is concave in its middle part. Between the coracoid facet and the base of the proscapular process is a marked depression limited above by a ridge. The proscapular process is more compressed antero-posteriorly than the shaft of the scapula; its greater diameter exceeds that of the scapular shaft at the fracture.

I have compared the present fossil specimen with a great number of scapulae of gigantic land-tortoises from the Galâpagos Islands, the Seychelles and Aldabra-Madagascar group, and the Mascarenes (Mauritius, Rèunion or Bourbon, and Rodriguez) in the Leiden Museum, the British Museum (Natural History) at London and the Zoological Museum at Tring. The specimens are listed below, and the numbers given to the specimens refer to those in the tables 1—3 and 5.

Galâpagos group

Testudo darwini Van Denburgh

1. Male skeleton. Tring Museum, no. 62.

Testudo ephippium Günther

2. Male skeleton. Leiden Museum, cat. a.
3. Id. Tring Museum, no. 25.

Testudo nigrita Dum. & Bibr.

4. Skeleton. British Museum, no. 76.10.23.2.
5. Id. British Museum, no. 47.3.5.27.
6. Male skeleton. Leiden Museum, from C. Blazer, 27-4-1928.
7. Id. Leiden Museum, from the Rotterdam Zoo, 24-1-1928.
8. Female skeleton. Leiden Museum, from the Rotterdam Zoo, 14-4-1928.
9. Male skeleton. Leiden Museum, from the Rotterdam Zoo, 30-5-1931.
10. Id. Leiden Museum, from the Rotterdam Zoo, 31-3-1927.
11. Female skeleton. Leiden Museum, reg. no. 7438.

Testudo vicina Günther

12. Male skeleton. Leiden Museum, reg. no. 7041.
13. Id. Tring Museum, no. 136.
14. Id. Tring Museum, no. 60.
15. Id. Tring Museum, no. 61.

Seychelles and Aldabra-Madagascar group
Testudo elephantina Dum. & Bibr.

16. Female skeleton. Leiden Museum, cat. b.
17. Male skeleton. Leiden Museum, cat. a.
18. Id. Tring Museum, no. 153.
19. Id. Leiden Museum, from the Rotterdam Zoo, July 1935.

Testudo elephantina Dum. & Bibr. × *T. daudinii* Dum. & Bibr.

20. Male skeleton. Tring Museum, no. 142.

Testudo daudinii Dum. & Bibr.

21. Female skeleton. Tring Museum, no. 148.
22. Male skeleton. Tring Museum, no. 184.

Testudo daudinii Dum. & Bibr. × *T. gigantea* Schweigger

23. Male skeleton. Tring Museum, no. 140.

Testudo gigantea Schweigger

24. Female skeleton. Tring Museum, no. 180.
25. Male skeleton. Tring Museum, no. 176.
26. Female skeleton. Leiden Museum, reg. no. 7440.

Testudo gouffei Rothschild

27. Male skeleton. Tring Museum, no. 144.

Testudo grandidieri Vaillant

28. Female skeleton, subfossil, from Madagascar. British Museum, R. 1974, described by BOULENGER (1894).

Mascarene group
Testudo vosmaeri Schoepff

29—43. Fifteen isolated scapulae from Rodriguez, secured by the Transit-of-Venus Expedition. British Museum, nos. 76.11.1.1 → (1947. 3.4.97).

Testudo inepta Günther

44—45. A left and a right scapula from Mauritius. British Museum, nos. 39938 and 39937 (1947.3.4.99).

Testudo leptocnemis Günther

46—47. Two right scapulae from Mauritius. British Museum, nos. 76.11.4.12 and 76.11.4.13.

Tortoises are distinguished chiefly by characters of their carapace and plastron. Monographs like those of VAN DENBURGH (1914), ROTHSCHILD (1915 a and b) and GARMAN (1917) give no information as to shoulder girdle, pelvis, or limb bones. The authors that have dealt with distinguishing characters of the scapula of gigantic land-tortoises will be cited below.

In his first paper on the Galápagos giants GÜNTHER (1875, pp. 265 and 279) calls attention to differences between the scapulae of *T. elephantopus* and *T. vicina* respectively. While in the first mentioned species the angle at which the scapula and the proscapular process (named acromium by GÜNTHER) meet is very obtuse, and the cross section of the scapular shaft in the middle is trihedral with the lateral (anterior in GÜNTHER's description) surface convex and much wider than the medial surface, in *T. vicina* the angle formed by scapula and proscapular process is much less obtuse, and the shaft of the scapula is elliptical in cross section with both its lateral and medial surface equally convex. In a Leiden Museum skeleton of *T. vicina* (reg. no. 7041), however, the proscapular process meets the scapula at an angle even more obtuse than that in the example of *T. elephantopus* figured by GÜNTHER (l.c., pl. 44 C). The cross section of the scapula at its middle indeed is rather more elliptical than triangular in the Leiden specimen, the medial surface being not much narrower than the lateral, but in skeletons of *T. vicina* in the Tring Museum (especially nos. 60 and 61) I found the scapulae to be distinctly triangular in cross section. One or both of the broader surfaces of the scapular shaft, the anterior and the posterior surfaces, are concave transversely, whilst the lateral surface, convex or flattened, is decidedly wider than the medial surface that mostly is not more than a ridge. This is the shape of cross section of the scapula most commonly found, the elliptical form of cross section being rather exceptional. The scapula of one of the skeletons of *T. daudinii* in the Tring Museum (no. 148) is elliptical in cross section, and in another skeleton of the same species (Tring Museum, no. 184) the scapula has the common triangular cross section.

In a subsequent paper by GÜNTHER (1877, pp. 67 and 75) we find the same remarks on the scapula of the Galápagos tortoises, but also descriptions of that bone in *T. elephantina* (l.c., p. 31) and *T. "ponderosa"* (l.c., p. 37; the specimen on which this species is founded is regarded as a hybrid of *T. gigantea* and *T. elephantina* by ROTHSCHILD, 1915 b, p. 426) which resemble the scapula of *T. elephantopus* in the shape of their cross section [1]). GÜNTHER gives the angle between scapula and proscapular process as about 100° in *T. elephantina*, and as 130° in *T. "ponderosa"*. The gigantic Leiden Museum skeleton of *T. elephantina*, however, has a scapula in which this angle is 130°. Unlike the Galápagos specimens and those of *T. elephantina* the scapula of *T. "ponderosa"* has the coracoid ankylosed (GÜNTHER, l.c., p. 37). This is an exceptional phenomenon in recent giant land-tortoises but it is found in the extinct Rodriguez form *Testudo vosmaeri* (GÜNTHER, l.c., p. 58/59). GÜNTHER states that in *T. vosmaeri* the coracoid becomes ankylosed to the scapula at an early age of the animal, but found one pair of scapulae with not-ankylosed coracoids. HADDON (1881, p. 160) found 32 out of 85 scapulae of *T. vosmaeri* with ankylosed coracoids and thinks it to be a characteristic of this species for the coracoid to be very irregular in its ankylosis with the scapula.

1) DEPÉRET and DONNEZAN (1893, p. 149) state the scapula of *T. elephantina* to be elliptical in cross section.

GÜNTHER (1877, pp. 46—47) also describes three slender types of scapulae from Mauritius, one with a triangular cross section, straight proscapular process, and ankylosed coracoid (*T. triserrata* and *T. leptocnemis*), the second with the cross section also triangular but with a compressed and curved proscapular process and not-ankylosed coracoid (*T. inepta*), and the third with a sub-rectangular cross section which is regarded as anomalous. HADDON (1881, p. 156) records a scapula from Mauritius with the coracoid ankylosed as in *T. triserrata* but with the compressed curved proscapular process of *T. inepta*. GADOW (1894, p. 321/22) emphasizes the great amount of variation in the shoulder girdle bones of these tortoises and regards the specific identity of the isolated scapulae from Mauritius as uncertain.

The measurements of the 47 scapulae used by me for comparison with that of *T. margae* nov. spec. are recorded in tables 1—3. Because of the enormous variation in size I give all dimensions also as percentages of the length. The variation ranges of these percentages will be found in table 4. It is evident that the scapulae in the Galâpagos and in the Seychelles and Aldabra-Madagascar groups are much alike, and that both groups are rather different in proportions from the scapulae of the Mascarene group.

The scapulae in the Galâpagos and in the Seychelles and Aldabra-Madagascar groups differ from our fossil in the shaft and proscapular process being less compressed antero-posteriorly. The coracoid facet is relatively shorter than that in the fossil species. While in the Galâpagos scapulae the glenoid cavity is more compressed antero-posteriorly that that in the fossil, the Seychelles and Aldabra-Madagascar scapulae have the glenoid cavity only less distended transversely. In both groups, again, the coracoid facet meets the vertical axis at a greater angle (65°—80°) than in the fossil (45°).

The scapulae of the Mascarene group of tortoises are invariably more slender than that of *Testudo margae* nov. spec., and agree with the latter only in the relative antero-posterior diameters of the shaft and of the proscapular process, just two points of difference between the fossil Celebean form and the Galâpagos and Seychelles and Aldabra-Madagascar groups. In all scapulae of the Mascarene group the coracoid is ankylosed except in one out of the fifteen specimens of *T. vosmaeri* (no. 41 of the table) and in the two of *T. inepta*. The angle between the coracoid facet and the vertical axis (55°—60°) is somewhat smaller in these three specimens than that in the other groups and thus is nearer to that in the fossil.

The above comparisons show that the fossil *Testudo* from Celebes combines characters found in the Galâpagos group, the Seychelles and Aldabra-Madagascar group and the Mascarene group. It is difficult to make out whether it is most closely related to the first or to the second of these groups; the Mascarene group is most clearly distinct from our fossil.

TABLE 1.

Comparative measurements of the scapula of *Testudo margae* nov. spec. and of *Testudo* species of the Galàpagos group.

No. of specimen	Celebes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Length from lateral lip of glenoid cavity	240	266	133	142	72	79	84	84	102	117	124	128	153	218	225	258
Middle width	38	54	20	25	13	11	12	14	17	21	26	25	32	43	47	50
Middle antero-posterior diameter	21	32	13	15	10	8	8	9	11	16	18	18	17	21	30	31
Transverse diameter of glenoid cavity (to middle of coracoid suture)	63	—	32	—	—	—	21	24	28	31	35	38	40	—	—	—
Antero-posterior diameter of idem	52	51	20	22	15	11	12	15	17	20	22	24	29	32	40	49
Greater diameter of coracoid facet	62	—	21	—	—	—	14	16	20	25	28	26	36	—	—	—
Smaller diameter of idem	40	—	20	—	—	—	12	13	17	22	25	23	28	—	—	—
Greater diameter of proscapular process	ca. 45	52	20	26	14	12	14	16	19	22	24	25	27	45	40	48
Smaller diameter of idem	14	34	14	18	10	8	9	11	12	16	18	19	21	34	30	32
Angle between coracoid facet and vertical axis of scapula	45°	—	75°	—	—	—	70°	75°	70°	75°	80°	70°	75°	—	—	—
Percentages of length: Middle width	16	20	15	18	18	14	14	17	17	18	21	20	21	20	21	19
Middle antero-posterior diameter	9	12	10	11	14	10	10	11	11	14	15	14	11	10	13	12
Glenoid cavity, transverse	26	—	24	—	—	—	25	29	27	27	28	30	26	—	—	—
Idem, antero-posterior	22	19	15	15	21	14	14	18	17	17	18	19	19	15	18	19
Coracoid facet, greater diameter	26	—	16	—	—	—	17	19	20	21	23	20	24	—	—	—
Idem, smaller diameter	17	—	15	—	—	—	14	15	17	19	20	18	18	—	—	—
Proscapular process, greater diameter	ca. 19	20	15	18	19	15	17	19	19	19	19	20	18	21	18	19
Idem, smaller diameter	6	13	11	13	14	10	11	13	11	14	15	15	14	16	13	12

TABLE 2.

Comparative measurements of the scapula of *Testudo margae* nov. spec. and of *Testudo* species of the Seychelles and Aldabra-Madagascar group.

	No. of specimen	Celebes	16	17	18	19	20	21	22	23	24	25	26	27	28
	Length from lateral lip of glenoid cavity	240	136	149	196	238	215	168	240	204	141	147	178	210	153
	Middle width	38	20	20	34	44	43	32	47	36	24	25	33	52	36
	Middle antero-posterior diameter	21	15	15	27	27	33	19	36	25	16	20	20	34	25
	Transverse diameter of glenoid cavity (to middle of coracoid suture)	63	33	36	—	53	—	—	—	—	—	—	42	—	—
	Antero-posterior diameter of idem	52	21	25	32	41	43	30	54	39	23	21	26	41	37
	Greater diameter of coracoid facet	62	24	28	—	49	—	—	—	—	—	—	35	—	—
	Smaller diameter of idem	40	23	23	—	39	—	—	—	—	—	—	32	—	—
	Greater diameter of proscapular process	ca. 45	22	25	40	34	45	26	47	36	26	28	29	40	38
	Smaller diameter of idem	14	12	12	24	25	30	16	26	28	16	19	19	22	21
	Angle between coracoid facet and vertical axis of scapula	45°	75°	65°	—	70°	—	—	—	—	—	—	70°	—	—
Percentages of length	Middle width	16	15	13	17	18	20	19	20	18	17	17	19	25	24
	Middle antero-posterior diameter	9	11	10	13	11	15	11	15	12	11	14	11	16	16
	Glenoid cavity, transverse	26	24	24	—	22	—	—	—	—	—	—	24	—	—
	Idem, antero-posterior	22	15	17	16	17	20	18	23	19	16	14	15	20	24
	Coracoid facet, greater diameter	26	18	19	—	21	—	—	—	—	—	—	20	—	—
	Idem, smaller diameter	17	17	15	—	16	—	—	—	—	—	—	18	—	—
	Proscapular process, greater diameter	ca. 19	16	17	20	14	21	15	20	18	18	19	16	19	25
	Idem, smaller diameter	6	9	8	12	11	14	10	11	14	11	13	11	10	14

TABLE 3.

Comparative measurements of the scapula of *Testudo margae* nov. spec. and of *Testudo* species of the Mascarene group.

	No. of specimen	Celebes	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
	Length from lateral lip of glenoid cavity	240	73	81	87	93	94	100	100	102	105	109	121	123	126	141	198	115	162	121	142
	Middle width	38	8	8	9	11	11	10	11	12	13	14	14	13	14	16	22	14	20	18	21
	Middle antero-posterior diameter	21	7	6	8	8	8	9	10	9	9	11	11	10	10	13	19	10	18	11	15
	Transverse diameter of glenoid cavity (to middle of coracoid suture)	63	—	—	—	—	—	—	—	—	—	—	—	—	25	—	—	23	35	—	—
	Antero-posterior diameter of idem	52	11	12	12	14	15	13	16	16	15	14	17	16	17	20	26	18	26	19	23
	Greater diameter of coracoid facet	62	—	—	—	—	—	—	—	—	—	—	—	—	20	—	—	21	34	—	—
	Smaller diameter of idem	40	—	—	—	—	—	—	—	—	—	—	—	—	13	—	—	17	21	—	—
	Greater diameter of proscapular process	ca. 45	10	9	12	15	13	13	15	16	15	15	16	16	16	20	24	16	25	18	23
	Smaller diameter of idem	14	4	4	5	6	6	6	7	6	8	8	9	8	8	9	14	8	13	10	10
	Angle between coracoid facet and vertical axis of scapula	45°	—	—	—	—	—	—	—	—	—	—	—	—	60°	—	—	55°	55°	—	—
Percentages of length	Middle width	16	11	10	10	12	12	10	11	12	12	13	12	11	11	11	11	12	12	15	15
	Middle antero-posterior diameter	9	10	7	9	9	9	9	10	9	9	10	9	8	8	9	10	9	11	9	11
	Glenoid cavity, transverse	26	—	—	—	—	—	—	—	—	—	—	—	—	20	—	—	20	22	—	—
	Idem, antero-posterior	22	15	15	14	15	16	13	16	16	14	13	14	13	14	14	13	16	16	16	16
	Coracoid facet, greater diameter	26	—	—	—	—	—	—	—	—	—	—	—	—	16	—	—	18	21	—	—
	Idem, smaller diameter	17	—	—	—		—	—	—	—	—	—	—	—	10	—	—	15	13	—	—
	Proscapular process, greater diameter	ca. 19	14	11	14	16	14	13	15	16	14	14	13	13	13	14	12	14	15	15	16
	Idem, smaller diameter	6	5	5	6	6	6	6	7	6	8	7	7	7	6	6	7	7	8	8	7

TABLE 4.

Variation ranges of dimensions of scapulae as percentages of their length in *Testudo margae* nov. spec. and other *Testudo* species

	Celebes	Galápagos group	Seychelles Aldabra-Madagascar group	Mascarene group
Middle width	16	14—21	13—25	10—15
Middle antero-posterior diameter	9	10—15	10—16	7—11
Glenoid cavity, transverse	26	24—30	22—24	20—22
Idem, antero-posterior	22	14—21	14—24	13—16
Coracoid facet, greater diameter	26	16—24	18—21	16—21
Idem, smaller diameter	17	14—20	15—18	10—15
Proscapular process, greater diameter	ca. 19	15—21	14—25	11—16
Idem, smaller diameter	6	10—16	8—14	5— 8

Since we know nothing as yet of the carapace or plastron of the fossil Celebean tortoise it is only possible to give estimates as to the actual size of this form. However, even the exact length of the carapace would be of limited value only because giant tortoises continue to grow during a considerable time of their life, though most rapidly so while they are young (TOWNSEND, 1931, p. 461). ROTHSCHILD (1915 a, p. 404) writes that a very large male of *Testudo darwini*, at least a hundred years old, was growing between the scutes to the day of his death. In table 5 the lengths of the

TABLE 5.

Scapula and carapace length in various individuals of giant species of *Testudo*.

	Scapula length (mm)	Straight carapace length (cm)	Scapula/ carapace ratio
Testudo darwini (no. 1)	266	123	0.22
Testudo vicina (no. 13)	218	98	0.22
Id. (no. 14)	225	114	0.20
Id. (no. 15)	258	117	0.22
Testudo elephantina (no. 19)	238	106	0.22
Testudo elephantina × *T. daudinii* (no. 20)	215	116	0.19
Testudo daudinii (no. 22)	240	133	0.18
Testudo daudinii × *T. gigantea* (no. 23)	204	103	0.20
Testudo gouffei (no. 27)	210	116	0.18
Testudo grandidieri (no. 28)	153	97	0.16

scapula and carapace are given for nine giant specimens of *Testudo* in which the scapula is longer than 200 mm, as well as those of the subfossil specimen of *T. grandidieri* from Madagascar in the British Museum. The ratio scapula length/carapace length is seen to vary from 0.22 in dome-shaped forms like *T. elephantina* to 0.16 in *T. grandidieri* which has the most depressed shell of any gigantic land-tortoise (ROTHSCHILD, 1915 b,

p. 437). We do not know whether *Testudo margae* nov. spec. has a dome-shaped or a depressed carapace but we may safely accept that the straight carapace length of the individual to which the Beru scapula has belonged was more than one metre. If dome-shelled like *T. elephantina* with a scapula/carapace length ratio of 0.22 the straight carapace length of the Celebean individual would be 109 cm, and if depressed-shelled like *T. grandidieri* with a scapula/carapace length ratio of 0.16 the carapace would even measure 150 cm in straight length which exceeds the maximum carapace length recorded in any living land-tortoise (the male *T. daudinii* purchased by Lord ROTHSCHILD, straight carapace length 52.25 inch or 133 cm (ROTHSCHILD, 1928, p. 660), now in the Tring Museum (no. 184); it is no. 22 of my list).

Fossil remains of gigantic land-tortoises may indicate animals even of larger dimensions. LYDEKKER (1885, p. 159) estimates the carapace length of *Testudo atlas* (Falconer et Cautley) from the Lower Pleistocene of the Siwaliks of India to be about 8 feet (244 cm). A reconstructed carapace exhibited in the British Museum and made onder FALCONER's supervision measures more than 250 cm. The most complete specimen of this form is in the American Museum of Natural History (BROWN, 1931) and is much smaller; the carapace length being about 180 cm. VON KOENIGSWALD (1935, p. 195) records a humerus 60 cm in length from the Upper Pliocene or Lower Pleistocene of Java as probably belonging to the gigantic Siwalik form, the humerus of which indeed is probably about that long (LYDEKKER, 1885, p. 160).

Gigantic land-tortoises have been recorded from the Pleistocene of East Africa by LEAKEY (1935) who states the animal to be ca. 6 feet long, and by ARAMBOURG (1948, p. 468). Large forms occur also in the Oligocene Vertebrate fauna of the Fayum in Egypt; one shell of *Testudo ammon* Andrews (1908, p. 284) is 88 cm long.

GARDNER and BATE (1937) have recorded remains of a giant land-tortoise from the Pleistocene of Palestine. During my recent visit to the British Museum (Natural History) Miss D. M. A. BATE showed me a humerus from Bethlehem which has a length of 42 cm. This bone is almost one-half longer than the humerus of the skeleton of *Testudo elephantina* in the Leiden Museum, 106 cm in straight carapace length. The humerus of the skeleton of *T. grandidieri* in the British Museum, 97 cm in carapace length, however, is only 21 cm long. The carapace of the individual to which the Bethlehem humerus has belonged thus even might have been about two metres in length.

A description of remains of giant land-tortoises from the Pleistocene of Malta by ADAMS permits of a closer comparison with *Testudo margae* nov. spec. In *T. robusta* Adams (1877, p. 179, pl. V figs. 2, 2a) the articulating surface for the scapula on the coracoid measures 45 by 40 mm. In the smaller form, *T. spratti* Adams (l.c., p. 180, pl. VI figs. 3, 3a) the coracoid facet on the scapula is 31 by 26 mm, or only about one-half as large as that in the type specimen of *Testudo margae* nov. spec. (62 by 40 mm). It will be observed that in the Maltese forms the coracoid facet is less elongated in shape than that in the Celebean species; the scapulae of the recent Galápagos and Seychelles tortoises differ from the latter by the same character. Additional remains of giant tortoises from Malta have been recorded by BATE (1914, p. 101, 1935, p. 250), and the same authoress has discovered and described a giant species (*Testudo gymnesicus* Bate) in the island of Menorca (BATE, 1914) the scapula of which is not preserved.

From France we have *Testudo gigas* Bravard from the Upper Oligocene of Bournoncle-St.-Pierre (Allier) the longest carapace of which measures 80 cm (GERVAIS, 1859, p. 436), *T. leberonis* Depéret from the Pontian of Mt. Lebéron 150 cm in length, and *T. perpiniana* Déperet from the Astian of Perpignan (carapace length 120 cm) reported upon by DEPÉRET and DONNEZAN (1893). Remains of a gigantic *Testudo* species have been found in the neighbourhood of Zürich in Switzerland (PEYER, 1940). ARAMBOURG and PIVETEAU (1929, p. 76) and SZALAI (1931) describe remains of giant land-tortoises from the Pontian of Salonica and from that of Samos respectively.

The Canary Islands yield fossil remains of *Testudo* the carapaces of which are about 50 cm and 80 cm long (AHL, 1925, BURCHARD and AHL, 1927).

Large fossil North American forms have been dealt with by HAY (1908) and others.

Gigantic land-tortoises survived up to the present day only on some oceanic islands devoid of any but small and harmless Mammals, but their once world-wide distribution in N. America, Europe, Africa and Asia shows that they must have been able to live under less favourable conditions too. So HAY (1908, p. 373) found that the large land-tortoises from the Lower Eocene to the Pliocene of N. America were exposed to the attacks of large carnivores. It seems, however, probable, as BATE (1914, p. 101) writes, that the extinction of a race of giant tortoises would be more easily brought about by the continued and wholesale destruction of the eggs and young. An interesting coincidence lies in the fact that in Malta, like in Celebes, the giant tortoise was associated with a pigmy elephant. While most of the insular Mammals are smaller than corresponding continental forms, among Reptiles the tendency to develop gigantic forms is about as frequently met with as that toward insular nanism (MERTENS, 1934, p. 67). It is quite imaginable, however, that *Testudo margae* nov. spec. is a small representative of *Testudo atlas* (Falconer et Cautley) from the Lower Pleistocene of the Siwaliks and perhaps Java. Whether or not the Pleistocene Celebean form eventually grew to these colossal dimensions is only to decide upon further discovery.

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Botany. — *Action de l'acide α-naphtylacétique contre la chute des fleurs et des fruits de la tomate et son influence sur la couche séparatrice des pédicelles.* (Avec Résumé en Anglais.) Par CORNELIA A. REINDERS-GOUWENTAK et FRANÇOISE BING. (Communicated by Prof. W. H. ARISZ.)

(Communicated at the meeting of October 30, 1948.)

§ 1. *Introduction.*

Dans certaines parties de l'Amérique, au climat tropical, la culture des tomates est difficile (FENNELL (2)). Souvent elle n'est possible que durant la saison humide, où elle est alors la proie facile de nombreuses maladies, spécialement de virus. Souvent dès le commencement de la saison sèche, les petits fruits, qui se trouvent formés, tombent; de nouveaux fruits peuvent encore se former parfois, mais tombent pareillement.

Selon REICHERT, MOELLER et PALTI (6) la culture des tomates en Palestine est rendue difficile durant l'été, probablement par suite des conditions simultanées de température élevée et d'humidité relative basse. (voir aussi SMITH (9)). Abstraction faite des maladies, il y a en outre une perte de récolte par suite de la chute de fleurs et de jeunes fruits. En considération de l'effet favorable obtenu sur des pommes en vue d'éviter la chute de septembre, ces auteurs traitèrent les plantes par des pulvérisations d'acide α-naphtylacétique, avec ou sans fungicides, à la concentration de 22×10^{-6}. Ils constatèrent une limitation de la chute des boutons.

Les expériences de WENT (15, 18) tendent à prouver que c'est une diminution du transport des sucres qui entraînerait ces chutes. Or les résultats communiqués par REICHERT, MOELLER et PALTI (6) indiquent que l'on pourrait y pallier par l'usage de substances de croissance.

Dans la note présente nous voulons faire connaître quelques résultats obtenus au début de l'étude du problème de la chute des fleurs et des fruits chez la tomate, que l'une de nous (BING) se proposa de poursuivre. Ces résultats se rapportent à l'étude de l'action contre la chute des fleurs et des fruits exercée par l'acide α-naphtylacétique et aux altérations morphologiques et anatomiques qui en résultent dans la couche séparatrice des pédicelles de fruits et dans les couches de tissu voisines.

§ 2. *Conditions de l'expérience.*

Les tomates furent d'abord cultivées sous châssis chez Mr le Professeur Dr Ir S. J. WELLENSIEK, qui a bien voulu nous laisser une partie de ses plantes lorsque l'une de nous (BING) arriva à Wageningen pour y travailler durant deux mois comme boursière des Echanges Culturels entre la France et la Hollande; nous prions Mr WELLENSIEK de trouver ici nos vifs remerciements.

Les plantes appartenaient à trois variétés: Vetomold 121, Tucqueen, Ailsa Craig, qui d'ailleurs se comportèrent différemment lors des expériences, Ailsa Craig se montrant la moins adaptable à un climat plus ou moins tropical, où elle présente un développement végétatif trop marqué aux dépens de la fructification.

Les plantes, dont la hauteur variait de 15 à 20 cm furent mises en pots fin avril et placées en partie dans notre serre expérimentale (que nous appellerons serre I), en partie dans les serres de Mr le Professeur Ir J. E. VAN DER STOK (serre II) et de Mr le Professeur Dr H. J. VENEMA (serre III). Nous adressons nos remerciements à Messieurs les Professeurs VAN DER STOK et VENEMA, qui ont bien voulu nous prêter une partie de leurs serres.

Fin avril les plantes avaient développé les boutons floraux de la première grappe; dans les serres II et III elles trouvèrent des conditions plus favorables à la floraison et au développement des jeunes fruits qu'en serre I, serre d'expérience où la température fut nuit et jour supérieure à 24° C.

En serres II et III, l'humidité relative fut en général de 60 à 80 % durant le premier mois, un peu plus élevée ensuite, et la température varia entre 31° C et plus le jour et un minimum nocturne toujours inférieur à 20° C (min. 9° C en serre II, max. 18° C en serre III; cette dernière serre étant une serre plus chaude que la serre II).

Lorsque les plantes eurent des fruits de 1 à 4 cm de diamètre, appartenant à la première grappe, et les fleurs de la deuxième grappe furent en partie ouvertes, elles furent traitées et immédiatement après une partie fut apportée avec des plantes témoins en serre I. Durant la journée, la température de cette serre varia selon les jours, la régulation n'étant pas possible; elle s'éleva en général à 34°, exceptés quelques jours très ensoleillés, où elle dépassa 40° C; à l'exception de deux jours, la température minimum nocturne fut de 24° C. Jusqu'au 11 juin le degré d'humidité relative varia de 75 % (jour) à 90 % (nuit), puis, à partir de cette date, de 50% (jour) à 80 % dans la nuit. Le 24 juin les expériences furent arrêtées.

Le traitement consista en une pulvérisation d'acide α-naphtylacétique à 25 mg/l appliquée avec le plus de soin possible sur les pédicelles des fruits. Les pulvérisations furent répétées parfois une à trois fois, à une semaine d'intervalle entre chaque.

§ 3. *Action sur la chute des fleurs et des fruits.*

Chez les plantes cultivées depuis le 29 avril en serre II, les premiers fruits apparurent vers le 12 mai. La fructification de cette première grappe fut complète, mais seuls 2 à 3 fruits se développèrent par grappe. WENT (14, 16, 17) signale que la plupart des variétés de tomate cultivées en Californie a besoin d'une température nocturne comprise entre 15° C et 20° C; ces températures non réalisées, les fruits ne se développent pas (WENT (13)). La température dans notre serre varia depuis le 12 mai de 11 °C à 15° C la nuit, température par conséquent inférieure à celle des expériences de WENT. Les variétés cultivées en Hollande étant adaptées à des températures plus basses que celles des régions subtropicales, cette température ne fut certainement pas trop basse. Au contraire elle fut trop élevée, puisqu'elle ne fut pas réalisée assez longtemps. La température montant dans cette même période parfois à plus de 40° C pendant la journée, la serre ne raffraîchissait pas assez et la température ne restait en général que 7 heures par nuit entre 10° et 15°. Ainsi le phénomène désigné par WENT (13) sous le nom de thermopériodicité se manifesta aussi chez les variétés Ailsa Craig, Vetomold 121 et Tucqueen, cultivées aux Pays Bas. La deuxième grappe florale se développa normalement, phénomène signalé aussi dans les expériences de WENT (13), l'initiation des fleurs étant indépendante de la température (voir aussi 10).

Lorsque les premières fleurs de cette deuxième grappe s'ouvrirent, on transporta une partie des plantes dans la serre I et laissa une partie en serre II, afin d'étudier l'effet de la température nocturne haute et basse (resp. de 24° C à 29° C et de 9° C à 15° C) sur la chute des fleurs et des fruits. Les plantes à ce moment portaient chacune 2 ou 3 fruits de 1 à 4 cm de diamètre appartenant à la première grappe et les fleurs de la deuxième grappe étaient en partie ouvertes.

Les plantes furent divisées en 2 lots: dans chaque serre un lot fut gardé comme témoin, l'autre fut traité deux fois par pulvérisation d'une solution à 25 mg/l d'acide α-naphtylacétique sur les pédicelles des fruits de la première grappe et des fleurs de la deuxième grappe.

Trois plantes de la variété Vetomold 121 furent traitées le 21 mai, dès leur arrivée en serre I, et une deuxième fois le 28 mai.

Trois plantes de la variété Tucqueen, un peu plus lente dans son développement, ne furent traitées que le 25 mai dès leur arrivée en serre chaude, et une deuxième fois le 1er juin.

Enfin trois plantes de la variété Ailsa Craig, au développement encore plus lent, furent laissées dans la serre II et traitées une première fois le 28 mai, une deuxième fois le 4 juin.

Chaque plante traitée avait un témoin (traité de l'eau pur) de même variété, laissé dans les mêmes conditions climatiques et, autant que possible, dans le même état de fructification.

En outre furent laissées en serre II trois plantes non-traitées de la variété Vetomold 121 et trois plantes non-traitées de la variété Tucqueen.

Nous avons observé tout d'abord l'épanouissement normal des fleurs de la deuxième grappe; les tiges furent alors secouées légèrement pour favoriser l'autofécondation. Néanmoins, dans les deux serres, la mise à fruit fut irrégulière, et quelques fleurs se desséchèrent sans former de fruits.

Chez les plantes traitées par l'acide α-naphtylacétique on observe le phénomène signalé dans les expériences de ZIMMERMAN et HITCHCOCK (19, 20) relatives à la formation de fruits parthénocarpiques par action de l'acide β-napthoxyacétique et des dérivés de l'acide phénoxyacétique:

en plusieurs cas la corolle ne tomba pas, mais resta dans le calice jusqu'à la fin des expériences (1e 23 juin), tout à fait desséchée et cependant résistant fermement à une petite tape du doigt.

Pour les deux serres réunies il y eut au total 58 fleurs et fruits, dont les pédicelles furent traités et 144 dont les pédicelles ne furent pas traités.

Dès le 17 juin nous avons pu observer le phénomène suivant: chez les plantes qui ne furent pas traitées par l'acide α-naphtylacétique, qu'elles soient dans l'une ou l'autre serre, des fleurs et des petits fruits commencèrent à tomber alors que chez les plantes qui furent traitées, dans l'une ou l'autre serre, aucune chute ne se produisit.

Le 24 juin, date de la fin des expériences, les 58 fleurs et petits fruits traités étaient encore tous là, tandis que chez les 144 non traités, il s'était produit une chute de 17 fleurs ou fruits en serre I et de 51 en serre II; les 144 fleurs et fruits non traités se répartissant comme suit: 41 en serre I et 103 en serre II, on peut dire que les chutes dans les deux serres furent respectivement de 41 % et 49 %.

A partir du 11 juin en serre I le degré d'humidité relative avait été abaissé à 50 % pendant la journée; on n'avait pu cependant empêcher que pendant la nuit l'humidité relative ne monta à 80 %. La raison de ce changement fut que nous n'avions pas une quatrième serre à notre disposition et que jusqu'alors il n'y avait eu nulle chute, ni en serre II (temp. élevée dans la journée, basse la nuit; humidité rel. environ 70 à 85 %), ni en serre I (temp. élevée jour et nuit; humidité rel. 80 % à 90 %). Si l'on avait pu prévoir, qu'il y eût une chute en serre II une semaine plus tard, on aurait laissé une humidité rel. élevée en serre I. Maintenant nous ne pouvons dire si les conditions premières de la serre I auraient pu empêcher une chute également chez les plantes non traitées.

Un fait est sûr, c'est que l'acide α-naphtylacétique a pu empêcher une chute, car ni en serre I, ni en serre II les plantes traitées ne perdirent de fleurs ni de petits fruits.

Un autre fait est intéressant à signaler. Les seuls fruits tombés furent les fruits tout petits par conséquent ceux de la deuxième grappe. Les deux ou trois fruits de la première grappe ne tombèrent pas, même chez les plantes non traitées. Il semble donc, que le phénomène de la thermopériodicité de WENT (13, 15, 16) ne concerne que des fruits tout petits; car pour le développement de ceux-ci une fraîcheur pendant la nuit est essentielle. Les fruits ayant déjà atteint une certaine dimension — dans nos expériences un diamètre de 1 cm — se développent bien, malgré une température constamment supérieure à 24°: le 23 juin les premiers furent mûrs, les autres le furent quelques jours plus tard.

On pourrait essayer d'expliquer ces résultats d'une autre manière. On pourrait dire qu'à cette température élevée, un transport de sucre insuffisant ne permet que la croissance de quelques fruits de la première grappe: dans les expériences de WENT et CARTER (18) la chute fut empêchée par pulvérisation de sucre. Mais au cours de notre expérience il y a des faits contradictoires à cette hypothèse. L'une de nous (BING) se proposa de poursuivre ce problème.

Nous devons ajouter que les plantes de la serre II à la fin des expériences n'étaient plus depuis une dizaine de jours dans un état sain. Les plantes, continuellement soumises aux conditions chaudes et humides de la serre I étaient bien moins atteintes par un virus. De toute manière l'état sanitaire était comparable pour les plantes traitées et non traitées; celui ci n'influence donc pas les résultats de traitement ou non-traitement par l'acide α-naphtylacétique.

§ 4. *La couche séparatrice.*

La structure morphologique et anatomique de la couche d'abscission fut étudiée chez les pédicelles non traités et traités par l'acide α-naphtylacétique. En ce qui concernait les altérations anatomiques, cette étude se limita aux pédicelles des fruits mûrs de la première grappe.

Les plantes utilisées ont déjà été signalées en § 3. On traita en outre le 4 juin d'une part 15 plantes de la serre III, transportées alors en serre I (appartenant aux trois variétés des expériences) et six plantes des variétés Vetomold et Tucqueen restant en serre III. De ces dernières plantes deux avaient déjà été traitées le 11, le 18 et le 25 mai par une solution à 25 mg/l, deux autres à une concentration deux fois moindre; deux autres étaient témoins.

Comme nous l'avons déjà signalé, les fruits, dès le début des expériences, avaient un diamètre de 1 à 4 cm. On voit alors nettement le sillon sur le pédicelle, d'abord zone de rupture (KENDALL (5)), qui changera de fonction et deviendra articulation à mâturité (SCHWARZ (8)).

Au bout d'une semaine, par conséquent après un seul traitement, nous avons observé sur le pédicelle du fruit que le sillon prenait un aspect tout à fait différent de celui des témoins (Pl. I, fig. 1 et 2). Il se produisit d'abord un gonflement de la partie immédiatement voisine du sillon, située entre celui ci et la tige de la plante. Ce bourrelet en avance sur celui qui s'observe à mâturité du fruit chez un sujet non traité est également plus développé; il blanchit, jaunit en son milieu, jusqu'à former un anneau très visible autour du pédicelle, aspect qu'on ne trouve jamais chez un sujet non traité de même âge.

Puis cet anneau régulier se modifie, la couleur blanchâtre s'accentue, tandis que la forme devient boursouflée. L'épaisseur n'est pas toujours la même tout autour du pédicelle, le gonflement prend souvent un aspect de pustules (Pl. II, fig. 5). Ces phénomènes se sont produits de la même manière dans les trois serres; la seule différence observée est une vitesse de transformation moindre dans le cas où le traitement ne fut effectué qu'une fois et dans celui où la solution utilisée ne contenait que 12,5 mg/l d'acide α-naphtylacétique.

Nous devons signaler également que chez une des plantes traitées trois fois, nous avons observé un aspect de pustules beaucoup plus prononcé, et sur la partie du pédicelle située entre l'anneau et le fruit des lignes longitudinales de petites protubérances blanchâtres; celles-ci étaient également dues à des proliférations de l'écorce et de l'épiderme.

L'aspect de la couche primaire d'abscission à la fin de l'expérience était donc devenu bien différent de celui observé chez les pédicelles non-traités. En outre, tandis que celle-ci, devenue articulation, reste assez fragile pour permettre à mâturité suffisante la cueillette du fruit à cet endroit, en appuyant parfois seulement du bout du doigt sur cette zone, l'articulation des pédicelles traités devient très dure. Cette région est extrêmement rigide et résistante; il est impossible de casser le pédicelle à cette hauteur. Aussi lorsque l'on exerce sur la tomate mûre un très léger effort pour la cueillir, se détache -t- elle aisément de son pédicelle à la base même du fruit, laissant la petite couronne carpellaire fixée au pédicelle. Il ne s'agit peut-être là que de l'effet d'une concentration trop élevée en acide α-naphtylacétique, effet qu'on a observé également chez des pommes et des poires traitées en vue d'empêcher la chute de septembre. Nous n'avions pas assez de sujets en expérience pour dire si cette abscission du fruit laisse toujours la tomate intacte ou si elle produit parfois des déchirures de la base du fruit, portes d'entrée aux moisissures, particulièrement lorsque les tomates doivent voyager.

Le traitement par hormone végétale de croissance a empêché ainsi la rupture dans la couche séparatrice. Même chez les pédicelles des fleurs, à un stade où la couche fonctionne encore comme zone séparatrice, le traitement ne causa pas de chute. Ceci est en accord avec les faits signalés par WARNE (11) et WASSCHER (12) sur la chute de fleurs chez le lupin et le bégonia, mais est un peu en contradiction avec l'opinion commune sur la première chute de pommes en juin.

En ce qui concerne la structure anatomique, nous pouvons signaler les faits suivants.

Nous avons examiné 10 pédicelles de tomates mûres qui avaient été traitées par l'acide α-naphtylacétique et 8 pédicelles témoins, en coupes longitudinales et transversales, sans coloration ou traitées par la phloroglucine et l'acide chlorhydrique. L'étude suivante concerne le niveau de la couche séparatrice et les tissus avoisinants dans les variétés Vétomold 121 et Tucqueen; nous n'avons pas de différences à signaler entre les deux variétés et par conséquent nous pouvons les discuter ensemble.

Pédicelles de tomates mûres non traités (Pl. I, fig. 3). Au niveau de l'articulation, la partie la plus épaisse du pédicelle, le diamètre varie de 3,0 à 4,0 mm. Dans l'écorce la couche séparatrice apparaît sous forme de cellules au lumen réduit (Pl. I, fig. 3, a), aux parois épaissies et collenchymateuses. Ce caractère des parois n'est pas signalé par SCHWARZ (8): sans doute cet auteur n'a examiné que des coupes longitudinales, où ces phénomènes ne sont pas frappants. Déjà dans les pédicelles jeunes la couche ne se trouvait pas toujours exactement perpendiculaire à l'axe longitudinal, mais était située un peu obliquement, condition accentuée encore lorsque la couche est devenue articulation. En chaque coupe transversale on ne retrouve alors que des groupes plus ou moins petits des cellules de la couche, selon la position plus ou moins oblique de celle-ci. La dilatation est effectuée alors essentiellement par les cellules normales de l'écorce, les petites cellules de la couche y participent, mais à un degré moindre. Lorsque la couche se trouve perpendiculaire à l'axe longitudinal, la dilatation est effectuée par les cellules euxmêmes de la couche, mais on retrouve alors par-ci, par-là des petits groupes de cellules non dilatées de la couche. En coupe longitudinale la zone d'articulation dans les pédicelles mûrs se présente encore sous forme d'une couche pluricellulaire d'éléments aplatis longitudinalement. A la place des cellules sous-épidermiques de parenchyme chlorophyllien qui, au voisinage de la couche séparatrice, sont normalement allongées radialement, se trouvent des cellules collenchymateuses, chlorophylliennes, dont le grand axe est dirigé tangentiellement.

Les fibres de sclérenchyme du péricycle, disposées au voisinage de la couche à intervalles petits et réguliers, se retrouvent dans la couche, à la fois collenchymateuses et sclérenchymateuses.

Le liber externe ne subit pas de modifications à la hauteur de la couche, comparé au liber au voisinage de celle-ci.

Le bois est continu et lignifié, à l'exception parfois de quelques bandes radiales (Pl. I, fig. 3, *b*), interrompues par endroits dans leur longueur. KENDALL (5) et SCHWARZ (8) signalent une discontinuité de la lignification du bois au niveau de la zone d'articulation; KENDALL ajoute que le bois formé par la zone génératrice est lignifié, mais que cette couche néoformée n'est que très mince. Nous n'avons observé ces conditions que lorsque les tomates sont très jeunes encore. Longtemps avant le stade de mâturité, cette discontinuité disparaît par suite d'une lignification attaquant les tissus restés cellulosiques au niveau de la couche séparatrice, à l'exception parfois des bandes radiales interrompues, signalées plus haut. Le bois formé par la zone génératrice se présente sous forme d'une couche épaisse dans la couche séparatrice et à son voisinage; jamais elle n'est une couche mince.

Le liber interne est entouré en partie ou totalement par des cellules lignifiées provenant du bois et de la moelle (Pl. I, fig. 3, *c*). En outre, dans la couche séparatrice le liber interne s'est épaissi par des divisions du tissu cambial interne et des cellules de la moelle, de sorte qu'il en résulte en sens radial deux ou trois parties de liber arrondies (voir aussi Pl. I, fig. 4, *a*, coupe transversale d'un pédicelle traité, où l'on trouve le même phénomène). Cet accroissement de liber manque au voisinage de la couche séparatrice ainsi que dans le reste du pédicelle.

La moelle se compose de cellules petites; elle a un diamètre plus grand dans la couche séparatrice qu'au voisinage de celle-ci. Aux abords de la couche, entre celle-ci et la tige, se trouve une zone centrale de cellules lignifiées, décrite en détail par SCHWARZ (voir aussi Pl. I, fig. 4, *d*). Nous nous bornerons à en faire mention. Par contre la cavité signalée par

SCHWARZ (8) à la hauteur du sillon ne se trouve pas dans les pédicelles de nos variétés.

Pédicelles de tomates mûres traités (Pl. I, fig. 4; Pl. II, fig. 5, 6, 7). Au niveau de l'articulation, la partie la plus épaisse du pédicelle, le diamètre varie de 4,5 à 6,0 mm, ce diamètre étant supérieur à celui des pédicelles de tomates mûres non traités de même taille.

Cet accroissement est dû au développement de la moelle, par néoformation de bois et prolifération des tissus extérieurs du cambium, l'écorce parfois étant devenu très volumineuse en sens radial. Ces faits sont en rapport avec les observations de l'une de nous et d'autres auteurs concernant l'écorce et la zone génératrice des arbres sous l'action de l'acide β-indole-acétique; pour la littérature voir GOUWENTAK (3, 4).

La partie du pédicelle la plus soumise à l'influence de l'acide α-naphtylacétique est sans doute la couche séparatrice et son voisinage. Or le pédicelle de part et d'autre du gonflement est un peu plus épais lui aussi chez les traités que chez les non-traités. N'ayant fait porter nos observations que sur des pédicelles mûrs, nous ne pouvons faire connaître que le résultat obtenu en fin d'expérience. La différence entre les traités et les non-traités n'étant pas très importante, on peut supposer que le diamètre du pédicelle mûr ne puisse pas dépasser une certaine grosseur. On aurait ainsi un phénomène comparable à celui que Mr et Madme BOUILLENNE (1) ont signalé et que REINDERS–GOUWENTAK (7) a confirmé, à savoir que la substance de croissance ne modifie pas le nombre total de racines formées, mais leur vitesse de néoformation. Reste ainsi l'hypothèse qu'au début des expériences, l'acide α-naphtylacétique provoque des divisions de la zone génératrice de telle manière que le résultat définitif, la grosseur plus ou moins fixe du pédicelle, soit atteinte plus tôt chez les pédicelles traités que chez les témoins. Nous pensons vérifier si cette hypothèse est fondée.

Dans l'écorce la dilatation est plus forte que dans les non traités et est effectuée parfois par les cellules normales (plus ou moins collenchymateuses), lorsque celles-ci se trouvent à la hauteur de la couche. Les cellules de la couche séparatrice peuvent participer à la dilatation, comme cela a été décrit chez les pédicelles non traités, et en général elles ont gardé leur caractère collenchymateux, bien que souvent elles se soient divisées plusieurs fois (Pl. II, fig. 6). Néanmoins les cellules périphériques de la couche séparatrice peuvent proliférer pour donner les pustules signalées plus haut en perdant leur caractère collenchymateux et en devenant des cellules parenchymateuses (Pl. II, fig. 5). De la même manière se forment les pustules au voisinage de la couche séparatrice aux dépens de cellules normales plus ou moins collenchymateuses de l'écorce.

Dans les coupes transversales nous retrouvons toujours la couche séparatrice autour du péricycle; son épaisseur est alors de six cellules environ, très petites, aux parois collenchymateuses (Pl. II, fig. 6). Néanmoins, en certains endroits, ces cellules sont allongées, ne montrant que rarement une paroi radiale de dilatation. Parfois les cellules ne se touchent que par des points de conjugaison.

Dans le cas des pustules l'ensemble de l'écorce est environ 2 à 3 fois plus épais que celui des témoins; dans le cas contraire $1\frac{1}{2}$ à 2 fois. En coupe longitudinale la zone d'articulation se présente sous forme d'une couche pluricellulaire d'éléments aplatis longitudinalement.

On ne retrouve qu'avec grande difficulté les fibres sclérenchymateuses du péricycle. Le liber ne diffère pas de celui des pédicelles non traités.

Sous l'action de la substance de croissance la zone génératrice de la couche séparatrice a produit une quantité de tissu ligneux plus grande que celle produite dans les pédicelles non traités, le bois ayant $1\frac{1}{2}$ à 2 fois le diamètre du bois des témoins.

Le bois à la hauteur de la couche séparatrice n'est presque pas interrompu; on n'y retrouve presque jamais, excepté auprès de la moelle, de bandes radiales non lignifiées décrites précédemment chez les témoins. Signalons que le bois, en ce qui concerne la lignification, ressemble plus à celui des non traités, alors que la prolifération de l'écorce est plus marquée.

Le liber interne est entouré plus ou moins complètement d'une couche de cellules lignifiées, qui, avant mâturité, appartenaient au tissu interfasciculaire de la moelle.

La moelle à la hauteur de la couche (Pl. II, fig. 7) n'est pas différente de celle des témoins à l'exception de son diamètre, qui est un peu plus large chez les pédicelles traités. On ne peut pas observer toujours la partie lignifiée (Pl. I, fig. 4, *d*) du centre de la moelle, signalée par SCHWARZ (8) et retrouvée dans nos coupes de pédicelles non traités. La partie lignifiée centrale se retrouve surtout lorsque les pédicelles traités ressemblent aux non traités en ce qui concerne le bois.

En résumé au point de vue de la structure anatomique, nous pouvons signaler que les tissus situés de part et d'autre du gonflement ont subi eux aussi l'influence de l'acide α-naphtylacétique. La substance de croissance, appliquée de la manière signalée, agit donc non seulement sur les tissus du sillon et ceux qui l'avoisinent, mais aussi sur les tissus situés de part et d'autre du gonflement.

Dans la couche séparatrice des pédicelles traités, il y a une augmentation en épaisseur de l'écorce, du bois et de la moelle. L'épaisseur du bois peut atteindre jusqu'à 2 fois celle du bois non traité. L'écorce subit une forte dilatation et est en outre beaucoup plus épaisse que celle des témoins (2 à 3 fois l'épaisseur de l'écorce des témoins). La couche des petites cellules dans la partie externe de l'écorce chez les pédicelles traités subit parfois des divisions tangentielles et radiales jusqu'en formant des pustules et leur caractère collenchymateux disparaît alors. Plus cette prolifération est marquée, plus le processus de lignification du bois ressemble à celui des non traités. Les 5 ou 6 petites cellules de la partie interne de l'écorce semblent garder toujours le caractère collenchymateux. La masse de tissu médullaire lignifié, présente dans les témoins au centre de la moelle, ne se forme pas toujours chez les pédicelles traités par la substance de croissance.

PLANCHE I.

Fig. 1. Pédicelles traités 2 fois par une solution de 25 mg/l d'acide α-naphtylacétique, photographiés 2 semaines après le dernier traitement. A la hauteur de la zone d'articulation un bourrelet s'est formé, qui est beaucoup plus prononcé que chez les non traités de la fig. 2.

Fig. 2. Pédicelles traités par l'eau pur. La zone d'articulation est normale, moins boursouflée que chez les pédicelles traités de la fig. 1.

Fig. 3. Coupe transversale d'un pédicelle non traité, à la hauteur de la couche séparatrice, laquelle en ce cas, se trouvait être perpendiculaire à l'axe longitudinale. *a.* petites cellules collenchymateuses de la couche; *b.* bandes non lignifiées dans le bois; *c.* cellules lignifiées provenant du bois et de la moelle; *d.* liber interne; à ce grossissement on ne peut pas distinguer le liber externe (*e*); *f.* cellules contenant de petites crystalles d'oxalate de calcium.

Fig. 4. Coupe transversale d'un pédicelle traité aux environs du liber interne et des tissus qui l'avoisinent. *a.* liber interne; *b.* bois; *c.* cellules lignifiées provenant du bois; *d.* zone centrale de cellules lignifiées de la moelle; *e.* cellules contenant de petites crystalles d'oxalate de calcium.

