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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
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PROCEEDINGS

VOLUME XXX

N^o. 1

President: Prof. F. A. F. C. WENT

Secretary: Prof. L. BOLK

(Translated from: "Verslag van de gewone vergaderingen der Afdeeling
Natuurkunde", Vol. XXXV)

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Botany. — *On the light-growth-reactions in different zones of the coleoptile of Avena sativa.* By C. VAN DILLEWIJN. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of May 29, 1926).

In a preceding communication (7) I determined the light-growth-reactions with various quantities of light for coleoptiles of *Avena*. In accordance with BLAAUW's theory (3) the phototropic curvatures, positive as well as negative, could be explained from the light-growth-reactions. Two distinct kinds of light-growth-reactions were found, which I have indicated by "short", when the growth-minimum occurs after 24—30 minutes, and by "long", if a longer time, about 80 minutes, is required.

Investigations by WENT (15) have proved that the long reaction is a function of the top. When the base is illuminated the short reaction only is observed. WENT therefore speaks of a top-reaction and a base-reaction. If an entire coleoptile is illuminated the light-growth-reaction is the resultant of top- and base-reaction. Until now the light-growth-reactions have generally been determined by illuminating whole coleoptiles and so represent a complex of light-growth-reactions of the different zones of the coleoptile. The question now arises how a plant reacts when a definite zone only is illuminated and the remaining part of the coleoptile is kept in the dark. By means of zonal illumination I hoped to be able to carry the analysis of the light-growth-reaction a little further.

Method: Three zones of 2 mm length were investigated, the topzone from 0 to 2 mm, indicated as zone 0, one just below it (zone II) and one from the strongly growing part, from 7 to 9 mm (zone VII). When it appeared later that the top-zone was by far the most important, also the extreme top from 0 to $\frac{1}{2}$ mm was investigated.

The method for measuring the growth-reactions was the same as in former experiments (7), with the difference that metal sheaths were applied to keep certain parts of the coleoptile in darkness. The light-growth-reactions of the different zones were determined for light-quantities of 800, 8000, 80.000 and 800.000 metre-candle-seconds, at a temperature of 20° C. and a degree of moistness of 80 %. Heat rays were absorbed by a layer of water. Of the light-growth-reactions only those of zone 0 will be given in full, for comparison also a reaction of zones II and VII will be given. More extensive data will be found in a subsequent paper. The figures here given are averages of several determinations.

Zone 0. If this zone is illuminated during 10 seconds with 80 m.c. the plant, after a latent period of about 15 minutes, shows a slow decrease of its growth-rate; after about 70 minutes the minimum is reached and growth slowly increases again (fig. 1d). After about 2 hours it has a feeble maximum and then becomes constant again. Investigations by PAAL (10), SÖDING (14), NIELSEN (9), BEYER (2) and others have proved that the top possesses a particular function, namely the formation of growth-regulating substances. These substances diffuse downwards and there induce growth. PAAL (10) supposed that by illumination the correlation between top and base is affected and among other things assumes that illumination influences the formation of these growth-regulating substances, either accelerating or retarding it. As long as the plant grows in darkness the top continuously produces growth-regulating substances. We may now imagine that by illuminating with 10×80 m.c. the process by which these substances are formed is retarded, so that a smaller quantity of them is formed. This will cause a growth-retardation, first in the upper zone, but, as the deficiency is transmitted to lower zones, these also will be retarded, so that the total growth-retardation continually increases. After some time, however, the equilibrium in the top will have been restored and more growth-regulating substances will again be formed, the result being a growth-acceleration, likewise starting in the upper zone and transmitted to

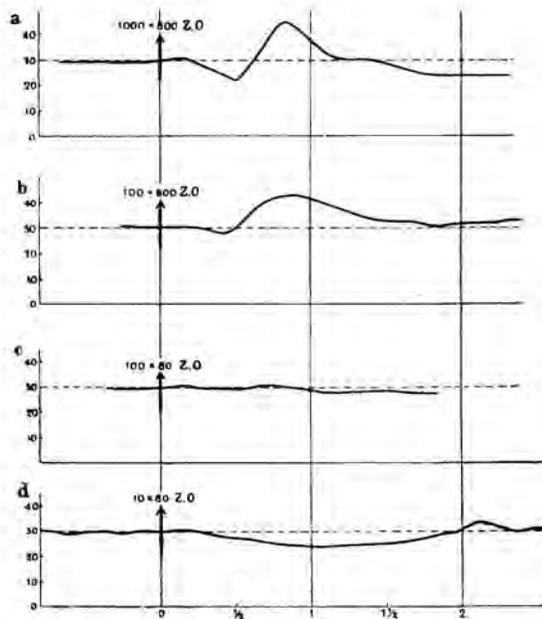


Fig. 1. Light-growth-reactions. The top is illuminated over a distance of 2 mm. The abscissa indicates the time in hours, the ordinate the growth-rate per minute. 30 divisions of the ordinate correspond to a growth-rate of 20μ per minute. The arrow indicates the moment of the beginning of the illumination.

the lower ones. In the light-growth-reaction this is reflected by a minimum after a certain time, followed by an acceleration. At this stage the growth-acceleration has therefore penetrated so far to the deeper zones that the total acceleration exceeds the total retardation. When in the light-growth-reaction the growth-rate has become normal again, this does not mean, however, that in all zones growth has become normal. It is quite possible that then the growth-retardation is still always being transmitted downwards, followed at a certain distance by the growth-acceleration. The behaviour of unilaterally illuminated coleoptiles even strongly supports this view. When a coleoptile is unilaterally illuminated with about 800 m.c.s. the curvature starts at the top and then spreads over the basal zones. But after some time stretching sets in at the top (autotropism). On this point we refer to ARISZ (1) and PISEK (11). So here a growth-acceleration has followed the unilateral retardation, and while this retardation is still being transmitted further in the base, as is manifested by a shifting of the phototropic curvature, a growth-acceleration starts from the top and is transmitted downwards, as is manifested by a stretching of the originally curved parts. It is very probable that these processes, occurring in unilaterally illuminated plants on the exposed side only, occur on all sides with all sided illumination. Now this means that of the light-growth-reaction only the part during the first hour after illumination represents a simple process, namely a retardation, progressing downwards, but that after the minimum the reaction consists of two antagonistic processes: a growth-retardation progressing downwards and an acceleration, following the retardation at some distance.

When zone 0 is illuminated during 100 seconds with 80 m.c. (fig. 1c) growth remains almost unaltered. In the single reactions a small retardation was often observed, followed by an equally slight acceleration, but normal growth was soon restored. So we must assume that the formation of growth-regulating substances undergoes no or hardly any influence from this light-quantity.

A very different aspect is found when we illuminate 100 seconds with 800 m.c. (fig. 1b). Leaving aside for the moment the small minimum, we may say that this quantity causes a distinct acceleration, reaching a maximum after about 54 minutes and then falling off again to about the normal rate. So this quantity of light promotes the formation of growth-regulating substances. The slight minimum may be interpreted as the beginning of a long reaction (cf. fig. 1 d) which is immediately superseded by an acceleration. This would indicate that the formation of growth-regulating substances is first retarded and then accelerated. We shall presently see, however, that also the basal zones give a similar minimum with this illumination and here a relation between the minimum and the formation of growth-regulating substances is of course excluded, since only the top forms them.

With an illumination of 800 m.c. during 1000 seconds (fig. 1 a) a reaction

occurs, differing herein from the preceding one that the minimum lies considerably deeper, while the following growth-acceleration sooner reaches its maximum and then rapidly falls to the normal rate and later even below it. As the growth-retardation and acceleration approximately annul each other, the final result of this illumination will be that the total growth after 2 hours is about equal to that of a non-illuminated plant in the same time.

Let us now take a summary view of the reactions as given in fig. 1. By illuminating with 800 m.c.s. growth is retarded. Using less than 800 m.c. the retardation is smaller, as I found in experiments which I will not mention here. With 8000 m.c.s. an indifferent stage is met with, i.e. growth remains the same as before the illumination. We may therefore say that between 0 and 800 m.c.s. the retardation becomes continually stronger by increasing quantities of light, that between 800 and 8000 m.c.s it reaches a maximum and then diminishes and entirely disappears at 8000 m.c.s. By still increasing the light an acceleration sets in, which at 80,000 m.c.s. is fairly considerable and with still more light also diminishes again, so that at 800,000 m.c.s. disregarding a transitory retardation and acceleration, an indifferent stage is again reached. The same behaviour was found for the uppermost top zone of $\frac{1}{2}$ mm., with this difference that both retardation and acceleration are somewhat smaller than in illuminating zone 0. This means that the top reactions take place almost entirely in the extreme $\frac{1}{2}$ mm of the top and the next $1\frac{1}{2}$ mm is only very little sensitive to these top reactions. This is in good agreement with the results of SIERP and SEYBOLD (3) for phototropic sensitiveness. They found namely that this is chiefly localised in the extreme $\frac{1}{4}$ mm of the top and then rapidly diminishes. As will be presently shown all basal zones lack the property of giving top reactions with any of the light-quantities used. What do these reactions tell us now about the nature of the phototropic curvature? When answering this question the difficulty is met with that we do not know how much light the extreme top lets through to its back. We may assume, however, that the back receives about $\frac{1}{3}$ to $\frac{1}{9}$ of the light, received on the front. For what follows an exact knowledge of this amount is not necessary, however.

We shall now consider what curvatures must occur if we illuminate unilaterally with quantities of 0—800 m.c.s. With 800 m.c.s. the front and the back will both give a growth-retardation which will be strongest at the front. So here a positive curvature will occur (fig. 1*d*). When illuminating with 8000 m.c.s. growth will remain constant at the front, whereas the back shows a retardation (fig. 1*c* and *d*). Here we must expect a negative curvature. Somewhere between 800 and 8000 m.c.s. it is possible that the reactions at the front and at the back compensate each other so that no curvature is observed. ARISZ (1) studied the curvatures, caused by illumination of the top and found a positive bending at 800 m.c.s. and a negative one at 8000 m.c.s. At about 4000 m.c.s. he did not obtain a distinct positive or negative curvature, so we have here the transition between

the two ranges. It will be seen that the curvatures, derived from the growth-reactions agree with the actual phototropic curvatures.

Between 8000 and 800.000 m.c.s. the reaction is principally a growth-acceleration, at first increasing, and diminishing again with larger light-quantities. With a certain quantity the acceleration will be at its maximum, lying in the neighbourhood of 80.000 m.c.s.

At 80.000 m.c.s. the front as well as the back will give an acceleration, which will be strongest at the front, so that a negative curvature will result. At 800.000 m.c.s. the front will show some oscillations, but the total growth will there in the 2 hours after starting the illumination be about equal to that in the same time in the dark. The back will give an acceleration, however, and the result will be a positive bending. Between 80.000 and 800.000 m.c.s. there must again be a transitional range, where the growths of front and back compensate each other and no distinct curvature is observed. Also these curvatures have been observed by ARISZ. The positive curvature at 800.000 m.c.s. belongs to the range of the second positive response, contrary to the curvature at 800 m.c.s. which belongs to the range of the first positive one.

It thus appears that the first positive curvature is caused by unequal growth-retardations at the front and back, while the second positive curvature is caused by unequal growth-accelerations at the front and back.

Negative responses may be caused in various ways. The front may present a retardation or an indifferent reaction, as well as an acceleration, while the back may also present this reaction, provided that the growth-rate is there smaller than at the front. I shall not develop this further, as it may easily be derived from the communicated light-growth-reactions.

BOYSEN JENSEN (4), contrary to PAAL (10), assumes that a positive curvature is caused by growth-acceleration at the non-illuminated side. From what precedes it appears, however, that both cases occur, with the first positive curvatures the front and back are both retarded, with the second positive ones both are accelerated. *Hence the theories of PAAL and BOYSEN JENSEN are not at variance, but supplement each other. PAAL's conception holding good for the first positive curvature, BOYSEN JENSEN's for the second.* I was able fully to confirm PAAL's conception of the negative curvature.

Zone II and VII. As a contrast to the reactions of the top zone I shall now give a few reactions of lower zones, for which purpose I choose the reactions with 80.000 m.c.s. of zones II and VII (fig. 2). Both are nearly similar, showing a growth-minimum after 25 minutes, followed by a maximum after 43 minutes. Then a retardation sets in again, but soon normal growth is restored. This confirms WENT's conception (15) that the reaction of the base is fundamentally different from that of the top. With less than 8000 m.c.s. I could scarcely detect the reaction in either zone. Now it is remarkable that also zone 0 shows the short reaction only

with light-quantities larger than 8000 m.c.s. (fig. 1, *a*, *b*). So the reactions of zone 0 above 8000 m.c.s. are presumably composed of a short and a long reaction. The minimum of diagram 1*b* is shallower than that of diagrams 2*a* and *b*, obtained with the same light-quantities, probably a growth-acceleration of the long reaction sets in simultaneously with the retardation of the short reaction. Diagram *a* of fig. 1 may be interpreted as being composed of an indifferent top reaction and a distinct short reaction. The latter is now very distinct, as it is not counteracted by the top reaction.

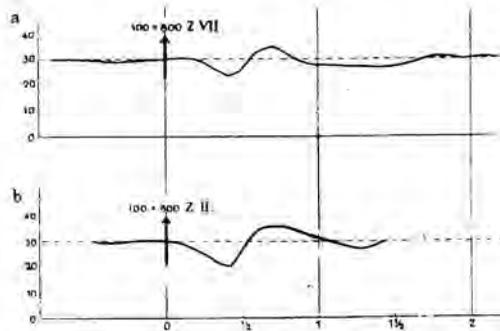


Fig. 2. Light-growth-reactions of basal zones.

a. of zone VII.

b. of zone II.

For the explanation see fig. 1.

We conclude that all the zones have the property of giving a short reaction with large light-quantities (above 8000 m.c.s.), while the property of giving a long reaction only belongs to the top.

Assuming that the long reaction is caused by the influence of light on the process of formation of growth-regulating substances, we can only say of the base reaction that it has nothing to do with this process. As the short reaction occurs as well in the top as in the base, i.e. as well in not-growing as in growing parts, it can stand in no relation to growth. It is much more probable that it is related to the turgor, for example. There is a great similarity between this reaction and the light-turgor-reaction, observed by BRAUNER (5) in the leaf-joints of *Phaseolus*.

This short reaction has a special bearing on the phototropic curvature. My experiments proved that, as the light-quantity increases, the minimum of this reaction lies deeper and the maximum higher. The coleoptile below the top receives at the back $\frac{1}{30}$ of the light, received by the front. With one-sided illumination with a quantity above 8000 m.c.s. the minimum will now lie deeper at the front than at the back, so that positive curving results; the following maximum will at the front lie higher than at the back, so that the curvature will be inverted. This is the explanation for the oscillations occurring with large light-quantities. When illuminating a zone of 2 mm the short reaction does not set in under 8000 m.c.s. But

since the amount of this reaction depends on the length of the illuminated surface, as was shown by WENT (15), totally illuminated coleoptiles will show these reactions already with smaller light-quantities. WENT found them already with 500 m.c.s. (temp. 25° C.). The oscillations observed by BURCKHARDT (6) are in agreement with this conception.

The oscillations have nothing to do with the permanent phototropic curvatures, no more than the short reaction is to any extent related to the light-growth-reaction proper. The short reaction is past after 1—1½ hour (see also WENT fig. 6), while the phototropic curvature still goes on then. The long reaction, on the other hand, has a longer duration and is transmitted to the base, it produces the real phototropic curvature.

So phototropic curvature with large light-quantities consists of two different components, namely of passing oscillations, caused by short reactions in all parts of the coleoptile, and of the phototropic curvature proper, caused by the long reaction, which occurs in the top only and from there proceeds downwards.

VON GUTTENBERG (8) also distinguishes two elements in the phototropic bending: the passing oscillation he explains as proto-phototropic curvatures, followed by the phototropic curvature proper, which occurs later.

SIERP (12) who has extensively studied the light-growth-reaction with *Avena*, also considers the first oscillations as less important and attaches great value to the later retardation or acceleration. He too is of opinion that only this latter is essential for the phototropic curving. Although SIERP exposed the total plants to illumination, the agreement between his and mine reactions is very good.

LITERATURE.

1. ARISZ, W. H., Untersuchungen über den Phototropismus. Rec. des Trav. bot. néerl. **12**, 1915.
2. BEYER, A., Untersuchungen über den Traumatotropismus der Pflanzen. Biolog. Zentralblatt **45**, 1925.
3. BLAAUW, A. H., Licht und Wachstum I. Zeitschrift f. Bot. **6**, 1914.
4. BOYSEN—JENSEN, P. und NIELS NIELSEN, Studien über die hormonalen Beziehungen zwischen Spitze und Basis der *Avena*-Coleoptile. Planta, Archiv für wissenschaftliche Botanik **1**, 1925.
5. BRAUNER, L., Phototropismus und Lichtturgorreaktion. Berichte d. Deutschen Bot. Ges. **42**, 1924.
6. BURCKHARDT, H., Untersuchungen über die Gültigkeit des Reizmengengesetzes für die Lichtkrümmung der Avenakoleoptile. Zeitschrift f. Botanik **18**, 1925.
7. DILLEWIJN, C. VAN, The connection between lightgrowthresponse and phototropical curvature of seedlings of *Avena sativa*. Proc. Royal Acad. of Sc. Amsterdam. Vol. **28**, 1925.
8. Von GUTTENBERG, H., Ueber das Verhalten von Hypokotylen bei schräger Beleuchtung. Beitr. zur Allg. Bot. **2**, 1923.
9. NIELSEN, N., Studies on the transmission of stimuli in the coleoptile of *Avena*. Dansk Botanisk Arkiv, **4**, 1924.
10. PAAL, A., Ueber phototropische Reizleitung. Jahrb. f. wiss. Bot. **58**, 1918.
11. PISEK, A., Untersuchungen über den Autotropismus der Haferkoleoptile etc. Jahrb. f. wiss. Bot. **65**, 1926.

12. SIERP, H., Untersuchungen über die durch Licht und Dunkelheit hervorgerufenen Wachstumsreaktionen. Zeitschrift f. Bot. **13**, 1921.

13. SIERP, H. und SEYBOLD A., Untersuchungen über die Lichtempfindlichkeit der Spitze und des Stumpfes in der Koleoptile von *Avena sativa*. Jahrb. f. wiss. Bot. **65**, 1926.

14. SÖDING H., Werden von der Spitze der Haferkoleoptile Wuchshormone gebildet? Ber. d. Deutsch. Bot. Ges. **41**, 1923.

15. WENT, F. W., Concerning the difference in sensibility of the tip and base of *Avena* to light. Proc. Roy. Ac. of Sc. Amsterdam. Vol. **29**. 1925.

Utrecht, May 1926,

Botanical Laboratory.

Botany. — *On growth-accelerating substances in the coleoptile of Avena sativa.* By F. W. WENT. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of June 26, 1926).

The following communication is a preliminary account of the results I obtained in an investigation on the formation, the character and the effect of the growth-accelerating substances, produced by the tip of a coleoptile of *Avena*.

PAAL (1919), on account of his experiments, has been the first to assume a regulation of the growth in the coleoptile of *Avena*. This regulation is seated in the tip and must be caused by the local formation of a substance which, partly by diffusion, moves in a basipetal direction. Arrived in the growing zone it there accelerates growth. This substance (or substances) I shall for the sake of brevity call growth-regulators. Of late years the growth-accelerating effect, exerted by the tip on the base, has been confirmed by direct growth-measurements (SÖDING 1923, NIELSEN 1924). PAAL also says: "phototropic perception is perhaps a growth-regulation, influenced by light." His idea is that the growth-regulators, which normally are continually being formed by the tip, are by illumination of the tip either prevented from being formed, or are photochemically destroyed, or perhaps have their motion impeded by illumination.

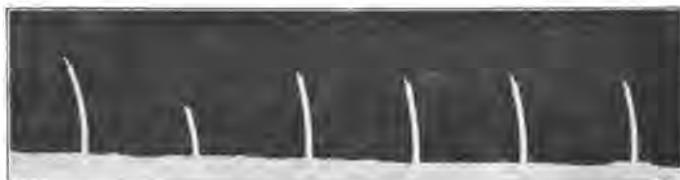
In a former publication (1925) I showed that the reaction of growth to light has two components, a tipresponse and a baseresponse. For both I tried to give an explanation. The one for the tipresponse essentially agrees with PAAL's conceptions. It occurs after the tip of a coleoptile has been illuminated and consists of a growth-retardation, at first small, then stronger, followed by a small acceleration. It is natural to think here of an influence of the light on the substances diffusing from the tip. In this way the reaction of growth to light is regarded as a special case of growth in the dark. If we wish to follow up the consequences of this conception the factors involved in growth must be known as well as possible. Among these we have in the first place the growth-regulators which will be dealt with in this paper. I succeeded namely in isolating them from coleoptile tips.

Miss SEUBERT (1925) tried to do this by mixing the expressed juice of coleoptiles, in which these regulators must be present, with agar. Little blocks of this jelly were placed on one side of decapitated coleoptiles and from the resulting curvature it might be concluded whether this expressed juice had an influence on growth. It was found, however, that the juice

expressed from triplets and that from coleoptile rings had the same effect and always caused a growth-retardation. Hence nothing could be concluded as to a growth-accelerating action of the tip. So I made a different attempt. PAAL's experiments had proved that diffusion of a phototropic stimulus also takes place through gelatine. I therefore placed a number of tips of coleoptiles of *Avena* closely together with their cut faces on a thin layer of gelatine. After about an hour I took off the tips again and placed the gelatine, cut into small blocks, on one side of decapitated plants. Already one hour later these plants began to show a negative curvature which after 3 hours had become strongly negative, sometimes amounting to an angle of 40° with the vertical. Gelatine on which no tips had been put, when placed on one side, gave no or a very feeble positive curvature, while gelatine on which coleoptile rings, cut off just below the tip, had stood, also produced no perceptible curvature. It was evident that I had obtained the more or less hypothetical growth-regulators. For the curvature produced by placing gelatine blocks on one side, indicates the difference in growth-rate on the two sides of the coleoptile. This difference is caused by the influence of the gelatine on the underlying half of the coleoptile. The more the growth of this half is accelerated the stronger the curvature will be. Already the first experiments showed that on this foundation a quantitative method could be worked out, in which the angle of curvature is the measure for the quantity of growth-regulators, diffusing from the triplets into the gelatine. For, if the unilaterally placed blocks were from the same portion of gelatine, also the curvatures of the stumps, expressed as angles, were pretty much the same. Taking an average of the angles with six plantlets, the mean error was as a rule not larger than 20 %, and sometimes much smaller. This is particularly striking, as phototropical curvatures often diverge widely. Two conclusions may be drawn: 1. the effect of a certain quantity of growth-regulators on different stumps is the same, and 2. the curvature of the stump is a means for quantitatively determining the amount of growth-regulators. So my experiments fall into two groups.

In the first place I am studying the formation and change of the growth-regulators by various external circumstances, the curvature of the stumps after one-sided application of the gelatine being used as the means of analysis, and secondly I am examining how the stump reacts on the presence of growth-regulators under different conditions. The method is briefly as follows: Coleoptiles are raised in a dark room at a constant temperature of 20° C. All manipulations are there made with red light. When the plants have attained the desired length (10—30 mm above the ground) the tips, having a length of 1 to 2 mm are cut off and placed for a few moments on moist filter-paper, in order to remove as well as possible the contents of the injured cells. They are then put close to each other on a thin slice of 10 % gelatine. It is important always to place the same number of tips on equal surfaces, the tips should therefore all be cut off at the same length. They are removed from the gelatine after about an hour.

The gelatine obtained in this way is then cut into small blocks of about equal size, which by means of melted gelatine are stuck on one side of coleoptile stumps. Decapitation is carried out in the manner indicated by STARK and DRECHSEL (1922), the length of the cut-off portion being about 5 mm. The primary leaf is partly pulled out, so that it will not lift the gelatine block from the stump when it grows larger. After this treatment the plantlets are left to themselves for about half an hour, because within this time the curvatures resulting from the operation manifest themselves; only those plants are used for the experiments that are still perfectly straight then. Already one hour after the gelatine with the growth-regulators has been put on, a distinct curvature is observed in the upper part of the stump. This is gradually transmitted downwards, so that after 3 hours a zone of ± 15 mm is curved. After 3 hours the plants are photographed by casting their shadow on sensitive paper. It is very easy to measure the curvature on the paper. A line is drawn on the photo parallel to the uncurved basal portion of the plant and a tangent to the extreme tip. The angle between these two lines is the total curvature. The figure is a reproduction of such a shadow-image. The plants have bent themselves from left to right over angles of 20, 19, 18, 17, 18 and 17°.



Small blocks of gelatine containing growth-regulators, were placed on the right side of stumps of coleoptiles of *Avena*. The resulting negative curvature was photographed after 3 hours.

This is an average of 18° with a mean error of 0.3°. In the tables will be found the average angle, with the mean error, of groups of 5 to 6 plants, 3 hours after the beginning of the experiment. These figures cannot claim absolute value, since the circumstances under which the experiments were made were rather unfavourable. But they may prove the value of the method and they also contain some new results. A number of control experiments may be mentioned only.

If pure gelatine is placed on one side negative curvatures never occur; 50% of the plantlets remain straight and 50% bend slightly in a positive direction. The same result is obtained when coleoptile rings have previously stood on the gelatine, and it makes practically no difference whether these rings are cut out close under the tip or from lower parts of the coleoptile. Hence we infer that neither from the sectional surface nor from the ordinary coleoptile cells substances, having a marked influence on growth, diffuse into the gelatine.

I had also to make out whether the tips of longer plants excrete less or more growth-regulators than those of shorter plants. My experiments have not proved as yet that there is a marked difference between them, so it is sufficient in practice if for a series of experiments the length of the plants does not diverge too much.

Another question was whether the length of the stump has any influence on the amount of the reaction. This appeared not to be the case; if the stumps were 10—15 mm long the curvature was $18^{\circ}.5 \pm 2^{\circ}.7$; at 20—30 mm length the curvature was $20^{\circ} \pm 2^{\circ}.3$. As the length of the stumps I employed always lay within these limits, we may state that the stumps are practically equally sensitive, independent of their length.

With a view to the regeneration of a new physiological tip, which was studied by DOLK (1926), I never decapitated my plantlets for analysis much longer than half an hour before using them. If they are decapitated 12 hours before, and if then the regenerated physiological tip is cut off before the experiments, the stumps so obtained react much more feebly than freshly operated stumps. From these preliminary experiments I now proceed to a discussion of some results.

In the first place a few words must be said on the constancy of the growth-regulators. If gelatine into which they have diffused is left to dry in the air (with a constant moistness of 85 %) they remain active, at least partly, also when kept for a longer period. After 2 days' drying of the gelatine they gave a curvature of $18^{\circ}.2 \pm 0^{\circ}.2$; after having been kept dry for 11 days the curvature was still $10^{\circ} \pm 1^{\circ}.2$, and even after 43 days it still produced negative curvatures. When diluting the gelatine containing the growth-regulators with an equal quantity of pure gelatine, its activity diminishes proportionally to the dilution. Undiluted it gave a curvature of $22^{\circ}.5 \pm 2^{\circ}.0$ (in another case $19^{\circ}.5 \pm 7^{\circ}.9$); diluted the curvature was $12^{\circ} \pm 1^{\circ}.6$ ($8^{\circ} \pm 1^{\circ}.6$). This also points to their individuality as substances.

The experiments of FITTING (1907), who found that no stimuli were transmitted through a coleoptile disc that had been heated to 43° C., induced me to heat the gelatine to 40° , 60° and 90° . These experiments have a qualitative value only, as I did not make sufficient control experiments. Gelatine that had been heated for 1 minute to 90° was still quite able to produce curvatures, even as strongly as gelatine heated for 1 minute to 40° . Also when heated to 60° very fine curvatures were produced. Hence in FITTING's experiments only the transport of the stimulating substances, in this case the growth-regulators, can have been stopped, without the substances themselves having been destroyed. And this transport must for the greater part be ascribed to the living cells, as diffusion is not nearly sufficient to explain the rate of transport. BRAUNER (1922) therefore regards the protoplasmic streaming observed by him as a chief factor of transport. A theory of PAAL to explain the transport-rate I consider less probable. Transport of the growth-regulators I understand as diffusion

from protoplast to protoplast through the cell-wall, and for the rest as a mechanical distribution in the cell by protoplasmic streaming.

If by heating above 41° C. the protoplasmic streaming has been irreversibly checked, no transport of growth-regulators can take place in perceptible quantities.

Illumination of gelatine containing growth-regulators does not check their activity, but I have no quantitatively certain results on this point as yet.

I will deal now with the influence of external circumstances on the formation of the growth-regulators in the tip. It has already been stated that the length of the plantlets has scarcely any influence on the quantity produced.

One of the principal questions that may be answered by the present method is: what influence has light on the formation of growth-regulators? The result of the experiments on this point might have been foreseen after VAN DILLEWIJN's latest publication (1926). The growth-retardation observed by him after illumination from three sides with 800 metre-candle-seconds and the growth-acceleration stated after 80.000 m.c.s. could only be caused by a smaller and by a larger formation of growth-regulators. RAMAER (1926) could prove this growth-retardation and acceleration in a different way. And finally SIERP (1921) came to the same conclusion from his experiments. Because the growth-regulators formed during the illumination want some time before they have been so far conveyed downwards that an increase or diminution can make itself felt in the gelatine, I waited 30—40 minutes after illumination before cutting the tips and then kept them 45 minutes on the gelatine. The results of a few series of experiments are given in table I. Only one series gave a completely discordant result, which I can only explain by assuming that the numbers of the experiments were interchanged by accident. I always illuminated vertically from above.

From these data it appears very clearly that during a time of 35 to 80 minutes less growth-regulators diffuse from the tips (having a length of

TABLE I (for explanation see text).

Quantity of light	Curvature of the stumps		
	Series I	Series II	Series III
0 M. C. S.	11.5 ± 1.9	11 ± 1.8	8 ± 0.7
1000 " " "	6 ± 1.7	5.5 ± 0.5	4.5 ± 1.4
10000 " " "			11 ± 1.3
100000 " " "	15.5 ± 1.5	16 ± 1.1	10 ± 0.5
1000000 " " "			14 ± 0.3

1—1.5 mm) into the gelatine after applying 1000 m.c.s., but more after 100.000 m.c.s.

It was now also possible to imitate phototropic curves. To do this I first placed, as in all former experiments, on one side of the stump gelatine, treated with tips to which an illumination of e.g. 100.000 m.c.s. had been applied. Then on the other side a gelatine block was placed, on which tips had stood that had been illuminated with ten times less light. In this way a gelatine system was placed on the stump which, according to BLAAUW's theory, as nearly as possible approached the unilaterally illuminated tip. The plantlets indeed bent themselves in perfect accordance with the figures obtained by ARISZ (1915). With a difference of 1000 versus 0 m.c.s. a positive curvature (in 7 out of 9 plants, 2 remained straight) occurred, reckoned towards the 1000 m.c.s. With 10.000 versus 1000 m.c.s. the curvature was negative. With 100.000 versus 10.000 m.c.s. I found a positive curvature again, i.e. towards the 100.000 m.c.s.

This method is an improvement on that of BOYSEN JENSEN and NIELSEN (1925), who placed a tip on both sides of a stump and illuminated only one of them. Another advantage of the present method is that while the gelatine stands on the stumps, no fresh growth-regulators are formed, which might spoil the result. RAMAER already showed in 1926 that JENSEN and NIELSEN's conclusions were wrong, especially because they did not take into account the quantity of light applied. I provided a few plants in the above described manner with two small pieces of gelatine, on which tips had stood, that had been illuminated with the same quantities of light which RAMAER had used. The result was that with 0 versus 1000 m.c.s. a curvature towards 1000 m.c.s. was observed, while with 0 versus 100.000 m.c.s. the curvature was reversed and turned away from the 100.000 m.c.s. This is in complete agreement with RAMAER's results.

More remarkable is perhaps the result I obtained concerning the influence of one-sided or many-sided geotropical stimulation of the tip of a coleoptile on the excretion of growth-regulators. The method was essentially the same as that for studying the influence of light. In this case, however, the plants were placed horizontally during 40 minutes or were rotated horizontally for 25—50 minutes on a clinostat, after which the tips were cut off and remained standing on the gelatine for 45 minutes. The results are given in table II.

TABLE II.

Stimulated tips		Control: unstimulated tips
40 min. horizontal	13 ± 3.3	11.5 ± 1.9
25 min. .. rotated	8.5 ± 0.5	10 ± 1.0
50 min.	14.5 ± 4.0	15 ± 0.3

From this table it is seen that the formation of growth-regulators is not stopped by geotropic stimulation. Hence we should not expect a geo-

growth-response, which must be based on a change in the quantity of growth-regulators. What has until now been published on this point is very contradictory (ZOLLIKOFER 1922, KONINGSBERGER 1922) so that no certain conclusions can be drawn from the geo-growth-response. I shall presently return briefly to this question.

We have now seen how the formation of growth-regulators depends on certain external factors and how they react themselves on light, heat, etc. I shall finally give some observations on some factors that may influence the transport of growth-regulators in the stump; I mean light and gravity.

I started from the fact that the typical action of the tip, namely the formation of growth-regulators which are conveyed downwards, can be imitated by a little gelatine block in which these substances are accumulated. If such a block is placed on the whole stump, growth-regulators will move downwards everywhere and will accelerate the growth of the whole coleoptile, just as happens in an uninjured plant. In this way we have at our disposal an *Avena* coleoptile which is entirely like a normal plant, but in which no perception of stimuli in the tip is possible any longer, as the tip has been replaced by a physical system. This affords particular advantages for investigating the phototropic and geotropic sensitiveness of the base of a coleoptile. The whole plant can always be stimulated without any perception in the tip.

As regards phototropy the base had already been studied long ago (by ROTHERT 1893) with unilateral illumination only; a curvature is observed although there can be no change in the quantity of growth-regulators. Also with a gelatine tip I obtained positive, though feeble, curvature towards unilateral light (continuous application of 32 or 50 m.c.). Control plants which had also been decapitated, but on which no gelatine with growth-regulators had been placed, showed no curvature at all. It thus appears that there must exist a difference in concentration if curvature is to occur. I supposed a diminution in the rate of conveyance of the growth-regulators at the strongly lighted proximal side of the coleoptile. I could confirm this supposition by placing gelatine blocks with growth-regulators unilaterally on stumps. One half of the operated plants were continuously illuminated bilaterally with about 20 m.c., the other half remained in the dark. Table III gives the curves obtained.

TABLE III.

	Series I	Series II	
	After 3 hours	After 1 $\frac{1}{4}$ hour	After 3 hours
plants in darkness	17 \pm 6.1	15 \pm 0.2	21 \pm 5.2
.. under cont. illum.	15 \pm 4.6	8 \pm 0.4	24 \pm 2.0

After 1 hour the plants of series I also showed a distinct difference in curvature, which, however, had disappeared after 3 hours. So it appears that during the first hour the rate of transport is greatly diminished by the illumination, after this it is equalised, so that at last it makes no difference whether the plants have been illuminated or not. The influence of light on a coleoptile of *Avena* therefore appears in this investigation as in the former to be twofold: in the first place it has an effect on the formation of growth-regulators in the tip, and secondly it temporarily diminishes the transport-rate of the growth-regulators.

The first phenomenon gives rise to the tip-response, the second probably has something to do with the base-response. But this requires further investigation. BRAUNER (1922) seeks in the same direction an explanation for curvatures of the base. He says, however, that light increases the permeability and so accelerates the transport of growth-retarding substances. Although his hypothesis will explain the positive curvature of the base, my experiments prove it to be untenable.

These facts also enable us to explain why with stronger intensities a positive bending precedes the negative one. From the outset more growth-regulators are formed at the front than at the back. During the first hour, however, the transport-rate is smaller at the front than at the back, so that a positive bending must occur first. Later, when the transport has become normal again, more growth-regulators will be present at the front than at the back and the plant then bends negatively. ARISZ (1915) also obtained purely negative curvatures, not preceded by a positive one. He then worked with illumination of the tip only, so that the rate of diffusion at the base was not altered and the acceleration could at once manifest itself. Viewed in this light the results obtained by Miss SEUBERT with unilateral illumination of stumps on which agar mixed with saliva had been stuck, for which she wanted several auxiliary hypotheses, are easily explained. We have again a case here in which a growth-accelerating substance is from the agar conveyed downwards in the coleoptile. By illumination the transport at the exposed side is impeded, so that a positive bending must result.

Lastly I have to speak of the geotropic stimulation of the base.

GUTTENBERG (1911) was already able to prove by PICCARD's method that also the base of a coleoptile can perceive centrifugal forces. BRAUNER (1923), SEUBERT (1925) and CHOLODNY (1924) tried more or less successfully to stimulate the base only, the first-named by replacing the cut-off tip on the stump after geotropic stimulation of the base, SEUBERT by placing agar blocks with saliva on the base, the last-named by sticking the tip on the base at an angle of 90° . The two former could show that the base really is geotropically sensitive; CHOLODNY's method presented difficulties that could not be surmounted.

I cut the tips of plants at various lengths, some at 1 mm, some at 5 mm. Instead of the cut tips a little block, containing growth-regulators, was

again put on the whole surface. The plants were then placed horizontally. After $1\frac{1}{2}$ hours all the plants with gelatine were geotropically curved, those that had been curtailed 5 mm only feebly, the others very strongly. The bending of these latter plants was about as strong as with uninjured plants. We may infer firstly that geotropic curvature depends on a concentration gradient of growth-regulators, and secondly that it is perceived by all zones of the coleoptile (perhaps with the exception of the extreme tip), while the greatest sensitiveness must lie a little below the tip.

On account of the above-mentioned facts I think we are justified in concluding that geotropic perception is caused by a polar alteration in the coleoptile cells. Instead of an evenly distributed rectilinear transmission of the growth-regulators from the tip to the base, the transport in each cell undergoes a polar alteration and instead of moving rectilinearly the growth-regulators are more strongly conveyed towards that side which under geotropic stimulation was turned downwards. It is not necessary to assume a direct transformation of energy by gravity, and each cell, either at the under or at the upper side, reacts in the same way. I hope to be able later to extend these ideas by further experiments.

LITERATURE CITED.

- ARISZ, W. H. 1915. Untersuchungen über Phototropismus. *Rec. d. Trav. bot. néerl.* **12**.
 BOYSEN JENSEN und NIELSEN 1925. Studien über die hormonalen Beziehungen zwischen Spitze und Basis der Avenacoleoptile. *Planta, Arch. f. wiss. Bot.* **1**.
 BRAUNER, L. 1922. Lichtkrümmung und Lichtwachstumsreaktion. *Zeitschr. f. Bot.* **14**.
 BRAUNER, L. 1923. Ueber den Einfluss der Koleoptilspitze auf die geotropische Reaktion der Avenakeimlinge. *Ber. d. d. bot. Ges.* **41**.
 CHOLODNY, N. 1924. Ueber die hormonale Wirkung der Organspitze bei der geotropischen Krümmung. *Ber. d. d. bot. Ges.* **42**.
 DILLEWYN, C. VAN, 1926. De lichtgroei-reacties van verschillende zones bij het coleoptiel van Avena. *Verslag der Kon. Ak. v. Wetensch. Amsterdam* **35**.
 DOLK, H. E. 1926. Concerning the sensibility of decapitated coleoptiles of *Avena sativa* for light and gravitation. *Proc. Royal Acad. Amsterdam, Vol.* **29**.
 FITTING, H. 1907. Die Leitung tropistischer Reize in parallelotropen Pflanzenteilen. *Jahrb. f. wiss. Bot.* **44**.
 GUTTENBERG, H. VON, 1911. Ueber die Verteilung der geotropischen Empfindlichkeit in der Coleoptile der Gramineen. *Jahrb. f. wiss. Bot.* **50**.
 KONINGSBERGER, V. J. 1922. Tropismus und Wachstum. *Rec. d. Trav. bot. néerl.* **19**.
 NIELSEN, N. 1924. Studies on the transmission of stimuli in the coleoptile of *Avena*. *Dansk bot. Arkiv.* **4**.
 PAAL, A. 1919. Ueber phototropische Reizleitung. *Jahrb. f. wiss. Bot.* **58**.
 RAMAER, H. 1926. Phototropical curvatures of seedlings of *Avena* which appear when reaction of the distal side is excluded. *Proc. Royal Acad. Amsterdam, Vol.* **29**.
 ROTHERT, W. 1893. Ueber Heliotropismus. *Cohn's Beitr. z. Biol. d. Pflanzen* **7**.
 SEUBERT, E. 1925. Ueber Wachstumsregulatoren in der Coleoptile von *Avena*. *Zeitschr. f. Bot.* **17**.
 SIERP, H. 1921. Untersuchungen über die durch Licht und Dunkelheit hervorgerufenen Wachstumsreaktionen. *Zeitschr. f. Bot.* **13**.
 SÖDING, H. 1923. Werden von der Spitze der Haferkoleoptile Wuchshormone gebildet? *Ber. d. d. bot. Ges.* **41**.

STARK und DREHSEL, 1922. Phototropische Reizleitungsvorgänge bei Unterbrechung des organischen Zusammenhangs. *Jahrb. f. wiss. Bot.* 50.

WENT, F. W. 1925. Concerning the difference in sensibility of the tip and base of *Avena* to light. *Proc. Royal Acad. Amsterdam*, Vol. 29.

ZOLLIKOFER, C. 1922. Ueber den Einfluss des Schwereizes auf das Wachstum der Koleoptile von *Avena sativa*. *Rec. d. Trav. bot. néerl.* 18.

Utrecht, June 1926.

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Physics. — *Note on regularities in the spectra of Fluorine and Chlorine.* By T. L. DE BRUIN. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of June 26, 1926).

Little progress has been made in the detection of regularity in the spectra of the halogens (Fl. Cl. Br. J). Data concerning a revision of the observations would seem to be necessary for the study of the structure of these spectra.

Fluorine. Measurements of the Fluorine spectrum have been made by PORLEZZA¹⁾ and by GALE and MONK²⁾. The spectrum as measured by them consisted of a group of some 50 lines in the red (probably F. I) and a group of about 30 lines in the violet (probably F. II).

I have investigated these lines on constant differences and have found:

Fl. I 145.5, 160.1 and 274.6

Fl. II narrow triplets with $\Delta\nu = \pm 12$ and 20.

There are also in the F. I spectrum groups of lines with the constant differences 160.1 and 274.6. From the 3 to 5 interval ratio of these differences we have deduced a quartetterm (innerquantumnumbers 1, 2, 3; ⁴P). Recently CARRAGAN³⁾ has made a study of the ZEEMANeffects for some of the lines of the red group. These ZEEMANeffects showed the reality of this ⁴P-term. CARRAGAN has deduced a ⁴D-term, forming with the ⁴P-term the multiplet λ 6708—6909.

This note records the detection of different quartet- and doubletterms in the spectrum of F. I. The structure of the spectrum resembles the spectra of N. I.⁴⁾ and O. II⁵⁾. In the light of the theory of complex spectra developed by HUND⁶⁾ the deepest lying term in the spectra of the neutral halogen atoms would be a ²P-term. The following terms would be expected too: ⁴P, ²P; ⁴D, ⁴P, ⁴S, ²D, ²P, ²S; ⁴F, ⁴D, ⁴P, ²F, ²P, ²D. The components of any one of the terms would be probably inverted i. e. the component with the largest *j* value will be the "deepest".

The detected termcombinations are recorded in the following tables.

1) PORLEZZA: *Gaz. Chim. Ital.* **42**, 42, 1912.

2) GALE and MONK: *Astr. Journ.* **59**, 125, 1924.

3) H. CARRAGAN: *Astr. Journ.* **63**, 145, 1926.

4) C. C. KIESS: *Journal Opt. Soc. America.* Vol. **11**, 1, 1925.

5) A. FOWLER: *Proc. R. S. London.* Vol. **110**, 476, 1926.

6) F. HUND: *Zeitschr. f. Phys.* **33**, 345, 1925.

1.

	4P_1	160.0	4P_2	274.5	4P_3
4D_4	—				40. 6856.01 14581.7
176.6					
4D_3	—		30. 6902.46 14483.7	274.6	20. 6773.96 14758.3
144.4					
4D_2	20. 6909.79 14468.2	-159.9	18. 6834.25 14628.1	274.5	1. 6708.36 14902.6
83.4					
4D_1	18. 6870.21 14551.6	160.0	3. 6795.47 14711.6		—

2.

	23. 6413.61		28. 6348.46		30. 6239.63
4S_2	15587.5	160.0	15747.5	274.6	16022.1

3.

${}^4P'_3$	—		5. 7552.20 13237.5	274.2	20. 7398.96 13511.7
122.7					
${}^4P'_2$	5. 7573.32 13200.6	159.4	5. 7482.95 13360.0	274.6	18. 7332.25 13634.6
102.1					
${}^4P'_1$	(13302.7)		12. 7426.2 13462.1		—

4.

	6.		5.		1.
	6762.91		6690.48		6569.66
	14782.1	160.3	14942.4	274.9	15217.3
	4.		3.		20.
	6580.36		6650.40		6773.96 ¹⁾
	15192.5	160.0	15032.5	274.2	14758.3
	3.		2.		4.
	6210.85		6149.77		6047.55
	16096.4	159.9	16256.3	274.7	16531.0

5.

	² P ₂		² P ₁		
² P' ₁	15.		5.		
	7202.34		7128.36		
	13880.6	144.0	14024.6		
325.6 ²⁾					
² P' ₂	50.		4.		
	7037.48		6966.33		
	14205.7	145.1	14350.8		

6.

	18.		3.		
	6834.25 ¹⁾		6766.98		
	14628.1	145.4	14773.5		
	3.		1.		
	6416.30		6356.87		
	15581.0	145.6	15726.6		
	0.		1.		
	6049.66		5996.76		
	16525.2	145.8	16671.0		
	3.		00.		
	5707.23		5659.02		
	17516.8	149.2	17666.0		

¹⁾ Also in multiplet 1.

²⁾ We will remark that the theoretical value for the doublet

$$\Delta\nu_L = 0.365(9-3.5)^4 = 333.9.$$

ZEEMANeffects.

The publication of CARRAGAN deals with only the ZEEMANtypes, not the observations in decimal form. The results here recorded are therefore somewhat uncertain.

Multiplets 1 and 2.

The lines of these groups have ZEEMANeffects consisted with the suggested arrangement.

Multiplet 3.

20.	λ 7398.96	calc. (0)	1.60				
		obs. (0)	1.66				
5.	λ 7552.20	calc. (0.07)	(0.20)	1.40	1.53	1.67	1.80
		obs. (0)			1.50		
5.	λ 7573.32	calc. (0.47)	1.27	2.20			
		obs. (0.50)	1.35				
18.	λ 7332.25	calc. (0.07)	(0.20)	1.40	1.53	1.67	1.80
		obs. (—	0.25)	1.35	—		
12.	λ 7426.2	calc. (0.47)	1.27	2.20			
		obs. (0.50)	1.30	2.30			

Multiplet 4.

These lines form a possible term combination with 4P .

Multiplet 5.

The observed ZEEMANeffect of the lines $\lambda\lambda$ 7202.34, 7037.48 and 6966.33 agrees with calculation. For the line λ 7128.36 the calculation gives $\frac{(0)2}{3}$; the observed patterns are, however, $\frac{(0)3}{4}$.

Multiplet 6.

These lines form a possible termcombination with 2P . ZEEMANeffect not observed.

It should be remarked that the interval rule is confirmed in 4P and 4D . For the $^4P'$ term, however, there is a deviation from this rule ¹⁾.

Only the strong line (15) 7311.40 with the ZEEMANeffect $\frac{(6)21.33}{20}$ could not be tabulated. The line has two close satellites $\lambda\lambda$ 7309.28 and 7314.52. The ZEEMANeffect suggests a 4D_3 2D_2 termcombination.

calc. (0.29)	0.51	(0.86)	1.09	1.66	2.23
obs. (0.30)			1.05	1.65	

Chlorine. Measurements of the spectra of Chlorine were made by EDER and VALENTA ²⁾, JEVONS ³⁾ and ANGERER ⁴⁾. NELTHORPE ⁵⁾ has

¹⁾ See also the spectra NI and O II.

²⁾ EDER und VALENTA: Wien. Denkschr. 68, 1, 1899.

³⁾ JEVONS: Proc. R. S. London. 103, 1923.

⁴⁾ ANGERER: Zeitschr. f. wiss. Photogr. 22, 1924.

⁵⁾ NELTHORPE: Astr. Journ. 41, 16, 1915.

found that many of the vacuum tube lines of Chlorine as tabulated by EDER and VALENTA, did not appear in "Grundspektrum" tubes. These lines are also due to Cl. I. It is probable that the observed spectrum contains lines of Cl. II and Cl. III too. (L. and E. BLOCH ¹).

The red lines (EDER and VALENTA) examined for constant differences show: 40.5; 67.2 (107.7) and 530.5. The differences 40.5 and 67.2 are already found by PAULSON ². PASCHEN ³) has identified these differences as a ⁵P term of Cl. II.

In connection with the work of NELTHORPE and L. and E. BLOCH, the difference 530.5 is probable a frequency difference in the Cl. I. spectrum. The following table contains a number of these doublets.

Int.	$\lambda_{(Row)}$	$\nu_{vac.}$	Int.	$\lambda_{(Row)}$	$\nu_{vac.}$	$\Delta\nu$
3	4740.51	21088.9	3	4624.23	21619.1	530.2
1/2	4649.1	21504.0	1/2	4537.0	22034.8	530.8
4	4601.19	21727.4	3	4491.25	22259.2	531.8
1/2	4504.50	22193.8	2	4399.37	22724.1	530.3
4	4475.50	22337.6	2	4371.72	22867.9	530.3
2	4446.35	22484.1	10	4343.82	23014.8	530.7
4	4438.72	22522.7	5	4336.38	23054.2	531.5
8	4380.08	22824.2	3	4280.61	23354.6	530.4
6	4369.68	22878.6	3	4270.72	23408.7	530.1
8	4363.46	22911.2	3	4264.75	23441.5	530.3
1	4333.12	23071.6	4	4235.68	23602.3	530.7
6	4323.53	23122.8	7	4226.58	23653.1	530.3
4	4304.21	23226.6	4	4208.16	23756.7	530.1
7	4226.58	23653.1	3	4133.83	24183.8	530.7

It should be remarked that some of the differences found in the F. I and Cl. I spectrum, have approximately the ratio of the squares of the atomnumbers i. e. 3.57.

		Ratio.
Fl. I 145.5	Cl. I 530.5	3.6
Fl. II 12; 20	Cl. II 40.5; 67.2	3.3

Laboratorium „Physica“, Amsterdam.

¹) L. et E. BLOCH: C. R. **180**, 1740, 1925. SUGIURA: Journ. de Phys. **6**, 10, 323, 1925.

²) PAULSON: Astr. Journ. **40**, 289, 1914.

³) PASCHEN: Ann. d. Phys. **71**, 559, 1923.

Microbiology.— *Experiments on the differential characters of Leptospira-strains with Introductory Remarks on the Process of Agglutination and Lysis.* By Prof. W. SCHÜFFNER and ACHMED MOCHTAR.

(Communicated at the meeting of October 30, 1926).

In order to ascertain whether a *Leptospira*-strain consists of a single species or of different races and species which are morphologically alike, the methods generally used in bacteriology cannot be adopted. *Leptospirae* do not grow on solid media, or if they should do so, they are difficult of demonstration as this would have to be done with the "dark field" illumination. Likewise the one-cell culture, which was given a trial in our Institute, was considerably handicapped and failed completely. Still, such experiments (UHLENHUTH¹) pointed to it in his latest paper) are essential to our knowledge of the *Leptospirae*. Therefore, we have made an attempt to approach the solution of the problem by another method. We were prompted to do so by our serological experiments with a rather considerable number of *Leptospira*-strains. Beforehand we want to state some particulars regarding these experiments.

When determining the agglutination and the lysis with immune sera, no matter whether they were derived from men or from animals, we invariably obtained in our experiments, and with our working-method, the results tabulated below.

In taking our readings of the reaction we also took note of the smallest details. Contrary to the simple GRUBER-WIDAL reaction, this reaction is a complex one, i.e. it determines at the same time the capacity of a serum for agglutination as well as for lysis. Both capacities have been registered, while the differences in degrees are indicated by from 1 to 4 crosses.

To check our results we have registered in the 3rd column the number of surviving unaffected *Leptospirae*. To this we attach great importance, it being easiest to ascertain objectively whether the number of parasites in the lytic serum differs from that found in the control test. In accordance with this the control presents 4 crosses, just as the highest dilution of the lytic serum, which did not affect the parasites any more. Intermediate between them are the different degrees and modes of the reaction.

The dilutions are made with drops from one and the same pipette.

Dishes to mix water colours such as are procured by GÜNTHER and WAGNER, are very serviceable for this purpose (Text-figure); they are sufficiently illustrated by the accompanying photo. With these dishes complete sterilization can be obtained. To prevent desiccation they are covered with a glassbell.

To examine them with dark field illumination a loop-full is taken from the various dilutions in their regular order and spread on a slide, care being taken that they be as much as possible of the same size. Low power magnifications are used (e.g.

¹) Klin. Woch., 1926, N^o. 25, S. 1113.

ZEISS obj. 20; ocular 10) with which the parasites can be easily distinguished and their movements can be completely observed. The advantage of a low power magnification



consists in that we can do without a cover slip, and yet the picture is sharply outlined. The next drop is smeared close beside the first, and placed in the cone of light by shifting the slide. In this way we can watch quite a succession of from 12 to 16 dilutions on one and the same slide which saves time, and economizes material.

The two reactions, the agglutination and the lysis, have shifted considerably with respect to each other. In the first dilutions only the agglutination accomplishes itself. With 1 : 10 the reaction is scarcely to be noted ; with 1 : 25 it is more distinctly observable, but every individual clump still consists of only few entangled *Leptospirae*. Only in the next-following dilutions do the clumps become larger, as if they form nets with beautiful meshes, in which entangled *Leptospirae* are still distinguishable and betray their being still alive by their movements. Such is the picture generally presented to us after 16—20 hours (the time taken up by the readings.) When the culture stands longer, the nets begin to sink, but so do also the smaller clumps. The number and the size of the clumps is, so far as we can judge, proportional to the original quantity of *Leptospirae*, that swim about by themselves ; lytic changes in these first dilutions are not observable at all. However, there are always some isolated specimens left, that have remained intact macroscopically and keep floating contrary to the clumped masses.

In the higher dilutions agglutination is superseded by lysis ; the boundary is sometimes sharply marked, sometimes it is indistinct. The *Leptospirae* are disintegrated — we will not enter into details here — and may ultimately be completely dissolved. At the culminating point of the process (e.g. dilution 1 : 5000 after from 16 to 20 hours) the following two pictures will

be presented to us : Either all has disappeared into the serum-, and the *Leptospira*-mixtures, so that the field of vision presents a blank with the exception of some apparently quite intact specimens, or the granulated *Leptospirae* have united into multiform masses of a dazzling white in the dark field, with rounding contour-lines, which are easy to distinguish from the nets of the agglutinated parasites by their invariable granular structure. At the periphery of the otherwise motionless mass may still be suspended a single living spirochaeta, whose slackened movements betray its being no longer quite normal.

In the formation of these clews agglutination comes first. But this condition is not permanent as in the first dilutions ; it is followed by the action of the lysins which metamorphose the agglutinated nets into the amorphous masses above-described. These clews also precipitate. A quicker action of the lysins, which may occur with still higher dilutions at the height of the reaction, does not form clews so that the empty field of vision presents itself.

When the dilution draws near to the titer-limit, this will be signalled distinctly by the increase of the isolated, swimming *Leptospirae*. The conclusion of the reaction almost coincides with that dilution with which the picture and the concentration of the control is attained again.

Not infrequently a 3rd form of accumulations occurs in this stage, that we are inclined to term "foci" 1).

The typical forms look like stars, having a solid nucleus of a snowy, lustrous appearance, and being provided with an aureole of *Leptospirae*, whose free extremities are briskly swinging about in their well-known way. On closer inspection also the nucleus is seen to move, which is a sign of its being streaked with living spirochaete. They can easily be discriminated from the disintegrated, dead masses by their shape as well as by their multifarious mobility ; so they can also readily be distinguished from the finely knitted nets of the agglutination. These "foci" are not seldom seen to arise spontaneously in the cultures 2), for which we could not find an explanation. So we have to ask whether, after the conclusion of the reaction proper, we still may conceive their occurrence in the dilutions, to be the last effect of the immune serum. In cases where a coincident control did not bear this out, it is difficult to abandon the idea of even a very remote influence.

The process here expounded, which was of common occurrence with potent human sera or animal sera, at a reading after 16—20 hours (after 2 hours the reaction is generally not quite finished) diverges in more than one respect from what has hitherto become known in the literature. (INADA

1) This appellation seems to us more appropriate than agglomeration, or conglomeration which processes are hardly distinguishable from agglutination.

2) It is above all our water-strains (*L. pseudoicterogenes* Lhlehuth-Zuelzer) that showed this tendency.

& IDO, UHLENHUTH & FROMME, HÄNDEL, UNGERMANN & JÄNISCH, DIETRICH, JACOBSTHAL, ZUELZER, MARTIN & PETTIT, BÄRMANN, and others.) Some fundamental points have already been discussed by WOLFF ¹⁾ and RADEN SOESILO ²⁾).

To begin with, it is striking that the lytic effect on the *Leptospirae* occurs without complement, a fact that has not always been sufficiently laid stress on. Strictly speaking this is not quite true, for we are working with active serum. But the quantum of complement, this available for the reaction is in the dilution where the lysis begins ($\pm 1 : 250$) already negligibly small.

To avoid even this negligible source of error, however, we also made control-reactions with sera which had been inactivated at 56° for half an hour; these reactions yielded completely the same results. Consequently the lysis and bactericidal action occurring with *Leptospirae*, cannot be a complex process in the sense of the reaction common with bacteria, but they must originate in another way, as they differ fundamentally from the latter.

HÄNDEL, UNGERMANN and JÄNISCH believed to have observed after inactivation of the serum not a complete inhibition, but still a weakening and a slackening of the lytic reaction. Meanwhile the difference was by no means significant, and does not take away from our observation, which moreover, was made exclusively with a low dilution of serum, while our experiments concern only the high, the thousandfold dilutions.

Furthermore it appears that the lytic power of the immune sera in the first dilutions is completely inhibited. The *Leptospirae* remain alive in the concentrated serum-solutions, only in the higher solutions do lysis and bactericidal actions take place. This inhibition is in its action in vitro equivalent to the phenomenon of NEISSER—WECHSBERG. Whether it is also of the same nature, we will leave undecided. Neither can we give an interpretation of the remarkable phenomenon. In the case of *Leptospirae* a deviation of the "Complementablenkung", cannot be applied as the lysis proceeds without complement. This method, ingenious though it may be, has proved untenable for bacteria. The only thing we know is that in vitro a high dilution is required energy of the specific substance into a kinetic, by dissociating the molecule of the specific substance and unchaining its specific force.

We now come to the main point, viz. the fact that with these reactions there are always a number of *Leptospirae* that escape from the effect of the specific substances. Even in the dilutions where the action of the lysis — which we now have in view — is most intense, some *Leptospirae* remain entirely unaltered, and retain their characteristic brisk mobility. This cannot depend on the concentration of the serum, for this mobility does not cease either in the preceding, or the following dilution. Most

¹⁾ WOLFF, S. W., Thesis, Amsterdam, 1924.

Arch. für Schiffs- und Tropen Hygiene, 1925.

²⁾ RADEN SOESILO, Thesis, Amsterdam, 1925.

likely these survivors escape from the attack of the immune bodies, because their receptor-apparatus does not contain a group that suits these bodies.

Our next proceeding now was to make fresh cultures with the survivors, and then to establish by repeated tests with the immune serum, whether and how far the daughter strain had become different from the mother-strain. True, we did not start from a single cell then, but from quite a number of germs. But all these parasites that could easily be isolated, had one property in common, viz. that of resistance. If the strain originally examined, was composed of more than one race, there could not be a more favourable opportunity for their separation than here, where in the liquid only the specimens with aberrant properties were left behind.

After what has been said here, the technique required for our experiments, is a very simple one; due precaution should, however, be exercised to ensure sterility. Well-grown cultures should be taken, i.e. the *Leptospirae* should be quite motile, and of a normal size, so they should not be grown into long threads, and not have formed "foci". Now immune serum is added, and we test out the required dilutions, so as to attain the highest possible lytic action. After from 18 to 20 hours the last remains of the disintegrating *Leptospirae* have sunk to the bottom, the supernatant fluid only contains the resistant forms in an even distribution. If necessary the separation can be accelerated by centrifugation (at a moderate rate e.g. 1600 turns). Living, quite motile *Leptospirae* can withstand a moderate rate of centrifugation, without being precipitated.

The survivors are transplanted into new culture-tubes¹⁾, care being taken that the dilution of the immune serum in the subculture does not exceed the titre-limit, otherwise the immune serum, that has been transferred to the subculture, together with the survivors would check their growth considerably. For the rest it struck us that the first growth of these daughter-cultures was more tardy, so that we had to wait longer than usual for growths copious enough for the reaction.

The result of the reaction fell short of our expectation. As the tables 1 and 2 illustrate, from the resistant forms of strain *BA* another strain had developed, *that showed the original properties of the mother strain*. No differences worth mentioning between mother-, and daughter-strain were observable either in quality or in degree. A repetition of the experiment with the "Franken"-strain gave the same pictures. So, though the result was negative, it led to the following consideration and conclusion.

1⁰. In cultures composed of one species and cultivated from diseased persons and animals a small number of *Leptospirae* are regularly developed that resist the antibodies, which destroy the bulk of the organism. They may be isolated and transplanted, but they do not inherit their individual property of resistance. The strain regains the properties of the original culture, from which it emanated, and in which resistant forms are the exception.

What we can watch here *in vitro*, gives us an insight into the process occurring *in vivo* in various Spirochaetal-, and Trepanosome-diseases.

¹⁾ VERVOORT'S Culture medium.

1⁰/₀₀ Pepton Witte, boiled with Normal Phosphoric acid brought to \pm P.H. 6.8; again boiled, then filtered. After this 5—10⁰/₀ rabbit's serum is added, and of this mixture 3.3 c.c. is put in small sterile tubes. Keep for 1/2 hour at 56°.

TABLE I.
Reaction with strain "BA" and its antiserum.

Dilution	Mother culture			Daughter culture		
	Aggluti- nation	Lysis	Survivors	Aggluti- nation	Lysis	Survivors
Control	0	0	++++	0	0	++++
1: 10	+++	0	+	+++	0	+
1: 25	+++	0	+	+++	0	+
1: 50	+++	0	+	+++	0	+
1: 100	+++	0	+	+++	0	+
1: 250	++	+	+	++	+	+
1: 500	+	++	+	+	++	+
1: 1000	0	+++	+	0	+++	+
1: 2500	0	+++	+	0	+++	+
1: 5000	0	+++	+	0	+++	+
1: 10000	0	++	++	0	+++	+
1: 25000	0	+	+++	0	++	++
1: 50000	0	0	++++	0	0	++++

TABLE II.
Reaction with "Franken" strain and its antiserum.

Control	0	0	++++	0	0	++++
1: 10	+++	0	+	+++	0	+
1: 25	+++	0	+	+++	0	+
1: 50	+++	0	+	++	+	+
1: 100	++	+	+	+	++	+
1: 250	+	++	+	+	++	+
1: 500	0	+++	+	0	+++	+
1: 1000	0	+++	+	0	+++	+
1: 2500	0	+++	+	0	+++	+
1: 5000	0	++	++	0	+++	+
1: 10000	0	++	++	0	++	++
1: 25000	0	+	+++	0	++	++
1: 50000	0	+	+++	0	+	+++
1: 100000	0	0	++++	0	0	++++

Under the influence of the immune bodies formed, a wholesale destruction of parasites take place, e.g. in *Recurrans*, towards the termination of the attack, also with the sleeping-sickness at definite intervals, however, without a complete purification of the body resulting from it. Here it is presumably also the resistant forms that cause a continuance of the infection in the body, until, after the diminution of the immunizing properties, they reach the phase again at which their increase is not inhibited, and they can incite symptoms of disease. Then it would not be necessary to have recourse to the hypothesis that surviving parasites keep hidden in nooks and corners inaccessible to the immune bodies. The same might happen in Weil's disease. Beyond the crisis of the disease the *Leptospirae* disappear from the circulation, and from the organs with the exception of a few viz. the resistant forms which can hardly be recognized in the animal experiment. (Inoculation with organ-emulsion.) That after the wholesale destruction some still remain in the body is proof of the often prolonged *Leptospiruria*, by which living parasites, highly virulent for animals, are secreted. They no longer exert a pathogenic influence, and their multiplication in the body remains limited, while of their progeny only the exceptional individuals i.e. the resistant forms, can maintain themselves. The remaining ones succumb soon after their coming into existence, and for some time keep up, and even promote the stimulant to form antibodies (by liberation of endotoxin?). This conception based upon our experiments, would render quite conceivable the ultimate disappearance of the infection from the body.

Still the comparison sustained in the above should not detract our attention from another possibility to explain the secretion of the living *Leptospirae*. The *Leptospirae* in the urine need not have emanated from the organs, or the circulations, but may as well (e.g. in the rat) originate from the lumen of the *tabuli contorti*. Once there, they have, strictly speaking, left the body already and live on the outside, where they, being out of the reach of the immune bodies of the blood and the tissues, can reproduce themselves undisturbedly as ectoparasites. The rat, as could be ascertained, becomes a permanent carrier. In man, however, we see, for reasons unknown to us, the infection disappear again after rather a short time, a few months at most.

2⁰. The fact that the progeny of the survivors, is perfectly agglutinable and lysable, we should like to consider as a proof of the marked persistence of the serological properties. For this reason we give the preference to the serological reaction *in vitro* as a means to differentiate the several *Leptospirae* series, on account of its constant and well-marked results. No doubt also the culture-method, used and recommended by UHLENHUTH ("Wachsthumverfahren") as well as the method adopted by OBA in Miss ZUELZER's laboratory, enables us to make a sharp differentiation, but these methods are complicated and take much time; perhaps they are also less reliable, as with the culture-medium a factor is introduced, that is not always quite known, or controllable. According to our experience the method of crossed immunization is hardly suitable. In numerous experiments we made the observation that to sustain the disease by one strain protected

likewise against other strains, which, serologically, did not show the least affinities with the first. Differences remained hidden here, that became evident by the lysis in vitro. This reaction requires our attention for the further elucidation of the *Leptospira*-problem. In our next paper we shall discuss these results in detail, which are of great significance for the diagnosis. Finally PFEIFFER's test proved distinctly inferior to the serological reactions of agglutination and lysis in vitro.

This conception does not quite agree with Miss ZUELZER's¹⁾ experiments with water strains. In these experiments two waterstrains became pathogenic after having been cultured continually for years, and (which is most remarkable) they assumed the serological properties of the Weil-strain. We repeated these experiments which are of fundamental importance, but up to now we only registered negative results, just as VAN THIEL²⁾ (Leyden) did: i.e. 3 waterstrains, isolated by us, retained their typical features, in spite of repeated transplantation. But we are still continuing these experiments after UHLENHUTH's modified method (cultivation on rat's serum). Also in our Weil- and rat-strains, of which we possess three varieties, we saw during years of transplantation no change in their typical, well marked, serological properties, while it is well-known that the virulence fluctuated within very wide limits. This is why we should like to stick provisionally to our view that a safe index (probably the safest) of differentiation is furnished by the serological properties of the *Leptospirae*.

Our method of cultivating strains from survivors might be successfully applied, when mixtures of different *Leptospirae* are the chief concern. As such are to be considered all inoculations from water that contains besides the typical, harmless water *Leptospirae* (*L. pseudoictero genes*) also species pathogenic for rats and men.

This presence of highly infectious *Leptospirae* was raised beyond all doubt in Holland lately by a number of cases of Weil's disease, which had resulted from an unintentional bath in very filthy water, by which an abundance of water was swallowed as it were experimentally. If a culture be drawn from suchlike questionable water, and if the saprophytic *Leptospira* should be precipitated by the admixture of antiserum, the homologous resistant forms among the survivors, as well as the heterologous *Leptospirae* (if they should be present at all), which have nothing to do with the serum, would be retained. By a repetition of this process it should be possible to eliminate the water strain and ultimately to extract the parasitic strain in a pure condition.

1) Centralbl. f. Bact. 89, 1923. S. 117.

2) Paper read in the Microbiological Society. Ned. Tijds. voor Hyg. en Microb. 1926.

Physics. — *The Duration of Metastable States of Neon, Argon, and Helium.* By Prof. H. B. DORGELO and T. P. K. WASHINGTON.
(Communicated by Dr. G. HOLST.)

(Communicated at the meeting of October 30, 1926).

1. *Introduction.*

In a former communication ¹⁾ a few experiments have already been described in which the order of magnitude of the duration of the metastable $2s_5$ - and $2s_3$ -states of neon was measured.

The results of continued measurements concerning the dependence of this duration on the temperature, and also results of measurements of the duration of the metastable 2^3P_0 - and 2^3P_2 -states of mercury were communicated in the journal "Physica" ²⁾.

As criterion of the presence of the metastable states of the atoms was used the absorption in consequence of these states of those lines that correspond with transitions, in which these metastable states are the final states. Measurements were made of the time that an absorption tube, after the interruption of the discharge through which the metastable states are excited, was still able to absorb the lines in question of another discharge tube.

In what follows we can now communicate the results of continued measurements on argon and helium. For neon we have also determined the dependence on the pressure of the duration of the s_5 -state at different temperatures, which enabled us to discuss more fully the before-mentioned suppositions of JOFFÉ and FRANCK ³⁾ concerning the process of the disappearance of the metastable s_5 -states.

In order to be able to study better the course of the disappearance of the metastable states in the different gases and under different circumstances, the method used before was modified in so far that quantitatively the disappearance of the absorption could be measured in its dependence on the time. The curves thus obtained give an idea of the slow disappearance of the metastable state in the absorption tube.

¹⁾ H. B. DORGELO, These Proc. Vol. 27. N^o. 1, p. 7. Zs. f. Phys. 34, 766, 1925.

²⁾ H. B. DORGELO, Physica 5, 429, 1925.

³⁾ For mercury further investigations were made on this point by FRANCK and CARIO, Z. f. Phys. 37, 619, 1926.

2. *The Method Used.*

In the investigations carried out now we have first used the method II described before ¹⁾, in which by the aid of two rectifiers one half period of a transformed alternating current was sent through an emission tube, the other half period through an absorption tube. The light of the emission tube was sent as a parallel beam through the absorption tube, and then by the aid of a second lens projected on a vacuum thermoelement of MOLL and BURGER ²⁾, which was connected with a galvanometer.

At a definite frequency of the alternating current the value of the absorption in consequence of the metastable atomic states in the absorption tube, when the same half period of the alternating current was sent through both tubes (emission tube and absorption tube), was compared with the value of the absorption, when one half period was sent through the absorption tube and the other half period through the emission tube.

For the ideal case that the passage of the current through the absorption tube and through the emission tube takes place exactly in the middle of the half period (hence at the maximum tension during the half period), the absorption in the absorption tube is measured (with an alternating current of 50 periods) in the first case at the moment when the metastable states of the atoms are excited, and in the second case $1/100$ sec. after their excitation (with the use of an alternating current of 500 periods $1/1000$ sec. after the excitation etc.). By repetition of these measurements with other frequencies of the alternating current, the slow decrease of the absorption with the lapse of time may be followed.

In reality we have not to do with the above-mentioned ideal case of passage of the current through the tubes, but the current will continue for a pretty great part of the half period (if no particular precautions have been taken). This circumstance determines, of course, in a great degree the accuracy of the values of the duration found by this method. Oscillographic records of the current through the emission and absorption tubes really showed that, with the arrangement used (method II), the duration of the current gusts through the tubes was not small in relation to the duration of the half period of the alternating current. (See Fig. 2a.)

We have then tried in various ways to render the time of the duration of the current gusts through the emission and absorption tubes as short as possible. We obtained the best results with the arrangement (to be denoted further as method III), which is represented diagrammatically in fig. 1, and which was used for the final measurements. The anode current of an emission lamp T_1 was sent through the emission tube E , that of an emission lamp T_2 through the absorption tube A . Two Philips triode-lamps Z. 4, anode tension 2000 V. were used.

¹⁾ Loc. cit. *Physica* 5, 433, 1925.

²⁾ W. J. H. MOLL and H. C. BURGER, *Phil. Mag.* 50, 618, 1925.

Alternating tension (of which the number of periods could be varied) was applied to the grating and in series with this a high negative continuous

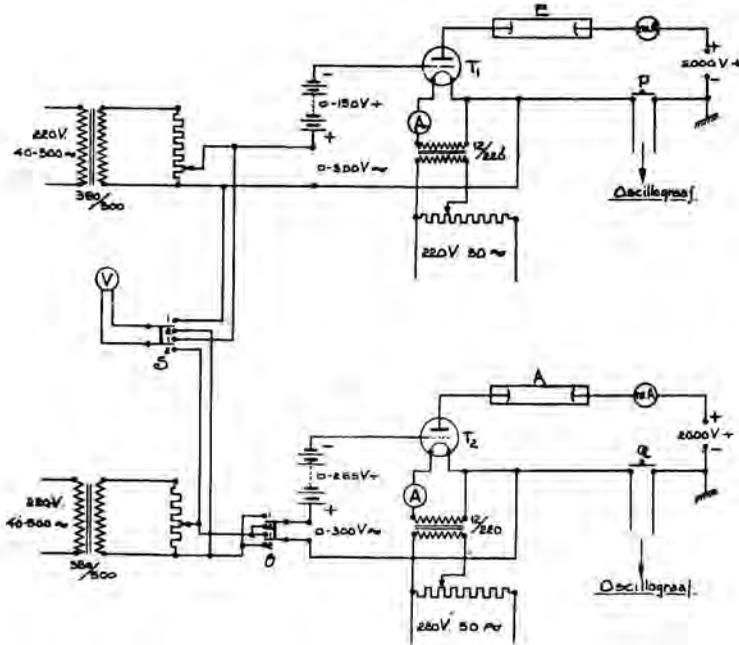


Fig. 1.

tension. By a suitable choice of the strength of the alternating tension and of the negative uniform tension it could be arranged that only during a short time of the half period during which the positive phase of the alternating tension was applied to the grating, the sum of the positive alternating tension and the negative tension of the battery was just sufficient to yield the desired anode current.

By cutting out the short-circuit switches *P* and *Q* the oscillograph could be placed in series with the discharge tubes.

By means of the commutator *O* it was possible to apply the same or else the opposite phases of the alternating tension to the two gratings.

Fig. 2 gives some reproductions of the oscillographic records of the current through the discharge tubes. In Fig. 2*a* the current-time curve through the emission tube is given at 20 mA, method II being used; Fig. 2*b* with use of method III, and at the same mean intensity of the current. The great difference in duration of the current gusts and the great advantage of method III is very apparent.

Fig. 2*c* gives an oscillographic representation of the currents through absorption and emission tube with opposite phases of the alternating tension at the gratings.

For the case that opposite phases were applied to the gratings, we could measure from the oscillographical records of the current through the

tubes, the length of the time interval between the end of the discharge through the absorption tube and the moment of maximum intensity of the current through the emission tube.

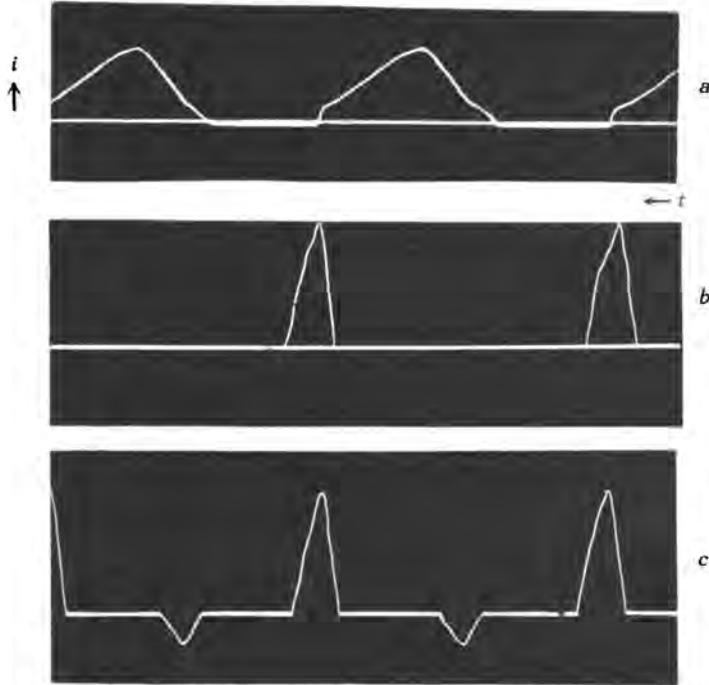


Fig. 2.

The voltmeter V served for the determination of the alternating tensions applied. This arrangement in combination with the before-mentioned thermo-electric-galvanometric method is very suitable for examining how the durations of the metastable states depend on the states in the absorption tube ¹⁾.

Besides we could now also determine the order of magnitude of the duration of the $2s$ -state of helium. As in this case the absorption of the ultra-red line 10830 \AA had to be used as criterion, photographic or visual observation was not possible.

3. Results.

Before mentioning the results obtained we refer to fig. 3, in which energy levels of helium, neon, argon and mercury are represented

¹⁾ Instead of the thermo-electric galvanometric method we could of course use now, as before, a spectral apparatus, and measure the absorption of different lines separately.

As it was, however, our purpose in this investigation to inquire more closely into the circumstances that determine the duration of the metastable circumstances, the method used now takes much less time.

diagrammatically (in volt scale). The metastable energy-levels occurring with these elements are denoted in these figures by enclosing the notation (e.g. s_5 and s_3 for neon) of the levels in a square. For the sake of clearness the splitting up of the s -levels of neon and argon is represented slightly enlarged. Such a combination of different schemata of levels in one figure gives a

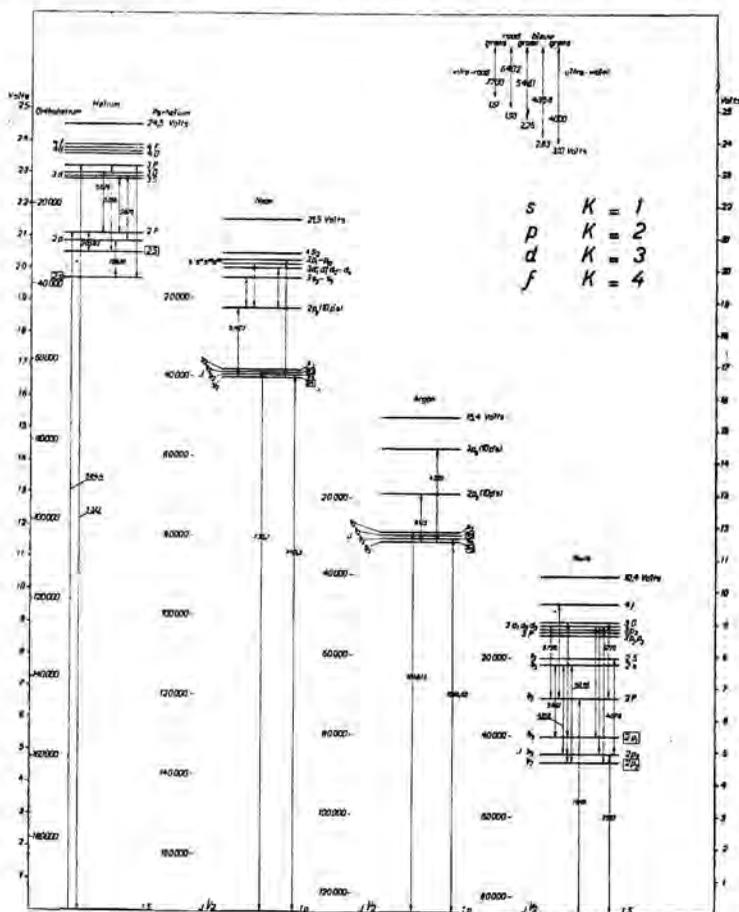


Fig. 3.

lucid survey of certain processes, in which metastable atomic states play a part. Among other things it is, e.g. clear that an excited helium atom in the 2^1S —($2S$)-state has sufficient energy to bring a neon atom into the $4s_2$ -state through a collision of the second kind¹⁾, and can thus excite the line $1p$ — $4s_2$.

a. Neon.

The results for neon already mentioned in our previous communications were now extended by examining the disappearance of the metastable

¹⁾ H. B. DORGELO and J. H. ABBINK, *Physica* 6, 153, 1926.

states of neon with the time elapsed and this at different pressures and temperatures.

In fig. 4 we have drawn as abscissae the values of t measured from the oscillographic records of the current-time curve which elapse on use of

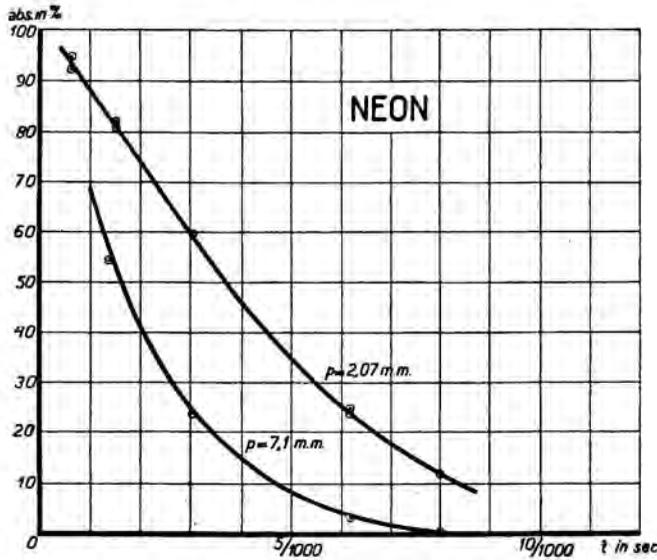


Fig. 4.

different frequencies between the end of the discharge through the absorption tube and the time of maximum intensity of the current through the emission tube ; as ordinates the ratio of the absorptions at the times t and those at the time $t = 0$. Both curves refer to the same absorption tube (diameter 46 mm.), for the curve A the neon pressure was 2.07 mm., for the curve B 7.1 mm.

It is clearly seen that the absorption left after a definite time, hence also the duration of the metastable states, is smaller at a pressure of 7.1 mm. than at a pressure of 2.07 mm.

Fig. 5 (result of measurements in which the pressure in the absorption tube was varied with constant frequency of the alternating tension at the gratings of the triode lamps) gives a still better idea of this dependence on the pressure of the duration of the metastable states of neon. The curves represented in this figure were obtained in the following way. Each curve expresses the results of a series of measurements at definite frequency of the alternating tension at the grating of the triode lamps, but at different pressures of the neon in the absorption tube. At the different pressures the following values were determined :

1. the value a of the absorption when equal phases of the alternating tension were applied at the gratings of the two emission lamps.
2. the value b of the absorption when opposed phases were applied.

The ratio $\frac{b}{a}$ is drawn in fig. 5 as ordinate. The greater this ratio, i.e. the greater the absorption left after a half period ¹⁾, the greater is the duration of the metastable states of the atoms.

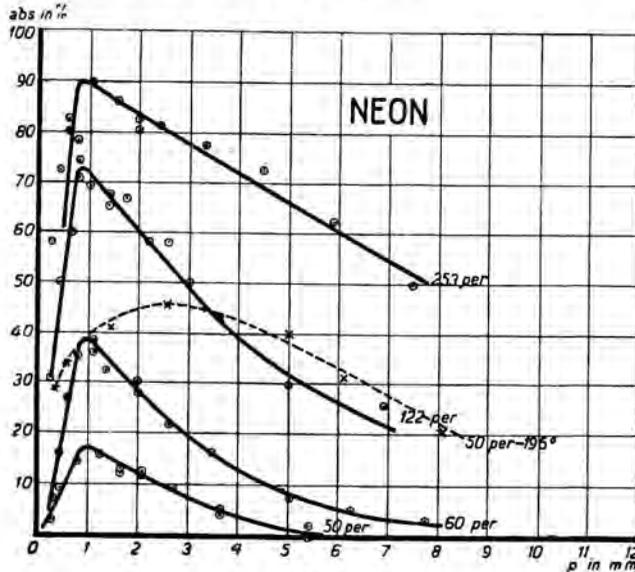


Fig. 5.

It is very clearly seen that *this duration* (passing from higher towards lower pressure) *first increases, reaches a maximum, and then again diminishes.*

At higher pressures (e.g. at 8 mm. pressure) the diffusion of the excited metastable conditions towards the wall plays as yet hardly any part. If, however, the number of collisions of the kind of the metastable states with normal neon atoms is considered, in which sufficient energy can be transferred to bring a metastable neon atom from the s_5 -state into the s_4 -state, this number decreases at lower pressures and consequently the duration becomes greater. If the pressure becomes much smaller, the influence of the diffusion of the metastable states towards the wall will gradually begin to preponderate, so that with a given diameter of the tube a maximum of the duration will be found, while at still smaller pressures this duration must rapidly diminish.

The Influence of the Diffusion and of the Collisions of the First Kind.

In fig. 6 the ordinates of the curve I represent for different pressures

¹⁾ As we already set forth at greater length under 2. in the description of the method used, the time interval between the end of the discharge through the absorption tube and the passage of the light of the emission tube through the absorption tube is actually shorter than a half period and may be derived from the obtained oscillograms of the currents through emission and absorption tube (see fig. 2).

in the absorption tube, the calculated percentage of the metastable neon atoms present at the time $t = 0$, which after the lapse of $1/250$ second would have been left, if the metastable atoms disappeared through diffusion to the wall; assuming in this that a metastable atom loses its energy on collision with the wall and that the metastable atoms are uniformly distributed over the tube at the time $t = 0$.

For the calculation of these curves use was made of the relation :

$$N_t = N_0 \sum 4 \cdot \frac{1}{x_m^2} e^{-\frac{\alpha^2 x_m^2 t}{d^2}}$$

in which N_0 = the number of metastable atoms in the tube at the time $t = 0$.

α^2 the diffusion constant,

x_m the roots of Bessel's function J_0 ,

and d the radius of the cylindrical absorption tube used.

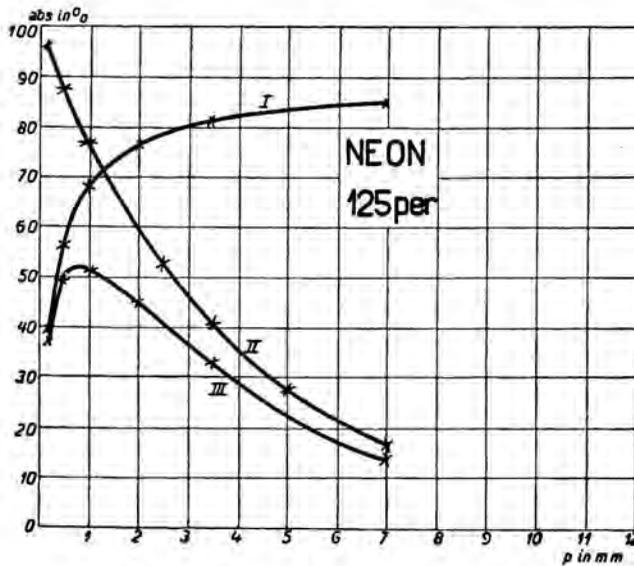


Fig. 6.

The ordinates of the curve II in this figure give the percentage that would have been left, if the metastable atoms disappeared exclusively by collisions of the first kind. The constant required for the calculation of this curve was derived from the experimentally derived curve of fig. 4 (valid for a pressure of 7.1 mm., at which pressure the diffusion does not play a part as yet within the time of $1/250$ second).

In order to obtain the result of the two processes (hence both of the diffusion towards the wall and of the collisions of the first kind), we have to multiply together the ordinates of the two curves. Curve III in fig. 6 represents the curve obtained in this way. We see that its shape is in

perfect harmony with those of the curves of fig. 5 found experimentally.

That experimentally (for 122 per.) we found a higher maximum than the theoretical curve of fig. 6 indicates, is owing to the fact that the disappearance of the metastable atoms present at the time $t = 0$ really proceeds more slowly, especially at the higher pressures, than is expressed by the curve I of fig. 6. As has been said, in the calculation of this curve it was assumed that the metastable atoms were distributed uniformly over the tube at the time $t = 0$.

With the absorption tube used we see clearly, however, that the atoms are excited over the full breadth of the tube on a discharge of the latter only at very low pressures; as the pressure becomes higher, the discharge is more and more confined to the central part of the tube. In consequence of this the disappearance of the excited metastable atoms through diffusion towards the wall will take place less rapidly than is rendered by the curve I of fig. 6. The maximum of the resulting curves will also lie higher¹⁾.

In connexion with these results Dr. HOLST remarked that, at very low temperatures, at which, as already appeared from our earlier measurements²⁾, the collisions of the first kind no longer play an important part, we ought to find that the diffusion would become predominant at higher pressures, and that therefore e.g. at the temperature of liquid nitrogen the maximum of the curves of fig. 5 ought to shift towards higher pressures.

The dotted curve of fig. 5 gives the results of measurements with an alternating tension of 50 periods, the absorption tube having been cooled with liquid nitrogen. We see that the maximum of this curve lies at greater pressure. We further see here that at -196° C. the curve for 50 periods lies much higher than the traced curve for 50 periods, which is valid for room-temperature, i.e. the number of metastable states that are still left in the tube after $1/100$ sec., is much greater at -196° C. than at room-temperature. This too, is in harmony with the conception of the destruction of the metastable states by collisions of the first kind with the other neon atoms in the normal state.

b. Argon.

From the investigations of K. W. MEISSNER³⁾ on the absorption of argon lines by excited argon-atoms and his analysis of the red argon spectrum and also by the investigations of the red argon spectrum

¹⁾ (As we have to do with the diffusion of excited atoms, hence with particles with greater diameter than the normal atoms, the diffusion constant will be smaller than that for normal atoms. In our calculation of the curve I of fig. 6 we have, somewhat arbitrarily, assumed a diffusion constant four times smaller than the gas-kinetic one for normal atoms).

²⁾ Loc. cit. *Physica* 5. 434. 1925.

³⁾ K. W. MEISSNER, *Phys. Zs.* 26, 686, 1925; *Z. f. Phys.* 37, 238, 1926; *ibid.* 39, 172. 1926.

in the extreme ultra-violet by LYMAN and SAUNDERS ¹⁾, HERTZ ²⁾ and DORGELO and ABBINK ³⁾, a close analogy has been established between the neon and the argon spectrum. Also argon has two metastable states (indicated in fig. 2 as the $2s_5$ - and the $2s_2$ -states of argon), which clearly appears from the fact that with argon there are also only two resonance lines λ 1048 and λ 1066 in the extreme ultra-violet. The transitions from the $2s_3$ - and $2s_5$ -states to the fundamental state do not take place.

In fig. 7 we give a reproduction of some photographs, which very clearly show the absorption by the excited metastable argon-atoms.

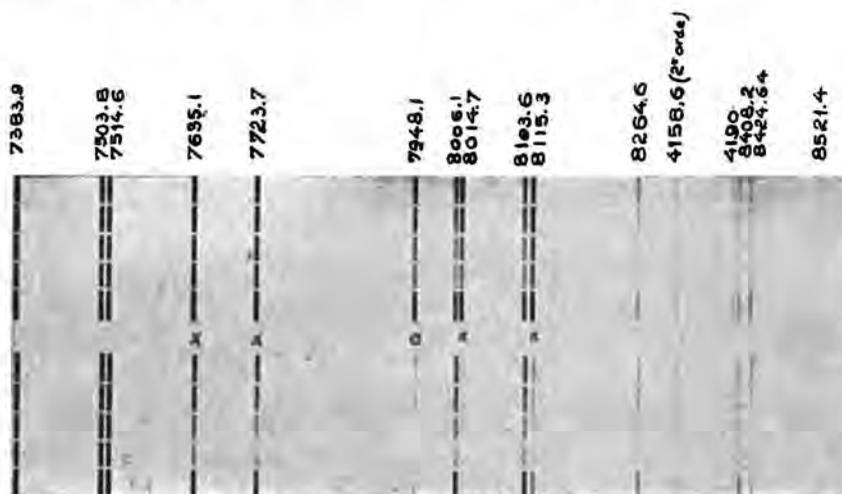


Fig. 7.

Spectrum I gives the reproduction of a photograph ⁴⁾ of part of the argon spectrum in the ultra-violet of an argon column, again without absorption of the light of this column by excited metastable states of argon, spectrum II with *absorption*. The great absorption of the lines marked with crosses (combinations with the $2s_5$ -state of argon) and with a circle (combination with the $2s_3$ -state of argon) is very apparent.

¹⁾ TH. LYMAN and F. A. SAUNDERS, Nature **116**, 358, 1925.

²⁾ G. HERTZ and J. H. ABBINK, Naturwiss. **14**, 648, 1926.

³⁾ H. B. DORGELO and J. H. ABBINK, Naturwiss. **14**, 755, 1926.

⁴⁾ These photos were taken with plates which had been sensitized with dycyanine, the stigmatic arrangement of the grating of our physical laboratory being used. The step-wise weakening of the spectrum lines was obtained by the aid of a quartz-platinum gradual weakener placed before the slit. We wish to draw attention to the efficiency of these step-by-step weakeners also in this part of the ultra red region of the spectrum examined by us. In the reproduced lines λ 8264 (1st order) and λ 4158,6 (2nd order) it is clearly seen that the weakening action of the different parts of the weakener is pretty well the same for both wave-lengths. Combining these results with earlier measurements in the ultra-violet, we may now say that these quartz-platinum weakeners are very efficient in a wave-length region of from 9000 Å to 2200 Å.

Just as with argon, this absorption by the metastable states could be used to determine the duration of these states in argon. For this purpose we used method III described in this paper.

It appeared at once that the duration of the metastable argon atom states depended in a great degree on the impurities present in the tube. In particular the addition of hydrogen appeared to exert a strongly destructive influence on the metastable states of argon¹⁾.

It was, therefore, of great importance to purify the argon as much as possible. We have applied several methods of purification to make the argon, which had been obtained by distillation of solid argon, still purer if possible. Among other methods the following were tried: purification by means of Na, or Mg, spluttering of Ca and Wa and circulation over heated CuO; further addition of small traces of O₂ during the discharge, in which the water vapour, if any had formed, was condensed with liquid air, and the excess of O₂ was removed by evaporation of Wo. The result of all these methods of purification was that at low pressure it was quite possible (especially by purification by means of Wo-vapour) to get the argon so pure that in a discharge through the tube only the argon lines appeared. At pressures of 1 cm. and higher the colour of the discharge of the column (current density about 1 mA/cm².) was always grey. If, however, at these high pressures the current density was increased, this grey discharge disappeared, and the line spectrum of argon reappeared.

Spectrographic records of the quartz spectrograph of this grey argon discharge²⁾ gave a continuous spectrum, which extended over the entire spectral region of the quartz spectrograph.

This grey "argon" discharge had already been observed before in this laboratory by Dr. PENNING, in his attempts at purifying the argon required for his measurements of the anode-heating. PENNING also, always obtained the grey column discharge at higher pressures, however carefully the gas had been purified.

We have, so far, not been able to ascertain whether this continuous spectrum, which appears at higher pressures of the argon in a column discharge, really belongs to argon, or whether it might, possibly, be due to very small traces of residual hydrogen. Different experiments on this point (which are still in progress) render it probable that the latter is really the case.

With regard to our results of the measurements of the duration for the metastable states of the argon atom, we may state that we had already been able to detect a perceptible absorption by the metastable states of

¹⁾ Very probably the Lyman bands of H₂ are excited by collisions of the second kind of the metastable states of the argon atom with H₂ molecules (Cf. E. E. WITMER, Proc. Nat. Acad. Amer. 12, 238, 1926).

²⁾ This column discharge was excited in a tube provided with a quartz window, which had been fused to glass. Hence the whole tube could be pumped at temperature ($\pm 350^\circ$). Besides, spluttered Mg was present in the tube.

the argon atom by means of method I (with rotating disc) ¹⁾ about $1/75$ sec. ²⁾ after interruption of the current by the absorption tube.

In fig. 8 is found a curve obtained by method III, representing the value of the remaining absorption in its dependence on the time (argon pressure

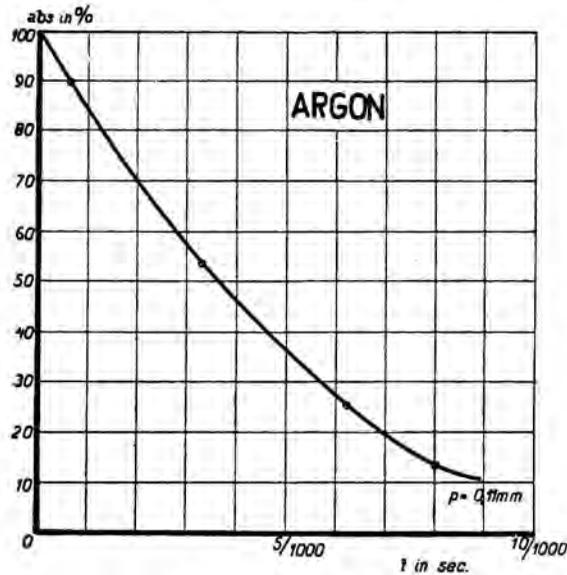


Fig. 8.

in absorption tube 0.11 mm.; inner diameter 46 mm.; abscissae and ordinates of fig. 7 analogous to those of fig. 4 for neon).

Just as for neon, we have also made measurements at different pressures of the argon in the absorption tube. We found a maximum duration of the metastable states of the atom with the given diameter ³⁾ of the tube at about 0.1 mm. pressure.

Repeated measurements yielded the same result.

Summarizing the results obtained for argon, we get the following:

1. With thoroughly purified gas durations of the metastable states are found which are still greater than those that are measured for neon.
2. The durations of the metastable states in argon are influenced by added impurities (especially hydrogen) in a much greater degree than they are in neon.

c. Helium.

As is shown in fig. 3, there appear in helium two metastable excited states ($2s$ - and $2S$ -states). The metastability of these states was proved by

¹⁾ Zs. f. Phys. **34**, 766, 1925; Physica **5**, 431, 1925.

²⁾ Communication at the XXIVth Flemish Physical and Medical congress, held at Ghent on April, 11th 1926. Compare Wis- en Natuurk. Tijdschrift (organ of the Flemish Physical and Medical Congress) III volume, p. 65, 1926.

³⁾ The curves of fig. 5 of Neon had been obtained with a tube of the same bore.

PASCHEN's well-known researches ¹⁾ on the absorption by excited helium atoms.

In order to determine the duration of the 2s-state by our method, the course of the absorption with time of the line 10830 Å in the ultra-red should be followed. This line was filtered out in the way indicated by PASCHEN by the aid of a water filter in combination with a Wratten-filter for ultra-red. As by our method III (triode lamp with 2000 V. anode tension) we could not get passage of the current in the helium tubes used, we give in fig. 9 curve I obtained by method II.

The helium pressure in the absorption tube used (diameter 32 mm.) was

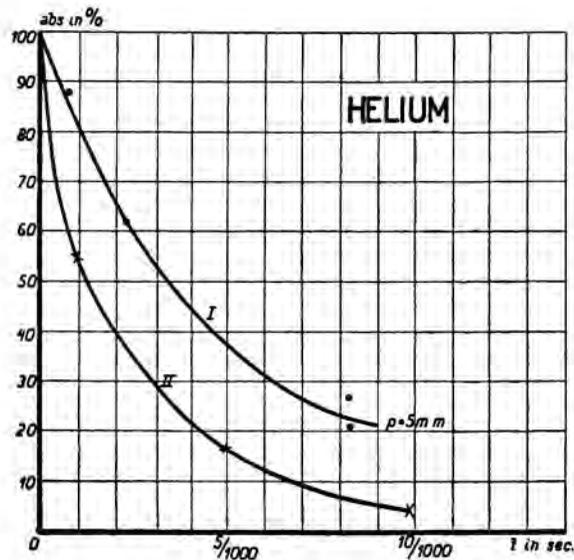


Fig. 9.

5 mm.²⁾ Curve II represents the theoretically calculated curve, which indicates the disappearance of the metastable states with course of the time through diffusion towards the wall. In this it was assumed that every atom in the metastable state loses its energy on collision with the wall; further as diffusion constant that calculated from the gas-kinetic data was assumed. As we have to do here with the diffusion of excited atom states, which have greater diameter than the atoms in the normal state, the curve will, in reality, have to be somewhat higher. In consequence of the drawbacks already mentioned, which belong to method II, curve I will have to lie slightly lower. From results concerning the disappearance of the

¹⁾ F. PASCHEN, Ann. d. Phys. 45, 625, 1914.

²⁾ The helium used was obtained according to HERTZ's diffusion method from a Ne-He mixture. In the column discharge exclusively He-lines appeared and no Ne-lines were any longer to be seen.

metastable state found for the $2s$ -state, we are, therefore, justified in concluding that the disappearance of the metastable $2s$ -states of He after interruption of the current through the absorption tube is entirely determined by the diffusion to the wall. On account of the great difference of energy in helium between the metastable $2s$ -state and the next higher energy-condition it is not possible that the metastable $2s$ -states are brought to a higher state of energy by collisions of the first kind with normal helium atoms.

Attempts to determine the duration of the $2s$ -states have so far yielded no result. For this it was required to filter out only the line 20852 Å from the radiation of the emission tube. According to PASCHEN biotite may be used for this purpose. It appeared to us, however, that the biotite kindly put at our disposal by Prof. RUTTEN of Utrecht, could not be used for this purpose.

N.V. PHILIPS *Gloeilampenfabrieken.*

Eindhoven, Oct. 13, 1925.

Physics. — *Influence of the Pressure on the Electric Conductivity of Gold.* (Preliminary communication). By A. MICHELS, P. GEELS, and Miss C. VERAART. 17th communication of the VAN DER WAALS-fund. (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of October 30, 1926)

In our previous communication¹⁾ we already drew attention to the desirability of determining anew the influence of the pressure on the electric conductivity of metals, and we recommended to do so at smaller pressure intervals than earlier observers.

By the kind assistance of Dr. C. HOITSEMA, Government Mint Master at Utrecht, we were put in possession of a quantum of very carefully purified gold, which was drawn out into wire of a diameter of $\frac{1}{10}$ mm. by the firm of HEREAUS at HANAU with particular precautions.

In order to compare the degree of purity with that of the material used by former investigators, — in which we had particularly in view P. W. BRIDGMAN²⁾ — we determined the influence of the temperature on the resistance of our gold wire.

If we put the resistance of the measuring wire at 0°, 26°, and 34° resp. at R_0 , R_{26} , and R_{34} , the temperature-coefficient was for our metal:

$$\frac{R_{34} - R_{26}}{R_0(34 - 26)} = 0.00391 (\pm 0.00001),$$

the corresponding coefficient of the gold used by P. W. BRIDGMAN being 0.00394.

From the equality of the two values we may conclude to a sufficiently equal degree of purity of the two metals.

We must, however, point out that a small difference still remains possible, because the hardness of the material may have an influence on the temperature coefficient, though it is not to be expected that this influence will be great for gold with its regular structure.

Secondly we thought we had to conclude from our values that also the temperature-coefficient experiences a slight influence from the fact that the gold wire was put under pressure.

The latter influence, too, lies at the limit of our accuracy of observation, so that we shall not be able to get a decisive answer to this question save by further measurements.

¹⁾ These proceedings 35, (1926), 578.

²⁾ Proc. Amer. Ac. 52 (1917), 601.

As BRIDGMAN subjected the gold wire to a pressure of 12000 atm., and we so far only reached 250 atm., it was possible that there too a small source of deviation was to be found.

Before we proceeded to the measurement of the influence of the pressure, an investigation was made into the validity of the known formula on the increase of temperature of the wire in consequence of the measuring current

$$dt = \frac{K \varrho i^2}{hd^3} \quad 1)$$

This investigation had, at the same time, the purpose of finding the value of the constant in the formula, in which k is the constant in question, ϱ the specific resistance, d the diameter of the wire, i the intensity of the current, and dt the increase of the temperature resulting therefrom.

This constant did not in itself interest us so much, but we wanted to ascertain what was admissible for the gold wire as a maximal current, if the increase of temperature of the wire was not to exert any injurious influence.

For these investigations too a measuring wire was used of $1/10$ mm diameter. Small contaminations could not be of importance here, so for the sake of saving the purer, we used less pure material, the temperature-coefficient of which was, accordingly 3.7×10^{-3} .

The results are recorded in the subjoined table.

i in milli-amperes	Resistance	Temperature
15.475	R	t
36.70	$1.000276 R$	$t + 0.0745^\circ$
67.175	$1.001026 R$	$t + 0.277^\circ$
89.575	$1.001881 R$	$t + 0.508^\circ$

By the aid of the formula cited above the intermediate temperatures were interpolated between the lowest and the highest temperatures of the measuring-wire, which is elucidated by the subjoined table and adjoined graph.

i^2	Δi^2	temper. experim.	theoret. interp.
2395	0	t	t
1347	11075	$t + 0.0745$	$t + 0.0723$
4513	42735	$t + 0.277$	$t + 0.279$
8024	77845	$t + 0.508$	$t + 0.508$

1) Cf. O. D. CHWOLSON, Lehrb. d. Phys. IV 1. 1908. Thesis A. MICHELS, p. 24.

Within the errors of observation the theoretically interpolated values are the same as the temperatures found experimentally. From the foregoing it is easy

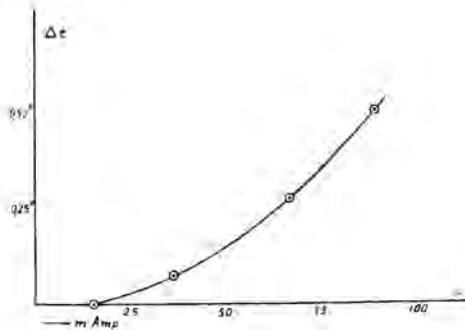


Fig. 2.

to calculate the constant c entering into the formula:

$$dt = ci^2$$

$$c = 0,0653$$

$$i \text{ in mA.}$$

$$dt \text{ in } \frac{1}{1000}^{\circ}.$$

The Measurement of the Influence of the Pressure.

The choice of our arrangement rendered absolute measurements unnecessary; so, different from the method used by us in determining the influence of the pressure on the electric conductivity of platinum, we could now use a WHEATSTONE bridge connecting scheme. This arrangement also enabled us to avoid a few sources of error.

Instead of one gold wire, we built two equal ones, enclosed in two identical mantles, and inserted each of the two gold wires into one of the two circuits of the WHEATSTONE bridge connecting scheme.

When now the two wires got an equal current, the influence of their increase of temperature was greatly diminished.

Our measuring current was about 6 mA., which according to our measurements given above, would yield an increase of temperature of $\frac{1}{500}^{\circ}$.

As we measured with an accuracy, where only temperature differences of $\frac{1}{1000}^{\circ}$ become troublesome, we may expect that the influence of selfheating was entirely eliminated.

A greater advantage, however, lay in the fact that it was possible to eliminate in this way the inevitable oscillations of the thermostat

temperature as the two mantles were inserted in the same thermostat closely side by side, and through their identical construction obtained an equal lag in temperature.

The oscillation of the temperature never amounted to more than $\frac{1}{200}^{\circ}$ in a period of about 4 hours.

With our arrangement it is practically impossible that there is ever a difference of temperature of $\frac{1}{1000}^{\circ}$ between the two wires.

The resistances of the gold wires amounted to about 26 ohms each, and the variable resistance in the circuit to 100 ohms. The high value of the latter, which had to compensate the influence of the pressure on the gold wire, had also the advantage that variations in the transition resistances of the plugs of the rheostat could be neglected.

For the determination of the galvanometer current the measuring current was commuted, through which at the same time the thermo-electric forces disappeared from the results.

For the arrangement of the experiment we refer to these Proceedings cited above.

The pressures were measured by the aid of a pressure balance, the accuracy of which far exceeded that of the electric measurement.¹⁾

The accompanying table and the graphical representation II give the results of three series of observations between 0 and 250 atm.

These resistance measurements agreed with each other with a margin corresponding to a difference of temperature of 0.001 degree of the measuring-wire.

$$\frac{-\Delta w}{w} \cdot 10^6.$$

Pressure in kg. per cm. ²	101.5	251.5
16°	5.03	11.8

Just as in the course of the observations made on platinum, it appears here again, that the decrease of resistance is not in linear relation to the increase of pressure.

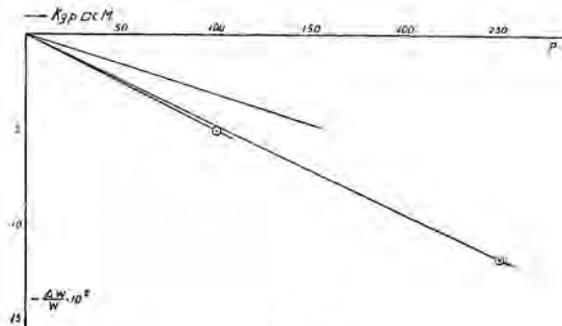


Fig. 2

¹⁾ Cf. Ann. d. Phys. 73. 1924.

The deviation of the straight line is, indeed, not so great as that found for platinum, but the distance of the point $101\frac{1}{2}$ atm. to the second line, is about five times as great as an error of observation.

In the graphs this is shown in this way that no curve has been drawn through the few points which have been determined, but by connecting these points with the origin.

The upper straight line in the graph represents the result at which BRIDGMAN arrives in the publication cited above.

BRIDGMAN states that he has found as a pressure-coefficient up to 1000 atm. 3.1×10^{-6} .

Though extrapolation to 1000 atm. is very unreliable here, it may be deemed doubtful whether at 1000 atm. it will be possible to us to obtain agreement with this value.

Here, too, it is, however, possible that the hardness of the material plays a part.

For the present no more points have been determined by us, as our purpose was only a preliminary investigation.

This investigation, however, has proved that it will be necessary to continue the measurements at least up to 1000 atm. in order to find more points of the curve, and also to examine the influence of the hardness.

Physics. — *The Three-Phase-Lines of the Systems: Water-Ortho-cresol, Water-Metacresol, and Water-Paracresol.* By A. MICHELS and E. C. F. TEN HAAF. 16th Communication of the VAN DER WAALS fund. (Communicated by Prof. J. D. VAN DER WAALS JR.)

(Communicated at the meeting of October 30, 1926).

The purpose of the following investigations was to examine in how far the mutual position of the CH_3 and OH groups has influence on the phenomenon of unmixing of cresol-water mixtures.

For the present we have confined ourselves to examining the three-phase curves by studying the unmixing in sealed glass tubes ¹⁾.

For the preparation of the said cresols the homonymous toluidines were started from, which were supplied to us as being pure. These toluidines were first distilled, then converted into the oxalic acid salt, and this salt was purified by repeated out-crystallisation in alcohol. The final product, which was assumed by us to be pure, amounted to $\frac{1}{4}$ of the quantity started with.

Then the toluidine was regained from those salts, and the cresol was prepared from the toluidine by diazotation.

The cresols prepared in this way were purified again by distillation in a nitrogen atmosphere. This distillation was repeated so often till the last fraction and the last but one gave no difference in their crystallisation points.

For para- and ortho-cresol a crystallisation point was found corresponding to that generally recorded in the literature; it appeared, however, possible to us to raise the melting-point of metacresol, by purification, considerably above that reached till now.

While usually the melting-point of meta-cresol is stated to be $\pm 4^\circ$, we obtained $\pm 10^\circ.6$.

It is perhaps worth mentioning that all three cresols appeared to be colourless liquids as clear as water, which, however, in course of time gradually assumed a yellow to brown colour, even when they were kept in the dark under nitrogen.

With the cresols obtained in this way the investigations on the three-phase curves were carried out. The results are recorded in the subjoined table, the accompanying diagram giving a graphical representation of the "Consolute temperature" as it varies with the concentration.

¹⁾ Compare inter alia also: A. MICHELS. Stiesel isobutyl-alcohol en water, Archives Néerlandaises. Sér. III^A T. VI (1923), p. 127.

№.	Ortho-cresol		Meta-cresol		Para-cresol	
	Conc.	Temp.	Conc.	Temp.	Conc.	Temp.
1	1.3	< 0	2.7	50.8	2.21	29 à 30°
2	2.9	46.2	3.6	78.7	3.74	82.1
3	4.0	86.7	4.5	92.2	5.4	105.0
4	4.5	104.5	10.8	121.7	6.9	118.5
5	6.9	121.0	14.0	140.4	9.2	127.9
6	7.0	123	23.2	147.5	16.4	138.0
7	8.7	134.0	29.7	148.7	32.1	142.6
8	16.4	157.9	42.9	148.7	32.3	142.5
9	17.5	159.6	48.9	147.6	43.7	140.7
10	36.4	167.3	58.0	141.2	50.2	139.5
11	42.6	168.9	59.3	141.9	56.9	136.6
12	45.9	168.3	62.2	137.7	66.6	124.4
13	50.4	167.9	65.9	133.2	71.3	110.8
14	56.5	163.7	70.6	124.4	79.5	71.0
15	64.7	160.1	73.1	120.0	81.3	59.5
16	74.8	139.2	76.7	103	83.7	37.1
17	75.9	135.4	79.7	90.2	85.9	below 21.8
18	82.9	92.8	80.0	85.2		
19	84.1	87.3	80.7	82.6		
20	86.2	50.5	82.6	67.9		
21	88.5	in ice hom.	85.9	36.2		

The upper critical endpoints appeared to be:

for ortho-cresol 168°₉ with a concentration of about 41 %.

for meta-cresol 148°₈ with a concentration of about 38 %.

for para-cresol 142°₆ with a concentration of about 36 %.

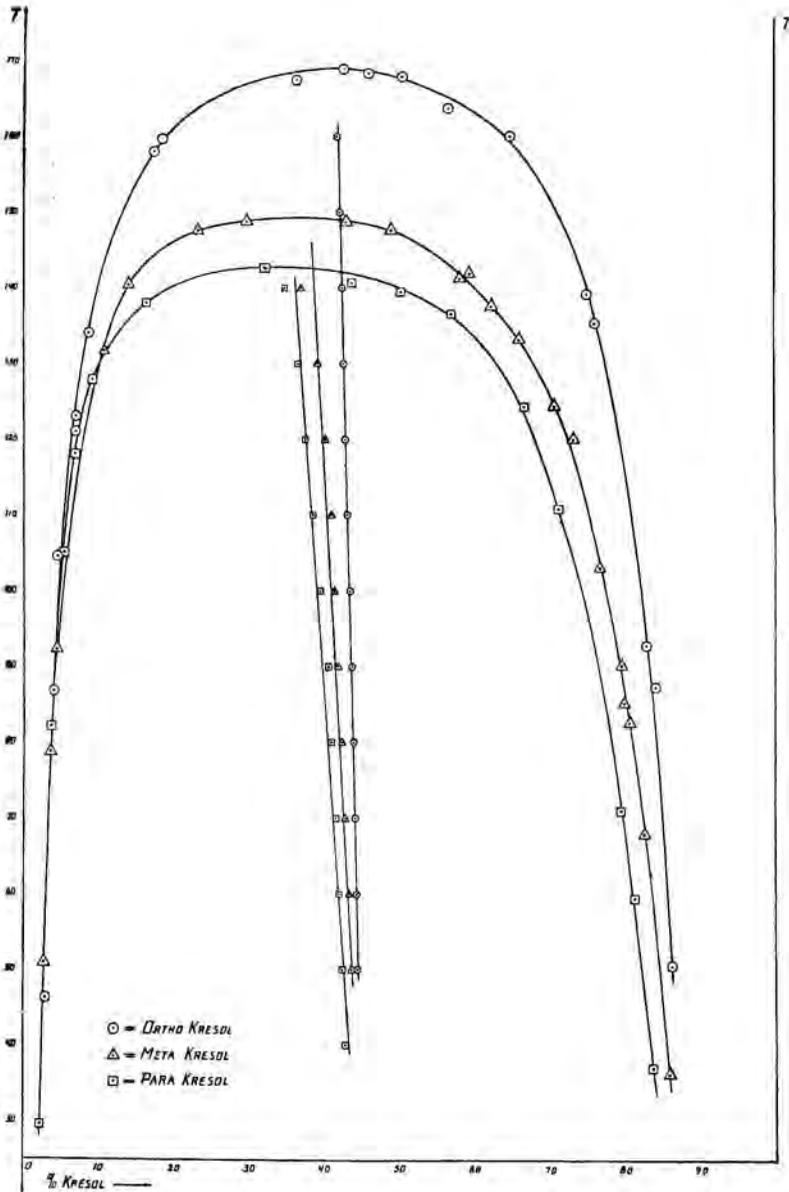
The values of these concentrations are graphically approximated by the aid of the known law of the rectilinear diameter.

In the diagram the points of the diameter are indicated, and the most probable straight line has been drawn through them. The deviation visible in the diagram is a very slight one, except at the top, where the accuracy of the measurement is much smaller.

It is also noteworthy that the phenomenon of barotropy occurred with all three cresols: while at low temperatures the phase rich in cresol was the heavier, the phase rich in water interchanged places with the other at a given higher temperature.

The exact transition temperature cannot be given with great precision,

because the viscosity was too great for the two phases to move easily through each other. Approximately the barotropic temperature for ortho-cresol lies at 145° , for meta-cresol at 148° , and for para-cresol at 138° .



Finally it may be stated that as soon as the cresols decompose sufficiently to change colour, this manifests itself at once in the position of the unmixing-points.

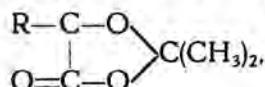
Chemistry. — *Some Remarks on the Condensation Products of Polyoxy-Compounds with Chloral.* By Prof. J. BÖESEKEN.

(Communicated at the meeting of November 27, 1926)

Our researches on the influence of poly-hydroxy-compounds on the conductivity of boric acid have led to closer examination of the complex compounds formed in the course of them. It then appeared that the mostly strongly acid compounds were often derivatives of boron, in which this atom had passed to a state of higher-valency (coordinate-value = 4). This is of hardly any consequence for the diagnosis of the position of the hydroxyl-groups in space, as also the formation of these higher-valency boron-compounds is dependent on the more or less easy appearance of five or six rings. Besides, it has been shown by KOLTHOFF ¹⁾ that in aqueous solutions these acids of the type $B(O_2 = R)_2H$ are for the greater part split up into $HOBO_2 = R_1$, in which a simple ring system occurs.

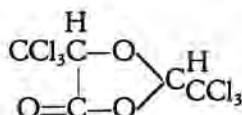
Besides to the investigation of these boric-acid compounds we have also undertaken the study of the acetone compounds, which remain simpler of structure as a matter of course.

Both the boric-acid and the acetone compounds are hydrolyzed exceedingly easily and rapidly by diluted acids, so that they can mostly only be preserved with exclusion of acids. R. WILLSTÄTTER ²⁾ has succeeded in showing that acetone condensed with α -oxy acids to volatile compounds of the same type as the simple boric acid compounds:



which, however, could only be separated and kept, when all the water was carefully removed and excluded.

Of the α -hydroxy acids the corresponding condensation products with chloral, the *chloralides*, have already long been known. The first representative of this class of compounds was discovered by STÄDELER ³⁾ as reaction-product of chloral with strong sulphuric acid; it is the trichlor lactic acid chloralide:



¹⁾ Recueil **45**, 607 (1926).

²⁾ W. und KÖNIGSBERGER. Berichte **56**, 2107 (1923).

³⁾ Ann. **61**, 104 (1847).

the constitution of which has been determined by WALLACH ¹⁾, and the formation of which was explained by me several years ago ²⁾.

WALLACH has made a number of similar compounds with α -hydroxy acids synthetically by heating these with anhydrous chloral in sealed tubes.

As these compounds on the whole crystallize very well, and are very stable, and as for this reason they seemed very suitable to me for the determination of the constitution of polyoxy-compounds, I have started by making an improvement in the way of formation (see below), and I have examined whether these chloralides might possibly also be formed easily from other polyoxy compounds, as α -glycols, which appeared actually to be the case.

The simplification in the method of preparation is this: the α -hydroxy acid or the glycol were gently heated with chloralhydrate and strong sulphuric acid at $\pm 80^\circ$ for some time, and the reaction mixture was extracted with ether, after the former had been diluted a little with water. In the case of hydroxy acids these first dissolve in the chloral sulphuric acid mixture, then the chloralide is separated, often already in crystallized state.

Besides the light that we expect the velocity of their formation or of their hydrolysis to throw in the future on the position of the hydroxyl groups, they are also of some interest from another stereo-chemical point of view. In consequence of the origin of a new asymmetric centre, the number of stereo-isomers becomes greater. Thus out of optically active lactic acid two optically active isomer dichloralides will be formed and from d-tartaric acids even three such isomeric dichloralides.

From pentaerythrite, if this possesses a tetrahedral configuration, a compound must be formed which in consequence of spirane-asymmetry will be decomposable into optical antipodes ³⁾ etc.

By the side of the acetone compounds, these chloral compounds, may probably become of great importance for sugar chemistry.

*The Preparation of Some Chloralides. Isomer d-Tartaric Acid
Chloralides (with Miss J. BLOK).*

Lactic acid chloralide. This was first made by WALLACH loc. cit. by heating lactic acid with anhydrous chloral in a sealed tube.

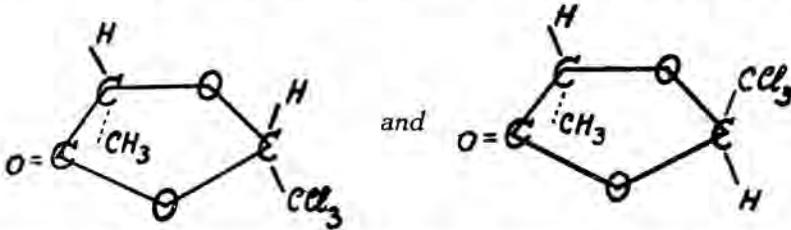
We have heated lactic acid (of 87 %) with the theoretic quantity of chloral hydrate, and an equal number of ccm of conc. H_2SO_4 at 35° for some hours, after first having come to the conclusion that these were the most favourable conditions. After the water had been poured off, a solid product was obtained which melted after distillation under diminished pressure at 45° . Yield 59 %.

¹⁾ Ann. 193, 1 (1875).

²⁾ Chem. Weekblad (1910), p. 123.

³⁾ If on the other hand it should be built up as a four-sided pyramid, it would have to be possible to separate two geometric isomers.

Though the lactic acid was feebly optically active, the compound appeared to be inactive, hence racemation has set in. We have got no



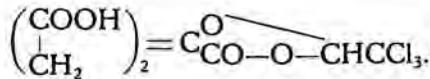
indication of the formation of cis-trans isomers, though this must be possible according to our spatial conceptions.

Citric acid chloralide. This was prepared by heating citric acid with a small excess of chloral hydrate with conc. H_2SO_4 (e.g. 50 gr. acid + 52 gr. chloralhydrate + 50 ccm. H_2SO_4) between $40^\circ-50^\circ$, and then diluting it with a little cold water. Then a semi solid mass is separated, which is washed with some ether, and is then recrystallized several times from chloroform. It appeared to be a mass consisting of blocks and needles (a total yield of 51%).

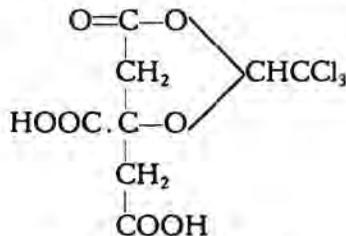
After repeated recrystallisation of the least soluble substance (the blocks) the melting-point was constant = 166° . It was strongly acid, dissolved in water, and also in the ordinary organic solvents in which citric acid does not dissolve.

				Found	Calc.
0.0719 gr.	equiv.	13.15 cc.	$AgNO_3$	0.0502 n.	$Cl\%$ 32.6
0.0685 "	"	12.60 "	"	"	33.0
				$Cl\%$	32.8

This was, accordingly, the citric acid monochloralide, and to all probability:



The yield of the pure product melting at 166° is pretty small; there are other substances among it, which will have to be studied further. The chlorine content of the mixture is, however, very near to the calculated value for a monochloralide, so that the possibility exists that by the side of the α -chloralide, also a β -chloralide is formed:



β citric acid chloralide.

From the mother liquid another mixture was obtained, the chlorine-content of which was considerably higher, $\pm 40\%$; there is, therefore, an indication that there has also been formed a dichloralide, which may be expected to dissolve more easily in chloroform.

d-Tartaric acid chloralides. In the action of chloral on d-tartaric acid the formation of the dichloralide was at once taken into account.

1 Mol. of d-tartaric acid + 2 mol. of chloral hydrate + conc. H_2SO_4 is shaken while heated to 50° . At first it is a turbid liquid, which gradually solidifies; the dichloralide is insoluble in water, hence all the acid parts and remaining not converted chloral can be removed by washing with cold water. Yield: 71% of a perfectly white product. Though the chlorine content was in conformity with that calculated for dichloralide (found 51.1, calculated 52.0), it was unmistakably a mixture; melting range $129^\circ-135^\circ$. This was to be expected; asymmetry centres are added; starting from d-tartaric acid, there are, therefore, three isomers to be expected:

D (of the tartaric acid)	+ d + d		or trans-trans
D	..	+ d - l	} for this case these are identical }
D	..	- l + d	
D	..	- l - l	

First the substance was sublimated in a locomotive according to EYKMAN: in 20 min. it already pretty rapidly went over at 130° ; needles are, therefore, formed, which however appeared not to be homogeneous. As a considerable part was decomposed in this, we have further tried to bring about the separation by recrystallisation from chloroform and benzene. In this way we have succeeded in obtaining two substances with constant melting-point and constant rotatory power.

The least soluble isomer, melting-point = $159^\circ-161^\circ$. Chlorine content = 52.0% . Rotation in chloroform, length of the tube = 2 dm.

$$\begin{aligned} 0.6759 \text{ gr. in } 25 \text{ cc. } CHCl_3 \text{ at } 16^\circ.2 \quad \alpha_D &= -2.92 \\ 0.6747 \text{ " " } 24 \text{ cc. " " " } \alpha_D &= -3.00 \\ [\alpha]_D &= -54^\circ.0 \text{ and } -53^\circ.7 \quad \text{Mean} = -53^\circ.9. \end{aligned}$$

Lowest-melting modification, melting-point = $116^\circ-118^\circ$. Chlorine content = 51.1% .

$$\begin{aligned} 0.6786 \text{ gr. in } 24 \text{ cc. } CHCl_3 \text{ at } 17^\circ.0 \quad \alpha_D &= -1^\circ.85 \\ 0.6271 \text{ " " } 22 \text{ " " " } 17^\circ.0 \quad \alpha_D &= -1^\circ.91 \\ [\alpha]_D &= -32^\circ.8 \text{ and } -33^\circ.5 \quad \text{Mean} = -33^\circ.1. \end{aligned}$$

It is noteworthy that both modifications are levo-rotatory; it is, however, not astonishing, because pure anhydrous d-tartaric acid is also

levo-rotatory¹⁾). As long as the third modification has not been found, it is not possible to draw further conclusions.

Racemic acid chloralide was prepared in a perfectly analogous way to that in which tartaric acid chloralide had been made. Obtained was 20 gr. chloralide from 8 gr. racemic acid, i. e. the theoretic quantity. Very readily soluble in all org. solvents, except petroleum ether, insoluble in H₂O.

Original melt. point = 118°—128°; after repeated solution in alcohol and precipitation with water 164°—166°; in this the melting-point was never constant and much was lost. There have, undoubtedly, been formed isomers; the number of possible isomers is three, as with d-tartaric acid, all racemates or racemic mixtures; they may be indicated by R-trans-trans, R-cis-trans, and R-cis-cis.

Analysis:

0.0341 gr. chloralide equiv.	9.82 cc. AgNO ₃	0.0502 n.	51.3%	Cl	Cal. 52.0%
0.0531	15.6	52.3 ..	Cl	

Of the anti tartaric acid the di-chloralide was likewise obtained with excellent yield. This, too, was clearly a mixture; there may possibly be again three isomers here: i-trans-trans, i-cis-trans, and i-cis-cis; one of them was obtained after repeated recrystallisation with constant melting-point = 171°; Cl = 52.6% Cal. 52.0%. No attempts were made to isolate the other isomers, the purpose set by us, to find a simple method to convert the aliphatic α -hydroxy acids into the corresponding chloralides, having been reached.

Salicylic acid chloralide. In the aromatic hydroxy acids a deviation may be expected, because the nucleus hydrogen atoms can be activated. This expectation proved true, for on application of the method described above, the product formed dissolved in diluted ammonia, which is not the case with the chloralide.

Hence a nucleus activator, as H₂SO₄ is to be excluded here. 15 gr. salicylic acid was boiled with 23 gr. anhydrous chloral (excess) in connection with a reflux condenser, in which beautiful crystals were deposited. After 5 hours' boiling the substance was poured out into H₂O, the mass was washed out with ammonia, and the residue was recrystallized from ether.

Melt. point = 124°—125°; 0.0718 gr. = 15.95 cc. AgNO₃ 0.0502 n. =
= 39.6% Cl; Cal. 39.8% Cl.

In view of this exceedingly easy six-ring formation one of the mono-chlorides observed in citric acid is very probably a β -chloralide (see before).

¹⁾ I found this at Assen already in 1899, it may, for the rest, also be derived from the course of the rotation change with the concentration on one side, and from the change of the rotatory power of the alkylesters with diminishing molecular weight.

Amygdalic acid chloralide. WALLACH has already prepared this chloralide; he states that it melted at 82° – 83° , and that he could obtain from the mother liquid a substance, melt. point 64° , with the chlorine content calculated for chloralide. We have prepared the substance by boiling amygdalic acid with anhydrous chloral (a small excess), as here too complications appeared with conc. H_2SO_4 . In this way a solid light yellow substance was obtained, which was recrystallized from petroleum ether after being washed with H_2O .

The substance appeared to be perfectly homogeneous, and always melted sharply at 59° – 60° , 0.06913 gr. = 14.55 cc. AgNO_3 0.0502 n. = = 37.5% Cl. Calculated 37.8%. Accordingly we cannot confirm WALLACH's communication.

It is advisable not to use H_2SO_4 as condensing substance; melting in with chloral is, however, not necessary; it may be boiled with a slight excess of anhydrous chloral.

Condensation of Glycols with chloral. (With Mr. J. STOK).

For this purpose the method with conc. H_2SO_4 appeared to yield excellent results at moderate temperature. In this way, i. e. as was described for citric acid and tartaric acid, glycerol and glycol were chloralized. The glycerol chloralide is a pretty viscid substance, which distils undecomposed, and is distinguished by a particularly great stability; it may be boiled without decomposition with 60% H_2SO_4 , and is also proof to the action of HNO_3 1.4 at boiling heat. By chromic acid in acid and KMnO_4 in alcalic solution oxidation sets in. Boiling point_{1 mm.} = 142° – 144° Cl % found = 48.8 % calc. = 48.1 %, d_{40}^{17} = = 1.5500, n_D^{17} = 1.5053. The simplest chloralide, the glycol chloralide, the prototype of all these compounds, is also exceedingly resistant; it crystallizes beautifully and has a pleasant smell of camphor; solidifying point = 40° .

The communication of LESTER YODER¹⁾, who prepared the chloralide of glycerol by means of ZnCl_2 for an entirely different purpose, led me to think it desirable to make some communications of our researches.

Delft, November 1926.

¹⁾ Journ. Amer. Chem. Soc. 45, 475, (1923).

Pathology. — *On anophelism without malaria around Amsterdam.*
(From the Institute of Tropical Hygiene, section of the Royal Colonial Institute at Amsterdam.) By N. H. SWELLENGREBEL, A. DE BUCK and E. SCHOUTE. (Communicated by Prof. W. A. P. SCHÜFFNER.)

(Communicated at the meeting of December 18, 1926).

Although we have no map showing the distribution of *Anopheles maculipennis* throughout the whole of the Netherlands, still we know that this mosquito is numerous in localities where malaria is rare or absent and was so likewise in the past. Perhaps this may be explained in this way. We do not contest that the presence of anopheles is a condition absolutely necessary for the existence of malaria in a given locality. But as this condition is fulfilled almost everywhere, the presence or absence of malaria does no longer depend on it, but is determined by wholly different factors. Still we should likewise consider the possibility of different biological races of *A. maculipennis* existing in malarious and non-malarious regions. It stands to reason that one's views regarding the practical means to deal with malaria cannot fail to be influenced by these widely divergent opinions, and it is of the utmost importance, to have the question answered which of them is the correct one.

An investigation to test the validity of the second opinion is the subject of this paper. It is limited moreover to a small region enclosing the southern part of the province of North-Holland and the adjacent portions of South-Holland and Utrecht. We are convinced that such fundamental problems can be solved only by numerous local investigations.

We have chosen this particular field because it is intersected by a line of demarcation, dividing it in two regions I and II, N^o I to the north where malaria is endemic, N^o II to the south where it is absent or extremely rare. The accompanying map shows the distribution of malaria as far as this was to be ascertained by the annual returns collected by the Malaria Commission in North-Holland.

A. maculipennis is numerous in both regions but more so in I than in II. Still there are so many in II, as compared with notorious malarious regions in Europe, where this mosquito is the local vector, that we cannot use this particular quantitative difference between I and II as an explanation of the absence of malaria in II.

The question to be answered is, whether there exists any difference between *A. maculipennis* in I and II. Only such differences are important which bear on the capability of anopheles to transmit malaria.

Consequently they should be biological. Morphological differences cannot interest us unless they show a correlation with biological ones. Still we commenced by looking for the existence of the former because they are easier to detect and to measure and because VAN THIEL's¹⁾ investigations already pointed in that direction.

It is quite out of the question to determine their origin (whether from region I or region II) by examining a single specimen or even a few ones. And still there is a difference. This became apparent by measuring the length of the thorax, the abdomen and the wing²⁾ and by counting the number of maxillary teeth, of hibernating³⁾ anopheles from a number of stations situated in both regions (likewise from elsewhere in the Netherlands and from other European countries).

For the sake of brevity we only consider the length of the wing, which we will indicate by the symbol "W."⁴⁾ For the same reason we omit to mention separately⁵⁾ the result of the measurement of more than 7000 mosquitoes (from 35 stations in region I, 18 in region II, 7 in Friesland and Zeeland and 8 outside the Netherlands) except for the particulars shown in the map.

These measurements show that the average length of the mosquitoes from II exceeds that from I.

The large *Anopheles* from II show an average W. above 122. At least 17 % of them, and usually 22—40 %, are „long wings" having a W. above 129. Among the small *Anophelines* in region I we make a distinction between the smallest type (either W. below 119 with 1—6 % long wings, or W. 119 but then with less than 3 % long wings) and the intermediate type (W. 119—121 and 3—16 %, usually 7—11 % long wings). As a rule (not without exceptions) this last type is to be found on the boundary between I and II. The results of the measurements, in the stations harbouring the large, intermediate and smallest types, taken together, yield the following average figures:

I. Large anopheles	W. = 125.53	Difference between I and III	: 9.21 ± 0.29
II. Intermediate	= 120.36	" .. I .. II	: 5.71 ± 0.29
III. Smallest	= 116.32	" .. II .. III	: 4.04 ± 0.27

All these differences are, consequently, valid because they exceed thrice the average error. In Friesland we examined Bergum, Tietjerk, Leeuwarden, Nieuwe Biltzjil and Bolsward and we found the large type in the first two, the smallest one in the last two and an intermediate condition in Leeuwarden.

1) Arch. f. Schiffs u. Trop. Hyg. Beihefte, XXX, 1926, 70.

2) Length of the wing: from the anterior margin of the alula to the apex of the wing. The length is expressed in units of 41.7 micra each.

3) Hibernating, because they are all, approximately, of the same age and generation. In summer, old and young ones are intermixed. For a comparative investigation it is necessary to start from a homogeneous material.

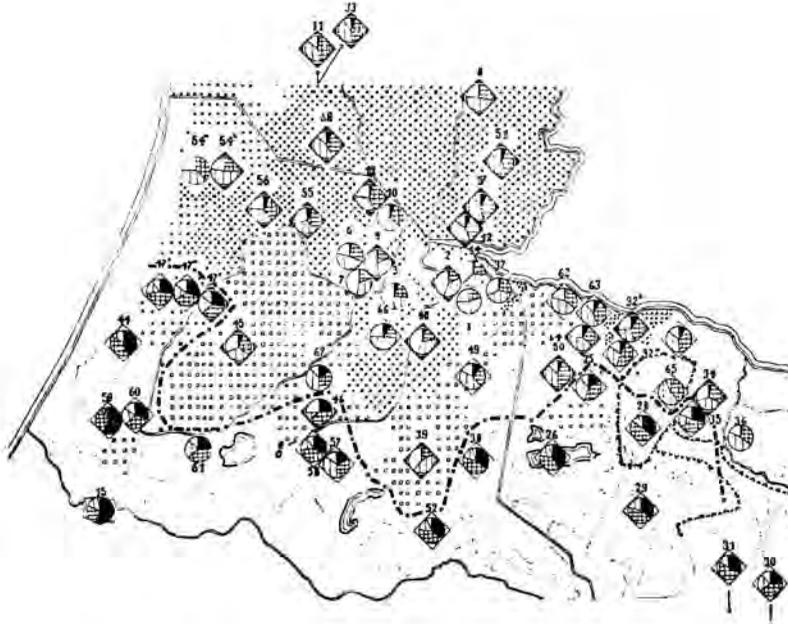
4) No objection can be raised against this limitation as the length of the wing and of the body show a high positive correlation, from + 0.804 ± 0.015 till + 0.820 ± 0.010.

5) For these and other particulars omitted here, we refer to our detailed account to be published elsewhere and to the thesis of one of us. (A. DE BUCK, Acad. Proefschrift. Amsterdam, 14 Dec. 1926).

As is shown in the map, the stations with large and small *Anopheles* are not mixed up together but show a distinct segregation, forming two separate

EXPLANATION OF THE MAP OF THE SURROUNDINGS OF AMSTERDAM.

SCALE 1 : 400.000.



The circles give the situation of the catching stations. The sectors indicate :

black :	‰ anopheles with a W. above 129
squares :	“ “ “ “ of 122—129
vertical lines :	“ “ “ “ 115—121
white :	“ “ “ “ under 115.

The angles of the squares enclosing the circles indicate the chlorine contents of the water in the breedingplaces :

black :	1000 mgr. Cl. p. Litre or more
squares :	500—999 “ “ “ “
white :	under 500 “ “ “ “

A. Area with black dots: approximately 20 ‰ or more of malaria.

B. Area with white dots: approximately 1—5 ‰ of malaria.

C. In the remaining areas the incidence of malaria is less than 1 ‰ or 0.

N^o. 11 and 33 are situated in an area like A, N^o. 30 and 31 in an area like C.

groups; the small ones to the north, the large ones to the south, in such a way that the bulk of the former are situated in the malarious region I and the majority of the latter in the non-malarious region II, in consequence of which these regions can be practically identified with the two harbouring the different forms of *A. maculipennis*.

But there are important exceptions outside our field of observation e. g. in South Beveland (Goes) and the adjacent parts of North-Brabant (Woensdrecht), where small anopheles occur in the absence of malaria. The coexistence of large anopheles and malaria we infer from the fact of mosquitoes we received from Oltenia (Rumania), Prizren (Macedonia)¹⁾ and Moscou, belonging to this last type.

All this holds true for the hibernating generation. In summer the situation changes of aspect: After the hatching of the first summer generation, the females in region I attain the length of the hibernating ones in region II (e.g. W. from 115 to 126). But in region II they have likewise increased in size (e.g. W. from 128 to 132) in consequence of which the difference between the two remains. Next autumn there was again a decrease in size, without however attaining the figures of the preceding winter (e.g. W. = 120 instead of 115 or 131 instead of 128).

Are these two types (the large one from region II and the small one from region I) to be regarded as local modifications, brought into existence by varying external circumstances, or are they to be taken as two hereditary varieties? ²⁾ Before endeavouring to answer this question we wish to refer to a very old theory stating that mixing of salt and fresh water favours the occurrence of severe marshfevers. This theory has been recently recalled to life by: 1^o. GRASSI³⁾ asserting that *A. maculipennis* bred in brackish water is of a more robust disposition than if bred in fresh water and therefore better able to transmit malaria; 2^o. ALESSANDRINI⁴⁾ whose opinion is quite inverse: the mosquitoes are larger and more robust in the fresh and clear water of the ricefields and this enables them to resist malarial infection, 3^o. VAN DER HOEVEN⁵⁾ who holds a similar opinion when ascribing to brackish water an influence rendering Anopheles, bred in it, weaker and of smaller size and more liable to become infected with malaria; 4^o. VAN THIEL⁶⁾ who sustains the same view by his findings of large mosquitoes from the freshwater around Leiden and of small ones from the brackish breedingplaces of Nieuwendam and Bolsward. Large or small mosquitoes, whether they be dangerous or innocuous, they are all considered to be bred under the influence of external circumstances.

Our findings would seem to corroborate this last named view. In the lowlying landreclamations of N.-Holland there are small Anophelines, in the higher districts of Gooi, S.-Holland and Utrecht the large ones occur. In Friesland we observe similar conditions. In localities where the districts, inhabited by small Anophelines, run like a wedge into those where the long ones are prevalent, one finds particularly lowlying areas (Mijdrecht, Haarlemmermeer). The chlorine contents of the breeding places near to

1) By the kindness of Messrs ZOTA, SFARCIC and MARZINOWSKI.

2) Or as two populations with a different hereditary composition.

3) Rendic. Re. Ac. Lincei, 1922, XXXI, 535.

4) La risicoltura etc. d'Italia. Roma 1925. ROUBAUD and SERGENT likewise find large anopheles in regions without malaria.

5) Cited after van THIEL, loc. cit.

our catching-stations was, as a rule, high in region I (54 % above 1000 mgr. Cl. per Litre and 17 % below 200 mgr.) and low in region II (none above 1000 and 76 % below 200 mgr. Cl. per L.) which is likewise shown in the map ¹⁾).

But we had to modify our opinion when we measured *Anopheles* hatched from pupae which we caught in breeding places showing a widely divergent chlorine contents, because these *Anopheles* did not differ much with regard to their size. The W. of *Anopheles* hatched from pupae caught in water with :

330—225 m.gr. Cl. p. L.	was 123	in 40 females	and 113	in 30 males,
4515—2955	" " " "	" " 125	" " 34	" " 107 " 32 " "
3620—1685	" " " "	" " 124	" " 30	" " 106 " 29 " "

One might even be tempted to assert that a higher chlorine contents increases the size, of the females at least. A similar negative result attended our breeding experiments in water from the "Zuiderzee" (3000 mgr. Cl. p. L.) and in tapwater with 0.25 % NaCl, which hatched 279 females with a W. (108.6) equal to that of 450 bred in fresh water (108.5), the males being even slightly longer (276 with W. = 98.1 against 287 with W. = 95.0).

Breeding at higher temperature ²⁾ (22°—25° against 17°—19°) caused a better marked difference in size. Bred at:

22°—25°	118 females	showed a W. of 101.9;	110 males	a W. of 86.2
17°—19°	134	" " " "	" " 106.6;	211 " " " " 91.7.

But this cannot help us to explain the difference in size of the mosquitoes from Santpoort and Sassenheim or from Diemen and Maariensdijk !

Still it would be an error to suppose that the size of the mosquitoes is independent from external conditions. But neither the chlorine contents nor the temperature are directly concerned, but the food is. In our breeding experiments we fed the larvae with green unicellular algae, which we scratched from the bark of trees ("tree-algae"). This diet suited them in so far that they pupated and that adults did hatch ; but these were subnormal in size (W. of the females 100—113 against 126—130). Even almost mature larvae, fed on this diet during the last 2—5 days of their larval existence only produced undersized adults, whereas pupae caught in nature and allowed to hatch in the laboratory did not suffer in this way. This distinctly points to the influence of the food. This opinion was

¹⁾ We find it difficult to imagine that the brackish water influences *Anopheles* unfavourably: In region I the average number of larvae per dip was 2.2, against 0.7 in region II. Within certain limits this number increased with increasing chlorine contents, an observation already recorded by SWELLENGREBEL (Ned. Tijdsch. v. Geneesk. 1922, 2nd. half. p. 350—359).

²⁾ In accordance with MARTINI's view (Arch. f. Sch. u. Trop. Hyg. XXVI, 1922 and XXVIII, 1924) that *A. maculipennis* is larger in the cool Northern Europe than in the hotter Southern Europe.

confirmed when we tried to breed larvae without tree-algae, by simply adding numerous waterplants (*Elodea canadensis*). We failed in this, unless we added the algae, except in those cases where the vegetation began to die. For then a rich growth of *Chilomonas paramaecium* occurred and this flagellate seems to be particularly suitable as food, for the larvae bred on this diet produced almost normal adults.

Supposing that the difference in size between the mosquitoes in region I and II is caused by a difference in natural diet¹⁾, we should succeed in rearing adults of the same average size when starting our breeding experiments with eggs deposited by two lots of adults, the large ones caught in a station of region II, the small ones from region I, on condition that temperature, P_H , chlorine contents (and other circumstances relating to the water) and food supply are the same in both sets of experiments. But this proved not to be the case. Bred under identical external conditions the eggs of mosquitoes from region II produced larger adults, than these from region I. Eggs from :

region I yielded 131 females with $W. = 101.8$ and 88 males with $W. = 87.7$
 „ II „ 241 „ „ $W. = 112.6$ „ 159 „ „ $W. = 98.4$.

Consequently we believe that the difference in size between the mosquitoes of both regions is not only caused by a dissimilarity of external conditions, but likewise by a difference of the hereditary constitution of the anopheline populations in both regions.

We have now to face the question whether these morphological differences are correlated with biological ones, which may influence the ability of anopheles to transmit malaria. The most obvious difference: dissimilarity in infectability, could not be detected: the mosquitoes of both regions showed the same degree (13 % with *Plasmodium vivax*) of infectability²⁾.

But we found two other biological differences which are of importance in this respect:

1⁰. We compared the inclination of mosquitoes from both regions to take human blood³⁾. In June and September those from region II were

1) Which is a likely supposition, because the vegetation of the breedingplaces of both regions is somewhat different (more *Elodea* and *Hydrocharis* in II and more *Potamogeton pectinatus* and *Myriophyllum* in I). Perhaps the chlorine contents influences the larvae indirectly by modifying the microflora and -fauna.

2) In a number of experiments conducted with 224 *A. maculipennis*, 131 from region I (17 infected) and 93 from region II (12 infected), each pair of experiments being performed with an identical gamete carrier.

3) In large wooden cages, with glass and wire-gauze walls and a hole to pass the human arm through. Feeding experiments were performed during the day-time and at night, but here we only take the latter into account. It was curious to see how this difference in voracity was almost completely obscured if the mosquitoes were kept in small glass jars covered by gauze and were allowed to feed by applying this gauze on to the human skin. In June the mosquitoes from region II were nearly as ready to feed in this way as those from region I, whereas they almost completely refused to do so in the large cages.

much less voracious than the mosquitoes from region I, but in August there was little difference between the two. Experiments, conducted with many hundreds of mosquitoes, showed the following percentages of anopheles taking human blood:

in June	19.4 ‰	for region I	and	1.9 ‰	for region II,
„ August	25.1 ‰	„ „ I	„	21.9 ‰	„ „ II,
„ September	25.2 ‰	„ „ I	„	8.5 ‰	„ „ II,
„ October	37.0 ‰	„ „ I	„	0 ‰	„ „ II.

20. When determining the "blood number" (i.e. the percentage of females with blood in their stomachs) of stable mosquitoes in region I and II, this number was found to be at its maximum (87 and 77 % respectively) in June. In region I it did not decrease much till November (29 %). In region II this occurred as early as September and in October it became 0. The "fat-number" (i.e. percentage of females with a well-developed fat-body) was high in region II since the beginning of autumn (sometimes above 90 %), but it was low in region I (1—20 %) ¹⁾.

To use GRASSI's terminology ²⁾, we have found "semihibernation" in region I and a complete hibernation in region II. Moreover it appears that the lack of appetite in our experiments, shown by the mosquitoes of region II in autumn, is not a sign of a disinclination to take human blood, but to take blood of any kind. But the same absence of appetite in June has quite a different meaning as it occurs at a time when the "bloodnumber" indicates a maximum of voracity with regard to animal blood. Here there is reason to assume a disinclination limited to human blood ³⁾.

In which way do these biological differences, between anopheles of region I and II, influence their ability to transmit malaria? To answer this question it should first be remembered, that the annual malaria epidemic in N.-Holland occurs in spring, with a maximum in May or June. According to KORTEWEG's ⁴⁾ and HONIG's ⁵⁾ investigations, primary cases occur as early as February and these authors hold the view that many of the spring-cases are a consequence of an infection incurred in the preceding autumn. This view is supported by our observations ⁶⁾ showing that the

1) With regard to the „bloodnumber" in region I, our experience since 1920 allows us to consider the statement mentioned here as a general rule. But our statements regarding the „bloodnumber" in region II and the „fatnumber" in both regions should not be taken this way until our present findings are confirmed by a more extensive examination.

2) Ann. d'Igiena 1923, XXXII, 438.

3) We make this statement with some reserve, because the anopheles from region I likewise show in June an appetite for human blood which is less than in the following months, whereas our experience teaches us that in human habitations the bloodnumber usually reaches its maximum in June, just as it does in stables.

4) Geneesk. Bladen, 22nd Series, No. 1, 1920.

5) Studie over de malaria te Nieuwendam en omgeving. Amsterdam 1922, 79 pp.

6) SWELLENGREBEL, Ned. Tijdschr. v. Geneesk. 1924, 2nd half, p. 750—763.

majority of anopheline infections are found in autumn, infections which they can transmit to man because bloodsucking in winter is never wholly discontinued. The complete hibernation of anopheles in region II prevents it from taking part in these winter infections. But perhaps the malaria cases in May and June are caused by infections incurred in spring? Even granted that this may be so, the anopheles of region II cannot take a share in it because of their disinclination towards human blood at that time of the year. The fact that later on they become less fastidious may cause them to incur a malaria infection in early autumn. But complete hibernation, setting in at that time, prevents them from transmitting it to man and even if they keep it through the whole length of the ensuing winter, they will spill it next spring into the blood of some animal.

The results of our continued investigations will have to decide whether or not this is the true explanation of "anophelism without malaria" in our field of observation. Even if they should completely confirm our present statements, we shall not be tempted to make them serve as a general explanation of this phenomenon. For we are convinced that by the term, mentioned above, quite a variety of phenomena is denoted with widely divergent causes. A long series of local investigations should precede any attempt to give a general explanation of this complex of phenomena.

Mathematics. — *On the Development of a Function of Two Real Variables into Series of Polynomials.* By W. P. THIJSEN.
(Communicated by Prof. J. C. KLUYVER.)

(Communicated at the meeting of November 27, 1926).

1. In *Statistics* use is sometimes made of an approximating function with five parameters

$$Z = \frac{hk\sqrt{1-r^2}}{\pi} e^{-E} \dots \dots \dots (1)$$

where

$$E = h^2(X-a)^2 + 2rhk(X-a)(Y-b) + k^2(Y-b)^2 \dots (1^a)$$

in order to be able to characterize in a more lucid way statistical relations which depend on two arguments.

Especially the *correlation-coefficient* r , which is computed from the original data, has appeared to be of importance.

If e.g. we want to examine whether there exists a more or less close connection between the lengths of fathers and sons, between the fertility of mothers and daughters, between the discount and the ratio of reserve and deposit at the american banks, the magnitude of the correlation-coefficient may throw some light on it. ¹⁾

In the *Theory of Errors* a similar characteristic function has been used of old, to wit the well known so called *law of errors* of GAUSS

$$Y = \frac{h}{\sqrt{\pi}} e^{-h^2(X-a)^2} \dots \dots \dots (2)$$

This contains only two parameters: the so called *true value* a and the *precision modulus* h .

The insight into the signification of this method was greatly deepened when it was found ²⁾ that for any real univalent function which satisfies the *conditions of DIRICHLET* and which, besides, is zero outside a certain finite interval, developments into series of polynomials may be found which converge to the value of the function, except in points where the function is discontinuous.

As a rule the function is first divided by the expression in the right-hand member of (2) where the parameters are chosen such that the second and the third term in the development vanish. It is no caprice

¹⁾ For more particulars we refer to the exposition in: G. H. YULE, *An introduction to the theory of statistics*, Londen, Griffin, 1922.

²⁾ H. GALBRUN. *Sur un développement d'une fonction à variable réelle en séries de polynômes.* Bull. de la Soc. Mathém. de France, Tome 41, 1913.

that we divide exactly by this function, — which we shall call *division-function*, — as it is closely connected with the above mentioned development of which the terms have the useful property of orthogonality.

The question arises whether in the problem of two dimensions there also exist such developments, which might give more insight into the said statistical method.

2. It appears that the above mentioned problems are closely connected with the so called *problem of the moments*.

The problem is in how far a function $f(x)$ is determined by its moments

$$\int_{-\infty}^{+\infty} dx x^p f(x) \dots \dots \dots (3)$$

on the supposition that these integrals exist.

If e.g. the function has a finite number of discontinuities, it is evident that the value of the function is not defined, at least not at these points.

For a function of two arguments we may in the same way consider moments of the form

$$\iint_{-\infty}^{+\infty} dx dy x^p y^q f(x, y) \dots \dots \dots (4)$$

In what follows we shall suppose that the functions occurring there are finite, that they have a limited number of discontinuities, that the integrals with which we have to do, exist, that the applied interchanges of the operations are allowed and that, besides, the following theorem holds good. This is only another form of the proposition of the moments, which form we have given to it as it is the fittest for our purpose.

I. If the relation

$$\int_{-\infty}^{+\infty} dx x^p f_m(x) = \int_{-\infty}^{+\infty} dx x^p f(x) \quad ; \quad p = 1, 2, \dots, m \dots \dots (5)$$

exists we have

$$\lim_{m \rightarrow \infty} f_m(x) = f(x) \dots \dots \dots (5^a)$$

except for values of x for which $f(x)$ is discontinuous.

For a function of two variables $f(x, y)$ we make the following suppositions. Let us consider the surface represented by

$$z = f(x, y)$$

If we cut this surface by planes that have the equation

$$x + ry = C_1$$

where the constant C_1 may assume any real value and r is a real number with modulus less than 1, we assume that the curve arising in this way is a function of the new variable

$$u = x + ry \quad \dots \dots \dots (6)$$

which also satisfies the above mentioned conditions.

Let us suppose the same for the intersection by planes with the equation

$$y = C_2$$

where we introduce the new variable

$$v = sy \quad \dots \dots \dots (7)$$

For abbreviation we have put

$$s = \sqrt{1-r^2} \quad \dots \dots \dots (8)$$

In this two-dimensional case the proposition of the moments reads quite analogously:

II. If the relation

$$\iint_{-\infty}^{+\infty} dx dy x^p y^q f_m(x, y) = \iint_{-\infty}^{+\infty} dx dy x^p y^q f(x, y) \quad ; \quad \begin{matrix} p = 1, 2, \dots, m; \\ q = 1, 2, \dots, m. \end{matrix} \quad (9)$$

exists, we have

$$\lim_{m \rightarrow \infty} f_m(x, y) = f(x, y). \quad \dots \dots \dots (9^a)$$

except for values for which $f(x, y)$ is discontinuous.

Among others STIELTJES¹⁾ and BOREL²⁾ have occupied themselves with the problem of the moments; as appears from recent prize questions of the *Wiskundig Genootschap* the study of it has not yet come to a theoretically satisfying conclusion so that we are still a long way from the necessary conditions for the validity of the proposition of the moments. Practically, however, there is no objection to include in the conditions that the function never becomes negative and that it is zero outside a certain finite interval (resp. region). A further discussion of sufficient conditions under which the proposition of the moments holds good, lies outside the scope of this paper.

3. Let $f(x)$ be a function which satisfies the above mentioned conditions. Let us consider

$$f_m(x) = c_0 P_0(x) + c_1 P_1(x) + \dots + c_m P_m(x) \quad \dots \dots (10)$$

where

$$c_n = \frac{1}{2^n n! \sqrt{\pi}} \int_a^b d\alpha e^{-\alpha^2} f(\alpha) P_n(\alpha) \quad \dots \dots \dots (10^a)$$

¹⁾ Annales de la Faculté des Sciences de Toulouse, Tome 8, 9, (1894, 1895).

²⁾ Leçons sur les séries divergentes, Paris, 1901, p. 66 e.v.

We can prove that the relation (5^a) holds good in the interval (a, b) with the above mentioned reservation.

The polynomials $P_n(x)$ are the so called *polynomials of HERMITE*, defined thus:

$$P_n(x) = (-1)^n e^{x^2} D^{(n)} e^{-x^2} \dots \dots \dots (11)$$

From this we can derive the formula

$$P_n(x) = (2x)^n - \frac{n(n-1)}{1!} (2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4} - \dots \dots (12)$$

We shall also want the following identity

$$(2x)^n = P_n(x) + \frac{n(n-1)}{1!} P_{n-2}(x) + \frac{n(n-1)(n-2)(n-3)}{2!} P_{n-4}(x) + \dots \dots (13)$$

The above mentioned property of orthogonality reads

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx e^{-x^2} P_m(x) P_n(x) = \begin{cases} 0, & \text{if } m \neq n; \\ 2^n n!, & \text{if } m = n. \end{cases} \dots \dots (14)$$

As all these formulas are well known we shall leave the proofs. According to (10^a) we may also write for (10)

$$f_m(x) = \frac{1}{\sqrt{\pi}} \int_a^b da e^{-a^2} f(a) \sum_{n=0}^m \frac{P_n(a) P_n(x)}{2^n n!}$$

If now we determine the moments of $f_m(x)e^{-x^2}$ we find for them according to (14) and (13)

$$\int_{-\infty}^{+\infty} dx x^p f_m(x) e^{-x^2} = \int_a^b da a^p f(a) e^{-a^2}; \quad p = 1, 2, \dots, m$$

If then we apply the proposition of the moments (5^a) holds good again with the known reservation, so that, disregarding this we get:

$$\lim_{m \rightarrow \infty} f_m(x) = \begin{cases} f(x), & \text{if } a < x < b. \\ 0, & \text{if } x \text{ lies outside the interval } (a, b) \end{cases} \quad (15)$$

In the meantime GALBRUN¹⁾ has proved in a rather subtle demonstration, independent from the proposition of the moments, that under the above mentioned conditions to which we must add the occurrence of a finite number of maxima and minima, $f_m(x)$ has the following values as limit:

$$\left. \begin{aligned} & \frac{1}{2} [f(x+) + f(x-)] && \text{if } x \text{ lies between } a \text{ and } b; \\ & \frac{1}{2} f(a) && \text{if } x \text{ is equal to } a; \\ & \frac{1}{2} f(b) && \text{if } x \text{ is equal to } b; \\ & 0 && \text{if } x \text{ lies outside the interval } (a, b). \end{aligned} \right\} \quad (15^a)$$

¹⁾ loc. cit.

4. Let us now introduce polynomials $P_{i,j}(x,y)$ defined thus:

$$P_{i,j}(x,y) = (-1)^{i+j} e^E \frac{\partial^{i+j}}{\partial x^i \partial y^j} e^{-E} \dots \dots \dots (16)$$

where

$$E = x^2 + 2 rxy + y^2 \dots \dots \dots (17)$$

In this way we have e.g.

$$\left. \begin{aligned} P_{0,0}(x,y) &= 1; & P_{1,0}(x,y) &= 2(x + ry); & P_{0,1}(x,y) &= 2(rx + y); \\ P_{2,0}(x,y) &= 4(x + ry)^2 - 2; & P_{1,1}(x,y) &= 4(x + ry)(rx + y) - 2r; \\ P_{0,2}(x,y) &= 4(rx + y)^2 - 2; & \text{etc.} \end{aligned} \right\} (18)$$

In order not to make the notation too complicated we have left r out of it; we shall also take this liberty in other cases when there is no danger of confusion.

5. In case $r \neq 0$ it appears to be useful to introduce the new variables u and v , which have already been indicated in the formulas (6) and (7).

This substitution transforms E (17) into a sum of two squares, which we shall indicate by E_1 in order to distinguish it.

$$E_1 = u^2 + v^2 \dots \dots \dots (19)$$

We have still to remark that the substitution determinant is

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \frac{1}{s} \dots \dots \dots (20)$$

If we transform the differentiations with respect to x and y into those with respect to u and v , this may be expressed symbolically by

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u}; \quad \frac{\partial}{\partial y} = r \frac{\partial}{\partial u} + s \frac{\partial}{\partial v} \dots \dots \dots (21)$$

From this we can derive

$$\frac{\partial^n}{\partial x^{n-i} \partial y^i} = \frac{\partial^{n-i}}{\partial u^{n-i}} \left(r \frac{\partial}{\partial u} + s \frac{\partial}{\partial v} \right)^i e^{-E}$$

which applied to gives

$$P_{n-i,i}(x,y) = r^i P_n(u) P_0(v) + i_1 r^{i-1} s P_{n-1}(u) P_1(v) + \dots + s^i P_{n-i}(u) P_i(v) (22)$$

according to (11) and (16).

6. Now by the aid of (22), (20) and (19) we can transform the following double integral

$$I_{k,l}^{m,n} = \frac{s}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy e^{-E} P_{m-k,k}(x,y) P_{n-l,l}(x,y) \dots \dots (23)$$

so that it passes into a linear expression of integrals of the form

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} du e^{-u^2} P_{m-p}(u) P_{n-q}(u) \int_{-\infty}^{+\infty} dv e^{-v^2} P_p(v) P_q(v) \dots \quad (24)$$

According to (14) these are always equal to 0, if $m \neq n$. For: if the indices of the two polynomials of HERMITE were equal in one integral, those in the other one would be different. Hence we have

$$I_{k,l}^{m,n} = 0; \quad m \neq n \dots \dots \dots (25)$$

If on the other hand $m = n$, the integrals (24) become 0 for $p \neq q$, but for $p = q$ they are equal to

$$2^n p!(n-p)! \dots \dots \dots (26)$$

according to (14).

Let us now consider

$$a_{k,l}^{(n)} = \frac{S}{\pi} \iint_{-\infty}^{+\infty} dx dy e^{-E} P_{n-k,k}(x,y) P_{n-l,l}(x,y) \dots \dots \dots (27)$$

If we suppose this integral to be again transformed and, accordingly, the formula (22) to be applied to the polynomials $P_{n-k,k}(x,y)$ and $P_{n-l,l}(x,y)$, it will be sufficient in the multiplication to take the products of the corresponding terms so that by the aid of (26) we finally arrive at

$$a_{k,l}^{(n)} = 2^n n! \left[\frac{k_0 l_0}{n_0} r^{k+l} + \frac{k_1 l_1}{n_1} r^{k+l-2} s^2 + \frac{k_2 l_2}{n_2} r^{k+l-4} s^4 + \dots \right] \dots \quad (28)$$

7. In order to deduce a relation which we shall want further below, it is necessary to consider a product of determinants.

Let us multiply the determinant

$$\begin{vmatrix} 0! 0_0 2^0 r^0 s^0 & 0 & 0 & \dots & 0 & 0 \\ 0! 1_0 2^0 r^1 s^0 & 1! 1_1 2^1 r^0 s^1 & 0 & \dots & 0 & 0 \\ 0! 2_0 2^0 r^2 s^0 & 1! 2_1 2^1 r^1 s^1 & 2! 2_2 2^2 r^0 s^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0! n_0 2^0 r^n s^0 & 1! n_1 2^1 r^{n-1} s^1 & 2! n_2 2^2 r^{n-2} s^2 \dots & n! n_n 2^n r^0 s^n & 0 & 0 \\ \frac{P_0(u) P_n(v)}{n! 2^n} & \frac{P_1(u) P_{n-1}(v)}{(n-1)! 2^{n-1}} & \frac{P_2(u) P_{n-2}(v)}{(n-2)! 2^{n-2}} \dots & \frac{P_n(u) P_0(v)}{0! 2^0} & 1 & \end{vmatrix} \quad (29)$$

by the following determinant that is evidently zero

$$\begin{vmatrix} n!0_0 2^n r^0 s^0 & 0 & 0 & \dots & 0 & 0 \\ n!1_0 2^n r^1 s^0 & (n-1)!1_1 2^{n-1} r^0 s^1 & 0 & \dots & 0 & 0 \\ n!2_0 2^n r^2 s^0 & (n-1)!2_1 2^{n-1} r^1 s^1 & (n-2)!2_2 2^{n-2} r^0 s^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n!n_0 2^n r^n s^0 & (n-1)!n_1 2^{n-1} r^{n-1} s^1 & (n-2)!n_2 2^{n-2} r^{n-2} s^2 \dots 0!n_n 2^0 r^n s^n & 0 & 0 & 0 \\ \frac{P_0(\varphi)P_n(\psi)}{0!2^0} & \frac{P_1(\varphi)P_{n-2}(\psi)}{1!2^1} & \frac{P_2(\varphi)P_{n-2}(\psi)}{2!2^2} & \dots & \frac{P_n(\varphi)P_0(\psi)}{n!2^n} & 0 \end{vmatrix} \quad (30)$$

by applying the well known rule for the multiplication to the rows. According to (22) and (28), if through the substitution $a, \beta; x, y$ are transformed into $\varphi, \psi; u, v$, there results

$$\begin{vmatrix} a_{0,0}^{(n)} & a_{0,1}^{(n)} & a_{0,2}^{(n)} & \dots & a_{0,n}^{(n)} & P_{n,0}(\alpha, \beta) \\ a_{1,0}^{(n)} & a_{1,1}^{(n)} & a_{1,2}^{(n)} & \dots & a_{1,n}^{(n)} & P_{n-1,1}(\alpha, \beta) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,0}^{(n)} & a_{n,1}^{(n)} & a_{n,2}^{(n)} & \dots & a_{n,n}^{(n)} & P_{0,n}(\alpha, \beta) \\ P_{n,0}(x,y) & P_{n-1,1}(x,y) & P_{n-2,2}(x,y) & \dots & P_{0,n}(x,y) & S_n \end{vmatrix} = 0 \quad (31)$$

where

$$S_n = \left. \begin{aligned} & \frac{P_0(\varphi)P_0(u)}{2^0 0!} \times \frac{P_n(\psi)P_n(v)}{2^n n!} + \frac{P_1(\varphi)P_1(u)}{2^1 1!} \times \\ & \times \frac{P_{n-1}(\psi)P_{n-1}(v)}{2^{n-1} (n-1)!} + \dots + \frac{P_n(\varphi)P_n(u)}{2^n n!} \times \frac{P_0(\psi)P_0(v)}{2^0 0!} \end{aligned} \right\} \quad (32)$$

If we indicate the following determinant by Δ_n

$$\Delta_n = \begin{vmatrix} a_{0,0}^{(n)} & a_{0,1}^{(n)} & \dots & a_{0,n}^{(n)} \\ a_{1,0}^{(n)} & a_{1,1}^{(n)} & \dots & a_{1,n}^{(n)} \\ \dots & \dots & \dots & \dots \\ a_{n,0}^{(n)} & a_{n,1}^{(n)} & \dots & a_{n,n}^{(n)} \end{vmatrix} \dots \dots \dots \quad (33)$$

we may also write (31) in the form

$$\begin{vmatrix} a_{0,0}^{(n)} & a_{0,1}^{(n)} & \dots & a_{0,n}^{(n)} & P_{n,0}(\alpha, \beta) \\ a_{1,0}^{(n)} & a_{1,1}^{(n)} & \dots & a_{1,n}^{(n)} & P_{n-1,1}(\alpha, \beta) \\ \dots & \dots & \dots & \dots & \dots \\ a_{n,0}^{(n)} & a_{n,1}^{(n)} & \dots & a_{n,n}^{(n)} & P_{0,n}(\alpha, \beta) \\ P_{n,0}(x,y) & P_{n-1,1}(x,y) & \dots & P_{0,n}(x,y) & 0 \end{vmatrix} = -\Delta_n S_n \quad (34)$$

If we omit the $(n+2)^{nd}$ row and the $(n+2)^{nd}$ column of the determinants (29) and (30) and if after that we multiply in the same way as before, we find:

$$\Delta_n = (0!1!2! \dots n!)^2 (2s)^{n(n+1)} \dots \dots \dots \quad (35)$$

(35^b) these systems of equations always admit of one solution for $|r| < 1$.

We shall now give another form to the coefficients c^{n-j} . With a view to this we introduce the quantities $A_{ij}^{(n)}$ which are the coefficients of the elements $a_{ij}^{(n)}$ in the determinant Δ_n of (33).

According to (38) we have

$$\Delta_n \times c_{n-j} = A_{0,j}^{(n)} C_0^{(n)} + A_{1,j}^{(n)} C_1^{(n)} + \dots + A_{n,j}^{(n)} C_n^{(n)}$$

According to (37) this may be written as

$$\frac{s}{\pi} \iint_G da d\beta e^{-E} f(\alpha, \beta) \sum_{i=0}^n A_{i,j}^{(n)} P_{n-i,i}(\alpha, \beta)$$

Hence (36) can be transformed to

$$f_m(x, y) = \frac{s}{\pi} \iint_G da d\beta e^{-E} f(\alpha, \beta) \sum_{n=0}^m \frac{1}{\Delta_n} \sum_{i=0}^n \sum_{j=0}^n A_{i,j} P_{n-i,i}(\alpha, \beta) P_{n-j,j}(x, y)$$

or

$$\frac{s}{\pi} \iint_G da d\beta e^{-E} f(\alpha, \beta) \sum_{n=0}^m \frac{1}{\Delta_n} \begin{vmatrix} a_{0,0}^{(n)} a_{0,1}^{(n)} & \dots & \dots & a_{0,n}^{(n)} P_{n,0}(\alpha, \beta) \\ a_{1,0}^{(n)} a_{1,1}^{(n)} & \dots & \dots & a_{1,n}^{(n)} P_{n-1,1}(\alpha, \beta) \\ \dots & \dots & \dots & \dots \\ a_{n,0}^{(n)} a_{n,1}^{(n)} & \dots & \dots & a_{n,n}^{(n)} P_{0,n}(\alpha, \beta) \\ P_{n,0}(x, y) P_{n-1,1}(x, y) & \dots & \dots & P_{0,n}(x, y) 0 \end{vmatrix}$$

We now introduce the new variables u and v considered above. According to (34) and (20) we finally arrive at

$$f_m(x, y) = \frac{1}{\pi} \iint_{G_1} d\varphi d\psi e^{-E_1} f_1(\varphi, \psi) \sum_{n=0}^m S_n \dots \dots \dots (39)$$

Here G_1, E_1 and $f_1(\varphi, \psi)$ are formed from G, E and $f(\alpha, \beta)$ through the transformation.

$f_m(x, y)$ can also be transformed, which we shall indicate by the notation $f_{m,1}(u, v)$.

We shall now consider the moments of the function

$$f_{m,1}(u, v) \times e^{-E_1}$$

Apart from the factor 2^{p+q} these moments are

$$\int_{-\infty}^{+\infty} du dv (2u)^p (2v)^q f_{m,1}(u, v) e^{-E_1}$$

If for $f_{m,1}(u, v)$ we write the expression in the right hand member of (39) and if by the aid of (13) we express $(2u)^p$ and $(2v)^q$ in polynomials

of HERMITE, we may change the order of the integrations and after that apply what was found in § 6.

After a few reductions we find by again applying formula (13) from right to left

$$\iint_{-\infty}^{+\infty} du dv (2u)^p (2v)^q f_{m,1}(u,v) e^{-E_1} = \iint_{G_1} d\varphi d\psi (2\varphi)^p (2\psi)^q f_1(\varphi,\psi) e^{-E_1}$$

$$p = 1, 2, \dots, m; \quad q = 1, 2, \dots, m.$$

Hence, according to the proposition of the moments

$$\lim_{m \rightarrow \infty} f_{m,1}(u,v) = \begin{cases} f_1(u,v) & \text{inside the region } G_1; \\ 0 & \text{outside the region } G_1, \end{cases}$$

except again for the values of the argument for which the function is discontinuous.

Getting back again to the former variables we find

$$\lim_{m \rightarrow \infty} f_m(x,y) = \begin{cases} f(x,y) & \text{inside the region } G; \\ 0 & \text{outside the region } G, \end{cases} \quad (40)$$

which was to be proved.

9. As we remarked before, in practice we get to work differently and make use of a division-function. However, this method may also be made to fit in the scheme of the above mentioned considerations.

Let $F(X, Y)$ be the function under consideration and let us suppose that

$$\iint_G dX dY F(X, Y) = 1. (41)$$

which may be ev. reached by a suitable constant.

Let us consider the linear substitution

$$X = a + \frac{x}{h}; \quad Y = b + \frac{y}{k} \quad (42)$$

through which the region G is transformed into g .

We have only to substitute in the above

$$f(x,y) = \frac{\pi}{hks} e^{+E} F\left(a + \frac{x}{h}, b + \frac{y}{k}\right) (43)$$

where

$$E = x^2 + 2 rxy + y^2 \quad (44)$$

For the present the constants, a, b, h, k appearing in these formulas are arbitrary.

Finally we arrive at a development of the form

$$F(X, Y) = \frac{hks}{\pi} e^{-E} \{ c_{0,0} + c_{1,0} P_{1,0}(x, y) + c_{0,1} P_{0,1}(x, y) + \dots \} \quad (45)$$

with

$$x = h(X-a); \quad y = k(Y-b) \quad \dots \quad (45^a)$$

For the coefficient $c_{0,0}$ we get the equation

$$a_{0,0}^{(0)} c_{0,0} = C_0^{(0)}$$

according to (38), which, considering (41) an (35^a), gives

$$c_{0,0} = 1 \dots \dots \dots (46)$$

Now we choose the constants a, b, h, k and r such that the coefficients $c_{1,0}; c_{0,1}; c_{2,0}; c_{1,1}$ and $c_{0,2}$ all become zero. According to (38) this means that the integrals $C_i^{(n)}$ must be zero for $n=1$ and 2.

If we get back to the original variables, these integrals are

$$C_i^{(n)} = \frac{1}{\pi} \int_G dX dY F(X, Y), P_{n-i,i}(hX-ha, kY-kb) \dots \quad (47)$$

After some transformation we find from this for $n=1$ in accordance with (18) and (41)

$$a = \iint_G dX dY X F(X, Y) ; \quad b = \iint_G dX dY Y F(X, Y) \quad \dots \quad (48)$$

Let us now consider the case $n=2$ and let us introduce the following quantities

$$\left. \begin{aligned} M_{2,0} &= \iint_G dX dY F(X, Y) (X-a)^2 \\ M_{1,1} &= \iint_G dX dY F(X, Y) (X-a) (Y-b) \\ M_{0,2} &= \iint_G dX dY F(X, Y) (Y-b)^2 \end{aligned} \right\} \dots \quad (49)$$

In connection with (18) and (41) we get the following equations to determine h, k and r

$$\left. \begin{aligned} M_{2,0} h^2 + 2 r M_{1,1} h k + r^2 M_{0,2} k^2 &= \frac{1}{2} \\ r M_{2,0} h^2 + (1+r^2) M_{1,1} h k + r M_{0,2} k^2 &= \frac{1}{2} r \\ r^2 M_{2,0} h^2 + 2 r M_{1,1} h k + M_{0,2} k^2 &= \frac{1}{2} \end{aligned} \right\} \dots \quad (50)$$

If we solve these equations we find

$$r = \frac{M_{1,1}}{\sqrt{(M_{2,0} \times M_{0,2})}} \dots \dots \dots (51)$$

and h and k are defined by

$$\frac{1}{h^2} = 2 s^2 M_{2,0} \quad ; \quad \frac{1}{k^2} = 2 s^2 M_{0,2} \quad \dots \dots \dots (52)$$

It would lead us too far if we wanted to bring to light all the peculiarities of the theory of correlations for this case although also the regression-lines may be investigated in connection with the development (45).

Let the remark suffice that if the further terms may not be neglected it will be convenient for practical calculation to introduce the auxiliary variables

$$U = h(X-a) + rk(Y-b) \quad ; \quad V = sk(Y-b) \quad \dots \dots (53)$$

For by the aid of them in the development (45) the polynomials $P_{i,j}$ of (22) may be replaced by sums and products of polynomials of HERMITE. And in that case we can make use of the tables that have been calculated by BRUNS ¹⁾.

It is hardly necessary to say that the two-dimensional problem which has been treated here, may be extended to more dimensions.

That the developments at which we arrive in this way are not without importance also from a purely mathematical point of view, follows already from the fact that we can base on formula (40) a method of integration that can be considered as an extension of the method with the discontinuous factor of LEJEUNE-DIRICHLET.

Nymegen, November 20, 1926.

¹⁾ H. BRUNS. Wahrscheinlichkeitsrechnung und Kollektivmasslehre. Leipzig und Berlin, 1906.

Anatomy. — *Indices for the anthropology of the brain applied to Chinese, dolicho- and brachycephalic Dutch, foetuses and neonati.*
By C. U. ARIËNS KAPPERS.

(Communicated at the meeting of November 27, 1926.)

The results attained hitherto in the study of the brains of different races and types are disappointing to students of the subject.

Even the variations in cerebral morphology, that have been described in types of the same country are based more on general impressions than on exact measurements.

The best researches in this respect are CALORI's¹⁾, who, besides giving the weight and surface figures of a great number of brachycephalic and dolichocephalic Italian brains, also tried to express the general type of those brains by means of the length-breadth index, the bitemporal diameter, and the shape of the fronto-occipital arch, which he indicated by its length, its cord and its perpendicular without, however, simplifying these in indices (l.c. p. 55.).

CALORI also examined the convolitional pattern in dolichocephalics and brachycephalics, but without obtaining important general results.

RÜDINGER²⁾ called attention to the shortening of the longitudinal convolutions in brachycephalic brains, associated with a prolongation of the transverse convolutions, and the frequent appearance of angular curves in the former.

MEYER³⁾ pleaded the mechanical cause of these phenomena on account of a study of a pathological case with growth repression of the posterior part of the skull, in consequence of which the transverse furrows in that region were more than usually pronounced, some of them (as the s. temporalis) even being opercularised⁴⁾.

ZUCKERKANDL⁵⁾ reached the same conclusions, considering a precocious closing of the sutures an important factor in shaping the brain.

Exact information about the general shape of the brain among non-European races is very rare. Most of the authors on that subject have also confined themselves to descriptions of furrows and convolutions.

Though there is occasional information about the general shape of the brain, especially concerning the Mongols, (PARKER and MILLS⁶⁾,

¹⁾ CALORI. Del cervello nei due tipi brachicefalo e dolicocefalo italiani. Memorie dell'Accademia delle Scienze dell'istituto di Bologna. Seconda serie, 10, 1870.

²⁾ RÜDINGER. Gehirnwindungen bei Lang- und Kurzköpfen und bei verschiedenen Geschlechtern. Korrespondenzbl. der Deutschen Gesellsch. f. Anthropol. u. Ethnol., 1877.

³⁾ MEYER. Ueber den Einfluss der Schädelform auf die Richtung der Grosshirnwindungen. Zentralblatt f. die med. Wiss., Jahrg. 14, 1876.

⁴⁾ I also made this observation a few times among my Chinese brains.

⁵⁾ ZUCKERKANDL. Ueber den Einfluss des Nahtwachstums und der Schädelform auf die Richtung der Gehirnwindungen. Wiener med. Jahrbücher, 1883.

⁶⁾ PARKER and MILLS. Preliminary studies of a Chinese brain. Journ. of nervous and mental diseases, Vol. 13, 1886.

DERCUM¹⁾, RETZIUS²⁾ and KURZ³⁾), indices, which enable us to make exact comparisons, are nowhere to be found.

The result is even more disappointing when one sees the poor state of preservation of some of the brains hitherto described as appears from in the pictures.

I do not blame the authors for this, as the brains they described have generally been preserved by others.

How shall we obtain anthropological data concerning the brain, approaching in exactness, the anthropological measurements of other parts of the body?

In the subsequent pages I will present some indices, that seem to be useful for this purpose.

In the first place, however, I must emphasize that more attention should be given to the fixation and preservation of the brain.

The best way to fix the brain is to inject 10% formalin into the carotids, before taking it out, and transporting it in the skull.

If, however, this cannot be done, one can, after taking the measurements of the skull, remove the brain under water, thus neutralizing its weight and avoiding lesions due to pressure.

The organ should be suspended by the arteria basilaris in a vessel, containing sufficient 10% formalin to prevent its touching the walls. It remains there at least one month, the liquid being weekly renewed. Then it should be photographed from the dorsal, ventral, frontal and lateral side, and subsequently cut in the sagittal midline with a long knife and photographed again laterally and medially, so that the axis of the photographic apparatus stands perpendicularly on the medial brainwall.