PLANCHE II.

Fig. 5. Coupe transversale d'un pédicelle traité. Prolifération de la couche séparatrice dans l'écorce, donnant une pustule. *a.* petites cellules de la couche de la partie interne de l'écorce.

Fig. 6. Coupe transversale d'un pédicelle traité, à la hauteur de la couche séparatrice dans l'écorce. La partie externe est plus ou moins dilatée; dans la partie interne des petites cellules ne sont pas dilatées; la photographie est prise de telle sorte que l'on y voit les deux couches de cellules aplaties l'une sur l'autre.

Fig. 7. Coupe transversale d'un pédicelle traité, au niveau de la couche séparatrice dans la moelle. *a.* petites cellules de la couche.

CORNELIA A. REINDERS-GOUWENTAK et FRANÇOISE BING:
Action de l'acide α-naphtylacétique contre la chute des fleurs et des fruits de la tomate et son influence sur la couche séparatrice des pédicelles.

PLANCHE I

Fig. 1.

Fig. 2.

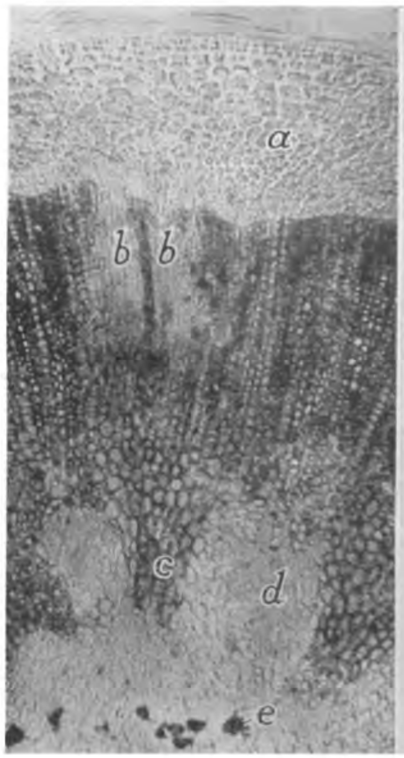

Fig. 3.

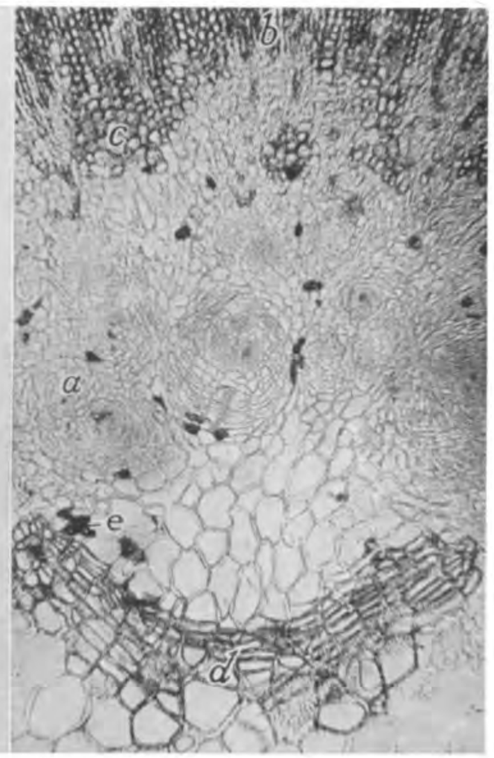

Fig. 4.

Photogr. B. W. Smit.

PLANCHE II

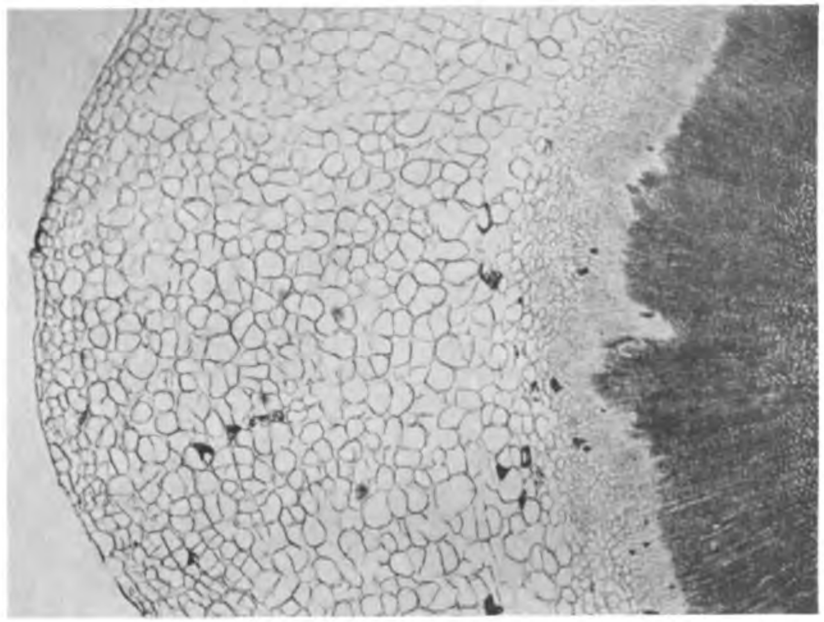

Fig. 5.

Fig. 6.

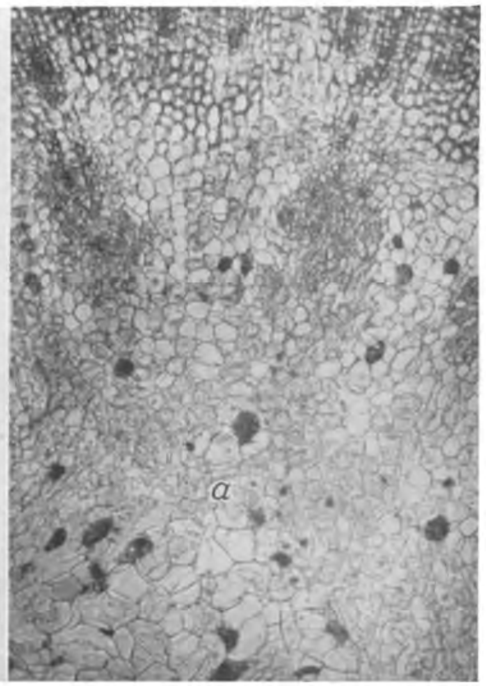

Fig. 7.

Photogr. B. W. Smit.

Summary.

Three varieties of tomato plants were grown under different conditions. The phenomenon of thermoperiodicity observed by WENT with Californian varieties holds also true for Vetomold 121, Tucqueen and Ailsa Craig, but these varieties require lower night temperatures; a night temperature of 10—15° even has to last more than 7 hours. Initiation of flowers takes place even under tropical conditions, but fruit set is poor.

A heavy fall of flowers and of young fruits (ca 50 %) occurred in all varieties tested, when grown under tropical or subtropical greenhouse conditions. It could be prevented by a once repeated spraying of the pedicels with a solution of α-naphthalene acetic acid in water, which contained 25 mg/l. Thermoperiodicity is only shown by small fruits; those which had attained a certain size — in our experiments a diameter of 1 cm — grew on even under tropical conditions without being dropped, even when they had not been treated with hormone. Two months afterwards these fruits were ripe.

The pedicels responded differently as to the development of the constriction zone, if treated or not treated. In pedicels which had been treated with hormone, this region made rapid growth and increased somewhat more in diameter (Pl. I, fig. 1 and 2) sooner becoming articulation zone than it does in the checks. Even after the fruits have ripened, the difference in size and type of the articulation zone is obvious: in the treated ones this zone is still advanced in diameter, the fruit cannot be broken off easily at this place and the original groove region is somewhat yellowish white in colour. Sometimes a callus is formed by a stormy division of the collenchyma cells of the separation layer (couche séparatrice of the figures) or of the parenchyma and collenchyma cells of the normal cortex proximal or distal of the groove (Pl. II, fig. 5), which sometimes results in the formation of outwardly visible pustules.

In treated pedicels cortex, wood and pith have a larger diameter than in untreated ones, due to the effect of the hormone on cell division in cortex, pith and the cambial tissue. But for this, the anatomy of these regions does not differ much from the anatomy of these regions in pedicels not treated by hormone, with the exception of the treated pedicels generally not showing as many unlignified areas in the wood (Pl. I, fig. 3, *b*) and just as often not the mass of lignified cells within the pith (Pl. I, fig. 4, *d*).

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CONTENTS

Biochemistry. — *Elastic-viscous oleate systems containing KCl.* I. *Measurements of the elastic properties. Dependence of period and damping on the temperature.* By H. G. BUNGENBERG DE JONG and H. J. V. D. BERG.

(Communicated at the meeting of November 27, 1948.)

1. *Introduction.*

During world-war II we started an orientating research on the viscous and elastic behaviour of oleate systems containing KCl, in which the KCl concentration is not high enough for coacervation [1]). In particular the influence of the first 6 terms of the normal primary alcohols on the behaviour of these systems was examined and it was observed that the viscous as well as the elastic properties could be strongly modified and that on the whole the same rules apply in this case as are valid for the influence of these alcohols on oleate coacervates, such as are formed from these elastic-viscous systems at higher KCl concentrations.

However, the experimental methods applied were rather primitive and we intended to investigate these systems more accurately after the war. The results will be published in a series of communications under the general title given above. In this first publication and in some following ones the elastic behaviour will be considered.

2. *Preliminary experiments with spherical vessels.*

The investigations mentioned in footnote 1) had been carried out with rather narrow cylindrical vessels. From a theoretical point of view it would be more accurate to take completely filled spherical vessels, which at the same time gives the advantage that the period becomes longer and as a consequence more exactly measurable.

It appeared further to be convenient to use a lower temperature than formerley (25° C), which caused a strong decrease of the damping (provisionally estimated from the reciprocal value of the total number of visible vibrations) under otherwise comparable circumstances.

In this first publication we restrict ourselves to oleate systems, comparable with those used formerly, and having a KCl concentration corresponding to the maximum number of visible vibrations. With such a system, prepared by mixing 2 units by volume KCl 3.8 N with 3 units by volume of a standard solution of oleate (20 gr Na-oleinicum medicinale pur. pulv. "Merck" in 1000 cm^3 H_2O + 200 cm^3 of KOH 2 N) [2]), first of

[1]) H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 849, 1011 and 1227 (1947).

[2]) The presence of the added KOH serves the purpose of preventing hydrolysis of the oleate.

all the influence of the radius of the sphere on the period of the oscillations was examined. The excitation of the vibrations happened in a very primitive way: the spherical vessels are placed on a cork-ring and this ring is suddenly turned over an angle of 60° about its vertical axis. The temperature was 18° C. With a stopwatch the time for 10 turning points (5 periods) was measured; the measurement was started after a few oscillations had already occurred.

Table I gives the average values of four measurements of the period, together with the measured capacities of the spherical reservoirs (in cm^3) and the cubic roots of these capacities.

TABLE I.

V = capacity of the spherical vessel (cm^3)	$V^{1/3}$	$10 \times T/2$ (sec)	$\frac{V^{1/3}}{10 \times T/2}$
1021	10.07	7.42	1.36
745	9.07	6.68	1.36
536	8.12	5.98	1.36
285	6.85	4.78	1.38
112	4.82	3.49	1.38

It appears from column 4 that the ratio $V^{1/3}/(10 \times T/2)$ is nearly constant, which proves that the period of the oscillations is directly proportional to the radius of the sphere.

With the same system a few days later we measured the influence of the degree of filling on the period, using the vessel of 536 cm^3 in order to prevent that the time needed for the series of measurements would become too long. In order to avoid the influences of a possible change of the temperature (which decreased from 19.3° C to 19.0° C during the experiment), at each degree of filling only two measurements were made of the time necessary for 10 turning points. The next table gives the averages of these measurements.

TABLE II.

Degree of filling in %	$10 \times \frac{T}{2}$ (sec)	Degree of filling in %	$10 \times \frac{T}{2}$ (sec)	Degree of filling in %	$10 \times \frac{T}{2}$ (sec)
10	3.41	50	5.92	80	6.49
20	4.41	55	6.12	85	6.46
30	5.04	60	6.25	90	6.27
35	5.32	65	6.37	95	6.06
40	5.53	70	6.42	100	5.96
45	5.70	75	6.51	100 + +	5.98

It will be seen that T increases with the degree of filling, a maximum occurring at a filling of 75—80 %, after which T decreases again. Filling of the neck of the vessel after the 100 % has been reached (indicated in the table as 100 ++) does not change the value of $10 \times T/2$ any further (the possible error of $10 \times T/2$ amounts to 0.02 sec). Hence in future measurements it is possible to fill the reservoirs into the neck, when experiments with completely filled spherical vessels are required.

An interesting point coming forward from the table is that the period at a degree of filling of 50 % is practically the same as at 100 % (the inevitable errors of measurement do not permit to consider the values 5.92 and 5.96 as really different). The result is what we must expect, if we assume that with this type of oscillation the deformation of the oleate system occurs in concentric spherical layers. In that case the equatorial plane is a plane of symmetry, and the period for the part of the sphere below this plane must be equal to the period for the part above, and hence also to that of the completely filled sphere. The filling of the neck of the reservoir, although slightly influencing the boundary conditions, seems to have very little effect. Experiments carried out later, with an improved method and greater accuracy, have demonstrated that T does not change when a glass tube with flat bottom is introduced into the neck, in such a way that the spherical surface becomes practically closed.

For the intended investigations of the influence of organic substances on the elastic behaviour, this is of much practical importance. It is advantageous to work with half filled spherical vessels, which makes it easy to add to the system certain admixtures, after which the reservoir is closed with the grounded glass stopper and is heavily shaken in order to dissolve the added substances completely.

3. *Influence of air bubbles on the elastic behaviour.*

Shaking introduces bubbles of air into the oleate system. The larger ones rise quickly and disappear, whereas the smaller and very small ones rise extremely slowly. For our system, after heavy shaking (which led to the formation of many small bubbles), this takes et least 6 to 8 hours at a temperature of 15° C. It is necessary to give much attention to this circumstance, as it has been found that the presence of a great number of small bubbles, and also the presence of a layer of bubbles floating on the surface of a half filled sphere, can have an appreciable influence on the elastic behaviour. For a given oleate system the value of $10 \times T/2$, after two periods of shaking of 3 minutes each, decreased from 11.75 through 11.38 to 10.68 sec. The presence of the air bubbles even has a procentually much greater influence on the damping. Whereas the system at first showed 30 visible oscillations, this number decreased through 25 until 21. When all bubbles had risen and had disappeared from the system (so that there was neither any foam on the surface), the original values for T and for the number of visible oscillations returned.

Hence when an oleate system has freshly been prepared, for which it is necessary to shake the mixture of oleate solution and of the KCl solution very heavily, one must not immediately start with the measurements, but allow the system to stand until the next morning. The reservoir is then carefully wheeled round for a short time, which procedure distributes a limited quantity of bubbles over the system, to be used as indicators of the elastic movements in the measurements.

4. *Slow heat exchange.*

Another property of the oleate system which must be taken account of in making measurements, is the very slow heat exchange between such a system and its surroundings. This is a consequence of its high viscosity, which practically prevents the appearance of convection currents, so that increase or decrease of temperature can be brought about only through heat conduction in consequence of a temperature gradient between consecutive spherical layers. It may take several hours (dependent on the volume of the sphere), before a system prepared *e.g.* at a temperature of 18° and placed in a thermostate of 15°, will have obtained the latter temperature at its centre. It is possible to speed up the heat exchange by stirring, but this must be done very carefully, as it is difficult to avoid the introduction of air bubbles, a circumstance which according to **3** is extremely dangerous. Nevertheless it is necessary to apply this method when it is desired to investigate the influence of the temperature on the elastic properties, as otherwise it is impossible to work sufficiently quickly.

The slow heat exchange has been a favourable property on the other hand in the case of measurements with very large vessels (capacity 6 and 3 litres), to be considered in details in the second communication. We did not have at our disposal a contrivance to excite the oscillations in such large vessels, while the latter are completely submersed in the thermostate. In this case we took the vessels out of the thermostate, in which they had remained for two days at a temperature of 15°, and made the measurements in the air, which had a temperature of 18°. During the relatively short duration of time necessary for the measurement (20 minutes), we may assume that there is no influence of heat exchange on the results, which thus will be the same as if the experiments had been performed within a thermostate.

5. *Rotational, quadrantal and meriodional oscillations.*

The type of oscillation, considered thus far, in which the elastic deformation was due to the relative displacement of concentric spherical layers, will be denoted as the "*rotational oscillation*". With every mode of production of this oscillation there simultaneously appeared another type, which we shall call the "*quadrantal oscillation*". When a plane through the axis of rotation is considered, the elastic deformations in the four quadrants form a symmetrical pattern (fig. 1 *A*).

Professor J. M. BURGERS pointed out that still another type of oscillation should be found, which we will call the "*meridional oscillation*" (fig. 1 *B*).

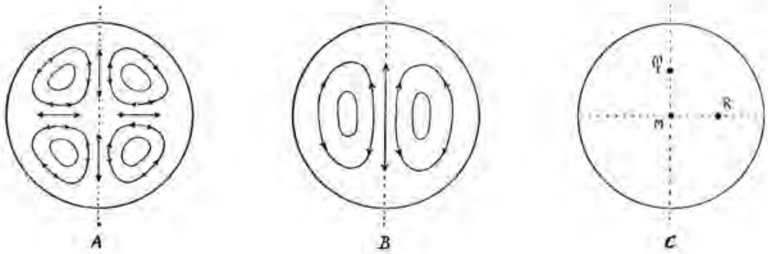

Fig. 1.

It proved to be possible to excite this oscillation by means of a particular contrivance (compare **6**).

From a theoretical point of view the quantitative comparison of the periods and logarithmic decrements of these three types of oscillation plays an important part. We come back to this subject in the next paper.

6. *Contrivances used for the excitation of the three types of oscillation.*

a) The "*turning table*" (fig. 2). In principle this is an axis carrying a wooden disc at its upper end, to which is fastened a cork ring. Spherical

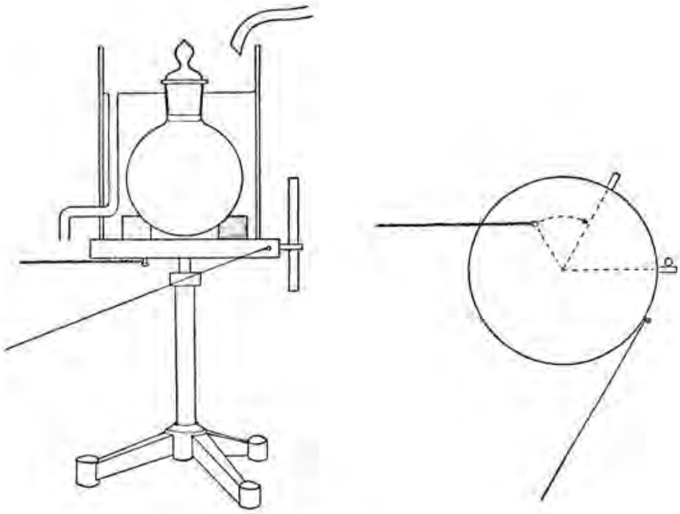

Fig. 2.

vessels with a capacity from 500 cm^3 to 6 litres can be placed on this ring [3]). A metal spiral spring, which can be extended to a variable degree, is

[3]) In the case, shown in fig. 2, in which the cork ring is fastened to the bottom of a cylindrical glass vessel (through which thermostate water flows) one can only use vessels up to a capacity of 1.5 litres.

connected to the lower side of the disc.. Two notches on the rim of the disc limit the angle of rotation to 60°. With the aid of a string the disc is turned as far as the angle of rotation desired, which already produces a certain deformation of the fluid. The string is then released and the spring draws the disc against the second notch, at the same time producing a deformation in the opposite direction. In counting the total number of visible oscillations of the ensuing motion, the first turning point which is reached, is numbered zero. Preliminary experiments had shown that the period of the oscillations is independent of the tension given to the spring, provided one does not start measuring from the first turning point, as the first oscillations have a period which may be from 10 % to 30 % too long. We usually measured the time necessary for 10 turning points, beginning with turning point no. 5 and continuing until no. 15. The oscillations are then found to be *isochronous* [4]).

Curiously enough it was found that the total number n of oscillations which could be counted with a given method of observation, likewise was independent of the tension given to the spring. The observable number of oscillations is greater when a telescope is used, than when observations are performed with the naked eye, but both numbers are remarkably constant for a given oleate system. Only in the case of a very weak initial impulse a smaller number of oscillations was found (compare **9**).

We have taken care that all measurements were performed with such a tension of the spring, that the number of observable oscillations was constant. It is not indifferent whether a heavy or a weak impulse is used in the domain of constant n. A weak impulse is advisable if it is desired to make observations on the rotational oscillation. The quadrantal oscillation, which always is superposed on the rotational one, becomes more pronounced with heavy impulses, which in certain cases can make the measurement of the rotational oscillation difficult, in particular the measurement of the damping.

b) Application of a pendulum for the excitation of the oscillation.

This apparatus (fig. 3) is mounted above the thermostate and allows the use of vessels of 500 and 750 cm^3 capacity, which are fixed in the grip A (formed of a fixed part, connected to the arm B, and a detachable part). The spherical vessels used were Jena or Pyrex "round bottoms", which were provided with a grounded glass stopper. The arm B can be moved through the screw C, which is rigidly connected to the axis D, supported by two ball bearings E and F. The spherical vessel can be fitted into the apparatus either completely filled or half filled. Behind this apparatus the

[4]) Later and more accurate measurements have shown that with certain systems with small decrement (large values of n) a larger number of oscillations must be allowed to pass before isochronous oscillations can be obtained. Whenever possible it is to be recommended therefore to determine the period between the turning points 10 to 20 or 15 to 25.

pendulum is suspended, consisting of a rod H, pivoting at its upper end on a horizontal axis, and carrying an exchangeable weight J at its lower

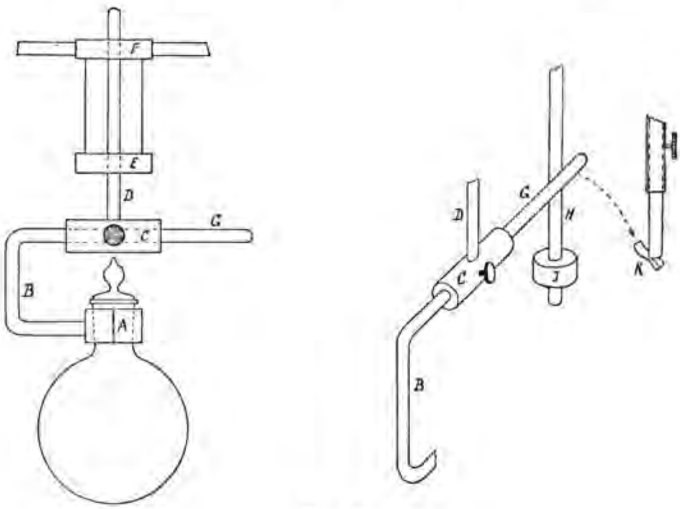

Fig. 3.

end. The end of G of the carrying rod B is put against the rod H: the pendulum is then moved out of its vertical position of equilibrium over a certain angle (read off from a scale), and is released. When it strikes against G, it causes the vessel to describe a rotational motion. Having turned through an angle of 90°, the rod G is caught by a spring clasp K; in this way the vessel is brought to rest, and the ensuing oscillations of the fluid can be measured in the same way as before. Here again it was found that the total number n of observable oscillations did not depend on the strength of the impulse. Only in the case of very weak impulses n slightly decreased.

Also in this case the quadrantal oscillation appeared in a more pronounced way when a heavy impulse was applied; with very heavy impulses the rotational oscillation even might prove to be nearly suppressed.

c) *Excitation of the meridional oscillation.*

For this purpose a glass rod is used, guided by a slightly wider glasstube, which can be lowered into the fluid (in the axis of the vessel), and can be raised again by means of a slightly heavier counterweight. In the case of the larger vessels it was necessary to move the rod up and down several times with the proper period, in order to produce a sufficient amplitude of the oscillation. It is necessary that in its position of rest the lower end of the rod must be drawn out of the spherical part of the vessel. It is important on the other hand, that the lower end of the rod shall remain immersed into the oleate system and is *not* drawn up into the air. For this purpose the neck of the vessel must be filled as far as possible with the oleate system. If the level of the liquid does not stand high enough, and if the lower end of the rod is drawn up into the air, a certain quantity of fluid will suddenly fall off the rod into the neck and causes irregularities in the meridional oscillation.

When the meridional oscillation is excited in the appropriate way, the two other forms practically do not appear. A disadvantage of the method is that it is not easy to obtain a sufficient amplitude.

7. *Measurement of the period, of the decrement and of the total number of visible oscillations.*

The measurements were carried out by observing the movement of small bubbles of air in the oleate system, by means of the telescope of a kathetometer, from a distance of about 1 meter. The telescope is focussed on that point in the interior of the vessel where the amplitude appears to be a maximum. For the rational oscillation this is in the equatorial plane, at a distance $\frac{1}{2}R$ from the axis of rotation (fig. 1 *C*, point R); for the meridional oscillation it is the centre of the spherical vessel (fig. 1 *C*, point M); and for the quadrantal oscillation it is on the axis of the vessel at a distance $\frac{1}{2}R$ from the centre (fig. 1 *C*, point G) [5]) [6]).

The measurement of the period and of the number n of visible oscillations does not require particular abilities of the observer. After having gained some experience, these quantities can be determined with a high degree of reproducibility [7]).

The measurement of the decrement, on the other hand, does not only require experience, but also special observational gifts. The telescope of the kathetometer is fitted with an ocular micrometer, which is turned horizontally for the measurement of the rotational oscillations, and vertically for the measurement of the quadrantal and the meridional oscillations. On this scale the observer must read off the positions of the image of a small

[5]) In the case of the rotational oscillation we have checked that when measuring the period, the decrement, and the total number of observable oscillations, it is indifferent on which point the telescope has been focussed, provided we do not choose a point too near to the wall of the vessel (*e.g.* at a distance of 0.6 cm or less, in the case of a vessel of 5.6 cm radius) or too near to the surface of a half filled vessel.

[6]) The positions mentioned are the most convenient, moreover, because — at least theoretically — the rotational and the quadrantal oscillations can be separated from each other in this way.

It should be mentioned that the images of the bubbles in the telescope, which always has a certain depth of focussing, do not move in straight lines, but describe elongated Lissajous figures. This is found in particular in the case of the quadrantal oscillation, for which ideal conditions are present only over a very short part of the axis of the sphere. In such a case the observer must concentrate his attention on one of two perpendicular components only.

[7]) In order to save time it is possible to determine these two quantities simultaneously. The system is brought into motion and the number of turning points is counted, while during the counting a stopwatch is pressed for instance at no. 5 and stopped at no. 15, the turning points being counted further until they are no longer observable.

oscillating bubble of air, at four consecutive turning points. As the turning points closely follow one another, the observer must be able to memorize these four readings and to interprete them subsequently in the form of numbers, which he must reproduce in good order after the measurement.

It is advisable that the observer, who also has the task of exciting the oscillation with the aid of the string, is aided by an assistant who notes down the numbers mentioned by the observer. For a full measurement of an oleate system it is necessary to observe 10 values of n, two groups of 10 observations of the time necessary for ten half periods, and two groups of ten observations, each consisting of 4 consecutive scale readings, for the determination of the decrement. It is possible to work out the averages for n and for the two groups of 10 observations of the period already during the measurements. A comparison between the mean values of the two groups of determinations of the period gives a check on the reliability of the observations.

When recording the groups of numbers representing each time four consecutive scale readings, there is just sufficient time to write down the differences between the readings 1, 2 and 3, 4. When 20 groups have been observed, the ratios of these differences (indicated in the tables as b_1/b_3) are calculated, which consequently gives 20 values for the damping ratio.

The mean of the first group of ten ratios is then compared with the mean deduced from the second group, in order to see if there is a satisfactory agreement. The record of a measurement of the elastic behaviour of an oleate system, obtained in this way, can be kept and can be used afterwards for the calculation of overall mean values and of mean errors.

With sufficient exercise the measurement and preliminary calculation of the mean values for n, $10 \times T/2$ and b_1/b_3 requires approximately $\frac{1}{2}$ hour. Hence in a single day it is possible to carry out measurements on a large number of systems, which enables us to investigate the influences of various factors.

8. *Influence of the temperature on T, b_1/b_3 and n.*

The elastic properties of an oleate system are dependent on a large number of variables. Here we restrict ourselves to a composition of 1.2 % oleate in 1.52 N KCl + 0.08 N.KOH.

We first consider measurements which had been performed with a slightly lower oleate concentration (1.18 %), with the object of determining the influence of the temperature in order to find an appropriate temperature for further work and to get information concerning temperature coefficients. For this purpose a vessel of 500 cm^3 capacity, half filled, was placed during one night in an ice chest, and measurements were carried out the following day after mounting the vessel on the turning table. The temperature was recorded every time immediately before and after the measurement of the period and of n, and the same was done immediately before and after the measurement of the damping. The quantities measured were assumed to refer to the average temperature. The reservoir gradually rises in temperature, until room temperature is reached, and it is necessary to stir the fluid gently with the thermometer during a few minutes before reading the thermometer. Measurements were also performed for temperatures above room temperature.

We shall restrict to representing the results in a diagram (see fig. 4). The value of $10 \times T/2$ has not been corrected for the damping; when this

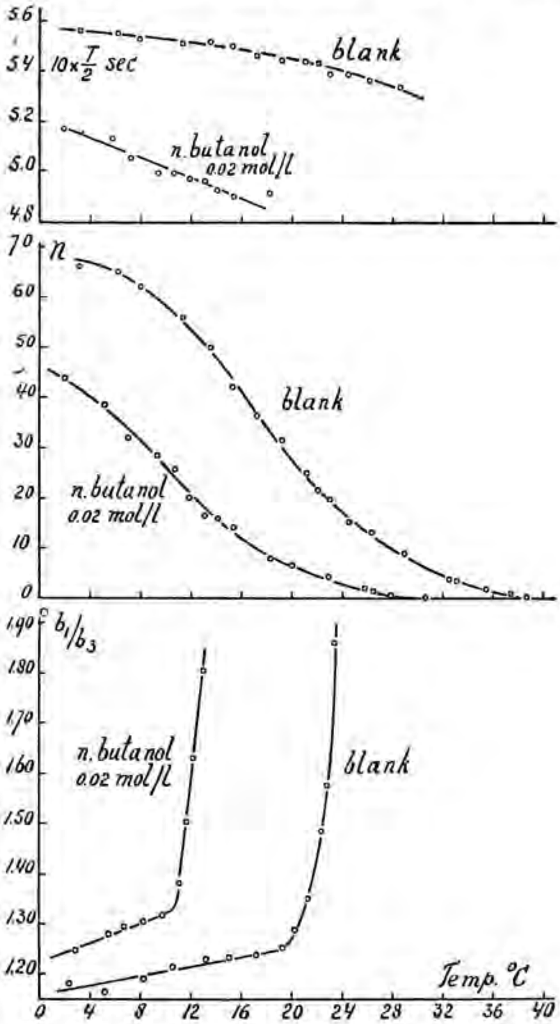

Fig. 4.

correction is introduced the curvature of the line at higher temperatures will become some what larger.

It will be seen that the period decreases only slightly when the temperature rises ($10 \times T/2 = 5{,}56$ sec at 3,2° C and $= 5{,}43$ at 19,45° C). On the contrary n is strongly dependent on the temperature (*e.g.* $n = 56{,}0$ at 11,35° and $n = 31{,}5$ at 19,45° C).

The curve for the damping presents two branches. In the domain of low temperatures (extending from 2° until 19° C) the damping increases only slightly with the temperature ($b_1/b_3 = 1{,}182$ at 2,35° C and 1,234 at 17,22° C). At temperatures exceeding this limit, however, a very rapid increase appears.

In the diagram there have been represented moreover the results obtained with an oleate system to which 0,02 mol per litre normal butyl alcohol had been added (the concentrations of oleate, KCl and KCH being unchanged). It will be seen that $T/2$ and n have decreased, b_1/b_3 has increased; apart from this there is no change in the form of the curve. The temperature at which the two branches of the b_1/b_3-curve meet has become lower (10.5°) than in the case of the curve for the system without alcohol (19.5°). We will come back to the interpretation of these curves after consideration of the measurements to be treated in the next communication.

9. *On the constancy of n.*

In our first paper on the elastic phenomena observed with oleate systems [8]) we have already drawn attention to the circumstance that n, the total number of visible oscillations, curiously enough appears to be independent of the intensity of the impulse used for producing the damped oscillation. The improved method described in the preceding sections again led to the same result. It is only in the case of very weak impulses that n becomes less.

In order to explain this fact it is necessary to suppose that the elastic deviation which leads to the appearance of the damped vibrations, automatically always assumes the same value. In this connection the following circumstance is of importance. When the oleate system in a half filled vessel has been brought into rotational motion by moving the vessel quickly around along a conical surface, and the vessel is put on the cork ring, it is seen that the fluid at first moves on in a single direction, with decreasing velocity. Suddenly, however, the whole mass comes to a standstill, and the damped oscillation sets in. We may suppose that something similar occurs in the case of the impulses, applied in the oscillation measurements. We shall come back to this point in one of our further communications of this series, in which we shall investigate the flow behaviour of the oleate systems.

10. *Correlation between n and damping.*

All results obtained with regard to the influence of various parameters (radius of the vessel, concentration of the oleate, concentration of KCl, temperature, admixtures) have shown that n decreases whenever b_1/b_3 increases and inversely. By way of example we refer to fig. 4. From this figure it is apparent, however, that there are differences

[8]) See note 1.

in the course of the two quantities; for instance, the well marked bend in the curves for b_1/b_3 (at 10.5° C for the system with 0.02 mol per litre butyl alcohol, at 19,5° C for the system without alcohol), does not show in the curves for n, which exhibit a more continuous course.

In one of the following communications we shal see that a useful picture of the influence of various organic admixtures can be obtained already from a consideration of the curves for n, so that it is possible to derive general conclusions concerning the behaviour of logarithmic decrement and relaxation time. This is of importance, as it is not always easy to find an observer who is able to perform the difficult measurement of the decrement. In such a case the determination of the period and of the maximum number of observable oscillations can help to find out the general qualitative behaviour.

11. *Some particularities of KCl containing oleate systems.*

When a reservoir containing an elastic oleate system is quickly moved around along a conical surface, taking care as far as possible to avoid the introduction of air bubbles, it is found that the system, which originally had the appearance of a clear liquid, becomes more or less turbid. When the reservoir is kept at rest, the turbidity disappears in a few seconds and the fluid again becomes clear. The intensity of the turbidity shows a marked increase with the concentration of the oleate.

The same effect can be observed in a Petri dish, in which a flat glass disc is mounted in such a way that it can be rotated in its plane (fig. 5).

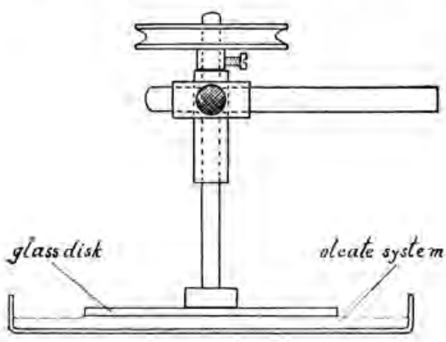

Fig. 5.

When the disc is rotated sufficiently quickly, the oleate in the narrow space between the disc and the bottom of the basin becomes turbid, whereas it remains clear outside the glass disc. When the apparatus is put under a microscope with crossed Nicol prisms, the system normally does not show any double refraction. Double refraction, however, becomes already apparent with small velocities of the glass disc. One of us (V. D. B.) intends to investigate this form of flow double refraction with the aid of more appropriate instruments.

12. *Phenomena observed upon the excitation of the rotational oscillation.* Certain oleate systems exhibit a small degree of opalescense, probably to be ascribed to a slight admixture of aniso-diametric particles. When the rotational oscillation is excited, *e.g.* with the aid of the pendulum apparatus, it is seen that a spherical wave of turbidity, detaching itself from the surface of the vessel, travels toward the centre. If the impulse is relatively weak, the wave distorts very little; in the case of stronger impulses, however, a marked distortion is observed. Some consecutive positions of this wave of turbidity are represented in fig. 6. The figures *d*, *e* and *f* may be compared with fig. 1 *A* giving the deformational movements in the case of the quadrantal oscillations.

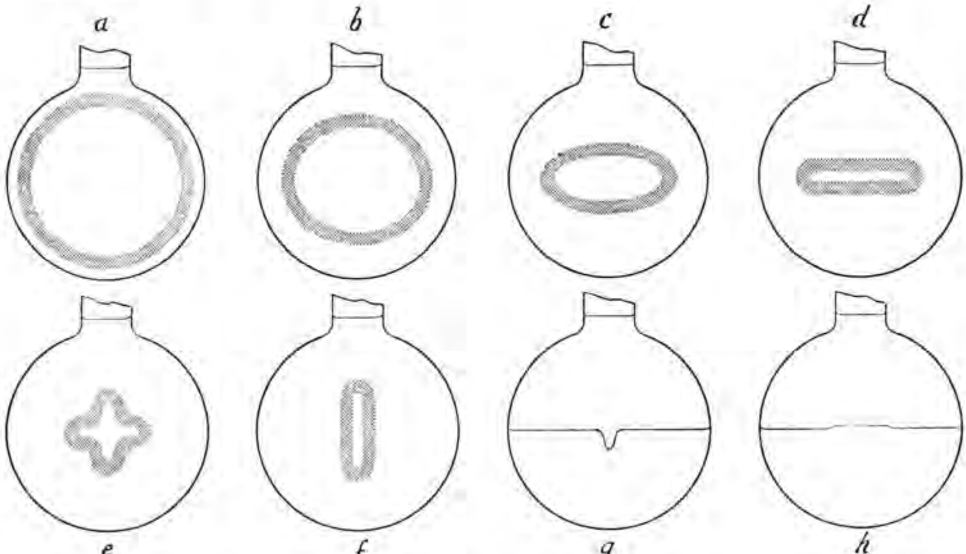

Fig. 6.

In the case of a half filled spherical vessel a surface wave can be seen, which travels towards the centre and back and even goes to and fro a few times. A dipple appears in the surface (fig. 6 *g*) at the instant corresponding to fig. 6 *f*. It is this spherical wave, moving forward and backward, which finally brings the whole system into rotational oscillating motion, while at the same time a part of the energy is given to the quadrantal oscillation, a part which becomes larger with stronger initial impulses. After stage fig. 6 *g* the surface shows a slight elevation (e.g. a disk at the centre) which may last for some seconds (see fig. 6 *h*).

Summary of Part I:

1. In a completely filled spherical vessel containing an oleate solution to which has been added KCl, three types of oscillation can be excited, denoted as the rotational, the meridional and the quadrantal oscillations.

2. The apparatus used for the excitation and the methods applied in measuring the period and the damping ratio have been described.

3. With the rotational oscillation it has been found that the period is directly proportional to the radius of the sphere, while the period for a half filled sphere is equal to that for a completely filled sphere.

4. The damping is dependent on the concentration of the KCl and has a minimum for a certain value of this concentration. Investigations have been carried out on the temperature dependence of the period and of the damping ratio with a system containing 1.2 % oleate ("Merck"), the KCl concentration being very near to that giving minimum damping, a slight amount of KOH having been added to prevent hydrolysis. The period decreases slightly with the temperature. The curve obtained for the damping ratio as a function of the temperature exhibits two branches, the ratio increasing relatively slightly with the temperature in the domain from 2°—19° C, and increasing much more strongly at temperatures above 19°.

5. Certain particularities exhibited by 1.2 % oleate systems are described (phenomena observed upon excitation of the oscillation in completely and in half filled vessels; occurrence of flow double refraction; occurrence of reversible turbidity).

Mechanics. — *Damped oscillations of a spherical mass of an elastic fluid.* By J. M. BURGERS. (Mededeling No. 59 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of November 27, 1948.)

1. *Introduction.* — Professor BUNGENBERG DE JONG had asked me if a theoretical treatment could be given which would throw light on the results of his investigations on the oscillatory movements presented by certain soap solutions [1]. Since the most striking features of his beautiful experiments are the regularity of the observed oscillations and the geometrical pattern of the motion, it is natural that one should turn to the theoretical work of LAMB on the oscillations of a viscous spheroid and on the vibrations of an elastic sphere [2]. It is the object of the following lines to consider in which way the theory developed by LAMB can be applied in a discussion of BUNGENBERG DE JONG's experimental results, and to get information concerning the phenomena responsible for the damping of the motion.

Although it is not possible to extend LAMB's classical investigations, it may be of help to the reader to substitute for his highly mathematical and rather abstract deductions a more simple and direct treatment, adapted to the particular cases investigated by BUNGENBERG DE JONG. These cases are:

a) motion in concentric spherical layers or shells;
b) axially symmetric motions in meridian planes.

It will be assumed that the amplitude of the oscillations is small, so that velocities and accelerations can be calculated by means of the partial derivatives with respect to the time. Hence if u is any component of displacement, the corresponding velocity will be $\partial u/\partial t$, the acceleration $\partial^2 u/\partial t^2$, where these two quantities refer to the same point of space as does u itself.

A few general remarks may precede the deduction of the equations.

The fact that isochronous oscillations are obtained, proves that elastic forces are operative which are linear functions of the deformations.

The general relation between the dimensions of the field of motion, that

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[1]) H. G. BUNGENBERG DE JONG, Elastic-viscous oleate systems, containing KCl (Part I), these Proceedings **51**, 1197—1210 (1948). Parts II and III will appear in the next issues.

[2]) H. LAMB, On the oscillations of a viscous spheroid, Proc. London Mathem. Soc., **13**, 51 (1881); On the vibrations of an elastic sphere, ibidem **13**, 189 (1882); On the motion of a viscous liquid contained in a spherical vessel, ibidem **16**, 27 (1884); Hydrodynamics (Cambridge 1932), art. **354** (p. 637) and **356** (p. 642). The elastic vibrations of a sphere are also treated in A. E. H. LOVE's Theory of Elasticity (Cambridge 1920), Ch. XII (p. 281).

is in the case considered the radius R of the spherical vessel, and the period T of the oscillation, can be deduced from the following argument. If all linear dimensions of a given field of motion are changed in the same ratio, so that angles and angular displacements remain unaltered, the magnitude of the deformations will remain unaltered likewise; consequently the elastic stresses per unit area will remain the same. The resulting moment, e.g. over a spherical surface, changes proportionally with R^3. As the moment of inertia of a spherical mass is proportional with R^5, it follows that the angular accelerations produced by the elastic reactions will change proportionally with R^{-2}. On the other hand, as angular displacements are not changed, angular accelerations must be proportional with T^{-2}. Hence we must conclude that the period of the elastic oscillation will be proportional to the radius R, as was found in the experiments.

If we keep to the case of oscillations which are not heavily damped, it will be evident that when viscous forces are present, depending on the rate of deformation, the viscous stresses per unit area will be proportional with T^{-1}. The moment of the frictional stresses will be proportional to $R^3 T^{-1}$; and the angular accelerations or decelerations produced by them will be proportional to $R^{-2} T^{-1}$, that is, to R^{-3}. Hence with increasing radius R the decelerations due to viscosity will decrease in comparison with the accelerations due to the elastic reactions, and it follows that the damping per period will decrease with increasing R. This has the consequence that the logarithmic decrement Λ of the oscillations will be proportional to R^{-1}.

Damping can also be due to relaxation of the elastic stresses. This phenomenon is characterised by a constant of the nature of a time, the relaxation time, which is a property of the fluid and does not depend on the period of the motion or the dimensions of the field. It follows that the effects produced by relaxation will increase with the period of the motion, and it is found that in such a case the logarithmic decrement becomes proportional to T, that is, to R.

Damping finally can be a consequence of slipping of the fluid (more accurately: of the elastic system in the fluid) along the wall of the vessel. As the angular displacements are supposed to be the same, the linear displacements are proportional with R and the velocity of slipping will be proportional to $R\,T^{-1}$. If we suppose that the frictional force per unit area called into play by the slipping is proportional to the latter quantity, its moment will be proportional with $R^4 T^{-1}$; the angular deceleration produced by it will be proportional with $R^{-1} T^{-1}$, that is, with R^{-2}. Hence in this case the deceleration will be proportional to the elastic accelerations and it is found that the logarithmic decrement becomes independent of R.

As will be seen from BUNGENBERG DE JONG's account of his observations, a logarithmic decrement independent of the radius of the vessel has been found in the case of certain dilute soap solutions, while more concentrated solutions show a decrement proportional to the radius.

2. *Motion in concentric spherical shells.* — In order to show that a motion in concentric spherical layers, performing rotational oscillations about a common axis, is possible, we denote the angular displacement of a particular layer by $\Phi(r, t)$. The shear along a parallel circle at the angular distance θ from the pole of the axis is then given by:

$$\gamma = r \sin \theta \cdot \partial \Phi / \partial r \quad \ldots \ldots \ldots \ldots \quad (1)$$

The shearing stress τ, acting across a spherical layer, in the direction of the parallel circles, will be a function of γ. When ordinary elastic behaviour is present we must take:

$$\tau = G \gamma \quad \ldots \ldots \ldots \ldots \quad (2a)$$

where G is the shear modulus. In the case where the elastic reaction is accompanied by viscous friction we may take:

$$\tau = G\gamma + \eta\,(\partial\gamma/\partial t) \quad \ldots \ldots \ldots \quad (2b)$$

with η = viscosity. If, instead of viscosity, relaxation of the elastic stresses makes itself felt, we must write the relation between τ and γ in the form [3]):

$$\partial\tau/\partial t = G\,(\partial\gamma/\partial t) - \tau/\lambda \quad \ldots \ldots \ldots \quad (2c)$$

where λ is the relaxation time.

In the case of harmonic, or damped harmonic, oscillations we can write:

$$\Phi = e^{\nu t} \cdot \Phi\,(r) \quad \ldots \ldots \ldots \ldots \quad (3)$$

with ν imaginary or complex. In that case: $\partial\Phi/\partial t = \nu\Phi$, from which $\partial\gamma/\partial t = \nu\gamma$. Equations $(2a)$—$(2c)$ can then be brought into the general form:

$$\tau = L\gamma \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (4)$$

with either:

$$L = G \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (5a)$$

or:

$$L = G + \nu\eta \quad \ldots \ldots \ldots \ldots \ldots \quad (5b)$$

or:

$$L = G\,(1 + 1/\nu\lambda)^{-1} \simeq G\,(1 - 1/\nu\lambda) \quad \ldots \ldots \quad (5c)$$

corresponding to the three cases represented by $(2a)$, $(2b)$, $(2c)$ respectively. The expression for L can also be adapted to more complicated cases; L will always be an algebraic function of ν, independent of r.

Introduction of (1) into (4) gives:

$$\tau = L\,r \sin\vartheta \cdot \partial\Phi/\partial r \quad \ldots \ldots \ldots \ldots \quad (6)$$

We now consider the motion of a ring-shaped mass of fluid, contained between two concentric spherical shells with radii r and $r + dr$, and two

[3]) Compare *e.g.* "First Report on Viscosity and Plasticity", Verhand. Kon. Ned. Akad. v. Wetensch., Amsterdam (1e sectie) vol. **15**, no. 3, p. 18 (1939).

conical surfaces with semi-angles θ and $\theta + d\theta$ (compare fig. 1). The

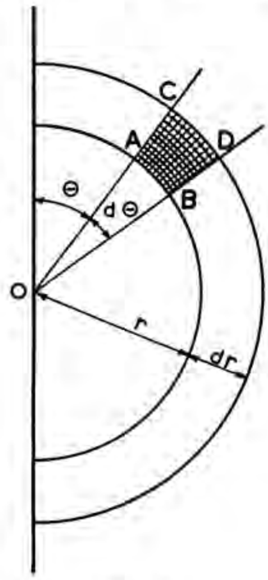

Fig. 1.

resulting moment of the shear stress acting on the exterior and interior surfaces (described by the arcs AB and CD) of this ring is:

$$\partial\,(2\pi r^3 \sin^2\theta \cdot \tau)/\partial r \cdot dr\, d\theta.$$

As the moment of inertia of the ring amounts to: $2\pi r^4 \sin^3\theta \cdot \varrho\, dr\, d\theta$, where ϱ is the density of the fluid, its equation of motion takes the form:

$$2\pi r^4 \sin^3\theta \cdot \varrho\,(\partial^2 \Phi/\partial t^2) = \partial\,(2\pi r^3 \sin^2\theta \cdot \tau)/\partial r.$$

When use is made of (3) and eq. (6) is substituted for τ, it will be seen that both the exponential factor and the factor $2\pi \sin^3\theta$ drop out, so that we obtain:

$$\varrho\, r^4 \nu^2 \Phi = L\, d\,(r^4\, d\Phi/dr)/dr. \quad \quad (7)$$

The fact that this equation does not contain the angle θ proves that the angular displacement Φ of each spherical shell can be independent of the polar distance, so that each shell can move as a whole.

We write:

$$a^2 = -\varrho \nu^2/L \quad \quad (8)$$

which makes it possible to bring (7) into the form:

$$\frac{d}{dr}\left(r^4 \frac{d\Phi}{dr}\right) + a^2 r^4 \Phi = 0. \quad \quad (9)$$

The solutions of this equation have been given by LAMB; with the omission of an arbitrary constant factor the solution applicable to the present case (field extending to $r = 0$) is:

$$\Phi = \frac{\sin ar}{(ar)^3} - \frac{\cos ar}{(ar)^2} \quad \text{. . . . .} \quad (10)$$

We assume that the wall of the spherical vessel (radius R) is at rest. When there is no slipping at the boundary, the function Φ must vanish for $r = R$. When slipping is possible, the angular displacement at the wall will be given by Φ, so that the linear displacement of the fluid relative to the wall is equal to $\Phi R \sin\theta$ and the velocity of slipping will be given by: $(\partial\Phi/\partial t)\, R \sin\theta = \nu\Phi R \sin\theta$. Introducing a friction coefficient $\varkappa$, we assume the relation:

$$(\tau)_R = -\varkappa\nu\Phi R \sin\theta. \quad \text{. . . . .} \quad (11)$$

Making use of (6) we find:

$$L\,(d\Phi/dr)_{r=R} + \varkappa\nu\,\Phi(R) = 0. \quad \text{. . . . .} \quad (12)$$

Inserting the expression (10) for Φ and writing, for shortness, $aR = \zeta$, this equation can be transformed into:

$$\mathrm{tg}\,\zeta - \zeta = -(L/\varkappa\nu R)\,(\zeta^2\,\mathrm{tg}\,\zeta - 3\,\mathrm{tg}\,\zeta + 3\zeta). \quad \text{. . . .} \quad (13)$$

We suppose that $\varkappa$ is large, so that slipping will be no more than a small disturbing effect. The solution of (13) will then differ only slightly from the solution of $\mathrm{tg}\,\zeta = \zeta$, the first root of which is 4,493. This first root corresponds to the most simple type of motion, which was observed by BUNGENBERG DE JONG in his "rotational oscillations". We shall distinguish the values of ζ, a, ν and T for this type of motion by the subscript $_0$. We then write $\zeta_0 = 4{,}493 + \triangle\zeta_0$ on the left hand side of (13) and $\zeta_0 = 4{,}493$ on the right hand side which has the large value of $\varkappa$ in the denominator; this gives:

$$\triangle\zeta_0 = -L\zeta_0/\varkappa\nu_0 R. \quad \text{. . . . .} \quad (13a)$$

With the aid of this result we obtain the following expression for the root of (13):

$$a_0 R = \zeta_0 = 4{,}493\,(1 - L/\varkappa\nu_0 R). \quad \text{. . . . . . .} \quad (14)$$

Now equation (8) gives us:

$$\frac{\varrho\nu_0^2}{L} = -a_0^2 = -\frac{(4{,}49)^2}{R^2}\left(1 - \frac{L}{\varkappa\nu_0 R}\right)^2. \quad \text{. . . . .} \quad (15)$$

Three cases will be considered.