After this, it may be forwarded in moist formalincotton so that the flat mesial side of the brain lies on the bottom of the container, separated from the bottom by a thin and even layer of cotton and covered plentifully with a similar, but heavier layer of moist cotton at the other sides.

On the box in which it is sent should be a label, stating which side is to remain up (the mesial side of the brain should always be down).

These instructions are so simple, and at the same time so necessary, that it seems superfluous to write them down, were it not that practice has proved that they are generally neglected⁴⁾.

After this the general morphology may be recorded in the following way :
The general relations to be measured on the brain itself, are the greatest

1) DERCUM. A description of two Chinese brains. Journ. of nervous and mental diseases, Vol. 16, 1889.

DERCUM. Note on a Chinese brain. Ibidem Vol. 19, 1892.

2) RETZIUS. Das Gehirn eines Lappländers. Internationale Beiträge zur wissenschaftlichen Medicin. Festschr. f. R. Virchow. Bnd. I, 1891.

3) KURZ. Zwei Chinesengehirne. Zeitschr. f. Morphologie und Anthropologie. Bnd. 16, 1913.

KURZ. Das Chinesengehirn. Zeitschr. f. Anatomie und Entwicklungsgeschichte. Bnd. 72, 1924.

4) The meninges are best removed after the brain has been in formalin for a few days. It may be a little easier immediately before fixation, but then there is a greater chance of disforming the brain.

The removal of the meninges should also be done under water.

transverse diameter and the greatest length (brainindex), the diameter between the triangular opercula (frontal diameter) and the greatest bitemporal diameter.

The other indices can be best measured on the photographs of the lateral and the medial sides. For these, the following lines (see figs. 1 2, 3, 4, 5, 6, 7, 8) should be drawn:

Laterally:

1. A line connecting the basis of the operculum orbitale with the basis of the lobus occipitalis: the *lateral horizontal*.

On this line the following perpendiculars are traced:

2. the perpendicular along the anterior pole of the frontal lobe: the *frontal perpendicular*.

3. the perpendicular along the anterior pole of the temporal lobe: the *insular perpendicular* (measured from the top to the lateral horizontal).

4. the perpendicular from the highest point of the parietal lobe: the *parietal perpendicular*.

5. the perpendicular along the posterior pole of the occipital lobe: the *occipital perpendicular*.

6. the perpendicular from the utmost ventral part of the temporallobe: *temporal perpendicular*.

Medially:

7. the line connecting the basis of the splenium with the basis of the genu corporis callosi: the *basal callosum line*.

8. the perpendicular from the highest point of the corpus callosum upon this line: the *callosum perpendicular*.

9. the line between the most frontal and most caudal points of the callosum: the *callosumlength*¹⁾.

10. a line parallel with the bottom of the fourth ventricle to the basal callosumline: the *stemaxis*.

By the use of these lines, which should be measured with the nonius, the following indices may be calculated:

A. The *general height index* of the brain, being the parietal perpendicular divided by the lateral horizontal.

B. The *occipital index*, being the parietal perpendicular, divided by its distance to the occipital perpendicular.

C. The *temporal index*, being the temporal perpendicular divided by the lateral horizontal.

D. The *frontal height index*, being the insular perpendicular, divided by its distance to the frontal perpendicular.

E. The *frontal length index*, being the distance from the insular to the frontal perpendicular divided by the lateral horizontal.

¹⁾ In my photographs this line is drawn over the whole cerebrum for reasons of no importance here.

F. The *callosum index*, being the callosum perpendicular, divided by the callosumlength.

G. The *stemangle*, being the frontal angle between the stemaxis and the basal callosumline.

The lateral horizontal, furthermore, permits a measurement of the ventral extension of the *rostrum orbitale*.

The following tables contain the figures thus found in some dolichocephalic Dutch, some Chinese, some brachycephalic Dutch and some foetuses and neonati.

Dolichocephalic Dutch brains.

Origin	L.B. index Brain	Height index	Occipital ind.	Tempo-ral ind.	Frontal height index	Frontal length index	Callosum height
W.G. 15296	75.2	0.487	1.15	0.142	1.91	0.227	0.315
W.G. 15302	76.8	0.457	1.37	0.130	1.74	0.233	0.261
Garens	77.0	0.486	1.15	0.130	1.84	0.224	0.327
W.G. 15297	77.2	0.477	1.08	0.175	1.85	0.214	0.314
W.G. 15258	78.5	0.520	1.14	0.136	1.88	0.241	0.300
W.G. 15256	79.8	0.507	1.28	0.162	1.85	0.252	0.370
W.G. 15244 A	80.3	0.501	1.18	0.143	1.90	0.230	0.357
Average:		0.491	1.19	0.145	1.85	0.231	0.321

That the number of brains mentioned in this table (I disposed in total of 22 Chinese brains and many more dolichocephalic Dutch brains and foetuses) is not larger, depends on the fact that I have used only the best preserved material for this purpose.

Comparing the average figures of the dolichocephalic Dutch with those of the Chinese, we see that all perpendicular indices in the Chinese are greater, the general height index (Chinese: 0.535, Dol. Dutch 0.491) as well as the occipital (Chin. 1.56, D. Dutch 1.19), temporal (Chin. 0.166, D. Dutch 0.145) and the frontal height index (Chin. 1.93, D. Dutch 1.85).

The average callosumheight is also larger in the Chinese than in the dolichocephalic Dutch (Chin. 0.383, D. Dutch 0.321), (thus confirming Dr. MA WEN CHAO's¹⁾ interesting observation).

The stemangle in the dolichocephalics here used is not recorded in my table, because I had removed the cerebella, for my researches on

¹⁾ A comparison of the form of the callosum and septum in the Chinese, Philippino and Dutch brains. These Proceedings, Vol. XXX, Febr. 1927.

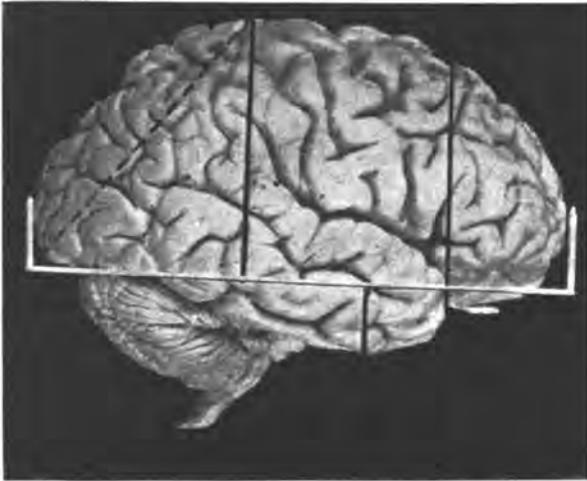


Fig. 1. Dolichocephalic Dutch cerebrum. Right hemisphere, lateral side (Garens).



Fig. 2. The same. Mesial side.

Chinese.

Origin	L.B. index Brain	Height index	Occipital ind.	Tempo-ral ind.	Frontal height index	Frontal length index	Callos. height	Stem-angle
N. Chin. 11	75.3	0.524	1.43	0.160	1.90	0.214	0.420	100°
N. Chin. 7	75.4	0.525	1.52	0.180	1.94	0.216 ^{1/2}	0.423	105°
N. Chin. 21	77.7	0.557	1.40	0.135	1.92	0.254	0.360	104°
N. Chin. 15	81.2	0.534	1.82	0.193	1.91	0.226	0.360	99°
N. Chin. 18	86.6	0.536	1.61	0.163	1.98	0.206	0.350	100°
Average:		0.535	1.56	0.166	1.93	0.223	0.383	101 ³ / ₅ °

the relative cerebellar weights ¹⁾, before photographing them, and I am not sure that the stemangle has not changed by this manipulation.



Fig. 3. Northern Chinese.
N^o. 18.
Left hemisphere, lateral side

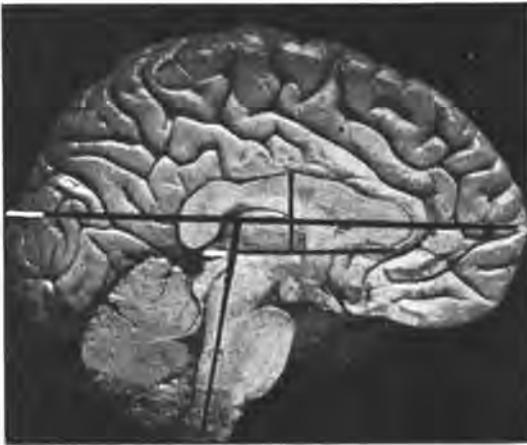


Fig. 4.
The same. Mesial side.

In other dolichocephalic material (among which one examined in the halved skull) I found an average stemangle of at least 106° , while the average in my Northern Chinese is only $101\frac{3}{5}^{\circ}$. The narrow fossa interpeduncularis thus arising in the Chinese also struck KURZ (l.c.).

These figures illustrate the fact which I pointed out before ²⁾, namely, that the explanation of this phenomenon is to be found especially in the antero-posterior shortness and dorsal height of the Chinese brains (and skulls); in other words in the hypsicephalic type of the Chinese.

¹⁾ These Proceedings, Vol. 28, 1925.

²⁾ ARIENS KAPPERS. Over de hersenen der Chineezzen. Ned. Tijdschr. v. Geneesk. 1926. — Eerste helft, no. 11, p. 8.

While the Central-Asiatics are more brachycephalic, the Chinese are more hypsicephalic. MOCHI¹⁾ called the Samoyedes platybrachycephalic (as also the Kalmucks, Kirgeese and Tartars) while he calls the Chinese hypsi-mesati-brachycephalic.

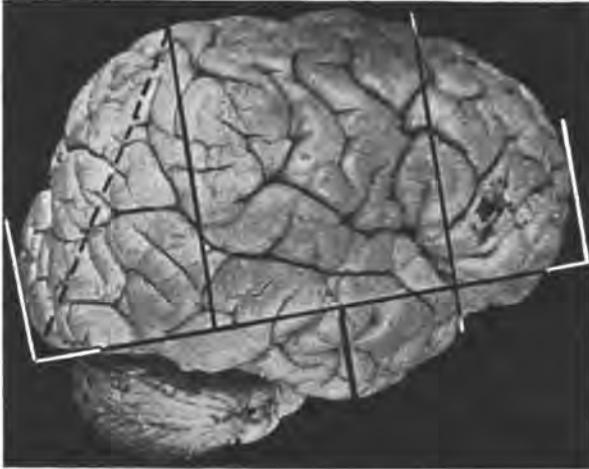


Fig. 5.
Dutch, brachycephalic.
Right hemisphere, lateral side.
Velt.

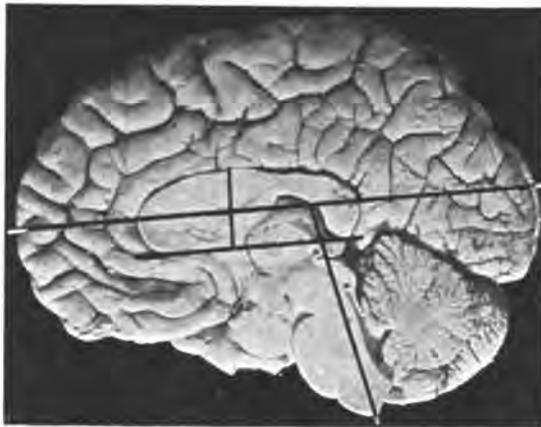


Fig. 6.
The same. Mesial side.

REICHER²⁾ gives as the breadth height index in Buriates 85,9 and in the Kalmucks 87,5, whereas according to the same author this index in the Chinese is 97,7 and, according to KOGANEI³⁾ even 100,2.

Now, reviewing the figures found in exquisite brachycephalic Dutch brains it is evident that they approach those of the Chinese.

1) MOCHI. Cranii cinesi e giapponesi. Arch. per l'Antropologia, Vol. 38, 1908 p. 324: "Dei cranil cinesi alcuni pochi sono simili per forma ai Samoiedi, ma la grande maggioranza ne differisce notevolmente per lo sviluppo in altezza".

2) REICHER. Untersuchungen über die Schädelform der alpenländischen und mongolischen Brachycephalen. Bnd. 15 u. 16 Zeitschr. f. Morphol. u. Anthropol. 1913, p. 59.

3) KOGANEI. Messungen an männlichen Chinesen Schädel. Zentralblatt für Anthropologie Bnd. 7, 1902, p. 129.

Dutch Brachycephalics.

Origin	L.B. index Brain.	Height index	Occipital ind.	Temporal ind.	Frontal height index	Frontal length index	Callos. height	Stem-angle.
Bo	84.8						0.369	97°
Le	85.5	0.451	1.46	0.135	1.73	0.241	0.337	st. cut off
Go	87	0.512	1.20	0.182	1.99	0.230	0.434	103°
Velt	89.87	0.536	1.64	0.164	1.90	0.240	0.366	101°
Krijt	90.24	0.591	1.50	0.160	1.93	0.262	0.402	st. cut off
Average:		0.522 $\frac{1}{2}$	1.45	0.160	1.88	0.243	0.382	100 $\frac{1}{3}$ °

Putting the average figures next each other we get the following review:

Brain-Indices	Chinese	Dutch Brachycephalics	Dutch Dolichocephalics
Height index	0.535	0.522 $\frac{1}{2}$	0.491
Occipital index	1.56	1.45	1.19
Temporal index	0.166	0.160	0.145
Frontal height index	1.93	1.88	1.85
Callosum height	0.383	0.382	0.321
Stemangle	101 $\frac{3}{5}$ °	100 $\frac{1}{3}$ °	106°

From this it is evident that in these relations the brachycephalic Dutch are nearer to the Chinese than the dolichocephalic Dutch. This is not strange, as in both the Chinese and the brachycephalic Dutch the skull is *relatively* short. Though in the Chinese this is compensated for particularly by the height, and in the brachycephalic Dutch more by the breadth of the brain (Dutch brachycephalic are not really hypsicephalic), it is apparent that the height indices have increased to some extent in Dutch brachycephalics also, though not nearly as much as in the Chinese.

If, however, I compare my most brachycephalic brains (*Velt* and *Krijt*) with those of the Chinese, these indices approach or even excel those of the Chinese.

Among the Dutch such indices are only found in strongly brachycephalic individuals, whereas they occur in mesocephalic Chinese.

Apparently this is to be explained by the fact that the brains of the

Brain-Indices	Average of my Northern Chinese	Velt 1)	Kryt 1)
L. B. Index	79.20	89.87	90.24
Height index	0.535	0.536	0.591
Occipital index	1.56	1.64	1.50
Temporal index	0.166	1.64	1.60
Frontal height index	1.93	1.90	1.93
Callosum height	0.383	0.366	0.402
Stemangle	101 ³ / ₅ °	101°	Stem cut off.

Dutch brachycephalics extend more laterally and less in height than those of the Chinese, so that only a part of the relative shortening of the Dutch brain is compensated for by an increase in height.

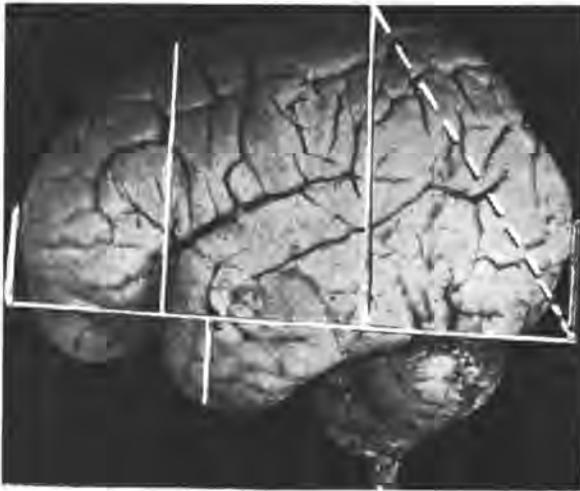


Fig. 7. Child two months of age. Left hemisphere, lateral side.

KURZ is inclined to derive the form of Chinese brain from that of the Orang Utang, which (in addition to the Gibbon) is the only Asiatic anthropoid. One need not go so far, however, as to accept a different anthropoid ancestry for the Chinese and the Caucasians.

As I pointed out already some years ago the general morphology of the human brain about the time of birth resembles that of the brachycephalic Dutch and especially that of the Chinese.

1) The skull indices of Velt and Kryt, according to the measurements, made in the pathological institute at Groningen, were 87, 87 and 88.

I wish to express my thanks to Prof. DEELMAN and Dr. DIJKSTRA for giving me these figures and these brains.

So calculating the above mentioned indices in foetuses shortly before

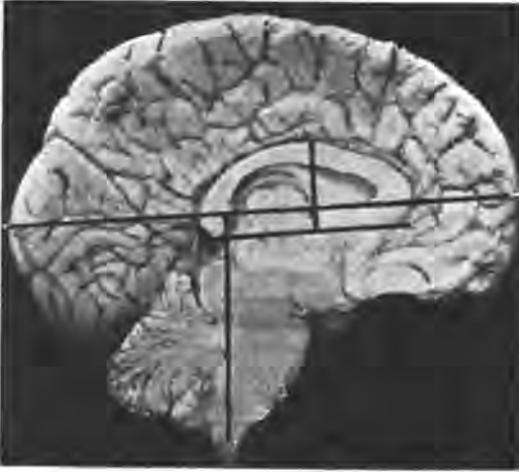


Fig. 8.
The same. Mesial side.

birth, in neonati and young individuals, again using only well preserved material, I get the following figures:

Foetus and Neonati.

Origin	Height index	Occipital ind.	Temporal ind.	Frontal height index	Frontal length index	Callos. height	Stem-angle
Foetus 45 cM.	0.622	1.54	0.154	1.91	0.290	0.434	cut off
Praemature (4 weeks)	0.596	1.54	0.142	1.86	0.284	0.424	92°
Neonatus nearly à terme	0.592	1.60	0.144	1.90	0.272	0.366	cut off
Neonatus à terme	0.564	1.33	0.181	1.71	0.286	0.355	93° 1)
Child 2 months	0.564	1.50	0.164	1.74	0.260	0.433	93°
Child 4 months	0.577	1.30	0.155	1.81	0.282	0.402	106°
Average:	0.586	1.47	0.157	1.82	0.279	0.402	96°

Arranging these average figures next to the average figures of the Chinese, brachycephalic and dolichocephalic Dutch, it appears that the indices found in foetuses and neonati, excepting the frontal height index²⁾ are larger than those in the dolichocephalic Dutch. They resemble those in the brachycephalic and Chinese, and their general height and callosum-indices, even surpass those of the Chinese.

1) Another neonatus measured in the skull had a stemangle of 98 0/0.

2) The frontal height-index of foetuses and neonati cannot be compared with those of the adult, as the temporal lobe still grows out in a frontal direction in postfoetal life, and thus this perpendicular in the adult is displaced in a frontal direction.

Brain-Indices	Chinese	Dutch Brachycephalics	Foetuses and Neonati	Dutch Dolichocephalics
Height index	0.535	0.522 ¹ / ₂	0.586	0.491
Occipital index	1.56	1.45	1.47	1.19
Temporal index	0.166	0.160	1.57	0.145
Frontal height index	1.93	1.88	1.82	1.85
Callosum height	0.383	0.382	0.402	0.321
Stemangle	101 ³ / ₅ °	100 ¹ / ₃ °	96°	106°

This result corresponds to the skull indices of neonati. Thus RETZIUS found Swedish foetal skulls inclining more to brachycephaly than full grown ones (quoted after MARTIN¹). Similarly RÖSE², in Swiss schoolboys in Schaffhausen stated a decrease from 84 to 82.5 between 8 and 18 years and FRETTS³ in Dutch boys found a decrease from 82.14 to 79.27 from the 1st. to the 20th year.

In this connection, it is not strange to find indices in Dutch perinatal brains, similar to those of the brachycephalic Dutch or even to those of the Chinese.

It is evident, however, that Dutch foetuses also have a more *hypsicephalic* character than full-grown Dutch⁴).

In view of these facts we should consider if it is not possible to use BOLK's theory of retardation to explain the general morphology of the Chinese brain, the more so as not only their height indices, but also their more rounded temporal lobes, their larger rostrum orbitale (c.f. fig. 3 and 7) and smaller stemangle remind us the fetal and neonatal characteristics of the Dutch brains.

To escape being called a partial critic I may remark that some of the characteristics of the Chinese brain can be accounted for without this theory, in as much it is a custom among the Chinese to force the babies, immediately after birth and later on, to rest their heads on the occiput in sleeping.

A flattening of that part which thus may arise, is favored, — so I have heard — sometimes by massage of the posterior part of the head.

This may lead to a fronto-occipital abbreviation and to an increase of the height of the skull, which again might explain the occipital and general height-indices of the brain.

As far as the transverse orbital ridge is concerned, this explanation is the more probable, as I found it most pronounced in a hydroptic Chinese cerebrum, in which the pressure in the skull is naturally greater.

Even the broader opening to the fossa sylvii and round temporal pole

1) MARTIN, Lehrbuch der Anthropologie, FISCHER Jena, 1914, p. 605.

2) RÖSE, Beiträge zur Europäischen Rassenkunde. Arch. für Rassenbiologie, Bnd 2 (p. 689), 1905; Bnd 3 (p. 42), 1906.

3) FRETTS, Heredity and Headform in man, Genetica, Vol. 3, p. 193.

4) REUTER noted an analogous phenomenon at the skulls of Pommerian children. See MARTIN, Lehrbuch der Anthropologie, p. 606.

already observed by KURZ (which occurred in three of my brachycephalic Dutch as well) may perhaps be thus explained, as these features, just as the transverse orbital ridge immediately in front of the temporal lobe (also present in RETZIUS' Laplander), may be produced by pressure against the wings of the sphenoid.

It is, however, more than doubtful whether the often occurring rostrum orbitale, the concave form of the orbital brain surface, and the smaller stemangle of the Chinese may thus be accounted for.

Besides we should be very careful about accepting a one-sided artificial external pressure¹⁾ as a sufficient explanation of the whole skull form¹⁾. It is well known that WALCHER²⁾ thus explained brachycephaly as a consequence of the infants sleeping occiput down and dolichocephaly by their sleeping with the lateral sides of their heads on the cushion.

Though the influence of these positions may be noted at times, VAN DEN BROEK's³⁾, observations upon his own twins do not favor this as a constant factor. Also the fact that in animals (Canidae, Suidae, Ovidae) domestication produces a shorter, broader skull⁴⁾, though pressure in the above named sense is excluded, should us make careful in this respect. Probably many factors besides the form of the brain are important in determining the form of the skull. So general metabolic influences (NATHUSIUS, NEHRING, HENSELER, BOAS)⁵⁾, or the action of vitamins (ECKSTEIN), and the influence of hormones (KEITH⁶⁾, BOLK⁷⁾, STOCKARD⁸⁾), must also be considered as well as general cultural relations (RETZIUS, AMMON) including the collective influence of all surrounding factors.

Finally the relative proportions between the head and the body of the individual are important, principally in animals (KLATT).

If, however, artificial pressure on the occiput is insufficient as an explanation, it is reasonable that we should view these phenomena in

1) Already MACALISTER (The causation of brachy- and dolichocephaly. Journ. of Anat. and Physiol., vol. 32, 1898 p. 334), criticized this standpoint. He considered the primary development of the central lobes as the principal cause of perinatal brachycephaly.

2) WALCHER. Ueber die Entstehung von Brachy- und Dolichocephalie durch willkürliche Beeinflüssung des kindlichen Schädels. Zentralblatt f. Gynaek., Bnd 29, 1904 and Korrespondenzbl. der Anthrop. Ges., Bnd. 36, 1905.

WALCHER. Weitere Erfahrungen etc. Münchener Med. Wochschr., 58, 1911.

3) VAN DEN BROEK. Korrespondenzbl. der Deutschen Gesellsch. f. Anthrop., Jahrg. 47, 1916, p. 68. Zur Frage der willkürlichen Beeinflüssung der Schädelform. One of his twins, born dolichocephalic (measured two days after birth its index was 72,4) put in the cradle gave its head a lateral position, the other, born brachycephalic (82,2) gave its head an occipital position. The dolichocephaly of the one and the brachycephaly of the other first increased, being 71 and 86,3 after 4 months, but after a year the dolichocephaly and brachycephaly both decreased (75 and 83) and after 3 years still more so (77,1 and 82,4). VAN DEN BROEK therefore doubts WALCHER's conclusion and asks whether this author does not reverse cause and consequence.

4) Concerning the influence of domestication and feeding, see KLATT. Studien zum Domestikationsproblem. Untersuchungen am Gehirn. Bibliotheca genetica, Bnd. 2, 1921, and BASLER, Die Beeinflüssung der Schädelform durch die Umwelt. Deutsche medizinische Wochenschrift, No. 43 und 44, 1925.

5) BOAS. Changes in bodily form of descendants of Immigrants. Immigration Committee Document No. 208, Washington.

6) KEITH, Differentiation of mankind etc. Ass. for Adv. of sciences, 1919.

7) BOLK, The part played by the endocrine glands etc. Lancet, 1921.

8) STOCKARD. Human types etc. Am. Journ. of Anatomy, Vol. 31, 1923, p. 261.

the light of BOLK's theory, and consider a further development of the skull in its fetal form as a basis of explanation particularly since analogous features are observed in brachycephalic Dutch, where artificial pressure as a constant factor cannot be adduced.

Proof might be attained if other parts of the Chinese body showed characteristics, which could be explained as persisting late fetal or early post-natal features. Now this actually occurs.

Thus BIRKNER¹⁾ found the size of the platysma-muscle to be unusually large, extending as a rather closed plate over the face, a feature which occurs only among the West-Europeans, in very young children.

Furthermore in the Chinese, the zygomaticus and the quadratus labii superioris persist as rather closed muscleplates, enlarged by irradiating bundles from the orbicularis oculi to the mouth-angles, another finding which also occurs in young European children. Similarly TOKUYASO KUDO²⁾ who compared the facial musculature in fifteen Japanese, three Chinese and five Europeans, stated (l. c. p. 669) „the Japanese and Chinese are separated from the Europeans by a somewhat smaller differentiation, a tendency of single muscles to fuse superficially in a single plate. This tendency is stronger in the Chinese than in the Japanese (l. c. p. 669—670). And „In Mongolians as a race the three parts of the quadratus labii superior fuse into a single plate, further, the caput zygomaticus, constantly present in Mongolians, is distinguishable with difficulty from the neighbouring muscles (l. c. p. 671). BOLK³⁾, though generally inclined to regard the white Homo nordicus as the most fetalised type has pointed out himself, however, that the Mongoloïd fold and epicanthus, characteristics of the Chinese upper eyelid, are encountered as a fetal phenomenon in Europeans. The deep noseroot and protusio bulbi should also be considered as foetal relics, according to this author.

SHELLSHEAR⁴⁾ (l. c. p. 7) has remarked that the parenchyma of the thymusgland persists until a later age than in the European. HAMMAR's⁵⁾ ages corresponding to that type are not in agreement with the findings in the Chinese. In them the young adult condition, instead of changing at 20, runs on till about 25 years of age.

The thymusgland undergoes its involution at a later period in the Chinese than in Europeans.

Finally SHIROKOGOROFF and APPLETON⁶⁾ point out (l. c. p. 110) that — especially among the northern Chinese — the hairgrowth is strongly retarded in comparison with that of Europeans. In men this holds good as well for the beard as for the pubic hair and in women it applies to the pubic and axillary hair. These writers say: „It is very characteristic in Chinese women that they have sometimes no axillary and pubic hair at all (l. c. p. 110).“

Dentition among the Chinese also seems to follow a different periodicity from that of Europeans: „there is a retardation of full dentition (except in the third molar) which

1) BIRKNER. Die Anthropologie der Mongolen. Arch. für Rassen u. Gesellschaftsbiologie. Jahrg. 1, 1904, p. 817.

2) TOKUYASO KUDO. The facial musculature of the Japanese. Journ. of. Morphology Vol. 32, 1919, p. 637.

3) BOLK. Over Mongolenplooi and Mongoloïde idiotie. Ned. Tijdschr. v. Geneesk. 1ste helft 1923. Afl. 3. See also: Over de oorzaak en de beteekenis van het niet sluiten der schedelnaden bij den mensch. Ibidem 1926, 2de helft N^o. 21.

4) SHELLSHEAR. The thymusgland in the Chinese. China med. Journal, August 1924.

5) HAMMAR. Die Menschenthymus Teil I. Das normale Organ, Akad. Verlag, Leipzig, 1926.

6) SHIROKOGOROFF, Process of physical growth among the Chinese. Volume 1. The Chinese of Chekiang and Kiangsu measured by Dr. APPLETON. The Commercial Press lim., Shanghai, China, 1925.

continues after the age of 15 years. On the other hand there is a premature (comparatively with Europeans) appearance of the second and third molars."

All of these factors point to a retardation, especially in the Northern Chinese (among the inhabitants of Kiangsu these phenomena are even more pronounced than in those of Chekiang), and seem to confirm SHELLSHEAR's statement (l.c. p. 10) „the anthropological type (of the Chinese) might almost be regarded from a European standpoint as a childlike type". They also support my view concerning the general brainform of the Northern Chinese, that we should explain this form from the standpoint of fetalisation.

A few words more about the *frontal length index* and the *stemangle*.

My figures show that the relative length of the part in front of the temporal lobe in the Chinese is smaller (0,223) than in the Dutch (Br. 0,243 and D. 0,231). This shortening of the frontbrain, is also expressed by the fact that the Chinese have such a great frontal height index.

On the contrary, the frontal length, that is the distance from the frontal point of the brain to the frontal pole of the temporal lobe, is very large in neonati, larger than in the Chinese. Consequently in this respect one cannot adduce a retarded proportion in this index among the Chinese, although the frontal lobe itself in fetuses is also relatively short. This fetal shortness of the frontal lobe is, however, not expressed by this index, since the insular perpendicular has no constant position in the fetus. Its site depends on the frontal outgrowth of the temporal lobe (the temporal operculum) and in as much as this operculum develops gradually, this perpendicular lies more caudally in the foetal than in the adult brain.

The *stemangle*, in the N. Chinese (similarly in RETZIUS' Laplander, also a Mongol) is smaller than in dolichocephalic Dutch it is also small in brachycephalics, but it is still smaller in the fetus.

In my opinion this depends on the location and the inclination of the foramen magnum in the skull.

BOLK¹⁾ has drawn attention to the fact that the foramen magnum occupies a relatively more frontal position at the end of fetal life than in the adult. As my stemaxis runs through the foramen magnum, it is not surprising, that in the fetus it runs steeper than in the adult, where it has a more backward inclination.

Though BOLK believes that the fetal condition has no direct connection with the brachycephaly of the fetal skulls, it is remarkable, that he also found the inclination of the foramen magnum to be turned more frontally in brachycephalic than in dolichocephalic Dutch, as I did for the *stemangle*.

Considering all the other fetal relations of the Chinese brain, it seems probable to me that also their smaller *stemangle* should also be regarded from this view point.

¹⁾ L. BOLK, On the position and displacement of the Foramen magnum in the Primates, These Proceedings, 12, p. 362, (1909); On the slope of the Foramen magnum in Primates, Ibidem, 525, (1909).

Physics. — *On the absorption-spectrum of chromium-oxy-chloride.* By
A. C. S. VAN HEEL. (Communicated by Prof. W. J. DE HAAS.)

(Communicated at the meeting of November 27, 1926).

The vapour of chromium-oxy-chloride, CrO_2Cl_2 , otherwise chromyl-chloride, absorbs all visible light with wave-lengths smaller than ca. 5000 AE. The limit of the range of absorption is very vague and depends greatly on the density of the vapour and on the length of the path, which the light travels. Besides this continuous absorption there is an absorption spectrum which exhibits many lines between ca. 5900 and less than 5000 AE. The liquid absorbs continuously all light with wave-lengths smaller than ca. 6000 AE.

These facts were already known by STONEY and REYNOLDS, who, according to KAYSER (*Handbuch der Spectroscopie*, III, page 367), observed this spectrum for the first time in 1871. Afterwards only F. KÄBITZ (Diss. Bonn, 1905) effected measurements, which have been communicated by KAYSER (l.c.). The precision of these determinations amounts to some Angström-units. The fact, that KÄBITZ could locate all observed lines in five series, induced us to undertake a closer examination. It is not surprising that there existed a rather constant difference of wavelength between succeeding lines of each of the series, the range of the absorption being small. The difference of the wave-numbers of the lines turned out to have a much more constant value.

In order to enable an accurate measurement of the wave-lengths it proved necessary to make use of low pressures of the vapour. Indeed the lines get fine by lowering the pressure and at the same time the intensity of the continuous absorption diminishes greatly, more than the intensity of the line absorption.

The pressure in the absorption-tube was regulated in the following way. A lateral tube with a not too small quantity of liquid chromyl-chloride was kept at constant temperature in a Dewar-vessel filled with alcohol. The alcohol was cooled by a stream of liquid air, which was siphoned out of a reservoir into a U-tube, placed in the alcohol, where it evaporated and flowed away in gaseous state.

The velocity of flow could be regulated, and thereby the temperature of the alcohol in the stationary state.

This apparatus, provided with a stirrer with a funnel (which was not even necessary for our research) is very suitable for keeping constant temperatures, which are not too low, to within some hundredths of a

degree; this is pointed out by HENNING in *Temperaturmessung*, 1915, page 221.

Many hours at a stretch the temperature (measured with a constantan-iron thermo-element, connected with a potentiometer of Cambridge and Paul's Co) was constant within a range of some tenths of a degree, quite sufficient for our purpose.

The filling of the absorption-tube caused some difficulty, as the chromyl-chloride is impaired or decomposed by all organic compounds and also by mercury and water.

Clouds of HCl are formed with aqueous vapour, preventing the appearance of the fine absorption-lines. The solid CrO_3 which is formed during this reaction, has practically no vapour, so it cannot influence the absorption appreciably.

The chromyl-chloride, produced by KAHLBAUM in small scaled off vessels each containing 2 grams, might also contain HCl. In order to remove this contamination the filling was done in the following way, represented schematically in figure 1.

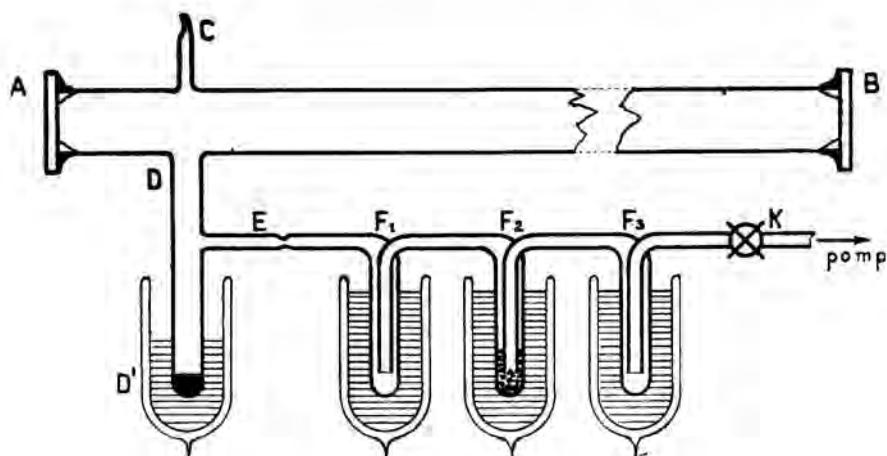


Fig. 1.

The absorption-tube AB had a lateral tube D , above which a small lateral tube C has been blown; by C a small quantity (e.g. 6 grams) of chromyl-chloride was introduced into D ; thereupon C was closed. The bottom of D was then plunged in liquid air and the tube was connected by the tap K with a Langmuir-pump, which evacuated the vessel provisionally.

Between D and K some moisture-traps F_1 , F_2 , F_3 had been blown on, each of which was plunged in liquid air, F_2 containing cocoa nut coal. These traps prevented the vapour of chromyl-chloride from penetrating to K .

Now the pumping was continued with the vessel *D'* containing liquid air, removed. The chromyl-chloride present in *D* melted (at ca. -100° Cent.), filled the tube with vapour and ended by boiling vehemently, in the meantime releasing the dissolved HCl. Thereupon *D* was again cooled with liquid air. This operation was repeated some times, till *D* retained no more than 1 cm^3 chromyl-chloride. The tube *E* was then blown off and the absorption tube was ready for observation.

To get a perceptible selective absorption at very low pressures, it was necessary to make the length of the absorption-tube very large. The best results were obtained with a tube of four metres. A diameter of 5 cm. was necessary to lead the light such a distance through the vapour. The ends had been provided with pieces of plate-glass to prevent loss of light by irregular refraction.

These pieces were too large to be blown in into the tube, so they had to be cemented on its ends. Piceine and other sealing-waxes are impaired strongly by the vapour of chromyl-chloride. Therefore they were cemented with heat-isolation material smeared on the outside with piceine. In this way the vacuum could be kept for several weeks; the piceine was protected sufficiently against the disintegrating influence of the vapour.

When the absorption-tube had been filled for a long time, the walls were covered by a brown layer, which however did not show any absorption-lines in the spectrum, as appeared from some exposures. So no inconvenience in studying absorption of the vapour had to be feared from an eventual covering of the pieces of plateglass by such a brown layer.

The absorption-spectrum was photographed with a Rowland-grating of 14438 lines to the inch (radius of curvature 10 feet) in stigmatic mounting. This mounting has been excellently constructed in the workshop of the laboratory under direction of Mr P. VAN DEN AKKER after the data of MEGGERS and BURNS (Sc. Pap. Bur. of Stand., No. 441, 1922); besides its stigmatism it has the advantage of yielding very luminous spectra, whose dispersion indeed is smaller than in Rowland's mounting, but whose resolving power is not less.

The resolving power of the grating in the first spectrum, which has been always used in this research, is 50000, its dispersion 11 \AA per mm. The width of the slit was .09 mm during most of the exposures.

At first use was made of a Philips tungsten arc lamp (35 Watts). The times of exposure were however too long with it, at least 28 hours. A small carbon-arc then was used. The surface-brightness of the crater was increased by feeding the lamp with 8 A instead of its normal current of 3 A.

The light was rendered parallel by an achromatic lens, then sent through the absorption tube and sent back through the tube by a mirror at the further end. The light now leaving the tube the second time was

concentrated by a total-reflecting prism and a second achromatic lens on the slit of the spectrograph. The path of the light in the vapour of chromyl-chloride thus amounted to eight metres.

The exposures were made on "Kodak Commercial Panchromatic Films" (5 by 30 cm). The measuring always took place by means of a comparison-spectrum of iron, obtained by changing one of the carbons of the arc for a small iron rod.

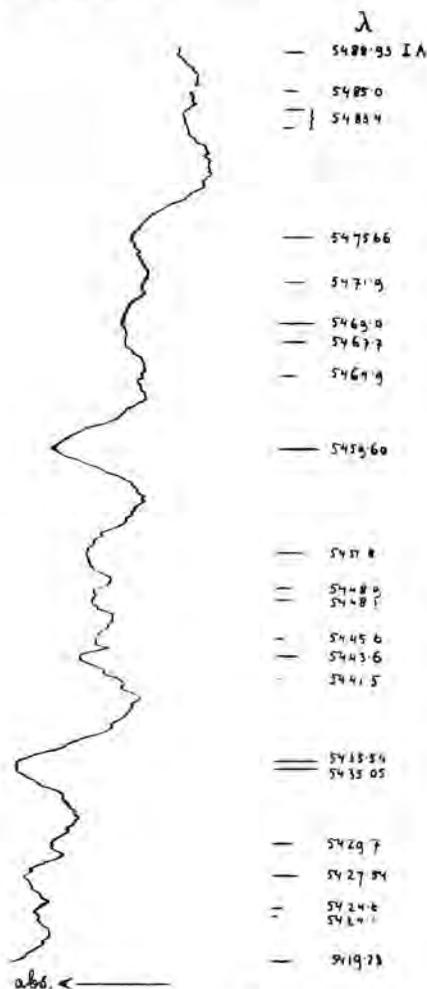


Fig. 2.

Photogram of a part of a negative

11 AE = 1 mm on the negative = 4 cm on the photogram = 19 mm on this reproduction.

The measuring of the negatives obtained was done on a comparator of the Soci t  G n voise; the errors of its screw are automatically corrected by an appropriate shift of the zeropoint of the vernier on which the readings are taken. The attainable accuracy is in the neighbourhood of .001 mm. corresponding at the given dispersion with ca. .011 AE.

TABLE.

Int.	λ in I. A.	1 : λ_{vac} .	Int.	λ in I. A.	1 : λ_{vac} .
2	5899·25	16946·6	2	5646·5	17705·2
2	5865·45	17044·3	1	5645·9	17707·1
2	5844·73	17104·7	1	5643·3	17715·2
2	5826·2	17159·1	1	5641·5	17720·9
2	5817·00	17186·2	8 }	5639·28	17727·8
2	5809·2	17209·3		5638·6	17730·0
5	5798·78	17240·2	1	5636·9	17735·3
3	5791·3	17262·5	2	5631·3	17753·0
3	5781·4	17292·1	1	5628·3	17762·4
3	5772·31	17319·3	1	5625·9	17770·0
8	5753·87	17374·8 ¹⁾	1	5623·7	17777·0
3	5744·9	17401·9	10 }	5622·61	17780·4
5	5735·1	17431·7		5622·17	17781·8
5	5726·97	17456·4		5621·52	17783·9
10	5709·26	17510·6 ²⁾	1	5619·2	17791·2
2	5701·1	17535·6	2	5616·8	17798·8
5	5690·4	17569·0	4	5615·50	17802·9
5	5682·86	17591·9	3	5614·35	17806·6
1	5674·7	17617·2	2	5610·6	17818·5
2	5667·3	17640·2	1	5608·0	17826·7
10 }	5665·77	17645·0	1	5607·40	17828·6
	5665·03	17647·3	1	5606·66	17831·0
1	5663·1	17653·3	1	5606·1	17832·8
1	5661·5	17658·3	3	5605·2	17835·6
3	5658·0	17669·1	3	5604·5	17837·9
3	5657·1	17671·1	3	5603·5	17841·0
1	5653·7	17682·6	1	5602·2	17845·2
2	5650·5	17692·6	2	5599·7	17853·2
1	5648·2	17699·9	1	5598·0	17858·6

¹⁾ Double, distance of the components: ·66 AE or 2·0 frequency units.

²⁾ Double, distance of the components: ·84 AE or 2·5 frequency units.

TABLE (Continued).

Int.	λ in I. A.	$l : \lambda_{vac.}$	Int.	λ in I. A.	$l : \lambda_{vac.}$	
10	5596·82	17862·3	1	5547·4	18021·5	
	5596·30	17864·0	1	5546·2	18025·4	
	5595·75	17865·8	1	5545·4	18028·0	
1	5594·2	17870·7	1	5544·3	18031·5	
1	5593·2	17873·9	1	5541·9	18039·4	
1	5591·1	17880·6	15	10	5541·14	18041·8
1	5590·1	17883·8		1	5539·2	18048·1
1	5589·3	17886·4		1	5538·5	18050·4
1	5588·1	17890·1	0	5537·8	18052·7	
1	5586·7	17894·7	2	5537·34	18054·2	
1	5583·8	17904·0	4	5532·7	18069·4	
3	5582·15	17909·3	5	5530·12	18077·8	
3	5580·38	17915·0	3	5524·62	18095·8	
10	5579·9	17916·5	2	5519·6	18112·2	
	5579·1	17919·1	5	5517·3	18119·8	
	5577·5	17924·2		5516·2	18123·4	
1	5574·6	17933·5	10	5512·9	18134·1	
1	5573·5	17937·1		5511·2	18139·8	
3	5572·45	17940·5	3	5509·0	18147·1	
3	5570·90	17945·4	2	5507·2	18153·0	
2	5564·6	17965·8	10	5500·6	18174·8	
4	5561·4	17976·1		5500·05	18176·6	
2	5558·7	17984·8	5	5491·19	18203·6	
2	5557·6	17988·4	5	5488·93	18213·4	
1	5556·1	17993·3	5	5485·0	18226·5	
1	5555·6	17994·9		5483·4	18231·8	
10	5554·92	17997·1	8	5475·66	18257·6	
	5552·93	18003·5	6	5471·9	18270·1	
2	5549·3	18015·8	8	5469·0	18279·8	
1	5548·5	18017·9		5467·7	18284·2	

TABLE (Continued).

Int.	λ in I. A.	1 : λ_{vac} .	Int.	λ in I. A.	1 : λ_{vac} .
3	5464·9	18293·5	4	5347·34	18695·7
10	5459·60	18311·3	10	5341·27	18716·9
5	5451·8	18337·5	8	5333·62	18743·8
4 {	5448·9	18347·2	5	5325·46	18772·5
	5448·1	18349·9	10	5317·93	18799·1
2	5445·6	18358·4	4	5309·18	18830·1
5	5443·6	18365·1	6	5302·51	18853·8
1	5441·5	18372·2	10	5295·07	18880·3
10 {	5435·54	18392·3	3	5284·92	18916·5
	5435·05	18394·0	10	5279·53	18935·8
1	5429·7	18412·1	5	5271·05	18966·3
2	5427·54	18419·3	10	5258·32	19012·7
4	5424·6	18429·4	8	5249·51	19044·1
2	5424·1	18431·1	8	5234·26	19099·6
10	5419·23	18447·7	10	5221·18	19147·4
1	5411·4	18474·4	2	5211·86	19181·7
5	5410·8	18476·4	3	5206·4	19201·8
3	5409·4	18481·2	8	5198·50	19231·0
3	5406·8	18490·1	8	5183·99	19284·8
1	5405·0	18496·3	3	5175·44	19316·7
3 {	5403·5	18501·4	10	5165·43	19354·1
	5402·2	18505·8	2	5161·70	19368·1
10 {	5396·07	18526·9	3	5151·3	19407·2
	5395·03	18530·4	5	5139·4	19452·1
5	5386·83	18558·6	5	5129·4	19490·0
5	5384·4	18567·0	5	5125·0	19506·8
10	5379·84	18582·8	4	5099·7	19603·5
8	5372·48	18608·2	4	5082·5	19669·9
6 {	5364·61	18635·5	5	5056·5	19771·0
	5364·04	18637·5	5	5036·4	19849·9
	5363·56	18639·2	5	5018·3	19921·5
10	5356·68	18663·1			

In the case of the sharp and strong absorption-lines an accuracy in the measurement of the wave-length of 1 : 50000 was within reach, as appears from the following independent determinations.

negative 5,6-I-1926	negative 2,21-IX-1926	difference
5500.07	5500.04	+ .3 : 50000
5459.56	5459.63	- .6 : 50000
5435.27	5435.33	- .5 : 50000
5419.30	5419.16	+ 1.3 : 50000
5379.85	5379.83	+ .2 : 50000

The wave-length was always determined by interpolation between two lines of the iron spectrum, whose wave-lengths were looked up in KAYSER'S Handbuch, VII, 1924.

Each independent measurement of an iron line or an absorption line represents six independent settings. In this way 58 lines of the absorption spectrum were determined in International Angström units.

The first visual observations of the absorption spectrum at low pressures proved already that many of the strong lines are doublets or triplets. The lowering of the pressure revealed the complexity of the lines. At the same time the components themselves also got weaker, so much that it appeared impossible to resolve them completely. We resolved therefore to produce photograms of the negatives by means of a micro-photometer after MOLL. At this instance I was glad to secure the help of Mr ir. W. DE JONG, who placed at my disposal the micro-photometer of the mineralogical laboratory of the Technical High School, made by him. Once more I desire to express my thanks for his kindness. Also to prof. ir. J. A. GRUTTERINK I am much indebted for his kind permission for the use of this instrument.

To distinguish casual peaks in the micro-photogram from real lines several photograms were made from five different negatives, altogether 18 photograms.

On some of these 1 mm. of the negative corresponds with 6 mm. of the photogram; on other ones the magnification was stronger viz. fortyfold.

The measuring of the photograms soon showed the possibility to determine with their aid the wave-lengths of all real peaks, using as standards the lines, whose wave-lengths had been determined directly on the negatives. The feebly and the strongly magnified photograms yielded wave-lengths which agreed well with each other, viz. 2 or 3 : 50000.

Only those peaks were measured (on the comparator) which occurred on photograms of at least three negatives.

The wave-lengths obtained in this way in International Angström units are given in the table. In the first column the estimated intensities are given, in column 2 the wave-lengths determined by direct measurements of the negatives and by the photograms.

Finally the wave-numbers in vacuo are mentioned in the last column. Their calculation was much facilitated by the well known "Tabelle der

Schwingungszahlen" of KAYSER (1925). The numbers printed in heavy type have a precision of 0.4 units, the other ones of 1 unit.

We hope to be able to communicate something before long about the classification of the absorption lines.

Finally I wish to express my thanks to prof. FOKKER for his interest and his valuable support during this research.

Physical Laboratory of the Technical High School.

Delft. Dec. 1926.

Hydrology. — *An hypothesis explaining some characteristics of clay.*
By Dr. J. VERSLUYS.

(Communicated at the meeting of December 18, 1926).

In 1916 I published an approximate calculation of the tensile strength or resistance to tensile stress of ideal soil (1, pp. 77—87). By ideal soil is understood that which consists of regularly piled up globules of the same size.

Tensile strength was calculated from the surface tension of what I call the *pendular water* (1, p. 14). An ideal soil in which water is found



Fig. 1.



Fig. 2.

in the *pendular* condition can be shown in diagram by fig. 1 or fig. 2. The small body of water, this is shown in grey, is found around the points of contact of the solid particles, those solid particles being shown in black.

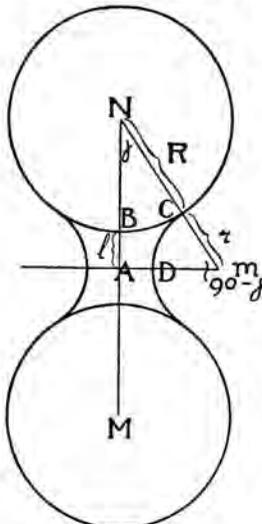


Fig. 3.

According as the *pendular* bodies become larger, the soil contains more water. The volume of this water can be deduced from the dimensions of the *pendular* bodies, while the calculation can also be made from these, concerning how much greater the liquid surface becomes when two solid particles are separated from each other by an infinitely small distance. In fig. 3 e. g. the volume of the *pendular* body and the liquid surface may be expressed in the radius R , half the distance between two grains l and the angle γ .

From this the proportion of water of the ideal soil may also be expressed in γ , provided l is equal to 0.

The tensile strength at every percentage of water was calculated in this manner. This is shown in a diagram (fig. 4), should the diameter of the small particles be 1 mm. The proportion of water is given in percentages of the total volume. The widest arrangement

of the grains, as in fig. 1, gives a pore space of $47\frac{1}{2}\%$, the closest (fig. 2) 26% of the total volume.

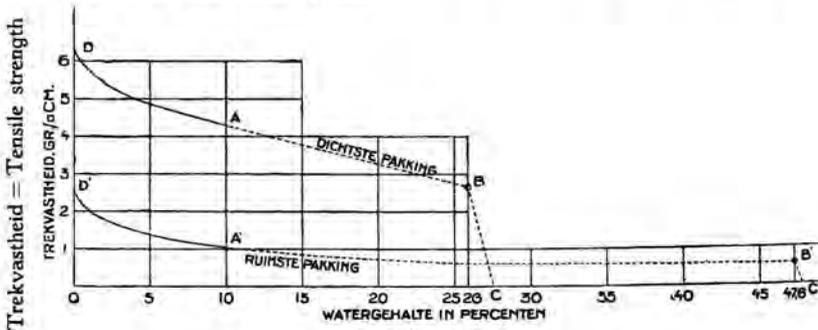


Fig. 4.

Dichtste pakking = Closest packing Ruimste pakking = Widest packing
 Watergehalte in percenten = Proportion of water in percentages

According to the calculations the tensile strength of ideal soil, with the same proportion of water, in inverse ratio, would be in proportion to the size of the grains.

In a treatise by S. JOHANSSON (II), published in 1913, diagrams of the tensile strength of different kinds of soil with various proportions of water are met with, determined experimentally. Those diagrams of experimental data exhibit a somewhat great resemblance to those computed by me. The units in which JOHANSSON expresses the tensile strength are other than those adduced by me. Hence the numbers cannot be compared with each other, though the diagrams can be so compared, they are given in figs 5, 6, 7 and 8. The similarity of the calculated diagram is very great and this indicates that the principle adopted by me, in 1916, possesses a great degree of probability.

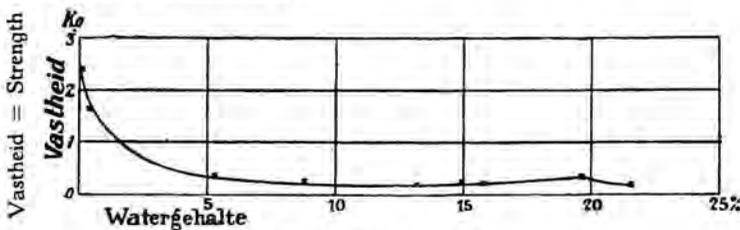


Fig. 5.

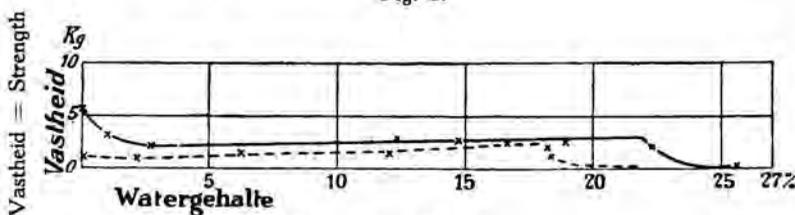


Fig. 6.

That principle explains the tensile strength of the various kinds of soil containing water, without being completely saturated.

With the aid of the tensile strength, considered in this manner and the friction of the solid grains among themselves, the resistance of masses of sand to deformation by external forces can be explained.

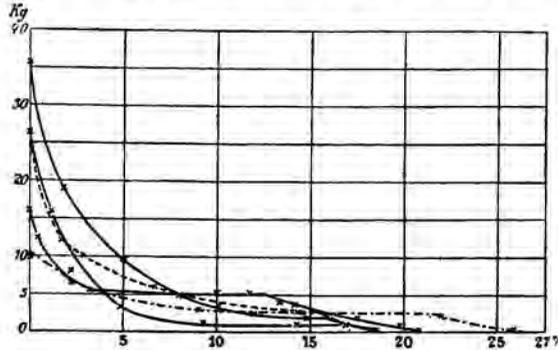


Fig. 7.

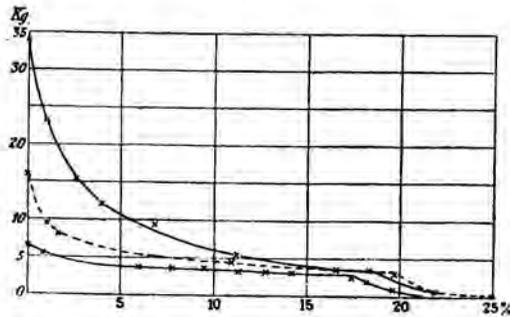


Fig. 8.

With clay other phenomena occur too, such as plasticity, expansion and contraction on change in the proportion of water. These can be partially accounted for by the contracting effect of the pendular and eventually of the funicular (I p. 14) water, for a full explanation of these phenomena, however, an expanding force is still required.

As has been explained, clay can shrink on drying and expand on the absorption of water. Clay that has become saturated with water can be compressed, provided the water is able to escape and the decline in volume is dependent upon the pressure. Clay compressed in this manner is again able to expand and by so doing to exert energy when it absorbs water. This has been proved by recent tests by K. TERZAGHI (III p.p. 82—87). According as the pressure becomes greater, the clay becomes more compressed and according as it again further expands, the pressure which it exercises declines.

Besides the contracting force which can be deduced from the surface tension of the pendular water, there is thus a power of expansion.

If the liquid adsorbed by the solid particles be taken to account for the power of expansion, other phenomena can also be explained, e.g. that clay is still more or less liquid while the surface tension of the water in it is already exercising the contracting force. With sand, as soon as this contracting force becomes manifest the internal friction of the mass is extremely great. With clay the process of drying must have progressed much further before the mass really becomes solid.

It may be taken that the liquid adsorbed envelops the solid particles with a thin layer which cannot be removed without labour. The adsorbed molecules of the liquid have a smaller potential energy than those lying in the midst of the liquid.

For the approaching of two solid particles it is necessary that a part of the adsorbed liquid becomes transferred to non-adsorbed, for which purpose energy must be exercised. A force is thus required to get those particles to approach each other. Whether the layer of adsorbed liquid is mono-molecular or is built up from layers, each to the thickness of one oriented molecule, is a question that has only to be answered in the second place. Much might be said for taking it that there are several layers, representing more potential energy per unit of volume, according as they are lying closer to the solid matter.

For simplicity's sake, let us first take it that all layers of the adsorbed liquid are under the same conditions. The expanding force may then be expressed as follows in the size of the grain D an unknown quantity, the potential energy a per unit of volume and the thickness h of the adsorbed stratum.

If two globules, each having a radius R are enveloped with a thin layer of adsorbed water having a thickness h , energy will have to be exerted so soon as the distance of the centres of the globules becomes smaller than $2R + 2h$.

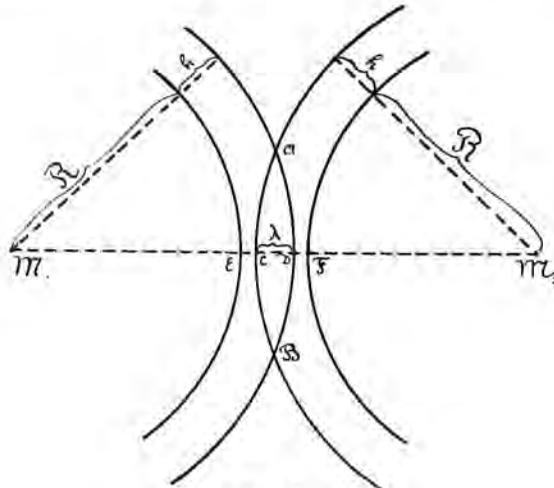


Fig. 9.

If the distance is $2R + 2h - \lambda$, the volume of the adsorbed layers becomes reduced by that of the body, the section of which (fig. 9) is the figure $ACBD$. The volume of this body is equal to the sum of two segments each having a height of $\frac{1}{2}\lambda$ and of globes with radii $R + h$.

The volume of each segment is:

$$\frac{1}{3} \pi \left(\frac{1}{2} \lambda\right)^2 (3R + 3h - \frac{1}{2} \lambda) \dots \dots \dots (1)$$

and the entire volume of the displaced adsorbed water is:

$$\frac{1}{3} \pi \lambda^2 (3R + 3h - \frac{1}{2} \lambda) \dots \dots \dots (2)$$

While if an energy a per unit of volume must be exerted, the total energy expended will be:

$$A = \frac{1}{3} \pi \lambda^2 (3R + 3h - \frac{1}{2} \lambda) a \dots \dots \dots (3)$$

The force required for this, with each value of λ is:

$$p = \frac{dA}{d\lambda} = \pi a \lambda (R + h - \frac{1}{4} \lambda) \dots \dots \dots (4)$$

If h be very small in proportion to R , then we have

$$p = \pi a \lambda R \dots \dots \dots (5)$$

On the globules coming into contact, λ is equal to $2h$ and the force thus increases from 0 to $2\pi a h R$ on the solid particles coming into contact, or $\pi a h D$, if D is the diameter of the grains. The force is therefore proportionate to the diameter of the particles or grains of the soil. The number of forces in the section, however, is proportionate, in inverse ratio, to the second power of the diameter and the power of expansion is therefore proportionate, in inverse ratio, to the size of the grain, just as the contracting force, which is responsible for the tensile strength.

If the adsorbed layers consist of several mono-molecular layers, representing greater potential energy according as they are closer to the solid matter, the power of expansion increases according to another law every time when a succeeding layer has to be displaced. It is only necessary to make a calculation of these changes in the law of the augmentation of force, if one knows anything concerning the value of the energy a in a unit of volume for the several layers. Hence such calculations may be left out of consideration now.

If the clay dries out, the contracting force increases from 0 to a certain amount and the force that is exercised on the adsorbed layer increases proportionately with this from 0 to a certain amount, according as the proportion of water becomes less.

With the driving out of the adsorbed layers near the points of contact of the solid particles, the power of expansion will increase from 0 to a certain amount, according as the solid particles approach each other.

Hence with clay that is drying out, two forces are at work, viz. the contracting, which becomes greater as the proportion of water declines and the expanding force, which continually affects the equilibrium.

Owing to the approaching of the particles to each other, the clay

then continually assumes a smaller volume. Only when the particles come into contact with each other there is friction of the solid bodies with each other set up with deformation. As long as that contact does not occur, with deformation there is only an internal friction of the fluid. It is conceivable that in adsorbed fluid, the internal friction is greater than in non-adsorbed, in any case, however, it is much less than that of solid substances with each other. It is upon this that is based the reducing of frictional resistance by lubrication with fluids suitable for that purpose.

The point of contact of two solid particles in a mass of clay may be shown consecutively in diagram by figs. 10, 11 and 12. With the

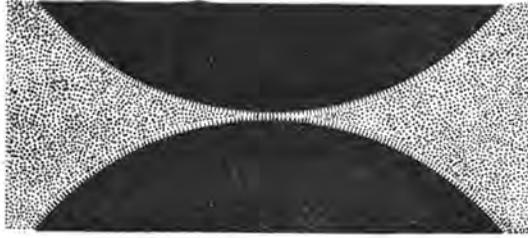


Fig. 10.

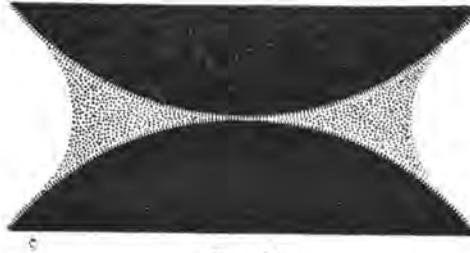


Fig. 11.

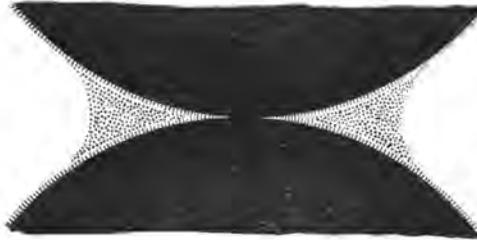


Fig. 12.

condition shown in fig. 10, the process of drying has progressed so far that the mass is still completely saturated with water, but in such manner that, on the outside of the mass curved liquid surfaces begin to form. The adsorbed waterlayers are shown by means of radial lines. These

layers touch each other in the conditions represented by fig. 10. With the condition shown in fig. 11, the process of drying out has progressed so far that a pendular condition has already set in. The pendular body is still large and the contracting force is still small owing to this. The adsorbed layers are already partially penetrating into each other. As the drying process proceeds further, the pendular bodies then become smaller, the contracting force greater and the solid particles then force a way through the adsorbed layers until they come into contact with each other, as is shown in fig. 12.

Should the clay contain more water than is shown in fig. 10, there is then not only adsorbed water but also non-adsorbed water between the particles. The clay is purely then a suspension. The power of resistance to deformation is proportionate to the velocity with which this deformation takes place. If drying out has progressed so far that the condition as shown in fig. 10 arises, then on deformation, friction of the adsorbed layers sets in and the mass becomes much tougher, its viscosity augments. If this process goes on still further, so that air must force its way into the pores, friction on deformation becomes still greater. With the condition as in fig. 11, however, there is only liquid friction on still but of adsorbed water.

When the mass has progressed so far as is shown in fig. 12, so that the solid particles are in contact with each other, there is then a friction of solid bodies and then other forces are at work with the deformations.

I have deduced that the power of expansion is proportionate, in inverse ratio, to the size of the particles or grains. If the grains are large, the weight of these particles themselves is sufficient to eject the adsorbed layer at the points of contact. With sand there is always friction of the solid parts over each other, except in the state of quicksand, which will be referred to presently.

If quartz particles, however, are so small as to have dimensions at which particles of clay already possess the characteristic qualities, quartz powder mixed with water then still conducts itself as an ordinary granular substance such as sand.

It must therefore be taken that adsorption with quartz only takes place to an extremely small degree. The weight of a grain is sufficient to cause the adsorbed layer to disappear at the points of contact of the grains even when these are extremely small.

With sand another phenomenon occurs, which we might call the quicksand phenomenon. A mass of sand saturated with water, owing to shock may pass into a liquid suspension, without enlarging the space occupied. Such a suspension, however, soon settles and is then firm again.

This phenomenon (IV) can be shown in diagram in figs. 13, 14, 15 and 16. The grains are first deposited with a great volume of pores, fig. 13, they rest, however, upon each other and on deformation there

is great friction as long as external pressure is exercised on the grains. If e.g. a horizontal force is being effected on the upper surface, in the figure from left to right, which is sufficiently great to overcome the friction, then the conditions shown in fig. 14 will arise. If a vibration reduces that pressure and thus the friction of the solid particles for a moment, a smaller force will then be sufficient to effect a change in this manner of the mutual position of the grains. The grains are held in



Fig. 13.



Fig. 14.



Fig. 15.



Fig. 16.

suspension only as long as the mass is stirred or in any way deformed by external forces. If no such influence is intervening, the grains soon settle. But as long as the grains are held in suspension and also at the beginning of the settling the grains occupy in the water the position shown in fig. 15. In this diagram the volume of free space between the grains which represents water is just as great as in figs. 13 and 14. In condition of fig. 15 the mass is liquid. The grains then again sink and form a closer deposit as shown in fig. 16. The condition shown in diagram by fig. 13 in which the sand has a great pore space occurs i.a. with quicksand on the coasts which is firm but on vibration occurring it becomes liquid. Sand in the condition of fig. 13 and in that of fig. 15 both is generally called quicksand, the fact that two conditions occur not being understood.

This phenomenon that with the same volume and the same content of water the sand may be either in solid condition (fig. 12) or in that of a suspension (fig. 14), though in this latter state only for a short

period, also occurs with fine quartz powder and with some other powders.

If A. ATTERBERG's works (V) are read carefully one acquires the impression that this authority also observed with loam this phenomenon that with the same content of water and the same volume two conditions are capable of prevailing side by side. ATTERBERG speaks of „*Fliesslehm*” and of a „*Stossfliess-grenze*”.

I therefore consider that it is not beyond the region of possibility that the difference between clay and loam will have to be sought in the adsorption. The prevalence of two conditions, the one of which may make way for the other owing to vibration in the same manner as with quicksand might be a characteristic of loam distinguishing it from clay.

LITERATURE.

1. J. VERSLUYS: „De capillaire werkingen in den bodem”. Amsterdam 1916.
 2. S. JOHANSSON: „Die Festigkeit der Bodenmassen bei verschiedenem Wassergehalt nebst Vorschlag zu einer Klassifikation”. *Sveriges Geologiska Undersökning*, Annual 7, N^o. 3 Series C, N^o. 256, 1913.
 3. K. TERZAGHI: „Erdbaumechanik”. Leipzig und Wien, 1925.
 4. J. VERSLUYS: „Loopzand en drijfzand”. *De Ingenieur*, pp. 589—598, 1926.
 5. A. ATTERBERG: see i.a. „Die Konsistenzlehre, eine neue physikalische Lehre”. *Kolloid-Zeitschrift*, Bd. 20, Heft 1, pp. 1—7, 1917.
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Zoology. — *The Nemas Anchylostoma and Necator*, By J. H. SCHUURMANS STEKHOVEN Jr. (Communicated by Prof. J. F. VAN BEMMELN.)

(Communicated at the meeting of December 18, 1926).

The Hookwormproblem will reach its final solution, when we will be able to distinguish the different earth-living larval stages of all *Anchylostoma*- and *Necator*species.

Until now our knowledge of the morphology of the hookwormlarvas still is superficial, notwithstanding LOOSS has done very prominent work on this subject. Therefore the finest technique, combined with the keenest observation are wanted to conquer all difficulties.

The present research considers the third, i.e. infective larval stage only. It will however be of great use to apply the same technique to the first and second stage larvae as well as to the hookworms, just after their arrival in the intestine or to the sexually mature individuals, as their histology has not been studied sufficiently.

Now the third stage larva, generally enveloped by a sheath, — the skin of the second stage larva, — is not only the most important stage, by its relation with mankind, but may be discriminated also from other freeliving nemas, according to its structure and movements. Chemical substances which destroy other earth-living larvae easily, do not attack the hookwormlarvae. Its snakelike movements are typical enough to identify the third stage larvae of the hookworm, although it is very difficult to give an adequate description. So I am quite convinced that BAERMANN did not mistake other larvae for hookwormlarvae, although he never gave an exact description of the incriminated larvae.

Moreover, it will be rather easy, I suppose, to gather sufficient knowledge about the social conditions in the examined regions, as for instance about the presence or absence of dogs to exclude in many cases a soilinfection with canine hookworm larvae. A decision, however, can be given by exact morphological research only.

Last years these larvae were studied by the SCHUURMANS STEKHOVEN's along biometrical lines, by VAN THIEL, who found some differences between the Genera *Anchylostoma* and *Necator*, although he is not always very clear in the interpretation of his observations, by SVENSSON, who paid special attention to the esophageal sphincter, by COBB,

who gave a minute description and beautiful figures of the head of *Necator americanus*.

During a Travelling Fellowship, granted to me by the ROCKEFELLER Education Board I got the opportunity to make a thorough study of some freeliving marine nemas in the office of Dr. STEINER, Nematologist at the Bureau of Plant Industry, Department of Agriculture, Washington D.C. U. S. A. and to gather more complete knowledge on the infective larvae of *Necator americanus* and *Anchylostoma caninum* during the last fortnight of my stay at Washington. Dr. STEINER was so kind to give me the necessary technical attendance, whereas Dr. SCHILLINGER furnished me with pure cultures of *Necator americanus* larvae, with human and canine excrementa and some living females of *Anchylostoma caninum*.

This research was made with the same technique as the nematologists of the Bureau of Plant Industry generally use. Confer COBB's and STEINER's paper. To isolate larvae from a soilculture I used wet, hairy tomato-seeds on which the larvae crept. So the attached larvae could be transported easily to a drop of water on a slide.

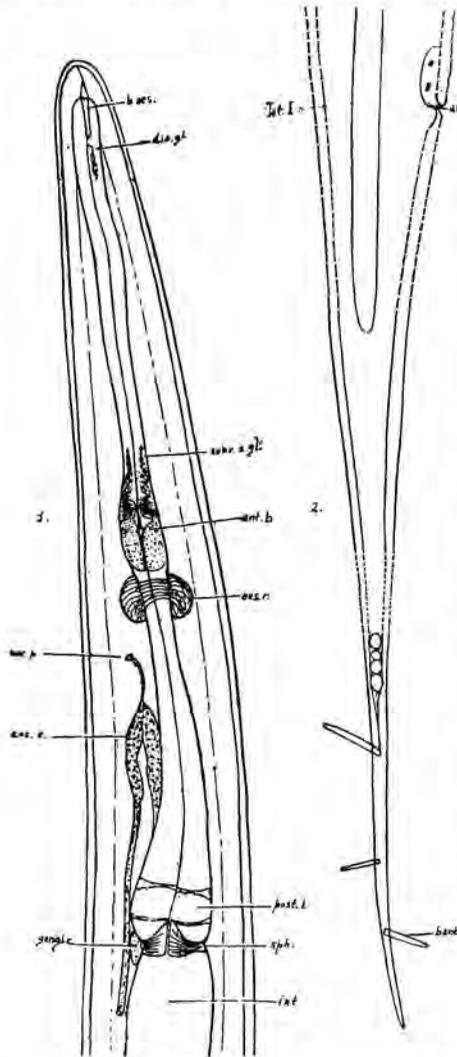
The third stage larva of Necator americanus.
Textfigure A., Plate I.

VAN THIEL emphasized the fact that the cuticular striations are much more pronounced and also more widely separated in the larvae of *Necator* than in those of *Anchylostoma*. This is true for the 3^d stage larva as well as for the second. Moreover the tail of the sheath is long and tapers to a very fine point, thus resembling a tube, pointed in a flame. To the empty skin of the larva (confer Textfigure A 2) some bacteria are attached. The head of the larva, when seen in lateral view (Pl. I, Fig. 2—4) shows no lips; this is still more evident when the head is seen on top (Pl. I, Fig. 1). Now the triangular mouth opens widely and marks the centre of the rectangular oral field. The oral aperture is encircled by the four papillae (pap.) and the lateral organs or amphids (amph.), both lying on the same level.

The latter near the oral opening very much, which is particularly clear, when the head is seen in lateral view (Pl. I, Fig. 2, 3). The amphids are composed of a short tubelike pouch (amph. p.) opening distally into a widening, where nervous fibrillae distinctly are to be seen, whereas its proximal end bears a median slit and resembles therefore an opening bud. The amphids are small in size.

The cylindrical, short oral tube passes into the beginning of the esophagus (b. oes.). In Literature some controversy exists about the beginning of the esophagus. According to COBB the cylindrical tube, lined with chitinised bars (oes. b.), to which the oral cavity passes, belongs to the esophagus. VAN THIEL describes the same as oral cavity. I think

we are allowed to assume, that the esophagus begins at the point where



Textfig. A.

Fig. 1. *Necator americanus*, third stage larva.

Enlarg. Öl Imm. 2 mM., Oc. 12, Tub.
140 mM. $\times 1/2$.

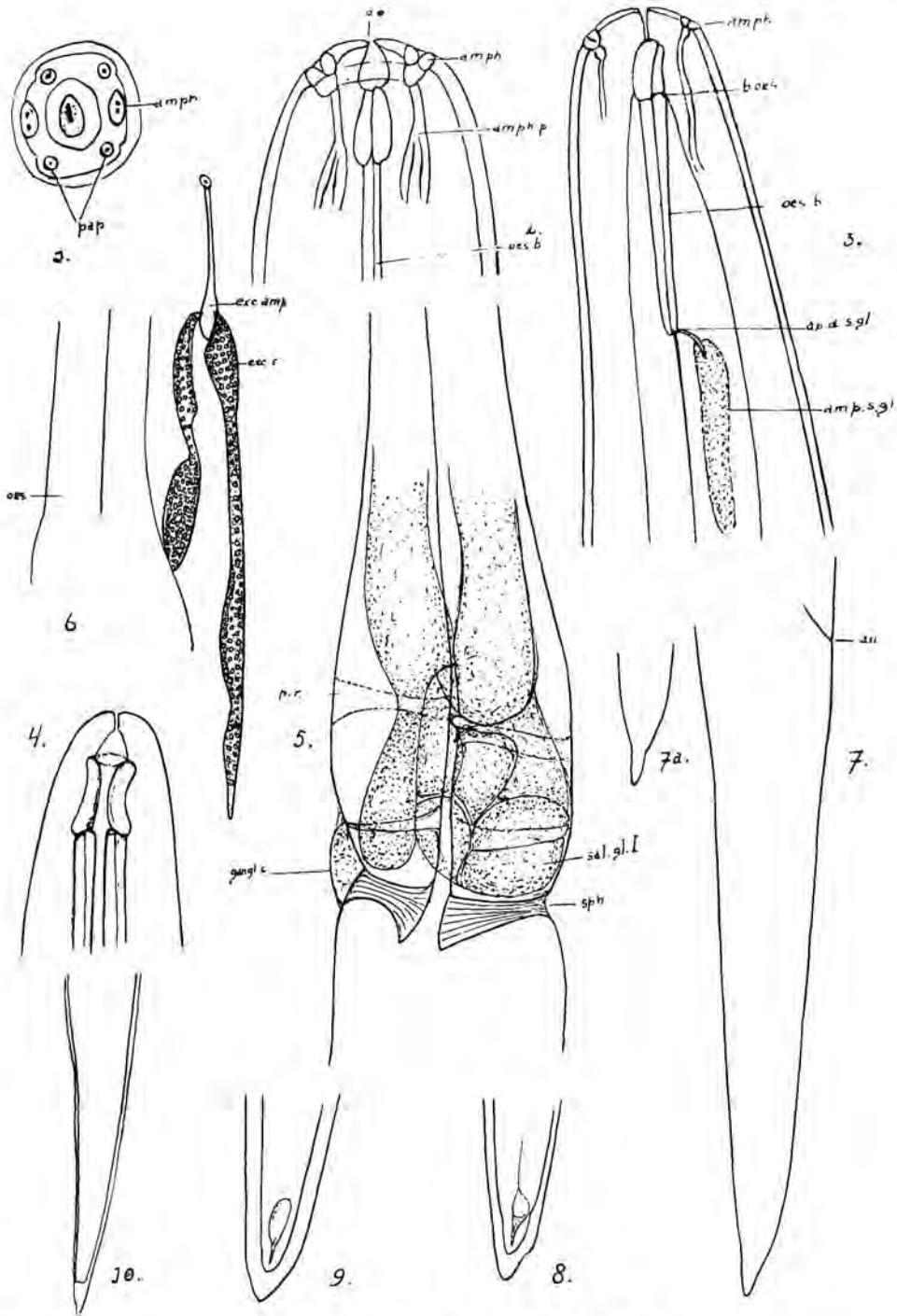
Fig. 2. The sheath (skin second stage larva)

Enlarg. Öl Imm. 2 mM., Oc. 6, Tubus
140 mM. $\times 1/2$.

ant. b. = anterior bulbus; an. = anus; b. oes. beginning of oesophagus; bact. = bacteria attached to the skin; d. s. gl. = dorsal salivary gland; exc. c. = excretory cell; exc. p. = excretory porus; gangl. c. = ganglion cell; int. = intestine; lat. l. lateral line; oes. n. = esophageal nerving; post. b. = posterior bulbus; sph. = sphincter; subv. s. gl. = subventral salivary gland.

spear of the Mermithids. Confer also Pl. I, Fig. 3.

the esophageal cells are attached to the intestinal tract at its proximal end (Pl. I, Fig. 3). In this case the mentioned part undoubtedly belongs to the esophagus and COBB is right, unless one supposes that the oral cavity, together with its chitinisations is pushed into the esophagus. In this I agree with COBB. The cuticularisations of the beginning of the esophagus are very characteristic. Here the esophagus is lined by two, equally thick parallel bars, each ending proximally into a curved clasp. These clasps are more or less bean-shaped, faintly curved outwards at their apices. The supposition seems logical that this clasp articulates with the long rod, although I could not prove this. In a former paper my wife and I compared this rod, together with its clasp with the stamen of a grass, bearing one pollensack only. If one turns the larva round its longitudinal axis, this whole apparatus gets a quite different appearance. If I compare a bar with its clasp with a spoon, both spoons, which sometimes touch each other almost with their apical ends (Pl. I, Fig. 2) — will be seen as individual objects in optical section (Pl. I, Fig. 4) in the case mentioned above; if however the animal is rolled over, the backside of one of the spoons will come up. Apparently COBB has figured this case in his paper, where he describes this apparatus as an onchium and compares it with the



Necator americanus, third stage larva.

- Fig. 1. Apex of head, enlarg. ÖI Imm. 2 mm., Oc. 12, Tubus 140 mm. $\times 2/3$.
 Fig. 2. The head in profile, enlarg. as 1. $\times 2/3$.
 Fig. 3 and 4, idem $\times 2/3$.
 Fig. 5. Esophagus, posterior bulb.
 Fig. 6. Intestine and excretory apparatus; enlarg. ÖI Imm. 2 mm., Oc. 4. Tub. 140 mm. $\times 2/3$.
 Fig. 7-9. Tailend.
 Fig. 10. Tail. enlarg. ÖI Imm. 2 mm., Oc. 4. Tubus 140 mm. $\times 2/3$.

amp. = amphot; amph. p. = amphidial pouch; amp. s. gl. = ampulla of the salivary gland; ap. d. s. gl. = aperture dorsal salivary gland; an. = anus; b. oes. = beginning of esophagus; exc. amp. = excretory ampulla; exc. c. = excretory cell; gangl. c. = ganglioncell; o. o. = oral opening; oes. = esophagus; oes. b. = esophageal rods; pap. = papilla; sal. gl. l. = lobus of salivary gland, sph. = sphincter.

With good reason VAN THIEL protests against this view. Former papers of the SCHUURMANS STEKHOVEN's pointed into the same direction.

When rolling the larva over, it may seem sometimes if one of the rods is thicker than the other; in reality this is not the case and I think this gives the solution of the controversy between COBB and VAN THIEL. But apart from the given morphological arguments my physiological observations speak against COBB's idea of the function of these structures.

Once I found the clasps protruding from the oral opening in a dead animal. This proves however nothing in favour of the real function of these parts, as changes in osmotic pressure of the bodyfluid may have occurred during the death of the animal as the cause of this.

Moreover, when a quiet ensheathed larva is observed for longer periods, one may have the chance to catch a larva in the act of shedding its sheath. The head curves backward, during which the skin of the third stage larva shrivels and folds; the mouth of the larva sucks at a definite spot the enveloping sheath into the oral aperture; the larva tugs at it violently, till it suddenly loses its hold and straightens again. Often the same manipulation is repeated, during which the rods do not change their place. It is not possible to conclude anything about the presence of chemical substances, helping to perforate the sheath at its apical end. I am however inclined to think at the possibility that the products of the salivary glands participate in the exsheathing of the larva. At any rate the larva suddenly lifts the lid of the sheath, which often remains connected with the tailend of the sheath at one side. The larva will attain its aim easiest, if it gets the opportunity to anchor itself in the sand, or if the distance between slide and coverslip becomes smaller and smaller, thus urging the larva to remain in the same position, or if the sheath sticks to a rough portion of the underlayer. So this phenomenon will be mainly mechanical.

If once the lid has been lifted, it is curious to observe the larva while turning round its longitudinal axis, and by doing this, freeing itself from its envelope. Sometimes the head glides backwards along the body as if to strip off the sheath. Each moment the larva emerges further; the empty skin falls down after each turn. Therefore the empty sheath presents a number of oblique folds just as in a towel turned round its axis in the same way. Textfigure C 6 gives an idea of such a sheath for a larva of *A. canium* in which things happen in the same manner.

In a waterculture of larvae a considerable number of animals had stripped off their sheath and I suppose that the same thing will happen in the soil. When a stain is added to the water of the culture, as for instance methylenblue or methylgreen the percentage of larvae, which sheds their sheath becomes larger.

The esophagus of the third stage larva presents 2 bulbi (Textfigure A 1) a rudimentary anterior bulb (ant. b.) which precedes the compar-

atively narrow nerve ring (oes. r), thus dividing the esophagus into 2 equal portions. This bulb was described already by VAN THIEL, which however did not identify it as such. The posterior bulb (post. b.) is very pronounced and marks the end of the esophagus.

Clear transverse (p. r.) rings divide the posterior bulb (Textfigure A 1 and Plate I, Fig. 5) into 3 strongly granulated portions. Here 3 esophageal glands (1 dorsal gland and 2 ventral glands), the salivary glands of other authors, originate. Both ventral glands open into the esophageal lumen just in front of the nervous ring (Textfigure 1, subv. s. gl.), whereas the dorsal gland forms an ampulla, (amp. d. s. gl.) just behind the mentioned cuticularisations of the anterior end of the esophagus, and opens into the latter by means of a short tube (Textfigure 1 and Plate I, Fig. 3 ap. d. s. gl.). The part of the salivary glands, lying in the esophageal bulb is lobed (Pl. I, Fig. 5 sal. gl. 1.). Next to the sphincter, see below, one finds a ganglionic cell. The posterior esophageal bulb does not show any thickenings or valvae at its lumen.

SVENSSON and CORT emphasize the existence of a clear space at the junction of the esophageal bulb and the intestine. SVENSSON is undoubtedly right to call this structure a sphincter (sph.). The muscular fibres are clearly visible. The sphincter is low at its junction to the intestinal wall and becomes higher at the lumen (Textfig. 1 and Pl. I, Fig. 5).

Its proximal side is perpendicular to the body-axis, therefore its distal portion protrudes into the intestine as a plug and resembles very much a cone, which is separated from the intestinal wall by a circular groove. As I will describe below for *Anchylostoma* this sphincter is the only part which really acts as such, providing the ingested food a passage to the intestine, whereas the bulb functions as a sucking apparatus. VAN THIEL is not very clear in this point. The lumen of the intestine (int.) is wide, the cells forming its wall are scanty granular.

The excretory apparatus, H-shaped in the typical representants of this group, has lost its anterior horns (Textfig. 1 and Pl. I, Fig. 6). The excretory ampulla (exc. amp.) is easily to be seen in front of the excretory cells (exc. c.) which open into the caudal end of the ampulla. These excretory cells lay astride over the intestine. Each excretory cell is rather narrow, its foremost portion resembles a peanut as it is narrowed in the middle, the caudal part of the excretory cells is narrow and parallel-sided; it reaches till behind the esophageal sphincter. Apparently a sphincter is found at the end of the curved tube, which leads from the excretory ampulla to the exterior. Another sphincter connects the ampulla and this tube. When the larva is alive it is very easy to observe the functioning of the excretory apparatus. Just after the ampulla has fallen down, having ejaculated its contents, it appears again in the same manner as a pulsatile vacuole of protozoa. I think it would be of great importance to compare the histology and the physiology of the pulsatile vacuole of protozoa with the excretory ampulla of nemas. I figured some phases of the ex-

cretory ampulla (Pl. III, Fig. 5 and 6) of the larve of *A. caninum*, which behaves in the same manner.

Finally I will point to the fact that the tail of the third stage larva, opposite to the tail of the second stage larva (sheath) is rather short and ends bluntly (Pl. I, Fig. 7—10).

The third stage larva of Anchylostoma caninum.

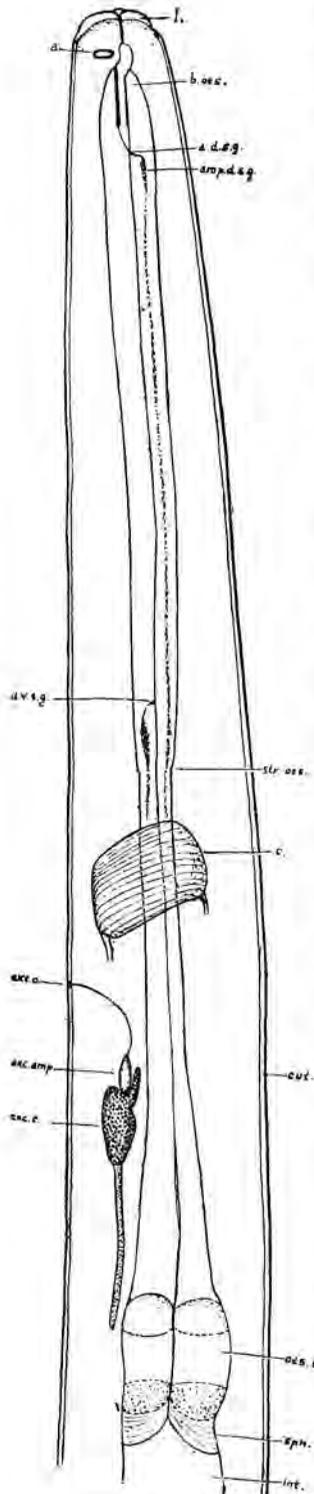
Textfigures B, C, Plate II en III.

Seen on top the larvae of *A. caninum* are easily to distinguish from those of *Necator americanus* as the first ones possess three lips, which reduce the oral aperture to a small triangular space. If once observed, these lips (*l*) are also very obvious, when seen from aside (Pl. 2, Fig. 2—5, Pl. 3, Fig. 1).

It is very interesting to note that PERRONCITO first found these lips in *A. duodenale*, whereas LOOSS denied their existence. The papillae (Pl. 2, Fig. 4 pap.) are situated on the lips; each dorsal lip bears one papilla only (Pl. 2, Fig. 1 d.p.), whereas the subventral lip is in the possession of 2 papillae (Pl. 2, Fig. 1, subv. p.). The amphids (amph.) are larger than in *Necator* (Pl. 2, Fig. 4—6), dont show a lateral slit, which character is very obvious, when the larva is observed from aside. Moreover they are shifted further caudal. Therefore when the papillae are focussed, the amphids are not to be seen (Pl. 2, Fig. 1). The amphidial pouch (amph. p.) is much wider in this species than in *Necator*, its wall is however thinner, which character is not figured in my drawings (Confer Textfig. C, Fig. 5).

The oral cavity is short and narrow (Pl. 2, Fig. 2—5). The apical end of the esophageal wall is provided with three rodlike thickenings, which VAN THIEL correctly described to be of unequal thickness. The ventral thickening is the thickest and at their apical end these rods are connected two by two. A widening of the oral wall forms the junction between the short oral cavity and the esophageal lumen. In this cavity the spoon-shaped parts, which articulate with the longitudinal rods, project. Thus the oral cavity gets the shape of an urn. Compare also VAN THIEL's description. The latter author did not understand the mutual relations of the described parts.

The esophagus differs in its shape and structure in many points from that of *Necator*. For instance a distinct anterior bulb is absent, although the oesophagus is slightly constricted just in front of the nerve ring (Textfig. B, C, Pl. 2, Fig. 8, Pl. 3, Fig. 4, oes. r.), which is broader than in *Necator*. The structure of the esophagus has not been changed however interiorly at this spot. Slightly more apical (Textfig. B, a. v. s. g.) the ventral salivary glands open into the esophageal lumen (Confer Pl. 2, Fig. 8) which LOOSS figured minutely in his well known monograph. Sometimes, viz. when the larvae are sufficiently transparent,



Textfigure B.

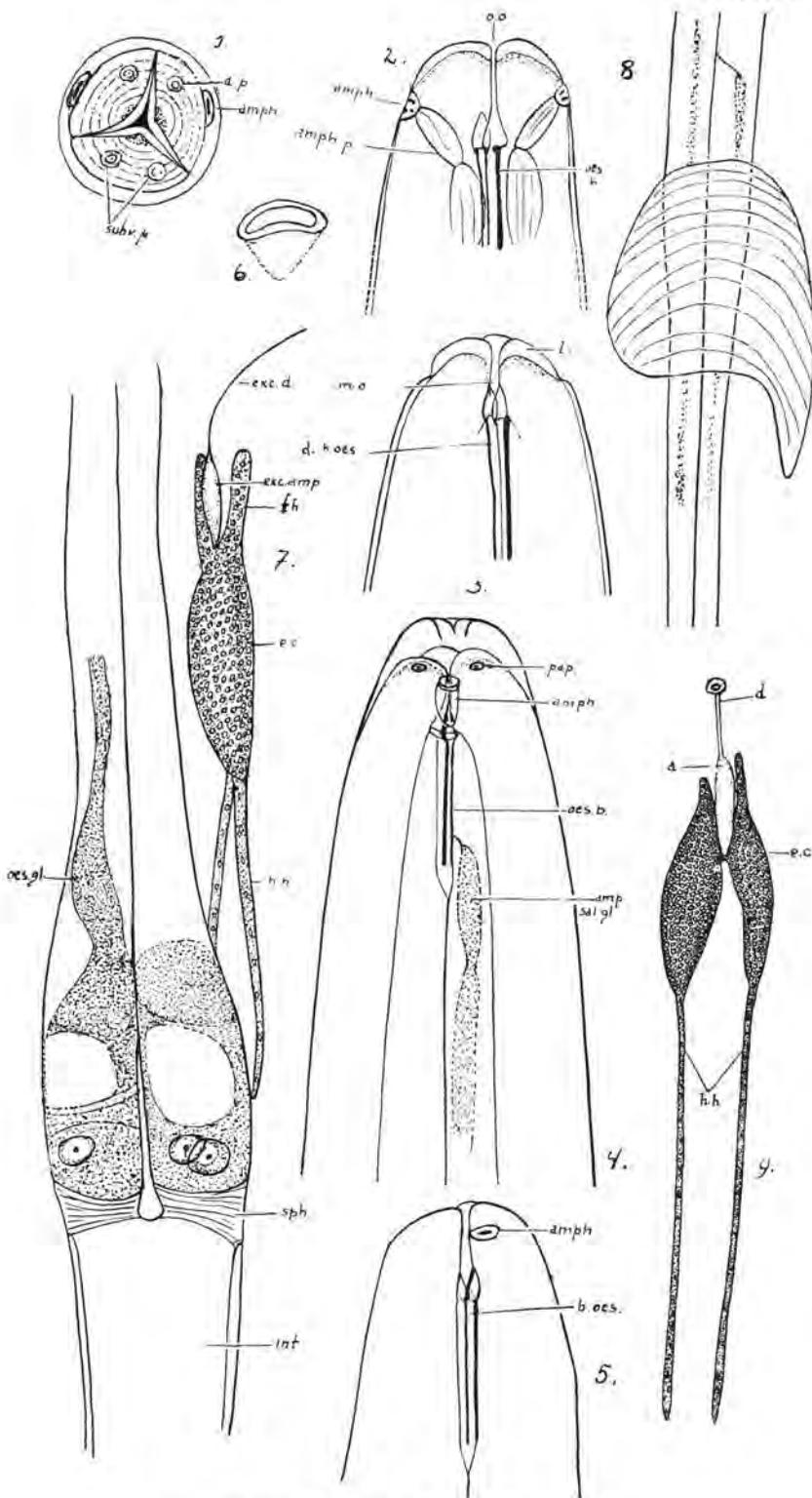
one is able to observe nerves (Pl. 3, Fig. 4 n) originating from the nerving and running in caudal and apical direction. The posterior bulbus (Textfig. B, oes. b.) misses the valvulae, LOOSS ascribes to it. Its lumen has somewhat thickened walls only. Compare Pl. 2, Fig. 7 and Pl. 3, Fig. 2 and 3. Moreover the bulbus is not sharply separated from the rest of the esophagus, but gradually passes into it. In structure this bulbus also differs from that of *Necator*. Its granulation is scanty. The clear rings (Pl. 2, Fig. 7), which were very obvious in *Necator* are rather inconspicuous here. The nuclei of the salivary glands however are more pronounced (Pl. 3, Fig. 2, n. oes. c.). The middle portion of the bulbus is vacuolised. I mentioned already the ventral salivary glands, the dorsal salivary gland stretches further apical, its ampulla (amp. d. sal. gl.) is narrower than that of *Necator*, whereas the efferent tube opens into the esophagus just in front of the proximal end of the esophageal cylinder (Textfig. B., a. d. s. g.). In exceptional cases only it is possible to follow the course of the esophageal glands (oes. gl. Pl. 2, Fig. 7) in the bulbus of the living animal. The larva of *Anchylostoma* possesses a similar sphincter (Textfig. B, Pl. 3, Fig. 7, Pl. 3, Fig. 2, 3, sph.) at the junction of the esophagus (l. oes.) and intestine (int.). Here the intestinal wall forms a continuous curved line with the sphincter, therefore this structure resembles the mediastinum of mammals. Once I had the opportunity to observe the sphincter,

Textfigure B.

Anchylostoma caninum, third stage larva.

Enlarg. Ö1 Imm. 1.5 mm, Oc. 8, Tubus 140 mm. $\times 1/2$.

a. = amphi; a. d. s. g. = aperture dorsal salivary gland; a. v. s. g. = aperture ventral salivary gland; amp. d. s. g. = ampulla dorsal salivary gland; c = oesophageal nerving; b. oes. = beginning of esophagus; exc. c. = excretory cell; exc. amp. = excretory ampulla; exc. o. = orifice excretory apparatus; int. = intestine; l. = lip; oes. b. = esophageal bulbus; sph. = sphincter; str. oes. = constriction of esophagus.

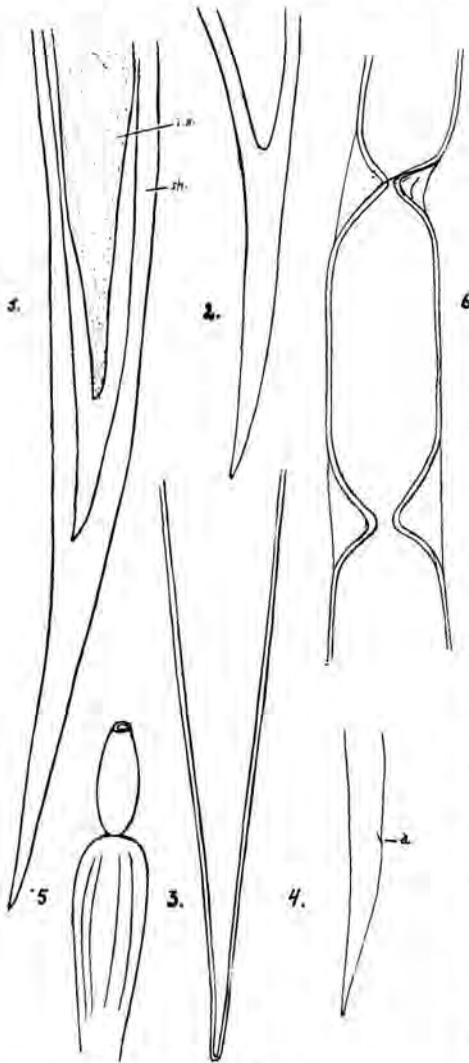


Anchylostoma caninum, third stage larva

- Fig. 1. The head, frontview, enl. Ö1 Imm. 2 mm., Oc. 12, Tubus 140 mm. $\times 2/3$.
 Fig. 2. Head in profile, Enlarg, Ö1 Imm. 1.5 mm., Oc. 12, Tubus 140 mm. $\times 2/3$.
 Fig. 3. enlarg. as in 2.
 Fig. 4. enlarg. as in 2.
 Fig. 5. enlarg. as in 1.
 Fig. 6. Amph., enlarg. as in 1.
 Fig. 7. Oesophagus with the excretory apparatus, enlarg. as in 1.
 Fig. 8. Oesophagus and nerving, enlarg. as in 1.
 Fig. 9. Excretory apparatus enlarg. Ö1 Imm. 1.5 mm., Oc. 8, Tubus 140 mm. $\times 2/3$.

a = ampulla excretory apparatus; amph. = amphid; amph. p. = amphidial pouch; amp. sal. gl. = ampulla of salivary gland; b. oes. = beginning of oesophagus; d = excretory porus; d p. = dorsal papilla; e. c. = excretory cell; exc. amp. = excretory ampulla; exc. d. = excretory duct; h.h. = posterior horns of excretory apparatus; f.h. = anterior horns of the same; int. = intestine; l = lip; m.o. = oral aperture; m.o. = mouthcavity; oes. b. = oesophageal rods; oes. gl. = oesophageal gland; pap. = papilla; d. b. oes. = dorsal oesophageal rod; subv. p. = subventral papilla.

on which the bulbus rests as a recipient in the ring of a water-bath



Textfigure C.

Anchylostoma caninum,

Fig. 1. Sheath enveloping the third stage larva
Enlarg.: Obj. 12 mm., Oc. 8, Tub. 140 mm. $\times 1/2$.

Fig. 2. Empty sheath, enlarg. as in 1.

Fig. 3. Tail of third stage larva; enlarg. as in 1.

Fig. 4. Tail of third stage larva; enlarg. Obj.
16 mm., Oc. 4, Tub. 140 mm. $\times 1/2$.

Fig. 5. Amphid; enlarg. ÖI Imm. 4 mm., Oc. 12
Tub. 140 mm. $\times 1/2$.

Fig. 6. Shedded skin of second stage larva;
enlarg. ÖI Imm. 2 mm. Oc. 4, Tub. 140 mm. $\times 1/2$;
a = anal opening; i. s. = third stage larva;
sh. = sheath.

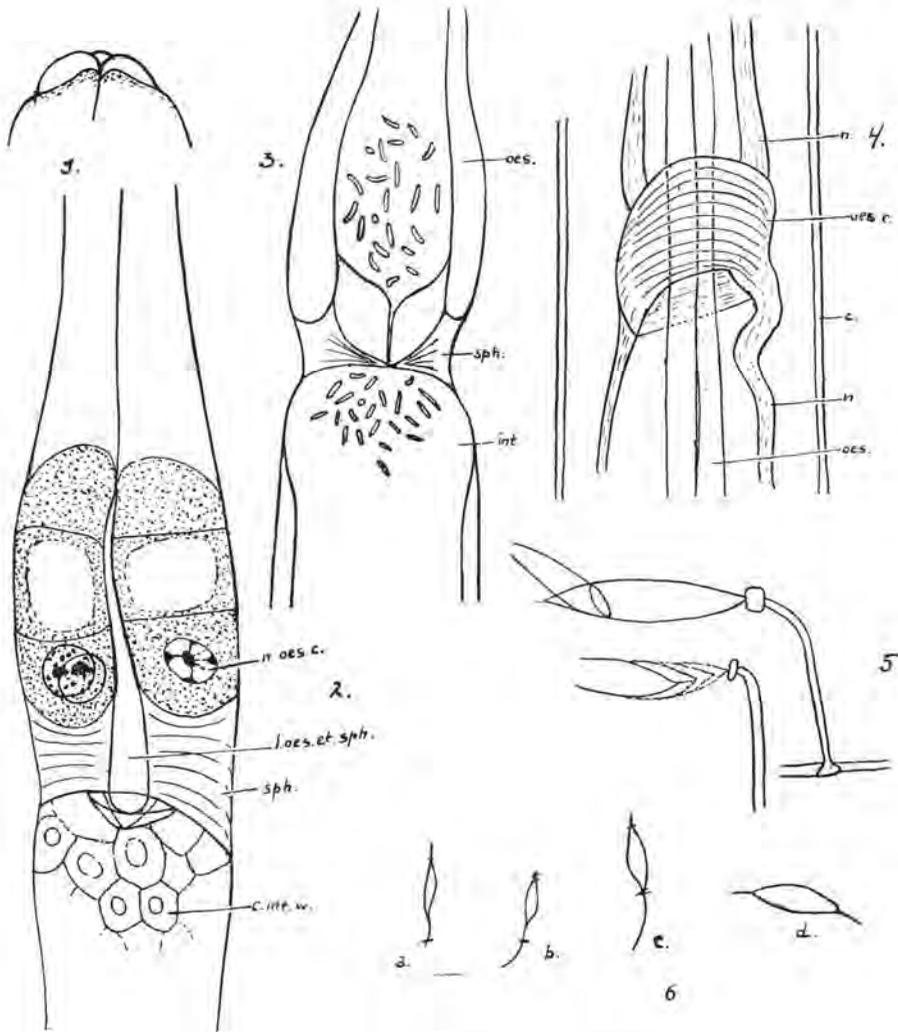
("au bain marie"), while functioning. The esophagus was filled with bacteria (Pl. 3, Fig. 3). Now and then the sphincter (sph.) opened and moved the esophageal contents to the intestine (int.). The latter has a rather narrow lumen in *Anchylostoma*, whereas its wall is thickly granulated. So SVENSSON's supposition has been proved. LOOSS described the junction of esophagus and intestine in *Anchylostoma duodenale* as follows:

"Die vordersten Zellen des Darmrohres heben sich deutlich von den folgenden ab, dadurch dass sie kürzer bleiben und in ihrem Innern keine Körnchen abschieden, auch sich so dicht an den Bulbus anlegen, dass sie eher zu diesem als zu dem Darne zu gehören scheinen. Sie representieren die Anlage des späteren zelligen Verschlussapparat am Ende des Oesophagus".

Now apparently LOOSS has not caught the true meaning of the sphincter, as the sphincter really belongs to the esophagus and contains muscular fibres. The intestine commences caudal from the sphincter. VAN THIEL speaks in this connection about some cells serving for the purpose of closing the entrance of the intestine ("Schliesszellen").

The excretory apparatus is H-shaped (Pl. 2, Fig. 7, 9, Textfig. B), the anterior horns (f. h.) are very short and embrace the excretory ampulla (exc. amp.), the posterior horns (p. h.) lay astride over the intestine. The apical por-

tion of this apparatus is wide, shows no constriction and reaches till about



Anchylostoma caninum, third stage larva.

Fig. 1. Head with papillae, Enlarg. ÖI Imm. 2 mm., Oc. 12, Tubus 140 mm. $\times 2/3$.

Fig. 2. Esophagus and esophageal bulbus Enlarg. ÖI Imm. 1.5 mm., Oc. 12, Tubus 140 mm. $\times 2/3$.

Fig. 3. Sphincter between intestine and oesophagus; Enlarg. ÖI Imm. 1.5 mm. Oc. 8, Tubus 140m m. $\times 2/3$.

Fig. 4. Esophageal nerving and nerves, enlarg. as in 3.

Fig. 5 & 6. Excretory apparatus while functioning, enlarg as in 2, the stripes indicate the points of maximum expansion.

c. = cuticula; c. int. w. = cells intestinal wall; int. = intestine; l. oes. et sph. = lumen of esophagus and sphincter; oes. = oesophagus; oes. r. = esophageal nerving; n. = nerve; n. oes. c. = nucleus of esophageal cell, sph. = sphincter.

the middle portion of the posterior esophageal bulbus. In Plate 3, Fig. 5 and 6 I have figured how the ampulla gradually enlarges till the utmost tension has been reached and an ejaculation follows; compare p. 118.

The tail of the sheath (Textfig. C, Fig. 1 and 2) is shorter than the same part in *Necator*. VAN THIEL compares its point with a sharpened pencil-point. The tail of the third stage larva (Textfig. C, Fig. 1, 3, 4), is considerably longer than the tail of the same stage larva of *Necator*, whereas its point is also finer. The transverse striations visible on the sheath as well as on the skin of the third stage larva are rather inconspicuous; striae much closer than in *Necator*.

One is inclined to ask what bearing these facts have on the relationship of these parasites with free-living nematodes. They differ widely from *Rhabditis* in the structure of the mouth cavity, number of papillae and structure of esophagus.

Concerning other free-living forms as *Rhabdolaimus* (especially *Rh. terrestris*) to which these show an unmistakable resemblance, we know too little to enable us to arrive at trustworthy conclusions. Moreover, the adult forms of *Necator*, as well as of *Anchylostoma*, up to the present have not been studied along these lines, so that further speculation must be postponed until at least a part of this program has been fulfilled.

SUMMARY.

An accurate morphological research on the larvae of two species of the Genera *Anchylostoma* and *Necator* gave a number of differences between the concerned nemas. The main differences were found to consist in the structure of the head and its organs, lips, papillae, amphids, in the covering of the apical end of the esophagus, the presence or absence of the anterior esophageal bulbus, in the structure of the posterior esophageal bulbus, the structure of the esophageal (salivary) glands, in the sphincter forming the junction between esophagus and intestine, in the structure and shape of the excretory apparatus, in the shape of tail and sheath of the third stage larva and in the distribution of the transverse striations on the skin of second and third stage larvae.

Now it is possible to distinguish between the larvae of *Necator* and those of *Anchylostoma*. In large parts of America, where *Necator americanus* is the only human hookworm it is rather easy to conclude at the presence or absence of human or canine hookworm larvae in infested soil.

As far as the Dutch East Indies are concerned we are not allowed to go so far in our conclusions, but as for instance *Anchylostoma duodenale* is a rather rare parasite in several provinces of Java the above mentioned facts may be sufficient for those regions to arrive at trustworthy conclusions. Further I expect that a new research taking into

consideration the larvae of *A. duodenale* and *A. ceylanicum* will provide us with new data, which will enable us to identify all concerned larvae by means of easy characteristics.

The present research emphasizes also that a study of free living nemas is indispensable for Parasitologists and that a uniformity in methods of research is quite necessary for both groups of Nematologists.

POSTSCRIPTUM.

When controlling my proofs I saw a paper of SVENSSON and KESSEL (Journal of Parasitology Vol. 13, Dec. 1926) which brings more extensive data about the larvae of *Necator americanus*, *Anchylostoma duodenale* and *A. caninum*. The differences which they emphasize especially are the so called protusile onchium of COBB, which they think acts as a spear in the sense of COBB and the esophageal sphincter. Some illustrations are added. With respect to the morphology this paper is only a somewhat more minute description of the facts contained in the previous paper of SVENSSON of 1925.

LITERATURE.

COBB, N. A. The Pharynx and alimentary Canal of the Hookworm larva, *Necator americanus*, Jl. of Agric. Research Vol. 25, p. 359, 1923.

CORT, Journal of Parasitology Vol. 11, p. 228, 1925.

LOOSS, A. The Anatomy and Life History of *Anchylostoma duodenale* Dub. I, Rec. Egypt. Government School of Medic. 1905.

PERRONCITO, E. La maladie des mineurs du St. Gothard au Simplon. Archives Italiennes de Biologie T. 3, p. 315, 1882, p. 7, 1883.

SCHUURMANS STEKHOVEN, J. H. Jr. en SCHUURMANS STEKHOVEN—MEIJER, A. W. Anchylostomiasisbekämpfung und Bodeninfektion, Verh. des 4ten Kongr. der F. E. A. T. M.

SCHUURMANS STEKHOVEN, J. H. Jr. en SCHUURMANS STEKHOVEN—MEIJER, A. W. Voorbereidende onderzoekingen met betrekking tot de differentiatie van de larven van de mijnwormen van mensch en hond, Tijdschr. v. Vergel. Geneesk. Dl. 10, p. 1, 1923.

SVENSSON, R. A morphological distinction between infective larvae of *Anchylostoma* and *Necator*. Proc. Soc. f. Experim. Biology and Medicine 22, 1925.

VAN THIEL, P. H. De differentieeldiagnose van de strongyloïde larven van *Necator americanus* en *Anchylostoma caninum*, Tijdschr. v. Vergel. Geneesk. 10, 1923.

VAN THIEL, P. H. Verdere onderzoekingen omtrent de differentieeldiagnose van de strongyloïde larven van *Necator americanus* en *Anchylostoma caninum*. Geneesk. Tijdschr. v. Nederl. Indië, Deel 66, p. 14, 1926.

VAN THIEL, P. H. Diagnostie différentiel des larves strongyloïdes du *Necator americanus* et de l'*Anchylostoma caninum*. Annales de parasitologie humaine et comparée T. 4, p. 228, 1926.

Utrecht, 21 November 1926.

Palaeontology.— *Report on a Fossil Found at Trinil in 1926, and erroneously considered as the remains of a skull of Pithecanthropus.*
By Dr. W. A. MIJSBERG and Dr. H. J. T. BIJLMER at Weltevreden.
(Communicated by Prof. L. BOLK.)

(Communicated at the meeting of December 18, 1926).

In the latter half of 1926 during a visit to Trinil in Java in the neighbourhood of the region where some time before DUBOIS had disinterred the remains of the Pithecanthropus, Dr. HEBERLEIN at Soerabaja received from a native a fossil, which the latter had found there. The find was presented by Dr. HEBERLEIN to the Government of the Dutch East Indies. As the fossil was considered as the remains of the cranium of a prehistoric man, the Government entrusted Dr. W. A. MIJSBERG, anatomist of the S. T. O. V. I. A. at Weltevreden, and Dr. H. J. T. BIJLMER, military doctor, charged with the carrying out of anthropological researches, likewise at Weltevreden, with the task of studying the remains, and of drawing up a scientific report of their conclusions.

On December 7th the fossil was put at our disposal. Already at the first view serious doubt arose whether we had actually the remains of a skull before us, and the thought occurred to us that the fossil was the extremity of a long piece of a skeleton adjoining an articulation of a large prehistoric animal.

The object is very heavy in proportion to its size. It consists chiefly of a hard mass, exhibiting a sponge-like construction. Save for two exceptions to be more fully described later, this mass has an irregular outline and a rough surface, for the pores and the partitions between them of the spongy mass reach to the surface. Evidently all along this surface the remains found are severed from their surroundings. On one side, however, the surface is smooth, the spongy substance being covered by a thin layer of a compact mass not quite $\frac{1}{2}$ mm. thick. This smooth surface is curved regularly; it is part of an almost spherical surface, of which in one direction a larger part is present than in the direction normal to it (Fig. 1). The greatest dimension of this part is about 22 cm. measured across the curvature, about 11 cm. in the direction at right angles to it. This latter dimension cannot be determined so accurately as the length, because on one side along part of the long edge, a piece about 2 cm. broad, fits on to the other. This piece is curved in the longitudinal direction, and presents in this direction the same curvature as the spherical part, whereas it is flat cross-wise, perhaps even bent slightly concave. Along the other long edge the spherically curved surface is connected over a length of only 12 mm.

with another region, where the spongy mass likewise does not reach the surface. The plane in which this second region lies, is about at right angles to the base of the spherically bent part. In this second region the spongy mass is covered by a thicker layer of compact substance; its surface is feebly corrugated, shows besides irregular nodular protuberances, and in the region nearest the edge of the spherically curved surface it has round pits, which give access to ducts that penetrate further inward. (Fig. 2.)

Successively the three parts which are to be distinguished in the object, the spongy mass, the spherically curved smooth part, and the part where the compact mass shows relief, will be described more in detail.

The hard *spongy mass* has a pale greyish colour; locally the colour varies; at some places there is present a whitish bye-colour, in many places a brownish one. Scattered all about there lie small particles which glitter brightly when the light strikes them in a favourable way. At the surface the mass is pretty brittle, so that in the manipulation small pieces are sometimes broken off; these can easily be rubbed fine. The pores of the spongy mass are at most a few mm. deep, but for the greater part shallower; if it is tried to penetrate deeper, one is stopped by a hard stony mass. The apertures are sometimes irregularly round, sometimes clearly angular, the structure resembling a honey-comb; sometimes, and this seems to be partly dependent on the direction of the surface of fracture, one dimension clearly preponderates. The greatest dimension of the spaces nowhere exceeds 2 mm.; this measurement, which occurs only by way of exception, is only found as longitudinal dimension of the longest among the elongated holes. The partitions between the holes are always narrower than the apertures themselves.

It is very significant that in the neighbourhood of the spherically curved smooth surface the holes are much smaller and closer to each other than elsewhere; close under the curved surface they are exceedingly fine. This is very clearly to be seen at some places along the borders of the spherically curved surface, where the thin layer of the compact mass has disappeared, and the most superficial layer of the spongy mass lies bare. Here the holes resemble pin's pricks exceedingly close together.

At the rough surface of the spongy mass end some canals, which are sharply distinguished from the pores of the spongy mass by their almost purely round form and their greater size. Opposite the side covered by the compact layer showing relief, there ends at the surface a round canal of a diameter of 2 mm., which extends in the direction of the said side coated with the compact mass, possesses an almost straight course, and can be probed 8 cm. deep. On the same side, but nearer the border of the curved surface there are four smaller similar apertures, in which it is, however, possible to penetrate only a small distance with the probe.

The general form and dimensions of the smooth, for the greater part spherically curved, surface have already been described. The smoothness

of this part is chiefly owing to the spongy mass being covered here with a thin layer of a compact mass not quite $\frac{1}{2}$ mm. thick. All along the edge

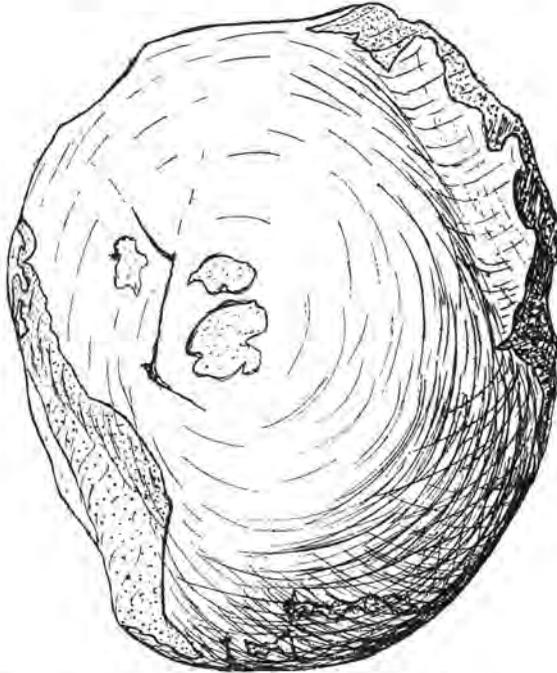


Fig. 1. ($\pm \frac{1}{2} \times$ Natural size)
The smooth, for the greater part spherically curved, surface.

of the smooth part the thickness of this layer is the same. Everywhere, except where the smooth surface bends and passes into the second compact part, which is still to be described more fully, the thickness can be measured accurately, as the covering layer stops everywhere abruptly. Evidently its edge is, therefore, an artificial, not a natural one; hence the piece of the curved compact layer present forms part of a larger compact mass, with which the spongy mass was coated. Only over the range of 12 mm., where the smooth curved surface bends and forms the surface at right angles to it, which shows relief, the natural border has been preserved. That for the rest the edge is an artificial free one, appears also from the facts that its form is exceedingly irregular, and that at some places the most superficial layer of the spongy mass continues the curvature of the smooth surface, the covering compact layer which undoubtedly had lain over it, being absent. These are the spots where the very fine structure of the most superficial parts of the spongy mass can be studied most successfully.

The smooth surface is not intact everywhere. About $\frac{1}{2}$ cm. from the border where the smooth curved surface passes into the second part where the spongy mass is covered with a compact layer, there is a very small defect where the spongy mass lies bare. It is owing to this defect that

the compacta of the part with the only natural border that the smooth curved surface possesses, is just not in connection with the rest of the compact layer. The separating defect is, however, very narrow, sometimes only $1\frac{1}{2}$ mm. broad, the bottom of the defect now and then still retaining remains of a compact substance, which prove the earlier connection of the regions now separated.

There is further on the smooth curved surface a Z-shaped scratch of a length of 6 cm., reaching to the spongy mass. Next to it there are two shallow defects near each other, one with a diameter somewhat more than 1 cm., the other of about $2\frac{1}{2}$ cm. diameter; they possess irregular edges. On the other side of the Z-shaped scratch there is another smaller, also very superficial defect. Close to the narrowest border of the curved surface there is still a long, narrow defect of a very irregular shape. The total length is $6\frac{1}{2}$ cm.; it consists of three parts, which do not run in the same direction; between the parts the compact mass is intact. The width of this defect varies greatly, maximum 9 mm.; its edges are, therefore, very irregular. The depth also differs; but it is nowhere more than 1 mm. The importance of this defect is due to the fact that it was considered as the remains of a cranial suture by the first owner of the object.

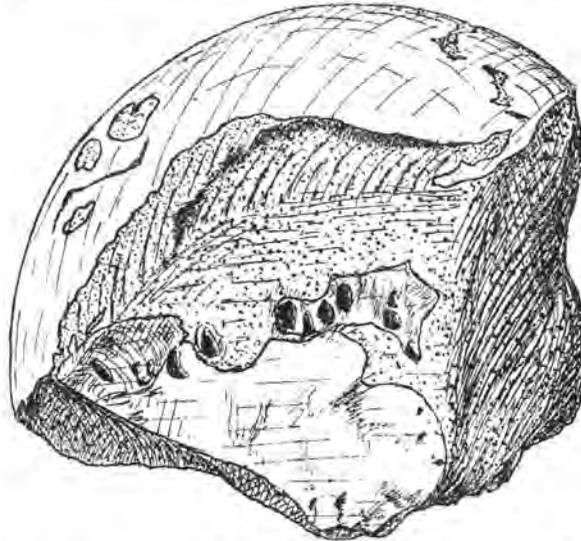


Fig. 2. The two regions where the spongy mass is covered with a compact layer.

The second part where a compact layer overlies the spongy mass, has the form of a trapezium. The longest side, which lies on the side of the border of the curved surface, has a length of about 11 cm., the height is about 6 cm. Only over a range of 12 mm., where as described before, this part bends round, and the smooth part with spherically curved surface begins, is the border natural. For the rest the edge has been formed artificially by fracture also of this compact part. The thickness of the

compact layer varies, on the whole it is greater than that of the spherically curved part ; at one spot I could measure a thickness of 5 mm. The surface of this part is not quite flat ; it is slightly corrugated as a whole, and possesses besides small nodular rugosities. Close to the narrow edge there are a few small pits to be seen, and along the long side of the trapezium, the side lying closest to the border of the spherically curved surface, there lie 13 large depressions. These depressions are almost round ; the diameter of the largest of them is $\frac{1}{2}$ cm. The depressions are smooth-walled, i.e. they are covered with a compact mass, their depth is about equal to their diameter.

At the bottom of each depression there is the entrance to one, sometimes two ducts ; the canals are narrower than the depressions themselves, they are about at right angles to the outer surface. As a rule they cannot be probed far, but in one of them it is possible to put in the probe 5 cm. deep.

Summarizing it can, therefore, be stated that the object found is of a stony substance, and consists for the greater part of a mass of a spongy structure. On one side the spongy mass is coated with a compact layer of $\frac{1}{2}$ mm. thickness, which has a smooth surface. The greater part of this surface is regularly curved, and forms part of an about spherical surface. The greatest dimension of this curved part is about 22 cm. measured across the curvature, 11 cm. measured in the direction normal to it. On one side this curved surface passes into a region about 2 cm. broad, which is curved in the same way lengthwise as the spherical part, but not curved convex breadthwise, perhaps it is even slightly concave. On the other side the curved surface, bending sharply, is connected over a length of 12 mm. with a second region, where a compact layer lies on the spongy mass. This second region is about at right angles to the basal surface of the curved part. The compact layer is thicker here, sometimes as much as 5 mm. thick ; the surface of this part is slightly arched, and besides provided with small nodular protuberances. In the region that lies nearest to the border of the spherically curved part there are found 13 round pits with smooth walls, the largest of which is about 5 mm. in diameter, and of the same depth. At the bottom of the pits, there are the mouths of one, sometimes two smaller canals penetrating into the spongy mass. One of these canals possesses a length of 5 cm. Some canals also end at places of the free surface of the spongy mass that is due to fracture ; one of them has a length of 8 cm. The spongy mass is much denser under the spherically curved surface, the pores in the mass are much smaller here, and lie closer together than elsewhere.

On the ground of the data communicated in what precedes it is possible to decide with certainty what the object is.

When the object is viewed obliquely from the side (this is figured in Fig. 3), the arching of the smooth surface actually bears some resemblance to that of a skull cap provided on the front side with bulging out orbital

arches fused to one ridge, as they are found in the skulls of *Homo Neanderthalensis*. The second part covered with a compact mass would then have to represent the somewhat flattened occipital part of the skull. There is, however, no more than a superficial resemblance to the form of the vault of a human skull. The curvature of the bent surface is not very probable for a skull. The view that the flat part should represent projecting orbital arches is supported by nothing, the place where these projecting ridges were to bend into the orbits being absent. There are no sutures; what has been taken as such is really a defect of the compact layer, where the spongy mass lies bare. Hence nothing pleads in favour of the diagnosis skull except a slight morphological resemblance. On the other hand important data exclude the validity of this diagnosis. The "skull cap", i.e. the thin compact layer with for the greater part spherically, arched surface, is $\frac{1}{2}$ mm. thick. So slight a thickness is undoubtedly excluded for a skull cap. Hence the original owner supposed that the skull

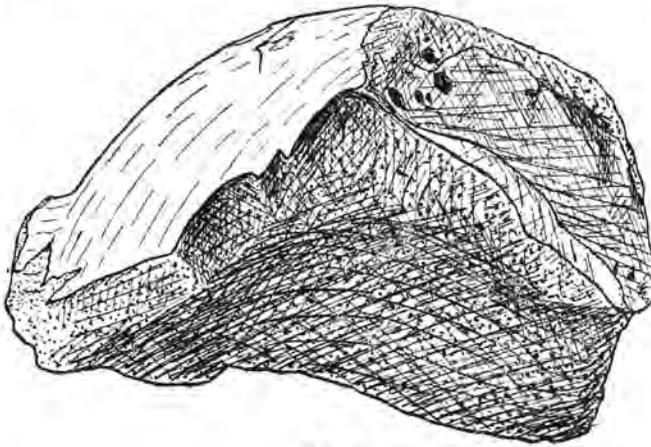


Fig. 3.

cap had become thinner, by being dissolved from within by a mass that had flowed into the cranial cavity, and now remains under the skull cap as the spongy mass described before. Apart from the question whether it would be possible that such a solution should have taken place, it must be pointed out here that it would have to be considered as very unlikely that such a solution should have led uniformly everywhere to the existence of a skull cap of so small a thickness, whereas the "flattened occipital part" would have become much thicker, and moreover of an irregular thickness; and it would be entirely inconceivable that this solution should also have taken place at the compacta of the frontal ridges, for as the cranial cavity is not continuous with the frontal ridges, it cannot be conceived how the mass that has penetrated into the cranial cavity, should have got under the compacta of these ridges. What further entirely disposes of the diagnosis "skull", is the circumstance that the "occipital

part" would exhibit 13 depressions, from where canals would pass inward into the spongius mass.

When we now proceed to deal with the question, what then *is* the object, first of all the question rises what is the stony spongius mass of which the greater part of the object consists. This mass might be thought to be a stony formation of spongius structure, but this view cannot explain, why the stony mass should be covered on one side with a compact layer, $\frac{1}{2}$ mm. thick with a perfectly smooth surface; why the cavities in the spongiosa get smaller towards this surface; why there are found canals there, some of which reach the second surface covered with a compact mass, and which are coated near their mouths with a continuation of this compact layer.

All these facts can, however, be accounted for from one point of view, when the spongius mass is considered as the fossilized spongius bony mass at the articular end of a fragment of a skeleton, and the compact layer of a thickness of $\frac{1}{2}$ mm. overlying it and spherically curved for the greater part, as the fossilized substantia compacta forming the articular surface. In harmony with this view is the fact that the spongius substance under the smooth superficial layer shows a very fine structure, since the substantia spongiosa are always greatly condensed under an articular surface. The for the greater part spherical curvature of the smooth surface and its great dimensions render it probable that the object is one of the extremities of a long piece of a skeleton of a large animal. Also the position of the second part consisting of a compact mass is in conformity with the view that it is the extremity of a long piece of a skeleton, for the direction of the second part is about at right angles to the curved surface. Hence this second piece is to be considered as part of the lateral surface of a long skeleton piece near its articular end. The somewhat irregular surface and the not uniform thickness of the in general considerably thicker compacta in this region are in conformity with the fact that a piece of a skeleton often exhibits rough processes near the articular extremity. In this region of a long skeletal part and chiefly close to the border of the articular surface vascular canals always penetrate into the spongius mass, which at their beginning are covered with a continuation of the compact substance. The canals in the spongius mass described before are, therefore, to be considered as vascular canals. All the anatomical characters that a long piece of a skeleton ought to possess at the articular extremity, are present, and everything that is to be seen in the object found, can be accounted for when it is considered as the end of a long piece of the skeleton.

In conclusion the question may be considered, what extremity of what long skeletal piece of what animal is present. In general such a palaeozoological problem is very difficult to solve for an anatomist and an anthropologist. In our particular case, however, there are data that enable

us to make the diagnosis with great probability. From the very large dimensions of the extant part of the articular surface we have already drawn the conclusion in what precedes, that the fragment must have belonged to a very large animal. It is known that the prehistoric fauna of Trinil is rich in large mammals, as elephants, among which also Stegodon and Rhinoceros. It was, therefore, natural to suppose that the fossil might come from one of these animals. The comparison of the fossil remains with the parts of the skeletons of the recent representatives of the said mammals present in the Zoological Museum at Buitenzorg, has taught us that the fossil fragment presents so great a resemblance in form to the proximal, i.e. the extremity adjoining the scapular articulation of the Humerus of the recent elephant that we do not hesitate to identify the fragment found as the fossilized upper end of the humerus of an elephant.

W. A. MIJSBERG.

H. J. T. BIJLMER.

Weltevreden, Dec. 15, 1926.

Palæontology. — *The So-Called New Pithecanthropus Skull.* (With one Plate). By Prof. EUG. DUBOIS.

(Concise contents of a communication at the meeting of the Academy of December 18th, 1926).

On September 27th 1926 the Dutch newspapers contained the telegraphic news from Batavia that Dr. C. E. J. HEBERLEIN, district government physician at Surabaya, had found a complete skull of *Pithecanthropus erectus* at Trinil.

The news which, as appeared later, did not rest on information furnished by Dr. HEBERLEIN, was soon spread all over the world.

On October 2nd an amendment and completion followed to the effect that it was not correct to speak of a skull of *Pithecanthropus*, but only of the front part of a human skull cap, from the apex to the line of the orbital arches, which projected somewhat more than in an ordinary human skull. Of the bone substance only a thin layer was preserved on the spongy rock mass of volcanic origin, of which the object for the rest entirely consisted.

If this description was assumed to correspond to the real state of things, the spongy rock mass of volcanic origin could only be a piece of *volcanic tuff*, the rock formation of which the Kendeng-deposits, which contain the bones, consist. The spongy condition of this piece, which in this case could not have been found at Trinil itself, would then have to be ascribed to irregular calcareous impregnation and incrustation.

But the description was still too vague to enable us to form an opinion on the real nature of the object. Especially photographs were required, and measurements if the object actually contained parts of a cranium.

To the telegraphic request to send some preliminary measurements and photographs intended for communication to a meeting of the Academy, directed by me to Dr. HEBERLEIN on the same October 2nd, I received an answer on the 16th of this month, saying that he did not venture to give measurements, as the skull was mutilated and deformed, but that photographs would soon follow, and that he would place the skull at the disposal of the government.

The Minister of Instruction, Arts, and Sciences then expressed the wish that if the fossil should be brought to Holland, it would be added to the DUBOIS Collection, and that it would be placed at my disposal for study and also preparation if necessary.

For this reason I thought myself justified in wiring to Dr. HEBERLEIN on November 2nd to ask when the Government in Holland could expect the skull which it would place in my hands, and to remind Dr. H. of the promised photographs.

On the sixth of December the photographs so eagerly expected, came into my possession. There were two, one of the right side and one of the left side of the "skull", (fig. 1 and fig. 2), and at the same time I received a kind, extensive letter from Dr. HEBERLEIN. The photographs had been despatched from Surabaya on November 4th. ¹⁾

In his letter Dr. HEBERLEIN describes the object as a core of rock ("Steinkern") of a porous lava-like formation, covered by a lamella as thin paper, as remains of the flat cranial bones.

He, therefore, considers the "spongiuous" and "porous" character of this lava-like rock as an intrinsic feature, in other words: he takes it for cellular *lava*.

His description of the "skull cap" itself agrees in most points with that which had been made known on October 2nd (and which, as I learned later from the news agency Aneta, rested on information supplied by Dr. HEBERLEIN himself). Now he adds, however, that the occiput, the left temporal bone, and the greater part of the left parietal bone is absent. He also elucidates what gave him the impression that the skull has been slightly deformed by a force acting from the back forward, "witness the relatively great breadth of the skull and an area which has been isolated and displaced on the backside of the rock core". He states besides that the fragment of the skull makes on him more the impression of a decided homide form than of a Pithecanthropus.

His explanation of the formation of a bone lamella as thin as paper spread uniformly over the rock core, by sulphuric acid corrosion of the

¹⁾ In the "Algemeen Handelsblad", evening paper, of December 8th 1926 and also in the edition of the "Nieuwe Rotterdamsche Courant" of the same date there appeared the following announcement:

"Professor EUG. DUBOIS writes to us:

"Yesterday (Dec. 6) I received the long expected photographs of Dr. HEBERLEIN's "sensational "new Pithecanthropus-find", two views, from the kind discoverer. They appeared "to give a sure foundation for a diagnosis. At a cursory glance the object really bears "some resemblance to a Pithecanthropus skull. On careful examination it appears to be "something quite different. It is not even a skull, but beyond doubt the greater part of "the articular head of a humerus, the right humerus, of an elephant, probably *Stegodon*, "of which genus many bones have been found at Trinil.

"May the object itself so widely discussed, which was put at the disposal of the "Netherlands-Indian Government by Dr. HEBERLEIN more than a month ago, soon arrive "in Holland".

On the same day I wired to Prof. G. ELLIOT SMITH, University College in London and to Prof. H. T. OSBORN, President American Museum of Natural History in New-York: "Photographs of HEBERLEIN's Pithecanthropus just received show caput humeri *Stegodon*".

On December 8th the news agency Aneta also telegraphed my diagnosis to the Indies.

On December 18th there appeared in the "Algemeen Handelsblad" and in the "Telegraaf" an extensive extract of my communication to the meeting of the Academy, illustrated by photographs. The contents were in main lines the same as this article.

On December 31st Aneta wired from Batavia: "The investigation by Dr. MIJSBERG and Dr. BIJLMER of the fossil found at Trinil has confirmed that we have to do here with the upper part (articular head) of the humerus of a fossil elephant."

cranial bones from the outside, struck me as unfounded. Many bones from Trinil, inter alia the skull cap of the Pithecanthropus disinterred in 1891, actually show on the outside that they were corroded by sulphuric acid from the rock (ground water), but this never took place in a smooth surface, but very unevenly, so that numerous pits and even holes were made in the bone. Moreover lava, which on solidifying still possessed a temperature above 1000° C., and besides emitted chlorine and hydrochloric acid vapours, would have entirely destroyed a skull it filled.

When removing the wrapper from the precious photographs I was struck at the first glance by the, albeit faint, resemblance they show to the skull cap of the Pithecanthropus, but immediately after by the closer resemblance to a neandertal human skull, which Dr. HEBERLEIN will, no doubt, also have realised. But when I looked more closely at the photos, the first thing that struck my eye was that the porous and spongy "lava-like rock" was in reality *spongy bone substance* as it is found in short bones and in the articular extremities of the long bones. In some parts, especially on the "left side", the structure of spongiosa could be recognized in all particulars with the magnifying glass; the whole clearly manifests itself as such and shows a world-wide difference with lava. As the thin, solid, evidently unchanged bone layer smoothly rounds off this spongy substance on one side, we have undoubtedly to do with an articular extremity. In accordance with this the edges of the fracture on "the left side" are more like the edges of broken porcelain, and those of a petrified articular head, than like those of a fossil skull.

Accordingly: the fragment is part of an articular head, and this of a very large bone (else it could not have been taken for a human skull) — hence of a large animal, — of the largest animal species found at Trinil and in the Kendeng fauna, — of an elephant. At Trinil only bones of the elephant genus *Stegodon* were found, but many of these. The supposition, therefore, suggested itself that the object must be a fragment of an articular head of *Stegodon*, and only the superior extremity of a humerus could show the form of a cranium. The same day in the DUBOIS Collection at Leyden I found back HEBERLEIN's more manlike than Pithecanthropus-like cranial form in the right humerus of a *Stegodon* (Fig. 3). The spongiosa resembles to minute details that which is to be seen in fragments of the homonymous *Stegodon* bone in the said collection. This undamaged caput humeri likewise presents the absence of the "occiput", the breadth which is relatively great for a skull, and this apparent "mutilation" and "deformation" (on the reverse side of the figure; also part of the collum humeri had been taken as part of the skull). For so far as the object found is intact, the profiles correspond entirely. The only difference consists in this that HEBERLEIN's object is a fragment, and defective also in subordinate parts. It was for the greater part the latter circumstance that led Dr. HEBERLEIN to ascribe an apeman-manlike character to the object.

EUG. DUBOIS: THE SO-CALLED NEW PITHECANTHROPUS SKULL.



Fig. 1.



Fig. 2.



Fig. 3.

Also the way in which the object has come into Dr. HEBERLEIN's hands, according to his letter, is not without significance for its history, and moreover instructive.

He obtained the "cranial fragment" on the dry right bank of the Bengawan at Trinil, on the first visit that he, with a small party of friends, paid to this "classical locality" in August 1st 1926, from one of the native boys, who are daily pottering about there. The boy had found it on this bank, almost directly downward from the Pithecanthropus-memorial (a small pillar placed by me in 1893 as guide post to the finding place). At the very same spot also the latest excavations were made by the German expeditions. Dr. HEBERLEIN writes: "It was not found in situ in the layers, but loosened, between shingle". Dr. HEBERLEIN says shingle ("rolsteenen"), but genuine shingle or gravel does not occur there, and on a photograph shown me by Prof. BOLK, on which the spot at which the "skull" had been found, was accurately indicated, I clearly saw the pieces of broken rock of an excavation, so well known to me. For the rest according to my experience obtained during five years, nowhere bones are to be found there which have naturally come forth out of their layer. Obviously the find, which created such a sensation, is nothing but a fragment left behind as worthless by those most recent excavators. There are more such fragments to be found there. They are sought by native boys loitering around, and exchanged for some money with visitors of Trinil¹⁾.

This case then appears to be an analogue to that of SCHEUCHZER, who, taking for a human pelvis the cranium of a giant salamander, imagined he had found a "Homo diluvii testis". The stone plate with the skeleton quite laid bare by CUVIER is now a famous specimen of historic interest in the Teyler Museum. Such an error of one not quite expert in the matter is human, and the more excusable as it proceeds from meritorious motives.

EXPLANATION OF THE PLATE.

- Fig. 1. Reproduction, at $\frac{4}{5}$, of Dr. HEBERLEIN's photograph, which represents the right side of the supposed skull.
- Fig. 2. Idem of the left side.
- Fig. 3. Upper end of the right humerus of a Stegodon (DUBOIS collection), at $\frac{2}{5}$ of the natural size, back and outside view, reduced to such an extent that the linear dimensions of the articular head correspond with those of the (reduced) photograph of the right side of the supposed skull. The dotted line indicates the defective part of the outline of the fragment.

¹⁾ On this rests also AL. HRDLICKA's statement after his visit to Trinil: "Here again there was a great field for discovery; but priceless material which might bear upon the history of man was being lost for ever as it washed out of the deposits and was thrown away or destroyed by the natives". ("Nature", October 10, 1925, Vol. 116, p. 557).

Mathematics. — *Das Hauptproblem über die dimensionelle Struktur der Räume.* By KARL MENGER. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of November 27, 1926).

Der Raum R heisst im Punkte p mindestens k -dimensional, wenn alle hinlänglich kleinen Umgebungen von p mindestens $(k-1)$ -dimensionale Begrenzungen besitzen, d.h. wenn eine Umgebung $Z(p)$ existiert, so dass die Begrenzung jeder Umgebung $U(p) < Z(p)$ mindestens $(k-1)$ -dimensional ist. Ist R ein vorgelegter Raum, so bezeichnen wir mit R^k die Menge aller Punkte des Raumes, in denen derselbe mindestens k -dimensional ist. Das Hauptproblem über die dimensionelle Struktur der Räume ist die Frage nach der Dimension der Mengen R^k . Im Folgenden soll diese Frage¹⁾ beantwortet werden.

Theorem I. *In einem kompakten oder halbkompakten²⁾ metrischen Raum R ist für jedes k die Menge R^k aller Punkte, in denen der Raum mindestens k -dimensional ist, eine in jedem ihrer Punkte mindestens k -dimensionale Menge. Insbesondere ist also in einem kompakten oder halbkompakten n -dimensionalen Raum, die Menge $\overset{n}{R}$ aller Punkte, in denen der Raum n -dimensional ist, eine homogen n -dimensionale (d.h. in jedem ihrer Punkte n -dimensionale) Menge. Die Menge $\overset{n}{R}$ ist in einem kompakten n -dimensionalen Raum R identisch mit der Summe aller homogen n -dimensionalen Teilmengen von R ³⁾.*

Beim Beweise wollen wir stets die Begrenzung der offenen Menge U mit $B(U)$ bezeichnen und wir verwenden die folgende *Additionsformel* für Umgebungsbegrenzungen:

$$B(U_1 + U_2) = [B(U_1) - U_2 \cdot B(U_1)] \\ + [B(U_2) - U_1 \cdot B(U_2)].$$

Diese Formel lässt sich offenbar für eine beliebige endliche Anzahl

¹⁾ Ueber meine Behandlung dieser Probleme im Jahre 1922 vgl. die demnächst in diesen Proceedings erscheinende zweite Mitteilung über die Entstehung meiner Arbeiten zur Dimensions- und Kurventheorie. Im Bericht über die Dimensionstheorie (Jahresber. d. deutschen Math. Ver. 35, S. 113) musste ich die Frage noch als ungelöst bezeichnen (Vgl. insbes. a. a. O. S. 138). URYSOHN (Fundam. Math. 8, S. 285) weist auf die besondere Schwierigkeit und Wichtigkeit dieses Problems hin.

²⁾ Ich nenne einen Raum *halbkompakt* (Vgl. Monatshefte f. Math. u. Phys. 34, 1924, S. 144) wenn er Summe von abzählbar vielen kompakten Räumen ist.

³⁾ Ueber die Begriffsbildungen der homogen n -dimensionalen Menge und der Summe alle homogen n -dimensionalen Teile einer Menge vgl. bereits meine (Proc. Ac. Amsterdam 29, Mai) abgedruckte Note vom Februar 1922.

von Summanden verallgemeinern, versagt aber für unendlich viele Summanden. Ich stütze mich nun beim Beweis des obigen Theorems auf ein Lemma, auf das ich in verschiedenen Formen schon zu wiederholten Malen bei dimensions- und kurventheoretischen Strukturuntersuchungen geführt worden bin¹⁾ und das trotz seiner Einfachheit immer wieder die Herleitung wichtiger Konsequenzen gestattet. Die Bedeutung des Lemmas beruht darauf, dass es für gewisse unendliche Umgebungssummen eine Verallgemeinerung der Additionsformel liefert. Ueberdeckt man eine Menge M irgendwie mit abzählbar vielen Umgebungen, so besteht zwischen der Begrenzung der Summe aller dieser Umgebungen und den Begrenzungen der Summanden im allgemeinen keinerlei additive Beziehung. Ueberdeckt man aber die Menge M mit einer Nullfolge von Umgebungen, deren jede mit M mindestens einen Punkt gemein hat, (d.h. entweder mit endlich vielen Umgebungen oder mit abzählbar vielen zu M nicht fremden Umgebungen, deren Durchmesser gegen Null konvergieren), dann lehrt das Fundamentallemma, dass die Begrenzung der Summe aller dieser Umgebungen ausser Punkten von den Begrenzungen der Summanden höchstens Punkte einer im Raum abgeschlossenen Teilmenge von $\overline{M-M}$ enthalten kann.²⁾

*Fundamentallemma.*³⁾ Ist A eine Menge eines separablen metrischen Raumes und $\{U_n\}$ ($n = 1, 2, \dots$) eine A überdeckende Folge von offenen Mengen mit gegen Null konvergierenden Durchmessern, deren jede mit A mindestens einen Punkt gemein hat, dann gilt

$$B\left(\sum_{n=1}^{\infty} U_n\right) \subset \sum_{n=1}^{\infty} B(U_n) + (\overline{A} - \overline{A}) \cdot \sum_{n=1}^{\infty} U_n.$$

Sei nun zum Beweis des Theorems I p ein Punkt von R^k , ein Punkt also, in dem der halbkompakte Raum R mindestens k -dimensional ist. Es existiert eine Umgebung $Z(p)$, so dass die Begrenzung von jeder Umgebung $U(p) \subset Z(p)$ mindestens $(k-1)$ -dimensional ist. Um zu zeigen, dass R^k in p mindestens k -dimensional ist, wollen wir gegen diese Definition von $Z(p)$ einen Widerspruch herleiten aus der Annahme, dass eine Umgebung $V(p) \ll Z(p)$ ⁴⁾ existiert, deren Begrenzung mit R^k einen höchstens $(k-2)$ -dimensionalen Durchschnitt hat. Bezeichnet B die Begrenzung von $V(p)$, so existiert nach einem Satz von TUMARKIN⁵⁾ eine

¹⁾ Vgl. Ueber die Dimension von Punktmengen, Monatshefte f. Math. u. Phys., I. Bd. 33, S. 150 f.; II. Bd. 34, S. 146 f.. Grundzüge einer Theorie der Kurven, Mathem. Annalen 95, S. 281 f.

²⁾ Mit \overline{M} bezeichnen wir in üblicher Weise die abgeschlossene Hülle von M .

³⁾ Der Beweis dieses Lemmas erfolgt durch dieselben Schlüsse, die ich an den auf dieser Seite sub ¹⁾ erwähnten Stellen durchgeführt habe.

⁴⁾ Ich schreibe $A \ll B$ gleichbedeutend mit $\overline{A} \subset B$.

⁵⁾ Vgl. Proc. Ac. Amsterdam, 28, S. 995. Man beweist diesen für $n = -1$ trivialen Satz am einfachsten nach HUREWICZ so: Angenommen, der Satz sei bewiesen für $n-1$ und es sei M eine n -dimensionale Menge. Sei dann $\{U_i\}$ ($i = 1, 2, \dots$) eine Folge von offenen Mengen, deren Begrenzungen B_i mit M höchstens $(n-1)$ dimensionale Durch-

höchstens $(k-2)$ -dimensionale G -Menge $P < B$, welche die höchstens $(k-2)$ -dimensionale Menge $B \cdot R^k$ enthält. In jedem Punkt von $B - B \cdot R^k$, insbesondere also in jedem Punkt von $B - P$, ist R höchstens $(k-2)$ -dimensional. Die Menge $B - P$ ist also ein F_σ des halbkompakten Raumes R , von dem jeder Punkt enthalten ist in beliebig kleinen Umgebungen mit höchstens $(k-2)$ -dimensionalen Begrenzungen. Nun gilt, wie ich an anderer Stelle¹⁾ bewiesen habe, folgende:

Verallgemeinerung des Borelschen Theorems: Ist die Menge A ein F_σ eines halbkompakten Raumes und ist S ein System von offenen Mengen, so dass jeder Punkt von A in beliebig kleinen offenen Mengen des Systems S enthalten ist, dann ist A für jedes $\varepsilon > 0$ überdeckbar mit abzählbar vielen offenen Mengen aus S , deren Durchmesser $< \varepsilon$ sind und gegen Null konvergieren.

Betrachten wir die F_σ -Menge $B - P$, ferner das System S aller offenen Mengen $< Z(p)$ mit höchstens $(k-2)$ -dimensionalen Begrenzungen. (Wegen $V(p) \subset Z(p)$ ist jeder Punkt von $B - P$ in einer solchen Umgebung enthalten). Die Verallgemeinerung des Borelschen Theorems lehrt, dass eine $B - P$ überdeckende Folge $\{U_n\}$ ($n = 1, 2, \dots$) von offenen Mengen mit höchstens $(k-2)$ -dimensionalen Begrenzungen B_n existiert, die sämtlich $< Z(p)$ sind, deren Durchmesser gegen Null konvergieren und deren jede mit $B - P$ mindestens einen Punkt gemein hat. Setzen wir

$U = \sum_{n=1}^{\infty} U_n$, dann ergibt das Fundamentallemma

$$(+)$$

$$B(U) < \sum_{n=1}^{\infty} B(U_n) + F,$$

wobei

$$F = \overline{B - P} - \overline{B - P \cdot \sum_{n=1}^{\infty} U_n}$$

gesetzt ist.

Mit Rücksicht auf diese Definition der abgeschlossenen Menge F gilt

$$F < B - B \cdot \sum_{n=1}^{\infty} U_n,$$

also, wegen $B - P < \sum_{n=1}^{\infty} U_n$, die Beziehung

$$F < P.$$

Mithin ist die abgeschlossene Menge F höchstens $(k-2)$ -dimensional. $B(U)$ ist wegen (+), als Summe der abzählbar vielen abgeschlossenen

schnitte haben, und so, dass sich auf jeden Punkt des Raumes eine Teilfolge dieser Folge zusammenzieht. Nach Annahme gibt es zu jedem i eine höchstens $(n-1)$ -dimensionale G_σ -Menge $G_i > M \cdot B_i$. Wir nennen G das Komplement der F_σ -Menge $\sum_{i=1}^{\infty} (B_i - G_i)$.

Diese G_σ -Menge enthält offenbar M und ist höchstens n -dimensional, denn für jedes i ist der Durchschnitt von B_i mit G Teil von G_i , also höchstens $(n-1)$ -dimensional.

¹⁾ Vgl. Einige Ueberdeckungssätze der Punktmengenlehre, Wiener Ber. 133, S. 423.

höchstens $(k-2)$ -dimensionalen Mengen $B(U_n)$ und F , höchstens $(k-2)$ -dimensional. Setzen wir nun

$$W(p) = V(p) + U,$$

und wenden wir auf diese Summe die Additionsformel für Begrenzungen an, so haben wir

$$B(W(p)) < B(U) + (B - U \cdot B)$$

Die abgeschlossene Menge $B - U \cdot B$ ist, wegen $B - P < U$ Teil von P , mithin höchstens $(k-2)$ -dimensional. $W(p)$ ist also eine Umgebung von $p < Z(p)$, deren Begrenzung als Summe zweier abgeschlossener höchstens $(k-2)$ -dimensionaler Mengen höchstens $(k-2)$ -dimensional ist. Damit ist der zum Beweis von Theorem I gewünschte Widerspruch hergestellt.

Theorem II. *Ist R ein separabler metrischer Raum und R^k die Menge aller Punkte, in denen R mindestens k -dimensional ist, dann ist jede nicht-leere in R^k offene Menge mindestens $(k-1)$ -dimensional. Insbesondere ist also in einem separablen n -dimensionalen Raum R die Menge $\overset{n}{R}$ entweder n -dimensional oder $(n-1)$ -dimensional. Beide Fälle können sich ereignen.*

Wir haben zum Beweis einen Widerspruch herzuleiten aus der Annahme, U sei eine in R offene Menge, die mit R^k einen nicht-leeren höchstens $(k-2)$ -dimensionalen Durchschnitt hat. Sei p ein (auf Grund dieser Annahme existierender) Punkt von $U \cdot R^k$. Da p Punkt von R^k ist, ist der Raum im p mindestens k -dimensional. Die Annahme, $U \cdot R^k$ sei höchstens $(k-2)$ -dimensional, ist demnach ad absurdum geführt, wenn mit ihrer Hilfe gezeigt werden kann, dass beliebig kleine Umgebungen von p mit höchstens $(k-2)$ -dimensionalen Begrenzungen existieren. Wir geben, um dies tatsächlich aus der Annahme herzuleiten, eine Umgebung $V(p) \ll U$ vor und eine Umgebung $Z(p) \ll V(p)$. Zu jedem Punkt von $U - U \cdot R^k$ existieren beliebig kleine Umgebungen mit höchstens $(k-2)$ -dimensionalen Begrenzungen. Wir ordnen jedem Punkt q der Menge $U - U \cdot R^k$ eine Umgebung $U(q)$ mit höchstens $(k-2)$ -dimensionaler Begrenzung zu, mit der einzigen Bedingung (β), dass jede Umgebung $U(q)$, welche mit $U - V(p)$ Punkte gemein hat, zu $Z(p)$ fremd ist. Abzählbar viele derartige Umgebungen, etwa die Umgebungen $\{U_n\}$ ($n = 1, 2, \dots$), überdecken U . Wir setzen

$$W_1 = U_1 \quad . \quad W_n = U_n - U_n \cdot \sum_{i=1}^{n-1} \overline{U_i} .$$

Die offenen Mengen W_n ($n = 1, 2, \dots$) sind paarweise fremd, besitzen höchstens $(k-2)$ -dimensionale Begrenzungen, und überdecken nebst ihren Begrenzungen die Menge $U - U \cdot R^k$. Ist B_n die Begrenzung von W_n dann ist

$$U - U \cdot R^k < \sum_{n=1}^{\infty} W_n + \sum_{n=1}^{\infty} B_n, \quad \text{also} \quad U < \sum_{n=1}^{\infty} W_n + \sum_{n=1}^{\infty} B_n + R^k. \quad (+)$$

Wir bilden nun die Teilfolge aller jenen Mengen der Folge $\{W_n\}$,

welche mit $U - V(p)$ Punkte gemein haben. Es sei $\{W_n\}$ ($n = 1, 2, \dots$) diese Teilfolge. Wir setzen

$$W(p) = U - U \cdot \sum_{n=1}^{\infty} \overline{W_n}.$$

$W(p)$ ist eine offene Menge. Wegen der Bedingung (β) gilt $W(p) > Z(p)$, also ist $W(p)$ eine Umgebung von p . Ferner gilt $W(p) < V(p)$, denn es ist $W(p) < U$ und jeder Punkt von $U - V(p)$ liegt in einer der Umgebungen W_n oder in mindestens einer der Begrenzungen B_n . Wir haben also, um unser Ziel zu erreichen, nur noch zu zeigen, dass die Begrenzung B von $W(p)$ höchstens $(k-2)$ -dimensional ist. Nun ist aber sicherlich kein Punkt von B in irgend einer der offenen Mengen W_n

enthalten. Es gilt mithin wegen (+) $B < \sum_{n=1}^{\infty} B_n + R^k$. Auf der rechten Seite dieser Ungleichung steht die Summe von abzählbar vielen abgeschlossenen höchstens $(k-2)$ -dimensionalen Mengen. Denn jede der Mengen B_n ist laut Voraussetzung höchstens $(k-2)$ -dimensional und, als Begrenzung eine offenen Menge, abgeschlossen. Die Menge R^k aber ist bekanntlich ein F_σ , d.h. Summe von abzählbar vielen abgeschlossenen Mengen, und sie ist laut Annahme höchstens $(k-2)$ -dimensional. Also ist die Menge $\sum_{n=1}^{\infty} B_n + R^k$ Summe von abzählbar vielen abgeschlossenen höchstens $(k-2)$ -dimensionalen Mengen des separablen Raumes R und daher nach der HUREWICZ-TUMARKIN'schen Verallgemeinerung des Summensatzes¹⁾ höchstens $(k-2)$ -dimensional. Die Begrenzung B ist als Teil dieser Menge höchstens $(k-2)$ -dimensional. Damit ist der Beweis, dass $U \cdot R^k$ nicht $(k-2)$ -dimensional sein kann, abgeschlossen.

Insbesondere ist also in einem n -dimensionalen separablen Raum R die Menge $\overset{n}{R}$ entweder n -dimensional oder $(n-1)$ -dimensional. SIERPIŃSKI²⁾ hat tatsächlich eine ausserordentlich merkwürdige eindimensionale Menge konstruiert, welche bloss in den Punkten einer abzählbaren, also null-dimensionalen Menge eindimensional ist.

Offen bleiben folgende Fragen:

1) Gibt es für jedes n einen separablen n -dimensionalen Raum, mit $(n-1)$ -dimensionaler Menge R^n ?³⁾

1) Vgl. HUREWICZ, Mathem. Annalen 96, TUMARKIN, Proc. Ac. Amsterdam, 28.

2) Vgl. Fundam. Math. 2, S. 81.

3) Es wäre wünschenswert, dass wenigstens für $n=2$ eine solche Menge durch Zurückgehen auf SIERPIŃSKI's Konstruktionsmethode angegeben würde. — Bezeichnet man in üblicher Weise als Produkt der Räume A und B die Menge aller Paare von Elementen, deren erstes aus A und deren zweites aus B entnommen ist, und für welche die bekannten Limesfestsetzungen getroffen sind, dann entsteht auch folgende Frage: Welche Dimension besitzt das Produkt M der SIERPIŃSKISchen Menge mit einer Strecke und welche Dimension besitzt, falls M zweidimensional ist, die Menge $\overset{2}{M}$? Welche Dimension besitzt ferner das Quadrat der SIERPIŃSKISchen Menge und die Menge aller Punkte höchster Dimension

2) Ist die Menge R^k , von der wir wissen, dass jeder ihrer nicht-leeren offenen Teile mindestens $(k-1)$ -dimensional ist, in jedem ihrer Punkte mindestens $(k-1)$ -dimensional, oder kann sie in manchen ihrer Punkte von geringerer Dimension sein?

Wir ziehen aus dem Theorem II noch eine Folgerung.

In einem separablen Raum R ist für $k \geq 2$ die Menge R^k , falls sie nicht leer ist, kondensiert, d.h. jeder ihrer relativ offenen Teile enthält un abzählbar viele ihrer Punkte.

Denn für $k \geq 2$ ist jede nicht-leere in R^k offene Menge mindestens eindimensional, also un abzählbar.

Theorem III lehrt, dass die Menge $\overset{n}{R}$ aller Punkte, in denen der separable Raum R n -dimensional ist, zwar mindestens $(n-1)$ -dimensional, aber nicht notwendig n -dimensional ist. Theorem I zufolge ist für das letzere Kompaktizität oder Halbkompaktizität des Raumes R hinreichend; aus einfachen Beispielen geht aber hervor, dass für die n -Dimensionalität der Menge $\overset{n}{R}$ Halbkompaktizität von R keineswegs notwendig ist. Es entsteht daher die wichtige Frage nach allgemeinen Bedingungen, welche die n -Dimensionalität von $\overset{n}{R}$ gewährleisten. Solche Bedingungen ergeben sich durch Berücksichtigung verschiedener Verallgemeinerungen des Borelschen Theorems.

Theorem III. Ist in einem separablen n -dimensionalen Raum R die Menge $R - \overset{n}{R}$ aller Punkte, in denen der Raum weniger als n -dimensional ist, für jedes $\varepsilon > 0$ überdeckbar mit mit abzählbar vielen offenen Mengen, deren Begrenzungen höchstens $(n-2)$ -dimensional sind und deren Durchmesser $< \varepsilon$ sind und gegen Null konvergieren, dann ist die Menge $\overset{n}{R}$ homogen n -dimensional. Insbesondere ist dies der Fall, wenn die Menge $R - \overset{n}{R}$, in sich betrachtet, ein halbkompakter Raum ist, oder wenn zu jedem Punkt von $R - \overset{n}{R}$ eine auf ihn sich zusammenziehende Folge von nicht allzu exzentrisch werdenden Umgebungen¹⁾ mit höchstens $(n-2)$ -dimensionalen Begrenzungen existiert.

Wir setzen voraus, dass die Menge $R - \overset{n}{R}$ die in Theorem III ange-

dieses Quadrates? Offenbar ist das Produkt eines n -dimensionalen und eines m -dimensionalen Raumes stets höchstens $(m+n)$ -dimensional. Die Frage ist aber noch offen, ob die Dimension eines solchen Produktes nicht unter Umständen weniger als $(m+n)$ -dimensional sein kann. Sollte dies möglich sein, so tritt der Fall vermutlich für das Produkt von in gewissem Sinn schwach n -dimensionalen und schwach m -dimensionalen Räumen ein, also vielleicht für Räume, die bloss in den Punkten einer $(n-1)$ -dimensionalen Menge n -dimensional sind.

¹⁾ Vgl. über diese Begriffsbildung meine erwähnte Arbeit, Wiener Ber. 133, S. 434 f. und Bericht über die Dimensionstheorie, S. 135 f.

gebene Ueberdeckbarkeitseigenschaft besitzt, und wollen einen Widerspruch herleiten aus der Annahme, die Menge $\overset{n}{R}$ sei in ihrem Punkt p höchstens $(n-1)$ -dimensional. So wie bei Beweis von Theorem I erhalten wir einen solchen Widerspruch, indem wir eine beliebige Umgebung $U(p)$ vorgeben und aus der fraglichen Annahme die Existenz einer Umgebung $W(p) < U(p)$ mit höchstens $(n-2)$ -dimensionaler Begrenzung herleiten. Sicher existiert der Annahme zufolge eine Umgebung $V(p) \subset U(p)$, deren Begrenzung B mit der Menge $\overset{n}{R}$ einen höchstens $(n-2)$ -dimensionalen Durchschnitt hat. Die Menge $B - B \cdot \overset{n}{R}$ kann aber (der Ueberdeckbarkeitsvoraussetzung zufolge) mit einer Nullfolge von offenen Mengen $< U(p)$ überdeckt werden, deren Begrenzungen höchstens $(n-2)$ -dimensional sind. Die Summe V' dieser Nullfolge ist nach dem Fundamentallemma eine offene Menge, deren Begrenzung höchstens $(n-2)$ -dimensional ist. Die Summe $W(p)$ von $V(p)$ und V' ist eine Umgebung $< U(p)$ von p , deren Begrenzung nach der Additionsformel höchstens $(n-2)$ -dimensional ist, womit der gesuchte Widerspruch hergestellt ist. — Dass die in der ersten Hälfte des Theorems III angegebene Ueberdeckbarkeitsbedingung in dem in der zweiten Hälfte des Theorems angeführten Fall tatsächlich erfüllt ist, habe ich an anderer Stelle ¹⁾ bewiesen.

Wir fassen die Resultate dieses Aufsatzes noch für die Teilmengen Euklidischer Räume zusammen. Die F_c -Mengen eines Euklidischen Raumes sind, in sich betrachtet, halbkompakte Räume. Jede beliebige Teilmenge eines Euklidischen Raumes ist, in sich betrachtet, ein separabler Raum. Wir können also sagen:

Für jede n -dimensionale Menge M eines Euklidischen Raumes ist die Menge $\overset{n}{M}$ aller Punkte von M , in denen M n -dimensional ist, entweder n -dimensional oder $(n-1)$ -dimensional. Für jede n -dimensionale abgeschlossene oder F_c -Menge M eines Euklidischen Raumes ist die Menge $\overset{n}{M}$ homogen n -dimensional, d. h. in allen ihren Punkten n -dimensional. Desgleichen ist auch für jede Menge M , die kein F_c ist, die Menge $\overset{n}{M}$ homogen n -dimensional, falls sich auf jeden Punkt von $M - \overset{n}{M}$ nicht allzu exzentrisch werdende Umgebungen, deren Begrenzungen mit M höchstens $(n-2)$ -dimensionale Durchschnitte haben, zusammenziehen.

¹⁾ Wiener Ber. 133, S. 434.

Physics. — *Investigations on the free energy of a mixture of ions.*
 By H. A. KRAMERS. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meeting of December 18, 1926).

I. Statement of the problem. Main results.

In 1923 DEBYE and HUECKEL ¹⁾ published a simple and elegant method of calculating the influence of the electrostatic forces acting between the ions in the solution of an electrolyte. They arrived at the result that the free energy ζ of a solution is decreased on account of the electrostatic forces by an amount which, for sufficiently small concentrations, can be represented by:

$$\Delta \zeta = - \sum_i \frac{N_i z_i^2 \epsilon^2 \kappa}{3 D} \frac{1}{1 + \kappa a_i} \dots \dots \dots (1)$$

In this formula N_1, \dots, N_s denote the numbers in which ions of the kind $1, \dots, i, \dots, s$ are present in the solution, while $z_i \epsilon$ is the charge ($\epsilon =$ elementary quantum of electricity) and a_i an "effective" radius belonging to an ion of the i^{th} kind. D is the dielectric constant of the solvent and finally, κ is a quantity of the dimensions of a reciprocal length, defined by:

$$\kappa^2 = \frac{4 \pi \epsilon^2}{D k T} \sum_i n_i z_i^2 \dots \dots \dots (2)$$

In this formula T is the absolute temperature, k BOLTZMANN's constant and n_i the number of ions of the kind i per cm^3 .

Imagining the ions as spheres of radius a_i plunged in a continuous medium of dielectric constant D , we can, by means of the methods of statistical mechanics, establish an exact expression for $\Delta \zeta$ in the form of a definite integral in which the integration has to be extended over the coordinates of every ion in the solution ²⁾:

$$V^N e^{-\frac{\Delta \zeta}{kT}} = \int \dots \int e^{-\frac{E}{kT}} dx_1 dy_1 dz_1 \dots dz_N \dots \dots \dots (3)$$

Here E denotes the electrostatical energy (if D depends on temperature, the electrostatical *free* energy) of a given configuration of the ions in the solution:

$$E = \frac{1}{D} \sum \sum \frac{\epsilon_k \epsilon_l}{r_{kl}} \dots \dots \dots (4)$$

¹⁾ Phys. Zs. **24**, 185, 1923.

²⁾ Comp. W. GIBBS, Principles in statistical mechanics, p. 33, form. 92, and Chapter XIV.

ε_k and ε_l denote the charges of the k^{th} and l^{th} ion, while r_{kl} is the distance between these ions. The integration variables x_1, \dots, z_N in (3) are the cartesian coordinates of the ions, and the integration has to be extended over all possible configurations in the total volume V . The fact that E is a homogeneous function of the coordinates of the degree -1 involves (at any rate if the influence of radii of the ions may be neglected) that the value of $\Delta\zeta/T$ for a mol of the solved substance depends on the temperature T and on the concentration $n = \sum n_i$ in the form of a function of n/T^3 .¹⁾

The explicit calculation of $\Delta\zeta$ by means of (3) meets with mathematical difficulties, mainly due to the fact that the ordinary method, development in decreasing powers of T , does not give a series the first terms of which suffice. In the case where the ionic radii are very small, the terms of that series become very large, although a simple consideration shows²⁾ that for sufficiently small concentrations $\Delta\zeta$ converges asymptotically to a function which does not depend on these radii. This property is clearly expressed in the formula (1) of DEBYE and HUECKEL. If we take the radii exactly equal to zero, however, the integral (3) diverges; the model involves in that case that the positive and negative ions will, in the state of equilibrium, be groupwise associated.

Starting from formulae equivalent to (3), MILNER, in 1912, tried by long numerical calculations to find the values of $\Delta\zeta$ for very small ionic radii. He introduced several approximations which were difficult to control, and as regards their practical applicability his results fall short of DEBYE's and HUECKEL's³⁾ elegant formulae.

DEBYE and HUECKEL do not start from the integral (3) but they apply an ingenuous artifice. They consider the statistical distribution of the ions surrounding a definite ion I and establish a partial differential equation which the mean potential at a distance r from I must obey. This method, however, could only give a first approximation. One might especially suspect that for not too small concentrations the distribution in the nearest neighbourhood of I is not accounted for with sufficient exactness.

In the next chapter we shall, starting from the methods of statistical mechanics, try to determine the influence of the electrostatic forces as far as it does not depend on the ionic radii, i.e. for small radii. For small concentrations we re-find the expression of DEBYE and HUECKEL. For larger concentrations, however, we find peculiar deviations. Thus our formulae seem to show that a *state of statistical distribution of the ions which is independent of the atomic radii* and which is established

¹⁾ O. KLEIN, Medd. fr. K. Vet. Nobelinstitut, 5, 1919. See also p. 148 of this article.

²⁾ S. R. MILNER, Phil. Mag. 23, 551, 1912, and especially N. BJERRUM, Copenhagen Academy, Mat. fys. Medd. 7, 9.

³⁾ Compare what MILNER himself states about it in NONHEBEL and HARTLEY, Phil. Mag. 7 2, 586, 1926.

if from the random distribution at very small concentrations we pass to larger concentrations *can only be realised below a certain maximum concentration*. For an electrolyte of the KCl type in water at 15° C. this concentration corresponds to about 0,03 mol. KCl per litre. The consequences of this seem to be 1. that, conforming with DEBYE's and HUECKEL's results, a description of the phenomena in which the radii are neglected, is only possible at extremely small concentrations, and 2. that is useless, for higher concentrations, to apply DEBYE's and HUECKEL's¹⁾ formulae as a correction factor.

II. Calculation of the free energy.

As our starting point we choose a canonical ensemble of a more general type than that on which (3) was based. A sample of the latter was specified by definite positions of the centres of the N ions in the volume V , each ion carrying its corresponding charge. Our generalized ensemble, however, will be such that, for a given configuration of the centres of the N ions, the charge on the k^{th} ion may still assume any value of which the charge of an arbitrary ion in the solution is capable. The chance that an arbitrary ion belongs to the kind i is expressed by the fraction N_i/N . We will therefore allow N^N samples of our ensemble to correspond with a given configuration of the ions in the volume under consideration, in such a way that for given charges of the ions $1, 2, \dots, k-1, k+1, \dots, N$, there will be N_i samples in which the k^{th} ion carries the charge $z^i \varepsilon$, i taking the values $1, \dots, s$. Our ensemble forms thus a part of the so-called grand-ensemble introduced by GIBBS²⁾, namely that part for which the number of ions in V just equals N .

If now we proceed to calculate mean values in our generalised ensemble we must remember that the integrals over all possible configurations of the N ions will diverge if the radii are taken infinitely small. This difficulty may be met in two ways. We may in the first place assume the radii to be finite (method *A*). In that case the extension over which the variables x_1, \dots, z_N are to be integrated will depend on the charges attributed to the ions. Formally all configurations of the centres of the N ions are permissible if we add to the energy expression (4) a function which becomes infinite if the centres of two ions with charges ε_i and ε_j are nearer to each other than $a_i + a_j$. Only if all the radii a_i were equal it would be permissible for all samples of the ensemble, to integrate over the same extension in the $x_1 \dots z_N$ space.

In the second place (method *B*) we might apply a less physical treatment, not comparing the ions to solid spheres, but modifying the

¹⁾ For the literature compare F. HUECKEL, Zur Theorie der Elektrolyte, Ergebnisse der exakten Naturwissenschaften III, 1925.

²⁾ Principles in statistical mechanics, Ch. XV. See especially p. 190. The quantities μ_i of GIBBS are in our terminology equal to $kT \log N_i/N$.

expression (4) in another way. This may be done so that the integrals representing the mean values over all configurations no longer diverge but that at the same time the property of the energy to be a homogeneous function of the coordinates of the degree -1 is preserved. For this purpose we may choose for the energy function:

$$E' = \frac{1}{D} \sum \sum \varepsilon_k \varepsilon_l g_{kl} \dots \dots \dots (5)$$

where g_{kl} is equal to

$$g_{kl} = \frac{1}{r_{kl}} (1 - e^{-\lambda h_{kl}}), \quad h^2 = \frac{r_{kl}^2}{4R_{kl}^2} = \frac{(x_k - x_l)^2 + (y_k - y_l)^2 + (z_k - z_l)^2}{(x_k + x_l)^2 + (y_k + y_l)^2 + (z_k + z_l)^2} \quad (6)$$

If the origin of the system of coordinates is taken somewhere outside the vessel so that R_{kl} is always of the order of magnitude of the linear dimensions of the vessel ($d \approx V^{1/3}$), λ can always be taken so large that the energy expression (5) practically coincides for all configurations with (4). Deviations will only occur if $\lambda \frac{r_{kl}}{d}$ is of the order 1 or smaller. As the mean distance of two neighbouring ions is of the order $dn^{-1/3}$ we need only take λ of the order $n^{1/3}$ or $n^{2/3}$.

We will now replace the expression (3) for $\Delta \zeta$ by the mean value over our generalised ensemble:

$$V^N e^{-\frac{\Delta \zeta}{kT}} = \int \dots \int N^{-N} \sum e^{-\frac{E'}{kT}} dx_1 \dots dz_N \dots \dots (7)$$

Here \sum denotes the summation over the N^N samples corresponding to a given configuration. The considerations of KLEIN cited above will now hold exactly if the method *B* is assumed, whereas they will hold approximately if *A* is applied. In fact if every coordinate is multiplied by an arbitrary factor f , and if, at the same time, the temperature is chosen f times smaller, the value of the exponent in (7) does not change. From this it follows that, by the introduction of the factor f , the value of integral has been multiplied by f^{3N} since the volume of the vessel has become f^3 times as large. If we remember, moreover, that for N sufficiently large and for a given concentration n (= number of ions in unit volume) the integral in (7) will depend on N only in the form of an N^{th} power (this holds exactly with method *A*, but approximately with method *B*), we see that (7) can be written in the form:

$$e^{-\frac{\Delta \zeta}{kT}} = [\varphi (n^{1/3}/T)]^N \dots \dots \dots (8)$$

The deduction shows that this result under all circumstances stands as an approximation, which can be applied only as far as a statistical

1) The occurrence in (5) of R_{kl} , the distance of the geometrical centre of two ions from the origin, means that this centre is attracted towards, or repelled from, the origin. Only for a pair of ions which lie very near to each other ($r_{kl} \approx R_{kl}/\lambda$) and which for small concentrations are very rare, will these forces have a perceptible influence.

distribution of the ions exists, which does not depend on the ionic radii.

We will now transform the integrand in (7).

We consider the expression (5) for E' as a quadratic function of the N variables $\varepsilon_1, \varepsilon_2 \dots \varepsilon_N$, and we imagine this function to be given the canonical form by a so-called "transformation of principal axes":

$$E' = \frac{1}{D} \sum \sum \varepsilon_k \varepsilon_l g_{kl} = 1/2 \sum_1^N b_m y_m^2 \dots \dots \dots (9)$$

The quantities b_m are the roots of the following equation of the N^{th} degree:

$$\Delta(b) = 0, \dots \dots \dots (10)$$

Δ being a determinant $|\Delta_{kl}|$ of the order N , the diagonal terms of which are equal to b whereas the other terms are given by $\Delta_{kl} = -g_{kl}/D$. Let the relation between the ε 's and the y 's be represented by:

$$y_m = \sum_k \gamma_{mk} \varepsilon_k \dots \dots \dots (11)$$

the coefficients γ_{mk} obeying the well known conditions of orthogonality:

$$\sum_k \gamma_{mk}^2 = 1, \quad \sum_k \gamma_{mk} \gamma_{m'k} = 0 \dots \dots \dots (12)$$

The b 's as well as the γ 's depend only on the ionic configuration. The summation in the integrand in (7) has now become a summation over the different values which the y 's can assume as a consequence of the different discrete ε_k -values in our ensemble. This summation, however, can be written in the form of an integral if we may assume that all quantities γ_{mk} are small compared with unity, an assumption which will be fulfilled, independent of the concentration, for the large majority of possible configurations. In that case the quantities y can practically assume all values between $-\infty$ and $+\infty$, and the probability that y lies between y_m and $y_m + dy_m$ will be given by a GAUSS error function. The mean value of y_m is equal to zero since the mean value of ε_k is equal to zero, due to the condition of neutrality $\sum N_i z_i = 0$. For the mean value of y_m^2 we find, using (12),

$$\overline{y_m^2} = \sum \gamma_{mk}^2 \overline{\varepsilon_k^2} = \varepsilon^2 (\sum \gamma_{mk}^2) \cdot \left(\sum_i \frac{N_i}{N} z_i^2 \right) = \varepsilon^2 \sum_i \frac{N_i}{N} z_i^2 = \eta$$

where η is an abbreviation. From (12) follows furthermore that the different y 's are statistically independent.

$$\overline{y_m y_{m'}} = \sum \gamma_{mk} \gamma_{m'k} \cdot \overline{\varepsilon_k^2} = \eta \sum \gamma_{mk} \gamma_{m'k} = 0$$

The chance that y_m lies between y_m and $y_m + dy_m$ ($m = 1, 2 \dots N$) therefore equal to

$$(2\pi\eta)^{-N/2} \exp \left[-\frac{\sum y_m^2}{2\eta} \right] dy_1 \dots dy_N \dots \dots \dots (13)$$

and the integrand in (7) can be calculated as follows:

$$N^{-N} \sum e^{-E'/kT} = (2\pi\eta)^{-N/2} \int \dots \int dy_1 \dots dy_N \cdot \exp \left[-1/2 \sum \left(\frac{1}{\eta} + \frac{b_m}{kT} \right) y_m^2 \right] = \\ = \left[\prod_1^N \left(1 + \frac{\eta}{kT} b_m \right) \right]^{-1/2} = \left[\left(\frac{-\eta}{kT} \right)^N \Delta \left(-\frac{kT}{\eta} \right) \right]^{-1/2} = A^{-1/2} \dots (14)$$

Here Δ denotes, just as in (10), the N^{th} degree polynomial the roots of which are the b 's and we see that A is an abbreviation for

$$A = \begin{vmatrix} 1 & ag_{12} & ag_{13} & \dots & ag_{1N} \\ ag_{21} & 1 & ag_{23} & \dots & ag_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ ag_{N1} & \dots & \dots & \dots & 1 \end{vmatrix} \dots \dots \dots (15)$$

where g_{kl} , according to what has been said with reference to (6), is practically everywhere equal to $1/r_{kl}$. The quantity a stands as an abbreviation

$$a = \frac{\eta}{DkT} = \frac{\varepsilon^2}{DkTN} \sum N_i z_i^2 = \frac{\varkappa^2}{4\pi n} \dots \dots \dots (16)$$

The quantity a is independent of the concentration n and has the dimensions of a length¹⁾. It is related in the way stated with DEBIJÉ's \varkappa^2 which is directly proportional to the concentration.

Before we proceed, some remarks must be made as regards the approximation tacitly involved in the results (14) and (15). In the first place it will be seen that the integration over a y -coordinate y_m is only permissible as long as $1 + b_m/kT$ is positive. Now b_m will often be negative, so that for every configuration T may always be so small that the mentioned quantity becomes negative. In that case the integration over y_m from $-\infty$ to $+\infty$ would lead to an infinitely large value. In order to meet this difficulty, we might extend the integration from $-M$ to $+M$, these quantities being the extreme values which y_m can assume, i.e. $M = \varepsilon_{\text{max}} \sqrt{N}$ where ε_{max} is the maximum ionic charge present. We will however content ourselves here with stating that $A^{-1/2}$ would, for a given configuration, exactly represent the mean value of $e^{-E'/kT}$ in our generalised ensemble if the possible values of ε_k for an ion were not limited to the discrete values $z_1\varepsilon, \dots, z_s\varepsilon$ but if ε_k could assume all values from $-\infty$ to $+\infty$, the chance for ε_k to lie between ε_k and $\varepsilon_k + d\varepsilon_k$ being proportional to $\exp[-\varepsilon_k^2/2\varepsilon] d\varepsilon_k$. That this holds for a continuous range of values a , including $a = 0$, for which A is positive, is easily verified, for instance by on one side developing $A^{-1/2}$ in a series of powers of a , and by on the other side developing $e^{-E'/kT}$ in a series of negative powers of T and

1) At $T = 300^\circ$ Kelvin, $D = 80$, and for a binary electrolyte with $z_1 = z_2 = 1$, we have $\varkappa = 6,9 \cdot 10^{-8}$ cm.

averaging over all possible ϵ_k values. The two series will coincide exactly provided the ϵ_k are not distributed discretely but continuously in the way just mentioned.

A closer investigation of the approximation involved in (14) would of course be very desirable; in the final result this approximation involves the neglect of terms essentially depending on the ionic radii and in the quantities $\overline{\epsilon_k^3}$, $\overline{\epsilon_k^4}$, etc. We will, therefore, not enter upon this question here.

By means of the value of the integrand in (7), given by (14) and (15), we find for the decrease of the free energy due to the interionic forces:

$$\Delta \zeta = -kT \log \overline{A^{-1/2}} \dots \dots \dots (17)$$

where the bar denotes that the mean value has to be taken over all configurations of the ions in the volume under consideration. Now, if (17) is to have a thermodynamical meaning, the relative fluctuations of A must be exceedingly small on account of the large number of ions. We may therefore replace $\overline{A^{-1/2}}$ by $\overline{A}^{-1/2}$.

A general expression for \overline{A} is derived in the following way. We denote the determinant (15) which refers to N ions in the volume V by A_N , while a determinant which is constructed in an analogous way but which refers to only $N-1$ ions in the same volume, will be called A_{N-1} . We may write:

$$\overline{A}_N = a^N \begin{vmatrix} 1/a & g_{12} & g_{13} & \dots \\ g_{21} & 1/a & g_{23} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \dots \dots \dots (18)$$

Differentiating with respect to a and remembering that the mean values of the subdeterminants of the diagonal terms in (18) are all equal we find:

$$\frac{\partial \overline{A}_N}{\partial a} = \frac{N}{a} \overline{A}_N - a \frac{N}{a^2} \overline{A}_{N-1} = \frac{N}{a} (\overline{A}_N - \overline{A}_{N-1}) \dots \dots (19)$$

Considering \overline{A}_N as a function of the continuous variable N and writing as an approximation

$$\overline{A}_N - \overline{A}_{N-1} = \frac{\partial \overline{A}}{\partial N} \dots \dots \dots (20)$$

(19) will assume the form:

$$a \frac{\partial \overline{A}}{\partial a} = N \frac{\partial \overline{A}}{\partial N}$$

From this we conclude at once that \overline{A} , for a given volume V , depends on N in the form of a function of aN :

$$\overline{A} = \psi(aN, V) \dots \dots \dots (21)$$

We compare this result with (8) the right hand term of which equals $\bar{A}^{-1/2}$. As T is inversely proportional to α , and n equal to N/V , we find:

$$\bar{A} = \psi(\alpha N, V) = \left[\omega \left(\frac{\alpha^3 N}{V} \right) \right]^N \dots \dots \dots (22)$$

This functional relation is only satisfied by:

$$\bar{A} = \exp[-2K(\alpha N)^{3/2} V^{-1/2}] = \exp[-2K\alpha^{3/2} n^{1/2} N] \dots (23)$$

From the relation

$$\Delta \zeta = 1/2 kT \log \bar{A} \dots \dots \dots (24)$$

it follows then that $\Delta \zeta$ has precisely the form which formula (1) of DEBYE and HUECKEL assumes for infinitely small radii.

Before calculating the constant K in (23) we will first investigate what becomes of \bar{A} if we do not introduce the approximation (20). In fact follows from (23) that for larger concentrations the ratio between \bar{A}_N and \bar{A}_{N-1} will deviate more and more from unity, so that (20) is no longer permissible. For the sake of simplicity we will in the following calculation assume $V=1$, i.e. $n=N$, so that we may write:

$$\bar{A}_n = [\omega(\alpha^3 n)]^n \dots \dots \dots (25)$$

For $\bar{A}_n - \bar{A}_{n-1}$ we get then, using (25):

$$\begin{aligned} \bar{A}_n - \bar{A}_{n-1} &= [\omega(\alpha^3 n)]^n - [\omega(\alpha^3(n-1))]^{n-1} = \\ &= [\omega(\alpha^3 n)]^{n-1} \left[\omega - \left(1 - \alpha^3 \frac{\omega'}{\omega} \right)^{n-1} \right] = [\omega(\alpha^3 n)]^{n-1} \left[\omega - e^{-\alpha^3 n \omega'/\omega} \right]. \end{aligned}$$

Here ω' denotes the differential coefficient $\frac{d\omega}{dx}$ of ω with respect to the argument

$$x = \alpha^3 n \dots \dots \dots (26)$$

The approximations which have been introduced are quite in conformity with the spirit of statistical mechanics and are certainly legitimate if the number of ions is sufficiently large¹⁾.

Equation (19) now assumes the form:

$$\frac{\partial \bar{A}_n}{\partial \alpha} = n \omega^{n-1} \cdot 3 \alpha^2 n \omega' = \frac{\pi}{\alpha} \omega^{n-1} (\omega - e^{-\alpha^3 n \omega'/\omega}).$$

This leads to the following differential equation for ω :

$$3 x \omega' = \omega - e^{-x^2/\omega} \dots \dots \dots (27)$$

¹⁾ Mr. ORNSTEIN had informed me that, in his dissertation (Leyden, 1908) analogous transformations were successfully applied in the theory of the equation of state.

The exact solution of this equation can be represented by means of a parameter t :

$$\omega = e^t (1 + 3t)^{-1} \quad K^2 x = t^2 (1 + 3t)^{-3} \quad \dots \quad (28)$$

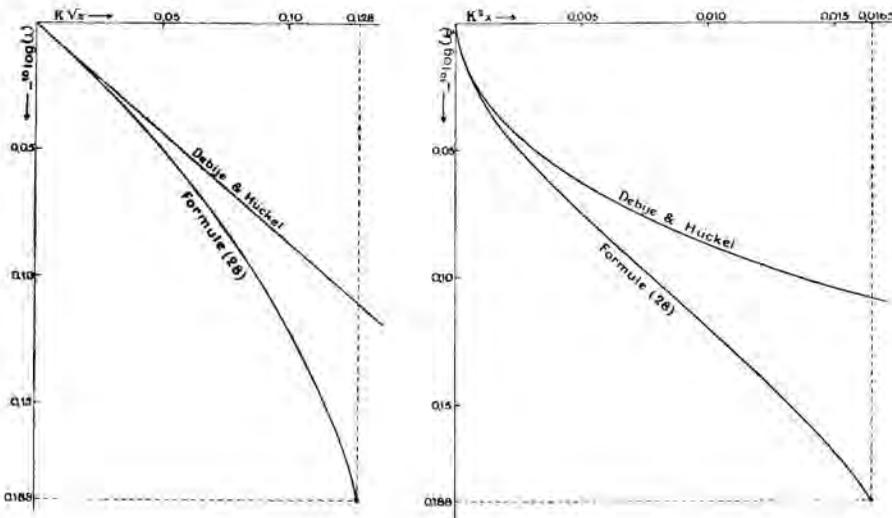
where K is a constant of integration.

For small t values we find:

$$\omega = \bar{A}^{1/n} = 1 - 2K\sqrt{x} \quad \text{or} \quad \bar{A}^{1/n} = 1 - 2K\alpha^{3/2}n^{1/2} \quad \dots \quad (29)$$

Thus K is the same constant as that entering in (23).

For larger values of x , ω changes in a peculiar way. The maximum value which x can take for real values of t (and only such values lead to real ω) correspond with $t = 2/3$, and just for that x value ($K^2 x = 4/243$) ω takes its minimum value. For t larger both x and $-\omega$ decrease. To such values, as well as to negative values for t , we shall attach no physical significance. The figures on this page give $-^{10}\log \omega$ as a function



of \sqrt{x} and of x . In both figures also the function $-^{10}\log \omega = 0,869 K\sqrt{x}$ which would correspond to DEBIJE's and HUECKEL's formula has been given.

In order to find the value of the constant K we might simply assume the formula of DEBIJE and HUECKEL to hold for small concentrations. For the sake of uniformity, however, we will start directly from the expression (15) for A , and by means of it compute the function ω in (25) for small concentrations. For simplicity we again put $V=1$, $N=n$. Let, for a given configuration A_{n-1} represent a determinant analogous to (15), but in which the ion number 1 is disregarded. Then putting:

$$\frac{A_{n-1}}{A_n} = \xi_1 \quad \dots \quad (30)$$

the quantity ξ_1 will be the solution of the following set of linear equations with n unknown quantities $\xi_1, \xi_2, \dots, \xi_n$:

$$\begin{aligned} \sum_k \xi_k b_{k1} &= 1/2 & (a) \\ \sum_k \xi_k b_{kl} &= 0 \quad (l=2, 3, \dots, n) & (b) \end{aligned} \quad (31)$$

where

$$b_{kl} = \frac{1}{r_{kl}} \quad (l \neq k) \quad b_{kk} = 1/2$$

For a configuration where the particles are practically distributed at random, the significance of ξ_1 can be simply illustrated in the case where a is small compared with the mean distance ρ between neighbouring ions. We imagine n metal spheres distributed at random in the volume $V=1$. The spheres carry charges $\xi_1, \xi_2, \dots, \xi_n$. The equations (31) then express that the potential at the surface of the first sphere equals $1/2$, whereas the potentials at the surface of the other spheres are equal to zero. This may be realized by earthing the spheres $2, 3, \dots, n$. Then ξ_1 is the charge which the first sphere must carry in order to possess a potential $1/2$ or, in other words, $\xi_1 a$ is the capacity of the system of conductors just described. The illustration holds only as long as the surface charge on each sphere is equally distributed, and in order that this be so the above mentioned condition $a \ll \rho$ is necessary.

The equations (31) may be solved by an artifice quite analogous to that applied by DEBIJE and HUECKEL. Using the terminology of the electrostatic system just described we introduce the mean potential φ existing at a distance r from the first sphere. Further we consider ξ_{kl} as a function of the distance r of the k^{th} to the first sphere. The mean space charge in unit volume is seen to be equal to ξn , and POISSON'S formula takes the form:

$$\Delta \varphi = -4\pi \xi n \dots \dots \dots (32)$$

Now, according to (31b), the value of φ is so as just to be compensated by the potential due to the charge ξ_k if from an arbitrary point in space we pass to a point lying on the surface of the k^{th} sphere. We therefore write:

$$\varphi = -\xi/2 \dots \dots \dots (33)$$

Eliminating ξ from (32) and (33) we arrive at a differential equation for φ quite analogous to that obtained by DEBIJE and HUECKEL:

$$\Delta \varphi - 4\pi \varphi a n = 0 \dots \dots \dots (34)$$

Its solution is:

$$\varphi = \xi_1 \frac{e^{-r\sqrt{4\pi a n}}}{r} \dots \dots \dots (35)$$

We have chosen ξ_1 as constant of integration, since for small r values, φ must behave as ξ_1/r . The charge ξ_1 can now be calculated for instance by replacing in (31a) the summation over the ions 2, 3, . . . n by an integral over the total volume. By means of (33) and (35) we obtain in this way:

$$\begin{aligned} 1/2 &= \xi_1/a + 4\pi \int_0^\infty \frac{\xi n}{r} r^2 dr = \xi_1/a - 4\pi n a \int_0^\infty \varphi r dr = \\ &= \xi_1/a - 4\pi \xi_1 a n \int_0^\infty e^{-r\sqrt{4\pi a n}} dr = \xi_1/a - \xi_1 \sqrt{4\pi n a} \end{aligned}$$

To the first approximation this gives:

$$\xi_1 = 1 + \sqrt{4\pi a^3 n} \dots \dots \dots (36)$$

For $4\pi a n$ sufficiently small, the value of the integral is mainly determined by a part of space which contains many spheres; this justifies our method of solving ξ_1 .

We have now calculated the right hand side of (30) for a random distribution of ions and for small concentrations. Since under these conditions the left hand side of (30) will be equal to \bar{A}_{n-1}/\bar{A}_n we find, using (25):

$$\bar{A}_{n-1}/\bar{A}_n = [\omega (a_3 (n-1))]^{n-1} / [\omega (a_3 n)] = 1/\omega e^{-a_3 n \omega / \omega} \dots \dots (37)$$

Introducing again the abbreviation (26), we find from (30), (36) and (37) the following differential equation for ω :

$$e^{-x \omega / \omega} = \omega (1 + \sqrt{4\pi x}) \dots \dots \dots (38)$$

Substituting for ω the solution (29) which holds for small x values, we obtain an equation which determines K :

$$\begin{aligned} e^{-x \omega / \omega} = e^{K \sqrt{x}} = 1 + K \sqrt{x} = \omega (1 + \sqrt{4\pi x}) = 1 - 2K \sqrt{x} + \sqrt{4\pi x} \\ K = 1/3 \sqrt{4\pi} \dots \dots \dots (39) \end{aligned}$$

Going back to the general case of N ions in a volume V we find from (17), (25), (26), (28) and (39):

$$\begin{aligned} \Delta \zeta &= 1/2 N k T \log nat \omega(x) \\ \omega &= e^t (1 + 3t)^{-1} , \quad \frac{4\pi}{9} x = t^2 (1 + 3t)^{-3} \dots \dots \dots (40) \\ x &= a^3 n = \left(\frac{\epsilon^2}{DkTn} \sum n_i z_i^2 \right)^3 n \end{aligned}$$

To the first approximation the following formulae hold ($x \ll 1$):

$$\left. \begin{aligned} \omega &= 1 - \frac{2}{3} \sqrt{4\pi x} - \frac{5}{18} 4\pi x \dots \\ -\log \omega &= \frac{2}{3} \sqrt{4\pi x} + \frac{1}{2} 4\pi x \dots \\ \Delta \zeta &= -\sum \frac{N_i z_i^2 \epsilon^2 \kappa}{3 D} - \dots \end{aligned} \right\} \dots \dots (41)$$

The last expression coincides exactly with DEBYE's and HUECKEL's formula (1) in the limiting case of small ionic radic.

III. Discussion of results.

By means of (40) we will first calculate some important thermodynamical quantities.

For the logarithm of the coefficient of activity $(f_a)_i$ of the ions of the kind i we find:

$$\log (f_a)_i = \frac{1}{kT} \frac{\partial \Delta \zeta}{\partial N_i} = \frac{3}{2} t \left(1 - z_i^2 \frac{N}{\sum N_i z_i^2} \right) - \frac{1}{2} \log (1 + 3t) \dots (42)$$

If only two kinds of ions are present for which $z_1 + z_2 = 0$ (KCl, MgSO₄), the first term of the right hand member becomes equal to zero and we find:

$$f_a = \frac{1}{\sqrt{1 + 3t}} \dots \dots \dots (43)$$

For the decrease Δp and $\Delta \psi$ in the values of the osmotic pressure p and the thermodynamical potential ψ due to the interionic forces we find:

$$\left. \begin{aligned} \Delta p &= -\frac{\partial \Delta \xi}{\partial V} = \frac{1}{2} N k T \frac{1}{\omega} \frac{d\omega}{dx} \frac{V}{x} = -\frac{NkT}{2V} t \\ \Delta \psi &= \Delta \zeta + V \Delta p = -\frac{1}{2} N k T \log (1 + 3t) \end{aligned} \right\} \dots \dots (44)$$

From the first of these formulae we obtain a simple interpretation of the quantity t . Since the osmotic pressure of the ideal solution is equal to $\frac{NkT}{V}$, the coefficient with which this expression must be multiplied in order to obtain the osmotic pressure corrected for interionic forces, i.e. the osmotic coefficient f_0 , is given by:

$$f_0 = 1 - \frac{t}{2} \dots \dots \dots (45)$$

The maximum concentration for which our formulae still have a meaning corresponds, according to what has been said on p. 9, to $t = \frac{2}{3}$ and is thus given by:

$$\frac{4\pi}{9} x = \frac{4\pi}{9} a^3 n = \frac{4}{243}$$

For $D = 80$, $T = 300$, a becomes equal to $6,9 \cdot 10^{-8} \frac{\sum N_i z_i^2}{N}$ and we find consequently that the number of gramions in a litre at that concentration is equal to:

$$\frac{1000 n}{6,06 \cdot 10^{23}} = 0,059 \left(\frac{N}{\sum N_i z_i^2} \right)^3 \quad (46)$$

For an electrolyte of the KCl type this corresponds to a concentration of 0,03 mol. KCl per litre, for an electrolyte of the MgSO_4 type with 0,0005 mol. MgSO_4 per litre, etc. For this maximum concentration the osmotic coefficient is always equal to $\sqrt[2]{3} = 0,67$ whereas the logarithm of the coefficient of activity has become equal to $1 - z_i^2 \frac{\sum N_i z_i^2}{N} - \frac{1}{2} \log 3$. For KCl and MgSO_4 the coefficient of activity itself has thus become equal to $\frac{1}{\sqrt{3}} = 0,58$.

It is certainly striking that our formulae have a physical meaning only in a relatively small range of concentrations. We might naturally ascribe this to the circumstance that we have neglected the finite dimensions of the ions, although the integral (7) which was our starting point, diverges for infinitely small radii. The mathematical expression for the compromise which has been made is obviously involved in formulae (8) and (14). As mentioned in the text these approximations involve the neglect of terms depending essentially on the ionic radii. The meaning of the peculiar behaviour of the function ω would then be as follows. Considering an isothermal compression of the system where we start from a very big volume V i. e. from very small concentrations, statistical distributions of the ions will, at the beginning of the process, be possible, which are continuously connected with the random distribution at very small concentrations. This might also be expressed by stating that, for sufficiently small concentrations, an "atmosphere of free ions" can exist. At larger concentrations, however, the tendency of ions with opposite charges to "associate" becomes so preponderant (large fluctuations of the quantity A in (15)), that at concentrations larger than our critical concentration, no distribution of ions can exist which is independent of the dimensions of the ions. Obviously this does not yet mean that these dimensions will not play a part even at smaller concentrations (partial „association" of the ions).

It will therefore be clear that a comparison of the theory with the experiments is rather useless so long as the influence of the ionic radii has not been investigated more closely. We hope to return to this question on a later occasion. Meanwhile I think we may conclude already that it is illegitimate, from a theoretical point of view, to treat the state of a mixture of ions, at greater concentrations and without neglecting

the ionic radii, in the way proposed by DEBIJE and HUECKEL. In fact it seems that the considerations of these authors are based too much on the picture of an atmosphere of free ions. The beautiful investigations of BJERRUM (*loc. cit.* p. 2) point in the same direction. In this work the electrolyte is considered as a mixture of "free" and "associated" ions (not chemical association in the sense of the old theory, of course) and in many cases more plausible values for the ionic radii are found than in DEBIJE's and HUECKEL's theory.

Mathematics — *Ueber stetige Bilder von Punktmengen.* (Zweite Mitteilung)¹⁾. By Dr. W. HUREWICZ. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 27, 1926)

Den nachstehenden Betrachtungen liegt folgende Fragestellung zugrunde: Sei ein n -dimensionaler Raum R eindeutig und beiderseits stetig²⁾ auf einen n^* -dimensionalen Raum R^* abgebildet, und es sei für jeden Punkt p^* von R^* mit $\text{Urb}(p^*)$ die *Urbildmenge* von p^* bezeichnet, d.h. die Menge aller Punkte von R , deren Bilder in den Punkt p^* fallen. Was lässt sich über die Mengen $\text{Urb}(p^*)$ auf Grund der Kenntnis der Dimensionzahlen n und n^* aussagen?

Wir beschäftigen uns zunächst mit den *dimensionserhöhenden* Abbildungen, d.h. wir setzen $n^* > n$ voraus. Der Spezialfall $n = 0$ wurde in meiner ersten Mitteilung über diesen Gegenstand ausführlich behandelt. Es hat sich ergeben, dass in diesem Fall unter den Mengen $\text{Urb}(p^*)$ auch solche auftreten müssen, die mindestens $n + 1$ verschiedene Punkte enthalten. Dies ist im folgenden allgemeinen Theorem enthalten:

Theorem I. *Ist der separable n -dimensionale Raum R eindeutig und beiderseits stetig auf den n^* -dimensionalen Raum R^* abgebildet, wobei $n^* \geq n$ gilt, dann gibt es in R^* Punkte, die Bilder von mindestens $n^* - n + 1$ verschiedenen Punkten von R sind.*

Angenommen jede der Mengen $\text{Urb}(p^*)$ enthielte höchstens k Punkte. Wir haben nachzuweisen, dass dann $n^* \leq n + k - 1$ gilt. Wir beweisen diese Behauptung durch doppelte Induktion nach n und k . Für $k = 1$ (und beliebiges n) ist der Satz sicher richtig, da er sich dann auf den Satz von der topologischen Invarianz der Dimensionszahl reduziert³⁾. Ferner ist der Satz trivial für $n = -1$ (das Bild einer leeren Mengen ist leer). Wir beweisen ihn für gegebenes n und k ($n \geq 0, k \geq 2$) unter der Annahme, er sei bereits bewiesen für die Kombinationen $(n, k-1)$ und $(n-1, k)$.

Auf Grund dieser Annahme zeigen wir zunächst: Haben in R zwei abge-

¹⁾ Meine erste Mitteilung über diesen Gegenstand ist in diesen Proceedings 29, 1926, S. 1014 erschienen. Eine ausführliche Darstellung der Resultate beider Noten wird in den Mathem. Annalen erscheinen.

²⁾ Bezüglich des Begriffes der mehrdeutigen stetigen Abbildung, worin als Spezialfall der Begriff der beiderseits stetigen Abbildung enthalten ist, vgl. die erste Mitteilung.

³⁾ Vgl. MENGER, Monatshefte f. Math. u. Phys. 34, S. 140 und URYSOHN, Fundam. Math. 7, S. 168.

geschlossene Mengen A und B einen höchstens $(n-1)$ -dimensionalen Durchschnitt, so ist der Durchschnitt der Bildmengen A^* und B^* von A , bzw. B höchstens $(n+k-2)$ -dimensional. Sei nämlich B^{**} die Urbildmenge von B^* , d.h. die Menge aller Punkte von R , deren Bilder in B^* liegen. Die Menge B^{**} ist in R abgeschlossen und enthält B als Teil. Die Menge $A \cdot B^{**} - A \cdot B$ ist Differenz zweier abgeschlossener Mengen, kann folglich als Summe einer Folge abgeschlossener Mengen $P_1, P_2, \dots, P_n, \dots$ dargestellt werden. Die Bildmenge von $A \cdot B^{**}$ ist, wie man sofort sieht, mit $A^* \cdot B^*$ identisch. Aus der Summendarstellung

$$A \cdot B^{**} = A \cdot B + \sum_{i=1}^{\infty} P_i$$

ergibt sich eine entsprechende Summendarstellung für die Bildmenge $A^* \cdot B^*$, nämlich:

$$A^* \cdot B^* = (A \cdot B)^* + \sum_{i=1}^{\infty} P_i^*$$

wobei $(A \cdot B)^*$ und P_i^* die Bildmenge von $A \cdot B$ bzw. von P_i bezeichnet. Die Mengen $(A \cdot B)^*$ und P_i^* sind in R^* abgeschlossen³⁾. Die Menge $(A \cdot B)^*$ ist eindeutiges beiderseits stetiges⁴⁾ Bild der höchstens $(n-1)$ -dimensionalen Menge $A \cdot B$ und nach dem als bewiesen vorausgesetzten höchstens $(n+k-2)$ -dimensional. Bei der eindeutigen und beiderseits stetigen⁴⁾ Abbildung $P_i \rightarrow P_i^*$ entsprechen jedem Punkt von P_i^* höchstens $k-1$ Urbilder in P_i ; denn unter den höchstens k Urbildern eines Punktes von $P_i^* \subset B^*$ bei der Gesamtabbildung $R \rightarrow R^*$ liegt mindestens einer in B , also ausserhalb von P_i . Da aber P_i höchstens n -dimensional ist, ist nach unserer Annahme P_i^* höchstens $(n+k-2)$ -dimensional. Die oben gegebene Summendarstellung liefert also eine Zerlegung von $A^* \cdot B^*$ in abzählbar viele abgeschlossene höchstens $(n+k-2)$ -dimensionale Mengen. Mithin ist auch $A^* \cdot B^*$ ⁵⁾ höchstens $(n+k-2)$ -dimensional.

Nun sind unter den separablen Räumen die n -dimensionalen durch folgende Eigenschaft charakterisiert: Es kann jeder natürlichen Zahl m eine Zerlegung des Raumes R in endlich viele abgeschlossene Mengen $R = \sum_{i=1}^{k_m} R_i^m$ zugeordnet werden, so dass 1^o. auf jeden Punkt von R eine Folge von Mengen $R_i^1, R_i^2, \dots, R_i^m, \dots$ sich zusammenzieht und dass 2^o. je zwei Mengen R_i^m und R_k^m für $i \neq k$ einen höchstens $(n-1)$ -dimen-

³⁾ Da die Abbildung beiderseits stetig ist, sind die Bilder von in R abgeschlossenen Mengen in R^* abgeschlossen. Vgl. die erste Mitteilung.

⁴⁾ Bei einer eindeutigen beiderseits stetigen Abbildung eines Raumes R wird, wie leicht ersichtlich, auch jeder abgeschlossene Teil von R beiderseits stetig auf einen Teil des Bildraumes abgebildet.

⁵⁾ Vgl. meine Arbeit: Normalbereiche und Dimensionstheorie, Math. Annalen 96, S. 760, sowie TUMARKIN, Zur allgemeinen Dimensionstheorie, Proc. Ak. Amst. 28, S. 995.

sionalen Durchschnitt haben.⁶⁾ Es sei eine diesen Bedingungen genügende Folge von Zerlegungen des Urbildraumes R gegeben. Sie induziert im Bildraum R^* eine Folge von Zerlegungen in abgeschlossene⁴⁾ Mengen, welche mit Rücksicht auf die Stetigkeit der Abbildung ebenfalls der Bedingung 1^o genügen. Aus der oben bewiesenen Behauptung folgt aber, dass je zwei Mengen $(R_i^m)^*$ und $(R_k^m)^*$ für $i \neq k$ höchstens $(n+k-2)$ -dimensionale Durchschnitte haben; also ist R^* höchstens $(n+k-1)$ -dimensional. Damit ist das Theorem I bewiesen. Aus ihm ergibt sich leicht das folgende

Korollar. Ist der separable n -dimensionale Raum R eindeutig und beiderseits stetig auf den Raum R^ abgebildet, dann ist die Menge aller Punkte von R^* , deren Vielfachheit $\cong k$ ist, (d. h. die Menge aller Punkte von R^* , die höchstens k Urbilder besitzen), höchstens $(n+k-1)$ -dimensional.*

Die Schranke $n^* - n + 1$ von Theorem I kann im allgemeinen nicht erhöht werden, denn es kann leicht gezeigt werden, dass der Euklidische m -dimensionale Raum R_m , oder ein Intervall des R_m sich für jedes $n < m$ als eindeutiges und beiderseits stetiges Bild einer n -dimensionalen Menge darstellen lässt derart, dass jeder seiner Punkte höchstens $m - n + 1$ Urbildpunkte besitzt⁷⁾. In einigen Spezialfällen ist indessen eine Erhöhung der Schranke möglich, vor allem dann, wenn der n -dimensionale Urbildraum R eine Teilmenge des R_n ist.

Theorem II. *Wird eine Teilmenge des Euklidischen n -dimensionalen Raumes eindeutig und beiderseits stetig auf einen n^* -dimensionalen Raum R^* abgebildet, wobei $n^* > n$ gilt, dann treten in R^* Punkte von einer Vielfachheit $\cong n^* - n + 2$ auf.*

Wir beweisen zunächst folgenden Spezialfall des Theorems II: Ist R^* eindeutiges und beiderseits stetiges Bild einer Teilmenge R des R_n , wobei jedem Punkt von R^* höchstens zwei Urbilder entsprechen, dann ist R^* höchstens n -dimensional. Bezeichnen wir als einen Euklidischen R_0 jede Menge, die nur aus einem Punkt besteht, so ist die Behauptung für $n = 0$ trivial. Wir nehmen an, sie sei für $n - 1$ bewiesen, und wollen ihre Gültigkeit für n herleiten. Wäre der Bildraum R^* mindestens $(n + 1)$ -dimensional, so könnten wir ohne Einschränkung der Allgemeinheit annehmen, R sei überall von der Dimension $> n$, d. h. jeder offene Teil von R^* sei

⁶⁾ Dies folgt aus dem Satze, dass die höchstens n -dimensionalen Teilmengen eines kompakten Raumes dadurch charakterisiert sind, dass sie sich als Summe von endlich vielen beliebig kleinen relativ abgeschlossenen Mengen darstellen lassen, die zu je zweien höchstens $(n-1)$ -dimensionale Durchschnitte haben (Vgl. meine unter ⁵⁾ zitierte Arbeit), in Verbindung mit dem URYSOHN'Schen Satz, dass jeder separable Raum mit einer Teilmenge eines kompakten Raumes homöomorph ist.

⁷⁾ Es entsteht die Frage ob bei vorgegebenen n^* und n ($n^* > n \cong 0$) sich jede n^* -dimensionale separable Menge als ein eindeutiges beiderseits stetiges Bild einer n -dimensionalen Menge mit höchstens $(n^* - n + 1)$ -fachen Punkten darstellen lässt. Für $n = 0$ ist diese Frage positiv zu beantworten, d. h. separable höchstens n -dimensionale Räume sind nichts anders als eindeutige beiderseits stetige Bilder von nulldimensionalen Mengen mit höchstens $(n+1)$ -fachen Punkten. (Vgl. erste Mitteilung.)

mindestens $(n+1)$ -dimensional. Dies folgt aus dem allgemeinem Satze, wonach jeder separable m -dimensionale Raum einen in ihm abgeschlossenen überall m -dimensionalen Teil enthält ⁸⁾.

Wir können weiter annehmen dass R in einem n -dimensionalen Würfel W enthalten ist (denn jede Menge des R_n ist mit einer beschränkten Menge des R_n homöomorph). Zerlegen wir W für jede natürliche Zahl m in endlich viele kongruente Würfel W_i^m ($i=1, 2, \dots, k_m$) von der Kantenlänge $< 1/m$ und setzen $R_i^m = R \cdot W_i^m$. Dann sind die Durchschnitte $R_i^m \cdot R_k^m$ Teilmengen des Euklidischen R_{n-1} . Die Annahme, dass R^* mindestens $(n+1)$ -dimensional sei, ist zum Widerspruch geführt, wenn wir zeigen, (Vgl. den Beweis von Theorem I) dass die Durchschnitte von je zwei Mengen $(R_i^m)^*$ und $(R_k^m)^*$ für $i \neq k$ höchstens $(n-1)$ -dimensional seien; dazu genügt es aber, folgendes nachzuweisen: Ist der Durchschnitt zweier in R abgeschlossener Mengen A und B Teilmenge eines Euklidischen R_{n-1} , dann ist der Durchschnitt der Bildmengen A^* und B^* höchstens $(n-1)$ -dimensional.

Wie beim Beweis von Theorem I und unter Beibehaltung der dortigen Bezeichnungen nehmen wir die Zerlegungen vor:

$$A \cdot B^{**} = A \cdot B + \sum_{i=1}^{\infty} P_i$$

$$A^* \cdot B^* = (A \cdot B)^* + \sum_{i=1}^{\infty} P_i^*$$

Die Menge $(A \cdot B)^*$ ist nach dem als bewiesen vorausgesetzten höchstens $(n-1)$ -dimensional. Wir brauchen also nur noch zu zeigen, dass jede der Mengen P_i^* höchstens $(n-1)$ -dimensional ist. Nun besitzt jeder Punkt von P_i^* genau einen Urbildpunkt in P_i , (denn es kann im ganzen höchstens zwei Urbildpunkte geben. Einer von ihnen muss aber mit Rücksicht auf $P_i^* < B^{**}$ in B , also ausserhalb von P_i liegen). Die Abbildung zwischen P_i und P_i^* ist somit ein-eindeutig und daher (vgl. Fussnote ⁴⁾) *topologisch*. Wäre also P_i^* mindestens n -dimensional, so müsste dasselbe von P_i gelten. Als Teilmenge des R_n enthielte dann P_i nach einem Theorem von MENGER und URYSOHN ein n -dimensionales offenes Intervall ⁹⁾, etwa J . Zu jedem Punkt p von J (wie überhaupt zu jedem Punkt von $B^* - B$) gibt es in B einen und nur einen Punkt p' , dessen Bild mit dem Bild von p zusammenfällt. Durchläuft p das n -dimensionale Gebiet J , so durchläuft p' eine, wie man leicht sieht, mit J homöomorphe Menge J' . Diese Menge J' ist nach dem BROUWERSCHEN Satz von der Gebietsinvarianz ebenfalls ein n -dimensionales Gebiet. Die Bildmengen von J und $R - (J + J')$ sind fremd und ihre Summe ist R^* . Die Bildmenge der abgeschlossenen Menge $R - (J + J')$ ist abgeschlossen, also ist ihr Komplement, d. h. die Bildmenge von

⁸⁾ Vgl. meine unter ⁵⁾ zitierte Arbeit, S. 762, Theorem VI.

⁹⁾ Vgl. MENGER, Monatshefte f. Math. u. Phys. 34, S. 157 und URYSOHN, Fund. Math. 7, S. 81.

¹⁰⁾ Vgl. LEBESGUE, Fundam. Math. 2, S. 280.

J in R^* offen; ferner ist diese Menge als topologisches Bild von J n -dimensional. R^* enthält also einen relativ offenen n -dimensionalen Teil im Widerspruch zur Annahme. Damit ist die Behauptung bewiesen.

Der allgemeine Beweis des Theorems II erfolgt auf Grund des soeben erledigten Spezialfalles in ganz analoger Weise, wie der Beweis des Theorems I. Als Ausgangspunkt der Induktion dienen unter Beibehaltung der Bezeichnungsweise von Theorem I die Spezialfälle: $k=2$ (beliebiges n) und $n=-1$ (beliebiges k). Es sind ferner im Beweis von Theorem I die Worte „ n -dimensional“, bzw. „ $(n-1)$ -dimensional“ überall durch die Worte „Teilmenge des R_n “, bzw. „Teilmenge des R_{n-1} “ zu ersetzen.

Durch das Theorem II ist die von LEBESGUE²⁾ aufgeworfene Frage betreffend die Vielfachheit der Punkte bei der eindeutigen stetigen Abbildung¹²⁾ eines n -dimensionalen auf einen m -dimensionalen Würfel ($n < m$) beantwortet. Von LEBESGUE wurde bewiesen, dass bei einer derartigen Abbildung Punkte von einer Vielfachheit $\cong \frac{n+1}{l}$ auftreten

müssen, wobei $l = \frac{m}{2}$, bzw. $\frac{m+1}{2}$ ist, je nachdem m gerade oder ungerade. Durch Theorem II wird diese Schranke auf $m-n+2$ erhöht. Weiter kann sie sicher nicht erhöht werden, da, wie LEBESGUE bemerkt hat¹¹⁾, der n -dimensionale Würfel auf den m -dimensionalen Würfel derart eindeutig und stetig abgebildet werden kann, dass höchstens $(m-n+2)$ -fache Punkte auftreten.

Aus dem Theorem II ergibt sich ferner, dass bei einer eindeutigen und beiderseits stetigen Abbildung einer Teilmenge des Euklidischen R_n die Menge aller Punkte des Bildraumes, welche höchstens k Urbildpunkte besitzen ($k \cong 2$), höchstens $(n+k-2)$ -dimensional ist.

Wir wenden uns nunmehr zu den dimensionserniedrigenden Abbildungen, setzen also voraus, dass die Dimension n^* des Bildraumes R^* kleiner sei als die Dimension n des Urbildraumes. Wir beschränken uns dabei auf kompakte Räume. Es gilt nun der folgende Fundamentalsatz:

Theorem III. *Ist der kompakte n -dimensionale Raum R eindeutig und stetig auf den n^* -dimensionalen Raum R^* abgebildet¹²⁾, wobei $n^* \leq n$ ist, dann gibt es in R^* Punkte, mit mindestens $n - n^*$ -dimensionalen Urbildmengen¹³⁾.*

¹¹⁾ LEBESGUE a.a.O. S. 285.

¹²⁾ Jede eindeutige stetige Abbildung eines kompakten Raumes ist beiderseits stetig.

¹³⁾ Ein Spezialfall dieses Satzes, nämlich der Fall $n=0$ wurde von TUMARKIN (diese Proceedings, 28, S. 1001) bewiesen. Ein anderer Spezialfall des Theorems, die Behauptung nämlich, dass die Dimension eines Raumes bei einer eindeutigen beiderseits stetigen Abbildung nicht erniedrigt werden kann, wofern jedem Punkt des Bildraumes eine null-dimensionale Menge von Urbildpunkten entspricht, wurde für allgemeine separable Räume in der ersten Mitteilung ausgesprochen (S. 1016, Fussnote¹⁾).

(Zusatz bei der Korrektur). Wie ich eben ersehe, wurde das Theorem III von ALEXANDROFF vermutet; vgl. seine nach Fertigstellung dieser Note erschienene Abhandlung, Math. Annalen 96, S. 570.

Der Beweis stützt sich auf das folgende Lemma, auf dessen Begründung wir hier nicht näher eingehen ¹⁴⁾.

Lemma. Sei n eine ganze nicht-negative Zahl, ε eine reelle Zahl > 0 . Gibt es zu jedem Punkt des kompakten Raumes R eine Umgebung mit höchstens $(n-1)$ -dimensionaler Begrenzung, so dass die abgeschlossene Hülle der Umgebung Summe ist von endlich vielen abgeschlossenen Mengen mit Durchmessern $< \varepsilon$, die zu je $n+2$ fremd sind, dann ist der ganze Raum R Summe von endlich vielen abgeschlossenen Mengen mit Durchmessern $< \varepsilon$, die zu je $n+2$ fremd sind.

Sind also die Voraussetzungen des Lemmas bei festem n für jedes $\varepsilon > 0$ erfüllt, dann ist R nach einem Theorem von URYSOHN ¹⁵⁾ höchstens n -dimensional.

Sei nun k die grösste unter den Dimensionen der Urbildmengen $\text{Urb}(p^*)$, wenn p^* alle Punkte von R^* durchläuft, und sei n^* gegeben. Es ist zu zeigen, dass $n \leq n^* + k$ gilt. Für $n^* = -1$ ist die Behauptung trivial. Wir wollen den Beweis für $n^* = m$ führen, unter der Annahme, die Behauptung gelte für $n^* < m$. Sei p ein Punkt von R , $-p^*$ sein Bild. Die Menge $\text{Urb}(p^*)$ ist höchstens k -dimensional, kann daher, wie aus einem Theorem von MENGER und URYSOHN folgt, bei vorgegebenem $\varepsilon > 0$ mit endlich vielen in R^* offenen Mengen mit Durchmessern $< \varepsilon$ überdeckt werden, deren abgeschlossene Hüllen zu je $n+2$ fremd sind ¹⁶⁾. Sei U die Summe dieser offenen Mengen. Da es zu p^* beliebig kleine Umgebungen mit höchstens $(m-1)$ -dimensionalen Begrenzungen gibt, so existiert insbesondere auch eine derartige Umgebung V , deren Urbildmenge V' in der Umgebung U von $\text{Urb}(p^*)$ liegt, folglich zerfällt $\overline{V'}$ in endlich viele abgeschlossene Teile $< \varepsilon$, die zu je $k+2$, also a fortiori zu je $k+m+2$, fremd sind. Nun ist aber V' eine Umgebung von p , und das Bild ihrer Begrenzung ist in der höchstens $(m-1)$ -dimensionalen Begrenzung von V enthalten; also ist die Begrenzung von V' nach dem für $m-1$ als gültig angenommenen Satz höchstens $(k+m-1)$ -dimensional. Der Raum R erfüllt mithin die Voraussetzungen unseres Lemmas für $n = k+m$ und für jedes $\varepsilon > 0$. Folglich ist R höchstens $(k+m)$ -dimensional. ¹⁷⁾

Es kann weiter bewiesen werden, dass es eine Zahl $k \leq n$ gibt, für

¹⁴⁾ Vgl. diesbezüglich meine angekündigte ausführliche Darstellung in den Mathem. Annalen.

¹⁵⁾ Fundam. Mathem. 8, S. 225.

¹⁶⁾ Vgl. MENGER, Monatshefte f. Math. u. Phys. 34, S. 148. URYSOHN, Fundam. Mathem. 8, S. 225.

¹⁷⁾ Die Frage, ob das Theorem III auch für allgemeine separable Räume gilt, (wobei natürlich in der Aussage des Theorems der Begriff „stetig“ durch „beiderseits stetig“ zu ersetzen ist) bleibt offen (mit Ausnahme des in der Fussnote ¹³⁾ erwähnten Spezialfalles). Die zum Beweis von Theorem III verwendete Methode kann auf separable Räume im allgemeinen nicht übertragen werden, solange nicht der in den Beweis eingehende URYSOHNsche Satz, (dass eine Menge, die als Summe von endlich vielen beliebig kleinen relativ abgeschlossenen zu je $n+2$ fremden Teilen dargestellt werden kann, höchstens n -dimensional ist) für beliebige separable Mengen bewiesen ist.

die die Menge aller Punkte von R^* denen mindestens k -dimensionale Urbildmengen entsprechen, mindestens $(n-k)$ -dimensional ist.

Die Theoreme I und III zeigen, dass die *dimensionserniedrigende* Wirkung einer eindeutigen und beiderseits stetigen Abbildung von der *Mächtigkeit* der Urbildmengen der einzelnen Punkte des Bildraumes abhängt, während die *dimensionserhöhende* Wirkung einer Abbildung von der *Dimension* dieser Urbildmengen abhängt.

Aus dem beim Beweis von Theorem III verwendeten Lemma kann auch ein Satz über die Struktur der n -dimensionalen kompakten Räume hergeleitet werden. Bezeichnen wir für jeden Punkt p des n -dimensionalen Raumes R mit $R^n(p)$ die Menge aller Punkte von R , die sich von p durch eine weniger als $(n-1)$ -dimensionale Menge nicht trennen lassen. Ich habe bewiesen¹⁸⁾, dass die Menge $R^n(p)$ entweder nur aus dem Punkte p besteht (dies tritt dann und nur dann ein, wenn R in p weniger als n -dimensional ist), oder ein Kontinuum ist. In diesem letzteren Fall ist R in allen Punkten von $R^n(p)$ n -dimensional. Wir zeigen nun:

*Unter den Mengen $R^n(p)$ des kompakten n -dimensionalen Raumes R existiert mindestens eine von der Dimension n .*¹⁹⁾

Sonst könnten wir analog wie beim Beweise des Theorem III zu jedem Punkt p von R und zu jedem $\varepsilon > 0$ eine Umgebung $U(p)$ von $R^n(p)$ angeben, deren abgeschlossenen Hülle in endlich viele abgeschlossene Teile $< \varepsilon$ zerfällt, die zu je $n+1$ fremd sind. Nun kann p von jedem Punkt des Komplementes der Menge $U(p)$ durch eine höchstens $(n-2)$ -dimensionale Menge getrennt werden; und daraus folgt nach dem Borelschen Theorem, dass eine Umgebung $V(p) < U(p)$ mit höchstens $(n-2)$ -dimensionaler Begrenzung existiert.²⁰⁾ Es sind also alle Voraussetzungen unseres Lemmas für $n-1$ erfüllt, und daher ist R höchstens $(n-1)$ -dimensional im Widerspruch zur Annahme.

Aus diesem Satz folgt das MENGERSche Ergebniss, wonach die Menge aller Punkte, in denen ein kompakter n -dimensionaler Raum n -dimensional ist, selbst die Dimension n besitzt.²¹⁾ Ferner enthält unser Satz das TURMARKINSche Ergebnis²²⁾, dass ein n -dimensionaler kompakter Raum ein n -dimensionales Teilkontinuum besitzt.

¹⁸⁾ Vgl. meine Arbeit in den Mathem. Annalen 96, S. 762, Theorem VII.

¹⁹⁾ Aus diesem Ergebniss kan in sehr einfacher Weise das Theorem III dieser Arbeit gefolgert werden.

²⁰⁾ Es gibt nämlich zu jedem punkt von $R-U(p)$ eine Umgebung mit höchstens $(n-2)$ -dimensionalen Begrenzung, so dass p in der abgeschlossenen Hülle dieser Umgebung nicht enthalten ist. Nach dem BORELSchen Theorem ist die abgeschlossene Menge $R-U(p)$ in der Summe (nennen wir sie W) von endlich vielen derartigen Umgebungen enthalten. Dann ist $R-W$ eine Umgebung $U(p)$ von p mit höchstens $(n-2)$ -dimensionaler Begrenzung.

²¹⁾ Vgl. MENGER, Das Hauptproblem über die dimensionale Struktur der Räume, diese Proceedings, 30, S. 138. MENGER beweist, dass in einem kompakten n -dimensionalen Raum die Menge aller Punkte, in denen der Raum n -dimensional ist, sogar *homogen* (d.h. in jedem ihrer Punkte) n -dimensional ist.

²²⁾ Vgl. TURMARKIN, diese Proceedings, 28, S. 1001.

Mathematics. — *Zum Beweise des zweiten Fundamentalsatzes der symbolischen Methode.* By R. WEITZENBÖCK.

(Communicated at the meeting of December 18, 1926).

Der bisher bekannte Beweis des zweiten Fundamentalsatzes der symbolischen Methode in der projektiven Invariantentheorie stützt sich auf Reihenentwicklung ¹⁾. Es ist mir jetzt gelungen, einen Satz über p -Relationen zu beweisen ²⁾. Auf Grund dieses Satzes kann dann der bisherige Beweis des zweiten Fundamentalsatzes bedeutend vereinfacht werden, worauf ich seinerzeit schon hingewiesen habe ³⁾. Macht man von diesem Satze über p -Relationen Gebrauch, so bleibt nur noch zu zeigen, dass alle fünf Typen der beim zweiten Fundamentalsatze auftretenden Identitäten II aus einer einzigen ableitbar sind. Dies ist im Wesentlichen ein schon von E. PASCAL ⁴⁾ gefundener Satz, dessen Beweis hier mit Verwendung von $(n-1)$ -fältigen Komplexsymbolen sehr vereinfacht wird.

§ 1.

Es sei J_0 eine ganze, rationale, identisch verschwindende projektive Invariante einer oder mehrerer n -ärer Grundformen:

$$J_0 = J_0(a'_{rst\dots}; a_{rst\dots}; p_{ikl\dots}, q'_{ikl\dots}) \dots \dots \dots (1)$$

J_0 ist hierbei linear in den $p_{ikl\dots}, q'_{ikl\dots}$. Stellen wir J_0 symbolisch dar ⁵⁾

$$J_0 = J_1(a', \dots; a, \dots; p, \dots, q', \dots), \dots \dots \dots (2)$$

so wird J_1 ein Polynom von Faktoren erster und zweiter Art mit den Reihen a', \dots, a, \dots von gewöhnlichen Symbolen und den Reihen p, \dots, q', \dots von Komplexsymbolen.

Aus J_1 leiten wir durch Polarisierung eine Invariante J_2 ab, die linear ist in allen Reihen $a', b', \dots, \alpha, \beta, \dots$:

$$J_2 = J_2(a', b', \dots; \alpha, \beta, \dots; p, \dots, q', \dots) \dots \dots \dots (3)$$

Das Zurückgehen von J_2 zu J_1 erfolgt eindeutig durch gleichsetzen mehrerer Reihen $a', b', \dots, \alpha, \beta, \dots$.

¹⁾ Vgl. die Darstellung in meiner „Invarianten-Theorie“, NOORDHOFF, Groningen (1923), p. 98 ff.

²⁾ Vgl. den demnächst in den Mathematischen Annalen erscheinenden Aufsatz „Ueber p -Relationen“.

³⁾ Invariantentheorie. Anmerkung p. 117.

⁴⁾ Memorie della R. Acc. dei Lincei IV, 3 und V, 4a (1888).

⁵⁾ Invariantentheorie p. 92.

Aus J_2 bekommen wir eine Invariante

$$J_3 = J_3(a', b', \dots; a, \dots; x, y, \dots, u', v', \dots) \dots \dots (4)$$

wenn wir

$$p_{ikl\dots} = (xy\dots z)_{ikl\dots}, q'_{ikl\dots} = (u'v'\dots w')_{ikl\dots} \dots \dots (5)$$

setzen. Auch J_3 ist ein Polynom von Faktoren erster und zweiter Art und das Zurückgehen von J_3 nach J_2 ist ebenfalls eindeutig möglich, indem die Reihen x, y, \dots, z durch p , die Reihen u', v', \dots, w' durch q' u.s.f. ersetzt werden, wobei auf die Reihenfolge der x, y, \dots in J_3 zu achten ist.

Aus J_3 eliminieren wir schliesslich alle gestrichenen Reihen $a', b', \dots, u', v', \dots$ durch Zerlegungen der Gestalt

$$a'_i = (rs\dots t)_{23\dots n} \dots \dots \dots (6)$$

und erhalten so eine Invariante

$$J_4 = J_4(a, \beta, \dots; x, y, \dots) \dots \dots \dots (7)$$

die nur noch ungestrichene Reihen, also nur noch Klammerfaktoren vom Typus $(a\beta\dots\gamma)$ enthält. Auch hier ist das Zurückgehen zu J_3 eindeutig, wobei analog mit (6) die $n-1$ Reihen r, s, \dots, t durch $(n-1)$ -fältige Komplexsymbole a zu ersetzen sind:

$$a'_i = (a^{n-1})_{23\dots n} = (n-1)! a_{23\dots n} \dots \dots \dots (8)$$

Wir haben auf diese Weise statt des ursprünglichen J_0 eine Invariante J_4 von N Linearformen a, β, \dots , die identisch bezüglich jeder Reihe a, β, \dots verschwindet und also eine p -Relation zwischen G_n -Koordinaten $(a\beta\dots\gamma)$ im G_N darstellt. Somit ist ²⁾:

$$J_4 = \Sigma_4 A \Pi_2, \dots \dots \dots (9)$$

wo Π_2 eine quadratische p -Relation ist, also die linke Seite einer Identität von der Gestalt

$$\Pi_2 = (a\beta\dots\gamma)(xy\dots z) - (x\beta\dots\gamma)(ay\dots z) + \dots \dots \dots (10)$$

In (9) haben wir links bei J_4 ein eindeutiges Zurückgehen zu J_3 , also auch rechts bei $\Sigma_4 A \Pi_2$. Wir müssen zeigen, dass bei diesem Zurückgehen die Gestalt $\Sigma A \Pi$ der rechten Seite erhalten bleibt, d. h. dass wir wieder einen Ausdruck $\Sigma_3 A \Pi$ erhalten.

§ 2.

Wir haben in jedem $A \Pi_2$ von $\Sigma_4 A \Pi_2$ $n-1$ bestimmte Reihen r, s, \dots, t , in dieser Reihenfolge aufgeschrieben, nach (6) und (8) durch $(n-1)$ -fältige Komplexsymbole a zu ersetzen und dann den "Uebergang $a \rightarrow a'$ " nach (8) auszuführen. Hierbei haben wir die verschiedenen Möglichkeiten zu untersuchen, auf welche Weise die $n-1$ Reihen a dann in $A \Pi_2$ verteilt sind ⁶⁾.

⁶⁾ Das Folgende ist nichts anderes als der vereinfachte Beweis des von E. PASCAL (l.c.) bewiesenen Satzes: Gilt der zweite Fundamentalsatz für die „Transformierte“, so gilt er auch für die ursprüngliche Invariante.

Π_2 hat die Gestalt (10). A ist ein Produkt von Klammerfaktoren $(\alpha\beta\dots\gamma)$. Enthält A wenigstens eine Reihe a , so steht diese in einem Klammerfaktor. In diesen können wir dann durch Umformen alle $(n-1)$ Reihen a vereinigen ⁷⁾, wodurch

$$(a^{n-1} a) = (n-1)! (a' a)$$

in A zu stehen kommt, während Π_2 seine Gestalt nicht ändert; es werden nur die eventuell in Π_2 vorhandenen Reihen a ersetzt durch neue, von a verschiedene Reihen α, β, \dots

Dieser "erste Uebergang $a \rightarrow a'$ " ergibt also hier aus $\Sigma_4 A\Pi_2$ wieder eine Summe $\Sigma'_4 A\Pi_2$, wobei jetzt in A neben Klammerfaktoren $(\alpha\beta\dots\gamma)$ auch Linearfaktoren (aa') vorkommen können. Sind aber alle $n-1$ Reihen a in Π_2 von $A\Pi_2$ anwesend, enthält also A keine Reihe a , so gibt der Uebergang $a \rightarrow a'$ Null oder $-$ und dies ist nur dann der Fall, wenn in (10) $y = \dots = z = a$ ist $-$ die Identität

$$\Pi_1 = (\alpha\beta\dots\gamma)(xa') - (x\beta\dots\gamma)(aa') + \dots \dots \dots (11)$$

Der erste Uebergang $a \rightarrow a'$ ergibt somit aus $\Sigma_4 A\Pi_2$ eine Summe

$$\Sigma'_4 A\Pi \dots \dots \dots (12)$$

wo Π eine Π_2 oder Π_1 ist.

Was entsteht nun bei den weiteren Uebergängen $b \rightarrow b', c \rightarrow c', \dots$ aus (12)? Wir wollen wieder a statt b, c, \dots schreiben. Enthält dann ein A von (12) wenigstens eine Reihe a in einem Klammerfaktor, dann vereinigen wir wie oben alle $n-1$ Reihen a in diesem und bekommen wieder einen Linearfaktor $(a'a)$. Sind alle Reihen a in A anwesend und zwar nur in Linearfaktoren, dann haben wir

$$(u' a)(v' a) \dots (w' a) = (a' u' v' \dots w')$$

und ein solcher Klammerfaktor hat bei den weiteren Uebergängen $a \rightarrow a'$ keinen Anteil mehr. Es bleiben also nur noch die beiden folgenden Möglichkeiten zu untersuchen:

1. Alle $n-1$ Reihen a sind in einem Π von (12) vorhanden.
2. Ein Π von (12) enthält k Reihen a ($1 \leq k \leq n-2$) und die übrigen $n-1-k$ Reihen sind in dem zugehörigen A nur in Linearfaktoren enthalten.

Ist im *ersten Falle* Π ein Π_2 , so entsteht bei $a \rightarrow a'$ nach dem Oben Gesagten entweder Null oder ein Π_1 , also nichts Neues. Ist aber Π ein Π_1 , so entsteht bei $a \rightarrow a'$ Null.

Im *zweiten Falle* sei erstens Π ein Π_2 . $A\Pi$ hat dann $-$ von den weiteren Faktoren ohne a abgesehen $-$ die Gestalt:

$$\Pi A = [(\alpha\beta\dots\gamma)(xy\dots z) - (x\beta\dots\gamma)(ay\dots z) + \dots](au')(av') \dots (aw'), (13)$$

wo in der eckigen Klammer k Reihen a anwesend sind, also ausserhalb $n-1-k$ Faktoren $(au'), \dots, (aw')$ stehen.

Hier können wir durch Umformung alle diese $n-1-k$ Reihen a in den zweiten Klammerfaktor von Π hineinziehen, was eine Summe von $\Pi A'$ ergibt, wobei jetzt wieder alle $n-1$ Reihen a in Π konzentriert sind.

⁷⁾ Inv. Theorie, p. 79.

Hiedurch kommen wir wieder auf die bereits aufgezählten Fälle zurück.

Wir können also voraussetzen: Π von (12) ist ein Π_1 und ΠA hat die Gestalt (13) und zwar:

$$\Pi A = [k(a^{k-1} \alpha \beta \dots \gamma)(a m') - (a^k \beta \dots \gamma)(a m') + \dots](a u')(a v') \dots (a w') \quad (14)$$

Bringen wir hier alle Reihen a von A in die Klammerfaktoren von Π und gehen zu a' über, so muss eine identisch verschwindende Invariante L entstehen mit den $n-k+1$ Reihen $\alpha, \beta, \dots, \gamma$ und den $n-k+1$ Reihen $m', u', v', \dots, w', a'$, wobei sich bei $k > 1$ L aus Linearfaktoren allein aufbaut. Ist also $k > 1$, so verschwindet L ⁸⁾. Bei $k=1$ aber haben wir

$$\Pi A = (\alpha \beta \dots \gamma)(a' m' u' v' \dots w') - \Sigma \pm (a' \alpha)(m' \beta) \dots (w' \gamma) = \Pi_3 \quad (15)$$

Wir erhalten somit den neuen Typus Π_3 .

§ 3.

Bei den folgenden Uebergängen $a \rightarrow a'$ haben wir also jetzt mit den Möglichkeiten zu rechnen, dass in $\Sigma A \Pi \Pi$ ein Π_1, Π_2 oder Π_3 ist. Die Fälle $\Pi = \Pi_1$ und $\Pi = \Pi_2$ sind bereits erledigt. Bei $\Pi = \Pi_3$ haben wir wieder in zwei Fällen etwas neues zu erwarten, nämlich:

- 1). Alle $n-1$ Reihen a sind in Π konzentriert.
- 2). In Π_3 sind k Reihen a vorhanden, $1 \leq k \leq n-2$, und in A die übrigen $n-1-k$ Reihen in Linearfaktoren.

Im ersten Falle wird nach dem Uebergang $a \rightarrow a'$ aus Π_3 die Identität

$$\Pi'_1 = (u' v' \dots w')(a' x) - (a' v' \dots w')(u' x) + \dots \dots \dots (16)$$

Im zweiten Falle hat ΠA die Gestalt

$$\Pi A = \left\{ \begin{aligned} & [(a^k a_{k+1} a_{k+2} \dots a_n)(u'_1 u'_2 \dots u'_n) - \Sigma \pm (a u'_1) \dots (a u'_k)(a_{k+1} u'_{k+1}) \dots \\ & \dots (a_n u'_n)] \cdot (a v'_1) \dots (a v'_{n-k-1}) = M - N. \end{aligned} \right\} \quad (17)$$

Bei M bringen wir die $n-k-1$ Reihen a von A durch Umformung in den Klammerfaktor $(a^k a_{k+1} \dots a_n)$ und erhalten:

$$\begin{aligned} M &= k! [(a' a_{k+1}) \cdot \Sigma (a_{k+2} \dots a_n)_{ikl\dots} (v'_1 \dots v'_{n-k-1})_{ikl\dots} - \\ & - (a' a_{k+2}) \cdot \Sigma (a_{k+1} \dots a_n)_{ikl\dots} (v'_1 \dots v'_{n-k-1})_{ikl\dots} + \dots] \cdot (u'_1 u'_2 \dots u'_n) \\ M &= k! (u'_1 \dots u'_n) \cdot \Sigma_{i k l} (a_{k+1} \dots a_n)_{ikl\dots} (a' v'_1 \dots v'_{n-k-1})_{ikl\dots} = k! (u'_1 \dots u'_n) \cdot \Sigma \quad (18) \end{aligned}$$

Bei N haben wir vorerst bei $a \rightarrow a'$:

$$\begin{aligned} & (a u'_1) \dots (a u'_k)(a_{k+1} u'_{k+1}) \dots (a_n u'_n)(a v'_1) \dots (a v'_{n-k-1}) = \\ & = (a' u'_1 \dots u'_k v'_1 \dots v'_{n-k-1})(a_{k+1} u'_{k+1}) \dots (a_n u'_n) \end{aligned}$$

und dies formen wir mit Hilfe von Π'_1 um:

$$\begin{aligned} & (a' u'_1 \dots u'_k v'_1 \dots v'_{n-k-1})(u'_{k+1} a_{k+1}) = (u'_{k+1} u'_1 \dots u'_k v'_1 \dots v'_{n-k-1})(a' a_{k+1}) - \\ & - (u'_{k+1} a' u'_2 \dots) (u'_1 a_{k+1}) + \dots \end{aligned}$$

⁸⁾ Vgl. Inv. Theorie, p. 103.

Bilden wir hier jetzt Σ_{\pm} bezgl. der Permutationen der u'_i , so werden k Glieder rechts mit N identisch und wir haben:

$$(k+1)N = \Sigma_{\pm} (u'_{k+1} u'_1 \dots u'_k v'_1 \dots v'_{n-k-1}) (a'_{k+1}) (u'_{k+2} a'_{k+2}) \dots (u'_n a'_n) - \\ - \Sigma_{\pm} (u'_{k+1} a' u'_2 \dots v'_{n-k-1}) (u'_1 a'_{k+1}) \dots (u'_n a'_n) + \dots$$

Hier formen wir neuerdings mit II'_1 um und setzen dies so oft fort bis alle $u'_{k+1}, u'_{k+2}, \dots, u'_n$ im Klammerfaktor vereinigt sind. Hiedurch wird schliesslich:

$$(k+1)(k+2)\dots n \cdot N = \Sigma_{\pm} (u'_1 \dots u'_n) \cdot \Sigma_{(a'_{k+1} \dots a'_n)_{ikl} \dots} (a' v'_1 \dots v'_{n-k-1})_{ikl} \dots \\ = n! (u'_1 \dots u'_n) \cdot \Sigma,$$

also

$$N = k! (u'_1 \dots u'_n) \cdot \Sigma.$$

Nach (18) ist $M - N$ also Null, d.h. das ursprüngliche IIA ist eine lineare Kombination von Identitäten (16) vom Typus II'_1 .

Wir haben somit jetzt noch zu ermitteln, was der Uebergang $a \rightarrow a'$ bei IIA ergibt, wenn II ein II'_1 ist und die Reihen a in A nur in Linearfaktoren auftreten. Dies gibt nach (16) nur die einzige Möglichkeit

$$IIA = [(u'v' \dots w')(a'a) - (a'v' \dots w')(u'a) + \dots] (a\beta') \dots (a\gamma'). \quad (19)$$

Hier führt $a \rightarrow a'$ auf den neuen und letzten Typus:

$$II'_2 = (u'v' \dots w')(a'\beta' \dots \gamma'a') - (a'v' \dots w')(u'\beta' \dots \gamma'a') + \dots \quad (20)$$

§ 4.

Gehen wir jetzt zurück auf Gleichung (9). In den beiden vorhergehenden §§ haben wir gezeigt, dass die Uebergänge $a \rightarrow a'$ aus $\Sigma_4 II_2 A$ eine $\Sigma_3 A II$ hervorbringen, wo jetzt II jeden der fünf möglichen Typen darstellen kann. Wir haben dann

$$J_3 = \Sigma_3 A II \dots \dots \dots (21)$$

und hier sind die durch (6) eingeführten Reihen r, s, \dots, t verschwunden.

Von J_3 gelangen wir zu J_2 zurück indem wir nach (5) einige Reihen durch Komplexsymbole p bzw. q' ersetzen. Hierdurch wird die Gestalt von Σ_3 nicht geändert, es entstehen höchstens die besonderen Formen der Identitäten II , die sich bei Komplexsymbolen einstellen:

$$J_2 = \Sigma_2 A II.$$

Und von hier aus kommen wir schliesslich auf die ursprüngliche symbolische Darstellung J_1 von J_0 zurück indem wir mehrere Reihen von gewöhnlichen Symbolen einander gleichsetzen. Dies gibt bei Σ_2 wieder keine Gestaltänderung, d.h. wir haben:

$$J_1 = \Sigma_1 A II$$

und die II enthalten jetzt nur die ursprünglich verwendeten Grössen- und Symbolreihen.

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Meteorology. — *Contribution to the Explanation of Complex Halos.*

By M. PINKHOF. (Communicated by Prof. E. VAN EVERDINGEN)

(Communicated at the meeting of June 26, 1926)

1. *Introduction.*

Though at present most of the separate halo phenomena have been explained in a satisfactory way, i. e. have been brought in connection with ice crystals of acceptable shape and orientation, it must be admitted that as yet comparatively little attention has been given to the theory of complex halos. The second edition of PERNTER's *Meteorologische Optik* revised by EXNER, does not yet deal with it. Nevertheless it is necessary also for the theory of the simple halo phenomena, to study the complex halos more closely. By the simultaneous appearance of different phenomena an explanation of one of them, based on the occurrence of very specially formed crystals, can be rendered less probable. It is beyond doubt that, in the case of complicated halos, there is very often a greater number of the component parts mutually related than could be contributed to by one and the same crystal at a fixed moment. In the latter case WEGENER (20) — who was the first to occupy himself with the complex halos — speaks of "verschwisterte Halos". All the other phenomena that make their appearance simultaneously, he calls "vergesellschaftet". I myself have pointed out afterwards (17, p. 66) that WEGENER's "vergesellschaftete Halos" should be divided in their turn into two groups:

1. phenomena arising in the same cloud;
2. phenomena that find their origin in different Ci or Ci-St clouds present at the same time.

As a classification into two, I should therefore prefer a group A of phenomena in one cloud ("verschwistert" in a wider sense) and a group B of phenomena in different clouds ("vergesellschaftet" in a narrower sense). In what follows here, an attempt is made to account for the way in which the phenomena of group A can be connected together.

On an earlier occasion I made an attempt to find a solution of this problem of the complex halos (16, p. 72; 17, p. 65). I started from the supposition that in one ice-cloud, arising as it does under very definite circumstances, a wide diversity is not possible either in the shape of the crystals, which is dependent on the temperature, or in their size, which is related with the vapour tension. From this there ensues amongst others that the simultaneous presence of the two fundamental shapes of ice crystals — the *plates* falling with vertical principal axes, and the

rods falling with horizontal principal axes — is not probable. Basing myself on DOBROWOLSKI's experiences (6), I drew, however, attention to the fact that, in consequence of the difference in size of their central cavity, some of the rods of the most prevalent form, i.e. the hemimorphous form, will fall with vertical, others with horizontal principal axes, (Fig. 1).

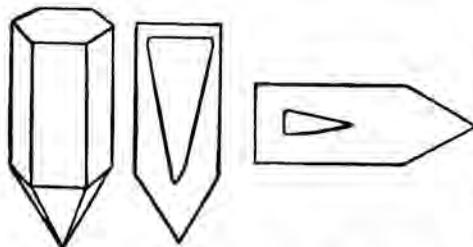


Fig. 1. Hemimorphous ice prisms.

Accordingly, if a cloud consists merely of hemimorphous prisms, a pretty complex halo can already appear in this way:

1. Ordinary ring caused by the practically never lacking not specially directed crystals;
2. A. Parhelia
B. Circumzenithal arc } caused by crystals with vertical principal axes.
3. Upper and lower tangent arc, due to crystals with horizontal principal axes.

Theoretically the following phenomena might be added to this:

- large ring ("verschwistert" with 1);
- arcs of Lowitz ("verschwistert" with 2);
- parhelic ring ("verschwistert" with 2 or 3);
- light column ("verschwistert" with 2 or 3);
- lateral tangent arcs to the large ring ("verschwistert" with 3).

From the fact, however, that these phenomena are much rarer than the first mentioned, there follows that they evidently require still other circumstances than those under which the general form of a complex halo so often occurs.

In my earlier publications (16, 17) I have not occupied myself with the very rare halo forms; HASTINGS, however, has tried to draw up a theory (11), in which also the possibility of their presence was taken into consideration. I do not, however, consider his attempt as successful:

1. because he uses for his explanation at the same time the two fundamental forms: plates and rods;

2. because for some of the phenomena he does not only require an orientation in a definite plane of the crystallographic principal axis, but also a fixed position for definite lateral faces.

However this may be, it remains necessary, in an explanation of complex halos, to take the less usual phenomena into account as well. The question then arises: under what circumstances can hemimorphous iceprisms cause, besides the more common phenomena, the rarer ones also? To these belong not only the above-mentioned phenomena, but also the *anthelion* and the *parantheta*, of which it is not even certain with what other halo forms they are "verschwistert". Before proceeding

to the discussion of this question, it is desirable to give a summary of the explanations of the anthelion c. a. published so far.

2. *The Anthelion and the Paranthelia.*

BRAVAIS (5, p. 189, cf. also 14, p. 424), who imagined the crystals, when falling, to orientate themselves in such a way that they meet with the least resistance, explained the anthelion by the aid of double interior total reflection against planes making an angle of 90° with each other (Fig. 2) and which are in a vertical position. This theory is untenable since it has become known that ice plates fall with their bases *horizontal*, and ice rods — supposing they do not rotate about their principal axes in falling — will not have an edge downward, but a side face, in consequence of which the side face corresponding with A of Fig. 2 will not be vertical.

Of the explanations of the paranthelia before BESSON's, that of SORET (cf. 14, p. 428) was considered as the least artificial (Fig. 3). Several

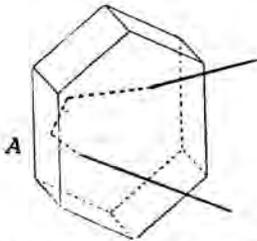


Fig. 2. BRAVAIS' explanation of the anthelion. (According to BESSON).

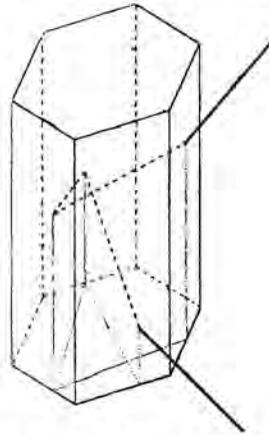


Fig. 3. SORET's explanation of the paranthelia. (According to BESSON).

objections, however, were advanced against it by BESSON (2, p. 77), and EXNER (14, p. 428).

As a substitute for the earlier theories BESSON (1; 2 p. 80) has given an explanation, according to which the anthelion and the paranthelia are due to aggregates of crystals. To the figures 4 and 5 nothing need be added to make it understood that the crystal faces which act as "double mirrors", ¹⁾ at an angle of 90° always change the direction of the horizontal projection of the rays 180° , and at an angle of 120° give it a deviation of 120° .

By the aid of the aggregates of figures 4 and 5 the phenomena in question are therefore, "easily" explained.

¹⁾ Mirrors including an angle α , reflect the light in a direction deviating $360^\circ - 2\alpha$ from the original, independent from the angle of incidence. *German*: Winkelspiegel.

Now the question is: 1. do the postulated aggregations occur in reality, and even in large quantities? 2. do they bear their reflecting side faces really vertically?

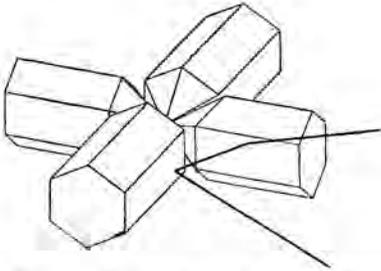


Fig. 4. BESSON's explanation of the anthelion. (According to BESSON).

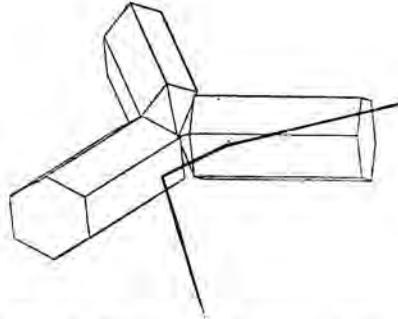


Fig. 5. BESSON's explanation of the paranthelia. (According to BESSON).

In answer to the first question it may be said that the very basis on which BESSON founds his theory is DOBROWOLSKI's experience (6) that the hemimorphous ice rods very often form radiary aggregates by uniting their tips. In 1903 already, DOBROWOLSKI gave a number of drawings of such aggregates in his extensive publication, both of completed ones and of those in *statu nascendi* (6, p. 32–37).

Not only these aggregates of 3 and 4 arise by union of hemimorphous ice rods. DOBROWOLSKI showed that also the so-called holohedric prisms have, after all, been formed by two joined hemimorphous ones. In 1916 DOBROWOLSKI (7) was able to publish micro-photographs of all these forms, made by F. HALLBERG. Four of these photos are reproduced in the figures 6–9¹⁾.

The answer to the second question cannot be given as yet with certainty. As may be seen from the photo, it is not to be avoided that the edges of the crystals soon melt off during the observation, so that the exact position of the faces cannot be ascertained. BESSON (4, p. 379), however, points out that from the fact that in the combinations of 2 the faces always lie exactly in each other's prolongation (twisted specimens have never been observed), it may be concluded that also with combinations of 3 or 4, the mutual position of the component crystals shows great regularity. In this case they would both turn either a face or an edge towards each other. If it is a face, they act as "double mirrors" with vertical planes.

In his treatise of 1923 BESSON complains that his theory of the anthelion c.a., published as early as 1907, has not been favourably received. "It has been thought improbable that shapes so complicated

¹⁾ I am greatly indebted to the Société Astronomique de France for the loan of the clichés and I gladly express my thanks to Dr. BESSON for his kind assistance.

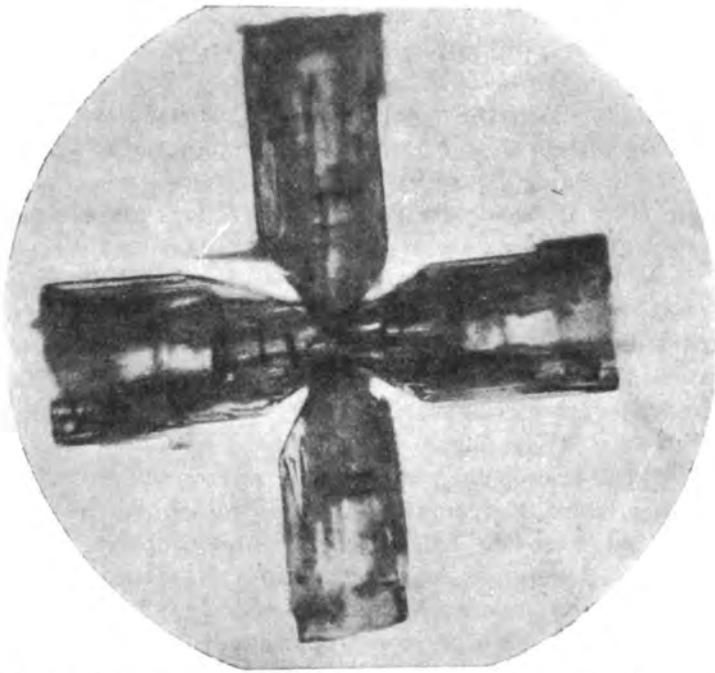


Fig. 6. Aggregate of 4 hemimorphous ice prisms (Photo F. HALLBERG).
(Cliché of the Société Astronomique de France).

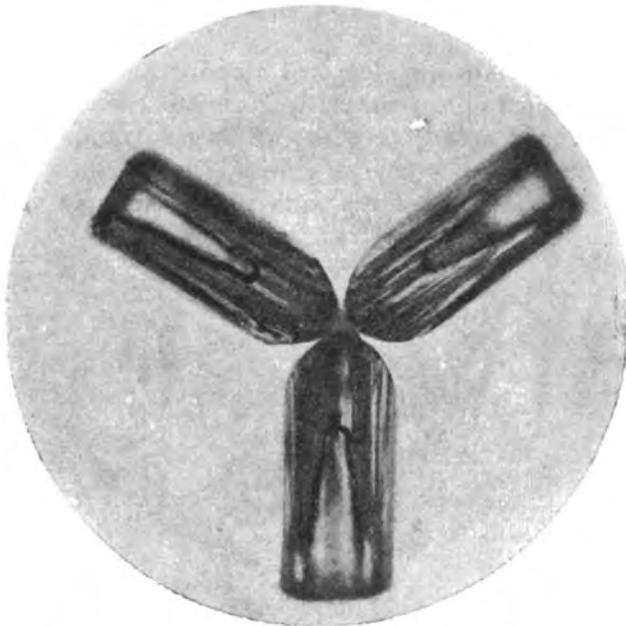


Fig. 7. Aggregate of 3 hemimorphous ice prisms (Photo F. HALLBERG).
(Cliché of the Société Astronomique de France).

could exist in the atmosphere in quantities large enough to give rise to a luminous phenomenon". The microphotos of HALLBERG reproduced

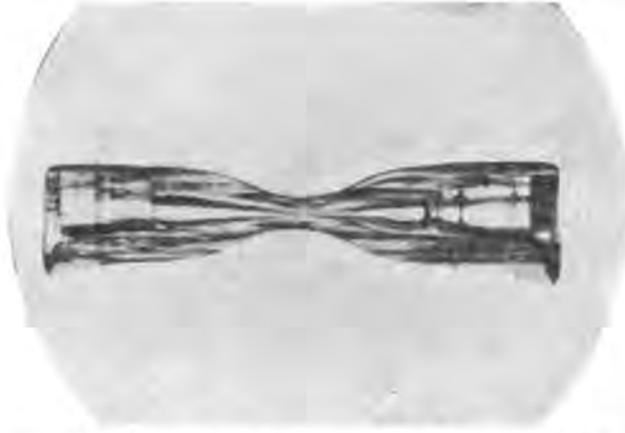


Fig. 8. Aggregate of 2 hemimorphous ice prisms in statu nascendi (Photo F. HALLBERG)
(Cliché of the Société Astronomique de France).

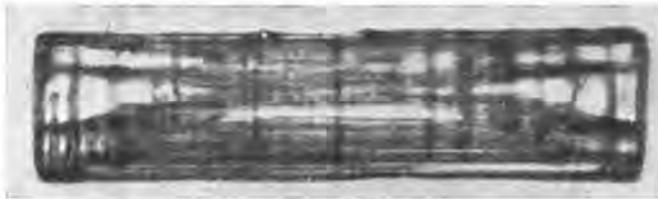


Fig. 9. Aggregate of 2 hemimorphous ice prisms (holohedric prism). (Photo F. HALLBERG).
(Cliché of the Société Astronomique de France).

here show irrefutably that radiary aggregates of ice crystals exist and are well defined crystalline forms.

It seems to me that the complex halo, observed by me at Amsterdam on December 23rd 1925, and described in the March number 1926 of "Hemel en Dampkring" (19), supplies an important support to BESSON's theory. At the same time the explanation of complex halos can get nearer to a satisfactory solution.

3. The Halo of Dec. 23, 1925.

A. The following phenomena were observed:

- | | |
|---------------------------------|--------------------------------------|
| 1. ordinary ring | faint |
| 2. upper tangent arc to same | exceedingly bright and extensive |
| 3. parhelia with arcs of LOWITZ | bright |
| 4. large ring | relatively very bright and extensive |
| 5. upper tangent arc to same | relatively faint |
| 6. anthelion | faint uncoloured little column. |

N^o. 1—5 are represented diagrammatically in Fig. 10, which has been borrowed from the above-mentioned article in "Hemel en Dampkring", to which we refer for the description of the course of the phenomenon.¹⁾

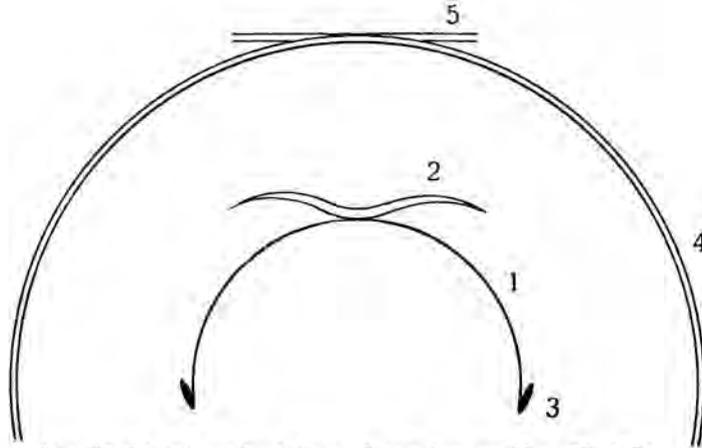


Fig. 10. Sketch of the halo at Amsterdam on Dec. 23, 1925.

1 = ordinary ring; 2 = upper tangent arc; 3 = parhelion; 4 = large ring;
5 = circumzenithal arc. The anthelion cannot be shown in this way of representation. (Cliché "Hemel en Dampkring").

Two things were particularly noteworthy in this halo: the strong development of the large ring and the presence of the anthelion in the absence of a parhelic ring. It should further be pointed out that of the other phenomena the upper tangent arc was the most intense, the small ring remaining faint.

B. The explanation of the large ring has only recently been made a point of closer investigation. After it had already been recognised as the equivalent of the ordinary ring for optical prisms of 90° by CAVENDISH (cf. 5, p. 79), little attention was further devoted to it, since the prisms in question were not to have any definite orientation. In my preceding publication I, accordingly, considered the large ring only from this point of view.

HASTINGS (11, p. 326), however, has pointed out that the possibility exists of an optical phenomenon which, whilst practically not to be distinguished from the large ring, yet arises in a different way, viz. in ice-prisms which lie almost or entirely horizontally. His explanation is very closely allied to that which BRAVAIS (5, p. 121, compare also 2, p. 72 and 13, p. 359; it is greatly to be regretted that EXNER in the second edition of PERNTNER'S handbook has omitted BRAVAIS' explanation) has given for the lateral tangent arcs to the large ring. HASTINGS points out that prisms in a horizontal plane have already refracting edges of 90° in all possible directions, so that with comparatively small

¹⁾ See also "Zur Erklärung der komplizierten Halos", published afterwards in the Meteorologische Zeitschrift, Nov. 1926, p. 411.

deviations from the strictly horizontal position of the principal axis, many of the optical prisms can already get a position of minimum deviation.

BRAVAIS' theory of the lateral tangent arcs to the large ring is in conformity with his explanation of the circumzenithal arc: to the latter all the prisms with an upturned basal surface contribute, thus giving a circle with the zenith as its centre. In the same way all the horizontal prisms with basal surfaces turned in one direction give a circle with its centre in this direction. There are, however, in the horizontal plane an infinite number of directions in which the prisms may be directed. Of the infinite number of circles which would theoretically be formed in this way, a certain number lie so close to each other that together and under favourable circumstances they produce a visible arc. Thus the lower lateral tangent arcs are formed by rays of light entering through a basal surface and leaving the crystal through a side face, and the upper lateral tangent arcs (sun's altitude below 30°) when the rays of light pass the faces in inverse order. These latter must turn their convex side towards the sun, and for this very reason already they coincide mainly with the large ring. In BRAVAIS' time they had not yet been observed. In his dissertation BESSON (2, p. 72) mentions two cases observed by himself and his colleague DUTHEIL. Afterwards he has recorded another very positive observation of these arcs (3, p. 5). I myself had for a long time already, considered the possibility that a number of cases, in which mention was made of the large ring, would really refer to the arcs in question. In my publication of 1919 (16, p. 31) I already alluded to this with reference to the halo of Dec. 29, 1914 (Fig. 11; taken from 8). In a publication in "Hemel en

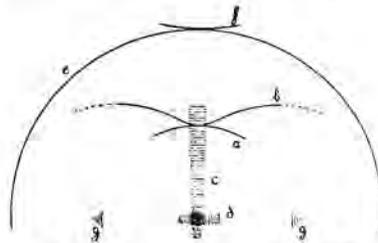


Fig. 11. Halo of Dec. 29, 1914.

a = ordinary ring; *b* = upper tangent arc; *c* = light column; *d* = parhelic ring;
e = large ring; *f* = circumzenithal arc; *g* = parhelion. (The parhelia and the
 ordinary ring were only seen by some of the observers). (Cliché K. N. M. I.)

Dampkring" (15, p. 37) I ascribed the large ring on Jan. 8 1919 (cf. 10, p. 51) to intensification caused by the upper lateral tangent arc. On Febr. 21, 1918 (cf. 9, p. 155), and on March 28, 1920 (cf. 18, p. 55) intensified portions of the large ring were clearly observed by me; in the latter case the height of the most intense part was estimated, and found to correspond to the height required by BRAVAIS'

theory. In most of these cases (with the exception of Jan. 8, 1919) the upper tangent arc to the ordinary ring was particularly strongly developed, both as regards intensity and extension. This suggests the great prevalence of *horizontal ice rods*, which, as said before, would have to be held responsible for the upper (and lower) lateral tangent arcs to the large ring.

Hence the conclusion should be drawn in my opinion that in a number of cases in which (with solar altitudes $< 20^\circ$) the large ring was observed in complex halos — accompanied by an intense tangent arc to the ordinary ring, this ring itself being faint — this large ring was caused by almost horizontal ice rods, and has to be explained by the theory of BRAVAIS for the upper lateral tangent arcs, or by that of HASTINGS. I cannot, however, agree with HASTINGS that this explanation of the large ring can replace the earlier one. The observations of the large ring at the same time with an intensive ordinary ring which I published before (16, p. 31, 36) can only be explained in the old way.

These considerations may elucidate to some extent, how, given the practical impossibility of seeing the large ring detached from the tangent arcs under consideration, these tangent arcs, which were predicted by the ingenious BRAVAIS, could remain considered as non-existent for so long a time.

Now recurring to the halo of Dec. 23 1925, we arrive at the conclusion that both the very intensive and extensive upper tangent arc to the ordinary ring and the intensive large ring point to a particular development of the horizontal ice rods.

C. Besides the striking brightness of the large ring and upper tangent arc, it was the presence of the anthelion *without parhelic ring* that made the halo of Dec. 23, 1925 so remarkable. We might also give the following account of this phenomenon: Among the infinite number of possible directions of reflection, the one with an angular difference of 180° was the privileged one. As we have seen, according to BESSON's theory there is question of such a privilege of the direction of 180° in the occurrence of combinations of 4 ice prisms fused together with their points and falling with vertical side faces: The "double mirrors" can be turned over a considerable angle in the horizontal plane, and all the same reflect the light from the direction of the anthelion. Every other reflecting vertical face will at the turning, send out the light every time in another direction.

These aggregates may, further, be the crystals that fall most undisturbedly in horizontal position: The air having an opportunity of escaping between the arms, the "Schaukeln" to which the ice plates are liable, need not occur here. By this circumstance it is rendered possible that the anthelion appears exactly at the sun's altitude.

If we now accept BESSON's theory, we must assume that with the

halos under consideration the particular circumstance presents itself, that the ice prisms have combined in large numbers to \dagger shaped aggregates. When the circumstances of temperature etc. in that cloud had to give rise to the occurrence of aggregates, the chance was still greater that also the combination of 2 would occur, i.e. the so called holohedric prisms. That these were actually present in great numbers, is rendered probable by the strong development of upper tangent arc and large ring; in consequence of the greater stability in the horizontal orientation which the holohedric prisms show compared with the hemimorphous ones, the more common phenomenon (in casu the upper tangent arc) becomes exceedingly intense and extensive, the necessarily less intense and consequently rarer phenomenon (in this case the upper lateral tangent arcs to the large ring) coming in the region of visibility. Another factor that can contribute to the greater intensity is the double length of the holohedric prisms. If once such a crystal has the right position to give a definite point of the halo, twice as much light passes as a hemimorphous prism would have given in this position.

4. The Very Complex Halos.

A. It seems to me that from the above a far-reaching conclusion may be drawn with reference to the explanation of complex halos:

Under rare circumstances the hemimorphous ice prisms — which already under ordinary circumstances are able to produce several halo-phenomena at the same time (cf. p. 173) — begin to join to combinations of 2, 3, and 4. If this is the case "the" rare halo-phenomena may be added to a halo. These are: the upper and lower lateral tangent arcs to the large ring, the paranthelia and the anthelion.

B. Besides in the observation of Dec, 23, 1925 and the other halos already mentioned, this hypothesis finds a firm support in the statistical data published by MEYER in 1925 (12, p. 13, cf. also 21, p. 191).

The part of the "Table of Relationship" referring to the 3 rare phenomena in question, is reprinted here. Every value has been calculated

by MEYER according to the formula $\frac{G^2}{N_1 N_2}$, in which G is the number

	Ordinary ring	Large ring	Upper and lower tangent arcs to the large ring	Lower lateral tangent arcs to the large ring	Parhelia	Circumzenithal arc	Light column	Parhelic ring	Paranthelia	Anthelion	Arcs through anthelion
Lower lateral tangent arcs to the large ring	1	2	2	—	3	0	0	7	24	8	13
Paranthelia	3	4	3	24	4	2	0	14	—	4	4
Anthelion	1	4	2	8	2	1	0	9	4	—	40

of observations in which the two phenomena the relationship of which is to be determined, occurred simultaneously, and N_1 and N_2 the total number of observations of each separately. The values thus obtained have been multiplied by 100.

Very recently in his discussion of MEYER's work in the *Meteor. Zeitschrift* (21) WEGENER pointed out that the anthelion according to MEYER's table shows close relationship to the lateral tangent arcs to the large ring. To this may be added that it is also much more closely allied to the large ring itself than to the ordinary ring. In the light of our hypothesis these relationships become very comprehensible. The relation to the large ring suggests that, among the circles observed simultaneously with the anthelion, there will probably be some not well distinguished upper lateral tangent arcs. That also the paranthelia (3-aggregates) present a very close relationship to the lower lateral tangent arcs is a surprising result. In contrast with WEGENER, who questions the validity of the theory of the paranthelia — because they are so closely allied to the halo-phenomena caused by refracting edges of 90° — I arrive at the conclusion that the theory (but then that of BESSON) is certainly valid. The relationship is, however, not to be looked for in the angles of the crystals, *but in the appearance of the phenomenon of "aggregation"*.

And this again makes it clear how the study of the halo-phenomena might lead to conclusions about the state of the atmosphere.

Postscript during the correction of the Dutch edition. After this communication had been presented to the Academy there appeared a publication by WEGENER (22), in which inter alia, a new theory is given of the arcs through the anthelion not discussed here by me. The anthelion itself is, then, considered as the luminous nodal point of the arcs. Though WEGENER's theory seems plausible for the explanation of the arcs in question, it cannot be accepted as the only correct one for the anthelion. This appears already directly from the fact that the anthelion of Dec. 23 1925 was observed with a solar altitude $< 14^\circ$ [i.e. between $12^\circ 15'$ (10.40 a.m.) and $13^\circ 21'$ (11.07 a.m.)], whereas the lowest limit of visibility according to WEGENER lies at a solar altitude of $14^\circ.1$. Hence the explanation of BESSON, ignored by WEGENER, will have to be maintained at least by the side of the new theory.

LITERATURE.

1. BESSON, L. (1907). Nouvelle Théorie de l'Anthélie, des Paranthélie, etc. *Comptes rendus de l'Acad. des Sciences*. Paris, CXLIV, p. 1190.
2. BESSON, L. (1909). Sur la Théorie des Halos. Dissertation. Paris.
3. BESSON, L. (1913). Notices sur divers phénomènes optiques observés à Paris. *Annales de l'Observatoire de Montsouris*. XI.
4. BESSON, L. (1923). L'Anthélie, les Paranthélie et les Halos blancs. *L'Astronomie*, XXXVII, p. 371.
5. BRAVAIS, A. (1847). Mémoire sur les Halos. *Journal de l'École polytechnique Paris*, 31e Cahier.
6. DOBROWOLSKI, A. (1903). La Neige et le Givre. *Resultats du Voyage du S. Y. Belgica (Météorologie)* Anvers.

7. DOBROWOLSKI, A. (1916). Les cristaux de glace. Arkiv for kemi mineralogi och geologi, VI, 7. Stockholm (cited after BESSON).
 8. EVERDINGEN, E. VAN. (1915). De halo van 29 Dec. 1914. Hemel en Dampkring **12**, p. 130 (cf. also: Onweders, opt. verschijnselen enz. in Nederland in 1914, p. 113).
 9. EVERDINGEN, E. VAN. (1918). Een feestdag voor Halowaarnemers (21 Febr. 1918). H. en D. **15**, p. 155 (cf. also: Onweders, opt. verschijnselen enz. in Nederland in 1918, p. 59).
 10. EVERDINGEN, E. VAN. (1921). Onweders, optische verschijnselen enz. in Nederland in 1919 (Halo van 8 Jan. 1919, p. 51, halo van 24 Maart 1919, p. 55).
 11. HASTINGS, CH. S. (1920). A General Theory of Halos. Monthly Weather Review, June 1920, p. 322—330.
 12. MEYER, R. (1925). Haloerscheinungen. Abh. des Herder-Institutes zu Riga. Band I, no. 5.
 13. PERNTER, J. M. (1906). Meteorologische Optik. 3. Abschnitt. Wien und Leipzig.
 14. PERNTER-EXNER. (1922). Meteorologische Optik. 2. Auflage. Wien und Leipzig.
 15. PINKHOF, M. (1919a). Naar aanleiding van de halo's van 8 Jan. en 24 Maart 1919. Hemel en Dampkring **17**, p. 36.
 16. PINKHOF, M. (1919b). Bijdrage tot de theorie der haloverschijnselen. Verhandelingen der Kon. Ak. v. Wetensch. te Amsterdam. Eerste Sectie, deel XIII, No. 1.
 17. PINKHOF, M. (1920a). Beiträge zur Halotheorie. Meteor. Zeitschr. XXXVII. Heft 3/4, p. 60.
 18. PINKHOF, M. (1920b). De halo's van Maart 1920. Hemel en Dampkring, **18**, p. 43 en 53. (Cf. also: Onweders, optische verschijnselen enz. in Nederland in 1920, p. 58).
 19. PINKHOF, M. (1926). De Halo van 23 Dec. 1925. Hemel en Dampkring **24**, p. 86.
 20. WEGENER, A. (1915). Verschwisterte und vergesellschaftete Halos. Meteor. Zeitschrift XXXII, Heft 12, p. 550.
 21. WEGENER, A. (1926a). Besprechung von R. MEYER, Haloerscheinungen. Meteor. Zeitschrift XLIII, Heft 5, p. 190.
 22. WEGENER, A. (1926b). Theorie der Haupthalos. Aus dem Archiv der deutschen Seewarte XLIII, N^o. 2, Hamburg.
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Physiology. — *Radiation-substances and Cardiac Hormones.* By Prof.
H. ZWAARDEMAKER.

(Communicated at the meeting of November 27, 1926.)

The potassium-salt that is invariably present in the circulation fluid of animal organisms, is an indispensable constituent. For a number of systems, however, — I believe for all automatic systems — it can be replaced by any radio equivalent dose (1916¹). It is particularly this radio equivalence (1 micro-erg per second, and per grm of heart-muscle) that in the last two or three years has been demonstrated by detailed ash-determinations²). The radio-active element (Potassium, Rubidium, Thorium, Uranium, Ionium, Radium, Radon) can also be substituted by corpuscular radiation, either alpha-, or beta. *Not*, however, by rays of light or Röntgen-rays, in the intensities used by us. In such experiments we started from isolated organs, from which $\frac{1}{3}$ of the original potassium had been washed out³). The fittest object is the heart of cold-blooded animals, which stops beating as soon as the removal of the potassium has proceeded so far. The standstill is definite, provided the perfusion with potassium-free Ringer's solution be continued uninterruptedly. The beta-rays of mesothorium or radium⁴), the alpha-rays of polonium⁵), likewise cathode-rays⁶), evoke in 90 % of the hearts thus radiated, new regular pulsations that can persist for hours⁷).

In all these cases, a threshold-value is found.

There is also a latent period. We were surprised to see that in cases of feeble radioactivity these latencies sometimes last for hours. The latter circumstance leads us to suppose that there is some indirect causal connection between the new radioactivity and recovery. It is reasonable to

¹) H. ZWAARDEMAKER, Verslag Kon. Akad. v. Wetensch. Amsterdam, 30 Sept. 1916; Proc. Vol. 19, p. 633; PFLÜGER's Arch. Bd. 173, p. 28, 1918; Ergebnisse der Physiol. Bd. 19, p. 326, 1921; PFLÜGER's Arch. Bd. 205, p. 20, 1924; Q. J. of exp. Physiol. Vol. 14, p. 338, 1924, etc.

²) G. M. STREEF. Thesis. Utrecht 1926.

³) A. J. CLARK J. of Pharm. a. exp. Therapeut., Vol. 18, p. 423, 1922; H. ZEEHUISEN, Ned. Tijdsch. v. Geneesk. 1926, I, p. 2123.

⁴) H. ZWAARDEMAKER, C. E. BENJAMINS, T. P. FEENSTRA, Ned. Tijd. v. Geneesk. 1916, II, p. 1923; H. ZWAARDEMAKER, Verslag Kon. Akad. v. Wetensch. Amsterdam, 31 Maart 1917; Proc. Vol. 19, p. 1161.

⁵) H. ZWAARDEMAKER en G. GRIJNS, Arch. néerl. de Physiol., t. 2, p. 500, 1918; Stralenterapie Bd. 21, p. 215, 1925; PFLÜGER's Arch. Bd. 213, p. 757, 1926.

⁶) H. ZWAARDEMAKER en T. P. FEENSTRA, Verslag Kon. Akad. v. Wetensch. Amsterdam, 27, Juni 1925; Proc. Vol. 28, p. 650. (or weak X-rays?)

⁷) H. ZWAARDEMAKER, PFLÜGER's Arch. Bd. 213, p. 763n. 1926.

assume that stimulating chemical substances are formed *by the radiation*, which substances might be called "automatins", (a term introduced by BENJAMINS already a twelvemonth ago), and which in their turn furnish the automatically moving organ with the so-called primary stimulus. Our "automatins" are purely hypothetical substances, they have never been seen in fact. Moreover, the existence of such substances was suggested by recent researches¹⁾, demonstrating the formation of so-called "ions", or carriers, also in liquids. The "ions" generated by the radiation in the tissue (some of which reconstruct the complete molecules from which they have originated, while others remain separate) are, as we believe, our hypothetical "automatins", or give them origin.

If we assume the automatins to be dialysable, they will be regularly washed out of the beating heart, whose lacunae are emptied out at every contraction, and replenished at every diastole. When, with potassium or with its substitutes, their formation remains too weak, it is easy to find an explanation for the groupings and the brief standstills. But it is not so easy to explain the very late after-effects we are confronted with in half the radiation-experiments. They almost make us believe in the cooperation of an autocatalysis. We will picture a case in which the pulsations were resumed in consequence of the action of cathode-rays (or weak X-rays) (Fig. 1). The later group lasted here about 24 hours (Fig. 2).

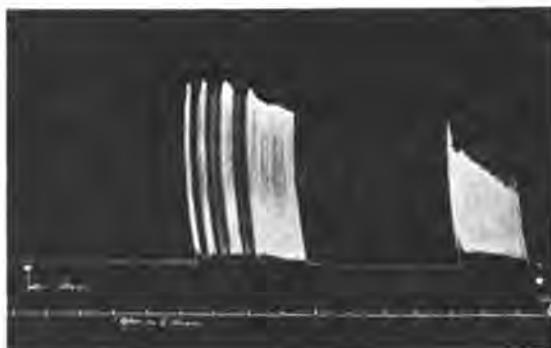


Fig. 1.

Of course, after the organ had ceased to contract through deprival of potassium we waited some time. There is a definite correlation between such a standstill, before starting the radiation and the latent period between the commencement of the radiation and the moment of revival. Those periods generally keep pace with each other. Thus the times before and after the beginning of the radiation present a sort of symmetry, with weak, though sufficient radiation (1.6 mgm Ra in 1 cm² of enamel) the latency

¹⁾ A. KAILAN, Liebig's Ann. Bd. 433, p. 272, 1923.

is somewhat longer than the time of waiting, whereas with somewhat intenser radiation (1.6 mgm Ra + 3.3 mgm from a distance of 0.5 cm)

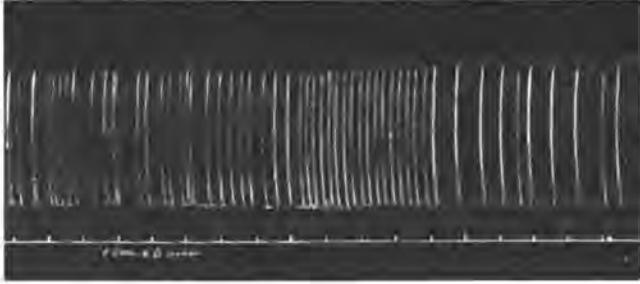


Fig. 2.

the periods are equal. This (it should be understood) is an average result of a large number of experiments, for the varieties are many.

The duration of the latent period depends not only on the time of waiting, but also on the dosis of the radiation. When, in former times we fixed it at f.i. 2 mgm-hours, the latent period amounted to only a few minutes, half-an-hour, or an hour; now that we have been working of late with threshold values of f.i. 1 mgm-hour, the revival will often manifest itself after hours.

The connection between the interval of waiting and latency, as well as that between the intensity of radiation and latency, lends support to the probability of existing intermediate chemical substances, which first have to accumulate and to penetrate as far as the receptors in the nodal tissue, before contractions can be evoked in the sinus or the atrioventricular node, which can propagate over the whole heart-muscle.

Our hypothesis that automatins originate *through radiation*, would have remained a dead letter, because the substances are not perceptible, if not new, unexpected investigations by others had offered a wider field of inquiry.

It so happened, namely, that the same conception was brought forward from two entirely different quarters.

L. HABERLANDT of Innsbruck has published a series of researches on a hormone of the cardiac movement ¹⁾, showing that in the sinus and in the atrioventricular node of the frog's heart there is a substance, extractable by means of Ringer's solution resp. absolute alcohol, which when administered to a ventricle which is badly beating, or standing still, heightens an existing frequency, improves a faulty rhythm, and restores the cardiac action. This substance is dialysable, insoluble in ether, and resistant to a temperature of 100°.

¹⁾ L. HABERLANDT. *Ergeb. der Physiol.* Bd. 25, p. 86, 1926.

At the same time J. DEMOOR ¹⁾ of Brussels made experiments, with which from the nodal tissue of mammalia, aqueous or alcoholic extracts were obtained, which, when administered to the isolated left auricle of the rabbit's heart, made it beat rhythmically and regularly, which does not happen without administering these extracts. The active substances, concerned here, are dialysable, they do not resist a temperature higher than 70° C., and are destroyed in a diluted solution within 24 hours.

Do these researches bear any relation to mine? Prof. NOYONS broached this subject and thereby gave a welcome impetus to the studies that occupy us. Moreover, Prof. HABERLANDT kindly imparted to me his view of the nature of his substances. He also suspected the possibility of a relation between our results. A possibility or a probability?

A very simple experimentum crucis appeared, by which this question might be set at rest. At once I availed myself of it.

A frog's heart on Kronecker's cannula is brought to a standstill through deprival of potassium. After a quarter of an hour I placed 1.6 mgm of Ra in 1 cm² enamel close to the ventricle, which is exposed to full radiation from that moment. After a latent period of half an hour the pulsations recommence.



Fig. 3.

During the pulsations, thus produced, we collect the out-flowing liquid, after having diminished the rate of perfusion (without interrupting it) as much as possible (underlined in the figure). With this liquid we perform HABERLANDT'S experiment on another ventricle on Straub's cannula. The ventricle that comes to a standstill on potassium-free Ringer's mixture, commences with grouped pulsations, and soon attains a regular-rhythm of about half an hour's duration.

We were thus able to perform quite a series of experiments. Not one of them turned out negative, while in control-tests a Straub-heart, arrested by deprival of potassium did not resume its beats on repeated renewal of its contents with ordinary potassium-free liquid.

¹⁾ J. DEMOOR, Arch. Int. de physiol., 20—23; résumé in: les transmissions humorales, Bruxelles médical, 13 Sept. 1925.

Besides by radiation with the beta-rays of Radium we have revived hearts also by alpha-rays of Polonium (from three sides, for the penetrating power

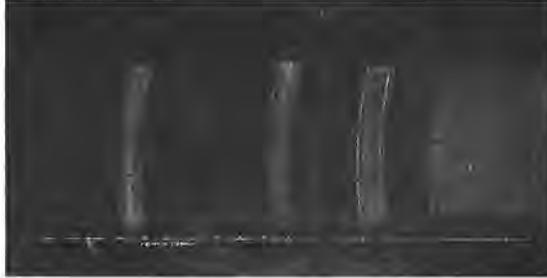


Fig. 4.

of polonium-rays is low (in water 32 microns). Well then, also the potassium-free Ringer's solution, that passes slowly through such a Polonium-heart, makes a heart which has ceased to contract (on a Straub-cannula) recover its beats in a short time (a few minutes) ¹⁾.

It is a fact, then, that automatins are formed as well by alpha-, as by beta-rays. Are they homogeneous, or antagonistic, as the physiological radio-antagonism leads us to suspect? ²⁾ This problem is being investigated. I may be allowed to confine myself for the present to the hypothesis that, on the basis of our actual experience, the beta-radiation substances are to be considered identical with HABERLANDT's and DEMOOR's hormones.

¹⁾ Future critics of experiments like those discussed in this paper, should remember that the revival through automatins after repeated renewal of the contents, takes place with a latency of a few minutes. Recovery after half an hour, without renewed filling, would not, of itself, prove anything, as this is not surprising in a Straub-heart, left to itself, in which sooner or later potassium is always set free.

²⁾ I refer to my contribution to GRIESBACH'S "Jubilee volume" Giessen, 1925, p. 135.

the light will be unequally distributed in consequence of the absorption of the light in the vessel, and it will diminish from I_0 to $I_0 e^{-\kappa c x}$, dependent on the absorption coefficient of the ketone (κ) and on its concentration (c) over the cross-section of the vessel (x). Accordingly at the moment of the illumination there will be a number of active molecules, the concentration of which decreases from the light side to the dark side according to this law.

Now the question may be raised, what happens immediately after this activation? If these molecules act as perfect mirrors, i.e. if they send back the light energy with the same velocity, we could only speak of reflecting or scattering, and the concentration of the active molecules would remain insignificant. The fact that in the absence of appreciable reflection and scattering a vigorous absorption is observed, suggests that the activated molecules have time to transfer their activity to themselves or to other molecules.

A. If we assume in the first place that this time is very short, they will only be able to bring about this transference at the place itself or very close to the place of their activation. The concentration in an arbitrary layer of light in active benzophenone molecules may then be calculated as follows. By the illumination a number of active ketone molecules is formed proportional to the quantity of light present there. If S is the illuminated surface, the quantity is:

$$= K_1 S I_0 e^{-\kappa c x} \cdot \kappa c dx.$$

Part of these will be converted; if we assume this to be a monomolecular reaction (that we may, therefore, disregard collisions of equal molecules) the number of disappearing molecules = $K_2 c_{\text{active-ketone}} S dx$.

A stationary state will result, when these two reactions are in equilibrium; from this ensues that in every layer of light the concentration of the active ketone mol. = $K_1 I_0 e^{-\kappa c x} \kappa c$. These active ketone molecules now get secondarily in contact with alcohol molecules, in which the ternary photoactive compound (see further) originates:

$$\frac{d[\text{tern. comp.}]}{dt} = K_2 c_{\text{act. ket.}} \cdot c_{\text{alcohol}} = K_2 K_1^2 I_0^2 e^{-2\kappa c x} \cdot \kappa^2 c^2 \cdot c_{\text{alcohol}}$$

From this the total quantity of this compound in the layer can be calculated by multiplication by $S dx$, and then by integration the total quantity in the vessel is:

$$\int_0^x S K_2 K_1^2 I_0^2 e^{-2\kappa c x} \kappa^2 c^2 c_{\text{alcohol}} \cdot dx = \frac{1}{2} S K_2 K_1^2 I_0^2 (1 - e^{-2\kappa c x}) \kappa c c_{\text{alcohol}}$$

$$\text{The concrete increase} = \frac{d[\text{comp.}]}{dt} = \frac{1}{2} \frac{K_2 K_1^2}{x} I_0^2 (1 - e^{-2\kappa c x}) \kappa c c_{\text{alcohol}}.$$

B. If we assume that the activated benzophenone molecules lose their activity comparatively slowly, so that in a thin layer and at not too

low temperature we may assume that they have time to spread evenly through the reaction space without having lost much of their activity, a stationary state will have established itself after a short space of time, which may be represented by:

$$\int_0^x K_1 S I_0 e^{-\nu cx} \cdot \nu c dx = K_1 I_0 (1 - e^{-\nu cx}) S = K_2 C_{\text{active mol.}} \cdot x \cdot S.$$

or

$$C_{\text{act. mol.}} = \frac{K_1}{K_2 \cdot x} I_0 (1 - e^{-\nu cx}).$$

If we, therefore, suppose a sufficient duration for the light-active molecules, and if we choose a small and equal thickness of the layer, since this is necessary for a sufficient absorption of oxygen, we may use for concentration of the active molecules the expression: $\frac{K}{x} I_0 (1 - e^{-\nu cx})$, and express the velocity reaction by:

$$\frac{d [\text{active comp.}]}{dt} = \frac{K^2 I_0^2}{x^2} (1 - e^{-\nu cx})^2 \cdot \text{Calcohol}.$$

Whatever image we may use here, always the square of the light intensity appears; a choice between the two views can only be made by the determination of the reaction velocity under different circumstances.

§ 2. *The photo-active region.* It had already been rendered probable by COHEN that the photo-active region lies in the extreme visible violet and that the ultra violet, for so far as the region of the mercury lines occurring in it is concerned, was inactive.

On absorption of the visible light by a solution of nitroso dimethylaniline, no reaction could be observed of this light in quartz tubes. For the rest there remained much uncertainty about the extension of the photo-active region.

One of us has defined this active area more closely with H. D. MULLER in the following way. We made use of another reaction, which, however, likewise ensues from the formation of the ternary compound: (photo-ketone)₂ alcohol.

If an alcoholic solution of $\text{HgCl}_2 + (\text{NH}_4)_2 \text{C}_2\text{O}_4$ is illuminated in the presence of the above-mentioned ketones, HgCl is deposited in the sunlight and in the light of the mercury lamp.

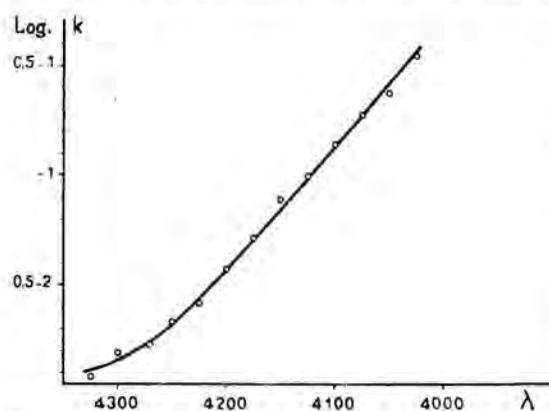
We then laid a series of capillary tubes filled with this solution, under the microscope, and illuminated them by means of a spectrum obtained by the aid of a strong PHILIPS microburner.

It could then clearly be observed that in the tube which was in the extreme violet between 400 and 410 $\mu\mu$, the first HgCl crystals appeared;

not until some days later were there also seen crystals on either side of this tube, both to the side of the red and to that of the ultra-violet in a greatly decreasing degree. These crystals being of about the same size and the tubes having the same diameter, their number was an approximate measure of the converted quantity of substance. Though this number decreases exceedingly rapidly, we may yet conclude that the photo-active region is not confined to a few light vibrations.

Then the question suggested itself if this region of photo-activity with a clearly-marked top coincided with a region of general absorption.

As the Technical University is not fully equipped with apparatus for accurate determinations, one of us has called in the hospitality of



the physical laboratory at Utrecht, which is under the direction of Prof. ORNSTEIN. Through the very kind assistance met with there, he has succeeded in defining the whole area of absorption of some ketones both in the visible light and the ultra violet part of the spectrum adjoining it.

In this there was not found the slightest indication of an absorption top in the violet part. We are here in the ascending part of the region of absorption, but with the three investigated ketones we are very far from the top, which lies in the ultra violet part of the spectrum, where we have not met with any reaction. The fact, indeed, that the photo-active region is narrowly limited indicates that it is quite surrounded by the region of total absorption.

We see that part of the light energy, and undoubtedly only a very small part, can be applied for this oxidation reaction, hence that the photo-chemical efficiency is small. It is certain that when the light is thought divided in separate vibrations, there are some among these vibrations that possess a very high efficiency; the more we are, however, struck then with the slight efficiency of the light vibrations from the rest of the absorption region.

If the photo-active region were confined to very few light vibrations, we might imagine the case that the photo absorption top was so sharp that it was not observed in the determination of the total absorption region. The experiments with the capillaries show, however, that the photo active region in question is not to exceedingly narrow.

It is possible that one single ray of light possesses an efficiency of 100%, and it would certainly be of great importance to refine the means of investigation to such an extent that this too could be found.

It follows from our investigations and those by COHEN that the tops of the photo-active regions for different ketones lie "in the neighbourhood" of the mercury lines in the violet, and that these regions certainly overlap for an important part, (see below). It is, however, a priori probable that the real top occupies a separate place in the spectrum for every ketone. For the rest the general absorption is very sensitive to admixtures; it was found by us that an alcoholic solution of the ketones yielded a spectrum which was turned more towards the red than a solution in petroleum ether.

When the question is now considered what happens with the rest of the absorbed energy, it is undoubtedly conceivable that *other chemical reactions* may be started by it, e.g.: $C_6H_5COC_6H_5 + C_2H_5OH \rightarrow C_6H_5COO_2H_5 + C_6H_6$, which then take place somewhere in the ultraviolet absorption region, etc. Under the circumstances chosen by us, the rest of this energy is probably entirely converted to heat.

With somewhat complicated molecules no theoretical light efficiency can in general be expected throughout the whole absorption region, nor is this, certainly, the case with the assimilation process. Here, too, the photochemical region is encompassed by the general absorption region of the chlorophyll. The maximum efficiency is not reached at the characteristic top in the red, (645—670 $\mu\mu$), it lies probably more in the yellow between 570—645 $\mu\mu$, hence here (O. WARBURG and E. NEGELEIN Z. phys. Ch. 102, 235 (1922)) past the top.

§ 3. Some mixtures of ketones in alcoholic solution were illuminated by W. D. COHEN in order to examine whether disturbances appeared then, and if so what disturbances. Because with somewhat considerable concentration of the ketone the reaction velocity seemed to be independent of this concentration, it was at first expected that the two reactions would not interfere with each other. This appeared to be by no means the case.

If we consider the velocity of formation of the pinakones:

$$-\frac{d[\text{ternary comp.}]}{dt} = +\frac{d[\text{pinakone}]}{dt} = \left[\frac{I_0 K}{x} (1 - e^{-x}) \right]^2_{\text{Alcohol}}$$

then with somewhat considerably concentration and with *perfect specific absorption* the increase of the concentration will really have little influence on the velocity of formation of each pinakone separately, as $(1 - e^{-x})$ rapidly approaches to the unit.

When they are mixed, however, and when the absorption regions coincide at the places of the photo-activity, the two ketones will divide this light according to their power of absorption and their concentration.

One ketone will partly be in the path of light of another ketone, so that we then might speak of an *interior filter action*. The law of this,

formulated by LUTHER and WEIGERT, Z. Phys. Ch. 53, 408 (1905), has been applied by one of us (LANGEDIJK) as follows:

If a vessel with two substances in the same solvent with the concentrations c_A and c_B is exposed to monochromatic light, the absorption coefficients being κ_A and κ_B , the intensity of the light after having traversed a layer dx is diminished by the amount:

$$\begin{aligned} -dI &= (\kappa_A c_A + \kappa_B c_B) I dx \\ \frac{dI}{I} &= -(\kappa_A c_A + \kappa_B c_B) dx \\ \ln I &= -(\kappa_A c_A + \kappa_B c_B) x + C \\ I &= C' \cdot e^{-(\kappa_A c_A + \kappa_B c_B) x}. \end{aligned}$$

For $x=0$ $I=I_0$ hence $C'=I_0$, from which follows:

$$I = I_0 e^{-(\kappa_A c_A + \kappa_B c_B) x}.$$

Then the total absorption, after having traversed a layer x , is then ¹⁾:

$$H = I_0 [1 - e^{-(\kappa_A c_A + \kappa_B c_B) x}].$$

If we further call H_A the quantity of light absorbed by A, and H_B that absorbed by B, the following equation is valid for a thin layer:

$$dH = dH_A + dH_B = -dI = (\kappa_A c_A + \kappa_B c_B) I \cdot dx$$

and we may think this distributed in this very thin layer as follows:

$$dH_A = \kappa_A c_A I dx \text{ and } dH_B = \kappa_B c_B I dx.$$

$$\text{Now } I dx = - \frac{dI}{\kappa_A c_A + \kappa_B c_B}$$

$$dH_A = - \frac{\kappa_A c_A}{\kappa_A c_A + \kappa_B c_B} dI \text{ and } dH_B = - \frac{\kappa_B c_B}{\kappa_A c_A + \kappa_B c_B} dI$$

If these values are integrated, we get as above:

$$H_A = \frac{\kappa_A c_A}{\kappa_A c_A + \kappa_B c_B} I_0 [1 - e^{-(\kappa_A c_A + \kappa_B c_B) x}]$$

$$H_B = \frac{\kappa_B c_B}{\kappa_A c_A + \kappa_B c_B} I_0 [1 - e^{-(\kappa_A c_A + \kappa_B c_B) x}].$$

If, therefore, there are some substances present in a solution, they divide the absorbed light of a definite wave-length — hence also the photo-active light — in proportion to the products of their concentrations and absorption coefficients.

Since with the reaction velocities of our ketone reductions the squares of this value come into play, it is clear why COHEN observed such strong

¹⁾ This derivation of the total absorption in the presence of two (and more) substances, has already been given by PESKOW, Journ. Russ. Chem. Ges. 47, 918(1915), Journ. Phys. Chem. 21, 382 (1917).

20 cc.) and illuminated for 30 hours in a light-thermostat with 30 volts and 3.5 ampères — the oxygen absorption being 0.012 cc. per minute. The alcohol distilled off exhibited *no rotation*. The experiment was repeated with methyl-ethyl carbinol. There were absorbed 0.14 cc. O₂ per minute. The carefully fractionated methyl-ethyl carbinol showed faint levo-rotation at 60 volts and 4.1 Ampères after 100 hours' illumination, in which, therefore, 6×0.14 litres O₂ = 0.84 l had been absorbed, equivalent to 2.5 grammes of alcohol. In a 2 dm. tube this amounted to $-0^{\circ}.04$; a repeated experiment gave $-0^{\circ}.03$. Though the amount is very small, it is certainly above the error of observation, and we may say with pretty great certainty that in principle a decomposition has been effected by this way.

New points of views are opened in so far that it must be possible that in nature in the presence of compounds at the same time sensitive to light and optically active, new optically active compounds are built up by photo-chemical way.

In the assimilation process the formation of optically active glucose and of amyllum is not necessarily connected with the light process. This may be restricted to the activation of the chlorophyllcarbonate to a product of high potential energy, that now further under the influence of enzymatic dark reaction can change into oxygen and into optically active compounds.

Delft, Dec. 1926.

Botany. — *On the favorable effect of 35° C. on the cell-formation in foliage-leaves of Hyacinthus orientalis.* (Communication N^o. 22. Laboratory for Plant-physiological Research, Wageningen.) By Miss IDA LUYTEN. (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of December 18, 1926).

§ 1. *Introduction and Method.*

In the research on the optimal temperature for the flower- and leaf-formation of *Hyacinthus Orientalis* (BLAAUW 1924) stress had already been laid on the fact, that the optimum for these two differs greatly. It appeared that if on shooting in spring one would be sure of the best flower-cluster the hyacinth-bulbs should after being lifted first be exposed to 26° C. for 8 weeks, next to 17° C. for 4½ weeks.

If however after being lifted the bulbs were exposed to 35° C. for 5 weeks and next to 17° C. (i.e. 17° for 7½ weeks till planting-time), it appeared that in spring the foliage was several cms. longer than in the other still favorable high temperature-exposures. The following question therefore was obvious: is this length of the foliage-leaves merely a result of greater stretching of the cells or have there more cells been formed on exposure to 35° for 5 weeks, so that the leaves grew longer, with equally long cells.

In order to trace this, cells must be counted in the longitudinal direction. The material for it was obtained as follows. We started with 140 bulbs "Queen of the Blues", circumference 80—90 mms, 2 years old, July 4, 1924. On this date at the beginning of the experiment, 60 bulbs were exposed to 35° C., and 60 bulbs to 17° C., and 20 bulbs were fixed in alcohol 96 %. After 5 weeks (Aug. 9, 1924) 20 bulbs were fixed from the two temperatures and the bulbs from 35° transferred to 17°. On Sept. 30, i.e. 12½ weeks from the beginning of the experiment, 20 specimens were fixed of the group exposed to 35° for 5 weeks, next to 17° for 7½ weeks. Likewise of the group constantly exposed to 17°. On this latter date the remaining bulbs (20 of each exposure) were planted; Febr. 11, 1925 they were lifted and fixed.

We subjoin a survey of the fixations.

July 4, 1924 Beginning of the experiment	Aug. 9, 1924 (after 5 weeks)	Sept. 30, 1924 (after 12½ weeks)	Febr. 11, 1925 (planted in the open on Oct. 1)
20 bulbs	20 bulbs from 35° 20 bulbs from 17°	20 bulbs from 35°—17° 20 bulbs from 17°	20 bulbs from 35°—17° 20 bulbs from 17°

Next each group of each date was examined in the following way.

The bulbs were put in alcohol 50 %, so that the tissues grew soft and supple, next they were peeled to the outermost foliage-leaf. On the data July 4, and Aug. 9 it was sometimes hard to see whether the outermost foliage-leaf had already been reached. For by that time there is but little difference outwardly between the last sheath-leaf and the first foliage-leaf. In doubtful cases a reagent may be used, consisting in the application of a solution of iodine and iodide of potassium to slight cuts in the tissue. In the sheath-leaf the incisions are stained blue on account of the starch they contain. In the foliage-leaf the colour is browner and disappears sooner. Under the simple dissecting stage-microscope of Zeiss the lengths of the outer foliage-leaves of the 20 bulbs were accurately measured and the average computed, except with the foliage-leaves on Febr. 11, that were already long. The 3 bulbs containing the foliage-leaf, the length of which approaches this average as near as possible, are kept apart. In these 3 bulbs the number of cells in the longitudinal direction of the outer foliage-leaves will be counted. For this purpose the entire terminal bud is cut from the bulb, and good care is taken, that besides the terminal bud some disc-tissue is removed, so that the outer foliage-leaf is not injured at the base.

Next the outermost foliage-leaf is removed and measured again. It frequently appears, that the leaf has grown slightly shorter in most cases sometimes a little longer; next sections are made by hand through the central-longitudinal axis of this foliage-leaf, as was done by Miss VERSLUYS (1925) for the foliage-leaves of the Hyacinth. This central part of the foliage-leaves of July 4 and August 9, 1924 could be cut in its entirety; on the later dates the leaves were too long and they were divided into pieces of ca 1 cm.; the 2 lowest pieces at the base were taken smaller, because they contained a great many cells. The series of longitudinal sections were made perpendicular to the plane of the epidermis, the leaf lying between elder-pith soaked in glycerine. The cells were counted at the backside, i.e. morphologically the undersurface, viz. the third row under the epidermis. When a vascular bundle has been struck, the backside may usually be recognised by the fact the phloem is always turned towards the backside of the leaf. The presence of a vascular bundle in the section being far from frequent, before cutting I always removed a piece of the epidermis with a sharp razor at the inside in the direction of the base, so that from this injury I could determine the backside and underside of that piece of the leaf. Before being stained the sections were tapped with the sides of a pincette; this was done in order to remove the starch from the lower parts of the leaves and besides to make the cell-walls extend normally, for these have been compressed too much in consequence of being jammed between the pieces of elder-pith. Next a 2 % solution of methylene-green is used for staining, so that the cellwalls are clearly visible. For the counting

a Zeiss microscope is used, provided with an objective A and a micrometer-ocular 2 of LEITZ. The entire scale of the micrometer (10 spaces) corresponds with 1.73 mm. of the preparation. Accordingly it may be counted, how many cells occur in 10 successive spaces.

§ 2. *On the length of the buds (outer foliage-leaves).*

Let us now first consider the results we get, when from each group on each date we measure the buds down to the base and compute the average. It was already mentioned above, that groups of 20 bulbs were fixed. Yet on a few dates the average has been taken from 18 and 19 bulbs, as a few bulbs could not be used, because either the apex of the outer foliage-leaf was injured or because the bulb contained two terminal buds instead of one. For the sake of accuracy this latter deviation was not included in the figures.

On the date of starting (July 4) no average was computed, though 20 bulbs were fixed, but three of an average length were selected by the eye. This group of bulbs had also to serve for a root-examination. If one wants to measure the bulbs accurately, the surrounding tissue should be cut away deeply, and consequently on account of the injury to the disc, the bulbs would have been of no value for the other research. On this date no average is given, but the lengths of all of the three buds measured.

Now it appears from table 1, that on August 9, 1924 there is not much

TABLE 1. Length of the terminal bud in mms.

July 4, '24 beginning of the ex- periments	Aug. 9, 1924 after 5 weeks		Sept. 30, 1924 after 12 $\frac{1}{2}$ weeks		Febr. 11, 1925 results after 30 $\frac{1}{2}$ weeks	
	after 5 weeks 35°	after 5 weeks 17°	5 w. 35° + 7 $\frac{1}{2}$ w. 17°	12 $\frac{1}{2}$ w. 17°	5 w. 35° + 7 $\frac{1}{2}$ w. 17°	12 $\frac{1}{2}$ w. 17°
3.3	<i>n</i> = 19	<i>n</i> = 18	<i>n</i> = 20	<i>n</i> = 20	<i>n</i> = 20	<i>n</i> = 20
3.2	3.11 ± 0.08	3.45 ± 0.13	23.13 ± 0.28	11.88 ± 0.41	180.3 ± 0.35	84.6 ± 0.20
3.3						

difference in length in the bulbs which were exposed either to 35° for 5 weeks or to 17° for 5 weeks; it might only be observed, that the length of the terminal bud in 17° is slightly increased, whilst the buds had not grown in 35° after this period.

September 30 however yields quite a different picture. After *both* groups have been exposed to 17° for another 7 $\frac{1}{2}$ weeks, we see that the length of the outer foliage-leaf, previously exposed to 35° for 5 weeks, is *nearly twice as long* as in the bulbs constantly exposed to 17°.

On Febr. 11, 1925 the outer foliage-leaves, exposed to 5 weeks $35^{\circ} + 7\frac{1}{2}$ weeks 17° are also more than twice as long as those exposed to $12\frac{1}{2}$ weeks 17° . It follows that *the high temperature has a long after-effect and influence on the length of the foliage-leaves*. BLAAUW (1924) already pointed this out. The results I get from these macroscopic external measurements of these foliage-leaves, tally with the figures BLAAUW (1924, p. 16) found for the foliage-leaves of the same variety. For comparison we subjoin them, as far as observed at the time.

TABLE 2. Length of the outermost foliage-leaf in mm.

July 7 1922 Beginning of the experi- ments	Aug. 11, 1922 after 5 weeks		Oct. 2, 1922 after $12\frac{1}{2}$ weeks		Jan. 4, 1923 after $25\frac{1}{2}$ weeks	
	5 w. 35°	5. w. 17°	5 w. $35^{\circ} + 7\frac{1}{2}$ w. 17°	$12\frac{1}{2}$ w. 17°	5 w. $35^{\circ} + 7\frac{1}{2}$ w. 17°	$12\frac{1}{2}$ w. 17°
2.99	3.43	4.00		12.39	70.2	38.8

From this table 2 we see, that in the bulbs exposed to 5 weeks 17° the length was more after 5 weeks, that after an exposure to 5 weeks 35° , whilst on the other hand on January 4 it appears that the foliage-leaves in this latter exposure have grown twice as long as in those which had been constantly exposed to 17° .

§ 3. *On the number of cells in the longitudinal axis of the outer foliage-leaves.*

Let us now consider the result of counting the cells in connection with table 3.

The number of cells found in each leaf is given in the third column. There the number of cells has been given, lying in the spaces counted between the micrometer-divisions.

The number of spaces (= sps.) multiplied by 1.73 gives the lengths of the foliage-leaves in mms. This also enables us to compare with one another the foliage-leaves cut in each treatment. It may be observed that the number of cells of these foliage-leaves shows very little mutual variation.

This must probably be attributed to the accurate selection of the specimens to be treated before starting the experiments, besides to the fact that afterwards only those leaves were cut the length of which corresponded as much as possible with the macroscopically determined average. On account of the very slight variation in these figures and the very conspicuous results, the cutting of only three foliage-leaves on July 4, August 9 and September 30 proved sufficient.

On discussing the results we shall always refer to the *averages* of these

TABLE 3. Number of cells counted (Sps. = spaces in the micrometer.)

Date	Exposure	Number of cells counted in each foliage-leaf	Macroscopic length of each foliage-leaf in mms (See table 1)	Average number of cells
Juli 4	Beginning of the experiments	18 sps. = 3.11 mms. = 71 cells	3.2	3.16 mms. = 69.3 cells
		18 sps. = 3.11 mms. = 70 cells	3.3	
		17 sps. = 2.94 mms. = 67 cells	3.0	
Aug. 9	5 w. 35°	19 sps. = 3.29 mms. = 71 cells	3.2	3.3 mms. = 73.5 cells
		19 sps. = 3.29 mms. = 75 cells	3.2	
		19.6 sps. = 3.63 mms. = 75 cells	3.5	
Aug. 9	5 w. 17°	21.5 sps. = 3.72 mms. = 87 cells	3.5	3.65 mms. = 92.5 cells
		24.0 sps. = 4.15 mms. = 105 cells	3.7	
		21.0 sps. = 3.63 mms. = 86 cells	3.5	
Sept. 30	5 w. 35° + 7½ w. 17°	127.5 sps. = 22.05 mms. = 418 cells	23.0	23 mms. = 426 cells
		120.6 sps. = 20.86 mms. = 437 cells	23.2	
		123.7 sps. = 21.40 mms. = 423 cells	22.9	
Sept. 30	12½ w. 17°	62 sps. = 10.72 mms. = 225 cells	11.1	11.06 mms. = 233 cells
		67 sps. = 11.59 mms. = 240 cells	10.9	
		62.5 sps. = 10.81 mms. = 233 cells	11.2	
Febr. 11	5 w. 35° + 7½ w. 17°	1039.6 sps. = 179.8 mms. = 1506.5 c.	178.0	178 mms. = 1506.5 cells
		460 sps. = 79.58 mms. = 1036 cells	83.0	
		458 sps. = 79.28 mms. = 968.5 cells	82.9	
	12½ w. 17°			82.95 mms. = 1002 cells

figures; we put them in the last column. In this column we also find the average length in mms. In every case it was computed from the macroscopic length of the foliage-leaves from each treatment, which were destined for counting the number of cells. These figures give a more correct picture of the length than the figures of the converted number of spaces in the micrometer. In consequence of the continual tapping of the sections and the tight squeezing between elder-piths the cells did not quite recover their initial shape: now the length got slightly shorter, now slightly longer. But of course this did not influence the *number* of cells in the whole length. The number of mms. and spaces in column 3 was only given in order to

give the reader a survey of the figures obtained. To judge of the exact number of cells in the lengths the macroscopically measured lengths of the loose foliage-leaves (column 5) destined for counting the cells were always taken as average lengths.

On starting the experiments (*July 4*) we counted 69.3 cells in 3.16 mms.

As early as *August 9* some difference in the two treatments is to be noticed. The leaves exposed to 17° for 5 weeks, contained more cells than those exposed to 35° for 5 weeks. In this latter treatment there probably have been added some cells since July 4, but the exposure to 17° shows a greater increase. What was suggested as probable in the macroscopic measuring (the increase in length p. 199) is beyond dispute on counting the cells. In these weeks the number increased from 69.3 to 92.5 cells.

On *Sept. 30* i.e. after $12\frac{1}{2}$ weeks we get quite a different picture. With the macroscopic measurements we already found that on this date the preliminary treatment 5 weeks 35° gives to the foliage-leaves twice the length of those exposed to 5 weeks 17° , but now it appears that the number of cells is also nearly twice as great. The greater length, therefore, is not due to longer cells, but to a greater number of cells: so the high temperature proves to have a very favorable effect on the cell-division; not however while the bulbs are in this high temperature, but especially after that period, i.e. as a very favorable after-effect. For while the bulbs were kept in 35° , the number of cells in the longitudinal direction was smaller than in the bulbs exposed to 17° ; when however the two exposed groups have been in 17° for a long time, we begin to notice the favorable aftereffect of 35° on the cell-division in the longitudinal direction.

On the thickness of the foliage-leaves these various temperatures have no influence, for it appeared that the thickness of the foliage-leaves always consisted of the same number of cells, viz. varied between 16—18.

The cutting of the leaves on *Sept. 30* already answered the question, what the greater length of the foliage-leaves exposed to 35° for 5 weeks, was due to. For the sake of completeness some leaves were cut on *February 11* after all. The leaves of the group exposed to 35° — 17° already having a length of 18 cms, on *February 11*, cutting 1 foliage-leaf of this lot was considered sufficient. The cutting of 2—3 foliage-leaves and the counting in parts would have taken up an exceedingly long time, whilst the favorable effect of 35° on the cell-formation was an ascertained fact. Of the group $12\frac{1}{2}$ weeks 17° the cells of two foliage-leaves were counted on this date.

From table 3 last column we see, that on *February 11* the group 35° — 17° still has $1\frac{1}{2} \times$ as many cells in the longitudinal direction of its foliage-leaves, as the group "always 17° ". The ratio of the number of cells of the groups is somewhat altered with respect to the previous observation. This is owing to the fact the group $12\frac{1}{2}$ weeks 17° is no more lagging behind so far in the formation of the cells. This we may read from table 4.

TABLE 4. The average number of cells formed per week.

Group	July 4 — Aug. 9	Aug. 9—Sept. 30	Sept. 30 — Febr. 11
5 w. 35° + 7½ w. 17°	0.8	47	60
12½ weeks 17°	4.6	18.7	42.1

It has been computed how many cells have been formed *per week* in the two groups in the periods July 4—August 9, August 9—September 30 and September 30—February 11. This was done by dividing each number added in those periods (Table 3) by the number of weeks. These figures give us a standard, on account of their being formed by the dividing zone at the base of the leaf and are therefore not influenced by the length or the number of cells of the foliage-leaf. It is clear that the increase in cell-formation does not occur by leaps and bounds, but gradually.

Now we see from table 4, that in the group 35°—17° from August 9—September 30, 47 cells are formed per week, whilst from September 30—February 11, 60 cells are added per week. The lot constantly exposed to 17° first forms but 18.7 cells in those periods, afterwards 42 cells per week. In both groups the cell-formation is still increasing, but in the weeks from September 30—February 11 the group "always 17°" shows a greater increase compared with the previous weeks. The number of cells added per week in 35°—17° and the number of "always 17°" from August 9—September 30 are in the ratio of $\pm 5 : 2$, while in the next period that ratio has become $10 : 7$. Yet it appears, that the high temperature 35° *still has a favorable after-effect*, since this group adds 60 cells per week, and the group "always 17°" does not add more than 42 cells per week. The fact that the lot "always 17°" catches up a little in the later months may be due to the shifting of the optimum for the length of the foliage-leaves from higher to lower temperatures (see BLAAUW 1924, p. 16) in the later weeks.

On February 11 it moreover appears, that the group 35°—17° has slightly larger cells in the upper part of the leaf, from which it may be concluded, that the cells are stretched slightly more than in "always 17°". In order to get a survey of the size of the cells, I subjoin a successive series of a foliage-leaf of 35°—17° and one of "always 17°".

By comparing these figures we may prove : 1°. that preliminary treatment with 35° 5 wks makes the organ twice as long as only in 17° ; 2°. that while on Sept. 30 there are indeed $2 \times$ as many cells, months later (Febr. 11) the length *does* remain twice as much after stretching, but the number of cells is but $1\frac{1}{2} \times$ as much ; 3°. that this follows from the fact that the cells higher up in the leaf have become larger after 5 wks 35° + 7½ wks 17° than after "12½ wks 17°" (so that there are fewer in an equal distance).

It would be interesting to trace when the high temperature which now has a favorable effect in these experiments, will become injurious to the

TABLE 5.

Febr. 11 5 wk. 35° + 7½ wk. 17°				Febr. 11 12½ wk. 17°			
base	29.3 sps. *)	=	148.5 cells	base	38.6 sps.	=	171 cells
	49.5 "	=	209.— "		59.— "	=	224 "
	57.8 "	=	170.— "		52.4 "	=	145 "
	64.— "	=	124.— "		53.3 "	=	106.5 "
	58.— "	=	100.— "		66.5 "	=	105.5 "
	66.— "	=	87.— "		66.— "	=	102.— "
	67.5 "	=	84.— "		65.2 "	=	92.5 "
	67.— "	=	62.5 "	apex	59.— "	=	89.5 "
	66.5 "	=	55.— "		<hr/>		
	66.5 "	=	56.5 "		460.0 sps.	=	1036.— cells
	67.— "	=	58.— "				
	62.— "	=	54.— "				
	67.— "	=	62.— "				
	69.— "	=	66.— "				
	64.5 "	=	59.— "				
	57.— "	=	52.— "				
apex	61.— "	=	59.— "				
	<hr/>						
	1039.6 sps.	=	1506.5 cells				

*) sps. = spaces of the eye-piece micrometer

cell-formation and have an inhibitory effect. For this subject we once more refer to BLAAUW (1924), who in *longer exposure* to 35° observed a *decrease* in length in the foliage-leaves. In a group, exposed to 5 wks 35° + 7½ wks 17° the length in Jan. was 70.2 mms.; in a lot exposed to 8 wks 35° + 4½ 17°, the length in Jan. was 63.4 mms., whereas the group exposed to 12½ wks 35° did not attain a greater length than 34.2 mms. in January.

Our research having proved, that the high temperature gives the difference in length through the different number of cells formed afterwards, it might be worth while transferring lots from a group of bulbs, exposed already to 35° for 5 weeks, at regular rather short intervals to 17°, and next fix all of them on a certain date. Then we could determine by counting the cells, when inhibition begins.

LITERATURE.

BLAAUW, A. H. (1924). The results of the temperature during flower-formation for the whole Hyacinth: First part. Verhandl. Koninkl. Akademie v. Wetensch. Amsterdam 2de Sectie, Dl. 23, N^o. 4 en Meded. N^o. 10 v. h. Labor. v. Plantenphysiol. Onderzoek, Wageningen.

VERSLUYS, MARTHA C. (1925). The mode of growing of foliage-leaves, sheath-leaves and bulb-disc in *Hyacinthus orientalis*. Recueil des travaux botaniques néerlandais Vol. 22, en Meded. N^o. 14 v. h. Laborat. v. Plantenphysiol. Onderzoek, Wageningen.

Botany. — *On the atmospheric humidity during the flower-formation of the Hyacinth.* By Prof. A. H. BLAAUW. (Communication N^o. 23. Laboratory for Plant-physiological Research, Wageningen.)

(Communicated at the meeting of January 29, 1927).

§ 1. *Introduction.*

Between the lifting and planting of the Hyacinths, i.e. from the beginning of July to the beginning of October, lies the important period, in which the bulbs pass on to flower-formation and in which a good flowering for the next year, — or if desired the early flowering in December — entirely depends on the adequate treatment. With regard to this we have first of all to deal with the temperature as most important factor. Hence the influence of the temperature was first investigated into, and part of these researches were already published, whilst further papers on this subject will follow after being worked out.

The only other important factor might be the hygrometric condition of the air. The bulbs have just received the assimilation-products from the leaves. It is often said — and frequently not unjustly — that the concentration of the assimilation-products enables a plant, during vigorous cell-division, to pass on to flower-formation. This makes us ask ourselves whether, when the concentration is forwarded by a dry atmosphere, so that the bulbs evaporate a great deal of water, the assimilation-products do not accumulate more inward and in this way the atmospheric humidity may influence the flower-formation.

In practice the growers often talk of "dry-storing" and the question rises whether drought is really favorable for the growing plant, in what degree and whether from July to October the drought should be equal or varied. Questions, therefore, essential both to the theory of flower-formation and leaf-development and to the application.

Based on experiments fully described and on the fact that the temperature is the most important factor after all, we stick to the optimal temperature-treatment for field-cultures; after the lifting till ca Sept. 1 (ca 8 weeks) 25° to 26°, next till the planting (ca 4½ weeks) 17° at most.

After having repeated experiments on the hygrometric condition of the air for 3 summers, I shall communicate their results here.

For reading the hygrometric condition of the air, expressed in percents of the saturation, we chose of various makes or systems as the most practical for our use the hair-hygrometers of RICHARD in the shape of an alarmclock (Hygromètre à cadran; section of dial 10 cms.). There are those among them which are unsuitable for some reason; therefore a

choice should be made and those taken into use which are sensitive enough and which in spite of a change of temperature or humidity, when reduced to the same circumstances almost point to the same hygrometric condition. The correct height of the Hygrometer is regulated by repeated comparison with the Assmanns-Aspiration-psychrometer, i.e. "gauged", which regulation is easily performed by adjusting the hand.

Next 4 experiments were compared each year, viz. the first hot period (26°) moist, the second cooler period (17°) dry, indicated *VD*, or first dry then moist (*DV*), or both periods dry (*DD*) or both periods moist (*VV*).

The bulbs were put in glass or china dishes in two zinc boxes. Those boxes were put on their sides, the open "upper side" turned towards the front, slightly sloping backwards, a glass-plate against them. The dry box contained (besides the bulbs and a hygrometer) a dish of unslacked lime for drying, which was renewed as soon as the humidity approached the fixed upper limit. The moist box was kept at the desired moisture by a little water at the bottom or by an evaporation-tube (of a radiator of the central heating). The chinks between the glass and zinc box were covered with moist cloths if necessary.

For the drought (*D*) the hygrometric condition was kept at 30—40 % in all three summers, for the moisture (*V*) in 1923 first at 80—90 %, in 1924 and 1925 even at 90—100 %, as a rule 95—100 %. On purpose strong contrasts were chosen. But as moderately moist and moderately dry might be more favorable than one of these four extremes, after the result of 1923 a 5th group was added in 1924 and 1925, in which fairly moist amounted to 70—80 % and fairly dry to 50—60 %. This 5th group has been indicated as "control group" in the following tables.

In the very moist treatment there naturally occurred mould, only however on the totally exhausted outer scales and the bottom disc-layer. On account of this these bulbs were cleaned a few times with a cloth. Rotting or injury to living parts did not occur in spite of the very high moisture-content.

§ 2. *Experiments in the year 1923—1924.*

In the summer of 1923 2 year old bulbs were chosen of a circumference of 80—90 cms. By that time the different experimental groups were selected only according to the circumference and not yet — as always happened in later years — divided into groups of equal weight.

Consequently in October after the treatment there was a certain variation between the weights of the 30 bulbs (per experiment), which variation is arbitrary and cannot just be attributed to the previous treatment (See Tab. 1).

Of the 30 bulbs treated per experiment some 9 were lifted afterwards in order to trace whether the humidity had any effect on the root-system.

Various data on the root-system, also in connection with different circumstances, have been examined and will be published in a separate treatise by Mrs. M. C. TROOST—VERSLUYS.

Results 1923—1924.

Though in February 1924 the group *VD* came up somewhat later than the three other groups, yet on March 26 all four groups were perfectly equal. Likewise on April 22nd all groups were uniform and in full flower, so that not the slightest distinction could be made as a result of the treatments.

An influence on the number of foliage-leaves may hardly be expected, since on lifting the foliage-leaf-formation ceases and at 26° the growing-point passes on to flower-formation. Yet with so severe a drought (e.g. in *DD* and *DV*) contrasted with a high moisture-content (in *VD* and *VV*) the leaf-formation might cease at different points of time, and at any rate an influence on the shooting or non-shooting of the last-formed leaflets was not unthinkable. In Tab. 1, first column, we see that after the treatments of 1923 the number of shooted foliage-leaves computed for 10 bulbs was identical (38) averagely in three groups, in *VV* slightly smaller (36), but this difference of upwards of 5% may be quite accidental. In *VV*, therefore, the influence on this is at most very slight; in the three other cases the shooting is at all events the same. We shall revert to this in the further experiments.

TABLE 1. Treated summer 1923, beginning of July 80—90 mms.

Treatment	Weight per 20 bulbs. Beginning of Oct. 1923 after treatment	Number of shooted foliage-leaves per 10 bulbs	Number of flowers per 10 plants ($n = 30$)	Average length of the foliage above ground in mms.	Weight per 20 bulbs July 6, 1924	Average circumference per bulb July 6, 1924.
1	2	3	4	5	6	7
VV	199 Grams	36	62.4 (± 2.7)	334.6 (± 4.3)	640	125.8 mms. (± 2.0)
DD	201 ..	38	62.0 (± 2.4)	313.9 (± 7.4)	586	118.1 .. (± 2.2)
DV	209 ..	38	63.7 (± 2.5)	311.9 (± 6.5)	596	121.9 .. (± 2.1)
VD	194 ..	38	66.0 (± 2.3)	342.4 (± 3.5)	702	131.7 .. (± 1.6)

In this first year there initially seemed to be not the slightest difference between the four groups. The four lots flowered particularly uniformly. The average number of flowers per cluster is still slight in these small bulbs. Column 4 gives the number per 10 plants as an average from 30 specimens after the different moisture-treatment in the previous summer. That number is also equal in the four groups. True, after *VD* the number seems slightly higher, but the difference does not exceed the amount possible according to the mean errors.

At the beginning of May after the roping of the flowers the foliage was also perfectly equal. Not before the end of May and the beginning of June the length of the foliage-leaves in the various groups became unequal. At the end of May the length of the foliage was measured above ground. Column 5 shows that there certainly is a difference in length here, even though the mean error is taken into account. The two groups *V V* and especially *V D* as averages of 20 observations are clearly 2 and 3 cms. longer than those which were first treated dry.

While the leaves were still growing out in June, no further measures could be taken, because by that time the leaf-apexes begin to dry up, though the rest of the leaf is still assimilating. It is however more important to trace the effect of the assimilation-period in the increase of weight (or increase of circumference).

In Tab. I columns 6 and 7 show us the following :

The weight per 20 bulbs rather varies at the beginning of Oct. 1923 ; this, however, does not prove anything with regard to more or less evaporation, since in July the groups had not yet been divided according to equal weights as later on (see 1924 and 1925).

The difference in increase of the bulbs after different moisture-treatment (previously between lifting and planting) was so striking in 1923—1924, that we understood that these experiments ought to be continued and repeated. The very bulbs which chanced to have the lowest weight at the beginning of Oct., i.e. after the treatment and before the planting, have by far the greater weight on July 6, 1924. Those are the two groups which were kept extremely moist for two months after the lifting, whilst the greatest increase in weight was shown by the one that was kept very dry at 17° after that.

So it appears from four groups, *that at all events during the flower-forming period a very moist atmosphere has a more favorable effect on the increase in thickness of the bulbs than a dry atmosphere in July and August*, whilst no detriment e.g. concerning diseases was noticed. This result we ourselves had not surmised at all ; it already proves that the idea "dry-storing" often used in practice, is put in a peculiar light. We shall however first have to consider the results of following years. In Tab. I the average circumference on July 6 has also been given. From this the same results can be read as from the column of the weights of the four groups. Apparently the circumferences do not diverge so much as the weights or the increases of weight. But these two criteria form an essential point for comparison of magnitudes as we repeatedly want them. *With the weights we have to deal with a comparison of the heaviness of certain bodies, i.e. we compare with each other three-dimensional magnitudes. With the circumferences we only compare a certain linear measure of those same bodies.* Therefore on our comparing the circumferences the difference of the real contents will be much less conspicuous than on our comparing groups of bulbs by their weights. Hence after having exclusively worked

with circumferences in the first years, we now in particular compare the weights in our experiments, wherever this is possible.

Meanwhile we have already got a great number of data on weight and circumference of bulbs, so that afterwards we shall probably revert to the connection between these two magnitudes,

§ 3. *Experiments of the years 1924—1925 and 1925—1926.*

In this § 3 the results of 1924—1925 and 1925—1926 will be discussed together. The experiments were started on July 9, 1924 with bulbs having a circumference of 110—120 mms., weighing 518 grams per 20; in 1925 the experiments were started on July 3 with bulbs of 115—120 mms., i.e. selected within slightly narrower limits, weighing 531 grams per 20, so somewhat heavier than in 1924, which corresponds with the limits of circumference which were chosen 115—120 mms. instead of 110—120 mms.

As was already stated in the conclusion of § 1, a control-group has been added of fairly moist ($25\frac{1}{2}^{\circ}$ C.) + fairly dry (17° C.).

Mid-September 1924 in *V V* the roots had already shooted so far, that this group had to be planted in the field with great care.

Thus *V V* of 1924 was planted with shooting roots in the field upwards of 14 days earlier than all other groups. In 1925 this did not occur in *V V*, though the root-whorl was already much developed on Oct. 1.

In the spring of both years all five groups first mutually behaved similarly, just as the four groups in the first year did.

Here the question first rises whether there is any influence on the number of flowers. In April 1925 the average is 13 to 14 per cluster in the control-group, *V V* and *V D*, compared to 15 to 16 in *DD* and *D V*; taking the mean error into consideration we might decide here on a slight difference in favour of *DD* and *D V*.

TABLE 2. Number of flowers per cluster in April after the different moisture-treatments in the previous summer.

	Circumference 110—120 mms. April 1925	Circumference 115—120 mms. April 1926
Control-group	12.9 (± 0.5)	12.8 (± 0.4)
<i>V V</i>	13.3 (± 0.8)	12.0 (± 0.4)
<i>DD</i>	15.2 (± 0.6)	12.5 (± 0.5)
<i>D V</i>	15.8 (± 0.6)	12.1 (± 0.5)
<i>V D</i>	14.2 (± 0.7)	12.8 (± 0.4)

On our considering the experiment of 1925—1926, we see that the number of flowers of an average of 12 to 13 per cluster differs very little in all 5 groups. The mean errors should be taken into account — and it should likewise be borne in mind, that in the more accurately selected material the mean error (computed for groups of 20) is also visibly slighter.

After having repeated these experiments for 3 years, we must conclude, *that the atmospheric moisture (between 30 % and 100 %) has no noticeable effect on the number of flowers of the cluster in the flower-forming period.*

Let us now consider as with the experiments of 1923—1924 how many foliage-leaves shoot and assimilate in the spring. For 1925 and 1926 this is found in Table 3. In the 5 groups of 1924—1925 this number varies from 48 to 51, in 1925—1926 from 51—55 per 10 plants. This difference per 10 plants is certainly slight. In this respect too we must conclude

TABLE 3. Number of foliage-leaves shooted per 10 bulbs.

	1925	1926
Control-group	51	55
VV	48	51
DD	49	52
DV	50	52
VD	49	53

after comparison of the results of 3 years from 4 and 5 groups, *that the hygrometric condition of the air between 30 % and 100 % has no noticeable effect on the shooting of the number of young leaves already formed, which are to assimilate next spring.*

Moreover we see that *during the flowering and a short time after in April the foliage is uniform in the various groups.*

It was already mentioned in the first series of experiments, how in June 1924 at last a striking dissimilarity was observed, because finally the foliage of group *VD* grew a good deal longer, which was corroborated in the results, because the increase of weight in this group was by far the greatest.

Table 4 gives the average length of the foliage in the beginning of June 1925 and 1926. Already in this case with the leaf-lengths, but still more so from the subsequent tables 5 and 6 on the increase in weight it appears, that 1925 was a more favorable year for assimilation than 1926.

TABLE 4. Average length of the foliage in mms. above ground, beginning of June, after different moisture-treatments in the previous summer.

	Circumference 110—120 mms. June 1925	Circumference 115—120 mms. June 1926
Control-group	293.1 (\pm 4.5)	283.0 (\pm 5.1)
VV	315.8 (\pm 7.9)	260.5 (\pm 7.0)
DD	307.0 (\pm 7.4)	286.1 (\pm 5.8)
DV	288.0 (\pm 9.3)	275.5 (\pm 9.5)
VD	320.7 (\pm 5.2)	291.6 (\pm 5.8)

With regard to the Hyacinth it may make a great difference whether the winter lasts long and is followed by a warm, dry early summer, or that we have a rather early mild spring passing into a moderately warm and rather moist early summer. The assimilation-period, as it is already very limited for the Hyacinth may be very short or rather long in consequence of this.

Hence the result of experiments can be so divergent one year and another, though we compare the effect of treatments applied to the bulbs in summer under completely controlled conditions.

In Tab. 4 we give the average length of the foliage in mms. above ground ca June 1. Little may be concluded from this. Considering the mean errors, the differences are rather slight in most cases. Yet in both

TABLE 5. Increase of weight in grams per 20 bulbs.
Treated summer 1924. Circumference 1924: 110—120 mms.

	Weight 20 bulbs July 9, 1924	Weight Sept. 30, 1924	Weight 20 bulbs July 8, 1925	Increase in a year
Contr.	518	448	933	415
VV	518	planted earlier see text	908	390
DD	518	421	890	372
DV	518	441	869	351
VD	518	433	941	423

TABLE 6. Increase of weight in grams per 20 bulbs.
Treated summer 1925. Circumference 1925: 115—120 mms.

	Weight 20 bulbs July 3, 1925	Weight Sept. 30, 1925	Weight Juni 28, 1926	Increase in a year
Contr.	531	413	779	248
V V	531	449	718	187
D D	531	397	746	215
D V	531	422	788	257
V D	531	420	821	290

years the foliage is longest in *V D*. In 1926 *V V* yields a fairly low figure, while this group succeeded *V D* in foliage-length in 1924 and 1925. Taking all together we should not attach too much value to this measure of the leaf-lengths; we can only say that the phenomenon of 1924 "that after the treatment *V D* the leaves finally attain a somewhat greater length" is confirmed in 1925 and 1926, though it is not so striking as in 1924.

Let us finally consider the increase of weight in 1925 and 1926. First of all these tables 5 and 6 prove that with slight differences in leaf-lengths as in Tab. 4 we should be careful and that these are not quite parallel with the increase in thickness of the bulbs. Only in case of somewhat greater differences we may rely to some extent on the foliage-lengths. So we find that in 1925 and 1926 the groups with the longest and with the shortest foliage are likewise the best and the worst group as regards the weight.

In Table 6 attention should be paid first of all to the loss of weight of the bulbs directly after the treatment. As regards the evaporation, of course this depends greatly on the moisture or drought of the storage-atmosphere. Starting from 531 grams the group *V V* has lost 82 grams or 15 %, the group *D D* 25 % (in 1925 but 19 %), while the other groups lost an amount between those two extremes. We see that even with 90—100 % moisture the loss of weight though much slighter, is yet considerable (15 %). This loss of weight will for the greater part be due to evaporation, but will partly be owing to oxydation. That part of the loss of weight which is due to respiration, may be expected to remain constant in this different atmospheric moisture but uniform temperature-treatment.

Both in 1925 and 1926 this increase of weight at the end of the assimilation-period is greatest after the treatment *V D*, though in 1924 the differences were greater (see Tab. 1). Of the remaining groups little can be said with certainty about the three years. In 1925 the control-group fairly

moist-fairly dry produces a favorable effect, in 1926 a rather favorable, it does not belong to the 2 worst groups. In the three years the worst figure is successively yielded by *DD*, *DV* and *VV*; the least increase but one by the groups *DV*, *DD* and *DD*. While in 1926 *VV* increases strikingly little, this group was fairly favorable in 1924 and 1925. So we see that the groups *DD* and *DV* yield one of the two lowest figures 5 of the 6 times in those three years, the group *VV* but once, while *VD* gives the greatest increase all the 3 times.

§ 4. Conclusion.

We have exposed Hyacinth-bulbs in one and the same temperature-treatment to very moist and very dry conditions and repeated this three years running.

The hygrometric condition (30 %—100 %) of the surrounding air in the period of storing has *no influence on the number of leaves that can shoot the next year.*

The hygrometric condition of the air during that flower-forming period has *no influence on the number and the good development of the flowers.*

Finally the applied moisture is noticeable next summer in the final length of the foliage, as very moist 25° C. followed by very dry 17° C. grows out a little more than all other groups, which was corroborated all the three years.

Probably it may be considered as an additional result that all the three years after this treatment the greatest increase of weight was found, exceeding the results of the other treatments sometimes more, sometimes less. Besides the application of very dry in the first months is by no means favorable, so that dry-dry and dry-moist are far behind moist-dry in increase of weight. Moist-moist is now fairly good, now unfavorable and should be avoided, because of the danger that the roots will shoot untimely in 17° in September.

A moisture of 90—100 % followed by 20—40 % is difficult to apply on a large scale. In great quantities the growing mouldy and the decay of old bulb-rests might yield a too great danger for the living parts of the bulb (which however could not be observed by us in small quantities). Moreover a drought of 30—40 % moisture is difficult to maintain in the large stores.

After the lifting of the bulbs we keep our rooms dry for a couple of days, until sand and root-rests get loose from the bulbs, next we raise the temperature to 26° C. and the moisture to 70—80 % for 8 weeks for the field-cultures. After 8 weeks we transfer to 17° C. with a moisture of 50—55 %.

Though on applying fairly moist and fairly dry the advantage of very moist—very dry is partly given up (see control-group in tables 5 and 6), the difficulties of applying very moist and very dry on a large scale as in our experiments, are avoided in that way.

Thus the hygrometric condition cannot act a great part in the application to considerable quantities. A moisture of 90—100 % followed by 30 % would certainly produce a greater increase of weight, but because of the difficulties of application we shall have to be satisfied with fairly moist (70—80 %) followed by fairly dry (50—55 %).

January 1927.

Mathematics. — *On metric connexions with absolute parallelism.* By E. BORTOLOTTI. (Communicated by Prof. JAN DE VRIES.)

(Communicated at the meeting of February 26, 1927).

I wish to indicate in this paper some results ¹⁾ concerning the metric connexions ²⁾ with zero curvature in an n -dimensional RIEMANNIAN manifold (V_n). Like results are in connexion with those which CARTAN and SCHOUTEN have established in two papers, published here in 1926. ³⁾

CARTAN and SCHOUTEN have determined the RIEMANNIAN spaces admitting some metric connexions with absolute parallelism (namely with zero curvature), which have the same geodesics as the RIEMANNIAN connexion (by CARTAN: connexion euclidienne de M. LEVI-CIVITA).

Omitting the last condition, in an arbitrary V_n we establish an absolute parallelism, with respect to an arbitrary orthogonal ennuple $\overset{i}{X}^y$ ⁴⁾ in V_n , if we assume that *all vectors parallel to one another make the same angles with the curves of the congruences $\overset{i}{X}^y$* . This parallelism has been studied, from an analytical point of view, by G. VITALI ⁵⁾; but previously also R. WEITZENBÖCK ⁶⁾ has considered the tensorial differentiation to which that parallelism corresponds.

¹⁾ Which have been more developed in an article ("*Parallelismi assoluti nelle V_n Riemanniane*") of the "Atti Ist. Veneto", t. 86. p. II, 1926—1927, p. 455—465.

²⁾ I call here a "*metric connexion*" the connexion which arises from an "*Ueberschiebungsinvariante metrische Uebertragung*" (J. A. SCHOUTEN, *Der Ricci-Kalkül*, Berlin, Springer 1924, p. 75). E. CARTAN calls this connexion a "*connexion euclidienne*"; while CARTAN's "*connexion métrique*" (which generalizes the WEYL's "*metrische Zusammenhang*"), corresponds to SCHOUTEN's "*Ueberschiebungsinvariante konforme Uebertragung*".

³⁾ E. CARTAN and J. A. SCHOUTEN: "*On the Geometry of the Group-manifold of simple and semi-simple groups*", Proceedings 29, p. 803—815, and "*On RIEMANNIAN Geometries admitting an absolute parallelism*". *ibid.*, p. 933—946. S. also E. CARTAN: "*La Géométrie des groupes de transformations*". (Journal de Mathématique, t. VI, 1927, p. 1—119) partic. p. 49—54.

⁴⁾ $\overset{i}{X}^y$ ($i = 1, 2, \dots, n$) denote here the contravariant components of n mutually orthogonal (unit-) vectors-field in V_n .

⁵⁾ G. VITALI: "*Una derivazione covariante formata coll'ausilio di n sistemi covarianti del 1° ordine*". (Atti Soc. Ligustica, vol. II, 1924, pag. 248—253).

⁶⁾ R. WEITZENBÖCK: "*Invariantentheorie*". P. Noordhoff, Groningen, 1923, p. 320. S. likewise SCHOUTEN, *Ricci-Kalkül*, p. 101 (Aufgabe 1); G. J. C. GRISS, "*Differentialinvarianten von Systemen von Vektoren*". Acad. Proefschrift; P. NOORDHOFF, Groningen, 1925), p. 10—14.

Let us call a WEITZENBÖCK-VITALI's connexion the metric one which arises from that parallelism. I have remarked that the equations of such a connexion may be expressed in the form:

$$\nabla_{\mu} \xi^{\nu} = \overset{0}{\nabla}_{\mu} \xi^{\nu} + T_{\lambda\mu}^{\nu} \xi^{\lambda} \quad (1)$$

where $\overset{0}{\nabla}$ is the symbol of RICCI's covariant derivative, and $T_{\lambda\mu}^{\nu}$ is given by

$$T_{\lambda\mu}^{\nu} = \sum_{i,j,l} \gamma_{ijl} X^i X^j X^l_{,\mu} \quad (2)$$

where γ_{ijl} are the coefficients of rotation of the ennuple X^{λ} ⁸⁾.

Let us now assume that, in form. (1), $T_{\lambda\mu\nu}$ denotes an arbitrary affiner, skew-symmetric with respect to λ, ν . Then by (1) is represented the most general metric connexion in V_n .

The necessary and sufficient condition that there exists an ennuple X^{λ} , with respect to which $T_{\lambda\mu}^{\nu}$ may be expressed in form (2), is that

$$R_{\alpha\beta\lambda}^{\nu} = \overset{0}{R}_{\alpha\beta\lambda}^{\nu} - \left\{ \overset{0}{\nabla}_{\omega} T_{\lambda\mu}^{\nu} - \overset{0}{\nabla}_{\mu} T_{\lambda\omega}^{\nu} + T_{\lambda\mu}^{\tau} T_{\tau\omega}^{\nu} - T_{\lambda\omega}^{\tau} T_{\tau\mu}^{\nu} \right\} = 0, \quad (3)$$

where $\overset{0}{R}_{\alpha\beta\lambda}^{\nu}$, $\overset{0}{R}_{\alpha\beta\lambda}^{\nu}$ are the RIEMANN-CHRISTOFFEL's affiners of the connexion (1), and of the RIEMANNIAN one in V_n .

Consequently:

The most general metric connexion with zero curvature in a RIEMANNIAN V_n is the WEITZENBÖCK-VITALI's connexion with respect to an arbitrary orthogonal ennuple in V_n .

Besides we have:

The geodesics of an arbitrary metric connexion with zero curvature in V_n may be ranged in $\infty^{\frac{n(n-1)}{2}}$ orthogonal ennuples, with respect to any of which the supposed connexion is a WEITZENBÖCK-VITALI's one.

In particular, the connexions which have been studied in the above cited CARTAN and SCHOUTEN's papers, are WEITZENBÖCK-VITALI's connexions with respect to some orthogonal ennuples with skew-symmetric coefficients of rotation.

7) I make use of some notations of the "Ricci-Kalkül" of SCHOUTEN, S. pag. 63, 73, 83. $\Delta_{\mu} \xi^{\nu}$ is called by WEITZENBÖCK the "kovariante Ableitung von ξ bezüglich der n Formen X^1, X^2, \dots, X^n "; by VITALI, the "primo sistema derivato covariante di ξ , rispetto agli n sistemi X^{λ} ".

8) RICCI et LEVI-CIVITA: "Méthodes de Calcul différentiel absolu et leur applications" (Mathematische Annalen, t. 54 (1900), p. 125—201) p. 149. It may be observed that from (2) arises a new signification for the coefficients of rotation of an orthogonal ennuple in V_n .

Let us say that an n -tuply system of mutually orthogonal congruences in V_n is a *canonical* one, where $n-1$ (whatsoever) of these congruences are the (RICCI's) canonical congruences⁹⁾ with respect to remainder one. Then we have:

The RIEMANNIAN, CARTAN and SCHOUTEN's spaces (that is: the spaces admitting some metric connexions with zero curvature, and with the same geodesics as the RIEMANNIAN connexion) are characterized by possessing an n -tuply canonical system of orthogonal geodesic congruences; and, therefore, infinite ones, two of which (inversely like, corresponding to the two opposite signs of γ_{ijl} , or $T_{\lambda\mu\nu}$) are (generally) determined by choosing an orthogonal ennuple of directions in a point of the space.

⁹⁾ RICCI et LEVI-CIVITA: "*Méthodes, . . .*", pag. 154.

Mathematics. — *Ueber Syzygien bei sechs binären Linearformen.*
By R. WEITZENBÖCK.

(Communicated at the meeting of January 29, 1927).

Wir zeigen in § 1 dass jede irreduzible p -Relation der G_d -Koordinaten $p_{ikl\dots}$ in einem G_n (= lineares Gebiet n -ter Stufe oder linearer, projektiver, $(n-1)$ -dimensionaler Raum) einem invarianten Gleichungssystem angehört, das durch Nullsetzen einer Komitante K des alternierenden Tensors $p_{ikl\dots}$ entsteht. In § 2 wird dies für die quadratischen p -Relationen näher ausgeführt.

Die p -Relationen der G_d -Koordinaten in einem G_n sind Syzygien $S=0$ zwischen den Invarianten J von n Punkten eines G_d . Umgekehrt ist auch jedes $S=0$ eine p -Relation und die Struktur dieser Gleichungen wird aufgedeckt durch die Ermittlung aller *irreduziblen* Syzygien S und die dazu gehörige *Syzygienkette*.

Dies ist bisher bekannt für $d=2, n=5$ (§ 3). Wir behandeln hier den Fall $d=2, n=6$, wo die Verhältnisse schon bedeutend komplizierter sind. Die zugehörige Syzygienkette bricht hier nach der vierten Art ab, d.h. es gibt noch (einundzwanzig) irreduzible Syzygien vierter Art aber keine von fünfter Art mehr (§§ 4 bis 9).

§ 1. *Irreduzible p -Relationen.*

Es seien x, y, \dots, z d linear-unabhängige Punkte eines G_d im G_n (G_m = linearen, projektiver, $(m-1)$ -dimensionaler Raum), deren homogene Koordinaten die Elemente der Matrix

$$\mathfrak{M} = \begin{vmatrix} x_1 & x_2 & \dots & \dots & \dots & \dots & x_n \\ y_1 & y_2 & \dots & \dots & \dots & \dots & y_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ z_1 & z_2 & \dots & \dots & \dots & \dots & z_n \end{vmatrix} \dots \dots \dots (1)$$

bilden. Die d -reihigen Determinanten.

$$p_{i_1 i_2 \dots i_d} = (xy \dots z)_{i_1 i_2 \dots i_d} = (i_1 i_2 \dots i_d) \dots \dots \dots (2)$$

aus \mathfrak{M} sind die $\binom{n}{d}$ homogenen „Punktkoordinaten des G_d “. Sie sind verknüpft durch eine Reihe von Gleichungen, die wir p -Relationen $R=0$ nennen, wenn R ein Polynom der $p_{ikl\dots}$ ist, das die beiden Bedingungen erfüllt: a) $R \equiv \equiv 0 \{p_{ikl\dots}\}$, d. h. R ist nicht Null, wenn die $p_{ikl\dots}$ als unab-

hängige Veränderliche betrachtet werden; β) $R \equiv 0 \{x, y, \dots, z\}$, d. h. R verschwindet identisch, wenn die $p_{ikl\dots}$ durch die x_i, y_k, \dots ausgedrückt werden.

Wir wollen weiters von einer *irreduziblen* p -Relation $R = 0$ sprechen, wenn erstens R nicht als Summe $R_1 + R_2 + \dots$ geschrieben werden kann, wo schon jedes $R_i = 0$ eine p -Relation darstellt und wenn zweitens R nicht (im Körper der rationalen Zahlen) in Faktoren zerlegbar ist: $R = R_1 R_2$, wo schon $R_1 = 0$ oder $R_2 = 0$ eine p -Relation ist.

Die einfachsten irreduziblen p -Relationen sind die *quadratischen* p -Relationen $\Pi = 0$. Hierbei ist der allgemeine Typus von Π gegeben durch:

$$\Pi = (i_1 i_2 \dots i_d) (k_1 k_2 \dots k_d) - (k_1 i_2 \dots i_d) (i_1 k_2 \dots k_d) + \dots + (-1)^d (k_1 i_1 i_2 \dots i_{d-1}) (i_d k_2 \dots k_d) = 0 \quad (3)$$

und es besteht der Satz ¹⁾, dass jede p -Relation $R = 0$ sich auf die Gestalt

$$R = \Sigma A_\nu \Pi_\nu = A_1 \Pi_1 + A_2 \Pi_2 + \dots = 0 \quad (4)$$

bringen lässt, wo die A_ν Polynome der $p_{ikl\dots}$ sind.

Die Gleichungen $\Pi = 0$ sind notwendig und hinreichend dafür, dass $\binom{n}{d}$ Grössen $p_{ikl\dots}$ als Punktkoordinaten eines G_d im G_n betrachtet werden können.

Wir wollen jetzt zeigen, dass jede irreduzible p -Relation $R = 0$ aus einer Komitante K des schiefsymmetrischen Tensors $p_{ikl\dots}$ abgeleitet werden kann durch Spezialisierung der in K auftretenden Reihen $\xi, \eta, \dots, u', v', \dots$.

Hiezu haben wir vor allem nachzuweisen, dass jedes irreduzible R homogen in den $p_{ikl\dots}$ und ausserdem homogen in allen auftretenden Indizes i, k, l, \dots ist. Wäre nämlich R nicht homogen in den $p_{ikl\dots}$, dann würde R additiv zerfallen in homogene Bestandteile $R = R_1 + R_2 + \dots$. Ersetzen wir hier x_i durch λx_i , dann erscheint jedes $p_{ikl\dots}$ mit λ multipliziert und aus $R_1 + R_2 + \dots = 0$ wird $\lambda^{m_1} R_1 + \lambda^{m_2} R_2 + \dots = 0$ mit $m_i \neq m_k$. Also wäre $R_i = 0$ bereits eine p -Relation.

Die Indizeshomogenität bei einer irreduziblen p -Relation beweist man genau so, indem man x_i, y_i, \dots, z_i durch $\lambda x_i, \lambda y_i, \dots, \lambda z_i$ (i fest) ersetzt.

Jetzt leiten wir aus R ein Polynom R' ab, das ebensoviele untereinander äquivalente Reihen $p_{ikl\dots}, q_{ikl\dots}, r_{ikl\dots}, \dots$ linear enthält, als der Grad von R in den $p_{ikl\dots}$ beträgt. Wir bilden hiezu die Polaren ($m = \text{Grad von } R \text{ in den } p_{ikl\dots}$):

$$R_1 = \frac{1}{m} \Sigma \frac{\partial R}{\partial p_{ikl\dots}} q_{ikl\dots}, \quad R_2 = \frac{1}{m-1} \Sigma \frac{\partial R_1}{\partial p_{ikl\dots}} r_{ikl\dots}, \dots, R_{m-1} = R'$$

Wir haben dann

$$(R')_{p_{ikl\dots}} = q_{ikl\dots} = \dots = R \quad (5)$$

¹⁾ Vgl. einen demnächst in den Mathem. Annalen (1927) erscheinenden Beweis.

Da R' linear in jeder Reihe $p_{ikl\dots}, q_{ikl\dots}, \dots$ ist, können wir jetzt diese Reihen symbolisch zerlegen: $p_{ikl\dots} = p_i p_k p_l \dots$. Nun wählen wir, den n Indizes $1, 2, \dots, n$ entsprechend, n Reihen von Raumkoordinaten u', v', \dots, w' und ersetzen in R' jedes p_i durch (pu') , jedes q_i durch (qu') , \dots jedes p_n durch (pw') u.s.w. Aus R' entsteht auf diese Weise eine Komitante

$$K = R'((pu'), (pv'), \dots, (qu'), (qv'), \dots) \dots \dots (6)$$

und von K kommen wir auf R' , d.h. auf R zurück, wenn wir die Matrix $((u' v' \dots w'))$ als Einheitsmatrix wählen.

Durch $K \equiv 0 \{u', v', \dots, w'\}$ wird also ein invariantes Gleichungssystem gegeben, wovon eine Gleichung die ursprüngliche p -Relation $R = 0$ darstellt. Es ist dies ein besonderer Fall des Satzes von GRAM¹⁾, denn alle irreduziblen p -Relationen desselben Grades bilden ein invariantes Gleichungssystem.

§ 2. Die quadratischen p -Relationen.

Wir führen diese Zusammenfassung der irreduziblen p -Relationen desselben Grades m durch Komitanten K für den einfachsten Fall $m = 2$ näher aus.

Ersetzen wir in (3) die ersten Faktoren durch $p_{ikl\dots}$, die zweiten durch $q_{ikl\dots}$, dann entsteht, wenn wir noch i_{d+1} an Stelle von k_1 schreiben:

$$\begin{aligned} \Pi &= p_{i_1 i_2 \dots i_d} \cdot q_{i_{d+1} k_2 \dots k_d} - p_{i_{d+1} i_2 \dots i_d} \cdot q_{i_1 k_2 \dots k_d} + \dots = \left. \right\} \dots (7) \\ &= (\Sigma \pm p_{i_1} p_{i_2} \dots p_{i_d} q_{i_{d+1}}) q_{k_2} \dots q_{k_d} \end{aligned}$$

oder, da die eingeklammerte Summe gleich der durch $d!$ geteilten Determinante

$$(p p \dots p q)_{i_1 i_2 \dots i_d i_{d+1}} = (p^d q)_{i_1 i_2 \dots i_d i_{d+1}}$$

ist:

$$d! \Pi = (p^d q)_{i_1 \dots i_{d+1}} q_{k_2} \dots q_{k_d} \dots \dots \dots (8)$$

Hieraus finden wir die Komitante

$$K = (p^d q \pi^{n-d-1}) (q \varrho')^{d-1} \dots \dots \dots (9)$$

wo $\pi_{j_1 \dots j_{n-d-1}}$ und $\varrho'_{l_2 \dots l_d}$ willkürliche alternierende Tensoren bedeuten. Von K gelangt man, bis auf einen von Null verschiedenen numerischen Faktor zur ursprünglichen quadratischen p -Relation zurück, indem man alle $\pi_{j_1 \dots j_{n-d-1}}$, bei denen die Indexgruppe $(j_1 \dots j_{n-d-1})$ vom Komplement von $(i_1 \dots i_{d+1})$ verschieden ist und ebenso alle $\varrho'_{l_2 \dots l_d}$ wo $(l_2 \dots l_d) \neq (k_2 \dots k_d)$ ist, gleich Null setzt.

$$K \equiv 0 \{ \pi_{ikl\dots}, \varrho'_{rst\dots} \} \dots \dots \dots (10)$$

ergibt dann ein invariantes Gleichungssystem das (7) enthält, d.h. das jede quadratische p -Relation enthält.

¹⁾ Vgl. Inv. Theorie, p. 160.

Bei (9) können wir statt der $p_{i_1 \dots i_d}$ auch die Raumkoordinaten $p'_{j_1 \dots j_{n-d}}$ einführen:

$$K = d! (p' q) (p' \pi)^{n-d-1} (q \varrho')^{d-1}$$

und vermöge (8) wird jetzt:

$$\Pi = (p' q) p'_{j_1 \dots j_{n-d-1}} q_{k_2 \dots k_d} \dots \dots \dots (11)$$

Dies ist eine besonders einfache symbolische Darstellung der p -Relationen (7). Sie kann jedoch noch durch eine andere ersetzt werden, nämlich durch Gleichungen der Gestalt

$$X_2 = (p' q)^2 p'_{j_1 \dots j_{n-d-2}} q_{k_1 \dots k_{d-2}} = 0, \dots \dots \dots (12)$$

die also entstehen aus Komitanten mit einem Klammerfaktor $(p^d q^2 \pi^{n-d-2})$.

Gehen wir nämlich von (9) aus, dann können wir durch identisches Umformen alle $d-1$ Reihen q von $(q \varrho')^{d-1}$ in den Klammerfaktor $(q p^d \pi^{n-d-1})$ hineinbringen, wodurch K in eine Summe von Komitanten der Gestalt

$$(q^d p^m \pi^{n-d-m}) (p \varrho')^{d-m} (\pi \varrho')^{m-1} \quad (m = 2, 3, \dots) \dots \dots (13)$$

verwandelt wird¹⁾. Diese Komitanten (13) haben aber alle die Gestalt ΣX_2 , wo X_2 durch (12) gegeben ist. Somit ist jede quadratische p -Relation in der Form

$$\Pi = \Sigma X_2 \dots \dots \dots (14)$$

darstellbar. Aber es ist auch umgekehrt jedes X_2 eine Summe von Ausdrücken Π :

$$X_2 = \Sigma \Pi \dots \dots \dots (15)$$

Dies sieht man sofort, wenn man in $X_2 = 0$ oder in

$$(p^d q^2)_{i_1 i_2 \dots i_{d+1} k_2} q_{k_3 \dots k_d} = 0$$

die Determinante $(q^d p^2)_{i_1 i_2 \dots i_{d+1} k_2}$ nach der letzten Zeile entwickelt, wodurch

$$\Sigma_{i_1, k_2} \pm (p^d q)_{i_1 \dots i_{d+1}} q_{k_2 k_3 \dots k_d} = 0$$

entsteht.

Das Studium der höheren p -Relationen (d.h. der vom dritten und höheren Grad in den $p_{ikl\dots}$) hat dann von den Komitanten (13) auszugehen. So erhält man z.B. alle p -Relationen dritten Grades, wenn man in (13) eine oder mehr Reihen π durch das mit p und q äquivalente r ersetzt oder analog ϱ' durch r' . Dies gibt Komitanten der beiden Typen

$$(p^d q^m r^h \pi^{n-d-m-h}) (q \varrho')^{d-m} (r \sigma')^{d-h}$$

$$(p^d q^m \pi^{n-d-m}) (q \varrho')^{d-m-h} (q r')^h (r' \sigma)^{d-h}.$$

Doch ist dies bisher nur für den einfachsten Fall $d=2$ bei beliebigem n näher ausgeführt

¹⁾ Vergl. Inv. Theorie. p. 86.

§ 3. Syzygien bei fünf binären Linearformen.

Nehmen wir jetzt $n=5, d=2$. Dann sind die zehn Grössen p_{ik} Linienkoordinaten im G_5 . Hier ist die Struktur der p -Relationen bekannt und von W. H. YOUNG aufgedeckt ¹⁾.

Transponieren wir die Matrix

$$\mathfrak{M} = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{vmatrix}, \text{ so entsteht } \mathfrak{M}' = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \end{vmatrix}$$

und hier kann jedes Paar x_i, y_i als homogene Koordinaten eines Punktes P_i in einem binären Gebiete betrachtet werden. Die p_{ik} sind dann die zehn relativen projektiven Invarianten der fünf Punkte P_i und eine irreduzible p -Relation wird zu einer Syzygie zwischen diesen Invarianten. Die Struktur aller p -Relationen aufdecken heisst jetzt: alle irreduziblen Syzygien erster, zweiter, . . . Art angeben, bis nach dem HILBERT'schen Satze ²⁾ die Syzygienkette abbricht.

Hiebei heisst eine Syzygie $S=0$ irreduzibel, wenn keine Darstellung $S=S^{(1)} + S^{(2)} + \dots$ möglich ist, wo jedes $S^{(i)}=0$ schon für sich eine Syzygie vorstellt. Eine Syzygie m -ter Art $S_m=0$ liegt vor, wenn

$$S_m = N_1 S_{m-1}^{(1)} + N_2 S_{m-1}^{(2)} + \dots = 0 \dots \dots \dots (16)$$

ist, wo die $S_{m-1}^{(i)}$ Syzygien $(m-1)$ -ter und die N_i Polynome der Invarianten p_{ik} sind, wobei die folgenden Bedingungen erfüllt werden:

- a. Die N_i sind nicht identisch Null, wenn die Invarianten durch unabhängig Veränderliche ersetzt werden.
- b. Drückt man die S_{m-1} in (16) durch S_{m-2} aus, sodass $S_m = \sum M_v S_{m-2}^{(v)}$ entsteht, dann ist jedes M_v , als Polynom der Invarianten betrachtet, identisch Null.
- c. Für eine Syzygie erster Art S_1 gilt: $S_1 \equiv / \equiv 0 \{p_{ik}\}, \equiv 0 \{x_i, y_i\}$.
- d. Drückt man in einer $S_2 = \sum N_v S_1^{(v)}$ die S_1 durch die Invarianten aus, so entsteht identisch Null.

In unserem Falle sind also die irreduziblen Syzygien erster Art S_1 gegeben durch die fünf quadratischen p -Relationen.

$$A'_i = (p q')^2 q'_i = 0 \dots \dots \dots (17)$$

Setzen wir nun, mit noch unbestimmten X_i , für die Syzygien S_2 an:

$$B = \sum X_i A'_i = (X A') = (p q')^2 (q' X),$$

dann sind wegen $A'_i \equiv / \equiv 0 \{p_{ik}\}$ die X_i nicht konstant und da $B=0$ aus einer Komitante entstehen muss, haben wir $X_i = r_i (r U')$ zu setzen. Also wird:

$$B = (A' r) (r U') = (p q')^2 (q' r) (r U').$$

¹⁾ Atti di Torino 34 (1899) p. 596—599. Vgl. auch Inv. Theorie, p. 176.

²⁾ Vgl. Inv. Theorie, p. 171.

Hier ist $r_i (rU') \equiv 0 \{r_{ik}\}$, dagegen $B \equiv 0 \{p_{ik}\}$ weil

$$(p q')^2 (q' r) r_i \equiv 0 \{p_{ik}\} \dots \dots \dots (18)$$

Deshalb sind durch:

$$B = (A' r) r_i = 0 \dots \dots \dots (19)$$

alle irreduziblen S_2 gegeben.

Bei S_3 haben wir schliesslich den Ansatz

$$C = \sum U'_i B_i = (A' r) (r U') \equiv 0 \{B_i\}, \equiv 0 \{A'_i\},$$

also $r_i (rU') \equiv 0 \{p_{ik}\}$, woraus nach (18) und (19) $U'_i = q'_i (q' p)^2 = A'_i$ folgt.

Daher ist

$$C = \sum A'_i B_i = (A' B) = 0 \dots \dots \dots (20)$$

die einzige Syzygie dritter Art. Hier endigt die Syzygienkette, denn $D = M \cdot C = \sum M A'_i B_i \equiv 0 \{B_i\}$ hat nur die Lösung $M = 0$.

§ 4. Syzygien bei sechs binären Linearformen.

Bedeutend komplizierter liegen die Verhältnisse bei $n=6, d=2$. Hier können wir die $\binom{6}{2} = 15$ Grössen p_{ik} als Invarianten von sechs binären Linearformen betrachten und haben dann als irreduzible Syzygien S_1 die 15 quadratischen p -Relationen

$$A'_{ik} = (p q')^2 q'_{ik} = 0 \dots \dots \dots (21)$$

Eine S_2 entspringt aus einer identisch verschwindenden Komitante K , die linear ist in den A'_{ik} und bei der man überdies voraussetzen kann, dass sie höchstens drei Reihen x, y, z und höchstens drei Reihen u', v', w' enthält. Sind nämlich in K z.B. sechs Reihen x, y, \dots, z vorhanden, so entwickeln wir K in eine Gordan-Capellische Reihe

$$K = K_0 + (x y \dots z) \cdot K_1 + (x y \dots z)^2 \cdot K_2 + \dots$$

und hier liefern bereits $K_0 = 0, K_1 = 0, \dots$ Syzygien, sodass die aus $K = 0$ entspringenden S nicht irreduzibel sind. Sind in K fünf Reihen x, y, \dots, z vorhanden, so ermöglicht eine analoge Reihenentwicklung die Zusammenfassung $u'_1 = (xy \dots z)_{23456}$ u.s.f. zu einer Reihe u' . Sind schliesslich vier Reihen x, y, z und t in K vorhanden, so kann man sie zu $(u'v')_{12} = (xyzt)_{3456}$ u.s.f. zusammenfassen.

Um die irreduziblen S_2 zu finden haben wir deshalb zuerst alle Komitanten K zu ermitteln, die linear in den $A'_{ik} = A'_{i,\mu,\nu,\tau}$ sind und höchstens je drei Reihen x, y, z und u', v', w' enthalten.

Zur Verfügung stehen die Reihen:

$$A', p', q', \dots, u', v', w'; A, p, q, \dots, x, y, z \dots \dots (22)$$

Hieraus sind Faktoren erster und zweiter Art zu bilden. Von letzteren können wir absehen, da bei Klammerfaktoren mit Komplexsymbolen (A', p', A und p) immer Umformung auf Produkte von Linearfaktoren möglich ist. Da ferner (Ap') auf $(A'p)$ reduziert werden kann und weiters (pq') wegen

$$(p q') (p u') (q' x) (q' y) (q' z) = \frac{1}{4} [(u' x) (A' y) (A' z) + (u' y) (A' z) (A' x) + (u' z) (A' x) (A' y)] \quad (23)$$

als Reduzent zu betrachten is, bleiben für den Aufbau von K die Linearfaktoren übrig :

$$(A' p), (A' x), (A' u'), (p' x), (p' u'), (u' x) \dots \dots \dots (24)$$

Hieraus ergeben sich dann :

$$\left. \begin{aligned} K_1 &= (A' p)^2, K_2 = (A' p) (A' x) (p' u'), K_3 = (A' x) (A' y), \\ K_4 &= (A' p) (p' u') (A' q) (q' v') \end{aligned} \right\} (25)$$

wozu noch die Typen

$$K_5 = (p' u') (p' v') \quad , \quad K_6 = (u' x) \dots \dots \dots (26)$$

als Komitanten ohne A'_{ik} kommen.