(1) *Damping through viscous forces* ($\lambda = \infty$; $\varkappa = \infty$), in which case L is given by (5*b*). Equation (15) becomes:

$$\nu_0^2 = -\frac{(4{,}49)^2}{R^2}\,\frac{G}{\varrho}\left(1 + \frac{\nu_0\eta}{G}\right).$$

from which the following approximate value of ν_0 is obtained:

$$\nu_0 = i\,\frac{4{,}49}{R}\sqrt{\frac{G}{\varrho}} - \frac{(4{,}49)^2\eta}{2R^2\varrho}.$$

The period T_0 and the logarithmic decrement Λ_0 become:

$$T_0 = \frac{2\pi R}{4{,}49}\sqrt{\frac{\varrho}{G}}\,;\quad \Lambda_0 = \frac{4{,}49\,\pi\eta}{R\sqrt{G\varrho}}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

(II) *Damping through relaxation* ($\eta = 0$; $\varkappa = \infty$), in which case L is given by (5*c*). Equation (15) becomes:

$$\nu_0^2 = -\frac{(4{,}49)^2}{R^2}\frac{G}{\varrho}\left(1 - \frac{1}{\nu_0\lambda}\right),$$

giving the approximate solution:

$$\nu_0 = i\,\frac{4{,}49}{R}\sqrt{\frac{G}{\varrho}} - \frac{1}{2\lambda}.$$

The period T_0 and the logarithmic decrement Λ_0 become [4]):

$$T_0 = \frac{2\pi R}{4{,}49}\sqrt{\frac{\varrho}{G}}\,;\quad \Lambda_0 = \frac{\pi R}{4{,}49\,\lambda}\sqrt{\frac{\varrho}{G}}. \quad . \quad . \quad . \quad . \quad (17)$$

(III) *Damping through slipping.* We take $\eta = 0$; $\lambda = \infty$ but retain $\varkappa$, and use eq. (5*a*) for *L*. In this case eq. (15) takes the form:

$$\nu_0^2 = -\frac{(4{,}49)^2}{R^2}\frac{G}{\varrho}\left(1 - \frac{G}{\varkappa\nu_0 R}\right)^2,$$

---

[4]) The accurate equation for the calculation of ν_0 in the case of damping through relaxation has the form:

$$\nu_0^2 + \frac{\nu_0}{\lambda} + \frac{(4{,}49)^2 G}{R^2\varrho} = 0.$$

Its roots are:

$$\nu_0 = \pm\, i\sqrt{\frac{(4{,}49)^2 G}{R^2\varrho} - \frac{1}{4\lambda^2}} - \frac{1}{2\lambda},$$

from which:

$$T_{\text{obs}} = 2\pi\left\{\frac{(4{,}49)^2 G^2}{R^2\varrho} - \frac{1}{4\lambda^2}\right\}^{-1/2};\ \Lambda_{\text{obs}} = T_{\text{obs}}/2\lambda.$$

If we write:

$$T_{\text{corr}} = \frac{2\pi R}{4{,}49}\sqrt{\frac{\varrho}{G}},$$

the following relation exists between T_{corr} and T_{obs}:

$$T_{\text{corr}} = T_{\text{obs}}\left\{1 + (\Lambda_{\text{obs}}/2\pi)^2\right\}^{-1/2}.$$

This formula has been applied by BUNGENBERG DE JONG in those cases where a more accurate determination of the shear modulus G is desired.

giving the approximate value:

$$\nu_0 = i\,\frac{4.49}{R}\sqrt{\frac{G}{\varrho}} - \frac{G}{\varkappa R}.$$

The period T_0 and the logarithmic decrement Λ_0 now become [5]):

$$T_0 = \frac{2\pi R}{4.49}\sqrt{\frac{\varrho}{G}}\,; \quad \Lambda_0 = \frac{2\pi}{4.49}\,\frac{\sqrt{G\varrho}}{\varkappa} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

3. *Motion in meridian planes.* — The discussion of this case is less simple than that of the former one. However, as the fluid can be considered as incompressible (the shear modulus proves to be very much lower than the modulus of compressibility can be expected to be), we may take the equations of motion of an elastic body in the form [6]):

$$\varrho\,(\partial^2 \omega_i/\partial t^2) = L \triangle \omega_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where the ω_i are the components of the rotation (not of the rotational velocity, or vorticity, which is given by $\partial\omega_i/\partial t$) of an element of volume of the fluid, with respect to rectangular coordinates; $\triangle$ is the Laplacian; and L is the same quantity as in (4) and (5*a*)—(5*c*).

Motion in meridian places can be described with the aid of a function $\psi(r, \theta, t)$, analogous to STOKES' stream function used in hydrodynamical problems with axial symmetry. The components of the linear displacement, defined with respect to spherical polar coordinates, are given by:

$$u_r = \frac{1}{r^2 \sin\theta}\,\frac{\partial\psi}{\partial\theta}\,;\; u_\theta = \frac{-1}{r\sin\theta}\,\frac{\partial\psi}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and the only component of rotation (rotation in the meridian plane about an axis perpendicular to that plane, that is, tangential to a parallel circle) becomes:

$$\omega = \frac{1}{r}\left\{\frac{\partial}{\partial r}(r\,u_\theta) - \frac{\partial u_r}{\partial\theta}\right\} = -\frac{1}{r\sin\theta}\left\{\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\,\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\,\frac{\partial\psi}{\partial\theta}\right)\right\}. \quad (21)$$

We assume:

$$\omega = e^{\nu t}\cdot\Omega\,(r, \theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

---

[5]) The deduction of more accurate formulae for this case leads to certain complications which require careful inspection. It is hoped to come back to this point in connection with the IIIrd part of BUNGENBERG DE JONG's paper (to be published in one of the following issues of these Proceedings).

[6]) This form corresponds to that used in hydrodynamics for a number of problems of slow motion, when terms of the second degree in the velocities can be neglected. The hydrodynamical equations for that case are obtained from (19) if L is replaced by $\eta\,(\partial/\partial t)$.

For several of the formulae used in the text the reader may be referred to the analogous hydrodynamical equations as given in S. GOLDSTEIN, Modern Developments of Fluid Dynamics (Oxford 1938), vol. I, pp. 103—105 and 114—115.

In applying eq. (19) to the system of spherical polar coordinates used here, we must keep in mind that the ω defined by (21) is directed along the tangent to a parallel circle and thus has a different meaning from the rectangular components ω_i used in (19). Whereas in the case of a scalar quantity ω the Laplacian would be given by:

$$\Delta\,\omega = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial \omega}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial \omega}{\partial \theta}\right),$$

we must now add to this expression the amount $-\omega/r^2\,\sin^2\theta$, corresponding to the derivative of the second order taken in a fixed direction normal to a particular meridian plane, of the component of ω along that normal [7]). With this addition eq. (19) takes the form:

$$\frac{\partial}{\partial r}\left(r^2\,\frac{\partial \Omega}{\partial r}\right) + \frac{1}{\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial \Omega}{\partial \theta}\right) - \frac{\Omega}{\sin^2\theta} + \alpha^2\,r^2\,\Omega = 0 \quad . \qquad (23)$$

where α^2 is the same quantity as defined by (8).

We look for solutions of the type: $\Omega = h(r)\cdot k(\theta)$. Introducing a number m to be determined afterwards, it is found that the function $k(\theta)$ must satisfy the equation:

$$k'' + k'\,\cot\theta - k/\sin^2\theta + m\,k = 0.$$

If we put: $k = dl/d\theta$, where $l(\theta)$ is another function of θ, we find that l must be a solution of the equation of the Legendre functions:

$$l'' + l'\,\cot\theta + m\,l = 0.$$

The constant m must have one of the values $n(n+1)$, n being an integer, if we desire solutions which are regular in the domain $0 \leq \theta \leq \pi$ with the endpoints included. The cases corresponding to BUNGENBERG DE JONG's "meridional oscillations" and "quadrantal oscillations" respectively, are obtained with:

$$n=1 \;\;;\;\; m=2 \;\;:\;\; l_1 = \cos\theta \qquad ; \;\; k_1 = -\sin\theta$$
$$n=2 \;\;;\;\; m=6 \;\;:\;\; l_2 = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\,; \;\; k_2 = -3\sin\theta\cos\theta.$$

The corresponding equation for the function h becomes:

$$r^2\,h'' + 2\,r\,h' + (\alpha^2\,r^2 - m)\,h = 0,$$

and its solutions for the cases mentioned are:

$$n=1 \;\;;\;\; m=2 \;\;:\;\; h_1 = \frac{\sin\alpha r}{(\alpha r)^2} - \frac{\cos\alpha r}{\alpha r}$$
$$n=2 \;\;;\;\; m=6 \;\;:\;\; h_2 = \frac{(3-\alpha^2 r^2)\sin\alpha r}{(\alpha r)^3} - \frac{3\cos\alpha r}{(\alpha r)^2}.$$

---

[7]) When the meridian plane is taken as x,y-plane, we can write $\omega_z = \omega\cos\varphi$ (φ being the position angle of a point, measured from this plane). We then have:

$$\Delta\,\omega_z = \cos\varphi\cdot\Delta\,\omega - \frac{1}{r^2\sin^2\theta}\left(2\sin\varphi\,\frac{\partial\omega}{\partial\varphi} + \omega\cos\varphi\right).$$

which reduces to the terms given in the text when φ is taken zero.

We must now find ψ from (21). To this end we write:

$$\psi = e^{\nu t} \cdot \Psi(r, \theta) = e^{\nu t} H(r) K(\theta). \quad . \quad . \quad . \quad . \quad . \quad (24)$$

After some calculations the following expressions are obtained:

$$\left.\begin{array}{lll} n=1: & K_1 = \sin^2\theta & ; \quad H_1 = -a\,r\,h_1 + C_1 (a\,r)^2 \\ n=2: & K_2 = 3\sin^2\theta\cos\theta; & H_2 = -a\,r\,h_2 + C_2 (a\,r)^3 \end{array}\right\} . \quad . \quad (25)$$

where C_1 and C_2 are integration constants.

The following boundary conditions must be observed. In the first place the radial velocity of the fluid must be zero at the wall of the spherical vessel; this requires Ψ to become a constant for $r = R$, which necessitates that $H(R)$ shall be zero. This condition fixes the values of the constants C_1, C_2.

The velocity of slipping along the wall is then given by:

$$\left(\frac{\partial u_\theta}{\partial t}\right)_{r=R} = -\left(\frac{\nu}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)_{r=R},$$

and the equivalent of equation (11) takes the form:

$$(\tau_{r\theta})_{r=R} = -\varkappa\left(\frac{\partial u_\theta}{\partial t}\right)_{r=R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The shearing stress $\tau_{r\theta}$ appearing in this equation is given by the formula:

$$\tau_{r\theta} = G\left\{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right\} = G\left\{-\frac{r}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right\}.$$

When the expression (24) is inserted for ψ and attention is given to the fact that the function $H(r)$ vanishes for $r = R$, the following equation is obtained, which takes the place of eq. (12) in the case of section **2**:

$$\frac{d^2H}{d(ar)^2} - \frac{2}{ar}\frac{dH}{d(ar)} + \frac{\varkappa\nu R}{L}\frac{1}{ar}\frac{dH}{d(ar)} = 0 \text{ (for } r = R). \quad . \quad . \quad (27)$$

The case of no slipping is obtained by making $\varkappa$ infinite, in which case the condition becomes: $dH/d(ar) = 0$. This gives:

for $n = 1$ (meridional oscillation) $\zeta_1 = 5{,}76$
for $n = 2$ (quadrantal oscillation) $\zeta_2 = 6{,}99$.

In the case of a finite (but large) value of $\varkappa$ we write: $\zeta_1 = 5{,}76 + \triangle\zeta_1$; $\zeta_2 = 6{,}99 + \triangle\zeta_2$. It is found that for both values of n the correction is given by the expression: $\triangle\zeta = -L\zeta/\varkappa\nu R$, so that the roots of eq. (27) become:

$$\left.\begin{array}{l} \text{for } n=1 : a_1 R = \zeta_1 = 5{,}76\,(1 - L/\varkappa\nu_1 R) \\ \text{for } n=2 : a_2 R = \zeta_2 = 6{,}99\,(1 - L/\varkappa\nu_2 R) \end{array}\right\} . \quad . \quad . \quad (28)$$

We can now calculate the values of ν_1 and ν_2 and the corresponding periods and logarithmic decrements, in the same way as was done at the end of section **2**. It will be seen that the only difference is the substitution of the numerical factor 5,76 (for the meridional oscillation) or 6,99 (for the quadrantal oscillation) in the place of the factor 4,49. It follows that the period of the oscillation is decreased in such a way that:

$$T_{\text{rot}}/T_{\text{mer}} = a_1/a_0 = 1{,}282$$
$$T_{\text{rot}}/T_{\text{quadr}} = a_2/a_0 = 1{,}556.$$

Having regard to (17) and (18) it is further seen that both in the case of damping through relaxation and in the case of damping through slipping the logarithmic decrement changes in the same ratio as the period. The way in which the decrement depends on the radius is not changed when we pass from the rotational oscillations to the meridional or the quadrantal oscillations.

4. *Magnitude of the shear stress.* — Numerical data concerning the shear modulus G, the relaxation time λ and the coefficient of friction $\varkappa$ operative in slipping will be given in BUNGENBERG DE JONG's papers.

It may be of interest to have an estimate of the magnitude of the elastic stresses active in the system. This can easily be obtained for the case of the rotational oscillation. The angular displacement is given by the formula:

$$\Phi = A\,\frac{\sin ar - ar\cos ar}{(ar)^3}\,e^{-t/2\lambda}\cos\frac{2\pi t}{T},$$

A being a coefficient determining the amplitude. Leaving aside the time factors, the linear displacement at $\theta = 90^\circ$ is determined by:

$$r\Phi = \frac{A}{a}\,\frac{\sin ar - ar\cos ar}{(ar)^2}.$$

The maximum of this expression is found in the neighbourhood of $r = \frac{1}{2}R$, that is $ar = 2{,}25$, giving $0{,}433\ A/a = 0{,}096\,A\,R$. Hence if we write a for the maximum deviation actually observed, we shall have: $A = 10{,}4\ a/R$.

The maximum value of the shearing stress is found at the wall of the vessel, at $\theta = 90^\circ$. Equation (2*a*) gives (when the time factor is again left aside):

$$\tau_{\max} = GR\,\frac{d}{dr}\left\{A\,\frac{\sin ar - ar\cos ar}{(ar)^3}\right\}_{r=R} = GA\,\frac{\sin aR}{aR} = 0{,}217\ GA.$$

With the value of A given above there results: $\tau_{\max} \cong 2{,}25\ Ga/R$.

According to a footnote to BUNGENBERG DE JONG's second paper, the deviation from the equilibrium position at the moment the determination of the damping ratio was started, amounted to ca. 3 mm, but larger deviations had been observed before that instant. If we choose $a = 5$ mm $= 0{,}5$ cm in a vessel of 7,5 cm radius, we find:

$$\tau_{\max} \cong 0{,}15\ G.$$

Résumé.

Afin d'obtenir des formules qui puissent élucider les résultats obtenus par BUNGENBERG DE JONG dans ses expériences sur les oscillations élastiques présentées par certaines solutions d'oléates (v. l'article précédent), il a été donné dans l'article ci-dessus une déduction directe de certaines formules de LAMB pour les oscillations d'un fluide élastique, contenu dans un réservoir sphérique. En même temps on a calculé le degré d'amortissement, provoqué soit par une résistance visqueuse, soit par rélaxation des tensions élastiques, soit par un glissement du fluide le long du paroi du réservoir.

On trouve que la période des oscillations est toujours proportionelle au rayon du réservoir. D'autre part le décrément logarithmique dans les trois cas se comporte différemment: le décrément est inversement proportionnel au rayon dans le cas d'une résistance visqueuse, directement proportionnel au rayon dans le cas de rélaxation, et indépendant du rayon dans le cas de glissement. Le deuxième cas est présenté par les résultats obtenus par BUNGENBERG DE JONG avec des solutions d'oléates à concentration supérieure à ca. 1,1 %, le troisième pour des solutions à concentration inférieure à ca. 0,9 %.

Les calculs ont trait aux trois formes d'oscillations observées: rotationelles, méridionales et quadrantales.

Resumo.

Por trovi formulojn kiuj povos klarigi la rezultojn de la esploroj de BUNGENBERG DE JONG rilate al la elastaj osciloj kiujn montras iuj solvaĵoj de oleatoj (vidu la antaŭan artikolon), oni donas en la ĉi-supra artikolo rektan derivadon de kelkaj formuloj de LAMB por la osciloj de elasta fluidaĵo entenata en sfera vazo. Samtempe oni kalkulas la gradon de amortizo, kaŭzitan ĉu per viskozeca rezisto, ĉu per perdo de elastaj strečoj, ĉu per glito de la fluidaĵo laŭ la pario de la vazo.

Oni trovas ke la periodo de la osciloj ĉiam estas rekte proporcia al la radio de la vazo. Kontraŭe, la logaritma dekremento kondutas diference en la tri kazoj: la dekremento estas inverse proporcia al la radio en la kazo de viskozeca rezisto, rekte proporcia al la radio en la kazo de strečoperdo, kaj nedependa de la radio en la kazo de glito. La duan kazon prezentas la rezultoj akiritaj de BUNGENBERG DE JONG kun solvaĵoj de oleatoj de koncentriteco pli ol cirkaŭ 1,1 % a, la trian tiuj kun solvaĵoj de koncentriteco malpli ol ĉirkaŭ 0,9 % a.

La kalkuloj rilatas al la tri observitaj formoj de osciloj: rotaciaj, meridianaj kaj kvadrantaj.

Aerodynamics. — *Spectral analysis of an irregular function.* By J. M. BURGERS. (Mededeling No. 58*b* uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft) *).

(Communicated at the meeting of November 27, 1948.)

5. As a supplement to the result of the preceding part of this article, we will briefly consider the fluctuations which may be expected to be shown by the indication of a galvanometer, connected to a thermo-couple through which is passed the signal $w(t)$, transmitted by the filtering circuit.

We assume that the indication z of the galvanometer depends on the signal w according to the equation:

$$z + \frac{1}{\beta}\frac{dz}{dt} = w^2 - \overline{w^2}. \quad \ldots \ldots \quad (13)$$

where $1/\beta$ is an adjustable damping coefficient. The mean value of z will be zero and we are interested in the mean of z^2. The solution of (13) is:

$$z = \beta \int_0^\infty d\tau \, \{ w^2(t-\tau) - \overline{w^2} \} \, e^{-\beta\tau} \quad \ldots \ldots \quad (14)$$

and we easily find:

$$\overline{z^2} = \beta^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \, \{ \overline{w^2(t-\tau_1) \, w^2(t-\tau_2)} - (\overline{w^2})^2 \} \, e^{-\beta(\tau_1+\tau_2)}. \quad (15)$$

Introducing $\sigma = \tau_2 + \tau_1$ and $\delta = \tau_2 - \tau_1$ as new variables, we can bring this integral into the form

$$\overline{z^2} = \beta \int_0^\infty d\delta \, \{ \overline{w^2(t) \, w^2(t+\delta)} - (\overline{w^2})^2 \} \, e^{-\beta\delta}. \quad \ldots \quad (15a)$$

The mean value $\overline{w^2(t) w^2(t+\delta)}$ must be calculated by making use of the integral for w given in (3). Before starting with this calculation, consider a quantity W which is the sum of a large number n of terms of the type: $\varepsilon \cos \theta_k$, where ε is a constant, while the θ_k $(k = 1, 2, \ldots n)$ are irregular functions of the time, in such a way that the mean value of each term $\varepsilon \cos \theta_k$ is zero. We easily find:

$$\overline{W} = 0$$

$$\overline{W^2} = n \, \varepsilon^2 \, \overline{\cos^2 \theta_k} = \tfrac{1}{2} \, n \, \varepsilon^2$$

$$\overline{W^4} = n \, \varepsilon^4 \, \overline{\cos^4 \theta_k} + 3 \, n \, (n-1) \, \varepsilon^4 \, \overline{\cos^2 \theta_k \cos^2 \theta_l} = \tfrac{3}{8} \, n \, \varepsilon^4 + \tfrac{3}{4} \, n \, (n-1) \, \varepsilon^4.$$

*) Part I has appeared in these Proceedings **51**, 1073 (1948).

Hence if n is very large and ε small, in such a way that $n\,\varepsilon^2$ is finite, we obtain the approximate relation:

$$\overline{W^4} \cong 3\,(\overline{W^2})^2.$$

On the other hand, when an interval of time η is taken, sufficiently long in order that no correlation will exist between $W(t)$ and $W(t+\eta)$, we shall have:

$$\overline{W^2(t)\,W^2(t+\eta)} = (\overline{W^2})^2.$$

From these relations we deduce:

$$\overline{W^2(t)\,W^2(t+\eta)} - (\overline{W^2})^2 = 2\,(\overline{W^2})^2 \qquad \text{for } \eta = 0$$

$$\overline{W^2(t)\,W^2(t+\eta)} - (\overline{W^2})^2 = 0 \qquad \text{for large values of } \eta.$$

We may expect that a similar relation will hold for the function $v(t)$, defining the signal introduced into the filtering system, that is to say, we may suppose that also in the case of this function the combined effect of second order correlations will be more important than that of the fourth order correlations. Then it is possible to calculate the value of $\overline{w^2(t)\;w^2(t+\eta)}$ (see next section), for which the following approximate result is obtained:

$$\overline{w^2(t)\,w^2(t+\eta)} - (\overline{w^2})^2 =$$
$$= (\overline{w^2})^2\, e^{-2p\omega\eta}\,(1+\cos 2\,\omega\,\eta) - p\,\omega\,\frac{d\,(\overline{w^2})^2}{d\omega}\, e^{-2p\omega\eta} \sin 2\,\omega\eta \qquad (16)$$

In deriving this formula it has been assumed that η is several times larger than θ. There is some indication that the domain of validity will extend further downward when p is taken smaller and smaller.

Assuming also β to be small, we find, upon insertion of (16) into (15*a*), the approximate formula:

$$\overline{z^2} = (\overline{w^2})^2 \left\{ \frac{\beta}{\beta + 2\,p\,\omega} + \frac{\beta\,(\beta + 2\,p\,\omega)}{4\,\omega^2} \right\} - \frac{\beta\,p}{2}\,\frac{d}{d\omega}\,(\overline{w^2})^2. \quad (17)$$

All three terms appearing here have the factor β, which expresses the fact that the fluctuations of the indication of the galvanometer will become smaller and smaller when the damping factor $1/\beta$ is taken larger.

From the first term it will be seen that β must become smaller than $2\,p\,\omega$ in order to make the damping effective. If for a given value of β we should decrease p in order to increase the selectivity of the filter, the ratio of $\overline{z^2}$ to $(\overline{w^2})^2$ will become larger, meaning that larger fluctuations of the galvanometer are to be expected. However, in consequence of the presence of the factor ω, this effect is not much to be feared with a filter of medium or low selectivity (p of the order of 0,1 or more).

A very interesting term is the last one (supposing that p is not very small), as this term contains the derivative of $\overline{w^2}$ with respect to ω, so that it may become significant in regions of the spectrum where the spectral intensity [the magnitude of the Fourier component $I_1(\omega)$] changes rapidly with ω. Such a phenomenon had been observed by BETCHOV, who found that the indication of the galvanometer showed appreciable oscillations of low frequency on both sides of the maximum of the curve giving the spectral energy as function of the frequency, approximately at the places where the curve descended most rapidly. The last term of (17), although it carries the factor p, may give a clue to the appearance of these oscillations (in the arrangement used by BETCHOV the selectivity of the filter corresponded approximately to the value 0,1 for p).

6. *Approximate calculation of* $\overline{w^2(t)\, w^2(t+\eta)}$.

Referring again to equation (3) it will be seen that:

$$\overline{w^2(t)\, w^2(t+\eta)} =$$
$$= 16\,p^2 \iiiint_0^\infty d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4\, V\, e^{-p(\tau_1+\tau_2+\tau_3+\tau_4)} \cdot$$
$$\cdot \cos(\tau_1+\varepsilon)\cos(\tau_2+\varepsilon)\cos(\tau_3+\varepsilon)\cos(\tau_4+\varepsilon) \qquad (18)$$

where V has been written for the mean value:

$$V = \overline{v(t-\tau_1/\omega)\, v(t-\tau_2/\omega)\, v(t+\eta-\tau_3/\omega)\, v(t+\eta-\tau_4/\omega)}\ . \qquad (18a)$$

while $\sqrt{1-p^2} = \cos\varepsilon$ has everywhere been replaced by unity. An exact calculation of the integral (18) would require knowledge concerning quadruple correlations in the function $v(t)$, which is not available. However, if we restrict ourselves to values of η which sufficiently exceed θ, we can arrive at an approximate evaluation by observing that in this case V will differ from zero only if the four factors occurring in (18*a*) can be arranged into two pairs, in such a way that there exists a certain correlation within each pair, while at the same time the two pairs are sufficiently separated from each other in order that no correlation shall be found between the factors of the first pair and those of the second pair. It is possible that this will not give a full solution of our problem, so that when the formula derived for large values of η is extrapolated downward to $\eta = 0$, the result may be in error. Nevertheless there are indications that, provided p is very small, the approximate method of integration will give the most important part of the quantity to be calculated and we shall try to estimate the possible errors involved.

We introduce the following auxiliary variables:

$$\left.\begin{aligned} \sigma_0 &= \tau_4+\tau_3+\tau_2+\tau_1 \\ \sigma &= \tau_4+\tau_3-\tau_2-\tau_1 \\ \delta_2 &= \tau_4-\tau_3 \\ \delta_1 &= \tau_2-\tau_1 \end{aligned}\right\} \text{ so that: } \left\{\begin{aligned} \tau_4 &= \sigma_0/4+\sigma/4+\delta_2/2 \\ \tau_3 &= \sigma_0/4+\sigma/4-\delta_2/2 \\ \tau_2 &= \sigma_0/4-\sigma/4+\delta_1/2 \\ \tau_1 &= \sigma_0/4-\sigma/4-\delta_1/2. \end{aligned}\right.$$

The functional determinant of the set $\sigma_0, \sigma, \delta_2, \delta_1$ with respect to the set $\tau_4, \tau_3, \tau_2, \tau_1$ has the value 8. With the abbreviation: $h = \eta - \sigma/2\,\omega$ the expression for V can be written:

$$V = \overline{v\left(-\tfrac{1}{2}h + \delta_1/2\omega\right) v\left(-\tfrac{1}{2}h - \delta_1/2\omega\right) v\left(\tfrac{1}{2}h + \delta_2/2\omega\right) v\left(\tfrac{1}{2}h - \delta_2/2\omega\right)}. \quad (18b)$$

Putting:

$$\alpha_2 = \tfrac{1}{2}(\sigma_0 + \sigma) + 2\,\varepsilon \quad ; \quad \alpha_1 = \tfrac{1}{2}(\sigma_0 - \sigma) + 2\,\varepsilon,$$

the integral (18) can now be brought into the form:

$$\overline{w^2(t)\,w^2(t+\eta)} =$$
$$= \tfrac{1}{2}p^2 \int d\sigma_0\, e^{-p\sigma_0} \int d\sigma \int d\delta_1 \int d\delta_2\, V(\cos\alpha_2 + \cos\delta_2)(\cos\alpha_1 + \cos\delta_1) \quad (19)$$

The integration with respect to σ_0 can be deferred to the last and a threedimensional picture can be used to illustrate the integrations with respect to σ, δ_1 and δ_2. The domains over which the latter three variables can vary are limited as follows:

$$-\sigma_0 < \sigma < +\sigma_0$$
$$-\tfrac{1}{2}(\sigma_0 - \sigma) < \delta_1 < +\tfrac{1}{2}(\sigma_0 - \sigma)$$
$$-\tfrac{1}{2}(\sigma_0 + \sigma) < \delta_2 < +\tfrac{1}{2}(\sigma_0 + \sigma).$$

When we look for pairs of factors in (18*b*) between which correlation can exist, it is found that V can differ from zero in the following three cases only:

(*a*) $|\delta_1| < \omega\,\theta$; $|\delta_2| < \omega\,\theta$, σ having an arbitrary value between $-\sigma_0$ and $+\sigma_0$; in this case:

$$V = A^4 R\left(\frac{\delta_1}{\omega}\right) R\left(\frac{\delta_2}{\omega}\right). \quad \ldots\ldots \quad (20a)$$

(*b*) $|h| < \theta$; $|\delta_2 - \delta_1| < 2\,\omega\theta$; in this case:

$$V = A^4 R\left(h + \frac{\delta_2 - \delta_1}{2\omega}\right) R\left(h - \frac{\delta_2 - \delta_1}{2\omega}\right) \quad \ldots \quad (20b)$$

(*c*) $|h| < \theta$; $|\delta_2 + \delta_1| < 2\,\omega\theta$; in this case:

$$V = A^4 \cdot R\left(h + \frac{\delta_2 + \delta_1}{2\omega}\right) \cdot R\left(h - \frac{\delta_2 + \delta_1}{2\omega}\right). \quad \ldots \quad (20c)$$

The three domains have been indicated schematically in fig. 1. Outside of the space thus defined V will be zero. It will be seen that the length of the various domains increases with the value of σ_0. Hence the main contribution to be derived from the triple integration can be expected to be of the order $\sigma_0(\omega\theta)^2$; as this is multiplied by $e^{-p\sigma_0}$, the corresponding contribution to the quadruple integral will be of the order $(\omega\theta)^2/p^2$. It must be observed that the product $\omega\theta$ in general will be a quantity of normal order of magnitude, as the interesting values of ω will be comparable to $1/\theta$. As the quadruple integral according to formula (19) is multiplied by the factor $\frac{1}{2}p^2$, the final result will be of order $(\omega\theta)^2$, as should be expected in connection with the expression (10) for $\overline{w^2}$, given in the preceding part of this paper.

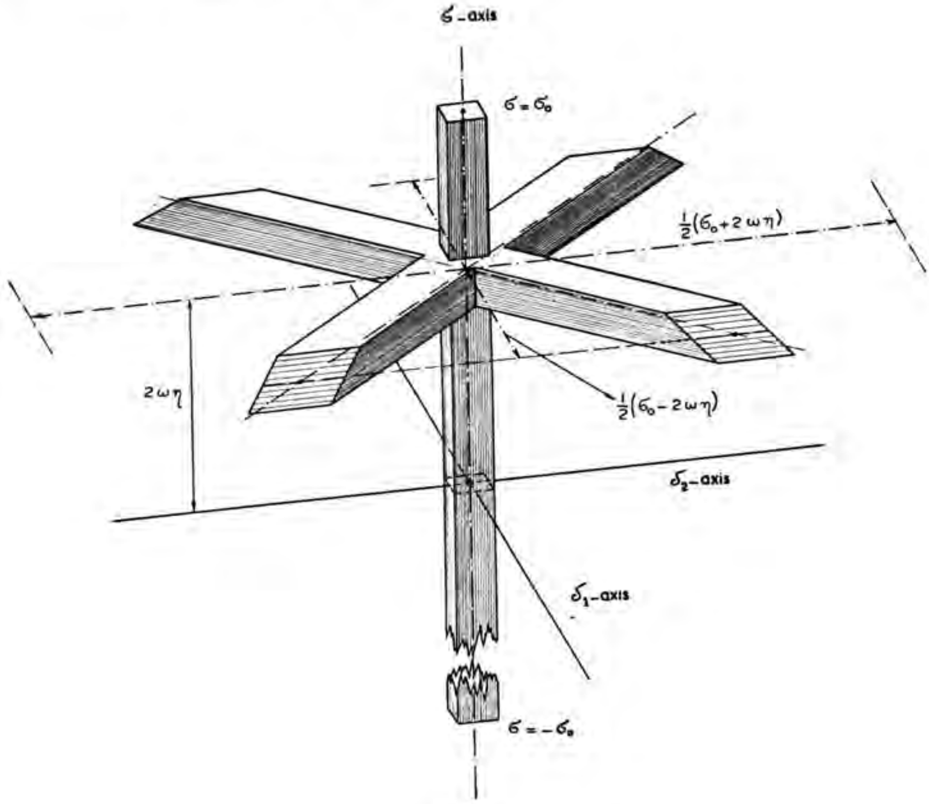

Fig. 1.

In calculating the triple integral we shall attempt also to take care of certain contributions of the order $(\omega\theta)^2$, giving contributions of the order $(\omega\theta)^2/p$ to the quadruple integral and of order $p(\omega\theta)^2$ in the final result. Although the presence of the factor p detracts somewhat from the importance of these contributions, they have been retained on account of the interesting form which they present in the final result.

The domains (a), (b), (c) overlap more or less in a region having a volume of the order $(\omega\theta)^3$. The fact that a contribution derived from the region of overlapping is counted in all three domains, is correct, as the three different forms of correlation indicated by the expressions (20a) — (20c) are actually to be found in this region and must be taken into account, each for itself. It can be brought forward that more intimate (fourfold) correlations between the four factors of the expression V might occur in this region. As nothing has been given concerning fourfold correlations in the course of the function $v(t)$, we have no data on this matter. However, if we suppose that $v(t)$ presents a character analogous to that of the quantity W, mentioned by way of example in the preceding section, we may expect that the contribution due to fourfold correlations will be small in comparison with that derived from the three pairs of double correlations. In that case the contribution from these fourfold correlations to the triple integral would be small in comparison with $(\omega\theta)^3$ and the corresponding contribution to the final result would be small in comparison with $p(\omega\theta)^3$. It seems possible therefore to leave this matter out of account. At any rate we may be sure that the main terms depending upon second order correlations have been collected without omissions.

It must further be observed that the domains (b) and (c) will have their complete forms only provided $\sigma_0 > 2\,\omega(\eta+\theta)$, while they will not exist at all when $\sigma_0 < 2\,\omega(\eta-\theta)$. In calculating the contributions from (b) and (c) we shall take $2\,\omega\,\eta$ as the lower limit in the integration with respect to σ_0 and shall reckon as if (b) and (c) are complete for all $\sigma_0 > 2\,\omega\,\eta$. The error introduced into the quadruple integral in this way is of the order $(\omega\,\theta)^4$ (see below), which can be neglected.

In consequence of the symmetry of the situation, the contributions from the domains (b) and (c) will be equal, so that we can restrict to the calculations for (b).

Contribution obtained from the domain (a).

As $R(\delta/\omega)$ becomes zero for $|\delta| > \omega\theta$, the integrations with respect to δ_1 and δ_2 can be written as if the limits are $-\infty$, $+\infty$.

We introduce:

$$\left.\begin{aligned} \int_{-\infty}^{+\infty} d\delta\, R(\delta/\omega) &= F \\ \int_{-\infty}^{+\infty} d\delta\, R(\delta/\omega)\cos\delta &= G \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (21)$$

For small values of p the quantity G is nearly equal to $2\,I_1$, where I_1 has been defined by (9a), the error being of the order p^2 in consequence of the symmetry of $R(\delta)$. Hence making use of (10) we shall assume:

$$A^2\,G \cong \overline{w^2}.$$

When the integrations with respect to δ_1 and δ_2 are carried out, the integrand becomes:

$$A^4\,(F\cos\alpha_2 + G)\,(F\cos\alpha_1 + G).$$

After working out the product and performing the integration with respect to σ, between the limits $-\sigma_0$ and $+\sigma_0$, the following result is obtained:

$$A^4\,[F^2\,\{\sigma_0\cos(\sigma_0+4\,\varepsilon)+\sin\sigma_0\} + 4\,FG\,\{\sin(\sigma_0+2\,\varepsilon)-\sin 2\,\varepsilon\} + 2\,\sigma_0\,G^2].$$

This expression must be multiplied by $e^{-p\sigma_0}$ and integrated with respect to σ_0 from 0 to ∞. It is found that the last term of the preceding expression gives a result of the order p^{-2}, whereas the others give results of order unity or of order p^2. Hence the last term is the only important one and it is found that the contribution derived from the domain (a) has the approximate value:

$$\tfrac{1}{2}\, p^2 \cdot 2 A^4\, G^2/p^2 = A^4\, G^2 \cong (\overline{w^2})^2. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Contribution obtained from the domain (b).

We again start with the integrations with respect to δ_2 and δ_1. The domain of integration for a given value of σ has been represented separately in fig. 2. As $h = \eta - \sigma/2\,\omega$ must be confined between $-\theta$ and $+\theta$, we

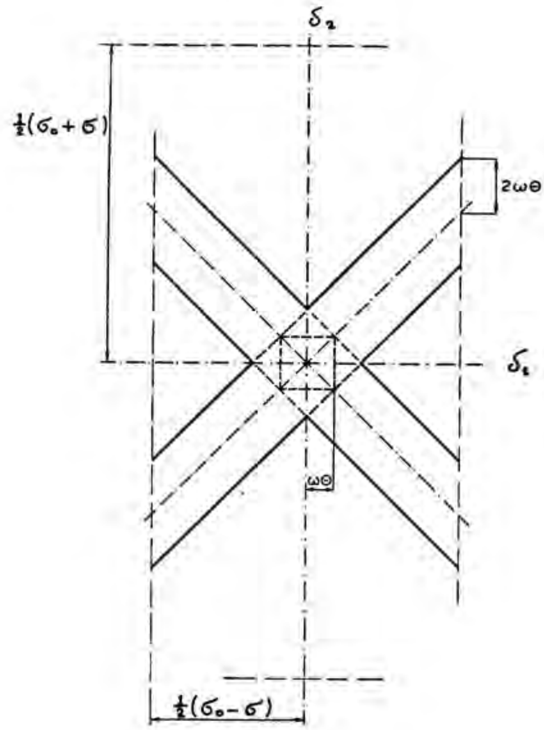

Fig. 2.

can restrict to the consideration of positive values of σ between $2\,\omega(\eta-\theta)$ and $2\,\omega(\eta+\theta)$, and it follows that the limits for δ_2 are wider than those for δ_2.

We take the integration with respect to δ_2 first and write $\delta_2 = \delta_1 + \delta'$; δ' will be taken as integrational variable instead of δ_2 (for constant δ_1). According to (20b) the value of V will be zero as soon as $|\,\delta'\,| > 2\,\omega\theta$; hence we may write $-\infty$, $+\infty$ as limits for the integration with respect to δ'.

We introduce:

$$\left.\begin{aligned}\int_{-\infty}^{+\infty} d\delta'\, R\,(h+\delta'/2\,\omega)\, R\,(h-\delta'/2\,\omega) &= P\,(h)\\ \int_{-\infty}^{+\infty} d\delta'\, R\,(h+\delta'/2\,\omega)\, R\,(h-\delta'/2\,\omega)\cos\delta' &= L\,(h)\end{aligned}\right\}\quad . \quad . \quad . \quad (23)$$

(the corresponding integral with sin δ' would be zero). The integration with respect to δ' then turns the integrand of (19) into:

$$A^4\,(P\cos\alpha_2 + L\cos\delta_1)\,(\cos\alpha_1 + \cos\delta_1).$$

A further integration, with respect to δ_1 between the limits $\pm\frac{1}{2}(\sigma_0-\sigma)$, gives the result:

$$\left.\begin{aligned}A^4\,[(\sigma_0-\sigma)\,(P\cos\alpha_1\cos\alpha_2+\tfrac{1}{2}\,L)+\tfrac{1}{2}\,L\sin(\sigma_0-\sigma)+\\ +\,2\,(P\cos\alpha_2+L\cos\alpha_1)\sin\tfrac{1}{2}\,(\sigma_0-\sigma)]\end{aligned}\right\}\quad (24)$$

This expression must next be integrated with respect to σ. We put $\sigma = 2\,\omega\eta - \sigma'$; the limits for σ' can be taken as $-\infty$, $+\infty$, as $P(h)$ and $L(h)$ both vanish when $|h| = |\eta - \sigma/2\,\omega| = |\sigma'/2\,\omega|$ exceeds θ [this supposes that $\sigma_0 > 2\,\omega(\eta+\theta)$]. We shall not write down the full result, but restrict to those terms which appear to furnish the main contributions in the integration with respect to σ_0 which is still to follow. These terms are:

$$\tfrac{1}{2}\,A^4\left[(\sigma_0-2\,\omega\,\eta)\,Q\,(1+\cos 2\,\omega\,\eta)+(Q^*-2Q)\sin 2\,\omega\,\eta\right]\quad . \quad . \quad (25)$$

where:

$$\left.\begin{aligned}Q &= \int_{-\infty}^{+\infty} d\sigma'\, P\,(\sigma'/2\,\omega)\cos\sigma' = \int_{-\infty}^{+\infty} d\sigma'\, L\,(\sigma'/\,2\,\omega)\\ Q^* &= \int_{-\infty}^{+\infty} d\sigma'\, P\,(\sigma'/2\,\omega)\,\sigma'\sin\sigma'\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (26)$$

The expression (25) is not exact when $\sigma_0 < 2\,\omega(\eta+\theta)$. Hence when the lower limit $2\,\omega\,\eta$ is used in the integration with respect to σ_0, we make an error of the order $(\omega\,\theta)^2\,Q$, that is of the order $(\omega\,\theta)^4$, in the full quadruple integral, as had been mentioned before.

After multiplication by $e^{-p\sigma_0}$ and integration with respect to σ_0 from $2\,\omega\,\eta$ to ∞ (it is convenient to put $\sigma_0 = 2\,\omega\,\eta + \sigma^*$) and having regard to the factor $\frac{1}{2}\,p^2$ before the integral in (19) the following expression is obtained for the contribution from the domain (b):

$$\tfrac{1}{4}\,A^4\,Q\,e^{-2p\omega\eta}\,(1+\cos 2\,\omega\,\eta)+\tfrac{1}{4}\,pA^4\,(Q^*-2\,Q)\,e^{-2p\omega\eta}\sin 2\,\omega\,\eta.\quad (27)$$

Now if we write $s = \frac{1}{2}(\sigma' + \delta')$; $r = \frac{1}{2}(\sigma' - \delta')$, the integrals (26), defining Q and Q^*, can be transformed as follows:

$$Q = \int_{-\infty}^{+\infty} d\sigma' \int_{-\infty}^{+\infty} d\delta' \, R\left(\frac{\sigma' + \delta'}{2\,\omega}\right) R\left(\frac{\sigma' - \delta'}{2\,\omega}\right) \cos\delta' =$$

$$= 2 \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dr \, R(s/\omega)\, R(r/\omega)\,(\cos s \cos r - \sin s \sin r) =$$

$$= 2 \left\{ \int_{-\infty}^{+\infty} ds \, R(s/\omega) \cos s \right\}^2 = 2\,G^2 \qquad (28a)$$

while Q^* becomes:

$$Q^* = 4\,G\,G^* \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28b)$$

where:

$$G^* = \int_{-\infty}^{+\infty} ds \, R(s/\omega)\, s \sin s \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

It is not difficult to prove the relation:

$$G^* = G - \omega \cdot dG/d\omega,$$

so that:

$$Q^* - 2\,Q = 4\,G\,(G^* - G) = -4\,\omega\,G \cdot dG/d\omega. \; . \; . \; . \quad (30)$$

Addition of the contributions from all three domains gives the final result:

$$\overline{w^2(t)\, w^2(t+\eta)} = A^4\,G^2 \{1 + e^{-2p\omega\eta}(1 + \cos 2\,\omega\eta)\} -$$
$$- 2\,p\,\omega\,A^4\,G\,\frac{dG}{d\omega}\,e^{-2p\omega\eta} \sin 2\,\omega\eta. \quad (31)$$

The second term has the factor p and consequently will be of subordinate importance. Nevertheless it may become influential in cases where G steeply changes with ω at some point of the spectrum.

Note. — It was pointed out to me that investigations on problems of similar nature have been published by S. G. RICE in two extensive papers: *"Filtered thermal noise — fluctuation of energy as function of interval length"*, Journ. Acoustical Society of America **14**, 216—227 (1943) and *"Mathematical analysis of random noise"*, The Bell System Technical Journal **23**, 282—332 (1944) and **24**, 46—156 (1945). The starting point of RICE's deductions and the mathematical methods applied by him differ greatly from those used in the article above, which makes a comparison difficult. A formula resembling the first term of (31), but without the exponential factor, is given in the second paper (Bell System Technical Journal **24**, p. 89, equation (3. 9—7)). I did not find an indication that RICE's formulae might explain the appearance of a high value of $\overline{z^2}$ when ω has such a value that $|dI_1/d\omega|$ is large, as signaled by BETCHOV. This may be connected with the circumstance that RICE's formulae apparently refer to a case for which p is zero.

Résumé.

Le but de cet article était d'obtenir une formule approchée pour la distribution de l'énergie dans le spectre d'une fonction aléatoire, sans qu'il soit nécessaire de supposer que cette fonction puisse être représentée par une intégrale de Fourier. On suppose qu'un signal électrique proportionnel à la fonction considérée est passé à travers d'un circuit de filtrage; il est possible alors de calculer le carré moyen du courant sortant du filtre, pourvu que le carré moyen et le coefficient de corrélation de la fonction aléatoire soient donnés. La formule obtenue, qui dépend encore d'un paramètre définissant le degré de sélectivité du filtre, tend vers la formule donnée par TAYLOR, quand la sélectivité devient infinie.

Dans la seconde partie de l'article on a étudié le carré moyen de l'indication d'un galvanomètre mesurant le courant donné par une thermocouple, auquel on a appliqué le courant sortant du filtre. L'expression trouvée donne une relation entre ce carré moyen et les paramètres définissant la sélectivité du filtre et l'amortissement du système thermocouple-galvanomètre; elle montre que pour un filtre de grande sélectivité et un système à amortissage faible ce carré moyen peut devenir relativement grand. On peut attendre des oscillations spéciales auprès des valeurs de la fréquence pour lesquelles l'intensité spectrale change rapidement.

Resumo.

Ĉi tiu artikolo celas trovi proksimuman formulon por la distribuo de la energio en la spektro de funkcio de hazarda karaktero, ne enkondukinte la supozon ke tiu funkcio povas esti prezentata per integraĵo laŭ Fourier. Oni supozas ke elektra signalo proporcia al la funkcio ekzamenata estas sendata tra filtro-cirkvito; tiuokaze estas eble kalkuli la mezan kvadraton de la kurento eliranta el la filtro, kondiĉe ke la meza kvadrato kaj la koeficiento de korelacio de la funkcio de hazarda karaktero estas konataj. La formulo trovita dependas de parametro difinanta la gradon de selektiveco de la filtro; se ĉi tiu selektiveco fariĝas infinita, la formulo egaliĝas al la formulo donita de TAYLOR.

En la dua parto de la artikolo oni studas la mezan kvadraton de la indiko de galvanometro mezuranta la kurenton produktatan de varmopilo al kiu oni aplikas la kurenton elirantan el la filtro.

La trovita esprimo donas rilaton inter ĉi tiu meza kvadrato kaj la parametroj difinantaj la selektivecon de la filtro kaj la amortizon de la sistemo varmopilo-galvanometro; ĝi montras ke por filtro tre selektiva kaj amortizo malgranda, ĉi tiu meza kvadrato povas fariĝi relative granda.

Specialaj osciloj povas aperi plue apud tiaj frekvencoj apud kiaj la spektra energio rapide modifiĝas kun la frekvenco.

Zoology. — *On the specificity of the lithium effect on the development of Limnaea stagnalis.* By CHR. P. RAVEN and MARIA A. SIMONS.
(Zoological Laboratory, University of Utrecht.)

(Communicated at the meeting of November 27, 1948.)

Previous investigations have shown that a treatment of the eggs of *Limnaea stagnalis* with solutions of LiCl at an early stage may cause serious disturbances of development (RAVEN 1942; RAVEN, KLOEK, KUIPER and DE JONG 1947; RAVEN and RIJVEN 1948). Either, the eggs develop to vesicular exogastrulae, or embryos with malformations of the head region (monophthalmic, synophthalmic, cyclopic or anophthalmic embryos) are produced in a certain percentage of cases.

It may be asked if the abnormalities resulting from a treatment with LiCl are specific for this salt. In order to study this question, the action of LiCl has been compared with that of other chlorides.

Material and methods.

The following chlorides have been used: NaCl, KCl, LiCl, $MgCl_2$ and $CaCl_2$. In order to get reliable results, the groups of eggs treated with each of these salts had to be strictly comparable with each other. The reaction of each egg to the treatment will depend on its sensibility. This may vary from egg to egg; furthermore, previous investigations have shown that great differences in sensibility between different egg-masses exist. The individual variability cannot be eliminated; by distributing the eggs at random over the groups, and making the number of eggs in each group sufficiently great, this error variation can be minimized, however. In order to prevent the differences between egg-masses to invalidate the comparisons between the salts, equal numbers of eggs from each egg-mass must be treated with each salt. Therefore, in all experiments the eggs of one egg-mass were divided into 6 equal groups; 5 of these were each treated with one of the salts, the 6th is a control group in tap water. Hence, these 6 groups are comparable, and also the 6 sum-totals of all similar groups.

It has been found in previous experiments that the sensibility of *Limnaea* eggs to the action of LiCl changes during development (RAVEN et al. 1947, RAVEN and RIJVEN 1948). Sensible periods for the production of head malformations were found to exist immediately after oviposition and at the 24-cell stage, whereas exogastrulae are most easily produced at the time of 2nd cleavage. It appeared possible that similar differences in susceptibility would exist with respect to the other salts. Therefore, each egg-mass was divided into 3 equal parts, treated at different stages of development, viz.:

Stage I : immediately after oviposition.
Stage II : during 2nd cleavage.
Stage III: at the 24-cell stage.

The totals of these 3 stages, and of each salt per stage, are comparable, since also in this case the differences in susceptibility between egg-masses are eliminated by the procedure.

Furthermore, the concentrations of the salt solutions and the times of exposure have been varied. 3 Concentrations have been used: 1/40 Mol, 1/100 Mol and 1/400 Mol; the exposure times were ½ hour, 1 hour and 2 hours. Hence, there are 9 combinations of concentration and exposure time. The eggs of one egg-mass have all been treated according to one of these combinations; each combination has been tested with 3 egg-masses. Hence, 27 egg-masses have been used in total. As 9 different egg-masses have been employed for each of the 3 concentrations, their results are not strictly comparable, since the differences in sensibility between egg-masses cannot be eliminated in this comparison; the same holds true for the comparison between exposure times.

Each experiment was arranged in the following way: From a freshly-laid egg-mass about ⅓ was cut off; the egg capsules of this part were freed from the surrounding jelly and distributed equally over 5 tubes with equimolar solutions of NaCl, KCl, LiCl, $MgCl_2$ and $CaCl_2$, and 1 control tube with tap water. The salt solutions had also been prepared with tap water. The remaining ⅔ of the egg-mass were kept at about 25° C; when the eggs had reached the phase of 2nd cleavage, about half of the remaining egg capsules were treated in the same way as before, and with the same concentration and exposure time. The remainder was cooled during the night with running water, in order to slow down somewhat the rate of development, and treated the next morning. In later experiments, this latter part was cooled from the beginning of the experiment on.

As a rule, each group of eggs consisted of 6 eggs (per salt per stage). Hence, theoretically 108 eggs $(3 \times 6 \times 6)$ per egg-mass were used; in total, the number of eggs employed should be $27 \times 108 = 2916$ which should be equally distributed as concerns salts (486 eggs each), stages, concentrations and exposure times (972 eggs each). In fact, slight variations of these numbers occur, as occasionally some eggs were lost in handling, whereas in some instances 7 eggs per group were used. These small deviations do not seriously interfere with the reliability of the results.

After exposure to the salt solutions the eggs were transferred to tap water, which was renewed several times; after 24 hours, they were placed in petri dishes on a bottom of 2½ % agar, and cultured in the usual way.

The embryos were reared till a stage, in which the eyes, in normal embryos, are well-developed (advanced "hippo"-stage); their mode of development was judged by their condition at this moment, without considering their previous history.

As "head malformations" all microphthalmic, monophthalmic, synophthalmic, triophthalmic, cyclopic and anophthalmic embryos are classed; furthermore, embryos with evaginations of the buccal organs are brought under this heading, as also this malformation seems to be caused by a reduction of median parts of the head; often, it is attended with abnormalities of the eyes.