Setzen wir :

$$J = \frac{1}{8} (p^2 q^2 r^2) = (p' q)^2 (p' r)^2 = (A' r)^2 \dots \dots \dots (27)$$

dann ist $K_1 = J$ und wir haben die Beziehung :

$$(A' r) (A' x) (r' u') = (p' q)^2 (q' r) (r' u') (q' x) = \frac{1}{8} \cdot J \cdot (u' x) \dots \dots (28)$$

Dies gibt :

$$(B' x) (B u') = (A' p) (A' x) (p' u') - \frac{1}{8} (A' p)^2 \cdot (u' x) \equiv 0 \{ x, u', p_{ik} \} \dots (29)$$

Hier ist $B'_k B_i \equiv \equiv 0 \{ A'_{ik} \}$, aber $\equiv 0 \{ p_{ik} \}$, d. h. wenn die A'_{ik} durch die p_{ik} ausgedrückt werden. Somit sind $B'_k B_i = 0$ irreduzible Syzygien S_2 wovon aber wegen (Verjüngung von (29)) :

$$(B' B) = (A' p)^2 - (A' p)^2 \equiv 0 \{ A'_{ik} \} \dots \dots \dots (30)$$

nur fünf von den sechs Gleichungen $B'_i B_i = 0$ zu den irreduziblen S_2 zu rechnen sind, sodass wir durch (29) $6^2 - 1 = 35$ irreduzible S_2 dargestellt erhalten :

$$B'_k B_i = (A' p) A'_k p_i - \frac{1}{8} (A' p)^2 \delta^k_i = 0 \dots \dots \dots (31)$$

Jetzt wollen wir noch beweisen, dass dies alle irreduziblen S_2 sind. K ist linear und homogen in den Komitanten der Typen (25). Wegen (29) können wir nach Aufzählung der Syzygien (31) K_2 und K_4 weglassen, da jetzt der Faktor $(A' p)$ als Reduzent zu betrachten ist. Somit ist K linear in $K_1 = (A' p)^2$ und $K_3 = A'_{yz}, A'_{zx}, A'_{xy}$ [Wir schreiben im Weiteren kurz

$$\begin{aligned} A'_{xy} &\text{ für } (A' x) (A' y), \\ q'_{xyz} &\text{ für } (q' x) (q' y) (q' z) \text{ u.s.w.} \end{aligned}$$

mit Koeffizienten φ und ψ_i , die Polynome in K_5 und K_6 von (26) sind :

$$K = (A' p)^2 \cdot \varphi + \sum_1^3 A'_{yz} \cdot \psi_1 \equiv 0 \{ p_{ik} \}.$$

Wählen wir hier p_{ik} halbspeziell, d. h. $J = (A' p)^2 = 0$, aber nicht alle A'_{ik} gleich Null, so folgt aus

$$A'_{yz} \cdot \psi_1 + A'_{zx} \cdot \psi_2 + A'_{xy} \cdot \psi_3 \equiv 0$$

für $z_i = p_i (p' \sigma')$, dass $\psi_3 \equiv 0$ wird, wo ψ_3 ein Polynom ist mit den

Argumenten $K_5 = p_{v'w'}, p_{w'u'}, p_{u'v'}$ und $K_6 = (u'x), (u'y), (u'p) (p\sigma')$ u.s.w. Hieraus folgt aber $\psi_3 \equiv 0 \{p_{ik}\}$; also ist auch $\varphi \equiv \{p_{ik}\}$ und daher $K \equiv 0 \{A'_{ik}\}$, es liegt als keine S_2 vor und durch (29) sind alle S_2 aufgezählt.

§ 5. Die Syzygien dritter Art S_3 .

Hier müssen wir die Komitanten L ermitteln, die linear sind in den durch (29) gegebenen $B'_k B_i$. L enthält also die Reihen

$$B', A', A_1, \dots, p', q', \dots, u', v', w'; B, A, A_1, \dots, p, q, \dots, x, y, z. \quad (32)$$

wo A'_1, A'_2, \dots äquivalent sind mit A' . Hieraus ergeben sich die Faktoren

$$\left. \begin{aligned} (B' B), (B' A), (B' p), (B' x) \\ (A' B), (A' A_1), (A' p), (A' x) \\ (p' B), (p' A), (p' q), (p' x) \\ (u' B), (u' A), (u' p), (u' x) \end{aligned} \right\} \dots \dots \dots (33)$$

(pq') ist ein Reduzent; $(p'A)$ führt auf (pA') und gibt nach (28) Reduktion auf $(A'p)^2$. $(B'B)$ ist nach (30) $\equiv 0 \{A'_{ik}\}$, ist aber trotzdem keine S_3 , da die Koeffizienten der $B'_k B_i$ konstant sind. Auch der Faktor $(A'A_1)$ ist ein Reduzent; es ist nämlich

$$(A'A_1)(A'x) = \frac{1}{3} (p^2 q^2 A_1 x)$$

und bringt man hier alle drei weiteren Reihen A_1 in den Klammerfaktor, so entstehen Glieder mit

$$(A_1^3 px) = 24 (A'p)(A'x),$$

was Reduktion auf $(A'p)^2$ ergibt.

Aus den übrigen Faktoren von (33) erhält man die folgenden Komitanten:

$$\left. \begin{aligned} L_1 = (B'x)(Bu') & \quad L_4 = (B'q)(qu')(Bv') \\ L_2 = (B'x)(Bq')q'_{xyz} & \quad L_5 = (B'q)(qu')(Br')r'_{xyz} \\ L_3 = (B'x)(BA')(A'y) & \quad L_6 = (B'q)(qu')(BA')(A'x) \\ L_7 = (B'A)A_{u'v'w'}(Bu') \\ L_8 = (B'A)A_{u'v'w'}(Bq')q'_{xyz} \\ L_9 = (B'A)A_{u'v'w'}(BA')(A'x) \end{aligned} \right\} \dots (34)$$

Dazu kommen noch die Komitanten ohne $B'_k B_i$, die schon bei (25) und (26) aufgezählt wurden, nämlich:

$$(A'p)^2, A'_{xy}, p_{u'v'}, (u'x) \dots \dots \dots (35)$$

Von (34) sind L_7, L_8 und L_9 reduzibel auf L_4, L_5 und L_6 , wenn wir in $(B'A)A_{u'v'w'}$ A^4 durch p^2q^2 ausdrücken.

Es sei nun $L = 0$ eine S_3 . L ist linear in den Komitanten L_1 bis L_6 mit Koeffizienten, die Polynome der Komitanten (35) sind. Da $L \equiv 0 \{A'_{ik}\}$ sein muss wenn wir die $B'_k B_i$ nach (29) durch die A'_{ik} ausdrücken, müssen wir

zuerst die L_i von (34) linear in den A'_{ik} schreiben. Wir wollen dann M'_{ik} statt derjenigen A'_{ik} schreiben, die von den $B'_k B'_i$ stammen. Aus diese Weise erhalten wir:

$$\begin{aligned}
 L_1 &= (M'p)(M'x)(pu') - \frac{1}{6}(M'p) \cdot (u'x) \\
 L_2 &= (M'p)(M'x)(pq')q'_{xyz} = \frac{1}{4} [(M'y)(M'x) \cdot A'_{xz} - (M'z)(M'x) \cdot A'_{xy}] \\
 L_3 &= (M'p)(M'x)(pA')(A'y) - \frac{1}{6}(M'p)^2 \cdot A'_{xy} = \frac{1}{6} J M'_{xy} - \frac{1}{6}(M'p)^2 \cdot A'_{xy} \\
 L_4 &= (M'p)(M'q)(pv')(qu') - \frac{1}{6}(M'p)^2 \cdot (qv')(qu') \\
 L_5 &= (M'p)(M'q)(qu')(pr')r'_{xyz} - \frac{1}{6}(M'p)^2 \cdot (qr')(qu')r'_{xyz} = \\
 &= \frac{1}{4} [(M'q)(M'x)(qu') \cdot A'_{yz} + (M'q)(M'y)(qu') \cdot A'_{zx} + \\
 &+ (M'q)(M'z)(qu') \cdot A'_{xy}] - \frac{1}{24}(M'p)^2 [(u'x)A'_{yz} + \\
 &+ (u'y)A'_{zx} + (u'z)A'_{xy}] \\
 L_6 &= (M'p)(M'q)(pA')(qu')(A'x) - \frac{1}{6}(M'p)^2 \cdot (A'q)(qu')(A'x) = \\
 &= \frac{1}{6} J [(M'q)(M'x)(qu') - \frac{1}{6}(M'p)^2 \cdot (u'x)].
 \end{aligned} \tag{36}$$

In diesen Formeln ist die Reduktion ausgeführt, wenn ein Faktor (pq') oder ein Faktor (pA') vorhanden war.

Nach (25) haben wir also:

$$\begin{aligned}
 L_1(x, u') &= K_2(x, u') - \frac{1}{6} K_1 \cdot (u'x) \\
 L_2(x, xyz) &= \frac{1}{4} [K_3(x, z) \cdot A'_{xy} - K_3(x, y) \cdot A'_{xz}] \\
 L_3(x, y) &= \frac{1}{6} J \cdot K_3(x, y) - \frac{1}{6} K_1 \cdot A'_{xy} \\
 L_4(u', v') &= K_4(v', u') - \frac{1}{6} K_1 \cdot p_{v'u'} \\
 L_5(u', xyz) &= \frac{1}{4} \{ [K_2(x, u') - \frac{1}{6} K_1 \cdot (u'x)] A'_{yz} + \\
 &+ [K_2(y, u') - \frac{1}{6} K_1 \cdot (u'y)] \cdot A'_{zx} + [K_2(z, u') - \frac{1}{6} K_1 \cdot (u'z)] \cdot A'_{xy} \} \\
 L_6(x, u') &= \frac{1}{6} J [K_2(x, u') - \frac{1}{6} K_1 \cdot (u'x)].
 \end{aligned} \tag{37}$$

Hieraus folgen:

$$L_6 - \frac{1}{6} J L_1 \equiv 0 \{ M'_{ik} \} \dots \tag{38}$$

$$L_5 - \frac{1}{4} [L_1(x, u') \cdot A'_{yz} + L_1(y, u') \cdot A'_{zx} + L_1(z, u') \cdot A'_{xy}] \equiv 0 \{ M'_{ik} \}. \tag{39}$$

Wir können also jedes L_5 und L_6 durch L_1 ausdrücken und haben die S_3 aufzuzählen, die entspringen aus:

$$(C'x)(Cu') = (B'q)(qu')(BA')(A'x) - \frac{1}{6} J (B'x)(Bu') = 0. \tag{40}$$

$$\begin{aligned}
 (Du') \Delta'_{xyz} &= (B'q)(qu')(Br')r'_{xyz} - \frac{1}{4} [(B'x)(Bu') \cdot A'_{yz} + \\
 &+ (B'y)(Bu') \cdot A'_{zx} + (B'z)(Bu') \cdot A'_{xy}] = 0.
 \end{aligned} \tag{41}$$

Jedes weitere L ist jetzt linear und homogen in L_1 bis L_4 . Bei L_1 haben wir entsprechend den sechs Reihen x, y, z, u', v', w' , neun Komitanten $L_1(x, u'), L_1(y, u'), \dots$. Bei L_2 haben wir drei Komitanten, die nach (37) verknüpft sind durch die Beziehung:

$$L_2(x, xyz) \cdot A'_{yz} + L_2(y, xyz) \cdot A'_{zx} + L_2(z, xyz) \cdot A'_{xy} \equiv 0 \{ K_3 \}. \tag{42}$$

Hier ergeben sich also wieder S_3 :

$$\left. \begin{aligned} (\alpha'x)^2 (\beta'y)^2 (\gamma'z)^2 &= (B'x)(Bq') q'_{xyz} \cdot A'_{yz} + \\ &+ (B'y)(Bq') q'_{xyz} \cdot A'_{zx} + (B'z)(Bq') q'_{xyz} \cdot A'_{xy} = 0. \end{aligned} \right\} (43)$$

Bei $L_3(x, y)$ haben wir nur drei Komitanten, da wegen $K_3(x, y) = M'_{xy}$ $K_3(x, x) = 0$ wird und also auch $L_3(x, x) \equiv 0 \{M'_{ik}\}$ ist, was wieder Syzygien dritter Art liefert, entspringend aus:

$$(R'x)^2 = L_3(x, x) = (B'x)(BA')(A'x) = 0 \dots (44)$$

Ebenso haben wir bei $L_4(u', v')$ nur drei Komitanten; denn auch hier ist wegen $K_4(u', v') = (M'p)(pu')(M'q)(qv') = -K_4(v', u')$:

$$(Ru')^2 = L_4(u', u') = (B'q)(qu')(Bu') = 0 \dots (45)$$

Machen wir jetzt den allgemeinen Ansatz

$$L = \sum_9 L_1(x, u') \cdot \alpha_{11} + \sum_3 L_2(x, xyz) \cdot \beta_1 + \sum_3 L_3(x, y) \cdot \gamma_3 + \sum_3 L_4(u', v') \cdot \delta_3, \quad (46)$$

dann muss dies $\equiv 0 \{M'_{ik}\}$ sein, wenn wir die L_i nach (37) durch die K_i ausdrücken und setzen:

$$\left. \begin{aligned} K_1 &= (M'p)^2, \quad K_2 = (M'p)(M'x)(pu'), \quad K_3 = M'_{xy}, \\ &K_4 = (M'p)(pu')(M'q)(qv'). \end{aligned} \right\} (47)$$

Die Koeffizienten α_{ik} , β_i , γ_i und δ_i in (46) sind Polynome der Typen:

$$J = (A'p)^2, \quad A'_{xy}, \quad p_{u'v'}, \quad (u'x) \dots (48)$$

Bei K_2 in (47) haben wir 9, bei K_3 und K_4 je drei Komitanten, (46) geht daher über in:

$$\left. \begin{aligned} (M'p)^2 \cdot \Phi + \sum_9 (M'p)(M'x)(pu') \cdot \alpha_{11} + \\ + \sum_3 M'_{xy} \cdot \chi_3 - \sum_3 (M'p)(pu')(M'q)(qv') \cdot \delta_3 \equiv 0 \{M'_{ik}\}. \end{aligned} \right\} (49)$$

Dabei ist nach (46):

$$\Phi = -\frac{1}{6} \sum_9 (u'x) \cdot \alpha_{11} - \frac{1}{6} \sum_3 A'_{xy} \cdot \gamma_3 + \frac{1}{6} \sum_3 p_{u'v'} \cdot \delta_3 \dots (50)$$

(49) muss gelten für alle M'_{ik} . Setzen wir $M'_{ik} = (\sigma' \tau')_{ik}$, dann erhalten wir:

$$\left. \begin{aligned} 2p_{\sigma'\tau'} \cdot \Phi + \sum_9 [p_{\sigma'u'}(\tau'x) - p_{\tau'u'}(\sigma'x)] \cdot \alpha_{11} + \sum_3 [(\sigma'x)(\tau'y) - (\sigma'y)(\tau'x)] \cdot \chi_3 - \\ - \sum_3 [p_{\sigma'u'} \cdot q_{\tau'v'} - p_{\tau'u'} \cdot q_{\sigma'v'}] \cdot \delta_3 \equiv 0 \{ \sigma', \tau' \}. \end{aligned} \right\} (51)$$

Wählen wir jetzt σ' so, dass das lineare Gebiet $G_5(\sigma')$ durch die 5 Punkte $x, y, z, p, (pu')$ und $q, (qv')$ geht, dass also:

$$\sigma'_i = (x y z p q)_{23456} (pu')(qv')$$

ist und setzen gleichzeitig:

$$\tau'_i = (x y z p q)_{23456} (pu')(qw'),$$

so ist erstens bei nicht-speziellen $p_{ik} : (p\sigma')(p\tau') \equiv \equiv 0$ und zweitens verschwinden in (51) alle Glieder bis auf das erste. Also ist $\Phi \equiv 0$.

Wählen wir dann σ' wieder so wie eben ausgeführt, hingegen τ' so, dass das $G_5(\tau')$ durch x, y, z und $p_i(pu')$ geht, so ergibt sich $\delta_3 = 0$; es verschwinden also auch alle δ_i . Wählen wir dann σ' so, dass

$$(\sigma'x) = (\sigma'y) = (\sigma'z) = 0$$

ist, so folgt $\alpha_{ik} = 0$ und es bleibt:

$$\sum_3 K_3(x, y) \cdot \gamma_3 = \sum_3 M'_{xy} \cdot \gamma_3 = 0, \quad \dots \quad (52)$$

woraus leicht $\gamma_i = 0$ gefolgert wird. Statt (46) kommt also jetzt:

$$L = \sum_3 L_2(x, x y z) \cdot \beta_1 + \sum_3 L_3(x, y) \cdot \gamma_3 = 0, \quad \dots \quad (53)$$

wobei wegen $\Phi = 0$ nach (50) die Gleichungen bestehen:

$$\sum_3 A'_{xy} \cdot \gamma_3 = 0 \quad \dots \quad (54)$$

$$\left. \begin{aligned} \chi_1 &= +\frac{1}{4} \beta_2 A'_{xy} - \frac{1}{4} \beta_3 A'_{zx} + \frac{1}{8} J \cdot \gamma_1 = 0 \\ \chi_2 &= +\frac{1}{4} \beta_3 A'_{yz} - \frac{1}{4} \beta_1 A'_{xy} + \frac{1}{8} J \cdot \gamma_2 = 0 \\ \chi_3 &= +\frac{1}{4} \beta_1 A'_{zx} - \frac{1}{4} \beta_2 A'_{yz} + \frac{1}{8} J \cdot \gamma_3 = 0 \end{aligned} \right\} \dots \quad (55)$$

Sind in (55) alle $\beta_i = 0$, so folgt $\gamma_i = 0$, wir erhalten keine S_3 . Ist $\beta_2 = \beta_3 = 0$, $\beta_1 \neq 0$, so folgt $\gamma_1 = 0$ und nach (55):

$$\frac{1}{4} \beta_1 \cdot A'_{xy} = \frac{1}{8} J \cdot \gamma_2 \quad , \quad \frac{1}{4} \beta_1 A'_{zx} = -\frac{1}{8} J \cdot \gamma_3$$

Aus (53) ergibt sich dann die Komitante

$$J \cdot L_2(x, x y z) - \frac{3}{2} [L_3(x, z) A'_{xy} - L_3(x, y) A'_{zx}] \equiv 0 \{M'_{ik}\}, \quad \dots \quad (56)$$

aus welcher wieder Syzygien dritter Art entstehen:

$$(E'x)^2 (F'y) (F'z) = J \cdot (B'x) (Bq') q'_{xyz} - \frac{3}{2} [(B'x) (BA') (A'z) \cdot A'_{xy} - (B'x) (BA') (A'y) \cdot A'_{zx}] \quad \left. \vphantom{(E'x)^2} \right\} \quad (57)$$

Sind in (55) schliesslich alle $\beta_i \neq 0$, so folgt $\gamma_i = 0$, da die Koeffizientendeterminante der β_i in diesen Gleichungen verschwindet. Es bleiben dann drei lineare und homogene Gleichungen für die β_i , deren Lösung auf die schon aufgezählte Beziehung (42) führt.

§ 6. Die irreduziblen S_3 .

Nach (44) haben wir

$$(R'x) (R'y) = \frac{1}{2} [(B'x) (BA') (A'y) + (B'y) (BA') (A'x)];$$

ersetzen wir hier y_i durch $p_i(pu')$, so entsteht:

$$\begin{aligned} (R'x) (R'p)(pu') &= \frac{1}{2} [(B'x) (BA') (A'p)(pu') + (B'p)(pu') (BA') (A'x)] = \\ &= \frac{1}{2} [-\frac{1}{8} J \cdot (B'x) (Bu') + L_6] = \frac{1}{2} [L_6 - \frac{1}{8} J L_1]. \end{aligned}$$

also nach (38) und (40):

$$(C'x)(Cu') = 2(R'x)(R'p)(pu') \dots \dots \dots (58)$$

und dies gilt identisch in allen $B'_k B'_i$. Die durch $C'_k C'_i = 0$ dargestellten Syzygien dritter Art sind also auf $R'_{ik} = 0$ reduzierbar.

Auf analoge Weise bekommen wir bei (45) aus

$$(Ru')(Rv') = \frac{1}{2} [(B'q)(qu')(Bv') + (B'q)(qv')(Bu')]$$

wenn wir $v' = p' p'_{xyz}$ setzen:

$$(Ru')(Rp') p'_{xyz} = \frac{1}{2} [(B'q)(Bu')(qp') p'_{xyz} + (B'q)(Bp')(qu') p'_{xyz}],$$

also nach (23), (34), (39) und (41):

$$(Du') \Delta'_{xyz} = 2(Ru')(Rp') p'_{xyz} \dots \dots \dots (59)$$

d. h. auch die Syzygien $D_i \Delta'_{klm} = 0$ sind reduzibel.

Ersetzen wir weiters in $(C'x)(C'u)$ die u' durch $p' p'_{xyz}$, so finden wir nach (40):

$$(C'x)(Cp') p'_{xyz} = (B'q)(BA')(A'x)(qp') p'_{xyz} - \frac{1}{6} J \cdot (B'x)(Bp') p'_{xyz}$$

und hier führt die Umformung von (qp') nach (57) zur Beziehung

$$\left. \begin{aligned} \frac{1}{6} (E'x)^2 F'_{yz} &= - (C'x)(Cp') p'_{xyz} - \frac{1}{4} (R'x)^2 \cdot A'_{yz} - \\ &\quad - \frac{1}{2} (R'x)(R'y) \cdot A'_{zx} - \frac{1}{2} (R'x)(R'z) \cdot A'_{xy} \end{aligned} \right\} (60)$$

d. h. nach (58):

$$(E'x)^2 F'_{yz} = \frac{3}{2} (R'x)^2 \cdot A'_{yz} \dots \dots \dots (61)$$

Es sind also auch alle Syzygien $E'_{jk} F'_{lm} = 0$ reduzibel.

Ersetzen wir schliesslich in $(Du') \Delta'_{xyz}$ die u' durch $p' p'_{xyz}$, so wird nach (41):

$$(Dp') p'_{xyz} \Delta'_{xyz} = (B'q)(qp') p'_{xyz} (Br') r'_{xyz} - \frac{1}{4} [(B'x)(Bp') p'_{xyz} \cdot A'_{yz} + \dots],$$

also nach (42) und (43), wenn wir im ersten Gliede rechts (qp') nach (23) umformen:

$$(a'x)^2 (\beta'y)^2 (\gamma'z)^2 = - 2 (Dp') p'_{xyz} \Delta'_{xyz} \dots \dots \dots (62)$$

Nach (59) ist also:

$$(a'x)^2 (\beta'y)^2 (\gamma'z)^2 = - 4 (Rp') p'_{xyz} (Rq') q'_{xyz} \dots \dots \dots (63)$$

wodurch auch die Syzygien $\alpha'_{ik} \beta'_{lm} \gamma'_{rs} = 0$ reduziert sind.

Es bleiben somit die 21 irreduziblen S_3 , die aus $(R'x)^2 \equiv 0$ entstehen:

$$R'_{ik} = \frac{1}{2} [B'_i (BA') A'_k + B'_k (BA') A'_i] \equiv 0 \{B'_k B'_i\}, \equiv 0 \{A'_{ik}\} \quad (64)$$

und die 21 irreduziblen S_3 , die sich aus $(Ru')^2 \equiv 0$ ergeben:

$$R_{ik} = \frac{1}{2} [(B'q) q_i B_k + (B'q) q_k B_i] \equiv 0 \{B'_k B'_i\}, \equiv 0 \{A'_{ik}\} \quad (65)$$

§ 8. Die Syzygien vierter Art.

Es sei $T = 0$ eine S_4 . T ist linear in den R'_{ik} und R_{lm} und verschwindet identisch, wenn wir diese nach (64) und (65) durch die $B'_k B'_i$ ausdrücken.

Die Typen von linear nicht-reduziblen Komitanten mit R_{ik} und R'_{ik} sind, ausgedrückt durch die L_i von (34):

$$\left. \begin{aligned} R_1 &= 2(Ru')(Rv') = L_4(u', v') + L_4(v', u') \\ R_2 &= 2(Ru')(Rp') p'_{xyz} = L_5 - \frac{1}{4}[L_1(x, u') \cdot A'_{yz} + L_1(y, u') \cdot A'_{zx} + \\ &\quad + L_1(z, u') \cdot A'_{xy}] \\ R_3 &= 2(Ru')(RA')(A'x) = L_6 - \frac{1}{6}J \cdot L_1(x, u') \\ R_4 &= 2(RA')(A'x)(Rq') q'_{xyz} = -\frac{1}{6}JL_2(x, xyz) - \frac{1}{4}[L_3(x, x) \cdot \\ &\quad \cdot A'_{yz} + L_3(y, x) \cdot A'_{zx} + L_3(z, x) \cdot A'_{xy}] \end{aligned} \right\} (66)$$

$$\left. \begin{aligned} R_5 &= 2(RA')(A'x)(RB')(B'y) = -\frac{1}{6}J[L_3(x, y) + L_3(y, x)] \\ R_6 &= 2(Rp') p'_{xyz} (Rq) q'_{xyz} = \\ &= -\frac{1}{2}[L_2(x, xyz) \cdot A'_{yz} + L_2(y, xyz) \cdot A'_{zx} + L_2(z, xyz) \cdot A'_{xy}] \\ R'_1 &= 2(R'x)(R'y) = L_3(x, y) + L_3(y, x) \\ R'_2 &= 2(R'x)(R'p)(pu') = L_6(x, u') - \frac{1}{6}J \cdot L_1(x, u') \\ R'_3 &= 2(R'p)(pu')(R'q)(qv') = -\frac{1}{6}J[L_4(u', v') + L_4(v', u')] \end{aligned} \right\} (67)$$

Aus der letzten der Gleichungen (67) und der ersten von (66) ergibt sich:

$$\left. \begin{aligned} T_1 &= (Tu')(Tv') = R'_3(u', v') + \frac{1}{6}J \cdot R_1(u', v') = \\ &= (R'p)(pu')(R'q)(qv') + \frac{1}{6}J(Ru')(Rv') \equiv \equiv 0 \{R_{ik}, R'_{em}, \equiv 0 \{B'_k B_i\} \} \end{aligned} \right\} (68)$$

woraus 21 S_4 entspringen. Wir wollen beweisen, dass dies alle irreduziblen Syzygien vierter Art sind.

Eine S_4 $T=0$ entsteht durch Elimination der L_i aus den Gleichungen (66) und (67). Die Elimination von L_4 gibt (68), die von L_6 gibt:

$$T_3 = (Tu')(TA')(A'x) = -\frac{1}{6}J \cdot R'_2(x, u') + \frac{1}{6}J \cdot R_3(x, u') \equiv 0 \{B'_k B_i\} \quad (69)$$

Die Elimination von L_3 gibt:

$$T_5 = (TA')(A'x)(TB')(B'y) = \frac{1}{6}J[R_5(x, y) + \frac{1}{6}J R'_1(x, y)] \equiv 0 \{B'_k B_i\}. \quad (70)$$

Setzen wir ferner in (68) $v' = p' p_{xyz}$, so entsteht:

$$\left. \begin{aligned} 4(Tu')(Tp') p'_{xyz} &= \frac{2}{3}J(Ru')(Rp') p'_{xyz} - \\ &- [(R'x)(R'p)(pu') \cdot A'_{yz} + (R'y)(R'p)(pu') \cdot A'_{zx} + (R'z)(R'p)(pu') \cdot A'_{xy}]. \end{aligned} \right\} (71)$$

Hier kann man rechter Hand nach (69) und (67)₂ die R' eliminieren und erhält:

$$\left. \begin{aligned} 4J(Tu')(Tp') p'_{xyz} - 6 \Sigma (Tu')(TA')(A'x) \cdot A'_{yz} &= \\ &= \frac{2}{3}J R_2(u', xyz) - [R_3(x, u') \cdot A'_{yz} + R_3(y, u') \cdot A'_{zx} + \\ &\quad + R_3(z, u') \cdot A'_{xy}] \equiv 0 \{B'_k B_i\}, \end{aligned} \right\} (72)$$

woraus sich für

$$\left. \begin{aligned} u' = A'(A'x) \text{ bzw. für } u' = q' q'_{xyz} \text{ ergeben:} \\ 4J(TA')(A'x)(Tp') p'_{xyz} - 6 \Sigma (TA')(A'x)(TB')(B'x) \cdot A'_{yz} &= \\ &= \frac{2}{3}J R_4(x, xyz) - [R_5(x, x) \cdot A'_{yz} + R_5(x, y) \cdot A'_{zx} + \\ &\quad + R_5(x, z) \cdot A'_{xy}] \equiv 0 \{B'_k B_i\} \end{aligned} \right\} (73)$$

$$\left. \begin{aligned} 4 J (T p') p'_{xyz} (T q') q'_{xyz} - 6 \Sigma (T A') (A' x) (T p') p'_{xyz} \cdot A'_{yz} = \\ = \frac{2}{3} J R_6 - [R_4 (x, x y z) \cdot A'_{yz} + R_4 (y, x y z) \cdot A'_{zx} + \\ + R_4 (z, x y z) \cdot A'_{xy}] \equiv 0 \{B'_k B_i\}. \end{aligned} \right\} (74)$$

Alle diese identisch in den $B'_k B_i$ geltenden Beziehungen sind S_4 die auf T_1 reduzibel sind.

Sei nun $T=0$ eine S_4 . T ist linear in den R_i und R'_k von (66) und (67). Multiplizieren wir T mit J^2 , so ist jedes R'_3 nach (68) durch R_1 , jedes R'_2 nach (69) durch R_3 , jedes JR'_1 nach (70) durch R_5 ausdrückbar, sodass in T keine R'_i mehr vorkommen.

Jetzt können wir auf dieselbe Weise vermöge (72), (73) und (74) R_2 , R_4 und R_6 eliminieren. Dies gibt schliesslich für T den Ansatz:

$$J^2 T = \sum_1^6 R_1 (v', w') \cdot a_{23} + \sum_1^9 R_3 (u', x) \cdot \beta_{11} + \sum_1^6 R_5 (y, z) \cdot \gamma_{23} \equiv 0 \{B'_k B_i\} \quad (75)$$

Hier sind die a_{ik} , β_{ik} und γ_{ik} Polynome in den Komitanten der Typen

$$J = (A' p)^2, \quad A'_{xy}, \quad p_{u'v'}, \quad (u' x).$$

Aus (75) folgt nun, dass alle a_{ik} , β_{ik} und γ_{ik} Null sein müssen, d.h. dass auch $T \equiv 0 \{R_{ik}\}$ gelten muss, d.h. (68) gibt die einzigen S_4 .

Das Verschwinden der Koeffizienten in (75) beweist man wie folgt. Wir wählen $B'_k B_i$ so, dass B' und B Grössenreihen sind:

$$B'_1 = (x y z p q)_{23456} (p u') (q v') \quad , \quad B_1 = (u' v' w' \sigma' \tau')_{23456}.$$

Dann haben wir:

$$(B' x) = 0, \quad (B' y) = 0, \quad (B' z) = 0, \quad (B' p) (p u') = 0, \quad (B' q) (q v') = 0$$

und

$$(B u') = 0, \quad (B v') = 0, \quad (B w') = 0.$$

Bei diesen Annahmen reduziert sich (75) auf

$$(B' q) (q w') \cdot [(B A') (A' x) \cdot \beta_{13} + (B A') (A' y) \cdot \beta_{23} + (B A') (A' z) \cdot \beta_{33}] \equiv 0,$$

woraus wegen

$$(B' q) (q w') = (x y z p q r) (p u') (p v') (r w') \equiv 0$$

$\beta_{ik} = 0$ folgt. Daher kommt statt (75):

$$\begin{aligned} T = \sum_1^6 [(B' p) (p v') (B w') + (B' p) (p w') (B v')] \cdot a_{23} - \\ - \frac{1}{6} J \cdot \sum_1^6 [(B' y) (B A') (A' z) + (B' z) (B A') (A' y)] \equiv 0 \{B'_k B_i\} \end{aligned}$$

Wählt man hier B' so, dass $(B' x) = 0$, $(B' y) = 0$ und B so, dass $(B u') = 0$, $(B v') = 0$ und $(B w') = 0$ ist, so folgt $\gamma_{ik} = 0$ und analog finden wir, dass auch $\alpha_{ik} = 0$ sein muss.

Wir haben somit 21 irreduzible S_4 , die aus (68) entspringen:

$$T_{ik} = (R' p) p_i (R' q) q_k + \frac{1}{6} J \cdot R_{ik} \equiv 0 \{R'_{ik}, R'_{lm}\}, \equiv 0 \{B'_k B_i\}$$

§ 9. Das Abbrechen der Syzygienkette.

Die in den T_{ik} linearen, linear-irreduziblen Komitanten sind analog zu (66) gegeben durch:

$$\left. \begin{aligned} T_1 &= (Tu')(Tv') = R'_3(u', v') + \frac{1}{6} JR_1(u', v') \\ T_2 &= (Tu')(Tp') p'_{xyz} = -\frac{1}{4} [R'_2(x, u') \cdot A'_{yz} + \dots] + \frac{1}{6} JR_2(u', xyz) \\ T_3 &= (Tu')(TA')(A'x) = -\frac{1}{6} JR'_2(x, u') + \frac{1}{6} JR_3(x, u') \\ T_4 &= (TA')(A'x)(Tp') p'_{xyz} = \frac{1}{24} J[R'_1(x, x) \cdot A'_{yz} + \dots] + \frac{1}{6} JR_4(x, xyz) \\ T_5 &= (TA')(A'x)(TB')(B'y) = \frac{1}{36} J^2 \cdot R'_1(x, y) + \frac{1}{6} JR_5(x, y) \\ T_6 &= (Tp') p'_{xyz} (Tq') q'_{xyz} = \frac{1}{16} \Sigma R'_1(x, x) \cdot A'_{yz} A'_{yz} + \frac{1}{6} J \cdot R_6. \end{aligned} \right\} (76)$$

Es sei $S=0$ eine S_5 . Dann ist S linear in den Komitanten (76). Wir wollen zeigen, dass man $J^2 \cdot S$ durch T_1, T_3 und T_5 allein ausdrücken kann. Hiezu ist nachzuweisen, dass die Gleichungen (72), (73) und (74) des vorigen § nicht nur für beliebige $B'_k B'_l$, sondern auch bei willkürlichen R_{jk} gelten. Dieser Nachweis kann wie folgt erbracht werden.

Formt man $(A'^2 B'^2 C'^2)(p' t)$ um, so entsteht:

$$(A'^2 B'^2 C'^2)(p' t) p'_{xyz} = -6(p' A' B'^2 C'^2) p'_{xyz} (A' t) \dots (77)$$

Da aber, wie leicht nachzurechnen:

$$(A'^2 B'^2 C'^2) = \frac{1}{3} J^2 \dots (78)$$

$$(p' A' B'^2 C'^2) p'_{xyz} (A' t) = \frac{1}{3} J [A'_{xt} A'_{yz} + A'_{yt} A'_{zx} + A'_{zt} A'_{xy}] \dots (79)$$

ist, wird aus (77), wenn wir $t=R$ setzen und mit (Ru') multiplizieren:

$$\frac{2}{3} J \cdot (Ru')(Rp') p'_{xyz} = \Sigma (Ru')(RA')(A'x) \cdot A'_{yz} \dots (80)$$

was Gleichung (72) mit beliebigem R_{ik} ist.

Wir können daher auch T_{ik} statt R_{ik} in (80) schreiben und dann in $J^2 S$ mit Hilfe der zu (72), (73) und (74) analogen Gleichungen jedes JT_2, JT_4 und JT_6 durch T_3 und T_5 ausdrücken. Es entsteht so:

$$\left. \begin{aligned} J^2 S &= \sum_1^6 T_{v'w'} \cdot \alpha_{23} + \sum_1^9 (Tu')(TA')(A'x) \cdot \beta_{11} + \\ &\quad + \sum_1^6 (TA')(A'y)(TB')(B'z) \cdot \gamma_{23} \end{aligned} \right\} (81)$$

und dies muss $\equiv 0 \{R'_{ik}, R_{lm}\}$ sein. Nach den Gleichungen (76) folgt dann, dass auch

$$\begin{aligned} \sum_1^6 R_{v'w'} \cdot \alpha_{23} + \sum_1^9 (Ru')(RA')(A'x) \cdot \beta_{11} + \\ + \sum_1^6 (RA')(A'y)(RB')(B'z) \cdot \gamma_{23} \equiv 0 \{R_{ik}\} \end{aligned}$$

sein muss. Hieraus folgt aber nach dem vorigen § das Verschwinden aller α_{ik}, β_{ik} und γ_{ik} , d.h. $S \equiv 0 \{T_{ik}\}$. Es gibt also keine S_5 mehr.

Wir stellen schliesslich noch die erhaltenen Syzygien übersichtlich zusammen, wobei wir von der Schreibweise mit hoch- und tiefstehenden Indizes Gebrauch machen, indem wir z. B. p^{iklm} statt p'_{iklm} schreiben:

- 15 irred. Syzygien erster Art $A^{ik} = p^{ik\lambda\mu} p_{\lambda\mu} = 0$
 35 „ „ zweiter „ $B_i^k = A^{\lambda k} p_{\lambda i} - \frac{1}{6} A^{\lambda\mu} p_{\lambda\mu} \delta_i^k = 0$
 42 „ „ dritter „ $\begin{cases} R^{ik} = \frac{1}{2} [B_\lambda^i A^{\lambda k} + B_\lambda^k A^{\lambda i}] = 0 \\ R_{ik} = \frac{1}{2} [B_i^\lambda p_{\lambda k} + B_k^\lambda p_{\lambda i}] = 0 \end{cases}$
 21 „ „ vierter „ $T_{ik} = R^{\lambda\mu} p_{\lambda i} p_{\mu k} + \frac{1}{6} J \cdot R_{ik} = 0$
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Physiology. — *The action of veratrin on skeletal muscles. At the same time an application of our knowledge of the segmentation of the frog's M. gastrocnemius on the problem of the refractory stage of skeletal muscles after veratrin poisoning.* By S. DE BOER.
(Communicated by Prof. R. MAGNUS.)

(Communicated at the meeting of January 29, 1927).

I performed this investigation in consequence of Dr. QUERIDO's dissertation (1). From my first investigation (2) I deduced an argument in favour of the theory of BOTTAZZI (3), who ascribes the contracture of striped muscles poisoned with veratrin to a slow contraction of the sarcoplasm. At present I should rather use the expression *tonus substratum* instead of *sarcoplasm* to indicate that the site of the muscle-tonus and the contractures is not yet determined. As for the rest I maintain the conclusion which I drew at that time. I discriminate therefore the *tonus substratum* from the *substratum* in which the twitches occur.

To prevent confusion I wish to point out, that I shall continue to give the determinations which were used formerly. I oppose the contractures to the tetanic contractions. Both form a steady and continuous shortening of the muscle. The mechanical effect for both is therefore steady. The tetanic contraction, however, is accompanied by intermittent electrical phenomena. Therefore the tetanic contractions consist of discontinuous processes. Such discontinuous electrical processes do not accompany the contractures. We find here a continuous deflection of the string in monophasic leading off in such a way that the tissue of the muscle in contracture is electro-negative to an injured part of the muscle or to the tendon. By muscle tonus I mean the tension, which is caused by a slight shortening and which is maintained reflectorily in the striped muscles. This slight shortening must be freed from the last traces of tetanus, which we conclude from the fact, that the discontinuous phenomena must be absent. I represented these questions in this way in my publications of 1913 and 1914. The introduction of new denominations cannot, in my opinion, make our insight in this matter clearer.

During my investigations of 1913 I found in the monophasic leading off a continuous deflection in the electrogram, accompanied particularly in the beginning of the contracture by discontinuous deflections. After this the curve of the electrogram became smooth, while the contracture still continued. I found also, that these discontinuous deflections were more distinct after the first stimulus. Repeating the experiment, the oscillations of the electrogram curve diminished or disappeared. These results differed very slightly if at all, from those which HOFFMANN (4) obtained after strong poisoning. HOFFMANN, however, only mentions oscillations in the beginning of the contracture, after which the electrogram curve became smooth. I hope to indicate the cause of this in my theoretical discussion.

We both found therefore discontinuous electrical phenomena, *and at the same time I found a continuous deflection in the monophasic leading off as well.*

Our conclusions however are quite different. HOFFMANN gave it as his opinion that the second shortening was a tetanic contraction. That the electrogram-curve became smooth while the maximum of shortening still continued, was not according to HOFFMANN of great importance. In any case it did not alter his conception. I have already mentioned my conception. In my opinion the oscillations of the electrogram were not of fundamental importance.

Dr. QUERIDO considers that activity of the same substratum causus both shortenings of the veratrin curve (the initial twitch and the contracture). His chief argument is that he could not see any response after an induction shock at the beginning of the contracture and ordinarily not until the contracture decreased. From this experiment he concluded that the muscle was refractory. Now after my investigations on the segmentation of the M gastrocnemius, I am able to prove that this conclusion is wrong. The scheme in fig. 1 shows a veratrin curve. The so-called refractory stage should have a duration from A to B. In any case Dr. QUERIDO did not

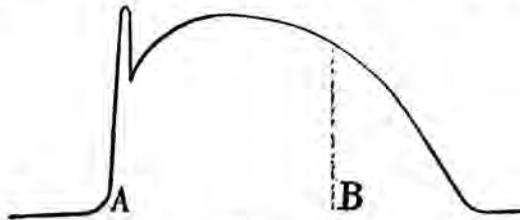


Fig. 1.

Veratrincurve. Duration of the so-called refractory stage according to Dr. QUERIDO from A to B.

obtain any effect during that period after the second stimulus. I emphasize that Dr. QUERIDO arrived at this conclusion after the experiments which he performed by means of mechanical registration of the shortening curves. To make my point of view more easily understood, I mention here an observation of BIEDERMANN (5), who put one half of a muscle in water, so that this part swelled. He then stimulated the part of the muscle which had absorbed water. This swollen part did not contract, but the other part did. From this experiment BIEDERMANN concluded that the half of the muscle that had absorbed water, propagated the excitation wave but could not contract. Contractility and conductivity according to BIEDERMANN are two different properties of the skeletal muscles.

ENGELMANN (6) performed similar experiments in the heart. He put the auricles of the hearts of frogs in water until these heartsections swelled. He then stimulated the auricles which did not contract. The ventricle however contracted after an A—V interval. ENGELMANN therefore came to the same conclusion for the heart-muscle. During an investigation in

1917 (7) I was able to confirm the experiments of BIEDERMANN and ENGELMANN. I rejected however their conclusion. I demonstrated indeed that the swollen part of the muscle or the auricles had already shortened on account of the edematous shortening already present and that therefore a stimulus did not have any effect. I proved this by means of experiments. In water rigor — owing to the swelling of the muscle-fibres — a contraction of the ventricle could not be recorded on account of edematous swelling. The electrograms of the ventricle however continued from which we may conclude that the processes of the systoles continued rhythmically. I performed similar experiments with the gastrocnemius muscle. From this investigation it was also evident that the mechanical and electrical responses were inseparable. The literature of this subject is mentioned in a communication by FULTON (8), who came to the same conclusion in an interesting investigation.

I proceeded from my above mentioned investigation and I arrived at the conclusion that Dr. QUERIDO did therefore not see any mechanical response after an induction shock during the contracture, *because at that moment the muscle was already shortened.*

This conclusion was quite right. I proved it in the following way:

In former investigations (9, 10, 11, 12 and 13) I showed that the M. gastrocnemius of the frog consists of two groups of muscle fibres. The one group is innervated by the 8th, the other by the 9th spinal nerve. Both nerves were tied and cut near their exit from the vertebrae. Both nerves were placed on separate electrodes, which were connected by means of a commutator to the secondary circuit of an induction apparatus. I was thus able to stimulate each of the nerves separately by means of an induction shock. The tendon of the gastrocnemius was connected with a recording lever for the registration of the contraction curves. Then three to five drops of acetate of veratrin 1 p. c. were injected into the dorsal lymph sac. As soon as the poisoning process had proceeded sufficiently, the experiment began. At first both roots were stimulated separately and when both curves arrived about at the same height, the experiment was continued. After a pause of five minutes, the 8th spinal nerve was stimulated by an induction shock. A distinct veratrin curve followed. During the contracture both nerves were stimulated alternately. The result is shown in Fig. 2. We see that at the maximum height of the contracture an induction shock of the 8th as well as of the 9th spinal nerve does not give any response. At a later period, when the height of the curve has decreased, an induction shock of either of both nerves does give response. I obtained the same effect if the ninth spinal nerve was stimulated first as is shown in the second part of Fig. 2. From this experiment it is evident, that during the veratrin contracture, produced after an induction shock of the 8th spinal nerve, an induction shock of the 9th does not give any mechanical response, because the muscle is then already shortened. If besides the already contracting fibres more

begin to contract, the height of the mechanical curve does not increase. This would certainly happen if the two groups of muscle fibres were arranged one above the other.

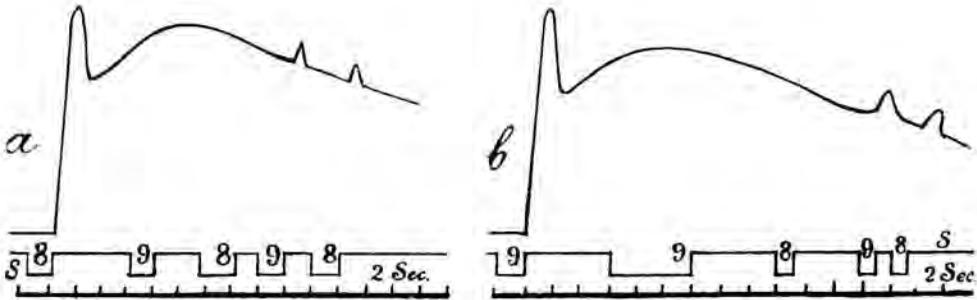


Fig. 2.

a. At 8 a veratrin contraction of the *M. gastrocnemius* is caused by an induction shock of the 8th spinal nerve. During the contracture the 8th and the 9th spinal nerve are stimulated alternately. At the maximum height of the contraction neither after an induction shock of the 8th nor after an induction shock of the 9th spinal nerve does any response in the mechanogram appear. But as soon as the height of the contracture decreases it occurs.

b. The experiment of a is repeated here with the same preparation; the result is the same. Now, however, the veratrin curve is registered after an induction shock of the 9th spinal nerve. An upward deflection of the signal arises when the primary circuit is broken. Only the break shocks are sent in. This also applies to fig. 3 and 4.

If however one of the spinal nerves innervates only a small part of the muscle fibres of the *M. gastrocnemius* and the other spinal nerve a large part, a stimulus of the first nerve gives a small contraction curve and the same stimulus of the second one yields a big contraction curve if the same weight is raised in both cases. If I repeat the experiment with a similar preparation, I easily obtain a confirmation of my experiment if I first register the big curve. If however I first register the small one e.g. after an induction shock of the 8th spinal nerve, then during the maximum height of the contracture an induction shock of the 8th spinal nerve has no effect. An induction shock of the 9th spinal nerve however yields a large veratrin curve, as is clearly shown in Fig 3.

Now one might suppose that during a veratrin contracture produced after an induction shock of the 8th spinal nerve not only the group of muscle fibres, innervated by this nerve would be refractory but also the remaining part innervated by the 9th spinal nerve. This possibility however is excluded, because, as I have proved, both groups of muscle fibres contract insulated chemically and electrically after poisoning with veratrin (9).

Both groups of muscles-fibres therefore work independently of each other in a chemical as well as in an electrical sense. If either spinal nerve is stimulated rhythmically until the contractions are those of great fatigue and the other nerve is then stimulated, the contraction caused by it shows no sign of fatigue. Further recovery from fatigue which has been produced by one nerve is unaffected by fatigue stimulation of the other nerve. It was

therefore evident from my former investigations that during activity of one group of muscle fibres the other group cannot be refractory.

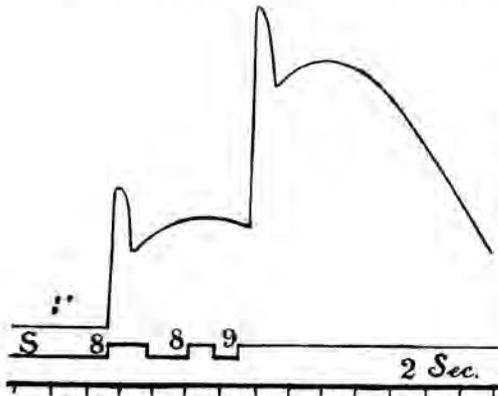


Fig. 3.

The same experiment as in fig. 2. but now the veratrin curve is small after an induction shock of the 8th spinal nerve. An induction shock of the 8th spinal nerve at the maximum height of the contracture does not have any effect now, an induction shock of the 9th spinal nerve however has effect because such a large veratrin curve is produced by it.

It is thus evident, that by means of registration of the mechanical curves, as Dr. QUERIDO did, it is impossible to decide, whether during the contracture the muscle is refractory or not. Indeed an eventual shortening of the muscle cannot be registered mechanically during the contracture.

I have therefore performed experiments, in which the mechanograms are registered as well as the electrograms. From these experiments it was clearly shown that during a maximum contracture induction shocks do not give any effect in the mechanogram but at the same time activity of the muscle does give a distinct deflection of the electrogram. As soon as the height of the contracture decreases, the mechanogram also shows a distinct shortening of the muscle after an induction shock. This effect is clearly shown in Fig. 4¹⁾. If however the contracture is

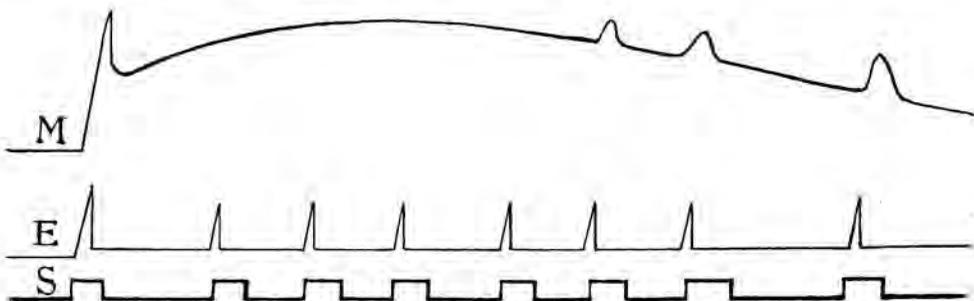


Fig. 4.

The sciatic plexus is placed on the electrode. At the top the mechanogram (M) is registered, at the bottom the electrogram curve (E). (One electrode is placed a little below the nervous aeqator, the other on the tendon, shifting of the electrodes is prevented by fixing them to the muscle). At the maximum height of the contracture the induction shocks have no effect on the mechanogram, they do have effect on the electrogram curve.

¹⁾ For this investigation I used a platina string. For the purpose of this investigation it was sufficient.

smaller, then after each stimulus the response is also produced in the mechanogram. We conclude therefore from these experiments that during the veratrin contracture the skeletal muscles are not refractory. Dr. QUERIDO's conclusion is therefore wrong ¹⁾).

I shall mention more particulars of this investigation in a more elaborate publication and I shall then demonstrate that already without registration of the electrograms I was able to prove that during the whole contracture the muscle is not refractory. I should like to call your attention to one particularity, viz. that in the mechanogram a shortening is produced after an induction shock not only at the descending part but also in the ascending part of the contracture at the same time when at the maximum height of the contracture a stimulus does not give any affect.

Theoretical discussion.

After the before mentioned investigations it is evident that the skeletal muscles are not refractory during the veratrin contracture. If an induction shock did not give any effect in the mechanogram, this was caused by the fact that at that moment the muscle was already shortened. Here we have to do with a similar experiment to that which was explained by me in 1917, with the skeletal muscles which had absorbed water. The conclusion of Dr. QUERIDO: "I consider, however, that the facts, which I mentioned about the refractory period of the muscle poisoned with veratrin, justify the conclusion that both parts of the curve are caused by the activity of one substratum and that consequently the theory of BOTTAZZI is not tenable", is therefore anticipated. If now, after my investigation, we should continue with Dr. QUERIDO's conception, we necessarily arrive at the opposite conclusion, viz. that the veratrin contracture is caused by the activity of another substratum than that in which the twitches take place. At the same time after my investigation all the conclusions drawn by Dr. QUERIDO about the absolute refractory stage, the relative refractory stage and about the determination of the size of the maximum stimulus by means of the refractory period are excluded.

Dr. QUERIDO draws a second argument in favour of his conception from the electrograms, which accompany the veratrin contracture. P. HOFFMANN (4) was the first who investigated the veratrin contracture by means of the string-galvanometer. He found during the contracture oscillations of the electrogram curve, which were most pronounced in the beginning of the contracture. These oscillations were much smaller after strong poisoning and then generally limited to the beginning of the contracture. The electrogram-curve afterwards became quite smooth, although the mechanical curve was still at the maximum. The oscillations,

¹⁾ The 24 curves which Dr. QUERIDO publishes on page 58 and 59 of his dissertation are therefore wrongly explained.

which P. HOFFMANN found after strong poisoning, were very small and became still smaller or disappeared when the stimulus was repeated several times.

In 1913 (2) my publication appeared on this subject. I also investigated the veratrin contracture by means of the string-galvanometer and *I only used a strong poisoning*. I used the monophasic leading off on purpose. To prevent shifting of the electrodes I registered the isometric curves besides the electrograms. I only made use of a silvered quartz string, which was very sensitive and which easily produced the rapid oscillations ¹⁾. Just as HOFFMANN I also found the rapid oscillations in the beginning of the contracture. These oscillations of my curves (2) were more pronounced than those of HOFFMANN's experiments in which he used a strong poisoning. These oscillations decreased or disappeared, when the poisoning was very strong or if the stimulus were repeated several times. Up to this point the results of HOFFMANN and myself agree. But I added one more to the before mentioned phenomena. I also found in my monophasic leading off a slow deflection of the string during the contracture. If we use the biphasic leading off, this slow deflection does not appear. I will explain this by a schema (in which I neglect the rapid oscillations, vide fig. 5). When we make use of the biphasic leading off, the muscle at first becomes electro-negative under the upper electrode and this electro-negativity continues as long as the contracture lasts. This causes the deflection A—B—C—D. A short time afterwards the muscle tissue under the other electrode becomes electronegative and this causes the deflection E—F—G—H in the opposite direction. The result is then a rapid biphasic deflection during the initial contraction and during the contracture the string remains in the zero in consequence of addition of both slow deflections. In the monophasic leading off therefore after the initial rapid deflection A—B the slow one C—D occurs because from this deflection the deflection E—F—G—H is not subtracted. (Vide fig. 5.) The

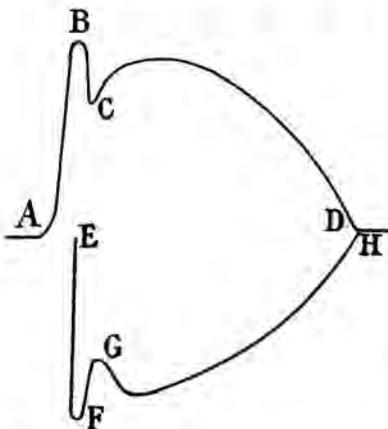


Fig. 5.

Schema of biphasical leading off in a muscle poisoned with veratrin. The electronegativity at the upper electrode is represented by the curve A—B—C—D, that at the lower one by the curve E—F—G—H. The result is a rapid biphasic curve in the beginning. Later on the string remains in the zero under the influence of two negativities with opposite sign.

¹⁾ Vide page 145 of my publication.

difference between the results of HOFFMANN and myself is only caused by the different leading off and not as Dr. QUERIDO, who also used the biphasic leading off suggests by the fact that the string which I used, was not sensitive enough ¹⁾. In his experiments Dr. QUERIDO like HOFFMANN and myself obtained oscillations which also were more distinct in the beginning of the contracture. The slow deflection was wanting of course, in any case it was not pronounced and always in the same direction. In some of his curves the string deflects a little in one direction, in other curves a little to the opposite direction. These different deviations are probably caused by impurities of the experiment.

Dr. QUERIDO rather remarkably does *not* mention my investigations of 1913 and he has forgotten to mention the rapid oscillations which I described in many of my curves ²⁾. Moreover I mention from the text of my previous publication (1913): "Die zweite langsame Schwankung bot gewöhnlich in ihrer ersten Strecke sekundäre Schwankungen, die bisweilen sehr ausgesprochen waren." I already mentioned in short, that the oscillations which HOFFMANN considered as the main point, were considered by me as accidental and I am still of this opinion. Muscle contractures indeed are often accompanied by fibrillar contractions and we also see them after poisoning with veratrin especially after weak poisoning. In my opinion these oscillations are caused by the fibrillar contractions. The magnified shadow of the muscle, which Dr. QUERIDO photographed obtains the small oscillations by the same cause. The fibrillar contractions are only more pronounced in weak poisoning with veratrin and less pronounced in strong poisoning. At the same time it is easily explainable that in strong poisoning the oscillations are less pronounced than in weak poisoning ³⁾. Dr. QUERIDO's second argument viz. the presence of oscillations is insufficient to justify his conception (tetanic contraction). This argument was not new and had already been used by HOFFMANN in his investigation, which I knew in 1913 and which I mentioned circumstantially.

LITERATURE.

1. QUERIDO A. Over de werking van veratrine op dwarsgestreept spierweefsel. Amsterdam. Em. Querido's Uitgevers-Maatschappij. 1926.
2. DE BOER S. Ueber das Elektrogram der veratrinisierten Muskeln. Zeitschr. f. Biologie Bd. 61. 1913. p. 143.

¹⁾ As I already mentioned, I used a sensitive silvered quartzstring, which could move very rapidly. The tension as described in my annotations was usually such that 1mV caused a deflection of 1 to 2mM. In some experiments 1 mV caused a deflection of 3 or 4 mM. I have already mentioned in my former communication (1913) that with the slack strings the slow deflection especially is more pronounced than the rapid one.

²⁾ I emphasize here that HOFFMANN as well as I saw that the oscillations disappeared under certain circumstances.

³⁾ The oscillations were still very distinct in the strong poisoning (4 to 10 drops of acetate of veratrin 1⁰/₀) which I used, but from the investigation of HOFFMANN we know that in very weak poisoning the oscillations are much more pronounced than in strong poisoning.

3. BOTTAZZI PH. Ueber die Wirkung des Veratrins und anderer Stoffe auf die quergestreifte atriale und glatte Muskulatur. Archiv f. Physiol. 1901. p. 377.
 4. HOFFMANN P. Ueber die Aktionströme des mit Veratrin vergifteten Muskels. Zeitschr. f. Biologie Bd. 58. 1912. p. 55.
 5. BIEDERMANN W. Beiträge zur allgemeinen Nerven- und Muskelphysiologie. Sitzungsber. der Math. Naturwissensch. Classe der Akademie der Wissensch. in Wien. Bd. 97. 3. Abt. 1888. p. 84.
 6. ENGELMANN TH. W. Beobachtungen und Versuche am suspendierten Herzen. II. Ueber die Leitung der Bewegungsreize im Herzen. Pflüger's Archiv Bd. 56. 1894. S. 149.
 7. DE BOER S. Are contractility and conductivity two separate properties of the skeleton-muscles and the heart? Verslag Kon. Akad. v. Wet. Versl. Deel 26 1917 p. 535 and Proc. Vol. 20 p. 554.
 8. FULTON J. F. The inseparability of mechanical and electrical Responses in skeletal Muscle. The americ. Journ. of Physiol. Vol. 75. p. 261. 1925.
 9. DE BOER S. The double innervation of the M. gastrocnemius. Journ. of Physiol. Vol. 60. 1925. p. 215.
 10. DE BOER S. The monosegmental innervation of the muscle fibers of the M. gastrocnemius of the frog. The Americ. Journ. of Physiol. Vol. 75. p. 285. 1925.
 11. DE BOER S. On the segmentation of skeletal muscles in the frog (*rana esculenta*) Royal Acad. Amsterdam, Proc. Vol. 28. p. 821. 1925 and Ueber die Verteilung der Muskel-segmente im M. gastrocnemius des Frosches. Pflüger's Archiv Bd. 211. p. 636. 1926.
 12. DE BOER S. Die segmentelle Innervation einiger Froschmuskeln. Zeitschr. für Biologie. Vol. 85, p. 471. 1927.
 13. DE BOER S. The Americ Journ. of Physiol. 1927.
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[That nevertheless with solid-vapour or liquid-vapour $\left(\frac{dp}{dt}\right)_{\text{coex.}}$ approaches to 0 at $T=0$, has also nothing whatever to do with the theorem of heat, but is only owing to this that in $\left(\frac{dp}{dt}\right)_{\text{coex.}} = \frac{L}{Tv_3}$ the vapour volume v_3 (v_1 or v_2 may be neglected by the side of v_3 at low temperatures) then approaches ∞ more strongly than T approaches 0. For with $pv_3 = RT$ follows $\frac{1}{p} \left(\frac{dp}{dt}\right)_{\text{coex.}} = \frac{L}{RT^2}$, hence (supposing L at these low temperatures independent of T) $\log p = -\frac{L}{RT} + C$, $p = C'e^{-L/RT}$, so that v_3 becomes $= RT : p = C''Te^{L/RT}$, in which L (the heat of sublimation or evaporation) remains finite throughout].

And the same thing applies to the melting heat Q , which can become $=0$, but not only at $T=0$ (cf. § 3). For this quantity is not only determined by the difference of the *thermic* Energies, as NERNST and others believe, but also by the much more important difference of the *potential* Energies. The latter is often overlooked.¹⁾

II. The shape of the melting-point line in the neighbourhood of $T=0$.

The difference of the *potential energies* just mentioned is never $=0$, unless the liquid and the solid state should be *identical*, but then there are of course not two coexisting phases. The truth on this point is, that it only appears from the experimental data that at about 1° abs. p passes through a *minimum*, where $Q=0$. But at $T=0$ Q will become $= aT \log T$ (as we shall see in § 3), so that (we henceforth omit the indication coex. with $\frac{dp}{dt}$)

$$\left(\frac{dp}{dt}\right)_{T=0} = \frac{Q_0}{T(v_2-v_1)} = -\infty$$

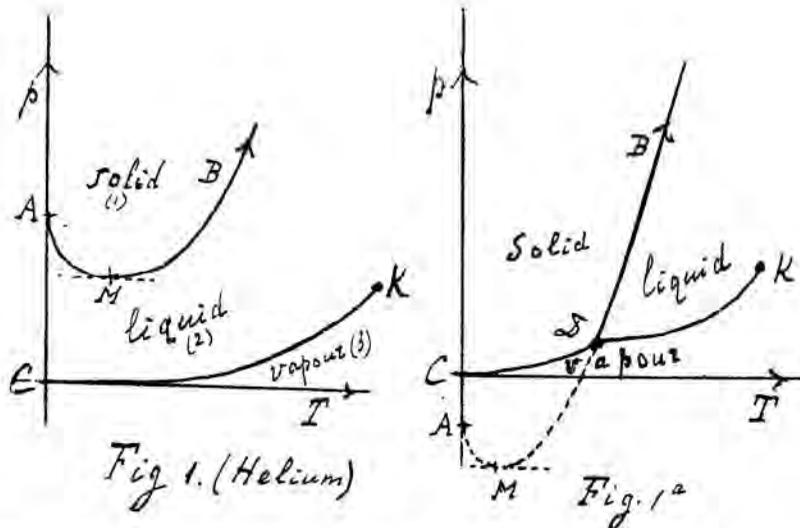
(see Fig. 1 A), since at $T=0$ v_2-v_1 (at finite pressure) either remains finite, or (accidentally) might also be $=0$.

For all other substances the melting-point line will intersect the vapour line, so that the ordinary course of Fig. 1^a will be obtained, with a triple-point at D , but in Helium *no* intersection takes place (in consequence of the circumstances set forth in my preceding Paper).

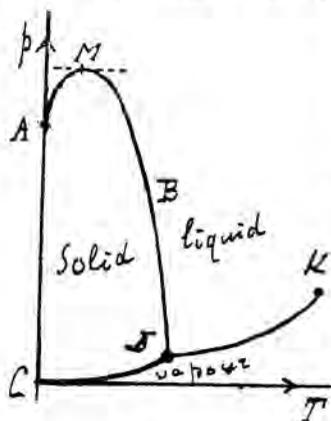
[At very high pressures v_2 will more and more approach to v_1 , hence $\frac{dp}{dt} = \frac{Q}{T(v_2-v_1)}$ will approach to ∞ , because at ordinary and higher tem-

¹⁾ See with regard to the use of these shortened formulae by NERNST and others some examples in my Paper in These Proc. 27, p. 897—913, especially § 7: *A thermodynamic sophism* (p. 911—913).

peratures Q is finite positive, which consequently — if T remains finite — gives an asymptotic course to the vertical line $T_{lim.}$, in which $T_{lim.}$ can be greater or smaller than the critical temperature].



We will point out here, that in the case $v_2 < v_1$, which sometimes occurs (the liquid density greater than that of the solid substance), the course of the melting-point line will be just the reverse, f.i. as drawn in Fig. 2. Then



$\left(\frac{dp}{dt}\right)_{T=0}$ is $= +\infty$, and there will occur a maximum at M . It is self-evident, that in this case the melting-point line always intersects the vapour line.

The shape of the melting-point line in this latter case was fully described and derived by me in a series of 7 papers already more than 17 years ago¹⁾.

III. The course of the melting heat Q .

At lower temperatures the following equation can evidently be written for this (the index 2 refers to the liquid), when E_0 is the zero-point energy of the solid Helium:

$$Q = E_2 - E_1 = (k_2 T - E_0 - \theta_1 T^4) + (A_2 - A_1) + (P_2 - P_1) + p(v_2 - v_1). \quad (3)$$

¹⁾ „On the Solid State”. Cf. III, and IV, These Proc. of June 1909, p. 133 (Fig. 6 of the Plate). Also Arch. Néerl. (2) 15, p. 1—56 (1910) (Plate III, Fig. 6); also Chemisch Weekbl. 1910, N^o. 50, p. 1—28 (Fig. 3) and 1912, N^o. 37, p. 744—755 (Fig. 2). With regard to the ordinary case $v_2 > v_1$, I at first assumed critical endpoints on the strength of the now abandoned association theory, which seems now rather improbable to me.

in which the difference of the potential energies of the attractive forces, viz. $A_2 - A_1$, is evidently $= \frac{a_1}{v_1} - \frac{a_2}{v_2}$. Further the difference $P_2 - P_1$ of the potential energies of the static repulsive forces will be $= 0 - P_1$, since this quantity does not exist in the liquid state. For there the molecules are not confined to definite positions of equilibrium. For $P_1 = \int_v^{\infty} \frac{a/v}{v-b} dv$ we find¹⁾ a somewhat complicated expression, which

of course disappears at $v = \infty$, and attains its greatest value, viz. $+\infty$, at $v = v_{00}$ ($p = \infty$). Also k_2 , the specific heat at (infinitely great) constant volume of the liquid helium, remains finite in the neighbourhood of $T = 0$, but $-$ in consequence of the decrease of a at decrease of the temperature²⁾ $-$ it is smaller than 3 at higher temperatures. For we have seen in our preceding paper on Helium, that in the case of liquid Helium no degradation of the Energy appears down to the lowest temperatures $-$ in opposition to what many investigators suppose $-$ so that the RT -law remains valid there. All the abnormal phenomena found for Helium can be accounted for by the dependence on the temperature of a and b .

If we now neglect $\theta_1 T^4$ by the side of $k_2 T$, we get:

$$Q = k_2 T + \left(\frac{a_1}{v_1} - \frac{a_2}{v_2} \right) - (P_1 + E_0) + p(v_2 - v_1),$$

for which we may write at very low temperatures:

$$Q = \underline{k_0 T - C + p(v_2 - v_1)}, \dots \dots \dots (3^a)$$

since $\left(\frac{a_1}{v_1} - \frac{a_2}{v_2} \right) - (P_1 + E_0)$ will vary very little with the temperature³⁾, as the volumes remain almost constant; k_0 having been written for the limiting value of the specific heat at low temperatures.

When at the melting equilibrium the thermodynamic potentials of the two phases are put equal, it follows from $E_1 - TS_1 = E_2 - TS_2$, that $Q = E_2 - E_1$ must also be $= T(S_2 - S_1)$.

Now $S_2 = k_2 \log T + R \log(v_2 - b_2) + (S_2)_0$, while $S_1 = \frac{4}{3} AT^3$. If $v_2 - b_2 = RT : (p + a/v_2^2) = RT : a/v_2^2$ is written, we get the following equation:

$$Q = T \left[k_2 + R \log T + R \log R - R \log a/v_2^2 + (S_2)_0 - \frac{4}{3} AT^3 \right],$$

¹⁾ Cf. These Proc. 29, N^o. 4, p. 502 (1926), Footnote 1. The quantity γ is the same as that which occurs in the equation of state (2).

²⁾ These Proc. 29 N^o. 10, p. 1303-1334.

³⁾ As a is a function of the temperature, and according to our preceding paper $= a_0 + \alpha T - \gamma T^2$, in the expression for the Energy $\frac{a}{v}$ must be replaced by $\frac{a - T a'}{v} = \frac{a_0 + \gamma T^2}{v}$, which will differ little from $\frac{a_0}{v_0}$ in the neighbourhood of $T = 0$ ($a' = \left(\frac{da}{dt} \right)_v$).

for which at very low temperatures may be written, neglecting the term with T^3 :

$$Q = T [(k_0 + R) \log T + C'], \dots \dots \dots (3^b)$$

when k_0 is the limiting value of the specific heat of the liquid Helium at $T=0$.

From this it is clearly seen, that at $T=0$ Q indeed approaches to $T(k_0+R)\log T$, i.e. to $0 \times -\infty = 0$, but $\frac{Q}{T}$ to $(k_0+R)\log T = -\infty$, so that also $\frac{dp}{dt}$ will approach $-\infty$ (see Fig. 1), and *not* to 0, as KEESOM assumes. At about 1° abs. Q is *again* $=0$, because then $(k_0+R)\log T$ is $= -C'$; but now also $\frac{Q}{T}$ is $=0$, hence $\frac{dp}{dt} = 0$, which has been found experimentally by KEESOM. At still higher values of T both Q and $\frac{dp}{dt}$ become positive.

The same thing follows, of course, also from (3^a), when the course of p and of Q according to (3^b) is taken into consideration. But it is better to use the equation (3^a) for the calculation of the vapour pressure at low temperatures. From

$$p = \frac{Q + C - k_0 T}{v_2 - v_1}$$

follows at $T=0$, when $Q = 0$, that $p_0 = C : (v_2 - v_1)$. Inversely the quantity $C = \left(\frac{a_1}{v_1} - \frac{a_2}{v_2} \right) - (P_1 + E_0)$ at $T=0$ might be calculated from p_0 and $v_2 - v_1$.

If, as is often done, the *potential* energies has been neglected, C would have become $= -E_0$, and in contradiction to observation, the limiting pressure p_0 would have become negative at $T=0$!

We see accordingly that here the theorem of heat (or rather the T^4 -law of DEBIJE) does not play any essential part, and that the fact, that $\frac{dp}{dt}$ becomes 0 at $\pm 1^\circ$ abs., by no means involves that $\frac{dp}{dt}$ would also have to disappear at $T=0$, which — we already pointed this out in § 1 — is not required by any theorem. If Prof. KEESOM should succeed in descending to still lower temperatures, he will very probably be able to observe after the minimum at $1^\circ,2$ abs. a change of the sign of $\frac{dp}{dt}$ in the melting point line (from M to A). This would at the same time be an interesting confirmation of my theory. ¹⁾

Tavel sur Clarens, Suisse, Dec. 2, 1926.

¹⁾ Cf. a later Article in a following number of these Proc. 1927, where the whole problem is treated in a more ample way.

Chemistry. — *Roentgenspectrographic Investigations on the Structure of the Artificial Ultramarines and the Problem concerning their Relations to the Minerals Hauyne, Nosean, Sodalite, Lazurite and Nephelite.* By Prof. Dr. F. M. JAEGER, Dr. H. G. K. WESTENBRINK and Mr. F. A. VAN MELLE.

(Communicated at the meeting of January 29, 1927).

§ 1. Notwithstanding the extensive technical experience gradually obtained in the preparation of the *ultramarine*-dyes during almost a hundred years, and notwithstanding the very great number of investigations made with the purpose to elucidate the true structure of these remarkable compounds, — it must be acknowledged, that the problem concerning the true chemical nature of the *ultramarines* still remains unsolved.

After CLEMENT and DESORMES ¹⁾ in 1806 gave the data of a complete analysis of the blue ultramarine, GUIMET ²⁾ and GMELIN ³⁾ in 1828 discovered the methods allowing the artificial preparation in truly unlimited quantities of the precious dye formerly only obtained from *lapis lazuli*. In this case also chemical synthesis has finally surpassed nature, because experience has taught, that ultramarines of all kinds of colour can be obtained ⁴⁾ by mere variations of the applied method, by which the colouring matter was prepared by heating with *sodium-sulphide* or with a mixture of *sodium-sulphate* and coal some *sodium-aluminium-silicates* of a composition analogous to that of *nephelite*. Now, besides the original *blue* ultramarine, also *yellow, green, pink, red* and *violet* ultramarines are prepared; and within each of these colour-limits, a great number of different modifications may, moreover, be obtained, which differ from each other only by slight gradations of colour or by their particularly warm glow.

The occurrence of the blue ultramarine-dye in *lazurite* and of the blue colour in many *hauynes, noseans* and in some *sodalites*, have since long drawn the attention of mineralogists to the fact, that presumably a closer constitutional relation may exist between the natural *ultramarine* on the one side, and the remarkable minerals mentioned on the other, which minerals hitherto were considered to belong to one and the same family. More particularly BRÖGGER and BÄCKSTRÖM ⁵⁾ emphasized

¹⁾ J. K. CLÉMENT and DESORMES, *Ann. de Chim. et Phys.* **57**, 317, (1806).

²⁾ J. B. GUIMET, *Bull. de la Soc. d'Encouragement, etc.*, (1828).

³⁾ G. C. GMELIN, *Abh. Naturw. Ges. Württemberg*, (1828), **2**, 191; SCHWEIGGER's *Polyt. Journ.* **54**, 360.

⁴⁾ See also R. HOFFMANN: „Ultramarin“, Braunschweig, (1902); L. BOCK, „Die Konstitution der Ultramarine“, Braunschweig, (1924); idem, „Die Fabrikation der Ultramarinfarben“, Halle, (1918); F. SINGER, „Die künstlichen Zeolithe“, Diss. Berlin, (1911). In these works and also in C. DOELTER's „Handbuch der Mineral-Chemie“, Band 2, 2, p. 295, 286 and 229—261, (1917) a detailed review is published of the literature concerning these compounds.

⁵⁾ W. C. BRÖGGER and K. BÄCKSTRÖM, *Zeitschr. f. Kryst. und Miner.* **18**, 209, (1890).

the mutual relations of these compounds as a result of their very detailed investigations on the *alkali-* and *calcium-garnets*. It was for this reason, that during our ROENTGENSpectrographic work on the artificial ultramarines, we simultaneously started the examination of the said minerals after the same experimental method. The results of all these investigations are published in the present paper.

According to BRÖGGER and BÄCKSTRÖM, *sodalite*, *hauyne* and *nosean* must all be placed in one and the same group of minerals, namely that of the *alkali-* and *alkali-calcium-garnets*. According to these authors, *sodalite* would surely not possess hexakisoctahedral symmetry; as well the corrosion-figures obtained by them, as also other particulars of the rhombicdodecahedrally shaped crystals, seem to indicate a hexacistetrahedral symmetry and a twin-formation parallel to $\{100\}$. They attribute the same symmetry, moreover, to *nosean* and *hauyne*, — which also crystallize in most cases in the form of rhombicdodecahedra, — by drawing attention to the great analogy which exists between these minerals and *sodalite*, and to the octahedral forms with furrowed edges, as observed by VOM RATH. This last argument, however, does not hold ¹⁾, as may be seen from the analogous phenomenon in the case of *diamond*. In the case of a colourless *nosean* we observed, moreover, that the crystal accidentally broke off exactly parallel to $\{111\}$, while *sodalite* shows a cleavage parallel to $\{110\}$; it is, however, yet hazardous to assume a cleavability parallel to $\{111\}$ in *nosean*. In the following page it will become clear, that *sodalite* has, indeed, a structure, which is quite different from that of *nosean* and *hauyne*; so that it can by no means be placed in the same group with the others, although BRÖGGER and BÄCKSTRÖM believed to have proved, that *lazurite* is a mixture of *sodalite*, *hauyne* and natural *ultramarine*.

As regards the artificial *ultramarines*, — microscopical examination has proved, that they are optically isotropous, like the natural product, and that they crystallize in the shape of microscopical, very minute rhombicdodecahedra ²⁾. Furthermore, the *sodium* of the ultramarines appears to be easily replaceable by other bases ³⁾, e.g. by the oxides of the metals *K*, *Li*, *Ca*, *Ba*, *Mg*, *Zn*, *Ag*, etc., — as is the case also in the *permutites* and some *zeolites*. As is the case in the *zeolites*, the *sodium*-atoms in these compounds are evidently highly mobile; moreover, SINGER ⁴⁾

¹⁾ Conf. F. M. JAEGER, „Inleiding tot de Studie der Kristalkunde, Groningen, (1924), p. 178, 179.

²⁾ See also F. FISCHER, DINGLER's Polyt. Journal, **221**, 562, (1876); T. MOREL, Mon. Scientifique, (1879), 785; C. GRÜNZWEIG and R. HOFFMANN, Ber. d. d. Chem. Ges., **9**, (1876), 864; W. C. BRÖGGER and H. BÄCKSTRÖM, loco cit., p. 242.

³⁾ K. HEUMANN, Ber. d. d. Chem. Ges. **10**, (1877), 991; P. G. SILBER, ibid., **14**, 941, (1881); DE FORCRAND and BALLIN, Bull. Soc. chim., **30**, 112, (1878); Compt. rend., **88**, 30, (1879); K. HEUMANN, Ann. d. Chem. u. Pharm., **201**, 262, (1880); J. SZILASI, Ann. der Chemie und Pharm., **251**, 97, (1889).

⁴⁾ F. SINGER, „Ueber künstliche Zeolithe“, Diss. Berlin, (1911).

demonstrated, by means of direct synthesis, that there exists a close relation between *permutites* and *zeolites* on the one hand, and the *ultramarines* on the other hand. Also derivatives of the ultramarines are known, in which *aluminium* is substituted by *titanium*, *silicon* by *boron*, etc., and in which *selenium* and *tellurium* hold the place of the *sulphur*-atoms¹⁾. On treating the ultramarines with acids, about one third of the *sulphur* present is made free as H_2S , while the remaining part is precipitated in the form of elementary *sulphur*.

COMPOSITION OF SOME STUDIED ULTRAMARINES.				
GUIMET I; Violet	GUIMET II; Red	GUIMET III; Blue	GUIMET IV; Green	Germ. Ultram. VIII; Blue
Specific weight : $d_{250} = 2.34$ Al = 14.25 % Si = 19.33 % S = 10.40 % Atomic ratio :	Specific weight : $d_{250} = 2.36$ Al = 15.53 % Si = 19.79 % S = 9.65 % Atomic ratio :	Specific weight : $d_{250} = 2.35$ Al = 13.10 % Si = 19.38 % S = — Atomic ratio :	Specific weight : $d_{250} = 2.27$ Al = 16.54 % Si = 17.21 % S = 7.98 % Atomic ratio :	Specific weight : $d_{250} = 2.37$ Al = 14.01 % Si = 18.33 % S = — Atomic ratio :
Al; Si; S = 6 : 6.86; 3.72 Al; Si; S = 6 : 7.30; 3.13 Al; Si; S = 6 : 6.86; 3.72 Al; Si; S = 6 : 7.30; 3.13				

Some specific weights of German ultramarines were, in addition, also determined :
D. V.; dark violet : $d_{250} = 2.41$. D. VI.; violet : $d_{250} = 2.37$.
D. VII.; pink : $d_{250} = 2.48$.

§ 2. For this investigation we had a number of differently coloured products at our disposal from the *Usines d'Outremer-GUIMET* in *Fleurieu s/Saône*, namely: GUIMET-green N^o. 9210; blue N^o. 7553; pink N^o. 8219 and violet OVR; furthermore, a number of red, green and blue preparations from the *Vereinigte Ultramarin-Fabriken* in *Cologne*, which most kindly were supplied to us through the medium of Dr. C. O. LEVERKUS; they were partially prepared after the *sulphate*-, partially after the *soda*-, partially after the *mixed sulphate-soda*-method. With the exception of the green ultramarines, all the products investigated appeared to belong to the series of compounds with *high silica-content*; a number of analytical data obtained with the carefully dried materials, and a series of determinations

¹⁾ E. GUIMET, *Bull. Soc. chim.*, **27**, 480 (1877); *Ann. d. Chim. et Phys.* (5), **13**, 102. (1878); D. MOREL, *ibid.* **28**, 522, (1878); J. HOFFMANN, *Koll. Zeits.*, **10**, 275, (1911).

of the specific weights of the same preparations at 25° C., are recorded here. (See table).

From these data, which were all obtained in repeated and agreeing analyses with the aid of most carefully dried ultramarine-species, it becomes clear, that there cannot be spoken here of a really „constant” composition of these products: even the two blue preparations *G. III* and *D. VIII*, — which contain both a high percentage of silica, — appear to deviate appreciably from each other in this respect. Neither is there a simple stoichiometrical ratio of the *Al*-, *Si*-, and *S*-atoms present, it being, therefore, impossible to give here rational molecular formulae for these compounds. In literature the following data are given for the blue ultramarines of high silica-content: *Al* = 12,70 till 13,93 %; *Si* = 19,07 till 19,53 %; *S* = 13,44 till 14,09 %; while for an ultramarine of the formula: $Al_4 Na_6 Si_6 S_4 O_{20}$ these numbers should be: 12,53 %, 19,49 % and 14,85 % respectively. It is evident that the chemical composition of these dyes must be variable within rather wide limits.

For the ultramarines with a high silica-content, the empirical formulae seem to oscillate more or less round: $Na_6 Al_4 Si_6 O_{20} S_4$ for the *blue* and $Na_3 Al_4 Si_6 O_{22} S_3$ for *red* ultramarine; for *green*, which belongs to a series of „moderate” sulphur-content, the composition oscillates round: $Na_9 Al_6 Si_6 O_{24} S_3$ or more correctly round: $Na_8 Al_6 Si_6 O_{24} S_2$. The sulphur-content is highest in the ultramarines prepared after the soda-method, lowest in those made by means of the sulphate-process.

With respect to the minerals studied here, it must be remarked, that they were taken from different localities: colourless *sodalite* from the *Monte Somma, Vesuvius*; blue *hauyne* from the same spot, from the *Laacher See* in the *Eifel* and from *Niedermendig*; colourless *nosean* from the two last mentioned localities. Some of these crystals were kindly put at our disposal by Prof. RUTTEN of Utrecht and Prof. ESCHER of Leyden, to whom we also in this place wish to express once more our gratitude for their kind help. The ROENTGENSpectrographic study of these minerals, and more particularly the interpretation of their rotation-spectrograms, proved to offer extraordinary difficulties, as a consequence of the fact, that all these crystals were not only very small and often inhomogeneous, but, moreover, that as well in the case of *hauyne*, as in that of *nosean* and *sodalite*, they were almost always twins. In connection with the great identity-distances of their structures in the direction of the axis of rotation, the last mentioned circumstance made a rigorous analysis of the diffraction-images obtained often very difficult and sometimes quite impossible. With the exception of the special case, that the revolution of the crystal occurred round an axis perpendicular or parallel to the twinning-plane, the analysis of the spectrograms obtained could even be considered as quite hopeless. If e.g. a twin parallel to (111) be rotated round an axis in a direction, which has the symbol, [111] for the first of the two apparent individuals of the twin, then no dist-

turbance of the spectrogram occurs, in so far as the determination of the identity-distance in that direction [111] by means of the successive hyperbolae is considered, because the said direction has the symbol $[1\bar{1}\bar{1}]$ in the second apparent individual of the twin. But if the same twin be rotated round axes in directions as [110], $[\bar{1}\bar{1}\bar{1}]$, $[\bar{1}\bar{1}\bar{1}]$, etc. in the first apparent individual, — these directions are resp. [114], [151], and [115] in the second individual. As the identity-distances in a direction $[h k l]$ in the cubic system are c. p. proportional to $\sqrt{h^2 + k^2 + l^2}$, — the succession of the hyperbolae in the spectrogram obtained will appear to be disturbed until hardly recognisable, if the crystal be rotated round axes in the directions mentioned above, and especially in the case of such minerals as are studied here, in which the mutual distances of the successive hyperbolae in the image are, after all, already very small.

§ 3. In the first place a series of powder-spectrograms of the *ultra-*

No. of line:	Intensity (estimated):	2 l in mm.:	λ :	$\frac{\theta}{2}$:	$\frac{\sin^2 \frac{\theta}{2}}{2}$ (observed):	$\frac{\sin^2 \frac{\theta}{2}}{2}$ (calculated):	Indices $\{h k l\}$
1	1	16.8	α	9°37'	0.0279	0.0284	(200)
2	1	19.0	β	10°53'	0.0356	0.0352	(211)
3	8	21.0	α	12° 2'	0.0433	0.0426	(211)
4	5	27.0	α	15°28'	0.0711	0.0710	(310)
5	5	29.7	α	17° 4'	0.0861	0.0852	(222)
6	1	34.5	α	19°49'	0.1149	0.1136	(400)
7	3	36.5	α	20°58'	0.1280	0.1278	(330)
8	3	44.5	α	25°33'	0.1860	0.1846	(431)
9	2	49.5	α	28°25'	0.2265	0.2272	(440)
10	1	51.3	α	29°27'	0.2417	0.2414	(530)
11	1	53.0	α	30°22'	0.2556	0.2556	(600)
12	1	54.8	α	31°24'	0.2715	0.2698	(532)
13	1	59.5	α	34° 9'	0.3151	0.3124	(622)

Radius of camera: 25 mm.; $V = \text{ca. } 55000$ Volt; time of exposure: 3 hours.
Wave-length: $\lambda_{Cu} = 1.540$ A.U. $\lambda_3 = 1,388$ A.U.

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.0071 (h^2 + k^2 + l^2)$; from which: follows:
 $a_0 = 9.13$ A.U.

marines GUIMET I—V and of a number of the ultramarines of German origin was prepared after HULL-DEBIJE's method, and, moreover, such of nosean, hauyne and sodalite. The remarkable result was obtained, that quite independent of the colour and the chemical composition of these ultramarines, all these spectrograms appeared to be completely identical, and, moreover, to be also quite the same as those for nosean and hauyne. On the other hand, sodalite gave a spectrogram, which, as well with respect to the situation of the diffraction-lines, as to their intensities, was quite different from them. Evidently no appreciable influence on the localisation and intensities of the diffraction-images is exerted here by the special circumstance, that, — as in hauyne, — the sodium-atoms of nosean are partially substituted (for about half their number) by calcium-atoms; or that the ultramarines investigated contain 3, 6, 8 or 9 atoms of sodium; or that the (SO_4) -group in nosean, in the transition to the ultramarines, is substituted by two, three or four sulphur-atoms. None of these changes appears to have any detectable influence whatsoever on the intensity or on the special situation of the diffraction-images.

A review of the complete analysis of these powder-spectrograms is given in the preceding table; the values of $\sin^2 \frac{\theta}{2}$ are calculated by means of the exact constant of the quadratic equation, as it was later-on accurately determined by means of BRAGG-spectrograms¹⁾.

From these data it becomes evident, that in no case $(h+k+l)$ is an odd number; the fundamental BRAVAIS' grating of the structure must, therefore, be the body-centred one. As will be shown, this conclusion is confirmed by all later experiments. As the specific weights of the ultramarines used are known (= about: 2,36), as well as those of nosean and hauyne (about: 2,40), — the number of "molecules" of these compounds per elementary cell can be calculated in the usual way.

For nosean of the simplest formula: $Na_5 Al_3 Si_3 O_{16} S$, this number appears to be: 2, and for the ultramarines of the formulae mentioned above, it is: 1; so that in every case the mass contained in the fundamental cell corresponds to six silicon-atoms in the different formulae.

§ 4. After the identity of the structures of nosean, hauyne and all ultramarines had been demonstrated in this way, we endeavoured to confirm the conclusions hitherto drawn with respect to the structure of these compounds, by means of rotation-spectrograms of nosean and hauyne, and to calculate accurately the dimensions of the grating by the aid of spectrograms prepared after BRAGG's method.

Originally hauyne gave in no case any useful rotation-spectrograms;

¹⁾ Still better spectrograms of ultramarine were afterwards obtained with a new and improved camera. No less than 32 different diffraction-images could be measured on these films. The results obtained, however, have fully confirmed the conclusions drawn in the above. In a following paper the new data will be discussed in detail.

but later-on very good spectrograms of this kind were obtained, which will be discussed in the next paper. The crystal used, which seemingly had the shape of a perfect rhombicdodecahedron, appeared to be a twin, which, if rotated round [111] or [110], gave no analysable rotation-spectrograms. However, a BRAGG-spectrogram on (110) could be obtained, *rocksalt* parallel to (100) being simultaneously used for the purpose of comparison. Observed value: $\frac{\theta}{2} = 6^{\circ}52'$; from which follows: $d_{(110)} = 6,44$ A.U., and, therefore, $d_{(100)} = 9,11$ A.U. That the "reflection" on (110) is really of the *first* order, was confirmed later-on experimentally.

The *nosean*-crystals were rhombicdodecahedra, which were strongly elongated parallel to one of their trigonal axes; they appeared to be twins parallel to (111). If such a crystal be rotated round its longest axis, a normal spectrogram may be expected beforehand, as was already emphasized in the above. The distance between the first hyperbola in the spectrogram to the centre of the photographic image was 8 mm.; as in this case the distance of the photographic plate to the crystal was: 40 mm. here, so $tg \mu = 0,20$ and, therefore, $\mu = 11^{\circ}20'$. From this value the identity-distance in the direction of a trigonal axis is found to be:

$$I_{[111]} = \frac{\lambda}{\sin \mu} = 7,84 \text{ A.U.}$$

A BRAGG-spectrogram on a face of {110} gave $\frac{\theta}{2} = 6^{\circ}51'$; from which $d_{(100)}$ is again calculated to be: 9,13 A.U.; the formerly observed diffraction-line must, indeed, be of the 1st order.

The diagonal of the cubic cell is calculated from $d_{(100)} = 9,13$ A.U. to be: 15,80 A.U., which is nearly $= 2 \times 7,84$ A.U. From this follows, that the spacings in the direction [111] are *halved*, — which proves that the cubic cell is really the *body-centred* one. If the image on (110) had been of the *second* order, then, in accordance with the data of the rotation-spectrogram obtained by turning the crystal round [111], the bodily diagonal of the cubic cell would have to be *quartered*. This case, however, is excluded in each of the three types of cubic gratings.

A complete analysis of the rotation-spectrogram of *nosean* on turning round [111] is given in the following table. The values of $\sin^2 \frac{\theta}{2}$ are calculated here by means of the quadratic equation: $\sin^2 \frac{\theta}{2} = 0,0071(h^2 + k^2 + l^2)$, — the coefficient: 0,0071 being determined from the BRAGG-spectrogram $\left(\frac{\theta}{2} = 6^{\circ}51'\right)$ already mentioned.

As the fundamental grating is the body-centred one and as rotation took place round [111], the sum of the indices of the principal spectrum must be equal to *zero*; that of the indices of the 1st hyperbola equal to 2; that of those of the 2nd hyperbola equal to 4; that of the indices of

the 3^d hyperbola equal to 6; etc. This now appears really to be the case; so once more the conclusions drawn previously are confirmed by this.

TABLE II.					
Analysis of the Rotation-spectrogram of Nosean, when rotated round [111].					
Distance between photographic plate and crystal = 40 mm.					
Distance between spot and centre of image in mm.:	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices $\{h k l\}$:	Number of hyperbolae:
18.0	12°7'	0.0441	0.0426	(2 1 1)	Principal spectrum
30.0	18°26'	0.1000	0.0994	(3 2 1)	
36.0	21°0'	0.1284	0.1278	(3 3 0)	
49.5	25°32'	0.1860	0.1846	(4 2 2)	
62.5	28°0'	0.2304	0.2272	(4 4 0)	
18.0	12°7'	0.0441	0.0426	(2 1 1)	I
24.0	15°29'	0.0731	0.0710	(3 0 1)	
41.5	23°2'	0.1531	0.1562	(3 2 3)	
49.5	25°32'	0.1860	0.1846	(4 1 3)	
66.0	29°23'	0.2407	0.2414	(334) or (503)	
17.5	11°49'	0.0419	0.0426	(2 1 1)	II
24.0	15°29'	0.0713	0.0710	(3 1 0)	
33.0	19°46'	0.1144	0.1136	(4 0 0)	
30.0	18°26'	0.1000	0.0994	(3 2 1)	III
36.0	21°0'	0.1284	0.1278	(3 3 0)	
49.5	25°32'	0.1860	0.1846	(4 3 1)	
41.5	23°2'	0.1531	0.1562	(3 3 2)	IV
49.5	25°32'	0.1860	0.1846	(4 3 1)	

Wave-length, $\lambda_{Cu} = 1.540$ A.U.; $V = 55000$ Volt; time of exposure: 5 hours.

Evidently all images $\{h k l\}$ have the intensity = 0 in *odd* orders, when $(h + k + l)$ is an *odd* number. This fact could be corroborated once more by means of a BRAGG-spectrogram on a cleavage-plane of

nosean parallel to (111); after a few minutes exposure the 2nd order of the diffraction-image was present already with a very considerable intensity, while even after exposure during a full hour, not the slightest trace of a 1st order-image could be detected.

§ 5. The next problem to be discussed, is that of the symmetry of *nosean* and *hauyne*, which, — as has already been said, — are commonly considered to possess hexacistetrahedral symmetry, an opinion also advocated by BRÖGGER and BÄCKSTRÖM. But there is no valid proof adduced for this, since the analogy of these minerals with *sodalite*, as brought to the fore by these authors as an argument, can no longer be maintained now. This will be seen from what follows, because we were able to prove, that the structure of *sodalite* is quite different from that of the two other minerals. As all arguments for the determination of the space-group of *nosean* and *hauyne* must be taken into account, it is, therefore, not allowed to make beforehand any restriction with respect to the special symmetry of these minerals, and it thus appears necessary to consider all crystal-classes of the cubic system as of the same importance.

It was proved, that the fundamental grating is the body-centred one, and that all spacings $\{hkl\}$ are halved, if $(h+k+l)$ is an *odd* number. If these facts be taken into account, then there are *ten* space-groups, whose special symmetry may eventually be attributed¹⁾ to both minerals considered here; namely

$$T^3; T^5; T_h^7; T_h^7; T_d^3; T_d^6; O^5; O^8; O_h^9; \text{ and } O_h^{10}.$$

in WYCKOFF's notation.

However, we can already immediately exclude from those the groups: T_h^7 , T_d^6 and O_h^{10} , because on the BRAGG-spectrograms of $\{110\}$ the *first* order image of $\{110\}$ was observed with full certainty. Moreover, since also (200) was observed (see Table I) the group O^8 too must be excluded.

Nosean and *hauyne* can, therefore, only have the symmetry of one of the following six groups:

$$T^3; T^5; T_h^5; T_d^3; O^5, \text{ or } O_h^9.$$

It must, furthermore, be remarked, that, — as the most probable formula for pure *nosean* is: $Na_5Al_3Si_3O_{16}S$, and as twice the number of atoms of this formula is present in the elementary cell, — the last one must contain a mass, the composition of which is in the simplest case expressed by: $Na_{10}Al_6Si_6O_{32}S_2$. In the analogously built *ultramarines* there is present in each case once the number of atoms which is expressed by the formulae for these compounds as given before; so that the elementary cell of all *ultramarines* also in each case contains *six* atoms of *silicon*.

Now, — what perhaps would appear to be the final choice between

¹⁾ W. T. ASTBURY and K. YARDLEY, Phil. Trans. Royal Soc. London, A, 224, (1924) p. 254, 255, 256.

the six space-groups mentioned above, — in every case it can be proved on closer examination¹⁾, that the possible arrangement of the equivalent *twofold* and *sixfold* places in the cell manifests itself in all six groups in *quite the same way*, — in T^5 excepted, — in which group such places do not occur at all. For it can be shown, that the coordinates of these places in all those groups are: $[000]$ and $[\frac{1}{2}\frac{1}{2}\frac{1}{2}]$ for the *two-fold* places, while for the *sixfold* ones they are: $[0\frac{1}{2}0]$, $[00\frac{1}{2}]$, $[\frac{1}{2}00]$, $[\frac{1}{2}0\frac{1}{2}]$, $[\frac{1}{2}\frac{1}{2}0]$ and $[0\frac{1}{2}\frac{1}{2}]$ (See Fig. 1). All these coordinates are simply fixed

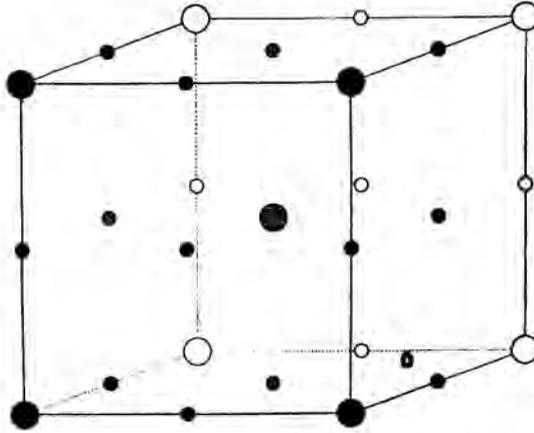


Fig. 1.

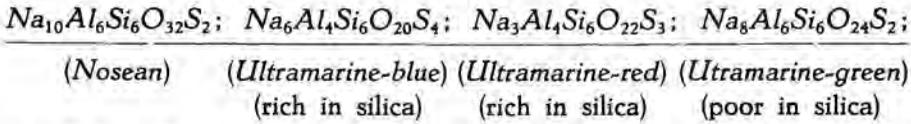
beforehand, because *they do not contain any variable parameter*. Moreover, in none of these groups can occur *univalent* or *fourfold* places; in all, however, there might be found *eight-, twelve- and twenty-four-fold* places, and in some of them also *sixteen-, forty-eight and ninety-six-fold* places. Now, because in the cells of *all* these compounds, *six silicon-atoms* appear to be present, there must certainly be present *sixfold* places in each case; so that we can also exclude beforehand the group T^5 from the six mentioned above.

But the fact that the coordinates of the necessary *six-fold* places in the cell are completely fixed and that they do not contain any variable parameter, involves the impossibility of the simultaneous occurrence of other *sixfold* places in the structure, besides just that single complex of six points, the coordinates of which are mentioned above. If these six places are once occupied by six identical atoms of a certain kind, it is, therefore, excluded *à priori*, that six atoms of a second or third kind could simultaneously be present on six equivalent places within the same elementary cell.

Now, as *all* elementary cells evidently contain 6 *Si-atoms*, it is obvious to make the supposition, that it is these six *silicon-atoms*, which really

¹⁾ R. W. G. WYCKOFF, Publ. CARNEGIE-Institut. Washington, N^o. 318, (1922), p. 122, 126, 129, 135, 148, 103.

occupy the only sixfold places available here, with the coordinates mentioned above¹⁾. If now, however, the composition of the masses present in the elementary cell of all these compounds be taken into consideration:



it will immediately be seen, that perhaps only the O-atoms possibly could be placed in the cell in some such way as: (12 + 8); (12 + 8 + 2); 24, (or 16 + 8); (24 + 8) atoms; etc. It must be remarked, however, that the numbers of O-atoms indicated, are not very accurate, as all errors of the chemical analysis are accumulated in these numbers.

If 24 O-atoms must be placed as (16 + 8) atoms, the space-groups T_d^3 and T_3 would be excluded here also; and if the compounds studied here, were really holohedral, then the space-group O_h^9 could most probably be attributed to them. But all this is mere speculation; any thing certain cannot be said about it.

However, — in what way the question may be considered or perhaps finally decided, — there is, after all, one fact, which is undeniably certain: namely, it is quite impossible to attribute to an appreciable number of components of the total mass included within each elementary cell any definite place in the structure. For these components: Na, Al, S, — and in the case of *hauyne* also for Ca, — no place whatsoever is any more available within the fundamental cell. While it is even quite uncertain, which atoms on the parameterless two- and sixfold places be really present, or which atoms are perhaps situated at the 8-, 12-, 16-, and 24-fold places, — it is, after all, fairly well possible to substitute the sixfold places by atoms of one and the same kind, e.g. by Si-atoms; but then every further possibility will simultaneously have disappeared for localising other complexes of six identical atoms, e.g. of Al_6 or Na_6 , because there are in reality no other sixfold places any more available within the cell, besides the six with the fixed coordinates, as mentioned in the above. Therefore, beyond all doubt there must be an appreciable number of atoms present within each fundamental cell, which can by no means be arranged in definite places of the periodical structure; they must in space be distributed at random, in a merely statistical way, and be scattered throughout the fixed, rigorously periodical structure of these crystals, which forms, as it were, their rigid skeleton.

In the same way GOLDSCHMIDT and THOMASSEN²⁾ in the case of

¹⁾ It would e.g. be obvious to suppose that the two S-atoms, which in *nosean* and in the *ultramarienes* are always present, be situated in the twofold places. However, our later experiments seem to indicate, that not these S-atoms, but rather two of the *sodium*-atoms must occupy these places.

²⁾ V. M. GOLDSCHMIDT en L. THOMASSEN, Vidensk. Skrift., Christiania, (1923).

cleveite and *bröggerite* always found, by means of the ROENTGEN-ographic method, the periodical structure of UO_2 , although the chemical composition of these minerals indicates, that they chiefly consist of U_3O_8 , with as accessory components: *Pb-*, *Tho-*, *Yt-*, *Erb-*, *Ce-*, and *Fe-*derivatives, besides some H_2O . A certain amount of oxygen must, with the other accessory compounds, therefore, be statistical distributed within the UO_2 -lattice, — exactly as occurs in our case of the ultramarines; so that the ROENTGEN-spectrograms do not reveal anything of the presence of these compounds, except the diffraction-lines, which correspond more particularly to the rigorously periodical structure of the UO_2 -framework. An analogous explanation must doubtlessly be given for the facts observed by WYCKOFF, GREIG and BOWEN, that *sillimanite* and *mullite*, and also for these observed by WYCKOFF and MOREY, that a series of *sodium-calcium-metasilicates* of different chemical compositions, gave identical or hardly discernable ROENTGEN-spectrograms¹⁾.

It is because of the special circumstances present in this case, — and more particularly because of the very favourable circumstance, that the coordinates of the *two-* and *sixfold* places of the structures considered here, are completely free from *indefinite* parameters u , v , etc., — that we are inevitably compelled to the conclusion, that in the minerals studied here, — which formerly were considered as being "molecular compounds" between *silicates* and Na_2SO_4 , — as well as in the *ultramarines*, — which were considered as analogous compounds between *silicates* and *Na-polysulphides*, — there can only be question of a true periodically built frame-work of the whole, presumably consisting here of *Si-*, *Na-* and *O-*atoms, — in which a great number of other components are distributed in a merely statistical way.

This supposition gives an explication, on the one hand, of the particular "mobility" of a part of the *Na-*atoms and of their being replaceable by other atoms, — even by *Ca* (as in *hauyne*), which as a consequence of its bivalency, must substitute 2 *Na-*atoms at the same time. Otherwise it would be quite inexplicable in the last case, that the *Ca-*atom could substitute 2 *Na-*atoms of a rigorously periodical structure, without that structure being influenced in the least respect by that substitution, as we have experimentally demonstrated it to be the case in *hauyne* and *nosean*. The conception of these "errant" components within the fundamental lattice explains, why F. SINGER and others were able to show an intimate relation between such *zeolites* (and *permutites*) on the one hand and the *ultramarines* on the other²⁾; just as is the case with the H_2O -molecules in some *zeolites*, which can reversibly be deshydrated, and for

¹⁾ R. W. G. WYCKOFF, J. W. GREIG and N. L. BOWEN, Amer. Journ. Science, 11, 459, (1926); R. W. G. WYCKOFF and G. W. MOREY, Amer. Journ. Science, 12, 419, (1926).

²⁾ That the ultramarines possess e.g. "hydraulic" properties, and that they are even able to increase the binding-power of cements, has first been shown by ROHLAND; vid.: Zeits. f. angew. Chemie, 17, 609, (1904).

which an analogous supposition about their constitution now must generally be adopted. It also permits of giving in some degree an explication of the factors, on which the coloration of the ultramarines depend, and of the fact, that all these differently coloured preparations, — yellow, red, green, blue or violet, — always give almost completely *identical* diffraction-images. That colour is probably determined by the greater or smaller complexes of *S*-atoms, or of *AlS*- and *NaS*-groups, etc., and of their special degree and mode of dispersion within the total bulk of the solid mass. It is now fairly certain, that the *ultramarines* can *not* any longer be considered as compounds of a constant stoichiometrical composition; — a fact, to which also is pointed by the deviating results of the chemical analysis in different cases. The attribution to them of empirical formulae, — and much less that of "structural" formulae, — has no longer any significance: these substances consist of a regular, periodical structure as bearer of an irregularly distributed company of other atomic individuals, amongst which *Na*, *Al*, and *S* are the most predominant, and the number and nature of which greatly depend on the special conditions of the preparative processes. The colour of the solid mass seems to be undeniably bound to the presence of the element *sulphur*, — perhaps in a way analogous to that recently described by VON WEIMARN¹⁾, or otherwise by the special degree of dispersion of *S*-containing components. The stability of the ultramarines at red heat, on the other hand, makes it rather improbable, that *free sulphur*²⁾ should be the cause of their colours, as some authors have suggested.

§ 6. In the beginning of this paper it has been said, that *sodalite* (*Monte Somma*) yielded quite another type of powder-spectrogram than *nosean*, *hauyne*, etc., — although formerly *sodalite* has always been considered to be a compound of a structure perfectly analogous to that of the others, *sodiumsulphate* in the latter being simply substituted by *sodiumchloride*, but in combination with the same fundamental silicate. This conception of the constitution of all the substances of this series seemed to be strongly corroborated by the results of the artificial syntheses³⁾ of these minerals.

From the investigations of BRÖGGER and BACKSTRÖM, who place all these minerals also in one and the same group, it follows with certainty, that the symmetry of *sodalite* is surely *not hexakisoctahedral*: the corrosion-figures obtained by them were, on the contrary, in better agreement

1) P. VON WEIMARN, Koll. Chem. Beihefte, 22, 38, (1926); J. HOFFMANN, Koll. Zeits., 10, 275, (1912); F. KNAPP, DINGLER's Polyt. Journ., 233, 479, (1879); Idem 229, 74, (1878).

2) Vid. i.a.: PATERNÒ and MAZZUCHELLI, Atti R. Acad. Lincei, (V), 16, I, 465, (1907).

3) J. LEMBERG, Zeits. d. Geol. Ges., 28, 603, (1876); 35, 583, (1883); J. MOROZEWICZ, Tscherm. Min. Mitt., 18, 143, (1899); O. MÜGGE, in ROSENBUSCH' „Mikrosk. Physioogr., I, 323, (1892); Z. WEYBERG, Z. Min. (1905), 653; CH. and G. FRIEDEL, Bull. Soc. min., 13, 183, (1890); S. J. THUGUTT, N. Jahrb. Min., etc., Beil. Bd. 9, 568, (1894).

with *hexacistetrahedral* symmetry. In that case, the apparently simple rhombicdodecahedra of *sodalite* must be considered to be twins parallel to $\{100\}$. This, indeed, is in perfect agreement with our experience in preparing rotation-spectrograms of a transparent crystal of rhombicdodecahedral shape: on rotating the crystals round $[100]$, the spectrograms obtained appeared to be quite normal; but on rotating them round $[111]$, they were completely irregular and not suited for analysis, just as it was before observed in the case of *hauyne*, — in reversed sequence, however.

In the first place in Table III are reviewed the data, relating to the powder-spectrogram of *sodalite*; the values of $\sin^2 \frac{\theta}{2}$ are calculated

No. of the Lines:	Intensities estimated:	$2l$ in mm.:	Wave-length:	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$: (observed):	$\sin^2 \frac{\theta}{2}$: (calculated):	Indices $\{h k l\}$:
1	1	23.1	λ_{β}	$13^{\circ}14'$	0.0524	0.0503	(220)
2	8	25.5	λ_{α}	$14^{\circ}37'$	0.0637	0.0608	(220)
3	2	31.2	λ_{α}	$17^{\circ}53'$	0.0943	0.0912	(222)
4	3	34.3	λ_{α}	$19^{\circ}39'$	0.1131	0.1064 or 0.1216	(321) or (400)
5	3	37.5	λ_{α}	$21^{\circ}29'$	0.1341	0.1368	(330)
6	3	41.3	λ_{α}	$23^{\circ}40'$	0.1611	0.1596	(421)
7	3	42.3	λ_{α}	$24^{\circ}14'$	0.1685	0.1672	(332)
8	1	50.1	λ_{α}	$28^{\circ}43'$	0.2309	0.2280	(521)
9	1	53.0	λ_{α}	$30^{\circ}25'$	0.2563	0.2584	(530)
10	1	56.3	λ_{α}	$32^{\circ}16'$	0.2850	0.2812 or 0.2888	(601) or (611)

Radius of camera: 25 mm. $V = 55000$ Volts; time of exposure: 3 hours.
 Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.0076 (h^2 + k^2 + l^2)$ for α -radiation of 1.540 A.U.
 From this a_0 calculated to be 8.81 A.U.

from the coefficient of the quadratic equation, as this was deduced from the numerical results of the BRAGG-spectrogram to be described later-on. Among the indices of Table III such symbols as: $\{321\}$, $\{421\}$, $\{332\}$, $\{521\}$, etc. are present, for which, therefore, $(k+l)$, or $(h+k)$, or $(h+l)$ are *odd*. Moreover, there are also symbols present, for which $(h+k+l)$ is *odd*. From all these facts it must be concluded, that the fundamental

BRAVAIS' lattice of the structure of *sodalite* is the *simple cubic* lattice, — as contrasted with what was found in the case of *nosean* and *hauyne*.

§ 7. These conclusions, now, were more particularly corroborated by the analysis of a number of rotation-spectrograms and of such obtained after BRAGG's method.

A BRAGG's spectrogram on a face of $\{110\}$, in combination with one on $\{100\}$ of *rocksalt* at the same distance, for the purpose of comparison, gave the value: 8,81 A.U. for $a_0 = d_{(110)} \cdot \sqrt{2}$; from this the coefficient of the quadratic equation for $\sin^2 \theta/2$ is calculated to be: 0,0076, which value was already formerly used in calculating the data of the powder-spectrogram.

A rotation-spectrogram obtained by turning the crystal round a principal axis [001], gave the results, which are recorded in Table IV (see next page).

The distance between the principal spectrum and the first hyperbola being 7.3 mm., $tg \mu$ was equal to 0,1791; from which follows: $\mu = 10^\circ 11'$, and for $I_{(100)}$, — being the identity-distance in the direction of the principal axis: $I_{(100)} = \frac{\lambda_x}{\sin \mu} = 8,72$ A.U., — which is in perfect accordance with the exact value: 8,81 A.U., calculated from the BRAGG-spectrogram at (110).

For the number n of atomic complexes of the formula: $Na_4 Al_3 Si_3 O_{12} Cl$, which are present per fundamental cell, we find, as the specific weight d is 2,30 here:

$$n = \frac{2,30 \times (8,81)^3}{485 \times 166} = 1.96 \text{ (i.e. } = 2\text{)}.$$

Thus in each elementary cell a total mass is present, whose composition in the simplest case is expressed by: $Na_8 Al_6 Si_6 O_{24} Cl_2$. If the strongly deviating data of the analysis of different *sodalites* be taken into account, as they are given in literature, the question may be raised, whether also in this case there can be properly spoken of a rational stoichiometrical composition or not? Nor is, in this case, the possibility by no means excluded, that there is only question of a chemical composition, oscillating within rather wide limits.

§ 8. If the *hexacistetrahedral* symmetry of *sodalite*, as suggested by BRÖGGER and BÄCKSTRÖM, be adopted, only two space-groups: T_d^1 and T_d^2 need be considered here, because the fundamental lattice was proved to be the simple cubic one. But in ROENTGEN-analysis the spacings of net-planes with the symbol $\{hhl\}$, in which l is an *odd* number, will in T_d^2 always appear to be halved ¹⁾. With absolute certainty, however, a spot (221) has been found on the rotation-spectrogram, although its

¹⁾ W. T. ASTBURY and K. YARDLEY, loco cit., p. 255.

TABLE IV.

Rotation-spectrogram of Sodalite from Monte Somma.

(Distance between plate and cristal: 40.5 mm.)

The axis of revolution here was the principal axis [001].

Distances of spots from the centre in mm.:	$\theta/2$:	$\sin^2 \theta/2$ (observed):	$\sin^2 \theta/2$ (calculated):	Indices $\{h k l\}$:	Sequence of hyperbolae:
25.3	16° 0'	0.0760	0.0760	(310)	Principal spectrum.
32.0	19° 9'	0.1076	0.1131	(330) λ_3	
38.9	21° 56'	0.1395	0.1368	(330) λ_{2c}	
42.0	23° 1'	0.1529	0.1520	(420)	
52.0	26° 3'	0.1927	0.1900	(500) of (430)	
64.1	28° 51'	0.2328	0.2204	(520)	
70.0	29° 59'	0.2496	0.2432	(440)	
18.7	12° 24'	0.0461	0.0456	(211)	I.
31.5	18° 56'	0.1053	0.1064	(321)	
39.1	22° 0'	0.1402	0.1368	(411)	
52.0	26° 3'	0.1928	0.1976	(501) or (431)	
18.7	12° 24'	0.0461	0.0456	(112)	II.
24.0	15° 9 1/2'	0.0684	0.0699	(212)	
28.0	17° 20'	0.0888	0.0912	(222)	
32.1	19° 12'	0.1081	0.1064	(312)	
39.0	23° 41'	0.1613	0.1596	(412)	
48.0	24° 55'	0.1775	0.1824	(422)	
62.5	28° 32'	0.2282	0.2204	(432)	
75.0	30° 49'	0.2625	0.2508	(522)	
82.0	31° 51'	0.2785	0.2736	(442)	
25.8	16° 15'	0.0783	0.0760	(103)	III.
31.8	19° 41'	0.1067	0.1064	(123)	
37.9	21° 33'	0.1348	0.1368	(303)	
45.6	24° 12'	0.1682	0.1672	(233)	
72.0	30° 20'	0.2553	0.2584	(433) or (503)	
48.6	25° 6'	0.1799	0.1824	(224)	IV.
55.8	27° 1'	0.2063	0.2010	(314)	
77.9	31° 16'	0.2694	0.2628	(334)	

Wave-length: $\lambda_{ca} = 1.540$ A.U.: $V = 55000$ Volts; time of exposure: 5 hours.Quadratic Equation: $\sin^2 \theta/2 = 0.0076 (h^2 + k^2 + l^2)$. Accordingly: $a_0 = 8.81$ A.U.

intensity was relatively weak. For that reason *sodalite* must necessarily ¹⁾ belong to the space-group T_d^1 .

In this space-group T_d^1 there occur *no twofold* places; but there are *two* different kinds of *sixfold* places, the coordinates of which all contain, however, a variable parameter u . Now evidently Si_6 , as well as Al_6 , can both be placed in those localities; so that, in contrast with the case of *nosean*, it is *not* absolutely certain here, whether also in *sodalite* such "errant" constituents must be supposed to be present in its structure, distributed in it at random in all directions. But even then, the empirical, coordinative or structural formulae, as occasionally proposed for *sodalite*, cannot be considered as being in real agreement with the different kinds of available places in its elementary cell.

§ 9. Some intensely coloured, dull blue and opaque individuals of *lazurite* from *Afghanistan*, which superficially had the appearance of rhombic-dodecahedra, were used for comparison with the substances hitherto investigated. However, on the apparent {110}-faces no BRAGG-spectrograms whatsoever could be obtained, notwithstanding several endeavours were made. Evidently the so-called "crystals" were no true individuals, but probably conglomerates having only an approximate resemblance with a rounded rhombicdodecahedron. Moreover, the blue masses were very inhomogeneous and they contained a number of alien and incrustated substances, i.a. *pyrite*.

A small piece of this blue *lazurite*, as homogeneous as possibly, could be separated out of the inner parts of the mass; it was finely powdered now and used for the preparation of a powder-spectrogram after HULL-DEBIJE'S method. The photograph obtained after five hours exposure, was rather weak, but the lines were sharp enough to be localised by means of a comparator. It was found, that by far the greater number of the diffraction-lines coincided with those of *hauyne* and the *ultramarines*; more especially the spectrogram of *lazurite* showed the following lines of those substances (numbers as in Table I): no's 2, 3, 4, 5, 6, 7, 9, 10, 11 and 13, while no's 8 and 12 were absent. Moreover, three very weak lines were observed, which must correspond to other constituents; although two of them rather exactly coincided with the lines no's 7 and 10 of *sodalite* (Table III), it is yet highly problematic, whether they can be attributed to that mineral, because the most conspicuous and intense line of *sodalite*, no. 2 of Table III, was certainly *not* present. There are undoubtedly other substances present in the blue mass as accessory products; but *it is quite certain now, that the spectrogram of lazurite is chiefly identical with that of hauyne and of the artificial ultramarines*. This result

¹⁾ TH. BARTH, Norsk Geol. Tidskr., 9, 40, (1926). The paper published by this author appeared, when the structure of *sodalite* had already been determined by us. Mr. BARTH was, moreover, not able to fix his choice between the two space-groups possible here; this problem is now settled here definitely.

is in partial agreement with BRÖGGER and BÄCKSTRÖM's supposition of *lazurite* being a mixture of *hauyne*-substance, natural *ultramarine* and other accompanying substances; but their suspicion, that *sodalite* would be the accessory mineral, is *not* confirmed beyond all doubt by the results described in this paper. As *sodalite* and *hauyne* have different structures and, most probably, also different symmetries, — it is quite improbable, that *lazurite* can be considered as a kind of mixed crystal of isomorphous components, as *hauyne* and *sodalite* by the Swedish authors are evidently and erroneously supposed to be.

§ 10. Finally we have kept in view the question, in how far there may exist any relation between the constitution of the *ultramarines* and *nephelite*, the composition of which is in the simplest case expressed by the formula: $Na Al Si O_4$. *Nephelite* itself is hexagonal-pyramidal; its axial ratio, according to VON BAUMHAUER, is: $a : c = 1 : 0,8314$. That there is really some relation between the so-called „ultramarine-base” and a *sodium-aluminium-silicate* of the composition as given above, is acknowledged by all authors, who have occupied themselves with the synthesis of the *ultramarines*. BRÖGGER and BÄCKSTRÖM¹⁾ indicate, how by a *moderate* heating during six hours of *kaoline* with somewhat more than the theoretical quantity of Na_2CO_3 , a silicate of the composition mentioned is obtained, which would be *optically-isotropic*. But, besides the hexagonal form of *nephelite* already mentioned, hitherto only a single modification of it is known: the *carnegiüte*, which is stable only at much higher temperatures²⁾, and which is certainly *not* cubic. A preparation of the composition: $Na Al Si O_4$, obtained by repeatedly sintering at $900^\circ C.$ of a mixture of $2 Si O_2$, $Na_2 CO_3$ and $Al_2 O_3$ or of $Na Al O_2$ and $Si O_2$, and homogenizing the product, appeared to be identical with natural *nephelite*; the powder-spectrogram was in every respect the same as that of natural *nephelite* from the *Monte Somma, Vesuvius*.

The crystals of this *nephelite* were prismatic, with $\{10\bar{1}0\}$ and $\{0001\}$, and occasionally with the pyramid $\{10\bar{1}1\}$; according to VON BAUMHAUER, they are twins or double twins after $\{10\bar{1}0\}$ and $\{0001\}$, as e.g. becomes clear from the lack of symmetry and the special shape of the corrosion-figures on $\{10\bar{1}0\}$. This twin-formation will, however, not disturb the rotation-spectrograms, obtained by turning the crystal round an axis $[0001]$, as has already been discussed.

In such a spectrogram, made with a cylindrical film, the distance between the principal spectrum and the top of the first hyperbola, was found to be: 9,5 mm., the distance between photographic plate and crystal being 50 mm.; therefore, $tg \mu = 0,190$, or $\mu = 10^\circ 46'$, from which follows: $I_{\{0001\}} = 8,3 A.U.^2)$.

1) W. C. BRÖGGER en H. BÄCKSTRÖM, Zeits. f. Kryst., 18, 242, (1891).

2) During the correction of this proof the said rotation-spectrogram was made with a new cylindrical camera and an intensifying screen. The new measurements made it necessary to improve the numbers of the Dutch publication.

This dimension could be determined more exactly by means of a BRAGG-spectrogram on (0001), $d_{(0001)}$ being found to be equal to: 4,19 A.U. $= \frac{1}{2} \times 8,38$ A.U. The dimension of the hexagonal cell in the direction of the c -axis is, therefore, equal to: 8,38 A.U., this spacing being halved apparently in ROENTGEN-analysis. Of all space-groups in the hexagonal system, where this can occur, only C_6^2 can be attributed to this mineral of hexagonal-pyramidal symmetry.

On rotating the crystal round a perpendicular to (10 $\bar{1}$ 0), a rotation-spectrogram was obtained, from which the identity-distance in that direction was deduced to be: 16,75 A.U. For the identity-distance in the direction of the a -axis, therefore, the value: 9,67 A.U. is obtained. A BRAGG-spectrogram on (10 $\bar{1}$ 0) showing two diffraction-lines, gave however: $d_{(10\bar{1}0)} = 2,85$ A.U. and 4,28 A.U. respectively; as: $3 \times 2,85 = 2 \times 4,28 = 8,56$ A.U., the diffraction-images observed here are evidently of the 3rd, respectively of the 2nd order. No reflexion of the 1st order was ever observed; and for the dimension of the cell in the direction of the a -axis, the exact value: 9,87 A.U. is immediately deduced. Thus the axial ratio $a : c$ is equal to: $9,87 : 8,38 = 1 : 0,849$; it appears, therefore, that to the form {10 $\bar{1}$ 1} in VON BAUMHAUER's measurements, really the symbol: {10 $\bar{1}$ 1} must be given.

As the specific gravity of the *nephelite* is: 2,62, the number n of the "molecules" of the composition: $NaAlSiO_4$, which are present within a hexagonal cell (i.e. in $\frac{1}{3}$ of the total combination of six triangular prisms), is calculated to be 8. In each triangular prism, therefore, the total mass present, must be written as: $Na_4Al_4Si_4O_{16}$. In the space-group C_6^2 only two- and sixfold places, the last ones with three variable parameters, — can be present; as the number of atoms per cell is so great and the values of the dimensions in the directions a and c are also very considerable, — it may be considered impossible to determine these parameters for all atoms present. The same is true for the powder-spectrogram obtained with the artificial silicate prepared from $2SiO_2$, Al_2O_3 and Na_2CO_3 , mentioned above. Of course, in no case can there be question to state already now a closer relation of this silicate with the ultramarines.

Whether it will appear to be possible to ascertain completely, which constituents form the rigid crystal-skeleton of the ultramarines and which are the "errant" ones, can only be decided by further experiments, which are already in progress.

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Mathematics. — *A complex of Conics.* By Prof. JAN DE VRIES.

(Communicated at the meeting of Januari 29, 1927).

The conics k^2 cutting the fixed conic a^2 in the plane a twice and resting on the lines b_1, b_2, b_3 , form a *complex*, Γ .

1. *Pairs of lines in Γ .* (a.) Any transversal t_{123} of the lines b forms a k^2 with any line r cutting it in the plane a ; to any r there correspond two t_{123} .

(b.) Any transversal t_{12} of b_1, b_2 and a^2 forms a k^2 with any line r that rests on t_{12}, b_3 and a^2 . The lines t_{12} form a scroll of the fourth degree; the lines r belong to a congruence [2,2]; any r defines three t_{12} .

(c.) Each of the four transversals a of a^2, b_1, b_2, b_3 , is completed to a k^2 by any ray r of the congruence [1,2] that has a and a^2 as directrices.

(d.) The line a_{12} lying in a and resting on b_1 and b_2 forms a k^2 with any line r , that cuts a_{12} and b_3 .

(e.) Any line r_{12} resting on b_1 and b_2 is completed to a k^2 by the line r in a that cuts r_{12} and b_3 .

Evidently each of (b), (d) and (e) represent three different systems.

2. *Conics through a given point.* The k^2 through a point C form a surface Ψ ; in order to determine the degree we shall investigate the intersection with the plane a .

Any plane through C and a point A of a^2 contains a k^2 ; these k^2 form a dimonoid of the fourth degree which has five points A^* outside A in common with a^2 , hence a^2 is a *fivefold curve* on Ψ . Through C there passes a transversal r of a_{ki} and b_m ; accordingly Ψ contains the lines a_{12}, a_{23}, a_{31} . The transversal of b_k and b_l through C yields a line of a that cuts it and b_m .

The intersection of Ψ and a consists, therefore, of the five-fold a^2 and six straight lines; consequently the surface is of the degree 16.

Its intersection with the plane Cb_1 consists of the following figures. In the first place a k^2 cutting b_1 twice, which must, therefore, be counted double. Through C there pass two lines r cutting b_1 and a^2 each of which defines three lines t_{23} and is, accordingly, a triple line of Ψ . Further Cb_1 contains three lines as component parts of pairs of lines of which one line lies in a (§ 1, *d* and *e*). And now it appears that b_1 must be a *triple line*.

This may be confirmed in the following way. The conics through C

and a point of b_1 which cut α^2 twice and rest on b_2 , form a cubic dimonoid; this contains, therefore, three k^2 that also cut b_3 .

Now the intersection with Cb_k shows that C is an *eleven-fold* point of Ψ^{16} .

Ψ contains the four transversals of α^2, b_1, b_2, b_3 , and the corresponding lines r through C . Further the 18 lines t_{ki} corresponding to the 6 above mentioned triple lines r .

3. *Surface of the k^2 that cut a given line l twice.* The k^2 that cut l in the point L , form a Ψ^{16} with an 11-fold point L ; hence there are 5 k^2 which cut l in another point L^* . As each plane through l contains one k^2 , the said surface is of the degree seven.

Its intersection with α consists of α^2 and five lines three of which rest on l and a line b whereas the other two cut l and a transversal of l and the three lines b_k . The lines b are single on Φ^7 .

There are ten k^2 that touch the line l .

4. *Surface of the k^2 that cut the lines b in projective point ranges.* The planes of these k^2 osculate a twisted cubic; α^2 is, therefore, a triple conic of this surface Ω . The intersection of b_k and α defines a pair of lines of which one lies in α and the other one rests on b_l and b_m . Consequently the degree of Ω is nine.

The scroll defined by two of the point ranges cuts α^2 four times; hence Ω^9 contains twelve pairs of lines of which one of the lines rests on two b_k , the other one on the third line b .

5. *Surface A of the k^2 that cut two more lines, b_4 and b_5 .* The k^2 of Γ that cut b_4 in a point B_4 , form a surface of the degree 16 (§ 2); accordingly there are 16 k^2 which also rest on b_5 . Hence on the surface A the 5 lines b are 16-fold.

α contains two lines that cut a line b and one of the two transversals of the other four lines b . Further α contains the lines which each cut two lines b and rest on a transversal of the other three lines b and which are, accordingly, double lines. Consequently in all 10 lines and 10 double lines of A lie in α .

The k^2 through a point A of α^2 resting on the 5 lines b , form a surface Φ of the degree 18 ($P\nu^6=18$). Its intersection with the plane Ab_5 consists of a k^2 that cuts b_5 twice and is, accordingly, a double curve, the quadruple line b_5 ($P^2\nu^4=4$), two lines through A that cut a transversal of b_1, b_2, b_3 and b_4 , and four double lines through A each of which cuts one of the lines b and two transversals of the other three. Accordingly a straight line through A in Ab_5 , has 6 points outside A in common with Φ^{18} . As this is also the case in each of the other planes Ab_k , A is a 12-fold point of Φ ; hence α^2 and Φ^{18} have 24 points outside A in common and α^2 is a 24-fold curve of A .

Finally a contains the k^2 which cuts the 5 lines b and which apparently must be counted six times. The entire intersection of A and a is, therefore, of the order $10 + 20 + 48 + 12$; hence A is a surface of the degree 90.

On A there lie $10 \times 4 \times 3$ pairs of lines (r, r') , of which r rests on a^2 and $3 b_k$, r' on a^2 , r , and the other $2 b$.

The surface Φ^7 of the k^2 of Γ that have b_4 as chord, has 7 points in common with b_5 . Hence A has besides $5 \times 7 = 35$ double curves k^2 the planes of which pass through one of the lines b .

6. Consequently there are 90 conics that cut a given conic twice and that rest on six given lines.

If we use the symbol k to indicate that a conic rests on a given conic, we may express the result found above by $k^2\nu^6 = 90$. This number (and other ones containing k) can be found in the following way by applying the principle of the conservation of the number.

In order to determine $P^2k^2\nu^2$ we place one of the lines, b_1 , in the plane of the given conic a^2 . The plane through the points P_1, P_2 and one of the points of intersection of b_1 and a^2 contains one k^2 which satisfies the conditions. The same also holds good for the configuration of $P_1 P_2$ and the line of a resting on it and on b_2 . Hence $P^2k^2\nu^2 = 3$ (cf. § 2).

In order to determine $Pk^2\nu^4$ we choose three lines b_1, b_2, b_3 in a plane φ^1). In this case there satisfy in the first place the 3×3 figures k^2 through P and one of the points $b_1 b_2, b_1 b_3, b_2 b_3$ which rest twice on a^2 and also on b_4 . Further 7 pairs of lines (r, r') of which r lies in φ and is a chord of a^2 or rests on a^2 and b_4 or cuts a^2 and a transversal r' of a^2 and b_4 . Consequently $Pk^2\nu^4 = 16$ (cf. § 2).

In order to find $k^2\nu^6$ we again choose b_1, b_2, b_3 in φ . In this case there satisfy in the first place the 3×16 k^2 through one of the points $b_1 b_2, b_1 b_3$ or $b_2 b_3$, which rest on the other four b_k and cut a^2 twice. Further the k^2 in φ that cuts a^2 twice and rests on b_4, b_5, b_6 ; evidently this must be counted eight times.

There are three chords of a^2 each of which is completed to a k^2 by a line of φ which cuts one of the lines b_4, b_5, b_6 and three chords of a^2 to which there belongs a transversal in φ of two of these lines b .

Each of the six lines r of φ that rest on a^2 and one of the lines b_4, b_5, b_6 is completed to pairs of lines by three transversals of r, a^2 and the other two of these lines b . The chord of a^2 in φ belongs to two pairs of lines. Finally each of the four transversals of a^2, b_4, b_5, b_6 forms a pair of lines with the line that joins its intersection with φ to one of the intersection with a^2 .

1) In this way I have again determined some time ago the known numbers $P\nu^6 = 18$ and $\nu^8 = 92$. (These Proceedings, 4, 181).

Consequently we find in all $48 + 8 + 3 + 3 + 18 + 2 + 8 = 90$ figures; hence $k^2\nu^6 = 90$.

7. A plane ϱ through b_1 has also a curve of the order 74 in common with the surface A^{90} . This cuts b_1 in the first place in the 7×2 points of intersection with the k^2 , that have b_1 as chord. In each of the remaining 60 points of intersection the plane ϱ is touched by a k^2 of Γ . Hence the locus of the points of contact of conics k^2 with ϱ is a curve of the order 60 and the tangent k^2 form a surface of the 120nd degree. Hence $k^2\nu^5\varrho = 120$.

Applying a similar reasoning to the surface Ψ^{16} (§ 2) we find the number $Pk^2\nu^3\varrho = 22$.

Also these results are easily verified by the method of § 6.

In the first place we find $P^2k^2\nu\varrho = 4$ by remarking that any plane through P_1P_2 contains two conics which cut a^2 twice, touch the plane ϱ and pass through P_1 and P_2 .

In order to arrive at $Pk^2\nu^3\varrho$ we again choose the three lines b in a plane φ . In this case there satisfy 3×4 k^2 through P and a point $b_k b_l$. Further the chord of a^2 in φ belongs to a pair of lines that must be counted twice. Finally there are four pairs of lines (r, r') to be counted twice, with a double point on $\varphi\varrho$ of which r passes through P and r' lies in φ . Hence $Pk^2\nu^3\varrho = 22$.

The number $k^2\nu^5\varrho$ is found in the following way. φ contains two conics each of which must be counted eight times. Through each point $b_k b_l$ in φ there pass 22 k^2 . The chord of a^2 in φ belongs to a pair of lines with double point on $\varphi\varrho$. There are eight pairs (r, r') of which r lies in φ and rests on a^2 and b_4 or b_5 and the double point lies on $\varphi\varrho$. Further there are eight pairs of which r' rests on a^2 , b_4 , b_5 and $\varphi\varrho$ and r lies in φ . Finally the two pairs (r, r') satisfy of which r' is a chord of a^2 and cuts the lines b_4 (b_5) and $\varphi\varrho$. Accordingly $16 + 66 + 2 + 16 + 16 + 4 = 120$.

Geology. — *Fractures and faults near the surface of moving geanticlines. IV. The elastic rebound of the earth crust in Central Honshu.* By Prof. H. A. BROUWER.

(Communicated at the meeting of January 29, 1927).

During the 3^d Pan Pacific Science Congress (October, November 1926) some of the members including myself were so fortunate to visit the regions that were devastated by the Mino-Owari earthquake of 1891 in Central Japan, guided by Prof. S. NAKAMURA of the University of Kyoto. This part of Central Japan is known for its numerous and important fractures and faults with horizontal and vertical displacements, formed during this earthquake, and which are clearly visible in the topography. The fault-scarp near Midori, which cuts a new road in two, showing clearly a vertical displacement of 6 m. and a horizontal one of 4 m., is reproduced in many handbooks as a typical example of movement along fractures during earthquakes.

Of this earthquake an extensive description is given by KOTO ¹⁾. He describes a fracture which can be followed over a distance of 112 km. from the Kisogawa to Fukui with striking regularity and clearness right through the mountains, valleys and plains of Central-Japan.

The origin of this great transverse fracture is of importance in connection with my views exposed a few years ago, about the relation between earthquakes and the movement of geanticlines in space ^{2) 3)}. It is clear, that such a regular and sharply defined transverse fracture, which can be followed nearly over the whole width of Central-Japan cannot be connected with local deformations and differences of velocity, but that here the deformation of the whole island Honshu has to be considered.

So we find a confirmation of my thesis, that *important transverse fractures near the surface of moving geanticlines will occur near the bending points of the horizontal projection of the geanticlinal axes*⁴⁾.

¹⁾ B. KOTO. On the cause of the great earthquake in Central-Japan. 1891. Journ. Coll. of Science. Imp. Univ. Japan. Vol. 5. Part IV, p. 295. Tokyo. 1893.

²⁾ H. A. BROUWER. Some relations of earthquakes to geological structure in the East-Indian archipelago. Bull. Seism. Soc. of America. 11, 1921, blz. 166.

³⁾ H. A. BROUWER. The geology of the Netherlands East-Indies. University of Michigan Studies. Scient. Ser. Vol. 3. p. 84-92.

⁴⁾ H. A. BROUWER. The horizontal movement of geanticlines and the fractures near their surface. Journ. of Geology. Vol. 29, 1921, p. 576.

Fig. 1 gives a schematic geanticlinal axis of the island Honshu and it is evident, that the great transverse fracture along which movement took place during the Mino-Owari earthquake intersects the island near this point. Thus follows, that the deformation and movement of the island



Fig. 1. Schematic representation of the geanticlinal axis of the island Honshu (2) and the fracture of the Mino-Owari earthquake of 1891 near the bending point of the geanticlinal axis (1).

Honshu take place in a way which are incidental to important differences in velocity in a horizontal direction f.i. near the fracture in the central part of the island. These slow deformations which take place continually, cause a strain in the crust which is occasionally relieved by sudden and rapid movements along the fractures, causing the earthquakes.¹⁾

The strength of the rocks will generally be exceeded in the first place where the differences in velocity of the slow movements are the greatest.

The displacements are shown in fig. 2. During an earthquake the crust suddenly springs back to positions of equilibrium as shown by the arrows 1 in fig. 2.

In applying this theory to the movements during the earthquake of 1891 in Central-Japan we find, that in contradiction to what could be concluded from the above mentioned considerations, the N.E. part along

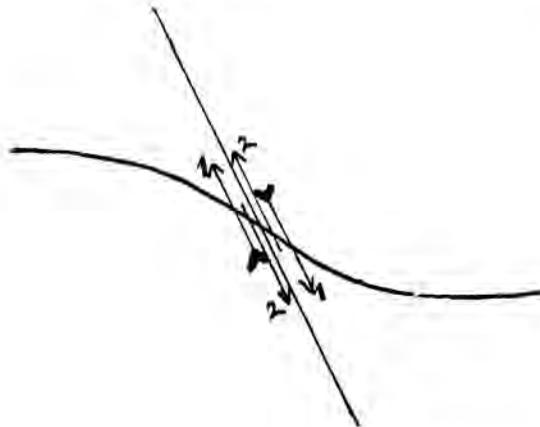


Fig. 2. Relative displacements along the fracture of the Mino-Owari earthquake. 1, derived from the shape of the geanticlinal axis of the island Honshu. 2, the observed relative displacements.

¹⁾ H. F. REID. The elastic rebound theory of earthquakes. University of Calif. Publ. Bull. Dept. Geol., Vol. 6, N^o. 19, 1911.

the fracture has been moved to the N.W., with regard to the S.W. part (2 in Fig. 2).

It could be concluded, that the movement of 1891 was an exception to the general rule, but in the following it will be shown that this needs not be the case and that the displacement at the surface can be in opposite direction of that at greater depth, the shape at the surface of the geanticlines being in the first place dependent on the movement at greater depths.

We did formerly already point out ¹⁾ the existence of tectonic zones and zones of movement at different depths. The velocity of horizontal movement at greater depth can show an important difference with that near the surface and if the horizontal movement is considered during a sufficiently long period the result will be, that points which in the beginning of the movement were situated on a vertical line, afterwards are on an irregular curve.

The deformation and the shear, tension and compression will change from place to place. In fig. 3 we have supposed for the sake of simplicity that of two adjacent parts of the crust one part deforms but slightly. In the other part the forces cause bending as well as compression. In the upper part, that is in the part nearer to the surface, the lengthening by bending is greater than the shortening by compression; in the lower

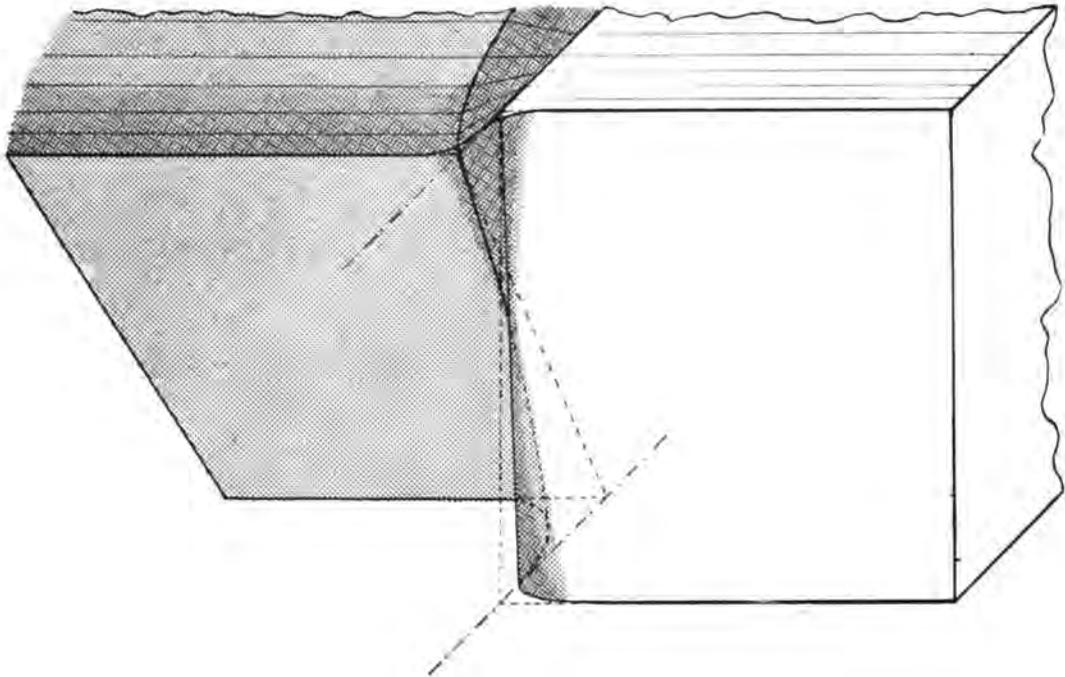


Fig. 3.

¹⁾ H. A. BROUWER. Fractures and Faults near the surface of moving geanticlines. II. Abnormal strikes near the bending-points of the geanticlines. *Proceed. Kon. Akad. v. Wet. Amsterdam*. Vol. 25, Nos 7 and 8, 1922.

part, that is in the part farther from the surface, shortening is due to bending as well as to compression.

When the strength is exceeded a sudden movement takes place along a fracture plane as is shown in exaggerated proportion in fig. 3. This movement causes a variable vertical displacement, the horizontal displacement in the upper part will have a direction *opposite* to that of the part situated farther from the surface.

The dimensions of the above mentioned parts of the crust can be left unconsidered. The distribution of the stress can vary very considerably for different depths. All this shows, that with a certain distribution of stress the movement described above can occur and that it is not permitted to make conclusions with regard to the direction of movement based on relative displacements at the surface.

When the horizontal projection of a geanticlinal axis shows a bending point, this proves differences of velocity at greater depths. The movements may be directed differently at both sides of the bending point, they can, however, also have the same direction but with a difference of velocity.

The strain near the bending point will differ from place to place and the amount of horizontal and vertical displacement will also differ from place to place. But the tendency of movement, which can be derived from the shape of the geanticline permits the supposition that a certain strain will predominate over greater distances near the fracture plane, so that definite vertical and horizontal relative displacements can also predominate.

Mathematics. — *The invariants of linear connexions with different transformations.* By Prof. J. A. SCHOUTEN. (Communicated by Prof. HENDRIK DE VRIES.)

(Communicated at the meeting of February 26, 1927).

A RIEMANNIAN connexion with fundamental tensor $g_{\lambda\mu}$ being transformed conformally

$${}'g_{\lambda\mu} = \sigma g_{\lambda\mu} \quad \dots \quad (1)$$

the quantity of curvature is transformed into

$${}'K_{\omega\mu\lambda\nu} = K_{\omega\mu\lambda\nu} - g_{[\omega} s_{\mu]} s_{\lambda} s_{\nu]} \quad \dots \quad (2)$$

being

$$s_{\mu\lambda} = 2 \nabla_{\mu} s_{\lambda} - s_{\mu} s_{\lambda} + \frac{1}{2} s_{\alpha} s^{\alpha} g_{\mu\lambda} \quad ; \quad s_{\lambda} = \partial_{\lambda} \log \sigma \quad \dots \quad (3)$$

Hence the connexion is then and only then conformal-euclidian if the quantity of curvature has the form

$$K_{\omega\mu\lambda\nu} = g_{[\omega} L'_{\mu]} s_{\lambda} s_{\nu]} \quad \dots \quad (4)$$

$L'_{\mu\lambda}$ being symmetrical, and if moreover a field s_{λ} can be found so that

$$2 \nabla_{\mu} s_{\lambda} - s_{\mu} s_{\lambda} + \frac{1}{2} s_{\alpha} s^{\alpha} g_{\mu\lambda} = L'_{\mu\lambda} \quad \dots \quad (5)$$

Now it is remarkable that the conditions of integrability of this equation are a consequence of (4) because of the identity of BIANCHI. Hence the equations (4) by themselves are necessary and sufficient to secure the conformal-euclidean property of the connexion²⁾. In other wise we can say that the quantity

$$C_{\omega\mu\lambda}^{\dots\nu} = K_{\omega\mu\lambda}^{\dots\nu} - g_{[\mu} L_{\lambda]} s_{\nu]} \quad ; \quad L_{\mu\lambda} = -K_{\mu\lambda} + \frac{1}{2(n-1)} K g_{\mu\lambda} \quad \dots \quad (6)$$

is an invariant with conformal transformations³⁾ and that the connexion is then and only then conformal-euclidean if this quantity vanishes.

WEYL⁴⁾ has shown that this peculiarity occurs also in the case of the transformations of an affine connexion that leaves the geodesics invariant. Also in that case the conditions of integrability are a consequence of the algebraical conditions and the identity of BIANCHI and the connexion is then and only then projective-euclidean if an invariant quantity of the fourth degree vanishes.

1) ∂_{λ} stands for $\frac{\partial}{\partial x^{\lambda}}$.

2) Math. Zeitschr. 11 (1921), p. 104—122; R.K. (Der Ricci Kalkül. Springer 1924), p. 169.

3) WEYL, Math. Zeitschr. 2 (1918), p. 384—411; Göttinger Nachrichten (1921), p. 99—112. Comp. R.K. p. 170.

4) Göttinger Nachrichten (1921), p. 99—112. Comp. R. K., p. 131.

In the following pages will be shown that the same peculiarity occurs in two other cases.

1. *Transformations leaving invariant all directions.*

By a linear connexion with $C_{\lambda\mu}^{\nu} = 0$ ¹⁾, but not necessary a symmetrical one, the local E_n in every point P of the X_n is represented linearly on every E_n belonging to a neighbouring point. Now we transform the connexion in such a way that the representation of all directions rests the same.

The formula for a general transformation is

$${}'G_{\lambda\mu}^{\nu} = G_{\lambda\mu}^{\nu} + A_{\lambda\mu}^{\nu} \dots \dots \dots (7)$$

$A_{\lambda\mu}^{\nu}$ being an arbitrary quantity of the third degree. If a field v^{ν} is chosen in such a way that in an arbitrary point P $\nabla_{\mu} v^{\nu}$ is zero, then the differential

$${}'\delta v^{\nu} = \delta v^{\nu} + A_{\lambda\mu}^{\nu} v^{\lambda} dx^{\mu} = A_{\lambda\mu}^{\nu} v^{\lambda} dx^{\mu} \dots \dots \dots (8)$$

must have the same direction as v^{ν} for every choice of v^{ν} and dx^{ν} . This is then and only then possible when $A_{\lambda\mu}^{\nu}$ has the form

$$A_{\lambda\mu}^{\nu} = A_{\lambda}^{\nu} p_{\mu} \dots \dots \dots (9)$$

p_{λ} being an arbitrary vectorfield. Hence the general formula for a transformation leaving invariant the representation of directions is (7) with $A_{\lambda\mu}^{\nu} = A_{\lambda}^{\nu} p_{\mu}$. The general formula for the transformed quantity of curvature is

$${}'R_{\omega\mu\lambda}^{\nu} = R_{\omega\mu\lambda}^{\nu} - 2 \nabla_{[\omega} A_{|\lambda|\mu]}^{\nu} - S_{\omega\mu}^{\alpha} A_{\lambda\alpha}^{\nu} - 2 A_{\alpha[\omega}^{\nu} A_{|\lambda|\mu]}^{\alpha} \dots (10)$$

Substituting (9) in this equation we get

$$\left. \begin{aligned} {}'R_{\omega\mu\lambda}^{\nu} &= R_{\omega\mu\lambda}^{\nu} - 2 A_{\lambda}^{\nu} \nabla_{[\omega} p_{\mu]} + 2 S_{\omega\mu}^{\alpha} p_{\alpha} A_{\lambda}^{\nu} - 2 A_{\lambda}^{\nu} p_{[\omega} p_{\mu]} \\ &= R_{\omega\mu\lambda}^{\nu} - 2 A_{\lambda}^{\nu} \partial_{[\omega} p_{\mu]} \end{aligned} \right\} (11)$$

and

$${}'V_{\omega\mu} = V_{\omega\mu} - 2 n \partial_{[\omega} p_{\mu]} ; \quad V_{\omega\mu} = R_{\omega\mu\alpha}^{\alpha} \dots \dots \dots (12)$$

hence

$${}'R_{\omega\mu\lambda}^{\nu} - \frac{1}{n} {}'V_{\omega\mu} A_{\lambda}^{\nu} = R_{\omega\mu\lambda}^{\nu} - \frac{1}{n} V_{\omega\mu} A_{\lambda}^{\nu} = \text{invariant.} \dots \dots (13)$$

Writing $A_{\omega\mu\lambda}^{\nu}$ for this invariant quantity we deduce immediately

$$A_{\omega\mu\lambda}^{\lambda} = 0 \dots \dots \dots (14)$$

and

$$A_{\alpha\mu\lambda}^{\alpha} = R_{\mu\lambda} - \frac{1}{n} V_{\lambda\mu} \dots \dots \dots (15)$$

¹⁾ R.K. p. 75.

From (15) we see that there exists also an invariant quantity of the second degree, which is in general not zero.

$A_{\omega\mu\lambda}^{\dots\nu}$ is zero if the transformed connexion is euclidean (= integrable). But vice versa, if $A_{\omega\mu\lambda}^{\dots\nu}$ vanishes it is possible to transform the connexion into an euclidean one by a transformation of the form (7,9).

Proof.

The supposition $A_{\omega\mu\lambda}^{\dots\nu} = 0$ is equivalent to the supposition that $R_{\omega\mu\lambda}^{\dots\nu}$ has the form

$$R_{\omega\mu\lambda}^{\dots\nu} = M_{\omega\mu}^{\cdot} A_{\lambda}^{\nu} \dots \dots \dots (16)$$

By contraction we get $M_{\omega\mu} = \frac{1}{n} R_{\omega\mu\alpha}^{\dots\alpha}$. Now we try to find a field p_{λ} so that

$$2 \partial_{[\xi} p_{\mu]} = M_{\omega\mu} \dots \dots \dots (17)$$

The conditions of integrability are

$$\partial_{[\xi} M_{\omega\mu]} = 0 \dots \dots \dots (18)$$

But by applying the identity of BIANCHI on (16) we get

$$\nabla_{[\xi} M_{\omega\mu]} A_{\lambda}^{\nu} = -S_{[\omega\mu}^{\dots\alpha} R_{\xi]}^{\dots\nu} = 0 \dots \dots \dots (19)$$

from which equation follows

$$\partial_{[\xi} M_{\omega\mu]} = -S_{[\omega\xi}^{\dots\alpha} M_{|\alpha|\mu]} - S_{[\mu\xi}^{\dots\alpha} M_{|\alpha|\omega]} - 2 S_{[\omega\mu}^{\dots\alpha} M_{\xi]\alpha} = 0 \quad (20)$$

Hence the conditions of integrability are a consequence of (16) and the identity of BIANCHI, and we have got the following theorem:

It is then and only then possible to transform a linear connexion (with $C_{\lambda\mu}^{\dots\nu} = 0$) into an euclidean one by a transformation leaving invariant all representations of directions, if the quantity of curvature has the form $M_{\omega\mu} A_{\lambda}^{\nu}$, in other words if the quantity $A_{\omega\mu\lambda}^{\dots\nu} = R_{\omega\mu\lambda}^{\dots\nu} - \frac{1}{n} R_{\omega\mu\alpha}^{\dots\alpha} A_{\lambda}^{\nu}$, which is an invariant with these transformations, vanishes.

Besides $A_{\omega\mu\lambda}^{\dots\nu}$ there exists another invariant quantity of the fourth degree, namely the projective quantity of curvature $P_{\omega\mu\lambda}^{\dots\nu}$ belonging to the symmetrical connexion with the parameters

$$\overset{0}{\Gamma}_{\lambda\mu}^{\nu} = \Gamma_{(\lambda\mu)}^{\nu} \dots \dots \dots (21)$$

By substituting $A_{\lambda\mu}^{\dots\nu} = -S_{\lambda\mu}^{\dots\nu}$ in the general formula (10) it follows that the quantity of curvature of this connexion is

$$\overset{0}{R}_{\omega\mu\lambda}^{\dots\nu} = R_{\omega\mu\lambda}^{\dots\nu} + 2 \nabla_{[\omega} S_{|\lambda|\mu]}^{\dots\nu} - 2 S_{\omega\mu}^{\dots\alpha} S_{\lambda\alpha}^{\dots\nu} - 2 S_{\alpha[\omega}^{\dots\nu} S_{|\lambda|\mu]}^{\dots\alpha} \dots (22)$$

and from $\overset{0}{R}_{\omega\mu\lambda}^{\dots\nu}$ we get $P_{\omega\mu\lambda}^{\dots\nu}$ in the well known way ¹⁾.

¹⁾ C.f. f.i. R.K. p. 131.

2. Transformations of the form $'I_{\lambda\mu}^{\nu} = I_{\lambda\mu}^{\nu} + q_{\lambda} A_{\mu}^{\nu}$.

In the second place we consider the case where the representation of the vectors in P on the E_n in a neighbouring point Q is changed in such a way that all extremities of vectors move parallel to δx^{ν} . If a field v^{ν} is chosen so that $\nabla_{\mu} v^{\nu}$ is zero in P , then $'\delta v^{\nu}$ must have the same direction as dx^{ν} . From this follows that $A_{\lambda\mu}^{\nu}$ has the form

$$A_{\lambda\mu}^{\nu} = q_{\lambda} A_{\mu}^{\nu} \dots \dots \dots (23)$$

q_{λ} being an arbitrary vectorfield. The transformed quantity of curvature is

$$\left. \begin{aligned} 'R_{\omega\mu\lambda}^{\nu} &= R_{\omega\mu\lambda}^{\nu} - 2 A_{[\mu}^{\nu} \nabla_{\omega]} q_{\lambda} + 2 S_{\omega\mu}^{\nu} q_{\lambda} - 2 A_{[\omega}^{\nu} q_{\mu]} q_{\lambda} \\ &= R_{\omega\mu\lambda}^{\nu} + 2 A_{[\omega}^{\nu} (\nabla_{\mu]} q_{\lambda} - q_{\mu]} q_{\lambda}) + 2 S_{\omega\mu}^{\nu} q_{\lambda} \end{aligned} \right\} (24)$$

hence

$$\left. \begin{aligned} 'R_{\mu\lambda} &= R_{\mu\lambda} + n \nabla_{\mu} q_{\lambda} - \nabla_{\mu} q_{\lambda} - n q_{\mu} q_{\lambda} + q_{\mu} q_{\lambda} + S_{\alpha\mu}^{\nu} q_{\lambda} \\ &= R_{\mu\lambda} + (n-1) (\nabla_{\mu} q_{\lambda} - q_{\mu} q_{\lambda}) + 2 S_{\alpha\mu}^{\nu} q_{\lambda} \end{aligned} \right\} (25)$$

and

$$'S_{\lambda\mu}^{\nu} = S_{\lambda\mu}^{\nu} + q_{[\lambda} A_{\mu]}^{\nu} ; 'S_{\alpha\mu}^{\nu} = S_{\alpha\mu}^{\nu} - 1/2 (n-1) q_{\mu} \dots (26)$$

From (24-26) we deduce that the quantity

$$\left. \begin{aligned} T_{\omega\mu\lambda}^{\nu} &= R_{\omega\mu\lambda}^{\nu} - \frac{2}{n-1} A_{[\omega}^{\nu} R_{\mu]\lambda} + \\ &\quad + \frac{4}{n-1} S_{\omega\nu}^{\nu} S_{\alpha\lambda}^{\nu} - \frac{8}{(n-1)^2} A_{[\omega}^{\nu} S_{|\alpha|\mu]} S_{\beta\lambda}^{\beta} \end{aligned} \right\} (27)$$

is invariant. From (27) follows

$$T_{\alpha\mu\lambda}^{\nu} = 0 \dots \dots \dots (28)$$

and

$$T_{\omega\mu\alpha}^{\nu} = V_{\omega\mu} + \frac{2}{n-1} R_{[\omega\mu]} + \frac{4}{n-1} S_{\omega\mu}^{\beta} S_{\alpha\beta}^{\nu} \dots \dots (29)$$

From (29) we see that there exists also an invariant quantity of the second degree, which is in general not zero. If the given connexion is halfsymmetrical

$$S_{\lambda\mu}^{\nu} = S_{[\lambda}^{\nu} A_{\mu]} \dots \dots \dots (30)$$

then the transformed connexion has the same property and $T_{\omega\mu\lambda}^{\nu}$ and $T_{\omega\mu\alpha}^{\nu}$ take the simple forms

$$T_{\omega\mu\lambda}^{\nu} = R_{\omega\mu\lambda}^{\nu} - \frac{2}{n-1} A_{[\omega}^{\nu} R_{\mu]\lambda} \dots \dots \dots (31)$$

$$T_{\omega\mu\alpha}^{\nu} = V_{\omega\mu} + \frac{2}{n-1} R_{[\omega\mu]} \dots \dots \dots (32)$$

1) R. K. p. 69.

From (24) follows that $T_{\omega\mu\lambda}^{\dots\nu}$ is zero when the transformed connexion is euclidean. But vice versa, if $T_{\omega\mu\lambda}^{\dots\nu}$ vanishes for a halfsymmetrical connexion it is for $n > 2$ always possible to transform the connexion into an euclidean one by a transformation with an $A_{\lambda\mu}^{\dots\nu}$ of the form (23).

PROOF.

The supposition $T_{\omega\mu\lambda}^{\dots\nu} = 0$ is aequivalent to the supposition that $R_{\omega\mu\lambda}^{\dots\nu}$ has the form

$$R_{\omega\mu\lambda}^{\dots\nu} = \frac{2}{n-2} A_{[\omega}^{\nu} R_{\mu]\lambda} \dots \dots \dots (33)$$

Now we try to find a field q_λ so that

$$R_{\mu\lambda} = -(n-1) (\nabla_{[\mu} q_{\lambda]} - q_{[\mu} q_{\lambda]} - S_{\mu} q_{\lambda]) \dots \dots \dots (34)$$

The conditions of integrability are

$$\left. \begin{aligned} 2 \nabla_{[\omega} R_{\mu]\nu} &= -(n-1) R_{\omega\mu\lambda}^{\dots\nu} q_\nu - 2(n-1) S_{[\omega} \nabla_{\mu]} q_\lambda + \\ &+ 2(n-1) \nabla_{[\omega} q_{\mu]} q_\lambda + 2(n-1) \nabla_{[\omega} S_{\mu]} q_\lambda \end{aligned} \right\} (35)$$

which equation passes into

$$\left. \begin{aligned} 2 \nabla_{[\omega} R_{\mu]\lambda} &= -2(n-1) q_{[\omega} S_{\mu]} q_\lambda - 4(n-1) S_{[\omega} \nabla_{\mu]} q_\lambda + \\ &+ 2(n-1) q_{[\omega} \nabla_{\mu]} q_\lambda + 2(n-1) q_\lambda \nabla_{[\omega} S_{\mu]} \end{aligned} \right\} (36)$$

by applying (34) and (35). But by applying the identity of BIANCHI on (33) we get

$$-2 S_{[\omega} R_{\xi\mu\lambda]}^{\dots\nu} = \frac{2}{n-1} A_{[\omega}^{\nu} \nabla_{\xi} R_{\mu]\lambda} \dots \dots \dots (37)$$

or, regarding (33)

$$-\frac{4}{n-1} S_{[\omega} A_{\xi}^{\nu} R_{\mu]\lambda} = \frac{2}{n-1} A_{[\omega}^{\nu} \nabla_{\xi} R_{\mu]\lambda} \dots \dots \dots (38)$$

from which follows for $n \neq 2$

$$2 \nabla_{[\omega} R_{\mu]\lambda} = 4 S_{[\omega} R_{\mu]\lambda} \dots \dots \dots (39)$$

or, in consequence of (34)

$$2 \nabla_{[\omega} R_{\mu]\lambda} = -4(n-1) S_{[\omega} \nabla_{\mu]} q_\lambda + 4(n-1) S_{[\omega} q_{\mu]} q_\lambda \dots (40)$$

Now the second identity ¹⁾ for a halfsymmetrical connexion is

$$R_{[\omega\mu\lambda]}^{\dots\nu} = 2 \nabla_{[\omega} S_{\mu} A_{\lambda]}^{\nu} \dots \dots \dots (41)$$

¹⁾ R. K. p. 88.

from which follows, regarding (33)

$$\frac{2}{n-1} A_{[\omega}^{\nu} R_{\mu\lambda]} = 2 A_{[\omega}^{\nu} \nabla_{\mu} A_{\lambda]}^{\nu} , (42)$$

for $n > 2$ aequivalent to

$$\nabla_{[\nu} S_{\lambda]} = \frac{1}{n-1} R_{[\mu\lambda]} = - \nabla_{[\mu} q_{\lambda]} + S_{[\mu} q_{\lambda]} (43)$$

In consequence of this equation the conditions of integrability (36) pass into the equation (39) which follows from the identity of BIANCHI, and we have got the following theorem:

It is for $n > 2$ then and only then possible to transform a half-symmetrical linear connexion (with $C_{\lambda\mu}^{\nu} = 0$) into an euclidean one by a transformation of the form $\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} + q_{\lambda} A_{\mu}^{\nu}$ if the quantity of curvature has the form $\frac{2}{n-1} A_{[\omega}^{\nu} R_{\mu\lambda]}$, in other words if the quantity $T_{\omega\mu\lambda}^{\nu} = R_{\omega\mu\lambda}^{\nu} - \frac{2}{n-1} A_{[\omega}^{\nu} R_{\mu\lambda]}$ vanishes.

The quantity $P_{\omega\mu\lambda}^{\nu}$ (§ 1) is also invariant with transformations characterised by (23).



Chemistry. — *Equilibria in systems in which phases, separated by a semi-permeable membrane. XIX.* By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of February 26, 1927).

Influence of the pressure on osmotic systems.

Of the many cases, which may occur, if we change the pressure of an osmotic system in which also occur solid phases, we only shall discuss some cases. If we assume that the saturationcurve wv and the isotonic curves wm and an of fig. 1 (communication III) are valid for the pressure P then we have a.o. the osmotic equilibria:

$$[Y + L_w \mid L_e]_P \quad [Y + L_a \mid L_n]_P \quad \text{fig. 1 III} \quad \dots \quad (1)$$

We now bring the pressure on the right side of the membrane on the higher pressure P_1 . As, like we have seen in the previous communication, then the *O.W.A.* of the liquid at the right side of the membrane decreases, we get the osmotic systems:

$$(Y + L_w)_P \leftarrow (L_e)_{P_1} \quad (Y + L_a)_P \leftarrow (L_n)_{P_1} \quad \dots \quad (2)$$

in which the water diffuses in the direction of the arrows. We now assume that a sufficient quantity of Y is present, so that the system at the left side of the membrane remains. Then the systems (2) pass into the osmotic equilibria:

$$(Y + L_w)_P \mid (L'_e)_{P_1} \quad (Y + L'_a)_P \mid (L'_n)_{P_1} \quad \dots \quad (3)$$

With the isotonic curve of the pressure P is conjugated an isotonic curve of the pressure P_1 , which must be situated further from the point W than curve wm ; we imagine this to be represented by curve bo . (fig. 1. III). Consequently liquid L'_e will be represented by the point of intersection l of the line We with this curve bo . The first of the osmotic equilibria (3) becomes, therefore:

$$(Y + L_w)_P \mid (L_l)_{P_1} \quad \text{fig. 1 III}$$

The result of increase of pressure is, therefore, that from liquid L_e so much water diffuses to the left, till it passes into the liquid L_l .

The composition of the liquids L'_a and L'_n in the second of the systems (3) depends on the ratio of the phases. However, it is sure that L'_a will be represented by a point of the saturationcurve between w and a or by the point w itself. In the latter case L'_n is represented by point o ; the second of the osmotic equilibria (3) becomes then:

$$(Y + L_w)_P \mid (L_o)_{P_1} \quad \text{fig. 1 III}$$

If, however, L'_n is represented by a point a' between w and a , then L'_n will be represented by a point n' on the side WX between o and X . Those points a' and n' , however, are situated not arbitrarily with respect to one another, but the isotonic curve of the pressure P going through point a' and the isotonic curve of the pressure P_1 going through point n' must be conjugated with one another.

We imagine in fig. 1 III two points q and r anywhere within the region $wmXv$; as the *O.W.A.* of each of those liquids L_q and L_r is larger than that of the solid substance Y under the same pressure, consequently an osmotic complex

$$[L_q \mid Y \mid L_r]_P \dots \dots \dots (4)$$

exists, in which no diffusion of water occurs. [For systems with two membranes, compare Comm. X]. We are able now, by bringing the pressure of the left liquid to a definite value P_1 and that of the right liquid to a definite value P_2 , to make the *O.W.A.* of both liquids equal to that of the solid substance Y under the pressure P . Then the osmotic complex (4) passes into an osmotic equilibrium, which, if we raise the pressure of both liquids still with ΔP , passes into the osmotic system:

$$(L_q)_{P_1+\Delta P} \overset{\uparrow}{\underset{\downarrow}{\rightleftharpoons}} Y_P \overset{\downarrow}{\underset{\uparrow}{\leftarrow}} (L_r)_{P_2+\Delta P} \dots \dots \dots (5)$$

in which the water diffuses from left and right towards the solid substance Y . The double-membrane $\mid Y \mid$ in system (4) impermeable for water, becomes permeable, therefore, if we raise the pressure of both liquids sufficiently only.

If a sufficient quantity of solid Y is present, then between the two membranes the system $Y + L_w$ is formed, and the right and the left liquid lose water as long as their *O.W.A.* becomes equal to that of the liquid L_w under the pressure P .

We take under the pressure P an equilibrium

$$E = (F + L)_P \dots \dots \dots (6)$$

in which F is an arbitrary liquid. We now bring the pressure to $P + dP$; keeping the total composition of this system E constant; then we get the new system:

$$E' = (F + L')_{P+dP} \dots \dots \dots (7)$$

in which the liquid L' differs a little in composition from L ; this difference in composition is completely defined by the change in pressure dP . With the transition of system E into E' the *O.W.A.* changes, as well on account of the change in pressure itself, as on account of the

change of the composition of the liquid; both influence may intensify or debilitate each other. In the previous communication we have seen that this total change $d\xi$ of the *O.W.A.* is defined by:

$$d\xi = -\Delta V_E \cdot dP \quad \dots \dots \dots (8)$$

if viz. $\Delta V_E \cdot \delta w$ is the change in volume, which occurs in the system *E* under the constant pressure *P*, if we add δw quantities of water. As ΔV_E , like we will see further, can be positive, negative and also zero, it follows from (8):

the *O.W.A.* of an equilibrium $E = F + L$ of constant composition decreases or increases at increase of pressure, according as ΔV_E is positive or negative; if $\Delta V_E = 0$, then the *O.W.A.* does not change with a small increase of pressure.

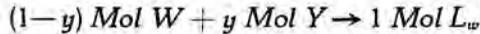
We will determinate for some cases this change of the *O.W.A.* Let us firstly take the equilibrium:

$$E = (Y + L_w)_P \quad \text{fig. 1 III} \quad \dots \dots \dots (9)$$

On increase of pressure this passes into the equilibrium:

$$E' = (Y + L'_w)_{P+dP}$$

in which L'_w is represented by a point w' which we must imagine in fig. 1 III in the vicinity of point w on the side *WY*. In order to apply formula (8) we have to determinate ΔV_E viz. the change in volume which occurs in system *E* (consequently under the constant pressure *P*) on addition of water. If we represent the composition of L_w , which remains unchanged on addition of water to system *E*, by y Mol *Y* + (1- y) Mol *W*, then, on addition of water the reaction:



occurs. If we represent the molecular volumina of L_w and the solid substance *Y* by *V* and V_y , then we find for ΔV_E and also with the aid of (8) for the change of the *O.W.A.*:

$$\Delta V_E = \frac{V - yV_y}{1 - y} \quad \dots \quad (10^a) \quad d\xi = -\frac{V - yV_y}{1 - y} dP \quad \dots \quad (10^b)$$

As we have seen in the previous communication, we can deduce the change of the *O.W.A.* yet also in another way. The *O.W.A.* of the liquid L_w of equilibrium *E* is viz. defined by:

$$\xi = -\zeta + y \frac{\partial \zeta}{\partial y} \quad \dots \dots \dots (11)$$

For the *O.W.A.*: $\xi + d\xi$ of liquid L'_w of equilibrium *E* is valid then:

$$d\xi = \left(-V + y \frac{\partial V}{\partial y}\right) dP + y t dy \quad \dots \dots \dots (12)$$

in which, however, dy is defined by dP . For the equilibrium E is true viz :

$$\zeta + (1-y) \frac{\partial \zeta}{\partial y} = \zeta_y \quad \dots \dots \dots (13)$$

from which follows for the equilibrium E' :

$$\left[V + (1-y) \frac{\partial V}{\partial y} - V_y \right] dP + (1-y) t dy = 0 \quad \dots \dots (14)$$

If we substitute this value of dy in (13) then we find (10^b).

As $1-y$ is always positive, the sign of ΔV_E is defined by the sign of the numerator of (10^a). If $V > V_y$ then the numerator is always positive; if $V < V_y$ then the sign depends on the value of y . With values of y , which do not differ much from 1, the numerator shall be negative then. We may say, therefore, that in general ΔV_E shall be positive, unless L_w should be a concentrated solution, which is formed from its components with contraction of volume.

In order to give another form to ΔV_E in (10^a) we put :

$$V + (1-y) \frac{\partial V}{\partial y} - V_y = \Delta V_y \quad V - y \frac{\partial V}{\partial y} = \Delta V_w \quad \dots (15)$$

so that $\Delta V_y \cdot \delta w$ represents the change in volume of the system E , when δw quantities of solid Y dissolve in the liquid; $\Delta V_w \cdot \delta w$ is the change in volume of the liquid, if this takes in δw quantities of water. We now can write for (10^a):

$$\Delta V_E = \Delta V_w + \frac{y}{1-y} \Delta V_y \quad \dots \dots \dots (16)$$

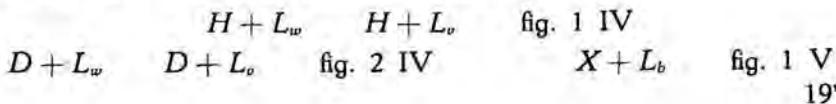
Of the many considerations to which this relation can give rise, we only take the case $\Delta V_y = 0$. Instead of (10^a) and (10^b) we then get

$$\Delta V_E = \Delta V_w \quad \dots (17^a) \quad d\xi = -\Delta V_w \cdot dP \quad \dots (17^b)$$

The change of the *O.W.A.* of system E on change of the pressure is then the same as if no solid Y but only the liquid L_w is present. This also follows from (14); if herein $\Delta V_y = 0$, then follows $dy = 0$, so that the composition of the liquid L_w remains constant with a change of pressure dP ; consequently the quantity of solid Y remains unchanged also.

As the *O.W.A.* of the solid substance Y under the pressure P is equal to that of the liquid L_w of system E , and under the pressure $P + dP$ equal to that of the liquid L'_w of the system E' , (10^b), therefore, defines also the change of the *O.W.A.* on a change of pressure of the solid substance Y .

Of course similar considerations are valid also for the change of the *O.W.A.* on a change of pressure of the equilibria :



and consequently also for the change of the *O.W.A.* on a change of pressure of the solid substances *H*, *D* and *X*.

Hence follows, therefore:

the *O.W.A.* of a solid substance *F* can as well increase as decrease on increase of pressure; this depends on the ΔV_E of the system $E = F + L$, in which *L* is the liquid, which is formed from the solid substance *F* by taking in (losing) water.

We now take under the pressure *P* the equilibrium:

$$E = (Y + L_a)_P \quad \text{fig. 1 III} \dots \dots \dots (18)$$

and we bring this, while we are keeping constant the total composition, to the pressure $P + dP$; we then get an equilibrium:

$$E' = (Y + L'_a)_{P+dP} \dots \dots \dots (19)$$

in which L'_a is represented by a point a' , which we must imagine in fig. 1 III in the vicinity of a on the line Ya (within or out of the region Ywv). In order to apply formula (8) we have to define ΔV_E , viz. the change in volume, which occurs in system (18) on addition of δw quantities of water. However, with this, the composition of the liquid L_a does not remain constant, as it was the case with liquid L_w in (9); L_a viz. changes its composition along curve wv starting from a in the direction towards w .

In order to simplify the calculations, we express the composition of the liquid L_a in *W*, *Y* and an arbitrary phase *Q*. We write its composition:

$$x \text{ quant. } W + y \text{ quant. } Y + (1 - x - y) \text{ quant. } Q \dots \dots (20)$$

so that we have a system of coordinates with *QW* as *X*-axis and *QY* as *Y*-axis. If the equilibrium *E* consists of *n* quantities of L_a and *m* quantities of *Y*, then we have:

$$\Delta V_E \cdot \delta w = d(nV + mV_y) = V dn + n \frac{\partial V}{\partial x} dx + n \frac{\partial V}{\partial y} dy + V_y dm \quad (21)$$

in which *V* represents the molecular-volume of the liquid L_a ; dx and dy are the changes in concentration, which occur in the liquid L_a , by the addition of δw quantities of water. As the quantity of water increases with δw , but the quantity of *Y* and *Q* rests unchanged, it follows from this:

$$\delta w = d(n x) = x dn + n dx \dots \dots \dots (22)$$

$$0 = d(n y + m) = y dn + n dy + dm \dots \dots \dots (23)$$

$$0 = dn(1 - x - y) = dn + dm - \delta w \dots \dots \dots (24)$$

in which (24) is simplified with the aid of (22) and (23). As the liquid L_a is in equilibrium with solid *Y*, the equation:

$$\zeta - x \frac{\partial \zeta}{\partial x} + (1 - y) \frac{\partial \zeta}{\partial y} = \zeta_y \dots \dots \dots (25)$$

is valid.

Hence follows:

$$[-rx + (1-y)s] dx + [-xs + (1-y)t] dy = 0 \dots (26)$$

Eliminating dn , dm , dx and dy from the equations (21)–(24) and (26) we find the value of ΔV_E . This elimination becomes more simple if we let coincide the arbitrary phase Q in (20) with the point a in fig. 1 III; then aW is the X -axis and aY the Y -axis of the system of coordinates; we now have $x=0$ and $y=0$. Putting those values in (22)–(24) and in (26) and eliminating dn , dm , dx and dy then we find after dividing by δw :

$$\Delta V_E = V + \frac{\partial V}{\partial x} + \frac{s}{t} \left(V_y - V - \frac{\partial V}{\partial y} \right) \dots (27)$$

so that the change of the *O.W.A.* is known with the aid of (8).

We are able to deduce this more plainly in the following way. We represent again the composition of the liquid L_a as in (20); its *O.W.A.* is then defined by

$$\xi = -\zeta - (1-x) \frac{\partial \zeta}{\partial y} + y \frac{\partial \zeta}{\partial y}$$

For an arbitrary liquid, which differs infinitely little from L_a then is under the pressure $P + dP$:

$$d\xi = - \left\{ \begin{aligned} & \left[V + (1-x) \frac{\partial V}{\partial x} - y \frac{\partial V}{\partial y} \right] dP + \\ & + [-(1-x)r + ys] dx + [-(1-x)s + yt] dy \end{aligned} \right\} (28)$$

If this new substance under the pressure $P + dP$ is always in equilibrium with solid Y , then follows from (25) that dx and dy must satisfy an equation, which we get by adding to (26) the term:

$$\left[V - x \frac{\partial V}{\partial x} + (1-y) \frac{\partial V}{\partial y} - V_y \right] dP \dots (29)$$

We call this (26^a). Just as above we now let coincide the arbitrary phase Q with the point a . As then x and y become zero, (28) and (26^a) pass into:

$$d\xi = - \left[V + \frac{\partial V}{\partial x} \right] dP - r dx - s dy \dots (30)$$

$$0 = - \left[V + \frac{\partial V}{\partial y} - V_y \right] dP + s dx + t dy \dots (31)$$

We now must define dx and dy in such a way that liquid L_a of equilibrium (18) passes into L'_a of (19); as, like we have noted already above L'_a must be situated on the line aY , consequently $dx=0$. If we substitute this in (30) and (31) and eliminating then dy we find:

$$d\xi = - \left[V + \frac{\partial V}{\partial x} + \frac{s}{t} \left(V_y - V - \frac{\partial V}{\partial y} \right) \right] dP \dots (32)$$

Consequently the same value for $d\xi$ as follows from (27).

In a similar way we can define the change of the *O.W.A.* in the systems:

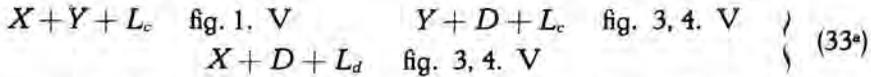


and other systems in which a liquid, saturated with a solid substance.

We now take a liquid, saturated with two solid substances, viz. the equilibrium:

$$E = (F + F_1 + L)_P \dots \dots \dots (33)$$

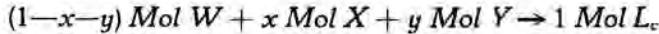
If we bring this, while its composition remains constant, under the pressure $P + dP$, then the liquid L gets a definite change in composition. If we take as simple example for E a ternary system, f.i. one of the equilibria:



then, on addition of water, the liquid does not change its composition, so that a phases-reaction occurs. If we represent the composition of the liquid L_c in the first of the equilibria mentioned above, by:

$$x \text{ Mol } X + y \text{ Mol } Y + (1-x-y) \text{ Mol } W \dots \dots (34)$$

then on addition of water the reaction:



occurs. Hence now follows:

$$\Delta V_E = \frac{V - xV_X - yV_Y}{1-x-y} \dots \dots \dots (35)$$

so that the change of the *O.W.A.* is defined. Corresponding considerations as for (10^a) are valid for this value of ΔV_E .

If we take the system:

$$E = (Y + H + L_d)_P \quad \text{fig. 2. V} \dots \dots \dots (36)$$

then, on addition of water, also herein a phases-reaction occurs in which, however, the liquid L_d does not participate. If we represent the composition of the hydrate H by: $\beta \text{ Mol } Y + (1-\beta) \text{ Mol } W$, then this reaction is:



Hence follows:

$$\Delta V_E = \frac{V_H - \beta V_Y}{1-\beta} \qquad d\xi = -\frac{V_H - \beta V_Y}{1-\beta} \cdot dP \dots \dots (38)$$

by which the influence of a change in pressure on the *O.W.A.* of system (36) is defined.

We also can deduce this again in the following way. If we bring the pressure of (36) to $P + dP$ then arises the equilibrium:

$$E' = (Y + H + L_d)_{P+dP} \dots \dots \dots (39)$$

in which L'_d is represented by a point d' , which we must imagine in fig. 2 V in the vicinity of point d . If we represent the composition of the liquid L_d by (34), then its *O.W.A.* is defined by:

$$\xi = -\zeta + x \frac{\partial \zeta}{\partial x} + y \frac{\partial \zeta}{\partial y} \dots \dots \dots (40)$$

As the liquid L_d is saturated with $Y + H$, are valid the equations:

$$\zeta - x \frac{\partial \zeta}{\partial x} + (1-y) \frac{\partial \zeta}{\partial y} = \zeta_Y \quad \zeta - x \frac{\partial \zeta}{\partial x} + (\beta - y) \frac{\partial \zeta}{\partial y} = \zeta_H \dots (41)$$

which we write, with the aid of (40) in the form:

$$\xi = \frac{\partial \zeta}{\partial y} - \zeta_Y \quad \xi = \beta \frac{\partial \zeta}{\partial y} - \zeta_H \dots \dots \dots (42)$$

For the liquid L'_d of equilibrium E' under the pressure $P + dP$ is valid then:

$$d\xi = \left(\frac{\partial V}{\partial y} - V_Y \right) dP + s dx + t dy$$

$$d\xi = \left(\beta \frac{\partial V}{\partial y} - V_H \right) dP + \beta (s dx + t dy)$$

from which follows by elimination of $s dx + t dy$:

$$d\xi = - \frac{V_H - \beta V_Y}{1 - \beta} \cdot dP \dots \dots \dots (43)$$

which is in accordance with (38).