Exogastrulae are large vesicular structures, as described by RAVEN (1942).

As "unspecific abnormalities" all malformations of body or head, which do not belong to one of the above-mentioned classes, are summarized; in all these cases the eyes are normal.

As "dead" all embryos have been counted, which did not reach the "hippo"-stage and had not developed to exogastrulae.

Results.

Table I shows the results, grouped according to salts.

From this table, the following conclusions can be drawn:

Both head malformations and exogastrulae are no strictly-specific lithium effects, as a few instances of these abnormalities are also produced by the other salts. However, the number of these malformations in the Li-treated embryos is much greater, and surpasses largely that in all other salts together; this difference is statistically significant, both as regards head malformations and exogastrulae. Hence, it is clear that lithium differs considerably, be it only gradually, from the other ions in its action on the eggs.

Among the other ions, potassium seems to be the most active in the production of these malformations; however, this difference is not statistically significant. In the controls neither head malformations nor exogastrulae appear.

As regards the "unspecific malformations" no significant differences exist between the action of the various salts and in comparison with the controls.

No significant differences in mortality between the groups exist.

In table II, the results of the experiments are grouped according to stage of treatment.

It appears from this table that both head malformations and exogastrulae are produced especially after a treatment at stage II. As regards unspecific malformations, no significant differences exist between the stages, but the mortality is significantly higher after a treatment at stage II.

Table III gives the number of head malformations and exogastrulae, grouped according to salt and stage (each group consisting of about 162 eggs).

It shows that the increased susceptibility of the eggs at stage II applies especially to a treatment with LiCl, whereas the head malformations and

TABLE I

Salts	Number of eggs	Head malformations		Exogastrulae		Unspecific malformations		Dead	
		Number	%	Number	%	Number	%	Number	%
NaCl	488	2	0.41 ± 0.289	0	0	11	2.25 ± 0.671	46	9.63 ± 1.335
KCl	479	4	0.84 ± 0.417	7	1.46 ± 0.548	6	1.25 ± 0.508	36	7.51 ± 1.204
LiCl	481	**37**	$\mathbf{7.69 \pm 1.215}$	**26**	$\mathbf{5.41 \pm 1.031}$	12	2.50 ± 0.712	50	10.40 ± 1.392
$MgCl_2$	488	1	0.20 ± 0.202	2	0.41 ± 0.289	7	1.43 ± 0.537	42	8.60 ± 1.269
$CaCl_2$	482	2	0.41 ± 0.291	1	0.21 ± 0.209	5	1.04 ± 0.462	48	9.96 ± 1.364
Controls	473	0	0	0	0	7	1.48 ± 0.555	45	9.52 ± 1.350
Total	2891	46		36		48		267	

TABLE II

Stage of treatment	Number of eggs	Head malformations		Exogastrulae		Unspecific malformations		Dead	
		Number	%	Number	%	Number	%	Number	%
I Uncleaved	977	3	0.31 ± 0.178	6	0.61 ± 0.249	19	1.94 ± 0.441	71	7.27 ± 0.831
II 2nd cleavage	963	**39**	$\mathbf{4.05 \pm 0.635}$	**26**	$\mathbf{2.70 \pm 0.522}$	11	1.14 ± 0.342	109	11.32 ± 1.021
III 24-cell stage	951	4	0.42 ± 0.210	4	0.42 ± 0.210	18	1.89 ± 0.442	87	9.15 ± 0.935
Total	2891	46		36		48		267	

exogastrulae produced with the other salts are not restricted to a special stage of treatment.

TABLE III

Salts	Head malformations				Exogastrulae			
	Stage I	Stage II	Stage III	Total	Stage I	Stage II	Stage III	Total
NaCl	1	1	0	2	0	0	0	0
KCl	2	2	0	4	4	2	1	7
LiCl	0	35	2	37	2	23	1	26
$MgCl_2$	0	0	1	1	0	1	1	2
$CaCl_2$	0	1	1	2	0	0	1	1
Total	3	39	4	46	6	26	4	36

Finally, table IV gives the number of head malformations and exogastrulae, produced by LiCl at stage II, grouped according to concentration and exposure time (each group consisting of 18 eggs from 3 egg-masses).

TABLE IV

Concentration	Head malformations				Exogastrulae			
	Exposure times				Exposure times			
	½ h.	1 h.	2 h.	Total	½ h.	1 h.	2 h.	Total
1/400 Mol	4	4	5	13	0	0	7	7
1/100 Mol	3	6	2	11	3	8	0	11
1/40 Mol	1	4	6	11	0	4	1	5
Total	8	14	13	35	3	12	8	23

Though the numbers are not strictly comparable, as the eggs of each group come from different egg-masses, the table shows that head malformations may arise with each combination of concentration and exposure time tested. The exogastrulae, on the other hand, show no such regular distribution among the groups; with the lowest concentration used (1/400 Mol), exogastrulae were obtained only in the group with 2 hours' exposure time. No definite conclusion can be drawn, however, the number of cases being too small.

Discussion.

1. Head malformations may be produced, besides by LiCl, also by NaCl, KCl, $MgCl_2$ and $CaCl_2$; exogastrulae by KCl, $MgCl_2$ and $CaCl_2$. In all these cases, however, the number of malformations produced is much less than with LiCl. We may say, therefore, that both kinds of malformations are relatively (but not absolutely) specific for the action of lithium. In no case these malformations have been observed thus far in control eggs.

2. The number of abnormalities, described as "unspecific malform-

ations", is not increased by any of these salts as compared with untreated controls. Hence, their denomination as "unspecific" seems to be justified for the time being.

3. Mortality is highest when the eggs are treated at the time of 2nd cleavage; it is evident that the general resistance of the eggs against injurious agents is low at this moment. This agrees with previous experiments where it has been found that the eggs at this stage of 2nd cleavage are very susceptible against hypotonicity and urea (RAVEN and KLOMP 1946), against thiourea (SOBELS 1948) and LiCl (RAVEN, KLOEK, KUIPER and DE JONG 1947).

4. With LiCl, both head malformations and exogastrulae have been produced in these experiments especially after a treatment at the stage of 2nd cleavage. After a treatment at the 24-cell stage, only a few of both malformations have been produced, whereas an exposure of the eggs to LiCl immediately after oviposition yielded only a few exogastrulae, but no head malformations at all. These results do not agree with those of RAVEN, KLOEK, KUIPER and DE JONG (1947) and of RAVEN and RIJVEN (1948), where it was shown that the eggs of *Limnaea* exhibit a phase-specific sensibility for the production of exogastrulae at the moment of 2nd cleavage, whereas maxima of sensibility for the production of head malformations were found to exist immediately after oviposition and again at the 24-cell stage.

In order to explain this discrepancy, the following points must be considered: It is fairly certain that the moment of treatment at stage I (immediately after oviposition) in the present experiments corresponded to the maximum of sensibility found in earlier experiments. As regards stage III (24-cell stage), we are not altogether sure in this respect. The results of RAVEN and RIJVEN (1948) show that a maximum of susceptibility is reached 2—3 hours after the beginning of the 24-cell stage, while a rather sudden drop in susceptibility occurs after this moment. Since these facts were not yet known to us, when our experiments were begun, the moment at which the eggs reached the 24-cell stage has not been determined. Hence, it is possible that in many cases the eggs had passed the maximum of susceptibility when they were treated with LiCl in the morning. Furthermore, it might be possible that their susceptibility has been influenced by the preceding cooling during the night.

Moreover, it has to be taken into account that in these experiments the concentrations of LiCl used differ from those employed in previous experiments. The combination of 1/400 Mol, 1 hour corresponds to the one used by RAVEN and RIJVEN (1948) and in part of the experiments of RAVEN et al. (1947); for the rest, the concentrations and exposure times differ, the former being, in general, much greater in the present experiments. Therefore, a strict correspondence might be expected to occur in these corresponding test groups only. When we further consider that each of these groups, as given in table IV, consists of 18 eggs only, coming from 3 egg-masses, and take into account the well-known capriciousness of the Li-effect (in RAVEN and RIJVEN's experiments only 25 out of 57 egg-masses yielded any head malformations), it is clear that this lack of agreement between experiments of such different design is not astonishing. We can only conclude, therefore, that (for some or other of the above-mentioned reasons) the susceptibility of the *Limnaea* egg to LiCl at stages I and III does not appear in these experiments, and that in stage II, besides exogastrulae, also head malformations can be induced under these circumstances.

5. The results of Table IV show that the induction of head malformations by LiCl can take place with various combinations of concentration and exposure time. The same holds good for exogastrulae, though some combinations yielded no exogastrulae in these experiments. In general, it can be said that the number and nature of malformations induced by LiCl is not very dependent on concentration and exposure time.

Summary.

1. Eggs of *Limnaea stagnalis* have been treated at 3 different stages with solutions of NaCl, KCl, LiCl, $MgCl_2$ and $CaCl_2$, of varying concentration and for different times of exposure.

2. Head malformations have been produced by all of these salts; exogastrulae by all except NaCl. The number of these malformations is much greater in LiCl than in the other salts, however. Both head malformations and exogastrulae are, therefore, relatively specific Li-effects.

3. Other malformations show no significant increase after treatment with these salts.

4. The resistance of the eggs against injurious agents is low at the moment of 2nd cleavage.

5. Both head malformations and exogastrulae have been induced by LiCl mainly at this stage of 2nd cleavage.

6. The number and nature of malformations induced by LiCl is not very dependent on concentration and exposure time.

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Mathematics. — *Opmerkingen over het beginsel van het uitgesloten derde en over negatieve asserties.* By L. E. J. BROUWER.

(Communicated at the meeting of November 27, 1948.)

§ 1.

Onder een *aanstuiving* verstaan we de vereniging γ van een positief convergente fundamentaalreeks van van elkaar verwijderd liggende reële getallen $c_1(\gamma), c_2(\gamma), \ldots$, de *telgetallen* der aanstuiving, en hun van iedere $c_\nu(\gamma)$ verwijderd ondersteld limiet $c(\gamma)$, de *kern* der aanstuiving. Zij α een nog niet toetsbaar gebleken wiskundige assertie. Dan kan het scheppende subject naar het volgende voorschrift een onbegrensd voortschrijdende sequentie $R(\gamma, \alpha)$ van reële getallen $c_1(\gamma, \alpha)$, $c_2(\gamma, \alpha), \ldots$ creëren: Zolang bij de keuze der $c_n(\gamma, \alpha)$ aan het scheppende subject nòch de juistheid, nòch de absurditeit van α is gebleken, wordt iedere $c_n(\gamma, \alpha)$ gelijk aan $c(\gamma)$ gekozen. Zodra echter tussen de keuze van $c_{r-1}(\gamma, \alpha)$ en die van $c_r(\gamma, \alpha)$ aan het scheppende subject hetzij de juistheid, hetzij de absurditeit van α is gebleken, wordt zowel $c_r(\gamma, \alpha)$ als voor ieder natuurlijk getal ν ook $c_{r+\nu}(\gamma, \alpha)$ gelijk aan $c_r(\gamma)$ gekozen. Deze sequentie $R(\gamma, \alpha)$ convergeert positief tot een reëel getal $D(\gamma, \alpha)$, dat we een *dempingsgetal van* γ *door* α noemen.

§ 2.

Het *enkelvoudige beginsel van het uitgesloten derde* luidt als volgt:

Een toewijzing τ van een eigenschap aan een wiskundige entiteit kan worden b e o o r d e e l d, d.w.z. hetzij haar juistheid, hetzij haar absurditeit kan worden bewezen.

Voor een enkele zodanige assertie τ is de formulering van dit beginsel niet-contradictoor. Immers was het contradictoor, dan zou de absurditeit van τ tegelijk juist en absurd zijn, hetgeen onmogelijk is. Een eenvoudige redenering leert, dat voor een *eindig* aantal zodanige asserties τ de simultane formulering van het beginsel eveneens niet-contradictoor is. Doch voor de simultane formulering van het beginsel voor alle elementen van een *willekeurige* soort van zodanige asserties τ blijft deze niet-contradictoriteit niet gehandhaafd.

Bijvoorbeeld kan uit de onderstelling voor een bepaald reëel getal c_1, dat de assertie: c_1 *is rationaal*, hetzij juist hetzij contradictoor is gebleken, geen contradictie worden afgeleid. Zijn verder $c_1, \ldots c_m$ reële getallen, dan kan uit de simultane onderstelling voor $\nu = 1, \ldots, \nu = m$, dat de assertie: c_ν *is rationaal*, hetzij juist hetzij contradictoor is gebleken, evenmin een contradictie worden afgeleid. Doch de simultane onderstelling voor *alle* reële getallen c, dat de assertie: *c is rationaal*, hetzij juist hetzij contradictoor is gebleken, kan wel degelijk ad absurdum worden gevoerd.

Formuleren we dus als volgt het *volledige beginsel van het uitgesloten derde:*

Ondersteld dat a, b en c soorten van wiskundige entiteiten zijn, dat a en b deelsoorten zijn van c en dat b bestaat uit de elementen van c die onmogelijk tot a kunnen behoren, dan is c identiek met de vereniging van a en b,

dan is dit laatste beginsel contradictoor.

§ 3.

Een corollarium van het *enkelvoudige* beginsel van het uitgesloten derde zegt dat *indien voor een toewijzing* τ *van een eigenschap aan een wiskundige entiteit de niet-contradictoriteit, d.w.z. de absurditeit der absurditeit is bewezen, eveneens de juistheid van* τ *kan worden gedemonstreerd.*

Het analoge corollarium van het *volledige* beginsel van het uitgesloten derde is het *beginsel van reciprociteit der complementariteit* [1]), luidende als volgt:

Ondersteld dat a, b en c soorten van wiskundige entiteiten zijn, dat a en b deelsoorten zijn van c en dat b bestaat uit de elementen van c die onmogelijk tot a kunnen behoren, dan bestaat a uit de elementen van c die onmogelijk tot b kunnen behoren.

Een tweede corollarium van het *enkelvoudige* beginsel van het uitgesloten derde is het *enkelvoudige beginsel van toetsbaarheid*, luidende aldus:

Een toewijzing τ *van een eigenschap aan een wiskundige entiteit kan worden g e t o e t s t, d.w.z. hetzij haar niet-contradictoriteit, hetzij haar absurditeit kan worden bewezen.*

Het analoge corollarium van het *volledige* beginsel van het uitgesloten derde is het volgende *volledige beginsel van toetsbaarheid:*

Ondersteld dat a, b, d en c soorten van wiskundige entiteiten zijn, dat a, b en d deelsoorten zijn van c, dat b bestaat uit de elementen van c die onmogelijk tot a kunnen behoren en dat d bestaat uit de elementen van c die onmogelijk tot b kunnen behoren, dan is c identiek met de vereniging van b en d.

Als we de assertie ener absurditeit een *negatieve assertie* noemen, dan is voor negatieve asserties binnen een gegeven soort van wiskundige entiteiten ten eerste het beginsel van reciprociteit der complementariteit steeds van

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[1]) Dit is zo te verstaan, dat indien binnen een gegeven soort van wiskundige entiteiten voor een gegeven eigenschap het beginsel van het uitgesloten derde geldt, binnen die soort voor die eigenschap eveneens het beginsel van reciprociteit der complementariteit van kracht is. De eveneens juiste, en bovendien, naar Jahresber. d. D. M. V. **33**, p. 252 (1924) werd opgemerkt, omkeerbare, interpretatie dat, zo voor alle wiskundige asserties het beginsel van het uitgesloten derde gold, eveneens voor alle wiskundige asserties het beginsel van reciprociteit der complementariteit van kracht zou zijn, is zonder wiskundige betekenis.

kracht, ten tweede het beginsel van toetsbaarheid æquivalent met het beginsel van het uitgesloten derde.

Zij namelijk α een negatieve assertie, uitsprekende de absurditeit der assertie β. Daar enerzijds de implicatie der waarheid ener assertie a door de waarheid ener assertie b de implicatie der absurditeit van b door de absurditeit van a impliceert, anderzijds de waarheid van β de absurditeit der absurditeit van β impliceert, wordt de absurditeit van β, d.w.z. α, geïmpliceerd door de absurditeit der absurditeit der absurditeit van β, d.w.z. door de non-contradictoriteit van α.

Verder, indien binnen een gegeven soort van wiskundige entiteiten voor α het beginsel van toetsbaarheid geldt, kan voor iedere wiskundige entiteit van deze soort hetzij de absurditeit der absurditeit van β hetzij de niet-contradictoriteit der absurditeit van β, d.w.z., op grond van de vorige alinea, hetzij de absurditeit der absurditeit van β hetzij de waarheid der absurditeit van β, d.w.z. hetzij de absurditeit hetzij de waarheid van α worden bewezen, zodat binnen de bedoelde soort voor α het beginsel van het uitgesloten derde van kracht is.

§ 4.

Klaarblijkelijk is het geldigheidsgebied van het beginsel van het uitgesloten derde identiek met de doorsnede der geldigheidsgebieden van het beginsel van toetsbaarheid en van het beginsel van reciprociteit der complementariteit. Verder is eerstgenoemd geldigheidsgebied een *eigenlijk* deelgebied van elk van beide laatstgenoemde geldigheidsgebieden, zoals uit de volgende voorbeelden blijkt:

Zij A de soort der dempingsgetallen van aanstuivingen met rationale telgetallen, B de soort der irrationale reële getallen, C de vereniging van A en B. Dan voldoen alle rationaliteitsasserties van een element van C aan het beginsel van toetsbaarheid, terwijl er rationaliteitsasserties van een element van C zijn, die niet aan het beginsel van het uitgesloten derde voldoen. Verder voldoen alle gelijkheidsasserties van twee reële getallen aan het beginsel van reciprociteit der complementariteit, terwijl er gelijkheidsasserties van twee reële getallen zijn, die niet aan het beginsel van het uitgesloten derde voldoen.

§ 5.

Voor wiskundige asserties is de absurditeitseigenschap, evenals de juistheidseigenschap, een *universeel additieve eigenschap*, d.w.z. als ze geldt voor ieder element α ener assertiesoort, geldt ze ook voor de assertie, die de vereniging der asserties α is. *Deze universele additiviteit bestaat niet voor de niet-contradictoriteitseigenschap.* Wèl bezit niet-contradictoriteit *eindige additiviteit*, d.w.z. als de asserties ϱ en σ niet-contradictoor zijn, is de assertie τ die de vereniging is van ϱ en σ, eveneens niet-contradictoor. Immers gaan wij uit van de onderstelling ω, dat τ contradictoor is, dan zou de juistheid van ϱ de contradictoriteit van σ ten gevolge hebben, die tegen de gegevens zou indruisen, zodat de juistheid van ϱ absurd is, d.w.z. ϱ is absurd. Maar deze consequentie der onderstelling ω is in strijd met de gegevens. Derhalve is de onderstelling ω contradictoor, d.w.z. τ is niet-contradictoor.

Door toepassing van dit theorema op de speciale niet-contradictore asserties, die door formuleringen van het beginsel van het uitgesloten derde voor een enkele assertie worden opgeleverd, blijkt de in § 2 vermelde niet-contradictoriteit der simultane formulering van genoemd beginsel voor een eindig aantal asserties.

§ 6.

Binnen een bepaalde soort van wiskundige entiteiten kunnen de absurditeiten van twee niet-æquivalente [2]) asserties æquivalent zijn. Bijvoorbeeld levert elk der volgende [3]) drie paren van niet-æquivalente asserties betreffende een reëel getal a:

I 1. $a = a$; I 2. hetzij $a \leq 0$, hetzij $a \geq 0$

II 1. $a \geq 0$; II 2. hetzij $a = 0$, hetzij $a \circ\!> 0$

III 1. $a > 0$; III 2. $a \circ\!> 0$

twee æquivalente absurditeiten.

§ 7.

Aan sommige absurditeiten van constructieve eigenschappen kan binnen een bepaalde soort van wiskundige entiteiten een constructieve vorm worden gegeven. Zo is voor elk natuurlijk getal a de absurditeit van het bestaan van twee van a en van 1 verschillende natuurlijke getallen, welker product a is, æquivalent met het optreden van een rest, wanneer a door een willekeurig van a en van 1 verschillend natuurlijk getal wordt gedeeld. Evenzo kan voor twee reële getallen a en b de in noot [3]) als absurditeit ener constructieve eigenschap ingevoerde relatie $a \geq b$ als volgt constructief worden geformuleerd: Als a en b achtereenvolgens zijn gedefinieerd door de convergente onbepaald voortschrijdende sequenties van rationale getallen $a_1, a_2, \ldots$ en $b_1, b_2, \ldots$, kan voor ieder natuurlijk getal n een zodanig natuurlijk getal m worden berekend dat $a_\nu - b_\nu \circ\!> -2^{-n}$ voor $\nu \geq m$.

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[2]) Onder niet-æquivalentie wordt verstaan absurditeit van æquivalentie, zoals onder niet-contradictoriteit wordt verstaan absurditeit van contradictoriteit.

[3]) Als voor twee reële getallen a en b, achtereenvolgens gedefinieerd door de convergente onbepaald voortschrijdende sequenties van rationale getallen $a_1, a_2, \ldots$ en $b_1, b_2, \ldots$ twee zodanige natuurlijke getallen m en n kunnen worden berekend dat $b_\nu - a_\nu > 2^{-n}$ voor $\nu \geq m$, schrijven wij $b \circ\!> a$ en $a <\!\circ\, b$. Absurditeit van $a = b$ wordt uitgedrukt door $a \neq b$, absurditeit van $a <\!\circ\, b$ door $a \geq b$, simultane absurditeit van $a = b$ en $a <\!\circ\, b$ door $a > b$. De absurditeiten van $a <\!\circ\, b$ en $a < b$ blijken dan onderling æquivalent en de absurditeit van $a \geq b$ blijkt æquivalent met $a < b$.

Doch voor sommige andere absurditeiten van constructieve eigenschappen schijnt er weinig hoop te zijn, dat ooit een constructief æquivalent zal worden gevonden. In een vorige mededeling [4]) is dit gedemonstreerd aan de relaties $a \neq b$ en $a > b$ bij reële getallen a en b. Voor de irrationaliteitseigenschap bij reële getallen blijkt er, als we haar constateren bij de dempingsgetallen van aanstuivingen met rationale kern en irrationale telgetallen, even weinig uitzicht op een constructief æquivalent te zijn.

§ 8.

Aan sommige niet-contradictoriteiten van constructieve eigenschappen ζ kan binnen een bepaalde soort van wiskundige entiteiten hetzij een constructieve vorm (al of niet ingevolge het bestaan voor ζ van reciprociteit der complementariteit), hetzij de vorm van absurditeit ener constructieve eigenschap worden gegeven. Zo is voor reële getallen a en b de niet-contradictoriteit van $a = b$ æquivalent met $a = b$ en de niet-contradictoriteit van: *hetzij* $a = b$ *hetzij* $a \circ\!> b$, æquivalent met $a \geq b$; verder de niet-contradictoriteit van $a \circ\!> b$ æquivalent zowel met de absurditeit van $a \leq b$, als met de absurditeit van: *hetzij* $a = b$ *hetzij* $a <\!\circ\, b$.

Doch voor sommige andere niet-contradictoriteiten van constructieve eigenschappen schijnt er weinig hoop te zijn, dat ooit een dergelijk æquivalent zal worden gevonden. Men denke bijvoorbeeld aan niet-contradictoriteit der rationaliteit bij reële getallen en constatere dan deze eigenschap bij de dempingsgetallen van aanstuivingen met rationale telgetallen.

§ 9.

Verstaan we onder *enkelvoudige absurditeit* der eigenschap η de absurditeit van η en onder $(n+1)$*-voudige absurditeit* van η de absurditeit der n-voudige absurditeit van η, dan brengt een in § 3 afgeleid theorema tot uitdrukking, dat *drievoudige absurditeit æquivalent is met enkelvoudige absurditeit*. En een corollarium van dit theorema zegt dat *n-voudige absurditeit æquivalent is met enkelvoudige of met dubbele absurditeit, al naarmate n oneven of even is* [5]).

[4]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, p. 963 (1948).

[5]) Vgl. Jahresber. d. D. M. V. **33**, p. 253 (1924).

Mathematics. — *On Differentiable Linesystems of one Dual Variable.* II.

By N. H. KUIPER (Princeton N.Y.) (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 30, 1948.)

9. *The strictionsurface of a non-degenerate D-system.*

The strictionpoint at a rule $\mathfrak{A}$ of a regulus is the intersection with the common perpendicular of $\mathfrak{A}$ and a rule infinitesimally near to $\mathfrak{A}$. It is the origin of the Blaschke-system of the regulus, which is also the invariant orthogonal system of the related D-system at the considered line. The strictionsurface of a D-system is the locus of strictionpoints of the reguli in the D-system. It is the locus of origins of first invariant orthogonal systems of the D-system.

We first determine the strictionpoints on the lines $\mathfrak{A}(T) = \mathfrak{A}(t) + \varepsilon \bar{t} \dot{\mathfrak{A}}(t) = \mathfrak{A}(t) + \varepsilon \bar{t} P(t)\, \mathfrak{A}_1(t)$ (t fixed, $\bar{t}$ variable) of a D-system. These strictionpoints lie in the plane $\mathfrak{A}(t), \mathfrak{A}_2(t)$, and on the lines (resp.)

$$\mathfrak{A}_1(t+\varepsilon\bar{t}) = \mathfrak{A}_1(t) + \varepsilon\bar{t}\dot{\mathfrak{A}}_1(t) = \mathfrak{A}_1 + \varepsilon\bar{t}(-P\mathfrak{A} + Q\mathfrak{A}_2) \quad ((9))$$

These strictionpoints are therefore the points of the line

$$\frac{q\,\mathfrak{A} + p\,\mathfrak{A}_2}{V(p^2+q^2)} = \cos\varphi \cdot \mathfrak{A} + \sin\varphi \cdot \mathfrak{A}_2 \quad (T=t).$$

The *strictionsurface of the D-system* is the ruled surface

$$\mathfrak{R}(t) = \frac{q\,\mathfrak{A} + p\,\mathfrak{A}_2}{V(p^2+q^2)} = \cos\varphi \cdot \mathfrak{A} + \sin\varphi \cdot \mathfrak{A}_2 \, . \quad . \quad . \quad . \quad (16)$$

The line $\mathfrak{A}_1(T)$ is at the line $\mathfrak{A}(T)$ perpendicular to any of the strictioncurves of the reguli in the D-system. Hence $\mathfrak{A}_1(T)$ is a normal of the strictionsurface. $\mathfrak{A}_1(t+\varepsilon\bar{t})$ (t fixed, $\bar{t}$ variable) has a constant direction $\mathfrak{a}_1(t)$. The surfacenormals along a rule of $\mathfrak{R}(t)$ have constant direction and so $\mathfrak{R}(t)$ is developable.

Theorem 7. *The strictionsurface of a non-degenerate D-system $\mathfrak{A}(T)$ is a developable regulus. The D-system $\mathfrak{A}_1(T)$ consists of the normals of this strictionsurface.*

10. *The intrinsic equations of a D-system* [5]).

$\mathfrak{A}(T)$, representing a D-system, was assumed to be analytic:

$$\mathfrak{A}(T) = \sum_{n=0}^{\infty} \frac{d^n \mathfrak{A}(0)}{d T^n} \frac{T^n}{n!} \qquad K_1 < T < K_2.$$

[5]) Compare: BLASCHKE [3], DWINGER [4].

From the equations (9), and equations obtained by derivation with respect to T from (9), $\mathfrak{A}(T)$ is seen to be determined by $\mathfrak{A}(0)$, $\mathfrak{A}_1(0)$, $\mathfrak{A}_2(0)$, $P(T)$ and $Q(T)$ (or $\Phi(T)$) · $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(0)$ can be transformed into any other equally oriented orthogonal system by a motion. The D-system is therefore, but for a motion, determined by the "intrinsic equations"

$$P = P(T),\ Q = Q(T) \quad . \ . \ . \ . \ . \ . \ . \ . \quad (17)$$

or also by

$$P = P(T),\ \Phi = \Phi(T) \qquad ((9) \text{ and } (11)) \quad . \ . \ . \ . \quad (17)$$

A non-degenerate D-system can be represented with the help of the invariant dual parameter. We get *one* intrinsic equation:

$$\Phi = \Phi(S) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (18)$$

In § 3 we stated that *D-systems are in better analogy with sphere-curves than reguli.* Indeed: a non-degenerate D-system can be characterised intrinsically by one equation (18), analogous to an intrinsic equation of a sphere curve. This is not true for a regulus (with invariant δ; § 6).

11. *The second invariant orthogonal system at a line of a D-system.*

In the theory of threedimensional curves an invariant orthogonal system γ at a point of a curve consists of the tangent, normal and binormal. The first invariant system of § 4 is not strictly analogous to this system. It was possible to choose it because we consider dual *unit* vectors. We call the system Γ analogous to γ, the second invariant orthogonal system at a line of a non-degenerate (assumed) D-system. The three mutually perpendicular and intersecting axes of this system are $(\mathfrak{A} = \mathfrak{A}(S))$

$$\dot{\mathfrak{A}} = \mathfrak{A}_1,\ \mathfrak{A}_2^* = \dot{\mathfrak{A}}_1/\sqrt{\dot{\mathfrak{A}}_1^2},\ \mathfrak{A}_3^* = \mathfrak{A}_1 \times \mathfrak{A}_2^* \quad . \ . \ . \ . \quad (19)$$

It is related with the first system as follows (11))

$$\left.\begin{aligned} \mathfrak{A}_2^* &= -\sin\Phi\cdot\mathfrak{A} + \cos\Phi\cdot\mathfrak{A}_2 \\ \mathfrak{A}_3^* &= -\cos\Phi\cdot\mathfrak{A} - \sin\Phi\cdot\mathfrak{A}_2\,(= -\mathfrak{S}) \end{aligned}\right\} \quad . \ . \ . \ . \quad (20)$$

The formulas analogous to the *formulas of* Frenet–Serret for threedimensional curves are ((11), (19), (20), (9))

$$\left.\begin{aligned} \dot{\mathfrak{A}}_1 &= \quad C\mathfrak{A}_2^* &&= \operatorname{cosec}\Phi\,\mathfrak{A}_2^* \\ \dot{\mathfrak{A}}_2^* &= -C\mathfrak{A}_1 + T\mathfrak{A}_3^* &&= -\operatorname{cosec}\Phi\cdot\mathfrak{A}_1 + \dot{\Phi}\mathfrak{A}_3^* \\ \dot{\mathfrak{A}}_3^* &= \quad -T\mathfrak{A}_2^* &&= -\dot{\Phi}\,\mathfrak{A}_2^* \end{aligned}\right\} \quad (21)$$

$C = \operatorname{cosec}\Phi$ is *the dual curvature* of the non-degenerate D-system; $T = \dot{\Phi}$ is *the dual torsion*. We have the formula:

$$\text{dual torsion} = \frac{d}{dS}\operatorname{cosec}^{-1}(\text{dual curvature}) \quad . \ . \ . \ . \quad (22)$$

D-systems for which the dual torsion vanishes, have a constant dual curvature. Their momentanous axis $\mathfrak{S} = -\mathfrak{A}_3^*$ is constant. They are analogous to circles, and each consists of lines which have a constant dual angle with a constant line. If this dual angle is $\Phi = \pi/2 + \varepsilon \cdot 0$ then consists $\mathfrak{A}(S)$ of the perpendiculars to a line. Such a D-system is analogous to a geodesic on the sphere. If we take two non-parallel non-intersecting lines of the "*geodesic D-system*", then there exists a doubly infinite set of lines in the D-system, for any of which in combination with the given lines the triangle-equality holds (6).

(21) can be considered as the system of equations (9) with respect to the D-system $\mathfrak{A}_3^* = -\mathfrak{S}$. The invariant parameter of $\mathfrak{A}_3^*$ is $-\Phi(+K)$. Under the assumption that $\mathfrak{A}_3^*$ is not degenerated, the equations (11) for $\mathfrak{A}_3^*$ become ((see (21))

$$\left.\begin{aligned}\left(\frac{d\mathfrak{S}}{d\Phi}=\right)\frac{d\mathfrak{A}_3^*}{d-\Phi} &= \mathfrak{A}_2^* \\ \frac{d\mathfrak{A}_2^*}{d-\Phi} &= -\mathfrak{A}_3^* + \frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\cdot\mathfrak{A}_1 \\ \frac{d\mathfrak{A}_1}{d-\Phi} &= -\frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\cdot\mathfrak{A}_2^*\end{aligned}\right\} . \quad (23)$$

The formulas (11) for the enveloped D-system $\mathfrak{S} = -\mathfrak{A}_3^*$ are

$$\left.\begin{aligned}\frac{d\mathfrak{S}}{d\Phi} &= \mathfrak{S}_1 \\ \frac{d\mathfrak{S}_1}{d\Phi} &= -\mathfrak{S} - \frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\cdot\mathfrak{S}_2 \\ \frac{d\mathfrak{S}_2}{d\Phi} &= \frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\,\mathfrak{S}_1\end{aligned}\right\} \quad . \quad . \quad . \quad (23')$$

The angle $\Phi^{(2)}$ between a rule of the "enveloped D-system" $\mathfrak{S}$ and its momentanous axis $\mathfrak{S}^{(2)}$, arclength of $\mathfrak{S}^{(2)}$, is obtained from (23') and (11):

$$\left.\begin{aligned}\operatorname{cotg}\Phi^{(2)} &= -\operatorname{cosec}\Phi\Big/\frac{d\Phi}{dS} \\ \operatorname{tg}\Phi^{(2)} &= \frac{d\cos\Phi}{dS}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

(24) is analogous to the following formula for a Euclidean plane curve

$$r' = r\,\frac{dr}{ds}$$

s is the arclength of the curve, r is the radius of curvature of the curve, r' is the radius of curvature of the evolute of the curve.

From (20), (23), (23') we get:

$$\left.\begin{aligned}\mathfrak{S} = \cos\Phi\cdot\mathfrak{A} + \sin\Phi\cdot\mathfrak{A}_1, \quad \mathfrak{S}_1 &= -\sin\Phi\cdot\mathfrak{A} + \cos\Phi\cdot\mathfrak{A}_2 \\ \mathfrak{A} = \cos\Phi\cdot\mathfrak{S} - \sin\Phi\cdot\mathfrak{S}_1 &\end{aligned}\right\} \quad (25)$$

12. *Osculating D-systems.*

In general one can define the n-th enveloped $\mathfrak{S}^{(n)}(T)$ of a D-system $\mathfrak{A}(T)$, if it exists, inductively by: $\mathfrak{S}^{(k+1)}(T)$ is the enveloped D-system with respect to the developed D-system $\mathfrak{S}^{(k)}(T) \cdot k = 0, 1, \dots n-1$. A generalisation of the formula (24) is

$$\operatorname{tg} \Phi^{(k+1)} = \frac{d \cos \Phi^{(k)}}{d \Phi^{(k-1)}} = -\sin \Phi^{(k)} \frac{d \Phi^{(k)}}{dT} \Big/ \frac{d \Phi^{(k-1)}}{dT} \quad . \quad . \quad . \quad (26)$$

$\Phi^{(i)}(T)$ is the dual arclength of $\mathfrak{S}^{(i)}(T)$.

The existence of the n-th enveloped at a line of a D-system, admits us to construct a rather simple D-system which osculates of order n with the given D-system at the considered line.

Two non-degenerate D-systems $\mathfrak{A}(S)$ and $\mathfrak{B}(S)$ are said to osculate of order n at the line $S = 0$, when (compare SABAN [9])

$$\mathfrak{A} = \mathfrak{B} \quad \text{and} \quad \frac{d^i \mathfrak{A}}{dS^i} = \frac{d^i \mathfrak{B}}{dS^i} \; i = 1, \dots, n+1 \quad S = 0,$$

equivalent to:

$$\mathfrak{A} = \mathfrak{B},\; \mathfrak{A}_1 = \mathfrak{B}_1,\; \Phi_a = \Phi_b,\; \frac{d^i \Phi_a}{dS} = \frac{d^i \Phi_b}{dS}\,;\, i = 1, \dots, n-1 \quad S = 0,$$

equivalent to ((25) and equations obtained by differentiation of (25)):

$$\mathfrak{A} = \mathfrak{B},\; \mathfrak{A}_1 = \mathfrak{B}_1,\; \Phi_a = \Phi_b,\; \Phi_a^{(i)} = \Phi_b^{(i)}\,;\, i = 2, \dots, n \quad S = 0,$$

equivalent to:

$$\mathfrak{A} = \mathfrak{B},\; \mathfrak{S}_a^{(i)} = \mathfrak{S}_b^{(i)}\,;\; i = 1, \dots, n \;\; S = 0.$$

A rather simple D-system which osculates of order n with a given D-system $\mathfrak{A}(S)$ with existing n-th developable at the line $S = 0$, is the D-system $\mathfrak{B}(S)$ for which:

$$\mathfrak{B}(0) = \mathfrak{A}(0),\; \mathfrak{S}_b^{(i)}(0) = \mathfrak{S}_a^{(i)}(0)\,;\, i = 1, \dots, n-1,\; \mathfrak{S}_b^{(\dot{n})}(S) = \mathfrak{S}_a^{(n)}(0),$$

hence also

$$\Phi_b^{(i)}(0) = \Phi_a^{(i)}(0)\,;\, i = 1, \dots, n-1,\; \Phi_b^{(\dot{n})}(S) = \Phi_a^{(n)}(0).$$

Examples.

1) *A simple D-system which osculates of order* 1 with a non-degenerate D-system $\mathfrak{A}(S)$ at the line $S = 0$, consists of the lines that make the same dual angle $K = \Phi(0)$ with $\mathfrak{S}(0)$ as $\mathfrak{A}(0)$.

Putting:

$$\mathfrak{S}(0) = (0, 0, 1), \quad \mathfrak{A}(0) = (0, \sin K, \cos K)$$

the required D-system is found to be:

$$\mathfrak{B}(S) = \left(\sin \frac{S}{\sin K} \cdot \sin K, \;\; \cos \frac{S}{\sin K} \cdot \sin K, \;\; \cos K \right) \quad . \quad (27)$$

2) $\mathfrak{A}(T)$ be a D-system with a non-degenerate enveloped D-system. We want to construct *a simple D-system which osculates of order* 2 with $\mathfrak{A}(T)$ at the line $T = 0$.

Let $\mathfrak{S}^{(2)}(0) = (0, 0, 1)$, $\mathfrak{S}(0) = (0, \sin \Lambda, \cos \Lambda)$, $\Lambda = \Phi^{(2)}(0)$.
The dual angle between $\mathfrak{A}(0)$ and $\mathfrak{S}(0)$ be K.
Then is the enveloped D-system of the required D-system:

$$\mathfrak{C}(\Phi) = \left(\sin \frac{\Phi - K}{\sin \Lambda} \cdot \sin \Lambda, \quad \cos \frac{\Phi - K}{\sin \Lambda} \sin \Lambda, \quad \cos \Lambda\right) \quad . \quad (28)$$

Φ is invariant parameter of $\mathfrak{C}$ and is also the dual angle between $\mathfrak{B}(\Phi)$ and $\mathfrak{C}(\Phi)$, where $\mathfrak{B}(\Phi)$ is the required D-system.
By differentiation of (28) we get:

$$\mathfrak{C}_1(\Phi) = \left(\cos \frac{\Phi - K}{\sin \Lambda}, \; -\sin \frac{\Phi - K}{\sin \Lambda}, \; 0\right).$$

The required D-system is found from (25):

$$\left.\begin{aligned} &\mathfrak{B}(\Phi) = \cos \Phi \cdot \mathfrak{C}(\Phi) - \sin \Phi \cdot \mathfrak{C}_1(\Phi) = \\ &\Bigg(\cos \Phi \cdot \sin \Lambda \cdot \sin \frac{\Phi - K}{\sin \Lambda} - \sin \Phi \cdot \cos \frac{\Phi - K}{\sin \Lambda}, \\ &\quad \cos \Phi \cdot \sin \Lambda \cdot \cos \frac{\Phi - K}{\sin \Lambda} \cdot \sin \Phi \cdot \sin \frac{\Phi - K}{\sin \Lambda}, \; \cos \Phi \cos \Lambda\Bigg)\Bigg) \end{aligned}\right\} . \quad (29)$$

The formula is also applicable when $\mathfrak{A}(T)$ is degenerate at $T = 0$ (the enveloped D-system of $\mathfrak{A}(T)$ however non-degenerate: $K = 0 + \varepsilon \bar{k}$, $\Lambda \neq 0 + \varepsilon \bar{\lambda}$).

13. *A formula of* EULER–SAVARY *and the analogue in linegeometry.*

A formula (f), like the formula of EULER–SAVARY (31) in the geometry on the sphere, reduces to a set (i) of real identities in real variables, when the involved entities (e) are replaced by their definitions (d). The functional identities (i), hold equally well for differentiable functions of dual variables (§ 1; KUIPER [7] Ch. 1), if only (assumption Z) we exclude those values of the dual variables for which a division by a zero-divisor would occur in the identities. Under assumption Z we can define entities (E) in linegeometry by definitions (D) analogous to (d). The defining formulas (D) are the differentiable dual continuations of the formulas (d). The formula (F), analogous by differentiable continuation to (f), holds true for the entities (E).

So here we have a method to construct, and at the same time to prove, formulas (theorems) on D-systems.

Examples of entities (E) are the dual curvature $\sin \Phi$ (21) and the momentanous axis $\mathfrak{S}$ of a D-system. An example of a formula is (24). According to the theory above, (24) follows without further proof from the formula for a curve on a unitsphere

$$\operatorname{tg} \varphi^{(2)} = \frac{d \cos \varphi}{d s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

s is the arclength in radials; tg φ, resp. tg $\varphi^{(2)}$, is the geodesic curvature of the curve, resp. of the evolute of the curve. (The evolute or developable is defined by a formula analogous to (13)).

We will conclude with another application of the theory, namely to the formula of EULER–SAVARY on the sphere:

$$\operatorname{cotg} \varphi_1 - \operatorname{cotg} \varphi = -(\operatorname{cotg} \varphi_f - \operatorname{cotg} \varphi_m) \operatorname{cosec} \psi. \quad . \quad . \quad (31)$$

This formula is related with the motion of a mobile unitsphere s_m containing a curve c_1, over a fixed concentric unitsphere s_f. The moving curve c_1 envelopes a curve c of s_f. Those momentanous invariant points of s_m, with respect to which the velocity of s_m is positive (compare the remark after (15); at each moment we have the choice between two invariant points), generate a curve c_f on s_f and a curve c_m on s_m. At the moment under consideration be p the invariant point of the motion, q the tangent point of c_1 and c, $\psi + \pi/2$ is the positive angle between the tangent at p to c_f and the tangent at q to c, both tangents equipped with the direction of increasing arclength, as seen from the direction of the outside-sphere-normal at p. φ_1, φ, φ_f, φ_m are the arclengths from p to the curvature-centres of c_1, c, c_f, c_m.

From (31) follows the analogous formula concerning linegeometry:

$$\operatorname{cotg} \Phi_1 - \operatorname{cotg} \Phi = -(\operatorname{cotg} \Phi_f - \operatorname{cotg} \Phi_m) \operatorname{cosec} \Psi, \quad . \quad . \quad (32)$$

equivalent to the two real formulas [6])

$$\left\{\begin{aligned} &\operatorname{cotg} \varphi_1 - \operatorname{cotg} \varphi = -(\operatorname{cotg} \varphi_f - \operatorname{cotg} \varphi_m) \operatorname{cosec} \Psi \\ &\left(\frac{\bar{\varphi}_1}{\sin^2\varphi_1} - \frac{\bar{\varphi}}{\sin^2\varphi}\right) \sin\psi = -\frac{\bar{\varphi}_f}{\sin^2\varphi_f} + \frac{\bar{\varphi}_m}{\sin^2\varphi_m} + \bar{\psi} \operatorname{cotg} \psi \,(\operatorname{cotg} \varphi_f - \operatorname{cotg} \varphi_m) \end{aligned}\right.$$

$$\left(\operatorname{cotg} \Phi = \operatorname{cotg} \varphi + \varepsilon \bar{\varphi}\, d \operatorname{cotg} \varphi / d\varphi = \operatorname{cotg} \varphi - \varepsilon \frac{\bar{\varphi}}{\sin^2 \varphi} \quad (2)\right)$$

This formula is related with the dual motion of a mobile Euclidean three-dimensional space S_m, containing a D-system ${}^1\mathfrak{A}$, with respect to a fixed space S_f. The moving ${}^1\mathfrak{A}$ is at each moment T at one of its lines "tangent" to a D-system $\mathfrak{A}$ in S_f (${}^1\mathfrak{A} = \mathfrak{A}$, ${}^1\mathfrak{A}_1 = \mathfrak{A}_1$). The moving ${}^1\mathfrak{A}$ "envelopes" $\mathfrak{A}$. The momentanous axes of the dual motion generate a D-system ${}^f\mathfrak{A}$ in S_f and a D-system ${}^m\mathfrak{A}$ in S_m. The dual motion can be considered as a development of the D-system ${}^m\mathfrak{A}$ along the D-system ${}^f\mathfrak{A}$. ${}^f\mathfrak{A}(0)$ is the momentanous axis at a moment under consideration ($T = 0$). $\mathfrak{A}(0)$ is the line at which ${}^1\mathfrak{A}$ and $\mathfrak{A}$ are tangent. $\pi/2 + \Psi$ is the dual angle between ${}^f\mathfrak{A}_1(0)$ and $\mathfrak{A}_1(0)$.

Φ_1, Φ, Φ_f, Φ_m are the dual angles between ${}^f\mathfrak{A}(0)$ and the momentanous axes of ${}^1\mathfrak{A}$, $\mathfrak{A}$, ${}^f\mathfrak{A}$, ${}^m\mathfrak{A}$.

If we restrict the dual variable, time T, to moments $T = T(u) = t(u) + \varepsilon \bar{t}(u)$ $(d\, t(u) / d\, u \neq 0)$, then u can be considered as an

[6]) Compare: VAN HAASTEREN [8] p. 59 formula 66; DISTELLI [2] p. 305 form. 74.

ordinary time-variable with respect to which an ordinary motion is determined. The results we may get are then stated in terms of reguli.

These results are also obtained in [2] [5] [8].

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Mathematics. — *Lattice points in non-convex regions*, II. By P. MULLENDER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 27, 1948.)

Let $x_1, \ldots, x_n$ denote the Cartesian coordinates of the points of an n-dimensional Euclidean space and let Λ be an n-dimensional lattice in that space with determinant $\triangle \neq 0$.

In this paper we consider the same problem as in the previous one under this title [1]), namely

Given a region R, find constants A, such that any lattice Λ with $|\triangle| \leqq A$ *has a point other than the origin contained in R.*

We use again the theorem we mentioned in I:

Theorem 1. *Let K be any bounded region of volume V_K and let be a closed region containing all the points $P_1 - P_2$ with P_1 and P_2 in K, then R contains a point other than the origin of any lattice Λ with $|\triangle| \leq V_K$.*

Our problem is now:

Given a region R, construct a region K, with the largest possible volume V_K, such that all points $P_1 - P_2$, with P_1 and P_2 in K, are points of R.

For, if K is such a region, then, according to theorem 1, we may take $A = V_K$. In this case we shall call K suitable.

So our probleem is in fact a geometrical one, and the solution we offer in this paper has also mainly geometrical interest. For, although it is quite obvious that the regions K we construct are much larger than those, which can be found by previous methods, it is often very difficult to calculate their volumes.

For a certain class of non-convex regions R we gave in I a method of constructing a suitable region K, which was in fact two-dimensional. The method was a development of ideas due to BLICHFELDT [2]) and MORDELL [3]). In this paper we consider non-convex regions R, for which we can give

II. An n-dimensional construction of the region K.

Replacing the function $f(x)$, which he used to define R, by a function $f(x_1, \ldots, x_{n-1})$ of $n-1$ variables and modifying the definition of K accordingly, MORDELL [4]) also generalised his method to n dimensions. In

[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 874—884 (1948). I refer to this paper as I.

[2]) H. F. BLICHFELDT, A New Principle in the Geometry of Numbers with some Applications, Trans. Am. Math. Soc. **15**, 227—235 (1914).

[3]) L. J. MORDELL, Latice points in some n-dimensional non-convex regions, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 773—792 (1946).

[4]) L.c. 782 et seq.

this generalisation, however, he had to put a restriction on the function $f(x_1, \dots, x_{n-1})$ similar to the inequality (7) in I, namely

$$f(\xi_1, \dots, \xi_{n-1}) + \sum_{k=1}^{n-1} \xi_k f'_{x_k}(\xi_1, \dots, \xi_{n-1}) > 0, \quad . \quad . \quad . \quad (41)$$

for a certain choice of the ξ's. For $n > 2$, this condition cannot always be satisfied. Hence it remains desirable to find also other generalisations for the cases to which MORDELL's generalisation does not apply. Such generalisations we intend to develop here, and we shall show that we can obtain a considerable improvement of MORDELL's results.

We only consider regions R with the property that, if $(X_1, \dots, X_n)$ is a point of R, then the same is true for all the points $(x_1, \dots, x_n)$ with

$$|x_1| \leq |X_1|, \dots, |x_n| \leq |X_n|.$$

Also the regions K we construct have that property.

That means that the shape of R or K is completely determined by that of the parts in the first (hyper)octant. Those parts we denote by $\bar{R}$ and $\bar{K}$ respectively. Then, of course, V_K is 2^n times the volume of $\bar{K}$.

Clearly we now can restrict ourselves to the consideration of the first octant only and so we can state our problem as follows:

Given a region $\bar{R}$, construct a region $\bar{K}$, such that all the points $P_1 + P_2$ with P_1 and P_2 in $\bar{K}$ are points of $\bar{R}$.

Such a region $\bar{K}$, which determines a suitable region K, we shall also call suitable.

To explain the idea we consider first a special case:

A. The product of n homogeneous linear forms.

1. The Problem. Let R be defined by

$$|x_1 \dots x_n| \leqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Then $\bar{R}$ is defined by

$$x_1 \geqq 0, \dots, x_n \geqq 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and

$$x_1 \dots x_n \leqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The problem was to find constants A, as large as possible, such that at least one point other than the origin of any lattice Λ with $|\Delta| \leq A$ satisfies (42).

The problem is now to construct regions $\bar{K}$, also as large as possible, such that any point $P_1 + P_2$ with P_1 and P_2 in $\bar{K}$ is a point of $\bar{R}$.

2. Previous Results. Since we need only consider the first (hyper)-octant, we may restrict ourselves to non-negative coordinates.

We obtain an almost trivial solution, defining $\bar{K} = \bar{K}_0$ by

$$x_1 + \dots + x_n \leqq \tfrac{1}{2} n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

For, if the points

$$P_1 = (x_1', \ldots, x_n') \quad , \quad P_2 = (x_1'', \ldots, x_n'')$$

both satisfy (45), then

$$(x_1' + x_1'') \ldots (x_n' + x_n'') \leqq \left(\frac{(x_1' + x_1'') + \ldots + (x_n' + x_n'')}{n}\right)^n \leqq \left(\frac{\frac{1}{2}n + \frac{1}{2}n}{n}\right)^n = 1$$

since the geometrical means of non-negative numbers is not greater than their arithmetical means.

Now we have

$$V_{K_0} = \frac{n^n}{n!}, \qquad \qquad (46)$$

and hence

$$A = \frac{n^n}{n!}, \qquad \qquad (47)$$

which is the well known result of MINKOWSKI [5]).

For $n = 2$ this region $\overline{K}_0$ is the largest suitable region, though the corresponding value of V_K is not the best possible result for A.

For $n > 2$, however, $\overline{K}_0$ can be replaced by larger regions.

This has been proved already by KOKSMA and MEULENBELD [6]), but they stated their result in a different form. In place of $\overline{R}$ they considered the regions $\overline{R'}$ given by

$$\left(\frac{x_1 + \ldots + x_p}{p}\right)^p \cdot \left(\frac{x_{p+1} + \ldots + x_n}{n-p}\right)^{n-p} \leqq 1, \qquad (48)$$

with $\frac{1}{2}n < p < n$. These regions are wholly contained in $\overline{R}$.