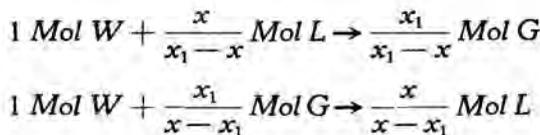
As the systems (33^a) and (39) define also the *O.W.A.* of the solid complexes $X + Y$, $Y + D$, $X + D$ and $Y + H$, we also know, therefore, the change of the *O.W.A.* of those solid complexes, if we bring their pressure from P to $P + dP$.

We now take under the pressure P an equilibrium:

$$E = (G + L)_P \dots \dots \dots (44)$$

with a vapour G and a liquid L . In order to make the calculation not too extensive, we assume that E is a binary system. We represent the composition of the vapour by x Mol $X + (1-x)$ Mol W and that of the liquid by x_1 Mol $X + (1-x_1)$ Mol W .

As the phases of this binary equilibrium have a constant composition at constant T and P , on addition of water, according as $x_1 - x$ or $x - x_1$ is positive, occurs one of the reactions:



If we represent the volume of the vapour by V and that of the liquid by V_1 then follows:

$$\Delta V_E = \frac{x_1 V - x V_1}{x_1 - x} \dots \dots \dots (45)$$

by which is defined the change of the *O.W.A.* of system E on a change of pressure.

We are able to find this in the following way also.

The *O.W.A.* of the vapour (and consequently also that of the liquid and that of the total system E) is defined by:

$$\xi = -\zeta + x \frac{\partial \zeta}{\partial x} \dots \dots \dots (46)$$

If we bring this system under the pressure $P + dP$, while its total composition remains constant, then we get the equilibrium:

$$E' = (L' + G')_{P+dP} \dots \dots \dots (47)$$

of which the *O.W.A.* is defined by

$$d\xi = \left(-V + x \frac{\partial V}{\partial x} \right) dP + x r dx \dots \dots \dots (48)$$

and in which dx has a value defined by dP . For the equilibrium E is true viz.:

$$\zeta - x \frac{\partial \zeta}{\partial x} = \zeta_1 - x_1 \frac{\partial \zeta_1}{\partial x_1} \quad \frac{\partial \zeta}{\partial x} = \frac{\partial \zeta_1}{\partial x_1} \dots \dots \dots (49)$$

from which follow for equilibrium E' the equations:

$$\begin{aligned} \left(V - x \frac{\partial V}{\partial x} \right) dP - x r dx &= \left(V_1 - x_1 \frac{\partial V_1}{\partial x_1} \right) dP - x_1 r_1 dx_1 \\ \frac{\partial V}{\partial x} \cdot dP + r dx &= \frac{\partial V_1}{\partial x_1} dP + r_1 dx_1 \end{aligned}$$

If we substitute, after having eliminated dx_1 , in (48) dx then we find:

$$d\xi = \frac{x_1 V - x V_1}{x - x_1} \cdot dP \dots \dots \dots (50)$$

which is corresponding with the value of $d\xi$ which follows from (45).

In order to define the sign of the coefficient of dP we distinguish two cases.

1. The vapour G has a smaller amount of water than the liquid L ; consequently we have $1 - x < 1 - x_1$ or $x > x_1$. As in general the volume V of the vapour is many hundred times as large as the volume V_1 of the liquid, $x_1 V - x V_1$ will be, therefore, generally positive, unless in the special case that x (viz. the concentration of the substance X in the vapour) is many hundred times as large as x_1 (the concentration of the substance X in the liquid).

If we exclude this case, then the coefficient of dP in (50) is positive, therefore.

2. The liquid L has a smaller amount of water than the vapour G ; consequently we have $1 - x_1 < 1 - x$ or $x_1 > x$. The numerator of (50) now always is positive, the denominator negative, so that the coefficient of dP is negative.

Consequently we can say:

the *O.W.A.* of the binary system $G + L$ becomes larger on increase of pressure, when the vapour has the smallest amount of water, and smaller if this is the case with the liquid.

We now take the osmotic equilibrium:

$$(Y + L)_P \mid (Y + L)_P \dots \dots \dots (51)$$

in which on both sides of the membrane the system (9) discussed above, under the pressure P . We distinguish two cases.

1. The solubility of the solid substance Y increases on increase of pressure.

If we assume that the total concentration of the substance Y is smaller at the left side of the membrane than on the right side, then, as on increase of pressure the quantity of solid Y decreases on both sides of the membrane, under a definite pressure P_1 ($P_1 > P$) the solid Y at the left of the membrane will disappear, while it is still present at the right side; then (51) passes into the osmotic equilibrium:

$$(L_1)_{P_1} \mid (Y + L_1)_{P_1} \dots \dots \dots (52)$$

in which the liquid has the same composition on both sides. If we increase the pressure still with ΔP , then the liquid at the left side of the membrane remains unchanged; at the right side of the membrane, however, it gets a somewhat other composition L'_1 . The *O.W.A.* at the left and at the right side of the membrane then increases with:

$$(d\xi)_l = -\Delta V_W \cdot dP \quad (d\xi)_r = -\Delta V_E \cdot dP \dots \dots (53)$$

in which ΔV_W and ΔV_E are defined by (10^a), (15) and (16). It now follows from (53):

$$(d\xi)_l - (d\xi)_r = \frac{y}{1-y} \cdot \Delta V_y \cdot dP \dots \dots \dots (54)$$

As the solubility of the substance Y increases on increase of pressure, ΔV_y must be negative, as is apparent from (14) and (15); consequently follows from (54):

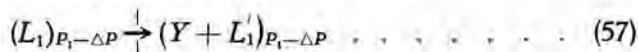
$$(d\xi)_l < (d\xi)_r \dots \dots \dots (55)$$

On an increase of pressure ΔP ($\Delta P > 0$) consequently (52) passes into the osmotic system:

$$(L_1)_{P_1+\Delta P} \mid (Y + L'_1)_{P_1+\Delta P} \dots \dots \dots (56)$$

in which the water diffuses in the direction of the arrow; this diffusion continues till the liquid L_1 gets the composition L'_1 .

2. The solubility of the solid substance Y decreases on increase of pressure. Then we must decrease the pressure, in order to make disappear the solid substance Y from the system $Y + L$; the osmotic equilibrium (52) then arises under a pressure P_1 smaller than P . As ΔV_y in (54) is positive on a decreased of pressure ΔP [in (54) then is $dP = -\Delta P$] (52) passes into the osmotic system:



in which the water diffuses in the same manner from left to right; the water-diffusion continues also here till the liquid L_1 gets the composition L'_1 .

(To be continued.)

Leiden, *Inorganic Chemical Laboratory.*

Zoology. — *On the Larval Forms of Rhizocephala.* By H. BOSCHMA.
(Communicated by Prof. C. PH. SLUITER.)

(Communicated at the meeting of February 26, 1927).

From a taxonomic point of view the larvae of the Rhizocephala are of special interest, as their structure gives sufficient evidence for the relationship of this group of parasitic animals with the Cirripedia. Like the nauplii of *Lepas* and *Balanus* the corresponding larval stages of *Sacculina* and *Peltogaster* possess well developed frontal horns, a peculiarity which is not found in other groups of Crustacea.

In all Rhizocephala the eggs after being hatched remain for a considerable time in the mantle cavity and here the first stages of development take place. The structure and the development of the larval stages of *Sacculina carcini* are well known (DELAGE, 1884). In this species the larvae leave the mantle cavity at the first nauplius stage. Between this stage and the cypris there are three more nauplius stages, each of which has a more or less different shape. Especially between the first and the second nauplius stages there is a considerable difference in structure (cf. DELAGE, 1884, Pl. XXII, figs. 1 and 2).

In general the specific characteristics of the adult Rhizocephala are not easily defined. Often the internal structure, especially the form and situation of the testes and the colleteric glands may give sufficient data for the description, but in other cases, as in the Sacculinidae, there are a great number of species which correspond completely in their internal structure. Now in this family many species are sufficiently characterized by the appendages of the external cuticle, which have a constant specific form, or are at least subject to very little variation in different specimens belonging to the same species. One would expect now that the larvae of different species of Sacculinidae would possess distinct specific characteristics also and even more conspicuously than the adult parasites.

Among the material of Rhizocephala from the collections of a number of zoological museums, which I have at present at my disposal, I have found the larvae of seven species of *Sacculina* in the mantle cavity of these parasites. They are therefore at the first nauplius stage, as also results from their structure. All these larvae, some of which still are in an excellent state of preservation, in general have the same form. The nauplii of three species are represented in fig. 1. One of these (fig. 1b) is the larva of *Sacculina carcini* (from a specimen attached to *Pachygrapsus marmoratus* from the Mediterranean Sea). The other, smaller specimen (fig. 1a) is from a small still undescribed species of *Sacculina* from Japan, the third (fig. 1c) is the nauplius of *Sacculina exarcuata* Kossm. from

Hong Kong. The general form of these three nauplii is very similar: they all have well developed frontal horns and small triangular ventral

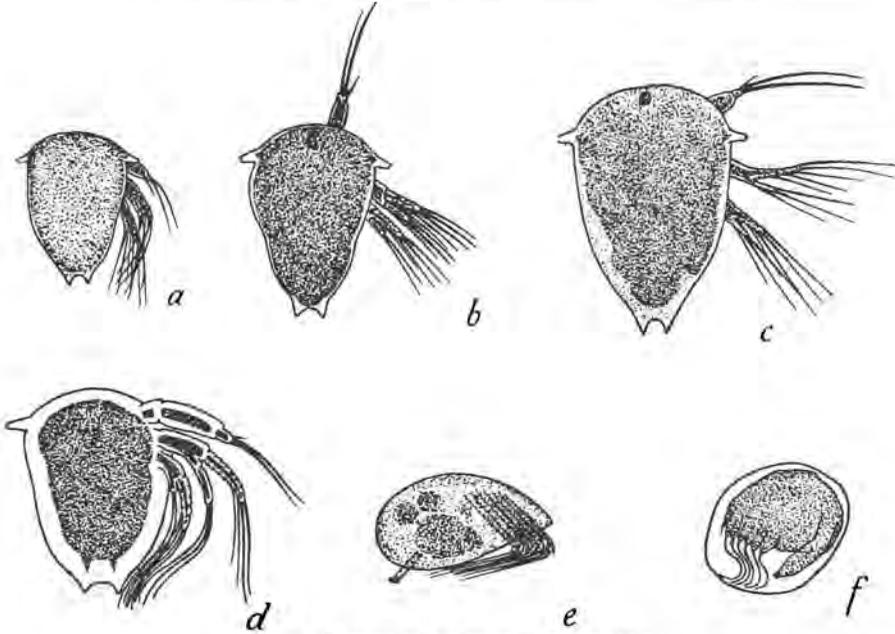


Fig. 1. Larvae of different Rhizocephala.

a—c, different species of *Sacculina*; d, *Peltogaster*; e, *Clistosaccus*; f, *Sylon*. $\times 130$.
In a—d the appendages of one half are omitted. For further particulars see text.

appendages at the extremity of the abdomen. In two of them (b, c) a distinct eye spot is to be seen; in the third species (a) this organ is completely wanting, which probably is a characteristic of the species. Another peculiarity, which, however, is subject to rather extensive individual variation, is the difference in size between the larvae of a certain species and those of another. The larvae of the Japanese species (fig. 1a) are much smaller than those of *Sacculina carcini* (fig. 1b), whilst the latter again are rather small in comparison with those of *Sacculina exarcuata* (fig. 1e). I did not find specific differences in the structure of the three paired appendages of these and other nauplii of *Sacculina*. But also in other respects the first nauplius stage in this genus gives very few data for a distinction of different species by characteristics found in the larvae. They may possess some peculiarities which certainly are of specific value, e.g., the presence or absence of an eye spot, but in general their differences are too inconspicuous to be of use for taxonomic questions. As to the differences in size found between corresponding larval stages of different species we cannot maintain that these are of any specific value. The eggs of large specimens of a certain species of *Sacculina* may be somewhat larger than those of small specimens belonging to the same species, and the resulting larvae in both cases may be of somewhat different size.

Probably the second and later nauplius stages of different species of *Sacculina* will show more striking differences. It is quite possible that their structure may give valuable data for the taxonomy. Unfortunately, however, the later nauplius stages of two species only are known, viz., those of *S. carcini* and of *S. neglecta*. The free swimming nauplius of the latter has been described by SMITH, and from this author's figure (SMITH, 1906, Pl. 4, fig. 17) results that really this larva differs from the corresponding stage of *S. carcini*. Especially the shape of the ventral abdominal appendages is different in both forms: in *S. neglecta* they are rather broad and are provided with dentations at the internal surface only, whilst in *S. carcini* they have a more slender shape and have lateral hairs at the external as well as at the internal surface.

In the Peltogastridae the larval stages may prove to be of very great interest for taxonomic purposes, e.g., in the case of *Peltogaster curvatus* Kossm. and *Peltogaster paguri* Rathke. The structure and anatomy of the adult forms of both of these species are very similar, though usually the external shape is somewhat different. The only characteristic of specific value in the adult animals is the presence of small papillae at the extremities of the mantle in *P. curvatus* (DUBOSCQ, 1912), which as a rule do not occur in *P. paguri*. There are, however, some remarks in the literature on *P. paguri* which point to the fact that also in the latter species sometimes these papillae may develop (cf. LILLJEBORG, 1859, GUÉRIN—GANIVET, 1911). But in the two species under consideration the larvae furnish a characteristic which in my opinion decides at once that we have to regard the above mentioned forms as different species. SMITH (1906) gives a short description and a figure of the free swimming nauplius of *Peltogaster curvatus*, which is characterized by much elongated frontal horns. These horns are directed forward (cf. SMITH, 1906, Pl. 4, fig. 18). SMITH used this peculiarity even as a characteristic in the diagnosis of the genus *Peltogaster*. Here, however, this author was not right, for in other species of this genus the frontal horns are much smaller and they are directed laterally or backward. The best known larvae are those of *P. paguri* and *P. sulcatus*, especially by the investigations of NILSSON—CANTELL (1921). On account of the different size and shape of the frontal horns these larvae may be at once distinguished from those of *P. curvatus* and this fact gives sufficient evidence for the view that *P. paguri* and *P. curvatus* are distinct species.

The first nauplius stage of different species of *Peltogaster* again often has a very similar form. Fig. 1d represents such a larva of *Peltogaster gracilis* Kr. from Chili, which in very subordinate peculiarities only differs from that of *P. sulcatus* (cf. NILSSON—CANTELL, 1921, fig. 14b). In the larva of *P. gracilis* also the frontal horns are small and have a lateral direction. The swimming hairs are somewhat longer than those of *P. sulcatus*.

As in *Sacculina* and in *Peltogaster* in most of the other Rhizocephala

the larva develops in the mantle cavity till the first nauplius stage. This is known from *Peltogasterella* (KRÜGER, 1914), *Lernaeodiscus* (MÜLLER, 1862), and *Parthenopea* (KOSSMANN, 1873). On the other hand in the genus *Thompsonia*, which in many other respects also differs from the other genera of Rhizocephala, the nauplius stage is omitted and the larva develops directly into the cypris stage (HÄFELE, 1912; POTTS, 1915). Besides this genus there is still another in which the eggs develop directly into cypris larvae, viz. *Sesarmaxenos* (ANNANDALE, 1911), a rather imperfectly known parasite from a fresh water crab found at the Adaman Islands.

Till now the larval stages of the species belonging to the genera *Sylon* and *Clistosaccus* (the name *Apeltes* is a synonym of the latter genus) were completely unknown. SMITH (1906) as well as HOEK (1909), the authors of the two most extensive publications on the Rhizocephala as a whole, express the opinion that there are in the literature no data concerning the larvae of these genera. In the material of the Copenhagen Zoological Museum, which I am studying for systematic purpose, I have found the larvae of both of the above mentioned genera. In *Clistosaccus* as well as in *Sylon* the development of the eggs results in the formation of the cypris larvae, here also a nauplius stage is omitted. The larva of *Clistosaccus paguri* (fig. 1e) is very similar to that of *Sesarmaxenos*; it has six pairs of pleopods with strong swimming hairs and a pair of well developed antennae. Though the larvae in my material are already fully developed many of them still are contained in the egg membrane, which proves that they have not developed from nauplii. Concerning the larvae of *Sylon hippolytes* (fig. 1f, the figured larva was taken from the mantle cavity of a parasite on *Spirontocaris lilljeborgi*) I want to state that the specimens of my material in all probability are not yet fully developed cypris larvae. Already a distinct pair of antennae and the six pleopods of each side are to be seen, but probably the larva of *Sylon* has to undergo some more changes before leaving the egg membrane. The swimming hairs of the pleopods are here also well developed.

Knowing now the chief points of the structure of the larvae of *Clistosaccus* and *Sylon* one can understand a few remarks published by KRÖYER (1855). This author has seen the larvae of three different species of Rhizocephala, which he could compare with the figure of the larva of *Sacculina* published by CAVOLINI (1787). KRÖYER examined the larvae of *Peltogaster gracilis*, of *Sylon hippolytes*, and of a species which he called "another species of *Peltogaster*". Now in former times the species *Clistosaccus paguri* has often been confounded with *Peltogaster paguri*, with which it often bears a strong external likeness. From another paper by KRÖYER (1842) we may deduce that he certainly has possessed specimens of the arctic form *Clistosaccus*. According to KRÖYER (1855) the larva of *Peltogaster gracilis* is very similar to that of *Sacculina*. This undoubtedly is true and is proved by a comparison of the figs. 1b and d in the present

paper. The larvae of *Sylon* and of "the other species of *Peltogaster*" were characterized by KRÖYER as forms which probably represented a later stage in the development. The larva of *Sylon* was very similar to that of "the other species of *Peltogaster*", but both differed strongly from those of *P. gracilis* and *Sacculina*. We may therefore safely conclude that KRÖYER as early as 1855 already knew the larvae of *Sylon* and *Clistosaccus* ("the other species of *Peltogaster*"), but owing to his short and incomplete remarks on these larvae they remained unknown in literature.

It is an interesting fact that protection of the brood, which occurs in so many arctic forms of animals, manifests itself also in the Rhizocephala. *Clistosaccus paguri* and *Sylon hippolytes* are true arctic species, whilst the Sacculinidae and the species of the genus *Peltogaster* do not occur in the far north. *Sesarmaxenos* is a tropical species, but it lives in fresh water and in this medium the development of the larvae also in other groups of animals is shorter than that in marine animals of the same group. The species of the genus *Thompsonia* are strongly specialized forms and differ structurally in many details from the other Rhizocephala. It is difficult to explain why also in this genus the eggs develop directly into a cypris stage.

LITERATURE.

- N. ANNANDALE, 1911. Note on a Rhizocephalous Crustacean from Fresh Water and on some specimens of the Order from Indian Seas. Rec. Ind. Mus., Vol. 6.
- F. CAVOLINI, 1787. Sulla Generazione dei Pesci e dei Granchi. Napoli. (This work was not accessible to me).
- Y. DELAGE, 1884. Évolution de la Sacculine (*Sacculina carcini* Thomps.). Arch. Zool. Exp. Gén. (2), T. 2.
- O. DUBOSCQ, 1912. Sur les Peltogastrides des Côtes de France. Arch. Zool. Exp. Gén. (5), T. 9, Notes et Revue.
- J. GUÉRIN—GANIVET, 1911. Contribution à l'Étude systématique et biologique des Rhizocéphales. Trav. Sc. du Lab. de Zool. et de Phys. Marit. de Concarneau, T. 3.
- F. HAFELE, 1912. Anatomie und Entwicklung eines neuen Rhizocephalen *Thompsonia japonica*. Abh. II. Kl. K. Ak. Wiss. München, Suppl. Bd. 2.
- P. P. C. HOEK, 1909. Die Cirripeden des nordischen Planktons. Nordisches Plankton, 8.
- R. KOSSMANN, 1873. Suctorina und Lepadidae. Habilitationsschrift, Würzburg. (also in: Arb. zool.-zoot. Inst. Würzburg, Bd. 1, 1874).
- H. KRÖYER, 1842. Monografisk Fremstilling af Slaegten *Hippolyte's* nordiske Arter. K. Dansk. Vidensk. Selsk. Naturvid. og math. Afh., Dl. 9.
- H. KRÖYER, 1855. Nogle Bemaerkninger om en højst ufulstaendigt bekjaendt Kraebdyr-gruppe. Overs. K. Dansk. Vidensk. Selsk. Forh. i. Aaret 1855.
- P. KRÜGER, 1914. Ueber ostasiatische Rhizocephalen. Abh. II. Kl. K. Ak. Wiss. München, Suppl. Bd. 2.
- W. LILLJEBORG, 1859. Les Genres *Liriope* et *Peltogaster*, H. Rathke. Nova Acta Reg. Soc. Scient. Upsal. (3), Vol. 3.
- F. MÜLLER, 1862. Die Rhizocephalen, eine neue Gruppe schmarotzender Kruster. Arch. Naturg., 28. Jahrg., Bd. 1.
- C. A. NILSSON—CANTELL, 1921. Cirripedenstudien. Zur Kenntnis der Biologie, Anatomie und Systematik dieser Gruppe. Zool. Bidr. f. Uppsala, Bd. 7.
- F. A. POTTS, 1915. On the Rhizocephalan Genus *Thompsonia* and its Relation to the Evolution of the Group. Papers Dep. Mar. Biol. Carnegie Inst. Washington, Vol. 8.
- G. SMITH, 1906. Rhizocephala. Fauna und Flora des Golfes von Neapel, 29. Monographie.

Anatomy. — *On the occurrence of Vascularized Enamel-organs.* By
J. MULLER. (Communicated by Prof. L. BOLK.)

(Communicated at the meeting of December 18, 1926).

Is the enamel-organ vascularized, or is it not? Up to this day opinions differ concerning this seemingly simple question.

In former years it was generally assumed that an enamel-organ was never vascular i.e. provided with bloodvessels. However, in later years some cases were described in which vascularization was beyond dispute. The present paper discusses a new case which Prof. BOLK submitted to me for examination. I shall preface the description of the case with a review of what has already been published on this vascularization.

When looking up the literature of the vascularization of the enamel-organ, it strikes us that this problem has but seldom received general notice, although it is of great importance for the nutrition of the ameloblasts. Most times that vascularization was treated of only incidentally; a systematic study has never been made of the occurrence of bloodvessels in the enamel-organ.

Indeed, it was generally known that in the later stages of the development, there is a closer relation between the bloodvessels and the stratum intermedium. When in the course of development the reduction of the enamel-pulp begins, the external epithelium leans up against the cells of the stratum intermedium, so that the bloodvessels are separated from the intermediate cell-layer only by this external epithelium. Nay, some researchers even hold that the external epithelium disappears altogether, and the capillaries come into direct contact with the stratum intermedium.

LEON WILLIAMS even describes, in this developmental stage of the *rat*, bloodvessels within the stratum intermedium. MUMMERY says of WILLIAMS: "he describes bloodvessels in the stratum intermedium at this stage (the later stages of enamel formation, when the stellate reticulum has disappeared), and that there are bloodvessels in this layer is fully evidenced by the photographs in illustration of his paper".

However, my object in this paper is to discuss the occurrence of vascularization in still fully developed enamel-organs, i.e. in stages in which the internal and the external epithelium are still separated by a well-developed pulp, and the latter does not show any sign of reduction. An instance of this was not described before 1889. Before entering upon my subject I will bring forward some reports regarding vascularization, that appeared before that date.

In the edition of 1870 WEDL wrote in his Manual in accordance with the then current opinion: "Das Schmelzorgan ist gefässlos."

In complete harmony with this pronouncement is LEGROS and MAGITOT's ('79) description of the enamelorgan in a fetus of a *horse* in the 3^d month of the development and in a newborn *dog*, in which they say: "cet organe étant absolument dépourvu aussi bien de système vasculaire que de système nerveux."

Just as the above researchers KLEIN and NOBLE SMITH ('80) detected that in a *dog* "no vessels are present in this middle membrane".

The first positive inquiry regarding a vascularized enamel-organ was made by POULTON ('89). This author studied the development of the dentition in *Ornithorhynchus paradoxus* and found here vascularization of the enamel-organ, as is borne out by the following quotation:

"I could not detect capillaries in the stratum intermedium — although they are certainly present in the stellate reticulum and are sometimes seen very near this layer (the stratum intermedium)", and a little further he says: "It is quite certain that blood-vessels are present in this layer (i.e. the enamel-pulp) and they extend into all parts of it." He could trace out the vessels from the adjacent connective tissue through the external epithelium down to the enamel-pulp, and observed that coincidentally connective tissue forced its way in. The capillaries looked like strands of spindle-shaped cells with a narrow lumen, often difficult to distinguish.

The author further remarks that HOWES has found bloodvessels in the enamel organ of the *rat* and showed him the preparations.

Strange to say, HILL and WILSON who, in connection with POULTON's study, examined more closely the dental development in *Ornithorhynchus*, do not make mention of a vascularization in their extensive description.

The next-following observation, recorded in the literature is of much later date (1911). In this long interval we only read, in a description given by ADDISON and APPLETON of a case of vascularization found by the authors themselves, that PAUL ('96) has observed the occurrence of vascularized enamel-organs in the *calves* and the *lamb*, but that he could not find in either bloodvessels in the enamel pulpa.

As stated just now, only in 1911 do we meet with the description of a second instance, and while POULTON found bloodvessels in the enamel organs of rudimentary teeth, such as are met with in *Ornithorhynchus*, HOPEWELL-SMITH and MARETT-TIMS now discovered bloodvessels in the enamel organ of teeth in a later developmental stage. They examined the development of the dentition in *Macropus billardieri*. According to their description the bloodvessels penetrate in several places through the external epithelium, and the section of these vessels was large enough to show blood-corpuscles. They did not find any bloodvessels in the stratum intermedium. The authors correlate this vascularization with the peculiar feature in the dental development of *Macropus*, that the ameloblasts begin to form enamel even prior to the formation of dentin. SMITH and TIMS write: "It is extremely probable that the unusual vascularity of the enamel-organs is correlated with the precocious development of the enamel."

A similar observation has been made by MUMMERY in an allied species, viz. in *Macropus rufus*. In his "The microscopical and general anatomy of the teeth" he says on this point: "bloodvessels are seen crossing the external epithelium and also lying within the stellate reticulum, an appearance seen in many sections."

An extensive description of a vascularized enamel-organ in another marsupial — viz. *Phascolarctos cinereus* — is given by BOLK ('15). He lays stress on the fact that a network of bloodvessels forms in the surrounding connective tissue at the transition from the enamel-septum to the external epithelium. From this network vascula force themselves into the enamel septum and thence into the pulpa, but also in other places bloodvessels grow through the external epithelium. Two layers are distinctly noticeable at the vascular reticulum in the pulpa the one more or less superficial, consisting of the larger vessels, and the other, situated deeper down, contiguous to the ameloblasts, and consisting of narrower vessels. The author adds that between the enamel pulp and the amelogenetic cells there lies only a thin layer of less differentiated cells, which should not be considered as a true stratum intermedium.

Also in the premolars of the *calf* vascularized enamel-organs were found by BROOMELL and FISCHER ('17), as is briefly indicated in a communication by ADDISON and APPLETON. A peculiar feature of this case is that the vessels could be traced as far as the stratum intermedium.

The observations of HOPEWELL-SMITH, MARETT TIMS and MUMMERY on the vascularization of the enamel-organs in the genus *Macropus* have been corroborated by CARTER. He says: "HOPEWELL-SMITH and MARETT TIMS referring to the vascularity of the enamel-organ in *Macropus billardieri*, state that they were unable to trace the capillaries beyond a point midway between the outer and inner enamel epithelium, but in my preparation vessels of a lumen from 20 μ to 30 μ in diameter are frequently seen lying in contact with the cells of the stratum intermedium."

CARTER found these relations in *Macropus ruficollis*.

ED. PASCALIS ('18) and SKILLEN ('21) worked with a technique quite new for this inquiry. They injected their preparations. PASCALIS studied the vascularization of the teeth during the development, and to this end he examined a *cat's*-embryo of 10 cm in length — a newborn *cat* — one of 1 week, one of 2, 3, 4, 5 weeks and a full-grown *cat*; also a newborn *dog* and a *human fetus*.

With all of them he finds: "les vaisseaux adamantins forment, à la surface de l'organe de l'émail un réseau comparable à celui de la pulpe, mais à mailles plus serrées et formées de capillaires de plus gros calibre."

SKILLEN also worked with much material and examined the development of the dentition in the *pig* — the *dog* — the *sheep*, the *rabbit*, the *cavia* and some earlier stages in *man*. He used injected and non-injected preparations, but could not detect any bloodvessels in the enamel-organ.

JORDAN's researches date from the same year. He studied an "anlage" of

a tooth from the lower jaw of a *cat* and felt convinced that vascularization of the enamel-organ does not occur here.

In a second publication he makes mention of his observations of an incisor of the *white rat* in a period from 1 to 7 days after birth and of the tooth-germ from the lower jaw of a newborn *cat*, and of cats, 1—2 and 3 weeks old. He finds that the incisor of the white rat is not furnished with bloodvessels, while in the *cat* the bloodvessels push the outer epithelium slightly forward. Hereby vascular papillae are generated, which were first described by WILLIAMS, and are called after him WILLIAMS'S papillary layer.

In 1922 ADDISON and APPLETON record the occurrence of vascularization of the enamel-organ of the first molar in the upper-, and the lower-jaw. Their material comprised different stages in the development of the *white rat*. On the 21st day of the intrauterine life vessels have really penetrated into the pulpa. In the following stages the vessels have grown deeper into the pulpa, and approach the stratum intermedium. In the three-day-old rat the vessels are contiguous to the stratum intermedium. It is interesting that "at the time of the entrance of the first bloodvessels into the enamel-organ (21 days' fetus) no enamel has yet been formed. As the number of entering bloodvessels increase, and with their deeper penetration into the stellate reticulum, amelogenesis begins. Not until the bloodvessels have penetrated to the vicinity of the stratum intermedium is there a well-defined layer of enamel to be seen."

The occurrence of perivascular lymph-spaces, first described by BOLK ('15) could be confirmed by them.

Although many of the above descriptions are conclusive for the occurrence of vascularized enamel-organs, still it is discredited by some observers.

JORDAN says that he is very sceptical as to the cases of vascularization described. He does not consider them conclusive.

PRENANT ('24) sides with JORDAN and holds that the would-be capillaries apparently found in the pulpa are nothing else but severed vascular papillae of WILLIAMS, so that the vessels are still enclosed by outer epithelium.

I do not agree with these authors, and feel convinced that bloodvessels may indeed occur in the enamel pulp, but that this depends on the species and the age of the animal. Vascularized enamel-organs are by no means present in every animal species, and they are found only in a definite developmental stage.

When summarizing the various publications on the vascularization of the enamel-organ it appears, that most authors deny its occurrence, but that the experience of some workers is that an entrance of bloodvessels into the enamel-organ can unmistakably be witnessed. What strikes us most is that the same positive results were obtained with various species of Marsupialia, viz. with *Phascolarctos cinereus*, *Macropus rufus*, *Macropus*

ruficollis and *Macropus billardieri*, in all of which bloodvessels were found in the enamel-pulpa. This indicates that with the species of Marsupialia the occurrence of this vascularization may reasonably be expected to be much more general than has hitherto been found.

Secondly the conviction grows upon us, that there is some connection between this supply of bloodvessels and amelogenesis.

HOPEWELL-SMITH and MARETT TIMS already pointed to this, as they found that the vascularization was attended with a precocious development of the enamel. ADDISON and APPLETON also came to the conclusion that: "The vascularization of the enamel-organ is closely connected chronologically with the beginning of amelogenesis.

Let us state in the third place that BOLK established the absence of a stratum intermedium in *Phascolarctos cinereus*, and correlated this with the precocious vascularization.

CARTER arrives at the same conclusion in his examination of *Macropus ruficollis*, when he says: "when growth is very rapid the stellate reticulum may be seen in contact with the ameloblasts without the intervention of the stratum intermedium."

That this is not always the case, is shown by the inquiry of ADDISON and APPLETON, in which the stratum intermedium was present in all the stages they studied.

In the fourth place we may ask why the vascular system comes so soon into contact with the stratum intermedium and the ameloblasts.

When there is no vascularized enamel-organ, we see that, after the reduction of the pulpa, the bloodvessels come into close contact with the ameloblasts. The pulpa provided nourishment for the ameloblasts, and after the reduction of the pulpa this function is assumed by the bloodvessels. What, however, should we think of those cases in which the enamel-organ is still fully developed and the bloodvessels penetrate into the pulpa and are contiguous to the stratum intermedium? At the conclusion of this paper I hope to recur to this point.

My own observation of vascularization of the enamel-organ was made on an embryo of *Dactilomys*.

Frontal sections of $20\ \mu$ of the preparation were stained with hematoxylin. Three tooth-germs were present in the upper- and the lower-jaw, on the left as well as on the right.

Dactilomys belongs to the Hystricomorpha, a subdivision of the Rodentia, whose complete dental formula is

$$I \frac{1}{1} P \frac{1}{1} M \frac{1}{1} \frac{2}{2} \frac{3}{3}.$$

There is no succession, so that the germs of I—P and M_1 are present.

Amelogenesis has not yet begun, but dentinification has. In the incisors the dentin already consists of a thick layer, as may be seen in fig. 1 representing a section through the "anlage" of an incisor from the

upper jaw. The enamel-organ has slightly receded from the adjacent connective tissue, also a fissure has arisen on either side of the dentin through the retraction of the dental papilla and the ameloblasts. A stratum intermedium is well-developed. The section has cut the tooth-band only a little near the mouth-epithelium.

However, what attracts our attention most is the occurrence of blood-vessels in the enamel-pulp. Vascularization can be seen in all the sections of this tooth. The bloodvessels may distinctly be traced from the adjacent connective tissue into the pulp. The vascula that appear as strands of

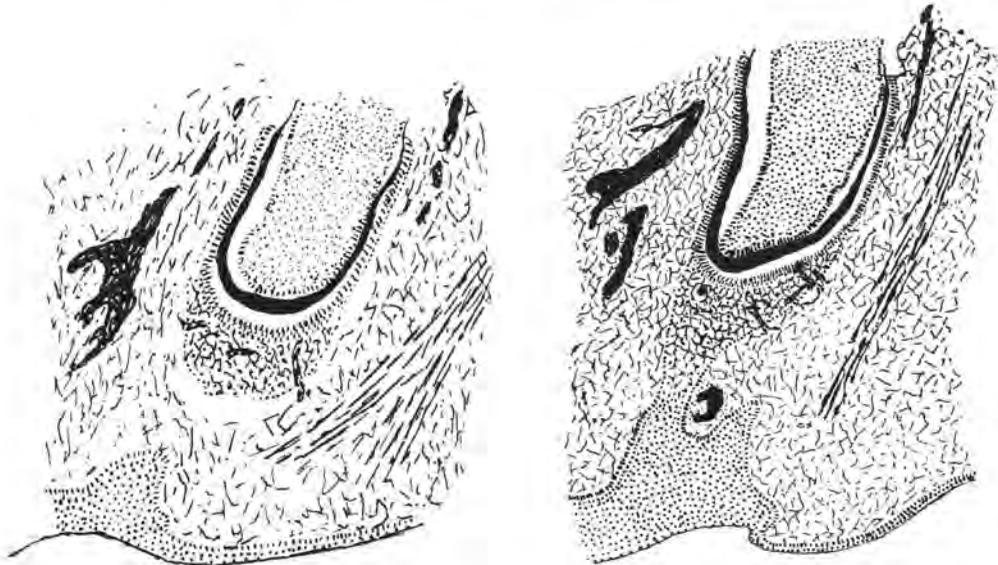


Fig. 1.

Fig. 2.

spindle-shaped cells, pass through the enamel-organ in all directions, but never enter the stratum intermedium. They can only be observed as far as this layer.

When proceeding to the following sections we notice (Fig. 2) another peculiarity of the vascularized enamel-organ, viz. that here vessels penetrate into the enamel-organ exclusively on the buccal side. This, however, is the exception, but when comparing the various sections of the incisivi, we shall observe that most vessels make their way into the enamel organ on the buccal side of the dental ledge. This is also the case with the premolars.

Something like this was also observed by ADDISON and APPLETON in their preparations. They ascribed it to the structure and the position of the enamel-organ. They correlated this phenomenon in the first place with the difference in the length of the lingual and the buccal sides of the organ, measured from the point of attachment of the dental ridge to this organ. The authors say: "This difference in length of the lingual and buccal portions, seems to be of importance in connection with the site of entrance

of the blood-vessels into the enamel-organ." This difference in length also occurs in my preparations.

But the relations in my case are the reverse of those found by ADDISON and APPLETON. This may be seen from a comparison of fig. 2 with what they write. In the figure the buccal area is considerably longer than the lingual area. The said authors write: "The fact that the palatal surface of the external epithelium is considerably wider than the buccal or lateral surface would (other things being equal) in itself afford opportunity for the entrance of more bloodvessels through the palatal surface, than through the buccal surface."

The authors have preceded these descriptions by the statement that the dental ledge is attached medially to the enamel-organ and say that: "the lingual surface of the dental ledge is almost in a straight line with the adjoining lingual surface of the external enamel epithelium", so that I may be allowed to conclude that this reversion of the relations is the consequence of a misconception, and that, therefore, the relations in their preparations agree with those I found in *Dactilomys*.

Secondly ADDISON and APPLETON ascribed it to the position of the dental germ relative to the alveolus of the tooth, as is expressed in the following quotation: "This advantage is further increased by the fact that the buccal surface of the external enamel epithelium is for almost its full extent in close contact with the osteogenetic membrane or with the developing bone itself of the maxilla. This proximity would seem to interfere with the free access of blood-vessels to this side of the enamel-organ."

After applying the correction alluded to above we see, then, that the relations are the same also here.

On close inspection of fig. 2 our attention is arrested by the presence of a rudimentary tooth. Buccally to the dental ledge we see it as a conoid mass of dentine. It is contiguous to the epithelium of the mouth-cavity and I could not detect an enamel-organ, but I observed that the connective tissue has thickened round the toothlet.

Similar rudimentary toothlets were present in the upper- and in the lower-jaw. They were always disposed labially to the germ of the incisors. They have often been described in papers on rodents and have been considered as the remains of a milk dentition.

Fig. 2 shows us still more. Hitherto we have always been speaking of "dental ledge". May we do so? In the figure the ledge is obviously many cells broad and it seems to me that the enamel-organ is rather suspended directly to the mouth-epithelium, without the intervention of a dental ledge. The epithelial cells are seen to pass gradually into the pulpal cells.

The germ of the incisivi from the lower-jaw is much smaller than of those of the upper-jaw. Fig. 3, representing a cross-section through such an incisor, shows us a veritable network of vessels in the enamel-pulp. This

vascular reticulum has penetrated into all the parts of the pulpa, and is supplied by vessels that have grown through the outer epithelium in several

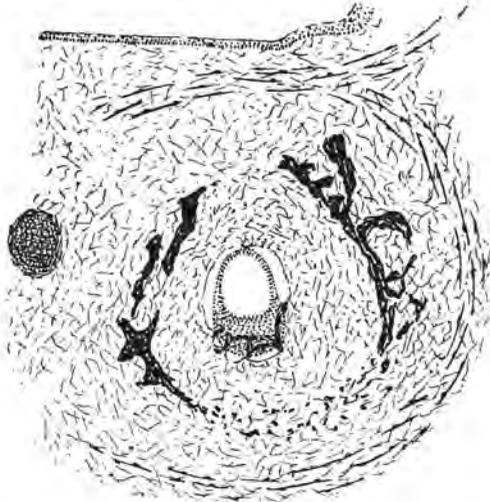


Fig. 3.

places. The stratum intermedium is very broad, the vessels do not enter here either. What we do see again, is that the vessels are seen to run up against this layer. The outer-, and the inner-epithelium are separated only on one side from the enamel-organ by pulpal tissue. On the other side they are adjoined. This is easy to understand, if we reflect that we have to do here with the rudiment of an incisor of a rodent, which displays amelogenesis only on the labial surface.

The section has not cut the dentine. Owing to the retraction it does not lie in the plane of the figure.

I fancied I had also found in this preparation the perivascular lymphspaces described by BOLK. We see them already in fig. 2, but better still in fig. 4. The vessels are surrounded here by broad spaces. Are these the same lumina that BOLK observed? The spaces shown by the figure may be artefacts. The enamel-pulp is extremely aqueous. Fixation of course makes it shrivel, thus causing the spaces. This view is still favoured by the fact that they are connected with that fissure round the dental germ, which is no doubt the result of the shrivelling.

These fissures in the incisivi I also found in the premolars. They also have a vascularized enamel-organ.

Fig. 5 represents a premolar from the upper-jaw. Here again we observe retraction through shrivelling. The odontoblasts have scarcely started the formation of the dentine. Only an extremely thin layer lines these cells. The vessels penetrate from the enclosing connective tissue into the enamel-organ, but they never reach the stratum intermedium, as was the case with the incisivi. There is always a layer of stellate reticulum between the

capillaries and the stratum intermedium. But here also the vessels cross and recross the pulpa, being most numerous between the folds of the molars.

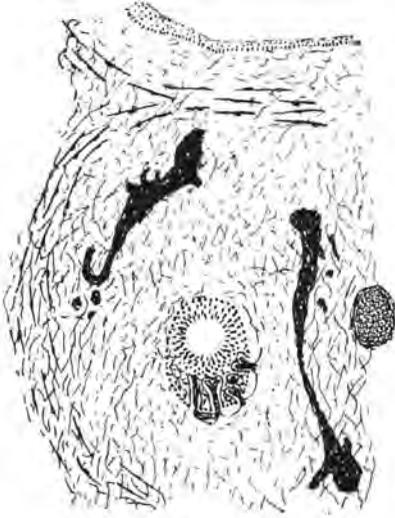


Fig. 4.

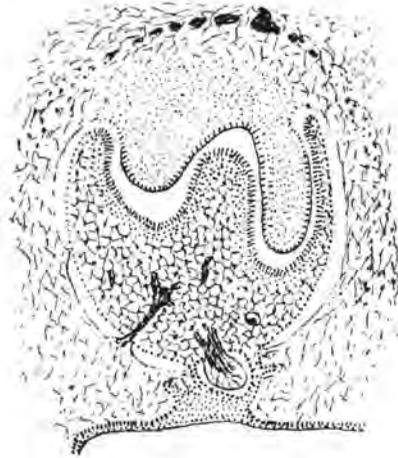


Fig. 5.

Another peculiarity is shown by this picture. There is namely an enamel-niche in the premolars of the upperjaw and in all the molars. In the false molars they have already developed into tunnels.

Whereas in the incisivi and the premolars the enamel-organ was completely developed, this appears not to be the case with the molars. The odontoblasts just begin to arrange themselves and as yet there is no question about dentinification. Neither has the enamel-pulp been differentiated, while the external epithelium is cubic. We do not meet with vascularization here either. I was going to say *as yet* for I am convinced that the bloodvessels enter into the enamel-organ only during the development. The incisivi that began to develop first, display a vascularization that is furthest advanced. The vessels close up to the cells of the stratum intermedium. In the premolars this process has not yet advanced so far. Here the vascularization is of a more recent date, whereas in the molars no bloodvessels have at all entered into the enamel-organ.

Amelogenesis begins only after the bloodvessels have penetrated far into the pulpa. Dentinification, however, begins about simultaneously with the entrance of the vessels, and has proceeded far when the vessels have reached the stratum intermedium.

But why do the bloodvessels enter into the enamel-organ at so early a stage? If what BOLK says, is true, viz. that in non-vascularized enamel-organs the enamel-pulp is intended for the nourishment of the ameloblasts, of which I am convinced, I am bound to conclude that in the case of vascularization of the enamel-organ the pulpa can no longer meet the demands of this nutrition, and that this higher function is assumed by the

bloodvessels themselves. If the enamel-organ is not vascularized, the bloodvessels come into contact with the ameloblasts only after reduction of the pulpa, but according as the demands of this nutrition increase, the bloodvessels are sure to assume sooner the task of the pulpa, i.e. they enter into the enamel-organ.

In Dactylomys we saw that amelogenesis occurs only after the bloodvessels have reached the stratum intermedium. So the bloodvessels grow into the enamel-organ at an early stage, and take over the nutrition of the ameloblasts. But I insist on it that here also the ameloblasts are fed first by the enamel-pulp and only later on by the bloodvessels.

LITERATURE.

- W. H. F. ADDISON and J. L. APPLETON Jr. The vascularity of the Enamel-organ in the Developing Molar of the Albino-Rat. *The Am. Journ. of Anat.* Vol. 31, Nov. 1922.
- L. BOLK. Ueber ein Gebiss mit vaskularisierten Schmelz-organen. *Anat. Anz.* Bd. 48, 1915.
- J. N. BROOMELL and PH. FISCHER. *Anatomy and histology of the mouth and teeth.* 5th edition 1917.
- J. TH. CARTER. The Cytomorphosis of the Marsupial Enamel-organ and its Significance in Relation to the structure of the completed Enamel. *Phil. Trans. Roy. Soc. Ser. B.* 208.
- A. HOPEWELL—SMITH and H. W. MARETT TIMS. *Tooth-germs in the Wallaby.* *Proc. Zoöl. Soc. London* 1911.
- A. E. JORDAN. Further evidence concerning the function of osteoclasts. *Febr.* 1921. *Anat. Rec.* Vol. 20.
- A. E. JORDAN. The comparative histology of the enamel organ of the mammalian tooth, with special reference to its blood-supply. *Am. Journ. of Anat.* Vol. 29, 1921.
- E. KLEIN and E. NOBLE SMITH. *Atlas of Histology* 1880.
- CH. LEGROS et E. MAGITOT. *Morphologie du follicule dentaire chez les Mammitères.* *Journal de l'anat. et de la Phys.* 1879.
- J. HOWARD MUMMERY. *The microscopical and general anatomy of the teeth.* Second edition 1924.
- ED. PASCALIS. *La vascularisation des Follicules dentaires des carnassiers et de l'homme.* Thèse 1918.
- F. T. PAUL. The enamel organ. *Dental Record* Vol. 16, 1896.
- EDW. P. POULTON. The true teeth and the horny plates of *Ornithorhynchus*. *Quart. Journ. Micr. sci.* Vol. 29, 1889.
- A. PRENANT. Contribution à l'histogénèse de l'email dentaire. *Arch. de Morph. Gén. et Exp.* 1924.
- WM. G. SKILLEN. A report on the formation of dentin and cementum relative to the structure of the root end. *Journ. Nat. Dent. Assoc.* Vol. 8, 1921.
- C. WEDL. *Pathologie der Zähne* 1870.
- J. T. WILSON and J. P. HILL. Observations on tooth development in *Ornithorhynchus*. *Quart. Journ. Micr. sci.* Vol. 51, Febr. 1907.

Zoology. — *On the structure, life-history and development of Hedrocystis pellucida* HERTW. & LESS. By H. R. HOOGENRAAD. (Communicated by Prof. J. F. VAN BEMMELEN.)

(Communicated at the meeting of February 26, 1927).

1. *Collection and conservation of Material.*

Those Heliozoa, whose protoplasmic body is enclosed in a homogeneous, spherical skeleton, generally provided with a stalk, and perforated by outlets for the pseudopodia, were in 1874 by R. HERTWIG and E. LESSER united in the group of the *Desmothoraca*. This group contains a very small number of genera, only one of which, the genus *Clathrulina*, has become more accurately known, and of this moreover but one single species *C. elegans* CIENK. It is true that SCHAUDINN in 1896 mentioned a second genus *Orbulinella* ENTZ Sr., as being rescribed with sufficient precision, but the author himself of this genus argues in favour of the view, that it occupies a separate position in the systematic arrangement, in the neighbourhood of the Foraminifera. In the same way PENARD, who however did not know the animal from personal observation, presumes that *Orbulinella* stands nearer to the Thecamoebic Rhizopods than to the Heliozoa.

The species belonging to the genera *Elaster* and *Choanocystis* until now have only been observed by their discoverers, and moreover in a very restricted number of widely spread specimens, so they must be considered as quite insufficiently known and therefore of very doubtful value for systematic arrangement.

Somewhat better stands the case for the genus *Hedriocystis*, which HERTWIG and LESSER proposed in 1874 for a species *H. pellucida*, discovered by them in the neighbourhood of Bonn. Later on PENARD described a second species *H. reticulata*, and afterwards BROWN distinguished a third one: *H. spinifera*.

Yet just like those named above the representatives of this genus are so rare and consequently so incompletely studied, that older investigators (ARCHER, SCHAUDINN, BÜTSCHLI), who were only acquainted with the species *H. pellucida*, considered even this to be a doubtful form, which possibly should be united with *Clathrulina*. As the more recent observations have not added much of importance to these older statements, the only form of the whole group *Desmothoraca*, that might be considered as characterized with sufficient accuracy, remained the species *Clathrulina elegans*.

In September 1926 I was able to collect material from the so called Wisselsche Veen near Epe o/d Veluwe, a locality, which for many years I have

highly valued as a rich emporium for Rhizopods and Heliozoa, amongst which were several of the rarest and most remarkable forms ever known. Of those that I have already mentioned on former occasions, I wish to refer to *Penardia mutabilis*, *Vampyrella lateritia*, *Hyalodiscus rubicundus*, *Nuclearia caulescens*, *Quadrula symmetrica*, *Heleopera petricola*, *Cryptodifflugia compressa*, *Paulinella chromatophora*, *Raphidiophrys elegans* and *viridis*, *Acanthocystis aculeata* and *turfacea*, *Clathrulina elegans*, *Clathrella Foreli*.

In the same way the spoils of this years collecting proved again rich in interesting forms, among which *Biomyxa vagans*, *Raphidiophrys pallida*, *Plagiophrys scutiformis*, *Gromia nigricans*, *Difflugia oviformis*, *Lieberkühnia Wagereri*, were new for the locality, the two latter moreover for the dutch fauna in general.

But what chiefly attracted my attention was the frequent occurrence of a great many specimens of one of the above-mentioned species of Heliozoa, namely *Hedriocystis pellucida* HERTW. & LESS. Since the discoverers, who observed it about 1870 in the neighbourhood of Bonn, this animal had been found by LEVANDER in Finland (1892), and by PENARD in a few localities near Geneva, about 1900, but nearly always in scanty numbers. PENARD is the only one, who gives a somewhat detailed description of its structure and life-habits, but its development has hitherto remained unknown.

The sample of water, in which the animals occurred, was taken from a shallow marshy pool, recently called into existence by the removal of the superficial layer of peat. I studied the animals in the usual coverglass-preparations, but always added a few algae and a little detritus to the liquor, with the view of keeping them as much as possible in natural conditions.

As experience proved, when the evaporated liquor was regularly replaced by distilled water, the animals kept alive for days and even for weeks in the same preparation, and went on propagating. In this way they could be studied at a magnification of 400—800 diameters.

2. Structure of the protoplasmic body and the shell.

The protoplasmic body of *Hedriocystis pellucida* (Fig. 1) is, in the same way as that of *Clathrulina elegans*, enveloped by a nearly spherical shell, which is fixed to the fragments of detritus lying in the water, by means of a long and thin stalk. The animals born in my preparations, often fixed this stalk on the coverglass or the slide.

On the surface of the shell conical projections are to be seen, either blunt or pointed, provided at the tip with exceedingly minute openings by which the pseudopodia emerge. I have never succeeded in observing these pores directly, but owing to a circumstance to be described later, I have been able to calculate their width at a diameter of $\pm 2 \mu$. By the presence of these protuberances the shell in optic sections has the appearance of a more or less regular polygon with 4, 5, 6 or 7 sides and with prominent angles ;

by elevating the object-lens to a higher level, the eminences on the surface of the shell assume the aspect of sharply defined, dark crescentic form lines, which give a very characteristic appearance to the animal.

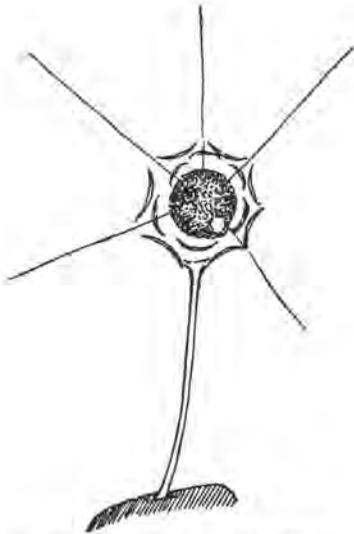


Fig. 1. *Hedriocystis pellucida* HERTW. et LESS. A specimen of the type. In the protoplasmic body at the left topside one sees the nucleus; at the right bottom-side a contractile vacuole. $\times 800$.

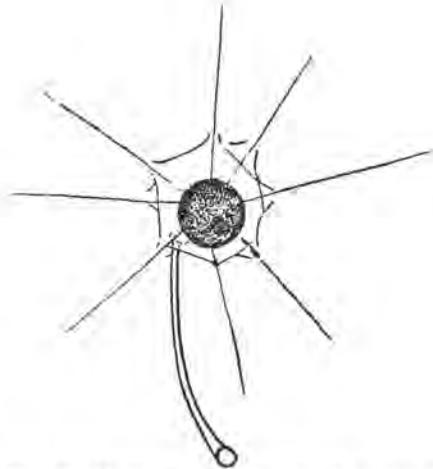


Fig. 2. *Hedriocystis pellucida*. A specimen of the deviating form. $\times 800$.

The above-described form represents the type, but besides this my material contains rather a considerable number of specimens of a second form, which though corresponding with the typical one in all other respects, is markedly distinguished from it by the structure of the shell. In this deviating form the thickness of the shell is much less, which makes the contourline of the optic section decidedly slimmer, to such an extent as to reduce it to a hardly visible line. In the second place the protuberances are less pronounced, so much so, that in high-level observation they never produce the above-described effect. The specimens showing this deviation were always completely colourless, those of the type being of a yellowish-red shade. PENARD considers this shade as a hue of the substance of the shell proper, but according to my view it seems more probable that it is caused by a phenomenon of interference. From a careful comparison of descriptions and illustrations I come to the conclusion, that the discoverers HERTWIG and LESSER have only seen what I have just now described as the deviating form ¹⁾, while the animals observed by PENARD

¹⁾ They compare the external appearance of the animal to a „Morgensternartige Keule“, meaning a mediaeval club, which at its swollen top is provided with spikes. This description corresponds much better indeed with the variety than with the type.

probably represented the type. Transitions between the two forms never came into my observation.

The diameter of my specimens fluctuated between 20 and 25 μ , their dimensions therefore correspond with that of the animals studied by former investigators.

As already remarked by PENARD the stalk is probably solid, not containing a central canal like that of *Clathrulina*. Its length was not in all cases easy to be measured, especially so, when it stood in a more or less oblique position. In the cases, where I was able to take its measure, its length varied between 50 and 90 μ , its thickness is $\pm 1 \mu$. The distal extremity of the stalk is either simply broadened, or split up into a small number of tiny fibres, giving it a more or less rooted appearance, as also sometimes occurs in *Clathrulina*.

As to the material of which the shell and its stalk are composed, nothing can be asserted with certainty; it seems to me to be rather more chitinous than that of *Clathrulina*, which is generally assumed to contain a certain amount of silicon. The protoplasmic body contained in this envelope has an average diameter of no more than 10—12 μ , and is freely suspended, probably by means of the pseudopodia, in the centre of the shell, or a little excentrically. In outward appearance it looks most like that of *Actinophrys sol.* Generally it shows no very distinct shade, but is tinted somewhat blueish-green. Its form is nearly spherical or somewhat flattened in a spheroidal way.

At its circumference it is sometimes lobed or denticulated in an irregular manner, but habitually it is rather purely rounded. The line of its contour is sharp and dark, which points in the direction of great optic density. Consequently the animals, even when observed with low powers, show a sharply defined appearance, making them immediately conspicuous, when once we have got acquainted with them.

Big food-morsels are generally absent, but small dark granules are always present, though in varying number.

Probably on account of the compact structure of the protoplasm, the nucleus is difficult to be distinguished. In cases where it is visible (Fig. 1, 6), it presents itself as a big, round spot, lying somewhat excentrically and provided with a large light-grey endosome.

Vacuoles seldom occur. Only when an uncommonly big prey has been swallowed (Fig. 6), a vacuole surrounding it can be clearly distinguished. Contractile vacuoles are always present, to the number of two or three. (Fig. 1, 6). They never form prominences on the surface, as in *Actinophrys*; their bulk size is variable and may attain $\frac{1}{3}$ of the diameter of the protoplasmic body. Their contractions are regular, with usually short periods; in one case where I was able to observe this process for a considerable time, the interval between two successive contractions amounted to ± 30 seconds (temperature 18°).

Excretion of undigested food-remains was never observed by me.

The pseudopodia are generally present in restricted numbers, in toto probably never more than 10 to 15, but sometimes only 6 to 8. In the normal condition they are thin, straight, unbranched and rather long, up to \pm four times the diameter of the shell; in most cases they start with a sharply defined, unbroadened base from the body of the protoplast. They are therefore different in several respects from the pseudopodia of *Clathrulina elegans*, which broaden considerably towards their base, and moreover are often forked, and sometimes form anastomoses. Eventually however, especially when food is taken in, the pseudopodia of *Hedriocystis* assume various irregular shapes by curving, broadening and by the appearance of vacuoles (Fig. 6). I never was able to observe a real current of granules; what might easily be considered to be such, in reality turned out to be nothing but the sliding movements of a microbe, absorbed as a food-object, and transported in the direction of the protoplasmic body.

3. *Life-history.*

In my specimens the food, conformally to what was observed by PENARD, chiefly consisted of bacteria, only in a few cases a small Flagellate or the swarm spore of some alga was captured. The absorption occurs in the usual way. The pseudopod, with which the prey comes in contact, generally forms a very spacious food-vacuole around it. Inside this the food-morsel becomes slowly moved towards the shell (Fig. 6). During the ingestion the pseudopod remains stretched: the prey, enclosed in its vacuole, glides along it with tolerable speed. It is clear that this movement points to the existence of a protoplasmic current in a centripetal direction.

Small objects speedily disappear into the interior of the protoplasmic body through the pores of the shell, bigger ones on the contrary remain for a long time fixed on its outside, surrounded by a protoplasmic mass, which itself is in communication with the rest of the body. After a certain lapse of time the prey is seen to become smaller and less sharply defined and at last it disappears completely in the protoplasm. In some cases I could observe that the last remnant slipped into the shell and sank into the protoplasmic body. Just as in *Clathrulina* part of the food at least is digested outside the shell.

Under normal circumstances the animals swallow rather a large quantity of food. It is not rare that several pseudopodia at a time are seen to carry bacteria towards the protoplasmic body. Digestion and absorption of the food seem to take place rapidly; occasionally I succeeded in following rather big food-morsels from the moment of their entrance into the protoplast up to that of their disappearance or becoming unrecognizable; the whole of this process lasted a couple of hours. By this rich nutrition the young animals grow rather rapidly: in two or three days they attained the normal size, and shortly afterwards the reproduction sets in.

As far as is known up till now, this latter always occurs by binary fission of the protoplasmic body; one of the daughter-individuals then escaping from the shell, to form a new specimen, which provides itself with a stalk, while the other remains behind, occupying the old shell. The processes of division and formation of the shell together take less than twelve hours; after this both animals enter into the period of food-ingestion and growth, then a new division sets in and so on. So the whole of the simple life-history of *Hedriocystis* takes place within three days, of which the greater part is devoted to feeding and growing, a much smaller period to propagation. The latter is — as far as known — purely vegetative, processes of impregnation or sexual differentiation have hitherto remained quite unknown for this animal, as for most Heliozoa in general.

4. *Reproduction.*

The older investigators do not seem to have observed the division. It is true that HERTWIG and LESSER, as well as PENARD, make mention of specimens, in which two protoplasmic bodies were seen inside one shell, and which therefore undoubtedly had just before passed through the act of division, but about this process itself they tell us nothing. I have been able to follow it several times from beginning to end; as far as I could see, it always takes the same course.

The division (Fig. 3) takes place during activity inside the shell, which

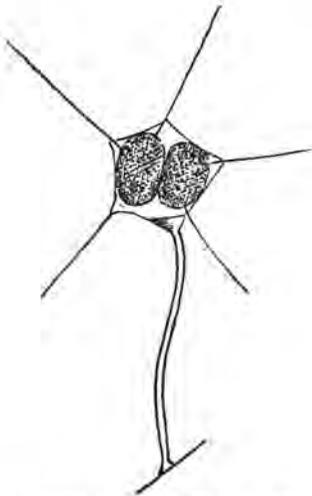


Fig. 3. *Hedriocystis pellucida*. Divided specimen of the deviating form. $\times 800$.

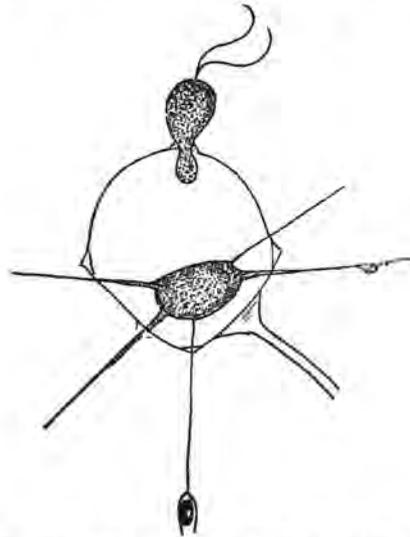


Fig. 4. *Hedriocystis pellucida*. Divided specimen of the deviating form, one of the daughter-individuals leaves the shell as a swarm spore with two flagella. The other animal is taking in food with two of its pseudopodia. $\times 1200$.

means, that the pseudopodia remain protruded. Nor is there the slightest indication of encystment or of a resting stage. I never saw cysts, as described by HERTWIG and LESSER and figured by PENARD; and I am inclined to explain their observations as simply caused by temporary retraction of the pseudopodia, a condition which I have observed more than once. PENARD himself states that the wall of the cyst was "lisse et peu apparente"; HERTWIG and LESSER express themselves in the same sense.

The divisional process proper shows nothing particular; it is completed in a few minutes. The behaviour of the nucleus could not be observed. During the division the capture and absorption of food goes on as usual.

The plane of division generally stands perpendicularly to the stalk, or at least to that part of it, which is in immediate contact with the shell.

After division is complete, the two daughter-individuals generally change their position, and lose their definite orientation inside the shell-cavity. They then assume a broad-elliptic shape, their long axes standing parallel, while they are a little flattened on the side where they are in contact with each other.

For how long the animals remain in this condition, I cannot tell, but I presume it is only for a short time, and certainly no more than 12 hours. At the end of this period one of the two individuals loses its rounded shape and begins to make amoeboid movements, at first at intervals, afterwards more continually. Shortly afterwards two flagella become visible, arising side by side on the surface of the body and soon they begin to show slow movements. They are exceedingly thin and therefore difficult to be seen, probably also because their optic density is but slightly different from that of water. Shortly after this, the pseudopodia are retracted, and the animal begins to force itself through one of the shell-pores (Fig. 4). At first a small drop of hyaline plasma appears on the external surface of the shell, in the course of time this slowly enlarges by continuous outpouring of plasma through the shell-opening; the part of the body still remaining inside the shell proportionately diminishing in bulk. During this act the animal is continually constricted by the extremely narrow shell-pore, and so has the form of a dumb-bell; the diameter of the constriction amounts to $2\ \mu$; therefore this will probably be the width of the shell-pore. From this extreme constriction, and the fact that its diameter remains unchanged during the whole act of escaping from the shell, we may conclude that the animal does not succeed in enlarging the pore through which it forces itself out; the shell therefore, however thin, must be of a rather rigid texture.

When the last part of the protoplast is still inside the shell, the two flagella have already passed out and their movements, at first slow, grow faster, and under their influence the free part of the animal, still fixed by the shell-pore, is brought into a rapid vibration. Soon afterwards the animal begins to rotate along its longitudinal axis, and thereby the last contact with the shell is broken. The animal then stretches and shortens itself and swims away as a zoospore with peculiar undulating and rotatory

movements. The other half, remaining in the shell, rounds itself and takes up a central position, so assuming the normal state for this animal.

The zoöspore (Fig. 5) has a cylindrical shape with rounded ends, sometimes the posterior pole is a little pointed. Its length is $15\ \mu$, its breadth $3\ \mu$. It is transparent, uncoloured and rather regularly granulated. In its anterior part is a nucleus, somewhat difficult to discern and at the posterior end a small but distinct contractile vacuole. The length of the flagella is probably somewhat greater than that of the spore. The rhythm of the swimming movements is so characteristic, that even with a low power it is easy to distinguish the zoöspore from similar organisms.

After a short time, usually only two or three minutes, the movements become slower, and soon the zoöspore fixes itself, and rounds itself off as a protoplasmic body of $\pm 10\ \mu$ in diameter.



Fig. 5. *Hedriocystis pellucida*. Swarm spore. Before the nucleus, behind the contractile vacuole. $\times 1200$.

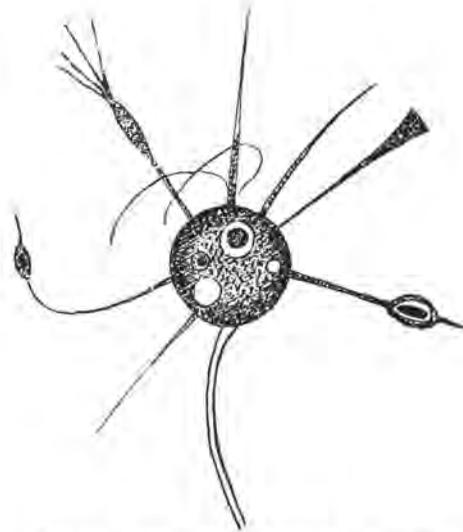


Fig. 6. *Hedriocystis pellucida*. Young specimen before the formation of the shell. At the top two flagella are still visible, at the bottom the outgrowing stalk. Abnormal pseudopodia forms two of the pseudopodia with a captured microbe. In the plasmic body at the top the nucleus, at the bottom, right and left side a contractile vacuole, at the left of the centre a digestion vacuole with a microbe. $\times 1500$.

The flagella, which, during the free-swimming period, remained invisible, on account of the rapidity of their movements, now become visible again. During a considerable lapse of time, at least a couple of hours, they execute peculiar, pendulating movements, but at last they disappear. What may be their final fate, to become dejected, or retracted, I am not sure; their

extraordinary thinness already necessitates to the greatest exertion to get a look at them, and even then one only succeeds from time to time just for a moment. Almost immediately afterwards the pseudopodia make their appearance, within a minute or so they have attained their normal length. In its general aspect the animal now rather resembles a miniature specimen of *Actinophrys sol* (Fig. 6). During the first period of their existence the pseudopodia appear to be in a very unstable condition; they often show deviations from the normal, straight and thin form which afterwards is typical for them, these consist in enlargements, sinuosities, ramifications and the formation of vacuoles. While the flagella are still present, the taking in of food already begins. Likewise the development of the stalk is a question of a few minutes only. In the same way as described by HERTWIG and LESSER for *Clathrulina* the stalk is formed by a protoplasmic protrusion of the body. The supposition of PENARD, that originally the stalk is nothing but a pseudopodium, as far as I can see may very well be correct. Be this as it may, soon afterwards the stalk is distinguished from a common pseudopodium by its superior thickness and stable diameter. Originally it stands rather strictly perpendicular to the surface on which the stalk fixed itself, probably in consequence of a thigmotactic stimulus. This may be stated with special ease when the support on which the organism fixes itself is the coverglass or the slide. The position of the stalk then being vertical, it becomes visible in projection as a sharply drawn ring of very small diameter, either above or under the protoplasmic body. By means of the scale on the micrometric screw the length of the stalk could be very accurately measured. Still easier could this be done by direct measuring, when the stalk was fixed to some object floating in the water, and thereby was occupying a horizontal position. After fixation the stalk elongates considerably just as in *Clathrulina*; after half an hour it attains its definite length. But long afterwards, for at least two hours, the upper part of the stalk, adjoining the protoplasmic body, shows amoeboid movements in the shape of blunt protuberances, which soon flatten out again. In a few instances I also was able to distinguish a current of granules in this part of the stalk, of the same character as always occur in the body. In the young stalk of *Clathrulina* HERTWIG and LESSER mention also such a granular current. All these facts plead distinctly for a genetic connection between stalk and pseudopodia, though perhaps the assertion that the stalk originally becomes formed as a true pseudopodium may not be quite accurate. In a phylogenetic sense however the view that the stalk of *Clathrulina* and *Hedriocystis* must be derived from a pseudopodium, seems to find support in the above-mentioned facts. A few hours after its formation the stalk begins to change its aspect; it becomes thinner and more transparent, the amoeboid motility disappears, probably on account of a change of its purely protoplasmic nature to the harder, more chitinous material, of which the definitive stalk consists.

An insoluble problem in this respect is the question, how the connection

of stalk and shell is brought about. For though the stalk undoubtedly is formed as a product of the protoplast, it afterwards has lost all ties with it, and passes exclusively into the shell. As far as I can judge, this question is not even completely solved for *Clathrulina*, though here it applies somewhat differently, because in *Clathrulina* the stalk contains a canal. It is true that also this stalk is originally formed as a purely protoplasmic stem, but then its outer layer hardens to a tube corresponding with the skeleton of the protoplasmic body, and, as HERTWIG and LESSER take it, the protoplasmic core then withdraws from the lumen.

In this connection we also may call to memory PENARD's communications, that in *Nuclearia caulescens* PEN., an organism of doubtful affinities but most probably connected with the Heliozoa¹⁾, both types of stalk seem to occur. One of them corresponds according to PENARD's description with the formation of the stalk in *Hedriocystis*.

About the formation of the shell I can make only a few remarks; nor is there much known about it in *Clathrulina*. As in this latter form the shell is always formed later than the stalk; HERTWIG and LESSER even saw a specimen dividing before the shell was formed, but I never met with this phenomenon.

At the moment the outline of the shell first becomes visible as an extraordinary thin stria, we should expect it to show itself immediately on the surface of the protoplast; this however is not the case, but it is seen from the beginning at the normal distance from this surface. This means that it is separated from it by a space amounting in diameter to that of the protoplast itself. The shell therefore seems to be formed independently of the body, so to say as a coagulum in the empty space. Up till now I cannot give an explanation of this curious appearance.

5. *The function of the zoöspores in Heliozoa.*

As we saw in the foregoing, *Hedriocystis pellucida* exclusively multiplies by fission — at least as far as we know up till now — and this division is followed by the formation of a zoöspore by one of the daughter-individuals. This mode of development likewise occurs in *Mikrogromia socialis* R. HERTW. This animal also splits up into two halves, of which one leaves the shell of the mother-individual, but in this case it may either assume the form of a zoöspore provided with two flagella, or that of an *Actinophrys*-like organism. The remaining history of these stages however is totally unknown.

In *Hedriocystis* the same process repeats itself periodically with one and the same animal, which thereby assumes the character of a mother-organism, producing a whole generation of daughter-individuals by serial division.

¹⁾ This organism was found by me on former occasions already, in a few specimens in the „Wisselsche Veen“.

The division causes the multiplication of individuals, the formation of the zoöspore favours the distribution of these individuals in space.

Both processes: binary division and formation of zoöspores, also occur in *Clathrulina*, though as far as we know the connection between them is less regular.

The alliance between this and *Hedriocystis*, which originally was only inferred from the structure of the shell and the protoplast, therefore finds a new support in development.

So we may state that the group of *Heliozoa desmothoraca* contains at least two forms, *Clathrulina elegans* and *Hedriocystis pellucida*, which in their structure and development are fairly well-known. The rest of the species brought under that heading, remain rather uncertain.

In the group of *Heliozoa*, apart from *Clathrulina elegans*, zoöspores had already in former days been observed in the case of *Acanthocystis*-species. This mode of reproduction likewise frequently occurs in other groups of Rhizopods, especially with Radiolarians. PASCHER, in his well-known treatise on the connections between Flagellates and Rhizopods, combats, especially on botanical grounds, the opinion of DOFLEIN, that in all departments of Rhizopods the forms with flagellate phases are the most primitive.

Now the systematic place of the *Heliozoa* as a whole is still rather uncertain, and the same is the case with the different smaller units which compose it, with regard to their mutual relationships; yet we may assume with a fair amount of probability that the *Desmothoraca*, to which *Clathrulina* and *Hedriocystis* belong, with their peculiar perforated and pedunculated shells, represent a less original condition than the naked *Aphrothoraca*, such as *Actinosphaerium*, and *Actinophrys*. The occurrence of zoöspores in the more highly differentiated types and their absence in the lower constitutes, as regards *Heliozoa*, an argument in favour of PASCHER's views.

PASCHER, who, in the above-cited study, proposes the thesis, that Flagellates are more primitive organisms, Rhizopods on the contrary more secondarily modified ones, affords a great phyletic value to zoöspores, in the sense of the theory of recapitulation.

Zoöspores should be temporary recollections of more primitive stages of development, through which the organism passed in former periods. Even in relatively high-differentiated forms they always still occur, be it as carriers of the sexual function, or as emigrants for the distribution of the species.

Even if this supposition should prove right, it affords no explanation of the fact, that the zoöspores, as reminiscences of more primitive flagellate organisms, may occur in higher specialised forms, while they have disappeared in less specialised, more original types. Their occurrence with sedentary organisms, as *Clathrulina* and *Hedriocystis*, where nothing is known about a possible sexual function for them, might however, as

PASCHER points out, be brought into connection with a chance of distribution in space, beyond the immediate neighbourhood of the mother-organism.

LITERATURE.

1. L. CIENKOWSKI, Ueber die *Clathrulina*, eine neue Actinophryen-Gattung (Arch. f. mikr. Anat. 3, 1867).
2. R. HERTWIG und E. LESSER, Ueber Rhizopoden und denselben nahestehenden Organismen (Arch. f. mikr. Anat. 10, 1874 Supplementheft).
3. R. HERTWIG, Ueber *Mikrogromia socialis* etc. (Arch. f. mikr. Anat. 10, 1874, Supplementheft).
4. G. ENTZ Sr., Ueber die Rhizopoden des Salzteiches von Szamosfalva (Termes. Füzetek 1, 1877).
5. J. LEIDY, Freshwater Rhizopoda of North America (U. S. Geol. Survey 1879).
6. S. M. FOULKE, Some Phenomena in the Life-history of *Clathrulina elegans* (Proc. Acad. Nat. Sc. Philadelphia, 1884).
7. F. SCHAUDINN, Heliozoa (in: Das Tierreich) 1896.
8. F. SCHAUDINN, Ueber das Centrankorn der Heliozoen etc. (Verh. Deutsch. Zool. Gesellsch. Bonn, 1896).
9. A. LANG, Lehrbuch der vergl. Anat. d. Wirbellosen Tiere. I. Protozoa. 1901.
10. E. PENARD, Les Héliozoaires d'eau douce. 1904.
11. F. DOFLEIN, Lehrbuch der Protozoenkunde. 2te Aufl. 1909.
12. A. PASCHER, Flagellaten und Rhizopoden etc. (Arch. f. Protistenkunde 38, 1917).

Deventer, December 1926.

Anthropology. — *On the Origin of human Races.* By Prof. L. BOLK.

(Communicated at the meeting of March 26, 1927).

In the anthropological literature many systematic divisions of mankind into races may be found. Now it strikes us that so little has been done concerning a systematic study of the origin of the human races. Leaving out of consideration the memoirs in which problems of general biology are treated, e.g. whether the origin of the races was a monogenetic or a polygenetic one, or whether they are not rather to be considered as subspecies, and suchlike questions, there is until a few years back scarcely any indication of a trial to explain the origin of races from a fundamental point of view. Of course, there are plenty of vague generalities, which can invariably be reduced to the little satisfactory standpoint occupied by DARWIN, when he summed up his opinion about the origin of species about as follows: "We may conclude that the variations giving origin to the human races, have been brought about either directly by exposure to various conditions of life, or indirectly by some form or other of selection". At the close of the chapter treating on the origin of races, in his "Descent of Man" he also admits the possibility that races may have evolved from spontaneous variations.

It is easy to understand that such generalities cannot furnish a basis for a methodical interpretation of even the most conspicuous somatic properties of the different races. DARWIN himself tries to do so with respect to the black skin of the negroes (to which I shall revert) but finally he declares that not one interpretation satisfies him.

In recent years, however, the problem of the origin of races has been studied on more sharply defined biological principles than those produced by DARWIN's theory. Starting from the consideration that the morphogenesis of the individual is regulated by the hormones of the endocrine system, now one is trying to retrace the somatic differences between the groups of individuals forming together one race, to a different agency of these substances. This reasoning is quite logical, and KEITH has been the first, who, in 1919, viewed the typical physical characters of the various races from that standpoint. After him several German authors have advocated the same opinion. The different hormones are supposed to be in a certain condition of equilibrium, a change in this equilibrium revealing itself in a more or less strong development of a somatic character. So every race would as PFUHL puts it, have its own type of hormonal equilibrium.

This idea, first unfolded by KEITH ¹⁾, as stated above, has already induced other investigators to study the degree of development of the endocrine glands in different races; and as yet some positive results have been obtained.

In their speculations on the origin of the human races I side with these authors the more readily as they are quite within the sphere of my own conceptions as to the origin of man. But, I'll go a step further, and not confine myself to the pronouncement that the racial qualities are indicative of a difference in, what I would term "the hormonistic index" of the individuals, but endeavour to show in what way certain racial characters result from these differences.

On the face of it this looks like a difficult task, yet it will appear that along this path it will be possible to comprehend the origin of racial qualities and consequently the origin of the races.

In my exposition I start from the hypothesis (in my opinion irrefutable), that the same cause from which arose the specific physical characters of mankind as a whole, also brought about the specific properties of the races, into which the genus differentiated. This, then, points to a common prime cause for the genus and for the races.

Some time ago I published a résumé of the principles of my theory of Anthropogenesis ²⁾. The leading thought of it was that man resulted from a retardation in the developmental tempo, and the course of life of the organism. Human life proceeds, to put it figuratively, like a retarded film. Now that retardation was, as regards the somatic development, not a regular one: certain physical characters were restrained more intensely in their evolution than the whole, of some the development was even completely arrested, i.e. suppressed. The consequence of this was that the form became characterized by the possession of properties of a fetal nature. This is what I termed the fetalisation-theory of our morphogenesis.

Now the study of racial features confirms the correctness of this theory, as it appears that just as the specific physical qualities of man are in general persisting fetal conditions, so also the specific racial features bear completely this character. In some sense, therefore, a race of men may be defined as a group of individuals in whom a definite somatic property or condition terminates its evolution at a certain stage, whereas in all other individuals its evolution is still proceeding till the very end. In other words: the specific racial features are fetalisation phenomena. The connection between this conclusion and the conception, just now alluded to, of the racial characteristics as manifestations of a different equilibrium of the hormones, is not far to seek. For, already in my discourse, just now referred to, I pointed out that the fetalisation

¹⁾ The differentiation of mankind into racial types. The Lancet 1919, Vol. 2.

²⁾ On the Problem of Anthropogenesis. Proceedings Kon. Akad. v. Wetensch., Vol. 29, 1926.

of the human body, as a whole, results from a checking agency emanating from the hormones. Racial characteristics, then, are merely manifestations of specific checks at work in the evolution of definite groups of men.

It is out of place here to fully discuss this principle, I must confine myself to unfolding the biological principle of the origin of races, and am now going to illustrate this by three of the most striking racial properties, viz. the specific characteristic of the Mongolian race, the colour of the skin, and the shape of the skull.

The so-called mongol-fold or Epicanthus is no doubt the most characterising feature of the Mongolian race: a skin-fold that, starting from the upper eye-lid, runs downward, and bridges over the inner angle of the palpebral fissure. This, however, is but one characteristic of the Mongolian physiognomy. When comparing the structure of the upper half of the face of a Mongol with that of a European, it will appear that the real differences extend farther, and that e.g. the face of a Japanese is typified by what I will designate by the comprehensive name of "Mongol-complex". This complex presents three anatomical peculiarities, viz. the above-named fold, the flattened nose-bridge, and the protrusion of the eye-balls. First let us dwell on the last two.

As ADACHI's investigations have borne out, the topography of the eye-ball of the adult Japanese, relative to the orbit, differs from that of the European. The deep-set eye, often met with among Europeans, does not occur with the Mongolian race. On the contrary, the eye-ball of the Mongol lies protruded to such a degree that nearly the half of it lies before the plane of entrance of the orbit, whereas with the European the eye-ball lies entirely within the orbit or at most for one third out of it. This fact, in conjunction with the flattened nosebridge, enables the Japanese and the Chinese to let a stick rest on both their eye-balls while it passes right across their nosebridge. This anatomical peculiarity is as typical of the Mongol as the occurrence of the epicanthus. Now, if we examine on this head the fetus of the European, it appears that at one stage of his evolution a transitory condition occurs like that which is characteristic of the adult Mongol. In the development of the European we also meet with a stage with flattened nosebridge and prominent eye-balls. But while with the Mongols this stage has become the terminal condition, it is for the European a transitional phase; for during further development the eyeball sinks deeper into the orbit, and the upper part of the nosebridge becomes prominent. Thus we see that with the Mongol these two components of the "mongol-complex" are persisting fetal conditions.

The same holds good for the third component of the complex, the epicanthus, or the mongolian fold. In order to demonstrate this, we have to make a short exposition of the development of the system of folds round the human palpebral fissure.

The different phases of this development after my personal investigations

are represented by figs.1 to 7. When the eyelids have been formed and cover the considerably prominent eyeball, there exists a marginal groove compassing the eyeball medially and upwards, while the medial angle of the

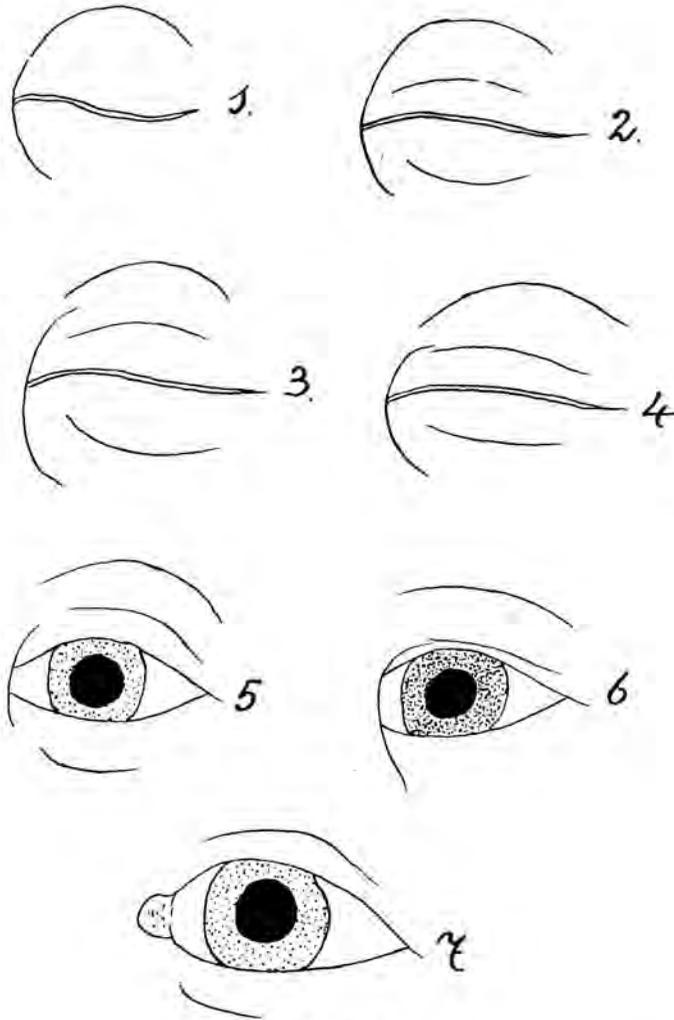


Fig. 1—7.

palpebral fissure extends to this groove (Fig. 1). After this a skinfold is evolved on the upper and the lower palpebra, which, however, does not reach the groove (Fig. 2). Subsequently the marginal fold splits up into two fragments, a superior and a medial one. Henceforth the latter forms a skinfold by itself, in the middle of which the medial end of the palpebral fissure terminates (Fig. 3). This medial fold slightly overlaps the nasal end of this fissure, so that in this phase the fetus has a true epicanthus (Fig. 4). Now, in case the eyes were to open in this stage of development, we should see a condition as illustrated in Fig. 5, in which the

medial canthus is overarched by the medial fold or epicanthus. It is commonly known that in neonati of the European this fold occurs occasionally, when the normal process of development, in which the epicanthus disappears, has been retarded. In normal cases this disappearance takes place in the European fetus during intrauterine life already, and the condition of fig. 7 is brought about. In the Mongolian race, however, the fetal epicanthus persists, and unites with the secondary skinfold on the upper eyelid (fig. 6). It is evident, therefore, that epicanthus and mongolian fold are not completely identical: the mongolian fold has originated from a combination of the fetal epicanthus with the secondary fold of the upper palpebra.

This brief description of the development of the skinfolds round the palpebral fissure goes to show that the epicanthus of the Mongols is a persisting fetal property.

The embryological investigation of the development of the system of folds round the palpebral fissure indicates that an epicanthus normally occurs during intrauterine life of Man. However, while as a rule this stage is transitory and the epicanthus disappears, the fetal condition persists in the Mongolian race. The "mongol-complex", as I termed it, which differentiates the Mongolian race so sharply from the other races, appears, indeed, to be a veritable fetalisation phenomenon. So with respect to this feature an interrelation exists between the races, corresponding to the relation of the general human characteristics to that of the other primates. Conditions, which in Primates are generally transitional, embryological stages, have become terminal stages in Man. It is just the same with the epicanthus: a condition, which in general is a transitional stage in the development of Man, has become a terminal stage with a definite race.

The fact that in every man the system of folds passes through an epicanthus-stage, elucidates two phenomena that I wish to accentuate here. Firstly the occurrence, already mentioned, of an epicanthus in neonati of the white race. In most cases this fold gradually disappears in the first years after birth. It is evident that the occurrence of such a fold is not at all due to some hereditary influence of a hypothetical Mongolian ancestor, as was formerly supposed; it is merely a pathological retardation of a normal evolutionary process. Occasionally this retardation gets so intense, that the fold persists also in Europeans, in this case the retardation has acquired a pathological character. This is one of the instances pointing to the fact that stabilization of a fetal condition being physiological for one race, may be pathological for another.

The second instance I wish to call attention to, and which becomes quite comprehensible to us now, is the frequent occurrence of an epicanthus among tribes of the negro-population of Africa. Until recently this was known only for the Hottentots, about 70% of them having a more

or less pronounced epicanthus. This led to the supposition of a hereditary relationship between Mongols and Hottentots, to which, however, objections have been raised. Another interpretation was brought forward, viz. that while racial characteristics may originate as spontaneous variations, the occurrence of an epicanthus in Mongols and Hottentots was evidence that such variations may arise independently among various groups of men. Subsequent reports stated that in Africa the epicanthus is not restricted to the Hottentots. It is especially among the negroes of Nigeria that it seems to occur frequently.¹⁾ And the first reporter, convinced that an epicanthus was typical of the Mongolian race, expressed his surprise at discovering Mongolian influence among the population of Central Africa.

Now the fetalisation-theory throws quite a different light upon the occurrence of the epicanthus among the inhabitants of Africa. The persistence of a fetal condition in human ontogenesis, characteristic of the population of Eastern Asia, also takes place frequently among certain tribes of the negro-group, as it does in individuals of the white races.

This can the more easily be understood, as in this case, just as with any other somatic condition, we have not to do with an acquired property. All human individuals have had an epicanthus; with most of them it disappears, with a minority it persists and now becomes a distinctive property of races. The problem why this stabilization occurs in definite groups, we are unable to solve. The retardation, and the fetalisation as phenomena and as evolutionary factors, are easy to demonstrate, but the selection exercised by the retardation we do not understand: it remains a puzzle.

For (and now I am going to discuss the second racial character) why has the pigmentation of the skin, the eyes, and the hair been retarded, even suppressed in a group of men? Am I justified in asking this question? Yes, for the ancestors of Man were black-haired, and black-skinned, the fair type issued from the dark. I must adduce my argument for this assertion.

With regard to the colour of the skin DARWIN takes a somewhat peculiar standpoint. He does not ask: what caused the difference in the colour of the skin, the eye and the hair, but sets himself the question: how is it that negroes have become black? and considers several possibilities, none of which, however, satisfied him.

I am of a different opinion; our ancestor was black-haired and black-skinned, just as the Gorilla and the Chimpanzee. My reason for this assertion is obvious. The genus *Homo*, as a whole, is characterized by black hair and brown to black eyes. Only the *Homo nordicus* possesses fair hair and blue eyes, a property acquired only in a later phase of Anthropogenesis. If, then, black hair and black eyes are characteristic of

¹⁾ SELIGMAN, A Pseudo-Mongolian Type in Negro-Africa. *Man*. Dec. 1924.

Man, we may conclude that also the skin was black, before it lost its hair. With some races this colour has persisted: Negroes, Papuans, Australians. The process of depigmentation was as follows: first the pigment in the skin was lessened: Mongolians, Malays; and afterwards the development was suppressed to such a degree that races with a fair skin were generated, but still in possession of black hair and dark eyes. Their representatives are the mediterranean, and the alpine races. In one race the depigmentation became complete, viz. the nordic race. In this race also the hair and the eyes lost their pigment, the eye became blue, the hair fair.

Now, if we consider the question: What is the nature of this process of depigmentation, causing a racial differentiation of Mankind to such a degree that it became the basis of the first classification, that of BLUMENTHAL? Evidently here also we have to do with different stages of a fetalisation-phenomenon. The faculty of producing pigment, so strong at the beginning that the neonatus was born with the intensely black colour of skin, hair, and eyes of his parents, was triggered in its development, and became less intense. This signified already an approach to the fetal condition, in which the skin is fair, the eyes blue, and the hair light. At first this retardation was limited to the pigmentation of the skin. The negroes show the first indication of it, the neonatus of this race being often of a considerably lighter colour than its parents, and acquiring its definitive colour some time after birth, at most after 6 months, according to records in the literature.

Now, the more the development of pigment in the skin is suppressed, the more the individual will approach the fetal condition: the colourless skin. In the Nordic race this condition persists most completely. In this race, furthermore, the depigmentation is progressive, iris and hair lose their dark colour as a manifestation of a fetalisation-phenomenon. This process of depigmentation is of just the same nature as that in the skin: the production of pigment is suppressed and retarded. Owing to the first circumstance the eyes and the hair become lighter, and as to the latter, we still observe its action in the well-known fact that with fair-haired children as a rule the hair gets darker after birth, a process varying individually considerably. This darkening of the hair and the eyes after birth seldom stays away in children of the fair race, which phenomenon is identical with the postnatal darkening of the skin in negro-children.

So, as to the colour of the skin, the eyes and the hair, the *Homo nordicus* appears to be more fetalized than any other race.

As the third racial character presenting itself as fetalisation-phenomenon, the form of the skull may be discussed. In this paper, in which only principal points can be discussed I shall confine myself for the sake of brevity to the two principal shapes of the skull; the brachycephalic and the dolichocephalic. There are brachycephalic and dolichocephalic races.

By some authors the cause of this difference is considered to be a telluric one, in such a sense namely that mountainous regions favour the evolution of a brachycephalic skull. However, the instances, at variance with this presumed correlation between environment and shape of the skull, are so many that this hypothesis may be repudiated as being faulty.

It was supposed by DARWIN that the increase of stature without a corresponding augmentation of the size of the brain, has provided some races with a dolichocephalic skull. I quite agree with DARWIN in supposing a correlation between increase of stature and dolichocephaly, but to my mind the additional factor, viz. the absence of a coinciding increase of the cerebrum, is superfluous.

However, the preliminary point we have to consider, is again: the shape of the skull of primitive man. This problem is rather difficult, because morphologically the term "primitive man" does not convey a distinct conception. There are no morphological criteria to refer a definite form still to the anthropoids, and to call another form *Homo*. This may be illustrated by the *Pithecanthropus*, which nobody will class as a primitive man. Yet, if of this form the calvarium were not known, and only the teeth had come into our possession, we should not hesitate in diagnosing them as the teeth of a primitive man.

When considering, however, that the skull of the *Pithecanthropus*, and likewise all skulls of the *Homo neanderthalensis*, known to us, and also the Talgay-skull, are dolichocephalic, while furthermore a brachycephalic skull appears only in a later period, it is within the bounds of probability that primitive man was dolichocephalic. Until recently one might have argued against this supposition that anthropoids are brachycephalic. This objection, however, has become invalid, since the existence of a dolichocephalic race of Gorilla was demonstrated¹⁾. The rule, however, that Anthropoids are brachycephalic, suggests the supposition, that the ancestor of Man also possessed a round skull, which during anthropogenesis was metamorphosed into a dolichocephalic one. When tracing the cause of this morphological change, it seems to me that a fair point of issue may be found in DARWIN's idea (just alluded to) that augmentation of body-length may be correlated with the development of dolichocephaly. Although no evidence can be adduced, it may still be assumed on the basis of several considerations, that anthropogenesis was attended with an increase of stature. This increase concerned particularly the trunk and the legs. The connection of this phenomenon with the development of dolichocephaly I imagine to be the following: The increase of the length of the trunk results from a prolonged persistence of the epiphyseal disks of the vertebrae, which is to be considered as a symptom of the

¹⁾ L. BOLK: "On the Existence of a Dolichocephalic Race of Gorilla. Proc. Kon. Akad. v. Wet., Amsterdam, Vol. 28, 1925.

H. A. HARRIS: Endocranial Forms of Gorilla skulls. American Journal of Physical Anthropology, Vol. 9, 1926.

retardation of the development in general. This prolonged function of the epiphyseal disks caused a vertical enlarging of the vertebrae, and brought about an increase in length of the trunk. But also the epiphyseal disk in the base of the skull persisted and functionated longer, so that the base of the skull was elongated. Add to this the rather considerable enlargement of the cerebellum by which the Os occipitale got protracted in sagittal direction, then we have two possible factors that render the transformation of brachycephaly into dolichocephaly, and the dolichocephalic skull of primitive man, comprehensible.

Now, how did in a later period the brachycephalic skull issue from this form? It would seem to me, that here we have to do again with a fetalisation phenomenon. It should be remembered that every man is brachycephalic during a rather long phase of his fetal life. According to the investigations by SCHULTZ (*Fetal Growth of Man and other Primates. Quaternal Review of Biology, Vol. I*) no dolichocephalic skulls are met with until the 6th month of the fetal growth, and only during the sixth or the seventh month the index cephalicus may be lowered so much that the skull has become dolichocephalic. Dolichocephaly, therefore, is a secondary form taking its origin during fetal growth. In brachycephalic individuals this transformation of the skull is suppressed, and the primary fetal form is retained. If we correlate this fact with brachycephaly as racial feature, we come to the conclusion that this characteristic is also due to the persistence of the cranium (now as a hereditary phenomenon) in the form that is typical of the first period of human fetal life. It follows, then, that brachycephaly is also a phenomenon of fetalisation.

It has thus been demonstrated, that there is one and the same cause for the origin of racial features and the genesis of the actual form of Man, as a whole.

Anatomy. — *A comparison of the form of the callosum and septum in Chinese, Philippino- and Dutch brains.* By Dr. MA (WEN CHAO), Peking. (Communicated by C. U. ARIËNS KAPPERS).

(Communicated at the meeting of February 26, 1927).

The observations in this paper are limited to certain features presented by the corpus callosum and the septum pellucidum in 65 human brains. Of these, 30 were Dutch¹⁾, 20 Chinese (Peking) and 15 Philippino (Manilla). All the brains were in good condition. They were cut in the median sagittal plane and tracings of the callosa and fornices were made on thin paper put on the callosum with a sheath of glass between.

The shape of the median sagittal sections of the corpora callosa and septa is seen in Fig. 1, 2 and 3.

I shall first deal with the form of the corpus callosum, discussing subsequently the three parts, in which it is generally divided: the knee (genu), the body (truncus) and the tail (splenium).

The *knee* comprises approximately the anterior fifth of the whole length. Frontally it curves downward and backward in front of the septum, and ends in the posteriorly tapering rostrum, which is connected below to the anterior commissure. The knee of the callosum is generally thicker than the body.

No constant difference was noted in the knee of the three groups, except that on the average in the Chinese it is a little rounder than in the European. In older brains its frontal pole may be more pointed, at least in my Dutch (in Fig. 2: ♀ 72, both ♀ 63, ♀ 60 with ♀ 17, ♂ 17 ♀ 25, both ♂ 34). More material would however be required to make sure whether this is an average fact.

The intermediate part of the callosum, the *truncus*, is slightly curved with the convexity upward and the grade of this curve differs in the various races, being more pronounced in my Chinese (Fig. 1) than in my Dutch brains (Fig. 2), as a glance at the figures shows.

This difference may be expressed in the height index of the callosum, divided by KAPPERS²⁾, being the greatest distance from the dorsum of the callosum to the *basal callosum line* (drawn in the figures) divided by the greatest distance between the most anterior and posterior point of

¹⁾ These brains, sent by the Central Institute for Brainresearch, Amsterdam, were from patients, who died (from non-nervous diseases) in the Wilhelmina gasthuis, Amsterdam.

²⁾ ARIËNS KAPPERS. Indices for the anthropology of the brain, applied to Chinese, dolicho- and brachycephalic Dutch, foetus and neonati. These Proceedings Vol. 29, 1926.

the callosum. The index thus calculated for each brain is written in my figures (sex and age are mentioned in the left and right hand corner of each case).

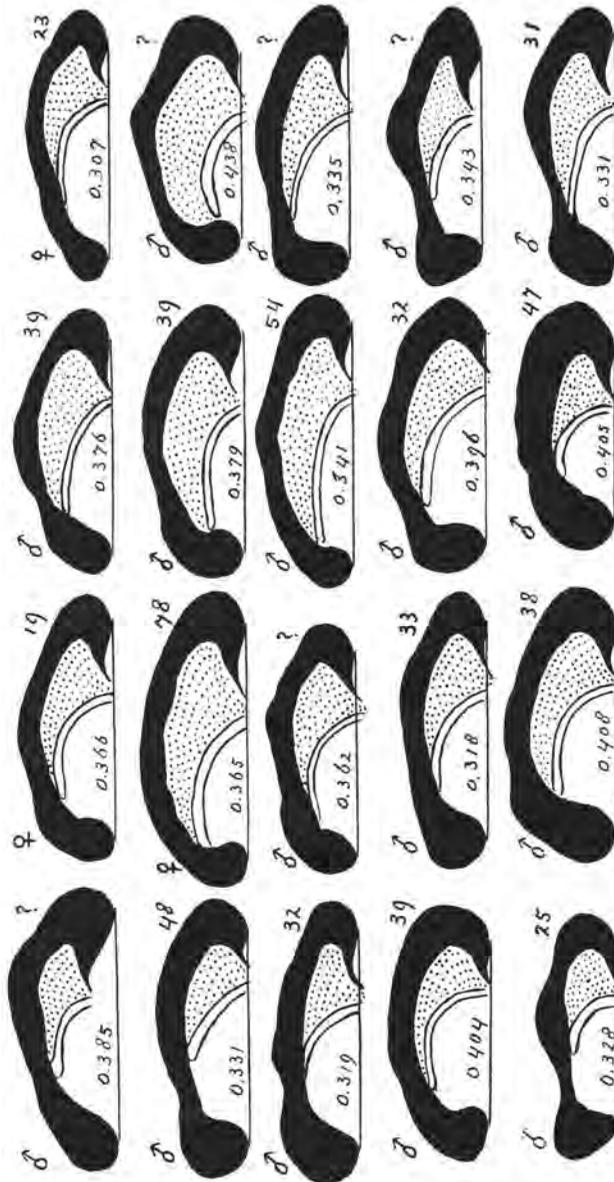


Fig. 1. Twenty corpora callosa of adult Chinese. — Average callosum height index 0.361.

From this results, that the callosum-height-index of my Dutch brains varies between 0.252 and 0.356, with an average of 0.300.

The callosum height-index in my Chinese varies between 0.307 and 0.438, with an average of 0.361.

While these figures are both a little less than those found by KAPPERS

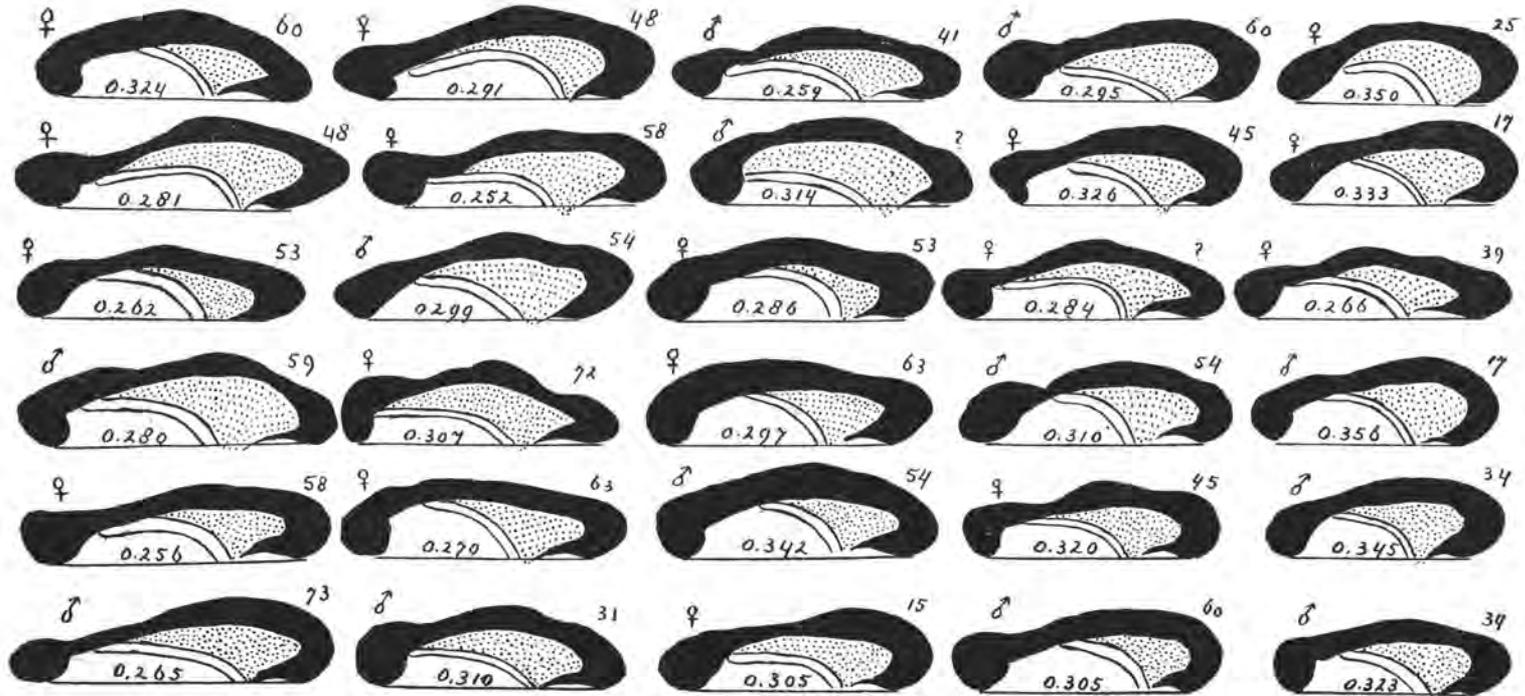


Fig. 2. Thirty corpora callosa of adult Dutch. Average callosum height index 0.300.

in his Chinese and dolichocephalic Dutch brains¹⁾, the difference is about the same in both series, KAPPERS finding 0.62, I 0.61 in favor of the Chinese.

I may add that the callosum curve is still more striking in the Chinese foetus and in early child life than in the adult.

The average callosum height index in my Philippino's was 0.340, being much nearer to that of my Chinese than to my Dutch brains.

This should not astonish us after KAPPERS' researches concerning the relationship between the hypsiccephalic and brachycephalic character of the skull and the form of the callosum (l.c.), as KOEZE²⁾ found the various races inhabiting the Philippino islands to be all hypsiccephalic (with the exception only of the Igorrotes). Besides several of them, the Baluga's and Negrito's, are brachycephalic, whereas the others are meso- or subbrachycephalic (c.f. also SULLIVAN³⁾ and BARROW⁴⁾).

Also the brain described by FALLOT⁵⁾ was brachycephalic (83.3). Besides, according to DENIKER⁶⁾, they are considerably mixed with Chinese.

The *diameter* of the trunk also shows a certain degree of variation in the different races. Generally speaking, I have the impression that the body of the callosum is comparatively thinner in my Dutch brains than in my Chinese.

This may be due to a fronto-caudal compression which also appears in its steeper curve in the Chinese (the distance from the frontpole of the callosum to its hindpole showing an average of 7.47 cm in my Dutch and of 7.23 cm in my Chinese).

The trunk is rarely of even thickness throughout its length, generally its middle-part is thicker than the transition to the genu and to the splenium. At its transition to the splenium there may be even a more or less distinct groove, the *impressio corporis callosi*⁷⁾.

This groove is more often observed among the white people of my series than among either the Philippino or the Chinese. In fetal and early child life it fails or is only slightly indicated.

The *splenium* also shows considerable variation among the races studied. In my Dutch brains the splenium is either round or prolonged backward along the direction of the trunk.

¹⁾ Also from Amsterdam post mortems.

²⁾ KOEZE. *Crania ethnica philippinica*, Veröffentlichungen des Niederländischen Reichsmuseums für Völkerkunde, Serie II N^o. 3, 1901—1904.

³⁾ SULLIVAN'S researches seem to confirm KOEZE'S findings, at least as far as concerns the length-breadth index (*Racial Types in the Philippine islands. Anthropological papers of the Amer. Museum of Nat. History, Vol. 23, Part 1, 1918*) and so do BARROW'S.

⁴⁾ The negrito and allied types in the Philippines. *Amer. Anthropologist* Vol. 12, 1910, p. 358.

⁵⁾ FALLOT. *Le cerveau d'un malais né à Manille. Bull. de la Société d'Anthropologie de Paris* 1880, p. 441.

⁶⁾ DENIKER. *Les Races et les peuples de la terre*. Paris, 1910, p. 558.

⁷⁾ ARIËNS KAPPERS. *Vergleichende Anatomie des Nervensystems der Wirbeltiere und des Menschen*, Vol. II, p. 1088, Fig. 572.

A pronounced downward curve of the splenium is rare among the



Fig. 3. Fifteen corpora callosa of adult Philippinos. Average callosum height index 0.340.

Dutch and also among my Philippino's, (only in one case — second row N^o. 3 —, it curves downward, but now forward).

In half of my Chinese, however, the splenium extends rather abruptly downwards, after which it may even turn forward.

It thus seems that the greater height-index of the callosum and the strong downward and eventual forward curve of the splenium may be considered an average racial distinction of the Chinese and that the first feature is also indicated — though in a less pronounced degree — in the Philippino's.

The *septum pellucidum*, extending between the inferior surface of the callosum and the superior surface of the fornix, consists of two layers, enclosing the so-called *ventriculus septi pellucidi*, a blind arachnoidal space, usually almost obliterated, but which at times may be rather extensive in size.

I found twice an enlargement of this space among the European and three times among Chinese.

In the case of one of the enlarged ventricles in the Chinese the 2 layers were separated from each other by about 2.5 cm and the anterior and posterior ends of the space are displaced, respectively forward and backward, apparently due to a hydropic condition.

The *septum pellucidum* may be divided for sake of description, into three parts, the anterior, the inferior and the posterior parts.

The *anterior portion* of the septum, situated immediately behind the posterior surface of the genu of the corpus callosum, is the largest. Its extension — specially its height — varies considerably in my Dutch brains, being larger in those where the genu is less pointed.

In my Philipinos this part of the septum shows no significant difference from the Dutch. In my Chinese, however, the anterior portion is generally higher (in both old and young brains) than in the Europeans or Philipinos, as may be expected from the greater rounding of the callosum.

The *inferior portion* of the septum situated between the posterior surface of the rostrum and the columnae anteriores fornicis is angular in shape, if the fornix and consequently the septum are not extended downward pathologically.

In both the Philippino and Chinese brains, the downward pointed angle of the inferior portion is more conspicuous than in the Dutch, i.e. the lower angle of the septum, which may be measured between a tangential on lowest parts of the columna anterior fornicis and the rostrum, on the average is sharper in the Chinese and Philipinos than in my Dutch brains.

It seems probable that this difference is correlated with the greater average curve and sagittal shortness of the Chinese and Philippino callosum.

The *posterior portion* of the septum — lying between the inferior surface of the posterior part of the callosum and the corresponding posterior part of the fornix — is generally very narrow and appears like a sharply pointed tail of the anterior portion, if the septum is not abnormally extended as in one case, that of the Dutch with a callosum-index of 0.314 (σ , ?) where the posterior portion has half the size of the anterior, and is prolonged as far as the free end of the splenium. Similar cases are my Chinese with a callosum index of 0.365 (φ , 78), 0.379 (σ , 39), 0.438 (σ , ?) and 0.341 (σ , 54).

It is most probable that this is due to a slight hydrocephalus of the lateral ventricles which, as we know, not only extends the nervous ventricular wall, but also pushes the fornix downwards, thus increasing the extension of the septum, sometimes eroding it entirely¹⁾.

In the Philipinos the posterior part of the septum shows practically no difference with the Dutch, being very small.

In most adult Chinese brains the posterior angle of the septum is generally a little wider, less acute than in either in the Dutch or Philipinos.

It may be that also this less sharp septum angle of the Chinese is due to the greater average curve of their callosum, by which the height of the septum is extended dorsally, also in its caudal division.

Finally I would remark that the course of the fornix — as far as can be judged from this material — is generally steeper in the Chinese and in the Philippino than in the Dutch brains here examined, a fact that may be also explained by the more hypsiccephalic character of the Chinese and Philippino races.

¹⁾ So that both lateral ventricles freely communicate over the fornices.

From the above observations I conclude:

1. Among the Chinese the corpus callosum has a greater curve than in the Dutch. The average callosum index of the Philipinos is nearer to that of the Chinese than of the Dutch.

2. The impressio corporis callosi, occurs more frequently in my Dutch than in my Philippino or Chinese brains.

3. A downward bending of the splenium is most frequent in the Chinese.

4. The inferior angle of the septum (between the rostrum and the lower part of the columnae anteriores fornicis), frequently is less obtuse in the Chinese than in the Dutch, whereas the posterior angle of the septum is generally wider in the Chinese.

The fornix may have a steeper course in the Chinese and in the Philippino, compared to the Dutch.

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Geology. — *On the Occurrence of the Danauf ormation in Martapura (S. E. Borneo).* By L. H. KROL. (Communicated by Prof. L. RUTTEN).

(Communicated at the meeting of October 30, 1926).

Dr. J. K. VAN GELDER kindly sent me an off-print of the Proceedings of the "Koninklijke Akademie van Wetenschappen" at Amsterdam (Section for Physics), containing a communication by Prof. L. RUTTEN on the subject expressed in the title of this paper. As I do not get off with a whole skin there, I may be allowed to put in a rejoinder.

First of all I must premise that I am now isolated in the "rimbu" of W. Borneo and secluded from all the literature and maps, so that I feel somewhat handicapped in my defence. Still, I will try to call to mind the material that has to back me up.

When residing in the Southern- and Eastern Division I often asked myself how HOOZE came to speak of Alino- and Waringin *clayrock*. Whenever I crossed this formation, I was confronted especially with siliceous rocks with very fine-grained weathering products.

I grant that some hand-specimens he broke off are indeed *clayrocks*, although the qualification "sticking to the tongue" cannot serve as the only criterion for the determination. Now it appears that of the 19 rocks determined more precisely by Prof. RUTTEN, no fewer than 14 are called *siliceous slate*, *silicified tuff* or *siliceous*.

By using the term *clayrock* HOOZE has caused confusion, and has led his successors somewhat astray.

For the present demonstration I start from the position of the Intrusiva of Martapura in the geological profile. At the time I came to the conviction that these rocks (namely in the Meratus mountains) are intracenanomanian, because they have metamorphosed, especially along the south, the cenomanian limes with *Orbitulina concava* into marble, where-with also magnetite originated (see KROL Jb. Mijnwezen 1918. Proceedings p. 330 and Verslagen en mededeelingen betreffende Indische Mineralen en hunne Toepassingen N^o. 9 — IJzerertsen van Pleihari).

It might be argued that only in one locality, viz. near the Batu Kora, both the contact phenomenon, and the presence of *Orbitulina* have been recognized, which would go against my conception by assuming, e.g. that the intrusiva of the Meratus mountains are older than Cenomanian and that this intrusiva rock of the Batu Kora is the only one that is intracenanomanian.

But if we mark all the localities of these iron-ores on the map from Talok-Takisung on sea in the south-west, as far as the Sei Pontain in

the north east, it will be seen that all these localities are situated on three parallel lines, which tectonic lines run again parallel to those of the Meratus-mountains. The limes have disappeared almost everywhere, and have fallen a victim to erosion. Only a few relics are extant of the vast and stupendous lime-rocks of olden time.

This, in my opinion, is sufficient proof of contact-metamorphous activity of the intrusiva during the Cenomanian over their whole length, so I wonder that Prof. RUTTEN has taken so little notice of my account, and calls my conclusions singular.

In those places where the lime has been destroyed by erosion, the iron-ore (assuming a vertical descent) lies on the rock on which the lime has been built up. I assume a vertical descent, not a rolling displacement because nearly all the iron-ores are situated on the tops of the hills (e.g. G. Tembaga near Pleihari). On that account these rocks are older than the Orbitulina-limes, but since they are also metamorphous, I take them to be the lowest layer of the Cenomanian.

Macroscopically and microscopically these rocks are quite similar to the Tambam-quartzites, already found by VERBEEK, for which reason they are referred by me to the Cenomanian. But when looked at they also resemble the rocks of G. Waringen.

Most often they are grey with pink tints and black veins, they have a very irregular cleavage, which reminds us of pressure phenomena, but the usual microscopic inspection did not reveal radiolaria ¹⁾.

Since I did not consider it probable that the mighty intrusiva should have generated contact-phenomena only on the south side, I supposed them to have occurred also on the north side. In the Meratus-mountains the crystalline rocks received consideration in this respect (see KROL p. 330 1st paragraph); in the Bobaris-mountains in the centre the crystalline slates; in the north the Alino- and Waringin-layers.

That these rocks are metamorphous, is also evident from the description by HOOZE and RETGERS.

Besides it is also apparent from the picture of the landscape, I admit, however, that also older rocks may have been metamorphosed by the intracenanomanian intrusiva. (See l. c. p. 331).

Now Prof. RUTTEN has been confronted with radiolaria in the Waringin-Alino-rocks, which he parallels with the Danauf ormation of MOLENGRAAFF in W. Borneo. Whether this is quite admissible over such a distance, I will leave undecided here; no doubt a closer palaeontological examination — such as HINDE's examination of MOLENGRAAFF's radiolaria — could certainly have yielded more positive evidence.

I take it, however, that similar stupendous movements of the soil have

¹⁾ During my geol. survey of the Paloh-district I found on Gg. Rasau (Serawak boundary) quite similar rocks, which on microscopical examination by Dr. GISOLF appeared to contain radiolaria and are on that account referred by me to the Danauf ormation.

not taken place within comparatively narrow limits, so that I shall have to revise my opinion about that part of Borneo.

Then, however, the consequence of RUTTEN's find is that this Danauf ormation will appear to occur over a very large tract in the part of Borneo mapped by HOOZE and myself, so that the major part of what I marked as Cenomanian has to be considered as Trias-Jura. (See KROL p. 354 bottom).

To sum up we come to the following conclusion:

1^o. The intrusiva of the Bobaris and Meratus-mountains are to be considered provisionally as intracenomanian, until other data shall be known.

2^o. These intrusiva have metamorphosed the sediments of the Cenomanian as well as those of the Danauf ormation.

In conclusion Prof. RUTTEN's find opens up singular perspectives relating to the origin and the locality of the Borneo-diamonds. The data, however, have not yet been sufficiently studied for further communications.

Ngabang-Landak, July 1926.

Physics. — *On the diffraction of Röntgen-rays in fused sodium and potassium.* By Prof. W. H. KEESOM. (Supplement No. 61a to the Communications from the Physical Laboratory at Leiden).

(Communicated at the meeting of December 18, 1926).

§ 1. *Introduction.* The communication of PRINS ¹⁾ about the diffraction of Röntgen-rays in mercury acted as an inducement to me to publish the results of a similar investigation about sodium and potassium in the fused state. This piece of research was carried out by me in 1923 in the Laboratory of Physics and Physical Chemistry of the Veterinary College as a part of an investigation on the diffraction of Röntgen-rays in solutions of liquid-mixtures ²⁾. The latter however has not been continued because of my departure to Leiden ³⁾.

§ 2. *Method.* The fused metal to be radiated, was brought into a thinwalled glass tube, in vacuo, in order to prevent as far as possible the inevitable formation of oxide. This tube was put into the middle of a DEBIJE—SCHERRER-camera, as described in Comm. N^o. 10 from the Laboratory of Physics and Physical Chemistry of the Veterinary College ⁴⁾. The lower part of the tube was immersed in a small beaker-glass filled with paraffine, which was kept, by means of an electrically heated resistance wire, at a temperature of about 25 degrees above the melting point of the metals (controlled with the aid of a thermoelement, brought into the paraffine). Separately we had made sure that then the whole mass of the metal present was molten. The metal was radiated (Cu-K-rays) some few millimeters above the part immersed in the paraffine. For K we worked with a HADDING tube for 23 hours with 22 mA; for Na 7½ hours with 17 mA.

§ 3. *Results.* The results obtained are given in table I. Here φ

TABLE I.

Sub-stance	Diameter diffractionring	φ	$\varphi_{corr.}$	a	δ	$1.33 \sqrt[3]{\frac{M}{\delta}}$
Na	27 mm.	28.6°	27.1°	4.04 Å	0.93 ⁵⁾	3.88 Å
K	21.5 ..	22.8	21.2	5.13	0.83 ⁵⁾	4.80

¹⁾ J. A. PRINS. *Physica* 6, 315, 1926.

²⁾ Some data about this are communicated in § 5.

³⁾ This investigation was already mentioned in *Comm. Leiden, Suppl. N^o. 53b, 1924, p. 17, note 2.*

⁴⁾ W. H. KEESOM and J. DE SMEDT, *These Proc.* 25, 118.

⁵⁾ VICENTINI and OMODEI 1888.

represents the half top angle of the cone formed by the diffracted Röntgen-rays, as this is found from the half diameter of the diffraction ring on the film and the radius of the cylinder inside which the film has been bent (27 mm). In $\varphi_{corr.}$ the correction for the thickness of the cylinder of molten metal (the tube had a diameter of 1.1 to 1.2 mm) has been applied after BIJL and KOLKMEIJER ¹⁾. a is the distance of the diffracting particles, calculated after the formula of EHRENFEST ²⁾: $a = 7.72 \lambda / 4 \pi \sin \frac{\varphi}{2}$; δ is the density of the liquid. The last column gives the distance of two neighbouring atoms, assuming that these are packed as closely as possible in the volume occupied by the liquid ³⁾.

§ 4. In the first place it appears from the approximating agreement of columns 5 and 7 of table I, that, just as in the cases of the liquids mentioned in the Communications N^o. 10 and 12 from the Laboratory of Physics and Physical Chemistry of the Veterinary College ⁴⁾, also with Na and K the distance of each two diffracting particles agrees approximately with the distance of two neighbouring particles, arranged in closest packing; the distance of two diffracting particles being derived from the diameter of the diffraction rings with the formula of EHRENFEST.

As in my calculations for Na and K the atomic weight is taken for M , it follows that these elements, just as A and Hg, are monatomic liquids.

It is interesting to note that the two numbers of column 7 are smaller than the corresponding ones of column 5, and although the difference is hardly beyond the limits of experimental error, it is perhaps significant that the same is found in the case of argon when one applies the correction of BIJL and KOLKMEIJER to the observations of Comm. N^o. 10 i.c. ⁵⁾ as is shown in the following table.

TABLE II.

Substance	Film N ^o .	Diameter diffractionring	φ	Diameter preparation	$\varphi_{corr.}$	a	$1.33 \sqrt[3]{\frac{M}{\delta}}$
A	37	26.75	28.4°	3.0 mm.	23.7°	4.61 Å	4.08
..	53	25.5	27.0	2.0 ..	24.0	4.55	4.08

¹⁾ A. J. BIJL and N. H. KOLKMEIJER, Comm. N^o. 2a from the Laboratory of Physics and Physical Chemistry of the Veterinary College; These Proc. 21, 496, note 1; A. J. BIJL, Thesis for the Doctorate, Utrecht 1918.

²⁾ P. EHRENFEST, These Proc. 17, 1184.

³⁾ Compare W. H. KEESOM and J. DE SMEDT, Comm. N^o. 10, i.c.

⁴⁾ These Proc. 25, 118 and 26, 112.

⁵⁾ For the other substances investigated there, which show a much smaller absorption-power for the Röntgen-rays, this correction will be much less.

That difference can be partly ascribed to the influence (comparatively larger for argon than for sodium and potassium) of the dimensions of the atom (comp. § 5 of the mentioned Comm. N^o. 10) but probably not completely.

§ 5. Though I will not connect conclusions with this, I may utilize the opportunity to communicate the results of another two Röntgenograms, namely those of a solution of NaCl in water, and of a mixture of water and aethylalcohol. The following table gives the half top angle of the cone formed by the diffracted Röntgen-rays for Cu-K-rays, with the liquids mentioned. To it are added the relative values for water and aethylalcohol as previously obtained (Comm. N^o. 10 l.c.)

water	$\varphi = 29^\circ$, further blackening to $\varphi = 46^\circ$	
alcohol	22° , " " " $\pm 43.5^\circ$ (limits	
30 gr. NaCl + 100 gr. H ₂ O	31°	unclear)
18 gr. H ₂ O + 46 gr. C ₂ H ₆ O	23° , second ring 42° .	

The ring of NaCl was (because of insufficient exposure) weak, but still clear; the film of the water-alcohol-mixture showed an intensive ring (extending itself from $\varphi = 20^\circ$ to $\varphi = 26^\circ$). Within and outside the ring there was a weak blackening, the last ending into a weak second ring at $\varphi = 42^\circ$.

Palaeontology. — *On a Dryasflora-bearing deposit in the Netherlands.*

By F. FLORSCHÜTZ. (Communicated by Prof. L. RUTTEN).

(Communicated at the meeting of January 29, 1927).

In 1891 the Swedish palaeobotanist A. G. NATHORST summarized his experiences about the occurrence of fossil remains of arctic plants within the area of the former North-European icesheet as follows:

"Bevor ich Norddeutschland verlasse, muss ausdrücklich betont werden, dass die arktischen Pflanzenreste in einer ganz bestimmten glacialen Süßwasserformation hier vorkommen. Diese Formation findet sich über die ganze Moränenlandschaft Norddeutschlands und Westrusslands verbreitet (ganz wie in Schonen und Dänemark) und man kann folglich dort die Zahl der Fundstätten nach Belieben vermehren" ¹⁾.

The said "Formation" forms part of a complex of layers always resting upon the moraine and consisting in descending sequence of peat, "gytja", peatlime and freshwater-clay and -sand of which either the "gytja" or the peatlime or both may be wanting. According to this author only *Betula nana* L. sometimes occurs in the peat; for the rest the remains of arctic plants are confined to the "gytja", the peatlime, the clay and the sand, especially to the last-mentioned two sediments ²⁾.

Directed by this information I went in search of fossil arctic plants in the Netherlands. First of all my attention was drawn to the spots where peatlime occurs at a short distance from boulderclay, as it is possible that the moraine extends under the lime.

Two places, one near Markelo in the province of Overijssel and the other south-west of Winterswijk, were chosen for further investigation.

An inundation in the last months of 1926 provisionally eliminated the marshes of Markelo. I therefore first confined myself to the bog of Blekkink in the vicinity of Winterswijk.

VAN BAREN records lime under peat on that spot ³⁾.

On the 28th of December 1926 a profile was dug near a brook intersecting the field. From the top downwards the following order was to be observed: surface soil 10 cm.; sandy loam 20 cm.; peat 30 cm.; peatlime 35 cm.; clay 30 cm.; grey sand 50 cm.

The floor of the sand was not reached at that time neither when

¹⁾ Ueber den gegenwärtigen Standpunkt unserer Kenntnis von dem Vorkommen fossiler Glacialpflanzen. Bihang till Kongl. Svenska Vetenskaps-Akademiens Handlingar. Band 17 Afd. III. N^o. 5. p. 20.

²⁾ Loco citato p. 5, 6 and 7.

³⁾ De bodem van Nederland, p. 716.

further efforts were made especially owing to the high level of the groundwater and the insufficient boring-apparatus. The presumption that under the sand boulderclay is to be found, as in places quite near it rises to the surface, could therefore not be confirmed at that time.

On examining the collected material a topfragment of a leaf of *Dryas octopetala* L. was found in the layer forming the transition from peatlime to clay.

A further investigation of this sediment consisting partly of lime partly of clay yielded a great many endocarps of *Potamogeton*, some seeds of *Myriophyllum spicatum* L. and of *Betula* (probably not of *Betula nana* L.), many epidermisfragments of *Phragmites*, multitudinous *Chara*-oogonia, Diatomeae, insects-rests, shells of *Ostracoda* and freshwatermolluscs.

Considering the following facts I may presume here a findingplace of the so-called *Dryasflora*.

NATHORST records as parts of this former vegetation besides *Dryas octopetala* L. especially the waterplants *Myriophyllum* and *Potamogeton* ¹⁾. Moreover he reports the occurrence of the following typical fauna accompanying the arctic plantrests: some *Pisidia*, *Anodonta* or *Unio*, *Limnaea ovata* and *Cyclas cornea* ²⁾.

VAN BAREN gives an enumeration of the shells found in the peatlime of Blekkink-bog, including some *Pisidia*, *Limnaea ovata* and *Sphaerium corneum* (syn. *Cyclas cornea*) according to the determination of M. SCHEPMAN ³⁾.

The review of the organic rests found in the "Dryashorizont" of North-Germany, given by PAUL RANGE, contains *Betulae* (also other ones besides *Betula nana* L.), *Phragmites* and *Charae*, and among the zoological remains *Ostracoda* ⁴⁾.

Besides the nature and order of the layers of the profile correspond with the general scheme composed by NATHORST.

The site of Blekkinkbog is about 40 m. above sealevel, 51° 55' N. and 6° 40' E. of Greenwich.

I am greatly indebted to Dr. A. TH. TEN HOUTEN and Docts. G. VAN EEKELN of Winterswijk and to Mr. W. KRUISELBRINK, agriculturist on Blekkinkhof, for their kind assistance during the exploration of the field.

Velp (G.), Holland, January 10th 1927.

¹⁾ Loco citato p. 12.

²⁾ " " p. 7.

³⁾ " " p. 716.

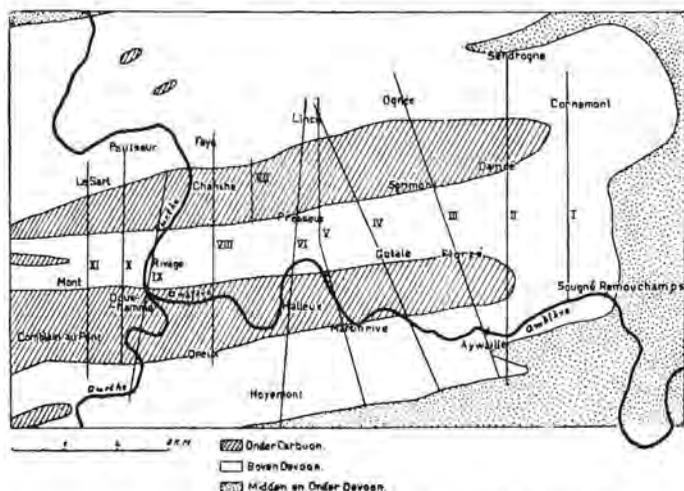
⁴⁾ Das Diluvialgebiet von Lübeck und seine Dryastone. Zeitschrift f. Naturwiss. Bd. 76. 1903. p. 232—241.

Geology. — *Profiles through the Synclines of Comblain au Pont and Chanxhe.* By P. ROGGEVEEN. (Communicated by Prof. L. RUTTEN).

(Communicated at the meeting of January 29, 1927).

Our knowledge of the structure of the region of the synclines of Comblain au Pont and Chanxhe, constituting the most eastern part of the Carbonian basin of Dinant, is still limited. An exception is afforded by the Ourthe-profile, which has already been described and mapped by LOHEST and FORIR¹⁾. We deemed it well worth consideration to give a description of these synclines, as it would add to our knowledge of a portion of the Belgian Ardennes, situated at a rather short distance to the south of the so-called "Faille Eifeliennne". According to FOURMARIER and to most of the other Belgian geologists this is to be considered together with the "Faille du Midi", as the northern limit of an important overthrust-sheet, so that the region under consideration will probably belong to it.

The body of facts collected last summer in the well-exposed territory, enabled us to reconstruct a number of profiles through the synclines. The accompanying map gives a survey of the territory we examined.



Onder Carboon = Lower Carboniferous
 Boven Devoon = Upper Devonian
 Midden en Onder Devoon = Middle- and Lower Devonian

The location of the profiles is indicated by the lines I—IX. The

¹⁾ Annales de la Société Géologique de Belgique, tome 22, 1894/95.

stratigraphical division in the profiles is the same as that of the Belgian geological map 1 : 40,000, though the stratigraphy that has been adopted in recent times, slightly departs from it. This, however, matters little for the description of the tectonic.

In our survey we adopted the subdivision of the Belgian geological map for the more detailed observations, as regards the Upper-Famennien; as regards the lower-Carbonian we used the subdivision given by LOHEST¹⁾.

The trend of the profiles is on the whole N.—S., which is normal to the general strike. The profiles have been orientated by giving the names of the places and the azimuth. The situation of the profiles I, II and VII, however, could not be indicated out with sufficient accuracy in this way. A few items concerning them may follow: At *A* in profile I point is a marked 750 m. west of Amblève-bridge at Sougné-Remouchamps. Point *B* in profile II is 375 m. east of Amblève-bridge at Aywaille. Point *C* in profile VII lies 1180 m. east of Ourthe-bridge at Chanxhe.

Profile IX, reconstructed from personal observations, corresponds with the Ourthe-profile, as given by LOHEST and FORIR.

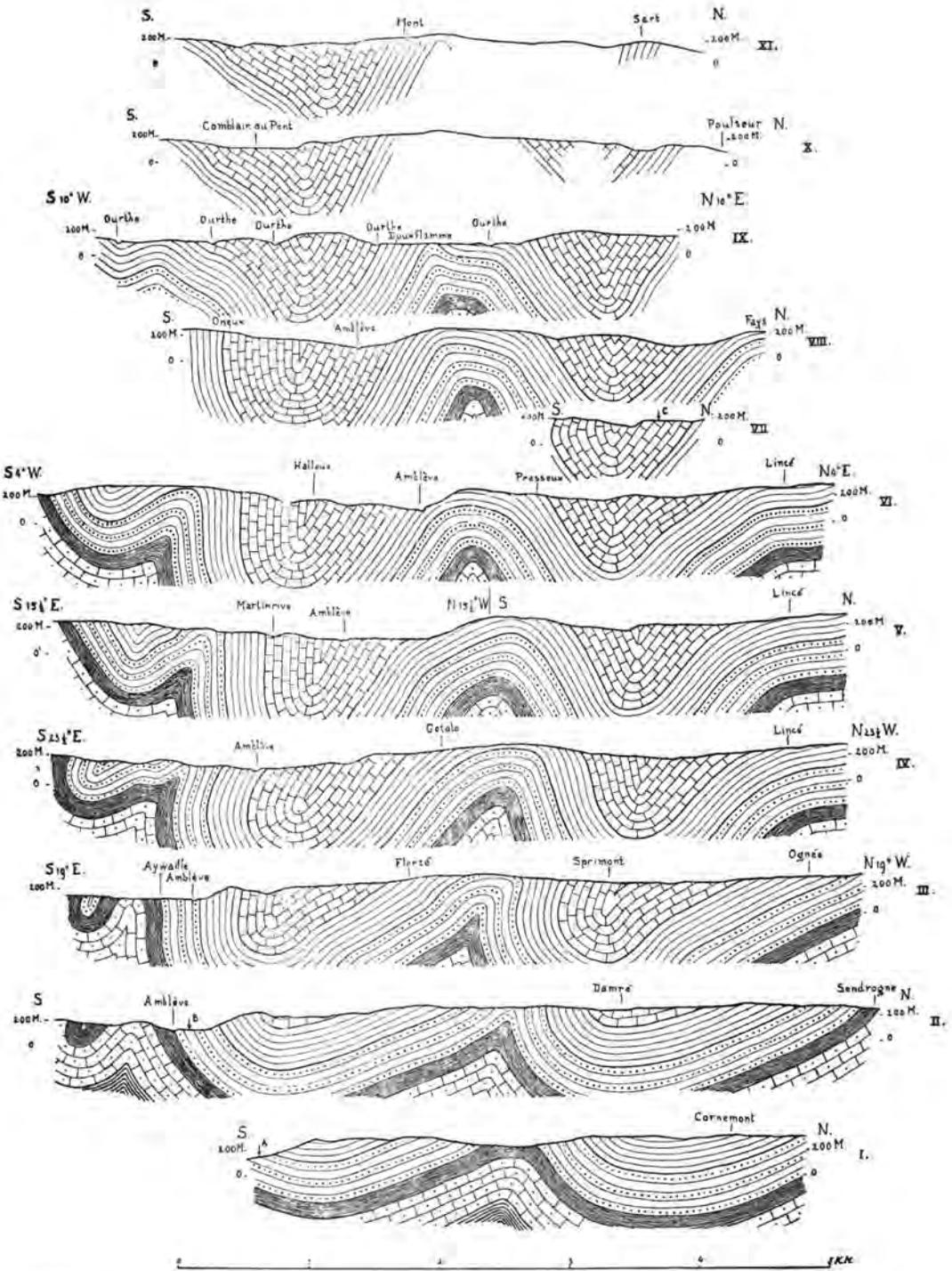
Localities, where complications appeared, which could not be explained, because of the too small number of observations, occurred: first, to the south of profile I, so that the reconstruction of this profile was discontinued; secondly, in profile X and XI in the syncline of Chanxhe and the adjacent anticline to the South.

Departures from the Belgian geological map were rarely met with. One instance is that in profile I — in the anticlinal nucleus south of Cornemont — Frasnien still crops out over a distance of certainly not less than 300 m., whereas the map indicates only Lower-Famennien. Probably the map is not correct either with respect to the localities, where the above-named complications occurred.

The eastern profiles represent the commencement of the two synclines. When the *Syncline of Comblain* becomes deeper, it soon presents a steep Southern-limb, which is in Profile III slightly overturned Northward. In Profile IV this overturning is more pronounced (a.o. 60° S. in the Southern-limb). Farther to the West the Southern-limb remains very steep, while the Northern-limb everywhere shows smaller dips than the Southern-limb. In the Ourthe-profile (IX) it is the other way about: there the Northern-limb is steeper than the Southern-limb, which is also the case in profiles X and XI.

In the east the *Syncline of Chanxhe* shows also in the Southern-limb larger dips than in the Northern-wing. Here such an asymmetry reveals itself, as is seen in the eastern portion of the syncline of Comblain au Pont: in profile II the Southern-limb is inappreciably overturned; in

¹⁾ LOHEST: Les facies du Dinantien, 2me partie: Régions orientales de la Belgique. Livret Guide du Congrès Géologique International, 1922, Excursion C3.



Onder Carboon = Lower Carboniferous Boven Famennien = Upper Famennian
 Onder Famennien = Lower Famennian Frasniën = Frasnian Givetien = Givetian Couvinien = Couvinian

profile III this overturning is more pronounced (dips of e.g. 70° S.). To the West of it the syncline yields more symmetrical sections, except in profile VII, where the South-limb dips locally to the South.

In the eastern profiles the anticline, lying between the two synclines, has larger dips in the Northern-wing than in the Southern. In profile VI it yields an approximately symmetrical section. More to the West the top gets flatter and at the Ourthe it sags. This proceeds probably more Westward into a small *Carboniferous* syncline.

The profiles II, III, IV, V, VI, and IX, which have been extended farther south than the others, represent sections of a syncline beginning in the east. There the Southern-limb dips southward (profile II, III, and IV).

To the west of this the syncline soon gets more normal, broader, and deeper.

The anticline between this syncline and that of Comblain au Port lays bare the Givetien in profile II and III. Then, however, it soon sinks rather rapidly and has subsided so deep in profile IX, that it forms only a feeble severance between the two synclines.

In fine it appeared from the profiles that the Frasnien and the Lower Famennien in the Southwest is considerably thinner than in the East and the North-east.

Nowhere, except in localities where the above-stated complications occurred, did we find any indication of faulting in the tectonic. We were invariably encountered with a normal sequence of subdivisions.

It strikes us that this territory which probably belongs to the overthrust-sheet, presents no faults of any significance, either longitudinal or transversal ones.

Physics. — *On Some Phenomena which are Caused by the Low Indices of Refraction of Alkali Boron Fluorides.* By J. H. DE BOER.
(Communicated by Dr. G. HOLST).

(Communicated at the meeting of February 26, 1927).

In the preparation of simple or complex fluorides it may often occur that already a considerable quantity of salt has crystallised or has been formed as a precipitate, the latter not being visible. The explanation is simply that the indices of refraction of most fluorides are so low that they are in the neighbourhood of their saturated solutions. The boundaries of separate crystals are consequently difficult to distinguish: powdery precipitates become transparent.

These phenomena were particularly marked in the preparation of the alkali boron fluorides. When the sparingly soluble potassium salt is precipitated by double transformation of boron fluor hydrogenic acid, HBF_4 , with a solution of potassium chloride, it is possible that the precipitate is hardly visible. It is stated in the literature that the salt is deposited as a gelatin, which does not pass into a crystalline form until later. The precipitate is, however, only apparently a gelatin; this is perceived when the salt is sucked off. The filtration proceeds exceedingly rapidly; as long as the mass is still wet, it is quite transparent; as soon however as the upmost layer of the salt is sucked dry, the filter is suddenly seen to be quite filled with a white crystalline substance.

Now the indices of refraction of the alkali boron fluorides are very low indeed, which was clearly demonstrated in the preparation of the cesium salt, CsBF_4 . A precipitate of the cesium salt, obtained by double transformation of a solution of HBF_4 (from fluor hydrogenic acid and boric acid) with cesiumchloride, was not only perfectly transparent, but displayed besides brilliant colour effects. In a definite solution, in which cesium salt was precipitated, green light was e.g. perfectly transmitted, the other colours being still more or less dispersed in the precipitate. Objects, viewed through the precipitate, were therefore coloured green, being surrounded by a red-violet border. The index of refraction of the cesium salt¹⁾ is, therefore, so low that in the green it is equal to that of the above solution containing hydrochloric acid, the colour effects being caused by the difference in dispersion between salt and solution.

¹⁾ The alkali boron fluorides are orthorhombic, hence they have, strictly speaking, three indices of refraction. This, however, is disregarded, and by index of refraction of the salt only a mean value is meant.

We find here a phenomenon analogous to that in potassium silico fluoride, K_2SiF_6 ¹⁾, of which BERZELIUS already states that during the precipitation rainbow colours are visible, and that the salt is deposited as a gelatin through which colours play.

The dispersion curve of the cesium boron fluoride has a less steep course than that of the solution, which is clear from the fact that by careful dilution of the solution or by gentle heating the colour of the transmitted light changed from green to blue, the dispersed colours changing from red-violet to brownish yellow.

The index of refraction of the above solution at 20° C. for the D-line appeared to be: 1.3502. Hence the "index of refraction" of $CsBF_4$ for the D-line lies somewhat higher than this value.

In another solution, in which the initial substances were brought together in more concentrated form, orange light was transmitted, the scattered light being greenish blue. The index of refraction of $CsBF_4$ for the D-line is, therefore, slightly *smaller* than the index of refraction for the D-line of this latter solution, which appeared to be 1.3563. Hence we know now:

$$1.3563 > n_D \text{ for } CsBF_4 > 1.3502.$$

This low value for the index of refraction of $CsBF_4$ might have been predicted on the ground of FAJANS and JOOS' considerations ²⁾. By the aid of definite suppositions as to the deformation of the electron layers in ions these investigators have succeeded in constructing a table of refractions for the ions, the latter being supposed as gaseous and undeformed. Thus they give as value for the triply charged boron ion 0.05, for the singly negatively charged fluor ion 2.5. Now in $(BF_4)'$ the boron ion will give the same value, the deformation under the influence of the fluor being only small. The fluor ions themselves are, however, strongly onesidedly deformed by the boron ion; their value of refraction is greatly lowered. We can, however, approximately evaluate how great this diminution is. While the free chlorine ion has a refraction 9.0, this value is diminished to 6.97 in BCl_3 , i.e. to 77 % of the original value. Likewise the free bromine ion has a value of 12.67, in BBr_3 9.97, hence only 79 %. Therefore we might say that in the neutral BF_3 the fluor had decreased to about 75 % of its value, and amounts therefore to 1.9. In the $(BF_4)'$ the deformation of the fluor ion will again be somewhat smaller, so that the value there will be > 1.9 . Hence we get a refraction > 7.65 for the $(BF_4)'$.

The free Cs' has a refraction 6.24. Contrary to many other salts we may assume that in $CsBF_4$ the ions influence each other only to a small extent. For the $(BF_4)'$ is little deformable (like ClO_4' , an example which FAJANS and JOOS cite), and Cs' is little deforming; also the reverse, i.e.

¹⁾ This is fully described by R. W. WOOD, Physical Optics 1921, p. 112

²⁾ K. FAJANS and G. JOOS, Zeitschr. f. Phys. 23, 1, 1924.

a deformation of Cs' by $(BF_4)'$ can practically be excluded. We therefore arrive at the conclusion that the refraction of $CsBF_4$ has a value $> 7.65 + 6.24$, hence > 13.89 .

As therefore

$$\frac{n^2 - 1}{n^2 + 2} \times \frac{M}{d} > 13.89$$

it is only necessary to know the density d to arrive at a valuation of n . The density proved to be 3.2, hence we find for the index of refraction:

$$n_D > 1.33.$$

On the ground of such considerations it would be expected that the indices of refraction of rubidium and potassium boron fluoride are still lower. We shall see that this is actually the case.

While the appearance of the colour effects in the precipitation of the cesium salt is more or less accidental (because the dispersion curve intersects that of the mother liquid which contains hydrochloric acid) the matter is simpler in the rubidium boron fluoride.

During the precipitation no colour effects are observed, evidently because the index of refraction lies lower than that of the solution. In the recrystallisation of $RbBF_4$ from pure water brilliant colours make their appearance. A solution of $RbBF_4$ saturated at room temperature (0.6 gr. in 100 cc.) appears to have the same index of refraction as pure water (1.3332). If solid $RbBF_4$ is brought into this solution, it again looks gelatinous; it appears that the yellow light is transmitted and that the dispersed colours cause blue borders. The index of refraction of $RbBF_4$ for the D-line appears, therefore, to be very near 1.33 indeed.

In $CsBF_4$ the play of colours could be varied at pleasure on dilution with water or on addition of HCl (on dilution of a very concentrated HCl-solution, which contained $CsBF_4$ and transmitted *red* light, we could pass through the whole spectrum, the complementary colour always appearing as border colour round the objects); this did, of course, not succeed in the case of rubidium salt. It is still possible to vary the colour, namely by heating the system; the indices of refraction of the solution diminish more rapidly than those of the salt, so that at higher temperatures the points of intersection of the curves shift more towards the violet.

It may be seen from the following table that in this salt really pretty well the whole spectrum can be traversed in this way.

The index of refraction of potassium boron fluoride lies still lower than that of the rubidium salt, hence also lower than that of water. A solution of the potassium salt saturated at room temperature (only 0.6 gr. in 100 cc.) had again practically the same index of refraction as that of water. The solid salt brought in this solution displays hardly any colours. Only a blue dispersion is very faintly perceptible; also the separation

between salt and solution is much clearer. Evidently the dispersion curve of the potassium salt does not intersect that of water, but it lies under

RbBF₄ in water

Temperature	Colours of the transmitted light	Colours of the dispersed light
0°	orange	green-blue
10°	orange yellow	blue
20°	yellow	blue
30°	green yellow	blue violet
40°	yellow green	violet with some red
50°	yellow green	faint red violet
55°	green	red violet
60°	green with some blue	violet red
70°	blue	brown orange
80°	indigo blue	orange yellow
90°	violet	yellow with some orange
100°	violet	yellow

the water-curve. On heating, the curve for the solution descends more rapidly than that for the salt; they intersect each other, and this causes again the pronounced colour effects to appear. As appears from the following table the colours are almost the same as those of the rubidium salt at about 30° to 40° lower temperature.

Temperature	Colours of the transmitted light	Colours of the dispersed light
40°	orange yellow	blue
50°	almost yellow	blue
60°	yellow	blue
70°	green yellow	blue violet
80°	yellow green	violet
90°	green	red violet
100°	blue	yellow brown

While the Cs-, Rb-, and K-salts of the boron fluor hydrogenic acid are sparingly soluble, the Na- and Li-salts dissolve readily in water¹⁾.

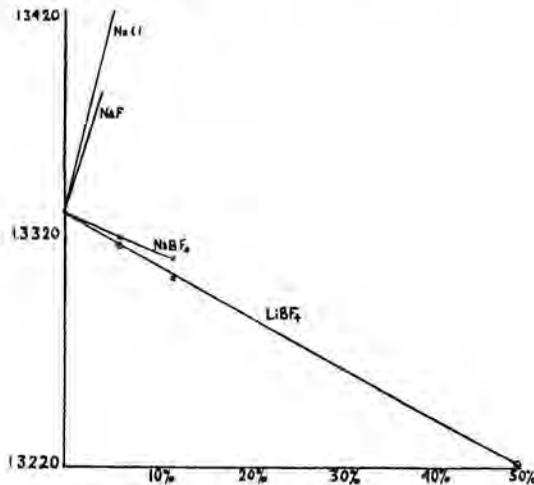
¹⁾ Cf. for the further properties: J. H. DE BOER and J. A. M. V. LIEMPT, *Rec. d. tr. chim. des Pays-Bas*, **46**, 124 (1927).

It was more or less to be expected that the indices of refraction of these salts should be still lower, and it might also be expected that the solutions of these salts had a lower index of refraction than that of water.

This appeared actually to be the case; thus the following values were found for the *D*-line at 20° C.:

	Water:	1.3332
NaBF ₄	6 ⁰ / ₀	solution: 1.3321
	12 ⁰ / ₀	.. : 1.3312
LiBF ₄	6 ⁰ / ₀	.. : 1.3318
	12 ⁰ / ₀	.. : 1.3303
	50 ⁰ / ₀	.. : 1.3221.

This last value is not so very certain, as the measurement had to be done very rapidly; there were indications that the glass was attacked. As is, however, clear from the adjoined figure, the points lie sufficiently accurately on a straight line.



By way of comparison also the increases of the index of refraction of water on dissolving NaCl or NaF have been drawn.

Also the boron fluor hydrogenic acid itself lowers the index of refraction of water. A solution containing about 20⁰/₀ of this acid gave a value of 1.3284.

Eindhoven, February 1927.

Natuurkundig Laboratorium
der N.V. Philips' Gloeilampenfabrieken.

Physics. — *The Structure of the Calcium Fluoride Band λ 6087.* By
S. GOUDSMIT. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of February 26, 1927).

The most accurate investigation of the structure of the calcium fluoride bands is that by DANJON¹⁾, who observed that these bands seem to show an entirely different structure from that of most of the other bands hitherto known. He could examine band λ 6087 most accurately, because this band is more easily resolvable than the other bands from this spectrum and is more simply constructed.

On cursory examination this band has the appearance of an ordinary group of bands, the band edges of which form a regular series. DANJON was, however, of the opinion that this could not be the case. For it appears that the bands do not overlap, as in an ordinary group of bands, but that every band seems to stop at the very place where the following band begins. Further DANJON particularly pointed out that the frequencies of the lines of the whole band can be represented by a single formula, with one parameter. According to DANJON we have to do here with one band which presents remarkable intensity anomalies. From the first band edge the intensity decreases regularly, till suddenly an intense line makes its appearance, after which the intensity again slowly diminishes. This is repeated a few times, the mutual distance of the lines at the same time becoming greater.

The purpose of our investigation was to examine whether DANJON'S view still appeared to be correct, if measurements were made with greater dispersion than was used by him.

The spectrum was photographed in the second and the third order of a concave grating with a radius of curvature of 6 meters and with 14438 rulings to the inch. The arrangement was that of EAGLE. In the region examined the dispersion in the second order amounted to about 12 Å per cm. and in the third order to about 8 Å per cm.

The greatest difficulty was to find a suitable source of light, and this has not been entirely overcome. The most satisfactory arrangement was an arc, the two carbons of which had a vertical position, the lower (positive) carbon having a perforation filled with calcium fluoride. Use was successfully made of a magnet in the well-known way to "blow" the arc in a definite direction, and to promote in this way the regular

¹⁾ A. DANJON, Diplôme d'Etudes sup. Paris (1914).

consumption of the calcium fluoride filling. The arc was operated with a direct current arc, which had to be continually regulated by the hand. Other sources gave the band to be investigated, which is one of the faintest of the spectrum, with too small intensity. The source used had, however, the very great drawback that the lines were fairly diffuse: it is to be regretted that our attempts to overcome this difficulty have not succeeded. The consequence of this was that the resolving power of the grating could not be entirely used; the limit for the resolution was defined by the diffuseness of the lines. In consequence of this the measurements in the third order have no advantages at all over those in the second, on the contrary, they required a longer time of exposure without giving better resolution.

With the times of exposure at which the band under investigation gave the best blackening, varying from 30 to 80 minutes, the neighbouring group of bands λ 6064 was already greatly over-exposed, and clearly showed reversals on most photographs.

Fig. 1 shows the intensities of the band as given by DANJON. This sketch is in conformity with his conception that the whole is one band with regular intensity anomalies. This conception finds expression in the figure: leaving the anomalies out of consideration, the intensity decreases regularly towards the tail of the band. This decrease is, however, not real; the appearance is caused by the fact that the lines towards the tail of the band lie further apart with diminishing wavelength, and that owing to this the blackening on the plate seems less. The real course of the intensity is represented by fig. 2. This is an intensity curve of one of the photographs in the second order of the grating made with a Moll-photometer ¹⁾. Roughly a curve would be obtained about as in fig. 3. The most striking feature is that in the first apparent band the intensity seems to decrease most rapidly, in the following bands this diminution proceeds more and more slowly and at the end all the lines have even about equal intensity so that it is not possible to define there the apparent edges with certainty, as in the beginning. When observing this it should, however, be taken into consideration that in the successive bands the lines lie farther and farther apart, and in consequence of this a seemingly slower decrease of the intensity is caused. As a matter of fact the number of lines in the bands should be taken into account in this comparison.

In view of what precedes it is improbable that DANJON's view should be correct. Nor does it seem that we have to do here with a number of bands each of which stops exactly where the next begins. Probably

¹⁾ The description of the arrangement for the application of the scale on the photogram, which was also used very successfully in this investigation, is found in KOK and ZEEMAN, *These Proc.* 27, 884, 1924.

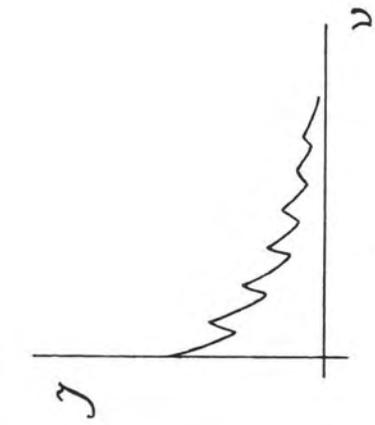


Fig. 1.

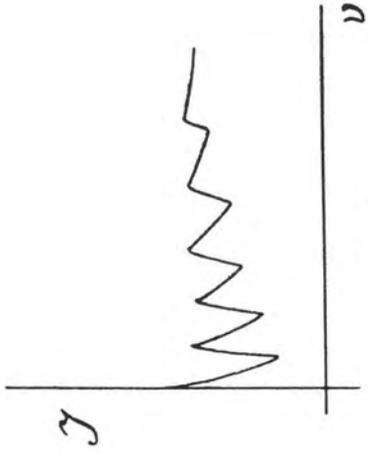


Fig. 3.

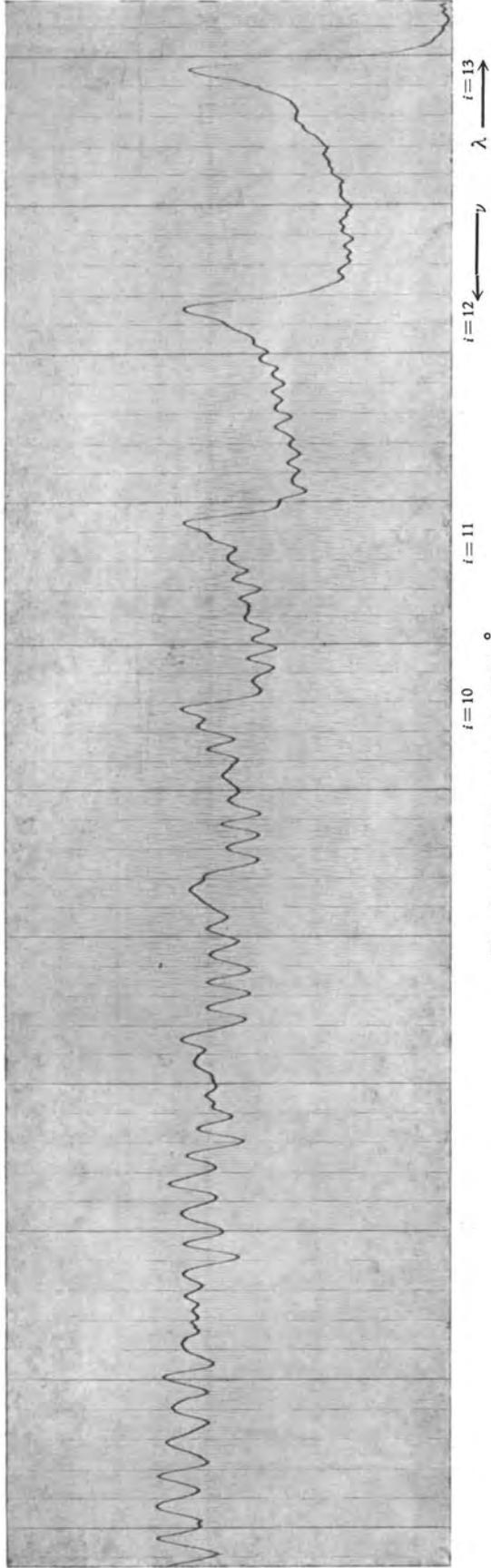


Fig. 2. 1 dividing line = 0.0244 Å

we have to do here with a group of bands of which the successive bands are superposed, but that in this case the lines from the different bands happen to coincide and consequently present the appearance of one continuous band. When this is assumed, it may be understood why the intensity in the successive bands diminishes more and more slowly. This is due to the superposition of lines from the different bands; in the first band the normal intensities are found, in the second the last lines of the first band coincide with those really belonging to the second band, changing their apparent intensities, in the third band lines from the second band produce the same effect, those of the first probably having no longer any influence, and so on. This superposition has not yet much influence for the second band, because the lines in question from the first band have a high number, greater than 33; in the third band, however, it is already very pronounced, because the overlapping extreme lines of the second band start already with the number 13 and are hence rather intense.

There is another reason indicating that probably we are not dealing here with one band. A close examination of the photographs and of the photometer curve reveals certain irregularities, which always become more and more pronounced towards the end of the band. It appears as if the coincidence of lines from the different bands does not take place so accurately as might be supposed at first sight. At different places the lines are diffuse, which seems to suggest "interference" of the lines from the different bands. Near the end of the group of bands this has such a great disturbing effect that it is no longer possible there to identify the lines with certainty.

Even if the conception of the structure of these bands given here, in contrast with that of DANJON, is correct, these bands do not present a normal structure, as will appear in the discussion of the band-formula to be given later. Inasmuch as this is the only known example of bands from this kind of a tri-atomic molecule, it is not yet possible to decide with certainty which interpretation is the correct one.

Before discussing the numerical data, we wish to point out that in consequence of the diffuseness of the lines their measurement could be made much better with the photometer than with the comparator. The edges, however, presented a difficulty. In consequence of there being a region of slight blackening on one side and a region of intense blackening on the other side, the maximum of the curve of intensity for the edges was always slightly displaced towards the latter region, especially at the beginning. The accuracy that could be reached, was no greater than about 0.015 Å.

According to DANJON all the lines of the group of bands can be represented by one formula, an ordinary zero-branch formula:

$$\nu = 16428,24 + 0,004497 n^2 + 0,0000010 n^3 \dots \dots (1)$$

In accordance with what precedes, our opinion is, however, that this fact is more or less accidental, and that in reality we have to do here with a number of different bands. Each band must then have a formula of its own. From the preceding formula we expect for these bands:

$$\nu = 16428,24 + 0,004497 (m + p)^2 \dots \dots \dots (2)$$

In this p is evidently the number that the first line from each band has in DANJON's formula. Hence p is equal to zero for the first band, and the ordinary zero-branch formula is obtained. The other bands seem to be zero-branch formulae of which the first $p-1$ lines are absent. They may, of course, also be taken as positive branches, but then an explanation is required why the negative branches are not present. The values of p are: 0, 34, 47, 56, 63, 69, 74.

Formula (2), however, still rests on the supposition that the lines from the different bands accurately coincide. It would not be easy to interpret the strange series of p -values. It seems more probable to us that for the derivation of the band formula another fact observed by FABRY ¹⁾ should be utilized. The band edges, the first lines of every band, themselves satisfy again a quadratic formula, namely: ²⁾

$$\nu_i = 16428,24 + 0,208,58 (13^2 - i^2) \dots \dots \dots (3)$$

Remarkable is here the appearance of the integer 13 in this formula. It might be held to be accidental, if the values 13.5, 14 and 14.5 did not occur in other groups of bands in this spectrum. If it is now assumed that formula (1) is valid approximately, and if this is combined with formula (3), we find for the bands:

$$\nu = 16428,24 + 0,004497 (m + 6,807 \sqrt{13^2 - i^2})^2 + \left. \begin{aligned} &+ 0,0000010 (m + 6,807 \sqrt{13^2 - i^2})^3 \end{aligned} \right\} \dots (4)$$

The coinciding of the different bands arises through the fact that the values p obtained as solutions of the equations

$$0,004497 p^2 + 0,0000010 p^3 = 0,20858 (13^2 - i^2),$$

come pretty near to whole numbers, especially in the beginning. As the deviation from a whole number must be pretty great to have a disturbing influence, this may very well be only accidental.

It should still be pointed out that the structure of the bands renders it probable that the values of m must be integers, also the numbering of the values i is integral in this band.

The table gives the wave-lengths in the first part of the band from new measurements in the region where the measurement is still pretty

¹⁾ Journ. de Phys. 4, 245, 1905.

²⁾ In this formula i passes through the values 13, 12 etc. to 0. If it is desired to have the numbering in inverse order, $(13-i)$ should be substituted for i .

n	m		n	m	
0	$i = 13$ 0	λ 6087.08 Å	43	$i = 12$ 9	λ 6083.98 Å
—	—		44	10	.83
—	—		45	11	.67
29	29	5.71	46	12	.52
30	30	.62	47	$i = 11$ 0	.37
31	31	.51	48	1	.21
32	32	.39	49	2	.05
33	33	.27	50	3	2.86
34	$i = 12$ 0	.16	51	4	.70
35	1	—	52	5	.52
36	2	—	53	6	.34
37	3	4.80	54	7	.16
38	4	.66	55	8	1.98
39	5	.55	56	$i = 10$ 0	.78
40	6	.41	57	1	.60
41	7	.27	58	2	.41
42	8	.12	59	3	.20

certain. The lines between $n = 0$ and $n = 29$ could not be resolved. In DANJON the resolution into lines did not begin before $n = 47$.

Let us finally consider the other bands investigated by DANJON. According to a communication by DANJON he has succeeded in resolving another group of bands, viz. λ 6051, also into lines. For the lines he gives the probable formula:

$$\nu = 16526,1 + 0,006233 n^2 + 0,0000083 n^3.$$

For the edges the formula is:

$$\nu_i = 16526,1 + 0,2030 (14^2 - i^2).$$

According to our conception the following formula would be written for this groups of bands:

$$\nu = 16526,1 + 0,006233 (m + 5,707 \sqrt{14^2 - i^2})^3 + 0,0000083 (m + 5,707 \sqrt{14^2 - i^2})^3.$$

For the other groups of bands only the edges have been measured. The formulae for these are:

$$\lambda 6064,61 : \quad \nu_i = 16489,1 + 0,1895 (15,5^2 - i^2)$$

$$\lambda 6037,17 : \quad \nu_i = 16564,1 + 0,2279 (13,5^2 - i^2).$$

If the formula (4) given here renders the correct conception of the bands, it would be probable that we have to do here with two moments of momentum, one with a quantum number m and one with the quantum number 13, of which the projection normal to m is equal to i , the projection along m is then $\sqrt{13^2 - i^2}$. Neither the theory nor the experiments are, however, sufficient to give, with certainty, a theoretical interpretation. Probably an electron moment of momentum also plays a part here, for, as is known, these bands, in contrast with most others, show a large ZEEMAN-effect.

I am greatly indebted to Prof. R. FORTRAT (Grenoble) for suggesting this investigation to me, and to Prof. P. ZEEMAN for his encouragement, advice, and for the necessary instruments which he placed at my disposal.

Amsterdam, Laboratory "Physica".

Physics. — *The vapour pressures of solid and liquid neon.* By C. A. CROMMELIN and R. O. GIBSON. Communication N^o. 185*b* from the Physical Laboratory at Leiden. (Communicated by Prof. W. H. KEESOM).

(Communicated at the meeting of February 26, 1927).

§ 1. *Introduction.* During the purification of some neon by a method similar to that already described by one of us (C. A. C.)¹⁾ it was observed that the vapour pressure of the solid neon at the temperature of boiling hydrogen ($\pm 20^{\circ}$ K.) was about 4 cm, whilst TRAVERS and JAQUEROD give ± 1.2 cm. As a matter of fact TRAVERS and JAQUEROD²⁾ give the following values

20 ^o .4 K.	12.8 mm Hg and
15 ^o .65 K.	2.4 mm Hg.

The latter were the results of preliminary measurements only and, as no others have been made, it seemed desirable to make an extended series of determinations of the vapour pressures of solid neon.

The vapour pressures of the liquid between the triple and critical points have been measured formerly in the Physical Laboratory at Leyden³⁾.

§ 2. *Details of the measurements.* The apparatus used was similar to that already described by CATH⁴⁾.

Two sets of apparatus were used, one filled with neon taken from a cylinder of the gas purified by the method mentioned above, and the other with neon that had been further fractionated by condensing some neon from the cylinder and allowing a middle fraction to evaporate directly into the previously washed and evacuated apparatus. The two sets were placed side by side in the cryostat and measured alternately. Only slight unsystematic differences, well within the experimental error, were found in the vapour pressures of the two samples of neon, indicating that they were of the same purity.

The measurements up to 20^o.4 K. were made in a liquid hydrogen cryostat, the temperatures being measured by means of one or two

¹⁾ C. A. CROMMELIN, *Rec. des Trav. chim. des Pays-Bas* (4) **42**, 814, 1923; *Comm. Leiden* N^o. 162c.

²⁾ M. W. TRAVERS and A. JAQUEROD, *Phil. Trans. A* **200**, 175, 1903.

³⁾ H. KAMERLINGH ONNES and C. A. CROMMELIN, *These Proceedings* **18**, 515; *Comm. Leiden* N^o. 147*d*; P. G. CATH and H. KAMERLINGH ONNES, *These Proceedings* **20**, 1160; *Comm. Leiden* N^o. 152*b*.

⁴⁾ P. G. CATH, *These Proceedings* **21**, 656; *Comm. Leiden* N^o. 152*d*.

platinum resistance thermometers. The pressures were measured by reading the heights of the mercury in the two arms of the manometer with a carefully levelled cathetometer and standard scale.

A hydrogen vapour cryostat was used for the measurements above the boiling point of hydrogen.¹⁾

§ 3. *Purity of the neon.* On several occasions vapour pressure measurements at the same temperature but with different volumes of the phases were made. The slight, if any, differences found were quite irregular and well inside the experimental error.

§ 4. *Experimental accuracy.* The accuracy of the temperature measurement was somewhat higher in the liquid cryostat than in the vapour cryostat (0.01 and 0.02 to 0.03 degree respectively), which appeared from the constancy of the temperatures and the concordance of the resistance thermometers. In the region of the higher temperatures the vapour pressures could only be determined to 1 mm, which becomes clear when one takes into consideration that e.g. at -248° C., 0.01 degree difference in temperature means 1.5 mm difference in pressure.

§ 5. *Results.* The results of the measurements in the solid and liquid regions, together with those previously published, are given in table I.

The temperatures are given in degrees KELVIN, while $-273^{\circ}.09$ C. is assumed for the absolute zero. The pressures are given in international cm mercury and in international atmospheres. At Leyden the international atmosphere is equivalent with 75.9529 local cm mercury.

§ 6. *Triple point.* From a large scale graph of the present results the following values of the triple point constants were found:

$$\theta = -248^{\circ}.52 \text{ C.}, \quad T = 24^{\circ}.57 \text{ K.}, \quad p = 32.35 \text{ int. cm.}$$

During one days measurements when special attention was fixed on the region round the triple point, it was observed that at one temperature the neon in one apparatus, that into which the neon had been distilled, was solid whilst that in the other, containing neon from the cylinder, was still liquid. Both were obtained solid on lowering the temperature still further, and on warming both appeared to melt at the same time. The phenomena was therefore due to supercooling and not, as was first thought, to impurities in the neon from the cylinder.

We are grateful to several assistants of the laboratory for their ready assistance in the measurement of temperatures, and one of us (R. O. G.) wishes to acknowledge his indebtedness to the SALTERS' Company of London for a Fellowship which made participation in this work possible.

¹⁾ H. KAMERLINGH ONNES, These Proceedings **19**, 1049; Comm. Leiden N^o. 151a; H. KAMERLINGH ONNES and C. A. CROMMELIN, These Proceedings **23**, 1185; Comm. Leiden N^o. 154c.

TABLE I.

	Θ	T	Pressure		
			int. cms Hg	int. atm.	
Solid	—257.62 C.	15.47 K.	0.055	0.00072	
	—256.79	16.30	0.13	0.0017	
	—256.46	16.63	0.21	0.0028	
	—255.79	17.30	0.37	0.0049	
	—255.43	17.66	0.51	0.0067	
	—254.92	18.17	0.78	0.0103	
	—254.63	18.46	0.98	0.0129	
	—254.07	19.02	1.49	0.0196	
	—253.64	19.45	2.00	0.0263	
	—253.30	19.79	2.53	0.0333	
	—253.22	19.87	2.71	0.0357	
	—253.16	19.93	2.82	0.0371	
	—252.62	20.47	4.00	0.0527	
	—251.24	21.85	9.1	0.120	
	—250.84	22.25	11.1	0.146	
	—250.22	22.87	14.8	0.195	
	—249.62	23.47	19.5	0.257	
	—249.09	24.00	25.0	0.329	
	—248.84	24.25	27.9	0.367	
	—248.58	24.51	31.5	0.415	
—248.56	24.53	31.7	0.417		
Liquid	Comm. 147d	—248.51	24.58	32.4	0.427
		—248.51	24.58	32.50	0.4276
		—248.29	24.80	35.0	0.461
		—248.10	24.99	37.3	0.491
		—247.82	25.27	41.0	0.540
	" 147d	—247.49	25.60	45.16	0.5942
		—247.33	25.76	48.6	0.640
	" 147d	—246.66	26.43	60.52	0.7963
	" 147d	—245.88	27.21	76.71	1.0094
		—245.79	27.30	79.1	1.041
	" 147d	—245.68	27.41	81.62	1.0740
	" 152b	—243.69	29.40		1.888
	" "	—241.77	31.32		2.980
	" "	—240.25	32.84		4.173
	" "	—236.82	36.27		7.970
	" "	—233.60	39.49		13.213
	" "	—231.71	41.38		17.428
	" "	—229.26	43.83		24.305
" "	—228.66	44.43		26.049	

Measurements of H. KAMERLINGH ONNES and C. A. CROMMELIN, These Proceedings 18, 515; Comm. Leiden N^o. 147d.
Measurements of P. G. CATH and H. KAMERLINGH ONNES, These Proceedings 20, 1160; Comm. Leiden N^o. 152b.

Mineralogy. — *A contribution regarding the Mineralogical Composition of some of the Soils of the Netherlands.* By J. H. DRUIF.
(Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of February 26, 1927).

I have for a considerable time been engaged on an inquiry into the mineralogical composition of a number of Netherlands soils and that of some of Belgium, with a view to collect data, that might aid an interpretation of the formation of the so-called Limburg Loess. It seemed to me useful to publish some results of my inquiry, although it is not quite ended yet.

My research concerned 46 samples, to wit: 14 Boulder clay from different provinces, 15 Limburg-Loess, 5 pre-glacial clays also from different provinces, 7 weathering soils from S. Limburg, and 5 Senonian rocks from S. Limburg, of which 2 Maastrichtien and 3 Gulpenien, (resp. Mb, Mc and Cr, étages).

The Belgian soils may be just as well left out of consideration.

The accompanying table gives a list of the minerals found, also the groups of soils in which they occurred, and a rough determination of their relative quantities. Besides this I specify the following results, as being the most important outcome of this part of my inquiry.

a. SCHROEDER VAN DER KOLK's quantitative relations¹⁾ between the heavy minerals in soils containing northern material and in soils exclusively built up of southern material, have been found by the present writer to be generally correct.

b. RETGERS's view,²⁾ originally also adopted, but ultimately relinquished, by SCHROEDER VAN DER KOLK³⁾ has after all been corroborated, for, as RETGERS suspected, typical northern minerals have been detected in our soils, while at least three specifically southern ones were met with. One mineral appeared to occur preferably and in larger quantities in southern soils.

c. There is, in my judgment, unmistakably a resemblance in the mineralogical composition of Boulderclay and the so-called Limburg Loess.

¹⁾ J. L. C. SCHROEDER VAN DER KOLK, Bijdrage tot de karteering onzer zandgronden I, K. A. v. W. Verh. Tweede Sectie Dl. 4, N^o. 4. 1895.

²⁾ J. W. RETGERS, De samenstelling van het Duinzand, K. A. v. W., Verh. Dl. 29 '91.

³⁾ J. L. C. SCHROEDER VAN DER KOLK, Bijdrage tot de karteering onzer zandgronden 3, K. A. v. W., Verh. Tweede Sectie Dl. 6, N^o. 4. '98.

Table of the Minerals found.

Name	Boulder-clay	Pre-glacial	So-called Loess	Chalk	Weathering soil
Aegirin	+	—	—	—	—
Anatase	l +	+	l +	—	+
Andalusite	+	—	+	—	+
Apatite	+	—	—	—	—
Arfvedsonite	+	—	l +	—	—
Barkevikite (?) ²⁾	l +	—	vl +	—	—
Biotite	+	+	+	—	—
Calcite	+	—	—	—	—
Cyanite ¹⁾	+	—	+	—	?
Disthene	m +	l +	m +	vl +	+
Dumortierite	—	—	—	vl +	vl +
Epidote	m +	l +	m +	—	l +
Glauconite	+	+	+	l +	l +
Glauco-phane	+	—	+	—	—
Granate	m +	+	+	vl +	vl-l +
Hornblende (gr.)	vm +	+	vm +	vl +	vl +
Hornblende (br.)	+	—	+	—	—
„ (no colour)	+	?	+	—	+
Hypersthene	+	—	—	—	—
Ilmenite	+	+	+	+	+
Korund	—	—	—	—	+
Quartz	vm +	vm +	vm +	+	m +
Lazulith	—	—	—	—	l +
Magnetite	+	+	+	+	+
Microklin	+	+	+	+	+
Muscovite	+	+	+	l +	l +
Ottrelite	—	—	—	—	+ ³⁾
Orthoclase	+	+	+	+	+
Piemontite	l +	—	l +	—	—
Plagioclase	+	+	+	+	+
Pyrite	+	+	—	+	—
Pyroxene	+	+	l +	—	—
Rutile	m +	+	m +	+	vm +
Sillimannite	+	—	+	—	—
Spinel	l +	—	l +	—	—
Staurolite	m +	+	m +	l +	m +
Thulite	+	—	l +	—	—
Titan-Augite	+	—	l +	—	—
Titanite	+	—	+	—	+
Vesuvian (?)	—	—	vl +	—	—
Zircon	vm +	m +	vm +	+	vm +
Zoisite	+	—	+	—	—

m = much; vm = very much; l = little; vl = very little; + = present; — = absent; ? = not quite certain.

1) Cyanite only for the distinctly blue variety.

2) It is not precluded that the mineral in question is a basaltic Hornblende.

3) In strictness, no weathering soil, but a river-deposit near Caberg.

The results of this inquiry appear to us to establish that the following minerals are specifically northern: Manganese-Epidote, which appears in two varieties, viz. Piemontite and Thulite; the Amphibole-variety Arfvedsonite (and incidentally Barkevikite); the Glaucofane, especially when there are many specimens in the sample, and the Titanous-Augite.

VAN BAREN¹⁾ records that he rarely recognized Glaucofane in the Senonian; however, I have not succeeded in detecting this mineral in any southern soil, (not in the Belgian soil either!) but at any rate it occurs in the northern soils and in rather large quantity even in Loess.

Only in southern soils did I find Lazulith (Klaprotite) (weathering soils), Dumortierite, and Ottrélite, while the mineral Anatase evinced a distinct preference for southern soils, again especially as to the number of specimens per sample.

From the Geol. Institute of the State-University at Utrecht.

Utrecht, February 1927.

¹⁾ J. VAN BAREN, *De bodem van Nederland*, Deel II, Kwartair, 19—22 stuk. 1926.

Mathematics. — *Fusion of the existing Theories of the Irrational Number into a New Theory.* By Prof. FRED. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

(Communicated at the meeting of February 26, 1927).

1. Introduction. In the existing theories of the irrational number as those of CANTOR, DEDEKIND, BAUDET and WEIERSTRASS (cf. my book: "Het Getalbegrip, in het bijzonder het Onmeetbare getal, met toepassingen op de Algebra, de Differentiaal- en de Integraalrekening") definitions are given of irrational numbers, of the operations addition and multiplication of real (rational or irrational) numbers and of the relation greater. It follows from these definitions that *for the real numbers the same calculation-rules (and the same properties connected with the relation greater) are valid as for the rational numbers* and that for the system of the real numbers the following *theorem of the upper boundary holds good*:

If S is a non-empty set of real numbers all of which are less than or equal to the same real number (bounded on the right), there exists a real number U (upper boundary of S), with the following two properties:

- 1. No number of S is $> U$;*
- 2. If U' is a real number $< U$, S contains at least one number which is $> U'$.*

It is at once clear that *there can only be one number which corresponds to the definition of upper boundary.*

As I have explained in my "Getalbegrip", for the application of the irrational numbers we have only to do with these results and not with the meaning of addition, etc. For this reason the theorem of the upper boundary is the keystone of each of the above mentioned theories of the irrational number and plays a fundamental part in the Algebra, the Differential and the Integral calculus. Instead of it we might also take a theorem that is equivalent to the theorem of the upper boundary, as the theorem of BOLZANO or the general principle of convergence of CAUCHY. However, the theorem of the upper boundary (or that of the lower boundary, which comes about to the same) seems to me to be the most suitable.

It would be very unpracticable to introduce the irrational number into Algebra and Analysis in another way than via the theorem of the upper boundary (or an equivalent theorem), e. g. by defining a^x for an irrational value of x as a convergent sequence of CANTOR or a section of DEDEKIND, or by proving the theorem about the zero-point of a

continuous function which can assume positive and negative values, by producing that zero-point of the function as a convergent sequence of CANTOR or a section of DEDEKIND. This would give the wrong impression that there exists an Algebra according to CANTOR and one according to DEDEKIND. As all the theories of the irrational number come together in the theorem of the upper boundary, in Algebra we have nothing to do with the chosen theory.

The purpose of this paper is to show *that we can make the different theories of the irrational number concur still more*, so that the part played by the theorem of the upper boundary becomes still greater. By so doing the theories of the irrational number approach each other more *because the definitions of addition and multiplication become the same*. The difference between the various theories of the irrational number only remains in the definition of these numbers, in the definition of "greater" and in a few proofs. In this way the theory of the irrational number (if we stop where it coincides with other theories) becomes considerably shorter and simpler, because addition and multiplication no longer belong to it but must be reckoned among the applications¹⁾.

2. Contents of a theory of the irrational number. We assume that the *rational numbers* have been introduced and that for them the well known fundamental properties (cf. my "Getalbegrip", § 18) have been proved.

A theory of the irrational number means (according to the ideas developed in this paper) an extension of the system of the rational numbers to that of the *real numbers* and such a definition of "greater" in the new system that the following properties are valid:

a. *For any two real numbers α and β one and only one of the following three relations holds good:*

$\alpha = \beta$ (both numbers are the same, also written as $\beta = \alpha$),

$\alpha > \beta$ (also written as $\beta < \alpha$),

$\beta > \alpha$ (also written as $\alpha < \beta$).

b. *If $\alpha > \beta$ and $\beta > \gamma$, we have $\alpha > \gamma$ (transitive property).*

c. *If α is a real number there always exists a rational number $> \alpha$.*

d. *If α is a real number, there always exists a rational number $< \alpha$.*

e. *If α and β are real numbers and if $\alpha < \beta$, there always exists a rational number c , so that $\alpha < c < \beta$.*

f. *For the system of the real numbers the theorem of the upper boundary holds good.*

The property a. includes that *the definition of "greater" given for the real numbers, if applied to two rational numbers, leads to the same result as the definition of "greater" for rational numbers.*

¹⁾ A more elaborate treatment and a discussion of the connection with the existing theories can be found in the periodical "Christiaan Huygens".

If in a definite theory (e.g. that of CANTOR) the five above mentioned properties have been proved ¹⁾, this theory may be considered as finished because its continuation is the same as for another theory.

3. Enclosure of a real number between two rational numbers.

Relating to this we shall prove the following property (always starting from the properties of N^0 . 2):

If a is a given real number and v a given positive rational number, we can always determine a rational number a so that $a < a < a + v$.

As the validity of this property is at once apparent if a is rational, we shall suppose a to be irrational. According to the properties *c.* and *d.* of N^0 . 2 there are rational numbers b and c so that $b < a < c$. The numbers $b + nv$, in which we can choose for n any positive integer or 0, contain only a finite number $< c$, hence also a finite number $< a$ (according to the property *b.* of N^0 . 2). Hence among the numbers $b + nv$ that are $< a$ there is one greatest. If this is a we have $a < a < a + v$, as $a = a + v$ is impossible, a being irrational.

4. If a is a given positive real number and v is a given rational number > 1 , we can always determine a positive rational number a so that $a < a < av$.

According to the property *e.* of N^0 . 2 there exists a rational number b , so that $0 < b < a$. According to N^0 . 3 there exists a rational number a , so that $a < a < a + b(v-1)$. Hence $b < a < bv$, if $a < b$, and $a < a < av$, if $b \leq a$.

5. Definition of the addition of real numbers. Let α and β be two (equal or unequal) real numbers; let A be the set of the rational numbers $a < \alpha$ and B the set of the rational numbers $b < \beta$. According to the property *d.* of N^0 . 2 neither A nor B is empty so that the set $A + B$ of the numbers $a + b$ is not empty either. Let a' be a rational number $> \alpha$ and b' a rational number $> \beta$. From $a < \alpha$ and $\alpha < a'$ it follows in connection with the transitive property *b.* of N^0 . 2, that $a < a'$. In the same way $b < b'$, hence $a + b < a' + b'$, so that the set $A + B$ is bounded on the right and therefore has an upper boundary. This upper boundary, which we shall call $\alpha + \beta$, defines the sum of the two real numbers α and β .