If a region $\overline{R'}$ is wholly contained in $\overline{R}$ and if for any two points P_1 and P_2 of a certain region $\overline{K}$ the point $P_1 + P_2$ belongs to $\overline{R'}$, then it is clear that $\overline{K}$ has the same property with respect to $\overline{R}$. Hence demonstrating their result for the regions $\overline{R'}$, KOKSMA and MEULENBELD obtained at the same time a result for $\overline{R}$.

They constructed suitable regions $\overline{K} = \overline{K}_1$, consisting of $\overline{K}_0$ and a small additional region $\overline{H}_1$ given by the inequalities

$$x_1 + \ldots + x_n > \tfrac{1}{2} n, \qquad (49)$$

$$x_{p+1} + \ldots + x_n > n - p, \qquad (50)$$

$$\left(\frac{x_1 + \ldots + x_p + \frac{1}{2}n}{p}\right)^p \cdot \left(\frac{x_{p+1} + \ldots + x_n}{n-p}\right)^{n-p} \leqq 1. \qquad (51)$$

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[5]) H. MINKOWSKI, Geometrie der Zahlen.

[6]) J. F. KOKSMA et B. MEULENBELD, Sur le théorème de MINKOWSKI, concernant un système de formes linéaires réelles, Proc. Ned. Akad. v. Wetensch., **45**, 256—262, 354—359, 471—478, 578—584 (1942).

The same result and even a slight improvement we may obtain on applying the methods we developed in I [7]).

It is not possible to apply MORDELL's n-dimensional method to the region $\overline{R}$ itself. It is possible to define $\overline{R}$ by

$$x_n \leqq f(x_1, \ldots, x_{n-1}),$$

with

$$f(u_1, \ldots, u_{n-1}) \equiv \frac{1}{u_1 \ldots u_{n-1}},$$

but then the condition (41) is not satisfied for any positive $\xi_1, \ldots, \xi_{n-1}$.

MORDELL [8]), however, generalising the result of KOKSMA and MEULENBELD, considered the regions $\overline{R}''$, given by

$$X^p\, Y^q\, Z^r \leqq 1, \quad \ldots \ldots \quad (52)$$

where

$$X = \frac{1}{p} \sum_{\lambda=1}^{p} x_\lambda,\ Y = \frac{1}{q} \sum_{\mu=1}^{q} x_{p+\mu},\ Z = \frac{1}{r} \sum_{\nu=1}^{r} x_{p+q+\nu}, \quad \ldots \quad (53)$$

with $p+q+r=n$ and $p > \frac{1}{2} n$. These regions are also contained in $\overline{R}$. Applying his method MORDELL constructed the regions $\overline{K}_2$ consisting of $\overline{K}_0$ and regions $\overline{H}_2$ defined by

$$x_1 + \ldots + x_n > \tfrac{1}{2} n, \quad \ldots \ldots \quad (49)$$

$$Y \geqq 1,\ Z \geqq 1, \quad \ldots \ldots \quad (54)$$

$$\left(X + \frac{n}{2p} \right)^p Y^q\, Z^r \leqq 1.. \quad \ldots \ldots \quad (55)$$

This is not the only possibility of application of MORDELL's method. Considering the regions $\overline{R}'''$ given by

$$\left(\frac{x_1 + \ldots + x_p}{p} \right)^p \cdot x_{p+1} \ldots x_n \leqq 1, \quad \ldots \ldots \quad (56)$$

again with $p > \frac{1}{2} n$, we find suitable regions $\overline{K} = \overline{K}_3$, consisting of $\overline{K}_0$ and a region $\overline{H}_3$ defined by

$$x_1 + \ldots + x_n > \tfrac{1}{2} n, \quad \ldots \ldots \quad (49)$$

$$x_{p+1} \geqq 1, \ldots, x_n \geqq 1, \quad \ldots \ldots \quad (57)$$

$$\left(\frac{x_1 + \ldots + x_p + \frac{1}{2} n}{p} \right)^n \cdot x_{p+1} \ldots x_n \leqq 1.. \quad \ldots \quad (58)$$

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[7]) See I, § 4. Writing

$$N = \frac{x_1 + \ldots + x_p}{p} \quad \text{and} \quad M = \frac{x_{p+1} + \ldots + x_n}{n-p},$$

(so that p becomes q and $n-p$ becomes p) we can define $\overline{R}'$ by (24) and then apply theorem 5.

[8]) L.c. 788. There is an error in MORDELL's paper: the exponents α, β and γ should be placed outside the brackets.

In all these cases an improvement of the trivial result is obtained by adding one region $\overline{H}$ to the original region $\overline{K}_0$. Now the question arises if it is possible to add more than one of those regions at the same time. As we shall show the answer is in the affirmative.

3. The Solution. Since $\overline{R}$ is symmetrical in $x_1, \ldots, x_n$, not only the regions $\overline{K}_3$, but also the regions we obtain from them by interchanging the coordinates are suitable. Our assertion is now that all the regions we obtain in this way form together another suitable region.

Let p be an integer and $1 \leqq p \leqq n$. We put $n - p = q$. We suppose that

$$k_1, \ldots, k_p, \; l_1, \ldots, l_q,$$

form a permutation of $1, \ldots, n$. If $p = n$, then we suppose $k_1, \ldots, k_n$ to be a permutation of $1, \ldots, n$.

We denote by $\overline{K}(k_1, \ldots, k_p)$ a region, which we define by the following inequalities

$$x_{k_1} \leqq 1, \ldots, x_{k_p} \leqq 1, \qquad (59)$$

$$x_{l_1} > 1, \ldots, x_{l_q} > 1, \qquad (60)$$

$$\left(\frac{x_{k_1} + \ldots + x_{k_p} + \frac{1}{2} n}{p} \right)^p \cdot x_{l_1} \ldots x_{l_q} \leqq 1. \qquad (61)$$

Clearly the region contains points, if and only if $p > \frac{1}{2} n$.

Further, we obtain the same region, if we replace $k_1, \ldots, k_p$ by the same indices in a different order, but a different region, if we replace $k_1, \ldots, k_p$ by any other set of indices. It follows immediately from (59) and (60) that none of those different regions overlap.

We define $\overline{K}$ as the sum of all the regions $\overline{K}(k_1, \ldots, k_p)$.

It is also possible to define $\overline{K}$ as the sum of a number of overlapping regions.

Omitting the condition (59) in the definition of $\overline{K}(k_1, \ldots, k_p)$ we define a region $\overline{K}^*(k_1, \ldots, k_p)$, in which $\overline{K}(k_1, \ldots, k_p)$ is wholly contained. When we deal with the general case we shall prove that the regions $\overline{K}^*(k_1, \ldots, k_p)$ do not contain points outside $\overline{K}$. Hence $\overline{K}$ can also be defined as the sum of these (overlapping) regions $\overline{K}^*(k_1, \ldots, k_p)$.

It is easy to see that the region $\overline{K}_0$ we defined before is the same as our region $\overline{K}^*(1, \ldots, n)$. Further, $\overline{H}_3$ is the same region as $\overline{K}^*(1, \ldots, p)$ minus the points of $\overline{K}^*(1, \ldots, p)$ which are already contained in $\overline{K}^*(1, \ldots, n) = \overline{K}_0$. Hence the sum $\overline{K}_3$ of $\overline{K}_0$ and $\overline{H}_3$ is wholly contained in $\overline{K}$, and so are all the regions we can obtain from $\overline{K}_3$ by interchanging the coordinates.

Further, $\overline{K}$ is also an improvement of the region $\overline{K}_1$ of KOKSMA and

MEULENBELD. For it can be shown that the volume of $\overline{K}_1$ is greatest when $p = n - 1$. For this value of p, however, the regions $\overline{K}_1$ and $\overline{K}_3$ are the same.

Similarly it seems probable that the volume of MORDELL's region $\overline{K}_2$ is also greatest when p is as large as possible, i.e. when $p = n - 2$, and for this value of p the regions $\overline{K}_2$ and $\overline{K}_3$ are the same. However, we shall also improve MORDELL's result in a different way.

To calculate V_K it is more convenient to use our first definition of $\overline{K}$ as the sum of the non-overlapping regions $\overline{K}(k_1, \ldots, k_p)$ with $p > \frac{1}{2}n$. However, we shall not give the actual calculations, but only the results.

It is easy to verify that the only region we have for $p = n$, namely $\overline{K}(1, \ldots, n)$ has the volume $\frac{1}{2}$ whatever the value of n may be.

Further, for a given p ($\frac{1}{2}n < p < n$), the $\binom{n}{p}$ regions $\overline{K}(k_1, \ldots, k_p)$ all obviously have the same volume as $\overline{K}(1, \ldots, p)$. Now, the volume of $\overline{K}(1, \ldots, p)$ can be given by the formula

$$V_{n,p} = \frac{p^n}{p!} \sum_{\lambda=0}^{p-[\frac{1}{2}n]-1} \sum_{\mu=1}^{p} (-1)^{\lambda+\mu+n-p} \binom{p}{\lambda} \binom{p}{\mu} \frac{1}{\mu^{n-p}} \left(\frac{n+2\lambda}{2p}\right)^{\mu} +$$

$$+ \frac{p^n}{p!} \sum_{\lambda=0}^{p-[\frac{1}{2}n]-1} \sum_{\mu=1}^{p} \sum_{\nu=0}^{n-p} (-1)^{\lambda+\mu+\nu+1} \binom{p}{\lambda} \binom{p}{\mu} \frac{\left(\log \frac{2p}{n+2\lambda}\right)^{n-p-\nu}}{\mu^{\nu}(n-p-\nu)!}.$$

Hence we can calculate

$$V_K = 2^n \left\{ \tfrac{1}{2} + \sum_{p=[\frac{1}{2}n]+1}^{n-1} \binom{n}{p} V_{n,p} \right\}.$$

B. The General Case.

1. The Region R. Let $F(x) \equiv F(x_1, \ldots, x_n)$ be a single valued, twice differentiable function of $x_1, \ldots, x_n$, defined for all non-negative values of the variables. We suppose that

$$F(x) > 0, \quad \text{for} \quad x_1, \ldots, x_n > 0. \qquad (62)$$

$$\frac{\partial}{\partial x_j} F(x) > 0, \quad \text{for} \quad x_1, \ldots, x_n > 0 \quad (j = 1, \ldots, n). \qquad (63)$$

$$F(x) + F(y) \leqq F(x+y). \qquad (64)$$

$$F(tx) = tF(x) \quad \text{for any} \quad t \geqq 0. \qquad (65)$$

Then it follows immediately from (64) and (65), that

$$\lambda F(x) + \mu F(y) \leqq F(\lambda x + \mu y) \quad \text{for any} \quad \lambda, \mu \geqq 0. \qquad (66)$$

Now we prove

Lemma 1. $\quad F(y) - F(x) \leqq \sum_{j=1}^{n} (y_j - x_j) \frac{\partial}{\partial x_j} F(x).$

P r o o f. We have, for $\lambda \geq 1$,

$$(\lambda - 1)F(x) + F(y) \leq F(\lambda x - x + y),$$

by (66), and hence, by (65),

$$F(y) - F(x) \leqq \lambda \left\{ F\left(x + \frac{y-x}{\lambda}\right) - F(x) \right\} =$$

$$= \sum_{j=1}^{n} (y_j - x_j) \frac{\partial}{\partial x_j} F\left(x + \vartheta \, \frac{y-x}{\lambda}\right),$$

with $0 < \vartheta < 1$.

The lemma follows on letting λ tend to infinity.

Further we suppose that the value of

$$\frac{\dfrac{\partial}{\partial x_k} F(x_1, \ldots, x_n)}{\dfrac{\partial}{\partial x_l} F(x_1, \ldots, x_n)}$$

only depends on the value of x_k/x_l and increases, when x_k/x_l decreases.

We now define R as the set of points $(x_1, \ldots, x_n)$, satisfying

$$F(|x_1|, \ldots, |x_n|) \leqq F(1, \ldots, 1) \quad . \quad . \quad . \quad . \quad . \quad (67)$$

and hence $\overline{R}$ as the set of points $(x_1, \ldots, x_n)$, with non-negative coordinates, satisfying the same inequality.

2. The Region K. Let $x_1, \ldots, x_n$ again denote non-negative coordinates only. We suppose p to be an integer and $1 \leqq p \leqq n$.

We write

$$\frac{\sum\limits_{j=1}^{n} \dfrac{\partial}{\partial x_j} F(1, \ldots, 1)}{2 \sum\limits_{\nu=1}^{p} \dfrac{\partial}{\partial x_\nu} F(1, \ldots, 1)} = \vartheta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

and

$$\frac{\sum\limits_{\nu=1}^{p} x_\nu \dfrac{\partial}{\partial x_\nu} F(1, \ldots, 1)}{\sum\limits_{\nu=1}^{p} \dfrac{\partial}{\partial x_\nu} F(1, \ldots, 1)} = x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Now we define a region, which we call $\overline{K}(1, \ldots, p)$, by the inequalities

$$x_1 \leqq 1, \ldots, x_p \leqq 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

$$x_{p+1} > 1, \ldots, x_n > 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$F(x + \vartheta, \ldots, x + \vartheta, x_{p+1}, \ldots, x_n) \leqq F(1, \ldots, 1). \quad . \quad . \quad (72)$$

It is easy to see that the region is not empty, i.e. that there are points satisfying the above conditions, if and only if $\vartheta < 1$. For it follows from (63) and (69) that x must be positive, and from (63), (71) and (72) that $x + \vartheta$ must be less than 1, and, conversely, if $x > 0$ and $x + \vartheta < 1$, then it is always possible to find suitable values for $x_1, \dots, x_p$ and $x_{p+1}, \dots, x_n$, such that the conditions are satisfied.

From this region $\overline{K}(1, \dots, p)$ we obtain a set of regions on interchanging the coordinates. So we have the following general definition:

First we put $n - p = q$ and as before we suppose

$$k_1, \dots, k_p, \; l_1, \dots, l_q$$

to be a permutation of $1, \dots, n$.

We write

$$\frac{\sum\limits_{j=1}^{n} \frac{\partial}{\partial x_j} F(1, \dots, 1)}{2 \sum\limits_{\nu=1}^{p} \frac{\partial}{\partial x_{k_\nu}} F(1, \dots, 1)} = \vartheta, \quad \dots \dots \quad (73)$$

and

$$\frac{\sum\limits_{\nu=1}^{p} x_{k_\nu} \frac{\partial}{\partial x_{k_\nu}} F(1, \dots, 1)}{\sum\limits_{\nu=1}^{p} \frac{\partial}{\partial x_{k_\nu}} F(1, \dots, 1)} = x, \quad \dots \dots \quad (74)$$

and now we define $\overline{K}(k_1, \dots, k_p)$ by the inequalities

$$x_{k_1} \leqq 1, \dots, x_{k_p} \leqq 1, \quad \dots \dots \quad (75)$$

$$x_{l_1} > 1, \dots, x_{l_q} > 1, \quad \dots \dots \quad (76)$$

and

$$F(u_1, \dots, u_n) \leqq F(1, \dots, 1), \quad \dots \dots \quad (77)$$

where

$$u_{k_1} = \dots = u_{k_p} = x + \vartheta, \quad \dots \dots \quad (78)$$

and

$$u_{l_1} = x_{l_1}, \dots, u_{l_q} = x_{l_q}. \quad \dots \dots \quad (79)$$

As before, the region is not empty, if and only if $\vartheta < 1$, i.e.

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} F(1, \dots, 1) < 2 \sum_{\nu=1}^{p} \frac{\partial}{\partial x_{k_\nu}} F(1, \dots, 1),$$

or

$$\sum_{\mu=1}^{q} \frac{\partial}{\partial x_{l_\mu}} F(1, \dots, 1) < \sum_{\nu=1}^{p} \frac{\partial}{\partial x_{k_\nu}} F(1, \dots, 1). \quad \dots \quad (80)$$

Further, as in the case of the homogeneous linear forms, we obtain the same region, if we replace $k_1, \dots, k_p$ by the same indices in a different order, but a different region, if we replace $k_1, \dots, k_p$ by any other set of indices, and those different regions do not overlap.

It follows from (80) that either $\overline{K}(k_1, \dots, k_p)$ or $\overline{K}(l_1, \dots, l_q)$ is empty, and, when the left and right hand sides are equal to each other, even both are empty. There fore we call those regions complementary regions. Since there are two possibilities for each index, whether it belongs to the k's or the l's, there are 2^n different regions possible. Hence at most 2^{n-1} different regions $\overline{K}(k_1, \dots, k_p)$ are not empty.

Finally we define $\overline{K}$ as the sum of all the regions $\overline{K}(k_1, \dots, k_p)$ and accordingly K as the set of all the points $(x_1, \dots, x_n)$, with arbitrary coordinates, for which $(|x_1|, \dots, |x_n|)$ is contained in $\overline{K}$.

Again we give an alternative definition of $\overline{K}$.

As before we define $\overline{K^*}(k_1, \dots, k_p)$ omitting the condition (75) in the definition of $\overline{K}(k_1, \dots, k_p)$.

We prove

Lemma 2. *If a point $P = (x_1, \dots, x_n)$ is contained in $\overline{K^*}(k_1, \dots, k_p)$, and $x_{k_p} > 1$, then it is also contained in $\overline{K^*}(k_1, \dots, k_{p-1})$.*

Proof. Without loss of generality we may take $k_1 = 1, \dots, k_p = p$.

We denote ϑ and x, defined with respect to $\overline{K^*}(1, \dots, p-1)$, and corresponding to the point P, by ϑ' and x' respectively.

We have to prove that

$$F(x' + \vartheta', \dots, x' + \vartheta', x_p, \dots, x_n) \leqq F(1, \dots, 1).$$

However, by lemma 1,

$$F(x' + \vartheta', \dots, x' + \vartheta', x_p, \dots, x_n) - F(x + \vartheta, \dots, x + \vartheta, x_{p+1}, \dots, x_n) \leqq$$
$$\leqq (x' + \vartheta' - x - \vartheta) \sum_{\nu=1}^{p-1} \frac{\partial}{\partial x_\nu} F(x + \vartheta, \dots, x + \vartheta, x_{p+1}, \dots, x_n) +$$
$$+ (x_p - x - \vartheta) \frac{\partial}{\partial x_p} F(x + \vartheta, \dots, x + \vartheta, x_{p+1}, \dots, x_n).$$

Hence, by (72), it suffices to prove that the right hand side of this inequality is not positive. That means, since

$$\frac{\dfrac{\partial}{\partial x_\nu} F(x + \vartheta, \dots, x + \vartheta, x_{p+1}, \dots, x_n)}{\dfrac{\partial}{\partial x_{\nu'}} F(x + \vartheta, \dots, x + \vartheta, x_{p+1}, \dots, x_n)} = \frac{\dfrac{\partial}{\partial x_\nu} F(1, \dots, 1)}{\dfrac{\partial}{\partial x_{\nu'}} F(1, \dots, 1)},$$

for $\nu, \nu' = 1, \dots, p$, we have to prove

$$(x' + \vartheta' - x - \vartheta) \sum_{\nu=1}^{p-1} \frac{\partial}{\partial x_\nu} F(1, \dots, 1) + (x_p - x - \vartheta) \frac{\partial}{\partial x_p} F(1, \dots, 1) \leqq 0. \quad (81)$$

By (69)

$$x' \sum_{\nu=1}^{p-1} \frac{\partial}{\partial x_\nu} F(1, \dots, 1) - \sum_{\nu=1}^{p-1} x_\nu \frac{\partial}{\partial x_\nu} F(1, \dots, 1) = 0,$$

and

$$x \sum_{\nu=1}^{p} \frac{\partial}{\partial x_\nu} F(1, \dots, 1) - \sum_{\nu=1}^{p} x_\nu \frac{\partial}{\partial x_\nu} F(1, \dots, 1) = 0.$$

Subtracting we obtain

$$(x'-x)\sum_{\nu=1}^{p-1}\frac{\partial}{\partial x_\nu}F(1,\ldots,1)+(x_p-x)\frac{\partial}{\partial x_p}F(1,\ldots,1)=0. \quad . \quad (82)$$

Further, by (68)

$$\vartheta-\vartheta'=\frac{\frac{\partial}{\partial x_p}F(1,\ldots,1)\sum_{j=1}^{n}\frac{\partial}{\partial x_j}F(1,\ldots,1)}{2\sum_{\nu=1}^{p-1}\frac{\partial}{\partial x_\nu}F(1,\ldots,1)\sum_{\nu=1}^{p}\frac{\partial}{\partial x_\nu}F(1,\ldots,1)}=\frac{\vartheta\frac{\partial}{\partial x_p}F(1,\ldots,1)}{\sum_{\nu=1}^{p-1}\frac{\partial}{\partial x_\nu}F(1,\ldots,1)},$$

and so

$$(\vartheta'-\vartheta)\sum_{\nu=1}^{p-1}\frac{\partial}{\partial x_\nu}F(1,\ldots,1)-\vartheta\frac{\partial}{\partial x_p}F(1,\ldots,1)=0. \quad . \quad . \quad (83)$$

Adding (82) and (83) we find (81) with equality sign.

From this lemma we conclude that every region $\overline{K}^*(k_1,\ldots,k_p)$ is wholly contained in $\overline{K}$:

For, suppose $P=(x_1,\ldots,x_n)$ is a point of $\overline{K}^*(k_1,\ldots,k_p)$ for which (75) is not satisfied. Then at least one of the x_k satisfies (75), for otherwise $x>1$, by (74), contrary to (76) and (77). Hence we may suppose

$$x_{k_1}\leqq 1,\ldots,x_{k_{p'}}\leqq 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

$$x_{k_{p'+1}}>1,\ldots,x_{k_p}>1\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Now it follows immediately from lemma 2, that P is contained in $\overline{K}^*(k_1,\ldots,k_{p'})$ and, by (84), also in $\overline{K}(k_1,\ldots,k_{p'})$, i.e. P is contained in $\overline{K}$.

Therefore we can also define $\overline{K}$ as the sum of the overlapping regions $\overline{K}^*(k_1,\ldots,k_p)$.

3. The Theorem. We prove, R and K being defined as in the previous two paragraphs.

Theorem 6. *If any two points P_1 and P_2 are contained in K, then P_1-P_2 is contained in R.*

From theorem 6 we derive

Theorem 7. *If V_K denotes the volume of K, then R contains a point other than the origin of any lattice Λ, with $|\triangle|\leqq V_K$.*

As we have stated before, theorem 6 is already proved, if we show that P_1+P_2 is contained in $\overline{R}$, when P_1 and P_2 are in $\overline{K}$.

It is obvious that theorem 6 remains true, if we replace $\overline{K}$ by any other region $\overline{K}'$, which is wholly contained in $\overline{K}$. However, if the volume of $\overline{K}'$ is less than that of $\overline{K}$, then we get a weaker result in theorem 7. On the other hand, taking such a smaller region $\overline{K}'$, we may be able to prove theorem 6 for a wider class of regions R.

In fact, replacing e.g. $\overline{K}$ by a region $\overline{K}'$, which only consists of $\overline{K}^*(1, \ldots, n)$, we need not assume any property of the function $F(x_1, \ldots, x_n)$, concerning the ratio of the derivatives, in order to prove theorem 7. The results we should find in this way, we could also obtain by the application of MINKOWSKI's theorem on convex regions.

Further we could extend our theorem to more regions R, though less than in the previous example, by taking $\overline{K}'$ as the sum of $\overline{K}^*(1, \ldots, n)$ and $\overline{K}^*(1)$ or $\overline{K}^*(2)$ etc., i.e. as the sum of two regions instead of 2^{n-1}. Thus we should get the same results as MORDELL, using his n-dimensional generalisation of BLICHFELDT's method.

(To be continued.)

Mathematics. — *On a problem in the theory of uniform distribution.* By P. ERDÖS and P. TURÁN. II. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 25, 1948.)

9. After this first reduction of our problem we transform it in the following way. Let for the polynomial $\psi(z)$ defined in (8.4)

$$M_\vartheta^* = \max_{|z|=\vartheta} |\psi(z)| = \max_{|z|=\vartheta} |\psi_1(z)| = \max_{|z|=\vartheta} \left| \prod_{\nu=1}^{n} (1 - z e^{-i\varphi_\nu}) \right|. \quad (9.1)$$

Then

$$\max_{|z|=\vartheta} \log |\psi_1(z)| = \log M_\vartheta^* . \quad \ldots \ldots \quad (9.2)$$

Since $\log \psi_1(z)$ is regular for $|z| = \vartheta$ we have by the classical theorem of HADAMARD–BOREL–CARATHEODORY that for $|z| \leqq \frac{\vartheta}{2}$, since $\log \psi_1(0) = 0$,

$$|\log \psi_1(z)| \leqq 2 \log M_\vartheta^*$$

i.e.

$$\left| \sum_{\nu=1}^{n} \log (1 - z e^{-i\varphi_\nu}) \right| \leqq 2 \log M_\vartheta^*$$

or denoting $\sum_{\nu=1}^{n} e^{-ki\varphi_\nu}$ by s_k,

$$\left| \sum_{k=1}^{\infty} \frac{z^k}{k} s_{-k} \right| \leqq 2 \log M_\vartheta^* , \quad |z| \leqq \frac{\vartheta}{2} . \quad \ldots \quad (9.3)$$

Then CAUCHY's estimation gives

$$\left.\begin{aligned} |s_k| = |s_{-k}| &\leqq k \frac{2 \log M_\vartheta^*}{\left(\frac{\vartheta}{2}\right)^k} = \\ &= 2k \left(\frac{2}{\vartheta}\right)^k \log M_\vartheta^* < \left(\frac{4}{\vartheta}\right)^k \log M_\vartheta^* = \left(\frac{4}{\vartheta}\right)^k \frac{n}{g^*(n, \vartheta)} . \end{aligned}\right\} . \quad (9.4)$$

Owing to the trivial inequality $|s_k| \leqq n$ the inequality (9.4) is restrictive only for those k's for which

$$k \leqq \frac{\log g^*(n, \vartheta)}{\log \frac{4}{\vartheta}} . \quad \ldots \ldots \quad (9.5)$$

Hence the proof of Theorem II is reduced to the question whether or not the inequalities (9.4) with the restriction (9.5) involve equidistribution of the φ_ν's mod 2π. In other words we reduced the proof of Theorem II to the proof of Theorem III with

$$\psi(k) = \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^k, \quad m = \left[\frac{\log g^*(n,\vartheta)}{2\log\frac{4}{\vartheta}}\right]. \quad . \quad . \quad (9.6)$$

Since

$$\sum_{\nu=1}^{m}\frac{\psi(k)}{k} < \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^{\frac{m}{2}}\left(1+\log\frac{m}{2}\right) + \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^m\cdot\frac{2}{m}\cdot\frac{m}{2} <$$

$$< \frac{n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^m\left\{1+2\left(\frac{\vartheta}{4}\right)^{\frac{m}{2}}\log\frac{m}{2}\right\} \leqq$$

$$\leqq \frac{2n}{g^*(n,\vartheta)}\left(\frac{4}{\vartheta}\right)^m \leqq \frac{2n}{\sqrt{g^*(n,\vartheta)}} < \frac{3n}{\log g^*(n,\vartheta)}$$

and

$$\frac{n}{m+1} < 2\log\frac{4}{\vartheta}\cdot\frac{n}{\log g^*(n,\vartheta)}$$

we obtain — anticipating Theorem III — that for every $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left|\sum_{\alpha\leqq\varphi_\nu\leqq\beta \bmod 2\pi} 1 - \frac{\beta-\alpha}{2\pi}n\right| < 5C\log\frac{4}{\vartheta}\cdot\frac{n}{\log g^*(n,\vartheta)}$$

i.e. Theorem II will be proved.

10. Before turning to the proof of Theorem III we sketch the corresponding reasoning for Theorem I. In this case — as we remarked in [1]) — the general case can be reduced to the case when all the roots lie on the unit circle. Then in (9.2) M_ϑ^* is replaced by $\max\limits_{|z|=1}|\psi(z)| = M$. Applying the theorem of HADAMARD–BOREL–CARATHEODORY to the interior circle $|z| = \varrho$, where we determine ϱ suitable later, we obtain

$$|\log\psi_1(z)| \leqq \frac{2\log M}{1-\varrho}, \qquad |z| \leqq \varrho$$

resp.

$$\left|\sum_{k=1}^{\infty}\frac{z^k}{k}s_{-k}\right| < \frac{2\log M}{1-\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.1)$$

CAUCHY's estimation gives

$$|s_k| = |s_{-k}| \leqq k\,\frac{2\log M}{1-\varrho}\cdot\frac{1}{\varrho^k}.$$

Choosing $\varrho = 1 - \dfrac{1}{k+1}$

$$\left.\begin{array}{c} |s_k| \leqq 20\, k^2 \log M \\ k = 1, 2, \ldots \end{array}\right\} \quad \ldots \ldots \quad (10.2)$$

This gives according to the corollary of Theorem III an error term $O(n^{2/3})$ only. Hence Theorem I seems to be much deeper. This seems to justify the use of more difficult analytical tools in the proof of [1]).

11. For the proof of theorem III we need some simple auxiliary considerations.

Let

$$R = \int_0^{2\pi} \left(\frac{\sin \dfrac{\left[\dfrac{m}{2}\right]+1}{2} t}{\sin \dfrac{t}{2}} \right)^4 dt \quad \ldots \ldots \quad (11.1)$$

Obviously

$$\left.\begin{array}{l} R > 4 \displaystyle\int_0^{2\pi} \left(\frac{\sin \dfrac{\left[\dfrac{m}{2}\right]+1}{2} t}{t} \right)^4 dt = 4 \left(\frac{\left[\dfrac{m}{2}\right]+1}{2} \right)^3 \int_0^{\pi\left(\left[\frac{m}{2}\right]+1\right)} \left(\frac{\sin y}{y} \right)^4 dy > \\ > \dfrac{1}{2} \left(\dfrac{m}{2} \right)^3 \displaystyle\int_0^{\frac{\pi}{2}} \left(\frac{\sin y}{y} \right)^4 dy > c_1 m^3 \end{array}\right\} \quad (11.2)$$

where c_1 and later c_2, ... denote numerical constants. Further

$$\left.\begin{array}{l} R = 2 \displaystyle\int_0^{\pi} \left(\frac{\sin \dfrac{\left[\dfrac{m}{2}\right]+1}{2} t}{\sin \dfrac{t}{2}} \right)^4 dt \leqq 2\pi^4 \int_0^{\pi} \left(\frac{\sin \dfrac{\left[\dfrac{m}{2}\right]+1}{2} t}{t} \right)^4 dt < \\ < \dfrac{\pi^4}{4} \left(\left[\dfrac{m}{2}\right]+1 \right)^3 \displaystyle\int_0^{\infty} \left(\frac{\sin y}{y} \right)^4 dy < c_2 m^3. \end{array}\right\} \quad (11.3)$$

12. Let a be a parameter subjected only to the restriction

$$\pi \geqq a \geqq \frac{10}{m+1} \quad \ldots \ldots \quad (12.1)$$

and let

$$\pi_m(x, a) = \frac{1}{R}\int_{-x}^{a-x}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{\sin\frac{t}{2}}\right)^4 dt. \quad . \quad . \quad . \quad (12.2)$$

We have also

$$\pi_m(x, a) = \frac{1}{R}\int_{0}^{a}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{t}(t-x)}{\sin\frac{t-x}{2}}\right)^4 dt \quad . \quad . \quad (12.3)$$

and since the integrand of (12.2) is an even function of t

$$\pi_m(x, a) = \frac{1}{R}\int_{x-a}^{x}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{\sin\frac{t}{2}}\right)^4 dt. \quad . \quad . \quad . \quad . \quad (12.4)$$

Since generally

$$\left(\frac{\sin\frac{k+1}{2}y}{\sin\frac{y}{2}}\right)^2 = (k+1) + k\cdot 2\cos y + (k-1)\,2\cos 2y + \ldots 1\cdot 2\cos k\,y,$$

we obtain at once that the integrand of (12.2), i.e. $\pi_m(x, a)$ itself, is a trigonometric polynomial of order $\leqq m$

$$\pi_m(x, a) = a_0(a) + \sum_{1\leqq \nu\leqq m}(a_\nu(a)\cos\nu x + b_\nu(a)\sin\nu x). \quad . \quad (12.5)$$

13. We need some information about the coefficients in (12.5). Evidently, using (12.3),

$$\left.\begin{aligned} a_0(a) &= \frac{1}{2\pi}\int_0^{2\pi}\pi_m(x, a)\,dx = \frac{1}{2\pi R}\int_0^a dt\int_0^{2\pi}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}(t-x)}{\sin\frac{t-x}{2}}\right)^4 dx = \\ &= \frac{1}{2\pi R}\int_0^a dt\cdot R = \frac{a}{2\pi}. \end{aligned}\right\} \quad (13.1)$$

Further using (12. 2)

$$a_\nu(a)=\frac{1}{\pi}\int_0^{2\pi}\pi_m(x,a)\cos\nu x\,dx=-\frac{1}{\pi\nu}\int_0^{2\pi}\frac{\partial\pi_m}{\partial x}\sin\nu x\,dx=$$

$$=-\frac{1}{\pi\nu R}\int_0^{2\pi}\sin\nu x\left\{\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}(a-x)}{\sin\frac{a-x}{2}}\right)^4+\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}x}{\sin\frac{x}{2}}\right)^4\right\}dx$$

i.e.

$$\left.\begin{aligned}|a_\nu(a)|\leqq\frac{1}{\pi\nu R}\int_0^{2\pi}\left\{\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}(a-x)}{\sin\frac{a-x}{2}}\right)^4\right.+\\+\left.\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}x}{\sin\frac{x}{2}}\right)^4\right\}dx=\frac{2}{\pi\nu},\qquad 1\leqq\nu\leqq m;\end{aligned}\right\}\quad .\quad (13.2)$$

similarly

$$\left.\begin{aligned}|b_\nu(a)|\leqq\frac{2}{\pi\nu}\\1\leqq\nu\leqq m.\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (13.3)$$

14. We need also some information about the shape of the graph of $\pi_m(x,a)$. The definition of R and representation (12. 2) give immediately for every real x

$$0\leqq\pi_m(x,a)\leqq 1.\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (14.1)$$

We consider $\pi_m(x,a)$ in the interval

$$\frac{2}{m+1}\leqq x\leqq a-\frac{2}{m+1}\ .\ .\ .\ .\ .\ .\quad (14.2)$$

(this has a meaning owing to (12. 1)). Using the representation (12. 2) and the estimation (11. 3) we have in the range (14. 2)

$$\left.\begin{aligned}\pi_m(x,a)>\frac{1}{c_2 m^3}\int_0^{\frac{1}{1+\left[\frac{m}{2}\right]}}\left(\frac{\sin\frac{\left[\frac{m}{2}\right]+1}{2}t}{\frac{t}{2}}\right)^4dt=\\=\frac{1}{c_2 m^3}\left(\frac{\left[\frac{m}{2}\right]+1}{2}\right)^3\int_0^{\frac{1}{2}}\left(\frac{\sin y}{y}\right)^4dy>c_3.\end{aligned}\right\}\quad .\ .\quad (14.3)$$

Further, for $a < x \leqq \frac{3}{2}\pi$ we have, using the estimation (11. 2) and the representation (12. 4)

$$\left.\begin{aligned} \pi_m(x, a) &< \frac{1}{c_1 m^3}\int_{x-a}^{\frac{3}{2}\pi}\left(\frac{\sin\dfrac{\left[\frac{m}{2}\right]+1}{2}t}{\sin\dfrac{t}{2}}\right)^4 dt < \frac{\pi^4}{c_1 m^3}\int_{x-a}^{\frac{3}{2}\pi}\left(\frac{\sin\dfrac{\left[\frac{m}{2}\right]+1}{2}t}{t}\right)^4 dt < \\ &< \frac{\pi^4}{c_1 m^3}\left(\frac{\left[\frac{m}{2}\right]+1}{2}\right)^3\int_{\frac{\left[\frac{m}{2}\right]+1}{2}(x-a)}^{\infty}\left(\frac{\sin y}{y}\right)^4 dy < c_4\int_{\frac{m}{4}(x-a)}^{\infty}\frac{dy}{y^4} < \frac{c_5}{(m(x-a))^3}. \end{aligned}\right\} \quad (14.4)$$

Finally, for $\frac{3}{2}\pi \leqq x \leqq 2\pi$ we have from the estimation (11. 2) and the representation (12. 4), since $x - a \geqq \frac{3}{2}\pi - \pi = \dfrac{\pi}{2}$

$$\left.\begin{aligned} \pi_m(x, a) &< \int_{\frac{\pi}{2}}^{x}\left(\frac{\sin\dfrac{\left[\frac{m}{2}\right]+1}{2}t}{\sin\dfrac{t}{2}}\right)^4 dt < c_6\int_{\frac{\pi}{2}}^{x}\left(\frac{\sin\dfrac{\left[\frac{m}{2}\right]+1}{2}t}{2\pi-t}\right)^4 dt = \\ &= c_6\int_{2\pi-x}^{\frac{3\pi}{2}}\left(\frac{\sin\dfrac{\left[\frac{m}{2}\right]+1}{2}y}{y}\right)^4 dy < \frac{c_7}{(m(2\pi-x))^3} \end{aligned}\right\} \quad (14.5)$$

15. Now we are going to prove the following

Lemma. Assuming (4. 1) and (4. 2) we have for the number N of the φ_ν's lying in an arbitrary interval of length $\dfrac{10}{m+1}$ with $m > 20$ the inequality

$$N < c_8\left(\frac{n}{m+1} + \sum_{\nu=1}^{m}\frac{\psi(\nu)}{\nu}\right). \quad . \quad . \quad . \quad . \quad . \quad (15.1)$$

Proof. Without loss of generality we may suppose that our interval is

$$\frac{2}{m+1} \leqq x \leqq \frac{12}{m+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.2)$$

We consider the polynomial $\pi_m(x, \gamma)$ where

$$\gamma = \frac{14}{m+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.3)$$

Replacing x in

$$\pi_m(x,\gamma)=a_0(\gamma)+\sum_{\nu=1}^{m}(a_\nu(\gamma)\cos\nu x+b_\nu(\gamma)\sin\nu x)$$

by $\varphi_1, \varphi_2, \ldots, \varphi_n$ and summing we obtain owing to (13. 1)

$$\sum_{l=1}^{n}\pi_m(\varphi_l,\gamma)=\frac{\gamma}{2\pi}n+\sum_{\nu=1}^{m}a_\nu(\gamma)\sum_{l=1}^{n}\cos\nu\varphi_l+\sum_{\nu=1}^{m}b_\nu(\gamma)\sum_{l=1}^{n}\sin\nu\varphi_l.$$

In the interval (15. 2) condition (14. 2) is satisfied i.e. from (14. 3) and the non-negativity of $\pi_m(x,\gamma)$ we have

$$c_3 N\leqq\sum_{l=1}^{n}\pi_m(\varphi_l,\gamma)\leqq\frac{7}{\pi}\cdot\frac{n}{m+1}+\sum_{\nu=1}^{m}(|a_\nu(\gamma)|+|b_\nu(\gamma)|)\left|\sum_{l=1}^{n}e^{\nu i\varphi_l}\right|.$$

Applying (4. 1), (13. 2) and (13. 3) we obtain further

$$c_3 N<\frac{7}{\pi}\cdot\frac{n}{m+1}+\frac{4}{\pi}\sum_{\nu=1}^{m}\frac{\psi(\nu)}{\nu}. \qquad \text{Q. e. d.}$$

16. Now we turn to the proof of theorem III. Let d be given satisfying

$$\frac{10}{m+1}\leqq d\leqq\pi \quad \ldots\ldots\ldots \quad (16.\ 1)$$

and consider the polynomial $\pi_m(x,d)$. Replacing x in

$$\pi_m(x,d)=\frac{d}{2\pi}+\sum_{\nu=1}^{m}(a_\nu(d)\cos\nu x+b_\nu(d)\sin\nu x)$$

by $\varphi_1, \varphi_2, \ldots, \varphi_n$ and summing we obtain

$$\sum_{j=1}^{n}\pi_m(\varphi_j,d)=\frac{d}{2\pi}n+\sum_{\nu=1}^{m}a_\nu(d)\sum_{j=1}^{n}\cos\nu\varphi_j+\sum_{\nu=1}^{m}b_\nu(d)\sum_{j=1}^{n}\sin\nu\varphi_j.$$

Arguing as before we obtain

$$\sum_{j=1}^{n}\pi_m(\varphi_j,d)>\frac{d}{2\pi}n-\frac{4}{\pi}\sum_{\nu=1}^{m}\frac{\psi(\nu)}{\nu}. \quad \ldots\ldots \quad (16.\ 2)$$

Now denoting the number of φ_ν's in $0\leq x\leq d$ by $N(d)$, the contribution of these φ_ν's is, owing to $\pi_m(x,d)\leq 1$,

$$\leqq N(d). \quad \ldots\ldots\ldots \quad (16.\ 3)$$

To obtain an upper estimation for the contribution of the other φ_ν's we construct successive contiguous intervals of length $\frac{10}{m+1}$ each starting from $x=d$ and covering the interval $d\leq x\leq\frac{3}{2}\pi$. The contributions of the φ_ν in the interval

$$D_k \ :\ d+k\frac{10}{m+1}\leqq x\leqq d+(k+1)\frac{10}{m+1}$$

is owing to the lemma

$$< c_8 \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right) \max_{x \in D_k} \pi_m(x, d)$$

which is, by (14. 4)

$$< c_8 \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right) \max_{x \in D_k} \frac{c_5}{(m(x-d))^3} <$$

$$< c_9 \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right) \cdot \frac{1}{k^3}.$$

Hence summing over k

$$\sum_{d \leqq \varphi_j \leqq \frac{3}{2}\pi} \pi_m(\varphi_j, d) < c_{10} \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right). \quad . \quad . \quad (16.\ 4)$$

The contribution of the φ_ν lying in the remaining interval $\frac{3\pi}{2} < x \leqq 2\pi$ we can estimate similarly. Combining (16. 2), (16. 3) and (16. 4) we obtain

$$N(d) > \frac{d}{2\pi} \cdot n - c_{11} \left(\frac{n}{m+1} + \sum_{\nu=1}^{m} \frac{\psi(\nu)}{\nu} \right). \quad . \quad . \quad . \quad (16.\ 5)$$

Obviously the same estimation holds for the number $N(c, c+d)$ of the φ_ν's for which $c \leqq \varphi_\nu \leqq c + d \mod 2\pi$. The restriction $d \leqq \pi$ is obviously unnecessary, if we replace c_{11} by $2c_{11} = c_{12}$.

To obtain the upper estimation of $N(c, c+d) - \frac{d}{2\pi} n$ we have, due to

$$N(c_1\, c + d) = n - N(0, c) - N(c + d, 2\pi) \qquad \text{Q. e. d.}$$

merely to apply (16. 5) twice.

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Mathematics. — *On the symbolical method.* I. By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

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§ 1. *Introduction.*

Denoting the homogeneous coordinates of a point X in $(n-1)$-dimensional projective space G_n by $(x_1, x_2, \dots, x_n)$ and the space-coordinates of a hyperplane U' by $(u_1', u_2', \dots u_n')$ one can use the short notations

$$(xy \dots z),\ (u'v' \dots w'),\ (u'x)$$

for determinants whose columns are the coordinates of the points $X, Y, \dots Z$, resp. of the spaces $U', V_{\prime}, \dots W'$ and the linear form

$$u_1'x_1 + u_2'u_2 + \dots u_n'x_n.$$

The first fundamental theorem of the theory of invariants asserts, that every rational projective invariant of a system of points and spaces can be expressed in polynomia of these symbols only. The theorem holds also if the points and spaces are merely symbolical, i.e. stand for variables, which are transformed cogredient or contragredient.

Between these symbols some relations are evident:

I. As the determinant

$$\begin{vmatrix} a_1 \dots a_n\,(au') \\ b_1 \dots b_n\,(bu') \\ \vdots \\ d_1 \dots d_n\,(du') \end{vmatrix} = \text{det. } a_1\, b_2 \dots (du'),$$

vanishes, we have

$$(ab \dots c)(du') \equiv (db \dots c)(au') - (da \dots c)(bu') \dots \pm (dab \dots)(cu').$$

This identity shows, that $(n+1)$ points in G_n are linearly dependent and furnishes the homogeneous coordinates of d, the fundamental simplex being $a, b, \dots c$.

II. $(ab \dots c)(u'v' \dots w') \equiv \text{det. } (au')(bv') \dots (cw')$.

III. Denoting the point of intersection of $(n-1)$ spaces $u', v', \dots w'$ by $x_i = (u'v' \dots w')_{\bar{i}}$ we have

$$(ab \dots cx) \equiv (ab \dots c\ (u'v' \dots w')) \equiv \text{det. } (au')(bv') \dots (cw').$$

The second fundamental theorem of the theory of invariants asserts, that every relation between invariants can be deduced with identities of this form only. In order to avoid confusion: in G_2 the determinants are written (ab), the linearform a_x and the identities are then

$$(ab)c_x \equiv (cb)a_x - (ca)b_x,$$
$$(ab)(xy) \equiv a_x b_y - a_y b_x.$$

This symbolical method has some disadvantages which can clearly be shown comparing the following examples.

1. The proof of DESARGUES-theorem for $n=3$, $n=4$.

$n=3$.

Two triangles being a, b, c and α, β, γ and denoting the opposite sides of $a, \alpha, \ldots$ with $a', \alpha', \ldots$ we have

$$\begin{aligned}((a'\alpha')(b'\beta')(c'\gamma')) &\equiv (((bc)(\beta\gamma))((ca)(\gamma\alpha))((ab)(\alpha\beta))) \equiv \\ &\equiv ((bc)(ca)(\gamma\alpha))((\beta\gamma)(ab)(\alpha\beta)) - ((\beta\gamma)(ca)(\gamma\alpha))((bc)(ab)(\alpha\beta)) \equiv \\ &\equiv (abc)(\alpha\beta\gamma)[(c\gamma\alpha)(b\beta a) - (c\gamma a)(b\beta\alpha)] \equiv (abc)(\alpha\beta\gamma)((c\gamma)\,b\beta)(a\alpha)).\end{aligned}$$

This shows, that if a, b, c and α, β, γ are not collinear triples, then the collinearity of the points of intersection of corresponding sides is equivalent to the concurrence of the lines joining corresponding vertices.

$n=4$.

If the lines of intersection of corresponding planes of two tetraedra a, b, c, d and $\alpha, \beta, \gamma, \delta$ are coplanar, the lines joining the corresponding vertices are concurrent. We have

$$\begin{aligned}(d\delta\, a\, \alpha) &\equiv ((a'b'c')(\alpha'\beta'\gamma')(b'c'd')(\beta'\gamma'\delta')) \equiv \\ &\equiv (a'b'c'd')[(b'\beta'\gamma'\delta')(c'\alpha'\beta'\gamma') - (c'\beta'\gamma'\delta')(b'\alpha'\beta'\gamma')] \equiv \\ &\equiv (a'b'c'd')(\alpha'\beta'\gamma'\delta')(b'\beta'c'\gamma'),\end{aligned}$$

which proves the theorem.

In these two deductions we find the highest degree of symmetry and no further simplification seems possible or necessary.

2. We ask for the equation of a conic through five given points a, b, c, d, e in G_3. The pencil of conics through a, b, c, d can be written

$$(abx)(cdx) + \lambda(acx)(bdx) = 0$$

To obtain a final result, λ has to be calculated from

$$(abe)(cde) + \lambda\,(ace)(bde) = 0$$

and so

$$(ace)(bde)(abx)(cdx) - (abe)(cde)(acx)(bdx) = 0$$

is *an* equation for this conic. The lefthand side is, however, invariant under the interchange of a and d or b and c. Therefore we obtain a large number of equivalent left-hand-sides by permutation of a, b, c, d, e, the equivalence of which can be proved by the identities quoted above. We can remark, that the left-hand-side of the equation is the determinant formed from the points of intersection of opposite sides of the hexagon $a\,c\,e\,b\,d\,x$, as this determinant is

$$(((ac)\;(bd))\;((ce)\;(dx))\;((eb)\;(xa))),$$

so the equation above proves PASCAL's theorem.

The equation of the conic is in each of the variables of the second degree, just as it has to be, and the only complication that remains is here the great number of equivalent formulae which may cause a difficulty: to choose the right identities necessary to prove the equivalence (in a most direct way).

3. We ask for the equation of a PASCAL-line of the conic a, b, c, d, e, f e.g. that joining the points of intersection of ac, bd and ce, df.

We have then

$$(((ac)(bd)) \quad ((ce)(df)) \quad x) \equiv$$
$$(ace)(cdf)(bdx) + (dce)(bdf)(acx) = 0.$$

Here again we obtain several equivalent lefthand-sides of the first degree in a, f, b, e of the second degree in c, d. As the PASCAL-lines describe a pencil of lines if one of the points moves along the conic, the other five remaining fixed there are one c and one d "too much". Here the situation is more serious: by identical transformations the degree of each of the variables cannot be changed. The only possibility to eliminate superfluous elements is to transform the equation so as to split off bracket-factors containing all superfluous elements. But with two elements c, d no not vanishing bracketfactor can be formed: *it is impossible* to eliminate the superfluous c, d in the above equation of the PASCAL-line by identical transformations.

In order to remove the dissymmetry we must have a method which splits up the ternary brackets into symbols containing *at most* two elements.

In higher dimensional spaces the abundance of equivalent forms becomes overwhelming and dissymmetries occur very often.

The splitting up of the n-air bracketfactors can be obtained from a normalcurve in G_n. Be the curve in parametric representation

$$P_t \equiv (AU')\, a_t^{n-1} = 0,$$

then the bracketfactor

$$(P_i\, P_k \dots P_l\, P_m) \equiv (AB \dots CD)\, a_i^{n-1}\, b_k^{n-1} \dots c_l^{n-1}\, d_m^{n-1}.$$

As this form vanishes for $i = k$, $i = l$, ... $l = m$ we have

$$(P_i P_k \dots P_l P_m) \equiv \text{const. } I. \ (ik)(il) \dots (im) \dots (lm)$$

where

$$I = (AB \dots CD) \ (ab) \dots (cd)$$

is the invariant, which vanishes if the curve lies in a G_k, $k \leq n-1$.

A normal curve defines a polarity (incident for $n =$ even, non-incident for $n =$ odd) and specifying the hyperplanes by the parameters of the points of intersection with the normalcurve and the points by the parameters of the points of intersection of the normalcurve with the polar-hyperplanes we can develop a symbolical method in which all the advantages are preserved but new possibilities created by the breaking up of the determinant factors.

§ 2. $(AU')a_t^2$.

If $A, \dots U'$ are ternary symbols, $a, \alpha, \dots$ binary, we have a parametric representation of a conic in

$$P_t \equiv (AU')\, a_t^2 = 0.$$

For a given t we have the equation of the point of the conic; a given line U' contains two points, which coincide if

$$(AU')\ (BU')\ (ab)^2 = 0,$$

the equation of the conic in linecoordinates.

A cord of the conic, joining the points i, k has the equation

$$(ABX)\, a_i^2\, b_k^2 \equiv (ABC)\, (ab)\, a_i\, b_k\, (ik) = 0.$$

Dividing by $(ik) \neq 0$ we find the equation of the tangent in the point t as

$$(ABX)\ (ab)\ a_t b_t = 0,$$

the lefthand-side of which we can replace by

$$(A'X)\, \alpha_t^2.$$

The equation of the conic in pointcoordinates is therefore

$$(\Omega' X)^2 \equiv (A'X)\ (B'X)\ (\alpha\beta)^2 = 0.$$

The conic degenerates if and only if three points are collinear i.e. if

$$A_1\, a_t^2 \qquad A_2\, a_t^2 \qquad A_3\, a_t^2$$

are linearly dependent. This means that the corresponding binary quadratic forms represent three pairs of an involution, which gives

$$I = (ABC)\ (ab)\ (bc)\ (ca) = 0.$$

Theorem:

The complete system of $(AU')a_t^2, \varphi_t^m, \psi_t^n, \dots$ consists of

$$I, (AU')\, a_t^2,\ (A'X)\, \alpha_t^2,\ (A'X)\, (B'X)\, (\alpha\beta)^2,\ (AU')\, BU')\, (ab)^2$$

and the comitants which are generated from the complete system of

$$m_t^2,\ \varphi_t^m,\ \psi_t^n, \dots$$

by replacement of m_t^2 by $(AU')a_t^2$ or $(A'X)\alpha_t^2$

Proof:

a) $(ABC) \dots$ is reducible to $I \dots$.

The form $(ABC)\ (ab)^2 \dots \equiv 0$.

The form $(ABC)\ (ab) a_x b_y c_z c_t$ is a polar form of $(ABC)\ (ab) a_x b_y c_z^2$

Changing A, B, C in B, C, A; C, A, B we have

$$(ABC)\, (ab)\, a_x\, b_y\, c_z^2 \equiv \tfrac{1}{3}\, (ABC)\, [(ab)\, a_x\, b_y\, c_z^2 + (bc)\, b_x\, c_y\, a_z^2 + (ca)\, c_x\, a_y\, b_z^2] \equiv$$
$$\equiv \tfrac{1}{3}\, (ABC)\, (ab)\, (bc)\, (ca) \cdot (zx)\, (yz).$$

The form $(ABC) a_x a_y b_z b_t c_u c_v \dots$ is a polar form of $(ABC) a_i^2 b_k^2 c_l^2 \equiv \equiv \frac{1}{3} I \cdot (ik)\ (kl)\ (li)$.

b) $K \equiv (ABX)\,(ab)\,(ac)\,b_x c_y (CU')\ldots$ is reducible to $I\ldots$.

We have, transforming the ternary symbols

$K \equiv (CBX)(ab)(ac)b_x c_y(AU') - (CAX)(ab)(ac)b_x c_y(BU')$ + reducible forms.

Interchanging A, C in the first and B, C in the second term we have

$$K \equiv (ABX)(CU')\,[(cb)(ca)\,b_x\,a_y + (ac)(ab)\,c_y\,b_x]\ldots + I\cdot(XU')\ldots$$
$$\equiv (ABX)(CU')(ab)(bc)(ca)(xy) + I\cdot(XU')\ldots \equiv I\cdot[(XU')\ldots + (XU')\ldots].$$

The only possible irreducible comitants containing $(ABX)\ldots$ are therefore of the form $(ABX)\,(ab)a_y b_z \ldots \equiv (A'X)\alpha_y\alpha_z\ldots$ in which y, z are *not* connected with $a_x^2\ldots$, and $(A'X)\,(B'X)\,(\alpha\beta)\ldots$ which is reducible to $(A'X)\,(B'X)\,(\alpha\beta)^2$; for interchanging A and B we can always obtain the factor $(ABX)\,(ab)\ldots$ *q.e.d.*

According to these formulae we have:

1. $$(A'B'U')(\alpha\beta)\,\alpha_\lambda\,\beta_\lambda \equiv {}^1/_2\,((AB)B'U')\,(ab)\,[(a\beta)\,b_\lambda + (b\beta)\,a_\lambda]\,\beta_\lambda \equiv$$
$$\equiv -\tfrac{1}{6}\,I\cdot(AU')\,a_\lambda^2.$$

This identity shows, that the line-equation obtained from the point-equation of a conic is the same as the original equation, provided that the conic is not degenerated. The importance of this identity lies in the fact that we can interchange point- and line-equation by a simple change of A, A' and a, α.

2. The bracketfactor $(P_i P_k P_l)$ of three points on the conic is split up in a cycle

$$\langle i\,k\,l\rangle = (ik)\,(kl)\,(li),$$

apart from a factor $\tfrac{1}{3}\,I$.

3. The equation of the line joining two points, the poles of the cords $P_p P_q$ and $P_r P_s$ is

$$(ABX)\,a_p a_q\,b_r b_s \equiv \tfrac{1}{2}\,(A'X)\,[(qr)\,\alpha_p\,\alpha_s + (ps)\,\alpha_q\,\alpha_r] \equiv$$
$$\equiv \tfrac{1}{2}\,(A'X)\,[(pr)\,\alpha_q\,\alpha_s + (qs)\,\alpha_p\,\alpha_r] = 0.$$

4. The linearform of the cord $P_p P_q$ and the point, being the pole of $P_r P_s$ is

$$(A'B)\alpha_p\alpha_q b_r b_s \equiv -\tfrac{1}{8}I\cdot[(ps)\,(qr) + (qs)\,(pr)].$$

This indentity shows, that if the cord $P_p P_q$ contains the pole of $P_r P_s$, then the cross-ratio $(pqrs) = -1$, provided $I \neq 0$.

5. $(A'C)\,(B'X)\,(\alpha\beta)^2 c_p\,c_q \equiv -\tfrac{1}{3}I(B'X)\beta_p\,\beta_q$.

which expresses the fact, that the polar line of the pole of the cord $P_p P_q$ is the cord itself, provided $I \neq 0$, and that the HESSIAN of the conic is apart from a constant I^2.

6. $(A'B)\,(B'C)\,(\alpha\beta)^2 b_p b_q c_r c_s =$ const. $I^2\,[(pr)\,(qs) + (qr)\,(ps)]$, which indicates that the pole of the cord $P_p P_q$ is conjugated to the pole of the cord $P_r P_s$ when the crossratio $(pqrs) = -1$, provided $I \neq 0$.

Denoting the cycle $(ik)\ (kl)\ (lm) \dots (qi)$ by

$$\langle\, i\,k\,l\,m \dots q\,\rangle$$

we have

$$\langle iklm \dots q\rangle \equiv (\mathbf{ik})(kl)(\mathbf{lm}) \dots (qi) \equiv -(lk)^2 \langle im \dots q\rangle - \langle ilkm \dots q\rangle$$
$$\langle iklm \dots q\rangle \equiv \langle klm \dots qi\rangle.$$

From this follows, that a cycle with an even number of bracketfactors is reducible to a sum of products of squares of these factors, as is clear from the iteration of the first formula, combined with cyclical permutation of the indices in the last cycle.

For cycles with six-brackets we have also

$$\langle iklmnp\rangle - \langle iknplm\rangle \equiv \langle inp\rangle\ \langle klm\rangle - \langle knp\rangle\ \langle ilm\rangle$$

as follows writing out the cycles and transforming identically the first and fourth brackets.

Because of the ternary interpretation it is evident that:

$$\langle inp\rangle\ \langle klm\rangle - \langle knp\rangle\ \langle ilm\rangle + \langle kip\rangle\ \langle nlm\rangle - \langle kin\rangle\ \langle plm\rangle \equiv 0.$$

S t a n d a r d e q u a t i o n s a n d s t a n d a r d f o r m s

The complete system of the form $(AU')a_t^2$ and a system of points with parameters $i, k, \dots$ contains the simultaneous invariant forms

$$(AU')a_i a_k \qquad (A'X)\alpha_i\alpha_k$$

only. As a standardequation of the pole of the cord P_iP_k we use

$$\underline{\{ik\} = \frac{(A\,U')\,a_i\,a_k}{(ik)} = 0}$$

and we denote the equation of the cord itself by

$$\underline{[ik] = \frac{(A'\,X)\,\alpha_i\,\alpha_k}{(ik)} = 0.}$$

We then have

$$\underline{\{ik\} \equiv -\{ki\}} \quad ; \quad \underline{[ik] = -[ki]}.$$

The point of intersection of the cords $[ik] = 0$ and $[lm] = 0$ is then:

$$\{im\} + \{kl\} = 0 \quad \text{or} \quad \{il\} + \{km\} = 0.$$

Denoting the standardequation with

$$\underline{P_{ik,lm} \equiv \{im\} + \{kl\}}$$

we have:

$$P_{ik,lm} \equiv P_{ki,ml} \equiv -P_{ml,ki} \equiv -P_{lm,ik}$$
$$(kl)\,(im)\,P_{ik,lm} \equiv (il)\,(km)\,P_{ik,ml}.$$

The linear form of the cord $[ik]$ and the point $\{lm\}$ is apart from a factor $-\frac{1}{6}l$:

$$(il)\ (km) + (im)\ (kl).$$

§ 3. Three points on a conic.

Be the three points P_i, P_k, P_l. Squaring the identity

$$(ki)a_l \equiv (li)a_k - (lk)a_i$$

we obtain at first

$$\langle ikl \rangle \{ik\} \equiv -\tfrac{1}{2}((li)^2 a_k^2 + (lk)^2 a_i^2 - (ik)^2 a_l^2).$$

Joining the point P_i with the pole of the cord $[kl] = 0$ we have

$$t_{i,kl} = [ki] + [li].$$

From this is evident

$$t_{i,kl} + t_{k,li} + t_{l,ik} \equiv 0,$$

which shows that the three lines are concurrent, the theorem of BRIANCHON for the triangle. The equation of the BRIANCHON-point $\triangle_{ikl}$ is obtained intersecting

$$[ki] + [li] = 0 \quad \text{and} \quad [lk] + [ik] = 0$$

which gives immediately

$$(lk)^2 P_i + (ik)^2 P_l + (il)^2 P_k = 0.$$

In virtue of the relation quoted above we obtain permutating cyclically and dividing by $< ikl >$ as a standardform of $\triangle_{ikl}$:

$$\triangle_{ikl} \equiv \{ik\} + \{kl\} + \{li\}.$$

According to the formula 6. we have immediately

$$(\Omega' \triangle_{ikl})^2 \equiv \text{const. } I^2,$$

which is independent of i, k, l.

§ 4. Four points on a conic.

The evident relations for $P_{ik,lm}$ given above can be completed by

$$\begin{aligned} \langle iklm \rangle P_{ik,lm} &\equiv -\langle klm \rangle P_i + \langle ilm \rangle P_k \\ &\equiv \quad \langle ikm \rangle P_l - \langle ikl \rangle P_m. \end{aligned}$$

Proof:

$$\begin{aligned} \langle iklm \rangle P_{ik,lm} &\equiv \langle iklm \rangle \{im\} + \langle iklm \rangle \{kl\} \equiv \\ &\equiv (lm)[-(\mathbf{ik})(kl)\, a_i\, \mathrm{a_m} + (\mathbf{i\,k})(mi)\, a_k\, \mathrm{a_l}] \equiv \\ &\equiv -\langle klm \rangle \mathrm{a}_i^2 + \langle ilm \rangle \mathrm{a}_k^2, \end{aligned}$$

omitting the ternary factor for sake of simplified notation. The second half of the theorem follows transforming the other factors.

As an immediate consequence of the standardformula we have in

$$P_{ik,lm} \equiv \{im\} + \{kl\} = 0 \qquad P_{il,km} \equiv \{im\} + \{lk\} = 0$$

the equations of the diagonalpoints of the tetragon. It is evident from these, that the diagonalpoints are collinear and harmonical with the poles of the cords $[im] = 0$, $[kl] = 0$.

Moreover we have

$$(\Omega' P_{ik,lm})(\Omega' P_{il,km}) \equiv (C' A)(D' B)(cd)^2 \left[\frac{a_l\, a_m\, b_l\, b_m}{(im)^2} - \frac{a_k\, a_l\, b_k\, b_l}{(kl)^2} \right] \equiv 0.$$

The diagonalpoints of the tetragon form a polar-triangle of $(\Omega' X)^2 = 0$.

Mathematics. — *On a combinatioral problem.* By N. G. DE BRUIJN and P. ERDÖS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 27, 1948.)

Let there be given n elements $a_1, a_2, \ldots, a_n$. By $A_1, A_2, \ldots, A_m$ we shall denote combinations of the a's. We assume that we have given a system of $m > 1$ combinations $A_1, A_2, \ldots, A_m$ so that each pair (a_i, a_j) is contained in one and only one A. Then we prove

Theorem 1. We have $m \geq n$ [1], with equality occurring only if either the system is of the type $A_1 = (a_1, a_2, \ldots, a_{n-1})$, $A_2 = (a_1, a_n)$, $A_3 = (a_2, a_n) \ldots A_n = (a_{n-1}, a_n)$, or if n is of the form $n = k(k-1) + 1$ and all the A's have k elements, and each a occurs in exactly k of the A's.

Corollary: If the elements a_i are points in the real projective plane the theorem can be stated as follows: Let there be given n points in the plane, not all on a line. Connect any two of these points. Then the number of lines in this system is $\geq n$. In this case equality occurs only if $n - 1$ of the points are on a line.

This corollary can be proved independently of Theorem 1 by aid of the following theorem of GALLAI (= GRÜNWALD) [2]:

Let there be given n points in the plane, not all on a line. Then there exists a line which goes through two and only two of the points.

Remark: The points of inflexion of the cubic show that it is essential that the points should all be real, thus GALLAI's theorem permits no projective and a fortiori no combinatorial formulation. Also the result clearly fails for infinitely many points.

We now give GALLAI's ingenious proof: Assume the theorem false. Then any line through two of the points also goes through a third. Project one of the points, say a_1 to infinity, and connect it with the other points. Thus we get a set of parallel lines each containing two or more points a_i (in the finite part of the plane). Consider the system of lines connecting any two of these points, and assume that the line $(a_i\, a_j\, a_k)$ forms the smallest angle with the parallel lines. (This line again contains at least three points). But the line connecting a_j with a_1 (at infinity) contains at least another (finite) point a_r, and clearly (see figure) either the line $(a_i\, a_r)$

[1]) This was also proved by G. SZEKERES but his proof was more complicated.

[2]) This theorem was first conjectured by SYLVESTER, GALLAI's proof appeared in the Amer. Math. Montly as a solution to a problem by P. ERDÖS. The corollary to Theorem 1 also appeared as a problem in the Monthly.

See also H. S. M. COXETER, Amer. Math. Monthly **55**, 26—28 (1948), where very simple proofs due to KELLY and STEINBERG are given.

or the line (a_r, a_k) forms a smaller angle with the parallel lines then $(a_i\, a_j\, a_k)$. This contradiction establishes the result.

Remark: Denote by $f(n)$ the minimum number of lines which go through exactly two points. It is not known whether $\lim f(n) = \infty$. All that we can show is that $f(n) \geqq 3$.

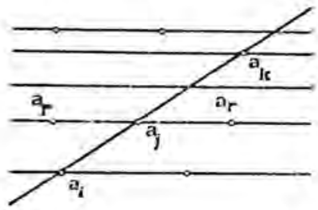

Fig. 1.

Now we prove the corollary as follows: We use induction. Assume that the number of lines determined by $n-1$ points, not all on a line, is $\geqq n-1$. Then we shall prove that n points, not all on a line, determine at least n lines.

Let (a_1, a_2) be a line going through two points only. Consider the points $a_2, a_3, \ldots, a_n$. If they are all on a line, then (a_1, a_i), $i = 2, 3, \ldots, n$ and $(a_2, a_3, \ldots, a_n)$ clearly determine n lines. If they do not all lie on a line, then they determine at least $n-1$ lines, and (a_1, a_2) is clearly not one of these lines. Thus together with (a_1, a_2) we again get at least n lines. The same induction argument shows that we get exactly n lines only if $n-1$ of the points lie on a line, q.e.d.

Proof of theorem 1. For simplicity we shall call the elements $a_1, a_2, \ldots, a_n$ points and the sets $A_1, A_2, \ldots, A_m$ lines. Denote by k_i the number of lines passing through the point a_i, and by s_j the number of points on the line A_j. We evidently find (by counting the number of incidences in two ways)

$$\sum_{j=1}^{m} s_j = \sum_{i=1}^{n} k_i \quad \ldots\ldots\ldots\ldots \quad (1)$$

Further if A_j does not pass through a_i, then

$$s_j \leqslant k_i \quad \ldots\ldots\ldots\ldots \quad (2)$$

(2) follows from the fact that a_i can be connected by a line (i.e. an A) to all the s_j points of A_j, and any two of these lines are different, since otherwise they would have two points in common.

Assume now that k_n is the smallest k_i and that $A_1, A_2, \ldots, A_{k_n}$ are the lines through a_n. We may suppose that each line contains at least two points, since otherwise it could be omitted. Also $k_n > 1$, for otherwise all the points are on a line. Thus we can find points a_i on A_i, $a_i \neq a_n$, $i = 1, 2, \ldots, k_n$. Also if $i \neq j$, $i \leq k_n$, $j \leq k_n$ then a_i is not on A_j (for otherwise A_i and A_j would have two points in common). Hence by (2) (putting $k_n = \nu$)

$$s_2 \leqslant k_1, \quad s_3 \leqslant k_2, \ldots, \quad s_\nu \leqslant k_{\nu-1}, \quad s_1 \leqslant k_\nu; \quad s_j \leqslant k_n \text{ for } j > \nu. \quad (3)$$

From (1), (3) and the minimum property of k_n we obtain $m \geq n$, which proves the first part of Theorem 1.

We now determine the cases where $m = n$. If $m = n$, then all the inequalities of (3) have to be equalities. Consequently we can renumerate the points so that $s_1 = k_1$, $s_2 = k_2$, ..., $s_n = k_n$. We may suppose that $k_1 \geq k_2 \geq \ldots \geq k_n > 1$. There are two cases:

a) $k_1 > k_2$. Hence by $s_1 = k_1 > k_i$ $(2 \leq i \leq n)$, (2) shows that all the a_i $(i \geq 2)$ lie on A_1. Of course a_1 does not lie on A_1 and we have the first case of Theorem 1.

b) $k_1 = k_2$. If no k_i is less than k_1 then clearly $k_i = s_j$ $(1 \leq i, j \leq n)$. We shall show that this is the only possibility. If $k_j < k_1$, then we have by (2) that a_j lies on both A_1 and A_2. Hence k_n is the only k which can be less than k_1. Now $s_n = k_n$ different lines contain a_n. Any line through a_n contains one further point and all but one contain two further points, since $k_1 = k_2 = \ldots = k_{n-1} > k_n \geq 2$. Thus there are at least two lines which do not contain a_n; for both of these lines we have by (2) $s_j \leq k_n$. This contradicts $s_1 = s_2 = \ldots = s_{n-1} > k_n$.

Apart from case a) we only have the case where $s_i = k_j = k$, $(1 \leq i, j \leq n)$. It is easily seen that then $n = k\,(k-1) + 1$, and also that any pair of lines has exactly one intersection point. For if A_i does not intersect A_j; and if a_l lies on A_i then we infer from (2) that $k_l \geq s_j + 1$ which is not possible since $k_l = s_j = k$. The two dimensional projective finite geometries with $k - 1 = p^\alpha$, p prime, are known to be systems of this type, but F. W. LEVI [3]) constructed a non-projective example with $k = 9$.

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[3]) F. W. LEVI, Finite geometrical systems, Calcutta 1942.

Mathematics. — *On the mutual inductance of two parallel coaxial circles of circular cross-section.* By C. J. BOUWKAMP. (Communicated by Prof. H. B. G. CASIMIR.)

(Communicated at the meeting of November 27, 1948.)

1. *Introduction.*

As is well known, the mutual inductance of two parallel coaxial circles of vanishing cross-section is expressible in terms of complete elliptic integrals. However, in practice the cross-sectional dimensions of a conductor are finite, and the question arises how to calculate the effect of a non-vanishing section. The solution of this problem is of value in standard measurements. A second possible application concerns the calculation of the mutual inductance of two coils both consisting of closely spaced parallel windings, it being understood that the transverse dimensions of the coils shall be small compared to the distance between their centres, while the currents flowing through the coils in the longitudinal direction are considered, to a first approximation, as being uniformly distributed over the respective cross-sections.

Numerous expressions have been derived for the inductances of circular coils of rectangular and square sections, for a survey of which the reader is referred to the publications of the Bureau of Standards (see refs 1—4 at the end of the paper). Surprisingly, however, expressions for the mutual inductance of two circular wires of *circular* cross-section do not seem to have ever been published. This may partly be due to the fact that practical coils are likely to have rectangular sections. Yet a natural extension of the problem of two circles of vanishing section is to assume these sections circular, because the wires used are mostly of that type. Therefore, in this paper we shall derive an expression for the mutual inductance of two parallel coaxial toroids, both of circular section, on the assumption that the current density in either ring is constant over the corresponding section, which implies that the skin effect is ignored.

Let R_1 and R_2 denote the radii of the toroids (see fig. 1), and let a_1, a_2 be the radii of the corresponding cross-sections, D the axial distance between the centres of the toroids. Moreover, let $\varepsilon_1 = a_1/R_1$, $\varepsilon_2 = a_2/R_2$.

In practical cases the parameters ε_1 and ε_2 are very small compared to 1, and at any rate not greater than 1. It will, therefore, be adequate to expand the mutual inductance, M, of the toroids into a double power series of the variables ε_1 and ε_2. As we shall see, only even powers occur in such an expansion, so that one may write

$$M = \sum_{n,m=0}^{\infty} M_{n,m}\, \varepsilon_1^{2n}\, \varepsilon_2^{2m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $M_{n,m}$ is independent of a_1 and a_2. The object of this paper, then, is to give an explicit expression for the coefficients in their dependence on D, R_1, R_2, n and m. These coefficients appear to be expressible in terms

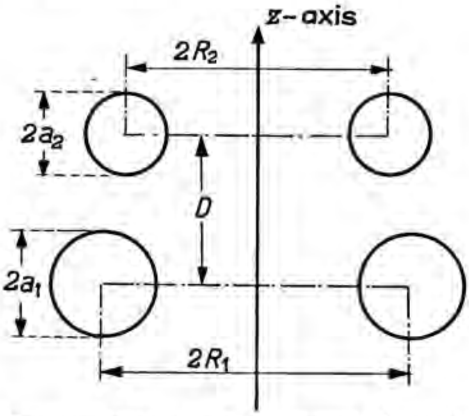

Fig. 1. Cross-sectional view of the system of coaxial toroids.

of a simple integral involving Bessel functions, which integral can be readily evaluated, at least for small values of $n + m$, leading to complete elliptic integrals of the first and second kinds.

The Bessel-function representation in question is a generalization of a known expression for $M_{0,0}$ due to HAVELOCK (ref. 5), viz.,

$$M_{0,0} = 4\pi^2 R_1 R_2 \int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, dt. \quad . \quad . \quad . \quad . \quad (2)$$

On the other hand, $M_{0,0}$ is known (since MAXWELL) in terms of elliptic integrals. For instance [1]),

$$M_{0,0} = 4\pi \sqrt{R_1 R_2} \left\{ \left(\frac{2}{k} - k \right) K - \frac{2}{k} E \right\}. \quad . \quad . \quad . \quad . \quad (3)$$

in which K and E are the complete elliptic integrals of the first and second kinds to modulus k, where

$$k^2 = \frac{4 R_1 R_2}{D^2 + (R_1 + R_2)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equivalent expressions are obtained if Landen's transformation of elliptic integrals is applied to (3); the simplest of them have been listed by GROVER (ref. 4).

2. *Generalization of* HAVELOCK'*s integral.*

First of all we calculate the mutual inductance of a circle of radius r of vanishing cross-section and the toroid (R_1, a_1), z being the axial distance

[1]) If R_1 and R_2 are the numerical values of the respective lengths measured in centimetres, then $M_{0,0}$ is the numerical value of the inductance also measured in centimetres. To make eq. (3) consistent in the rationalized m.k.s. units, the right-hand member has to be multiplied by 10^{-7} (empty space).

between their centres (see fig. 2). To that end we have to integrate (2), with $R_1 = r_1$, $R_2 = r$, $D = |z - z_1|$, over the cross-section of the toroid.

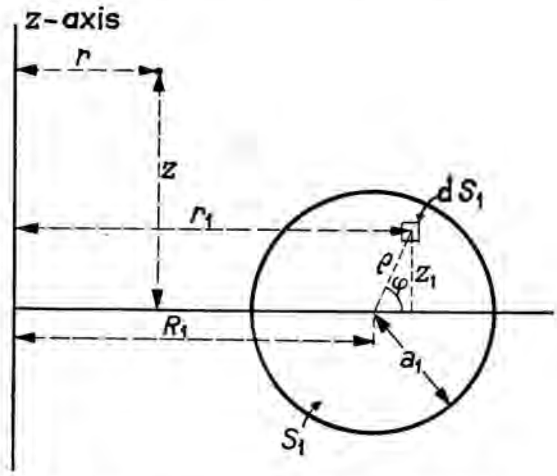

Fig. 2. Illustrating the integration over the cross-section of the first toroid.

Thus, if $|z_1| \leq a_1 \leq z$,

$$M' = 4\pi^2 r \int_0^\infty e^{-zt} J_1(rt)\, dt \, \frac{1}{\pi a_1^2} \iint_{S_1} e^{z_1 t} J_1(r_1 t)\, r_1\, dS_1. \quad . \quad . \quad . \quad (5)$$

In order to obtain for M' an expansion in powers of ε_1 analogous to (1), we must expand the surface integral, S, occurring in (5) in that manner. This is most easily achieved by introducing polar coordinates ϱ, φ around the centre of the cross-section, by means of which the surface integral is transformed into

$$S = \frac{1}{\pi a_1^2} \int_0^{a_1} \varrho\, d\varrho \int_{-\pi}^{\pi} (R_1 + \varrho \cos\varphi)\, J_1(R_1 t + \varrho t \cos\varphi)\, e^{\varrho t \sin\varphi}\, d\varphi \quad . \quad . \quad (6)$$

However, to make S tractable, we first replace the Bessel function in the integrand according to the integral representation

$$J_1(x) = \frac{i}{2\pi} \int_{-\pi}^{\pi} e^{-ix\sin\theta} \sin\theta\, d\theta,$$

which leads to

$$S = \frac{i}{2\pi^2 a_1^2} \int_{-\pi}^{\pi} e^{-iR_1 t \sin\theta} \sin\theta\, d\theta \int_0^{a_1} \varrho\, d\varrho \int_{-\pi}^{\pi} (R_1 + \varrho\cos\varphi)\, e^{\varrho t(\sin\varphi - i\sin\theta\cos\varphi)}\, d\varphi. \quad (7)$$

The integrations with respect to φ and ϱ can be carried out, giving simple Bessel functions, as will now be shown. For that purpose, let us consider the integral

$$F = \int_{-\pi}^{\pi} e^{x(\sin\varphi + u\cos\varphi)}\, d\varphi.$$

For positive values of u one has

$$\sin\varphi + u\cos\varphi = \sqrt{1+u^2}\,\sin(\varphi + \arctan u).$$

Now, since the integration over φ has to be performed over a full period of $\sin\varphi$, the phase component $\arctan u$ is of no consequence. That is to say,

$$F = \int_{-\pi}^{\pi} e^{x\sqrt{1+u^2}\sin\varphi}\,\mathrm{d}\varphi = 2\pi I_0(x\sqrt{1+u^2}),$$

in which I denotes the modified Bessel function of imaginary argument. The function F is analytic in u, in spite of the radical. On account of the principle of analytic continuation we may apply our result to $u = i\sin\theta$; thus

$$\int_{-\pi}^{\pi} e^{x(\sin\varphi - i\sin\theta\cos\varphi)}\,\mathrm{d}\varphi = 2\pi I_0(x\cos\theta). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

By differentiating (8) with respect to θ we further obtain

$$\int_{-\pi}^{\pi} e^{x(\sin\varphi - i\sin\theta\cos\varphi)}\cos\varphi\,\mathrm{d}\varphi = -2\pi i\tan\theta\, I_1(x\cos\theta). \quad . \quad . \quad (9)$$

The results (8) and (9) enable us to evaluate the φ-integral occurring in (7), viz.,

$$\int_{-\pi}^{\pi}(R_1 + \varrho\cos\varphi)\,e^{\varrho t(\sin\varphi - i\sin\theta\cos\varphi)}\,\mathrm{d}\varphi = 2\pi\{R_1 I_0(\varrho t\cos\theta) - i\varrho\tan\theta\, I_1(\varrho t\cos\theta)\}.$$

The next step is to perform the integration with respect to ϱ. This can be accomplished in virtue of

$$\int_0^{a_1}\varrho\, I_0(\varrho t\cos\theta)\,d\varrho = \frac{a_1 I_1(a_1 t\cos\theta)}{t\cos\theta},$$

$$\int_0^{a_1}\varrho^2 I_1(\varrho t\cos\theta)\,\mathrm{d}\varrho = \frac{a_1^2 I_2(a_1 t\cos\theta)}{t\cos\theta}.$$

The result of the combined integrations with respect to ϱ and φ in (7) then becomes

$$\int_0^{a_1}\varrho\,\mathrm{d}\varrho\int_{-\pi}^{\pi}(R_1 + \varrho\cos\varphi)\,e^{\varrho t(\sin\varphi - i\sin\theta\cos\varphi)}\,\mathrm{d}\varphi =$$
$$= \frac{2\pi a_1 R_1}{t\cos\theta}\{I_1(a_1 t\cos\theta) - i\varepsilon_1\tan\theta\, I_2(a_1 t\cos\theta)\}.$$

If this is substituted in (7) we obtain

$$S = \frac{i}{\pi\varepsilon_1 t}\int_{-\pi}^{\pi}\{I_1(\varepsilon_1 R_1 t\cos\theta) - i\varepsilon_1\tan\theta\, I_2(\varepsilon_1 R_1 t\cos\theta)\}\,e^{-iR_1 t\sin\theta}\tan\theta\,\mathrm{d}\theta. \quad (10)$$

It does not seem possible to evaluate the integral (10) in finite terms. On the other hand, it possesses a remarkably simple Neumann-series expansion. This expansion may be obtained by first developing the integrand in powers of ε_1, which is easily done by means of the known power series for the Bessel functions. This procedure leads to

$$S=\sum_{n=0}^{\infty}\varepsilon_1^{2n}\left[\frac{R_1}{2\pi}\frac{(R_1\,t/2)^{2n-1}}{n!\,(n+1)!}\int_{-\pi}^{\pi}\{n\sin^2\theta\cos^{2n-2}\theta+\right.$$
$$\left.+\frac{i}{2}R_1\,t\sin\theta\cos^{2n}\theta\}\,e^{-iR_1t\sin\theta}\,d\theta\right].$$

Secondly, the term between brackets is developed according to ascending powers of R_1t. The relevant coefficients are simple trigonometric integrals which can be expressed by means of gamma functions. When the terms are properly arranged it appears that the power series obtained is nothing more than that of the Bessel function $J_{n-1}\ (R_1t)$, except for an elementary factor. Omitting further details, we obtain for the final result

$$S=\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{(R_1/2)^{n+1}}{n!\,(n+1)!}\varepsilon_1^{2n}\,t^n J_{n-1}(R_1\,t). \quad . \quad . \quad . \quad (11)$$

Substitution of (11) in (5) then leads to

$$M'=4\pi^2r\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{(R_1/2)^{n+1}}{n!\,(n+1)!}\varepsilon_1^{2n}\int_0^{\infty}e^{-zt}J_1(rt)\,J_{n-1}(R_1t)\,t^n\,dt. \quad (12)$$

As to the convergence of the series (12), it should be remarked that, though we obtained (12) on the restriction $z>a_1$, the series is certainly convergent if $z>a_1{}^2/2R_1$ (this condition is less stringent than the former since $a_1/2R_1$ does not exceed $1/2$). To prove this we observe that the absolute magnitude of the integral in (12) is less than

$$\int_0^{\infty}e^{-zt}\,t^n\,dt=n!\,z^{-n-1}.$$

Further, $\Gamma(n-\frac{1}{2})/(n+1)!$ is uniformly bounded for $n=0,1,2,\ldots$. Therefore, a majorant of (12), except for a constant factor depending on R_1 and z, is given by the geometric series

$$\sum_{n=0}^{\infty}(a_1^2/2\,z\,R_1)^2,$$

which is convergent if $z>a_1{}^2/2R_1$. The actual domain of convergence of (12) may be wider still.

It is now easy to obtain the Bessel-function representation of $M_{n,m}$ in question. To calculate M we simply have to integrate M' over the cross-section of the second toroid (R_2,a_2). Thus

$$M=4\pi^2\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{(R_1/2)^{n+1}}{n!\,(n+1)!}\varepsilon_1^{2n}\int_0^{\infty}e^{-Dt}J_{n-1}(R_1\,t)\,t^n\,dt\,\frac{1}{\pi a_2^2}\iint_{S_2}e^{z_2t}J_1(r_2\,t)\,r_2\,dS_2.$$

The surface integral is of exactly the same form as before; consequently

$$\frac{1}{\pi a_2^2}\iint_{S_2}e^{z_2t}J_1(r_2t)\,r_2\,dS_2=\sum_{m=0}^{\infty}\frac{\Gamma(m-\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{(R_2/2)^{m+1}}{m!\,(m+1)!}\varepsilon_2^{2m}\,t^m J_{m-1}(R_2t).$$

Inserting this in the preceding equation, we at once obtain the generalization of HAVELOCK's expression (2), viz.,

$$M_{n,m}=4\pi\frac{\Gamma(n-\frac{1}{2})\,\Gamma(m-\frac{1}{2})}{n!\,m!\,(n+1)!\,(m+1)!}\left(\frac{R_1}{2}\right)^{n+1}\left(\frac{R_2}{2}\right)^{m+1}\int_0^\infty e^{-Dt}J_{n-1}(R_1 t)J_{m-1}(R_2 t)\,t^{n+m}\,\mathrm{d}t. \quad (13)$$

As was to be anticipated, $M_{n,m}$ is symmetric in the pairs of variables (n, R_1) and (m, R_2).

In deriving equation (1), with coefficients given by (13), we tacitly assumed $D > a_1 + a_2$. However, by using a similar argument as in the case of (12), we may readily show that the convergence of (1) with coefficients specified by (13) is guaranteed if $D > \frac{1}{2}\left(\frac{a_1^2}{R_1}+\frac{a_2^2}{R_2}\right)$. The actual domain of convergence may be larger, because the type of expansion (1) is expected to exist whenever the cross-sections of the toroids do not overlap and a_1 and a_2 are small enough, including $D = 0$ (see section 4).

3. *Evaluation of the Bessel-function integrals.*

We shall now express the coefficients $M_{n,m}$ in terms of elliptic integrals, at least for $n + m \leq 2$. For the sake of convenience we write down all coefficients of the order $(n + m)$ up to and including 2.

Order zero:

$$M_{0,0}=4\,\pi^2 R_1 R_2\int_0^\infty e^{-Dt} J_1(R_1 t)\, J_1(R_2 t)\,\mathrm{d}t;$$

Order one:

$$M_{1,0}=\tfrac{1}{2}\,\pi^2 R_1^2 R_2\int_0^\infty e^{-Dt} J_0(R_1 t)\, J_1(R_2 t)\, t\,\mathrm{d}t,$$

$$M_{0,1}=\tfrac{1}{2}\,\pi^2 R_1 R_2^2\int_0^\infty e^{-Dt} J_1(R_1 t)\, J_0(R_2 t)\, t\,\mathrm{d}t;$$

Order two:

$$M_{2,0}=\tfrac{1}{48}\,\pi^2 R_1^3 R_2\int_0^\infty e^{-Dt} J_1(R_1 t)\, J_1(R_2 t)\, t^2\,\mathrm{d}t,$$

$$M_{1,1}=\tfrac{1}{16}\,\pi^2 R_1^2 R_2^2\int_0^\infty e^{-Dt} J_0(R_1 t)\, J_0(R_2 t)\, t^2\,\mathrm{d}t,$$

$$M_{0,2}=\tfrac{1}{48}\,\pi^2 R_1 R_2^3\int_0^\infty e^{-Dt} J_1(R_1 t)\, J_1(R_2 t)\, t^2\,\mathrm{d}t.$$

As we have seen, $M_{0,0}$ is expressible in terms of complete elliptic integrals. We shall now show that the same holds for the first-order coefficients. To start with we have

$$\int_0^\infty e^{-Dt} J_0(R_1 t)\, J_1(R_2 t)\, t\,\mathrm{d}t=-\frac{\partial}{\partial R_2}\int_0^\infty e^{-Dt} J_0(R_1 t)\, J_0(R_2 t)\,\mathrm{d}t=$$

$$=-\frac{\partial}{\partial R_2}\int_0^\infty e^{-Dt}\,\mathrm{d}t\,\frac{1}{2\pi}\int_0^{\infty} J_0(t\sqrt{R_1^2-2\,R_1\,R_2\cos\vartheta+R_2^2})\,\mathrm{d}\vartheta=$$

$$=-\frac{\partial}{\partial R_2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\vartheta}{\sqrt{D^2+R_1^2-2\,R_1\,R_2\cos\vartheta+R_2^2}}=-\frac{\partial}{\partial R_2}\left\{\frac{k\,K}{\pi\sqrt{R_1\,R_2}}\right\}.$$

in which k is given by (4). In this we have used some well-known properties of Bessel functions, including the addition theorem, while the last integral has been transformed into the standard form of the elliptic integral of the first kind by choosing a new variable of integration, $\varphi = (\pi - \vartheta)/2$.

Now, by differentiating equation (4), we have

$$\frac{\partial}{\partial R_2} = \frac{k}{2R_2}\left\{1 - \frac{R_1 + R_2}{2R_1}k^2\right\}\frac{\partial}{\partial k}. \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Further, a known equation of the theory of elliptic integrals reads

$$\frac{\mathrm{d}K}{\mathrm{d}k} = \frac{E}{k(1-k^2)} - \frac{K}{k}.$$

With these auxiliary relations the differentiation with respect to R_2 can readily be performed; the result is found to be

$$\int_0^\infty e^{-Dt} J_0(R_1 t) J_1(R_2 t)\, t\, \mathrm{d}t = \frac{k}{2\pi R_2 \sqrt{R_1 R_2}}\left[K - E + \frac{\frac{1}{2}k^2 E}{1-k^2}\left(\frac{R_2}{R_1} - 1\right)\right]. \qquad (15)$$

The corresponding expression for the first-order coefficient then is

$$M_{1,0} = \frac{\pi}{4}\frac{R_1}{R_2}\sqrt{R_1 R_2}\, k\left[K - E + \frac{\frac{1}{2}k^2 E}{1-k^2}\left(\frac{R_2}{R_1} - 1\right)\right]. \quad . \quad . \qquad (16)$$

while $M_{0,1}$ is obtained from (16) by interchanging R_1 and R_2.

We next turn to the second-order coefficients. By differentiating the identity [compare eqs (2) and (3)]

$$\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, \mathrm{d}t = \frac{1}{\pi\sqrt{R_1 R_2}}\left[\left(\frac{2}{k} - k\right)K - \frac{2}{k}E\right] \quad . \qquad (17)$$

with respect to D, we find that

$$\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, t\, \mathrm{d}t = -\frac{1}{\pi\sqrt{R_1 R_2}}\frac{\partial}{\partial D}\left[\left(\frac{2}{k} - k\right)K - \frac{2}{k}E\right].$$

Further, analogous to (14),

$$\frac{\partial}{\partial D} = -\frac{k^3 D}{4R_1 R_2}\frac{\partial}{\partial k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

while, in addition to the expression given for $\mathrm{d}K/\mathrm{d}k$, one has

$$\frac{\mathrm{d}E}{\mathrm{d}k} = \frac{E-K}{k}.$$

Carrying out all transformations, we finally arrive at

$$\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, t\, \mathrm{d}t = \frac{Dk}{2\pi (R_1 R_2)^{3/2}}\left[\frac{1 - \frac{1}{2}k^2}{1-k^2}E - K\right]. \quad . \qquad (19)$$

A second differentiation with respect to D yields

$$\left.\begin{aligned} &\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, t^2\, \mathrm{d}t = \\ &\frac{k^3}{4\pi (R_1 R_2)^{3/2}}\left[K - \frac{1-2k^2}{1-k^2}E - \frac{1}{2}\left\{\frac{1-k^2+k^4}{(1-k^2)^2}E - \frac{1-\frac{1}{2}k^2}{1-k^2}K\right\}\left\{\sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}}\right\}^2\right]. \end{aligned}\right\} \qquad (20)$$

in which use is made of the identity

$$\frac{k^2 D^2}{4 R_1 R_2} = 1 - k^2 - \tfrac{1}{4} k^2 \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Consequently,

$$M_{2,0} = \frac{\pi}{192} \frac{R_1}{R_2} \sqrt{R_1 R_2}\, k^3 \left[K - \frac{1-2k^2}{1-k^2} E + \tfrac{1}{2} \left\{ \frac{1-\frac{1}{2}k^2}{1-k^2} K - \frac{1-k^2+k^4}{(1-k^2)^2} \right\} \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2 \right]. \quad (22)$$

This equation might also have been obtained by differentiating (15) with respect to R_1. Again, $M_{0,2}$ is found from (22) by interchanging R_1 and R_2.

The remaining second-order coefficient, $M_{1,1}$, is determined in a similar way. Omitting details, we only mention the Bessel-function integrals occurring in the course of calculation:

$$\int_0^\infty e^{-Dt} J_0(R_1 t) J_0(R_2 t)\, t \,\mathrm{d} t = \frac{D}{4\pi (R_1 R_2)^{3/2}} \frac{k^3 E}{1-k^2}, \quad . \quad . \quad . \quad (23)$$

$$\left. \begin{aligned} &\int_0^\infty e^{-Dt} J_0(R_1 t) J_0(R_2 t)\, t^2 \,\mathrm{d} t = \\ &= \frac{k^3}{4\pi (R_1 R_2)^{3/2}} \left[\frac{3-2k^2}{1-k^2} E - K + \frac{\frac{1}{4}k^2}{1-k^2} \left\{ K - \frac{4-2k^2}{1-k^2} E \right\} \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2 \right] \end{aligned} \right\}. \quad (24)$$

The last equation immediately leads to

$$M_{1,1} = \frac{\pi}{64} \sqrt{R_1 R_2}\, k^3 \left[\frac{3-2k^2}{1-k^2} E - K + \frac{\frac{1}{4}k^2}{1-k^2} \left\{ K - \frac{4-2k^2}{1-k^2} E \right\} \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2 \right]. \quad (25)$$

As might have been expected from the foregoing analysis, the coefficients of higher order are likewise expressible in terms of complete elliptic integrals, though it seems difficult to obtain the relevant explicit expressions. Therefore we shall not aim at giving these explicit expressions but shall indicate how the general coefficient $M_{n,m}$ may be obtained from $M_{1,0}$ and $M_{1,1}$ by a process of differentiation.

On account of symmetry we may assume that $n \geq m$. First of all we consider the case $m = 0$, $n = \nu + 1$, where $\nu = 1, 2, 3, \ldots$. Obviously $M_{\nu+1,0}$ is proportional to

$$\int_0^\infty e^{-Dt} J_\nu(R_1 t) J_1(R_2 t)\, t^{\nu+1} \,\mathrm{d} t.$$

Further, when $x = R_1 t$ is substituted in the known identity

$$\left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d} x} \right)^\nu J_0(x) = (-1)^\nu \frac{J_\nu(x)}{x^\nu},$$

it is seen that

$$R_1^\nu \left(-\frac{1}{R_1} \frac{\partial}{\partial R_1} \right)^\nu J_0(R_1 t) = t^\nu J_\nu(R_1 t).$$

Consequently, $M_{\nu+1,0}$ is proportional to

$$R_1^\nu \left(-\frac{1}{R_1} \frac{\partial}{\partial R_1} \right)^\nu \int_0^\infty e^{-Dt} J_0(R_1 t) J_1(R_2 t)\, t \,\mathrm{d} t.$$

The last integral, however, is proportional to $M_{1,0}$. Therefore, $M_{\nu+1,0}$ is obtained from $M_{1,0}$ by successive differentiations with respect to R_1. The corresponding explicit equation is easily found to be

$$M_{\nu+1,0} = \frac{2^{1-\nu}\,\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\,(\nu+1)!\,(\nu+2)!}\,R_1^{2\nu+2}\left(-\frac{1}{R_1}\frac{\partial}{\partial R_1}\right)^{\nu}\left(\frac{M_{1,0}}{R_1^2}\right). \quad . \quad (26)$$

The same holds for general values of n and m, in that $M_{\nu+1,\mu+1}$ ($\nu > 0$, $\mu \geq 0$) can be obtained from $M_{1,1}$ by means of successive differentiations with respect to R_1 and R_2, viz.,

$$\left.\begin{aligned} M_{\nu+1,\mu+1} = {} & \frac{2^{2-\nu-\mu}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\mu+\frac{1}{2})}{\pi\,(\nu+1)!\,(\mu+1)!\,(\nu+2)!\,(\mu+2)!}\,R_1^{2\nu+2}\,R_2^{2\mu+2}\,\cdot \\ & \cdot\left(-\frac{1}{R_1}\frac{\partial}{\partial R_1}\right)^{\nu}\left(-\frac{1}{R_2}\frac{\partial}{\partial R_2}\right)^{\mu}\left(\frac{M_{1,1}}{R_1^2 R_2^2}\right). \end{aligned}\right\} \quad . \quad . \quad (27)$$

It will now be evident that $M_{n,m}$ is expressible in terms of elementary functions and complete elliptic integrals of the first and second kinds to modulus k [see eq. (4)] since $M_{1,0}$, $M_{1,1}$, $\mathrm{d}K/\mathrm{d}k$, $\mathrm{d}E/\mathrm{d}k$, $\partial k/\partial R_1$ and $\partial k/\partial R_2$ are all expressible in that way.

There exist numerous other relations between the M-functions. For instance,

$$M_{\nu+1,\mu+1} = \frac{(\nu-\frac{1}{2})\,(\mu-\frac{1}{2})}{4\,(\nu+1)(\nu+2)\,(\mu+1)(\mu+2)}\,R_1^{2\nu+1}\,R_2^{2\mu+1}\,\frac{\partial^2}{\partial R_1\,\partial R_2}\left\{R_1^{-2\nu}\,R_2^{-2\mu}\,M_{\nu,\mu}\right\}, \quad (28)$$

which is easy to derive from (27). We shall no longer deal with these questions, however.

Finally, it may be noted that the results (15), (17), (19), (20), (23), and (24) are not all new; some of these Bessel-function integrals have been studied by VAN WIJNGAARDEN (ref. 6), while others are to be found in WATSON's book (ref. 7) for $R_1 = R_2$.

4. *Approximate expressions for the mutual inductance.*

We shall now develop an approximate expression for M including terms of the fourth order in the radii of the cross-sections. The term independent of the radii a_1 and a_2 is $M_{0,0}$. The first-order correction includes terms with a_1^2 and a_2^2; it is represented by

$$M_{1,0}\,(a_1/R_1)^2 + M_{0,1}\,(a_2/R_2)^2,$$

while the next correction term is given by

$$M_{2,0}\,(a_1/R_1)^4 + M_{1,1}\,(a_1/R_1)^2\,(a_2/R_2)^2 + M_{0,2}\,(a_2/R_2)^4.$$

Inserting the values of the coefficients as found in the preceding section, we have [2])

$$M \approx 4\,\pi\,\sqrt{R_1\,R_2}\,\{F_0 + F_1 + F_2\}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

[2]) See preceding footnote.

in which

$$F_0 = \left(\frac{2}{k} - k\right) K - \frac{2}{k} E, \quad \ldots \ldots \ldots \ldots \quad (30)$$

$$F_1 = \frac{k}{16} \left\{ \frac{a_1^2 + a_2^2}{R_1 R_2} (K - E) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \left(\frac{a_1^2}{R_1} - \frac{a_2^2}{R_2}\right) \frac{\frac{1}{2} k^2 E}{1 - k^2} \right\}. \quad (31)$$

$$F_2 = \frac{k^3}{768} \left\{ A + B \left(\sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right)^2 \right\}, \quad \ldots \ldots \quad (32)$$

$$A = \frac{1}{R_1 R_2} \left(\frac{a_1^4}{R_1^2} + \frac{a_2^4}{R_2^2} \right) \left(K - \frac{1 - 2k^2}{1 - k^2} E \right) + \frac{3 a_1^2 a_2^2}{R_1^2 R_2^2} \left(\frac{3 - 2k^2}{1 - k^2} E - K \right), \quad (33)$$

$$\left. \begin{aligned} B = \frac{1}{2 R_1 R_2} \left(\frac{a_1^4}{R_1^2} + \frac{a_2^4}{R_2^2} \right) \left(\frac{1 - \frac{1}{2} k^2}{1 - k^2} K - \frac{1 - k^2 + k^4}{(1 - k^2)^2} E \right) + \\ + \frac{3 a_1^2 a_2^2}{4 R_1^2 R_2^2} \frac{k^2}{1 - k^2} \left(K - \frac{4 - 2k^2}{1 - k^2} E \right). \end{aligned} \right\} \quad (34)$$

These expressions are simplified when the radii of the toroids are chosen equal, $R_1 = R_2 = R$. They become very simple if in addition the radii of the cross-sections are the same, $a_1 = a_2 = a$. In the latter case we obtain, when $D > 2a$,

$$\left. \begin{aligned} M^* = 4\pi R \left[\left(\frac{2}{k} - k\right) K - \frac{2}{k} E + \frac{k}{8} (K - E) \left(\frac{a}{R}\right)^2 + \right. \\ \left. + \frac{k^3}{768} \left(\frac{7 - 2k^2}{1 - k^2} E - K \right) \left(\frac{a}{R}\right)^4 + \ldots \right]. \end{aligned} \right\} \quad (35)$$

where the modulus of the elliptic integrals now is given by

$$k = \{1 + (D/2R)^2\}^{-1/2}. \quad \ldots \ldots \quad (36)$$

Finally, the approximate expression (29), which has been derived on the assumption that $D > a_1 + a_2$, R_1 and R_2 arbitrary, remains valid if $D \geq 0$ provided a_1 and a_2 are small enough and $|R_1 - R_2| > a_1 + a_2$, because the type of expansion (1) exists whenever the toroids do not overlap and a_1, a_2 are sufficiently small. To realize this, we observe that the coefficients $M_{n,m}$ might have been obtained by expanding $M_{0,0}$ in a Taylor series of four variables (the coordinates pertaining to the cross-sections) and integrating over the cross-sections. The result then would have been a double power series in a_1 and a_2 with coefficients determined by partial derivatives of $M_{0,0}$ with respect to D, R_1 and R_2, and of non-vanishing domain of convergence.

The same conclusion is reached by observing that our analysis might also have been based on the left part of the identity

$$\frac{2}{\pi} \int_0^\infty \cos zt \, K_0(rt) \, dt = \int_0^\infty e^{-|z|t} J_0(rt) \, dt,$$

instead of on the right part, both integrals representing the function $(z^2 + r^2)^{-1/2}$. For example, the coefficient $M_{0,0}$ would become

$$M_{0,0} = \begin{cases} 8\pi R_1 R_2 \int_0^\infty \cos Dt \, I_1(R_1 t) \, K_1(R_2 t) \, dt, & (R_1 < R_2) \\ 8\pi R_1 R_2 \int_0^\infty \cos Dt \, K_1(R_1 t) \, I_1(R_2 t) \, dt. & (R_1 > R_2) \end{cases}$$

Similar expressions hold for $M_{n,m}$, which may be considered as analytic continuations of (13) to $D = 0$. Whether we use the first representation or the second, the final result is in any case that given by equations (29) through (34).

In concluding the paper, we remark that in the case of coplanar toroids ($D = 0$) expression (32) is simplified to

$$F_2 = \frac{k}{768} \left[\frac{1}{R_1 R_2} \left(\frac{a_1^4}{R_1^2} + \frac{a_2^4}{R_2^2} \right) \left(2K - \frac{2-k^2}{1-k^2} E \right) - \frac{3a_1^2 a_2^2}{R_1^2 R_2^2} \frac{k^2}{1-k^2} E \right], \quad (37)$$

in which now

$$k = \frac{2\sqrt{R_1 R_2}}{R_2 + R_2}. \quad \ldots \ldots \ldots \quad (38)$$

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Mathematics. — *On multivectors in a V_n.* I. By WŁODZIMIERZ WRONA. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of November 27, 1948.)

Summary.

In this paper [1]) the curvature affinor $K_{\varkappa\lambda\mu\nu}$ and the fundamental tensor $a_{\lambda\mu}$ in a V_n are used to define some new affinors with $2\,m$ indices, where $2 \leq m \leq n$. By means of these affinors we can give a new form to the conditions that the space should be either EINSTEINian or conformal to a flat space, enabling us to interpret these conditions geometrically in terms of the scalar curvature of an m-direction. Further, the generalized SCHUR theorem [2]) appears as an almost automatic consequence of the new expression for the scalar curvature. Many of these results were obtained by a different method in papers by HAANTJES and myself [3]).