6. If $\alpha + \beta$ is a number of the set $A + B$ of N^0 . 5 and a_1 is a rational number so that $a < a_1 < \alpha$, also $a_1 + \beta$ is a number of $A + B$. We have, therefore, $a_1 + \beta \leq \alpha + \beta$ (as $\alpha + \beta$ is the upper boundary of $A + B$), hence $a + b < \alpha + \beta$ (according to $a + b < a_1 + \beta$ and the transitivity).

Let inversely c be a rational number $< \alpha + \beta$. If d is a rational

¹⁾ The proofs of these properties are not given in this paper as they are also given in existing theories. They may be found in a paper destined for "Christiaan Huygens".

number so that $c < d < a + \beta$, according to the property of N^0 . 3 we can define the rational numbers a and b such that

$$a < a < a + \frac{1}{2}(d - c), \quad b < \beta < b + \frac{1}{2}(d - c).$$

If we represent the third members of these inequalities resp. by a' and b' , we have $a' + b' = a + b + d - c$. From $a' > a$, $b' > \beta$ it follows that a' is greater than any number of A and b' greater than any number of B , hence $a' + b'$ greater than any number of $A + B$. Consequently $a' + b' \cong a + \beta$ (according to the definition of the upper boundary). As $d < a + \beta$, we have, therefore, $a' + b' > d$, hence $a + b > c$. Consequently $c - a < b$, hence $c - a < \beta$ (as $b < \beta$), so that $c - a$ is a number of B . As $c = a + (c - a)$, c is a number of $A + B$. We find accordingly:

The set $A + B$ of N^0 . 5 (formed by the numbers $a + b$, where a is an arbitrary rational number $< a$ and b an arbitrary rational number $< \beta$) is the same as that of the rational numbers $< a + \beta$.

As an immediate consequence $a' + b' \cong a + \beta$ if a' is a rational number $\cong a$ and b' a rational number $\cong \beta$. For $a' + b'$ is greater than any number of $A + B$; hence it is not a number of $A + B$ and does not satisfy the relation $a' + b' < a + \beta$.

7. Fundamental properties of addition. These are:

I. *From any two real numbers a and β we can derive one and only one real number $a + \beta$ by addition.*

II. $a + \beta = \beta + a$ (commutativity of addition).

III. $(a + \beta) + \gamma = a + (\beta + \gamma)$ (associativity of addition).

IV. $a + 0 = a$ (modulus-property of addition).

V. *If $a < \beta$, then $a + \gamma < \beta + \gamma$.*

The validity of I appears from the fact that the definition of addition of real numbers given in N^0 . 5, applied to two rational numbers, leads to the same result as the earlier addition of rational numbers.

The validity of II is at once evident from the corresponding property of the rational numbers.

The same holds good for III. For if we represent a rational number $< a$, $< \beta$ or $< \gamma$ resp. by a , b , c , according to the property of N^0 . 6 a rational number $< a + \beta$ is the same as a number of the form $a + b$. Consequently $(a + \beta) + \gamma$ is the upper boundary of the set of the numbers $(a + b) + c$. In the same way $a + (\beta + \gamma)$ is the upper boundary of the set of the numbers $a + (b + c)$. Both sets are the same and have, therefore, the same upper boundary.

If a is a real number and if a and b are rational numbers so that $a < a$, $b < 0$, we have $a + b < a$, hence $a + b < a$. If, inversely, c is a rational number $< a$ and a a rational number so that $c < a < a$, we have $c = a + (c - a)$ where $a < a$ and $c - a < 0$. The set of the numbers $a + b$ (a rational and $< a$, b rational and < 0), that has $a + 0$ as upper boundary, is, therefore, the same as the set of the rational numbers $< a$. Hence $a + 0 = a$. In this way IV is proved.

If $a < \beta$, we can determine the rational numbers p and q so that $a < p < q < \beta$. If γ is a given real number, according to the property of N^0 . 3 we can determine the rational number c so that $c < \gamma < c + q - p$. According to the property of N^0 . 6 $\beta + \gamma > q + c$, and from the remark at the end of N^0 . 6 it follows that $a + \gamma \leq p + (c + q - p) = q + c$, whence $a + \gamma < \beta + \gamma$. In this way V is proved.

8. Subtraction of real numbers. The possibility and unambiguity of subtraction rests on the fundamental properties I—IV of N^0 . 7 and

VI. If a is a given real number, there exists at least one real number ξ which satisfies $a + \xi = 0$ (possibility of subtraction from 0).

If we represent a rational number $> a$ by a' , the set S of the numbers $-a'$ is bounded on the right (as all these numbers are $< -a$, if a is a rational number $< a$). Hence the set has an upper boundary. We shall call this ξ . If a' and p are rational numbers so that $a' > p > a$, we have $-p \leq \xi$ (as $-p$ is a number of S); as $-a' < -p$ we may write $-a' < \xi$. If inversely b is a rational number $< \xi$, there exists such a rational number a' that $a' > a$ and $-a' > b$ (according to the definition of upper boundary); consequently $-b > a'$, hence $-b > a$, so that b is the opposite of a rational number $> a$. The set of the rational numbers $< \xi$ is, therefore, the same as that of the numbers of the form $-a'$ where a' is rational and $> a$. From this follows that $a + \xi$ is the upper boundary of the set of the numbers $a + (-a') = -(a' - a)$ (a rational and $< a$, a' rational and $> a$). As $a' - a$ is positive and can assume any positive value (according to the property of N^0 . 3), $a + \xi$ is the upper boundary of the set of all negative rational numbers, whence $a + \xi = 0$. In this way VI is proved.

From VI we can further deduce that $\beta - a = \beta + (-a)$ where $-a$ (the opposite of a) is an abbreviation for $0 - a$.

9. From $a + \xi = 0$ and $a > 0$ follows $0 = a + \xi > 0 + \xi = \xi$, hence $0 > \xi$, so that the opposite of a positive real number is negative. In the same way it appears that the opposite of a negative real number is positive.

From

$$\{-a + (-\beta)\} + (a + \beta) = (-a + a) + (-\beta + \beta) = 0 + 0 = 0$$

there follows:

$$-(a + \beta) = -a + (-\beta),$$

so that the opposite of the sum of two real numbers is the sum of the opposites of these numbers.

10. Definition of the multiplication of real numbers. Let a and β be two positive real numbers, A the set of the positive rational numbers $a < a$ and B the set of the positive rational numbers $b < \beta$. According to the property e. of N^0 . 2 neither A nor B is empty so that the set AB of the numbers ab is not empty either. If a' and b' are rational numbers that are $> a$ resp. $> \beta$, we have $a < a'$ and $b < b'$, hence

$ab < a'b'$. Accordingly the set AB is bounded on the right and hence has an upper boundary. We shall call this $a\beta$ and define in this way the product of two positive real numbers.

Supplementary definitions. If a is positive and β negative (hence $-\beta$ positive), we define $a\beta$ as $-\{a(-\beta)\}$. Likewise we define $a\beta = -\{(-a)\beta\}$, as a negative and β positive. If a and β are both negative we define $a\beta = (-a)(-\beta)$.

If $a = 0$ or $\beta = 0$, $a\beta$ is defined as 0.

11. In a similar way as in N^0 . 6 it appears that a number ab of the set AB of N^0 . 10 is less than $a\beta$ (where a and β are positive real numbers). Let c be a positive rational number $< a\beta$. If d is a rational number so that $c < d < a\beta$ and e is a rational number so that $1 < e < \frac{d}{c}$, we have $\frac{d}{c} = ef$ where f is a rational number > 1 . According to the property of N^0 . 4 we can determine the positive rational numbers a and b such that

$$a < a < ae, \quad b < \beta < bf.$$

Consequently ae is greater than any number of A and bf greater than any number of B , hence $abef$ is greater than any number of AB so that $abef \geq a\beta$. As $d < a\beta$ we have $abef > d$, hence $ab > c$. Accordingly $\frac{c}{a} < b$, hence $\frac{c}{a} < \beta$, so that $\frac{c}{a}$ is a number of B . Owing to $c = a \cdot \frac{c}{a}$, c is a number of AB .

Accordingly :

If a and β are positive real numbers, the set AB of the numbers ab (where a and b are arbitrary positive rational numbers resp. $< a$ and $< \beta$) is the same as that of the positive rational numbers $< a\beta$.

12. Fundamental properties of multiplication. These are :

VII. From any two real numbers a and β one and only one real number $a\beta$ may be derived through multiplication.

VIII. $a\beta = \beta a$ (commutativity of multiplication).

IX. $(a\beta)\gamma = a(\beta\gamma)$ (associativity of multiplication).

X. $a(\beta + \gamma) = a\beta + a\gamma$ (distributive property).

XI. $a \cdot 1 = a$ (modulus-property of multiplication).

XII. If $a > 0$ and $\beta > 0$, then $a\beta > 0$.

Proof of VII. The definition of multiplication of positive real numbers given in N^0 . 10, if applied to two positive rational numbers, leads to the same result as the earlier multiplication of positive rational numbers. As also the supplementary definitions of N^0 . 10 are in accordance with the properties of the rational numbers, multiplication is an unambiguous operation.

Proof of VIII. This rests on the corresponding property of rational numbers.

Proof of IX. If α, β, γ are all positive, the proof is quite analogous to that of the fundamental property III (cf. N^o. 7). If one (or more) of the numbers α, β, γ is zero, $(\alpha\beta)\gamma$ as well as $\alpha(\beta\gamma)$ is zero. If α, β, γ are all $\neq 0$ but not all positive, the validity of $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ follows from the validity for positive real numbers and the supplementary definitions of N^o. 10.

Proof of X. We shall first suppose that α, β and γ are all three positive. It appears from N^o. 6 that $\alpha(\beta + \gamma)$ is the upper boundary of the set S of the numbers $a(b + c)$ where a, b and c are arbitrary positive rational numbers resp. $< \alpha, < \beta$ and $< \gamma$. From the property of N^o. 11 it is further evident that $\alpha\beta + \alpha\gamma$ is the upper boundary of the set T of the numbers $a_1b + a_2c$ where a_1 and a_2 are rational numbers between 0 and α . As $a(b + c) = ab + ac$, any number of S is also a number of T . If $a_1 \leq a_2$ we have $a_1b + a_2c < a_2(b + c)$, so that to any number t of T there corresponds a number of S that is $\geq t$. Hence S and T have the same upper boundary so that $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

If $\alpha = 0, \beta = 0$ or $\gamma = 0$, both $\alpha(\beta + \gamma)$ and $\alpha\beta + \alpha\gamma$ are resp. 0, $\alpha\gamma$ or $\alpha\beta$, so that in this case X holds good. We may therefore suppose α, β and γ all three $\neq 0$.

If α and β are positive, γ negative and $\beta + \gamma \geq 0$, then $-\gamma$ is positive (cf. N^o. 9). As X holds good for $\alpha > 0, \beta \geq 0, \gamma > 0$, we have:

$$\begin{aligned} \alpha(\beta + \gamma) &= \{ \alpha(\beta + \gamma) - \alpha\gamma \} + \alpha\gamma = \{ \alpha(\beta + \gamma) + \alpha(-\gamma) \} + \alpha\gamma = \\ &= \alpha\{(\beta + \gamma) + (-\gamma)\} + \alpha\gamma = \alpha\beta + \alpha\gamma. \end{aligned}$$

In connection with II (cf. N^o. 7) this implies the validity of X in any case where $\alpha > 0$ and $\beta + \gamma \geq 0$.

Taking N^o. 9 into account, we derive from this the validity of X in the case that $\alpha > 0$ and $\beta + \gamma < 0$. For in this case $-(\beta + \gamma) > 0$, hence:

$$\begin{aligned} \alpha(\beta + \gamma) &= -\alpha\{-(\beta + \gamma)\} = -\alpha\{-\beta + (-\gamma)\} = \\ &= -\{\alpha(-\beta) + \alpha(-\gamma)\} = -\{-\alpha\beta + (-\alpha\gamma)\} = \alpha\beta + \alpha\gamma. \end{aligned}$$

Consequently X always holds good for $\alpha > 0$. From this the validity for $\alpha < 0$ ensues thus:

$$\begin{aligned} \alpha(\beta + \gamma) &= -(-\alpha)(\beta + \gamma) = -\{(-\alpha)\beta + (-\alpha)\gamma\} = \\ &= -\{-\alpha\beta + (-\alpha\gamma)\} = \alpha\beta + \alpha\gamma. \end{aligned}$$

Proof of XI. If α is a positive real number and if a and b are rational numbers satisfying $0 < a < \alpha, 0 < b < 1$, we have $ab < a$, hence $ab < \alpha$. If, inversely, c is a positive rational number $< \alpha$ and a a rational number so that $c < a < \alpha$, we have $c = a \cdot \frac{c}{a}$ where $0 < a < \alpha$ and $0 < \frac{c}{a} < 1$. The set of the numbers ab (a and b rational, $0 < a < \alpha$,

$0 < b < 1$), that has $a \cdot 1$ as upper boundary, is, therefore, identical with the set of the positive rational numbers $< a$, so that $a \cdot 1 = a$.

$a \cdot 1 = a$ also holds good for $a = 0$ as in this case both members are 0 and the validity for $a < 0$ follows from $a \cdot 1 = -\{(-a) \cdot 1\} = -(-a) = a$.

Proof of XII. If $a > 0$ and $\beta > 0$ and if a and b are positive rational numbers resp. $< a$ and $< \beta$, we have $ab \equiv a\beta$ (according to the definition of upper boundary). Consequently from $ab > 0$ follows $a\beta > 0$.

13. Division of real numbers. *The possibility and unambiguity of division except by 0, rests on the fundamental properties VII, VIII, IX and XI of N^o. 12 and*

XIII. *If α is a given real number different from zero, there is at least one real number ξ so that $\alpha\xi = 1$ (possibility of division in 1 except by 0).*

We shall first suppose $\alpha > 0$. We represent a rational number $> \alpha$ by a' . The set of the numbers $\frac{1}{a'}$ has an upper boundary (as all these numbers are $< \frac{1}{a}$ where a is a positive rational number $< \alpha$); let this upper boundary be ξ . If a' and p are rational numbers so that $a' > p > \alpha$, we have $\frac{1}{p} \equiv \xi$, hence $\frac{1}{a'} < \xi$. If, inversely, b is a positive rational number $< \xi$, there exists a rational number a' so that $a' > \alpha$ and $\frac{1}{a'} > b$; consequently $\frac{1}{b} > a'$, and therefore $\frac{1}{b} > \alpha$. *The set of the positive rational numbers $< \xi$ is, accordingly, identical with the set of the numbers of the form $\frac{1}{a'}$ where a' is rational and $> \alpha$.* Consequently $\alpha\xi$ is the upper boundary of the set of the numbers $a \cdot \frac{1}{a'} = \frac{a}{a'}$ (a positive rational and $< \alpha$, a' rational and $> \alpha$). As $\frac{a}{a'}$ can assume any positive value < 1 (according to the property of N^o. 4), this upper boundary is 1 so that $\alpha\xi = 1$, hence $\xi = \frac{1}{\alpha}$.

If $\alpha < 0$ we have $\alpha \cdot \left(-\frac{1}{-\alpha}\right) = (-\alpha) \cdot \frac{1}{-\alpha} = 1$, so that XIII is also proved for negative values of α .

From XIII we find in the known way that $\frac{\beta}{\alpha} = \beta \cdot \frac{1}{\alpha}$ ($\alpha \neq 0$).

Physiology. — *Experiments with JANSEN and DONATH's Antiberiberi-vitamin.* By Prof. Dr. C. EIJKMAN.

(Communicated at the meeting of March 26, 1927).

About thirty years ago I published in the Proceedings of the session of 29 May 1897 a communication from the Medical Laboratory of Weltevreden (Batavia) entitled „The Combating of Beriberi”. It included among other things a report about feeding experiments with gallinaceans, the results of which promised to throw a new light on the etiology of the disease to which so many Orientals succumbed, also in the Dutch Indies.

It had been demonstrated among others by WINKLER, that Beriberi brings on a degeneration of the whole peripheral nervous system, a polyneuritis or rather a polyneuratrophy. The symptoms of the disease tend to the same conclusion.

As to the etiology of the disease, however, researchers were still much in the dark. Truly, various theories had been advanced, but none of them gave satisfaction, for the prophylactic and the curative measures based on them yielded no or only little and temporary success.

At the time it appeared that a one-sided diet of polished rice could produce polyneuritis in gallinaceans, whereas unpolished or undermilled rice, the so-called silverlayer-rice, produced a sheltering as well as a remedial effect.

It has been the great merit of Dr. VORDERMAN, Medical Inspector in Java, that, after having got cognizance of the above-named results, he has instituted a special inquiry into the occurrence of beriberi in the native prisons, in connection with the feeding. It brought to light that in prisons, where polished rice was administered the number of cases of beriberi exceeded about 300 times the number revealed in prisons where unpolished rice was the staple article of diet.

VORDERMAN'S experience has since been corroborated by numerous observations, and even by advisedly undertaken experiments on man, also by researchers out of the Dutch Indies (British India, Japan).

The conclusion I felt confident to deduce from the animal-experiments, continued and extended by GRIJNS after my departure in the same laboratory, was that there must be a substance, or substances in the rice-polishings, responsible for the favourable effect. Indeed, from the polishings an active extract could be obtained that was amenable to further purification. After many vain trials by different researchers it was FUNK, who seemed at length to have succeeded in isolating the

C. EIJKMAN: EXPERIMENTS WITH THE ANTIBERIBERI-VITAMINE OF JANSEN AND DONATH.



Fig. 1. Crystals of antiberiberi-vitaminchloride.

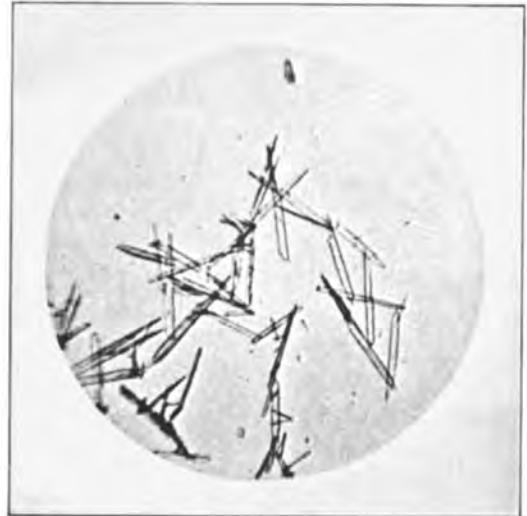


Fig. 2. Golddoublesalt.

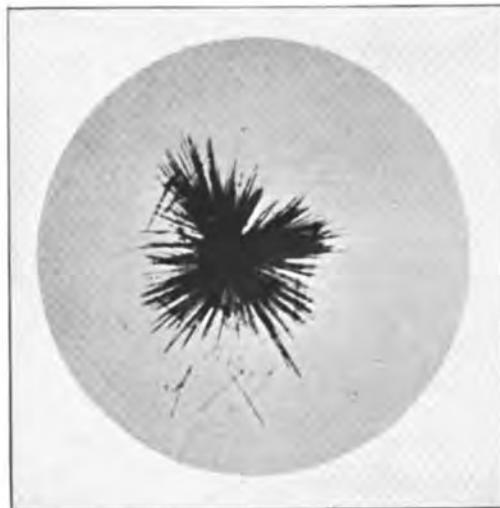


Fig. 3. Vitamin-picrolonate.

(From JANSEN and DONATH).

Enlargement 120 ×

active component in a pure, crystalline condition. He imagined it to be an organic nitrogen base of the formula $C_{17}H_{20}N_2O_7$. Its administration to an amount of 20 mgrs to pigeons, suffering from polyneuritis, had according to him a rapid curative effect. But afterwards neither he himself nor others have been able to confirm this. Moreover it has been proved that a favourable though only transitory effect can be produced by a number of substances in pigeons suffering from the disease, especially on timely interference, and specially when the substance is given not per os, but subcutaneously. However, these substances have no protective properties.

Meantime the name vitamin that FUNK has given to his substance, has become a generic name. A number of vitamins have now been distinguished, a deficiency of each of which in the diet is supposed to induce as many diseases, the so-called deficiency-diseases or avitaminoses.

Now, it may be called a happy coincidence, that in the same laboratory, where the first researches were made in this field, two of our countrymen JANSEN and his coworker DONATH have apparently succeeded, after many years of persistent endeavours, in isolating the antineuritis or, as they term it, the anti-beriberi vitamin from rice-polishings¹⁾.

They report that the addition of as little as 2 mgm., of the vitamin to 1 k.g. of polished rice, ergo in the ratio of 1:500.000, wards off polyneuritis gallinarum. I am in a position to confirm this in virtue of my own experiments with their preparation, to which I shall revert presently. At this rate 1 mgm of vitamin added to $\frac{1}{2}$ k.g. of polished rice would protect man from an outbreak of beriberi.

Some months back JANSEN and DONATH kindly sent me 40 mgms of their preparation, a white powder, consisting of microscopical needle-shaped crystals (Figs 1—3), well soluble in water, slightly less so in alcohol. The behaviour of the substance towards various reagents used in physiologic chemistry, in connection also with its elementary composition ($C_6H_{10}ON_2$), induces them to think it highly probably, that it contains either an imidazol-, or a pyrimidin-nucleus. The amount of the substance at my disposition being very limited I had to abstain from verifying these reactions. It seemed to me to be more interesting to supplement their experiments by applying the test of FOLIN and DENIS. This test, originally intended for a colorimetric determination of acidum uricum (blue stain with phosphotungstic acid and soda) gives, according to EDDY²⁾, also a positive result with extracts containing the antineuritic vitamin. It now appeared that a 1 0/0 solution of the anti-beriberi vitamin yielded with the reagent in question a blue stain, much weaker though than a uric-acid solution of the same concentration.

¹⁾ These Proceedings Vol. 29, No. 10, 1926. Erratum p. 1392 line 19 from bottom read 3 kg instead of 300 kg.

Geneesk. Tijdschr. v. Ned.-Indië, LXVI afl. 6, 1926.

²⁾ Proc. Soc. Exp. Biol. Med. 14, 164, 1917.

Now, in my feeding experiments I had to do without ricebirds, of which the researchers at Weltevreden could avail themselves. These birds yield the advantage that they react rather quickly on a deficiency of the vitamin in question with typical symptoms of polyneuritis, and that, owing to their light bodyweight, they require much less food, consequently also less vitamin than fowls and pigeons, which are generally used for similar experiments. On this account ricebirds were specially adapted to establish the vitamin-content of the various fractions obtained in the isolation-process. Besides, for examining the pure crystalline preparation JANSEN and DONATH have also taken pigeons. Their experiments were exclusively prophylactic, so that they had only to demonstrate that animals on polished rice, to which a certain minimum amount of their vitamin had been added, did not develop polyneuritis, even when they were submitted to a sufficiently prolonged experimentation.

I made curative as well as preventive experiments. According to my experience JANSEN and DONATH's objection, that a number of substances (which certainly do not contain any vitamin) appeared to have a slightly curative effect, applies only to a superficial improvement of short duration, at the most of a few days. The evidence is conclusive, however, when the experiment is continued for a longer space of time and complete recovery is the result. Contrary to them I, therefore, did not desist from curative experiments, especially with a view to their application to man. Fowls yielded hardly any favourable results with such substances. Also in other respects I consider fowls to be in casu first-rate experimental animals, although they require more food than pigeons, which is a drawback. With the latter the symptoms are rather variable. Most often the most conspicuous symptoms at the outset are convulsions and spasms of the neckmuscle, and not, as in cases of beriberi, paresis of the legs. Not rarely, however, do the pigeons succumb unexpectedly without the typical symptoms. As to the behaviour of ricebirds in this respect I must leave it undecided. In fowls the aspect of the disease is more uniform than in pigeons, it reminds us rather of that of beriberi in man, as KUENEN ¹⁾ also maintains. The progress of the disease is slower than in pigeons, so that there is less chance of the animal dying pending the experiment. Another objection raised by JANSEN and DONATH to the use of cocks, is that it often takes months before they manifest symptoms of polyneuritis. My experience is that this is not the case with half-grown cocks, as will be shown by the experiments described lower down, in which the incubation lasted with four cocks, fed on polished rice without vitamin, respectively 21, 23, 24 and 28 days. This, as we shall see, is even a shorter interval of time than the maximum with our pigeons.

Still, in pursuing the line of research of JANSEN and DONATH I first

¹⁾ Ned. T. v. Geneesk. 1917 I p. 2091 sqq.

of all experimented with pigeons. Whether the quantity of vitamin left after this, should be sufficient to experiment also with cocks, would appear afterwards.

Experiments with pigeons.

Nine pigeons were divided into three groups, in such a way that the total weight of the several groups inter se was about equal.

Group I serves as a test-group. Their trough is filled only with polished rice without vitamin. Bodyweight at the commencement of the experiment, resp. 285, 325, and 450 gms., with an empty crop, total weight 1060 gms.

Group II gets polished rice to which 2 mgms of vitamin-hydrochlorid per kg is added. Bodyweight respectively 325, 345 and 375, total weight 1045 grms.

Group III gets polished rice with 4 mgrs of vitamin-hydrochlorid per kg. Bodyweight resp. 295, 330 and 395 grms, total weight 1020 grms.

Every Wednesday morning the pigeons were weighed, after the food had been removed from the pen on the previous evening, as a full crop increases the bodyweight rather considerably. All are placed in separate pens.

On Dr. JANSEN's suggestion all the 9 pigeons were given, besides the rice ad libitum, every day (except Sundays) a pap consisting of $\frac{1}{2}$ gram of meat powder, profusely extracted with water, besides this a few drops of codliver oil some salt-solution and water. The meat for the supply of protein, the oil to prevent xerophthalmia.

JANSEN and DONATH have pointed out that the behaviour of the pigeons fed with rice + vitamin-hydrochlorid, is quite different from that of the pigeons fed with rice alone. The latter practically cease eating the polished rice after a few days, making a mess of their diet, while it looks as if they are rummaging for some grains to their taste, while the others finish their allowance with great relish.

This observation was substantiated by our experiments with pigeons. The pigeons of Group I which at first ingested about 20 gms per day, soon began to scatter the rice about and had to be fed forcibly, as they took hardly any food spontaneously. In spite of the forced feeding and a heightened meat-diet their bodyweight diminished steadily. In the second week one of the pigeons of this group has lost its usual energy and brightness, it sits huddled up in its pen, its feathers are ruffled. Already on the 16th day unmistakable signs of polyneuritis: spasms and convulsions of the muscles of the neck, paresis of the legs. The bodyweight has diminished from 285 to 215 gms. Now the animal is used for a curative experiment. To this end it is given 0.2 mg of vitamin-hydrochlorid per os. The apparent cures, effected by indifferent substances to which we alluded higher up, occur mostly with subcutaneous injection: So JANSEN and DONATH in following the example of THEILER found, that

even an injection of distilled water into diseased pigeons brought about a regression of the symptoms.

The small amount of vitamin was administered by the mouth at 12 o'clock in the day; at 2 o'clock the symptoms were still worse, but at 4.30 the cramps had ceased altogether. Now a trough with rice of group III (4 mg. of vitamin per kg.) is placed in the cage. Next morning the trough is empty, to all appearance the pigeon is restored to health. When this diet is continued, the appetite keeps good at first, the body-weight rises in the course of 17 days to that at the beginning of the experiment, then remains constant for about a fortnight, and then diminishes slowly, which is not surprising, considering such a prolonged monotonous diet. Then the rice of group II (2 mg. per kg.) is passed into the crop daily. After this curative experiment has been continued for 7 weeks in all, it is discontinued. The bodyweight has now fallen to 210 gm., without new symptoms of polyneuritis manifesting themselves. Subsequently the ordinary mixed pigeon-food; within a few weeks the bodyweight has risen above the initial weight.

It is a well-known fact that pigeons as well as fowls are very particular about alteration of food; they may enjoy their food ever so much at the beginning, in the long run they lose their appetite for a monotonous diet, though it may consist of the best foodstuffs.

At the beginning the other two pigeons of the test-group went the same way as the first, only more slowly, for although they sat huddled up, listless, and with ruffled feathers for more than a week already, no typical symptoms were yet visible on the 30th day of the experiment. Seeing that the bodyweight of either of them had fallen considerably in spite of the forced feeding (resp. from 450 to 300 grms and from 325 to 200 grms) and since we were afraid that they should succumb to inanition, a curative experiment was decided on in this stage. They were now given rice of group III (4 mgs per kg.). The result was an immediate recovery of the appetite, greater vitality, and a gradual increase of bodyweight. But after some time they fared worse, just as the first control-pigeon: less appetite, decrease of bodyweight, but polyneuritis did not make its appearance and finally improvement in every respect was brought on by ordinary foodstuffs.

The course of our experiment with the groups II and III differed widely from that with the control-pigeons. It is true, that in the long run their appetite also diminished, but the animals ate sufficiently to keep their bodyweight up to the mark in the 5 weeks that they were fed with vitamin-rice. For the rest they kept in good health. After this they got only polished rice without vitamin. The consequence was that the pigeons of group II revealed the typical symptoms of polyneuritis resp. after 20, 26, and 17 days. They were then submitted to the same curative experiment as the first control-pigeon, which again resulted in complete recovery.

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JANSEN AND DONATH.



Fig. 4.

Cock having been fed for 4 weeks with polished rice + 1:500.000 vitaminchloride.



Fig. 5.

Cock with polyneuritis, after having been fed for 23 days with polished rice without vitamin. Gets then $\frac{1}{2}$ mg. vitamin per os.



Fig. 6.

The same cock as in fig. 5, next day. Gets then polished rice + 1:500.000 vitaminchloride.



Fig. 7.

The same cock as in fig. 6, next day.

The pigeons of group III were more resistant than those of group II, very likely because they could lay up a larger store of vitamin in their organism. After they had been deprived of vitamin for 4 weeks, the experiment was discontinued. Indeed the pigeons were less thriving after that time, and their weight had diminished from 1050 in toto to 760 grms in spite of the forced feeding, but the typical signs of the disease have not appeared.

Experiments with cocks.

After the preceding experiments our store of vitamin had run so short that we had to economize.

Group I. Four young cocks serving for control. They get polished rice without vitamin. Bodyweight resp. 1400, 1575, 1735 and 1850.

Group II. Two young cocks. They are given polished rice with 2 mg. of vitamin-hydrochlorid per kg. Bodyweight resp. 1525 and 1540 grms.

In addition all the 6 cocks were given from the 16th experimental day 5 grms of meat-powder, 1 c.c. of codliver oil and some salt-mixture, in the way described in our experiments with pigeons.

Just as the pigeons in the preceding experiments the two groups of cocks show a marked difference in their behaviour with respect to the food, already in the second week. The control-birds eat their rice hardly any more of their own accord, they waste it and have to be forcibly fed. Nevertheless the bodyweight decreases, the crop discharges itself slowly, comb and gills hang loose and are pale. After an incubation of from 3 to 4 weeks, the four control cocks develop the typical paralytic form of polyneuritis. Their total bodyweight amounts to 5710 grms, against 6560 at the outset of the experiment. But for forced feeding the difference would undoubtedly have been considerably greater. The small remainder of our store of vitamin still sufficed to subject the cock that was the first to contract the disease, to the beginning of a curative experiment. On the 23^d day of the experiment, when the animal had been ill for 2 days, $\frac{1}{2}$ mgrm vitamin-hydrochlorid was administered per os (fig. 5). An appreciable improvement the next day; the bird is strong enough to stand again, but its gait is still unsteady (fig. 6); then rice of group II (2 mgrs of vitamin per kg) is passed into the crop and the next day the animal is slightly better again (fig. 7). In my wide experience with fowls the same recuperation never occurred spontaneously. I am convinced, that without the vitamin the condition would have deteriorated steadily, and the animal would probably have been dying or would have been dead already on the last named day. It was much to be regretted that, owing to the exhaustion of my supply of vitamin, its administration had to be put a stop to. The cock was then gradually restored to health by supplying yeast and the ordinary poultry food.

The two cocks of group II presented during the experiment quite a

different aspect from that of the control-animals, as had also been the case with the pigeons. They kept thriving, gained in weight although their appetite had diminished a little, forced feeding appeared to be unnecessary; comb and gills of a bright red and full of turgor. Such was their permanent condition until the 18th day of the experiment, the day on which the last cock of group I got ill. Thenceforth group II was supplied with rice alone, i. e. without vitamin. It was passed into the crop with 5 grms of meat-powder, 1 cc of codliver-oil etc. With this diet the cocks contracted polyneuritis after resp. 13 and 20 days. The predisposition of these animals to the disease, to which they had previously offered such a stout resistance, as compared with the control-animals, has thus been established. The comparatively brief incubation after deprivation of vitamin agrees in this respect with the behaviour of the pigeons of group II. Had our supply of vitamin been sufficient for feeding 4 mgrs per kg of rice to a third group of cocks, just as with the pigeons, the incubation period, after the deprivation of the vitamin would most likely have been longer also here.

All things considered we get an impression, that in the long run a supply of 2 mgrs per kg of polished rice would not suffice to keep up the store of vitamin proper to the healthy animal organism; 4 mgrs seems to be quite sufficient.

We find great satisfaction in stating that our experiments tend to corroborate the highly important results of JANSEN and DONATH's accurate researches. There is no doubt but that their isolated substance has the power to ward off polyneuritis gallinarum. We were able also to demonstrate its remedial power, as was to be expected. The authors have remarked with justice, that one cannot reasonably doubt of the purity of their substance, of which already a wonderfully small dosis yields the desired effect, and which could be isolated in the form of crystals, while the mother-lye proved much less active, I readily admit, that I had come to think (just as others had), that the protection from polyneuritis could not be ascribed to a single substance, but to a combination of two or more substances, notably because in our efforts to purify extracts, acting antineuritically, we ultimately came to a dead stop, where continued purification abolished the remedial action. This conception has been splendidly negated by the two chemico-physiologists of the Medical Laboratory of Weltevreden.

Hygienic Institute of the Utrecht University.

Physics. — *On the Heats, resp. Pressures of Evaporation, Sublimation and Melting, also in the Neighbourhood of the Absolute Zero; in Connection with NERNST's So-Called Heat-Theorem. (On the Equation of State of Solid Substances, IV. (Conclusion)).*
By Dr. J. J. VAN LAAR, (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of March 26, 1927).

I. Introduction.

In the first of three papers ¹⁾ I drew up an equation of state for *solid* substances, which, as regards form, corresponds to that of VAN DER WAALS for liquids and gases. I showed that not only the values of the coefficients of expansion and compressibility, calculated from this equation of state, are in very good harmony with the values found experimentally, but also the coefficients of the pressure and the temperature of this latter coefficient.

In the second paper I derived the relation which, according to a known thermodynamic formula, must exist at high and low temperatures between the second member of the equation of state (the temperature part) and the temperature part of the expression for the *Energy*.

Finally in the third paper the expressions for Energy and Entropy were more closely considered, also at very low temperatures; and the known expression for the Entropy constant in the *RT*-region was derived from the value of the so-called elementary volume, in connection with the zero-point energy.

As was already announced on p. 698 of the third paper, it remains reserved for this fourth (concluding) paper to draw up the different equations of the pressure in connection with the different heats of transformation in case of coexistence of liquids resp. solid substances with their vapour, of liquids with solid substances, and of solid substances inter se.

Let us first state that I think I have proved in the third of the papers cited, that the so-called "*energy-degeneration*" at *very low* temperatures, in consequence of which *RT* becomes $A'T^4$, resp. AT^4 both in the equation of state and in the equation of energy, is found only with *solid* substances, in which the molecules (atoms) move round fixed positions of equilibrium; but that liquids and gases (unless at exceedingly high pressures) remain in the *RT*-region down to the lowest temperatures, i.e. do *not* degenerate. For gases this is clear, because there at very low

¹⁾ These Proc. 29, 95—112, 497—514 and 683—698 (1926).

temperatures the quantity θ (the so-called "characteristic" temperature of DEBYE) would approach 0 (cf. p. 684—685 loc. cit.) in consequence of the exceedingly great volume, that is if there could be question of degeneration in gases. And for liquids (with the exception of Helium), which can only exist at higher temperatures (above the melting point), this is, after all, a matter of course.

All the same — even at high temperatures — the assistance of the energy-degeneration has been very often called in (especially in Germany) to account for deviations in liquids, which could of course *not* be explained by the aid of the equation of state of V. D. WAALS with a and b constant, but which already found a ready explanation by the assumption of $b=f(v)$.

With regard to what has been adduced above, we may point out that PALACIOS MARTINEZ and K. ONNES (Comm. 164, 1923) have shown already before for gaseous H_2 and He , that even at 20,05 abs. there is no question as yet of any quantum effect. Besides it was very recently proved by me in two papers ¹⁾, that also *liquid Helium* does *not* degenerate down to the very lowest temperatures, and that *all* the abnormal phenomena appearing there — e. g. maximum density at 2^o,3 abs., minimum density at 0^o,5 abs., maximum internal heat of evaporation at 3^o,4 abs., minimum *ibid.* at 1^o,5 abs., etc. — can be explained in a natural way by the assumption that a and b are also functions of the temperature. This latter is, of course, likewise the case for other substances, but there the two decreases with T counteract each other in the equation of state, whereas in Helium b also decreases, but a increases with T , on account of the exceedingly low temperatures below the critical temperature (5^o,2 abs.). (c.f. p. 1305 loc. cit., fig. 3).

The formulae for a and b , viz.

$$\left. \begin{aligned} 10^6 a &= 39,515 + 11,178 T - 1,1177 T^2 \text{ (valid to slightly past } T_k) \\ 10^6 b &= 1223,4 - 118,45 T + 10,356 T^2 \text{ (valid to } \pm 2^{\circ},3 \text{ abs.)} \end{aligned} \right\}$$

have been derived from known *experimental* data, whereas our considerations concerning λ and L are further based — besides on this material — on the *experimental* vapour-pressure observations, which go as far down as 1^o,3 abs. Hence the *minimum* of λ at 1^o,5 has been experimentally established, though the direct confirmation through more accurate measurements of the heat of evaporation L has at present not yet been extended so far (loc. cit. p. 1333—1334). And as a minimum at 1^o,5 abs. is certain ($\pm 16,5$ gr. cal., cf. p. 1331 loc. cit.), it is clear that λ and L will approach at $T=0$ to a slightly greater **finite** limiting value (= 17,5 gr. cal.).

This is in contradiction with the opinion of some recent writers, who on the ground of an *erroneous* interpretation of the so-called NERNST-Theorem (which is only valid in the case of coexistence of two

¹⁾ These Proc. 29, 1303—1316, 1317—1334 (1926).

solid phases, and even then only in the neighbourhood of $T=0$, see § VI) have come to the conclusion that at $T=0$ the quantities λ and L would approach 0 even with the equilibria liquid-vapour (in *He*) and solid-vapour! ¹⁾)

II. Equations of state, and expressions for E and S .

We will recapitulate here some formulae, of which repeatedly use will be made in the future. (see in particular the third of the papers cited of 1926, presented to the Academy).

a. **Liquids and gases for all temperatures** (if not at extremely high pressures, see § IV).

$$\left. \begin{aligned} p + \frac{a}{v^2} &= \frac{RT}{v-b} \\ E &= kT + \left(\frac{a_{00}}{v_{00}} - \frac{a}{v} \right) \\ S &= k \log T + R \log (v-b) + S_0 \end{aligned} \right\}$$

For the sake of simplicity it has been assumed here, that a and b are no functions of T , for else a term with $\frac{db}{dt}$ would have to be added in E , and $a - T \frac{da}{dt}$ would have to be substituted for a . Besides b has been supposed to be no function of v either, for else a somewhat more complicated expression would have come in the place of $\log(v-b)$ in S . But for our purpose it is perfectly unnecessary to introduce these complications, as we wish to develop only general points of view, which are quite independent of the said complications, which would needlessly render the formulae unmanageable and difficult to survey — only for the sake of a quantitative accuracy unnecessary here.

In the term a_{00}/v_{00} the limiting volume at $T=0$, $p=\infty$ is represented by v_{00} . As at high pressures the quantity a may possibly be dependent on v , and at all events dependent on T , we have written a_{00} at $T=0$, $p=\infty$, and we shall write a_0 at $T=0$, $p=0$, to which the volume v_0 corresponds.

For the so-called *Entropy constant* S_0 the following expression is found:

$$\underline{S_0 = -k \log \theta - R \log \omega + k,}$$

¹⁾ See among others an article by DE KOLOSSOWSKY of Leningrad in the J. de Ch. ph. of Oct. 25, 1926, (23, N^o. 8), p. 728—732. The writer makes there, nota bene, use of a perfectly wrong formula for $\frac{dL}{dT} - \frac{L}{T}$. Compare also my article to refute this in the J. de Ch. ph. of Febr. 25, 1927 (24, N^o. 2), p. 115—119, and an article by VERSCHAFFELT (Ibid. March 25, 1926 (23, N^o. 3), p. 238—241) refuting an earlier paper of DE KOLOSSOWSKY (Ibid. 22, 77—79, 1925). To this we may add a still fooler article in the Zeitschr. f. Physik, 43, 509 (Heft 7).

through which the expression for S becomes *homogeneous*, viz.

$$\underline{S = k \log \frac{T}{\theta} + R \log \frac{v-b}{\omega} + k.}$$

In this θ is the so-called „characteristic” temperature in the *solid* state at $v = v_0$ ($p = 0$), and ω the so-called “elementary volume”, i.e. the joint volume within the paths which the molecule centres describe round the positions of equilibrium at $T = 0$ and $p = 0$ in consequence of the still remaining “zero-point energy”. This in consequence of the alternate action of the mutual attractions and repulsions of molecule nuclei and electrons, which still continues to exist also at $T = 0$ (zero-point paths). For mon-atomic substances (see my derivation in the third of the papers cited, p. 696) ω is represented by (k' is $R:N$):

$$\underline{\omega = \frac{9 N h^3}{16 \sqrt{\pi} (k' \theta)^{3/2} (2\pi m)^{3/2}}.}$$

The quantity k in the expressions for S and S_0 is everywhere the specific heat at infinitely large constant volume. Its value for mon-atomic substances is $= \frac{3}{2} R = 3$ gr. cal.

b. Solid substances at higher temperatures (if not at extremely high pressures).

$$\left. \begin{aligned} p + \frac{a}{v^2} - \beta &= \frac{RT}{v-b} \\ E &= kT + \left(\frac{a_{00}}{v_{00}} - \frac{a}{v} \right) + P \\ S &= k \log T + R \log (v-b) + S_0 \end{aligned} \right\}$$

The same remark as above concerning the simplifications with respect to a and b .

In the equation of state (see the first of the papers cited) a term $-\beta$ has now been introduced, which relates to the (static) repulsive virial. I gave it the form $\beta = \frac{\lambda}{v-b}$, but this special form is of no consequence for what follows¹⁾. The quantity P in the expression for E is evidently

$$= \int_v^{\infty} \beta dv.$$

The quantity S_0 has the value indicated above; only the double value $3R = 6$ gr. cal. (for mon-atomic substances) must now be put for k , just as in kT and $k \log T$ in E and S , whereas ω has the same value for all the states of aggregation.

¹⁾ GRÜNEISEN and others assumed the form λ/v^n .

c. Solid substances at very low temperatures.

$$\left. \begin{aligned} p + \frac{a}{v^2} - \beta &= \frac{A'T^4}{v-b} \\ E &= AT^4 + \left(\frac{a_{00}}{v_{00}} - \frac{a}{v} \right) + P + E_0 \\ S &= \frac{4}{3} AT^3 \end{aligned} \right\}$$

The quantity E_0 is the so-called „zero-point energy” (see above). Further A' in the equation of state is related to A in the equation of the Energy (compare for this the second of the papers cited). Both are functions of v . The variability of S with the volume, which at higher temperatures was chiefly embodied in the term $R \log (v-b)$, has now got into the coefficient A .

III. Equilibrium liquid-vapour.

(Above the melting-point; only in Helium down to $T=0$).

If in what follows we always denote the solid state by the index 1, the liquid by 2, and the vapour by 3, the following equation follows immediately from the equalization of the thermodynamic potentials $Z = E + pV - TS$ of liquid and vapour:

$$(E_3 - E_2) + p(v_3 - v_2) = T(S_3 - S_2), \text{ i.e. } L = T(S_3 - S_2),$$

when $L = (E_3 - E_2) + p(v_3 - v_2)$ represents the *total* heat of evaporation. Hence we have according to a. of § II, because the limiting values a_{00}/v_{00} for the smallest volume v_{00} possible are of course identical for liquid and vapour at $T=0$, $p=\infty$:

$$L = (k_3 - k_2) T + \left(\frac{a_2}{v_2} - \frac{a_3}{v_3} \right) + p(v_3 - v_2).$$

But also $k_2 = k_3$, because both refer to $v = \infty$. Further at comparatively low temperatures (i.e. far enough from the critical temperature, e.g. below the boiling-point) the liquid volume v_2 may be neglected by the side of the vapour volume v_3 (and therefore $1/v_3$ by the side of $1/v_2$), so that, as $p v_3 = RT$ may then be put, finally simply

$$L = \frac{a_2}{v_2} + RT \dots \dots \dots (1)$$

is found. In this a_2/v_2 is the *internal* heat of evaporation and RT the external work.

From $L = T(S_2 - S_1)$ follows:

$$L = RT \log \frac{v_3 - b_3}{v_2 - b_2} = RT \log \frac{p + a_2/v_2^2}{p + a_3/v_3^2}$$

as k_2 is again $= k_3$ in $(k_3 - k_2) \log T$, and also $(S_0)_2$ will be $= (S_0)_3$, the fraction after the \log -sign having been transformed by means of the equation of state. At comparatively low temperatures (see above) p is, however, neglected by the side of a_2/v_2^2 , but for vapour exactly inversely a_3/v_3^2 with respect to p (in consequence of the great value of v_3), so that we get:

$$L = RT \log \frac{a_2/v_2^2}{p} \dots \dots \dots (2)$$

The equation (1) serves to determine L ; equation (2) for the calculation of p .

It follows immediately from (1), that at $T=0$ [this is realizable in Helium; in all other substances only through extrapolation below the melting-point] L approaches the finite value a_0/v_0^1 , in which a_0 and v_0 refer to $T=0$ and the corresponding vapour pressure $p=0$.

From the equating of (1) and (2) follows after division by RT :

$$\log p = - \frac{a_2/v_2 + RT}{RT} + \log \frac{a_2}{v_2^2} = - \frac{L}{RT} + \log \frac{a_2}{v_2^2} \dots \dots (3)$$

that is to say at "comparatively" low temperatures, where the above mentioned simplifications are valid. We see that the extrapolated vapour pressure exponentially approaches 0 at $T=0$, this also being the case with dp/dt . The latter follows, indeed, also from $\frac{dp}{dt} = \frac{L}{T(v_3 - v_2)} = \frac{L}{Tv_3}$, in which v_3 (exponentially) approaches more strongly to ∞ than T to 0.

When (3) is written in the form

$$\log p = - \frac{L_0}{RT} + C + DT + \dots = - \frac{a_0/v_0}{RT} + \log \frac{a_0}{v_0^2} - a_0 T \dots \dots (3 \text{ bis})$$

by the expansion of a_2/v_2 and $\log a_2/v_2^2$ into a series with respect to T^2 , $L_0 = a_0/v_0$ can be easily determined from vapour pressure observations, and therefore, if v_0 is known at $T=0$, $p=0$ by extrapolation, also

1) We saw already in § I that DE KOLOSSOWSKY lately came to the amazing discovery, that also at this equilibrium L would have to approach 0! In the liquid also the constant of attraction a would then have to approach 0, which is of course, nonsense. We might communicate more discoveries of this kind; some Italian authors have in view no less than the substitution of an entirely "new" theory of Thermodynamics for the earlier "antiquated" one. Also these authors arrive at the most startling, and (it need hardly be said) the most absurd results!

2) When a and b are assumed to be constant, one finds easily by the aid of the equation of state $\frac{a}{v} = \frac{a_0}{v_0} - RT - \alpha_0 RT^2 \dots$, in which α_0 is the coefficient of expansion at $T=0$, i.e. $\frac{1}{v_0} \left(\frac{dv}{dt} \right)_0 = \frac{Rv_0}{a}$. Further $\log \frac{a}{v^2}$ is then $= \log \frac{a_0}{v_0^2} - 2\alpha_0 T \dots$

a_0 is known, i.e. the *extrapolated* limiting value of a_2 at very low temperatures. In this way the quantity a has been calculated by me in many cases (also for melted metals and salts, where a varies very little with the temperature).

If $b = f(v)$ had been taken into consideration, a term with $\log T$ would have been added in the second member in (3), and also the vapour pressure constant $\log a_0/v_0^2$ would have become somewhat more complicated, especially when also a and b are considered as functions of T . But in any case these constants C at the equilibrium liquid-vapour have nothing whatever to do with the "chemical" constants, introduced by NERNST, which are in relation with the constants of entropy¹⁾. In the case $b = f(v)$ the coefficient of $\log T$ will always be **negative**, and depend in a simple way on the degree of variability of b with v . (Compare also my "Zustandsgl.", p. 265–278). As $\frac{dL}{dt}$ is $= (c_p)_3 - (c_p)_2^2$, i.e. $L = L_0 + [(c_p)_3 - (c_p)_2]_0 T \dots$ it follows from $\frac{dp}{dt} = \frac{L}{Tv_3}$ or $\frac{1}{p} \frac{dp}{dt} = \frac{L}{RT^2}$, that $\log p$ will be $= -\frac{L_0}{RT} + \frac{[(c_p)_3 - (c_p)_2]_0}{R} \log T + \text{etc.}$ Now (when a and b are no functions of T) $(c_p)_3$ is always $= (c_p)_2$; but when $b = f(v)$, $(c_p)_3$ in the vapour will always be considerably smaller than $(c_p)_2$ in the liquid, so that the coefficient of $\log T$ $[(c_p)_3 - (c_p)_2$ at $T=0$ (extrapolated) differs but little from this quantity at higher temperatures] is always *negative*. Hence never $= +1.75$, as NERNST c.s. on the ground of a certain Theorem of Heat, which is absolutely misapplied here, would try to make us believe — even at 4000 to 5000°!

IV. Equilibrium solid-vapour.

a. At higher temperatures.

As long as no degeneration of the energy is perceptible in the solid phase, and we are, therefore, still in the RT -region, there is no essential difference with the equilibrium liquid-vapour. The difference is only, that in the expression (1) for L (the total *sublimation heat*) a term $-P_1$ of the (static) repulsive forces is added, according to b. of § II, and k_3 (vapour) $= \frac{3}{2} R$ for mon-atomic gases will now not be $= k_1$ (solid) $= 3R$. We thus obtain:

$$L = (k_3 - k_1) T + \left(\frac{a_1}{v_1} - P_1 \right) + RT, \quad . \quad . \quad . \quad (1^*)$$

¹⁾ The vapour-pressure constants $C = \log a_0/v_0^2$ at a and b constant are evidently in relation to the logarithms of the *critical pressures* $\left(p_k = \frac{1}{27} \lambda \frac{a_k}{b_k^2} \right)$.

²⁾ This formula, too, is only valid at "comparatively" low temperatures. Cf. Zustandsgl. p. 121–123, and the Article in the J. Chim. phys. against DE KOLOSSOWSKY cited in § I.

(2) becoming :

$$L = T \left[(k_3 - k_1) \log T + R \log \frac{a_1/v_1^2 - \beta_1}{p} + (S_0)_3 - (S_0)_1 \right],$$

which with $(S_0)_3 - (S_0)_1 = -(k_3 - k_1) \log \theta + (k_3 - k_1)$ according to § II passes into

$$L = T \left[(k_3 - k_1) \left(1 + \log \frac{T}{\theta} \right) + R \log \frac{a_1/v_1^2 - \beta_1}{p} \right], \quad \dots (2^a)$$

in which $a_1/v_1^2 - \beta_1$ will now approach 0 at $T=0$ ($p=0$). For with $\beta = \frac{\lambda/v}{v-b}$ the equation of state, neglecting p , becomes $\frac{a}{v^2} = \frac{\lambda/v + RT}{v-b}$;

so that (extrapolated) $\frac{a}{v^2}$ becomes $= \frac{\lambda/v}{v-b} = \beta$ at $T=0$, i.e. $a/v^2 - \beta = 0$.

And since λ/v remains finite at $T=0$, $p=0$, this must also necessarily be the case with $v-b$. (Not before $p=\infty$ will v become $=b$). As the extrapolated value of $a_1/v_1^2 - \beta_1$ will approach 0 at $T=0$, $p=0$, this expression must be replaced by $RT:(v-b)$ in (2^a) , in consequence of which we get:

$$\log p = -\frac{L}{RT} + \frac{k_3 - k_1}{R} \left(1 + \log \frac{T}{\theta} \right) + \log T + \log \frac{R}{v-b}, \quad \dots (3^a)$$

in which L is given by (1^a) , and $k_3 - k_1$ will be *negative*. Again both p and dp/dt (extrapolated) will approach 0 at $T=0$, as $L_0 = a_0/v_0 - P_0$ remains *finite*.

Let us write (3^a) in the form

$$\log p = -\frac{a_1/v_1 - P_1}{RT} + \frac{k_3 - k_1 + R}{R} \log T - \frac{k_3 - k_1 + R}{R} + \frac{k_3 - k_1}{R} (1 - \log \theta) + \log \frac{R}{v-b},$$

as $(k_3 - k_1) T + RT$ of L , divided by RT , yields the constant term $-(k_3 - k_1 + R):R$. Hence we get:

$$\log p = -\frac{a_1/v_1 - P_1}{RT} + \frac{k_3 - k_1 + R}{R} \log T + \left[\log \frac{R}{v-b} - \left(1 + \frac{k_3 - k_1}{R} \log \theta \right) \right],$$

in which $a_1/v_1 - P_1$, expanded into a series with regard to T , will yield no term with T , so that — divided by RT — no further constant term is to be expected¹⁾. Accordingly again no "chemical" constants occur in

¹⁾ For from $\frac{a}{v^2} = \frac{\lambda/v + RT}{v-b}$ follows $-\frac{2a}{v^3} \frac{dv}{dt} = -\frac{\lambda/v + RT}{(v-b)^2} \frac{dv}{dt} - \frac{\lambda/v^2}{v-b} \frac{dv}{dt} + \frac{R}{v-b}$,
i.e. $\frac{dv}{dt} \left[-\frac{2a}{v^3} + \frac{a}{v^2} \frac{1}{v-b} + \frac{\lambda/v^2}{v-b} \right] = \frac{R}{v-b}$, when $\frac{a}{v^2} (v-b)$ is put for $\lambda/v + RT$. Hence
we get $\frac{dv}{dt} \frac{a}{v^2} \left[-2 \frac{v-b}{v} + 1 + \frac{\lambda}{a} \right] = R$. Now according to the equation of state

the vapour-pressure constant C , no more than in the case liquid-vapour.

If now $a_1/v_1 - P_1 = (a_0/v_0 - P_0) + aT^2$ (see Note 1) is written, and further $\log \frac{R}{v-b} = \log \frac{R}{v_0-b_0} + \gamma T$, in which according to the subjoined Note $v_0 - b_0$ is $= v_0 \times \frac{\lambda}{a_0}$, so that $\log \frac{R}{v_0-b_0}$ becomes $= \log \left(\frac{R a_0}{v_0 \lambda} \right)$, we get finally:

$$\log p = \underbrace{-\frac{a_0/v_0 - P_0}{RT} + \frac{k_3 - k_1 + R}{R} \log T + \left[\log \left(\frac{R a_0}{v_0 \lambda} \right) - \right.}_{- \left(1 + \frac{k_3 - k_1}{R} \log \theta \right) + DT + \text{etc.}} \left. \right\} \quad (3^{\text{a}} \text{ bis})$$

so that the extrapolated limiting value $L_0 = a_0/v_0 - P_0$ can be calculated from vapour pressure observations. At present a_0 can only be determined, if P_0 should be known.

The coefficient of $\log T$ agrees with $dL/dt = k_3 - k_1 + R$ according to (1^a); which also ensues from $dL/dt = (c_p)_3 - (c_p)_1 = (k_3 + R) - k_1$, so that also on integration of $\frac{d \log p}{dt} = \frac{L}{RT^2}$ a term $\frac{k_3 - k_1 + R}{R} \log T$ duly occurs.

If b is still dependent on v (in solid substances in a much smaller degree than in liquids), some supplementary terms must be added. But in any case the coefficient of $\log T$ will be *negative* again (in monatomic substances $= \frac{5}{2}R - 3R = -\frac{1}{2}R$, or greater negative, when $b = f(v)$ is assumed); but never $+1.75$. For it is entirely disregarded by NERNST and many others, that the expansions into series, holding at *very low* temperatures in the T^4 -region, are *not* valid at *higher* temperatures in the RT -region, and vice versa. In the intermediate region, where neither of the expansions into series are valid, only the complete DEBYE-formula for E and S can be used. The impermissible application of expansions into series, holding only at *very low* temperatures, also at much higher temperatures; and besides the assumption of Energy-degradation also for liquids and gases, have been the two greatest errors of the adherents of the Theorem of Heat.

Besides, it was a serious error of NERNST's to think that *merely* by

evidently $\frac{v_0 - b_0}{v_0}$ is $= \frac{\lambda}{a}$ at $T=0$, so that $\left(\frac{dv}{dt} \right)_0 = \frac{Rv_0^2}{a} \frac{1}{1 - \lambda/a} = \frac{Rv_0^2 v_0}{a b_0}$. And as $\frac{1}{v} = \frac{1}{v_0} - \frac{1}{v_0^2} \left(\frac{dv}{dt} \right)_0 T \dots$, $\frac{a}{v}$ becomes $= \frac{a}{v_0} - \frac{v_0}{b_0} RT$. Further $P = P_0 + \left(\frac{dP}{dv} \frac{dv}{dt} \right)_0 T \dots$, which with $P = \int_v^{\infty} \frac{\lambda/v}{v-b} dv$, hence $\left(\frac{dP}{dv} \right)_0 = -\frac{\lambda/v_0}{v_0 - b_0} = -\frac{a}{v_0^2}$, becomes $P = P_0 - \frac{v_0}{b_0} RT$.

For $\frac{a}{v} - P$ we find, therefore, $\frac{a_0}{v_0} - P_0 +$ terms with T^2 etc., but not with T .

the application of his Theorem of Heat it would be possible to get to know something concerning the vapour pressure constants, i.e. of the integration constant of the equation $\frac{d \log p}{dt} = \frac{L}{RT^2}$. He overlooked, that the *direct* calculation of $\log p$ from the equating of the thermodynamic potentials, which leads to (3) and (3^a), immediately yields the vapour pressure constant; i.e. it yields the *true* value of this constant ¹⁾, and not the *wrong* value in consequence of the application of a theorem, which is *not valid* at higher temperatures!

b. At very low temperatures (does not occur in Helium).

As soon as there is question of degeneration in the solid phase, the matter is entirely different. According to c. of § II the following equation will be found for L — that is to say when the T^4 -region is entered:

$$L = \left[k_3 T + \left(\frac{a_{00}}{v_{00}} - \frac{a_3}{v_3} \right) \right] - \left[A_1 T^4 + \left(\frac{a_{00}}{v_{00}} - \frac{a_1}{v_1} \right) + P_1 + E_0 \right] + p(v_3 - v_1),$$

in which the first part refers to the vapour. When $A_1 T^4$ is neglected with respect to $k_3 T$, a_3/v_3 by the side of a_1/v_1 , v_1 by the side of v_3 , while $p v_3 = RT$ is put, we find:

$$L = \left(\frac{a_1}{v_1} - P_1 - E_0 \right) + (k_3 + R) T \quad . \quad . \quad . \quad (1^b)$$

The heat of sublimation continues, therefore, to be finite unto $T=0$, as then evidently $L_0 = a_0/v_0 - P_0 - E_0$.

From $L = T(S_3 - S_1)$ follows:

$$L = T \left[\{ k_3 \log T + R \log (v_3 - b_3) + (S_0)_3 \} - \frac{1}{3} A_1 T^3 \right],$$

i.e. when the term with T^3 is neglected, and with $v_3 - b_3 = v_3 = RT/p$:

$$L = T \left[(k_3 + R) \log T - R \log p + R \log R + (S_0)_3 \right] \quad . \quad . \quad (2^b)$$

And from (1^b) and (2^b) follows for p :

$$\log p = -\frac{L}{RT} + \frac{k_3 + R}{R} \log T + \left(\log R + \frac{(S_0)_3}{R} \right) \quad . \quad . \quad (3^b)$$

which, at $T=0$, again gives $p=0$, $dp/dt=0$.

The form of the vapour pressure formula has remained the same, but *not until now* has the coefficient of $\log T$ become *positive*, viz. $c_p : R$ in the vapour; i.e. $5/2$ in monatomic vapours, $7/2$ in diatomic vapours, etc. — but all the same *never* + 1.75, as NERNST c.s. always give.

¹⁾ When in 1906 NERNST published his first communication in reference to this, I immediately brought forward my objections to his views. The paper had, already, been printed and corrected, when the Editor of the Z. Elektrochem., ABEGG, requested me kindly but urgently, to withdraw my paper, because NERNST had "energetically" protested against its insertion. The corrected Revision of the paper, written Nov. 10 (eingegangen Nov. 22) 1906, has lain in a drawer of my writing-desk for 20 years now.

If for L its value according to (1^b) is written, bearing in mind that $a_1/v_1 - P_1 - E_0$ again does not yield a term with T^1), we get the following equation, taking into consideration that $S_0 = -k \log \theta - R \log \omega + k$ (cf. a. of § II):

$$\log p = -\frac{a_0/v_0 - P_0 - E_0}{RT} + \frac{k_3 + R}{R} \log T + \left[-\frac{k_3 + R}{R} + \log R - \frac{k_3}{R} \log \theta - \log \omega + \frac{k_3}{R} \right] + DT^4 \dots$$

i.e.

$$\log p = \underbrace{-\frac{a_0/v_0 - P_0 - E_0}{RT} + \frac{k_3 + R}{R} \log T + \left[\log \frac{R}{\omega} - \left(1 + \frac{k_3}{R} \log \theta \right) \right]}_{(3^b \text{ bis})} + DT^4 + \text{etc.}$$

so that only now the vapour pressure constant C is in connection with the so-called "chemical" constant through $(S_0)_3$ or ω . From vapour-pressure observations (if possible, see below) $(S_0)_3$ might be determined.

As according to § 2 $\log \omega = \log \omega_0 - \frac{3}{2} \log \theta - \frac{3}{2} \log m$ may be written for $\log \omega$, in which, therefore, ω_0 has the same value for all substances, we may write for C in monatomic substances $C = \log \frac{R}{\omega_0} + \frac{3}{2} \log \theta + \frac{3}{2} \log m - 1 - \frac{3}{2} \log \theta, \frac{k_3}{R}$ then being $= \frac{3}{2}$.

Accordingly we get in this case:

$$C = \left(\log \frac{R}{\omega_0} - 1 \right) + \frac{3}{2} \log m = C_0 + \frac{3}{2} \log m,$$

in which m is the atomic weight of the (monatomic) substance. (In multi-atomic substances likewise C' will be $= C'_0 + \frac{3}{2} \log m$, but then C'_0 is not $= C_0$ in monatomic substances, and besides C'_0 will depend on θ , which quantity will be different in every substance).²⁾

If in solid substances at such low temperatures as those at which (3^{b bis}) is valid, vapour-pressure observations are possible (the vapour pressures will mostly be so low then, that they are inaccessible to direct measurement; at any rate so far they have not yet been measured

1) Even no terms with T^2 and T^3 , because $\frac{dv}{dt}$ is now of the order of T^3 , hence $1/v - 1/v_0$ and $P - P_0$ of the order of T^4 . (See also the Note in a.).

2) With regard to the homogeneity of the different formulae it may be pointed out, that e.g. in (3^{b bis}) ω is a volume, hence it has the dimensions $R\theta : \pi$. $\log(R : \omega)$ is therefore of the dimensions $\log \pi - \log \theta$, so that we may write $\log \frac{p}{\pi} = -\frac{a_0/v_0 \text{ etc.}}{RT} + \frac{k_3 + R}{R} \log \frac{T}{\theta} - 1$, which is perfectly homogeneous.

in any substance), the limiting value $L_0 = a_0/v_0 - P_0 - E_0$ could be calculated. And since from vapour-pressure observations at higher temperatures (see under a.) the extrapolated value of $a_0/v_0 - P_0$ can be determined, E_0 could in this way be found experimentally.

V. Equilibrium solid-liquid.

a. At higher temperatures.

This occurs with all substances, from the triple point to not too high pressures; and also in *Helium* as continuation of the low-temperature curve.

For the *melting-heat* $Q = (E_2 - E_1) + p(v_2 - v_1)$ the following equation is found:

$$Q = (k_2 - k_1) T + \left(\frac{a_1}{v_1} - \frac{a_2}{v_2} - P_1 \right) + p(v_2 - v_1). \quad (4^a)$$

in which $k_2 - k_1$ is negative. $Q = T(S_2 - S_1)$ gives further:

$$Q = T \left[(k_2 - k_1) \log T + R \log \frac{v_2 - b_2}{v_1 - b_1} + (S_0)_2 - (S_0)_1 \right].$$

But $(S_0)_2 - (S_0)_1 = (k_2 - k_1)(1 - \log \theta)$, hence we get:

$$Q = T \left[(k_2 - k_1) \left(1 + \log \frac{T}{\theta} \right) + R \log \frac{v_2 - b_2}{v_1 - b_1} \right]. \quad (5^a)$$

As in liquids $v_2 - b_2 = RT : (p + a_2/v_2^2)$ is always greater than $v_1 - b_1 = RT : (p + a_1/v_1^2 - \beta)$ in the solid phase, because a_1 is much greater than a_2 , v_1 and v_2 differing only comparatively little, so that even $a_1/v_1^2 - \beta$ will be $> a_2/v_2^2$ also if v_2 should be $< v_1$; and as T soon approaches to a limiting value above or below the triple point (according as v_2 is $> v_1$ or $< v_1$), the second term of (5^a) can be slightly greater than the first (negative), and Q can be small positive. Experimentally comparatively small positive values are actually always found, much smaller than the much greater heats of evaporation and sublimation.

It also follows from (4^a) that, as also $a_1/v_1 - P_1$ will always be $> a_2/v_2$, Q can become small positive. The negative term $(k_2 - k_1) T$ has little influence on this, no more than $p(v_2 - v_1)$ for the case that v_2 should be $< v_1$, at least at not too high values of the pressure.

But at very high pressures the matter is different. Then in consequence of the characteristic temperature θ becoming higher and higher (see the third of the paper cited, p. 683; the expression for θ has, in the denominator, the root from the coefficient of compressibility, which will approach 0), T/θ becomes smaller and smaller, and as the expansions into series for E and S used, holding for comparatively great values of $T : \theta$, are no longer valid, and we then enter the T^4 -region, other expressions will have to be substituted for (4^a) and (5^a). With very great values of

p it is then, however, not immediately to be seen from the modified equation (4^a), whether Q can still remain positive with $v_2 < v_1$. But it appears with the greatest clearness from $Q = T(S_2 - S_1) = T^{4/3} A_2 T^3 - 4/3 A_1 T^3 = 4/3 (A_2 - A_1) T^4$, in which A_2 and A_1 are both very small (these quantities have θ_2^3 resp. θ_1^3 in the denominator), while A_2 in the liquid (where the high-pressure degradation is not so far advanced as in the solid phase) will always be $> A_1$, that Q always remains small positive; and not before $p = \infty$, when A_2 would become $= A_1$, does it approach 0.

It follows then further from the relation of CLAPEYRON $\frac{dp}{dt} = \frac{Q}{T(v_2 - v_1)}$, that with $v_2 > v_1$ (the most frequently occurring case) $\frac{dp}{dt}$ will always be positive, and will approach to $+\infty$ at $p = \infty$. For Q then approaches $(a_2 - a_1) \frac{T^4}{\theta^3}$, $v_2 - v_1$ approaching to $(\gamma_2 \sigma_2 - \gamma_1 \sigma_1) \frac{T^4}{\theta^3}$, in which σ_2 and σ_1 represent the coefficients of compressibility. For

$$\left(\frac{dv}{dt}\right)_p = \left(\frac{dp}{dt}\right)_v \times - \left(\frac{dv}{dp}\right)_t = \gamma \frac{T^3}{\theta^3} \sigma,$$

because A' in the equation of state (as regards $\left(\frac{dp}{dt}\right)_v$) has likewise θ^3 in the denominator, so that $v = v_0 + \gamma \frac{T^4}{\theta^3} \sigma$. Hence CLAPEYRON's equation becomes:

$$\frac{dp}{dt} = \frac{a_2 - a_1}{\gamma_2 \sigma_2 - \gamma_1 \sigma_1} \times \frac{1}{T},$$

in which T remains finite, σ_2 and σ_1 both verging on 0, so that dp/dt will approach ∞ , because a and γ are of the same order of magnitude according to (24) of the second of the papers cited (p. 511). Q and Δv then approach both 0 at $p = \infty$, but Δv (on account of σ) much more strongly than Q . In the case $v_2 > v_1$ the melting-point line has, therefore, a vertical asymptote at a finite value of T above the triple point.

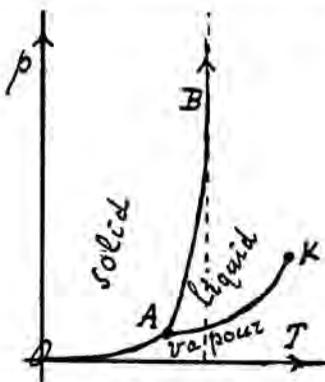


Fig. 1 ($v_2 > v_1$)

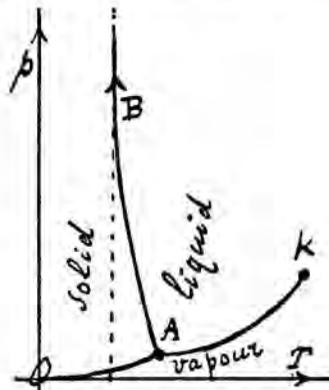


Fig. 2 ($v_2 < v_1$)

If $v_2 > v_1$ (retrogressive melting-point line), $\frac{dp}{dt}$ will approach $-\infty$, and there will be a vertical asymptote at a temperature below A .

The case might, however, also present itself, that in the latter case the melting-point line is so strongly retrogressive, or that the triple point is situated so low, that it enters the T^3 -region already at comparatively low, i. e. not too high pressures. Hence not because in $T: \theta$ the quantity θ approaches ∞ at very high pressures at a temperature that remains finite, but because T itself will approach 0 with θ remaining finite. But this case will presently be considered more closely under **b**.

From (4^a) follows for p (Q is given by (5^a)):

$$p = \frac{Q - \left(\frac{a_1}{v_1} - \frac{a_1}{v_2} - P_1 \right) + (k_1 - k_2) T}{v_2 - v_1}, \dots \dots (6^a)$$

which will be of the form $p = -c + \gamma T$ (6^a bis) with $v_2 > v_1$, and of the form $p = c - \gamma T$ with $v_2 < v_1$, i. e. at not too high pressures.

b. At very low temperatures (occurs with $v_2 > v_1$ only for Helium). Though we have already devoted a special article to this case¹⁾, we will briefly return to this subject in connection with what we have said above. We now find for Q :

$$Q = \left[k_2 T + \left(\frac{a_{00}}{v_{00}} - \frac{a_2}{v_2} \right) \right] - \left[A_1 T^4 + \left(\frac{a_{00}}{v_{00}} - \frac{a_1}{v_1} \right) + P_1 + E_0 \right] + p(v_2 - v_1),$$

i. e. with neglect of $A_1 T^4$ with respect $k_2 T$:

$$Q = k_2 T + \left(\frac{a_1}{v_1} - \frac{a_2}{v_2} - P_1 - E_0 \right) + p(v_2 - v_1) \dots \dots (4^b)$$

It also follows from $Q = T(S_2 - S_1)$, that

$$Q = T \left[(k_2 \log T + R \log (v_2 - b_2) + (S_0)_2) - \frac{4}{3} A_1 T^3 \right],$$

or with $v_2 - b_2 = RT$: $(p + a_2/v_2^2) = RT : a_2/v_2^2$ (because here the comparatively low values of p can be neglected with regard to a_2/v_2^2) and with omission of the term with T^4 :

$$Q + T \left[(k_2 + R) \log T - R \log a_2/v_2^2 + R \log R + (S_0)_2 \right] \dots \dots (5^b)$$

From this follows that at $T=0$ Q will approach to $(k_2 + R)T \log T$, i. e. to -0 . At temperatures somewhat above 0° abs. the *negative* value of Q will however (in consequence of $\log T$) soon pass through zero, after which it will become and remain *positive*.

Hence it follows from $\frac{dp}{dt} = \frac{Q}{T(v_2 - v_1)}$, that at $T=0$ $\frac{dp}{dt}$ will approach

¹⁾ These Proc. 30, 244-248 (1927).

$\frac{(k_2 + R) \log T}{v_2 - v_1}$, i. e. to $-\infty$ at $v_2 > v_1$ (Helium; see Fig. 3), and to $+\infty$ at $v_2 < v_1$ (retrogressive melting-point lines at low temperatures; see above the conclusion of a. and Fig. 4). The further course will now be at once evident from the two subjoined figures and from (5^b). The position of the minimum, resp. maximum, at M will in both cases be given by $Q = 0$, i. e. $(k_2 + R) \log T_m = R \log \frac{a_2}{v_2} - R \log R - (S_0)_2$.

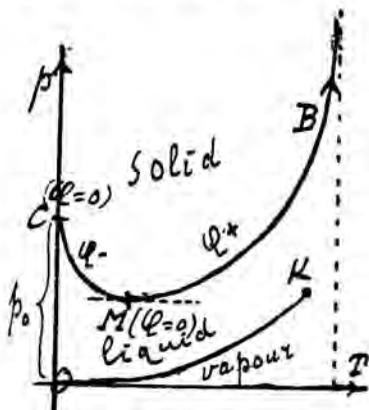


Fig. 3 (Helium) ($v_2 > v_1$)

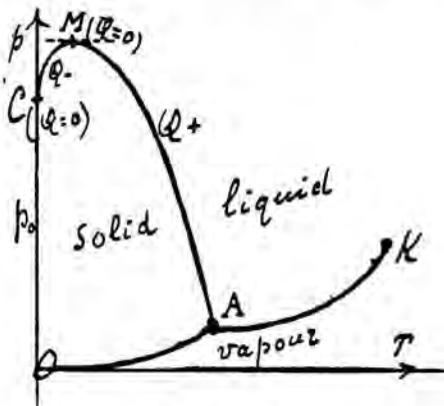


Fig. 4 ($v_2 < v_1$)

From (4^b) follows for p :

$$p = \frac{Q - k_2 T - \left(\frac{a_1}{v_1} - \frac{a_2}{v_2} - P_1 - E_0 \right)}{v_2 - v_1} - \frac{Q - k_2 T + C}{v_2 - v_1} \quad (6^b)$$

as $a_1/v_1 - a_2/v_2 - P_1 - E_0$ will be negative owing to E_0 . And as at very low temperatures $Q = T [k_2 + R] \log T - C'$ according to (5^b), p assumes ($v_2 > v_1$) the form

$$p = a T \log T - \beta T + \gamma \dots \dots \dots (6^b \text{ bis})$$

The latest very interesting observations bij KEESOM¹⁾ extend only to 1°.2 abs., so that the minimum at about 1°.14 abs. and the subsequent increase of pressure at still lower temperatures (portion MC in Fig. 3) have not yet been confirmed experimentally. But the observations, and the Fig. on p. 1142, lead us to expect a minimum somewhat below 1°.19 with pretty great certainty.

The following values namely have been found.

T	0	0.5	0.7	0.8	0.9	1	1.19	1.42	1.60	1.83	(boiling p.) (crit.)							
$p^{atm.}$	—	—	—	—	—	—	25.3	26.5	27.4	29.8	35.7	48.6	62.8	81.5	108.8	140.5	—	—
p (calc.)	54.1	31.0	27.8	26.7	25.7	25.5	25.3	26.1	27.4	29.8	35.7	49.6	63.1	81.5	106.5	140.5	205	—

¹⁾ These Proc. 29, 1136—1145 (1926).

For the *higher* temperatures from 2° abs. the "calculated" values of p have been determined according to the formule $p = -c + \gamma T$ (see (6^a bis)) above at a.), to which still a term with T^2 has been added, viz.

$$p^{atm.} = -16,4 + 14,54 T + 5,40 T^2 \text{ (Helium, 2° abs. and higher, provided the pressures be not too high).}$$

This formula is in good agreement, but would give entirely wrong results below 2° abs. (gives much too low values for p then), because then we gradually get into the T^4 -region of the solid state. Then the formula $p = aT \log T - \beta T + \gamma$ must be used, i.e. in our case:

$$p^{atm.} = 58,4 T \log^{10} T - 28,6 T + 54,1 \text{ (Helium, below 2° abs.),}$$

which formula would yield $p = 54,1$ atm. at $T = 0$ ($dp/dt = -\infty$) and presents a minimum at $T = 1^{\circ},14$, as follows from

$$dp/dt = 58,4 (0,4343 + \log^{10} T) - 28,6 = 58,4 \log^{10} T - 3,24,$$

which is zero at $\log^{10} T = 0,0554$, $T_{min.} = 1,136$, giving $p_{min.} = 25,26$ atm. = 25,3 atm.

Experiment will be able to establish an appreciable rise of the pressure not before about $T = 0^{\circ},8$ abs., which will probably increase to more than 50 atm. at $T = 0^1$).

Remark I. If — what we consider improbable on the ground of several facts (cf. Chapter I) — liquid Helium were degenerated between 1°₁₉ and 1°₈₃, dp/dt would have to decrease in direct ratio to T^3 on decrease of temperature. Now the quantities dp/dt between 1°₁₉ and 1°₄₂, 1°₄₂ and 1°₆₀, etc. are proportional as $\frac{0,8}{0,23} : \frac{1,3}{0,18} : \frac{2,4}{0,23}$, i.e. as 3,2 : 7,2 : 10,5 or as 1 : 2,25 : 3,3, whereas the third powers of the middle temperatures 1°₃₀₅, 1°₅₁, 1°₇₁₅ are in the ratio of 2,22 : 3,44 : 5,04 or as 1 : 1,55 : 2,3. Hence the decrease follows in reality another law, in casu the $T \log T$ -law, which is valid when the liquid Helium does not degenerate.

Remark II. According to (5^b) at $T = 0$ also $Q = 0$, and according to (6^b) p_0 becomes $= -\frac{(a_1/v_1 - a_2/v_2) - (P_1 + E_0)}{v_2 - v_1}$. Now both in the case of $v_2 > v_1$

¹⁾ Note added in the English text. In a recent Article (These Proc. 30 (1927)) KEESOM has doubt of the existence of a minimum in the neighbourhood of 1° abs. (or a little lower, for it is very good possible that at that temperature we are still not entirely entered in the T^4 -region), and believes that $\frac{dp}{dt}$ will approach to 0 at $T = 0$. But I think, one cannot conclude this with some certainty from his Figure on the page 1142 l.c. All things are still possible, and a decision will be reserved to further experiments.

I still remark, that KEESOM has calculated the coefficients of my formula (6^b bis) from experimental data at 1°₂, 1°₈ and 2°₄ abs. and has — of course — found no agreement (see the Fig. 2 in his recent Article). For this formula is only valid at *very low* temperatures; for Helium certainly not higher than 1°₈, so that his coefficients are entirely wrong and must be substituted by those, given above.

and of $v_2 < v_1$, p_0 is positive. In the first case the numerator must be negative, in the second case, however, positive. This can only be accounted for, when we assume that $-a_1$ being greater than a_2 — in the first case (Helium) $a_1/v_1 - a_2/v_2$ yet remains smaller than $P_1 + E_0$; whereas in the second case $P_1 + E_0 < a_1/v_1 - a_2/v_2$.

VI. Equilibrium between two solid phases.

Here for the first time — in consequence of the degeneration as regards the thermic energy of *both* the phases at very low temperatures — we meet with something, that can be brought in connection with a "Theorem of Heat". But we prefer, also here, to speak of the T^4 -law of DEBYE, according to which Q/T becomes $= 0$ at $T = 0$, and accordingly the line of equilibrium will present a *horizontal tangent* at C .

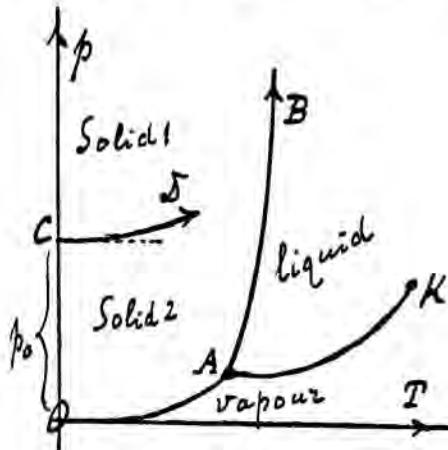


Fig. 5

a. At higher temperatures (e.g. at D).

We then find for Q at the transition $1 \rightarrow 2$:

$$Q = (k_2 - k_1) T + \left[\left(\frac{a_1}{v_1} - \frac{a_2}{v_2} \right) - (P_1 - P_2) \right] + p(v_2 - v_1) \quad (7^a)$$

and also

$$Q = T \left[(k_2 - k_1) \log T + R \log \frac{v_2 - b_2}{v_1 - b_1} \right] + (S_0)_2 - (S_0)_1 \quad (8^a)$$

the course of p along the transformation line according to (7^a) being determined by

$$p = \frac{Q - (k_2 - k_1) T - \left[\left(\frac{a_1}{v_1} - \frac{a_2}{v_2} \right) - (P_1 - P_2) \right]}{v_2 - v_1} \quad (9^a)$$

in which Q is given by (8^a) . From $\frac{dp}{dt} = \frac{Q}{T(v_2 - v_1)}$ follows, that p is ascending or descending according as Q and Δv have equal or opposite signs. We shall, however, see presently that only the first case will occur.

b. At very low temperatures (e.g. in the neighbourhood of C).

In this case we have:

$$Q = (A_2 - A_1)T^4 + \left[\left(\frac{a_1}{v_1} - \frac{a_2}{v_2} \right) - (P_1 - P_2) - \left\{ (E_0)_1 - (E_0)_2 \right\} \right] + p(v_2 - v_1). \quad (7^b)$$

and also

$$Q = \frac{1}{3}(A_2 - A_1) T^4, \quad \dots \dots \dots (8^b)$$

so that now Q will approach 0 proportionally to T^4 , hence $Q/T \therefore T^3$, in consequence of which in this special case (equilibrium between two *solid* phases) $\frac{dp}{dt}$ will approach 0, since $v_2 - v_1$ remains finite to the end.

There is, therefore, a horizontal tangent at C.

We find for p :

$$p = \frac{\frac{1}{3}(A_2 - A_1) T^4 - [\]}{v_2 - v_1}, \quad \dots \dots \dots (9^b)$$

According to the formula of CLAPEYRON the pressure will again be ascending or descending, according as Q (i.e. $A_2 - A_1$) and $v_2 - v_1$ have the same or opposite signs. Now $A = \frac{3}{5} R \pi^4 : \theta^3$, in which θ (cf. equation (24) in the second of the papers cited) is proportional to $v^{-3/2}$. In consequence of this A will be $\therefore v^{3/2}$, and thus $A_2 - A_1$ and $v_2 - v_1$ will always possess the same sign, so that the line of transformation will always be an *ascending* one. (See Fig. 5).

As at $T=0$ in Fig. 5 we have assumed p_0 positive, necessarily again according to (9^b) $(a_1/v_1 - a_2/v_2) - (P_1 - P_2) - ((E_0)_1 - (E_0)_2)$ and $v_2 - v_1$ will have to possess opposite signs. In Fig. 5 v_2 (under lower pressure) is probably $> v_1$, hence $(a_1/v_1 - a_2/v_2) - \text{etc.}$ negative (see also the conclusion of § V under **b.**).

And now all possible cases of lines of equilibrium between gaseous, liquid and solid phases have been treated. The NERNST-theorem is nowhere to be found, except at the equilibrium between two *solid* phases, and then only at very low temperatures. The transference of the formulae holding at these temperatures to higher temperatures, as e.g. NERNST does in the case of the two sulphur modifications at $\pm 100^\circ \text{C.}$ (273°abs.), is absolutely impermissible, for these formulae are no longer valid then. Nor is it permissible to write $+1.75 \log T$ at higher temperatures in the vapour-pressure formulae, the coefficient for $\log T$ always being evidently *negative* then! (See § III and IV). Even at very low temperatures the coefficient mentioned is only positive at the equilibrium solid-vapour, and even then never $+1.75$, but at least $+2.5$ (in monatomic substances).

Tavel sur Clarens (Suisse), Febr.-March 1927.

Chemistry. — *Equilibria in systems, in which phases, separated by a semi-permeable membrane.* XX. By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of March 26, 1927).

Influence of the pressure on osmotic systems and on the osmotic pressure.

An osmotic equilibrium, in which d diffusing substances, and in which occur at the one side of the membrane n_1 components in r_1 phases and at the other side n_2 components in r_2 phases, has, as we have seen in Communication VII, at constant temperature

$$n_1 + n_2 - (r_1 + r_2) + 2 - d$$

licences. Consequently an osmotic equilibrium with one diffusing substance has:

$$n_1 + n_2 - (r_1 + r_2) + 1$$

licences. As at constant temperature the equilibrium at the one side of the membrane has $n_1 - r_1 + 1$ licences and that at the other side has $n_2 - r_2 + 1$ licences, we can also say, therefore, that the number of licences at constant temperature is one smaller than the sum of the licences of the two systems.

We now take a system E , the composition of which we further keep constant; under given pressure the phases of this system are completely defined; they have viz. such composition that the total ζ of the system under that given P (and T) is a minimum. The system E is invariant under given pressure, therefore, so that it has one licence, if the pressure can change also.

If we take now an osmotic equilibrium:

$$(E_1)_{P_1} \mid (E_2)_{P_2} \dots \dots \dots (1)$$

in which both systems have a constant composition, then this has, in accordance with the membrane-phase-rule above-mentioned: $1 + 1 - 1 = 1$ licence. If we put still one condition, f.i. that the pressure must be equal on both sides of the membrane, so that we get the osmotic equilibrium:

$$(E_1)_P \mid (E_2)_P \dots \dots \dots (2)$$

then this has no more licence, therefore; it can only exist under a definite pressure P .

As system (1) has one licence only, P_1 and P_2 are dependent on one another; consequently we can choose only one of those pressures

arbitrarily. With every definite pressure P_1 at the one side of the membrane, a definite other pressure P_2 at the other side is corresponding, therefore; if P_1 increases with a definite value dP_1 , then P_2 must increase with a definite other value dP_2 . Between those increments a relation exists, which we can find as follows. If (1) passes into the osmotic equilibrium:

$$(E_1)_{P_1+dP_1} \parallel (E_2)_{P_2+dP_2} \dots \dots \dots (3)$$

then the *O.W.A.* of both systems increases with:

$$d\xi_1 = -\Delta V_{E_1} \cdot dP_1 \quad d\xi_2 = -\Delta V_{E_2} \cdot dP_2 \dots \dots (4)$$

As, however, the *O.W.A.* must remain the same on both sides of the membrane, it follows:

$$\Delta V_{E_1} \cdot dP_1 = \Delta V_{E_2} \cdot dP_2 \quad \frac{dP_2}{dP_1} = \frac{\Delta V_{E_1}}{\Delta V_{E_2}} \dots \dots (5)$$

If ΔV_{E_1} and ΔV_{E_2} have the same (opposite) sign, then we must, in order to keep both systems in osmotic equilibrium with one another, change the pressure on both sides of the membrane in the same (opposite) direction.

If we draw in a system of coordinates the pressure P_1 on one of the axes and on the other axis the corresponding pressure P_2 , then we have a $P_1 P_2$ curve, the direction of which is defined in every point by (5). If in a point $\Delta V_{E_1} = 0$, then the curve is in this point parallel to the P -axis; with a small change in pressure of system E_1 we must keep constant then the pressure of the system E_2 .

If we draw in this diagram a straight line, which makes equal angles with the two axes, then the points of intersection of this line with the $P_1 P_2$ curve represent the pressure P , under which the osmotic equilibrium (2) exists.

In the special case that E_1 and E_2 represent two gases or vapours, which follow the gas-laws, (5) passes into:

$$\frac{dP_2}{dP_1} = \frac{V_1}{V_2} = \frac{P_2}{P_1} \dots \dots \dots (6)$$

in which V_1 and V_2 represent the molecular volumina of those gases; hence follows:

$$P_2 = C P_1 \dots \dots \dots (7)$$

in which C is a constant. In order to keep both gases in osmotic equilibrium with one another, we have to change therefore, the two pressures in such a way, that their ratio remains constant. Then the $P_1 P_2$ curve is a straight line.

If we have chosen the composition of the two vapours in such a way, that they have the same *O.W.A.* under a certain pressure P_0 , then they have it also under every other pressure. Then the constant C in (7) is 1; the equilibrium (2) then exists not only under a single P ,

but under all pressures P . This is the case when both vapours have a same amount of water.

We now take the osmotic equilibrium :

$$(water)_{P_0} | E_P \dots \dots \dots (8)$$

in which at the left of the membrane pure water in liquid state under the pressure P_0 and at the right of the membrane an arbitrary system E under the pressure P . As $\pi = P - P_0$ represents the osmotic pressure of the system E with respect to water under the pressure P_0 , we shall write for (8):

$$(water)_{P_0} | E_{P_0+\pi} \dots \dots \dots (9)$$

If we represent the molecular volume of the water under the pressure P_0 by v_0 , then follows from (4):

$$d\xi_0 = -v_0 \cdot dP_0 \dots (10^a) \quad d\xi = -\Delta V_E \cdot dP \dots (10^b)$$

Consequently we find for the osmotic equilibrium (8):

$$v_0 dP_0 = \Delta V_E \cdot dP \quad dP = \frac{v_0}{\Delta V_E} \cdot dP_0 \dots (11)$$

As $d\pi = dP - dP_0$ follows from (11):

$$d\pi = \left(\frac{v_0}{\Delta V_E} - 1 \right) dP_0 \dots \dots \dots (12)$$

The osmotic pressure of an arbitrary system E is dependent, therefore, on the pressure P_0 under which the water is; (12) defines the change $d\pi$ of the osmotic pressure at a small change dP_0 of the pressure of the water. In formula (13) of Communication I we have deduced already this formula (12) in a somewhat other form for the special case that the system E consists only of one binary liquid.

If we replace in (8) and (9) the water by water-vapour, then we get the osmotic equilibrium:

$$(watervapour)_{P_0} | E_P \quad (watervapour)_{P_0} | E_{P_0+\pi} \dots \dots (13)$$

in which π now represents the osmotic pressure of the system E with respect to watervapour of the pressure P_0 . The change of the *O.W.A.*, of the osmotic pressure, etc. is also defined by (10)–(12). However, in this we have to replace the volume v_0 of the water by the volume V_0 of the vapour.

In order to apply the previous considerations to a definite system E we must know, therefore, ΔV_E . If we take for E one of the equilibria:

$$\left. \begin{array}{ll} (Y + L_w)_P & \text{fig. 1. III} \\ (X + Y + L_c)_P & \text{fig. 1. V} \end{array} \right\} \quad \left. \begin{array}{ll} (Y + L_a)_P & \text{fig. 1. III} \\ (Y + H + L_d)_P & \text{fig. 2. V} \end{array} \right\} (14)$$

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pressure $P_0 = O a_0$, then system E has an osmotic pressure $\pi = a_0 a$ and consequently a total pressure $P = O a_0 + a_0 a$.

We now shall consider more in detail the cases, above-mentioned.

I. If we change the pressure of the water with dP_0 then the changes dP and $d\pi$ of the total and osmotic pressure of the system E are defined by:

$$dP = \frac{v_0}{\Delta V_E} \cdot dP_0 \quad \dots \quad (15^a) \qquad d\pi = - \left(1 - \frac{v_0}{\Delta V_E} \right) dP_0 \quad \dots \quad (15^b)$$

in which the term between parentheses is positive and smaller than 1. If we raise the pressure P_0 of the water, then it appears that the total pressure P of the system E becomes larger also, but that its osmotic pressure becomes smaller. Further it is apparent that as well the increase of the total pressure as the decrease of the osmotic pressure, is smaller than the change of pressure which we give to the water. Consequently we get a $P_0\pi$ -curve as branch ab or ik (fig. 1) the first of which is convex to the abscis and the second concave to the abscis.

In fig. 1 are drawn some dotted lines, which make an angle of 45° with the P_0 -axis (in the following viz. we indicate only the angles, smaller than 90°). It follows from (15^b) that the tangent in every point of branches ab and ik makes with the P_0 -axis an angle smaller than 45° . If we go along branch ab from a towards b , then this is situated, therefore, firstly below and afterwards above the dotted line, which is drawn through a point of this branch; the same is valid for the branch ik .

As in an osmotic equilibrium the *O.W.A.* is the same on both sides of the membrane, we can call this the *O.W.A.* of this equilibrium. The change of this *O.W.A.*, when an osmotic equilibrium goes along branch ab or ik , is defined by (10^a) or (10^b). It follows from (10^a) that the *O.W.A.* increases with decreasing values of P_0 ; consequently the arrows on ab and ik indicate the direction, in which the *O.W.A.* increases.

If we represent the osmotic equilibria, which are indicated by the points a and b by:

$$(water \mid E)_a \qquad (water \mid E)_b \qquad \text{fig. 1} \dots \dots (16)$$

then is in the first equilibrium:

- the pressure P_0 of the water smaller,
- the osmotic pressure π of system E larger,
- the total pressure P of the system E smaller,
- and the *O.W.A.* larger than in the second equilibrium.

II. In the case, mentioned sub II the coefficient of dP_0 in the two equations:

$$dP = \frac{v_0}{\Delta V_E} \cdot dP_0 \quad \dots \quad (17^a) \qquad d\pi = \left(\frac{v_0}{\Delta V_E} - 1 \right) dP_0 \quad \dots \quad (17^b)$$

is positive; in (17^a) this coefficient is larger than 1, in (17^b) this can be as well larger as smaller than 1. If we raise the pressure P_0 of the water, then the total and the osmotic pressure of the system increases, therefore. The change dP of the total pressure is larger, the change $d\pi$ of the osmotic pressure can be greater also, but also smaller than the change dP_0 , which we give to the pressure of the water.

Consequently in a $P_0\pi$ -diagram we get a curve, as branch be or gi in fig. 1. The angle, which makes a tangent to those branches with the P_0 -axis can be as well larger as smaller than 45° .

With the aid of (10^a) we find again, that the *O.W.A.* increases along those branches in the direction of the arrows.

III. We now put $\Delta V_E = -\Delta$, so that Δ is positive. The change of the total and of the osmotic pressure are defined by:

$$dP = -\frac{v_0}{\Delta} \cdot dP_0 \quad d\pi = -\left(\frac{v_0}{\Delta} + 1\right) dP_0 \quad . . . \quad (18)$$

in which the term between parentheses is positive and larger than 1. Just as in the case, discussed sub I, the osmotic pressure decreases with increasing values of P_0 ; in contradiction with I, however, this decrease now is larger than the increase of pressure dP_0 which we give to the water; also the total pressure P of the system E decreases with increasing values of P_0 .

Hence follows that now we get a $P_0\pi$ -curve, as branch eg which has a falling direction at increasing values of P_0 just as ab and ik . In contradiction with the branches ab and ik the tangent in every point of this curve eg makes, however, an angle, which is greater than 45° .

It follows from (10^a) that the *O.W.A.* increases along this branch eg in the direction of the arrows.

IV. With the transition of I into II becomes $\Delta V_E = v_0$; we then get at first approximation:

$$dP = dP_0 \quad d\pi = 0 \quad \quad (19)$$

In point b , where the branches ab and eb pass into one another, and in the point i , where this is the case with the branches gi and ki , the tangent is horizontal, therefore. Consequently a small change in pressure dP_0 of the water at first approximation has no influence on the osmotic pressure of the system E ; this is a minimum then (point b) or a maximum (point i).

In order to examine this more in detail, we take in (10^a) and (10^b) still terms of the second order; then we get:

$$d\xi_0 = -v_0 dP_0 + a_0 dP_0^2 \quad d\xi = -\Delta V_E dP + a dP^2 \quad . \quad (20)$$

As those changes of the *O.W.A.* on both sides of the membrane must be equal, follows:

$$-v_0 dP_0 + a_0 dP_0^2 = -\Delta V_E dP + a dP^2 \quad \quad (21)$$

If we put in this $\Delta V_E = v_0$ and $dP = dP_0 + d\pi$ then follows:

$$d\pi = \frac{\alpha - \alpha_0}{v_0} \cdot dP_0^2 \dots \dots \dots (22)$$

in which the terms, which are infinitely small with respect to those, already written, are omitted. Hence is apparent that the osmotic pressure for $\alpha > \alpha_0$ is a minimum and for $\alpha < \alpha_0$ a maximum; also it appears that the curve is parabolic in the vicinity of b and i .

The change of the *O.W.A.* has nothing particular in those points b and i .

V. With the transition of II into III becomes $\Delta V_E = 0$; then follows from (11) and (12)

$$dP = \infty \quad d\pi = \infty \quad \dots \dots \dots (23)$$

so that the tangent in the points e and g is vertical. As $\Delta V_E = 0$, we write for (20) and (21):

$$\left. \begin{aligned} d\xi_0 &= -v_0 dP_0 & d\xi &= \beta \cdot dP^2 \\ -v_0 dP_0 &= \beta \cdot dP^2 \end{aligned} \right\} \dots \dots \dots (24)$$

If we put $dP = dP_0 + d\pi$ then we find:

$$d\xi_0 = d\xi = \beta \cdot d\pi^2 \dots \dots (25^a) \quad d\pi = \pm \sqrt{-\frac{v_0}{\beta} \cdot dP_0} \dots (25^b)$$

in which the terms, which are infinitely small with respect to those, already written, are omitted. We now distinguish two cases.

If β is negative, then we can satisfy (25^b) only by positive values of dP_0 ; with every positive increase of dP_0 then are corresponding a same increase and decrease of the osmotic pressure. Then the curve is situated as in point g , viz. at the right side of the vertical tangent going through this point. In accordance with the arrows follows from (25^a), that the *O.W.A.* decreases along this curve starting from point g in both directions.

The osmotic equilibrium has, therefore, in point g a minimum-pressure of the water, while the osmotic pressure is maximum.

If β is positive, then we can only satisfy (25^b) by negative values of dP_0 ; then the curve is situated as in point e , viz. at the left of the vertical tangent going through this point.

Consequently the osmotic equilibrium gets in the point e a maximum-pressure of the water, while its osmotic pressure is a minimum.

Further it appears from (25^b) that the curve is parabolic in the vicinity of the points e and g .

The vertical line $c_0 h$ intersects the $P_0 \pi$ -curves in three points, which represent the osmotic equilibria:

$$(water \mid E)_e \quad (water \mid E)_f \quad (water \mid E)_h \quad \text{fig. 1.} \dots (26)$$

as, although their osmotic pressures are different, those equilibria yet exist under the same pressure $P_0 = O c_0$ of the water, they have, therefore, also the same *O.W.A.* In general it follows from this: osmotic equilibria, which are situated in the $P_0 \pi$ -diagram on a same vertical line have the same *O.W.A.*

It is apparent from the previous considerations that every osmotic equilibrium has a definite $P_0\pi$ -curve, and that it depends on the ΔV_E of the system E , how many and which branches of the curve drawn in fig. 1 will occur.

If we change the pressure P of an equilibrium E of definite composition, then this can pass into a new system E' under a definite pressure P_s , either by the occurrence of one or more new phases or by the disappearance of one or more of the present phases. This pressure P_s is dependent on the composition of the system E . Let us take f.i. the system:

$$E = (Y + L)_P \dots \dots \dots (27)$$

When the solubility of the solid substance Y increases with increase of pressure, then the quantity of the solid substance in this system will decrease, therefore; under a definite pressure P_s the solid substance totally disappears and a new system E' arises, which only consists of a liquid L of definite composition. In this special case the pressure P_s is greater, according as the total concentration of the substance Y in (27) being greater. If we have f.i. the system:

$$E = (X + Y + L)_P \dots \dots \dots (28)$$

in which at increase of pressure the solubility of X as well as of Y increases, then two new systems can arise from this. The first arises when under a pressure P_s one of the solid phases disappears; the second when under a higher pressure P'_s also the other solid substance disappears and consequently only the liquid remains.

If the system E passes into a new system E' under a pressure P_s , then the osmotic equilibrium

$$(water)_{P_0} \mid E_{P=P_0+\pi} \dots \dots \dots (29)$$

passes in a definite point s of its $P_0\pi$ -curve into the new osmotic equilibrium:

$$(water)_{P_0} \mid E'_{P=P_0+\pi} \dots \dots \dots (30)$$

then there are two $P_0\pi$ -curves, which go both through the point s . If we imagine the first to be represented by curve ak of fig. 1, then the second can be indicated by the dotted curve tfu , when $P_s = P_f$. It depends on the ΔV_E of the new system, of how many and of which branches this curve tfu will consist and which angle it makes in f with the first curve.

We take in fig. 1 a point q which is not situated on the $P_0\pi$ -curve, this point represents an osmotic system (q) viz.:

$$(q) = (water \mid E)_q \quad \text{fig. 1} \dots \dots \dots (31)$$

in which the water has a pressure $P_0 = Oc_0$ and E a pressure $P = Oc_0 + c_0q$. As, however, q is not situated on the curve of equilibrium (31) is not

an osmotic equilibrium, so that water must diffuse from the one side of the membrane towards the other. In order to define the direction of this diffusion, we have to compare the *O. W. A.* of the water with that of the system *E*; we represent the first by $\xi_0(q)$ and the second by $\xi(q)$.

As the water has the same pressure $P_0 = Oc_0$ as well in the osmotic system (*q*) as in the osmotic equilibrium (*c*), the *O. W. A.* of the water is also the same, therefore, in both systems; consequently we have $\xi(q) = \xi_0(c)$. As, however, *c* is an osmotic equilibrium, so that the *O. W. A.* is equal on both sides of the membrane, we have also, therefore, $\xi_0(c) = \xi(c)$; consequently we have:

$$\xi_0(q) = \xi_0(c) = \xi(c) \dots \dots \dots (32)$$

We now compare the osmotic system (*q*) with the osmotic equilibrium (*d*). As the line *q d* makes an angle of 45° with the P_0 -axis is:

$$Oc_0 + c_0 q = Od_0 + d_0 d \dots \dots \dots (33)$$

This means that *E* has the same pressure *P* and consequently also the same *O. W. A.*, as well in the osmotic system (*q*) as in the osmotic equilibrium (*d*). Consequently we have:

$$\xi(q) = \xi(d) = \xi_0(d) \dots \dots \dots (34)$$

It is apparent from the direction of the arrows in fig. 1 that $\xi_0(c) = \xi(c)$ is greater than $\xi_0(d) = \xi(d)$. It now follows from (32) and (33):

$$\xi_0(q) > \xi(q) \dots \dots \dots (35)$$

This means that in the osmotic system (*q*) the *O. W. A.* of the water is larger than that of the system *E*. Instead of (31) we write, therefore:

$$(water \leftarrow E)_q \quad \text{fig. 1} \dots \dots \dots (36)$$

in which the arrow indicates the direction, in which the water diffuses. We find the same for the osmotic systems (*m*) and (*t*).

In the same way as above we find that the water in the osmotic system:

$$(water \rightarrow E)_r \quad \text{fig. 1} \dots \dots \dots (37)$$

must diffuse in the direction of the arrow; this is valid also for the osmotic systems (*n*) and (*u*).

In order to define more in detail the different behaviour of the systems (*q*) and (*r*) we place ourselves in fig. 1 in the point *O* and we consider the lines, which make an angle of 45° with the P_0 -axis. Then *q* is situated on the line *q d r* at the left of the point *d* and *r* is at the right; on the line *m h n* is situated *m* at the left and *n* at the right of the point *h*; etc. Therefore the $P_0 \pi$ -curve divides the diagram into two parts, of which we shall say that the one is situated at the left and the other at the right side of this curve.

We can summarise the previous results in the following way.

If the osmotic system

$$(water)_{P_0} \mid E_P \quad \text{fig. 1} \dots \dots \dots (38)$$

is represented in the diagram:

1. by a point of the $P_0 \pi$ -curve, then is $\xi_0 = \xi$; then it is an osmotic equilibrium with the osmotic pressure $\pi = P - P_0$.
2. by a point at the right side of the $P_0 \pi$ -curve, then is $\xi_0 < \xi$; then it is not an osmotic equilibrium but an osmotic system, in which the water diffuses towards the system E .
3. by a point at the left of the $P_0 \pi$ -curve, then is $\xi_0 > \xi$; then the water diffuses away from system E .

We are able to deduce this in the following way also. We imagine anywhere in fig. 1 on the $P_0 \pi$ -curve a point s , through which we draw a horizontal line; on this line we take a point s' in the vicinity of s . Then the point s represents an osmotic equilibrium:

$$(s) = (\text{water})_{P_0} \mid E_P \dots \dots \dots (39)$$

and the point s' an osmotic system:

$$(s') = (\text{water})_{P_0 + dP_0} \mid E_{P+dP_0} \dots \dots \dots (40)$$

the latter arises from the first by changing the pressure on both sides of the membrane with the same amount dP_0 . It now follows from (10^a) and (10^b):

$$\xi_0(s') = \xi_0(s) - v_0 dP_0 \quad \xi(s') = \xi(s) - \Delta V_E \cdot dP_0 \dots (41)$$

As (39) is an osmotic equilibrium, $\xi_0(s)$ viz. the *O. W. A.* of the water is equal to $\xi(s)$ viz. the *O. W. A.* of system E ; consequently it follows from (41):

$$\xi_0(s') - \xi(s') = (\Delta V_E - v_0) \cdot dP_0 \dots \dots \dots (42)$$

We now imagine the point s' in the region which is situated at the right side of the $P_0 \pi$ -curve (viz. at the right, according to the definition given above); we distinguish two cases.

a. Point s is situated on one of the branches *be*, *eg* or *gi*, so that $\Delta V_E < v_0$. In order to come from s in the point s' we have to take dP_0 positive, therefore. It now follows from (42):

$$\xi_0(s') < \xi(s') \dots \dots \dots (43)$$

b. Point s is situated on one of the branches *ab* or *ik*, so that $\Delta V_E > v_0$. In order to come from s in s' we have to take now dP_0 negative; hence follows again (43).

As, therefore, (43) is valid for every point s' which is situated at the right side of the $P_0 \pi$ -curve, the rule mentioned above sub 2. follows from this.

If we take point s' at the left side of the curve, then we find, just as above:

$$\xi_0(s') > \xi(s') \dots \dots \dots (44)$$

from which follows the rule mentioned above sub 3.

(To be continued).

Palaeontology. — *Discoasteridae Incertae Sedis.* By TAN SIN HOK.
(Communicated by Prof. H. A. BROUWER).

(Communicated at the meeting of February 26, 1927)

In this communication the stellate bodies of aragonite, mentioned only incidentally in the present author's article on the coccolith-limestone of Bebalain, will be discussed more fully.

First the author wants to express thanks to all those, who have directly or indirectly cooperated to find the identification of these problematic forms. Successively preparations of these *Discoasters* have been sent to Prof. S. SQUINABOL in Torino, Dr. A. FORTI in Naples, Prof. L. CAYEUX in Paris, Prof. Dr. J. SCHILLER in Vienna, and Prof. Dr. H. LOHMANN, in Hamburg, whose kindness has put the present author under great obligations.

The forms are not novel to science. EHRENBERG mentioned them already in his „Mikrogeologie” in 1854 ¹⁾, found in marls and limestones, originating from Placca di Furni, Zante, from Aegina, from the catacombs of Thebes, from the Anti-Libanon, from Sicily, from Caltanissetta, from Timor, from Carthago, from Barbados, from Kuh-Daëna and Dalaki in South-Persia.

He gave them the general name „Crystalldrusen”, „Scheibensternchen” or „Crystalloids”. As appears also from these names, his opinion was that their origin was inorganic. Besides, he found asterisks ²⁾, called *Actiniscus*, attributed to the „*Polygastern*”. These asterisks are probably *Dyctiochs* (*Silicoflagellates*), partly also spicules of sponges and are to be distinguished from the „Crystalldrusen”, on account that they are composed of silice. Originally the lime-asterisks too were called *Actiniscus* (1843), but in later works they were referred to „Unorganisches”.

Among the „Crystalldrusen” were reckoned stellate as well as rosette-shaped „crystalloids”.

SORBY found rosette-shaped „Crystalldrusen” in the English Chalk. According to him they are decidedly of inorganic origin ³⁾.

In Barbados HILL found „Crystalloids” in the calciferous oceanic deposits

¹⁾ EHRENBERG. Mikrogeologie. Pl. XXIV—66, 67; Pl. XXV B.—12, 13, 14, 15; Pl. XXVI—54a, b, c, d.; XXXVI B.—69, 70; p. 115, 156 e. o.

²⁾ Ibid. Pl. XIX—46, 47; Pl. XX—47.

³⁾ SORBY. Crystalloids of the Chalk, p. 197.

lying under the famous radiolarian rocks, besides in the marls, lying directly under the raised coral-reefs ¹⁾).

HAUPT described them from a marl found in the neighbourhood of Finschhafen in former German New-Guinea ²⁾).

JUKES-BROWNE and HARRISON mention them from the west coast of Java (op. cit. p. 215).

The CHALLENGER found them in the globigerina-ooze of station 338 in the Atlantic ³⁾, however, they are not mentioned in the text.

According to JUKES-BROWNE and HARRISON the asterisks have been found in every depth from 600—2000 fathoms. (op. cit. p. 201).

In the specimens described by HILL and HAUPT these asterisks form with the coccoliths an essential part of the rock.

Granted that the rock-fragment used for the count, gives a general notion of the whole, in the rock of Bebalain (168*) they occur in a number of 55×10^{14} a cubic meter.

In the marl-specimens from other islands of the Moluccas too, they were met with in great number.

In the marine sediments collected by the Siboga-Expedition, they are not mentioned.

Notwithstanding the general distribution, geographically as well as numerically, these *Discoasters* have not drawn attention and have remained as good as unknown.

On the name Discoaster.

In literature these asterisks have not got a special name. EHRENBERG called them "Crystalldrusen, Crystalloids, Scheibensternchen", JUKES-BROWNE and HARRISON called them "stellate bodies, crystalloids".

According to ZITTEL ⁴⁾, EHRENBERG's "Crystalldrusen" might be spicules of *Holothurians*.

However, the *Discoasters* could not be identified, either with spicules of the *Holothurians* collected by the Siboga-Expedition ⁵⁾, or with the fossil rests, described by SCHLUMBERGER ⁶⁾.

Among the spicules of the *Holothurians*, there are also regular forms, however, their appearance is quite different; they are small perforated plates, anchors, wheels, etc. Besides, generally their dimensions are more important, the *Discoasters* not exceeding 35μ , those of Barbados even not exceeding 15μ .

In the many rock-specimens of the Moluccas containing *Discoasters*,

¹⁾ JUKES-BROWNE & HARRISON. Barbados II. p. 177 & 216.

²⁾ HAUPT. Ein Kreide-ähnlicher Mergel, u. s. w.

³⁾ MURRAY & RENARD. Deepsea-Deposits, Pl. XI—4).

⁴⁾ ZITTEL. Handbuch der Palaeontologie p. 560.

⁵⁾ SLUITER. Holothurien der Siboga-Expedition.

⁶⁾ SCHLUMBERGER. Holothuridées du Calcaire Grossier.

spicules, which might with certainty be attributed to Holothurians, have *not* been found, what would be inexplicable, if the *Discoasters* belonged to the Holothuroids.

In my opinion, therefore, HAECKEL's identification might be dropped and the introduction of a new name has been satisfactorily accounted for.

In the spongiology names with the suffix-"aster" are usual for certain *microscleres*: sterraster, disco-hexaster, etc., but as far as I know, not in a composition "*disco-aster*", where the number of arms is left out of consideration, so that the name proposed in the previous communication¹⁾ may be kept without any risk of confusion.

Structure and mineralogical composition of the Discoaster.

The *Discoasters* are soluble in acids; the highest value of the index of refraction is about 1,69 (Method SCHROEDER VAN DER KOLK).

The birefracton varies dependent of the position towards the axis of the microscope. The highest value occurs in a position, where the plane of the *Discoaster* is placed parallelly with the axis of the microscope (white of higher order appears as interference-colour). In a position perpendicular on this, the *Discoasters* are almost isotropic. In an intermediate position every value between these two extremes occurs.

The colouring with MOHR's salt is positive for aragonite.

From the homogeneous extinction appears that the *Discoaster* consists of one homogeneous aragonite crystal. In the stellate *Discoasters* one of the directions of extinction always lies according to an arm. The arm, (in forms of an odd number of radians) or the arms (in forms with an even number of radians), lying in this direction, may be called the main arm(s).

In profile the *Discoasters* extinct parallelly.

From the values of elasticity of aragonite follows that the crystallographic *c*-axis is placed perpendicularly on the plane of the *Discoaster*, the *b*- and *a*-axis being placed in that plane and probably in this way, that *b* is in the direction of the main arm. (This observation could be only executed with little accuracy, as $b = \gamma = 1,686$; and $a = \beta = 1,682$ and $\gamma - \beta = 0,004$ is, given the minute thickness of the *Discoaster*, too small to be able to fix the positive or negative character with accuracy. The fact mentioned by HILL that in the same preparation the *Discoaster* shows two extremes of double refraction has its origin in the fact, that they are lying in different position towards the axis of the microscope.

If a *Discoaster* is rotated round an axis in the plane of the arms, then the interference-colours are seen to pass from one extreme to the other.

The organic origin of the Discoaster.

According to EHRENBURG and SORBY, as mentioned above, these proble-

¹⁾ TAN SIN HOK. Young-Tertiary Limestone of Rotti.

matica have nothing to do with organisms. SORBY's opinion (op. cit. p. 197) was that they are nothing but crystals, radially arranged round a fragment of a foraminiferal shell as a nucleus.

The hexaradiati often occurring in the Moluccas should make one suppose that they are nothing but sixfold twins of aragonite, consequently belonging to mineralogy.

From the behaviour between crossed nicols appears quite clearly, that we have to do with one homogeneous crystal. Further, the varying number of arms, the form of the separate arms, the convexo-concavely curved plane of the *Discoaster* gives conclusively proof of an organic origin.

An inorganic origin consequently may be considered out of question. That the *Discoaster* is constructed of aragonite is nothing peculiar, aragonite regularly occurs as building-stone of skeletons.

System of the Discoaster.

Fam. *Discoasteridae* incert. sed. nov. fam.

Actiniscus septenarius Ehrenberg, 1843. Monatsber. Ak. Wiss. Berl.

The rosette-shaped as well as the stellate bodies are considered as belonging to this family. They have in common their mineralogical composition of aragonite, the orientation of the axes of elasticity in the disks. By transitions the *Helio-discoasters* are connected with the *Eu-discoasters*, the *Eu-discoasters* with the *Hemidiscoasters*, so that it is motivated to attribute the rosette-shaped disks with the asterisks to the *Discoasteridae*.

As far as known, their diameter does not exceed 35μ . Three groups can be distinguished, which are looked upon as genera.

1. Genus: *Helio-discoaster* nov. gen.

To this are attributed all rosette-shaped *Discoasters*. The arms (sectors) are not separated: they fill the disk entirely. In the center a nucleus may be present.

In preparations of the Moluccas the *Helio-discoasters* are not frequent. In Barbados they occur in great numbers, the forms with nucleus being regular. These nuclei are thought to be caused, because in the center, where the arms meet, they are drawn into the direction of the disk axis¹⁾.

Forms in which this structure could be examined, have not been found in the Moluccas. The nuclei are small here, they are to be considered as rests of the nuclei of the *Helio-Discoaster* of Barbados, in many cases they are entirely absent.

The disk of the *Helio-discoaster* is flat, not curved convex-concavely as in the next two groups.

Distinguished are:

¹⁾ JUKES-BROWNE & HARRISON. Loc. cit. p. 178 fig. 4, 5 en 6.

Discoaster Barbadiensis nov. spec. typ.

EHRENBERG, Mikrogeologie 1854. Pl. XXIV — 67, and A.; Pl. XXV — 13, 14, 15, p. 155.

SORBY, Crystalloids of the Chalk, Ann. Mag. Nat. Hist. Vol. VIII, 1861, p. 197.

JUKES-BROWNE & HARRISON, Barbados II, Quart. Journ. Geol. Soc., London. Vol. XLVIII, 1892, p. 178, fig. 4, 5 and 6.

With 11—21 arms (sectors) filling the disk entirely. In the center the arms are drawn axially by which a nucleus originates. Size generally $15\ \mu$ and less.

Discoaster Barbadiensis var. *Bebalaini*, fig. 4.

Differs from the previous in the strongly reduced nucleus. As only a few specimen could be examined, it could not be traced, if *D. Barbadiensis* typ. was present. The variety *Bebalaini* is a transition to the

Discoaster Ehrenbergi nov. spec.

With 11—21 arms without nucleus. Named after the discoverer of these problematical bodies.

2. Genus: *Eu-Discoaster* nov. gen.

To this genus belong all stellate *Discoasters*, the arms of which can be distinguished from each other until the center of the disk.

They can be derived from the *Helio-Discoasters* without the least difficulty (comp. figs. 2, 3, 5, 6, etc.)

The number of arms varies from 5—8. The species are distinguished by the number of arms. The plane of the *Discoaster* is curved convexo-concavely, fig. 8a. With some a nucleus could be found in the center, analogical with that of *D. Barbadiensis*.

Discoaster Hilli nov. spec.

JUKES-BROWNE & HARRISON, Barbados II, p. 178, fig. 3.

With 8 cylindrical arms, subordinate in *Bebalaini*; a form which often occurs in Barbados.

Discoaster Brouweri nov. spec. typ, Figs. 8a, 8b.

With 6 cylindrical arms, predominant in the Moluccas. Figs. 8b is the *D. Brouweri* in profile.

D. Brouweri var. α Fig. 13.

The arms are thickened distally, not ramified.

D. Brouweri var. β Fig. 6.

The arms are distally not thickened, provided with a notch.

D. Brouweri var. γ Fig. 7.

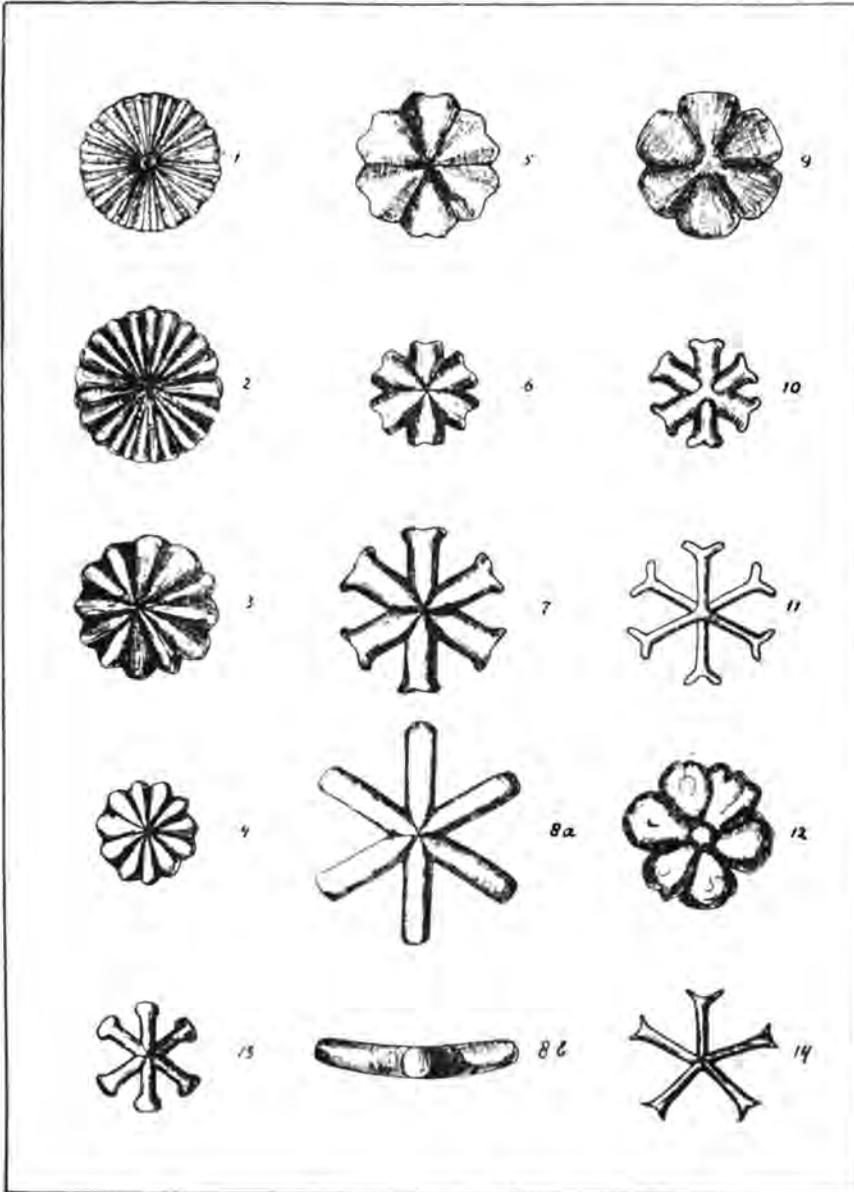
The arms are thickened distally and ramified.

D. Brouweri var. δ Fig. 5.

The arms are petaloid. Not frequent.

Discoaster pentaradiatus nov. spec. var. γ Fig. 14.

5 arms. The same varieties can be distinguished as mentioned above.



Enlargement: $2000 \times$ ¹⁾.

3. Genus; *Hemi-Discoaster* nov. gen.

Hereby those *Discoasters* are meant, where groups of arms have grown together without joints. In the preparations of the Moluccas only the hexa- and tri-radiati of this group have been found. Here the arms which form an interval of 120° , have grown two and two together.

¹⁾ As to the explanation, see page 419.

In connection with the joining of the arms the *Hemi-Discoasters* have less possibilities of variety in the number of arms. As to the varieties of the shape of the radii, they form a parallel series with the *Eu-Discoasters*.

They can be derived from the *Eu-Discoasters* (Fig. 10). The plane in which the arms are situated is curved convex-concavely.

Discoaster Molengraaffi nov. spec. typ.

MURRAY & RENARD, Deep-sea Deposits, Pl. XI — 4.

The arms have the same form as the *D. Brouweri* typ. The same varieties as mentioned above can be distinguished.

D. Molengraaffi and *D. Brouweri* are the predominant forms in the Moluccas. Of both species the variety δ is least frequent.

D. triradiatus nov. spec. typ.

With 3 cylindrical arms; α - and β -varieties are found. This species occurs sporadically.

Remark: The *Hemi-Discoasters* are easily to be derived from the *Eu-Discoasters* by the growing together of the 1st, 3^d and 5th arm. Transitions from the *Eu-* to the *Hemi-*joining are found, fig. 10.

As to the forms of the arms the *Discoasters* of the two genera form parallel series.

Comparison with the Discoasteridae of Barbados.

In Barbados the *Helio-Discoasters* are frequent, the forms with a nucleus are regular. Among the stellate *Discoasters* the 8-radiati are predominant, forms with a nucleus are not met with, the other species are less common. The *Discoasters* of Barbados are smaller on the whole, the largest is 15 μ .

In the Moluccas the stellate *Discoasters* are the predominant forms, generally they have the same size as those of Barbados; the hexa-radiati occur most frequently; forms with a nucleus are found, 8-radiati are subordinate. *Helio-Discoasters* are rare.

Remarkable is that relations are as good as complimentary.

Discoasteridae and Coccolithophoridae.

From the figures, appears that in some cases in outward appearance, a striking resemblance with the coccoliths may be found. *D. Ehrenbergi* (fig. 2) resembles strikingly a placolith (fig. 1), however, the axis-channel is missing and, besides, the structure of a coccolith is radially-fibered, so that in the latter, between crossed nicols an interference-cross becomes visible.

The *D. Molengraaffi* var. δ shows much resemblance to the "coccolith" figured by SCHMIDT¹⁾, found in the (. . . ?) sediments of Brindisi, which,

¹⁾ SCHMIDT. Coccolithen und Rhabdolithen, p. 675, Pl. I, fig. 10.

however, Prof. LOHMANN, is not willing to reckon among the *Coccolithophoridae*. (Private communication).

However great the resemblance, Prof. LOHMANN and Prof. SCHILLER think any relation to the *Coccolithophoridae* excluded.

The way of living of the Discoasteridae.

They are certainly marine organisms, until now the skeletons have only been found in marine deposits. From the small size of the *Discoasters* and the convexo-concave curving of the plane in which the arms are situated, some supposition may be derived, regarding size and form of these organisms.

They must have been microscopically small and may have a spherical form. Possibly, analogical with the coccoliths, the *Discoasters* act as covering of the cell.

The finding of these *Discoasters* exclusively in oceanic sediments or in sediments with oceanic character may be connected with a planctonic mode of living.

Geological age of the Discoasters.

Besides the finding of the *D. Barbadiensis* by SORBY in English Chalk, the stellate *Discoasters* have only been found in Young-Tertiary and Quaternary deposits, as far as the age of the deposits mentioned is known to me. Their composition of labile aragonite, their delicate forms make the finding in older sediments improbable.

LITERATURE.

1. BÖGGILD, O. B. Meeresgrundproben der Siboga-Expedition. Monographie XLV. Siboga-Expeditie, Leiden, 1916.
2. CAYEUX, L. Introduction à l'étude pétrographique des roches sédimentaires. Mém. Carte géol. France, Paris 1916.
3. EHRENBERG, C. G. Mikrogeologie, Leipzig, 1854.
4. HAUPT, O. Ein Kreide-ähnlicher, wahrscheinlich Jung-Tertiärer Mergel aus Kaiser Wilhelms-Land. (Deutsch Neu-Guinea). Zeitschr. Deutschen Geol. Ges. 57, 1906, p. 565.
5. JUKES-BROWNE, A. J. & HARRISON, J. B. The Geology of Barbados. Part. II. The Oceanic Deposits. Quart. Journ. Geol. Soc. London. 48, 1892. p. 170 et seq.
6. MURRAY, J. & RENARD, A. F. Report on Deep-Sea Deposits. Challenger-Expedition, London, 1891.
7. SCHLUMBERGER, CH. Note sur les Holothuridées du Calcaire Grossier Bull. Soc. Géol. France. Sér. III, Tome XVI, 1888, p. 347—441.
8. SCHLUMBERGER, CH. Seconde Note sur les Holothuridées fossiles du Calcaire Grossier. Ibid. Tome XVIII, 1890, p. 191—206.
9. SCHMIDT, O. Ueber Coccolithen und Rhabdolithen. Sitz. Ber. math. naturw. Cl. K. Akad. Wiss. Wien, Bd. LXII, Abt. I. 1870, p. 669.
10. SLUITER, C. PH. Die Holothurien der Siboga-Expedition. Monographie XLIV der Siboga-Expeditie.
11. SORBY, H. C. On the Origin of the so-called „Crystalloids” of the Chalk. Ann. & Mag. Natural History, Vol. VIII, Ser. 3, p. 193, London (1861).

12. TAN SIN HOK, On a Young-Tertiary Lime-Stone of the Isle of Rotti with Coccoliths, Calci- and Manganese-peroxyde Spherulites. Proc. Kon. Akad. Wetenschappen Amsterdam, Vol. 29, No. 8, 1926, p. 1095.

13. ZITTEL, K. A. VON, Handbuch der Palaeontologie. Bd. I, 1876—80, p. 560.

EXPLANATION: (Enlargement 2000 \times).

- Fig. 1. Placolith (*Coccolithophora leptophora* MURR. & BLACKM.)
 .. 2. Discoaster Ehrenbergi nov. spec.
 .. 3. " Ehrenbergi nov. spec.
 .. 4. " Barbadiensis nov. spec., var. Bebalaini.
 .. 5. " Brouweri nov. spec., var. δ transition from the Helio- to the Eu-
 discoaster.
 .. 6. " Brouweri nov. spec., var. β .
 .. 7. " Brouweri nov. spec. var. γ .
 .. 8a. " Brouweri nov. spec. typ.
 .. 8b. " Brouweri nov. spec. typ. in profile.
 .. 9. " Molengraaffi nov. spec., var. δ .
 .. 10. " Molengraaffi nov. spec. var. γ . transition from Eu- to Hemi-
 discoaster.
 .. 11. " Molengraaffi nov. spec., var. γ .
 .. 12. " "Coccolith" from Brindisi, SCHMIDT. op. cit. Pl. I, fig. 10.
 .. 13. " Brouweri nov. spec., var. α .
 .. 14. " pentaradiatus nov. spec., var. γ .

Delft, Laboratory for Historical Geology and Palaeontology.

February 1927.

Physiology. — *On some Properties of the Radiation-substances (with weak radiation).* By Prof. H. ZWAARDEMAKER.

(Communicated at the meeting of March 26, 1927).

In the November meeting of the Koninklijke Akademie van Wetenschappen the present writer demonstrated that radio-active radiation (radium rays of weak penetrating power, polonium rays) generates in the heart-muscle peculiar substances, which restore contractions of the heart that have ceased through deprivation of potassium ¹). Such an experiment succeeds best, when the organ is connected up in a small circulation of only 30 c.c. of liquid. This liquid (Ringer's potassium free solution) can be kept running by introducing air into a vertical portion of the course, along a small tube placed in a line with the direction of the stream. The air-bubbles carry the liquid upwards until it reaches the glass sphere open at the top, and can descend to the heart (eel-heart on glass canula in sinus or frog-heart on Kronecker canula. From the heart this liquid passes to an open receptacle, from which the liquid is again pumped up etc. ²).

When such a heart has first come to a standstill through Ringer's solution without potassium (occasionally automatic), i. e. without washing out $\frac{1}{3}$ of the potassium, we proceed to radiation from three sides, e. g. with radiofères of enamel 1.6 + 1.8 + 1.9 or plates covered with polonium of one square cm. ³). After a latent period of from $\frac{1}{2}$ to 1 hour the heart begins to beat. Gradually the radiation substances in question, of which we are going to describe the properties, will then appear in the liquid.

In a second circulation such a liquid can also set another heart beating that has beforehand been treated in the same way, thanks to the presence of the radiation substance. The experiment may, however, also be carried out without resorting to a renewed circulation of the old liquid, by connecting a second heart, prepared in the same manner, in series with the first organ in the old circulation. Then the same liquid first perfuses the radiated heart, and subsequently a non-radiated one. The result is that after the radiated heart has begun to pulsate, the second non-radiated organ will also begin. The two hearts can be made to record the curve on one and the same smoked surface.

¹) H. ZWAARDEMAKER, Kon. Akad. v. Wet., Amsterdam, 27 Nov. 1926.

²) NOYONS et BELEHRADEK, C. R. Soc. de biol. T. 88, p. 621, 1920.

³) H. ZWAARDEMAKER, Pflüger's Archiv Bd. 213, p. 757, 1926; *ibid.* Bd. 215, p. 460, 1927.

Such a graph is represented in our figure. The bottom curve registers the pulsation of the radiated heart (enamel radiofères), the top one that of the non-radiated organ. Initially the rhythm of the former heart, which commenced earlier, is quicker than that of the latter, but this changes as the experiment advances and the pulsations, not evoked by direct radiation, become better. Thus things go on for hours. All the while, besides radiation-substances some potassium-ions, also appear in the circulating fluid. These ions issue from the so-called fixed potassium of the muscles. Obviously their quantity is only small, so that they cannot of themselves evoke pulsations ¹⁾. The latent period is of interest. It is essential to the effect of the radiation, but also to that of the radiation-substance. It stands to reason that when the hearts are equally sensitive, the effect of the radiation substance follows that of the radiation, since

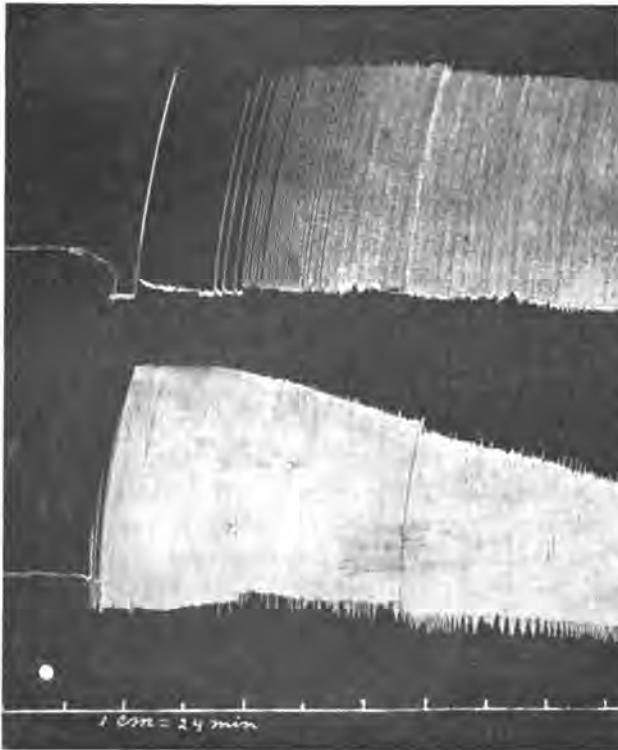


Fig. 1.

2 eel-hearts in one and the same circulation of 30 cc. of liquid.

Bottom-curve : the organ under direct radiation.

Top-curve : the organ under indirect radiation. At the site of the speck radiation was started. (1 scale mark of the time-line represents 24 min.).

¹⁾ These potassium-ions can emanate only from the muscles. Measured by the standard of the frog-tissue the two hearts contain 2 pro mille, or in 200 mgrs of muscle 0.4 mg K. Of this quantity only 0.12 mg can pass into the fluid.

this substance appears first in the heart subjected to direct radiation, subsequently in the liquid and finally in the second heart.

It is no trouble next morning to examine in every direction the liquid, which has been kept in circulation a whole night with the results illustrated above for both hearts.

Thanks to the valuable aid of Mr. Ph. ARONS, we were able in a few months to observe the properties of a large number of similar perfusion fluids. It appeared that the radiation-substance present in these fluids is

dialysable,
ultrafilterable,
thermostabile,
adsorbable to carbon and magnesium-silicate,
soluble in alcohol.

This is the case with the radiation-substance originating under the influence of polonium-radiation, as well as with the substance coming forth under the influence of radium-radiation.

Moreover the said radio-substances do not appreciably lower the surface-tension of water (dynamic measurement)¹⁾. Neither do they yield an adsorption-band in the ultraviolet, which other similar substances do.

With intenser and quicker radiation other radiation-substances will presumably develop, whose effect we suspected in the radiated heart.

Consequently this heart very often does not beat so well as the heart subjected to indirect radiation.

The long latency (48 minutes for direct radiation, 20 minutes after the addition of the radiation-substance in the perfusion-fluid) renders it probable that we have to do here with substances acting catalytically. They are probably identical with the „substances actives“ previously described by J. DEMOOR²⁾, resp. the heart hormones of L. HABERLANDT³⁾ in normal life. As we have to differentiate two independent radiation-substances, the one produced by the α -rays, and the other by the β -rays, the catalytic substances, concerned here, have in this paper always been spoken of in the plural.

¹⁾ R. BRINKMAN, Abderhalden's Hdb. d. Biol. Arbeitsmethoden, Abt. IV, Teil 4, Heft 6, 1927.

²⁾ J. DEMOOR, Arch. int. de Physiol. t. 20, p. 29, 446, 1922; t. 23, p. 121, 1924; Bruxelles méd. 13 Sept. 1925.

³⁾ L. HABERLANDT, Klin. Wochenschr. 1924, N^o. 36, Das Hormon der Herzbewegung, Berlin—Wien 1927.

Mathematics. — *The Definition of Euclidean Measure in the Projective Plane.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of March 26, 1927).

Euclidean metrics in the plane are fixed by choosing 2 conjugate complex points I_1 and I_2 , the isotropic points.

The projective transformations that leave these points invariant, form the group of similitude. The definition of *angular measure* rests immediately on the projective invariant par excellence, the *anharmonic ratio*. For, if A , B and C are three points of the plane, according to the formula of LAGUERRE:

$$\angle BAC = \frac{1}{2i} \log A(I_1 I_2 BC).$$

where $A(I_1 I_2 BC)$ is the anharmonic ratio of the rays AI_1 , AI_2 , AB and AC .

The definition of *distance* is less simple. It is often given as the limit of the expression which holds good in non-euclidean geometry, where angle and distance are entirely dualistic and the latter is likewise defined by the aid of an anharmonic ratio.

In what follows we shall give a direct definition of distance that is entirely based on anharmonic ratios.

We must notice that by choosing I_1 and I_2 only *ratios* of distances will be defined. For the definition of ratios of distances it is further sufficient to restrict ourselves to the comparison of segments that have a common extremity as after introducing the straight line at infinity, we can already ascertain the equality of parallel line segments.

If \overline{AB} and \overline{AC} represent the lengths of the segments we give the following definition:

$$\overline{AB}^2 : \overline{AC}^2 = I_1(I_2 ABC) \cdot I_2(I_1 ABC).$$

The square of the ratio of the distances is the product of two anharmonic ratios.

In order to prove that this definition agrees with the usual definition of length we choose the homogeneous coordinates of I_1 and I_2 resp. $(1, i, 0)$ and $(1, -i, 0)$, and the Cartesian coordinates of A , B and C resp. (a_1, a_2) , (b_1, b_2) , (c_1, c_2) . The four lines $I_1 I_2$, $I_1 A$, $I_1 B$, $I_1 C$ cut the Y -axis at four points of which the Y -coordinates are resp.:

$$\infty, (a_2 - ia_1), (b_2 - ib_1), (c_2 - ic_1),$$

so that

$$I_1(I_2 ABC) = \frac{(b_2 - ib_1) - (a_2 - ia_1)}{(c_2 - ic_1) - (a_2 - ia_1)} = \frac{(b_2 - a_2) - i(b_1 - a_1)}{(c_2 - a_2) - i(c_1 - a_1)},$$

and for $I_2(I_1 ABC)$ the conjugate complex value is found.

We find, therefore, in fact:

$$I_1(I_2 ABC) \cdot I_2(I_1 ABC) = \frac{(b_1 - a_1)^2 + (b_2 - a_2)^2}{(c_1 - a_1)^2 + (c_2 - a_2)^2} = \frac{\overline{AB}^2}{AC^2}.$$

The sides and the angles of a triangle ABC are now defined in a similar way, viz. as functions of the anharmonic ratios of the four rays drawn out of each of the five points I_1, I_2, A, B, C to each of the other four. These anharmonic ratios are the most simple projective invariants of the quintuple. They are for the rest equal to the anharmonic ratios of any four of the points of the conic which can be passed through the quintuple, which, speaking metrically, is the circumscribed circle about the triangle A, B, C . Not every function of the anharmonic ratios is an invariant of the triangle; it is necessary that I_1 and I_2 appear symmetrically.

The definitions of angle and distance agree still more if we write the anharmonic ratio that appears in the angular measure, likewise as the product of two anharmonic ratios. So e.g.:

$$A(I_1 I_2 BC) = B(I_2 I_1 CA) \cdot C(I_2 I_1 AB).$$

The given definitions may be used to prove simple theorems of Euclidean geometry by the aid of projectivity.

Mathematics. — *Ueber das Verhältnis separabler Räume zu kompakten Räumen* ¹⁾. By Dr. W. HUREWICZ. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of March 26, 1927).

Sei R ein separabler metrisierbarer topologischer Raum, ein Raum also, der laut einem Theorem von URYSOHN ²⁾ als Teilmenge eines kompakten metrisierbaren Raumes betrachtet werden kann. Es entsteht die Frage, ob es unter den R enthaltenden kompakten metrisierbaren Räumen stets auch solche gibt, deren Dimension mit der Dimension von R übereinstimmt. Wir wollen zeigen, dass dies tatsächlich der Fall ist. Es gilt m. a. W.:

Theorem. *Jeder separable metrische Raum R ist homöomorph mit einer Teilmenge eines kompakten metrischen Raumes, welcher dieselbe Dimension besitzt wie R .* ³⁾

Den Beweis stützen wir auf die folgende Verallgemeinerung eines bekannten dimensionstheoretischen Satzes von MENGER und URYSOHN ⁴⁾:

I. Ist R ein separabler ⁵⁾ Raum von der endlichen Dimension n , und \mathfrak{S} ein endliches System offener Mengen G_1, G_2, \dots, G_p , deren Summe R ist, so gibt es ein System abgeschlossener Mengen F_1, F_2, \dots, F_q mit den folgenden Eigenschaften:

1. Jeder Punkt von R liegt in mindestens einer und in höchstens $n+1$ Mengen F_i ;
2. Jedes F_i ist einer Menge G_k enthalten, ⁶⁾

¹⁾ Eine ausführliche Darstellung wird in den Math. Am. erscheinen.

²⁾ Vgl. Math. Ann. 92, S. 303.

³⁾ In diesem Theorem ist auch das Ergebnis von TUMARKIN (vgl. diese Proceedings 28, S. 996) enthalten, wonach jeder separable metrische Raum mit einer Teilmenge eines dieselbe Dimension besitzenden vollständigen Raumes homöomorph ist.

⁴⁾ Nämlich, des Satzes, dass ein kompakter n -dimensionaler Raum für jede positive Zahl ε in endlich viele abgeschlossene Mengen mit Durchmessern $< \varepsilon$ zerlegt werden kann, die zu je $n+2$ leere Durchschnitte haben. Vgl. MENGER, Monatshefte f. Math. u. Phys. 34, S. 153. URYSOHN, Fund. Math. 8, S. 292.

⁵⁾ Es werden hier, wie auch überall im Folgenden, ausschliesslich *metrische* bzw. metrisierbare Räume betrachtet.

⁶⁾ Falls R kompakt ist, gibt es bei vorgegebenen Mengen G_i , wie leicht ersichtlich, eine Zahl $\varepsilon > 0$, derart, dass jede Menge mit Durchmesser $< \varepsilon$ in einer der Mengen G_i enthalten ist. Für den Fall kompakter Räume ist daher Satz I mit dem sub ⁴⁾ angeführten Satz von MENGER und URYSOHN äquivalent.

Zum Beweise ist es bequem Satz I. in diese Gestalt zu bringen:

I'. Ist die in einem separablen Raum gelegene n -dimensionale Menge M in der Summe der endlich vielen offenen Mengen G_1, G_2, \dots, G_p enthalten, so kann M mit endlich vielen offenen Mengen U_1, U_2, \dots, U_q überdeckt werden, deren jede in einem G_i enthalten ist, und die zu je $n+2$ leere Durchschnitte haben.

Aus *I'*. folgt sofort *I.*: Man bilde nämlich zu den Mengen U_i von Satz *I'*. ein System von q in M abgeschlossenen Mengen F_i , so dass $F_i \subset U_i$ ($i=1, 2, \dots, q$) und $M = \sum_{i=1}^q F_i$,⁷⁾ so befriedigen die Mengen F_i die Forderungen des Satzes *I* in Bezug auf das System der in M offenen Mengen G_i . (Ebenso einfach kann man zeigen, dass umgekehrt aus *I.* *I'*. folgt).

Wir beweisen die Behauptung *I'*. zuerst für $n=0$. Zu jedem Punkt von M gibt es dann eine Umgebung, deren Begrenzung zu M fremd ist, und die in einer der p Mengen G_i enthalten ist. M wird (mit Rücksicht auf die Separabilität des Raumes) von abzählbar vielen derartigen Umgebungen V_1, V_2, V_3, \dots überdeckt. Setzen wir $W_1 = V_1$ und $W_m = V_m - (\bar{V}_1 + \bar{V}_2 + \dots + \bar{V}_{m-1})$ ($m=2, 3, \dots$) und bezeichnen wir mit U_i ($i=1, 2, \dots, p$) die Summe aller derjenigen W_m , die in G_i , aber in keinem G_k für $k < i$ enthalten sind, so genügen die p offenen Mengen U_i , wie man leicht sieht, den Forderungen des Satzes *I'*, der somit für den Fall $n=0$ bewiesen ist.

Nehmen wir jetzt an, Satz *I'* sei richtig, wenn man in ihm die Zahl n durch eine kleinere ersetzt, und zerspalten wir die gegebene n -dimensionale Menge M in einen $(n-1)$ -dimensionalen Teil ^{n-1}M und einen 0 -dimensionalen Teil 0M .⁸⁾ Nach Annahme lässt sich ^{n-1}M (bzw. 0M) mit endlich vielen offenen Mengen U_1, U_2, \dots, U_q (bzw. U'_1, U'_2, \dots, U'_r) überdecken, die Teilmengen der G_i sind und zu je $n+1$ (bzw. zu je zwei) leere Durchschnitte haben. Die $q+r$ Mengen U_i und U'_i bilden zusammen ein System von den im Satz *I'* verlangten Eigenschaften.

Sei jetzt R separabler Raum von der endlichen Dimension n ⁹⁾ den wir auf eine Teilmenge eines n -dimensionalen kompakten Raumes topologisch abbilden wollen. Wir setzen voraus (auf Grund des erwähnten URYSOHN'schen Theorems), R sei bereits als Teilmenge eines kompakten Raumes (von einer unbekanntenen Dimension) gegeben. Dann ist bekanntlich R für jedes $\varepsilon > 0$ Summe von endlich vielen in R offenen Mengen mit Durchmessern $< \varepsilon$ ¹⁰⁾.

Wir betrachten eine beliebige gegen 0 konvergierende Folge positiver Zahlen ε_m und bestimmen für jede natürliche Zahl m ein System abgeschlossener Mengen

$$\mathfrak{C}_m = \{F_1^m, F_2^m, \dots, F_{km}^m\}$$

gemäss den folgenden Bedingungen:

1. Es gilt $R = \sum_{i=1}^{km} F_i^m$, ($m=1, 2, \dots$) und jeder Punkt von R gehört (bei festem m) höchstens $n+1$ Mengen F_i^m an.

⁷⁾ In jedem metrischen Raum gilt der Satz: Ist M eine beliebige Menge und G_1, G_2, \dots, G_p offene Mengen, so dass $M \subset \sum_{i=1}^p G_i$, so gibt es p in M abgeschlossene Mengen F_i ($i=1, 2, \dots, p$) mit den Eigenschaften: $F_i \subset G_i$; $M = \sum_{i=1}^p F_i$.

⁸⁾ Vgl. HUREWICZ, Math. Ann. 96, S. 761, sowie TUMARKIN, a.a.O.

⁹⁾ Ist R ∞ -dimensional, so hat jeder R enthaltende topologische Raum dieselbe Dimension wie R .

¹⁰⁾ Vgl. HAUSDORFF, Grundzüge der Mengenlehre (Leipzig, 1914), S. 313—315.

2. Die Durchmesser aller F_i^m sind $< \varepsilon_m$.

3. Ist (für $m \geq 2$) \mathfrak{S}'_{m-1} das Vereinigungssystem der Systeme $\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_{m-1}$ (d. h. das System aller Mengen, die einem der Systeme \mathfrak{S}_i ($i < m$) angehören), und sind A_1, A_2, \dots, A_r irgend welche Mengen aus \mathfrak{S}'_{m-1} , die keinen gemeinsamen Punkt besitzen, so hat keine Menge des Systems \mathfrak{S}_m mit sämtlichen A_i Punkte gemein.

Dass man die Systeme \mathfrak{S}_m in Einklang mit diesen Bedingungen wirklich bestimmen kann, ist eine Folgerung aus Satz I. Sei nämlich m eine natürliche Zahl; denken wir uns die Systeme $\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_{m-1}$ und folglich aus das System \mathfrak{S}'_{m-1} bereits definiert, und bezeichnen wir für jeden Punkt p von R mit $A_{m-1}(p)$ die Summe aller Mengen des Systems \mathfrak{S}'_{m-1} , die p nicht enthalten. (Die Mengen $A_{m-1}(p)$ bilden natürlich ein *endliches* System). Sodann zerlegen wir R in endlich viele offene Mengen $U_1^m, U_2^m, \dots, U_{k_m}^m$ mit Durchmessern $< \varepsilon_m$ und bestimmen auf Grund von I. die Mengen F_i^m in der Weise, dass die Bedingung 1 besteht und dass jedes F_i^m in einer der endlich vielen offenen Mengen $U_k^m (R - A_{m-1}(p))$ (die in ihrer Gesamtheit R überdecken), enthalten sei. Dann sind, wie man sofort bestätigt, auch die Bedingungen 2 und 3 erfüllt.

Setzen wir der Einfachheit halber $F_i^m = 0$ für $i > k_m$, und bezeichnen wir für jedes System α aus endlich vielen verschiedenen natürlichen Zahlen m_1, m_2, \dots, m_r mit F_α^i den Durchschnitt $F_{m_1}^i \cdot F_{m_2}^i \cdot \dots \cdot F_{m_r}^i$. Eine unendliche Folge:

$$\alpha_1, \alpha_2, \alpha_3, \dots$$

von Systemen natürlicher Zahlen nennen wir eine *Kette*¹¹⁾, wenn folgendes gilt:

(a) Für jedes m ist

$$\prod_{i=1}^m F_{\alpha_i}^i \neq 0;$$

(b) Die Folge $\{\alpha_i\}$ verliert die Eigenschaft (a), sobald irgend eines ihrer Glieder α_i durch ein System natürlicher Zahlen α'_i ersetzt wird, in dem α_i als echtes Teilsystem enthalten ist. Es gilt m.a.W. für jede nat. Zahl p , die kein Element ist von α_i :

$$(x) \quad F_p^i \cdot \prod_{j=1}^m F_{\alpha_j}^j = 0$$

für mindestens einen Index m .

Aus 3. ergibt sich leicht:

(') Sind $\{\alpha_i\}$ und $\{\beta_i\}$ zwei *verschiedene* Ketten, (d. h. ist für mindestens einen Index i α_i von β_i verschieden), so sind die Systeme α_i und β_i für *alle hinreichend grosse* i elementenfremd.

In der Tat sei p eine natürliche Zahl, die in β_i , aber nicht in α_i vorkommt. Dann ist die Gleichung (x) für einen Index m erfüllt. Ist nun m' grösser als die beiden Zahlen i und m , so gibt es nach 3) keine Menge $F_p^{m'}$, die mit den beiden Mengen

$$(o) \quad F_p^i \cdot \prod_{j=1}^m F_{\alpha_j}^j$$

¹¹⁾ Vgl. ALEXANDROFF, Math. Ann. 96, S. 489. Die folgenden Ueberlegungen lehnen sich eng an den Gedankengang dieser ALEXANDROFF'schen Abhandlung an.

Punkte gemein hat. Ist aber q ein Element von $\beta_{m'}$ (bzw. von $\alpha_{m'}$), so hat $F_q^{m'}$, per Definitionseigenschaft (a) der Ketten zufolge, mit der ersten (bzw. mit der zweiten) der Mengen ⁽⁹⁾ Punkte gemein. Folglich haben die Systeme $\alpha_{m'}$ und $\beta_{m'}$ keine gemeinsame Elemente.

Sei jetzt $\{a_i\}$ eine beliebige Kette, m eine natürliche Zahl. Die Gesamtheit aller Ketten $\{\beta_i\}$, für die $\beta_1, \beta_2, \dots, \beta_m$ Untersysteme sind von a_1 , bzw. a_2, \dots , bzw. a_m , nennen wir die m -te *Umgebung* der Kette $\{a_i\}$. Durch diese Festsetzung wird, wie sich auf Grund von (') bei Berufung auf einen Satz von ALEXANDROFF ergibt ¹²⁾, ein *kompakter metrisierbarer topologischer R^* von der Dimension n* definiert, dessen Elementevorrat aus sämtlichen Ketten besteht.

Wir wollen jetzt eine topologische Abbildung zwischen R und einem Teil des soeben definierten Raumes R^* herstellen. Zu diesem Zwecke betrachten wir einen Punkt x von R und bezeichnen für jedes m mit $\alpha_m(x)$ das System aller natürlicher Zahlen n_i für die der Punkt x in F_{n_i} enthalten ist. Die Folge $\{\alpha_i(x)\}$ ist, wie man leicht einsieht, eine Kette.

Die Bedingung (a) ist nämlich in trivialer Weise erfüllt. Die Bedingung (b) folgt daraus, dass die Mengen $F_{\alpha_m}^m$ (wo $\alpha_m = \alpha_m(x)$ gesetzt ist) mit wachsendem m nach 1) unendlich klein werden und sich daher auf den Punkt x zusammenziehen. Ist also die Zahl p kein Element von α_i , d.h. liegt x in der offenen Menge $R - F_p^i$, so gilt bei hinreichend grossem m : $F_{\alpha_m}^m \subset R - F_p^i$, somit $F_{\alpha_m}^m \cap F_p^i = 0$.

Jedem Punkt von R ist also eine Kette zugeordnet. D.h. es liegt eine Abbildung des Raumes R auf einen Teil des höchstens n -dimensionalen kompakten Raumes R^* vor, und man zeigt leicht, dass diese Abbildung umkehrbar eindeutig und beiderseits stetig ist ¹³⁾. Damit ist der Beweis unseres Theorems erbracht.

Aus dem Bewiesenen ergeben sich vor allem wichtige Folgerungen hinsichtlich der MENGER'schen grundlegenden Untersuchungen über das

¹²⁾ Vgl. ALEXANDROFF, a.a.O. Um an die ALEXANDROFF'sche Terminologie anzuknüpfen, müssen wir ein System α natürlicher Zahlen mit der Eigenschaft $F_\alpha^m \neq 0$ als einen *Simplex des m -ten Komplexes* bezeichnen (Für jedes m besteht der m -te Komplex aus einer nur endlichen Anzahl von Simplizes) und unter einer *Gruppe* eine endliche Folge

$$[\alpha_1, \alpha_2, \dots, \alpha_m]$$

verstehen, wo α_i ein Simplex des i -ten Komplexes ist ($i = 1, 2, \dots, m$), und $\prod_{i=1}^m F_{\alpha_i}^i \neq 0$ gilt. (Eine Kette ist demnach in Übereinstimmung mit der im Text gegebenen Definition eine unendliche Folge $\alpha_1, \alpha_2, \alpha_3, \dots$ deren jeder Abschnitt eine Gruppe ist, die aber diese Eigenschaft verliert, wenn man für irgend ein m α_m durch einen umfassenderen Simplex des m -ten Komplexes ersetzt). Es liegt dann ein *höchstens n -dimensionales approximierendes Spektrum* vor (Vgl. ALEXANDROFF, a.a.O., S. 493): *höchstens n -dimensional*, weil jeder Simplex infolge von 1) aus höchstens $n + 1$ Elementen besteht; *approximierend* — mit Rücksicht auf ('). Der durch unser Spektrum "approximierte" Raum ist der im Text definierte Raum R^* . Mit Hilfe eines URYSOHN'schen Satzes (Vgl. unten Fussnote ¹⁸⁾) beweist ALEXANDROFF, dass R^* kompakt, metrisierbar und höchstens n -dimensional ist.

¹³⁾ Daraus folgt natürlich sofort, dass die Dimension von R^* *genau* n ist.

Verhältnis von allgemeinen Räumen zu den Euklidischen Zahlenräumen. Aus dem MENGER'schen Ergebniss, dass jeder ein-dimensionale kompakte Raum mit einer Teilmenge des drei-dimensionalen Euklidischen Raumes homöomorph ist ¹⁴⁾, folgt bei Heranziehung des hier bewiesenen Theorems, dass (wie dies auch von MENGER vermutet wurde ¹⁵⁾) allgemeiner jeder ein-dimensionale *separable* Raum (insbesondere also jede eindimensionale Teilmenge eines Euklidischen Raumes von beliebiger Dimension) mit einer Teilmenge des Euklidischen R_3 homöomorph ist. Ebenso lässt sich der MENGER'sche Fundamentalsatz ¹⁶⁾, wonach jeder endlichdimensionale kompakte Raum auf eine Teilmenge eines Euklidischen Raumes topologisch abbildbar ist, ohne weiteres auf separable Räume übertragen ¹⁷⁾.

Eine andere Folgerung ergibt sich aus der folgenden Bemerkung: Die Voraussetzung, dass R n -dimensional ist, ist in den Beweis unseres Theorems nur durch die Vermittlung der Tatsache eingegangen, dass R die Behauptung des Satzes I erfüllt. Jeder Raum, welcher dieser letzten Bedingung genügt, ist also mit einer Teilmenge eines höchstens n -dimensionalen kompakten Raumes homöomorph und a fortiori selbst höchstens n -dimensional. Es gilt somit neben Satz I auch die Umkehrung:

II. Ist n eine ganze nicht negative Zahl, — R ein separabler Raum, und gibt es zu jedem endlichen System \mathfrak{S} offener Mengen, die R überdecken, eine Zerlegung von R in endlich viele abgeschlossene Mengen, deren jede in einer Menge aus \mathfrak{S} enthalten ist und die zu je $n+2$ leere Durchschnitte haben, so ist R höchstens n -dimensional.

Dieser Satz ist für kompakte Räume mit einem Ergebniss von URYSOHN ¹⁸⁾ äquivalent und ist ebenso wie das letztere zahlreicher Anwendungen fähig.

Das Hauptergebnis dieser Arbeit gestattet eine wesentliche Verallgemeinerung: Es stellt sich heraus, dass man *abzählbar viele* separable Mengen, die in ihrer Summe abgeschlossen sind, *simultan* in kompakte Räume von entsprechenden Dimensionen einbetten kann. Es gilt genau gesprochen:

Seien $M_1, M_2, M_3 \dots$ abzählbar viele in einem separablen Raum R gelegene abgeschlossene Mengen. Als topologischer Raum betrachtet, lässt sich dann R zu einem kompakten metrisierbaren Raum R^* erweitern,

¹⁴⁾ Vgl. MENGER, diese Proceedings 29, S. 476.

¹⁵⁾ Vgl. MENGER, a.a.O., S. 482.

¹⁶⁾ Vgl. MENGER, a.a.O., S. 482 sowie Jahresbericht d. D. Math.-Ver. 35, S. 149.

¹⁷⁾ Von MENGER wurde dies vermutet. Vgl. Jahresbericht d. D. Math.-Ver. 35, S. 149, Fussnote 4).

¹⁸⁾ Mit dem Satze nämlich, dass ein kompakter Raum, der für jedes $\varepsilon > 0$ in endlich viele abgeschlossene Mengen mit Durchmesser $< \varepsilon$ zerlegbar ist, die zu je $n+2$ fremd sind, eine Dimension $\leq n$ besitzt. Vgl. URYSOHN, Fund. Math. 8, S. 294. Dieser Satz ist implicite in den Beweis des Haupttheorems dieser Arbeit eingegangen, da er die Grundlage für den ALEXANDROFF'schen Satz bildet, auf den wir uns berufen haben (Vgl. Fussnote ¹²⁾).

so dass, wenn \overline{M}_n die abgeschlossene Hülle von M_n in R^* bedeutet, die Dimension von \overline{M}_n für jedes n mit der Dimension von M_n übereinstimmt.

Der Gedankengang des Beweises bleibt in wesentlichen derselbe wie früher. Nur wird an Stelle von Satz I der folgende Satz benützt:

In einem separablen Raum R sei eine Folge von abgeschlossenen Mengen M_1, M_2, M_3, \dots gegeben. Zu jeder Zerlegung von R in endlich viele offene Mengen G_1, G_2, \dots, G_n gibt es dann eine Zerlegung von R in endlich viele abgeschlossene Mengen F_1, F_2, \dots, F_q , die Teilmengen der G_k sind, wobei jeder Punkt von M_i ($i = 1, 2, \dots$) in höchstens $n_i + 1$ Mengen F_k liegt, wenn n_i die Dimension von M_i bedeutet.

Es entsteht die Frage, ob ein separabler Raum R stets, in der Weise zu einem kompakten (metrisierbaren) Raume R^* erweitert werden kann, dass jede in R abgeschlossene Menge M ihre Dimension nach der Abschliessung in R^* bewahrt?

Betrachtet man anstatt der Gesamtdimension des Raumes seine Dimension in den einzelnen Punkten, so gilt das folgende Theorem:

Jeder separable metrisierbare topologische Raum R ist Teilmenge eines kompakten metrisierbaren Raumes R^* , so dass in jedem Punkte x von R gilt:

$$\dim_x R = \dim_x R^* \text{ }^{19)},$$

(unter $\dim_x R$, bzw. $\dim_x R^*$ die Dimension von R bzw. von R^* im Punkte x verstanden.

Beim Beweise geht man folgendermassen vor: Zuerst zeigt man leicht mit Hilfe des verallgemeinerten BOREL'schen Theorems: Es existiert ein System \mathfrak{S} von abzählbar vielen in R offenen Mengen, so dass sich auf jeden Punkt von R eine Folge von Mengen aus zusammenzieht, deren Begrenzungen (in R) $(n(x)-1)$ -dimensional sind, wo $n(x)$ die Dimension von R in x bedeutet. Den Raum R^* wird jetzt so bestimmt, dass jede beliebige Menge U aus \mathfrak{S} Durchschnitt ist von R mit einer in R^* offenen Menge U^* , deren Begrenzung in R^* mindestens die gleiche Dimension hat, wie die Begrenzung von U in R .

Auch an diesen Satz knüpft sich eine Fragestellung an: Bleibt die Behauptung richtig, wenn man verlangt, dass R^* in allen seinen Punkten (nicht nur in den Punkten von R) dieselbe Dimension besitze wie R ?

¹⁹⁾ Man kann noch zeigen, dass es einen R enthaltenden kompakten Raum gibt, der die soeben formulierte Bedingung erfüllt und überdies dieselbe Gesamtdimension besitzt wie R .

Physiology. — *An Experimental Study of the Cerebellar Control of the Vocal Organs.* By LAURETTA BENDER. (Communicated by Prof. G. VAN RIJNBERK and Prof. B. BROUWER).

This paper will be published in a following number of these Proceedings.

ERRATUM

In Prof. BORTOLOTTI's note, p. 218, lines 9 and 10 from the top to omit:

"two of which (inversely like, corresponding to the two opposite signs of γ_{ijl} or $T_{\lambda\mu\nu}$) are . . ."

and to read:

"each of which — when the connexion is done — is . . ."

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM

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VOLUME XXX

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Physics. — *The Specific Heats of Solid Substances at the Temperatures Attainable with the Help of Liquid Helium. I. Measurements of the Atomic Heat of Lead.* By Prof. W. H. KEESOM and Dr. DONALD H. ANDREWS ¹⁾. (Communication No. 185a from the Physical Laboratory, Leiden.

(Communicated at the meeting of January 29, 1927).

§ 1. *Introduction.* In the range of temperatures which can be reached with the help of liquid helium there have been scarcely any measurements of the specific heats of solids. KAMERLINGH ONNES and HOLST ²⁾ published a few preliminary measurements of the specific heat of mercury, which were made in order to find out whether a discontinuous change in the specific heat occurs at the temperature at which mercury becomes super-conductive. KAMERLINGH ONNES and HOLST concluded from these measurements, which however were not very exact, that no such discontinuity occurs.

In the present communication there are discussed some measurements of the specific heat of lead between the temperatures 2° K. and 20° K.

We selected lead as the initial substance for the study of specific heat at extremely low temperatures, primarily because lead belongs to those elements, whose specific heat falls off least rapidly with the lowering of the temperature, thus enhancing the accuracy of observation; besides this it seemed desirable because some exact observations ³⁾ of the specific heat of lead had been made in the region of temperature just above the range which we planned to investigate.

Particularly at the lowest temperatures, our measurements do not have the accuracy which we hoped they would and we plan to repeat them and continue them; but in the meantime by the departure of one of us it seems desirable to publish the results which we have already secured.

§ 2. *Method.* We employed the method of NERNST and EUCKEN for determining specific heat, in essentials in the way already followed by

¹⁾ National Research Fellow (U. S. A.) with a fellowship of the International Education Board.

²⁾ These Proc. 17, 760, 1914—1915; Comm. Leiden N^o. 142c.

³⁾ W. H. KEESOM and H. KAMERLINGH ONNES. These Proc. 17, 894, 1914—1915; Comm. Leiden N^o. 143. Cf. also F. SIMON, Zs. physik. chem. 110, 572, 1924.

KEESOM and KAMERLINGH ONNES¹⁾ for determining specific heats at the temperatures which can be reached with the help of liquid hydrogen.

For these experiments there was constructed a combination unit for measuring temperature and for heating (see Fig. 2, Comm. N^o. 143), consisting of two concentric cylinders of copper, the outer being threaded on the outside. On the inner cylinder there was wound bifilarly a coil of insulated constantan wire, and the remaining space between the two cylinders was filled with amalgam as described in Comm. No. 143.

This constantan coil served for the heating as well as for measuring temperature. It consisted of three parts, one of one ohm, one of one hundred ohms and one of ten ohms. The wires leading to the coil were so arranged that any one of these parts could be used at will. In these experiments only the 100 ohm coil was used.

The measurements of resistance for determining the course of temperature before and after heating were made with the help of a potentiometer. During the heating this was disconnected.

The heat imparted was calculated from the mean resistance of the heating coil, from the current strength as read on a milliammeter, and from the time as adjusted by the switching device described in Comm. No. 143, Fig. 7. The switch was actually controlled by an electric impulse sent out by the clock as was the case in the measurements of Comm. No. 143. For a control the time was read from a watch.

The current used for measuring temperature was about 0.5 mA. and that for heating from 10 to 50 mA. The heat from the measuring current was taken into consideration as was necessary.

The vacuum within the calorimeter jacket was secured and maintained by a three stage diffusion pump from LEYBOLD.

§ 3. *Accuracy of the Measurements.* a. A factor of uncertainty is introduced in that the resistance of the constantan heating coil during heating is taken to be the mean of the resistances before and after heating. Since there is but a very small change in the resistance of the constantan for a rise even of ten degrees, the error due to the temporary rise in temperature during heating cannot exceed some few percents.

The temperature-time curve gave no reason for concluding that the temperature of the wire during heating rose very high above that of the surroundings, moreover our results at higher temperatures are in accord with those of KEESOM and KAMERLINGH ONNES, indicating that the above assumption is permissible within the calculated limit of error.

Considering that at lower temperatures much smaller currents were used for heating, it does not appear likely that this sort of error could have been greater in that region.

¹⁾ These Proc. 17, 894, 1914—1915; Comm. Leiden N^o. 143. Also These Proc. 18, 484, 1915—1916; Comm. Leiden N^o. 147a.

b. The apparent decrease in the resistance immediately after heating at the lowest temperatures (see § 5) makes the measurements below 4° K. somewhat uncertain. Above 4° K. this cause of uncertainty does not appear to be significant.

c. The measurements of the heat capacity of the core are less accurate apparently because its heat capacity is so small. For just this reason however, the core has a relatively unimportant part of the total heat capacity in the measurements of the specific heat of lead.

d. A number of measurements is uncertain because the course of the temperature after heating was not observed over a long enough period to permit an accurate calculation of the temperature rise following the heating.

e. The following table gives an estimate of the order of magnitude of the different sources of error expressed in percent of the atomic heat capacity of lead.

Factor	2 — 4° K.	4 — 8° K.	8 — 16° K.
1. Weight of the lead block	0.05 %	0.05 %	0.05 %
2. Temperature rise	10.0	3.0	2.0
3. Energy imparted			
(current) ²	1.0	1.0	1.0
Resistance	1.0	1.0	3.0
Time	0.1	0.1	0.1
4. Part of the heat capacity due to the core	7.0	4.0	0.1
Total	19 %	9 %	6 %
Average deviation of the results from the mean	20 %	7 %	2 %
Uncertainty in the heat capacity of the core	50 %	20 %	3 %

For some measurements the deviation is considerably greater without there being any evident cause.

§ 4. *The resistance thermometer.* The data for the calibration of the constantan resistance thermometer are given in Tables Ia and Ib.

The temperature of the liquid helium was read off a graph based on

the formula for the vapor pressure by KAMERLINGH ONNES and WEBER ¹⁾, excepting the lowest three temperatures which were calculated from the formula of VERSCHAFFELT, given in Comm. Leiden, N^o. 181, p. 18, note 1 ²⁾.

TABLE Ia.			
Calibration of constantan thermometer in liquid helium			
Date	Vapor pressure of He mm. Hg	Temperature °K.	Resistance Ohms
20 November 1925	874.5	4.351	90.4244?
	845.1	4.313	.4233
	822.1	4.283	.4205
	784.5	4.233	.4136
	751.0	4.188	.4095
	744.8	4.180	.4088
	736.1	4.168	.4110?
	614.9	3.996	.3877
	393.5	3.599	.3425
	166.9	3.008	.2769
	59.8	2.481	.2219
	29.9	2.192	.1947
	5.21	1.578	.1522
	3.63	1.448	.1482
	3.44	1.436	.1478
17 March 1926	767.6	4.210	.4121 ³⁾
	.767,6	4.210	.4128

Between liquid helium and liquid hydrogen temperatures interpolation was made with the help of a graph.

¹⁾ These Proc. 18, 493, 1915—1916; Comm. Leiden N^o. 147b § 7.

²⁾ The newer formula of VERSCHAFFELT, Comm. Leiden Suppl. N^o. 49, p. 26, gives temperatures, which are 0.06° higher.

³⁾ During the hour preceding this measurement there was no current passing through the constantan wire; for the fifteen minutes just before the next measurement there was a current of 1 milliampere. This explains the somewhat higher value of the resistance according to the second measurement, as the result of the slightly higher temperature in the thermometer core due to the heating effect of the current.

TABLE 1b.			
Calibration Constantan Thermometer in liquid hydrogen			
Date	Vapor Pressure H ₂ mm. Hg	Temperature °K.	Resistance Ohms
5 November 1925	—	14.112 *)	91.6667
22 April 1926	754.9	20.338 **)	92.4507
" " "	60.0	14.115 **)	91.6678

*) Secured from simultaneous readings with Platinum thermometers Pt₃₂ and Pt₃₆.
 **) Secured from the vapor pressure of hydrogen according to Comm. Leiden N^o. 156b 1).

§ 5. *The Heat Capacity of the Heater-Thermometer-Core.*

Table II gives the data secured from the measurements of the heat capacity of the heater-thermometer-core.

A number of observations, not included in the table, did not give good results. For several of these, especially at the lowest temperatures, the energy added was too small to give a good measurable temperature rise. In some cases the behaviour of the galvanometer was as if the temperature was actually lowered by heating. The cause of this irregularity is as yet unknown to us.

For a number of measurements shown in the table, the deduction of the corrected final temperature after heating from the different successive readings of the galvanometer was uncertain. These measurements are marked with an *. In this connection it should be noted that, after the current for heating has passed, the heating wire has a temperature appreciably higher than that of the surrounding metal. There is in most cases about ten minutes before the temperature of this wire, serving also as thermometer, is the same as the temperature of the block of metal. Only when the temperature of the bath (in which it was impossible to use a stirrer²⁾) remained sufficiently constant over a time long enough to provide a regular change in temperature for the block, was it possible to get the final temperature reading for the calorimetric calculation with sufficient accuracy.

These remarks apply also to the measurements of the heat capacity of the lead block communicated in § 6.

1) J. PALACIOS MARTINEZ and H. KAMERLINGH ONNES. Arch. Néerl. (3A) 6, 31, 1922. Comm. Leiden N^o. 156b.

2) To promote uniformity of temperature in the bath, the calorimeter vessel was encased in a copper cylinder, cp. These Proc. 20, p. 1000; Comm. Leiden, N^o. 153a, § 1.

TABLE II.

Date No.	Current mA.	Mean Resistance Ohms	Time sec.	Temperature rise degrees K.	Mean Temperature °K.	Heat Capacity Cal./degr. K.	Heat Capacity per mol lead in the block
21 May 1926							
A XIV	18.82	90.29	36	1.98	3.07	0.1389	0.0411
XV	7.66	90.62	30	0.96	5.94	0.0398	118*
XVI	7.66	90.71	30	0.72	6.66	0.053	163*
B XII	7.67	90.33	90	1.86	3.43	0.0616	182*
XIII	7.66	90.57	30	0.34	5.58	0.1122	[332]
XIV	7.66	90.62	30	0.29	5.94	0.1316	391*
XV	7.66	90.70	60	0.81	6.56	0.0943	278*
XVI	7.65	90.90	180	2.61	8.14	0.0879	261*
XVII	7.65	91.26	300	4.26	10.99	0.0901	266
XVIII	7.65	91.63	300	3.51	13.79	0.1098	357
XIX	7.65	91.91	300	2.68	16.03	0.1440	426
XX	7.65	92.20	481	3.12	18.37	0.1989	592
XXI	7.64	92.33	599	2.99	20.95	0.2588	767
2 June 1926							
C XVIII	7.69	90.29	60	0.23	3.13	0.3081	912*
XX	7.52	90.67	180	2.93	6.30	0.0749	222*
XXI	7.50	90.91	90	1.71	8.22	0.0642	190*
D III	7.51	90.20	120	0.45	2.29	0.3231	957**
VI	7.50	90.65	90	1.21	6.18	0.0887	263*
VII	7.49	90.82	90	1.99	7.87	0.0550	163*
VIII	7.49	90.49	180	1.65	4.87	0.1322	391*

For the bracketed readings in Tables II and III there were other reasons for uncertainty, e. g. that for the measurement the temperature rise was not sufficiently large.

The results are shown in Fig. 1. The points within broken circles are those indicated in Table II as less accurate. For the calculation of the specific heat of lead in § 6 we used the line drawn through the points, (shown in the uncertain region as a dotted line).

§ 6. *The Atomic Heat of Lead.* The results of the measurements are given in Table III. The lead block ¹⁾ weighed 699.73 grams.

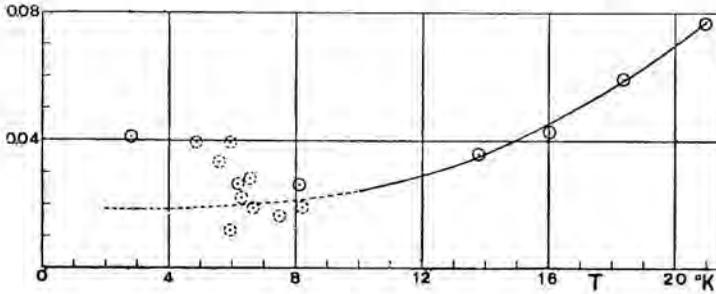


Fig. 1.

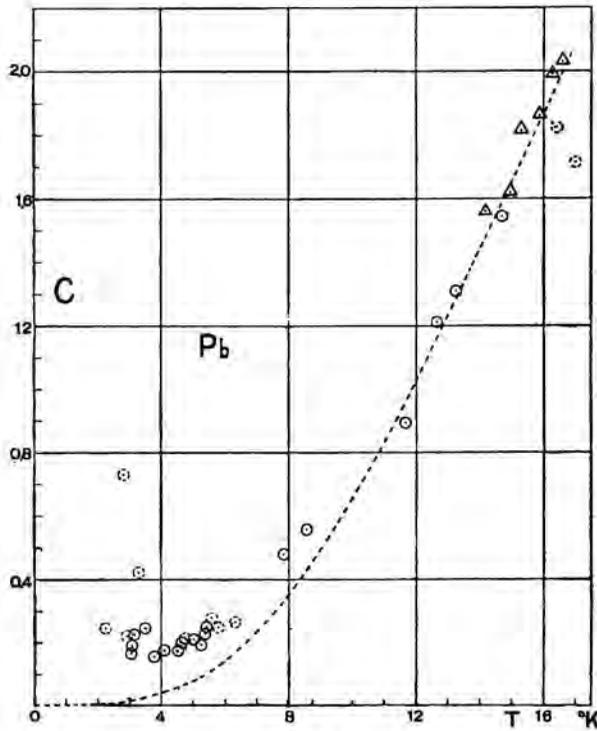


Fig. 2.

For remarks on the order of certainty of the results cf. § 5.

The results are shown graphically in Fig. 2 (dotted circles indicating less accuracy).

The \triangle \triangle mark the points secured from the accurate measurements of

¹⁾ The same block as that which served for the measurements of these Proc. 17, 894, 1914—1915; Comm. Leiden N^o. 143, diminished by a strip which served for the resistance measurements given in these Proc. 18, 487, note 3, 1915—1916; Comm. Leiden N^o. 147a, p. 6, note 3.

TABLE III.
Atomic Heat Capacity.

Date No.	Current mA.	Mean Resistance Ohms	Time sec.	Temperature-rise degrees K.	Mean Temperature °K.	Heat Capacity of core per mol lead	Atomic heat lead ¹⁾
30 March 1926	In bath of liquid helium.						
II	27.03	90.86	64	0.600	7.84	0.021	0.480
III	27.02	90.96	120	0.975	8.57	0.022	0.557
IV	26.92	91.48	240	0.907	12.66	0.031	1.211
V	26.92	91.96	360	0.908	16.40	0.047	1.823*
VI	16.47	90.27	60	0.139	2.81	0.018	[0.728]
VII	27.20	90.33	30	0.540	3.485	0.018	0.244
VIII	27.29	90.46	60	1.41	4.61	0.019	0.183*
IX	27.17	90.67	120	2.00	6.33	0.020	0.264*
22 April	In bath of solid or liquid hydrogen						
I	47.33	91.36	60	0.94	11.66	0.028	0.893
II	55.60	91.56	68	1.03	13.26	0.033	1.308
III	60.76	91.69	120	1.82	14.69	0.038	1.543
IV	varied	91.98	239	2.79	17.00	0.050	[1.713]
V	..	92.32	180	1.81	19.30	0.065	[2.300]
30 April	In bath of liquid helium.						
III	23.81	90.25	60	0.916	2.91	0.018	0.219*
IV	24.08	90.36	30	0.613	3.79	0.019	0.157
V	23.74	90.46	45	0.742	4.64	0.019	0.200
VI	23.75	90.55	60	0.81	5.41	0.019	0.249
IX	35.37	90.29	60	2.302	3.07	0.018	0.190
XI	23.82	90.29	120	2.355	3.04	0.018	0.166
XII	23.60	90.45	30	0.556	4.52	0.019	0.174
XIII	23.60	90.50	30	0.47	5.01	0.019	0.209
XIV	23.62	90.57	60	0.73	5.58	0.019	0.276*
7 May	In bath of liquid helium.						
VI	25.68	90.28	60	1.04	3.16	0.018	0.225
VIII	25.66	90.53	30	0.60	5.24	0.019	0.192
IX	25.62	90.60	30	0.47	5.78	0.019	0.250*
XII	25.86	90.20	60	0.976	2.24	0.018	0.244*
XIII	25.88	90.31	30	0.292	3.28	0.018	[0.422*]
XIV	25.81	90.40	30	0.663	4.09	0.018	0.175
XV	25.77	90.47	30	0.55	4.75	0.019	0.213*
XVI	25.75	90.54	30	0.52	5.34	0.019	0.226

¹⁾ The difference between c_p and c_v is negligible.

KEESOM and KAMERLINGH ONNES in the liquid hydrogen range. The agreement is relatively satisfactory, considering our calculated accuracy.

The broken line denotes the values for the atomic heat capacity (at constant volume) as calculated from the formula of DEBIJE with $\theta = 88$.

In the first place we note that with lead there is no indication that at the transition to the super-conductive state there occurs a discontinuous change in the atomic heat, even as KAMERLINGH ONNES and HOLST found with mercury.

Further it appears from our measurements that the atomic heat of lead at the lower temperatures is much greater than that calculated from the formula of DEBIJE, employing a value of θ which reproduces approximately the observed values at higher temperatures.

The first indication of this deviation in the case of lead was observed in the results of the measurements of KEESOM and KAMERLINGH ONNES (l.c.) which at the lowest temperatures obtainable with liquid hydrogen indicated that the atomic heat of lead is greater than would be calculated from the formula of DEBIJE with $\theta = 88$ (see their Fig. 10, as well as Fig. 2 of this communication); or otherwise expressed, the value of θ below 18° K. gets somewhat smaller than 88 (see Fig. 11, Communication No. 143). Also SIMON l.c. at the temperatures 9.85 to 13.37° K., finds θ smaller than 88.

At the temperature of liquid helium this deviation has become so great that the atomic heat there is several times greater than that calculated from the formula of DEBIJE with $\theta = 88$.

A similar behaviour in the case of mercury, since confirmed at temperatures above 9° K. by SIMON ¹⁾, was found by KAMERLINGH ONNES and HOLST (l.c.), in connection with the measurements of POLLITZER ²⁾. A further investigation will be immediately undertaken to see whether this behaviour is also characteristic of substances which are not super-conductive, for example, bismuth.

It is not clear from our measurements whether one would expect that, in approaching 0° K., the atomic heat capacity approaches zero or a positive value. It still seems likely from our measurements that the atomic heat capacity approaches zero but at a considerably smaller rate than the T^3 rule of DEBIJE would lead us to expect.