In the last section we define principal m-vectors in V_n and investigate some of their properties in order to point to the possibility of a new way of classification of RIEMANNian spaces. Similar results for a V_4 have been given by STRUIK, CHURCHILL and RUSE [4]).

Section 1. Introduction.

In a V_n with fundamental tensor $a_{\lambda\mu}$ where $|\,a_{\lambda\mu}\,| \neq 0$ we may consider the following set of affinors:

$$\overset{m}{a}_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} = m!\; a_{[\lambda_1 [\mu_1}\, a_{\lambda_2 \mu_2} \dots a_{\lambda_m] \mu_m]}; \qquad m = 2, 3 \dots n \quad (1.\,1)$$

If $f^{\lambda_1 \dots \lambda_m}$ is an m-vector we call the scalar

$$f^{(2)} = \left(\frac{1}{m!}\right)^2 \overset{m}{a}_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} f^{\lambda_1 \dots \lambda_m} f^{\mu_1 \dots \mu_m} \quad . \quad . \quad . \quad (1.\,2)$$

its *norm*.

If $f^{(2)} = \pm 1$ the m-vector $f^{\lambda_1 \dots \lambda_m}$ will be called a *unit m-vector*.

If $f^{(2)} = 0$ the m-vector is called *singular*. The unit n-vector $I^{\lambda_1 \dots \lambda_n}$ we sometimes call the *dualizing affinor*.

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1) I wish to express my gratitude to the British Council for giving me the opportunity to visit England where this paper was written. I am grateful to Prof. H. S. RUSE; thanks to his kind help and advice I was able during my relatively short stay in Leeds to get acquainted with the great development of English mathematics. I wish also to express my thanks to Prof. J. A. SCHOUTEN and Prof. J. HAANTJES for their valuable remarks on reading the manuscript of this paper.

2) Ref. 13.

3) Ref. 4, 5, 13, 14.

4) Ref. 1, 6, 7, 8, 9, 12.

By means of the n-vector $I^{\lambda_1\cdots\lambda_n}$ we can establish a one to one correspondence between m-vectors and $(n-m)$-vectors using the formula [5])

$$ {}^{\circ}f_{\lambda_{m+1}\cdots\lambda_n} = \frac{\varrho}{m!} f^{\lambda_1\cdots\lambda_m} I_{\lambda_1\cdots\lambda_n}, \quad (1.3) $$

where

$$ \varrho = (-1)^{\frac{1}{2}(n-m)m} \quad (1.4) $$

The $(n-m)$-vector ${}^{\circ}f^{\lambda_{m+1}\cdots\lambda_n}$ will be called the *dual* of $f^{\lambda_1\cdots\lambda_m}$. From *(1.2)*, *(1.3)* and *(1.1)* it follows that

$$ {}^{\circ}f^{(2)} = \varrho^2 f^{(2)}. \quad (1.5) $$

Hence: *the dual of a unit m-vector is a unit $(n-m)$-vector and similarly the dual of a singular m-vector is a singular $(n-m)$-vector.*

From *(1.3)* and *(1.4)* we have

$$ \left.\begin{aligned} {}^{\circ\circ}f^{\varrho_1\cdots\varrho_m} &= \frac{\varrho}{(n-m)!}\, {}^{\circ}f_{\lambda_{m+1}\cdots\lambda_n} I^{\lambda_{m+1}\cdots\lambda_n \varrho_1\cdots\varrho_m} = \\ &= \frac{\varrho^2}{m!\,(n-m)!} f^{\lambda_1\cdots\lambda_m} I_{\lambda_1\cdots\lambda_n} I^{\lambda_{m+1}\cdots\lambda_n \varrho_1\cdots\varrho_m} = \\ &= f^{\varrho_1\cdots\varrho_m}. \end{aligned}\right\} \quad (1.6) $$

If in a V_{2m}

$$ {}^{\circ}f^{\lambda_1\cdots\lambda_m} = \Theta f^{\lambda_1\cdots\lambda_m} \quad (1.7) $$

then $\Theta = \pm 1$ and we call the m-vector $f^{\lambda_1\cdots\lambda_m}$ *self-dual* [6]).

When an m-vector $F^{\lambda_1\cdots\lambda_m}$ satisfies relations

$$ {}^{\circ}F_{\lambda_{m+1}\cdots\lambda_{n-1}\alpha} F^{\lambda_1\cdots\lambda_{m-1}\alpha} = 0 \quad (1.8) $$

it is called *simple* and defines an m-direction. Let $\underset{a}{v}^{\lambda}$, $(a = 1, 2 \ldots m)$, be m arbitrary linearly independent vectors in this m-direction. Then we have

$$ F^{\lambda_1\cdots\lambda_m} = m!\, \mu\, \underset{1}{v}^{[\lambda_1} \cdots \underset{m}{v}^{\lambda_m]}, \quad (1.9) $$

μ being a scalar which depends on the choice of $\underset{a}{v}^{\lambda}$.

If a simple m-vector is non-singular, its corresponding m-direction is also called *non-singular*. The $(n-m)$-vector ${}^{\circ}F^{\lambda_{m+1}\cdots\lambda_n}$ then defines a non-singular $(n-m)$-direction which is absolutely perpendicular to the m-direction of $F^{\lambda_1\cdots\lambda_m}$.

Since an m-vector at a given point of a V_n has $\binom{n}{m}$ linearly independent components it may be interpreted as a point in the projective $\left(N = \binom{n}{m} - 1\right)$-

[5]) Ref. 3, p. 363.

[6]) For V_4 and V_6 see Ref. 8, p. 412 and 418.

space, P_N. The points of P_N, which correspond to simple m-vectors, create an $m(n-m)$-dimensional GRASSMANNian manifold.

Every system of $N+1$ linearly independent m-vectors

$$\underset{\alpha}{v}^{\lambda_1\ldots\lambda_m}, (\alpha=1,2\ldots N+1),$$

defines homogenous coordinates (α) in P_N. We call the $\underset{\alpha}{v}^{\lambda_1\ldots\lambda_m}$ the *coordinate m-vectors* of the coordinate system (α) in P_N.

Let $\overset{\alpha}{v}_{\lambda_1\ldots\lambda_m}$ be the reciprocal set with respect to $\underset{\alpha}{v}^{\lambda_1\ldots\lambda_m}$ i.e. satisfying the conditions

$$\frac{1}{m!}\underset{\alpha}{v}^{\lambda_1\ldots\lambda_m}\overset{\beta}{v}_{\lambda_1\ldots\lambda_m}=\delta_\alpha^\beta\begin{cases}=1 \text{ for } \alpha=\beta;\\=0 \text{ for } \alpha\neq\beta.\end{cases} \quad \ldots \quad (1.10)$$

Then we call the expressions

$$f^\alpha=\frac{1}{m!}f^{\lambda_1\ldots\lambda_m}\overset{\alpha}{v}_{\lambda_1\ldots\lambda_m}, \quad \ldots\ldots \quad (1.11)$$

$$f_\beta=\frac{1}{m!}f_{\lambda_1\ldots\lambda_m}\underset{\beta}{v}^{\lambda_1\ldots\lambda_m}, \quad \ldots\ldots \quad (1.12)$$

$$\overset{m}{a}_{\alpha\beta}=\left(\frac{1}{m!}\right)^2\overset{m}{a}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m}\underset{\alpha}{v}^{\lambda_1\ldots\lambda_m}\underset{\beta}{v}^{\mu_1\ldots\mu_m}, \quad \ldots \quad (1.13)$$

$$\overset{m}{a}{}^{\alpha\beta}=\left(\frac{1}{m!}\right)^2\overset{m}{a}{}^{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m}\overset{\alpha}{v}_{\lambda_1\ldots\lambda_m}\overset{\beta}{v}_{\mu_1\ldots\mu_m}, \quad \ldots \quad (1.14)$$

the *P_N-components* of the affinors

$$f^{\lambda_1\ldots\lambda_m}, f_{\lambda_1\ldots\lambda_m}, \overset{m}{a}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} \text{ and } \overset{m}{a}{}^{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m}.$$

From *(1.11)*, *(1.12)*, *(1.13)* and *(1.14)* it follows that

$$f^\alpha\overset{m}{a}_{\alpha\beta}=f_\beta \quad \ldots\ldots \quad (1.15)$$

and

$$\overset{m}{a}_{\alpha\beta}\overset{m}{a}{}^{\beta\gamma}=\delta_\alpha^\gamma \quad \ldots\ldots \quad (1.16)$$

When the m-vectors $\underset{\alpha}{v}^{\lambda_1\ldots\lambda_m}$ are unit m-vectors and simple each corresponds to an m-direction. It will be assumed that these m-directions are obtained by taking all sets of m-vectors of an orthogonal ennuple at the point under consideration in V_n. The coordinate system defined by them in P_N will be called *basic* and the orthogonal ennuple its *basis* [7]).

Section 2. Deviation of an m-vector in V_n.

Consider in V_n the affinors

$$U_{\varkappa\lambda\mu\nu}=K_{\varkappa\lambda\mu\nu}+\varkappa\overset{2}{a}_{\varkappa\lambda\mu\nu}, \quad \ldots\ldots \quad (2.1)$$

$$U_{\lambda\mu}=U_{\varkappa\lambda\mu\nu}a^{\varkappa\nu}=K_{\lambda\mu}+\varkappa(1-n)a_{\lambda\mu}, \quad \ldots \quad (2.2)$$

---

7) For V_4 see Ref. 7, p. 6.

$K_{\varkappa\lambda\mu\nu}$ being the RIEMANN curvature affinor and $\varkappa$ the scalar curvature of V_n.

From (2.2) we have

$$U = U_{\lambda\mu}\, a^{\lambda\mu} = 0\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

$U_{\varkappa\lambda\mu\nu}$ satisfies the same algebraic identities as $K_{\varkappa\lambda\mu\nu}$ [8]).

In terms of $U_{\varkappa\lambda\mu\nu}$ the necessary and sufficient conditions for V_n to be

(*a*) of constant curvature, S_n,

(*b*) EINSTEINian,

(*c*) conformal to a flat space, C_n,

are

$$(a) \qquad U_{\varkappa\lambda\mu\nu} = 0 \quad \text{and} \quad n > 2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

$$(b) \qquad U_{\lambda\mu} = 0 \quad \text{and} \quad n > 2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

$$(c) \qquad U_{\varkappa\lambda\mu\nu} = -\frac{4}{n-2}\, U_{[\varkappa[\mu}\, a_{\lambda]\nu]} \quad \text{and} \quad n > 3. \quad . \quad . \quad . \quad . \quad (2.6)$$

Consider further in V_n the set of affinors

$$\overset{m}{U}_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} = \frac{m!}{2}\, U_{[\lambda_1 \lambda_2 [\mu_1 \mu_2}\, a_{\lambda_3 \mu_3} \dots a_{\lambda_m] \mu_m]} \quad . \quad . \quad . \quad (2.7)$$

for $2 \leq m \leq n$. From this definition it follows that

$$\overset{n}{U}_{\lambda_1 \dots \lambda_n \mu_1 \dots \mu_n} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.8)$$

For the space of constant curvature, S_n, $(n > 2)$, we get from (2.4)

$$\overset{m}{U}_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.9)$$

In the case $2 \leq m \leq n-2$, multiplying (2.9) by $a^{\lambda_3\mu_3} \dots a^{\lambda_m\mu_m}$ and summing for $\lambda_2 \dots \lambda_m\ \mu_2 \dots \mu_m$ we obtain

$$U_{\lambda_1 \mu_1} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.10)$$

Multiplying now (2.9) by $a^{\lambda_3\mu_3} \dots a^{\lambda_m\mu_m}$, summing for $\lambda_3 \dots \lambda_m\ \mu_3 \dots \mu_m$ and using (2.10) we get further

$$U_{\lambda_1 \lambda_2 \mu_1 \mu_2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.11)$$

Thus *for $2 \leq m \leq n-2$ equation (2.9) gives a necessary and sufficient condition that the space should be of constant curvature.*

Similarly we can prove: *A necessary and sufficient condition that the space V_n, $(n \geq 3)$, be* EINSTEIN*ian is that*

$$\overset{n-1}{U}_{\lambda_1 \dots \lambda_{n-1} \mu_1 \dots \mu_{n-1}} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.12)$$

In fact, multiplying (2.12) by $a^{\lambda_2\mu_2} \dots a^{\lambda_{n-1}\mu_{n-1}}$ and summing for $\lambda_2 \dots \lambda_{n-1}\ \mu_2 \dots \mu_{n-1}$ we obtain

$$U_{\lambda_1 \mu_1} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.13)$$

8) Ref. 10, V. I, p. 113, f. (11. 23).

and that means that the space is EINSTEINian. Inversely for an EINSTEINian space we get multiplying (2.8) by $a^{\lambda_n \mu_n}$ and summing for λ_n and μ_n

$$\varphi \overset{n-1}{U}_{\lambda_1 \ldots \lambda_{n-1} \mu_1 \ldots \mu_{n-1}} + \psi\, U_{[\lambda_1 [\mu_1}\, a_{\lambda_2 \mu_2 \ldots}\quad a_{\lambda_{n-1}] \mu_{n-1}]} = 0, \quad . \quad (2.14)$$

where $\varphi \neq 0$ and ψ are two scalar factors. From (2.14) and (2.13) we obtain (2.12).

We call the scalar

$$\underset{(m)}{\omega} = -\frac{1}{(m!)^2 f^{(2)}} \overset{m}{U}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} f^{\lambda_1 \ldots \lambda_m} f^{\mu_1 \ldots \mu_m} \quad . \quad . \quad (2.15)$$

the *deviation* of the non-singular m-vector $f^{\lambda_1 \ldots \lambda_m}$.

Consider now a simple non-singular m-vector $F^{\lambda_1 \ldots \lambda_m}$ at a point of V_n and let $'a_{\lambda\mu}$ be the fundamental tensor of the induced metric in the subspace V_m tangent to the m-direction of $F^{\lambda_1 \ldots \lambda_m}$ and geodesic at the point under consideration. The scalar curvature, $\underset{(m)}{\varkappa}$, of this geodesic suspace, V_m, at the point as above is called [9]) the scalar curvature of the m-direction defined by $F^{\lambda_1 \ldots \lambda_m}$. By simple calculation we obtain [10])

$$K_{[\lambda_1 \lambda_2 [\mu_1 \mu_2}\, a_{\lambda_3 \mu_3 \ldots}\, a_{\lambda_m] \mu_m]}\, {}'a^{\lambda_1 \mu_1} \ldots {}'a^{\lambda_m \mu_m} = -2 \underset{(m)}{\varkappa}, \quad . \quad . \quad (2.16)$$

$$a_{[\lambda_1 [\mu_1 \ldots}\, a_{\lambda_m] \mu_m]}\, {}'a^{\lambda_1 \mu_1} \ldots {}'a^{\lambda_m \mu_m} = 1 \quad . \quad . \quad . \quad . \quad (2.17)$$

and

$$F^{\lambda_1 \ldots \lambda_m} F^{\mu_1 \ldots \mu_m} = m!\; F^{(2)}\, {}'a^{[\lambda_1 [\mu_1} \ldots {}'a^{\lambda_m] \mu_m]} \quad . \quad . \quad . \quad (2.18)$$

From (2.16) and (2.18) we have

$$\underset{(m)}{\varkappa} = -\frac{1}{2\,m!\,F^{(2)}} K_{[\lambda_1 \lambda_2 [\mu_1 \mu_2}\, a_{\lambda_3 \mu_3 \ldots}\, a_{\lambda_m] \mu_m]} F^{\lambda_1 \ldots \lambda_m} F^{\mu_1 \ldots \mu_m}. \quad (2.19)$$

By (2.1) and (2.7) formula (2.15) may be written

$$\left.\begin{aligned} \underset{(m)}{\omega} = -\frac{1}{2\,m!\,F^{(2)}} \big(&K_{[\lambda_1 \lambda_2 [\mu_1 \mu_2}\, a_{\lambda_3 \mu_3 \ldots}\, a_{\lambda_m] \mu_m]} + \\ &+ 2 \varkappa\, a_{[\lambda_1 [\mu_1 \ldots}\, a_{\lambda_m] \mu_m]}\big) F^{\lambda_1 \ldots \lambda_m} F^{\mu_1 \ldots \mu_m}, \end{aligned}\right\} (2.20)$$

$\varkappa$ being the scalar curvature of V_n. By (2.18), (2.17) and (2.19) we get from (2.20)

$$\underset{(m)}{\omega} = \varkappa - \underset{(m)}{\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.21)$$

We shall prove now the following

Lemma 1. *If the affinor* $W_{\varkappa\lambda\mu\nu}$ *satisfies the same algebraic identities as the* RIEMANN *curvature affinor* $K_{\varkappa\lambda\mu\nu}$ [11]) *and if for every simple non-singular m-vector* $F^{\lambda_1 \ldots \lambda_m}$

$$W_{\lambda_1 \ldots \lambda_m, \mu_1 \ldots \mu_m} F^{\lambda_1 \ldots \lambda_m} F^{\mu_1 \ldots \mu_m} = 0, \quad . \quad . \quad . \quad . \quad (2.22)$$

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[9]) Ref. 4, p. 626, Ref. 14, p. 46.

[10]) In order to derive formula (2. 16) we must use the theorem from Ref. 10, V. II, p. 133.

[11]) Ref. 10, V. I, p. 113, (11.23).

where

$$W_{\lambda_1\ldots\lambda_m,\mu_1\ldots\mu_m} = W_{[\lambda_1\lambda_2[\mu_1\mu_2}\, a_{\lambda_3\mu_3}\ldots a_{\lambda_m]\mu_m]} \quad . \quad . \quad . \quad (2.23)$$

then

$$W_{\lambda_1\ldots\lambda_m,\mu_1\ldots\mu_m} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.24)$$

We notice that from our assumption it follows

$$W_{\lambda_1\ldots\lambda_m,\mu_1\ldots\mu_m} = W_{\mu_1\ldots\mu_m,\lambda_1\ldots\lambda_m} \quad . \quad . \quad . \quad . \quad (2.25)$$

and

$$W_{[\lambda_1\ldots\lambda_m,\mu_1]\mu_2\ldots\mu_m} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.26)$$

In order to prove the lemma consider an orthogonal set of unit vectors $\underset{1}{i}^{\lambda}, \underset{2}{i}^{\lambda}, \ldots \underset{n}{i}^{\lambda}$. Let $h_1, h_2, \ldots h_n$ be an arbitrary permutation of the set $1, 2, \ldots n$ and $\alpha \underset{h_m}{i}^{\lambda} + \beta \underset{h_{m+1}}{i}^{\lambda}$ a unit vector.

If we consider now a non-singular m-vector

$$F^{\lambda_1\ldots\lambda_m} = m!\, \underset{h_1}{i}^{[\lambda_1}\, \underset{h_2}{i}^{\lambda_2} \ldots \underset{h_{m-1}}{i}^{\lambda_{m-1}} \left(\alpha \underset{h_m}{i}^{\lambda_m]} + \beta \underset{h_{m+1}}{i}^{\lambda_m]}\right). \quad . \quad . \quad (2.27)$$

and assume $\beta = 0$ we get from (2.22)

$$W_{h_1\ldots h_m,h_1\ldots h_m} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.28)$$

Assuming $\alpha\beta \neq 0$ we have from (*2.22*), (*2.27*) and (*2.28*)

$$W_{h_1\ldots h_{m-1}h_m,h_1\ldots h_{m-1}h_{m+1}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.29)$$

Considering now the simple non-singular m-vector

$$F^{\lambda_1\ldots\lambda_m} = \underset{h_1}{i}^{[\lambda_1} \ldots \underset{h_{m-2}}{i}^{\lambda_{m-2}} \left(\gamma \underset{h_{m-1}}{i}^{\lambda_{m-1}} + \delta \underset{h_{m+1}}{i}^{\lambda_{m-1}}\right)\left(\varepsilon \underset{h_m}{i}^{\lambda_m]} + \vartheta \underset{h_{m+2}}{i}^{\lambda_m]}\right), \quad (2.30)$$

where

$$\left(\gamma \underset{h_{m-1}}{i}^{\lambda} + \delta \underset{h_{m+1}}{i}^{\lambda}\right) \text{ and } \left(\varepsilon \underset{h_m}{i}^{\lambda} + \vartheta \underset{h_{m+2}}{i}^{\lambda}\right)$$

are two unit vectors and $\gamma\delta\varepsilon\vartheta \neq 0$ we obtain from (*2.22*), (*2.30*), (*2.28*) and (*2.29*)

$$\left.\begin{aligned} &W_{h_1\ldots h_{m-2}h_{m-1}h_m,h_1\ldots h_{m-2}h_{m+1}h_{m+2}} + \\ &\qquad + W_{h_1\ldots h_{m-2}h_{m-1}h_{m+2},h_1\ldots h_{m-2}h_{m+1}h_m} = 0 \end{aligned}\right\} (2.31)$$

and similarly by interchanging h_{m+1} and h_{m+2}

$$\left.\begin{aligned} &-W_{h_1\ldots h_{m-2}h_{m-1}h_m,h_1\ldots h_{m-2}h_{m+2}h_{m+1}} - \\ &\qquad - W_{h_1\ldots h_{m-2}h_{m-1}h_{m+1},h_1\ldots h_{m-2}h_{m+2}h_m} = 0. \end{aligned}\right\} (2.32)$$

But from (*2.26*) and (*2.23*) we have

$$W_{h_1\ldots h_{m-2}[h_{m-1}h_m,h_{m+1}]h_{m+2}h_1\ldots h_{m-2}} = 0 \quad . \quad . \quad . \quad (2.33)$$

Adding the last three equations we get by (*2.25*)

$$3\, W_{h_1\ldots h_{m-2}h_{m-1}h_m,h_{m+1}h_{m+2}h_1\ldots h_{m-2}} = 0 \quad . \quad . \quad . \quad (2.34)$$

(2.28), (2.29) and (2.34) imply that the orthogonal components, $W_{k_1\ldots k_m, l_1\ldots l_m}$, of the affinor (2.23) vanish if the sets $(k_1\ldots k_m)$ and $(l_1\ldots l_m)$ differ mostly in two indices. It is easily seen from (2.23) that all orthogonal components for which these sets differ in three or more indices also vanish. Hence the affinor $W_{\lambda_1\ldots\lambda_m, \mu_1\ldots\mu_m}$ vanishes identically.

If we assume that the deviation at a point is the same for every m-vector we have from (2.15) the identity

$$\left(\frac{1}{m!}\right)^2 [\underset{(m)}{\omega} \overset{m}{a}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} + \overset{m}{U}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m}] f^{\lambda_1\ldots\lambda_m} f^{\mu_1\ldots\mu_m} = 0 \quad (2.35)$$

from which it follows

$$\underset{(m)}{\omega} \overset{m}{a}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} + \overset{m}{U}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} = 0. \quad . \quad . \quad . \quad (2.36)$$

Multiplying the last identity by $a^{\lambda_1\mu_1}\ldots a^{\lambda_m\mu_m}$ and summing for all indices we obtain

$$\underset{(m)}{\omega} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.37)$$

Thus we have the theorem: *If the deviation at each point of V_n is the same for every m-vector it vanishes identically in the whole space.*

In this case (2.36) takes the form

$$\overset{m}{U}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.38)$$

and from (2.9), (2.12) we have that for $2 \leq m \leq n-2$ the space is of constant curvature and for $m = n-1$ EINSTEINian.

If we assume now that the deviation at each point is the same for all simple m-vectors $F^{\lambda_1\ldots\lambda_m}$, then writing (2.35) in the form

$$\frac{1}{2m!} W_{[\lambda_1\lambda_2[\mu_1\mu_2} a_{\lambda_3\mu_3}\ldots a_{\lambda_m]\mu_m]} F^{\lambda_1\ldots\lambda_m} F^{\mu_1\ldots\mu_m} = 0, \quad . \quad . \quad (2.39)$$

where

$$W_{\lambda_1\lambda_2\mu_1\mu_2} = \underset{(m)}{\omega} \overset{2}{a}_{\lambda_1\lambda_2\mu_1\mu_2} + U_{\lambda_1\lambda_2\mu_1\mu_2}. \quad . \quad . \quad . \quad . \quad (2.40)$$

and applying the lemma 1 we get (2.36) and (2.37), from which we obtain by (2.21)

$$\underset{(m)}{\varkappa} = \varkappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.41)$$

$\underset{(m)}{\varkappa}$ being the scalar curvature of the m-direction defined by the m-vector, $F^{\lambda_1\ldots\lambda_m}$, and $\varkappa$ the scalar curvature of V_n.

Thus the last theorem implies at once the generalized SCHUR's theorem [12]: *If the scalar curvature at each point of a space V_n is the same*

[12]) See Ref. 13.

for every m-direction it does not vary from point to point; the space is then of constant curvature for $2 \leq m \leq n-2$ and EINSTEINian for $m = n-1$.

If $n \geq 4$ and $2 \leq m \leq n-2$, we call the affinor

$$\circ \overset{m}{U}{}^{\lambda_{m+1}\cdots\lambda_n \mu_{m+1}\cdots\mu_n} = \frac{\varrho^2}{(m!)^2} \overset{m}{U}{}_{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m} I^{\lambda_1\cdots\lambda_n} I^{\mu_1\cdots\mu_n} \quad (2.42)$$

the dual of $\overset{m}{U}{}^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m}$. From (2.42) it follows that

$$\circ(\circ \overset{m}{U})^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m} = \overset{m}{U}{}^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m} \quad . \quad . \quad . \quad . \quad (2.43)$$

and further we have by (2.7)

$$\left.\begin{aligned} \circ \overset{m}{U}{}^{\lambda_{m+1}\cdots\lambda_n \mu_{m+1}\cdots\mu_n} &= \frac{\varrho^2}{2m!} U_{\lambda_1\lambda_2\mu_1\mu_2} a_{\lambda_3\mu_3}\ldots a_{\lambda_m\mu_m} I^{\lambda_1\cdots\lambda_n} I^{\mu_1\cdots\mu_n} = \\ &= \frac{\varrho^2}{2m!} U^{\lambda_1\lambda_2}{}_{\mu_1\mu_2} I_{\lambda_1\lambda_2\mu_3\cdots\mu_m \varrho_{m+1}\cdots\varrho_n} I^{\mu_1\cdots\mu_n} a^{\varrho_{m+1}\lambda_{m+1}} \ldots a^{\varrho_n\lambda_n} = \\ &= \frac{\varrho^2}{2m!} (m-2)!\,(n-m+2)!\, U^{\lambda_1\lambda_2\mu_1\mu_2} A^{[\mu_1\mu_2\mu_{m+1}\cdots\mu_n]}_{[\lambda_1\lambda_2\varrho_{m+1}\cdots\varrho_n]} a^{\varrho_{m+1}\lambda_{m+1}} \ldots a^{\varrho_n\lambda_n} = \\ &= \frac{\varrho^2}{2(m-1)m} (n-m+2)!\, U^{[\mu_1\mu_2\mu_1\mu_2} a^{\mu_{m+1}|\lambda_{m+1}|} \ldots a^{\mu_n]\lambda_n} = \\ &= \frac{\varrho^2}{2(m-1)m} (n-m)!\,[(n-m)(n-m-1)\, U^{[\lambda_{m+1}\lambda_{m+2}[\mu_{m+1}\mu_{m+2}} a^{\lambda_{m+3}\mu_{m+3}} \ldots a^{\lambda_n]\mu_n]} + \\ &\qquad + 4(n-m)\, U^{[\lambda_{m+1}[\mu_{m+1}} a^{\lambda_{m+2}\mu_{m+2}} \ldots a^{\lambda_n]\mu_n]}] \end{aligned}\right\} (2.44)$$

from which we obtain by (2.7)

$$\left.\begin{aligned} &\circ \overset{m}{U}{}^{\lambda_{m+1}\cdots\lambda_n \mu_{m+1}\cdots\mu_n} = \\ &= \frac{\varrho^2}{\binom{m}{2}} \left[\binom{n-m}{2} \overset{n-m}{U}{}^{\lambda_{m+1}\cdots\lambda_n \mu_{m+1}\cdots\mu_n} + (n-m)\cdot(n-m)!\, U^{[\lambda_{m+1}[\mu_{m+1}} a^{\lambda_{m+2}\mu_{m+2}} \ldots a^{\lambda_n]\mu_n]} \right] \end{aligned}\right\} (2.45)$$

or equivalently

$$\left.\begin{aligned} &\circ \overset{n-m}{U}{}^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m} = \\ &\qquad = \frac{\varrho^2}{\binom{n-m}{2}} \left[\binom{m}{2} \overset{m}{U}{}^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m} + m \cdot m!\, U^{[\lambda_1[\mu_1} a^{\lambda_2\mu_2} \ldots a^{\lambda_m]\mu_m]} \right] \end{aligned}\right\} (2.46)$$

We call the set of affinors $\overset{m}{U}{}^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m}$, $(2 \leq m \leq n-2)$, self dual if

$$\circ \overset{n-m}{U}{}^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m} = \overset{m}{\Theta} \overset{m}{U}{}^{\lambda_1\cdots\lambda_m \mu_1\cdots\mu_m}, \quad . \quad . \quad . \quad . \quad (2.47)$$

$\overset{m}{\Theta}$ being scalar factors.

In this case we get from (*2.46*)

$$\left[\overset{m}{\Theta}-\frac{\varrho^2\binom{m}{2}}{\binom{n-m}{2}}\right]\overset{m}{U}^{\lambda_1\ldots\lambda_m\,\mu_1\ldots\mu_m}=\frac{\varrho^2\,m\cdot m!}{\binom{n-m}{2}}\,U^{[\lambda_1[\mu_1}\,a^{\lambda_2\mu_2}\ldots a^{\lambda_m]\mu_m]}. \quad (2.48)$$

If the space is of constant curvature i.e. if $U^{\lambda_1\lambda_2\mu_1\mu_2}=0$ the last equation is satisfied for all values of $\overset{m}{\Theta}$.

If we assume that the space is not of constant curvature there are two possibilities, namely:

I.
$$\overset{m}{\Theta}=\frac{\varrho^2\binom{m}{2}}{\binom{n-m}{2}} \quad \ldots\ldots\ldots \quad (2.49)$$

In this case we have from (*2.48*)

$$U^{[\lambda_1[\mu_1}\,a^{\lambda_2\mu_2}\ldots a^{\lambda_m]\mu_m]}=0 \quad \ldots\ldots \quad (2.50)$$

Multiplying the last equation by $a_{\lambda_2\mu_2}\ldots a_{\lambda_m\mu_m}$ and summing fo. $\lambda_2\ldots\lambda_m\,\mu_2\ldots\mu_m$ we get

$$U^{\lambda_1\mu_1}=0. \quad \ldots\ldots\ldots \quad (2.51)$$

and the space is EINSTEINian.

II.
$$\overset{m}{\Theta}\neq\frac{\varrho^2\binom{m}{2}}{\binom{n-m}{2}}.$$

Then from (*2.48*) we obtain

$$\overset{m}{U}^{\lambda_1\ldots\lambda_m\,\mu_1\ldots\mu_m}=\lambda\,U^{[\lambda_1[\mu_1}\,a^{\lambda_2\mu_2}\ldots a^{\lambda_m]\mu_m]}, \quad \ldots \quad (2.52)$$

where

$$\lambda=\frac{\varrho^2\,m\cdot m!}{\binom{n-m}{2}\overset{m}{\Theta}-\varrho^2\binom{m}{2}} \quad \ldots\ldots \quad (2.53)$$

As equation (*2.52*) may be put in the form

$$V^{[\lambda_1\lambda_2[\mu_1\mu_2}\,a^{\lambda_3\mu_3}\ldots a^{\lambda_m]\mu_m]}=0 \quad \ldots\ldots \quad (2.54)$$

we get from (*2.9*), (*2.11*)

$$V^{\lambda_1\lambda_2\mu_1\mu_2}=\frac{m!}{2}\,U^{\lambda_1\lambda_2\mu_1\mu_2}-\lambda\,U^{[\lambda_1[\mu_1}\,a^{\lambda_2]\mu_2]}=0$$

and

$$\frac{m!}{2}\,U^{\lambda_1\lambda_2\mu_1\mu_2}=\lambda\,U^{[\lambda_1[\mu_1}\,a^{\lambda_2]\mu_2]} \quad \ldots\ldots \quad (2.55)$$

Multiplying the last equation by $a^{\lambda_1\mu_1}$ and summing for λ_1 and μ_1, we obtain

$$-\frac{m!}{2}\,U^{\lambda_2\mu_2}=\frac{\lambda}{4}(n-2)\,U^{\lambda_2\mu_2}.$$

Hence

$$\lambda=\frac{-2m!}{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.56)$$

from which we have by (2.53)

$$\overset{m}{\Theta}=\frac{-\varrho^2 m}{n-m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.57)$$

From (2.55) and (2.56) we obtain

$$U^{\lambda_1\lambda_2\mu_1\mu_2}=-\frac{4}{n-2}\,U^{[\lambda_1[\mu_1}\,a^{\lambda_2]\mu_2]}. \quad . \quad . \quad . \quad . \quad (2.58)$$

Therefore in this case the V_n is a C_n.

Thus we get the following two theorems for $2\leq m\leq n-2$.

I. *The identity*

$$\overset{n-m}{{}^{\circ}U}{}^{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m}=\varrho^2\,\frac{\binom{m}{2}}{\binom{n-m}{2}}\,\overset{m}{U}{}^{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} \quad . \quad . \quad . \quad (2.59)$$

is equivalent to the EINSTEIN*ian equation* (2.5).

II. *The identity*

$$\overset{n-m}{{}^{\circ}U}{}^{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m}=-\varrho^2\,\frac{m}{n-m}\,\overset{m}{U}{}^{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} \quad . \quad . \quad (2.60)$$

is equivalent to equation (2.6).

Multiplying (2.59) by $-\dfrac{1}{f^{(2)}(m!)^2}\,f_{\lambda_1\ldots\lambda_m}f_{\mu_1\ldots\mu_m}$ and summing for $\lambda_1\ldots\lambda_m\,\mu_1\ldots\mu_m$, we have

$$\underset{(n-m)}{{}^{\circ}\omega}=\frac{m(m-1)}{(n-m)(n-m-1)}\underset{(m)}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.61)$$

or

$$\binom{n-m}{2}\underset{(n-m)}{{}^{\circ}\omega}-\binom{m}{2}\underset{(m)}{\omega}=0\,. \quad . \quad . \quad . \quad . \quad . \quad (2.62)$$

where $\underset{(m)}{\omega}$ and $\underset{(n-m)}{{}^{\circ}\omega}$ are the deviations of the multivectors $f^{\lambda_1\ldots\lambda_m}$ and ${}^{\circ}f^{\lambda_{m+1}\ldots\lambda_n}$ respectively.

Inversely if we assume that we have at each point for every non-singular m-vector

$$\binom{n-m}{2} \underset{(n-m)}{{}^{\circ}\omega} - \binom{m}{2} \underset{(m)}{\omega} = \mu, \quad \ldots \ldots \quad (2.63)$$

μ being a scalar independent of the choice of $f^{\lambda_1 \ldots \lambda_m}$, at every point we obtain from (2.15)

$$\left.\begin{aligned} &\binom{n-m}{2} \frac{1}{{}^{\circ}f^{(2)}\,[(n-m)!]^2} \overset{n-m}{U}{}^{\lambda_{m+1} \ldots \lambda_n \mu_{m+1} \ldots \mu_n}\, {}^{\circ}f_{\lambda_{m+1} \ldots \lambda_n}\, {}^{\circ}f_{\mu_{m+1} \ldots \mu_n} - \\ &\qquad - \binom{m}{2} \frac{1}{f^{(2)}\,(m!)^2} \overset{m}{U}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} f^{\lambda_1 \ldots \lambda_m} f^{\mu_1 \ldots \mu_m} = \mu \end{aligned}\right\} (2.64)$$

from which by (1.3), (2.42), (1.5) and (1.2) we get

$$\left.\begin{aligned} &\left[\binom{n-m}{2} \overset{n-m}{{}^{\circ}U}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} - \right. \\ &\left. - \varrho^2 \binom{m}{2} \overset{m}{U}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} + \varrho^2\, \mu\, \overset{m}{a}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} \right] f^{\lambda_1 \ldots \lambda_m} f^{\mu_1 \ldots \mu_m} = 0. \end{aligned}\right\} (2.65)$$

Hence

$$\left.\begin{aligned} &\binom{n-m}{2} \overset{n-m}{{}^{\circ}U}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} - \\ &\qquad - \varrho^2 \binom{m}{2} \overset{m}{U}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} + \varrho^2\, \mu\, \overset{m}{a}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} = 0 \end{aligned}\right\} (2.66)$$

from which by (2.46)

$$\mu = 0$$

and consequently we get (2.59) from (2.66).

Thus we have: *A necessary and sufficient condition that the space be* EINSTEIN*ian is that the expression*

$$\binom{n-m}{2} \underset{(n-m)}{{}^{\circ}\omega} - \binom{n}{m} \underset{(m)}{\omega}$$

be independent of the choice of the m-vector $f^{\lambda_1 \ldots \lambda_m}$ at each point of V_n.

Zoology. — *A mathematical method for the determination of the state of activity of the thyroid gland.* (Preliminary note.) By J. LEVER. (Zoological Laboratory, Dept. of Endocrinology, University of Utrecht.) [1]) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of November 27, 1948.)

It is a well-known fact that the thyrotrophic hormone induces an increase of activity of the thyroid gland. After administration of antithyroid drugs there is also an increase of activity: the formation of active thyroid hormone is prevented by these drugs, which results in a decrease of the thyroid-hormone level in blood; this stimulates the output of thyrotrophic hormone in the pituitary, thus bringing about all the histological and cytological changes found in the thyroid.

These changes, visible in fig. 1, are

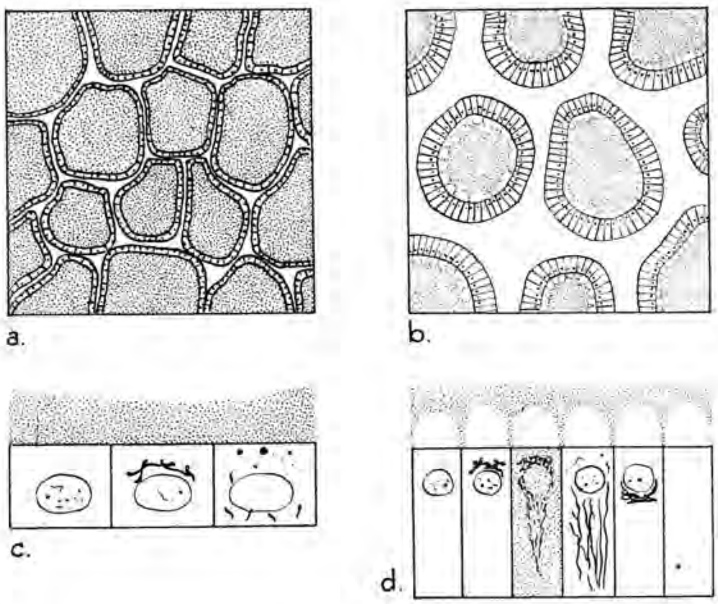

Fig. 1. Upper row: schematic drawings of sections of *a*) an inactive and *b*) an active thyroid. Bottom row: schematic drawings, showing cytological details of thyroid epithelium cells: *c*) inactive stage; *d*) active stage

1. an increase in epithelium height;
2. an absorption of colloid;

[1]) 25th Communication of the "Werkgemeenschap voor Endocrinologie", part of the "National Council for Agricultural Research T.N.O.".

3. an increase of the interfollicular spaces, caused by enlargement of the bloodvessels;
4. an increase of the number of epithelium cells (cf. SEVERINGHAUS, 1933; ELMER, 1938; PASCHKIS et al., 1945; ŘERÁBEK and ŘERÁBEK, 1947);
5. a change in the position of the nucleus from the base of the cell to its apex;
6. a change in the position of the Golgi-apparatus from the apical side of the cell to its base (COWDRY, 1922; LEVER, 1947);
7. an outgrowth of the mitochondria into long slender filaments (CRAMER and LUDFORD, 1926).

As it is easy to change the activity of the thyroid gland, is was desirable to trace the different phases phases of histological activity. Then this activity might be used as a bioassay for the active substances. Moreover, it might be useful in describing accurately the normal differences of thyroid activity. HEYL and LAQUEUR (1935) distinguished several phases of activity in the thyroid of the Guinea-pig, characterized by the form of the cells and of the nuclei, by the position of the nucleus in the cell and by the quantity of cell protoplasm.

In recent years the assay of active substances is especially based on the changes in thyroid weight. The increase in thyroid weight is generally caused by the increase in blood content of the gland (fig. 1). However, as the follicles are the internally secreting parts of the gland, only methods based on changes in the follicles can tell us something about its activity.

Several methods founded on this principle have been described:

1. The *cell height* of the thyroid epithelium is determined by averaging the cell height of a great number of follicles. This has been applied e.g. by ADAMS and BEEMANN (1942) and by LARSON et al. (1945). Against this method the objection may be raised that neither the colloid content of the follicle nor the mitotic activity of the follicle cells are considered. Moreover, it appears that in sections large follicles generally show a higher epithelium than small ones, as a result of unavoidable errors of observation.

2. VAN ECK (1940) determined planimetrically the relation between the colloid areas and the rest of the thyroid section. With the aid of a simple formula the percentage of *colloid content* of the gland was calculated. In the same way STEIN (1940) determined the colloid-contents in every single follicle of a human thyroid.

Here the same objection arises: very important deviations are sometimes caused by a highly developed hyperaemia of the gland.

3. DE ROBERTIS and DEL CONTE (1944) counted the number of *intracellular colloid droplets* in the thyroid epithelium after the administration of thyrotrophic hormone. GRASSO (1946) used this method for the investigation of the influence of antithyroid drugs, thus establishing a "*cytological coefficient*". A modification of this method was published by

DVOSKIN (1947), who calculated the number of colloid droplets in the thyroid epithelium of each animal in totalling the number of droplets in 25 cross sections through the middle part of the gland.

The last two methods, however, do not take into consideration the epithelium height, the colloid content and the increase in the total number of cells. Moreover, none of the methods described above enables us to get an idea of the laws controlling the variations in activity of the gland.

We have therefore tried to find a method by which not one but three characteristics of the follicle, and also the laws of their variation are considered.

As in a circle the circumference is related to the diameter [$c = f(d)$] and in a thyroid-follicle each epithelium cell has the same size, it is easily understood that in follicles, circular in section, the number of cells must be related to the diameter. As the follicles of a given thyroid are all in the same functional phase, we can accept this relation for all follicles, large as well as small ones.

In fig. 2 such hypothetical cases are drawn for 4 follicles of different

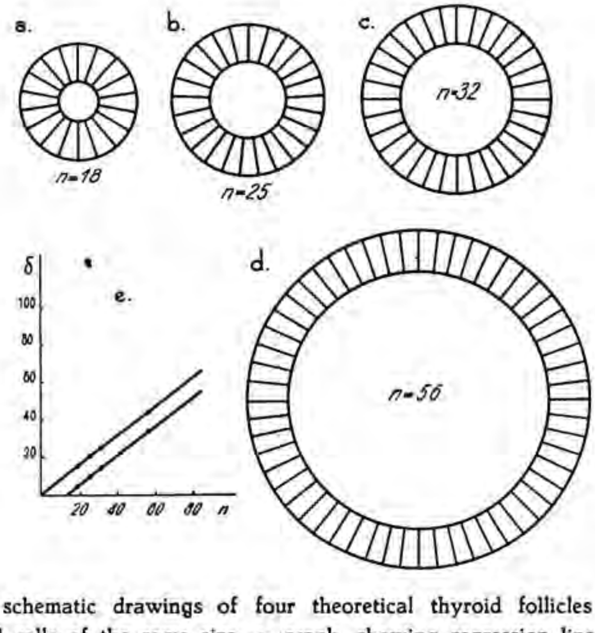

Fig. 2. *a—d:* schematic drawings of four theoretical thyroid follicles with different cell-number and cells of the same size. *e:* graph, showing regression lines in which the outer and inner diameter of the thyroid follicles are plotted against the cell numbers.

size, but in which the epithelium cells are equally large. In the graph of fig. 2 in the upper regression line the diameter of the whole follicle and in the bottom line the diameter of the follicular cavity is plotted against the number of epithelium cells. (Further on these diameters will be called outer and inner diameter respectively).

In fig. 3 similar graphs are calculated from three inactive thyroids of control cockerels (upper row) and from three thyroids of cockerels treated with antithyroid drugs (bottom row). The following details are distinctly visible:

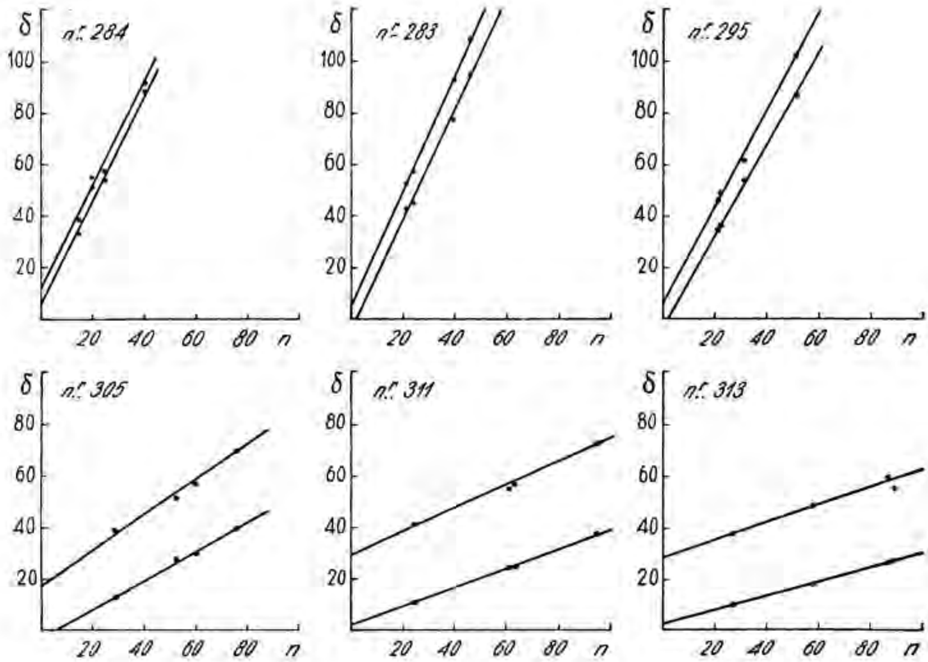

Fig. 3. Upper row: regression lines of inactive thyroids; bottom row: the same active thyroids. (Ordinates: diameters in μ, abscisses: cell numbers.)

1. In the upper row (inactive thyroids) the perpendicular distances between the almost parallel lines are very small. This distance demonstrates the difference between the outer and the inner diameter and equals twice the height of the epithelium cells. In the bottom row the distances between these 2 lines are large and demonstrate the increase in height of the epithelium cells.

2. The inclination of the regression lines in the upper and bottom graphs is very different and demonstrates the relation between the diameter of the follicles and the cell number. Comparing e.g. follicles with an outer diameter of 60 μ in the graphs 284 and 305, we see immediately that the first contains 25, the second 60 cells.

Moreover, if we presume that the height of the epithelium cells is increased by activation in only one direction, e.g. towards the lumen of the follicle, the outer diameter remaining constant, we can conclude from these graphs that the cell number in sections of follicles with a diameter op 60 μ is more than doubled. However, by measuring hundreds of follicles in inactive and active thyroids, we have found that the epithelium height increases in both directions after activation, which became evident by studying frequency-diagrams of the outer diameters.

3. The distance between the bottom line and the abscis demonstrates the colloid content in the follicles. As in the graphs of activated thyroids the bottom line runs much more horizontally, the colloid content of the thyroid follicles decreases by activation.

Consequently the graphs resulting from our method give an insight in

1. the height of the epithelium cells of the thyroid;
2. het number of epithelium cells of the follicles, and
3. the quantity of colloid, present in the follicles.

Therefore they show simultaneously the three principal histological facts, concerning the activity of the thyroid follicle.

Our method is easy to apply as sections stained by the ordinary laboratory routine techniques can be used. The one thing required is that the sections possess the same thickness; 3 μ-sections are the best as they contain only one cell-layer. Generally it suffices to count the number of epithelium cells of four follicles, and to measure their outer and inner diameters. In the case of a follicle being more elliptic than circular, the average of the largest and smallest diameter is sufficient.

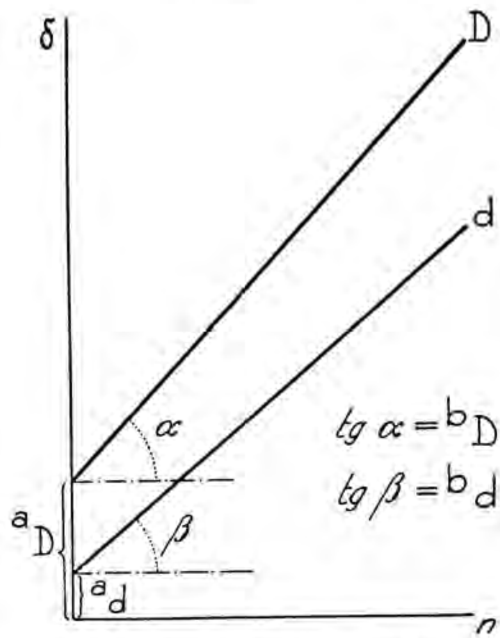

Fig. 4. Graph, showing the coefficients a_d, a_D, b_d and b_D (cf. text).

The lines of the graphs, delineated above, may be described by the general formula $y = a + bx$ (fig. 4). In this formula is y the ordinate, x

the abscis, a the part of the ordinate cut off by the regression line and b the tangens of the angle of inclination.

As each thyroid is represented by 2 regression lines, relating to the outer and inner diameters, D and d respectively, the 2 formulae:

$$y_D = a_D + b_D x \quad \text{and} \quad y_d = a_d + b_d x$$

represent the whole gland.

The distance between both lines — i.e. the difference between the outer and inner diameter of the follicle, or twice the epithelium height — in follicles of different sizes is the same in inactive thyroids only. In these b_d and b_D are practically equal.

Therefore in this case we can derive from these 2 lines one with the formula:

$$y_{dD} = A + Bx$$

in which $A = a_D - a_d$ and $B = \frac{b_D + b_d}{2}$, A being related to the epithelium height (cf. fig. 4) and B to the number of cells dependent on the diameter of the follicle.

Consequently, if the two lines run nearly parallel, we can use in stead of two factors A and B only one coefficient

$$\frac{A}{B} = \frac{2(a_D - a_d)}{(b_D + b_d)}.$$

In practice we have found that it is possible to simplify the method still further. As a matter of fact the value of a_d always fluctuates around the graph's zero, suggesting that in reality in a thyroid the value $\frac{d}{n}$ is the same in follicles of all sizes. This quotient is generally the same as b_d, but it is much easier to calculate. Moreover, it was experimentally stated that the coefficient $\frac{d}{n}$ is the best in expressing thyroid activity, for by activation the numerator (d) diminishes and the denominator (n) increases. Consequently the lower the value of $\frac{d}{n}$, the more active the thyroid is. In practice we have found that in chickens $\frac{d}{n}$ varies between $\pm$ 2.7 and $\pm$ 0.2.

We will give here only one example:

In 2 groups of 9 cockerels which were treated with thyroid activating substances it was not possible, when applying one of the above mentioned methods based only on epithelium height, on colloid content or on a- and b-factors, to demonstrate a distinct difference between these 2 active groups. But when using for both groups the values of $\frac{d}{n}$ for every observed follicle and the average value $\overline{\frac{d}{n}}$, we could easily separate them (cf. table);

this quotient is directly related to the colloid content and indirectly to the height of the epithelium cells, as the inner diameter decreases with the increase of the epithelium height.

Table, showing values of $\frac{d}{n}$ and $\frac{\bar{d}}{n}$ for two groups of cockerels, treated with thyroid activating substances. Group 1—9 is more active than group 10—18.

Number of animal	$\frac{d}{n}$	$\frac{\bar{d}}{n}$	Number of animal	$\frac{d}{n}$	$\frac{\bar{d}}{n}$
1	0.6, 0.5, 0.4, 0.5	0.5	10	1.4, 1.3, 1.4, 1.2	1.3
2	0.5, 0.6, 0.5, 0.4	0.5	11	0.9, 0.9, 0.8, 0.8	0.9
3	0.6, 0.6, 0.7, 0.7	0.7	12	0.9, 1.0, 0.9, 0.9	0.9
4	0.7, 0.7, 0.6, 0.7	0.7	13	0.8, 1.1, 0.9, 0.9	0.9
5	0.7, 0.7, 0.7, 0.7	0.7	14	0.8, 0.9, 0.9, 0.9	0.9
6	0.7, 0.8, 0.7, 0.6	0.7	15	1.0, 1.1, 0.9, 0.9	1.0
7	0.7, 0.8, 0.7, 0.7	0.7	16	1.0, 0.9, 0.8, 1.0	0.9
8	0.8, 0.7, 0.7, 0.8	0.8	17	1.0, 1.2, 0.9, 1.4	1.1
9	0.7, 0.7, 0.7, 0.7	0.7	18	0.9, 0.8, 1.0, 0.9	0.9

In exceptional cases it is possible to combine also the last factor with the formula by using the quotient $\frac{\frac{E}{\bar{d}}}{n} = \frac{nE}{\bar{d}}$, in which E is the epithelium height. As mentioned above this is only allowed in those cases in which the epithelium has the same height in all follicles, i.e. in inactive thyroid glands.

This method was successfully applied in sectioned thyroids of fishes (*Scylliorhinus canicula, Onos mustela, Callionymus lyra* and *Orthagoriscus mola*), of *Rana esculenta*, of the fowl and of mammals (Guinea pig, rabbit, pig, cattle, blue and sperm-whales).

Summary.

The principles of a new mathematical method in determining the state of activity of the thyroid gland are described, based on the relations between the cell-number, the outer and inner diameter and the height of the epithelium cells of the follicle. In applying this method not only the laws which govern the histological changes of thyroid activity, but also the activity of several antithyroid drugs can be studied.

Acknowledgements.

I wish to thank Prof. G. J. VAN OORDT and Prof. CHR. P. RAVEN for their kind advice and encouragement during this investigation.

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Zoology. — *On the thickness of the layer of blubber in Antarctic Blue and Fin Whales.* III. By E. J. SLIJPER (Institute of Veterinary Anatomy, State University, Utrecht). (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of September 25, 1948.)

Fig. 25—27 show that, according to the catch of "Willem Barendsz" up to the second half of January the percentage of immature Blue Whales was very high and that of the pregnant females very low. In the second half of January the curve of pregnant Blue Whales rises sharply. Probably this phenomenon is caused by the return of the majority of

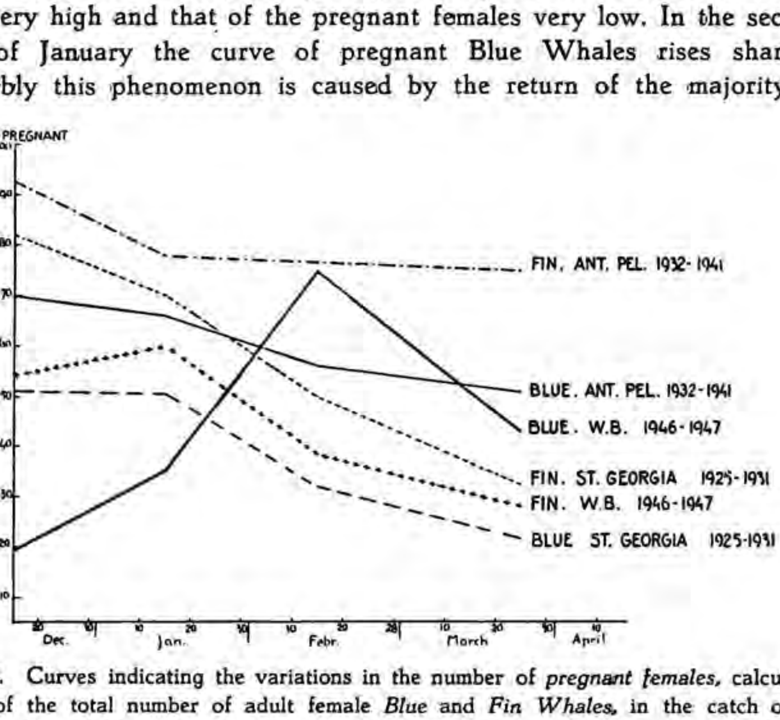

Fig. 24. Curves indicating the variations in the number of *pregnant females*, calculated in % of the total number of adult female *Blue* and *Fin Whales*, in the catch of f.f. "*Willem Barendsz*" during the season *1946—1947*. Average data calculated per month. The other curves bear on average data from *South Georgia* and *Antarctic pelagic whaling* published by MACKINTOSH (1942).

pregnant Blue Whales from the ice, since the curve falls again in February and shows no rise in March, when the expedition could penetrate into the outer zone of the pack-ice. Apparently then most of the pregnant females were gone. The decrease of the percentage of immature whales with a little rise and fall of the curve in the second half of February, may be explained by the above-mentioned increase of pregnant females and by the fact that probably the majority of immature Blue Whales does not go far into the ice. So the distinct increase of the average length and the percentage of pregnant females in the Blue Whale may be responsible for the sharp rise of the output-curve in January of both seasons.

Now an explanation, however, is still wanting for the fact that in both seasons from the beginning of February the course of the output-curve is

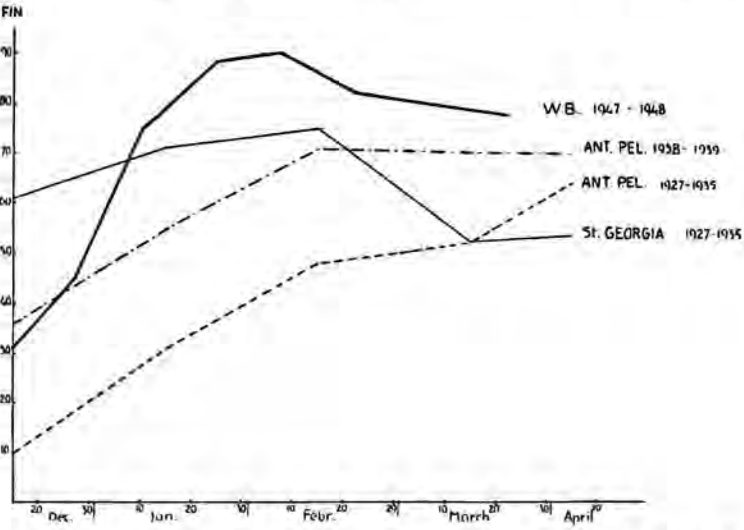

Fig. 25. Curves indicating the variations in the *percentage* of *Fin Whales* in the catch of f.f. "*Willem Barendsz*" during the season *1947—1948*. For explanation of other curves see fig. 21.

almost horizontal, although the general curve for Antarctic pelagic whaling still rises until the end of the season and although the same is shown by the special output-curves for 1946—1947 and 1947—1948. Fig. 22—27 show that normally during the second half of the season an increase of the

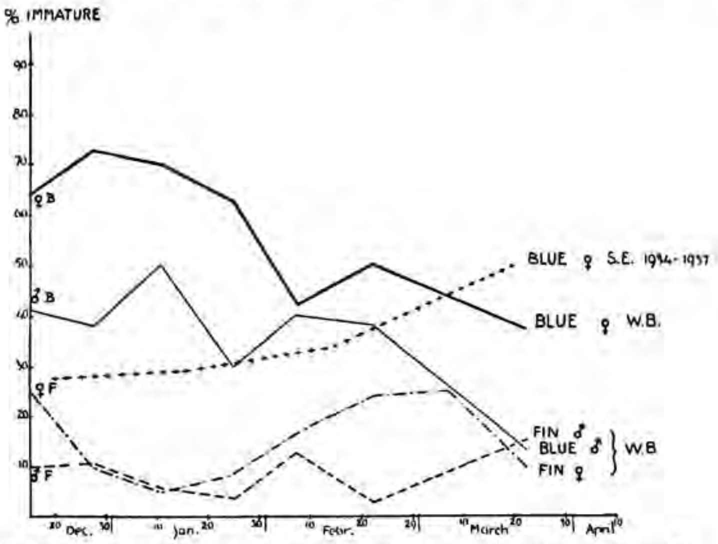

Fig. 26. Curves indicating the variations in the percentage of *immature Blue* and *Fin Whales* in the catch of f.f. "*Willem Barendsz*" during the season *1947—1948*. For further explanation see fig. 22.

percentage of immature and a decrease of the percentage of pregnant whales may be observed. This means a decrease of the percentage of fat whales, but it may be quite possible that the normal increase in fatness of

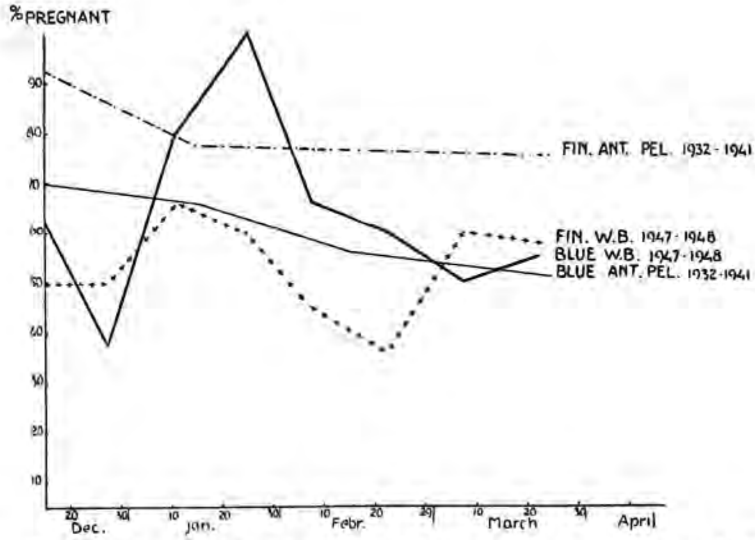

Fig. 27. Curves indicating the variations in the percentage of *pregnant female Blue* and *Fin Whales* in the catch of f.f. "*Willem Barendsz*" during the season *1947—1948*. For further explanation see fig. 24.

the whales is so high that nevertheless the output in barrels per B.W.U. increases. The curves, however, show also that with regard to the catch of "Willem Barendsz" in both seasons the decrease of the percentage of fat whales has been distinctly greater than in normal years, or that perhaps the increase in January has brought the output-curve on an abnormally high level. Consequently it may be supposed that the general increase in fatness is just compensated by the special decrease in the number of fat individuals. There have been some other seasons in which the general output-curve for Antarctic pelagic whaling shows a horizontal or nearly horizontal course from February onward. Fig. 2 in the paper of HJORT, LIE and RUUD (1938) shows that this has been the case in 1930—1931 and 1932—1933. Unfortunately no sufficient data are available about weather-conditions and other circumstances, nor about the composition of the catch during those seasons.

Now according to the curves for the thickness of blubber (fig. 17, 18) it appears that the left part of these curves and especially their sharp rise may be explained in the same way, as it has been done above for the output-curves, viz, by the special circumstances under which whaling was carried on by "Willem Barendsz". It may be supposed that from the moment at which the southward migrating whales have reached their final feeding-grounds, the amount of fat stored in the different parts of their bodies gradually increases up to the moment at which they migrate northward again. From the horizontal course of the right part of the blubber-curves the conclusion may be drawn for the present, however, that from about the middle or the last part of February onward, there is hardly any further increase in thickness of the blubber. It appears to be highly probable that in the first part of the season the fat is chiefly stored in the blubber, whereas in the second part it is chiefly stored in the meat, bone and internal organs. We got a vague impression that storage in the internal organs chiefly takes place at the very end of the season.

6. *The migrations of Whales.*

MACKINTOSH and WHEELER (1929), KELLOGG (1929), RAYNER (1940) and many other authors have shown that in the Southern spring the whales migrate southward and in the Southern autumn northward. With the exception of a small stock that remains in the surroundings of South Georgia or perhaps also at other places, the Antarctic whales live during the winter in the subtropical and tropical waters. In these waters, however, there is practically no food for them and the decrease in thickness of their layer of blubber has been clearly demonstrated by MACKINTOSH and WHEELER (1929). Consequently it might be possible to get a certain impression about the migration of the whales from the thickness of their blubber, since generally it may be supposed that a lean whale must be a whale that has been quite recently arrived from the northern waters.

MACKINTOSH and WHEELER (1929), RISTING (1929), HARMER (1931), WHEELER (1934) and MACKINTOSH (1942) have collected a number of data about the migration of whales by comparing the catches of South Georgia (54° S.), South Shetlands (62°) and Antarctic pelagic whaling (57°—68° S.). According to these researches it appears that the southward migration begins already at the end of September and certainly continues until the end of January. Although the first Blue and Fin Whales appear almost at the same time in the Antarctic waters, it is highly probable that the majority of the Blue Whales migrates earlier in the season than the majority of Fin Whales. The Blue Whales immediately go as far South as ice-conditions permit and then live chiefly in the outer zone of the pack-ice. The majority of the Fin Whales does not go so far South in the first months and although a number of them can be also observed in the outer zone of the pack-ice they chiefly live in a zone just North of the ice-boundary. There are some indications, relating to the Blue as well as to the Fin Whales, that the adult whales and among them especially the big animals and the pregnant females migrate to the South first. They are followed by the smaller adult animals and the non-pregnant females. Finally the immature Whales appear in the Antarctic waters. The majority of these immature animals do not go so far South as their adult relatives. There are also some indications that the pregnant females and the big adult animals go northward at an earlier date than the other whales, the pregnant females leaving the ice already from February onward. Most of these statements are supported by the data given in fig. 21—27.

The very thin blubber, the small average size of the Blue Whales, their high percentage of immature and their low percentage of pregnant animals during the first period of the season 1946—1947 (56°—58° S.), as well as the sudden increase in thickness of the blubber at 63° S. indicate that most of the big and pregnant Blue Whales had already passed the zone at 56°—58° S. and that they were already living then in the ice. The majority of the Blue Whales that were caught belonged to the rear-guard of small and immature animals. The fact that in the first period of 1947—1948 just north of the pack-ice the Blue Whales were lean, as well as the composition of the catch during this period, are in perfect accordance with the above-mentioned facts about the migration of this species.

It appears also from the curves that from the end of January onward the big Blue Whales and especially the pregnant females leave the ice and migrate northward. If the increase of pregnant and big whales at the end of January 1948 had been caused by an invasion of southward migrating animals, the blubber-curve and the output-curve would not have shown such a big and sudden increase in fatness of the whales. This opinion is supported by the fact that when the ship could go further southward in March 1948, no increase of the number of pregnant females was observed. The curves also show that the majority of the immature Blue Whales remains outside the boundary of the pack-ice.

From the shape of the curves for the percentage of Fin Whales the conclusion might be drawn that their main body arrives later in the Antarctic waters than that of the Blue Whales and that they live chiefly North of the pack-ice. If the curves for pregnant and immature whales, however, are compared with the curves for blubber-thickness, they show that, at least in area II, there is no such distinct succession of different groups of animals (pregnant, adult, immature) as may be observed in the migration of the Blue Whale. In future special attention should be given to the remarkable course of the curve for immature Fin Whales (fig. 23 and 26) and to the increase of the percentage of pregnant animals in the beginning of March. Apparently pregnant Fin Whales stay in the Antarctic waters until a later date than pregnant Blue Whales. This would be quite in accordance with the fact that Fin Whales give birth about one month later in the winter-season than Blue Whales (MACKINTOSH and WHEELER, 1929; fig. 149—150). This opinion is also supported by the nearly horizontal course of the curve for pregnant Fin Whales in Antarctic pelagic whaling during the second part of the season and by the distinct decline of the curve for South Georgia.

7. *Summary and conclusions.*

During the Antarctic whaling seasons 1946—1947 and 1947—1948 data about the thickness of the layer of blubber at several different points of the body of Blue and Fin Whales were collected on board the Dutch floating factory "Willem Barendsz". In both seasons the expedition operated in area II and III (from about 17° E.—36° W.), but according to the latitude South at which whaling took place, the circumstances were not quite normal. During the first period of 1946—1947 the position of the ship was too far to the North whereas in 1947—1948 up to the beginning of March it operated outside the northern limit of the pack-ice.

The mutual relation of the blubber-thickness at various points of the body appears to be almost the same in both species and in any part of the season. The great differences in thickness over the body (fig. 1) depend on demands made by the general body-outlines. Measurements taken at the dorso-median line just cranially of the dorsal fin or at the ventro-median line cranially of the anus, give the best impression about the general blubber-thickness of the animals. Foetuses and calves show a greater relative thickness of blubber than young and adult animals. This may be connected with the absence of fat or with its low percentage.

In young and adult animals the relative thickness of the blubber increases with increasing size of the body. This may be explained by changes in the metabolism. The blubber is thickest in pregnant, thinnest in lactating females. The difference (14 %) in the relative thickness between Blue and Fin Whales may be entirely ascribed to their average difference in size. At present there are no striking indications to doubt that according to their average output of oil one Blue Whale may be put on a level with two Fin Whales.

Throughout Antarctic there are comparatively large seasonal variations in fatness and blubber-thickness of the whales, probably connected with general feeding-conditions. Seasonal variations may be also connected with the average length of the whales that are caught and with the percentage of pregnant females. These facts may also be held chiefly responsible for local variations in fatness and blubber-thickness. Consequently it would be very important, if in future statistical publications on whales the output of oil would not only be calculated per Blue Whale Unit, but also per length-unit of the animals (for example the so-called "calculated whale").

The manner of increase in thickness of the blubber during the whaling season (December—April) appears to be the same in Blue and Fin Whales as well as in the different sex- and size-groups. During the first part of the season the thickness of the blubber may give a fairly good idea about the fatness of the whales. From the middle or last part of February onward, however, there seems to be hardly any further increase in thickness of the blubber. Storage of fat probably takes place chiefly in the layer of blubber first, but afterwards chiefly in the meat, bone and internal organs.

It has been shown that measurements of the layer of blubber may give a better insight in the migration of whales, since it may be generally supposed that a lean whale is a whale that has just arrived from the northern waters. Thus several conclusions according to the migration of the different kinds (size, maturity, pregnancy) of Blue Whales, previously made by other authors, could be confirmed. With regard to the Fin Whales, however, it was shown that at least in area II, this species does not show the same succession according to the different groups as the Blue Whale. Pregnant Fin Whales stay in the Antarctic waters up to a later date than pregnant Blue Whales. This may be connected with the fact that they give birth about one month later.

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Botany. — *On the interaction of 2.3.5.-triiodobenzoic acid and indole-3-acetic acid in growth processes.* By JEANNE DE WAARD and P. A. FLORSCHÜTZ. (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of November 27, 1948.)

Introduction.

ZIMMERMANN and HITCHCOCK (1942) were the first to apply triiodobenzoic acid to tomato plants. This substance proved to influence flowering, growth, as well as correlation phenomena. The growing parts of treated plants instead of continuing the normal sympodial growth, developed terminal flowering clusters on the main stem and axillary shoots. Similar experiments were performed in this laboratory. Very young tomato plants started to produce flowerbuds immediately after solutions of triiodobenzoic acid had been sprayed on them. By treating seeds, the seedlings developed flowerbuds when only three leaf primordia were present (DE WAARD and ROODENBURG, 1947). Because the vegetationpoint was "used up" in this reaction, no new leaves were initiated. The growth of leaf primordia already present when the plants were treated, was hampered and they developed into abnormal little leaves, the internodes remaining very short too. The older internodes and petioles, however, already stretching at the time of treatment, showed a stimulated growth.

These results show that the influence of triiodobenzoic acid on the growth of tomato plants is rather complicated, bringing about stimulation and inhibition of growth at the same time in different parts of the plant. This effect suggests an interaction between triiodobenzoic acid and the growth substances of the plant.

GALSTON (1947) carried out experiments with triiodobenzoic acid in the standard *Avena* test. He concluded that this substance, itself without auxin activity, antagonized and might completely negate the effect of indole-3-acetic acid. He supposed this effect to be due to a competition between these two substances, or to a destruction of the latter substance by the former.

COMMONER and THIMANN (1941) found a similar inhibitory action upon growth to be caused by iodoacetate. This substance also inhibits respiration; various 4-carbon dicarboxylic acids were found to release its effect. By repeating and extending these experiments THIMANN and BONNER (1948*b*) conclude that iodoacetate combines with "something within the cell". Since iodoacetate combines stoechiometrically with SH-groups, the authors suggest that SH-bearing enzymes or coenzymes will play a role in growth processes. This, however, is a mere working hypothesis. A similar reaction could be suggested in the case of triiodobenzoic acid. This is in accordance with the ideas of SKOOG, SCHNEIDER and MALAN (1942), who suppose growth substances, hemi- and pseudo-auxins to act as coenzymes, which combine with apoenzymes present in abundance (e.g. proteins).

In the present investigation an attempt has been made to find decisive arguments for one of the explanations of the effect of triiodobenzoic acid on growth, suggested by GALSTON. Our experiments being almost concluded, THIMANN and BONNER's paper (1948*a*) on the same subject became available. It will appear from the following that the results presented in the present paper do not agree in every respect with theirs

Methods.

In the standard *Avena* test (cf. WENT and THIMANN, 1937) the influence of triiodobenzoic acid on the effect of indole acetic acid was investigated quantitatively. Two hours after application of the agarblocks, which had been soaked in the solution to be tested, the curvatures were determined.

Experimental.

The effect of triiodobenzoic acid was studied in concentrations of 50, 5, 0,5 and 0,05 mg/l respectively, dissolved in a 0,05 mg/l heteroauxin solution. The results are shown in table I.

TABLE I.

Concentration in mg/l of triiodobenzoic acid in mixture with indole-3-acetic acid solution (0,05 mg/l)	Average curvature in degrees		
	Experiment		
	A	*B*	*C*
50	0,0	0,0	0,0
5	3,0	4,0	9,0
0,5	6,0	9,5	11,0
0,05	6,0	13,0	11,0
0,00	6,0	—	14,5

A distinct inhibition is caused by triiodobenzoic acid in concentrations of 5 and 50 mg/l. In experiment *B* a concentration of 0,5 proved to be active too. In no case however, a stimulation of growth by the smaller quantities could be observed. In this respect our results do not agree with those of THIMANN and BONNER (1948*a*). It is not clear on which factors this disagreement might be based.

The curvatures proved to vary considerably when the above mixtures were applied; the more so when using relatively high concentrations of triiodobenzoic acid. This is shown by the probable error ($\sqrt{\frac{\Sigma d^2}{n(n-1)}}$, in which $d =$ deviation of each curvature from the average curvature; $n =$ number of curvatures).

As an example we present experiment *A* of table I (see table II).

TABLE II.

Concentration in mg/l of triiodobenzoic acid in mixture with indole-3-acetic acid solution (0,05 mg/l)	Curvature in degrees; average of 20 plants	Probable error	Probable error in % of the average curvature
50	0,0	—	—
5	3,0	1,0	30
0,5	6,0	0,6	10
0,05	6,0	0,5	8
0,00	6,0	0,5	8

However, from the increase of the probable error with increasing concentration of triiodobenzoic acid, it was deduced that the competition of both substances is not due to a concentration factor merely. To check this supposition the method was modified by placing agarblocks with triiodobenzoic acid and indole-3-acetic acid on the decapitated coleoptiles in succession. For instance: agarblocks with triiodobenzoic acid (conc. 1 mg/l) were left on the coleoptiles for one hour. Then they were removed and immediately replaced by agarblocks with indole-3-acetic acid (conc. 0.05 mg/l), for another hour. Also agarblocks with triidobenzoic acid were placed on the coleoptiles after a one hour's treatment with indole-3-acetic acid. Control tests with plain agarblocks were run in every experiment. The data are given in table III.

TABLE III.

Substance applied during		Average curvature in degrees		
		Experiment		
first hour	second hour	*D*	*E*	*F*
triiodobenzoic acid 1 mg/l	indole acetic acid 0,05 mg/l	0,0	0,0	0,0
water	indole acetic acid 0,05 mg/l	—	5,0	6,0
indole acetic acid 0,05 mg/l	triiodobenzoic acid 1 mg/l	10,0	6,0	8,0
indole acetic acid 0,05 mg/l	water	9,0	5,0	9,0
first ½ hour	second 1½ hour			
triiodobenzoic acid 1 mg/l	indole acetic acid 0,05 mg/l	6,0	—	5,0
water	indole acetic acid 0,05 mg/l	6,5	—	5,0
triiodobenzoic acid 100 mg/l	indole acetic acid 0,05 mg/l	—	—	0,0
indole acetic acid 0,05 mg/l applied during two hours.		9,0	6,0	7,0

These results show that triiodobenzoic acid, when applied during one hour in a concentration of 1 mg/l, inhibits the subsequent action of indole-3-acetic acid. When applied during half an hour it shows this effect only in a higher concentration (100 mg/l).

When indole-3-acetic acid is applied first, the resulting curvature is not influenced by a subsequent treatment with triiodobenzoic acid.

Discussion.

The high probable error calculated in table II is an indication of inconstant curvatures resulting from treatment with mixtures of triiodobenzoic acid and indole-3-acetic acid. This fact seems to be incompatible with GALSTON's idea that the growth substance might be partly destroyed by the other acid. If so, the remaining part of the growth substance should give constant curvatures. Moreover, if destruction would occur, the curvatures of coleoptiles provided with indole-3-acetic acid should be impaired by subsequent treatment with triiodobenzoic acid. In table III was shown that this treatment has no effect. All facts, however, can be readily explained by assuming a competition between the two substances in reacting with certain cell constituents.

From the action of both substances when applied in succession we may conclude that indole-3-acetic acid as well as triiodobenzoic acid combine irreversibly with active groups; triiodobenzoic acid is unable to release such a bond with indole-3-acetic acid; likewise the latter substance is incapable to dislodge the triiodobenzoic acid.

The explanation given by THIMANN and BONNER (1948 *a* and *b*) for the competitive effects of both substances agrees with our conception. These authors, however, when applying mixtures of the two substances in the standard *Avena* test, in aequimolar concentration (i.c. triiodobenzoic acid 0.05 mg/l, indole-3-acetic acid 0.02 mg/l) found a stimulation, which never was observed in our experiments.

According to the hypothesis of SKOOG, SCHNEIDER and MALAN (1942) triiodobenzoic acid acts as a pseudo-auxin in occupying active spots in such a way, that the coenzyme (heteroauxin) is prevented from combining. According to THIMANN and BONNER (1948 *b*) triiodobenzoic acid reacts with groups of an enzyme or coenzyme. The authors conclude that the inhibition is restricted to one phase of the growth process. If one accepts this point of view, one may conclude from the above experiments, that the growth activity of heteroauxin is localized to the same phase.

Summary.

The inhibitory effect of triiodobenzoic acid in growth processes was studied in the *Avena* test.

Triiodobenzoic acid proved to antagonize heteroauxin activity: 5—50 mg/l of the former substance inhibits the action of indole acetic acid.

No growth promoting activity of very low concentrations of triiodobenzoic acid could be observed.

From experiments in which both substances were applied to the coleoptiles in succession it was deduced, that triiodobenzoic acid as well as indole acetic acid combine irreversibly with reactive groups present in the cell.

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Paleontology. — *Pleistocene Vertebrates from Celebes. III. Anoa depressicornis (Smith) subsp., and Babyrousa babyrussa beruensis nov. subsp.* By D. A. HOOIJER. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of November 27, 1948.)

The collection of Pleistocene Vertebrates from S. Celebes made by Mr. H. R. VAN HEEKEREN and entrusted to me by Prof. Dr. A. J. BERNET KEMPERS, Head of the Archaeological Survey of the Dutch East Indies, I have already reported upon (HOOIJER, 1948*b*, 1948*c*) contains a small number of teeth referable to two living endemic forms, the anoa and the babirusa. The association of these recent species with forms like *Celebochoerus heekereni* Hooijer, a pigmy archidiskodont elephant, and a gigantic land tortoise (*Testudo margae* Hooijer) which are now extinct, is far from surprising since virtually all living species date back into Pleistocene times. The fossil as well as the prehistoric remains belonging to recent species most often are only larger, in the average, than the recent, and in a number of cases examined it could be proven that the former are subspecifically distinct form the latter. The prehistoric and fossil remains of recent *Dicerorhinus sumatrensis* (Fischer) and *Rhinoceros sondaïcus* Desmarest (HOOIJER, 1946*a*, 1946*b*), *Tapirus indicus* Desmarest (HOOIJER, 1947*b*), *Acanthion brachyurus* (L.) (HOOIJER, 1946*c*), *Panthera tigris* (L.) (HOOIJER, 1947*a*) and *Pongo pygmaeus* (Hoppius) (HOOIJER, 1948*a*) bear evidence of this rule.

Neither the anoa nor the babirusa are living today in the region where their fossil remains were found. In the South-western peninsula of Celebes the anoa still only lives on the Peak of Bonthain in the extreme S. (WEBER, 1890, p. 112; SARASIN, 1905, p. 32), while the babirusa has vanished from the whole of the South-western peninsula of the island (SARASIN, l.c., p. 41). The occurrence of both species in the prehistoric collections from caves near Lamontjong in S. Bone (ca. 60 km E.N.E. of Macassar) and from the neighbourhood of Tjani (Lamontjong), Watampone (Central Bone, ca. 120 km N.E. of Macassar) and Bonthain on the S. coast, described by SARASIN (1905) and DAMMERMAN (1939) respectively, has already established that these forms were more generally distributed over the island in former times than they are now. SARASIN (1905, p. 39/40) reported that the subfossil teeth of the babirusa agree well in size with the recent, but the subfossil teeth of the anoa (l.c., p. 30) average smaller than the recent. From the latter fact SARASIN inferred that the Southern Celebes anoa may have been smaller than the northern form. DAMMERMAN (1939, p. 64) states to have found no noticeable difference in size between the prehistoric and the recent molars of the anoa, however. The Pleistocene fossils are described and discussed below.

Anoa depressicornis (Smith) subsp.

Our Pleistocene material of the anoa consists of four teeth, viz., P_4–M_3 dext. (pl. I, fig. 1). They are embedded in a matrix consisting of calcite grains of irregular form and containing some grains of quartz and alkaline felspar. The specimen originates from Sompoh near Tjabengè (Sopeng district), about 100 km N.E. of Macassar in S. Celebes.

Parts of the external surfaces of the teeth only were exposed when I received the specimen. The external lobes of the teeth have split vertically and the fissures are filled with matrix. Cement has completely weathered off. The crown surfaces of the teeth are incomplete; the V-shaped fossae in the median antero-posterior line of the crowns were found but the higher internal surfaces of the inner cusps are missing. The specimen is much distorted too: M_2 is on a lower level than M_3 while M_1 and P_4 are displaced inward. The posterior portion of M_1 and the anterior portion of M_2 are missing.

Of the P_4 the whole of the outer surface of the crown is preserved. The broad anterior lobe or protoconid exhibits a few vertical cracks, whereas the postero-external pillar or hypoconid has split all over its length. The protoconid is at least 21 mm high from the base to the worn edge, the roots are missing. It is regularly convex antero-posteriorly above and becomes more flattened at the base, especially anteriorly where a very small rudiment of a cingulum is seen. From above downward the protoconid is very slightly concave. Apparently it did not decrease in antero-posterior diameter (about 10 mm) up to the crown edge which is highest in the middle. It is separated from the hypoconid by a narrow vertical groove in which cement has completely gone. At the lower as well as at the upper border of the crown the groove flattens out; it is, therefore, deepest in the middle of its course. The hypoconid forms a straight and narrow vertical ridge, 19 mm in height and about 2.5 mm wide (it measures 3 mm antero-posteriorly but the vertical fissure is about 0.5 mm wide).

A small portion of the P_3 is preserved; it consists of the upper part of the hypoconid ridge, only about 6 mm high, and a very small adjacent portion of the protoconid surface. The groove in front of the hypoconid is decidedly less deep than that in P_4.

Of the M_1 the protoconid is broken and distorted. Its height may have been about 21 mm, the antero-posterior diameter 8 mm. It is regularly convex from before backward with no indication of a median rib. The base of an accessory column in the vertical groove separating it from the hypoconid is preserved. The enamel figure on the crown shows the paraconid to have possessed a strong median rib on its outer surface, but the inner surface of the paraconid is not preserved. Of the hypoconid only part of the anterior surface is seen, the main body of it is missing as well as that of the protoconid of M_2. The M_2 is displaced outward relative to M_1 and P_4 and is much crushed. The posterior surface of the protoconid is broken and in the vertical groove separating it from the hypoconid there is no accessory column. The hypoconid presents a vertical fissure that widens toward the roots which are again missing. Its total height is about 28 mm; antero-posteriorly it measures about 9 mm. The posterior surface is flattened and there is no trace of a median vertical rib on the outer side. From above downward the hypoconid may have been slightly concave.

The M_3 consists of three lobes, the anterior and the middle of which are badly broken; along the vertical fissures that run through them parts of the enamel have broken off especially toward the worn edge. The height of the crown is about 33 mm. The anterior surface of the protoconid as well as the posterior surface of the hypoconid are flattened, and the convexity of these lobes is most marked in the center without, however, forming median ribs. The groove between protoconid and hypoconid is narrow and deep. The fossae on the crown show the inner cusps to have possessed outer median vertical ribs; the internal surface of the molar is lost. The talonid is broken above and below and is shorter antero-posteriorly than the two main lobes. It is separated from the hypoconid by a wide groove. The talonid is well convex antero-posteriorly, narrows toward its summit and possesses a posterior keel, most marked above.

I can find no characters to distinguish the present fossil teeth from those of the living anoa. The lengths of P_4 and M_3 also are within the range of variation of these measurements in four mandibles of the latter species (table 1). In *Bubalus mindorensis* Heude, which is very near in

TABLE 1.
Measurements of fossil and recent molars of *Anoa depressicornis* (Smith) subsp. and of recent *Bubalus mindorensis* Heude.

	Anoa depressicornis (Smith) subsp. Leiden Museum					*Bubalus mindorensis* Heude Leiden Mus.	
	Sompoh	cat. a	cat. b	cat. c	cat. d	cat. a	cat. b
Length of P_4	13.5	14.5	14	12.5	13	15.5	16
Length of M_3	25	27	25	23.5	24	30	30

dimensions to the fossil, all lower molars have well-developed accessory outer columns instead of M_1 exclusively as is the case in our fossil specimen, which is typical of the anoa (HELLER, 1889, p. 17). Accessory outer columns, however, may occasionally develop in M_2 and M_3 of the anoa too (HELLER, l.c.), and in two of the Leiden Museum specimens (cat. a and b) these columns have slightly developed. In *Bubalus mindorensis* the protoconid of the P_4 possesses a median outer rib especially marked off posteriorly; this makes the vertical groove between protoconid and hypoconid $\sqcup$ shaped instead of V shaped as is the case in the anoa in which the protoconid is gradually convex from before backward. The molars of the anoa and our fossil specimen are distinguishable from those of *Bubalus mindorensis* by the same character: the outer lobes are not pinched in anteriorly and posteriorly as are those of *Bubalus*, thereby producing median ribs on protoconid and hypoconid in *Bubalus* that have not developed in *Anoa*. Unfortunately it is impossible to ascertain whether the anterior transverse valley on the internal side of P_4 is blocked or open, as it is in *Anoa* and *Bubalus* respectively (PILGRIM, 1939, p. 255). The distinguishing character I found is not mentioned by PILGRIM, and the presence of median ribs on the outer folds of the lower molars indeed does not seem to be a hard and fast rule in the genus *Bubalus*; in some specimens of *Bubalus bubalis* (L.) and *Bubalus palaeokerabau* Dubois they have hardly developed. Since the fossil teeth are absolutely indistinguishable from those of *Anoa depressicornis* (Smith) they may be classed with that species. I should have expected the fossil teeth to be larger than the recent, as is evident from what has been stated in the introduction of the present paper.

As remarked already above, SARASIN found the subfossil molars of the anoa to average smaller than the recent, while DAMMERMAN found no difference in size between the recent and his subfossil material of the anoa. Neither SARASIN nor DAMMERMAN remark upon the subspecies of *Anoa depressicornis* to which their material for comparison belongs. Besides *A. d. depressicornis* (Smith) typically from the Northern peninsula of Celebes there is also a smaller race, *A. d. fergusoni* (Lydekker), unfortunately without exact type locality, that has been recorded from the high forested mountains of the central region of Toradja and from the high districts of Binuwang on the W. coast of the island, at about lat. 3° 30′ S. and to which probably also the anoas reported from the Peak of Bonthain belong (HARPER, 1945, pp. 550—554). Skull measurements of the smaller race are not known, but if *A. d. fergusoni* is represented in the series of mostly unlocalized skulls of the anoa preserved in various European Museums recorded by HELLER (1889, p. 24) the difference in size is not considerable. HELLER gives the measurements of 25 skulls; the series is arranged from the largest to the smallest but only upon no. XIX they become markedly smaller. Nos. XXII and XXIII are in the Leiden Museum (cat. g and f) and the latter skulls are not adult, M_3 being still unerupted. Nos. XX and XXI, from German Museums, being very near in size to the Leiden Museum skulls, most probably are not yet adult too. The basal length of the 19 remaining and all probably adult skulls varies from 260 to 290 mm only, the zygomatic width from 120 to 141 mm. Skulls a and b in the Leiden Museum, which are nos. I and VII of HELLER's list, originate from Tondano and Menado respectively and they may be taken as representing *A. d. depressicornis*. Their teeth are larger than those of the fossil specimen. Skulls c and d of the Leiden Museum collection (nos. XVIII and IX in HELLER's table) were collected by REINWARDT and are without a record for the exact locality. They have teeth that are smaller than their fossil homologues (table 1). If the latter skulls represent *A. d. fergusoni*, this would link our fossil specimen up with the latter race rather than with the larger *A. d. depressicornis*.

Babyrousa babyrussa beruensis nov. subsp.

Diagnosis: Molars identical in specific characters to those of recent *Babyrousa babyrussa babyrussa* (L.) from Buru and of *B. b. alfurus* (Lesson) from Celebes, but most often distinctly larger (see measurements).

Holotype: An M_3 dext. described and figured in the present paper (pl. I, fig. 2).

Paratypes: Two specimens of M_3 (B and C), an M^2 dext. and an M^3 sin.

Locality: Holotype, and M_3 specimen C, as well as M^3: Beru; M_3 specimen B and M^2: Sompoh, 12 km N. of Beru, near Tjabengè (Sopeng district), about 100 km N.E. of Macassar, S. Celebes.

Age: Pleistocene.

The best preserved specimen is the holotype from Beru. It is an M_3 of the right side, complete, in a fragment of the ramus (pl. I, fig. 2). The plane of wear is oblique, the external cusps (protoconid and hypoconid) are much more worn down than the internal (metaconid and entoconid). These four main cusps are arranged in two transverse pairs, behind which there is a median lobe of the hypoconulid followed by a semicircular posterior portion of the hypoconulid or "talon".

The metaconid is the biggest of the main cusps, and is, like the entoconid behind it, modelled by wear into a chisel-shaped column with the transverse cutting edges falling off toward the outer side. Between these two inner cusps there is a wide valley that opens to the inner surface down to 2.5 mm above the crown base. The transverse valley is blocked in the middle portion of the tooth by ridges connecting the metaconid with the hypoconid, but it is as deep as its inner portion again between the outer cusps, and even wider due to the latter having less well developed than the inner cusps. The groove separating the metaconid from the protoconid has not yet disappeared by wear, but entoconid and hypoconid have just become confluent in the median line. The ridge between metaconid and hypoconid is crossed by a groove indicating the limit between the two cusps. The entoconid narrows distinctly toward the median line of the crown, as shown by grooves marking off this cusp in front and behind. The tooth is slightly wider over the entoconid and the hypoconid than in front but narrows distinctly backward. The semicircular portion of the hypoconulid, or "talon", that is subdivided by an antero-posterior groove into two parts of which the external is the larger, is placed at some distance behind the posterior pair of main cusps, and between them is the central lobe of the hypoconulid, partly wedged in between entoconid and hypoconid. The whole of the crown is surrounded by a feebly developed cingulum, some 2.5 to 3 mm high but hardly visible at some places. In the front surface of the molar there is a hollowed facet caused by interproximal wear with M_2. The dimensions of the present and of the following specimens will be found in table 2 below.

D. A. HOOIJER: *Pleistocene Vertebrates from Celebes. III. Anoa depressicornis (Smith) subsp., and Babyrousa babyrussa beruensis nov. subsp.*

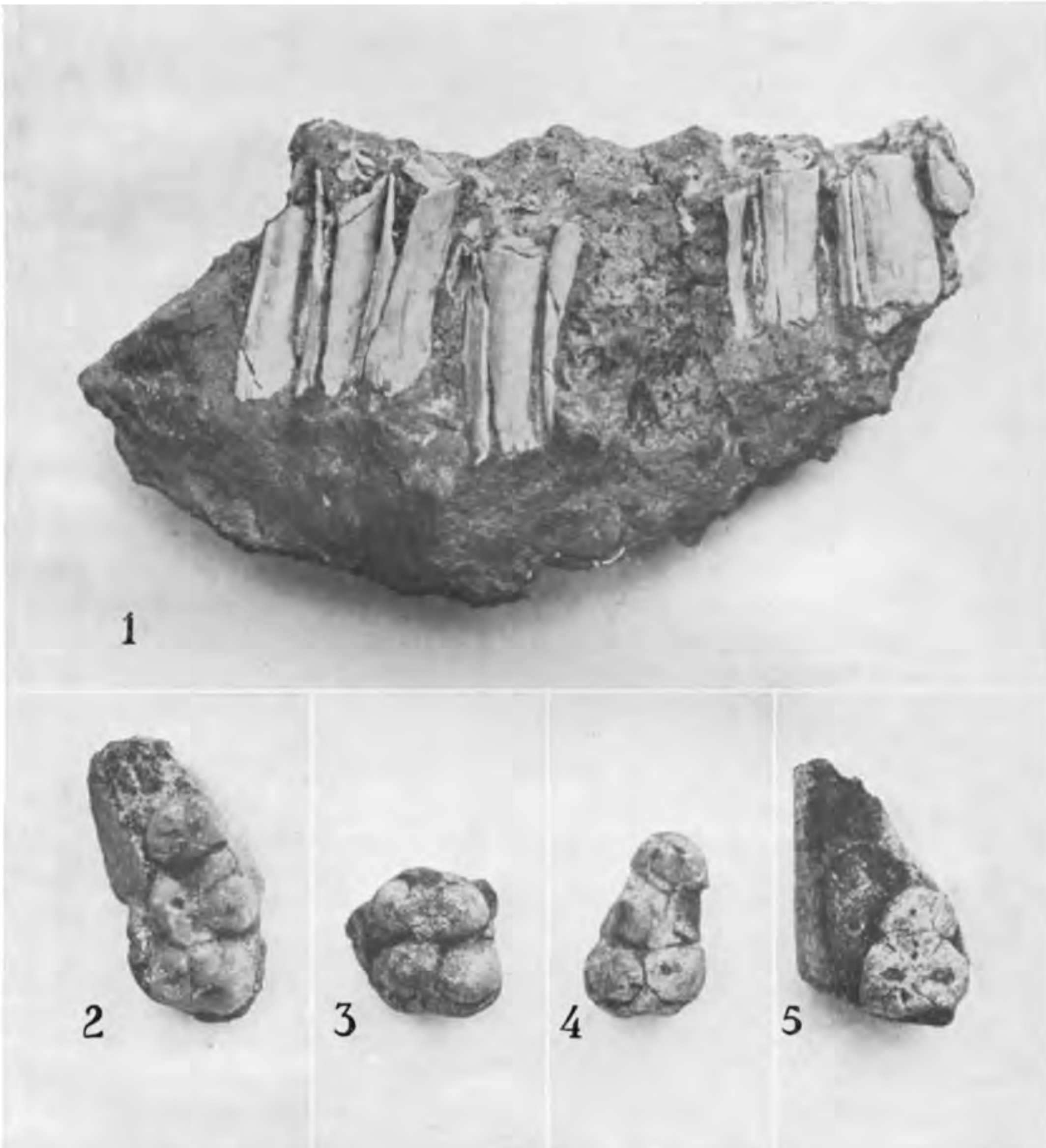

Fig. 1. *Anoa depressicornis* (Smith) subsp.; P_4—M_3 dext., Sompoh, S. Celebes, outer view.

Figs. 2—5. *Babyrousa babyrussa beruensis* nov. subsp.; Fig. 2, M_3 dext. (holotype), Beru, S. Celebes, crown view; Fig. 3, M^2 dext., Sompoh, S. Celebes, crown view; Fig. 4, M^3 sin., Beru, S. Celebes, crown view; Fig. 5, M_3 dext., (specimen C), Beru, S. Celebes, crown view. All figures natural size.

The second tooth to be described (specimen B) is also an M_3 dext., but originating from Sompoh. The protoconid has broken off and the stage of wear is more advanced than that of the holotype, but in all its characters the present specimen is similar to the last with the exception of the shape of the central hypoconulid lobe. The latter has less well developed in the Sompoh molar and consequently the "talon" is shorter. The united antero-posterior diameter of the two internal cusps is the same as that in the holotype, viz., 20 mm, but the length of the whole molar is somewhat shorter. The cingulum is less well marked than that in the holotype.

Our third specimen (C) of lower last molars (pl. I, fig. 5) is again of the right side, and originates from Beru. The anterior pair of cusps has broken off. The limiting grooves of the hypoconid and the entoconid, both narrowing almost to a point in the midline of the tooth, are well seen. The central lobe of the hypoconulid is intermediate in development between those of the foregoing molars; it is defined by grooves as a triangular structure separating the main cusps from the semicircular posterior portion. The latter is crossed by a groove distinctly to the internal side of the median axis of the crown. Dimensions are given in table 2.

From the upper jaw we have two fossil molars, an M^2 dext. and an M^3 sin. The former originates from Sompoh and is entire, though somewhat weathered (pl. I, fig. 3). It has just been touched by wear and has an interproximal wear facet only in front. The cusps are strong and simply built without marked vertical grooving of the enamel. The internal (protocone and metaconule) are higher than the external (paracone and metacone) and the transverse pairs formed by these cusps are slightly oblique: the internal cusps are placed more posteriorly than the external. The anterior cingulum is high in the middle and wide at the base; at the antero-external and antero-internal angles of the crown it terminates hook-shapedly turned up. In the median line its highest point is almost on a level with the saddle between protocone and paracone. The posterior cingulum is narrower but even slightly higher than that at the anterior side. The transverse valley which is blocked in the middle has accessory cusplets both at its internal and at its external entrance, evidently formations of the cingulum that is otherwise lacking at the inner and outer sides of the crown. The dimensions are given in table 2.

The last specimen is an M^3 sin., from Beru (pl. I, fig. 4). It is shorter and wider than the M_3 described above and narrows distinctly from front to back. The greater part of the metaconule is lost. Of the two anterior cusps the protocone is the larger; both are rounded and not grooved. The metacone is placed more inward than the paracone and is apparently the smallest cusp; it is separated from the paracone in front by a deep valley. A central antero-posteriorly elongated lobe separates the metacone from the metaconule and also partly fills the valley between metaconule and protocone. The central lobe turns outward behind the metacone and forms a distinct accessory cusplet above the external cingulum between the metacone and the talon. The latter is small but has an accessory outer tubercle. The cingulum forms a distinct ledge around the crown and rises in front into a high central point like that in the M^2.

TABLE 2.
Measurements of fossil and recent molars of *Babyrousa babyrussa* (L.)

	B.b. beruensis nov. subsp.			*B.b. babyrussa* (L.) (Buru)	*B.b. alfurus* (Lesson) (Celebes)
	holo-type	B	C		
M_3					
Length	30.5	28.2	—	22.7—26.7	24.5—27.4 (29.6)
Ant. width	14.6	—	—	12.9—14.1	12.9—14.7 (15.8)
Post. width	15.3	14.6	14.3	11.7—14.2	12.0—14.6 (14.9)
M^2					
Length		20.2		15.1—17.6	16.8—19.4 (20.0)
Ant. width		18.0		13.8—15.5	15.0—16.2 (17.5)
Post. width		17.6		12.4—15.9	13.6—16.6 (17.2)
M^3					
Length		25.4		20.4—23.4	22.7—25.7
Ant. width		17.1		14.5—16.7	15.0—16.7 (17.7)

The present fossil molars correspond so exactly with the homologous teeth of the babirusa that there can be no doubt in referring them to the same species. There is, however, a difference in size. As is evident from the inspection of table 2 the dimensions of the fossil molars are invariably greater than those found in the Buru babirusa (a series of 13 skulls in the Leiden Museum), that is slightly smaller than the recent Celebean form. The variation limits of the tooth dimensions in a series of 9 skulls of the babirusa from Celebes is given in the last column of table 2, and in parentheses I added the maximum figures found in a series of 19 babirusa skulls which have no record for the exact locality. The holotype M_3 is longer and has a greater width over entoconid and hypoconid than all recent specimens of M_3, and in its anterior width it is exceeded only by 5 out of the 40 recent homologues. The Sompoh M_3 (specimen B) is longer than all but one and wider than all but another out of the latter, while the smallest of the fossil specimens of M_3 (C) is still wider than 38 out of the 40 recent specimens. The M^2 is larger in all dimensions than its homologue in all of the 41 recent skulls. The M^3 is longer than all but 2 of the recent M^3, and is exceeded in width likewise only by 2 (other than those that are longer) out of the 40 recent specimens of M^3.

Thus four out of the five molars of the Pleistocene Celebean form of babirusa are larger than their homologues in a series of 40 recent skulls and the fifth molar (a fragment) still stands third for the greatest width in this long series. I have no babirusa material from the Sula Islands, but this form, *Babyrousa babyrussa frosti* (Thomas, 1920, p. 187) is stated to be slightly smaller even than the Buru race. Consequently it is rendered certain that the Pleistocene Celebean babirusa averages larger than the largest of the recent races of the species, viz., *B. b. alfurus* (Lesson) (= *B. b. celebensis* (Deninger, 1909, p. 185)), and therefore it may be described as a separate subspecies. *Babyrousa babyrussa beruensis* nov. subsp. bears evidence of the diminution in size which the babirusa has undergone in the island of Celebes since Pleistocene times, perfectly in accordance with the rule formulated on p. 1322. As has been stated above it is not yet certain whether the anoa presents an exception to this rule. In the island of Buru the decrease in dimensions has proceeded further, the Buru babirusa being smaller, in the average, than the Celebean race. Similar differences in size are well known to exist between insular races of one and the same species.

A number of mammalian species occurring with the Pleistocene, prehistoric and recent faunae in the islands of Java and Sumatra as well as on the Asiatic continent have now been studied intensively enough to enable us to state that the recent subspecies evolved in situ from racially distinct populations that existed already in the Pleistocene (HOOIJER, 1946b, 1946c, p. 265; 1947a, p. 12—13; 1947b, p. 288; 1948a, pp. 279, 292). For Celebes and the Eastern part of the Malay Archipelago we are merely at the beginning of this work. But as a result of the lucky finds of Mr. VAN HEEKEREN at Beru and at Sompoh in S. Celebes we know now already more of the Pleistocene fauna of this island than of that of the large continental islands of Sumatra and Borneo! The latter islands, because of their position on the Sunda shelf certainly yield much of the fossil species of Vertebrates found in Java. Celebes appears to have as faunistically distinct from the other Greater Sunda Islands in the Pleistocene as it is at the present day; two of its endemic forms have already come to light, and others will doubtless turn up in the collections to be made. The fossil teeth and bones found by Mr. VAN HEEKEREN consist of isolated specimens washed out of their deposits. The determination of the exact geologic age of the fossils when found in situ will be a matter of great evolutionary interest.

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