Our results do not show clearly whether the case of lead is explained by the assumption of SIMON ³⁾ that the deviation, which occurs in the case of gray tin and silicon, is due in these atoms to the existence of two different quantum states with small energy difference. Should this be the case then it would follow from our measurements that $\theta_u = 2.25$ approximately. This would give at 1° K. $\Delta C = 0.87$, at 2° K. 0.55 , at 3° K. 0.37 . Now it should be observed that several of the more uncertain

¹⁾ F. SIMON, Zs. f. physik. Chem. **107**, 279, 1923.

²⁾ F. POLLITZER, Zs. f. Elektrochemie **17**, 5, 1911; see also **19**, 513, 1913.

³⁾ F. SIMON, Berlin Sitz. Ber. 1926, p. 477.

measurements at about 3° K. give rather large values for the heat capacity, 0.7 and 0.4 (see also the values for the core, fig. 1). On the other hand, the more certain measurements give values of about 0.2. It is desirable to secure new measurements in order to be sure of this point.

It is a pleasure to record our hearty thanks to Mr. CHESTER W. CLARK of the University of California for his valuable help in securing the measurements and in calculating the results.

Microbiology.— *On the disappearance of Bacillus typhi from Water.* By Mrs. N. L. WIBAUT—ISEBREE MOENS. (Communicated by Prof. C. PH. SLUITER.)

(Communicated at the meeting of January 29, 1927).

It is a well known fact that when a culture of bacilli typhi is brought into water the bacteria vanish from the water after a certain time. However no definite information is to be found in the literature about this question as several authors give widely different figures for the period of time after which the bacteria have disappeared. Different circumstances may influence the survival of bacteria in water :

a. physico-chemical conditions : the water may be considered as a medium, in which the growth of the *Bacillus typhi* is inhibited or aided by the amount of dissolved organic matter, the amount of dissolved oxygen and carbonic acid, the salts, the temperature of the water, the sunlight that penetrates into it, etc. ;

b. biological conditions : in the water there may be organisms devouring the bacteria or acting upon them in one way or another. These organisms will in their turn be affected by the physico-chemical conditions mentioned sub a. The oxygen, for instance, may act directly upon the typhus bacilli and also upon the protozoa, which devour the bacteria. 1)

If the difference in lifetime of *Bacilli typhi* are caused by internal conditions such as oxygen, light etc., then it would be expected that in water of different origin the bacteria should behave in the same way other circumstances being the same. On the other side if the bacteria do not show the same behaviour in different samples of water, the cause of the vanishing of the bacteria is not to be found in external circumstances but in the special properties of the water itself.

During our investigation on water purification we had occasion to study the importance of certain protozoa for the purification process. We got the impression that previous investigators had not recognized sufficiently the rôle of protozoa in the phenomenon of the vanishing of typhus bacteria. We therefore started a series of experiments where in water samples of different origin were inoculated with an equal amount of bacteria other circumstances being alike. At regular intervals the amount

1) VAN LOGHEM en RUYSS, Tijdschrift voor Geneeskunde, 1926, November, Septic tank en typhophage protozoën.

of bacilli typhi was established at the same time the number and the species of the protozoa were determined by FRANÇOISE VERSCHAFFELT, assisting me for this biological work.

On 8 February four samples of water of different origin were taken :

- a. from the swimming pool Heiligeweg, Amsterdam ;
- b. from the Vecht-watersupply ;
- c. from a ditch in the province Noord-Holland ;
- d. from a cistern, indicated as rainwater.

These four samples of water had been equally infected by a suspension of typhoid bacilli, recently isolated from faeces. In each flask of a capacity of 2 litres and filled with water we put so much of the suspension that the liquid was faintly opaque. We had previously determined the titre of protozoa in each flask.

Every three days after the inoculation we determined the number and species of protozoa; at the same times the presence of bacilli typhi was tested by Dr. H. PLANTEN in this way that 10 cc. of the water was centrifugated during 10 minutes, the precipitate being sown on an endoplate. When on a certain date no bacteria could be found the testing was repeated after another three days and only if this second test was negative we concluded that the bacteria had really vanished.

The subjoined tables shows the content of protozoa of each of the water-samples and the species being present in the water before the inoculation. In rainwater protozoa could be made out in 0.01 cc. before the commencement of the experiment; in ditchwater and the water from the swimming pool (Heiligeweg) only in 1 cc. while none were found in 1 cc. of the watersupply Vecht, in 10 and 40 cc. however, they were present.

On the 8th of February the flasks were inoculated, kept on room temperature and in daylight. On the 12th the cultures were inspected for the first time; in each sample typhus bacilli were present but the number of some species of protozoa had allready increased while other ones had disappeared. In the swimming water and in the Vechtwater they have augmented, likewise in the ditchwater, but in the rainwater the number, as well as the species had diminished. Table II shows the species of protozoa for each sample of water; the following symbols have been used : . for a few, .. for several; ... for frequent; for many; :::: for abundant. Tables I and II present a clear survey of the results of the protozoa-calculation on 8, 12, 15, 18, 22, 25 of February, 1, 5, 8 and 12 of March. Total absence of Bacillus typhi is designated by ooooooooo.

It must be kept in mind studying the results of these experiments that mainly the bacteriophage species of protozoa are detected. Of course besides the bacteriophage protozoa there may occur other plankton forms. But they dont effect the results of this research. As the typhus bacilli brought into the water are an excellent food for the bacteriophage protozoa they will increase abundantly, when there are no more bacteria available they will decrease.

TABLE I. Febr. 8 — March 12 1926.

	Protozoa present in:	February						March			
		8	12	15	18	22	25	1	5	8	12
Water from Swimming pool Heiligeweg *)	1 cc	+	+	+	+	+	+	+			
	0.1	-	+	+	-	±	-	-			
	0.01	-	+	-	-	-	-	-			
	0.001	-	-	-	-	-	-	-			
Water from the Vechtwater supply	1 cc	-	+	+	+	+	+	+			
	0.1	-	-	+	+	+	-	+			
	0.01	-	-	+	+	-	-	-			
	0.001	-	-	+	+	-	-	-			
Rain-water	1 cc	+	+	+	+	+	+	+			
	0.1	+	+	+	+	+	+	+			
	0.01	+	-	-	+	+	+	-			
	0.001	-	-	-	+	-	-	-			
Ditch-water	1 cc	+	+	+	+	+	+	+	+	+	+
	0.1	-	+	+	+	+	+	-	±	±	-
	0.01	-	-	+	-	-	-	-	-	-	-
	0.001	-	-	-	-	-	-	-	-	-	-

*) Swimming pool replenished February 3.

In water from the *Vechtwatersupply*, although being poor in bacteria as well as in protozoa the bacilli typhi have very soon disappeared: after ten days the water was found to be free from them. Table I shows protozoa being absent in 1 cc. of water at the beginning on 8 February, increasing on 12th and 15th till more than 1000 in 1 cc. Comparing these figures with those on Table II it is clear that *Cercobodo bodo* 1) and *Alexeieffii* and *Oicomonas termo* have been important. They decrease as soon as *Bacilli typhi* have disappeared, whereas *Pleuromonas jaculans* being a non bacteriophage protozoa persist.

Swimming-water behaves towards typhus bacilli still more aggressively. After a week the bacteria have already vanished. Here also the *Oicomonas termo* and *Cyclidium glaucoma* have increased, *Cercobodo Alexeieffii*

1) For figures see: N. L. WIBAUT, De ontwikkeling van protozoën in hooiinfusie, and also: Protozoën in tankwater van stoomschepen. Tijdschr. v. Microbiologie 1927.

flourishing more afterwards (table II). After bacilli typhi have disappeared, *Oicomonas termo* can no more be detected, *Cercobodo Alexeieffii* soon declines. *Amoeba limicola*, is to be found in the absence of bacillus typhi.

In *ditch-water* the typhus bacilli survive during a longer time than in the

TABLE II. Febr. 8 — March 12 1926.

Species	February						March			
	8	12	15	18	22	25	1	5	8	12
Swimming water	40cc and 10cc	1cc								
<i>Amoeba limicola</i> *)		:::			
<i>Actinophrys vesiculata</i>						...				
<i>Bodo minimus</i>	...									
<i>Cercobodo Alexeieffii</i>			:::	..					
<i>Cinetochilum margaritacea</i>	.									
<i>Collodictyon triciliatum</i>	..									
<i>Cryptochilum nigricans</i>									
<i>Cyclidium glaucoma</i>	:::			
<i>Glaucoma scintillans</i>	.									
<i>Halteria grandinella</i>	.									
<i>Holophrya simplex</i>	.									
<i>Loxophyllum lamella</i>	.									
<i>Monosiga ovata</i>	.									
<i>Oicomonas termo</i>	:::	:::							
<i>Sphaerophrya magna</i>	.									
Vechtwater supply										
<i>Amoeba limicola</i>	.									
<i>Bodo minimus</i>	.									
<i>Cercobodo Alexeieffii</i>								
.. bodo	:::				...				
<i>Cyclidium glaucoma</i>									
<i>Oicomonas termo</i>			:::	:::		...				
<i>Pleuromonas jaculans</i>			..			:::	:::		

*) The species occurring in the cultures slightly differs from the description in the literature.

other samples of water. In this experiment they have vanished after 28 days. Among the different protozoa being present in the water at the beginning on 8th of February some bacteriophage species are seen to gain the upper hand. On the 15th of February *Cercobodo Alexeieffii* and *Cyclidium glaucoma* are still numerous, as table II shows whereas *Oicomonas termo*, which appeared so active in the other three samples of water has already disappeared. On the first of March typhus-bacilli can still be made out and *Cyclidium glaucoma* is present as sole bacteriophage protozoa. It is not quite clear why *Bacillus typhi* holds on until 8 March, while the number and species of protozoa remains the same.

On the 2^d of March similar experiments were arranged with other samples of water of the same origin in order to see whether the results should confirm those of February.

Tables III and IV illustrate the behaviour of the four samples of water which is nearly the same as that of the previous experiments.

B. typhi disappeared first in *Vechtwater* viz. within six days; again a regeneration of *Oicomonas termo* is to be observed and a disappearance of this species as soon as the typhus bacilli have gone, as well as a persistence of *Pleuromonas jaculans* the non-bacteriophage flagellate.

The *swimming-water*, which is by the way originally *Vechtwater*, after being inoculated, reveals a revival of two species, *Cyclidium glaucoma* and *Cryptochilum nigricans*, both good bacteriophages. It strikes us that these species maintain themselves also in the absence of the typhus bacilli. This can be accounted for only by assuming that the water contained a sufficient number of coli-like bacteria from the beginning. These bacilli still present after bacilli *typhi* have vanished keep, we suppose, the protozoa alive.

Rainwater after being inoculated on the 2^d of March, does not contain any typhus bacilli after ten days. *Oicomonas termo* and *Bodo Alexeieffii* have augmented and disappear together with the typhus bacilli. *Amoeba limicola* and *Pleuromonas jaculans* persist as in the other experiment.

Another experiment was arranged for *ditchwater* viz. a second flask was filled with water which was filtered and had been inspected for the absence of protozoa. The flasks with *ditchwater* and with filtered *ditchwater*, the later being absolutely free from protozoa, were now inoculated with typhus bacilli. From the *ditchwater* the bacteria had disappeared after 13 days. Especially *Cercobodo Alexeieffii* and *Cyclidium glaucoma* have developed. They persist even after typhus bacilli have vanished. Out of the *filtered ditchwater* however the bacilli *typhi* have vanished after 17 days. One should expect that if the rôle of the protozoa is really important the bacilli would survive much longer in water without these organisms. As it is evident that in the three other samples of water protozoa are really the deciding factor one must put the question otherwise i.e. why do bacilli *typhi* vanish so slowly in normal *ditchwater* in spite of the abundance of protozoa, so that there is a difference of only four days with the behaviour of the bacilli *typhi* in filtered water where there are no protozoa at all at

work? It seemed reasonable to suppose that the property of the water itself has to do something with this phenomenon.

Supposing this we took water from different ditches in Noord-Holland. One ditch was wellknown to us as there live along people under which febris typhoidea is endemic, on the other ditch this was not so. Two samples from this origin were treated just the same as above. I tabulate the results although they have not as once solved the propounded problem. It appears from table V that the water from these ditches is much more

TABLE III. March 2 — 24 1926.

Water	Protozoa present in:	March					
		2	8	12	15	19	24
Swimming water *)	1 cc	+	+	+	+	+	
	0.1	-	-	+	-	+	
	0.01	-	-	-	-	-	
	0.001	-	-	-	-	-	
Vechtwater supply	1 cc	+	+	+	+		
	0.1	-	+	+	+		
	0.01	-	+	+	+		
	0.001	-	-	-	-		
Rain-water	1 cc	+	+	+	+	+	
	0.1	+	+	+	+	+	
	0.01	+	+	+	-	+	
	0.001	-	+	-	-	-	
Ditch-water	1 cc	+	+	+	+	+	+
	0.1	±	+	+	+	+	+
	0.01	-	-	+	-	+	-
	0.001	-	-	-	-	-	-
Ditch-water (filtered)	1 cc	-	-	-	-	-	-
	0.1	-	-	-	-	-	-
	0.01	-	-	-	-	-	-
	0.001	-	-	-	-	-	-

*) Swimming-pool had been refilled on February 17.

TABLE IV. March 2 — 24 1926.

Species	March						Water
	2	8	12	15	19	24	
<i>Cyclidium glaucoma</i>				Swimmingwater (Heiligeweg)
<i>Cryptochilum nigricans</i>				
<i>Amoeba limax</i>				Vechtwater
<i>Cercobodo Alexeieffii</i>					
<i>Oicomonas termo</i>					
<i>Pleuromonas jaculans</i>				
<i>Amoeba limicola</i>					Rain-water
<i>Bodo Alexeieffii</i>					
<i>Oicomonas termo</i>			
<i>Pleuromonas jaculans</i>			
<i>Amoeba limicola</i>			Ditch-water
<i>Cercobodo Alexeieffii</i>			
<i>Cercobodo Bodo</i>						...	
<i>Cyclidium glaucoma</i>		
<i>Paramaecium aurelia</i>		...					
<i>Rhaphidiophrys intermedia</i>					
<i>Urostyla Weissei</i>					

aggressive than that of the experiments on 8th February and 2^d March. Ditchwater from *Watergang*, where febris typhoidea is endemic, destroys the inoculated typhus bacilli in 5 days, that of Ransdorp after 9 days.

The water from *Watergang* contains five typhus-eaters indicated on table VI by the letter *T* behind the name. There are three in the water from Ransdorp. In both samples they suffice to destroy the typhus-bacteria, anyhow they are more aggressive than in the experiment of 2 March.

The different behaviour of the ditchwater in February and March and in May cannot be explained without further experiments.

One fact is clear from the tables of the grouping of the species of protozoa in these experiments (table II, IV, VI) viz. that in ditchwater approximately the same number and the same species of bacteriophage protozoa survive, even after the disappearance of the typhus bacilli. In rainwater and in Vechtwater not only their number soon decreases but most often they

TABLE V. May 6—29 1926.

	Protozoa present in:	May									
		6	8	10	11	12	15	19	22	29	
Ransdorp	1 cc	+	typhus inoculation	+	+	+	+	+	+	+	
	0.1	+		+	+	+	+	+	+	±	
	0.01	—		—	+	+	+	+	±	—	
	0.001	—		—	—	—	+	—	—	—	
Watergang	1 cc	+		typhus inoculation	+	+	+	+	+	+	+
	0.1	+			+	+	+	+	+	±	±
	0.01	+			+	+	+	+	+	—	—
	0.001	—			±	—	±	+	—	—	—

disappear absolutely in absence of bacilli typhi. It is evident that here the inoculated bacteria are a welcome and superfluous feeding for the bacteriophage species, always present, though not frequent at the beginning in this water, being very poor on other bacteria. In swimming-water which has been infected with a great number of bacteria by the swimmers, we noticed also a survival of bacteriophage protozoa after the disappearance of typhus bacilli. In a water originally poor in bacteria the inoculated bacteria will act more stimulating on the hungry bacteriophage individuals, than in ditchwater or swimmingwater in which other bacteria are abundant. Here the inoculated bacteria make no such an effect on the bacteriophage species as they had already more than sufficient other bacteria to feed on.

In the water from Ransdorp and Watergang we therefore determined the total number of bacteria being present before inoculating with typhus bacilli. It is to be noticed in the following state that their number is really

Number of bacteria in 1 cc. of ditchwater.

	Cultivated on agar at 37° C.	Cultivated on agar at 22° C.	Cultivated on gelatine at 22° C.
Watergang	7,400	22,100	790,000
Ransdorp	41,600	15,400	innumerable

very high and may be only slightly increased by the inoculated bacilli typhi. On the other hand if bacillus typhi has vanished from the ditch-water the protozoa find still a great number of other bacteria to feed on. It is not probable that the protozoa in ditchwater, overcharged as it is with all kinds of bacteria, may be able to select the bacilli typhi.

TABLE VI. May 6 — June 3 1926.
Ransdorp.

Species	May												June
	6	8	10	11	12	14	15	17	19	22	25	29	3
<i>Amoeba limax</i>	.		*	*	*	*	
<i>Cercobodo Alexeieffii</i>	T
<i>Coleps hirtus</i>						
<i>Cyclidium glaucoma</i>	T
<i>Oicomonas termo</i>	T
<i>Pleuromonas jaculans</i>						
<i>Rhynchomonas nasuta</i>						

Watergang.

Species	May												June
	6	8	10	11	12	14	15	17	19	22	25	29	3
<i>Amoeba limax</i>	...		*	*	*	
<i>Anisonema variabile</i>	T
<i>Cercobodo Alexeieffii</i>	T
<i>Chilodon uncinatus</i>											
<i>Coleps hirtus</i>										
<i>Collodictyon triciliatum</i>	
<i>Cyclidium glaucoma</i>	T
<i>Halteria grandinella</i>								T
<i>Loxophyllum lamella</i>										
<i>Paramecium aurelia</i>								T
<i>Pleuromonas jaculans</i>						

* not tested.

CONCLUSION.

10. Typhus bacilli do not disappear with equal rapidity in water from different origin brought under the same condition.

20. The disappearance of typhus bacilli is attended with a multiplication of definite species of protozoa (*Oicomonas termo*, *Cyclidium glaucoma* and *Cercobodo Alexeieffii*).

30. The rate of disappearance appears to be less dependent on the total number of protozoa present at the beginning of the experiment than on the presence of certain bacteriophage species.

40. Ditchwater from Noord-Holland behaves less aggressively towards inoculated typhus bacilli than swimming-, rain-, or pipewater.

We purpose to ascertain whether the cause of this aberrant behaviour of ditchwater is to be looked for in its physico-chemical or in its biological properties.

*From the Laboratory of the Municipal Service of
Public Health at Amsterdam.*

Amsterdam, Dec. 1926.

Physiology. — *An Anatomico-physiological study of the Supra-vestibular Tractus in Columba domestica.* By L. J. J. MUSKENS. (Communicated by Prof. J. W. VAN WIJHE).

(Communicated at the meeting of March 26, 1927).

§ 1. *The function of the tr. strio-mesencephalicus and of the nuc. spiriformis.*

In pursuing the line of research of LONGET, SCHIFF, and others we established in a previous publication ¹⁾, that the commissura posterior in mammals (cat, dog, rabbit) marks the point, at which the physiological effect of the hemisection of the brain-stem is reversed. This may be observed best when, performing in a rabbit a series of hemisections of the brain-stem, starting in the vestibular region, and terminating in the thalamus; then it will be seen that e.g. after severance of the right side of the brain-stem the animal performs a circus-movement towards the left, while the rolling-movement (and also the so-called Magendie Hertwig strabismus, and the tendency to fall down) takes place towards the impaired side. With hemisection, oral to the posterior commissura, the circus-movement takes place towards the unimpaired side. By comparative experiments with subsequent osmic acid staining of the preparations it could be determined in quadrupeds, that only the middle-, and the lateral part of the posterior longitudinal bundle represents the anatomical substrate of these forced movements. However, with lesion oral to the commissura posterior these forced movements and positions occur only if the tertiary vestibular tracts, demonstrated by the present writer. (which connect the supravestibular-commissural nuclei with the globus pallidus and other palaeostriatal nuclei) have been injured. MARCHI's staining presents great difficulties for a precise decision as to which of the superposed nuclei from the region of the commissura-posterior, are the carriers of these functions. These nuclei had been examined purely anatomically by CAJAL, KÖLLIKER, KAPPERS, MESDAG, BECCARI, CASTALDI. In virtue of a number of lesions in the commissural-region in the cat, the present writer established with such a lesion the regular occurrence of bundles descending into the longitudinal bundles, viz. 1^o. of the tr. commissuro-medullaris and 2^o. of the tr. interstitio-spinalis.

A comparative study of the consequences of a number of lesions induced us to correlate the lesion of the first tract with the circus-movement (to

¹⁾ Verhandelingen der Koninkl. Akad. van Wetensch. 1902, Deel 8, N^o. 5; BRAIN, 1914 and 1922.

the impaired side), that of the second with the rolling-movement (towards the unimpaired side).

These data, which were in part new, postulating *inter alia* a vestibular function of the globus pallidus, necessitated a more extensive study of the commissuro-pallidal tracts, as well in their anatomical as in their physiologico-anatomical aspects. As to the former RIESE issued from the Senckenberg Institute at Frankfort an inquiry into the relations in aquatic mammals. The connection of the globus pallidus with the nuclei in the region of the posterior commissure, which had been demonstrated by dint of painstaking physiological experiments in the rabbit, the dog and the cat, was found very distinctly in these animals in the bundle H. 2. As RIESE has it, RIESE'S finding in the aquatic mammals supports "vom vergleichend-anatomischen Gesichtspunkt, die neueren Ergebnisse, zu den MUSKENS auf Grund experimenteller Untersuchungen gelangt ist" ¹⁾.

On this ground pursuance of the same anatomo-physiological research into the bird's brain promised a priori to be successful, because in this group of animals with their double mode of locomotion the system of superposed vestibular nuclei in the neighbourhood of the commissura posterior must be extremely developed. I was all the more inclined to do this, because the very accurate investigations by WALLENBERG, and MESDAG'S painstaking degeneration work, served as a guide for my experimentation.

While passing by in silence lesions in various parts of the corpus striatum, which did not reveal forced movements, I intend to report my findings with a pigeon that displayed until death circus-movements towards the impaired side. In this animal I had injured the occipito-parietal portion of the forebrain ²⁾ (Fig. 1 and 2 top on the right hand) the portion, that, according to FERRIER, BOYCE and WARRINGTON ³⁾ and KALISCHER ⁴⁾, MAC KENDRICK, JASTROWITZ, GALLERANI and LUSSANNA, elicits on an electric stimulus an eye-movement towards the opposite side. It was especially the tr. strio-mesencephalicus that was found thickly degenerated, (Fig. 1—4) ⁵⁾.

The degenerated bundle helps to constitute the ventral part of the brain-stem, bends in the oral part of the thalamus in dorso-lateral direction round the tractus thalamo-tectalis and the round nucleus. It is made up of rather thick fibres, and soonest develops medullary sheaths (SCHROEDER).

Already on this level degeneration is more extensive in the lateral (macrocellular) portion of the nuc. spiriformis, into which nucleus the

1) Zeitschr. f. d. Ges. Neur. u. Psych. 90, 1924, p. 597.

2) The small lesion at the bottom fig. 1 marks the most caudal point of the lesion.

3) Philos. Transactions Royal Society London B, 1899, Vol. 191, p. 308.

4) Abhandl. der Akad. der Wissenschaften, Berlin 1905, p. 69.]

5) This case instances a whole set of experiments. Similar results were achieved in other pigeons, in which a puncture in the forebrain had pierced only or principally the tr. strio-mesencephalicus.

degenerated bundle merges there where the round nucleus begins to disappear, at the oral side of the commissura posterior. In the same areal weak products of decomposition may be found distad, as far as the level

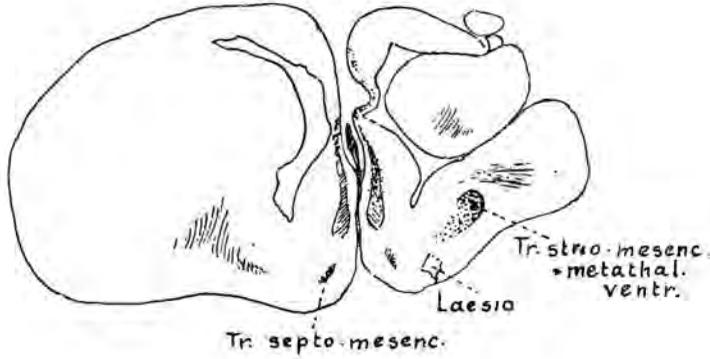


Fig. 1.



Fig. 2.

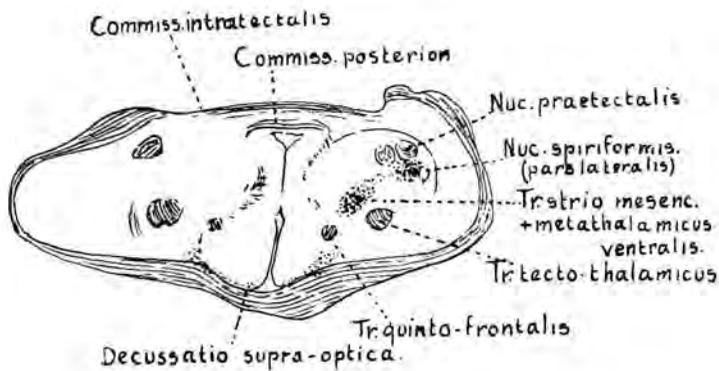


Fig. 3.

of the sensible Trigemini nucleus, while some of them seem to disappear into the tectum. This descending degeneration (tr. strio-reticularis) was described also by WALLENBERG.

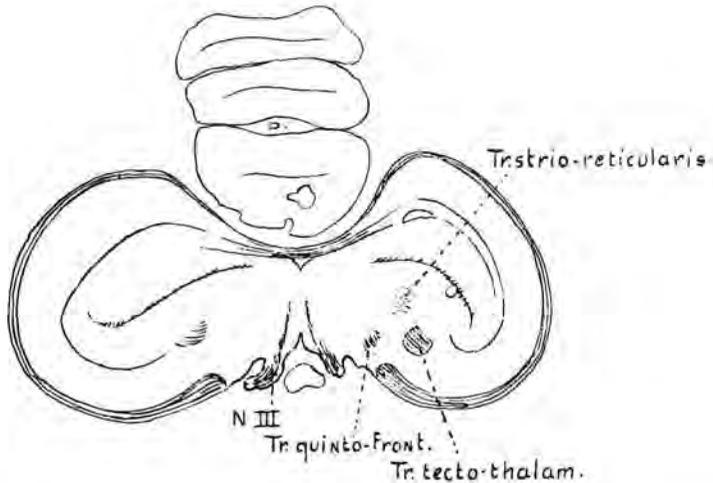


Fig. 4.

In a pigeon (273) the right nuc. spiriformis itself was punctured by a needle. For four days this animal showed a tendency to turn to the right. In the posterior longitudinal bundle no descending degeneration-fibres were encountered.

If this conjecture is correct (partial identity of the nuc. spiriformis of birds with the nuc. commissurae posterior of mammals), this case would warrant the conclusion, that the origin of the fibres in the longitudinal bundle, descending down to the nuc. abducens, is not to be found in the lateral portion of the nucleus.

Neither in my MARCHI-material, nor in the CAJAL-preparations, placed at my disposition by my colleague BOK, do I possess material to disprove WALLENBERG's hypothesis, that also from his nucl. dorso-ventralis (JELGERSMA's nucleus anterior) longitudinal bundle fibres take their origin. Nor is it altogether impossible, that contrary to the quadrupeds, birds with their double mode of locomotion, should have at their disposition two separate mechanisms for these functions.

It seems to me that these findings are to be interpreted in the way alluded to by WALLENBERG and MESDAG (if I am not mistaken), though they do not express that supposition, viz. that the nuc. spiriformis forms part of the system of commissural nuclei.

In birds these nuclei take up a much larger space than in quadrupeds, which tallies with the observation by EDINGER¹⁾, and WALLENBERG that the fibres of the posterior longitudinal bundle disappear into the nucleus spiriformis. It may be remembered here that already long ago KAPPERS

¹⁾ Anat. Anzeiger 33, 1908, p. 329. EDINGER thought of a connection with the visual function, which may be deemed reasonable.

suspected, that in birds the eye-movements are regulated in the palaeostriatum (mesostriatum).

In conclusion I think that it may reasonably be assumed that the tractus mesostriato-spiriformis represents one of the two tertiary vestibular bundles, alluded to above, which have been recognized in quadrupeds, namely in their function, and in their descending and ascending fibres, not yet exactly in their initial-, and terminal nuclei in the globus pallidus and in the commissural region.

According to my personal observations made on the cat the fibres of the nuc. posterior thalami (where according to D'HOLLANDER in the rabbit most cortico-thalamic fibres terminate), are directly connected with the two sources of the tractus, which descend in the posterior longitudinal bundle, (viz. nucleus commissurae posterioris and nuc. interstitialis). If, thus, the nuc. spiriformis in birds should occupy partly the place of the nuc. commissurae posterioris of mammals, the question arises, whether in that case the nuc. posterior thalami of mammals may be a newly acquired property of mammals, as a motor nucleus inserted into the cortical tract, (especially subservient to the voluntary eye-movements).

Just as the pallido-commissural bundle in quadrupeds the tr. strio-mesencephalicus of birds degenerates as well in ascending (but in a smaller degree) as in descending direction, which fact had already been established by EDINGER in 1903 and by GROEBBELS. My preparations do not furnish evidence for a partial crossing of this tractus, neither for a connection with the tectum (GROEBBELS).

The wider extent of the area of the commissural vestibular nuclei seems to render it possible that further experiments with various birds may be of use exactly for this more extended analysis, for further localisation of the superposed vestibular nuclei and further homologizing of the mesostriatum and of the commissural region in birds and quadrupeds.

As for our knowledge of the functions of these nuclei, after the function of one of the most significant nuclei has been identified, a way is opened up along which step by step progress can be expected, also with respect to the "Stell reflexe" or righting-reflexes. For it has become evident that a lesion of the secondary, and also of the tertiary ascending vestibular tractus in mammals, as well as in birds entail symptoms, that are comparatively easy to recognize.

§ 2. *Significance of the tracts descending from the mid-brain into the posterior longitudinal bundle, and their relative position in the areal.*

Forced movement in the horizontal plane (circus-, and clock-hand movement with lateral conjugated deviation of the eyes) and forced movement in the plane normal to the long-axis of the animal (rolling movement and skew-deviation of Magendie-Hertwig, tendency to fall down and to lie on one side) are in all animals commonly found to appear

coincidentally after lesion of secondary and tertiary vestibular tractus (just as with insects and octopods¹⁾ after otolith-excision and lesion of the supra-oesophageal ganglion).

This coincidence consists in the circus-movement towards the one side occurring simultaneously with the rolling movement to the other²⁾; this is very evident in frogs and tadpoles, and also in fish. As to the latter this may be ascribed to a mechanic, internal connection, (centrifugal force in connection with the labile equilibrium of these animals), for also steamers, when taking a turn on their path to the one side, may be seen to heel over to the other.

The investigation of warm-blooded animals has taught us, that the ascending vestibular tracts, are so disposed within the posterior longitudinal bundle in the areal of this bundle that the fasciculus vestibulo-mesencephalicus homolateralis and cruciatus in the middle-third part, the fasc. vestibulo segmentalis med. and lat. in the lateral part runs upwards. The middle third part, and that the most medial zone of the areal, is occupied by the descending tr. commissuro-medullaris and the tr. interstitio-spinalis.

When comparing WALLENBERG's³⁾ classic description of the composition of the posterior longitudinal bundle in pigeons with the later descriptions of that in the rabbit by VAN GEHUCHTEN⁴⁾, and of the same in the cat by the present writer⁵⁾, one is struck by the great similarity e.g. with respect to the increase and the decrease of the bundle in various regions, by the comparative strength of the crossed and the uncrossed bundles, by the ratio of the number of descending fibres from the acusticus-field, even by the shifting of the bundles in the medulla more ventrad, and by the relative position of the bundles (the longest excentric).

However, there is one considerable difference between quadrupeds and birds, in that in birds no trace has yet been found of the absolutely medial position, in quadrupeds, of the bundles descending from the mid-brain.

In connection with the termination of the tr. strio-mesencephalicus in and near the nuc. spiriformis, which on physiological grounds we consider to perform a similar function to that of the nuc. commissurae posterioris in quadrupeds, and in connection with the most lateral position of that nucleus in the formatio reticularis of the mid-brain in birds, an other light is thrown on this discordance. This is borne out by the fact that, whereas in quadrupeds the nuc. commissurae posterioris and the nucleus interstitialis (the two principal sources of the fibres descending into the longitudinal bundle), relative to the smaller extent of this region, are not lying far from the raphe, this is, at least partly, quite different in birds.

1) *Archiv. f. (Anat. u.) Physiol.*, 1904, p. 51.

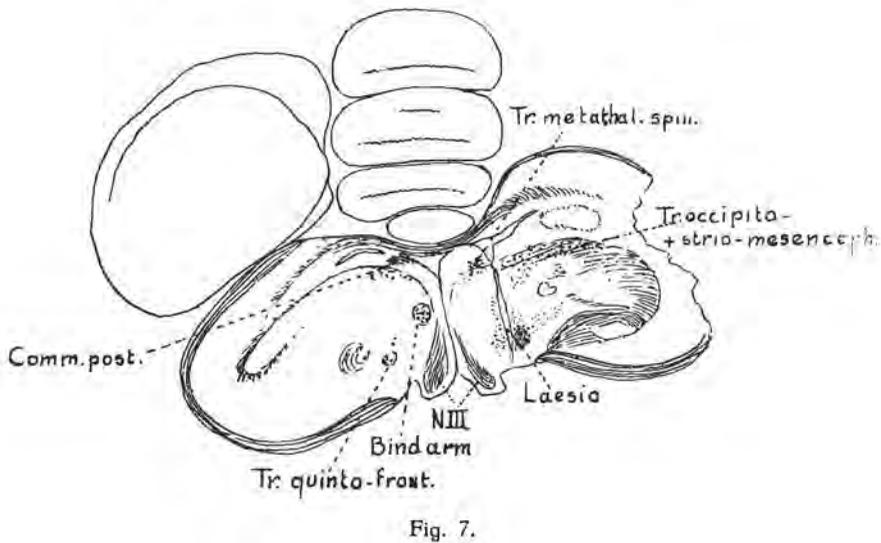
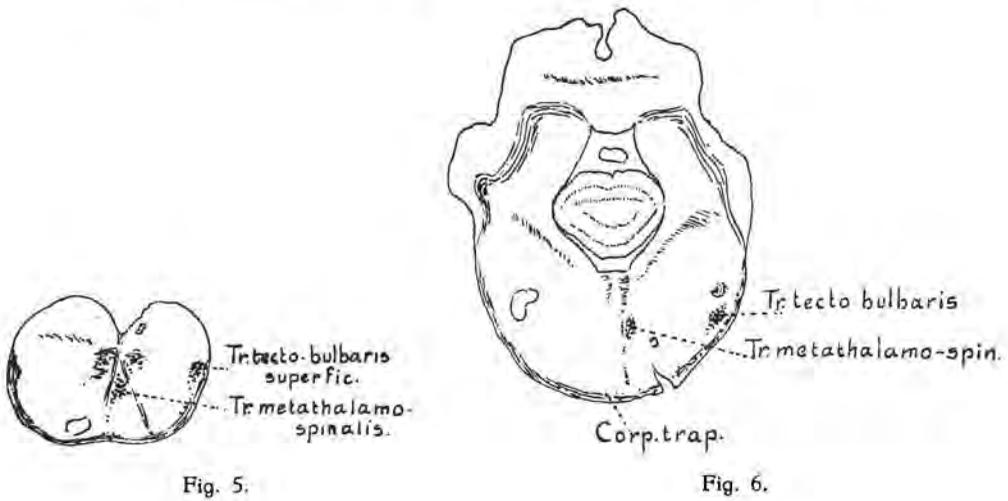
2) *Journal of Physiology*, 31, 1904, p. 205.

3) *Anatomischer Anzeiger*, 24, 1903, p. 142.

4) *Neuraxe*, 4, 1904, p. 63.

5) Cf. *BRAIN*, 1914, fig. 7 and 8, p. 370—373; *Verslag der zittingen van de Koninkl. Akademie Amsterdam*, 19 April 1913, p. 1474.

Here the nuclei in question are larger, and owing to the excentric position of some primary nuclei the fibres descending from them cannot possibly reach in the higher regions the most medial zone in the longitudinal bundle, and only on a lower level do they appear immediately beside the raphe. Figs. 5—9 have been derived from a pigeon, in which the rather considerably lateral incision severed the origin of a bundle, descending into the posterior longitudinal bundle, 10 days before death. They illustrate the anomalous dorsal position of these descending bundles in the pigeon.



From this observation we infer an extremely lateral source of thick longitudinal-bundle fibres. They descend far into the spinal cord just as the *tr. interstitio-spinalis*, and emanate from a nucleus situated near the

nuc. spiriformis. This, however, cannot take away from the fact, that there exists near the median line a source of similar thick fibres, according to CAJAL.

This is substantiated by BOK's preparations of very young chick-embryos, in which I could verify this latter source of descending bundles-

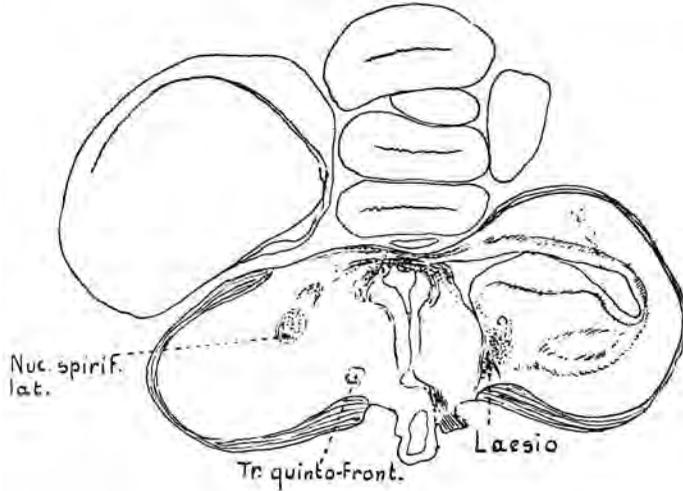


Fig. 8.

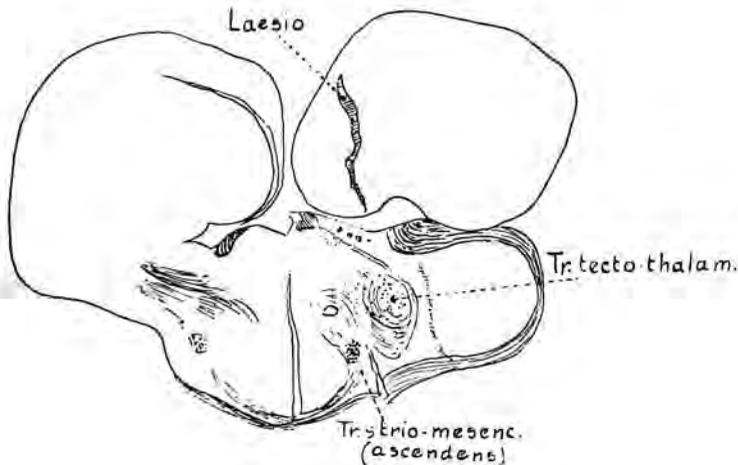


Fig. 9.

called also nuc. interstitialis in birds. This observation also unmistakably bespeaks in birds a double source of descending longitudinal bundle fibres, in connection also with their twofold mode of locomotion.

In connection with the greater extent of these commissural-nuclei in birds there is another physiological point of importance. Theoretically speaking, it should be possible also in warm-blooded animals to make

a lesion in the commissure-region in such a way, that the two forced movements, discussed above, should appear separately, or also in such a way that not the ordinary difference in direction should appear, so that circus-movement to the one side and rolling movement to the *same* side should coincide. Experience, however, teaches, that this does not practically occur in animals, very likely on the one hand because the discussed, ascending vestibular tractus in the longitudinal bundle are lying in the same areal, on the other hand, because the nuc. commissurae posterioris and the nuc. interstitialis are, like their descending bundles, disposed very medially at a short distance from each other. We are justified in assuming that an antero-posterior stab in birds can cut one of the commissural-nuclei, as well as (more caudad) one of the ascending bundles.

This is really the case in the above-named pigeon 22, which received a stab in the commissure from above on the right side. The animal succumbed after 10 days, after it had continually displayed a tendency to move *and* roll towards the left. The sections were stained with osmic-acid, and cut in orderly series.

In consequence of the operation under consideration we find, (figs 7, 6, 5) degenerated at the side operated upon, the bundle (whose course agrees with the tr. interstitio-spinalis (Boyce's bundle), to be much richer in fibres in the pigeon than in the cat. This degeneration proves that, in spite of the extremely lateral lesion, still a nucleus, partly homologous to nuc. interstitialis in the cat, has either been injured through the incision, or is situated more lateral than the lesion, anyhow is severed by the stab into the longitudinal bundle. Quite in keeping with the consequence of such an operation on the right side in cats, the animal invariably rolled towards the left after the dextrolateral operation.

Until the site of origin is exactly determined, this bundle in the pigeon we propose to designate by the name of tr. metathalamo-spinalis.

Whereas after a similar lesion in the cat and in the rabbit circus-movement towards the *impaired* side takes place, as a rule, concomitantly with the rolling towards the *unimpaired* side (and degeneration of the tr. commissuro-medullaris), with this pigeon the circus-movement (and the conjugate deviation of the eyes) had likewise been observed towards the *healthy* side, at the same time no proof was afforded of an injury done to the left tr. commissuro-medullaris. Still, here no doubt that part of the commissura posterior (middle layer) has been severed, in which according to the experiment, as well in birds as in mammals, those secondary ascending vestibular bundles are running of which a lesion always engenders circus-movement towards the uninjured side.

These degenerated commissural bundles can be traced out in the preparations towards the left up to the region of the nuc. spiriformis and the nuc. praetectalis.

From the region of the scar a bundle runs orally to the mesostriatum in the areal of tr. strio-mesencephalicus. In the light of what has been

said in the first paragraph, it is quite natural that we should suppose to have to do here with degeneration of the tr. strio-mesencephalicus, but now of the ascending fibres. These fibres are thinner, less numerous and slightly more ventral in the areal (MUNZER and WIENER) than the descending ones.

As to fish it should be borne in mind, that WALLENBERG¹⁾ recognized in Selachians termination of the tr. strio-mesencephalicus near the oral end of the fasc. longitudinalis posterior. In cyprinus auratus he traced out a descending bundle from the nuc. lentiformis thalami²⁾ down to the caudal part of the medulla oblongata.

About the relations in man of the ascending and the descending vestibular bundles in the longitudinal bundle and in the commissure, little is known thus far. We do know that here also, in contradistinction to the quadruped vertebrates, the bundles descending from the commissural nuclei are not disposed medially. As the assumption is warrantable of the existence, also in man, of a wider development of the superposed, vestibular nuclei, and of the commissural region (erect posture, complicate relations of the eye-movements), it is also reasonable to suppose, that here also considerable extension is answerable for the anomalous position of the descending bundles in the longitudinal bundle.

§ 3. *Some of the principal connections of the nuc. spiriformis.*

In the pigeon we have identified the nuc. spiriformis as the nucleus, at least as one of the nuclei, that have been inserted into the centrifugal bundle originating in the forebrain and governing the unilateral movement (section of the bundle causes movement towards the side of the lesion). It will now be worth while ascertaining what the anatomical pioneers of the bird's brain have to say about the morphological features of the nucleus, round which our interest has centred in this experiment. Whereas WALLENBERG was struck only by the fact, that the nuc. spiriformis is a terminal stage of longitudinal bundle fibres as well as of a majority of the fibres of the tr. strio-mesencephalicus, it is interesting to observe that MESDAG was obviously struck by the large quantity (6) of the connections of this nucleus found by him, so that he devotes more space to the description of that nucleus than to any of the others, and gives us many particulars concerning them by photographs, and by pictures³⁾. The sand-glass-shaped nucleus, then, "consists of medium-sized, triangular or multangular ganglion-cells with exceedingly ramified protoplasm-processes. The cells are packed closely. The axis-cylinders in the centre of the nucleus are not distinguishable from the protoplasm-processes. The nucleus stains very deep with the Cajal method, so that it is very

¹⁾ Anatomischer Anzeiger, 31, 1907, p. 395.

²⁾ This term is applied to a meta-thalamic nucleus of fishes.

³⁾ Loc. cit, p. 107.

conspicuous in all sorts of series (see photographs 5 and 6). The cells with the deepest stain lie on the anterior margin. The nucleus is joined by all sorts of fibre-tracts to various parts of the mid-brain".

And further on MESDAG says: "There is no doubt but that the Nuc. Spiriformis is extremely significant for the mid-brain. In the paragraph on the optic tract this point will be resumed".

I have devised a diagram exclusively of the tracts, that could be verified by MESDAG after CAJAL's method in chick-embryos, and by WALLENBERG after the MARCHI method in pigeons, and by myself after either method.

Undoubtedly the tracts for the vestibular- and the optic impressions are foremost among the reflex-forming connections which govern the direction of the eyes and of the movement to one side, i. e. the movement in the horizontal plane. As regards the former we see for instance rising out of the left vestibular nucleus into the longitudinal

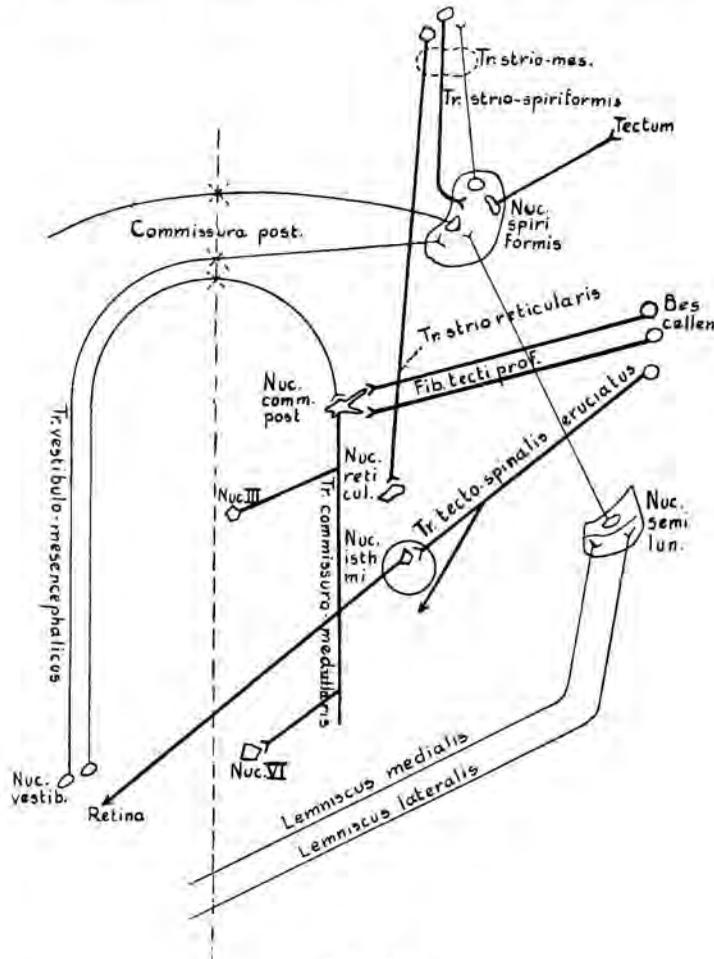


Fig 10.

bundle, a tract that (according to CAJAL-preparations of a six-day-old chick embryo) crosses in the commissura posterior, and terminates in the right nuc. spiriformis, and the right nuc. commissurae posterioris. Severance of that bundle (section of the left longitudinal bundle) will render the stimulation of the nuclei concerned insufficient, and will on that account induce circus-movement to the one side, which fits in with the physiological observation (section of the left longitudinal bundle yields circus to the right).

As for the optic stimuli, we know a tractus of the deep tectum fibres, whose axis-cylinders, according to MESDAG are disposed towards the protoplasm-processes of the ganglion-cells, that constitute the origin of descending fibres in the longitudinal bundle. Where we know, that *lesion* of that nucleus (homologon of the nuc. commissurae post. of mammals) causes the animal to move and look towards the operated side — so *mutatis mutandis* a *stimulation* of that nucleus: moving and looking towards the other side, thus towards the illuminated field of vision — so here also the anatomical connections tally with the physiological observation, for blinding of one eye yields for a short period a deviation towards the seeing side. We should not omit mentioning here the ganglion Isthmi-retina connection (reflex cone-movements of VAN GELDEREN STORT ¹⁾) according to the amount of illumination. The ventral part of the ganglion isthmi is *pierced* by the tr. tectospinalis non cruciatus, whereas, according to WALLENBERG the Isthmoretinal bundle ends in the crossed retina. So, no contrast here either.

As to the acoustic tractus, CAJAL and MESDAG see the lemniscus medialis terminate at the caudal wall of the tectum opticum, imparting fibres to the nuc. semilunaris medialis. The lemniscus lateralis also enters the nuc. semilunaris medialis, so that given the known tractus of the lobus opticus with the commissural nuclei, the acustical stimuli and tactile stimuli to one side of the body will induce the head and the eyes to move by reflex towards the side of the stimulation.

Anyhow, so far as we know, there is for the moving of head and eyes in the horizontal plane — the only one considered here — no contradiction between the anatomical and physiological data that have thus far been published.

¹⁾ Archiv. Néerlandaises T., 21, 1883.

Mathematics. — *Comparison of the Simplified Theories of the Irrational Number.* By Prof. FRED. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

(Communicated at the meeting of March 26, 1927).

1. Introduction. In an earlier paper (Fusion of the existing Theories of the Irrational Number into a New Theory) I have shown that the different theories of the irrational number may be simplified so that they only consist of the proofs of the following properties:

a. For any two real numbers a and β one and only one of the three relations $a = \beta$, $a > \beta$, $\beta > a$ holds good.

b. If $a > \beta$ and $\beta > \gamma$, we have $a > \gamma$.

c. If a is a real number, there always exists a rational number $> a$.

d. If a is a real number, there always exists a rational number $< a$.

e. If $a > \beta$, there always exists a rational number c , so that $a > c > \beta$.

f. For the system of the real numbers the theorem of the upper boundary holds good.

In this way the further discussion of addition and multiplication is made independent of any special theory of the irrational number and is based on the above mentioned properties.

The aim of this paper is to show how the different theories of the irrational number are connected and to prove the completeness of the system of the real numbers. We shall also give a simple proof of the possibility to extend the system of the rational numbers in such a way that the properties a.—f. hold good (cf. n^o. 2—5); this proof, slightly altered, may serve at the same time to demonstrate the completeness of the system of the real numbers (cf. n^o. 13).

2. Building of a system of real numbers. In several ways the system of the rational numbers can be extended to a system for which the properties a.—f. mentioned in n^o. 1 hold good. We shall do it here in a way which differs from the usual theories of the irrational number and which has some resemblance to the theories of DEDEKIND and BAUDET.

By a **class** we understand a set of rational numbers that is not empty, does not contain every rational number, has no greatest number and has the property that a rational number which is less than a number of the set, likewise belongs to the set. From this it follows immediately that a number of a class is less than a rational number that does not belong to the class and that a rational number which is greater than a

rational number that does not belong to the class, does not belong to the class either.

If one of the rational numbers that do not belong to the class is the smallest and if a is this smallest number, the class consists of the rational numbers $< a$. In this case we call the class **rational**.

We consider the classes as numbers (*real numbers*) and different classes as different numbers. By assuming that a rational class must be identified with the smallest rational number that does not belong to it, also the rational numbers are contained in the new system of numbers.

3. We call a class K **greater** than a class L , if K contains a number a that does not belong to L . In this case a number b of L is $< a$, hence a number of K , so that the relation $L > K$ is not satisfied. Accordingly for two classes K and L one and only one of the three relations $K = L$ (which means that the classes K and L are identical), $K > L$, $K < L$ holds good.

If the classes K and L are both rational, with k , resp. l , as smallest rational numbers that do not belong to them, and if $k > l$, l is a number of K that does not belong to L . Consequently $K > L$, so that the new definition of greater applied to two rational numbers, leads to the same result as the old definition.

4. A class K is built of the rational numbers $< K$.

For if a is a number of K , $a < K$, as a does not belong to the class formed by the rational numbers $< a$. If, inversely, a is a rational number $< K$, there exists a rational number b which belongs to K but not to the class of the rational numbers $< a$; accordingly $a \equiv b$, so that a belongs to K (because b belongs to it).

5. **Proofs of the properties mentioned in n^o. 1.** *Proof of a.* The validity of a . is apparent from n^o. 3.

Proof of b. If K , L and M are classes that satisfy the relations $K > L$ and $L > M$, there exists a rational number a which belongs to K but not to L and a rational number b which belongs to L but not to M . $b < a$ so that b belongs to K . As b does not belong to M we have $K > M$.

Proof of c. Let K be a class, a a rational number that does not belong to K , and b a rational number $> a$. In this case the number a belongs to the class of the rational numbers $< b$ but not to K , so that $b > K$.

Proof of d. If K is a class and c a number of K , we have $c < K$ (see the property of n^o. 4).

Proof of e. Let K and L be classes that satisfy $K > L$ and let a be a number of K that does not belong to L . As K has no greatest number, there is a rational number c that is $> a$ and belongs to K . This number c does not belong to L . As a belongs to the class of the rational numbers

$< c$, we have $c > L$. From the property of $n^0.4$ it ensues further that $c < K$.

Proof of f. Let V be a non-empty set of real numbers all of which are smaller than the real number M . We form the set B of the rational numbers that are smaller than a number of V . According to d , this set is not empty. According to c , there is a rational number $> M$; according to b , this does not belong to B . If b is a number of B and v a number of V which is $> b$, according to e , there is a rational number c so that $b < c < v$; as c is a number of B , B has no greatest number. As moreover a rational number which is smaller than a number of B , evidently also belongs to B , B is a class. We shall now show that B is the upper boundary of V .

Let v be a number of V which is $> B$. Then there is a rational number c so that $B < c < v$. This number c belongs to the class B (as $c < v$) so that $B < c$ is in conflict with the property of $n^0.4$. Accordingly a number of V which is $> B$, does not exist.

Let C be a real number $< B$. There is a rational number a so that $C < a < B$. According to the property of $n^0.4$ a is a number of the class B so that a is smaller than a number of V . This number of V is also greater than C .

6. Similar systems. A system of real numbers which contains all rational numbers and for which the properties $a.-e.$ mentioned in $n^0.1$ hold good, we shall briefly call a **system**.

We shall call two systems S and S' **similar** if such a $(1, 1)$ -correspondence can be established between the numbers of S and those of S' that the following two properties are valid:

1^o. the rational numbers correspond to themselves;

2^o. if α and β are two numbers of S so that $\alpha > \beta$, and if α' and β' are the corresponding numbers of S' , we have $\alpha' > \beta'$.

We shall call a correspondence that has this property, a **representation of one system on the other**.

The similitude of systems is *commutative* and *transitive*. Every system is similar to itself as we can make every number of the system correspond to itself (**identical representation**).

7. If S and S' are two similar systems, in the representation of S on S' the common numbers of the two systems correspond to themselves. We suppose that for two common numbers the relation "greater" is the same in the two systems.

Let a be a number that belongs to the two systems and a' the number of S' which correspond to a (as number of S). As a and a' both belong to S' , we have $a' > a$, $a' < a$ or $a' = a$. If $a' > a$ there is a rational number b so that $a' > b > a$. From $a < b$ (numbers of S) follows $a' < b$ (corresponding numbers of S') in contradiction to $a' > b$. In the same

way $\alpha' < \alpha$ leads to an absurdity so that $\alpha' = \alpha$ and, accordingly, α corresponds to itself.

Immediate consequences of this theorem are:

the only representation of a system on itself is the identical one;
two systems cannot be represented on each other in more than one way;
a system is not similar to a real part of itself.

8. Owing to the commutativity and the transitivity of similitude we can form a set of systems any two of which are similar. We can unite the corresponding numbers of these systems to one conception and consider this as a "number" of which the said corresponding numbers are the representatives.

For according to what was found in n^o. 7 no two of these corresponding numbers are different numbers of the same system so that a "number" in the new sense does not get two unequal numbers of an already existing system as representatives. As the relation "greater" remains unchanged by the representation, it does not matter from which system we derive the representatives if we have to judge which number is greater.

In the indicated way similar systems may be united to one system. By so doing the difference between similar systems disappears entirely as this difference only consists in the names that are given to the numbers or in the way in which they are indicated.

9. Systems for which the theorem of the upper boundary is valid.

We prove:

two systems S and S' for which the theorem of the upper boundary is valid, are similar.

Let a be a number of S and A the set of the rational numbers $< a$. In S the number a is the upper boundary of A ; for in the first place A has no number $> a$; if γ is a number of S that is $< a$, there is a rational number a so that $\gamma < a < a$; hence a is a number of A that is $> \gamma$. The set A has also an upper boundary in S' ; we shall call this a' . We shall make this number a' correspond to a .

Let β be another number of S and suppose $a < \beta$. In the same way from β we can derive a number β' of S' , which we shall make correspond to β . Now we can choose the rational numbers p and q such that $a < p < q < \beta$. As β' is the upper boundary in S' of the set B of the rational numbers $< \beta$, and as q belongs to B , we have $q \equiv \beta'$. As $p > a$ a rational number $> p$ is also $> a$, hence not a number of A ; accordingly A does not contain any number $> p$, so that $p \equiv a'$; for $p < a'$ is contradictory to the fact that a' is the upper boundary of A in S' . From $q \equiv \beta'$, $p \equiv a'$ and $p < q$ we may further conclude that $a' < \beta'$. Consequently to unequal numbers of S there correspond unequal numbers of S' .

A number a of A is $\leq a'$; however, $a = a'$ is excluded as A has no greatest number; a number of A is, therefore, $< a'$. If, inversely, a is a rational number $< a'$, A contains such a number a_1 that $a_1 > a$ (as a' is the upper boundary of A in S'); this number a_1 satisfies the relation $a < a_1 < a$, so that $a < a$ and a is a number of A . The set A is, accordingly, the same as that of the rational numbers $< a'$. Hence to any number a' of S' there corresponds a number a of S , viz. the upper boundary in S of the set of the rational numbers $< a'$.

In the indicated way we get a (1,1)-correspondence between the numbers of S and those of S' for which the property 2^o of n^o. 6 holds good. If a is a rational number, a is the upper boundary in S' of the set of the rational numbers $< a$; hence the correspondence has also the property 1^o of n^o. 6, so that this correspondence is a representation of S on S' .

10. *If we extend the system of the rational numbers to a system S for which the properties a.—f. of n^o. 1 hold good, and if we omit irrational numbers and add irrational numbers so that a system S' arises for which the said properties likewise hold good, the systems S and S' can be represented on each other (according to the theorem of n^o. 9). According to the theorem of n^o. 7 the omitted numbers correspond to the added ones. This shows that the transition from S to S' only consists in this that numbers are omitted and afterwards added again under another name, for instance that the irrational numbers according to the theory of CANTOR are omitted and the irrational numbers according to the theory of DEDEKIND are added.*

11. Completeness of a system of numbers for which the theorem of the upper boundary holds good. We have:

to a system S for which the theorem of the upper boundary holds good, not a single number can be added if we want the properties a.—e. of n^o. 1 to remain valid after the addition and also the relation "greater" to remain unchanged for two numbers of S .

Suppose that S may be extended to the system S' (with conservation of the properties a.—e. of n^o. 1). Let a' be a number of S' that does not belong to S and A the set of the rational numbers $< a'$. This set has an upper boundary a in S ; this number a also belongs to S' . As a' does not belong to S , we have $a < a'$ or $a > a'$.

If $a < a'$ and if p is a rational number so that $a < p < a'$, p is a number of A and hence $p > a$ is contradictory to the fact that a is the upper boundary of A in S . If $a > a'$ and if p is a rational number so that $a > p > a'$, it follows from $p < a$ that there is a number a of A greater than p ; from $a > p$ and $p > a'$ follows $a > a'$, in contradiction to the definition of A . Hence in both cases we arrive at an absurdity.

12. The theorem of n^0 . 11 shows that we cannot omit any number from the system S considered there, if we want the properties a.—f. of n^0 . 1 to remain valid. For the application of the theorem of n^0 . 11 to the system that arises from S through the omission of numbers would lead to a contradiction.

13. Theorem of completeness. We understand by this:

for a system that is not liable to extension if we want the properties a.—e. of n^0 . 1 to remain valid, the theorem of the upper boundary holds good.

We mean, of course, such an extension that the relation "greater" for the numbers of the original system remains unchanged.

We shall extend the system S by forming classes. In deviation from n^0 . 2 we shall now understand by a **class** a set of numbers of S which is not empty, does not contain every number of S , has no greatest number and has the property that a number of S which is smaller than a number of the set, likewise belongs to the set.

If among the numbers of S that do not belong to the class there is a smallest a , the class consists of the numbers of S which are $< a$.

We shall consider the classes formed in this way as numbers of a new system S' where we identify the class of the numbers of S which are $< a$ (a is a number of S) with a . In this way the numbers of S are also contained in S' .

If K and L are two classes, we give the same definition of $K > L$ as in n^0 . 3. In the same way as there it appears that the definition of greater which is valid in S' , applied to two numbers of S , leads to the same result as the definition which holds good in S . Accordingly for the system S' the property a. of n^0 . 1 is valid.

The validity of the property b. of n^0 . 1 for the system S' is proved as in n^0 . 5.

If K is a class, b a number of S that does not belong to K , and c a rational number $> b$, we have $c > K$ (property c. of n^0 . 1). If a is a number of K and d a rational number $< a$, we have $d < K$ (property d. of n^0 . 1).

Let K and L be two classes so that $K > L$ and let a be a number of K that does not belong to L . The class K contains a number $b > a$. If c is a rational number so that $b > c > a$, we have $K > c > L$ (property e. of n^0 . 1).

Let V be a non-empty set of numbers of S' all of which are smaller than the number M of S' . We shall now build the set B of the numbers of S that are smaller than a number of V . Just as in n^0 . 5 it appears that B is a class so that B is a number of S' . If v is a number of V which is $> B$, there is a rational number c (hence a number of S) so that $B < c < v$; according to $c < v$ c is a number of B and from $B < c$ it follows that c is not a number of B ; consequently $v > B$ is impossible.

If C is a number of S' which is $< B$ and if a is a rational number so that $C < a < B$, a is a number of B , hence a , and also C , is smaller than a number of V . This shows that B is the upper boundary of V in S' , so that for the system S' also the property f . of n^0 . 1 (theorem of the upper boundary) holds good.

As for the system S' the properties $a.$ — $e.$ of n^0 . 1 are valid, S' is a system as defined in n^0 . 6. It is further evident from the suppositions of the theorem of completeness, that the systems S and S' are identical. As the theorem of the upper boundary holds good for S' , this is also valid for S .

14. The theorem of n^0 . 13 together with that of n^0 . 11 (of which it is the converse) shows, that it amounts to the same if we say of a system that the theorem of the upper boundary holds good for it or that a further extension of the system is impossible. We can also say that the condition that is necessary and sufficient for the impossibility of a further extension of the system, is the validity for that system of the theorem of the upper boundary. It follows further from the theorem of n^0 . 9 that two systems which are not liable to extension (with conservation of the properties $a.$ — $e.$ of n^0 . 1) are similar.

15. A system S can be completed in the way indicated in n^0 . 13 to a system for which the theorem of the upper boundary holds good. From this in connection with the above it is evident that:

a system S is a part of the system of the real numbers and a real part or not according as an extension of S (with conservation of the properties $a.$ — $e.$ of n^0 . 1) is possible or impossible, hence according as the theorem of the upper boundary holds not good for S or holds good.

This means that S is similar to a part of any system for which the theorem of the upper boundary holds good.

Mathematics. — *Analytical Introduction of the Trigonometrical Functions, adapted to the Geometrical Introduction.* By Prof. FRED. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

(Communicated at the meeting of March 26, 1927).

1. Introduction. In the geometrical introduction of the trigonometrical functions (by the aid of a circle with radius 1) the idea "length of an arc of a circle" is supposed to be defined. This length of an arc can be defined as the limit of the perimeter of a *regular inscribed polygon* $A_0 A_1 A_2 \dots A_n$ if n increases. We can also indefinitely replace n by its double.

With advantage we can replace $A_0 A_1 A_2 \dots A_n$ by the *circumscribed polygon* $A_0 B_1 B_2 \dots B_n A_n$ formed by the tangents at the points A_0, A_1, \dots, A_n . In this way we get a simpler formula for y_{2n} , to wit $y_{2n} = \frac{4ny_n}{2n + \sqrt{4n^2 + y_n^2}}$, where y_n is the length of $A_0 B_1 \dots B_n A_n$. By putting $y_{2^n} = x_{n+1}$ this becomes:

$$x_{n+1} = \frac{2^{n+1}x_n}{2^n + \sqrt{4^n + x_n^2}} \dots \dots \dots (1)$$

The length s of the arc $A_0 A_n$ (which is supposed $< \frac{1}{2} \pi$) is $\lim_{n \rightarrow \infty} x_n$. If we put $\text{tg } s = x = x_0$, (1) is also satisfied for $n = 0$. We have, therefore, $\lim_{n \rightarrow \infty} x_n = \text{arc tg } x$ where x_n is defined by (1) and $x_0 = x$.

Inversely we can define arc tg x as $\lim_{n \rightarrow \infty} x_n$, which is a purely analytical way of introducing the cyclometrical and trigonometrical functions. In what follows we shall discuss this method more fully.

2. Arc tg x defined as limit of a sequence. We consider the *real sequence* $\{x_n\}$ which is defined by (1) and $x_0 = x$ where x is a given real number. In order to indicate to which value of x the number x_n corresponds, we also write $x_n(x)$ in stead of x_n .

We have $x_n(0) = 0$ and $x_n(-x) = -x_n(x)$, $x_n(x) > 0$ for $x > 0$. If $x > 0$, we have $x_{n+1} < x_n$; accordingly the sequence $\{x_n\}$ decreases monotonely; as in this case $x_n > 0$ for any index n , the sequence is convergent. The latter holds also good for $x \leq 0$, hence for any real value of x . We call the limit of this sequence *arc tg x*; accordingly:

$$\text{arc tg } x = \lim_{n \rightarrow \infty} x_n(x) \dots \dots \dots (2)$$

We have $\text{arc tg } 0 = 0$ and $\text{arc tg } (-x) = -\text{arc tg } x$.

3. The derivative of arc tg x. From (1) ensues:

$$\frac{dx_{n+1}}{dx_n} = \frac{2^{2n+1}}{\sqrt{4^n + x_n^2} (2 + \sqrt{4^n + x_n^2})} = \frac{4^{n+1} + x_{n+1}^2}{4(4^n + x_n^2)},$$

hence:

$$x'_n(x) = \frac{dx_n}{dx} = \frac{dx_n}{dx_{n-1}} \frac{dx_{n-1}}{dx_{n-2}} \cdots \frac{dx_2}{dx_1} \frac{dx_1}{dx} = \frac{1}{1+x^2} + \frac{x_n^2}{4^n(1+x^2)}. \quad (3)$$

consequently:

$$x'_n(x) = \frac{1}{1+x^2} + \frac{x_n^2}{4^n(1+x^2)} \leq \frac{x^2}{4^n(1+x^2)} < \frac{1}{4^n},$$

so that $x'_n(x)$ tends uniformly to the limit $\frac{1}{1+x^2}$ (in the interval formed by all real values of x) if n increases indefinitely. As according to (3) $x'_n(x)$ is a continuous function of x for any index n and any value of x , there follows from (2):

$$\frac{d \text{arc tg } x}{dx} = \lim_{n \rightarrow \infty} x'_n(x) = \frac{1}{1+x^2} \dots \dots \dots (4)$$

Consequently arc tg x is a differentiable function of x .

The validity of (4) for $x=0$ means that $\lim_{x \rightarrow 0} \frac{\text{arc tg } x}{x} = 1$.

4. According to (4) the derivative of arc tg x is always > 0 ; hence arc tg x is a monotonely increasing function of x . From this, in connection with arc tg $0 = 0$, it follows that arc tg x is positive when x is positive and negative when x is negative.

5. Formula for arc tg $a + \text{arc tg } b$. We consider the function

$$\varphi(x) = \text{arc tg } x - \text{arc tg } \frac{x-b}{1+bx};$$

if $b \neq 0$, the value $-\frac{1}{b}$ of x , for which $1+bx=0$, must be excluded.

According to (4) we find $\varphi'(x) = 0$, so that $\varphi(x)$ is constant in an interval where $1+bx$ does not assume the value 0. If $ab > -1$, $a \leq x \leq b$ or $a \geq x \geq b$ is such an interval, so that in this case $\varphi(a) = \varphi(b)$. This gives:

if $ab > -1$ we have:

$$\text{arc tg } a - \text{arc tg } b = \text{arc tg } \frac{a-b}{1+ab} \dots \dots \dots (5)$$

If we replace b by $-b$ this becomes:

if $ab < 1$ we have:

$$\text{arc tg } a + \text{arc tg } b = \text{arc tg } \frac{a+b}{1-ab} \dots \dots \dots (6)$$

6. The number $4 \operatorname{arc} \operatorname{tg} 1$ is represented by π . By putting in (5) $b = 1$ we find for $x > -1$:

$$\operatorname{arc} \operatorname{tg} x = \frac{1}{4} \pi + \operatorname{arc} \operatorname{tg} \frac{x-1}{x+1}.$$

In connection with the continuity of $\operatorname{arc} \operatorname{tg} x$ this leads to:

$$\lim_{x \rightarrow \infty} \operatorname{arc} \operatorname{tg} x = \frac{1}{2} \pi, \dots \dots \dots (7)$$

whence:

$$\lim_{x \rightarrow -\infty} \operatorname{arc} \operatorname{tg} x = -\frac{1}{2} \pi \dots \dots \dots (8)$$

7. If in (6) $a > 0$ and if b increases and tends to $\frac{1}{a}$, $\frac{a+b}{1-ab}$ increases indefinitely. Hence according to (7):

$$\operatorname{arc} \operatorname{tg} a + \operatorname{arc} \operatorname{tg} \frac{1}{a} = \frac{1}{2} \pi \quad (a > 0) \dots \dots \dots (9)$$

Further:

$$\operatorname{arc} \operatorname{tg} a + \operatorname{arc} \operatorname{tg} \frac{1}{a} = -\frac{1}{2} \pi \quad (a < 0) \dots \dots \dots (10)$$

From (9) and (10) there follows:

if $-\frac{1}{2} \pi < \operatorname{arc} \operatorname{tg} a + \operatorname{arc} \operatorname{tg} b < \frac{1}{2} \pi$, we have $ab < 1$.

In the first place this is the case for $a = 0$. If $a > 0$ it follows from $\operatorname{arc} \operatorname{tg} a + \operatorname{arc} \operatorname{tg} b < \frac{1}{2} \pi$ in connection with (9), that $\operatorname{arc} \operatorname{tg} b < \operatorname{arc} \operatorname{tg} \frac{1}{a}$, hence $b < \frac{1}{a}$, hence $ab < 1$. If $a < 0$ it follows from $\operatorname{arc} \operatorname{tg} a + \operatorname{arc} \operatorname{tg} b > -\frac{1}{2} \pi$, in connection with (10), that $\operatorname{arc} \operatorname{tg} b > \operatorname{arc} \operatorname{tg} \frac{1}{a}$, hence $b > \frac{1}{a}$, hence $ab < 1$.

Inversely from $ab < 1$ we can conclude $-\frac{1}{2} \pi < \operatorname{arc} \operatorname{tg} a + \operatorname{arc} \operatorname{tg} b < \frac{1}{2} \pi$; this ensues immediately from the formula (6) which holds good for $ab < 1$.

8. Definition of $\operatorname{tg} x$. As $\operatorname{arc} \operatorname{tg} x$ is an increasing function, it follows from (7) and (8) that $\operatorname{arc} \operatorname{tg} x$ does not assume any value $\cong \frac{1}{2} \pi$ and no value $\cong -\frac{1}{2} \pi$. $\operatorname{Arc} \operatorname{tg} x$ assumes any value y between $-\frac{1}{2} \pi$ and $\frac{1}{2} \pi$ for one and only one value of x (as $\operatorname{arc} \operatorname{tg} x$ is continuous and monotone). This value of x is called $\operatorname{tg} y$.

For values of x so that $-\frac{1}{2} \pi < x < \frac{1}{2} \pi$, $\operatorname{tg} x$ is a continuous function of x , increasing monotonely. We have $\operatorname{tg} 0 = 0$, $\operatorname{tg}(-x) = -\operatorname{tg} x$, $\operatorname{tg} \frac{1}{4} \pi = 1$ and:

$$\lim_{x \rightarrow \frac{1}{2} \pi} \operatorname{tg} x = +\infty, \quad \lim_{x \rightarrow -\frac{1}{2} \pi} \operatorname{tg} x = -\infty \dots \dots \dots (11)$$

We suppose that x while increasing tends to $\frac{1}{2}\pi$ and while decreasing to $-\frac{1}{2}\pi$ as $\operatorname{tg} x$ has as yet only been defined for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

From $\lim_{x \rightarrow 0} \frac{\operatorname{arc} \operatorname{tg} x}{x} = 1$ it follows further that $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x} = 1$.

9. Addition-theorem for $\operatorname{tg} x$. By assuming $\operatorname{arc} \operatorname{tg} a = p$ and $\operatorname{arc} \operatorname{tg} b = q$ we find from (6):

$$\operatorname{tg}(p + q) = \frac{\operatorname{tg} p + \operatorname{tg} q}{1 - \operatorname{tg} p \operatorname{tg} q}, \quad \dots \dots \dots (12)$$

where p, q and $p + q$ are supposed to lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. If this is the case we have $-\frac{1}{2}\pi < \operatorname{arc} \operatorname{tg} a + \operatorname{arc} \operatorname{tg} b < \frac{1}{2}\pi$, hence $ab < 1$ (see $n^0. 7$), so that the formula (6) holds good, hence also (12).

We find, accordingly:

for any set of values of p and q for which p, q and $p + q$ all lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ the formula (12) is valid.

10. Supplement of the definition of $\operatorname{tg} x$. If q tends to $-\frac{1}{2}\pi$ while decreasing, (12), in connection with the second formula (11), gives:

$$\operatorname{tg}(p - \frac{1}{2}\pi) = -\frac{1}{\operatorname{tg} p} \quad \dots \dots \dots (13)$$

on the supposition that $0 < p < \frac{1}{2}\pi$.

We shall now extend the definition of $\operatorname{tg} x$ to arbitrary real values of x so that (13) becomes of general validity. As it ensues from (13) that $\operatorname{tg}(p - \pi) = \operatorname{tg} p$, this gives:

if x is a number which is not of the form $(n + \frac{1}{2})\pi$ (n integer), according to definition $\operatorname{tg} x = \operatorname{tg} x'$; x' is here the number between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ for which $\frac{x - x'}{\pi}$ is an integer and $\operatorname{tg} x$ has the meaning indicated in $n^0. 8$.

This definition leaves $\operatorname{tg} x$ an odd function of x .

11. According to $n^0. 10$ $\operatorname{tg} x$ is periodical with a period π .

In the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ we have $\operatorname{tg} x = a$, where a is a given number, for only one value x_1 of x . Owing to the periodicity of $\operatorname{tg} x$ we have $\operatorname{tg} x = a$ for $x = x_1 + n\pi$ (n integer) and for no other value of x . We may also express this thus: *it follows from $\operatorname{tg} x = \operatorname{tg} y$ that $x - y$ is a multiple of π .*

12. *If none of the numbers $p, q, p + q$ is of the form $(n + \frac{1}{2})\pi$ (n integer), the formula (12) holds good.*

In order to prove this we may, owing to the periodicity of $\operatorname{tg} x$, assume that p and q lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The case that also $p + q$ lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ is that of $n^0. 9$. We may, therefore,

assume that $p + q$ lies between $1/2\pi$ and π (in which case p and q lie between 0 and $1/2\pi$), as the case $-\pi < p + q < -1/2\pi$ can be reduced to this because $\operatorname{tg} x$ is an odd function. Taking (13) into account we have in this case:

$$\begin{aligned} \operatorname{tg}(p+q) &= \operatorname{tg}\{(p-1/2\pi) + (q-1/2\pi)\} = \frac{\operatorname{tg}(p-1/2\pi) + \operatorname{tg}(q-1/2\pi)}{1 - \operatorname{tg}(p-1/2\pi)\operatorname{tg}(q-1/2\pi)} = \\ &= \frac{1}{\operatorname{tg} p} \frac{1}{\operatorname{tg} q} = \frac{\operatorname{tg} p + \operatorname{tg} q}{1 - \operatorname{tg} p \operatorname{tg} q}. \end{aligned}$$

13. Definitions of $\sin x$ and $\cos x$. As definitions we give:

$$\sin x = \frac{2 \operatorname{tg} 1/2 x}{1 + \operatorname{tg}^2 1/2 x}, \quad \cos x = \frac{1 - \operatorname{tg}^2 1/2 x}{1 + \operatorname{tg}^2 1/2 x} \quad \dots \quad (14)$$

These definitions fall short if x is of the form $(2n+1)\pi$ (n integer). Leaving the *continuity* of $\sin x$ and $\cos x$ intact, also for $x = (2n+1)\pi$, we find from (14) by taking the limit:

$$\sin(2n+1)\pi = 0, \quad \cos(2n+1)\pi = -1.$$

It ensues from (14) that $\sin x$ and $\cos x$ are periodical functions with a period 2π , and further, in connection with (12), that $\frac{\sin x}{\cos x} = \operatorname{tg} x$;

$\lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x} = 1$ leads further to $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

From (12) and (14) we also find the formulas for $\sin(a+b)$ and $\cos(a+b)$ and in the well known way the other properties of $\sin x$ and $\cos x$ can be derived from them.

14. Application of the trigonometrical functions to angles. We can easily prove:

if the numbers a and b satisfy the relation $a^2 + b^2 = 1$ we can always find one and only one number x so that $\cos x = a$, $\sin x = b$, $-\pi < x \leq \pi$.

From this we can a posteriori derive the connection between trigonometrical functions and angles (which in the elementary trigonometry is the starting point for the definition of these functions). See for this my book: "Het getalbegrip, in het bijzonder het Onmeetbare getal, met toepassingen op de Algebra, de Differentiaal- en de Integraalrekening," §§ 285—290.

Chemistry. — *Investigations into the Constitution of Artificial Ultramarines. II. On Ultramarine-blue with high Silica-content and on Silver-, Silversodium-, Selenium- and Silverselenium-Ultramarines.*
By Prof. Dr. F. M. JAEGER and Mr. F. A. VAN MELLE.

(Communicated at the meeting of May 28, 1927).

§ 1. In connection with our paper, already published in these Proceedings¹⁾, concerning the constitution of artificial and natural *ultramarines*, it was planned by us to substitute successively a part or all of the mobile constituents of these compounds, e.g. their *sodium*-atoms by *Li*, *K*, *Rb*, *Cs*, *Zn*, *Ba*, *Sr*, *Ag*, etc., or their *sulphur*-atoms by *selenium*, — with the purpose to elucidate the influence of such substitutions upon the ROENTGEN-spectrograms obtained with these substances. In this way it would perhaps be possible to ascertain, which components among those were a real part of the „rigid, periodical structure” of these compounds and which were merely dispersed in a statistical way within that rigid structure.

To proceed quite systematically in this, we have restricted our studies to one species of *ultramarine* only, chosen as primary material, namely to the *ultramarine-blue*-GUIMET N^o. 7553, which in a pure, thoroughly washed and unground condition was most kindly given us in a considerable quantity by the *Usines d'Outremer*-GUIMET in *Fleurieu sur Saône*. Its composition appeared to correspond almost exactly to the formula: $Na_6Al_4Si_6O_{23}S_3$; its colour was a vivid, brilliant blue²⁾.

For the sake of completeness, in Table I the results of the analysis of a powder-spectrogram of this preparation are put together. As in all sequent cases, an improved cylindrical camera was used, having a diameter of 88,8 m.m., and allowing the photographic film to be attached to its outer side; the camera is completely inclosed in a copper box. This variation of the older instrument has satisfied us in all respects.

In total 33 interference-images were measured; the grating-constant is here: $a_0 = 9,06$ A.U.

¹⁾ These Proceedings, 30, 249, (1927).

²⁾ The ultramarine used here is an ultramarine with high silica- and moderate sulphur-content; its spec. weight at (25° C.) was: 2.35. Analysis furnished: 15,4% *Na*; 13,0% *Al*; 19,2% *Si*; 10,6% *S*; 41,8% *O*, in which is included 2,78% clay-residue. For $Na_6Al_4Si_6O_{23}S_3$ is necessary: 15,6% *Na*; 12,3% *Al*; 19,3% *Si*; 10,9% *S*; 41,8% *O*. In no case the substance can correspond, as HOFFMANN supposed, to the aluminium-sodium-silicate: $Na_4Al_4Si_6O_{20}$; neither does it correspond to the *Marienberger blue* used by HEUMANN, because this was a product poor in silica.

TABLE I. Powder-spectrogram of Ultramarine-blue: $Na_6 Al_4 Si_6 O_{23} S_3$
 (GUIMET—III, N^o. 7553). (Fig. 1A).

N ^o . of Image	2l in mm.:	Estim. Intensities:	λ :	$\Sigma(h^2)$:	θ 2' :	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices:
1	(21.4)	4	α	2	6°54'	0.0144	0.0144	(110) only once observed
2	37.1	10	α	6	11°58'	0.0430	0.0432	(211)
3	43.6	1	β	10	14°4'	0.0591	0.0590	(310)
4	48.2	7	α	10	15°33'	0.0719	0.0720	(310)
5	53.0	8	α	12	17°6'	0.0865	0.0864	(222)
6	55.4	1 ⁻	β	16	17°52'	0.0941	0.0944	(400)
7	57.1	2	α	14	18°25'	0.0999	0.1008	(321)
8	59.0	1 ⁻	β	18	19°2'	0.1063	0.1062	(330); (411)
9	61.5	4	α	16	19°50'	0.1151	0.1152	(400)
10	65.4	5	α	18	21°6'	0.1296	0.1296	(330); (411)
11	71.3	1	β	26	23°0'	0.1528	0.1534	(431); (510)
12	72.9	1	α	22	23°31'	0.1590	0.1584	(332)
13	79.6	5	α	26	25°41'	0.1877	0.1872	(431); (510)
14	82.4	1 ⁻	β	34	26°35'	0.2004	0.2006	(433); (530)
15	86.0	1	α	30	27°44'	0.2165	0.2160	(521)
16	89.2	3	α	32	28°47'	0.2317	0.2304	as β -line (600) (440)
17	92.0	2	α	34	29°41'	0.2451	0.2448	(433); (530)
18	95.0	2	α	36	30°39'	0.2599	0.2592	(600)
19	97.9	2	α	38	31°35'	0.2742	0.2736	(532); (611)
20	106.4	3	α	44	34°20'	0.3180	0.3168	(622)
21	109.1	1 ⁻	β	56	35°12'	0.3322	0.3304	(642)
22	111.9	2	α	48	36°6'	0.3471	0.3456	as α -line (631) (444)
23	114.7	1	α	50	37°0'	0.3621	0.3600	(543); (505); (710)
24	120.0	2	α	54	38°43'	0.3910	0.3888	(633); (721)
25	122.1	1	α	56	39°23'	0.4026	0.4032	(642)
26	124.8	1	α	58	40°16'	0.4178	0.4176	(730)
27	130.2	1	α	62	42°0'	0.4476	0.4464	(732)
28	132.8	1	α	64	42°50'	0.4620	0.4608	(800)
29	135.0	1	α	66	43°33'	0.4747	0.4752	(554); (741); (811)
30	145.5	1	α	74	46°56'	0.5337	0.5328	(750); (743); (831)
31	148.4	1	α	76	47°51'	0.5498	0.5472	(662)
32	150.9	1	α	78	48°41'	0.5640	0.5616	(752)
33	153.5	1	α	80	49°31'	0.5785	0.5760	(840)

 Radius of the Camera: 44.4 mm. $V = 55000$ Volt.

 Exposure: 2 hours. $\lambda_x = 1.540$ A.E.; $\lambda_\beta = 1.388$ A.U.

 Quadratic Equation: $\sin^2 \frac{\theta}{2} = 0.0072 (h^2 + k^2 + l^2)$ (α -radiation)

 $\sin^2 \frac{\theta}{2} = 0.0059 (h^2 + k^2 + l^2)$ (β -radiation)

§ 2. To corroborate our conclusions formerly drawn about the type of the fundamental grating and about the space-groups of these compounds, we wish to give here some additional and very persuasive complementary data concerning the minerals *noseane* and *hauyne*. We happened to prepare two rotation-spectrograms of some small, really simple crystals of both minerals, which spectrograms were splendidly developed, containing a great number of interference-images, which could accurately be measured. Moreover, we succeeded in rotating the crystals in these experiments round a principal axis [100], — an experiment which formerly had been

TABLE II. Rotation-Spectrograms of *Noseane* and *Hauyne*, if rotated round a principal axis [100].

A. <i>Noseane</i> . The following images were observed:	B. <i>Hauyne</i> . The following images were observed:
Principal spectrum: $\lambda_{\alpha} : (\overline{031}) ; (\overline{051}) ; (\overline{040}) ; (\overline{031}) ; (\overline{033}) ; (\overline{051})$.	Principal spectrum: $\lambda_{\alpha} : (\overline{013}) ; (\overline{004}) ; (\overline{015}) ; (\overline{033}) ; (\overline{053})$.
$\lambda_{\beta} : (\overline{053}) ; (\overline{053}) ; (\overline{033})$.	$\lambda_{\beta} : (\overline{004}) ; (\overline{035}) ; (\overline{033})$.
1st Accessory spectrum:	1st Accessory spectrum:
$\lambda_{\alpha} : (\overline{132}) ; (\overline{150}) ; (\overline{121}) ; (\overline{132}) ; (\overline{150}) ; (\overline{152})$	$\lambda_{\alpha} : (\overline{113}) ; (\overline{105}) ; (\overline{152}) ; (\overline{121}) ; (\overline{132}) ; (\overline{143})$.
$\lambda_{\beta} : (\overline{150})$.	$\lambda_{\beta} : (\overline{121}) ; (\overline{143})$.
— 1st Accessory spectrum:	— 1st Accessory spectrum:
$\lambda_{\alpha} : (\overline{150}) ; (\overline{132}) ; (\overline{150}) ; (\overline{152})$.	$\lambda_{\alpha} : (\overline{113}) ; (\overline{105}) ; (\overline{152}) ; (\overline{121}) ; (\overline{132}) ; (\overline{143})$.
$\lambda_{\beta} : (\overline{121}) ; (\overline{150})$.	$\lambda_{\beta} : (\overline{143})$.
2nd Accessory spectrum:	2nd Accessory spectrum:
$\lambda_{\alpha} : (\overline{231}) ; (\overline{233}) ; (\overline{242})$.	$\lambda_{\alpha} : (\overline{213}) ; (\overline{215}) ; (\overline{211}) ; (\overline{222}) ; (\overline{233}) ; (\overline{242})$.
$\lambda_{\beta} : (\overline{222})$.	$\lambda_{\beta} : (\overline{253}) ; (\overline{235})$.
— 2nd Accessory spectrum:	— 2nd Accessory spectrum:
$\lambda_{\alpha} : (\overline{231}) ; (\overline{220}) ; (\overline{233}) ; (\overline{242})$.	$\lambda_{\alpha} : (\overline{213}) ; (\overline{215}) ; (\overline{222}) ; (\overline{233}) ; (\overline{242})$.
$\lambda_{\beta} : (\overline{222})$.	$\lambda_{\beta} : (\overline{222}) ; (\overline{253})$.
3rd Accessory spectrum:	3rd Accessory spectrum:
$\lambda_{\alpha} : (\overline{332}) ; (\overline{310}) ; (\overline{321}) ; (\overline{332})$.	$\lambda_{\alpha} : (\overline{312}) ; (\overline{303}) ; (\overline{314}) ; (\overline{310}) ; (\overline{321}) ; (\overline{332}) ; (\overline{350})$.
$\lambda_{\beta} : (\overline{350}) ; (\overline{352})$.	$\lambda_{\beta} : (\overline{352}) ; (\overline{350})$.
— 3rd Accessory spectrum:	— 3rd Accessory spectrum:
$\lambda_{\alpha} : (\overline{341}) ; (\overline{310}) ; (\overline{350}) ; (\overline{352})$.	$\lambda_{\alpha} : (\overline{312}) ; (\overline{303}) ; (\overline{314}) ; (\overline{305}) ; (\overline{310}) ; (\overline{321}) ; (\overline{332}) ; (\overline{350})$.
$\lambda_{\beta} : (\overline{331})$.	$\lambda_{\beta} : (\overline{303}) ; (\overline{350}) ; (\overline{352})$.
4th Accessory spectrum:	4th Accessory spectrum:
$\lambda_{\alpha} : (\overline{431}) ; (\overline{431})$.	$\lambda_{\alpha} : (\overline{422}) ; (\overline{413}) ; (\overline{431}) ; (\overline{420})$.
$\lambda_{\beta} : (\overline{440})$.	$\lambda_{\beta} : (\overline{404}) ; (\overline{431})$.
— 4th Accessory spectrum:	— 4th Accessory spectrum:
$\lambda_{\alpha} : (\overline{431}) ; (\overline{431}) ; (\overline{420}) ; (\overline{422})$.	$\lambda_{\alpha} : (\overline{422}) ; (\overline{413}) ; (\overline{420}) ; (\overline{422}) ; (\overline{431})$.
$\lambda_{\beta} : (\overline{440})$.	$\lambda_{\beta} : (\overline{431}) ; (\overline{413}) ; (\overline{404})$.

impracticable. The rotation-spectrograms obtained were graphically analysed according to BERNAL's method¹⁾: thus in the case of *noseane* up to 50, in that of *haugne* even up to 72 spots of the film, indices were attributed in this way. The data obtained, arranged after the successive hyperbolae, are reproduced in Table II.

TABLE III. Powder-spectrograms of Silver-Ultramarine N ^o . 3 with 46.5% Ag, prepared from Ultramarineblue-GUIMET III; 1 st Fraction.							
N ^o . of Image	Estim. intensities:	Radiation:	2l in mm.:	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices:
1	5	α	37.2	12°0'	0.0432	0.0440	(211)
2	5	β ; or α	43.6	14°4'	0.0591	0.0596	(310)
3	9	α	48.6	15°39'	0.0728	0.0733	as α -line (220) (310)
4	5	α	53.4	17°14'	0.0878	0.0880	(222)
5	4	α	57.5	18°33'	0.1012	0.1025	(321)
6	2	β	59.2	19°6'	0.1071	0.1073	(330) or (411)
7	9	α	66.1	21°19'	0.1321	0.1319	(330) or (411)
8	1	β ; or α	81.1	26°16'	0.1958	0.1907	(440)
9	2	β	87.6	28°16'	0.2243	0.2282	as α -line (431); (510) (611) or (532)
10	3	α	90.0	29°3'	0.2358	0.2346	(440)
11	5	α	93.0	30°0'	0.2500	0.2493	(433) or (530)
12	3	α	95.7	30°52 $\frac{1}{2}$ '	0.2633	0.2638	(600)
13	4	α	98.6	31°48 $\frac{1}{2}$ '	0.2778	0.2785	(611) or (532)
14	2	α	101.6	32°46 $\frac{1}{2}$ '	0.2931	0.2932	(620)
15	1	β	107.4	34°39'	0.3233	0.3218	(633) or (721)
16	1	β	112.4	36°16'	0.3499	0.3458	(730)
17	5	α	120.9	39°0'	0.3960	0.3958	as α -line (444) (633) or (721)
18	2	α	126.1	40°41'	0.4250	0.4251	(730)

Radius of the Camera: 44.4 mm. $V = 55000$ Volt. Time of exposure: 3 hours.

Quadratic Equation (α -radiation): $\sin^2 \frac{\theta}{2} = 0.00733 (h^2 + k^2 + l^2)$.

(β -radiation): $\sin^2 \frac{\theta}{2} = 0.00596 (h^2 + k^2 + l^2)$.

$\lambda_{\alpha} = 1.540$ A.U.; $\lambda_{\beta} = 1.388$ A.U. $\alpha_0 = 8.99$ A.U.

Body-centred, cubic grating. The colour of the compound is pale yellow.

¹⁾ J. D. BERNAL, Proceed. R. Soc. London, A, 113, p. 117, (1927).

The conclusions formerly drawn are fully confirmed by these data, as in no case the sum ($h + k + l$) appears to be an odd number. Undoubtedly a bodily-centred grating is, therefore, present in this structure, with $a_0 =$ about 9,1 A.U.

§ 3. Starting from this *ultramarine-blue*-GUIMET n^o. 7553, now the corresponding *silver-ultramarine* was first prepared in the way described by HEUMANN ¹⁾. 15 grammes of the *sodium-ultramarine* were constantly heated in sealed glass-tubes during 15 hours at 120° C. with 2⁵ grammes *silvernitate* and 20 ccm. water. The yellowish mass obtained was suspended in water, the solution of *silvernitate* filtered-off, the residu washed ²⁾ thoroughly several times and during some time treated on the waterbath with a solution of a little *iodine* in *potassium-iodide*. After removing this solution by filtering, and having washed the residue, this was at 90° treated several times with a 10 %-solution of *potassium-cyanide*. This salt being removed by thoroughly washing, the powder was suspended in several litres of distilled water and repeatedly fractionated by sedimentation. In this way the *silversulphide* which has been formed, having much greater specific weight than the derived compound, can be removed almost completely, if this treatment be continued as long as black particles are observed; the fraction which remains in suspension during the longest time, is the purest and was always used in the final experiments. Analysis of this yellow compound gave (n^o. 3); 46,5 % Ag; calculated for $Ag_6Al_4Si_6O_{23}S_3$; 46,6 % Ag. ³⁾

Of this *silver-ultramarine* a powder-spectrogram was obtained, which showed a great number of very sharp interference-images: the results of its analysis are collected in Table III.

§ 4. For the sake of comparison, we have in the same way prepared some other *silver-ultramarienes* also, starting from other *ultramarienes*. Thus, starting from *ultramarine-pink* GUIMET OVR (with 14,25 % Al; 19,33 % Si; 10,4 % S; spec. wgt.: 2,34 at 25° C.) a greyish looking *silver-compound* with 34,4 % Ag; from *ultramarine-blue* N^o. 13 of the *Ver-einigte Ultramarin-fabriken*, a *silver-ultramarine* with 34,2 % Ag; and from an *ultramarine-green* N^o. X of the same origin, a *silver-derivative*

¹⁾ K. HEUMANN, Ann. der Chemie, 199, 253, (1879); 201, 262, (1880); 203, 174, (1881).

²⁾ This purification appears to be very troublesome, because the *silver-ultramarine* in contact with water, changes, like finely divided clay, into a colloidal, plastic mass, which after some time is transformed into hard, strongly conglomerating clumps. It may, therefore, be used for the fabrication of pottery, which on strongly heating, however, gets brown under loss of sulphur. Perhaps it would be possible, by heating it in the vapours of *alkali-chlorides*, to stain it superficially with a blue colour. This will be tried out afterwards.

³⁾ Later-on we will publish the complete analysis of the *silver-compound*. It must be remarked, that the Ag-content appears to be somewhat variable, probably because of no complete homogeneity of the preparation.

TABLE IV. Powderspectrogram of Silver-Ultramarine from Ultramarine-green V. U. F. N^o. X. (Fig. 1B)

N ^o . of Image	2l in mm.:	Intensities	λ :	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices:
1	34.6	2	β	11°10'	0.0374	0.0362	(211)
2	37.5	5	α	12°6'	0.0439	0.0446	(211)
3	43.7	3	β	14°6'	0.0593	0.0604	(310)
4	49.7	8	α	16°2'	0.0763	0.0743	as α -line (220) (310)
5	53.1	1	β	17°8'	0.0868	0.0846	(321)
6	54.5	4	α	17°35'	0.0913	0.0892	(222)
7	58.9	3	α	18°58'	0.1056	0.1040	(321)
8	60.3	2	α	19°27'	0.1108	0.1179	(400)
9	64.3	1	β	20°44'	0.1252	0.1208	(420)
10	67.0	8	α	21°37'	0.1358	0.1337	(330) and (411)
11	70.8	3	α	22°50'	0.1505	0.1486	(420)
12	74.4	2	α	24°0'	0.1655	0.1635	(332)
13	77.9	4	α	25°8'	0.1804	0.1783	(422)
14	81.2	4	α	26°12'	0.1950	0.1922	(431) and (510)
15	84.1	2	β	27°8'	0.2080	0.2054	(433) and (530)
16	86.3	1	β	27°50'	0.2180	0.2174	(600)
17	90.8	3	α	29°17'	0.2391	0.2378	(440)
18	93.8	5	α	30°16'	0.2541	0.2526	(433) and (530)
19	97.0	3	α	31°18'	0.2699	0.2675	(600)
20	99.7	4	α	32°10'	0.2835	0.2823	(532) and (611)
21	102.6	2	α	33°6'	0.2983	0.2972	(620)
22	105.6	2	α	34°4'	0.3138	0.3121	(541)
23	108.4	2	α	34°58'	0.3284	0.3269	(622)
24	110.7	1	α	35°43'	0.3408	0.3418	(631)
25	113.7	1	α	36°41'	0.3569	0.3566	(444)
26	115.6	1	α	37°17'	0.3670	0.3715	(543); (505); (701)
27	121.9	5	α	39°19'	0.4014	0.4012	(633) and (721)
28	127.3	2	α	41°4'	0.4315	0.4309	(730)
29	132.5	1	α	42°44'	0.4605	0.4607	(732)

TABLE IV (Continued).							
No. of Image:	2l in mm.:	Intensities:	λ :	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices:
30	137.8	1	α	44°28'	0.4906	0.4904	(554); (741); (811)
31	140.4	1	α	45°17'	0.5049	0.5052	(644) and (820)
32	143.2	1	α	46°12'	0.5210	0.5201	(653)
33	145.5	1	α	46°56'	0.5338	0.5350	(660); (822)
34	148.4	1	α	47°53'	0.5503	0.5499	(750); (743); (831)
35	150.9	1	α	48°41'	0.5641	0.5647	(662)
36	153.7	1	α	49°35'	0.5796	0.5795	(752)
37	155.9	1	α	50°17'	0.5917	0.5944	(840)
38	158.8	1	α	51°14'	0.6080	0.6093	(833)
39	164.4	1	α	53°2'	0.6383	0.6390	(655)
40	169.7	1	α	54°45'	0.6669	0.6687	(851)

Radius of the camera: 44.4 mm.
Wavelength: $\lambda_\alpha = 1.540$ A.U.; $\lambda_\beta = 1.381$ A.U.; $a_0 = 8.93$ A.U.
V = 55000 Volt. Time of exposure: 3 hours.

Quadr. equation: $\sin^2 \frac{\theta}{2} = 0.00743 (h^2 + k^2 + l^2)$. (α -radiation).
 $\sin^2 \frac{\theta}{2} = 0.00604 (h^2 + k^2 + l^2)$. (β -radiation).

with 51.0% Ag. Notwithstanding the differences in silver-content, the powder-spectrograms of all these preparations were *rigorously identical*. In Table IV the results of the analysis of the last mentioned spectrogram are reviewed ($a_0 = 8.93$ A.U.), because this showed the greatest number of sharp lines. The relative intensities of the most important images are, — as far as so rough an estimation allows us to judge it, — analogous to those of Table III.

From these data the following conclusions may be drawn:

1°. Notwithstanding the different Ag-content of the compounds studied, their ROENTGEN-spectrograms appear to be practically identical.

2°. The original periodical structure of the *ultramarines* is preserved also after substitution.

3°. By the replacement of Na by Ag, the constant a_0 of the fundamental grating is *diminished*.

4°. Notwithstanding all the analogies mentioned, the relative intensi-

ties of corresponding interference-images in the spectrograms of the *ultramarines* and of the *silver-ultramarines* are appreciably modified. (Fig. 1; to compare *A* and *B*).

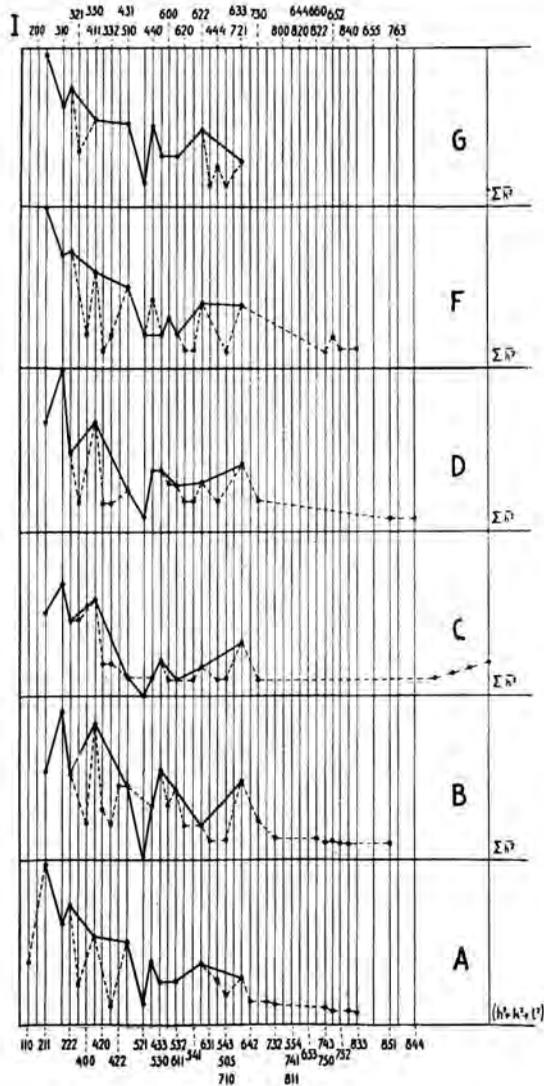


Fig. 1.

§ 5. The fact that the gradual replacement of *Na* in *ultramarine-blue* by *Ag* is really connected with a *gradual diminution* of the edge a_0 of the fundamental cell, could, moreover, also be proved in the following way.

If *ultramarine-blue* be digested during several days on the waterbath with an ammoniacal solution of *silverchloride*, then, as HEUMANN¹⁾ stated

¹⁾ K. HEUMANN, loco cit., 201, 282, (1880).

already, the Na of the original compound gets partially substituted by Ag . HEUMANN thought that in this way *two thirds* of the Na -atoms were replaced by Ag . In our experiments the original *ultramarine-blue* was heated during 6 days on the waterbath with an ammoniacal solution of *silver-chloride*, which was three times refreshed: the product obtained (N^o. 19ⁱ) appeared yet to contain 39,4% Ag . This product was again the same way during 10 subsequent days: now the carefully purified treated in product N^o. 19ⁱⁱ), gave on analysis: 31,6% Ag . A third treatment, however, gave again a product (N^o. 19ⁱⁱⁱ) with 34,2% Ag ; evidently the substitution of Na by Ag does not stop at a limiting stoichiometrical proportion of both constituents, as HEUMANN believed, but the reaction goes on until a state of *equilibrium* is reached, — in the same way as occurs in treating the *permutites* with salt-solutions of varied concentrations.

The results of the analysis of the ROENTGEN-spectrograms obtained with these preparations, are collected in the Tables V and VI.

Moreover, a product was prepared by heating, during 50 hours in a flask with reflux-cooler, the original *silver-ultramarine* with a strong solution of 20 times its weight of *sodium-chloride*. This blueish coloured product (N^o. 9) still contained, besides sodium, 7,75% Ag ; its ROENTGEN-spectrum gave the results collected in Table VII (Compare Fig. 1 F).

From the results of the analysis, it becomes evident, that besides the original *silver-compound*, there is nowhere present here a pure compound formed by substitution in simple stoichiometrical proportions:

For: $Ag_6Al_4Si_6O_{23}S_3$,	the <i>silver-content</i> is calculated at :	46,6% Ag ;	obs.: 46,5% Ag .	(3)
For: $Ag_5NaAl_4Si_6O_{23}S_3$,	"	41,3% Ag ;	} obs.: 39,4% Ag . (19 ⁱ)	
For: $Ag_4Na(NH_4)Al_4Si_6O_{23}S_3$,	"	35,5% Ag ;		
For: $Ag_3Na(NH_4)_2Al_4Si_6O_{23}S_3$,	"	28,7% Ag ;		
For: $Ag_2Na(NH_4)_3Al_4Si_6O_{23}S_3$,	"	20,8% Ag ;	(19 ⁱⁱⁱ)	
For: $AgNa_5Al_4Si_6O_{23}S_3$ (in exp. N ^o . 9),	"	11,2% Ag ;	obs.: 7,75% Ag .	(9)

Moreover, if the original *silver-compound* n^o. 3 be heated in a porcelain crucible with an excess of *sodium-iodide* just above the meltingpoint of this salt, and the molten mass be kept at this temperature for a longer time, then, after purification, an *ultramarine-blue* (n^o. 4) is formed, which, however, appears still to contain 6% Ag ; its powder-spectrogram (see Table VIII) is almost identical with that of preparation n^o. 9, also with respect to the relative intensities of the successive diffraction-lines, and with that of the original *GUIMET-blue* itself.

If the extremely weak influence on the size of the grating-constant a_0 , which is exerted, as we shall see later-on, by the substitution of Na by the (NH_4) -radicle, be now left aside for a moment, the constant a_0 of the preparations n^o. 3, 4, 9, 19ⁱ, 19ⁱⁱ, 19ⁱⁱⁱ, etc., appears to vary as follows:

TABLE V. Powderspectrogram of Silver-sodium-Ultramarine No. 19¹,
(Contains: 39.4 % Ag; Fig. 1 C).

No. of Image:	2l in m.m.:	λ :	Estim. intens.:	$\Sigma(h^2)$:	$\frac{\theta}{2}$:	$\frac{\sin^2 \theta}{2}$ (observed):	$\frac{\sin^2 \theta}{2}$ (calculated):	Indices:
1	37.5	α	6	6	12°6'	0.0439	0.0439	(211)
2	43.7	β	3	10	14°6'	0.0593	0.0595	(310)
3	48.8	α	7	10	15°45'	0.0737	0.0732	(310)
4	53.5	α	5	12	17°16'	0.0881	0.0878	(222)
5	58.3	β	1	18	18°50'	0.1042	0.1071	(330) and (411)
6	58.9	α	5	14	19°0	0.1059	0.1025	(321)
7	64.2	β	1	22	20°43'	0.1251	0.1310	(332)
8	65.7	α	6	18	21°13'	0.1310	0.1318	(330) and (411)
9	68.9	α	2	20	22°14'	0.1432	0.1464	(420)
10	74.0*)	α	3	22	23°56'	0.1646	0.1610	(332)
11	79.5	α	1	26	25°39'	0.1874	0.1903	(431) and (510)
12	89.6	α	1	32	28°54'	0.2336	0.2342	(440)
13	92.6	α	2	34	29°53'	0.2482	0.2489	(433) and (530)
14	95.5	α	1	36	30°52'	0.2622	0.2635	(600)
15	98.5	α	1	38	31°46'	0.2774	0.2782	(532) and (611)
16	100.8	α	1	40	32°31'	0.2889	0.2928	(620)
17	104.1	α	1	42	33°35'	0.3060	0.3074	(541)
18	107.0	α	2	44	34°31'	0.3211	0.3221	(622)
19	112.0	α	1	48	36°8'	0.3477	0.3514	(444)
20	115.5	α	1	50	37°16'	0.3667	0.3660	(543); (505); (710)
21	120.6	α	3	54	38°54'	0.3940	0.3953	(633) and (721)
22	126.3	α	1	58	40°44'	0.4258	0.4246	(730)
23	178.6	α	1	98	57°35'	0.7130	0.7125	(853)
24	208.0	α	2	116	67°6'	0.8486	0.8490	(864) and (10.04)

*) Only observed at one side of the central spot.

Radius of the camera: 44.4 mm. $V = 55000$ Volt; Time of exposure: 2 hours.
 $\lambda_{\alpha} = 1.540$ A.U.; $\lambda_{\beta} = 1.388$ A.U.; $a_0 = 9.00$ A.U.

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.00732 (h^2 + k^2 + l^2)$, for the α -radiation.

$\sin^2 \frac{\theta}{2} = 0.00595 (h^2 + k^2 + l^2)$, for the β -radiation.

TABLE VI. Powder-spectrogram of Silver-sodium-ammonium-Ultramarine No. 19^{II}.
(Analysis: 31,6⁰/₀ Ag; Fig. 1 D).

No. of Image:	2l in mm.	Estim. intensities:	$\overline{\Sigma(h^2)}$:	λ :	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (ob- served):	$\sin^2 \frac{\theta}{2}$ (cal- culated):	Indices:
1	37.7	6	6	α	12° 9'	0.0443	0.0439	(211)
2	44.2	2	10	β	14° 15'	0.0605	0.0599	(310)
3	48.9	9	10	α	15° 46'	0.0738	0.0732	(310)
4	53.7	4	12	α	17° 19'	0.0886	0.0878	(222)
5	58.2	1	14	α	18° 46'	0.1035	0.1025	(321)
6	59.8	2	18	β	19° 17'	0.1090	0.1062	(330); (411)
7	66.2	6	18	α	21° 21'	0.1325	0.1318	(330); (411)
8	69.1	1	20	α	22° 17'	0.1438	0.1464	(420)
9	73.8	1	22	α	23° 48'	0.1628	0.1610	(332)
10	80.0	2	26	α	25° 48'	0.1894	0.1903	(431); (510)
11	83.1	1	34	β	26° 48'	0.2033	0.2006	(433); (530)
12	90.0	3	32	α	29° 2'	0.2356	0.2342	(440)
13	93.1	3	34	α	30° 1'	0.2502	0.2490	(433); (530)
14	95.9	2	36	α	30° 56'	0.2642	0.2635	(600)
15	99.1	2	38	α	31° 58'	0.2803	0.2782	(611); (532)
16	100.5	1	40	α	32° 25'	0.2874	0.2928	(620)
17	104.5	1	42	α	33° 42'	0.3078	0.3074	(541)
18	107.2	2	44	α	34° 35'	0.3222	0.3221	(622)
19	112.5	1	48	α	36° 17'	0.3502	0.3514	(444)
20	120.6	3	54	α	38° 54'	0.3943	0.3953	(633); (721)
21	126.6	1	58	α	40° 50'	0.4275	0.4246	(730)
22	172.5	1	94	α	55° 39'	0.6816	0.6881	(763)
23	179.0	1	98	α	57° 45'	0.7152	0.7174	(853)

Radius of the camera; 44.4 mm. $V = 55000$ Volt. Time of exposure: 3 hours.

$\lambda_{\alpha} = 1.540$ A.U.; $\lambda_{\beta} = 1.388$ A.U.; $a_0 = 9.00$ A.U.

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.00732 (h^2 + k^2 + l^2)$, for the α -radiation.

$\sin^2 \frac{\theta}{2} = 0.00599 (h^2 + k^2 + l^2)$, for the β -radiation.

TABLE VII. Powderspectrogram of a Silver-sodium-Ultramarine N^o. 9.
 (From Silver-Ultramarine by heating with a solution of NaCl during 50 hours.)
 Contains: 7.75 % Ag; Fig. 1 F.

N ^o . of Image:	2l in mm.:	λ :	Estim. intensities:	$\Sigma(h^2)$:	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices:
1	37.5	α	10	6	12°6'	0.0439	0.0436	(211)
2	43.7	β	1	10	14°6'	0.0593	0.0591	(310)
3	48.6	α	7	10	15°41'	0.0731	0.0727	(310)
4	53.2	α	7	12	17°10'	0.0871	0.0872	(222)
5	58.9	α	4	14	19°0'	0.1059	0.1018	(321)
6	61.6	α	2	16	19°52'	0.1155	0.1163	(400)
7	64.9 *)	β	1	22	20°56'	0.1278	0.1300	(332)
8	65.9	α	6	18	21°16'	0.1315	0.1309	(330) and (411)
9	68.7	α	1	20	22°14'	0.1426	0.1454	(420)
10	73.5	α	2	22	23°43'	0.1618	0.1599	(332)
11	80.2	α	5	26	25°52'	0.1903	0.1890	(431) and (510)
12	82.6	β	1	34	26°39'	0.2012	0.2009	(433) and (530)
13	86.4	α	2	30	27°52'	0.2185	0.2181	(521)
14	89.4	α	4	32	28°50'	0.2326	0.2326	(440)
15	92.4	α	2	34	29°49'	0.2473	0.2472	(433) and (530)
16	95.6	α	3	36	30°50'	0.2627	0.2617	(600)
17	98.3	α	2	38	31°43'	0.2764	0.2763	(532) and (611)
18	100.8	α	1	40	32°31'	0.2889	0.2908	(620)
19	104.5 *)	α	1	42	33°43'	0.2081	0.3053	(541)
20	107.0	α	4	44	34°31'	0.3211	0.3199	(622)
21	112.0	α	2	48	36°8'	0.3477	0.3490	(444)
22	115.0	α	1	50	37°6'	0.3639	0.3635	(543); (505); (710)
23	120.5	α	4	54	38°52'	0.3938	0.3926	(633) and (721)
24	145.9 *)	α	1	74	47°5'	0.5361	0.5380	(743); (750); (831)
25	149.4 *)	α	2	76	48°12'	0.5557	0.5525	(662)
26	151.8 *)	α	1	78	48°58'	0.5690	0.5671	(752)
27	154.0 *)	α	1	80	49°41'	0.5814	0.5816	(840)

*) Only observed at one side of the central spot.

Radius of the camera: 44.4 mm. $V = 55000$ Volt; Time of exposition: 2 hours.

$\lambda_x = 1.540$ A.U.; $\lambda_\beta = 1.388$ A.U.; $a_0 = 9.03$ A.U.

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.00727 (h^2 + k^2 + l^2)$, for the α -radiation.

$\sin^2 \frac{\theta}{2} = 0.00591 (h^2 + k^2 + l^2)$, for the β -radiation.

If the *silver*-content is:

- 0 % *Ag*: $a_0 = 9.06$ A.U.
 7.75% *Ag*: $a_0 = 9.03$ A.U.
 31.6% *Ag*: $a_0 = 9.00$ A.U.
 39.4% *Ag*: $a_0 = 9.00$ A.U.
 46.5% *Ag*: $a_0 = 8.96-8.99$ A.U.
 51 % *Ag*: $a_0 = 8.93$ A.U.

TABLE VIII. Powderspectrogram of *NaAg*-Ultramarine No. 4.
 Contains about 6% *Ag*. (Fig. 1 G)

No. of Image:	$2l$ in mm.:	Estim. intensities:	λ :	$\Sigma(h^2)$:	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices:
1	37.1	9	α	6	11°58'	0.0430	0.0431	(211)
2	47.9	6	α	10	15°27'	0.0710	0.0718	(310)
3	52.9	7	α	12	17° 4'	0.0861	0.0862	(222)
4	54.6	3	α	14	17°37'	0.0916	0.1004	(321)
5	57.4	1	β	18	18°31'	0.1010	0.1049	(330) and (411)
6	65.4	5	α	18	21° 6'	0.1296	0.1292	(330) and (411)
7	79.3	5	α	26	25°35'	0.1864	0.1867	(431) and (510)
8	81.0	1	β	34	26° 8'	0.1940	0.1982	(433) and (530)
9	85.7	1	α	30	27°39'	0.2153	0.2154	(521)
10	88.7	4	α	32	28°37'	0.2294	0.2298	(440)
11	91.5	3	α	34	29°31'	0.2427	0.2441	(433) and (530)
12	94.6	3	α	36	30°31'	0.2578	0.2585	(600)
13	97.4	3	α	38	31°25'	0.2717	0.2729	(532) and (611)
14	106.3	4	α	44	34°18'	0.3176	0.3159	(622)
15	108.1	1	α	46	34°53'	0.3272	0.3303	(631)
16	111.6	2	α	48	36° 0'	0.3455	0.3446	(444)
17	114.4	1	α	50	36°54'	0.3605	0.3590	(543):(505):(710)
18	119.2	3	α	54	38°27'	0.3867	0.3877	(633) and (721)

Radius of the camera: 44.4 mm. $V = 55000$ Volt; Time of exposure: 3 hours.
 Wavelength: $\lambda_{\alpha} = 1.540$ A.U.; $\lambda_{\beta} = 1.388$ A.U.; $a_0 = 9.09$ A.U.

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.00718 (h^2 + k^2 + l^2)$, for the α -radiation.

$$\sin^2 \frac{\theta}{2} = 0.00583 (h^2 + k^2 + l^2), \text{ for the } \beta\text{-radiation.}$$

From these data it becomes very evident that the constant a_0 has really a tendency to *decrease*, if the *silver*-content of the preparation *increases*.

Indeed, the atomic radius for *silver*, which differs, after all, only slightly from that of *sodium*, seems, in most cases, to be somewhat *smaller* than the last one. The influence of the substitution is strongest on the side of the pure *Na*-compound: a content of 6 0/0, or 8 0/0 *Ag* is in this respect not yet very appreciable, but becomes discernible only, if perhaps more than a 15 0/0 *Ag* is introduced. This may be seen e.g. from a graph, the abscissae of which are the *silver*-contents. The same phenomenon is observed with respect to the relative intensities. For a dozen interference-images common to both kinds of *ultramarines*, namely: (211); (310); (222); (321); (330), resp. (411); (431), resp. (510); (521); (440); (433), resp. (530); (532), resp. (611); (622); and (633), resp. (721), — the most important differences in this respect for the *silver*-compound in comparison with the *sodium*-compound are the following.

For the *Na*-compound $I_{(211)}$ is always *greater* than $I_{(310)}$; also $I_{(222)}$ always is *greater* than $I_{(310)}$; $I_{(330), (411)}$ is *smaller* than $I_{(322)}$; $I_{(521)}$ has yet a discernible size; $I_{(440)}$ is *greater* than $I_{(433), (530)}$; and the last-one is, although almost equal, also *greater* than $I_{(532), (611)}$; and $I_{(622)}$ is *greater* than $I_{(532), (611)}$ and also *greater* than $I_{(633), (721)}$. For the *Ag*-compound it is just the reverse in these respects.

From Fig. 1, in which on the horizontal axis the subsequent values of $(h^2 + k^2 + l^2)$ are graphically drawn, all this may easily be seen in comparing the figures 1^A, 1^B, 1^G; furthermore, in comparing the figures 1^C and 1^D for the *silver-sodium*-compounds, which have, — with the exception of a few images like (532) in 1^C and (433) in 1^D, — yet completely the character of Fig. 1^B. On the contrary, Fig. 1^F shows already the type of the *Na-ultramarines*, although the compound yet contains 8 0/0 *Ag*. The characteristic intensities of the *silver*-compound seem to become discernible, as soon as more than 12 or 15 0/0 *Ag* has taken the place of the equivalent quantity of *Na*.

§ 6. GUIMET ¹⁾ first demonstrated, that the *sulphur* in the *ultramarines* may be substituted by *selenium* or *tellurium*. Through kind intervention of M. A. GUILLOCHIN, director-general of the *Usines d'Outremer*-GUIMET in *Fleurieu s/Saône*, who put a small quantity of GUIMET's original, scarlet-red *selenium-ultramarine* at our disposal, — we were able to study this substance and to compare its properties with those of the corresponding *sulphur*-derivatives. The brilliantly red powder was purified and freed from the last traces of adhering *selenates* or *selenites* by repeatedly washing with distilled water; the final product (N^o. 80) was used in these experiments and partially converted into the corresponding *silver-selenium*-

¹⁾ E. GUIMET, Bull. de la Soc. chim. Paris, 27, 480, (1877); Ann. de Chim. et Phys., (5), 13, 102, (1878). Here also we wish once more to express our sincere thanks to M. GUILLOCHIN.

TABLE IX. Powder-Spectrogram of Selenium-Ultramarine-red-GUIMET (N^o. 80).

No. of Image:	2l in mm.:	λ :	Estim. intensities:	$\Sigma(h^2)$:	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (observed):	$\sin^2 \frac{\theta}{2}$ (calculated):	Indices:
1	37.4	α	10	6	12°4'	0.0437	0.0430	(211)
2	42.0	β	1	10	13°33'	0.0550	0.0582	(310)
3	48.0	α	4	10	15°29'	0.0712	0.0716	(310)
4	53.1	α	9	12	17°8'	0.0868	0.0859	(222)
5	58.0	β	1	18	18°43'	0.1030	0.1048	(330); (411)
6	61.4	α	3	16	19°48'	0.1148	0.1146	(400)
7	65.1	α	8	18	21°0'	0.1284	0.1289	(330); (411)
8	71.0	β	1	26	22°55'	0.1516	0.1513	(431); (510)
9	72.9	α	2	22	23°31'	0.1592	0.1576	(332)
10	76.3	α	2	24	23°36'	0.1733	0.1718	(422)
11	79.3	α	8	26	25°34'	0.1862	0.1862	(431); (510)
12	82.0	β	1	34	26°27'	0.1984	0.1979	(433); (530)
13	85.8	α	2	30	27°40'	0.2156	0.2148	(521)
14	88.6	α	4	32	28°34'	0.2286	0.2291	(440)
15	91.7	α	4	34	29°35'	0.2437	0.2434	(433); (530)
16	94.6	α	3	36	30°31'	0.2578	0.2578	(600)
17	97.6	α	3	38	31°29'	0.2727	0.2721	(532); (611)
18	105.7	α	6	44	34°6'	0.3143	0.3150	(622)
19	108.3	α	2	46	34°56'	0.3279	0.3294	(631)
20	111.1	α	3	48	35°50'	0.3427	0.3437	(444)
21	114.0	α	3	50	36°46'	0.3583	0.3580	(543); (505); (710)
22	119.1	α	5	54	38°25'	0.3861	0.3866	(633); (721)
23	121.3	α	1	56	39°7'	0.3981	0.3999	(642)
24	124.6	α	1	58	40°12'	0.4166	0.4153	(730)
25	129.2	α	1	62	41°40'	0.4419	0.4439	(732)
26	131.4	α	1	64	42°23'	0.4544	0.4582	(800)
27	135.1	α	1	66	43°35'	0.4753	0.4726	(554); (741); (811)
28	145.0	α	1	74	46°48'	0.5314	0.5298	(750); (743); (831)
29	147.4	α	2	76	47°33'	0.5444	0.5442	(662)
30	149.5	α	1	78	48°13'	0.5560	0.5585	(752)
31	152.5	α	2	80	49°12'	0.5730	0.5728	(840)
32	165.5	α	1	90	53°23'	0.6442	0.6444	(851)
33	171.0	α	1	94	55°10'	0.6737	0.6730	(763)
34	173.6	α	1	96	56°1'	0.6876	0.6873	(844)

Radius of the camera: 44.4 mm. $V = 55000$ Volt. Time of exposure: 3 hours.
Wavelength: $\lambda_\alpha = 1.540$ A.U.; $\lambda_\beta = 1.388$ A.U. $a_0 = 9.10$ A.U.

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.00716 (h^2 + h^2 + l^2)$, for the α -radiation.

$\sin^2 \frac{\theta}{2} = 0.00582 (h^2 + k^2 + l^2)$, for the β -radiation.

TABLE X. Powderspectrogram of Silverselenium-Ultramarine.
(Analysis: 25.8% Ag).

No. of Image:	2l in mm.:	Estim. intensities:	$\Sigma(h^2)$:	λ :	$\frac{\theta}{2}$:	$\sin^2 \frac{\theta}{2}$ (ob- served):	$\sin^2 \frac{\theta}{2}$ (cal- culated):	Indices:
1	37.9	8	6	α	12°14'	0.0449	0.0445	(211)
2	43.9	1	10	β	14°10'	0.0600	0.0603	(310)
3	49.3	7	10	α	15°54'	0.0750	0.0742	(310)
4	54.1	5	12	α	17°25'	0.0896	0.0890	(222)
5	57.3	3	14	α	18°29'	0.1005	0.1039	(321)
6	66.7	7	18	α	21°31'	0.1345	0.1336	(330) and (411)
7	70.0	1	20	α	22°35'	0.1475	0.1484	(420)
8	74.2	1	22	α	23°56'	0.1646	0.1632	(332)
9	78.0	1	24	α	25° 7'	0.1801	0.1781	(422)
10	81.1	2	26	α	26°10'	0.1945	0.1930	(431) and (510)
11	83.7	1	34	β	27° 0'	0.2061	0.2050	(433) and (530)
12	90.7	3	32	α	29°15'	0.2387	0.2374	(440)
13	93.7	4	34	α	30°13'	0.2533	0.2523	(433) and (530)
14	96.6	2	36	α	31°10'	0.2678	0.2671	(600)
15	99.7	3	38	α	32° 9'	0.2832	0.2820	(532) and (611)
16	102.0	1	40	α	32°55'	0.2953	0.2968	(620)
17	105.5	1	42	α	34° 2'	0.3132	0.3116	(541)
18	108.3	2	44	α	34°56'	0.3279	0.3265	(622)
19	110.6	1	46	α	35°41'	0.3403	0.3413	(631)
20	113.6	1	48	α	36°39'	0.3563	0.3562	(444)
21	116.5	1	50	α	37°35'	0.3720	0.3710	(543);(505);(710)
22	122.1	4	54	α	39°23'	0.4026	0.4007	(633) and (721)
23	127.3	2	58	α	41° 4'	0.4316	0.4304	(730)
24	132.7	3	62	α	42°48'	0.4616	0.4600	(732)
25	137.9	1	66	α	44°29'	0.4910	0.4897	(554);(741);(811)
26	140.6	1	68	α	45°21'	0.5061	0.5046	(644) and (820)
27	150.9	2	76	α	48°41'	0.5641	0.5639	(662)
28	156.7	2	80	α	50°33'	0.5963	0.5936	(840)
29	159.2	1	82	α	51°21'	0.6099	0.6084	(833)
30	164.5	1	86	α	53° 4'	0.6389	0.6381	(655)
31	169.8	1	90	α	54°47'	0.6675	0.6678	(851)
32	193.1	1	106	α	62°18'	0.7839	0.7865	(950)
33	197.2	2	108	α	63°37'	0.8025	0.8014	(666)
34	200.1	1	110	α	64°33'	0.8153	0.8162	(10. 3. 1)
35	206.9	1	114	α	66°45'	0.8442	0.8459	(774) and (855)
36	211.1	1	116	α	68° 6'	0.8609	0.8607	(864) and (10.0.4)

Radius of the camera: 44.4 mm. $V = 55000$ Volt. Time of exposure: 2 hours.

$\lambda_{\alpha} = 1.540$ A.U.; $\lambda_{\beta} = 1.388$ A.U.; $a_0 = 8.94$ A.U.

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.00742 (h^2 + k^2 + l^2)$, for the α -radiation.

and $\sin^2 \frac{\theta}{2} = 0.00603 (h^2 + k^2 + l^2)$, for the β -radiation.

ultramarine (N^o. 81) (with 25,8% Ag, according to its analysis), whose colour is orange-yellow or brownish-yellow in this case, and which preparation was purified by frequent decantation and sedimentation. Of both these substances, which are poor in Ag-content, powder-spectrograms were made; the results of their analysis are collected in the Tables IX and X.

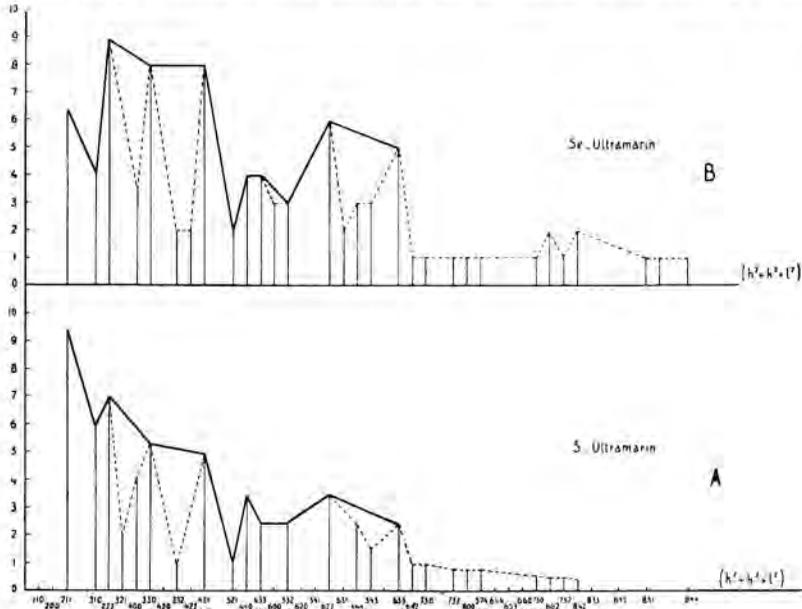


Fig. 2.

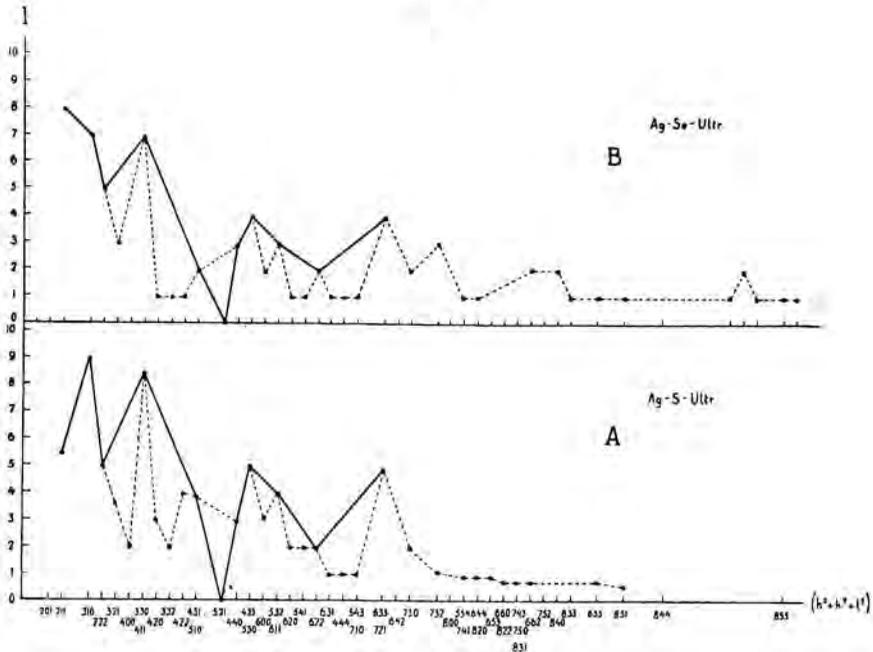


Fig. 3.

With the same time of exposure, viz. 3 hours, the background of the photographic film in the case of *selenium-ultramarine* appeared to be appreciably more and very homogeneously clouded, than in the case of the ordinary *sulphur-ultramarines*. The intensity of the diffraction-images was great and the different lines were very sharp.

From the data thus obtained, as well on comparing Fig. 2 and Fig. 3, — which in the same way as Fig. 1, give the estimated, relative intensities of corresponding diffraction-images in both patterns, — it may be deduced, that:

1stly. The substitution of S in the ordinary *ultramarines* by Se leaves the general structure unchanged. Under comparable conditions of exposure the film is, in the case of the *selenium-derivatives*, more intensely and generally clouded: which doubtlessly is caused by the greater diffracting power of the Se-atoms, in comparison with that of the S-atoms. However, the general character of the relative intensities of corresponding diffraction-images and their sequence, appears to remain the same; only a few exceptions are present, e.g. that $I_{(222)}$ in the case of the Se-compound is *greater* than $I_{(211)}$; and that the same occurs for $I_{(433), (530)}$ in comparison with $I_{(522), (611)}$ and for $I_{(422)}$, which last intensity in the case of the *S-ultramarine* is practically equal to zero. The grating-constant a_0 of the Se-derivative ($a_0 = 9,10 \text{ A.U.}$) is only *very slightly* greater than that of *S-ultramarine*.

2ndly. The substitution of S by Se in *silver-ultramarine* does neither change its structure to any appreciable extent, while also the relative intensities of the successive corresponding images appear to be quite analogous in both cases, with the only exception (Fig. 3) that $I_{(211)}$ of the *Se-Ag-ultramarine* is *greater* than $I_{(310)}$, and the reverse occurs in the *S-Ag-derivative*. Also in this case both grating-constants a_0 do not differ appreciably.

§ 7. From the investigations described it must become clear now, that the substitution of the atoms of the alkali-metal in *GUIMET-blue* by more than about 15% *Ag*, has an appreciably greater influence on the relative intensity-sequence of the diffraction-images in the corresponding ROENTGEN-spectograms, than has the substitution of the S-atoms in it by Se; this last substitution hardly, and only in some few respects, influencing the said intensities for some of the diffracting sets of planes, but chiefly causing only a general clouding of the background of the photographic films. In this an indication might be found that the Na- and Ag-atoms take a more important part in the formation of the "rigid" periodical structure of the *ultramarines*, than do the S- or Se-atoms. Notwithstanding this, it cannot yet be considered to be finally proved, that the S- (or Se)-atoms are, therefore, to be supposed to be statistically dispersed throughout the whole fixed structure, while the Na- (or Ag)-atoms would, at least partially, represent the true building-

stones in the fixed, invariable, periodical structure of these remarkable substances. The experience hitherto obtained can only give a few more arguments in favour of the probability of such a conception, but as yet no complete certainty about it. Further experiments in this direction have already been started.

§ 8. In the course of these investigations again a second tentative was made to find amongst the cubic zeolites a type of compound showing the peculiarities of the ROENTGENSpectrograms, as observed in *haugne*, *noseane* and the *ultramarines*. In this case *analcite* was chosen as the object of such a study; but also this beautifully crystallized and cubic *sodium-aluminium-silicate*, chemically closely related to the said silicates, appeared to possess quite another structure than the *ultramarines*. Although we intend to publish the more detailed data about the structure of *analcite* in the near future, — when at the same time we shall have an opportunity to discuss the question of the presence of “errant” constituents, namely of H_2O -molecules; in it, — we wish to communicate here already shortly some of the principal results of this investigation.

At our disposal were splendidly built, translucent crystals from the Cyclopic Isles, which were bordered principally by the forms $\{100\}$ and $\{211\}$.

With the aid of powder-spectrograms, of BRAGG's spectrograms on the faces of $\{100\}$ and $\{211\}$, and of rotation-spectrograms in which the crystals were turned round the directions $[100]$, $[110]$ and $[111]$, — we were able to demonstrate, that this mineral has a structure to be described by the space-group O_h^{10} . In each cubic cell, the edge of which is equal to 13.71 A.U., 16 molecules are present of the composition: $NaAlSi_2O_6, H_2O$. If to the 16 Na-, and the 16 Al-atoms the two only possible, parameterless 16-fold places be attributed:

$$\begin{array}{l}
 [0 \ 0 \ 0]; [\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}]; [\frac{3}{4} \ \frac{3}{4} \ \frac{3}{4}]; [\frac{1}{2} \ \frac{1}{2} \ \frac{1}{2}]; \\
 [\frac{1}{2} \ \frac{1}{2} \ 0]; [\frac{3}{4} \ \frac{3}{4} \ \frac{1}{4}]; [\frac{1}{4} \ \frac{1}{4} \ \frac{3}{4}]; [0 \ 0 \ \frac{1}{2}]; \\
 [\frac{1}{2} \ 0 \ \frac{1}{2}]; [\frac{3}{4} \ \frac{1}{4} \ \frac{3}{4}]; \text{ etc.}
 \end{array}
 \left| \begin{array}{l}
 [\frac{1}{8} \ \frac{1}{8} \ \frac{1}{8}]; [\frac{1}{8} \ \frac{7}{8} \ \frac{3}{8}]; [\frac{3}{8} \ \frac{1}{8} \ \frac{7}{8}]; [\frac{7}{8} \ \frac{3}{8} \ \frac{1}{8}]; \\
 [\frac{1}{8} \ \frac{7}{8} \ \frac{7}{8}]; [\frac{7}{8} \ \frac{1}{8} \ \frac{5}{8}]; [\frac{5}{8} \ \frac{7}{8} \ \frac{1}{8}]; [\frac{1}{8} \ \frac{5}{8} \ \frac{7}{8}]; \\
 [\frac{3}{8} \ \frac{3}{8} \ \frac{3}{8}]; [\frac{1}{8} \ \frac{3}{8} \ \frac{5}{8}]; \text{ etc.}
 \end{array} \right.$$

to one of both kinds of atoms, to the other of both kinds of atoms,

and if to the Si-atoms the 32-fold position $32f$ (after WIJCKOFF's notation; it has one variable parameter u) be given, — then there are yet available only two 24-fold, parameterless positions, two 48-fold positions with a single variable parameter v , and one 96-fold position with three variable parameters. Indeed, in this way it appears possible also to fix the positions of the 96 O-atoms of the complex $Na_{16}Al_{16}Si_{32}O_{96} + H_{32}O_{16}$; but there is no place any more available for the 16 O-atoms of the water-molecules, and it seems inevitable, to assume these very

"mobile" crystalwater-molecules to be also "errant" constituents of the structure. In short we hope to resume this topic in detail. As formerly already pointed out, for the *ultramarines* the space-group O_h^{10} cannot be taken into account.

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Pathology. — *On the isolation of a substance with carcinolytic properties [from the reticulo-endothelial system*¹⁾. By N. WATERMAN and L. DE KROMME. (Communicated by Prof. G. VAN RIJNBERK.)

(Communicated at the meeting of March 26, 1927).

In an earlier memoir we demonstrated what FREUND and KAMINER had previously pointed out, namely that the normal blood-serum and the normal skin contain a substance capable of dissolving cancer-cells in vitro.

This property diminishes, or is totally lost, when a carcinomatous process arises.

Experimental research has opened up the possibility of studying the changes in this property on the dorsal skin of the tarred white mouse.

It has now been found the smearing with tar actually diminishes this lytic property of the dorsal skin of the mouse, and that this property disappears altogether on the appearance of the carcinoma.

Further, the influence of Röntgen-rays upon this property appeared to be of great importance. If, for instance, skin-extracts possessed of a certain lytic power are subjected to Röntgen-rays, this power may be increased or diminished, according to the *length of time of the irradiation* (180 K.V., 2 m.A., 25 cm. unfiltered).

Inversely, a non-lytic extract from a carcinomatous skin may be made lytic again by röntgenization, and the same oscillations in the lytic action will be dependent upon the duration of the irradiation.

That this lytic power must have something to do with the disappearance of tumor-cells under the influence of irradiation, is evident from the fact that no destruction of the cells by the rays takes place in an inorganic environment, but does, on the contrary, occur when they are found in an originally non-lytic carcinomatous serum or organ-extract. (Biochem. Zeitschr. Vol. 182, p. 377, and Fortschr. a. d. Geb. der Röntgenstrahlen, Vol. 35, part. 4).

Whereas our results so far are more or less in agreement with those of FREUND and KAMINER, we differ from them widely on some points.

In particular their view that the lytic substance must be considered to be a certain saturated dicarbonic acid, we were unable to substantiate.

Neither succinic nor suberic acid, both of which may be taken as representatives of the above type of acids (as indicated by FREUND and KAMINER), exhibited any lytic action in our experiments.

¹⁾ After researches carried out in the Anthonie van Leeuwenhoekhuis at Amsterdam.

We are of opinion that the researchers in Vienna have been led astray by by-products.

We therefore felt constrained to take in hand once more the isolation of the lytic substance, and to do so after our own method.

Our hypothesis was that in the organism there must be a system which governs the generation and the regeneration of all the cells of the body, and which must be specially active wherever strong regenerative processes are continually at work.

We, therefore, determined in which organs the lytic element is present in the greatest quantity, and endeavoured to find a method of preparation whereby that element may be isolated from the organs. The organs examined were the liver, spleen, pancreas, thymus, adrenals and lymphatic glands.

Previous researches by FREUND and KAMINER had demonstrated that the active matter must be soluble in ether, and this was confirmed by us. But the experience with modern hormone preparations does not appear to warrant the conclusion that the substance has a lipoid character.

Now, while the ether extracts of almost all organs were found to possess a more or less lytic action, the extracts of the lymphatic glands, spleen and thymus possessed this in a much higher degree than that of the other organs. This was in accordance with the views of various researchers who have attributed the power of resistance to carcinoma in particular to the quality of the connective tissue and to the reticulo-endothelial and lymphocytic apparatus.

First of all, however, the method of extraction had to be improved.

An extraction of calves' spleens and lymphatic glands was made with 10 to 15 times the quantity of ether under return-flow cooling for 8 hours. The ether extract was then partially evaporated and the last remains of the ether driven off by the air-current. The watery residue was treated with an excess of acetone. This caused a white flocculent precipitate, which was washed several times with acetone and finally dried on the filter. This dried precipitate was then rubbed down with 96 % alcohol (or methyl-alcohol), whereby the greater part is dissolved. After being filtered, however, a whitish-grey substance is retained, which is separated from the last traces of alcohol in the filter at 37 %. This whitish-grey powder is finally shaken briskly with a phosphate-buffer in phys. NaCl solution (1 part buffer-solution after SÖRENSEN of P_H 7.7 with 5 parts phys. NaCl solution). Hereby the active matter is dissolved.

In this way an acetone-precipitate of several hundreds of milligrammes is obtained from 500 Gr. of organ. After washing with alcohol, only fractions of one milligramme pass into the watery solution. This preparation, however, must be considered as still very impure. As a rule, a clear solution of the active matter can be obtained by this method. The presence of any traces of lipoids hinders the determination of the lytic power. The solution

contains no ferment (trypsin: Fuld-Gross, or lipase: tributyrine test according to the stalagmometric method) and is devoid of albumen.

In determining the lytic power, account should be taken of the P_H of the environment in which this determination takes place. It appeared that an acid reaction counteracts the lysis; an alkaline (P_H of blood serum), on the other hand, promotes it. We therefore carried out all our determinations with a P_H of 7.1.

By this method a substance can be obtained from the spleen and lymphatic glands of the calf, which is able to dissolve cancer cells. The lytic power thus far has been brought up to 81 %. Further purification, which is very necessary, will probably raise this percentage still higher.

The question now was to see whether this artificially prepared extract would also be affected by the X-rays in the same way as serum or skin extract. This proved to be the case.

EXAMPLE:

Extract of lymphatic glands	Lytic power in %
Non-radiated	81
After 1 hour's irradiation	61
.. 2 hours' ..	46
.. 4	62

SUMMARY.

1. Normal blood-serum and normal organs contain a substance capable of dissolving cancer-cells.

This substance is influenced by Röntgen-rays.

Extracts of carcinomatous organs lack this power, but obtain it by irradiation.

2. We have succeeded in isolating this substance from the reticulo-endothelial system, but as yet only in an impure state.

The solutions obtained contain neither tryptic nor lipatic ferment, and are free from albumen.

The lytic power of these solutions changes under the influence of Röntgen-rays.

Botany. — *Rapid flowering of Early Tulips ("Van der Neer").*
(Communication N^o. 24 of the Laboratory for Plant-physiological
Research, Wageningen, Holland.) By Miss IDA LUYTEN.
(Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of March 26, 1927).

§ 1. *Introduction.*

In connection with the research of BLAAUW (1926) on the rapid bringing into bloom of Darwin-tulips, this year experiments were made on the early forcing of "Early Tulips". The influence of the various temperature-combinations in summer having only been traced on the Darwin-tulip Pride of Haarlem (BLAAUW and VERSLUYS 1925, LUYTEN, JOUSTRA and BLAAUW 1925, MULDER and BLAAUW 1925), the plans for the "Early tulips" were based on the results of these papers combined with the experience gained by BLAAUW (1926) with the rapid flowering of these tulips, in the hopes that the early tulip would not differ too much from the late Darwin-tulip.

It has appeared to us, that the Single Early Tulip *Van der Neer* comes very near to the Darwin-tulip in many respects; from the researches it also followed however, that among varieties of the "Single Early Tulip" as *Proserpine*, *La Remarquable* and *Van der Neer* there exist great differences in response to the external factors examined, such as: the various temperatures and light and dark. With respect to the "Early Tulips" therefore we cannot draw up one definite scheme; to the definite varieties separate rules apply. Subjoined we give the results obtained in forcing the variety *Van der Neer*.

§ 2. *The temperature-treatment during the summer till planting-time.*

As summer-treatment of the Darwin-tulip for early flowering 3 weeks 20° + 8 weeks 9° was chosen by BLAAUW (1926). Indeed it had appeared at the time (communications 17, 18, 19), that with respect to the rate of forming the flower 17° was as favorable as 20°, but 17° gave an increase of the number of floral parts and transitions between tepals and foliage-leaves. Especially this latter always increases the chance of the flower growing deformed or crooked.

This year however for *Van der Neer* a temperature of 17° was used by the side of 20°, in order to trace whether this offered a more favourable combination with 9° with regard to flowering earlier.

On June 25 the 220 tulips arrived at our laboratory after lifting straight from the field. The tulips were divided *into equal numbers and weights*, so that we might be sure afterwards, that every part of the research was based on the same initial weight. We placed 8×12 individuals at 17° C., whilst 8×12 and 1×18 were put at 20° C. Every group of 12 individuals weighed 405 grams; the 18 had $1\frac{1}{2} \times$ that weight. At the same time on June 25th 10 bulbs, weighing consequently $337\frac{1}{2}$ grs., were opened in order to trace how the condition was at that moment. It appeared that all bulbs were in *Stage I*, i.e. that the terminal vegetation-point was still splitting off foliage-leaves (Table I), 2 to 3 foliage-leaves had been formed per bulb. As the table shows, the length of the outer foliage-leaf was measured; this is still very small on this date, viz. averagely 1.15 mms. ± 0.016 . As already fully described in previous publications, staining in an aqueous solution of iodine and iodide of potassium is required to clearly see all the parts split off by the vegetation-point. For an accurate description of the stages we refer to MULDER (1927).

We wanted to transfer the tulips to 9° , when the flower would have reached *Stage III*, i.e. when the first whorl of tepals would be found as individual protuberances. Every time we wanted to state this, bulbs had to be opened to judge of the floral growing-point; for this control 2×12 bulbs had been reserved.

From the Tables I and II we see, that the bulbs were not opened until July 13; in 17° the terminal vegetation-point of 2 bulbs was respectively in stage II, i.e. the vegetation-point was raising itself, so it was passing on to flower-formation, and in stage II $\dot{\bar{a}}$ III, i.e. there was a beginning of the first whorl of tepals. In 20° both bulbs were in stage II. The opened bulbs having not yet reached the desired stage, the opening of 2 bulbs from each temperature was deemed sufficient. On July 16, 5 bulbs from 17° (Table I) all appear to have reached the desired stages III or III⁺; all sets from 17° therefore, are then transferred to 9° .

On this date the bulbs from 20° appeared to have progressed no farther than stages II and II — (Table II); on July 20 however these groups could also be transferred to 9° , since by that time 4 opened bulbs showed stages which had passed III. Probably these bulbs had already attained stage III on July 19.

$4\frac{1}{2}$ weeks after being transferred to 9° , the bulbs were *planted*, whilst on those days respectively Aug. 16 and Aug. 20 (Tables I and II) groups of 11 and 10 bulbs of both sets were controlled with regard to the fact whether the flower had really been finished in this period: indeed in both groups the flower appeared to be quite finished, except one flower of the exposure 17° — 9° , which was in stage VI⁺. At the same time some measures were taken, which we can read from the tables.

The tables moreover show: 1^o. that the successive flower-stages are reached in a very rapid tempo, from which it follows, that the control-days should follow very closely; 2^o. that the length of the outer foliage-

TABLE I. Condition from the lifting to the planting.
 Exposure: 21 days 17° C. + 4½ weeks 9° C.

Date	Length outer foliage-leaf in mms.		Number of foliage-leaves per 10 bulbs	Stage	Length apex of flower to base of stem in mms.	
	M	m			M	m
June 25, 1926 Beginning of the experiments	1.15 ± 0.016 n = 10		24	I n = 10		
July 13	2.— 2.25			II and II to III	—	
July 16 transferred to 9°	2.61 ± 0.12 n = 5			III and III+	—	
Aug. 16, 1926 Day of planting	10.18 ± 0.42 n = 11		42		6.16 ± 0.22 n = 11	

TABLE II. Condition from the lifting to the planting.
 Exposure: 25 days 20° C. + 4½ weeks 9° C.

Date	Length outer foliage-leaf in mms.		Number of foliage-leaves	Stage	Length apex of flower to base of stem in mms.	
	M	m			M	m
June 25, 1926 Beginning of the experiments	1.15 ± 0.016 n = 10		24	I n = 10		
July 13	1.55 1.55			II	—	
July 16	1.65 1.75			II and II —	—	
July 20 transferred to 9°	2.28 ± 0.08 n = 4			IV; V—; VI+; VI—VII.	—	
Aug. 20, 1926 Day of planting	10.46 ± 0.23 n = 10		40		7.02 ± 0.26 n = 10	

leaf on the day of transferring to 9° is greater in the group from 17° (Table I, July 16, 2.61 mms.) than in the group from 20° (Table II, July 20, 2.28 mms.), but that the bulbs with the preliminary treatment of 20° have caught up on the day of planting (Aug. 20) and now show an outer foliage-leaf about equally long as in the group 17°—9° on Aug. 16. The average length from the apex of the flower to the base of the stem is even somewhat greater now in 20° than in 17°.

§ 3. *From the planting to the transfer to the hothouses.*

All tulips were stripped of their outer, dry brown scale before the planting; this happened to prevent the tulips as much as possible from raising themselves during the shooting of the roots. Moreover the axillary buds were removed from this brown scale. As plant-boxes we used boxes measuring 22 × 20 cms. clear diameter and having a depth of 18 cms.

Since BLAAUW (1926) in the early flowering of the Darwin-tulips was troubled with the drying out and also with the tipping of the stems and we did not know whether these phenomena would occur in this early tulip and whether they might be due to an excess of humidity, certain precautions were taken in these experiments in order to trace this.

For this purpose the group of 18 tulips with the preliminary treatment 20°—9° (we shall indicate the groups which were in the thermostat in 20° C. and next in 9° C. in this way, whilst we shall designate the groups from the other temperature-exposure 17°—9°) was planted into 3 boxes, each having 10 air-holes in the side-walls. In that way the Darwin tulips had been planted; they had been liberally watered on the surface-soil. This group would be treated similarly. All other groups from 20°—9° and from 17°—9° were planted into boxes, the wood of which was entirely coated with paraffin: partly to prevent desiccation partly to preclude a possible influence of the wood. No holes were made in the side-walls, but 7 holes in the bottom. These boxes would never be watered on top, but were given a certain quantity at set times by immersing them in a cistern of water. The bottom of all the boxes was covered with a layer of fine gravel 1 cm. thick, whilst dune-sand was chosen as soil. Groups of six bulbs were placed in a box, the root-wall at 10 cms. below the edge of the box; the box was filled in with dune-soil up to 2 cms. above the tip of the bulb. For the present, therefore, little soil was added, to facilitate measuring the "noses"; afterwards when the plants were flowering the boxes were filled up to 4 cms. above the tip of the bulb.

Each group of 12 therefore was divided over two boxes. Accordingly we planted 6 groups each of 12 bulbs with a preliminary treatment of 17°—9°; 6 groups of 12 bulbs each, and 1 group of 18 bulbs having had a preliminary treatment of 20°—9°. How the groups were divided afterwards, we shall discuss in § 4 in connection with their removal to the hothouses.

Now that the tulips have been planted, they are still kept at 9° ; but as they are now in a room which is being cooled, the temperature fluctuates more, that is between 8.5° and 10° C. *in the ground*. The soil in the boxes is covered with thick moist filter-paper to prevent it from drying out. The soil of the boxes coated with paraffin remains much moister than of the non-paraffined.

On *November 11* the average length of the "nose", i.e. that part of the foliage-leaves which shows from the bulb and is measured from the apex of the bulb to the apex of the outer leaf appeared to be 4.63 cms. ± 0.14 in the 71 bulbs 17° — 9° C. (Table III). 72 bulbs had been planted; one however had to be left on account of disease. This whole set 17° — 9° C. was transferred to the hothouses, i.e. to the higher temperatures on this date. For the early flowering of the Darwin-tulip BLAAUW (1926) found ± 6 cms. as most desirable length for removal. Since the plant of the early tulip however remains much smaller than that of the Darwin-tulip, an *average of 4.5 cms.* was chosen in this case as average length of the "nose" as a standard for removal to the hothouses, that is to higher temperatures.

The "noses" of the groups 20° — 9° were not yet so long on *November 11*; the average length ($n=71$) was 3.44 cms. ± 0.10 on this date (Table IV). Not until *November 18* the average length was 4.59 cms. ± 0.12 , so that by that time they too could be transferred to the higher temperatures in the hothouses.

We call attention to the fact that the start *the group* 17° — 9° had in its development of 20° — 9° and which amounted to 4 days on *July 16* (cf. Tables I and II), has now increased to 7 days.

§ 4. *The effect of the various temperatures and of light and dark on the rapid flowering.*

The tulips in the paraffined boxes were transferred to 3 different temperatures in the hothouses, viz. 22° — 23° C., $19\frac{1}{2}^{\circ}$ — $20\frac{1}{2}^{\circ}$ C. and 16° — 17° C. Accordingly in every hothouse there were put two groups of each temperature-treatment on the tablets. These two groups were again divided by placing one group in the dark for the present, the other directly in the light. Our intention was to trace whether a temporal darkening affected the stretching and the rapid coming into bloom of the tulips. For this purpose black paper coverings were put on the two boxes of the one group; they were 25 cms. higher than the boxes, so that the plants had room and plenty of air to grow. The 18 bulbs of the treatment 20° — 9° planted into 3 non-paraffined boxes, were placed in a hothouse kept at $19\frac{1}{2}^{\circ}$ — $20\frac{1}{2}^{\circ}$ C. As contrasted with the 3 other hothouses which were kept dry and the moisture of which fluctuated between 35 %—50 %, this hothouse was frequently syringed in consequence of which the moisture was kept at 75 %. The boxes in the moist hothouse were watered on the surface every day, whilst as already mentioned in § 3, to the boxes in the

other hothouses such a quantity of water was given by immersing them, that in these dry hothouses the moisture of the soil was the same in all boxes and much less than in the moist hothouse. In this system of immersing, in which a definite period was fixed for every hothouse, the covers need not be taken from the boxes and consequently the dark was not interrupted. From the tables III and IV we can read the division once more. Let us now first discuss the groups which have had

a preliminary treatment with 17°—9° C.

and which were transferred to the hothouses on November 11, 1926. Through the influence of the different hothouse-temperatures we soon see a difference in length in the groups which are in the light. In the groups in the dark we cannot state this for the present, since it is not allowed to take off the covers. In the groups in the light, the nose-length is greatest in the hottest house. On our measuring the nose-length in these groups on *November 22*, it appears that besides this outer foliage-leaf part of the stem found under this leaf has also been pushed from the bulb. From table III we see, that the length of these parts together averagely amounts to 19.18 cms. \pm 0.91 (group B) in 22°—23° C., in 19½°—20½° averagely 15.89 cms. \pm 0.52 (group D), and in 17°—18° it is not more than 13.03 cms. \pm 0.61 (group F). Fig. 3 shows half of group B, that is from hothouse 22°—23°, whereas the box of Fig. 1 clearly shows, that the group from hothouse 19½°—20½° is less advanced; the foliage-leaves are still rolled up in this case.

On Nov. 22 when we measured the groups which were in the light, we took the covers from the groups kept in the dark for the first time. It appeared, that the period of 11 days during which they had been in the dark, had already been too long, for practically all groups were etiolated; and the groups kept in the higher temperatures, showed this most. Fig. 2 pictures to us what the group C from the hothouses 19½°—20½° C. looked like; it likewise enables us to compare it with the group kept in the light in the same hothouse (Fig. 1). It appears that darkening especially promotes the vigorous growth of the part of the stem under the outer foliage-leaf. On comparing the average lengths of the outer foliage-leaves in Table III, we see, that they are fairly identical; the averages of the parts of the stem under this leaf however show marked differences between the groups 'light' and 'dark', viz. between A and B 7.08 cms., between C and D 7.28 cms. and between E and F 6.07 cms. The long parts of the stem were no more erect, but nearly all of them were bent and crooked, which is clearly visible in the box from the dark in Fig. 2; this box may also be compared with the group D, placed in the *light* in the same hothouse (Fig. 1).

An other disadvantage of darkening is, that in all groups the leaves had remained firmly closed, while the groups in the light had started unfolding

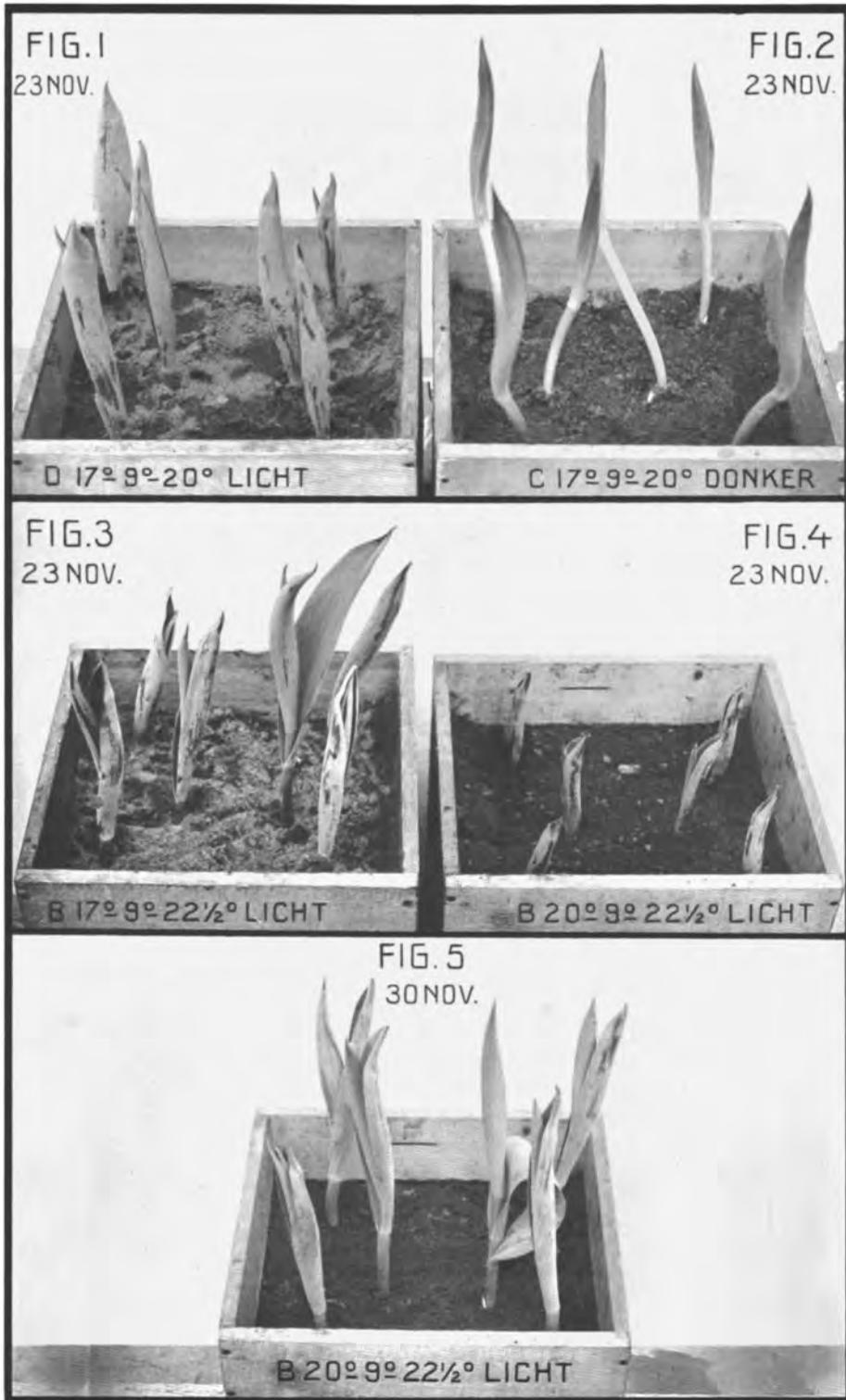
TABLE III. The temperature-exposure 17° — 9°.

Group	Nov. 11		Division		Nov. 22		Nov. 22		Nov. 22	Come into bloom	Number of flowers
	Average length of nose in cms.		Temperatures	Light and dark	Average length of part of stem + outer foliage leaf in cms.		Average length of part of stem in cms.				
M	m	M			m	M	m	M	m		
A	3.67 ± 0.15		22° — 23°	dark	22.8 ± 0.68		10.49 ± 0.42		12.40	Dec. 3	3 in bloom, 5 withered, 3 wh.-edged, 1 bulb diseased
B	5.62 ± 0.50		22° — 23°	light	19.18 ± 0.91		3.41 ± 0.45		15.77	Dec. 2	11 " " 1 " " 1 bulb diseased
C	4.75 ± 0.25		19.5° — 20.5°	dark	22.82 ± 0.82		9.22 ± 0.53		13.60	Dec. 3	5 " " 6 " 1 " "
D	4.33 ± 0.23		19.5° — 20.5°	light	15.89 ± 0.52		1.94 ± 0.26		13.95	Dec. 3	10 " " 1 " 1 " "
E	5.25 ± 0.22		17° — 18°	dark	19.92 ± 0.55		6.92 ± 0.34		13.0	Dec. 9	11 " " 1 " "
F	4.14 ± 0.24		17° — 18°	light	13.03 ± 0.61		0.85 ± 0.19		12.18	Dec. 9	10 " " 2 " "

TABLE IV. The temperature-exposure 20° — 9°.

Group	Nov. 11		Nov. 18		Division		Nov. 22		Come into bloom	Number of flowers
	Average length of nose in cms.		Average length of nose in cms.		Temperatures	Light and dark	Average length of nose in cms.			
M	m	M	m	M			m	M	m	
A	3.40 ± 0.18		4.27 ± 0.20		22° — 23°	dark	9.33 ± 0.36		Dec. 9	10 in bloom, 2 white-edged.
B	3.48 ± 0.24		4.80 ± 0.33		22° — 23°	light	9.15 ± 0.33		Dec. 7	12 " "
C	3.32 ± 0.10		4.85 ± 0.52		19.5° — 20.5°	dark	9.53 ± 0.82		Dec. 13	11 " " 1 coloured
D	3.69 ± 0.14		4.72 ± 0.18		19.5° — 20.5°	light	8.52 ± 0.31		Dec. 9	10 " " 1 wh.-edged. 1 bulb diseased
E	3.32 ± 0.20		4.30 ± 0.23		17° — 18°	dark	7.07 ± 0.36		Dec. 13	12 " "
F	3.52 ± 0.10		4.64 ± 0.18		17° — 18°	light	7.59 ± 0.25		Dec. 13	11 " " 1 withered
Non paraffined	3.02 ± 0.22		3.95 ± 0.25		19.5° — 20.5°	light	— —		Dec. 9	11 " " 1 " 5 white-edged

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on reaching a sufficient length. So we see, that in the group *B* from 22° — 23° , when the part of the stem between the tip of the bulb and the bottom-leaf is $3\frac{1}{2}$ cms., the outer foliage-leaf has already unfolded, so that the second foliage-leaf is usually revealed along its entire length (Fig. 3), while the group *D* from hothouse $19\frac{1}{2}^{\circ}$ — $20\frac{1}{2}^{\circ}$ (Fig. 1) starts unfolding when the part of the stem mentioned above average 1.94 cms., so that some apexes of the second foliage-leaves are already visible. In fig. 2 we notice that the foliage-leaves of the tulips kept in the dark still firmly keep together, in spite of the basal part of the stem averaging 10.5 cms.

From this moment all groups of the temperature-exposure 17° — 9° are put in the light. The groups which had been in the dark, grow green and the foliage-leaves are going to unfold.

On *Nov. 25* some flowers of the group *B* 22° — 23° light (Table III) are going to colour, on *Nov. 26* two are quite coloured and this group is transferred to 17° — 18° , to prevent the high temperature from withering the unfolded flowers too rapidly. On *Dec. 2* 11 of the 12 tulips are in flower; the 12th flower is dried up. With regard to the subjoined description it should be borne in mind, that the groups from hothouse 22° — 23° when 1—2 flowers start colouring and those from hothouse $19\frac{1}{2}^{\circ}$ — $20\frac{1}{2}^{\circ}$ when 2—3 flowers begin, are transferred to the hothouse 17° — 18° .

From the group exposed to 22° — 23° dark and subsequently to light (Table III A) 3 flowers are unfolded, 5 withered, 3 remain white-edged and 1 bulb is diseased on *Dec. 3*. This bad result of this group may be attributed with certainty to the 11 days' darkening, which made the basal part of the stem stretch too much.

Of the group *D*, $19\frac{1}{2}^{\circ}$ — $20\frac{1}{2}^{\circ}$ light (Table III) 10 flowers are in bloom on *Dec. 3*. Whereas from the group which had been kept in the light only 1 flower dried up, 6 flowers withered from the group which had been kept in the dark for some time. To give an impression of the bloom of these early tulips, the one box of the group 17° — 19° , next $19\frac{1}{2}^{\circ}$ — $20\frac{1}{2}^{\circ}$ C. light (Table III, group *D*) was photographed on *Dec. 1* (Fig. 6), while Fig. 7 shows the same box on *Dec. 22*, 1926, when the stalks have stretched still further and the colour has grown more intense. As a standard of the lengths the plants attain, it should be mentioned, that from the tip of the bulb to the upperside of the flower the length amounted to 39.0, 36.5, 39.5, 39.0, 37.0 and 35.0 cms. In the groups (Table III, *E* and *F*) which had been in the hothouse 17° — 18° the flowers were open on *Dec. 9*; in the group *F*, which was throughout kept in the light, two flowers were withered, in the group *E*, which had partly been kept in the dark, 1 flower. As this group was not so much etiolated, when the covers were removed, it might be expected, that the influence of the dark would not be so conspicuous in this case.

Looking backwards it appears that of the experimental plants which were kept in the light, 31 of the 35 bulbs flowered, while 4 flowers withered;

the groups which had been in the dark for 11 days, did indeed nearly all flower simultaneously with the groups which had only been exposed to the light, but of the 34 tulips only 9 flowered; 12 withered and 3 had white tips on the tepals.

The temperature-exposure 20°—9°.

From § 3 it already appeared, that the tulips which had been subjected to this preliminary treatment did not attain a nose-length of $4\frac{1}{2}$ cms. (averagely 4.59 cms. \pm 0.12), until Nov. 18, i.e. a *week* later than when exposed to 17°—9°, so that these groups were not distributed over the 4 different hothouses with the temperatures of 22°—23° C., $19\frac{1}{2}$ °— $20\frac{1}{2}$ ° C., and 17°—18° C. before that date (See Tab. IV).

On Nov. 22, when the groups 17°—9° were measured for the sake of comparison the nose-length was also determined in the groups 20°—9°. They cannot yet be distinctly discriminated, it might only be remarked, that the noses of those groups exposed to the higher temperatures, are already slightly longer than those kept in the hothouses 17°—18°. In this short period of time "the keeping in the dark and in the light" has not caused any difference. As in the set 17°—9° we had removed the covers a little late, so that the parts of the stem had grown too vigorously, we now took off the covers of these groups also on this date to prevent them from stretching too much. For mutual comparison figs. 3 and 4 give two boxes on Nov. 23: both have been exposed to the same hothouse-temperatures and "light", but the tulips of the box in Fig. 3 have been exposed before to 17°—9°: those of Fig. 4 to 20°—9°. The former box was kept in the hothouse for 12 days, the latter for 5 days. Accordingly the relatively inhibitory effect of 20°—9° is evident. When the group 20°—9° has also been in the hothouse for 12 days (Fig. 5), it is, on our comparing this picture with the box in Fig. 3 (the exposure to 17°—9°), equally far advanced as this group was on Nov. 23. The treatment 17°—9° therefore still has a start of 6—7 days.

When the tulips are going to colour in these groups too the boxes are directly transferred to the hothouse 17°—18° C., as it was done with the groups 17°—9°. The non-paraffined boxes which had been kept in a moist hothouse of $19\frac{1}{2}$ °— $20\frac{1}{2}$ °, were then put in a moist hothouse of 17°—18° C. In these groups too those sets which were kept in 22°—23° C., were the first to flower, viz. group *B* on Dec. 7 and group *A* on Dec. 9 (see table IV); from the hothouse $19\frac{1}{2}$ °— $20\frac{1}{2}$ ° group *D* (light and non-paraffined) followed on Dec. 9 and group *C* (dark) on Dec. 13, whilst *E* and *F* from the hothouses 17°—18° also flowered on Dec. 13. From this it follows (see also Table IV), that *keeping dark is not profitable with regard to the period of flowering*; on the contrary in the higher hothouse-temperatures its influence is rather somewhat *inhibitory* in this respect. Now that we have kept these groups of tulips dark for a shorter period,

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they do not suffer from withering or white tips. From the 35 tulips kept in the light in the dry hothouses, 33 came into bloom, 1 withered and 1 showed white tips at the tepals; in the 36 tulips which had been kept temporarily dark, 33 came into bloom, 2 flowers withered and 1 bore white edges. In the experiments with the preliminary treatment 17° — 9° we already noticed, that keeping dark somewhat too long has a detrimental effect due to the stretching of the basal part of the stem, which makes the tulips grow crooked and afterwards in the flowering period causes a strong withering of the flowers or a partial lack of pigmentation, i.e. appearance of white spots. As a period of "dark" does not appear to be a favorable factor for the rapid beautiful flowering of the early tulip *Van der Neer*, the best thing is to put the tulips in the then desired temperature in the light, when they have attained a nose-length of averagely $4\frac{1}{2}$ cms.

On our comparing those groups which have been exposed to a preliminary treatment of 17° — 9° and 20° — 9° , and which have throughout been kept in the light in the hothouses, it appears (Tables III and IV), that the groups 17° — 9° flowered earlier, viz. respectively those from 22° — 23° , 5 days, those from $19\frac{1}{2}^{\circ}$ — $20\frac{1}{2}^{\circ}$, 6 days and those from 17° — 18° , 4 days. From which we must conclude that a temperature-exposure to 17° at the beginning of the flower-formation has an accelerating effect and choosing this temperature is to be recommended.

It has likewise appeared that these early tulips are not subject to "tipping"; neither in the dry nor in the moist hothouse did this phenomenon occur in any specimen. The flowers in the moist hothouse 20° were practically as beautiful as in the other hothouses; from table IV we might be inclined to conclude, that the moist hothouse did not give such fine flowers; but this must probably be attributed to being kept long in the higher temperature, as at the beginning no moist hothouse 17° — 18° was available. Humidity however does not offer any advantages, therefore the temperature of the air and of the boxes need not be kept extra-wet.

In § 1 we called attention to the fact, that for the Darwintulip the temperature-exposure to 17° in summer for early flowering had previously not been chosen, because it would sooner give rise to deviations, and transitions from tepals and leaves might lead to crooked growth. In *Van der Neer*, as far as we could trace, no transitions of different leaves and floral parts occurred neither in 17° nor in 20° . As to the deviation of the normal number of floral parts: 663, this appeared to be by no means greater in 17° than in 20° . In 17° we counted in 27 flowers: 22 of 663 and 5 deviations (viz. 3 with 884, 1 with 763 and 1 with 773); on exposure to 20° on the other hand we counted in 39 flowers 30 flowers with 663 and 9 with deviations (viz. 1 with 884, 4 with 773, 3 with 673 and 1 with 774). So it may be, that the various temperatures have an effect on the number of floral parts slightly different from that in the Darwin-tulips, where an increase of the number occurred in the lower temperatures, which difference

was already marked in 23°, 20° and 17°. In a separate research we shall further ascertain the correlation between the various temperatures in summer and the number of floral parts in this "early Tulip".

I wish to conclude with some remarks on the time of planting. Some rests of the group, destined for the control of the stages, were kept dry for some time at a temperature of 9½°—10½° C. In this way we had got 4 tulips of the preliminary treatment 17°—9° and 5 tulips of that of 20°—9°. Not until *September 6* they were potted up in earthen cylinders and placed at 8½°—9½°. Simultaneously with the temperature-treatment to which they belonged, they were removed to the hothouse 19½°—20½°, and now it appeared, that they came into bloom simultaneously with the others in the hothouse 19½°—20½°. From this it might be concluded that planting later is possible. A subsequent year it will be further investigated whether this might lead to some change in the treatment, which may prove whether the time of planting need be chosen so accurately as we have done now.

§ 5. Conclusion.

From the above it has sufficiently appeared, that with a correct treatment the "Early Single Tulip" *Van der Neer* may flower about *December 1—2*. On account of its vigorous flower-stalk this tulip is particularly fit for forcing; this flower-stalk *does not react upon the light at all*, so that the stalks of all tulips continue as straight as arrows and never show a single curve.

If, therefore, these tulips in the first period of flower-formation (i.e. till Stage III has been reached) are put in 17°, this causes coming into bloom 5—7 days earlier than if they had been put in 20° C. during this first period. If Stage III has been reached, the tulips are transferred to 9° and they are planted after a month (in 9°). When the noses have attained a length of 4½ cms., they may be transferred to the various temperatures in the hothouses. Good care should be taken, that they are put in the light on the tablets, for this is conducive to earlier and finer flowering.

According as higher temperatures are chosen, respectively 22°—23°, 19½°—20½°, 17°—18°, they will sooner come into bloom; we should see to it, that in the higher temperatures they are left in these temperatures until they colour, next they should be placed cooler to prevent the flowers from wrinkling. Besides it appeared that the hothouses need not be kept moist to ensure good flowers.

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Wageningen, February 1927.

LITERATURE.

BLAAUW, A. H. (1926). Rapid flowering of Darwin-tulips I. Proceedings Royal Acad. of Sciences. Amsterdam. Vol. 29 and Communication N^o. 21 of the Laboratory for Plant-phys. Research. Wageningen).

BLAAUW, A. H. and Miss VERSLUYS, M. C. (1925). The results of the temperature-treatment in summer for the Darwin-tulip. First part. Proceedings Royal Acad. of Sciences. Amsterdam. Vol. 28 and Communication N^o. 17 of the Laboratory for Plant-phys. Research. Wageningen.

LUYTEN, I., JOUSTRA, G. and BLAAUW, A. H. (1925). Idem. Second part. Proceedings Royal Acad. of Sciences. Amsterdam Vol. 29 and Communication N^o. 18 of the Laboratory for Plant-phys. Research. Wageningen.

MULDER, R. en BLAAUW, A. H. (1925). Idem. Third part. Proceedings Royal Acad. of Sciences. Amsterdam. Vol. 29 and Communication N^o. 19 of the Laboratory for Plant-phys. Research. Wageningen.

MULDER, R. 1927. The Periodical development of the Darwin-tulip. Communication N^o. 16 of the Laboratory for Plant-phys. Research. Wageningen. (To be published shortly).

Physics. — *On the Luminescence produced by bombarding solidified Gases with electric Rays at the Temperature of liquid Helium.*
By L. VEGARD and W. H. KEESOM. Communication N^o. 186
from the Physical Laboratory at Leiden.

(Communicated at the meeting of April 30, 1927.)

Luminescence from solid Hydrogen.

§ 1. *Aim and Method of the Experiments.* In previous communications¹⁾ results were given of experiments on the luminescence from solid nitrogen, and argon and neon mixed with nitrogen at the temperature of liquid helium.

At the same time it was our object to study the luminescence of solidified hydrogen, but owing to the very low freezing point of this substance, these experiments met with considerable difficulties.

During the experiment a current of hydrogen flows from the discharge tube in which the electric rays are produced, through a boring in a metallic cylinder, through which the rays had to pass, and into the emission chamber, where the gas was condensed on the cold copper surface.

When now the rays were put on, the conditions appeared to be somewhat unstable. The heat transport to the cold surface became so large that a marked increase of temperature resulting in a less effective condensation took place. When the condensation became less effective, the pressure of the observation chamber and the heat transport increased, and in a short time the condensation might cease and the liquid helium would evaporate within 15–20 minutes.

Already in 1924 we were able to examine the luminescence from solid hydrogen produced by swift cathode rays spectroscopically, and it was found to be dominated by continuous bands, and no lines could be seen. The liquid helium, however, did not last sufficiently long for obtaining photographs of the spectrum.

Even with the improved apparatus described in Comm. N^o. 183 fig. 1, which was used with success for the study of neon-nitrogen luminescence, we were not able to proceed further with solid hydrogen.

The experimental arrangement might be improved in the following way: 1. By increasing the isolation power of the cooling apparatus and by securing good contact between the copper rod and the liquid. 2. By

¹⁾ L. VEGARD, H. KAMERLINGH ONNES and W. H. KEESOM. These Proceedings **28**, 467; Comm. Leiden N^o. 173d; C. R. **180**, 1084, 1925; Comm. Leiden N^o. 183, 1926.

diminishing the current of hydrogen necessary for the running of the bulb.

With regard to the last possibility we tried to use a thermionic current — by employing a glowing cathode — but as the heated cathode must be well protected from the cold surface, our attempts in this direction were not successful.

We therefore returned to the ordinary discharge, but tried to reduce the necessary gas current by using a long and narrow boring in the cylinder, which connects the discharge tube with the emission chamber.

In order to increase the isolation power, a new cooling apparatus was made, the construction of which is shown in fig. 1a and b. The apparatus consists entirely of metal and contains three vacuum receivers, one outside the other. The inner receiver (1) contains liquid helium, the second one (2) is to contain liquid hydrogen, and this is surrounded by one (3) which is to be filled with liquid air. Each of the two inner receivers is provided with a tube for filling and one for the evaporated gas and a floater arrangement, which makes it possible to determine the stand of the liquid. The spaces inside and outside the hydrogen receiver (2) communicate and may be evacuated through a side tube. The space outside the liquid air receiver communicates with the emission chamber.

The layer of condensed gas to be bombarded with electric rays is formed on the end of the copper rod (4), which now passes

into the inner receiver, so as to secure an intimate contact with liquid helium. Perhaps the most important point in connection with the cooling

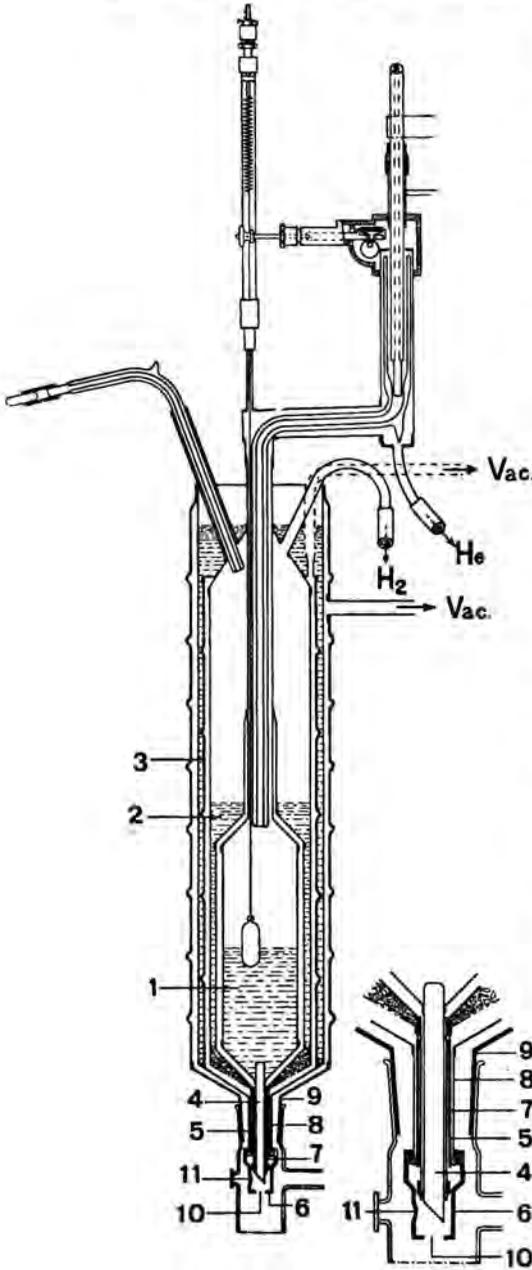


Fig. 1a.

Fig. 1b.

apparatus and that which has caused us the most trouble is the protection of the copper rod (4) from receiving heat from the surroundings.

Already for the apparatus previously described (fig. 1 Comm. N^o 183) a fairly satisfactory construction was found, but for the last apparatus the isolation of the copper rod was considerably improved.

In fig. 1 *b* is shown on a larger scale the construction of the bottom part of the cooling apparatus. The thick brass tube (5) is soldered onto the bottom of the liquid hydrogen receiver (2). A brass cap (6) is screwed onto the lower end of the tube (5). Thus the inner copper rod is surrounded by the tube (5) and cap (6), which during the experiment is maintained at a temperature approximately equal to that of liquid hydrogen.

In order to close the inner vacuum system the brass tube (5) is on the inside connected with the copper rod (4) and on the outside with the liquid air receiver by means of thin tubes of German silver (7) and (8).

The cap (6) is provided with an opening (10) at the bottom for the passage of the rays, and one (11) at the side for the observation of the luminescence.

The discharge tube is fixed to the cooling apparatus by means of a ground joint (9).

By means of this apparatus hydrogen was solidified very effectively on the copper surface, and by a proper choice of the canal that leads to the emission chamber and of the strength of the electric rays a surface of solid hydrogen could be bombarded for several hours. Under favourable circumstances one filling with liquid helium would last for 5–6 hours.

Experimental Results.

§ 2. *Luminescence from solid Hydrogen produced by swift Cathode Rays.* Several experiments have been made to investigate the light emitted when solid hydrogen is bombarded with swift cathode rays produced by induction coil and kenotron rectifier. The rays produced a luminescence of greenish blue colour and of moderate strength. The luminescence was seen to last for at least one minute after the cathode rays were cut off. Consequently a luminescence was produced, which was characteristic of solid hydrogen. In the spectroscope only a continuous spectrum was seen. It might, however, be possible that lines appeared, which were masked by the continuous spectrum. In order to see whether possibly also lines were present, the small spectrograph with high light power was used. The spectrograph was carefully adjusted, so that the continuous spectrum could be seen by means of an auxiliary lens.

Already in the spring 1926 we made in this way an exposure, which lasted for more than one hour. But there was no impression at all to be seen on the plate after development.

Similar experiments have been undertaken at the beginning of this year. In one experiment the exposure lasted for nearly two hours, in another for more than one hour, but with the same negative result.

If we take into account the fact that much less intense sources of light giving a line spectrum would have produced an overexposed spectrogram we can conclude that the line spectrum — if any — emitted from solid hydrogen exposed to swift cathode rays must be extremely faint as compared with the energy emitted in the continuous part.

The absence of any impression on the plate is explained from the continuous character of the spectrum, because the total luminescence, which is of moderate strength is distributed over the whole spectrum.

§ 3. *Luminescence from solid Hydrogen produced by Hydrogen Canal Rays.* Investigations on the luminescence produced when solid hydrogen is bombarded with canal rays were undertaken in June 1926. Bombardment with hydrogen canal rays with velocities corresponding to a parallel sparc gap 3 mm. produced a marked luminescence on the layer of solid hydrogen.

Although the total strength of the luminescence was much smaller than in the case of swift cathode rays, we obtained by means of the same small spectrograph a very good spectrogram by an exposure of about two hours. In both cases we used the same sort of plates (Imp. pancromatic *B*).

The greater spectrographic effect is due to the fact that the canal rays gave a *line spectrum*.

An enlarged reproduction of the spectrum is given on Fig. 2 N^o. 1.

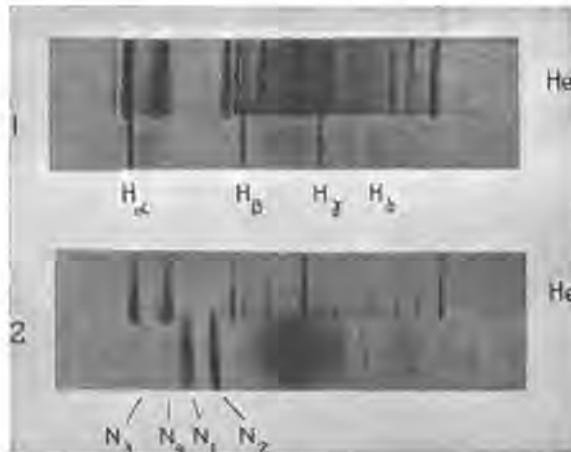


Fig. 2.

As the dispersion is small, the wavelength determinations cannot claim any great accuracy. Still the accuracy is sufficient to show that the very strong lines belong to the Balmer series. In between these lines we notice a number of faint lines.

All lines obtained on the spectrogram (26 in all) are given in the

table. The accuracy of measurement in the case of isolated strong lines may be estimated by means of the values found for the lines of the BALMER series, the identification of which is certain. In the case of faint

TABLE I.

No.	Solid hydrogen		Gaseous hydrogen	
	λ	I	λ	Designation
1	6559.0	100	6563.0	H_{α}
2	6324.9	5	6327.1	3 α 3 Q (1)
3	6214.5	5	6224.8	2 α 2 Q (1)
4	6106.8	10	6121.8	1 α 1 Q (1)
5	6018.6	2	6018.3	0 α 0 Q (1)
6	(5952.7)	2 (?)	5950.2	
7	(5796.0)	1 (?)	5796.8	
8	4861.7	80	4861.6	H_{β}
9	4680.5	10	4680.4	3 β 3 Q (1)
10	4629.5	20	4617.5	2 β 2 Q (1)
11	4578.5	5	4577	1 β 1 Q (m) (?)
12	4557.6	5	4554	1 β 1 Q (1)
13	4490.8	10	4490.6	0 β 0 Q (1)
14	4453.1	10	4453.2	
15	4418.0	5	4417.5	
16	4340.3	80	4340.8	H_{γ}
17	4308.8	10	4308.6	4 β 3 Q (1)
18	4204.0	10	4205.3	
19	4179.3	10	4182.1	3 γ 3 Q (1)
20	4161.0	5	4159.3	3 β 2 Q (1)
21	4101.5	35	4101.9	H_{δ}
22	4085.2	2	4085.2	{ 1 β 0 Q (1) 1 γ 1 Q (3)
23	4068.0	15	4069.8 (67.1)	
24	4025.8	5	4024.8	0 γ 0 Q (1)
25	3995.3	5	3995.3	
26	3970.8	5	3970.8	H_{γ}

lines and lines not well separated the error may be considerably greater than in the case of the BALMER lines.

In the third column is given the relative intensities of the lines as they were roughly estimated from the spectrogram.

With regard to the fairly small number of faint lines appearing in between the strong BALMER lines, they might either be lines characteristic of the solid state or a certain selection of lines belonging to the ordinary second hydrogen spectrum.

In the latter case we should on account of the extremely low temperature expect to find either rotational zero lines or lines corresponding to a small rotational quantumnumber.

Among the lines of the second hydrogen spectrum we find actually lines, which within the limit of possible errors coincide with those observed from solid hydrogen.

Although the second spectrum of hydrogen has a large number of lines, we cannot assume all coincidences to be accidental, and the correctness of the interpretation is greatly strengthened when we consider the series symbols of the lines appearing.

A great number of the lines of the second hydrogen spectrum has now been classified into band series and interpreted in terms of the general theory of band spectra. A summary of the present state of these investigations has been given by RICHARDSON¹⁾.

From the theory of band spectra we know that the frequency depends on the following conditions:

1. An electron jump between stationary orbits. For the class of lines here considered RICHARDSON finds that the electron jumps correspond to those of the BALMER-series, which take place towards a stationary orbit corresponding to a principal quantumnumber 2. These electronic jumps are denoted by α, β, γ , corresponding to $H_\alpha, H_\beta, H_\gamma$ of the BALMER series.

2. The number of oscillatory quanta possessed by the system at the beginning and end of the emission process. The oscillatory quantumnumber at the start RICHARDSON denotes by s , and that at the end by r .

3. The rotational energy at the beginning and end of the emission process. The rotational quantumnumber (m) cannot change by more than one unity.

With the denomination introduced by RICHARDSON lines of the second hydrogen spectrum are indicated by symbols of the form:

$$r \alpha s Q(m) \text{ or} \\ r \beta s Q(m).$$

The interpretation of the observed lines in terms of the symbols of RICHARDSON is given in the last column of the table.

¹⁾ O. W. RICHARDSON, Proc. Roy. Soc. 113, 368, 1926.

II

§ 4. *Luminescence from solid Nitrogen produced by Canal Rays at liquid Helium Temperature.* In previous communications one of us ¹⁾ has described results of experiments undertaken at the temperature of

TABLE II.

6379	N_3
6004.2	
5944.9	N_4
5765.9	
5628.0	} N_1
5548.5	
5230.3	N_2
4790 max.	A_6
4610 max.	A_5
4315	
4255 max.	A_4
(4169.1)	diffuse
4057.4	2. P.B. I
3996.1	" " II
3881.3	N.B. II
3870.1	
3849.3	
3802.5	2. P.B. I
3750.7	

liquid hydrogen, where solidified gases have been exposed to canal rays, In the case of solid nitrogen it was found that the luminescence consisted of the four bands $N_1N_2N_3N_4$, a diffuse band series (A), and some other lines and bands.

Similar experiments have been made at the temperature of liquid Helium.

In order to see whether the diffuse (A) bands were emitted at this very low temperature, we used the small glass spectrograph. A reproduction of the spectrum obtained is shown in N^o 2 of Fig. 2. The position of the lines and bands will appear from table II.

It appears that the four bands $N_1N_2N_3N_4$ are emitted. The series A also appears, but with an intensity distribution of the bands different from that found at the temperature of liquid hydrogen. In the case of liquid helium there is one maximum at 4610 of great intensity, but the intensity of the other bands is relatively much smaller than in the case of liquid hydrogen.

Some of the other lines and bands given in II have also been observed at the temperature of liquid hydrogen, but we also notice some new lines such as 6004, 5766, 3870, 3849. As to the nature of these lines,

some of them seem to belong to the positive band spectrum of nitrogen. These are indicated in the table with the series symbols used in H. KAYSER, „Handbuch der Spektroskopie“.

In conclusion we wish to express our sincere thanks to Mr. N. H. MOXNES (Oslo) and Mr. S. STENSHOLT (Oslo) for their valuable assistance in connection with the present investigations. We are also greatly indebted to Mr. G. FLIM and several others of the Cryogenic Laboratory for their most important assistance in connection with the technical part of the work.

¹⁾ L. VEGARD, Comm. Leiden N^o. 175; Det Norske Vid. Akad. Skr. 1925, N^o. 9; Ann. d. Phys. (4) 79, 377, 1926.

Chemistry. — *Equilibria in systems, in which phases, separated by a semi-permeable membrane.* XXI. By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of April 30, 1927).

Influence of the pressure on osmotic systems and on the osmotic pressure.

In the previous communication we have deduced some properties (a.o. the change of the osmotic pressure of a system E at change of the pressure of the water) with the aid of the $P_0\pi$ -diagram. Of the several other diagrams, which we can use for this also, we shall take now the $P\xi$ -diagram; consequently we draw on the one axis the pressure P and on the other axis the ξ viz. the *O.W.A.* of a system.

If we take f.i. in fig. 1 the point d , then this represents a system which has a pressure $P(d) = Ou$ and an osmotic water-attraction $\xi(d) = ud$. [Of course the length of the line ud does not represent the absolute value of $\xi(d)$, but this value, increased or decreased with an amount, which is the same for all systems in the same diagram. By this the difference between two ξ 's f.i. between $\xi(d)$ and $\xi(n)$ or between $\xi(d)$ and $\xi(k)$ does not change; consequently this has no influence on our considerations, in which we only use the difference of two *O.W.A.*'s].

If we represent the *O.W.A.* and the molecular volume of water by ξ_0 and v_0 , then we have:

$$d\xi_0 = -v_0 \cdot dP \dots \dots \dots (1)$$

so that the *O.W.A.* of the water decreases on increase of pressure. If we draw on the P -axis the pressure and on the ξ -axis the corresponding *O.W.A.* of the water, then we get, therefore, a $P\xi$ -curve like mr in fig. 1. As v_0 decreases at increase of pressure, but as this decrease is very small only, mr is a curve convex to the abscis, which is a right line at approximation.

We now assume that the temperature, for which fig. 1 is valid, is higher than the melting-point of ice under its own vapour-pressure. Under a definite pressure $P(m)$ the water then passes into:

$$\text{water} + \text{water-vapour} \dots \dots \dots (2)$$

which passes into water-vapour on further decrease of pressure. The $P\xi$ -curve of the water terminates on decrease of pressure, therefore, in the point m (fig. 1) where the $P\xi$ -curve ma of the water-vapour begins. This is defined in each point by:

$$d\xi_0 = -V_0 \cdot dP \dots \dots \dots (3)$$

in which ξ_0 and V_0 represent the O.W.A. and the molecular volume of the water-vapour. As $V_0 > v_0$ the two curves are situated in the point m , with respect to one another, as is drawn in the figure. This

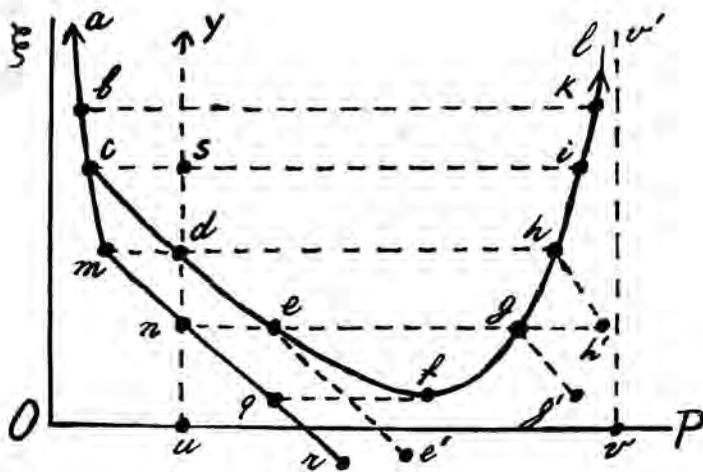


Fig. 1.

curve ma , of which we imagine the point a at infinite distance, is convex to the abscis and approaches asymptotically the ξ -axis; the O.W.A. of the water-vapour increases continuously on decrease of pressure, in order to become infinitely large under the extremely small pressure $P(a)$.

The $P\xi$ -curve of the water consists, therefore, of the two branches mr and ma , the first of which indicates the O.W.A. of the liquid water and the second that of the water-vapour; the point m represents the O.W.A. of the system (2) occurring under the pressure $P(m)$.

If we compress, starting from the pressure $P(a)$, water-vapour to water of the pressure $P(r)$ then we pass along the $P\xi$ -curve from a towards r , so that the O.W.A. decreases from $\xi(a)$ till $\xi(r)$. This change of the O.W.A. is continuous along the branches, in the point m , however, it is discontinuous.

We now take a binary system E , consisting of the components: $Y + \text{water}$. We assume that Y is a solid substance with such a small vapour-pressure, that it does not occur in the vapour under the considered pressures. Under a very small pressure we then have the system:

$$E_1 = Y + \text{water-vapour} \dots \dots \dots (4)$$

in which Y occurs as solid substance. If we raise the pressure, then this system passes under a definite pressure $P(c)$ which is smaller than $P(m)$ into the system:

$$E_2 = (Y + L + \text{water-vapour}) \dots \dots \dots (5)$$

As this system has no freedoms (2 components in 3 phases, while the

temperature is constant) it exists under a definite pressure $P(c)$ only and the liquid L has a definite composition. At further increase of pressure is formed a system:

$$E_3 = Y + L \dots \dots \dots (6)$$

We now shall assume that the solubility of the solid substance Y increases at increase of pressure. With increase of pressure then is formed under a definite pressure $P(s)$ the system:

$$E_4 = L \dots \dots \dots (7)$$

in which the liquid L has a definite composition and which remains on further increase of pressure. (We assume viz. that no new phase will be formed on further increase of pressure, as f. i. a hydrate of Y or a second liquid).

While the pressure $P(c)$ under which E_1 passes into E_2 and E_2 into E_3 , is independent on the ratio of the quantities Y and water, this is not the case with the pressure $P(s)$. Viz. $P(s)$ is larger, according as the system E_1 contains more solid Y and less water-vapour.

Consequently the system E occurs, according to its pressure, in several states, viz. as:

$$\left. \begin{aligned} E_1 &= Y + \text{water-vapour} && \text{when } P \leq P(c) \\ E_2 &= Y + L + \text{water-vapour} && \text{when } P = P(c) \\ E_3 &= Y + L && \text{when } P \geq P(c) \text{ and } P < P(s) \\ E_4 &= L && \text{when } P \geq P(s) \end{aligned} \right\} \dots (8)$$

As the system E_1 contains water-vapour, its *O.W.A.* is the same as that of water-vapour only; the $P\xi$ -curve of system E_1 coincides, therefore, with curve am of fig. 1. As the highest pressure $P(c)$ under which system E_1 still consists, is lower than the pressure $P(m)$, this curve does not continue till in m , but it terminates already in c . Consequently the $P\xi$ -curve of the system E_1 consists of curve ac , the final-point c of which represents also the *O.W.A.* of the system E_2 .

In this point c also begins the $P\xi$ -curve of the system E_3 , the direction of which is defined in every point by:

$$(d\xi)_3 = -\Delta V_E \cdot dP \dots \dots \dots (9)$$

If we represent the molecular volume of the solid substance Y and of the liquid L by V_y and V , then we have [compare formula's (10^a) — (16) of Communication XIX]:

$$\Delta V_E = \frac{V - yV_y}{1 - y} = \Delta V_w + \frac{y}{1 - y} \cdot \Delta V_y \dots \dots (10)$$

wherein for the sake of abbreviation:

$$\Delta V_w = V - y \frac{\partial V}{\partial y} \quad \Delta V_y = V + (1 - y) \frac{\partial V}{\partial y} - V_y \dots (11)$$

Now we shall show that the $P\xi$ -curve of system E_3 is represented by a curve cfi (fig. 1) which has a minimum in f and, if we represent

the melting-pressure of the substance Y by $P = Ov$, approaches asymptotically the vertical line vv' ; we imagine, therefore, the point l of this curve, on infinite distance.

We have assumed viz. that the solubility of the substance Y increases with increase of pressure; ΔV_y is negative, therefore [this follows f. i. from formula (15) of Comm. XIX]. Just as previously we shall take ΔV_w positive. It now appears from (10) that ΔV_E is certainly negative for values of y which differ no much from the unity, consequently under pressures which differ no much from the melting-pressure $P = Ov$ of the substance Y . Under lower pressure ΔV_E can be as well positive as negative; if, as we shall assume, Y is sufficiently small under low pressures, then ΔV_E is positive, therefore.

It follows from (9) that $(\delta\xi)_3$ is negative under lower pressures and positive under higher pressures. Starting from point c the $P\xi$ -curve must, therefore, firstly fall till in a point f , where the $O.W.A.$ is a minimum and it must rise afterwards, in order to approach the line vv' asymptotically.

The branches ac and cf form an angle with one another in the point c ; this follows from (3) and (9), the first of which defines the direction of ac and the second that of cf in the point c . [Of course we have to give to V_0 and ΔV_E the values, which they have in the point c , consequently under the pressure $P(c)$].

Above we have seen that system E_3 passes into system $E_4 = L$ under a pressure $P(s)$ and that $P(s)$ is larger, according as system E_1 contains more solid Y and less water-vapour.

We now imagine this point s anywhere on curve cf f. i. in the point g ; the $P\xi$ -curve of the system $E_4 = L$, starting from this point is then defined by:

$$(d\xi)_4 = -\Delta V_w \cdot dP \quad \dots \dots \dots (12)$$

of which the value of ΔV_w is given by (11). As we take ΔV_w positive and consequently the $O.W.A.$ of the system decreases with increase of pressure, we may represent the $P\xi$ -curve, therefore, by the dotted curve gg' . If we consider, as is done in fig. 1, stable states, only, then this curve goes, starting from g only towards higher and not towards lower pressures; the system $E_4 = L$ is viz. metastable under pressures smaller than $P(g)$ and it is converted into $E_3 = Y + L$.

As point s can coincide with every arbitrary point of curve cf , from each point of cf a similar curve is starting, three of which, viz. ee' , gg' and hh' are drawn.

It is clear, without more, that the curves gg' and hh' are situated below the curve cf ; it follows from (9) and (12) that this is also the case for ee' . If we consider viz. the value of ΔV_E from (10) then follows:

$$(d\xi)_3 - (d\xi)_4 = -\frac{y}{1-y} \cdot \Delta V_y \cdot dP \quad \dots \dots \dots (13)$$

As ΔV_y is negative in every point of curve cfl , follows for positive values of dP :

$$(d\xi)_3 > (d\xi)_4 \dots \dots \dots (14)$$

from which the above mentioned follows. Of course the same is true for the curve, starting from point c , which we have to imagine below cfl .

We are able to deduce the above still also in the following way. For this we use the rule, deduced in Communication XVI, that the *O.W.A.* of a system (at constant T and P) becomes smaller, when taking water in.

We imagine viz. in fig. 1 to be drawn a straight line, which intersects curve ef in a point x and ee' in a point x' . Point x represents a system $Y+L$ and x' a liquid L ; both systems have the same pressure. As the liquid L' must have a greater amount of water than the system $Y+L$, consequently L' can arise from $Y+L$ by taking water in; therefore, L' must have also a smaller *O.W.A.* than $Y+L$. Consequently the point x' must be situated in fig. 1 below point x and, therefore, curve ee' below curve ef .

If we compress, starting from the low pressure $P(a)$, the system E_1 , then this passes under the pressure $P(c)$ firstly into E_2 and afterwards into E_3 , which is converted into $E_4=L$ under a pressure $P(s)$.

If we take $P(s)=P(g)$ then the system goes along the curves ac , cfg and gg' ; consequently its *O.W.A.* firstly decreases from a till in f , where it becomes a minimum, then it increases till in g , in order to decrease again at further increase of pressure.

Of the many osmotic equilibria, which we can deduce from fig. 1, we only shall indicate some. If we draw in fig. 1 a horizontal line, then its points of intersection with the $P\xi$ -curve represent different states of a system E which have the same *O.W.A.* and may be, therefore, in osmotic equilibrium with one another. We have f.i. the osmotic equilibria:

$$(Y + \text{water-vapour})_b \mid (Y + L)_k \dots \dots \dots (15)$$

$$(Y + L + \text{water-vapour})_c \mid (Y + L)_l \dots \dots \dots (16)$$

in which the system at the left side of the membrane has a low pressure and the other a high pressure. Further we have f.i. still the osmotic equilibria:

$$(Y + L)_d \mid (Y + L)_h$$

$$(Y + L)_e \mid (Y + L)_g \mid (Y + L)_h$$

in which also the pressure at the left side of a membrane is smaller than at the right side.

With the aid of fig. 1 we can also deduce the osmotic pressure of a system and the change which this endures, if we change the pressure of the water. We then get the $P_0\pi$ -diagram ($P_0 =$ the pressure of the

water and π = the osmotic pressure of the system) discussed in the previous communication. Here however, we shall mean with osmotic pressure not only the difference in pressure on both sides of the membrane, if water is at one of the sides, but also if there occurs water-vapour instead of water.

As is apparent from fig. 1 we can have f.i. the osmotic equilibrium:

$$(water)_n \mid (Y+L)_e \quad \pi = P(e) - P(n) = en \dots (17)$$

in which at the left side of the membrane is a pressure $P(n)$ and at the right side a pressure $P(e)$. As, therefore, the osmotic pressure of this system is $\pi = P(e) - P(n)$ this is represented in fig. 1 by the line en .

We can represent system (17) in the $P_0\pi$ -diagram (fig. 2) by a point e ; the pressure $P(n)$ of the water is then indicated by On and the osmotic pressure π by en (equal to en in fig. 1).

It appears from fig. 1 that water under the pressure $P(n)$ yet also can be in osmotic equilibrium with the system $Y+L$ under the pressure $P(g)$. Consequently we also have the osmotic equilibrium:

$$(water)_n \mid (Y+L)_g \quad \pi = P(g) - P(n) = gn \dots (18)$$

The pressure $P_0 = P(n)$ of the water and the osmotic pressure $\pi = gn$ of the system are represented, therefore, in fig. 2 by the point g .

If we take f.i. the osmotic equilibria:

$$(water + water-vapour)_m \mid (Y+L)_d \quad \pi = P(d) - P(m) = dm \dots (19)$$

$$(water + water-vapour)_m \mid (Y+L)_h \quad \pi = P(h) - P(m) = hm \dots (20)$$

from fig. 1, then we can represent them in fig. 2 by the points d and h .

If we do in corresponding way with the other osmotic equilibria of

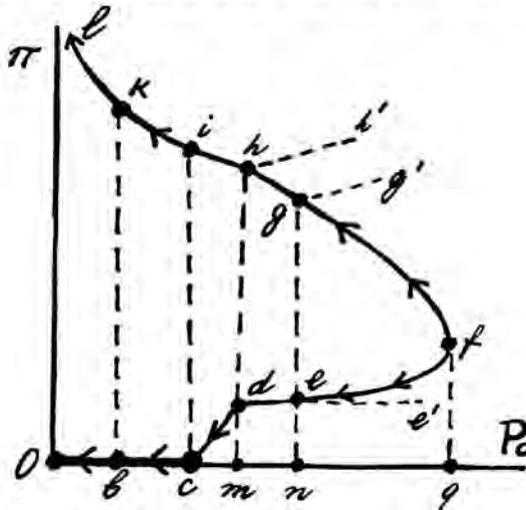


Fig. 2.

fig. 1, then we find a $P_0\pi$ -diagram as in fig. 2; the corresponding points in those figures are indicated by the same letters.

We now take from fig. 1 the osmotic equilibrium :

$$(water-vapour)_b \mid (Y + water-vapour)_b \quad \pi = 0 \quad . \quad . \quad . \quad (21)$$

in which the pressure must be the same on both sides of the membrane ; as, therefore, the osmotic pressure is zero, consequently it is represented in fig. 2 by the point *b* situated on the P_0 -axis.

If we heighten the pressure at the one side of the membrane, then we must heighten the pressure at the other side with a same amount till under a definite pressure $P(c)$ the osmotic equilibrium :

$$(water-vapour)_c \mid (Y + L + water-vapour)_c \quad \pi = 0 \quad . \quad . \quad (22)$$

occurs. Consequently the equilibrium goes in fig. 1 along the curve *bc* and in fig. 2 along the straight line *bc*.

We now imagine in fig. 1 a horizontal line, which intersects *cm* in a point *m'* and *cd* in a point *d'*. We then have the osmotic equilibrium :

$$(water-vapour)_{m'} \mid (Y + L)_{d'} \quad \pi = d'm' \quad . \quad . \quad . \quad (23)$$

in which the pressure $P(d')$ at the right side of the membrane is larger than the pressure $P(m')$ at the left side. Consequently the system has, as is indicated in (23) an osmotic pressure $\pi = d'm'$; the point *d'* (not drawn) is situated, therefore, in the P_0 π -diagram no more on the P_0 -axis, but above it. If in fig. 1 *m'* goes along curve *cm* (and, therefore, *d'* along curve *cd*), then therefore in fig. 2 *m'* goes along the line *cm* and *d'* along the curve *cd*.

If we bring the pressure at the left side of the membrane to $P(m)$, then we have to bring the pressure at the right side of the membrane to $P(d)$; we then get the osmotic equilibrium (19) with its osmotic pressure $\pi = dm$.

If we raise the pressure at the left side of the membrane, still further f.i. to $P(n)$, then we must raise the pressure at the other side to $P(e)$; we then get the osmotic equilibrium (17) with the osmotic pressure $\pi = en$.

With still further increase of pressure, arises the osmotic equilibrium :

$$(water)_q \mid (Y + L)_f \quad \pi = fq \quad . \quad . \quad . \quad (24)$$

represented in the two diagrams by *f* and *q*.

If we raise the pressure of the water till above $P(q)$ f.i. to $P(r)$ then, independent which pressure we make at the right side of the membrane, no more an osmotic equilibrium :

$$(water)_{P(r)} \mid (Y + L)_P \quad \pi = P - P(r) \quad . \quad . \quad . \quad (25)$$

can exist ; the difference in pressure $P - P(r)$ is, therefore, no osmotic pressure. It appears viz. from the $P\xi$ -diagram (fig. 1) that the *O.W.A.* of the water under the pressure $P(r)$ is smaller than that of the system $Y + L$ under every arbitrary pressure P ; consequently water must diffuse from left to right. Therefore, we can raise the pressure of the water till

a definite pressure $P(q)$ only; therefore, above this pressure the osmotic equilibria, discussed above, can exist no more.

As the system $Y + L$ goes along curve fl of fig. 1 with further increase of pressure, with which its $O.W.A.$ increases, we must lower the pressure of the water, therefore. We then get f.i. the osmotic equilibrium (18) with the osmotic pressure $\pi = gn$. The ascending branch fl in fig. 1 becomes, therefore, in fig. 2 a branch, descending from point f .

If we raise the pressure at the right side of the membrane to $P(h)$, then it must decrease at the left to $P(m)$; we then get the osmotic equilibrium (20) with the osmotic pressure $\pi = hm$.

If we raise still further the pressure at the right side of the membrane, then it must fall again at the left side of the membrane, so that the water passes into vapour. We then get f.i. the osmotic equilibria:

$$(waternapour)_c \mid (Y + L)_i \quad \pi = ic \dots \dots \dots (26)$$

$$(waternapour)_b \mid (Y + L)_k \quad \pi = kb \dots \dots \dots (27)$$

in which at the left side of the membrane is a very small pressure, but at the right side a very large pressure, so that the osmotic pressure also is very large. Both systems are indicated in the $P_0\pi$ diagram by the points i and k .

The $P_0\pi$ curve of the system consists, therefore, of four parts viz. of the straight line Oc and the three curves cd , $d'fh$ and hl . With the aid of fig. 1 we find that the $O.W.A.$ of the system must increase along the $P_0\pi$ -curve of fig. 2 in the direction of the arrows.

The dotted curves ee' , gg' and hh' of fig. 1 indicate, as we have seen above, the $O.W.A.$ of systems $E_4 = L$; if we represent also their osmotic pressures in fig. 2, then we get herein the homonymous and also dotted curves; starting from their beginning-points e , g and h they ascend or descend a little.

If we compare the $P_0\xi$ -diagram of this communication with that of the previous, which is deduced in a total other way then we see that both, although they correspond in substance, yet show difference. The diagram of the previous communication is viz. deduced for an entirely arbitrary system; however, fig. 2 of this communication is valid for a special case viz. if the system passes successively the states, indicated in (8).

We now take the osmotic system:

$$\boxed{\leftarrow} \mid (Y + L)_k \dots \dots \dots (28)$$

in which at the left side of the membrane a closed empty space, the volume of which we keep constant; at the right side is the system $Y + L$ under the constant pressure $P(k)$ represented by point k of fig. 1. Without more it is clear that water will diffuse towards the left; of course we can also say that the $O.W.A.$ at the left of the membrane (in fig. 1 represented by point a) is very large with respect to the

$O.W.A. = \xi(k)$ which is on the right. According as more water-vapour comes, however, in the closed space, the tension of this vapour increases and its $O.W.A.$ decreases, therefore; if this tension has increased to $P(b)$ and consequently the $O.W.A.$ has decreased to $\xi(b) = \xi(k)$, then the diffusion of the water-vapour stops and the osmotic equilibrium:

$$\boxed{\text{water-vapour}}_b \left| (Y + L)_k \quad \pi = k b (29)$$

with the osmotic pressure $\pi = k b$ is formed.

We now take instead of (28) the osmotic system:

$$\boxed{\leftarrow}_g (Y + L)_g (30)$$

which has on the right a pressure $P(g)$ and an $O.W.A. = \xi(g)$ (fig. 1). If so much water-vapour has diffused towards the left, that the closed space is saturated with water-vapour, then is at the leftside of the membrane the pressure $P(m)$ and the $O.W.A. = \xi(m)$; as, however, $\xi(m)$ is still greater than $\xi(g)$, the whole space must be filled with liquid water, the pressure of which will rise till its $O.W.A.$ is fallen to $\xi(g)$. It appears from fig. 1 that this is the case, if the water gets a pressure $= P(n)$ and an $O.W.A. = \xi(n) = \xi(g)$, (30) now passes into the osmotic equilibrium:

$$\boxed{\text{water}}_n \left| (Y + L)_g \quad \pi = g n (31)$$

If we take, therefore, an osmotic system:

$$\boxed{\leftarrow}_x (Y + L)_x (32)$$

then it depends on the pressure $P(x)$ at the right side of the membrane, which of the two osmotic equilibria:

$$\boxed{\text{water-vapour}}_x \left| (Y + L)_x \quad \boxed{\text{water}}_x \left| (Y + L)_x . . . (33)$$

will be formed from this. The first is formed with very small and also with very large values of $P(x)$, viz., as is apparent from fig. 1, if $P(x)$ is smaller than $P(d)$ or larger than $P(h)$. The second arises if we take $P(x)$ between $P(d)$ and $P(h)$.

If we bring into the closed space the solid substance Y , so that we have the osmotic system:

$$\boxed{Y \leftarrow}_x (Y + L)_x (34)$$

then this passes into one of the osmotic equilibria

$$\boxed{Y + \text{water-vapour}}_x \left| (Y + L)_x \quad \boxed{Y + L}_x \left| (Y + L)_x . . . (35)$$

The first is formed only with very high values of $P(x)$ viz. if $P(x)$ is greater than $P(i)$; the second arises when $P(x)$ is situated between $P(c)$ and $P(i)$.

With the second of the equilibria (35) we can distinguish two cases. The point x' viz. is always situated on branch cf while x can be situated as well on cf as on fi . If x is situated on cf then is formed, therefore, at the left of the membrane the same system as is present at the right side and $P(x')$ becomes $=P(x)$. If, however, x is situated on fi , then both systems are different and $P(x') < P(x)$.

We now imagine the closed space to be filled with water-vapour, we put the pressure $=P_0$, the thermodynamical potential of the water-vapour $=C_0$ and its *O.W.A.* $=\xi_0$. We then have:

$$\xi_0 = -C_0 \dots \dots \dots (36)$$

We bring into this space still q Mol $X + r$ Mol $Y \dots$, in which X and Y are substances which pass into vapour, f.i. alcohol, acetone etc. If there were already n Mol. water-vapour, then, if this vapour follows the gas-laws, in this space the pressure will rise to

$$P = \frac{n + q + r \dots}{n} P_0 \dots \dots \dots (37)$$

This vapour G has then a composition:

$$x \text{ Mol } X + y \text{ Mol } Y + \dots + (1 - x - y \dots) \text{ Mol } W \dots \dots (38)$$

in which $x, y \dots$ have a definite value and

$$1 - x - y \dots = \frac{n}{n + q + r \dots} \dots \dots \dots (39)$$

The addition of those new substance will enlarge the *O.W.A.* of the vapour, if we keep the pressure $=P_0$; the increase of the pressure from P_0 to P causes, however, the falling of this *O.W.A.*; we now shall show that both influences compensate one another, so that the *O.W.A.* remains unchanged.

If we take the vapour G under a pressure P_0 then its *O.W.A.* is defined by:

$$\xi(P_0) = -\zeta + x \frac{\partial \zeta}{\partial x} + y \frac{\partial \zeta}{\partial y} \dots \dots \dots (40)$$

In accordance with formula (4^a) of communication XVII we can write for this:

$$\xi(P_0) = -C_0 - RT \log(1 - x - y \dots) \dots \dots \dots (41)$$

As $\log(1 - x - y \dots)$ is negative, it appears from this and from (36) that the *O.W.A.* increases, if we keep the pressure P_0 . We now bring the pressure from P_0 to P . If we represent the molecular volume of the water-vapour by V_0 , then we have:

$$d\xi = -V_0 \cdot dP \dots \dots \dots (42)$$

Then follows for the *O.W.A.* of the vapour *G* under the pressure P_1

$$\xi = \xi(P_0) - \int_{P_0}^P V_0 \cdot dP = \xi(P_0) - RT \log \frac{P}{P_0} \quad \dots \quad (43)$$

so that we may write with the aid of (41)

$$\xi = -C_0 - RT \log(1 - x - y, \dots) - RT \log \frac{P}{P_0} \quad \dots \quad (44)$$

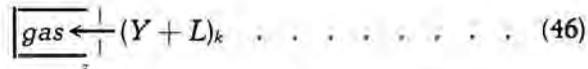
If we substitute in this the values of $1 - x - y$ from (39) and that of $P : P_0$ from (37) then follows:

$$\xi = -C_0 \quad \dots \quad (45)$$

It appears, therefore, from (36) and (45) that the *O.W.A.* has not changed. Consequently it follows from this for a vapour which follows the gas-laws:

the *O.W.A.* of a vapour, confined in a space of constant volume, does not change, if we add to this or withdraw from this one or more substances (excepted water).

Above we have seen that the osmotic system (28) passes into (29) with the occurrence of the state of equilibrium. We now fill the closed space of (28) with a gas or mixture of gases of a pressure $P(z)$ which does not contain water-vapour, (f. i. air, alcohol, acetone etc.). We then have the osmotic system:



in which watervapour will diffuse towards the left till is formed the osmotic equilibrium:



With the transition of (28) into (29) the pressure at the left side of the membrane rises from zero to $P(b)$; it now follows from our previous considerations that the pressure must increase with the same amount at the transition of (46) into (47); consequently we have: $P(z') = P(z) + P(b)$. The quantity of water-vapour, which diffuses towards the left, is, therefore dependent only on the volume of the closed space, independent whether this was firstly vacuum or filled with a mixture of gases.

(To be continued.)

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Crystallography. — On M. BEREK's *Methode der Charakteristischen Gangunterschiedsverhältnisse*. By W. NIEUWENKAMP. (Communicated by Prof. L. RUTTEN).

(Communicated at the meeting of January 29, 1927).

In his excellent work „Mikroskopische Mineralbestimmung mit Hilfe der Universaldrehtischmethoden“ M. BEREK points out a new method for determining the optic axial angle, the optical character and the position of the optic axes of symmetry of a mineral, requiring only the measurement of some six or seven values of wave differences by means of the compensator after BEREK. These measurements are simple and convenient, but BEREK derives from them the axial angle by means of rather long and tedious calculations. I hope I have shown in this paper that these calculations can be simplified without diminishing the accuracy, and that in this way I have made this really fine method more efficient.

For a clear understanding it is desirable to describe first shortly the instrumental part of the method: The slide is clamped on a Federow-stage, between two hemispheres of glass, whose refractive index is as much as possible like that of the mineral under test, and we place in the ordinary way an optic plane of symmetry perpendicular to the axis A_4 , i.e. the second (always) horizontal axis of rotation. Then the whole stage is rotated about the axis A_5 , (the axis of the microscope) for 45° , so that the direction of the vibration of the fast ray in the mineral and in the compensator are at right angles to one another, and we can start our measurements. They are taken in a series of different directions (all of them in the plane of symmetry) by placing the mineral by rotation about A_4 in different positions, e.g. successively on the readings: $a_4 = -50^\circ, -40^\circ, -20^\circ, 0^\circ, 20^\circ, 40^\circ, 50^\circ$. In this way we get a series of readings at the compensator, and in the table *Kompensatorfunktion* 10 000 $f(i)$, (p. 164 *Drehtischmethoden*) we can find a series of values proportional with the wave differences. They need not be multiplied with the constant of the compensator, for their rationes will be the same as those of the actual wave differences, and, as is suggested by its name, this method makes only use of these rationes. When we wish to intercompare the values of $f(i)$, they must first be reduced to the same length of way of the light in the mineral.

BEREK reduces them to the real thickness of the slide. It is unnecessary to be so scrupulous; the only condition is that the length of the way of the light has to be the same for every observation. The reduction becomes very simple when we reduce the observation to the interval

the ray passes when A_4 stands in its initial position ($\alpha_4 = 0^\circ$). This is shown in fig. 1.

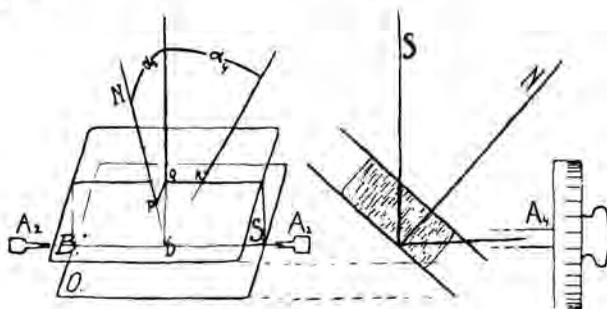


Fig. 1.

Here O , and B represent the under and upper surface of the slide, N is drawn perpendicular to it. S is the plane of symmetry in which we measure the wave differences; so the ray passes always through the mineral along one of the lines of S ; when A_4 stands in its initial position ($\alpha_4 = 0^\circ$) along the line DQ , when the stage is inclined, along a line DR .

In the initial position the axis A_2 is horizontal. Consequently in that position also the line QR , i.e. the trace of the plane of symmetry runs horizontal, for our first manipulation was placing this plane parallel to A_2 . The ray, in this case DQ is vertical, so DQ is at right angles with QR , and $QD:RD = \cos \alpha_4$; therefore in order to reduce the way travelled over by the light to DQ we have only to multiply by $\cos \alpha_4$.

In BEREK's procedure the reduction runs as follows: he begins by plotting a curve with the measured wave differences as ordinates, and α_4 as abscissae. On the same abscissae he also plots the ordinates that represent the angle RDP , corresponding with every α_4 , i.e. the angle between the ray and the normal on the slide. In a stereographic projection in which the plane of symmetry and the ray for every α_4 are represented, these angles are measured by means of a WULLF's net, and so we can multiply the ordinates of every point of the first curve by the cos of the corresponding angle RDP thus reducing indeed, all the wave differences to the real thickness of the slide.

In this way then BEREK obtains the reduced wave difference-curve. In this curve a maximum or a minimum has to be looked for; it is however not easy to find where it lies exactly. According to BEREK one begins by assuming a presumptive value for the abscissa of the maximum or minimum, and by marking out distances from it of 30° , 45° and if possible 60° , 75° and 90° , then the corresponding ordinates are divided by the ordinates of the maximum or minimum, and, we try to find in 5 diagrams, constructed by BEREK, and added to his work, by means of these 5 quotients 5 values for the axial angle. If these values are not equal, if their curve runs up or falls then the choice of the abscissa of the minimum is wrong and we have to do it over again. If now their

curve moves in the other direction, we can find by means of interpolations and by other trials an abscissa for which the axial angle is the same for all quotients; this then is the real axial angle and the value of the abscissa of the maximum or minimum indicates the position of an optical axis of symmetry.

I think I can substitute the latter operation of trial and error by moving a curve projected on transparent paper, and obtained from the observations, over a set of curves on a diagram until the best possible agreement is attained.

That this can really be done is borne out by the following consideration: in our computation of the axial angle we availed ourselves of ratios of the wave differences; it is these ratios that determine the axial angle. When plotting the logarithms of the wave differences the shape of the curve will depend on the axial angle. The lengths of the ordinates representing the logarithms of the wave differences indeed also depend on the thickness of the slide and the compensator constant, but the shape of the curve is determined by the differences of the ordinates, and these differences represent the logarithms of the quotients of the wave differences and consequently depend only on the axial angle. So with each axial angle there is a definite shape of the curve of the log. wave differences. Now the later have been drawn on the diagrams for axial angles varying from 0° — 90° . The absolute lengths of the ordinates are chosen so as to be most convenient for the construction and the selection of a definite curve.

From our theoretical discussions at the close of this paper it will appear that 2 diagrams are required, one for the optic axial plane and one for the 2 planes of symmetry perpendicular to it, but first we shall illustrate the use of the diagrams by a few examples.

1	2	3	4	5	6
α_1	Wave differences	$i = \frac{a-b}{2}$	log wave differences	log cos α_1	Reduced log wave differences
50	193.5	25.55	9.287	0.809—1	9.096
40	183.0	24.83	9.263	0.884—1	9.147
20	176.1	24.35	9.246	0.973—1	9.219
0	173.0	24.12	9.238		9.238
—20	170.6	23.95	9.232	0.973—1	9.205
—40	168.9	23.83	9.228	0.884—1	9.112
—50	175.2	24.28	9.243	0.809—1	9.052

In BEREK's work we read on p. 121 that when measuring a plagioklase he found corresponding with the α_1 from column 1 the wave

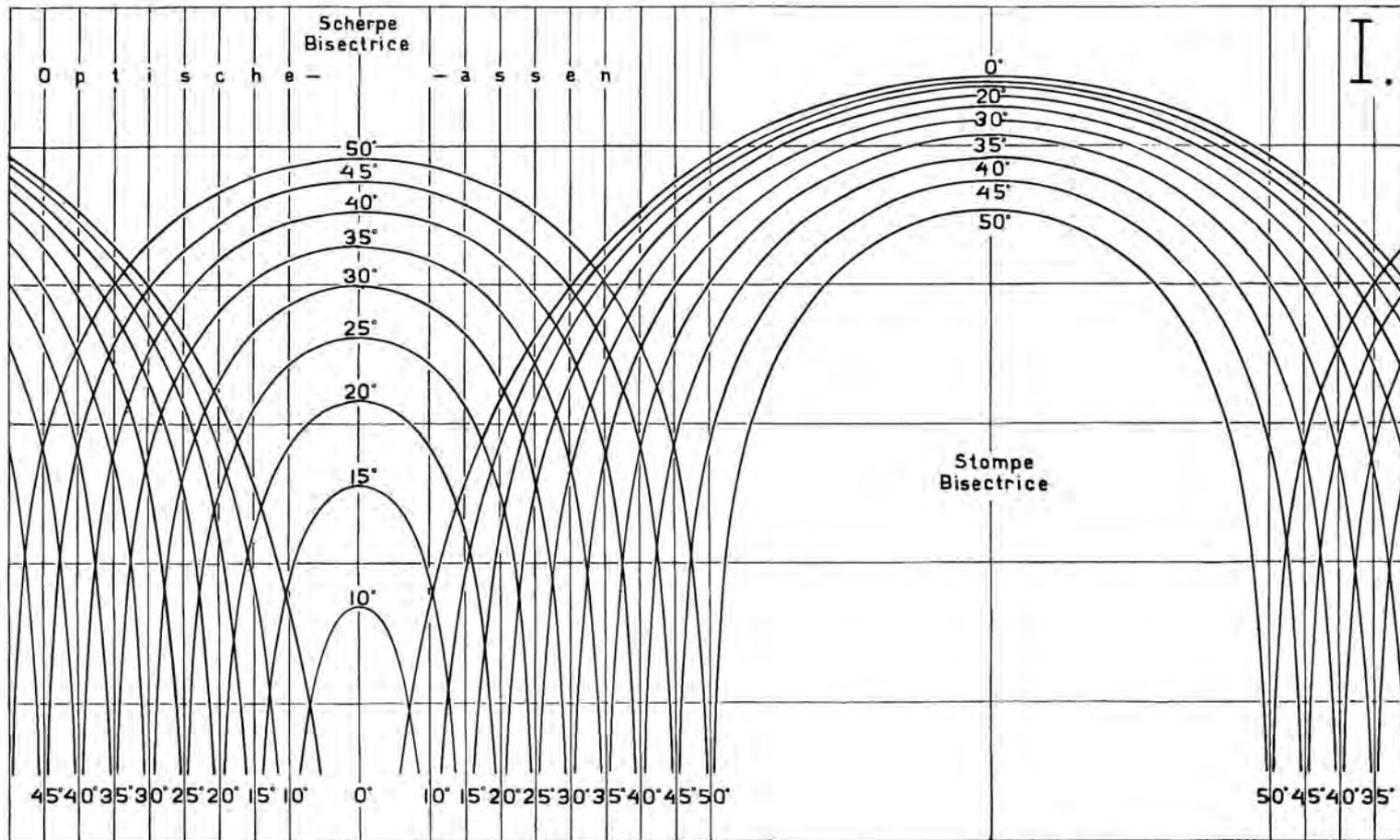
differences as indicated in column 2. These are not the real wave differences but the values (proportional with the wave differences) of table: "Kompensatorfunktion 10 000 $f(i)$ ", and we can find the $i = \frac{a-b}{2}$ (not given in his work) with which BEREK started his calculations, and upon which we will base our example. So we take the i of column 3 for measured with the compensator, and in the table "Für logarithmisches rechnen, Kompensatorfunktion $\log f(i)$ ", we look up the $\log f(i)$. They are given in column 4, but have still to be reduced to the same length of the way of the light. Therefore we have only to add the $\log \cos a_4$. $\log \cos a_1$ is given in column 5, by addition we get the values of column 6.

Now the calculations are already finished, and we can plot the curve. To get it on the same scale as the accompanying diagrams we have to mark off on the axis of abscissae 1 cm for every 10° of a_4 ; on the axis of ordinates 5 cm for every unit of the logarithm. On a sheet of millimeter paper this scale for abscissa and ordinate should be marked; if we cover it with a sheet of transparent paper the dotting of the curve on the right scale is an easy operation. Care should be taken to indicate at the same time the direction of the axis of abscissae, which may be done by drawing a horizontal line across the transparent paper. Care should then be taken that while moving the paper over the diagram, we always keep this line parallel to the horizontal lines of the diagram.

The shape of the curve of our example shows directly that it must fit to one of the curves of diagram 2, and is intermediate between the curve for $V 40^\circ$, and that for 45° . If the curve bends so little as in the present case we may for convenience' sake magnify the ordinates 10 times and then use diagram 2^A, a magnification of the top part of diagram 2. With both diagrams the result is of course the same. The correspondence to the curve for $42\frac{1}{2}^\circ$ is not yet quite sufficient. We see that our points are lying in a curve slightly less arched, we have therefore to go up a little, and shall than find that the best correspondence is obtained with a curve between 43° and 44° . So the total axial angle is established at 87° .

When in plotting the curve we have also indicated the scale of the abscissa we are able to indicate directly the position of an optic axis of symmetry, (in this case the optical normal, with $a_4 = 3^\circ$). The bisectrix lying in the contemplated plan of symmetry, is that which has the angle V , indicated in the diagram, on either side, so it is acute bisectrix for the curves $0^\circ - 45^\circ$, and obtuse bisectrix for those from $45^\circ - 90^\circ$ (in our case acute bisectrix). We now also know the third axis of symmetry and consequently we can also indicate the optical character of the mineral, when we know from our compensator the directions of the vibration of the slow and fast ray.

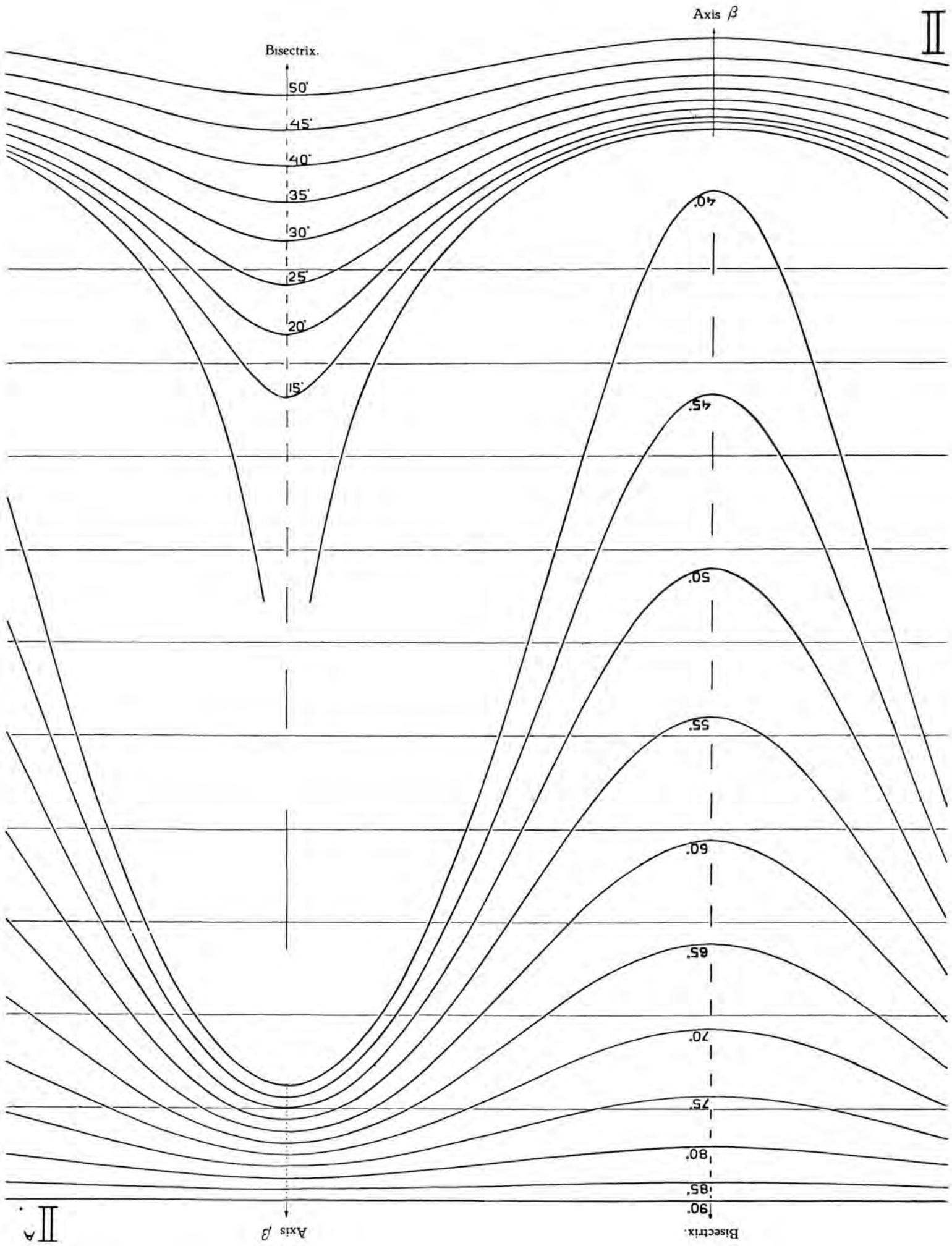
An example of the use of diagram 1 is to be found on page 116 of



Scherpe Bisectrice = Acute bisectrix.
 Optische assen = Optic axes.

Stompe Bisectrice = Obtuse bisectrix.

W. NIEUWENKAMP: ON M. BEREK'S "METHODE DER CHARAKTERISTICHEN
GANGUNTERSCHIEDSVERHÄLTNISSE".



the "Drehtischmethoden". This is a measurement made in the optic axial plane. We have not taken measures close to the optical axis, because there the small wave differences render them too inaccurate; on the other hand we have a determination of the abscissa of the optical axis; on the transparent sheet it is marked by a vertical line with $\alpha_4 = 21,2^\circ$, that is the α_4 with which BEREK has found the optical axis. For the other points of the curve we find in the same way as in the first example from BEREK's wave differences:

$\alpha_4 =$	50°	40°	10°	0°	-20°	-40°	-50°
<i>log wave diff.</i> =	8,706	8,618	8,529	8,811	9,057	9,105	9,077

It follows from the fact that the measurements have been made in the plane of the axes, that diagram I is to be used, which moreover is shown clearly by the shape of the curve. When fitting the curve on the diagram care should be taken that the same angle V is found on either branch of the curve and also that the same V stands at the line marking the position of the optical axis. We can place the right part of our curve first on the curve for 35° , but then the left branch lies at 32° (anyhow if we have laid the line for the optical axis along that for 35°). So we have to shove the paper upward slantingly to the left. With 33° there is a fair concordance, with 32° it decreases already. So we find $V = 32\frac{1}{2}^\circ$ ($2V = 65^\circ$), in complete accordance with BEREK's result.

Here also we can easily mark where in our mineral an optic axis of symmetry is to be found, and which of the axes it is, as well as the remaining 2 axes and the optical character.

One more remark about this graphical arithmetical method:

The points dotted on the transparent paper have not been connected by a continuous line; this is quite irrelevant as we do something like it when shoving the paper over the diagrams. Nay, it is better even to let the points alone, because otherwise we should most likely draw a curve of a shape that cannot occur with perfectly correct measurements, and is impossible with any axial angle, whereas in the shoving process the points are connected by a curve that is theoretically possible. In this case therefore we make a selection from curves that of themselves may occur, though perhaps with another axial angle, whereas in the first case we can hit upon a good curve only with perfectly correct measurements. Generally however we shall find one that will give trouble rather than pleasure.

The data for the construction of the diagram are found in the following way: According to BEREK (p. 154 sqq. of his Drehtischmethoden), when our measurements are taken in an optical plane of symmetry which is not the axial plane:

$$\frac{\sin(r, n_x)}{\sqrt{\frac{\Delta_r}{\Delta_x} - 1}} = \operatorname{tg} V \quad \text{or} \quad \frac{\sin(r, n_x)}{\sqrt{1 - \frac{\Delta_r}{\Delta_x}}} = \sec V.$$

Here r is the direction of the ray, n_x an optical axis of symmetry, Δ_r , Δ_x the wave differences in the direction r resp. n_x . If furthermore V is counted from 0° — 90° , one trigonometrical function will do (it is then needless to mention: or $\cotg V$, resp. $\operatorname{cosec} V$).

From the first formula follows:

$$\frac{\Delta_r}{\Delta_x} = \frac{\sin^2(r, n_x)}{\operatorname{tg}^2 V} + 1 = \frac{\sin^2(r, n_x) + \operatorname{tg}^2 V}{\operatorname{tg}^2 V}.$$

$$\lg \Delta_r = \lg \{ \sin^2(r, n_x) + \operatorname{tg}^2 V \} - \lg \operatorname{tg}^2 V + \lg \Delta_x.$$

FOR DIAGRAM I, $\lg |\sin^2(r, n_x) - \sin^2 V|$

V (r, n_x)	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°
0°		7.881	8.479	8.826	9.068	9.252	9.398	9.517	9.616	9.699
5°	7.881		8.352	8.774	9.039	9.233	9.384	9.507	9.608	9.692
10°	8.479	8.352		8.567	8.939	9.172	9.342	9.476	9.583	9.672
15°	8.826	8.774	8.567		8.699	9.048	9.262	9.418	9.539	9.636
20°	9.068	9.039	8.939	8.699		8.790	9.123	9.326	9.472	9.583
25°	9.252	9.233	9.172	9.048	8.790		8.854	9.177	9.370	9.507
30°	9.398	9.384	9.342	9.262	9.123	8.854		8.898	9.213	9.398
35°	9.517	9.507	9.476	9.418	9.326	9.177	8.898		8.925	9.233
40°	9.616	9.608	9.583	9.539	9.472	9.370	9.213	8.925		8.938
45°	9.699	9.692	9.672	9.636	9.583	9.507	9.398	9.233	8.938	
50°	9.768	9.763	9.746	9.716	9.672	9.611	9.527	9.411	9.239	8.938
55°	9.827	9.822	9.807	9.781	9.744	9.692	9.624	9.534	9.411	9.233
60°	9.875	9.871	9.857	9.834	9.801	9.757	9.699	9.624	9.527	9.398
65°	9.915	9.910	9.898	9.878	9.848	9.808	9.757	9.692	9.611	9.507
70°	9.946	9.942	9.931	9.912	9.884	9.848	9.801	9.743	9.672	9.583
75°	9.970	9.966	9.956	9.937	9.912	9.878	9.834	9.781	9.716	9.636
80°	9.987	9.983	9.973	9.956	9.931	9.898	9.857	9.807	9.746	9.672
85°	9.997	9.993	9.983	9.966	9.942	9.911	9.871	9.822	9.763	9.692
90°	10	9.997	9.987	9.970	9.946	9.915	9.875	9.827	9.769	9.699

FOR DIAGRAM II, $I_g \{ \sin^2(r, n_x) + tg^2 V \}$.

$V \backslash (r, n_x)$	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°
0°		0.8839—3	0.4926—2	0.8561—2	0.1221—1	0.3373—1	0.5230—1	0.6905—1	0.8476—1	0	0.1524	0.3095	0.4771	0.6627	0.8779	1.1439	1.5075	2.1159
5°	0.8806—3	0.1833—2	0.5876—2	0.8998—2	0.1464—1	0.3523—1	0.5327—1	0.6971—1	0.8523—1	0.0033	0.1547	0.3111	0.4782	0.6635	0.8784	1.1442	1.5076	2.1160
10°	0.4793—2	0.5776—2	0.7878—2	0.0084—1	0.2111—1	0.3937—1	0.5595—1	0.7164—1	0.8658—1	0.0129	0.1615	0.3159	0.4815	0.6656	0.8797	1.1449	1.5079	2.1161
15°	0.8260—2	0.8730—2	0.9917—2	0.1423—1	0.2998—1	0.4540—1	0.6025—1	0.7461—1	0.8870—1	0.0282	0.1724	0.3235	0.4867	0.6690	0.8818	1.1460	1.5084	2.1162
20°	0.0681—1	0.0956—1	0.1706—1	0.2759—1	0.3970—1	0.5243—1	0.6535—1	0.7834—1	0.9144—1	0.0481	0.1867	0.3337	0.4937	0.6737	0.8846	1.1476	1.5090	2.1163
25°	0.2519—1	0.2701—1	0.3216—1	0.3986—1	0.4929—1	0.5977—1	0.7092—1	0.8254—1	0.9458—1	0.0714	0.2038	0.3460	0.5022	0.6793	0.8881	1.1495	1.5098	2.1165
30°	0.3979—1	0.4110—1	0.4489—1	0.5076—1	0.5826—1	0.6697—1	0.7659—1	0.8694—1	0.9796—1	0.0969	0.2228	0.3597	0.5119	0.6857	0.8921	1.1517	1.5108	2.1167
35°	0.5172—1	0.5272—1	0.5564—1	0.6029—1	0.6642—1	0.7375—1	0.8211—1	0.9134—1	0.0141	0.1235	0.2429	0.3745	0.5223	0.6928	0.8965	1.1541	1.5119	2.1170
40°	0.6161—1	0.6241—1	0.6477—1	0.6857—1	0.7370—1	0.7997—1	0.8730—1	0.9559—1	0.0482	0.1502	0.2633	0.3896	0.5332	0.7001	0.9011	1.1566	1.5130	2.1173
45°	0.6990—1	0.7056—1	0.7252—1	0.7572—1	0.8011—1	0.8558—1	0.9208—1	0.9958—1	0.0807	0.1761	0.2834	0.4047	0.5441	0.7076	0.9058	1.1593	1.5142	2.1176
50°	0.7685—1	0.7741—1	0.7909—1	0.8186—1	0.8569—1	0.9054—1	0.9638—1	0.0323	0.1109	0.2005	0.3025	0.4193	0.5547	0.7149	0.9105	1.1619	1.5153	2.1179
55°	0.8267—1	0.8317—1	0.8465—1	0.8709—1	0.9050—1	0.9487—1	0.0019	0.0650	0.1384	0.2230	0.3204	0.4330	0.5648	0.7219	0.9149	1.1644	1.5164	2.1182
60°	0.8751—1	0.8794—1	0.8927—1	0.9148—1	0.9457—1	0.9856—1	0.0348	0.0935	0.1626	0.2430	0.3365	0.4455	0.5740	0.7284	0.9191	1.1667	1.5174	2.1184
65°	0.9146—1	0.9186—1	0.9307—1	0.9509—1	0.9795—1	0.0165	0.0625	0.1178	0.1834	0.2604	0.3506	0.4565	0.5822	0.7341	0.9228	1.1688	1.5184	2.1187
70°	0.9460—1	0.9497—1	0.9610—1	0.9800—1	0.0067	0.0416	0.0851	0.1378	0.2006	0.2749	0.3623	0.4658	0.5892	0.7390	0.9260	1.1706	1.5192	2.1189
75°	0.9699—1	0.9734—1	0.9841—1	0.0021	0.0276	0.0609	0.1025	0.1533	0.2141	0.2862	0.3717	0.4731	0.5947	0.7430	0.9285	1.1721	1.5199	2.1190
80°	0.9867—1	0.9901—1	0.0004	0.0177	0.0423	0.0745	0.1150	0.1644	0.2237	0.2944	0.3784	0.4785	0.5988	0.7458	0.9304	1.1732	1.5203	2.1191
85°	0.9967—1	0.0000	0.0101	0.0270	0.0511	0.0827	0.1225	0.1711	0.2295	0.2994	0.3825	0.4817	0.6012	0.7476	0.9316	1.1738	1.5206	2.1192
90°	0.	0.0033	0.0133	0.0301	0.0540	0.0854	0.1249	0.1733	0.2315	0.3010	0.3839	0.4828	0.6021	0.7482	0.9320	1.1741	1.5207	2.1193

From the second formula follows:

$$\frac{\Delta_r}{\Delta_x} = \frac{-\sin^2(r, n_x)}{\sec^2 V} + 1 = \frac{\cos^2(r, n_x) - 1 + (1 + \operatorname{tg}^2 V)}{1 + \operatorname{tg}^2 V},$$

$$\lg \Delta_r = \lg [\sin^2 \{ (r, n_x) \pm 90^\circ \} \operatorname{tg}^2 V] - \lg (1 + \operatorname{tg}^2 V) + \lg \Delta_x.$$

The shape of the curve for a definite V is now in the first case the same as in the second, so one curve suffices for both cases, if only in the second case we begin to measure the angle (r, n_x) from a point that differs 90° from the n_x in the first case.

In the axial plane (see BEREK l. c.) is:

$$\frac{\sin^2(r, n_x)}{\sqrt{1 \pm \frac{\Delta_r}{\Delta_x}}} = \sin V, \quad (V \text{ counted from } 0^\circ - 90^\circ) \text{ from which follows:}$$

$$\frac{\Delta_r}{\Delta_x} = \left| \frac{\sin^2(r, n_x)}{\sin^2 V} - 1 \right| = \frac{|\sin^2(r, n_x) - \sin^2 V|}{\sin^2 V}$$

$$\lg \Delta_r = \lg |\sin^2(r, n_x) - \sin^2 V| - \lg \sin^2 V + \lg \Delta_x.$$

For the construction of the diagrams, which have to indicate for each definite V , the differences between $\log \Delta_r$ when the angle (r, n_x) varies, only that part of the formula is of consequence in which (r, n_x) occurs; it does not matter if a constant is added or not. So for the construction we require the formulae:

$$\lg \Delta_r = \lg \{ \sin^2(r, n_x) + \operatorname{tg}^2 V \} \quad \text{resp.} \quad \lg \Delta_r = \lg |\sin^2(r, n_x) - \sin^2 V|$$

For those who should wish to attain a greater accuracy than is possible with the small diagrams reproduced here, and would construct them on a larger scale, the results have been tabulated here.

That as will be seen in formula 2, the $\log \Delta_r$ becomes $-\infty$ for $(r, n_x) = V$, need not discompose us very much, because (on BEREK's suggestion) we do not take any measurements in the neighbourhood of an optical axis, so that there we have no points to be fitted on the curve, and consequently a part of it can be missing without vitiating the results.

Comparative Pathology. — *On the mutual Immunization in Aphthae epizooticae and Vaccine.* (First very concise memoir). By Prof. C. F. VAN OIJEN. (Communicated by Prof. C. EIJKMAN).

(Communicated at the meeting of March 26, 1927).

It has been recorded repeatedly for more than a quarter of a century in the veterinary literature, that animals previously treated with "vaccine", are in a measure immune from Foot-and-mouth disease.

BOULLAND¹⁾ (1900) reports that in his private practise he observed that cows which had suffered from the genuine cowpox ("variola"), were not taken with foot-and-mouth disease, when this broke out in the same stable.

SEIBERT²⁾ (1907) stated that a calf which had served for the preparation of "vaccine" was not affected, when the other cows of the stable in which it was brought afterwards, were seriously afflicted with aphthae. He translated these observations into actions by inoculating calves with 20—25 pocks. After this the calves are said to have been immune against foot-and-mouth disease.

ORY³⁾ (1907) inoculated cows with lymph, derived from horses, which had been inoculated with "vaccine", and observed, that several of these animals did not get ill, when they were seriously infested with substances containing aphthae-virus (saliva etc.).

ANKER⁴⁾ (1907) (Oudewater) could not fully confirm ORY's findings. Only a group of young animals (1½ years) being inoculated with pocks and subsequently exposed to infection with aphthae, remained free from this disease.

Neither could UHLENHUTH and BIEBER⁵⁾ establish the immunity in question for all cases, although the critical reader will observe that it did exist in some vaccinated animals.

For a complete survey of the relative experiments we refer to GINS

¹⁾ BOULLAND. Le cow-pox et la fièvre aphteuse. Le répertoire de police sanitaire vétérinaire et d'hygiène publique. Vol. 16 1900. 15 Sept. N^o. 9. p. 377.

²⁾ SEIBERT. Kuhpocken Impfung als Schützmittel gegen Maul- und Klauenseuche. Wochenschrift für Tierheilkunde und Viehzucht. 1 Oct. 1907 N^o. 39.

³⁾ ORY. Essai d'un mode prophylactique de la fièvre aphteuse par la vaccination. La Semaine Vétérinaire 1907 p. 279.

⁴⁾ ANKER. Maul- und Klauenseuche Impfung nach ORY. Berliner Tierärztliche Wochenschrift 1907 p. 882, 883.

⁵⁾ UHLENHUTH und BIEBER. Untersuchungen zur Frage der wechselseitigen Vakzine- und Maul- und Klauenseuche Immunität bei Rindern und Meerschweinchen. Zeitschrift für Immunitätsforschung. Originale Bd. 35 1923 Heft 4 p. 10.

and KRAUSE¹⁾: "Zur Pathologie der Maul- und Klauenseuche sub. IX Pocken und Maul- und Klauenseuche".

Also the present author was in a position to observe in his private practice that two calves, which had served to prepare "vaccine", did not get ill when housed in the same stable with animals with foot- and -mouth disease.

These observations, based on practice, should not be underrated. Therefore, GINS and KRAUSE and others advocate the urgency of an experimental inquiry into the power of a vaccine-inoculation to protect from foot- and -mouth disease. This is all the more desirable, as in 1926 a report was received from France that very good results were obtained by inoculation of a definite mixture of vaccine and apthae-virus (Belin).²⁾

We purpose to prosecute this inquiry with cows and calves. But these experiments are expensive, while it is very difficult to avoid technical errors. For this reason we also made experiments with infected caviae. The experiments with cows will be published afterwards.

However, the cavia-experiments have reached a stage, at which they can for the present be concluded. A brief account of these experiments follows:

Detailed reports with protocols etc. will soon be published.

The investigation was divided as follows:

- I. Description of the symptoms of foot- and -mouth disease in caviae.
- II. Idem of the symptoms of vaccine ("cowpox") in caviae.
- III. Observations regarding the immunity of caviae from vaccine after inoculation with this virus.
- IV. Observations on the immunity from apthae-virus after inoculation with this infectious matter.
- V. Observations on the immunity of caviae from foot- and -mouth disease after inoculation with vaccine.
- VI. Observations on the immunity of caviae from vaccine after inoculation with apthae virus.

Graphs have been plotted for vaccine- as well as for apthae-infections, in which the succession of the symptoms, their intensity and their duration may be read at a glance. The line *AP* indicates the changes at the hindlegs, *VP* those at the forelegs, *T* at the tongue.

I. Description of the *symptoms of foot- and mouth disease* in caviae.

We observed:

1^o. The body-temperature. After 1 to 2 days' incubation the temperature

¹⁾ GINS und KRAUSE. Zur Pathologie der Maul- und Klauenseuche Ergebnisse der Allgemeinen Pathologie und Patholog. Anatomie des Menschen und der Tiere 1924 vol. II Abt. II.

²⁾ BELIN. Base scientifique d'une méthode de Vaccination antiaphteuse. Recueil de médecine-vétérinaire 1926. Tome 102, N^o. 14 p. 307.

rises in typical cases and is followed by a fall, occasionally below the normal temperatures. It is soon restored to a slightly higher level. For a long time the temperature remains variable (Fig. 1).

2°. The body-weight. In all the cases there is a heavy loss of body-weight (Fig. 1) from which the animals recover but slowly. Generally the initial weight is not regained before the lesions at the site of inoculation are cured.

3°. The development and healing of the aphthae-blains at the site of inoculation (sole of the foot metatarsus hindlegs). After the incubation blains appear, then their regression, formation of scabs, shedding of these scales, appearance of a skin-lesion, its healing.

4°. Alterations at the soles of the fore-legs. Here we observe redness, swelling, afterwards exfoliation of the skin and ultimate healing.

5°. Changes of the mucous membrane of the mouth. Blains appear in the mouth, especially on the tongue, that bring on a loss of epithelium and often excoriation all over the tongue. Cure, as a rule, proceeds rapidly.

6°. Some other manifestations, so staring that they, so to speak, obtrude themselves to be taken up in the protocols, such as considerable falling off of the hair, hyper secretions from the salivary glands and of intense salivation.

II, Description of the *symptoms* after inoculation with *vaccine* in *caviae* (Fig. 2).

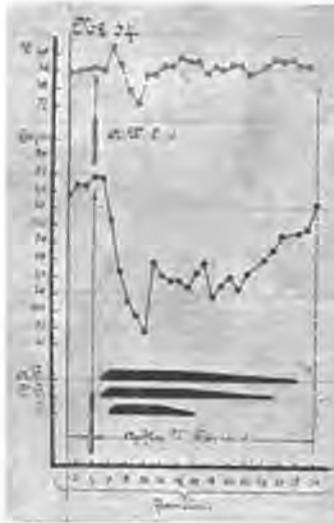


Fig. 1.



Fig. 2.

We observed:

1°. Changes in the body-temperature. In most cases fever ensues from

3 to 5 days after the inoculation. Gradually a fall succeeds and a varying temperature for some days. A fall below the "normal" temperature was not observed.

2°. Changes in the body-weight. Here also a considerable loss is often seen after the incubation period, truly, not so heavy as with apthae, but sufficient to discern that the inoculation with vaccine has not the effect of a localized inflammation but influences the general condition of the animal. It makes the animal ill.

3°. Growth of the blain and healing of the area of inoculation. A simple formation of blains exists here, that heals under a scab. After the first scales have fallen off there may appear a wound-surface, which heals in the usual way.

4°. Changes at the soles of the fore-legs. Only redness and swelling are observed here. No other symptoms appeared after the ordinary inoculation with vaccine.

III. Observations regarding the *immunity* of *caviae* from *vaccine* after inoculation with *this virus*.

A vaccine-injection imparts to a *cavia* complete immunity against a re-inoculation with this virus, administered immediately after recovery. It does not matter, whether for the re-inoculation the same vaccine-strain is employed or an other. A transit through *cavia* or calf is of no influence.

From the fact that the symptoms mentioned in II, were totally absent, we inferred the existence of this complete immunity. Fig. 3 illustrates this. It will be seen, that in the control-*cavia* N°. 26 the symptoms have been

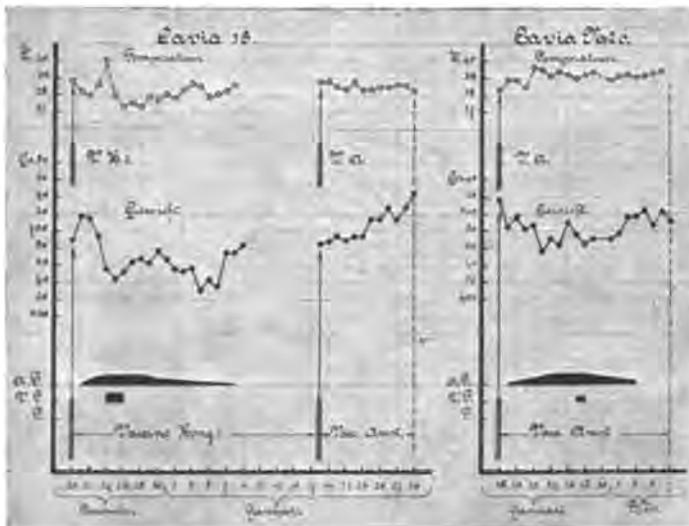


Fig. 3.

fully developed, while in the cavia N^o. 18 no anomalies could be recognized. This immunity could be established in 5 caviae.

IV. *Immunity* in caviae from *aphthae epizoöticae* after injection of *this virus*.

A single infection with a very virulent aphthae-strain is sufficient to evoke complete immunity against this virus. This is instanced by cavia N^o. 3 (Fig. 4). The accompanying curve shows distinctly the difference between cavia 3 after re-inoculation and cavia 10 after the first injection with the same infectious matter. The same immunity appears against other strains of equal virulence. Cavia N^o. 21 (control-cavia 43).

Yet there is a large difference regarding the facts found with vaccine.

When a cavia is inoculated with an aphthae-strain that is only slightly virulent for this animal, the symptoms will be confined to formation of blains at the hindlegs, redness at the fore-legs, loss of weight and an irregular temperature (Fig. 5, cavia N^o. 7). At a following inoculation



Fig. 4.

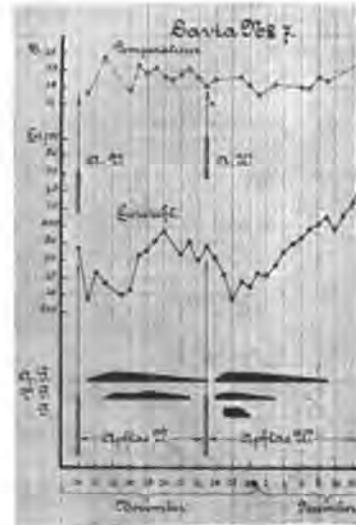
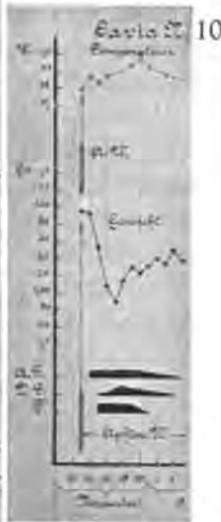


Fig. 5.

with a very virulent strain it appears that no immunity from this virus has been effected. Heavy loss of weight, affections at the hindlegs, forelegs and tongue appear just as with a cavia that has not been treated previously. Only there is a tendency to quicker healing (allergic reaction GINS and WEBER¹⁾).

This remarkable "plurality" of the aphthae-strains could be established in three caviae. Here we wish to refer to the observations made by

¹⁾ GINS und WEBER. Ueber experimentelle Maul- und Klauenseuche. Centralblatt f. Bakteriologie etc. Erste Abt. Originale Bnd. 88, 1922, Heft 3 p. 180.

VALLÉE and CARRÉ ¹⁾, WALDMANN and TRAUTWEIN ²⁾ and STOCKMANN and MINETT ³⁾ in which the same symptoms were recognized.

V. *Immunity against apthae after inoculation with vaccine.*

In the foregoing we have dwelt on the nature and the scope of the immunity from a specific virus after inoculation with vaccine and apthae. Now we could pass on to an inquiry into the mutual immunity with cross-inoculation. From this it appeared that *inoculation with vaccine does not induce immunity against apthae when administered directly after the healing.*

— In [fig. 6 we see that a cavia tolerates a vigorous vaccine infection

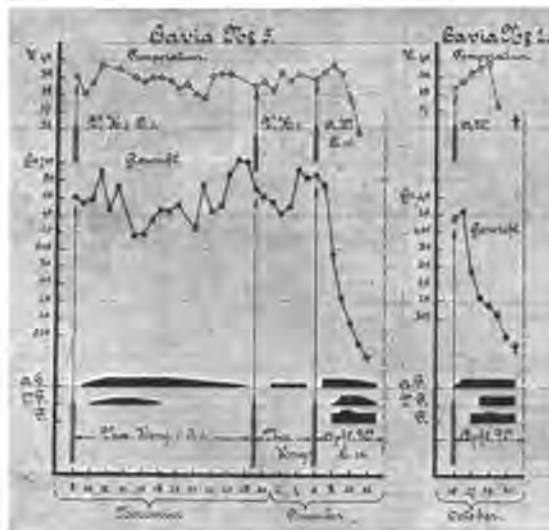


Fig. 6.

with fever, loss of weight, anomalies at the fore-, and hindlegs. Then it appears to be entirely immune against a repeated vaccine-inoculation.

But the subsequent inoculation with apthae-virus induces death within 6 days. The figure shows that the process is identical with the result obtained with cavia N^o. 2, which had not been treated beforehand. It was also observed in cavia N^o. 1 that a previous inoculation with vaccine acted no influence upon the course of the subsequent inoculation with apthae, though the latter did not induce death.

¹⁾ VALLÉE H. et CARRÉ H. Sur l'immunité antiaphteuse Compt. rend. de l'Acad. des Sciences 1922, T. 174, p. 207.

²⁾ WALDMANN and TRAUTWEIN. Experimentelle Untersuchungen über die Pluralität der Maul- und Klauenseuche Virus (Vorl. Mitteil.) Berl. Tierarztl. Wochenschr. Jrg. 42, N^o. 35.

³⁾ ST. STOCKMANN and F. C. MINETT. Experiments on Foot- and Mouth-disease. The journal of Comp. Path. and Therapeutics, Vol. 39, Part 3.

These observations stand in harmony with the findings established by UHLENHUTH and BIEBER¹⁾

VI. It still remains for us to watch the effect of a preceding inoculation with aphthae upon a vaccine-inoculation following it immediately.

In ten caviae it could be established that *sustaining a serious general attack of aphthae epizoöticae engenders an almost complete immunity against an infection with vaccine succeeding the attack directly.*

Graphs of cavia 10 and cavia 4 go to illustrate this.

It may be seen that a single inoculation with aphthae into cavia 10 (fig. 7) results in almost complete immunity from vaccine. Of all the

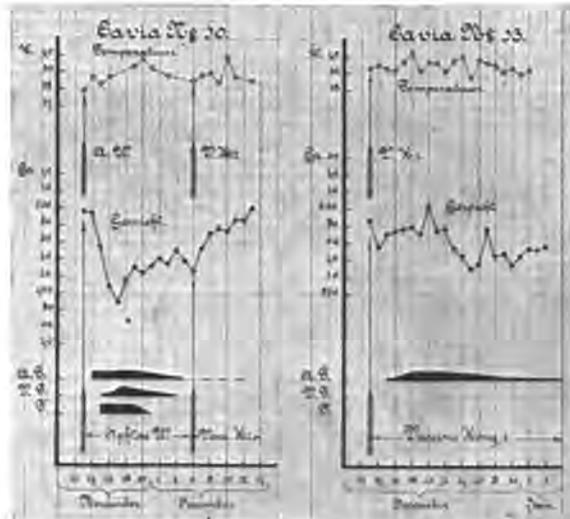


Fig. 7.

symptoms mentioned under II the only ones we notice here, is a rise of the temperature and very insignificant symptoms at the site of inoculation on the 4th day after the vaccine-inoculation. The graph of the control-cavia N^o. 13 shows that the inoculation-matter was sufficiently virulent (Fig. 7).

The same symptoms are distinguished in Cavia 4. This animal had been infected twice with aphthae, the first time with the slightly virulent strain S., the second time with the highly virulent strain W. After inoculation with "vaccine" the animal hardly displays any symptoms: we only detect a very slight reaction at the site of injection, which lasts two days (see fig. 8).

The control-cavia N^o. 12, however, reveals with the same infectious matter the complete aspect of the disease described in II.

¹⁾ See note p. 543.

This almost complete immunity could be established for three different vaccine-strains. The results of these experiments are illustrated in the subjoined table I.

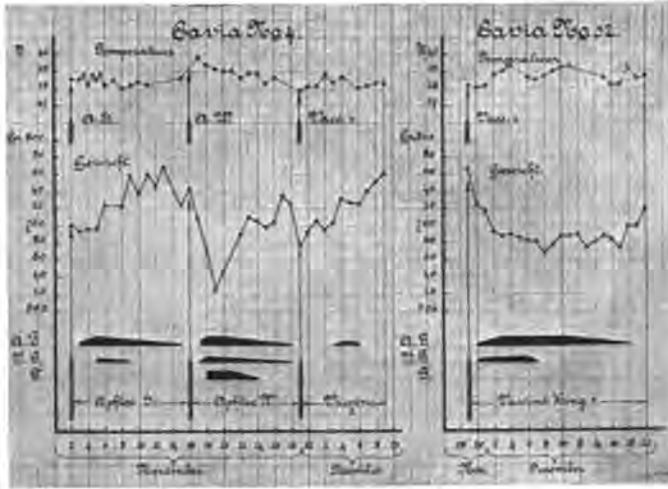


Fig. 8.

In one cavia the immunity appeared to be less complete. Here it should be remembered that this animal:

- 1^o. could endure only faintly infectious apthae,
- 2^o. was already rather old, so that perhaps it had almost lost the capacity of appropriating this immunity.

This experiment, however, cannot on any account be brought forward to detract much from the significance of the other nine observations.

CONCLUSION.

The above-described results are significant for the following reasons.

The close relation between the apthae- and the vaccine-virus is proved for the first time by scientific experiments.

The immunity from the latter, after inoculation with the former, points to a relationship between these two viruses.

It matters little that in these experiments vaccine-inoculation does not produce immunity against apthae, as we could also establish that inoculation with different "apthae-strains" did not yield immunity against other virulent foot- and -mouth disease-virus. Nevertheless, many facts point to a relationship between the two viruses.

This puts the observations of practitioners in a different light from that in which they were regarded up to the present.

On the basis of the relationship alluded to above, it may very well be possible that an intracutaneous vaccine-injection, properly carried out,

would yield a sufficient immunity against "contact"-infections with "aphthae-virus" to a degree that would make it serviceable for practice.

TABLE N^o. 1.

N ^o .	Weight	1 st Inoculation	2 ^d Inoculation	Vaccine	Résumé
8	860	Apht. W.		Hong. II	Almost complete immunity
9	500	Apht. W.		Hong. II	Almost complete immunity
19	420	Apht. W.		Hong. II	Very high degree of immunity
11	565	Apht. W. cavia 10		Hong. II	Very high degree of immunity
17	620	Apht. W. cavia 15		Amsterdam	Very high degree of immunity
18	900	Apht. W. cavia 17		Amsterdam	High degree of immunity
3	380	Apht. W.	Apht. W.	Hong. I	Almost complete immunity
4	700	Apht. S.	Apht. W.	Hong. I	Almost complete immunity
7	575	Apht. V.	Apht. W.	Hong. II	Almost complete immunity
6	680	Apht. S. cavia 4	Apht. W.	Hong. II	High degree of immunity

In connection with these experiments it is quite expedient that they should be continued in the direction indicated ¹⁾.

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¹⁾ In two advisedly arranged experiments it could be established, that two caviae treated beforehand with vaccine and inoculated two, resp. four months afterwards with very virulent aphthae-virus, possessed a much greater resistance against this virus than the animals that had not been treated previously. If this finding should be confirmed in other cases, it will be recorded in a following communication.

Mathematics. — *A Representation on R_8 of the Projectivities of Point-Fields and Ray-Fields.* By Prof. H. J. VAN VEEN. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of March 26, 1927).

§ 1. If in the relations:

$$\left. \begin{aligned} \varrho x'_1 &= a_1 x_1 + a_2 x_2 + a_3 x_3 \\ \varrho x'_2 &= a_4 x_1 + a_5 x_2 + a_6 x_3 \\ \varrho x'_3 &= a_7 x_1 + a_8 x_2 + a_9 x_3 \end{aligned} \right\} \dots \dots \dots (1)$$

the x_i and the x'_i ($i=1, 2, 3$) are considered as homogeneous point coordinates with respect to the same triangle of reference Δ in the plane τ . (1) associates a point x'_i to any point x_i of τ . We shall say of the set $[x'_i]$ of the points x'_i that through (1) it is in *projective (collinear) correspondence* with the point-field $[x_i]$. If the determinant of the 2nd members of (1) $|a| \neq 0$, also $[x_i]$ is projective with $[x'_i]$.

If the coordinates x_i of a point S make the 2nd members of (1) simultaneously zero, the associated point of S is indefinite; to such a *singular point* S we shall associate all points of τ .

The equations which we get by equating the 2nd members of (1) to zero, define 3 straight lines. They have either no common point or one common point S or they coincide in a line s . In the first case there is no singular point. In the second case S is singular; in this case we shall call the *projectivity (collineation) singular of the first kind*. In the third case any point of s is singular; let the line s be called a *singular line* and the projectivity *singular of the 2nd kind*.

For all singularities $|a| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 0$; for that of the 2nd kind at

the same time $\|a\| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 0$,

For a *singular collineation of the 1st kind* it is possible to choose the constants α , β and γ such that $\alpha (a_1 x_1 + a_2 x_2 + a_3 x_3) + \beta (a_4 x_1 + a_5 x_2 + a_6 x_3) + \gamma (a_7 x_1 + a_8 x_2 + a_9 x_3) \equiv 0$, hence for any point x'_i : $\alpha x'_1 + \beta x'_2 + \gamma x'_3 = 0$ or: to an arbitrary point x_i there corresponds a point x'_i of a fixed straight line; we shall call this the *singular line s'* of $[x'_i]$.

In a *singular collineation of the 2nd kind* the 2nd members of (1) are the same except for a constant factor; the ratios of the x'_i are, therefore, constant, or: *to an arbitrary point x_i a fixed point is associated, the singular point S' of $[x'_i]$.*

In what follows we shall indicate the singular collineations by (S, s') and (s, S') .

§ 2. I consider the 9 coefficients a_j ($j=1 \dots 9$) of (1) as homogeneous coordinates of a point in an 8-dimensional space R_8 . To each collineation there corresponds a definite point of R_8 and inversely: each point of R_8 defines a projective correspondence of the points x'_i and x_i (also the direction of the correspondence).

To a system of ∞^m collineations there corresponds an m -dimensional variety V_m of R_8 . A similar variety is defined by nine equations of the form $a_j = f_j(p_1 \dots p_m)$ or by $8-m$ homogeneous equations of the form $F(a_j) = 0$; in the latter case we may consider V_m as the intersection of $8-m$ hypersurfaces. In a following paper we shall deal more elaborately with *systems of collineations*; now we shall only mention that through linear equations a linear space in R_8 is defined to which a linear system of collineations corresponds; to a straight line (a plane) of R_8 there corresponds a linear system of ∞^1 (∞^2) collineations, a *pencil* (*net*).

§ 3. The equation of the *locus of the image-points of the singular collineations* is:

$$|a| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 0$$

It is, therefore, a cubic hypersurface V_7^3 .

We remark that we get this equation by eliminating the parameters out of the equations

$$\left. \begin{aligned} \alpha a_1 + \beta a_2 + \gamma a_3 &= 0 \\ \alpha a_4 + \beta a_5 + \gamma a_6 &= 0 \\ \alpha a_7 + \beta a_8 + \gamma a_9 &= 0 \end{aligned} \right\}$$

or out of

$$\left. \begin{aligned} \lambda a_1 + \mu a_4 + \nu a_7 &= 0 \\ \lambda a_2 + \mu a_5 + \nu a_8 &= 0 \\ \lambda a_3 + \mu a_6 + \nu a_9 &= 0 \end{aligned} \right\}$$

so that V_7^3 can be built in two ways of the five-dimensional linear spaces of intersection of homologous elements of three projective nets of hyperplanes.

Any point P of V_7^3 is the image of a singular collineation of the 1st kind with singular point S and singular line s' . In order to investigate the system of all the collineations that have S as singular point, we

consider S as the angular point $0/0/1$ of the triangle of reference (change of the system of coordinates gives rise to a projective transformation in R_8). In our relations (1) we must now choose the coefficients such that for $x_1=0$, $x_2=0$ and $x_3=1$ the 2nd members all become zero; hence $a_3=a_6=a_9=0$. These are the equations of three hyperplanes which together define a five-dimensional linear space R_5 that passes through P and lies on V_7^3 .

In order to investigate the system of the collineations that have s' as singular line, we consider s' as side $x_3=0$ of the triangle of reference. Now in (1) we must find $x_3=0$ for any system of values of the x_i ; hence $a_7=a_8=a_9=0$. Accordingly in R_8 we again find a five-dimensional space R'_5 through P lying on V_7^3 . R_5 and R'_5 have an R_3 as intersection, which is the image of all collineations that have S as singular point and s' as singular line: V_7^3 contains 2 systems of ∞^2 five-dimensional linear spaces $[R_5]$ and $[R'_5]$; the spaces R_5 are images of degenerate collineations with a fixed singular point S , the spaces R'_5 of degenerate collineations with a fixed singular line s' . Through any point of V_7^3 there passes one individual of each system (these spaces correspond to those mentioned in the 2nd paragraph of this section).

§ 4. The tangent hyperplane R_7 at a point P of V_7^3 contains the R_5 and the R'_5 that pass through this point, and is defined by them. At every point of the common space R_3 of R_5 and R'_5 V_7^3 has the same tangent hyperplane, hence: V_7^3 is touched by its tangent hyperplanes along an R_3 and for this reason we might call it a developable hypersurface of the 3rd kind.

Any R_5 (R'_5) is carrier of a net $[R_7]$ ($[R'_7]$) of ∞^2 tangent hyperplanes; all the tangent hyperplanes can be united in 2 ways to a system of ∞^2 nets.

§ 5. A singular collineation of the 2nd kind is defined by its singular line s and its singular point S' . There exist, therefore, ∞^4 such collineations; their image is a 4-dimensional variety, which can be fixed by the equations:

$$\|a\| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 0.$$

The order of this variety is the number of its points of intersection with an R_4 , which we can define by 4 linear equations:

$$\sum p_j a_j = 0, \quad \sum q_j a_j = 0, \quad \sum r_j a_j = 0 \quad \text{and} \quad \sum s_j a_j = 0.$$

If we put:

$$\frac{a_1}{a_4} = \frac{a_2}{a_5} = \frac{a_3}{a_6} = \frac{1}{\lambda}$$

and

$$\frac{a_1}{a_7} = \frac{a_2}{a_8} = \frac{a_3}{a_9} = \frac{1}{\mu}$$

we have:

$$\begin{aligned} (p_1 + p_4 \lambda + p_7 \mu) a_1 + (p_2 + p_5 \lambda + p_8 \mu) a_2 + (p_3 + p_6 \lambda + p_9 \mu) a_3 &= 0 \\ (q_1 + q_4 \lambda + q_7 \mu) a_1 + (q_2 + q_5 \lambda + q_8 \mu) a_2 + (q_3 + q_6 \lambda + q_9 \mu) a_3 &= 0 \\ (r_1 + r_4 \lambda + r_7 \mu) a_1 + (r_2 + r_5 \lambda + r_8 \mu) a_2 + (r_3 + r_6 \lambda + r_9 \mu) a_3 &= 0 \\ (s_1 + s_4 \lambda + s_7 \mu) a_1 + (s_2 + s_5 \lambda + s_8 \mu) a_2 + (s_3 + s_6 \lambda + s_9 \mu) a_3 &= 0 \end{aligned}$$

If we consider λ and μ as non-homogeneous point coordinates, a_1 , a_2 and a_3 as parameters, these are the equations of 4 projective fields of rays. Now it happens 6 times that 4 corresponding lines pass through one point; hence our equations admit of 6 solutions: *the image of the singular collineations of the 2nd kind is a four-dimensional variety of the sixth order, V_4^6 .*

§ 6. A point P on V_4^6 is the image of a singular collineation of the 2nd kind (s, S') . We shall first consider all collineations that have s as singular line. With a view to this we choose s as side $x_3 = 0$ of the triangle of reference. As for any system of values of the x_i where $x_3 = 0$, all x'_i must be zero, in this case we find in (1), $a_1 = a_2 = a_4 = a_5 = a_7 = a_8 = 0$. Consequently these collineations have as image a plane α through P lying on V_4^6 . In a similar way we find that all singular collineations of the 2nd kind that have the point S' as singular point, are represented by a plane α' on V_4^6 : V_4^6 contains 2 systems of ∞^2 planes $[\alpha]$ and $[\alpha']$; the α -planes are images of the singular collineations with a fixed singular line s , the α' -planes of the collineations with a fixed singular point S' . Through any point of V_4^6 there passes one plane of each of the two systems; two planes of the same kind have no point in common, two planes of different kinds have one point in common.

§ 7. Let $a_1 a_2 a_3$ be three planes of the 1st kind, α' an arbitrary plane of the 2nd kind, $S_1 S_2$ and S_3 the points of intersection of α' with $a_1 a_2$ and a_3 , and let S' be the common singular point of the collineations represented by α' . The fields of points $[S_1]$, $[S_2]$ and $[S_3]$ are in a [1,1] correspondence with the field of points $[S']$ in τ and with each other. Let α_1 be the image of all collineations that have the line s as singular line. Choose s again as $x_3 = 0$; then the latter collineations are defined by the relations:

$$\left. \begin{aligned} \varrho x'_1 &= a_3 x_3 \\ \varrho x'_2 &= a_6 x_3 \\ \varrho x'_3 &= a_9 x_3 \end{aligned} \right\}$$

The singular point S' has for coordinates $a_3/a_6/a_9$. To any point S' in τ there corresponds an a' -plane, which cuts α_1 in a point S_1 . If S' moves on a line $p_1 x_1 + p_2 x_2 + p_3 x_3 = 0$, S_1 moves on the line of intersection of the plane α_1 with the hyperplane $p_1 a_3 + p_2 a_6 + p_3 a_9 = 0$; consequently the point-fields $[S_1]$, $[S_2]$ and $[S_3]$ are each in projective correspondence with the field $[S']$, hence also with each other, or: V_4^6 is the locus of the planes through the homologous points of three projective fields of points. As also the α -planes cut three arbitrary a' -planes in projective point fields, V_4^6 can be produced in two similar ways.

Now we may consider V_7^3 as built (§ 3) of the 5-dimensional linear spaces along which the homologous elements of 3 projective nets of hyperplanes cut each other; consequently V_7^3 and V_4^6 are dualistic figures: V_7^3 is of the 6th class and V_4^6 of the 3rd class, V_7^3 has ∞^4 tangent hyperplanes 6 of which pass through an R_3 , V_4^6 has ∞^7 tangent hyperplanes (in each point ∞^3) 3 of which pass through an R_6 .

§ 8. The polar hyperplane of the point $P(a_j^0)$ relative to V_7^3 has for equation $\Sigma a_j A_j^0 = 0$; here the A_j^0 are the coefficients of the elements a_j in the development of the determinant $|a|$ when the a_j have been replaced by the a_j^0 . If we choose P on V_4^6 the A_j^0 all become zero, the polar hyperplane is indefinite, hence P is a double point of V_7^3 : V_4^6 is a double variety of V_7^3 .

The tangent hyperplane R_7 at a point P of V_7^3 passes through an R_5 and an R_5' (§ 4). R_5 (S fixed and s' arbitrary) contains ∞^1 α -planes: the images of the collineations that have as singular line s a ray of the plane pencil about S ; R_5' (s' fixed and S arbitrary) contains ∞^1 a' -planes: the images of the collineations that have a singular point S' on s' . In order to determine the locus of the points of intersection of these α - and a' -planes, we choose the line s' as side $x_3 = 0$ and the point S as angular point $0/0/1$ of the triangle of reference. The relations of the collineation are in this case:

$$\begin{cases} \varrho x'_1 = a_1 x_1 + a_2 x_2 \\ \varrho x'_2 = a_4 x_1 + a_5 x_2 \\ \varrho x'_3 = 0 \end{cases}$$

where $\frac{a_1}{a_4} = \frac{a_2}{a_5}$, and $a_3 = a_6 = a_7 = a_8 = a_9 = 0$.

The locus in question is, therefore, a quadratic surface V_2^2 . Let P be a point of V_2^2 ; the α - and the a' -plane through P define the tangent space R_4 at P ; any hyperplane through R_4 touches V_4^6 at P , hence: the tangent hyperplanes of V_7^3 are double tangent hyperplanes of V_4^6 ; the points of contact of such a hyperplane form a quadratic surface.

§ 9. We shall now consider the system of the singular collineations

of the 1st kind where the singular point S lies on a given straight line. If we choose this line as side $x_3=0$ of the triangle of reference, the 2nd members of (1) for $x_3=0$ must agree except for a constant factor, hence:

$$\frac{a_1}{a_2} = \frac{a_4}{a_5} = \frac{a_7}{a_8}.$$

The image of these collineations, which consists of ∞^1 spaces R_5 , is, therefore, a six-dimensional cubic variety V_6^3 . V_7^3 contains ∞^2 such varieties; through any two points of the hypersurface there passes one. Each of the systems of collineations under consideration contains all the singular collineations of the 2nd kind.

The latter is also the case for each of the systems of singular collineations where the singular line s' is a ray of a given plane pencil; these systems have as images ∞^2 varieties V_6^3 on V_7^3 , hence: V_7^3 contains 2 systems of ∞^2 six-dimensional cubic varieties $[V_6^3]$ and $[V_6^3]$; the varieties of the 1st kind are images of all the singular collineations where S lies on a given straight line, those of the 2nd kind of the singular collineations where s' passes through a given point. All these varieties pass through V_4^6 .

§ 10. Through the relations

$$\left. \begin{aligned} \sigma \xi'_1 &= a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3 \\ \sigma \xi'_2 &= a_4 \xi_1 + a_5 \xi_2 + a_6 \xi_3 \\ \sigma \xi'_3 &= a_7 \xi_1 + a_8 \xi_2 + a_9 \xi_3 \end{aligned} \right\} \dots \dots \dots (2)$$

where ξ_i and the ξ'_i are considered as homogeneous ray coordinates relative to the triangle of reference Δ (§ 1), a line ξ'_i is associated to any line ξ_i of τ and if $|\alpha| \neq 0$ also a line ξ_i to any line ξ'_i . The system $[\xi'_i]$ of the rays ξ'_i is again in a projective correspondence with the field of rays $[\xi_i]$ through (2).

Of these projectivities of rays we may make similar considerations as of projectivities of points.

We shall now consider the a_i as homogeneous coordinates of a hyperplane in R_8 . To the singular projectivities of rays of the 1st kind — i.e., accordingly, the projectivities through which to any ray ξ_i a definite ray ξ'_i (ray of the plane pencil about a point Σ') is associated except to one singular ray σ of which the associated line is indefinite — there corresponds a system of ∞^7 hyperplanes 3W_7 3 of which pass through an R_6 and which has as equation:

$$|\alpha| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 0.$$

To the singular projectivities of the 2nd kind — in which to any ray ξ_i one fixed ray ξ'_i is associated, which will be indicated by σ' , except

to the rays of the plane pencil about a point Σ for which the associated ray is indefinite — there corresponds a system of ∞^4 hyperplanes 6W_4 , 6 of which pass through an R_3 .

§ 11. From the point-projectivity where the coefficients in (1) have the values a_j^0 , there generally follows a projectivity of rays where the coefficients in (2) are equal to A_j^0 and the A_j^0 have the signification indicated in § 8.

If P is the image of the point-projectivity, the polar hyperplane π of P relative to V_7^3 has for equation $\Sigma a_j A_j^0 = 0$, hence for coordinates A_j^0 , or: a point (hyperplane) and a hyperplane (point) are images of a point-(ray-) projectivity and of the corresponding ray- (point-) projectivity if they are pole (hyperplane) and polar hyperplane (pole) relative to V_7^3 (3W_7).

If P lies on V_7^3 and is, therefore, the image of a singular point-projectivity of the 1st kind with singular point S and singular line s' , π is the tangent hyperplane at P and the corresponding ray-projectivity is singular of the 2nd kind. For to an arbitrary line as carrier of its points there corresponds the fixed line s' and for the associated line of a line ξ_i through S any line of τ may be chosen, because any point of τ is associated to the point S of the line ξ_i (such in deviation from what is assumed by SEGRE, HIRST, STURM a.o. ¹⁾). If the point-projectivity is defined by the coefficients a_j^0 , also in this case the coefficients for the ray-projectivity are A_j^0 . Accordingly the tangent hyperplanes of V_7^3 are identical with the hyperplanes of the system 6W_4 ; in the same way the variety enveloped by the hyperplanes of 3W_7 is identical with V_4^6 .

We still remark that a singular point- (ray-) projectivity of the 2nd kind does not give rise to a ray- (point-) projectivity.

§ 12. Summarizing we find:

If in the indicated way the point-projectivities are represented on the points and the ray-projectivities on the hyperplanes of an R_8 , R_8 contains a hypersurface V_7^3 of the 3rd degree the ∞^7 points of which are images of singular point-projectivities; this hypersurface is of the 6th class; its ∞^4 tangent hyperplanes are images of singular ray-projectivities of the 2nd kind. V_7^3 contains a double variety V_4^6 of the 6th order; its ∞^4 points are images of singular point-projectivities of the 2nd kind; this variety is of the 3rd class; its ∞^7 tangent hyperplanes are images of singular ray-projectivities of the 1st kind.

¹⁾ C. SEGRE. Sulla teoria e sulla classificazione delle omografie in uno spazio lineare ad un numero qualunque di dimensioni. Atti dei Lincei, 1884, (5), 19, 127.

T. A. HIRST. On the correlation of two planes. Proc. London Math. Soc. 5, 42.

R. STURM. Die Lehre v. d. Geometr. Verwandtschaften. 2, 219.

Mathematics. — *Systems of Plane Collineations.* By Prof. H. J. VAN VEEN. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of April 30, 1927).

§ 1. Making use of "A Representation on R_8 of the Projectivities of Point-Fields and Ray-Fields" ¹⁾ in the following paper I shall derive properties of systems of collinear point-fields. (I¹⁾, § 2).

By a system of ∞^m collineations to the field $[x_i]$, *the identical field* or *the auxiliary field*, ∞^m point-fields are associated, which in what follows we shall indicate by $[A']_m$. As a rule any of these fields is in a projective correspondence with any of the others and that by means of their relation to the auxiliary field; in this way we arrive at a system of ∞^{2m} projectivities.

If we make the condition that a collineation must associate a point of the given line $a'(\sum \xi_i x_i = 0)$ to a given point $A(x_i^0)$, the a_j (I, § 2) must satisfy the equation:

$$\xi_1 (a_1 x_1^0 + a_2 x_2^0 + a_3 x_3^0) + \xi_2 (a_4 x_1^0 + a_5 x_2^0 + a_6 x_3^0) + \xi_3 (a_7 x_1^0 + a_8 x_2^0 + a_9 x_3^0) = 0 \quad (1)$$

and inversely.

Accordingly the image of the collineations that satisfy the condition, is a hyperplane of R_8 .

Now among others there belong to this system of collineations:

1. all collineations that have A as singular point S (I, § 1 in connection with II²⁾, equation (1)); the image of these is an R_5 of V_7^3 (I, § 3);
2. all collineations that have a' as singular line s' ; these have as image an R_5 on V_7^3 (I, § 3).

R_5 and R_5' have the R_3 in common that represents the collineations with A as singular point S and a' as singular line s' ; they define a hyperplane (I, § 4): *the collineations through which a point of the line a' is associated to A — including the singular collineations where A is the singular point S — have as image the hyperplane that touches V_7^3 along the R_3 which represents the singular collineations that have A as singular point S and a' as singular line s' .*

If R_5 (R_5') is the image of all collineations that have A (a') as singular point (singular line), any hyperplane of the net (I, § 4, 2nd par.) about R_5 (R_5') is the image of the collineations for which the associated point of A either lies on a given line (the associated point of a given point lies on a') or is indefinite.

¹⁾ These Proceedings 30, p. 552.

²⁾ This paper.

§ 2. Through a point of R_8 there pass ∞^3 tangent hyperplanes of V_7^3 ; accordingly for *one collineation* it happens ∞^3 times that the associated point of a point x_i lies on a given line.

If the associated points of 8 given points A_k ($k = 1, \dots, 8$) must lie on 8 given lines a'_k , we get in R_8 eight tangent hyperplanes of V_7^3 as images of these conditions. As a rule these have one point in common so that there is *one collineation* that satisfies the condition.

If the 8 hyperplanes have a line or a linear space of more dimensions in common, we might call the conditions *dependent*; in this case a linear system of collineations satisfies the conditions.

§ 3. Two points P_1 and P_2 of R_8 define a line R_1 , two collineations define a *linear system of ∞^1 collineations, a pencil*.

Through both points P_1 and P_2 , hence also through the line R_1 , there pass ∞^2 tangent hyperplanes of V_7^3 , one of any net of the two systems formed by these hyperplanes (I, § 4, 2nd par.). Consequently (II, § 1, last par.) for all collineations of the pencil the associated points of any point of τ lie on one straight line and any line of τ contains all the associated points of one point. (For the indefinitenesses we refer to II, § 9).

If we make the condition that *the associated points of 7 given points must lie on 7 given lines*, the image in R_8 of the collineations that satisfy this condition, is the common space of 7 tangent hyperplanes of V_7^3 , hence as a rule a straight line. Accordingly these collineations form a *pencil*.

Concerning the singular collineations we remark that *a pencil contains three singular collineations of the 1st kind*, as the image R_1 of a pencil has 3 points P_i in common with V_7^3 ; two or three of these collineations can coincide. If R_1 cuts V_4^6 the pencil contains *a singular collineation of the 2nd kind and one of the 1st kind*. If R_1 is a bisecant of V_4^6 it lies on V_7^3 , hence: *if a pencil contains two singular collineations of the 2nd kind, it consists entirely of singular collineations*.

§ 4. Three points of R_8 that do not lie on one line, define a plane R_2 , three collineations that do not belong to one pencil, define a *linear system of ∞^2 collineations, a net*. Through R_2 there pass ∞^1 tangent hyperplanes of V_7^3 , hence there are ∞^1 points x_i of which the associated points lie on one line. (For the indefinitenesses, the 5th par. of this §).

R_2 cuts V_7^3 along a curve of the 3rd order k : *a net contains ∞^1 singular collineations of the 1st kind, but as a rule no singular collineations of the 2nd kind*.

The plane R_2 is cut by each of the varieties V_6^3 (V_6^3) — I, § 9 — in 3 points, hence: *in a net of collineations the locus of the singular points S is a curve of the 3rd order k^3 , the singular lines s' envelop a curve of the 3rd class 3k* .

Let A be a point of τ that is not singular for any of the collineations of our net, R_5 the image of all collineations that have A as singular point S , then R_5 has no point in common with our R_2 . Accordingly none of the tangent hyperplanes through R_5 of V_7^3 contains R_2 . Through any line of R_2 there passes one of the hyperplanes of each net $[R_7]$: for each of the ∞^2 pencils in our net A' describes one of the rays of the field in τ . To any point P of R_2 as image of a collineation one point A' of τ corresponding to A is associated and inversely; if P describes a line A does the same; the fields of points $[P]$ and $[A']$ are projective.

We shall now consider a point S of τ that is singular for one of the collineations of our net. In this case the R_5 mentioned in the preceding section has one point, lying on k , in common with R_2 and one of the hyperplanes through R_5 contains R_2 . The associated points of S lie, therefore, as a rule on one line. If s' is a singular line for any of the collineations under consideration, the R_5 (s' singular) has also a point in common with R_2 so that through this R_5 there also passes an R_7 that contains R_2 : all the associated points of a point S of the curve k^3 — except for the collineation for which S is a singular point, where there is an indefiniteness — lie on a tangent s' to the curve 3k and inversely: any tangent s' to 3k contains the associated points of a point S of k^3 .

The collineations for which the associated points of 6 given points lie on 6 given lines (or are indefinite), generally form a net, for the image of these collineations is the plane of intersection of 6 tangent hyperplanes of V_7^3 .

§ 5. Four points of R_8 that do not lie in one plane, define an R_3 , four collineations that do not belong to one net, define a linear system of ∞^3 collineations, a linear complex.

Through an R_3 (I, § 7, last paragraph) there pass six tangent hyperplanes of V_7^3 , hence: a linear complex of collineations contains six points for each of which the associated points are collinear. (See below the cases of indefiniteness).

Now all collineations for which all the associated points of 5 given points lie on 5 given lines, form a linear complex, because the image in R_8 is the space of intersection R_3 of 5 tangent hyperplanes of V_7^3 , hence: in all the collineations for which the points associated to five given points lie on five given lines, there is a sixth point for which the associated points are also collinear¹⁾. However we must remark that our R_3 has a point in common with any R_5 so that any point x_i of τ is singular for one of the collineations in question and its associated point indefinite. Also any line of τ is the singular line s' for one of these collineations. Moreover our R_3 has a line in common with the

¹⁾ A. CLEBSCH, Ueber ein neues Grundgebilde der analytischen Geometrie der Ebene. Math. Annalen 6, p. 203, 1873.

J. ROSANES, Ueber linear abhängige Punktsysteme. Journ. f. Mathem. Bd. 88, p. 241, 1880.

A. VOSS, Zur Theorie der linearen Connexe. Math. Ann. 15, 355, 1879.

R. STURM, Ueber Collineation und Correlation. Math. Ann. 22, 569, 1883.

R_5 (R'_5) with which it lies in the same tangent hyperplane of V_7^3 , hence: each of the six points of which the associated points generally lie on six fixed lines, is singular point S for a pencil of collineations: the six fixed lines are then the singular lines s' .

As a rule a linear complex does not contain any singular collineations of the 2nd kind. Such can be the case for special linear complexes.

§ 6. Five points that do not lie in one R_3 , define an R_4 , five collineations that do not belong to a linear complex, define a linear system of ∞^4 collineations.

Through an arbitrary R_4 there do not pass any tangent hyperplanes of V_7^3 , so that as a rule it does not happen in a linear complex that all the associated points of one point are collinear; in special cases, however, this is possible; the collineations for which the associated points of 4 given points lie on 4 given lines, (or are indefinite), form a — special — linear system of ∞^4 collineations.

In a linear system of ∞^4 collineations any point (any line) of τ is singular for a pencil of collineations.

An R_4 has 6 points of intersection with V_4^6 , hence: a linear system of ∞^4 collineations contains six singular collineations of the 2nd kind ¹⁾.

In the same way we can derive properties of linear systems of ∞^5 , ∞^6 and ∞^7 collineations.

§ 7. We shall now consider the ∞^1 collineations for which the associated points A'_l ($l=1 \dots 5$) of 5 given points A_l lie on 5 given lines a'_l and the associated points B' of the point B on a curve k of the n^{th} order.

The condition that the points A'_l must lie on the lines a'_l , defines a net of collineations which has as image in R_8 a plane R_2 . In § 4 we have shown that for these collineations the associated points of any point of a certain cubic k^3 lie on a tangent to a curve of the third class 3k .

At the same time it follows from the projectivity mentioned in the 4th paragraph of § 4, that the image of the collineations which satisfy all the conditions, is a curve projective with k ; consequently that all the associated points of any point of τ that is not singular for any of the collineations of the net, and which, therefore, does not lie on k^3 , lie on a curve projective with k . We also remark that our system contains $3n$ singular collineations, hence: if the associated points of 5 given points lie on 5 given lines and those of a sixth point on a curve k of the n^{th} order, τ contains a curve of the 3rd order k^3 with the following properties: k^3 contains $3n$ points for each of which the associated point is indefinite for one of the collineations in question. With the exception of these points all the associated points of any point of k^3 lie on a tangent to a curve of the third class 3k . All the associated points of any other point of τ lie on a curve that is projective with k .

¹⁾ STURM, p. 581.

It follows from § 11 that as a rule τ does not contain any points to which in the collineations under consideration a fixed point is associated.

§ 8. The collineations for which the associated point of a given point A must lie on a given line a'_1 , have as image in R_8 an R_7 through R_5 representing the collineations for which A is singular point S ; the same is the case with the collineations for which the associated point of A lies on the line a'_2 . Accordingly the collineations for which A corresponds to the point of intersection A' of a'_1 and a'_2 , have an R_6 through the said R_5 as image; among them there are ∞^5 collineations for which the associated point of A is indefinite. If now we consider all fields of points $[A']_6$ that correspond to the auxiliary field (II, § 1) by means of our collineations, these projective fields have A' as *invariant point*.

Each of the ∞^2 spaces R_6 through R_5 is the image of the collineations that make A correspond to a given point of τ . We get a system of ∞^4 spaces R_6 , which may be combined to ∞^2 nets $[R_6]$ each of which has one of the ∞^2 spaces R_5 on V_7^q (I, § 4) as carrier.

In a similar way we can derive the existence of a system of ∞^4 spaces R'_6 — these may be combined to ∞^2 nets $[R'_6]$ each of which has one of the spaces R'_5 (I, § 4) as carrier — that are the images of the collineations which associate a fixed line to a given line. In this case the point-fields corresponding to the auxiliary field have an *invariant line*.

Two of the said spaces R_6 (R'_6) have an R_4 (R'_4) in common, which is the image of all collineations that associate ∞^4 projective point-fields with *two invariant points (lines)* to the auxiliary field. In the same way three such spaces R_6 (R'_6) have a plane R_2 (R'_2) in common, which represents the collineations that associate ∞^2 projective point-fields with *three invariant points (lines)* to the auxiliary field. Four of the spaces R_6 (R'_6) generally have one point in common; there is only one collineation that associates 4 points (lines) of $[x]$ to 4 given points (lines).

If we also consider the intersections of spaces R_6 with spaces R'_6 we get the images of all *fundamental groups with invariant points and lines*¹⁾.

§ 9. Through a point P of R_8 there pass ∞^2 of the spaces R_6 (R'_6), one of each net; in fact for *one collineation* the element associated to an arbitrary point (arbitrary line) is fixed.

We shall now consider a *pencil of collineations*; this generally contains

¹⁾ LIE—SCHEFFERS, *Continuierliche Gruppen* p. 288.

H. B. NEWSON, A new theory of Collineations and their LIE-groups. *American Journal of Mathematics*, 24, p. 169, 1902.

H. B. NEWSON, Report on the theory of collineations. *Proceedings of the American Association*, Pittsburg, 1902, p. 579.

3 singular collineations (II, § 3) with singular points A_i and singular lines a'_i ; their images in R_3 are the points of intersection P_i of the image R_1 of the pencil with V_7^3 .

Each of the points P_i lies in one $R_5 (R_5')$; this is the carrier of ∞^2 spaces $R_6 (R_6')$ one of which passes through R_1 : a pencil of collineations contains three points A'_i that have a fixed associated point A'_i — except in the collineations for which they are singular elements, where the associated point is indefinite —; in the same way there are three lines a_i to each of which a fixed line a'_i is associated; these lines a'_i are the singular lines of the 3 singular collineations of the pencil.

A pencil of projective point-fields contains three invariant points A_i and three invariant lines a'_i .

The R_5 through P_1 lies in one tangent hyperplane of V_7^3 with the R_5' through P_2 (and likewise with the R_5' through P_3); this hyperplane contains R_1 , hence: the points A'_i are the angular points of the triangle of which the sides lie on the lines a'_i .

§ 10. If the image line R_1 of a pencil is a tangent to V_7^3 , $P_{1,2}$ its point of contact and P_3 its point of intersection, the pencil contains a singular collineation with singular point $A_{1,2}$ and singular line $a'_{1,2}$ and a singular collineation with singular point A_3 and singular line a'_3 . To $A_{1,2}$ there corresponds a fixed point $A'_{1,2}$ which again lies on a'_3 but at the same time on $a'_{1,2}$, because R_1 lies in the tangent plane at $P_{1,2}$ to V_7^3 . The fixed point A'_3 associated to A_3 lies likewise on $a'_{1,2}$.

If the image line R_1 is an inflexional tangent to V_7^3 , the pencil contains only one singular collineation (A, a') ; the point A' associated to A is fixed and lies on a' .

Invariant points and invariant lines of two projective point fields are also invariant elements of the pencil of projective fields defined by them. Making use of the notation of SEGRE¹⁾ we may, therefore, say: a collineation belongs to the type $[1, 1, 1]$, $[2, 1]$ or $[3]$ according as the line joining its image point to the image point of the identical collineation cuts V_7^3 in 3 different points or is a common tangent or an inflexional tangent of the hypersurface. (Cf. further § 12).

§ 11. As a rule through a plane there does not pass any of the spaces $R_6 (R_6')$, hence: a net of collinear point-fields does not generally contain any invariant point. If, however, the plane lies in one of the spaces $R_6 (R_6')$, it is the image of the collineations that associate a net of projective point-fields with one invariant point (line) to the auxiliary field. A plane in the common space R_4 of 2 spaces $R_6 (R_6')$ associates a net of projective fields with two invariant points (lines) to the auxiliary field. A net with 3 invariant points has already been mentioned in § 8 and

¹⁾ C. SEGRE, Sulla teoria e sulla classificazione delle omografie in uno spazio lineare ad un numero qualunque di dimensioni. Atti dei Lincei, 1884, (5), 19, 127.

from the discussions there we can derive all possibilities — concerning the invariant elements — for *linear systems of* ∞^3 , ∞^4 , ∞^5 and ∞^6 *projective fields.*

§ 12. If P_1 is a point of intersection of an arbitrary line R_1 with V_7^3 , only one of the spaces $R_5 (R'_5)$ passes through P_1 and only one of the spaces $R_6 (R'_6)$ of the net (II, § 8) that has this $R_5 (R'_5)$ as carrier.

If, however, R_1 cuts V_7^3 in a point $P (s = a, S' = A')$ of V_4^0 , this point lies in ∞^1 spaces $R_5 (S \text{ on } a)$ and in ∞^1 spaces $R'_5 (s' \text{ through } A')$; each of these $R_5 (R'_5)$ is the carrier of a space $R_6 (R'_6)$ through R_1 . Consequently we have in our collineations (cf. II, § 9): to any point of the line a there corresponds a fixed point of a line a' and to any line of a plane pencil about a point A there corresponds a fixed ray of the plane pencil about A' ; A is the singular point S and a' is the singular line s' for the collineation that has the 2nd point of intersection of R_1 and V_7^3 as image: *if a pencil of collineations contains a singular collineation of the 2nd kind (s, S') , the point fields $[A']_1$ are generally central-collinear (perspective); in this case the pencil has also one degeneration of the 1st kind (S, s') ; the singular elements S' and s' are the centre and the axis of these perspectivities.*

If R_1 is a tangent to V_7^3 at a point of V_6^1 , there correspond to this image-line perspectivities in which the centre lies on the axis.

In connection with § 10, last paragraph, we remark: a collineation belongs to the type $[(1,1), 1]$ or to the type $[(2,1)]$ according as the line that joins its image point to the image point of the identical collineation, cuts V_4^0 or touches V_7^3 at a point of V_4^0 .

§ 13. If by a central-collineation we understand a collineation through which to the auxiliary field a field is associated that is central-collinear with it, and if we call the image point of the identical collineation H , we find: *the image of all central-collineations is formed by the five-dimensional cone of the sixth degree K_5^0 that has H as vertex and V_4^0 as director variety.*

An arbitrary R_3 cuts K_5^0 in six points; if R_3 passes through H it has only this point (which is sixfold for K_5^0) in common with the cone; as any field of the system of projective fields $[A']_3$ corresponding to R_3 may be chosen as auxiliary field, we have: *a linear system of ∞^3 projective point-fields $[A']_3$ generally contains 6 fields that are perspective with the auxiliary field¹⁾; but no two of these fields are perspective with each other.*

An R_4 cuts V_4^0 in 6 points; a line of R_4 through one of these points defines a pencil of perspective point-fields: *a linear system of ∞^4 projective point-fields contains 6 systems of ∞^3 pencils of perspective fields.*

¹⁾ STURM p. 574.

Mathematics. — *A Congruence of Plane Monoids.* By Prof. JAN DE VRIES.

(Communicated at the meeting of April 30, 1927).

1. The plane curves of the order n that have an $(n-1)$ -fold point at D and rest on $2n$ straight lines b , form a congruence.

Let the number of these "plane monoids" that also rest on a line l and lie in planes a through a line a containing D , be indicated by $f(n)$. Any plane a contains a curve $\mu^n (D^{n-1})$; hence a is a multiple line and each of its points lies on $f(n)-n$ curves. The curves μ^n that cut a in a point A and rest on $2n-1$ lines b , form, therefore, a surface of the degree $f(n)-n$ on which a is a multiple line of the order $f(n)-2n$. Consequently this surface contains $f(n)-2n$ figures that consist of a and a $\mu^{n-1} (D^{n-2})$; but this number is also represented by $f(n-1)$. Accordingly

$$\begin{aligned} f(n) &= f(n-1) + 2n \\ f(n-1) &= f(n-2) + 2(n-1) \\ &\dots \dots \dots \\ f(3) &= f(2) + 2 \times 3. \end{aligned}$$

As $f(2)=6$ there follows from this $f(n)=n(n+1)$.

Consequently there are $n(n+1)$ curves $\mu^n (D^{n-1})$ in planes a through a that rest on $2n+1$ straight lines.

2. Let $\psi(n)$ represent the number of plane monoids that have their $(n-1)$ -fold point at D and rest on $2n+2$ lines b ; this number is at the same time the degree of the surface of the monoids that cut $2n+1$ lines.

Any plane through the transversal a_{12} through D of b_1 and b_2 contains a monoid $\mu^{n-1} (D^{n-2})$ which, together with a_{12} , forms a μ^n of the system. All these μ^{n-1} form a surface Γ_{12} of the degree $(n-1)n$.

The plane Db_3 contains a pencil (μ^n) of which each curve cuts b_3 in n points.

There are $\psi(n-1)$ curves $\mu^{n-1} (D^{n-2})$ that rest on the $2n$ lines b ; each of them is completed to a μ^n by any ray of the pencil about D in its plane.

If the point ranges (B_1) and (B_2) of b_1 and b_2 are brought into projective correspondence with two coplanar pencils (f_1) and (f_2) about the points F_1 and F_2 , let the point $f_1 f_2$ be considered as the image of the monoid defined by B_1 and B_2 .

The curves of Γ_{12} are represented in one, *singular*, point S_{12} , the pencils (μ^n) in the planes Db_k ($k=3$ to $2n$) in the *singular* points S_k . There are $\psi(n-1)$ more *singular* points S , each of which represents a pencil of which any individual consists of a fixed μ^{n-1} and a variable line.

3. The monoids resting on a line l form a surface Λ and are represented on a curve λ . Through a point B_1 of b_1 there pass $f(n) - n$, hence n^2 , curves μ^n ; λ is, therefore, the product of two plane pencils in a correspondence (n^2, n^2) , accordingly a λ^{2n^2} , with n^2 -fold points F_1 and F_2 .

It has an $n(n-1)$ -fold point in S_{12} and n -fold points in each of the $2(n-1)$ points S_k ; it also contains the $\psi(n-1)$ points S .

Any non singular point of intersection of two curves λ corresponding to the lines l and l' , is the image of a μ^n that rests on l and l' . As the number of these monoids is represented by $\psi(n)$ the relation

$$2n^4 + n^2(n-1)^2 + 2(n-1)n^2 + \psi(n-1) + \psi(n) = 4n^4$$

holds good, whence

$$\psi(n) + \psi(n-1) = n^2(n^2 + 1).$$

As $n^2(n^2 + 1) = \frac{1}{2}n^2(n+1)^2 + \frac{1}{2}(n-1)^2n^2$, it appears that

$$\psi(n) = \frac{1}{2}n^2(n+1)^2$$

Consequently on $2n+2$ lines there rest $\frac{1}{2}n^2(n+1)^2$ monoids that have their $(n-1)$ -fold point in D .

4. The monoids that rest on $2n$ lines b and touch a plane φ , form a surface Φ .

The monoid M of the order n^2 which contains the curves μ^n (D^{n-1}) through B_1 , has an $n(n-1)$ -fold line in DB_1 . Accordingly through the point of intersection of DB_1 and φ there pass $n^2(n^2-1) - n(n-1)(n^2-n+1)$ or $n(n-1)(2n-1)$ tangents to the intersection of M and φ . Through B_1 there pass, therefore, $n(n-1)(2n-1)$ curves μ^n which touch the plane φ ; consequently the image curve of the system of the curves touching φ has the order $2n(n-1)(2n-1)$.

The monoid Γ_{12} of the order $n(n-1)$ has an $(n-1)^2$ -fold line in a_{12} ; hence $n(n-1)(n^2-n-1) - (n-1)^2(n^2-2n+2)$ curves μ^{n-1} touching φ lie on Γ_{12} . Through the point of intersection of a_{12} and φ there pass $(n-1)^2$ monoids consisting of a_{12} and a μ^{n-1} , each of which has that point as a double improper point of contact with φ .

In all Γ_{12} contains, therefore, $(n-1)(n-2)(2n-1) + 2(n-1)^2$ or $n(n-1)(2n-3)$ monoids touching φ ; hence the image curve of the system has an $n(n-1)(2n-3)$ -fold point in S_{12} .

The plane Db_k contains $2(n-1)$ monoids μ^n that touch φ ; as they cut b_k n times, S_k is a $2n(n-1)$ -fold point.

Each of the $\psi(n-1) = \frac{1}{2}(n-1)^2 n^2$ planes which contain a fixed μ^{n-1} and a pencil about D , gives rise to $n-1$ composite μ^n of our system that must be counted double; accordingly the corresponding singular point S is $2(n-1)$ -fold on the image curve. This may, therefore, be indicated by the symbol

$$\varphi^{2n(n-1)(2n-1)} (F_{1,2}^{n(n-1)(2n-1)}, S_{12}^{n(n-1)(2n-3)}, 2(n-1) S_k^{2n(n-1)}, \frac{1}{2}(n-1)^2 n^2 S^{2(n-1)})$$

It has as many non-singular points in common with the image curve

$$i^{2n^2} (F_{1,2}^{n^2}, S_{12}^{n(n-1)}, 2(n-1) S_k^n, \frac{1}{2}(n-1)^2 n^2 S)$$

of the monoids that rest on l as the number of the μ^n , that touch φ and cut l .

Now

$$4n^3(n-1)(2n-1) - \{ 2n^3(n-1)(2n-1) + n^2(n-1)^2(2n-3) + 4(n-1)^2 n^2 + (n-1)^3 n^2 \}$$

is equal to $n^3(n^2-1)$. Consequently there are $n^3(n^2-1)$ monoids that touch φ and cut l ; the monoids which touch φ and cut $2n+1$ lines, form a surface of the degree $n^3(n^2-1)$.

The combination of two image curves φ gives: there are $n^2(n-1)^2(2n^2-1)$ monoids with an $(n-1)$ -fold point in D which touch two planes and cut $2n$ straight lines.

Biochemistry. — *The so-called co-enzyme of alcoholic fermentation.* By
A. J. KLUYVER and A. P. STRUYK.

(Communicated at the meeting of April 30, 1927).

I. In 1905 it was shown by HARDEN and YOUNG¹⁾ that it is possible to divide zymase containing liquids by ultrafiltration into two parts each of which misses the power to ferment glucose, whereas by reuniting both this power is again restored. In the same year it was proved by BUCHNER and ANTONI²⁾ that a similar separation could be effected also by dialysis of yeast-juice. It soon appeared from further experiments that contrary to what counts for the remaining mass in the ultrafilter or in the dialysator, the filtrate and the dialysate could be heated to 100° C. without causing the least change in the observed effect. The possibility that the inactivity of the residue should have to be reduced exclusively to the removal of the soluble alkaliphosphates proved indispensable to fermentation, was excluded by further experiments of HARDEN. Hence this investigator concluded that for the fermentation of the hexoses not only the presence of the zymase proper — and of alkaliphosphates — was required, but also the presence of an ultrafiltrating, dialysable, thermostable substance, which he gave the name of "co-enzyme"³⁾.

The interest for this co-enzyme has risen considerably in the last years, since namely O. MEYERHOF proved in 1917 that the occurrence of this principle was by no means confined to the yeast-cell; on the contrary the co-enzyme was also found in various animal tissues. Moreover MEYERHOF made also acceptable that the co-enzyme was not only acting in fermentation but also in respiration processes.

Notwithstanding the fact that the co-enzyme has been the subject of many investigations in recent years, one has not yet succeeded in elucidating the nature of this so mysterious substance. So there are to be found assembled in recent summaries of our knowledge in this domain much varying and to a certain extent hardly combinable opinions about the nature of the co-enzyme⁴⁾.

Characteristic however for the mode of view of all investigators is that they have not the least objections to accept the necessity of the co-operation of an unknown factor for the realisation of the alcoholic fermentation of the

1) Compare: A. HARDEN, *Alcoholic Fermentation*, 3rd Ed., 1923, p. 61.

2) E. BUCHNER und W. ANTONI, *Zeitschr. f. physiol. Chemie*, Bd. 46, p. 136, (1905).

3) In literature the names co-enzyme, coferment, cozymase and codiastase are mixed up.

4) For the sake of brevity this assertion will not be further documented here; compare the wellknown summaries of FUCHS, NEUBERG, HARDEN, VON EULER and SCHOEN.

hexoses. This cannot wonder when one realizes that biochemists are still generally used to reduce the chemical conversions, caused by the living cell, to a chain of reactions each of which passes under the influence of its own specific catalysator. Under these circumstances there is little objection to the acceptance of one single mysterious agent — the co-enzyme — more.

However this becomes different when one puts himself on the standpoint taken in by DONKER and one of us (KLUYVER)¹⁾ according to which the biochemical conversions — apart from some hydrolyses and esterifications, which are for the greater part of a preparatory nature — are to be reduced to a chain of reactions which may all be considered as a catalytic transference of hydrogen and which pass in each cell under the influence of one and the same agent.

Under these circumstances biochemistry is reduced for the greater part to a series of heterogeneous catalytic processes the model of which is to be found in organic chemistry in the coupled dehydrogenations and hydrogenations under the influence of finely divided nickel.

The simplification of insight brought along with this view renders the acceptance of the necessity of the collaboration of a co-enzyme for the fermentation process much more difficult and it seemed therefore desirable to us to consider in how far the necessity of an agent, besides the zymase proper, was to be derived from our special scheme for alcoholic fermentation based on the general theory referred to above.

II. For the scheme mentioned, we think we may refer to the last cited paper of KLUYVER and DONKER, and also to a former paper of ours²⁾. In view of the fact that an esterification namely the formation of a phosphoric ester occurs as the preliminary reaction, it seemed in the first place advisable to consider whether the co-enzyme was the agent, the phosphatase (resp. phosphatase) catalysing this reaction, resp. the hydrolysis of the triose phosphoric ester. Meanwhile this idea has been rejected at once on account of the fact that it is quite unacceptable to attribute the properties of easy dialysability and especially large thermostability to an enzyme such as the phosphatase³⁾.

However, a further consideration of the fermentation scheme leads to the conclusion that still another factor than only the zymase in its restricted sense and the phosphatase is required for the setting in of fermentation. As known the dehydrogenation of the methylglyoxal hydrate occurs in a normal way with acetaldehyde as acceptor whereby meanwhile the acetaldehyde is formed by the reaction ensuing therefrom (the decarboxylation of the pyruvic acid). So this situation implies the presence of a substance necessary for the commencement of fermentation, which

¹⁾ A. J. KLUYVER and H. J. L. DONKER, *These Proceedings* 28, 297. (1925); 28, 605. (1925) and *Chemie der Zelle und Gewebe*, Bd. 13, p. 134, (1926).

²⁾ *These Proceedings* 29, p. 322, (1926).

³⁾ As stated before it was already early shown by HARDEN that the effect of the co-enzyme does not rest exclusively upon the removal of the alkaliphosphates.

substance takes over the function of the at first missing acetaldehyde. This consideration made it possible that the co-enzyme should only act as a hydrogen acceptor and looking at this hypothesis we decided to approach experimentally the co-enzyme problem.

III. Before giving a short review of the result of our own experiments we must point out that some experiments were already made by NEUBERG ¹⁾ as well as by HARDEN ²⁾ the results of which could be interpreted at least partly in the described way. We will not discuss these results any further here, but will only remark that on the one side HARDEN ends his summary given in 1923 ³⁾ with the words: "It seems highly probable that the co-enzyme fulfils a similar function and it will be found like aldehyde, to be a substance capable of reduction in presence of the enzymes of yeast" whilst on the other side MEYERHOF, on the ground of his own experiments firmly denies the possibility of the substitution of the co-enzyme by aldehydes.

When making our own experiments we decided first of all to use the technic originally followed by HARDEN, with the difference, however, that maceration extract was used instead of BUCHNER's yeast-juice. Small quantities thereof were divided into two parts in an ultrafiltration apparatus according to BECHHOLD, after which the material remaining on the filter was washed with a little water. Further it was investigated how far the fermenting power of the inactive residue, which caused strong fermentation after addition of the ultrafiltrate, was also restored by adding small quantities of acetaldehyde accompanied by suitable quantities of alkaliphosphates.

The results of a series of investigations in this direction have been varying: sometimes it has been possible indeed to lead an otherwise wholly negative remaining mixture to strong fermentation by the addition of acetaldehyde and sometimes this addition remained without any effect, although, even then activation by ultrafiltrate occurred. The nature of the yeast used for the preparation of the maceration extract as well as the duration of the filtration and the intensity of washing of the residue seemed to influence the result.

In the meantime it could be said with certainty that the function of the co-enzyme was not only that of an introductory hydrogen acceptor as otherwise substitution by acetaldehyde should have been possible in all cases.

IV. With this series of experiments we had purposely not yet taken into account a possible complication to which MEYERHOF ⁴⁾ drew the attention by his important investigations about the co-enzyme. Starting

¹⁾ C. NEUBERG und E. SCHWENK, *Biochem. Zeitschr.*, Bd. 71, p. 135, (1915).

²⁾ A. HARDEN, *Biochemical Journal*, Vol. 11, p. 64, (1917).

³⁾ A. HARDEN, *Alcoholic Fermentation*, 3rd Ed. London, 1923, p. 72.

⁴⁾ O. MEYERHOF, *Zeitschr. f. physiol. Chemie*, Bd. 101, p. 165, (1918), *Ibid.* 102, p. 1, (1918), *Ibid.* 102, p. 185, (1918).

from the phenomenon of induction shown by certain maceration extracts, discovered by LEBEDEFF¹⁾, MEYERHOF points to the part which this phenomenon can play with experiments about the co-enzyme made with normal maceration extract.

We can summarise MEYERHOF's theories on this point by saying that he shows the necessity of making difference between the apparent inactivity of a zymase preparation — which is a consequence of the induction — and a real inactivity caused by the absence of the co-enzyme.

It is possible to distinguish between these two phenomena because of MEYERHOF's important discovery that the induction contrary to inactivity caused by lack of co-enzyme can be eliminated by addition of a small amount of hexosephosphate. So the result is that no zymase preparation, which does not remain inactive after addition of hexosephosphate, may be considered free from co-enzyme and MEYERHOF points out that to obtain this effect continued washing on an ultrafilter is indispensable.

Entirely in accordance with this it appeared to us from a new series of experiments — whereby we used like MEYERHOF the more simple ultrafiltrating apparatus of ZSIGMONDY — that the zymase preparations inactivated by too little washing, were only apparently inactive, in so far, that the addition of hexosephosphate²⁾ was sufficient to bring about a strong fermenting power. Further we found also that continued washing to a 200 fold dilution of the materials accompanying the zymase — led indeed to a zymase preparation which could be activated by the usual co-enzyme solutions, such as boiled maceration extract, but not by hexosephosphate.

Thus it had now become necessary to repeat the experiments concerning the possibility of activation by acetaldehyde of a zymase preparation really freed from co-enzyme. These experiments, however, gave without exception a negative result, so that the conclusion had to be drawn that except the hexosephosphate, still another substance — the co-enzyme proper — is required to make the zymase capable of producing sugar fermentation.

V. Herewith the way to a closer understanding of the function of the co-enzyme seemed to be blocked up for the present. After ample deliberation it became however alluring to connect the problem with phenomena in a somewhat further off area which — although full attention was given to it by the first zymase investigators — had been left later on with a single exception, without any consideration. We allude to the fact that it was already proved by HAEHN in 1898 that zymase preparations always possess a strong proteolytic power and in connection herewith are also subject to a strong autolysis by which the zymase itself is also destroyed. Hereto were added the later observations of BUCHNER and his

¹⁾ A. LEBEDEFF, *Annales de l'Institut Pasteur*, T. 26, p. 8, (1912).

²⁾ When in this paper is spoken of hexosephosphate always the with natriumoxalate conversed "Candiolin BAYER" or the corresponding "hexosediphosphorsaures Natrium", from the same firm is meant. We want to express here also our best thanks to the named firm, which was so kind to put the mentioned preparations at our disposal.

collaborators, including a.o. HAEHN, who proved that boiled maceration extract possesses an undeniable antiproteolytic action¹⁾). The possibility lying at hand that the co-enzyme function of boiled zymase preparations should have to be reduced to the supposed presence of a so-called "antiprotease", in other words to the zymase protecting action of these juices was meanwhile rejected by the named investigators, a.o. after observation that by continued boiling the co-enzyme function can be destroyed, whilst the "antiprotease" function remains then for the greater part preserved. The same conclusion was drawn by HAEHN and SCHIFFERDECKER²⁾ who published in 1923 a detailed investigation on this point.

Now we specially noted the fact that whilst on the one side MEYERHOF in his former cited investigations does not pay any attention to the studies of BUCHNER and collaborators about the proteolytic function of the zymase preparations, on the other side HAEHN and SCHIFFERDECKER seem to have remained ignorant of the 6 years earlier by MEYERHOF published results and thus had not taken into consideration at all the according to MEYERHOF secondary influence of a possible want of hexosephosphate. This situation seemed to us to imply the possibility that the principle considered by BUCHNER and SCHIFFERDECKER as "co-enzyme" might not be anything else than hexosephosphate which is destroyed in small concentrations by continued boiling whereas the by MEYERHOF as "co-enzyme" (Cofersment) indicated principle should have to be identified with the antiproteolytic factor present in the boiled juices.

Numerous observations in the series of experiments of the different investigators proved to be in perfect accordance with this hypothesis, but for shortness' sake we will not go further into this here.

An experimental test of the hypothesis given above led to a satisfactory result. It appeared to be possible indeed according to HAEHN and SCHIFFERDECKER to free a solution of co-enzyme by seven hours boiling entirely of its original activating power and to regenerate this by simple addition of some hexosephosphate. On the other hand a zymase preparation remaining inactive with hexosephosphate, could be activated by simple addition of the meant "Schutzsaft", so that there stood nothing in the way of identifying MEYERHOF's co-enzyme with BUCHNER's "antiprotease".

In the meantime as long as the nature of the antiproteolytically acting substances was not further ascertained, the above mentioned experiments did not exclude all doubts regarding the correctness of this last contention. It might be possible that besides these substances still another substance, the co-enzyme proper, had also endured without difficulty the long process of boiling, although this seemed improbable since it follows also from

¹⁾ E. BUCHNER und H. HAEHN, *Biochem. Zeitschr.*, Bd. 19, p. 198, (1909); *Ibid.* Bd. 26, p. 171, (1910).

²⁾ H. HAEHN und H. SCHIFFERDECKER, *Biochem. Zeitschr.*, Bd. 138, p. 209, (1923).

THOLIN's¹⁾ experiments that the co-enzyme is practically entirely destroyed by heating to 100° C. during seven hours.

Meanwhile we did our best to consolidate our view concerning the nature of the co-enzyme of MEYERHOF. Therefore it was necessary to form a more precise idea regarding the nature of the antiproteolytic action. It seemed probable to ascribe this action to proteins which might act as substrates of the endotryptase and in this way function to a certain extent as "lightning conductor" for the zymase. However great difficulties were in the way of an experimental test of this hypothesis, as protein extracts from much varying vegetable and animal organs were just tested by MEYERHOF on their activating function with positive result, where from this investigator concluded to the very general occurrence of the "co-enzyme".

Thus it was of importance for us to find a protein containing substance which was beyond any suspicion of containing MEYERHOF's co-enzyme. Such a material was only to be found, in the form co-enzyme, according to MEYERHOF's prescriptions, freed zymase. It seemed probable that a zymase preparation free of co-enzyme and subjected to a moderated autolysis after boiling up, in other words after destroying the zymase and the endotryptase should contain smaller complexes of proteins which in a second portion of zymase preparation free of co-enzyme, would protect the zymase against the endotryptase action.

In fact it now appeared that a similar effect could be observed so that our theory regarding the nature of MEYERHOF's co-enzyme was thus further confirmed.

VI. We may now presume from the above mentioned experiments that the preparations, used by the various investigators of which the activating function on inactivated zymase is indicated by them as "co-enzyme" action, contain, dependent on the different applicated methods of inactivation, different activating principles.

On the one hand inactivation might be a consequence of withdrawal of the introductory hydrogen acceptor, on the other hand of hexosephosphate or in the end also of the antiproteolytic acting substances. Not seldom these factors will act simultaneously and this in dependence of the followed mode of operation, the nature of the used zymase preparation etc. in varying degrees. We are of the opinion that this is very much the case with the extensive investigations of H. VON EULER and his collaborators, not discussed here, but we will not go further into this now.

Thus from all the foregoing the conclusion could be drawn that it would be advisable to distinguish henceforth more co-enzymes, which might be indicated as the co-enzyme of HARDEN (introductory hydrogen acceptor) that of BUCHNER and HAEHN (hexosephosphate) and that of MEYERHOF (antiprotease).

¹⁾ TH. THOLIN, Zeitschr. f. physiol. Chemie, Bd. 115, p. 235, (1921).

We should, however, prefer to strike out the whole conception of co-enzyme from the biochemistry of alcoholic fermentation and replace it by the insight that for fermentation of the hexoses by the usual zymase preparations a certain number of factors must be realized, namely : except the presence of alkaliphosphates, that of an introductory hydrogen acceptor and of hexosephosphate¹⁾, whilst further there must be reckoned with the autolytic properties of the named preparations.

For a further documentation of this preliminary communication we refer to a thesis to be shortly forthcoming from one of us (Str.) as also to an extensive paper to be published elsewhere.

*Laboratory for Microbiology of the
Technical University.*

Delft, April 1927.

¹⁾ In the meantime we have strong evidence that hexosephosphate on the condition that it is present in a somewhat higher concentration, can itself fulfil the function of introductory hydrogen acceptor.

Physics. — *The thermal expansion of silver between +101° C. and —253° C.* By W. H. KEESOM and Miss A. F. J. JANSEN.
(Communication N^o. 185c from the Physical Laboratory at Leiden.)

(Communicated at the meeting of May 28, 1927).

§ 1. *Introduction.* As a continuation of our measurements of the expansion of copper ¹⁾ in the temperature range of +101° C. to —253° C. we now have investigated silver in the same temperature range.

§ 2. *Method and apparatus.* We followed the same method as described in Comm. N^o. 182a. The only change to the apparatus was that the copper inner tube was replaced by a silver one, constructed by the "Zilverfabriek" at Voorschoten. For the silver alloy is given 99.8 %.

The tube has a longitudinal seam, which is soldered with as few silver-solder as possible of a high silver alloy.

The arrangement of tube, standard meter and kathetometers was not changed.

The distance of the platinum points was determined repeatedly at room temperatures. The temperatures of the inner and outer walls were read off with mercury thermometers, graduated in tenths of a degree.

In order to reduce this results to the temperature of 0° C. for both inner and outer walls, for silver we made use of the expansion coefficient found by SCHEEL ²⁾ and for copper of our measurements (l.c.).

Besides those at room temperature, measurements were made at the boiling points of water, ethylene, oxygen and hydrogen.

In order to obtain a temperature of about 100° C., boiling distilled water was poured into the tube, and then a heating coil introduced to keep the water boiling. The temperature of the water at the top was derived from the reading of the barometer; that of the tube at the bottom by taking into account the hydrostatic pressure of the water column; the average being taken.

The temperatures of the ethylene, oxygen, and hydrogen baths were determined by means of a *Pt*-resistance thermometer, which had been calibrated several times with the helium thermometer, and which was placed half-way between the platinum points.

Before the tube was filled with a liquefied gas, it was cooled by means

¹⁾ These Proceedings 29, 786, 1926; Comm. Leiden N^o. 182a.

²⁾ K. SCHEEL, Zs. f. Physik., 5, 167, 1921.

of liquid air. This appeared to be particularly necessary in the case of hydrogen ; in this case it was very difficult to get the tube filled.

§ 3. *Results.* The following results were obtained.

TABLE I.

Date	Temperature		$L_{\text{outer}} - L_{\text{inner}}$
	Inner tube	Outer tube	
December 18 '26	reduced to 0° C.	reduced to 0° C.	2.002 mm.
January 18 '27	" 0° "	" "	2.002 "
" 22 '27	" 0° "	" "	2.004 "
" 24 '27	" 0° "	" "	2.002 "
February 4 '27	" 0° "	" "	1.999 "
" 21 '27	" 0° "	" "	2.000 "
" 24 '27	-182.76° "	" "	4.938 "
March 10 '27	" 0° "	" "	2.022 "
" 11 '27	" 0° "	" "	2.015 "
" 14 '27	" 0° "	" "	2.019 "
" 15 '27	" 0° "	" "	2.023 "
" 17 '27	-102.47° "	" "	3.731 "
" 22 '27	" 0° "	" "	2.017 "
" 30 '27	-252.78° "	" "	5.587 "
<p>After the measurements in hydrogen the little point at the bottom of the inner tube appeared to be somewhat distorted, so that both points of inner- and outer tube could not be focussed at the same time. Therefore the lowest window was taken off, and this point distorted a little, so that now the distance between the points of inner and outer tube is changed. The following results were then obtained.</p>			
April 11 1927	reduced to 0° C.	reduced to 0° C.	5.127 mm.
" 12 1927	" 0° "	" "	5.128 "
" 26 1927	" 0° "	" "	5.128 "
May 2 1927	+101.19° "	" "	3.295 "
" 5 1927	+101.10° "	" "	3.301 "
" 6 1927	" 0° "	" "	5.128 "

From this the following changes of length were derived :

Temperature range	0° C. to +101.10° C.	expansion	1.827 mm.
"	" 0° C. " -102.47° C.	contraction	1.712 mm.
"	-102.47° C. " -182.76° C.	"	1.207 mm.
"	-182.76° C. " -252.78° C.	"	0.649 mm.

Length of the inner tube (from point to point) = 945.30 mm.

The change of length as a function of the temperature is represented in figure 1.

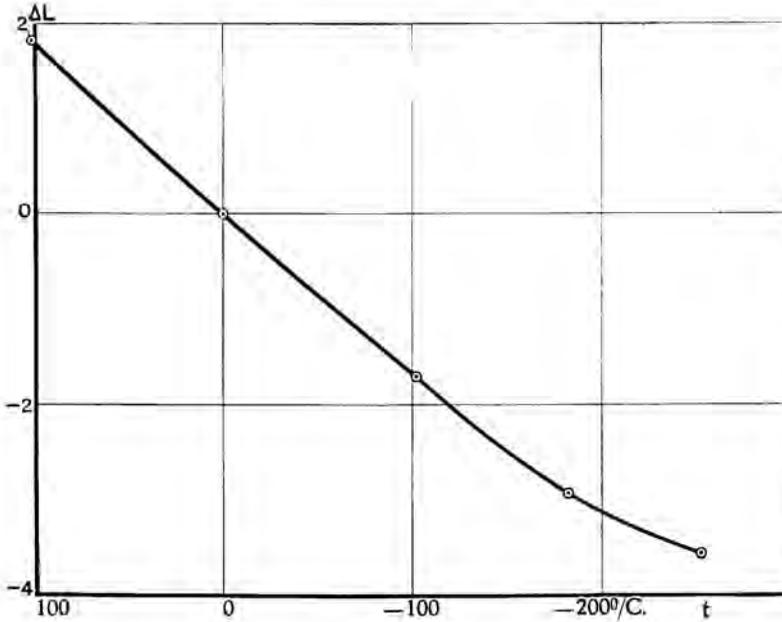


Fig. 1.

§ 4. *The expansion coefficient.* From the results the following mean expansion coefficients were calculated.

Temperature range	$\bar{\alpha} \times 10^6$
+101.10° C. to 0° C.	19.15
0° " " -102.47° "	17.61
-102.47° " " -182.76° "	15.89
-182.76° " " -252.78° "	9.81

In Fig. 2 we derived graphically $\frac{1}{l} \frac{dl}{dt}$ by drawing a smooth curve, so

that, for the temperature ranges mentioned above,

$$\int_{t_1}^{t_2} a dt = \bar{a} (t_2 - t_1).$$

The equality of the areas in question was tested with the aid of a planimeter.

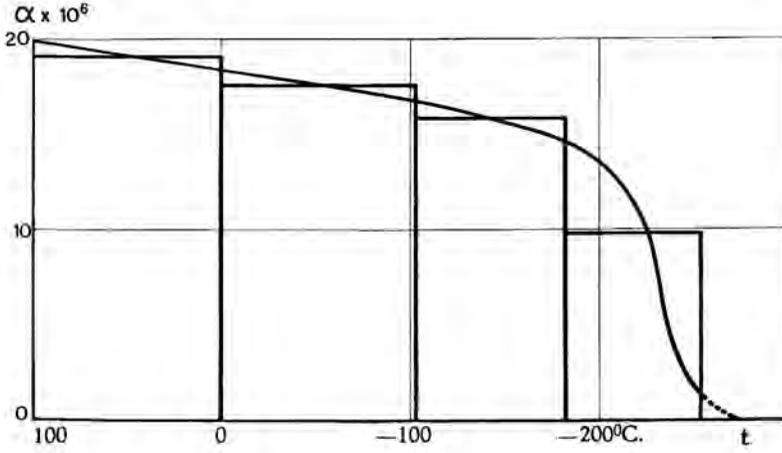


Fig. 2.

§ 5. *Representation in a formula.* In analogy to the formula for the expansion of copper, a fourth degree equation was made of the form :

$$L_t = L_0 \left[1 + \left\{ a \left(\frac{t}{100} \right) + b \left(\frac{t}{100} \right)^2 + c \left(\frac{t}{100} \right)^3 + d \left(\frac{t}{100} \right)^4 \right\} 10^{-6} \right].$$

The coefficients were calculated from the observations ; the following values were obtained :

$$\begin{aligned} a &= 1779.066 \\ b &= 34.641 \\ c &= 59.343 \\ d &= 35.754 \end{aligned}$$

§ 6. *Comparison with previous observations.* Previous investigations on the expansion of silver below 100° C. were made by FIZEAU ¹⁾ in the temperature range from 0° C. to 100° C., AYRES ²⁾ in the temperature range from +99.3 C. to -184° C., SIMPSON ³⁾ in the temperature range from

¹⁾ H. FIZEAU, C.R. 68, 1125, 1869.

²⁾ H. D. AYRES, Phys. Rev., 20, 38, 1905.

³⁾ Cf. J. S. SHEARER, Phys. Rev., 20, 52, 1905.

0° C. to -180° C., DORSEY ¹⁾ in the temperature range from +10° C. to -180° C., HENNING ²⁾ in the temperature range from +16° C. to -191° C., SCHEEL ³⁾ in the temperature range from 0° C. to 500° C., LINDEMANN ⁴⁾ in the temperature range from +19° C. to -253° C.

The measurements of LINDEMANN do not allow a direct comparison as they are relative to fused quartz, of which the expansion coefficient is not known.

For the other observers we have calculated from their results the mean expansion coefficients for the following temperature ranges, and compared these coefficients with ours in table II.

TABLE II.

$\bar{\alpha} \times 10^6$					
temp. range observers	0° C. to +100° C.	0° C. to -100° C.	0° C. to -180° C.	-100° C. to -180° C.	-180° C. to -250° C.
FIZEAU	19.36				
AYRES	18.90	18.80	17.67	16.10	
SIMPSON		18.82	17.78	16.47	
DORSEY		18.43	17.24	15.74	
HENNING			17.05		
SCHEEL	19.68				
KESOM and Miss JANSEN	19.14	17.63	16.89	15.94	10.37

¹⁾ H. G. DORSEY, Phys. Rev., 25, 88, 1907.

²⁾ F. HENNING, Ann. d. Phys., 22, 631, 1907.

³⁾ K. SCHEEL, Zs. f. Phys., 5, 167, 1921.

⁴⁾ CH. L. LINDEMANN, Physik. Zs., 12, 1197, 1911.

Zoology. — *The Nemas Anchylostoma and Necator. II. New researches on the larvae of Anchylostoma duodenale DUB. and Necator americanus STILES.* By J. H. SCHUURMANS STEKHOVEN Jr. (Communicated by Prof. W. SCHUEFFNER).

(Communicated at the meeting of April 30, 1927).

Sooner than expected, thanks to the kindness of Prof. W. SCHUEFFNER, Director of the Institute for Tropical Hygiene at Amsterdam, I was afforded another opportunity to continue my studies on hookwormlarvae. I am glad to acknowledge here with thanks Prof. SCHUEFFNER for his valuable help, which facilitated my work very much.

On the 26th of February he mailed me 3 samples of stools, produced by Javanese, working on board of one of the ships of the Steamship Cie "Nederland". All stools contained a good quantity of hookwormeggs: moreover eggs of *Ascaris* and *Trichocephalus* were present.

On the 28th of the same month, that is, immediately after the receipt of the samples, I prepared my cultures by mixing a small quantity of the stools with sterilised soil. These I placed in an incubator at 27° C. After the appearance of the first larvae I daily made a series of observations on living hookwormlarvae. For that purpose I isolated them in a drop of water on a slide, covered the drop with a circular coverglass and sealed the whole with wax of a taper, making thus a temporary mount. In the case the larvae made vehement movements, I warmed the slide cautiously above a flame, till the larvae came into a condition of "Wärmestarre".

With this research I aimed: firstly to give a correct, practical diagnose of the third stage, infectious larva of *Anchylostoma duodenale*, and secondly to complete my observations upon the first stage and second stage larvae of *Necator americanus* and possibly also of *A. duodenale*.

Since 90% of all hookworms, which Javanese harbour, belong to the Genus *Necator*, I could be quite sure to encounter larvae of the latter Genus. So I found indeed larvae of *Necator* in all cases, whereas one of the amples only produced a sufficient quantity of larvae of *A. duodenale*.

1. *The third stage larva of A. duodenale DUB.*

"We could divide the hookwormlarvae, cultivated from human stools—where eventually both species of hookwormlarvae may be present, although the chance is greater, at least at Weltevreden, that we have to do with larvae of *Necator americanus* — into two groups. In the larvae, belonging to the first group, the hooks at the distal end of the

cylinder run parallel (Fig. 7); the same structures enclose a vase-shaped cavity in the other group of larvae (Fig. 9). In the majority of the cases these rods with their hooklets did run parallel, as is also true for larvae, derived from pure cultures of *Necator americanus*. Almost all larvae, cultivated from canine stools presented cylinders with diverging hooklets. No parallelity was found", wrote my wife and I as a result of our researches at Weltevreden in 1921 (published in 1923).

At that moment we were not far from the final solution of the problem of diagnosis of hookwormlarvae. Yet the collected data were not sufficient to draw decisive conclusions from our observations. My studies in Washington allowed me to fill out the gaps.

If the word "almost" is dashed out from the cited sentence: "almost all larvae etc.", it is clear, that the observations, which we both made in 1921 were quite correct and that the cited alinea points to one of the most important differences, which exists between the larvae of the Genera *Anchylostoma* and *Necator*.

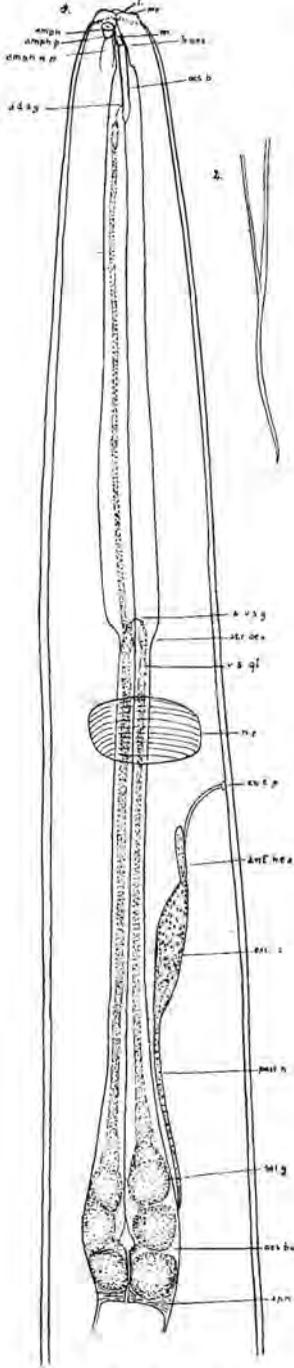
Our expression "gespreide haakjes", "diverging hooklets" was misunderstood by VAN THIEL. If he had confronted the latter expression with the indication "sikkelvormige haakjes", "falciform hooklets", which we used on page 11 of the mentioned paper for *Anchylostoma* larvae, he would not have misunderstood what we meant with the word "gespreid".

If the figures, accompanying this paper, are compared minutely with those, printed in the cited paper of my wife and me, it will be quite clear, that the median sides of the rods with their hooks run parallel in the larvae of *Necator*, whereas in all *Anchylostoma* species, known thus far, the hooklets or apical clasps diverge.

The head of the third stage larva (Textfig. 1; Pl. I) possesses, just the same as that of *A. caninum* 3 lips (*l.*), behind which the bodywall has been constricted slightly (Pl. I, Fig. 2 c). LOOSS speaks about "a simple ring-shaped lip which is marked off from the rest of the body by a slight annular constriction not always clearly visible (Pag. 349)", when he treats the first stage larva of the same species. PERRONCITO and ALESSANDRINI however observed apparently the lips, since the former speaks of a "testa triloba".

These lips are, just as in *A. caninum*, more transparent than the next plasmatic part of the head. Therefore VAN THIEL's microphotogram 2 (1924) depicts them as a clear space. LOOSS gives in his monograph (Pl. XIV, Fig. 204) a schematical figure of the head of *A. duodenale* seen on top; here 6 papillae encircle the oral opening, whereas the circumference of the head is broken at two opposite places by a couple of slanting lines. Possibly the latter indicate the amphids, although LOOSS denies the existence of cervical organs opposite to PERRONCITO. If this supposition holds true, a controversy arises between LOOSS's observations and mine, since the former found 6 papillae, whereas I found 4 papillae only, when the head was seen on top. Lack of time forced me to

postpone the making of headcuttings of larvae of *A duodenale*. I think however I may safely assume that both species of *Anchylostoma* larvae have the same number (4-) of labial papillae. In profile both subdorsal papillae present one papilla only.



Textfigure 1.

It is interesting to note, that LOOSS, although he denied the existence of the various organs at the head of *A. duodenale*, apparently has seen them. Pl. XVI, Fig. 205, 207 and probably also Fig. 202 depicts the amphids together with the nerve fibres, which run from here in caudal direction. Confer also Fig. 199. STEINER came to the same conclusion. The amphids, their pipe-shaped amphidial pouches, which contain the threadlike terminals, as also the widened parts, which come next, are likewise to be found on VAN THIEL's microphotogram 2 (1924) of a larva of *A. caninum*, although he does not mention these organs in his text.

In *A. duodenale* the amphids (*amph.*) have a similar structure. Their aperture (*amph. o.*) is elliptical and wide; just behind the constriction of the head they communicate with the exterior. The amphidial pouch (*amph. p.*) is rather wide and contains 2 terminals (Pl. I, Fig. 5); the next part of this sense-organ, separated from the former by a constriction, widens somewhat more (*amph. n. p.*) and stretches along the esophagus till it meets with the nerve ring (*n. r.*). Here again it is

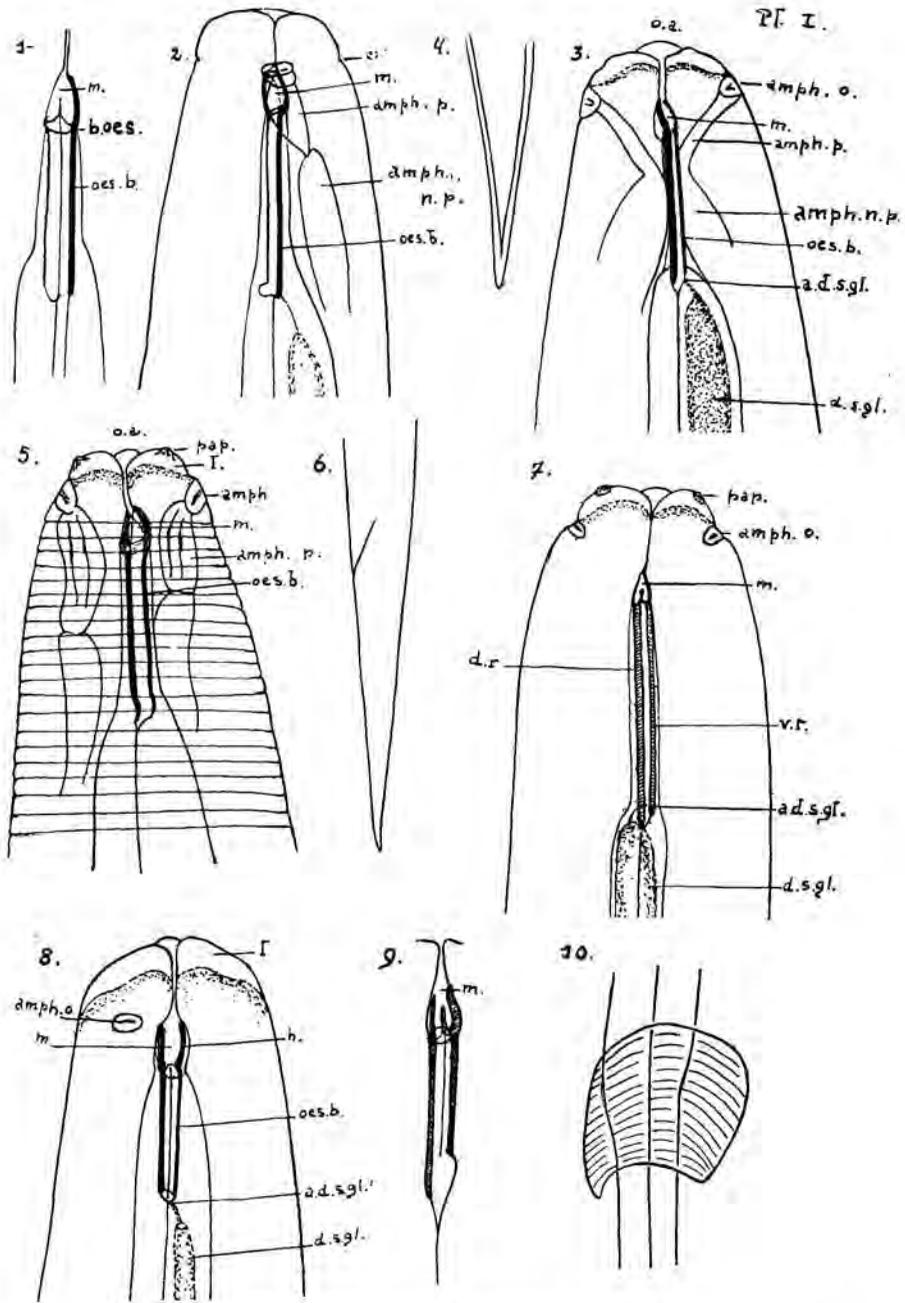
Textfigure 1.

Anchylostoma duodenale, third stage larva.

Fig. 1. The headend, Enlarg. Ö Imm. 1.25 mm., Oc. 12, Tubus 140 mm. $\times 2/5$.

Fig. 2. Tailend of third stage larva, Enlarg. Ö Imm. 1.25 mm., Oc. 1, Tubus 140 mm. $\times 2/5$.

a. d. s. gl. = aperture of dorsal salivary glands; *ant. h.* = anterior horns of excretory apparatus; *a. v. s. g.* = aperture of subventral salivary glands; *amph.* = amphid; *amph. p.* = amphidial pouch; *b. oes.* = beginning of esophagus; *exc. c.* = excretory cell; *exc. c. p.* = porus of excretory cell; *l.* = lip; *m.* = oral cavity; *n. r.* = nerve ring; *oes. b.* = esophageal rods; *oes. bu.* = bulbus esophagi; *pap.* = papilla; *post. h.* = posterior horns of excretory apparatus; *sal. gl.* = salivary glands; *sph.* = sphincter; *str. oes.* = constriction of esophagus; *v. s. gl.* = ventral salivary glands.



Anchylostoma duodenale, third stage larva.

Fig. 1. Esophagus cylinder, Enlarg. Ö Imm.

1.25 mm., Oc. 12, Tubus 140 mm. $\times \frac{2}{3}$.

Fig. 2. Head, profile, Enlarg. as in 1.

Fig. 3. Head, dorsolateral, as in 1.

Fig. 4. Tail of third stage larva, as in 1.

Fig. 5. Head, dorsal view, Enlarg. Ö Imm.

1.25 mm., Oc. 18, Tubus 140 mm. $\times \frac{2}{3}$.

Fig. 6. Tail, third stage larva, Enlarg. Ö Imm.

1.25 mm., Oc. 1, Tubus 140 mm.

$\times \frac{2}{3}$.

Fig. 7. Head, dorsal, as in 1.

Fig. 8. Head, lateral, as in 1.

Fig. 9. Esophagus cylinder, as in 1.

Fig. 10. Nerve ring, as in 1.

a. d. s. gl. = aperture of dorsal salivary gland; *amph. o.* = amphidial opening; *amph. p.* = amphidial pouch; *amph. n. p.* = amphidial nerve part; *b. oes.* = beginning of esophagus; *c.* = constriction; *d. s. gl.* = dorsal salivary gland; *d. r.* = dorsal rod; *h.* = hooklet; *l.* = lip; *m.* = oral cavity; *o. a.* = oral aperture; *oes. b.* = rod shaped thickenings of the esophagus cylinder; *pap.* = papilla; *v. r.* = ventral rod.

obvious that the amphidial pouches are placed at right angles to each other; the proximal part of the amphids however runs parallel with the intestine (Compare also *A. caninum* l. c. Pl. I, Fig. 2 and 3). Now it is clear that the shape and structure of the amphids in *Necator* deviates distinctly from the aforementioned scheme.

The oral cavity (*m.*) and the distal portion of the esophagus cylinder distinctly differ from what I found in *A. caninum*. The entrance of the oral cavity is short and cylindrical, but soon it widens and assumes the shape of a vase. A circular aperture at the basal end of the vase opens into a cylindrical tube, which is lined with parallel running rods (*oes. b.*). The dorsal rod (*d. r.*) is thicker than the ventral one (Pl. I, Fig. 7 *v. r.*) which moreover is slightly shorter than the former, whereas the third rod (Confer *A. caninum*) is reduced to a fine cuticularised line. Both dorsal and ventral rods are prolonged at their proximal ends into curved clasps (Pl. I, Fig. 8, *h.*) which double the curvature of the vase; each clasp, which is almost as strong as the hooklets in *Necator*, is connected with the fine cuticularised third line, whereas fine lines connect the top of each clasp with the base of its antagonist (Compare Pl. I, Fig. 1, 2, 3, 5, 7, 9). Where rod and clasp connect, the cavity is bordered by a ring-shaped thickening of its wall.

It is of importance I think to mention the fact that both clasps are not placed in the same plane, but are found in planes which stand at right angles to each other (Pl. I, Fig. 3).

At its caudal end the cuticularised esophageal cylinder, (the rods ought to be considered as thickenings of the wall of the latter) is closed by an irregular ring.

In rare instances, but in dead larvae only, the clasps protrude from the oral opening. SVENSSON and KESSEL pointed to the circumstance that in *A. caninum* "the difference in density of the dorsal and ventral lines is less pronounced than in *A. duodenale*", which observation I could affirm. Their Figure 8 may serve to prove the aforementioned fact, that rods and hooks show more resemblance with the same structures in *Necator* than with those in *A. caninum*. In this figure one of the hooks is depicted as a dotted line, which proves that these authors have observed the difference in situation of the clasps I described above.

The esophagus is attached to the ring, found at the base of the oral cavity (Pl. I, Fig. 1). Its outer wall runs at first parallel with the described rods, but afterwards it widens just behind the proximal ends of these cuticularisations. At this point (*a. d. s. gl.*). (Compare also LOOSS: Fig. 196, 197a, 202, 205) the excreta of the dorsal salivary gland are poured out into the lumen of the esophagus.

The dorsal salivary gland (*d. s. gl.*) is voluminous and may be traced easily along its whole course. The subventral salivary glands (*v. s. gl.*) are likewise very obvious. At the proximal end of the anterior portion of the esophagus, just in front of the nervering, these glands open into

the esophageal lumen with a wide ampulla (Textfig. 1, *a. v. s. gl.*), just as in *Necator*. Therefore the salivary glands of *A. duodenale* show a greater resemblance to the corresponding organs of *Necator* than to the same of *A. caninum*. This resemblance becomes still greater if we compare the glandular portion of the bulbus esophagi of *A. duodenale* with that of *Necator*. PERRONCITO gives a clear picture of the same (Fig. 24). This figure answers better to the reality than the figures of LOOSS, in which the glandular parts are depicted as clear vacuoles. As in *Necator*, *A. duodenale* presents lobular, strongly constricted salivary glands (Textfig. I, *s. gl.*). These constrictions divide the bulbus esophagi into 3 distinct pale rings. I never observed in *A. duodenale* a vacuolised middle portion as was found in *A. caninum*. The narrowed part of the esophagus gradually widens to the bulbus esophagi, just as in *A. caninum*. The sphincter (*sph.*) is in shape and in the mode of junction with the esophagus quite similar to the same structure in *A. caninum*. The only difference with the latter consists in the fact that here this part is more granular and therefore does contrast less with the bulbus. The lumen of the bulbus widens towards the sphincter (Textfig. 1, *sph.*); at this level the wall of the lumen is thickened at two opposite places. It is not always easy to observe this fact, since the strongly granular glands prohibit to see it through. Similar, although less pronounced thickenings I found in *A. caninum*. It may be thought possible that they do represent the remainder of a valvular apparatus, such as is found in first and second stage larvae, but I doubt, that it still functions as such.

LOOSS describes it on page 359: "The valves of the bulb are not actually reduced but only elongated and flattened; at the same time they loose their glistening chitinous covering, so that very close inspection is needed in order to recognize them in their altered shape".

The excretory apparatus of *A. duodenale* is not so distinctly visible as in *A. caninum*, but shape and structure are identical. The middle portion of the excretory cell (*exc. c.*) is swollen, the posterior horns (Textfig. 1, *post. h.*) are very slender, whereas the anterior horns (*ant. h. e. a.*) are short and wide.

The tail of the third stage larva (Pl. I, Fig. 4, 6) tapers, just as in *A. caninum*. The tail of the sheath is very long, finely pointed (Textfig. 1, 2) and resembles therefore the same part in *Necator*; so LOOSS describes the tail of the third stage larva correctly as strongly shortened in comparison with the second stage larva. The skin is finely ringed (Pl. I, Fig. 5); the distance between two rings is 0.9μ . Finally some dimensions may be mentioned. Now it appears that *Anchylostoma* larvae are somewhat longer on the average than the corresponding larvae of *Necator*. If we have to do with a mixture of both larvae, the *Anchylostoma* larvae are generally the longer ones. Again I compared the data collected by my wife and me with the dimensions of newly measured larvae. As our knowledge of the factors controlling growth is

quite insufficient, it is impossible to draw any definite conclusions from these data. Here again it was clear that *Necator* is somewhat shorter on the average than *Anchylostoma*, but at the other hand both curves overlap, which fact lessens the value of similar data.

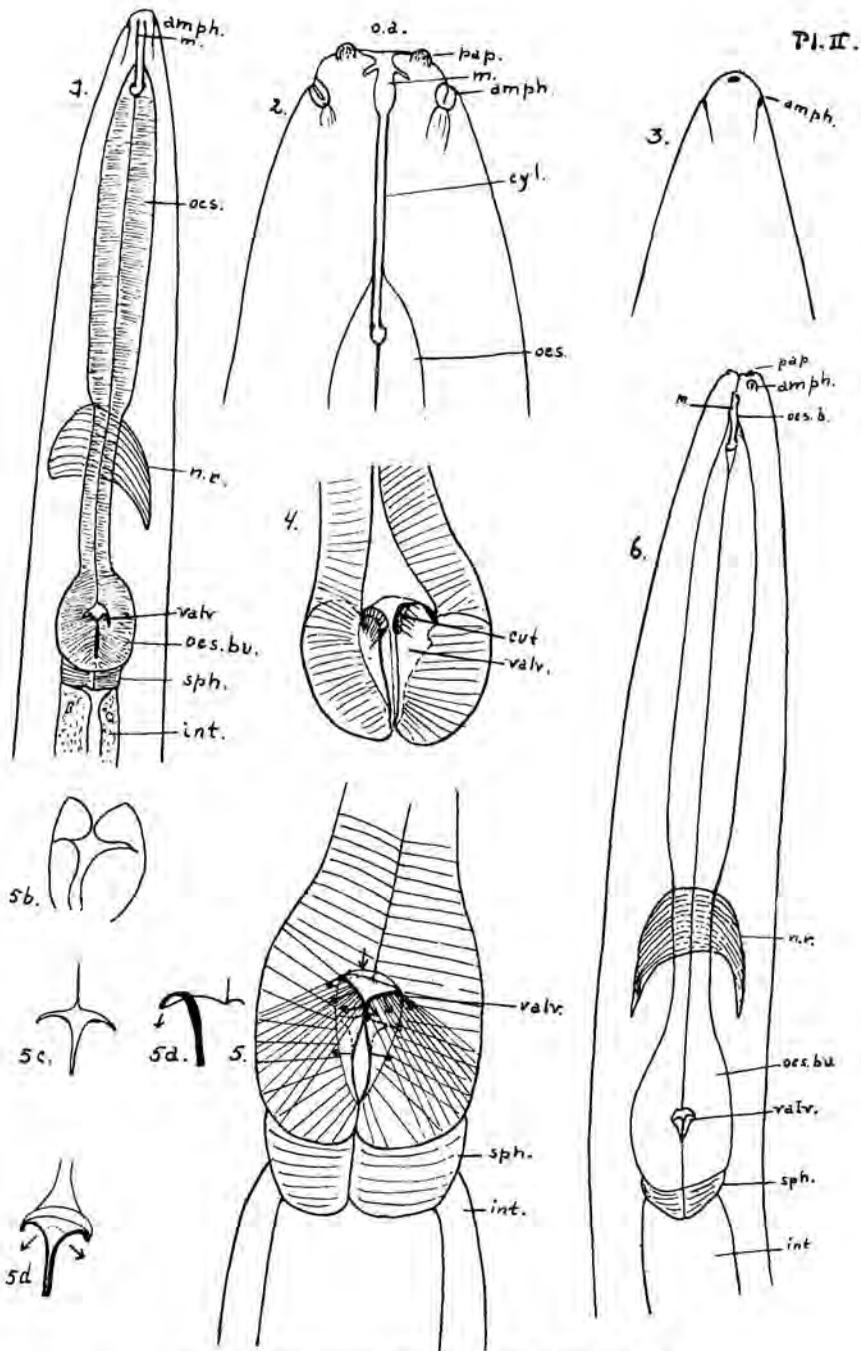
Dimensions of larvae	A. duodenale			Necator americanus		
	Age	3rd stage	2nd stage	Age	3rd stage	2nd stage
Total length Utrecht	11 d.	608—627 μ	—	1 d.	—	—
.. .. Batavia	5—8 d.	384—624 μ	—	5—8 d.	399—642 μ	—
.. .. Utrecht	1 d.	—	665—684 μ	1 d.	—	532—634 μ
.. .. Batavia	5—8 d.	—	471—654 μ	5—8 d.	—	462—666 μ
Head to end esoph. Batavia and Utrecht	5—11 d.	135—165 μ	—	5—11 d.	102—177 μ	—
Tail = anus till point, Batavia	5—8 d.	36—75 μ	—	5—8 d.	38—90 μ	—
.. Utrecht	11 d.	62—78 μ	—	—	—	—
.. ..	11 d.	—	138 μ	11 d.	—	—
.. Batavia	5—8 d.	—	69—141 μ	5—8 d.	—	81—150 μ

In length of tail and sheath both species agree almost wholly; comparatively the tail of *A. duodenale* is even slightly longer than that of *Necator americanus*, a fact predicted by VAN THIEL.

As a summary it is clear that the third stage larvae of *A. duodenale* have many points in common with *A. caninum*, but differ i.a. in the density of the crescent-shaped clasps, in structure and volumen of the salivary glands and their efferent ducts and in the length of the tail of the second stage larva (sheath). In these features the third stage larvae of *A. duodenale* resemble the corresponding larvae of *Necator americanus*.

2. The first stage larva of *Necator americanus*.

Under the present circumstances it was impossible for me to make pure cultures, starting from uteruseggs of each of the species of nemas, which I wished to diagnose. Since I was able to identify the larvae of my cultures after they had reached the infectious stage and one of my cultures yielded third stage larvae of *Necator americanus* only, I could be almost sure that the younger larvae, found in the same culture, ought to be reckoned to this species. Now I will describe here the



Necator americanus, first and second stage larva.

- Fig. 1. *Necator*, first larva, Enlarg. Ö1 Imm. 1.25 mm., Oc. 6, Tubus 140 mm. $\times \frac{2}{3}$.
- Fig. 2. Head of same larva, Enlarg. Ö1 Imm. 1.25 mm. Oc. 18, Tubus 140 mm. $\times \frac{2}{3}$.
- Fig. 3. First stage larva, empty sheath.
- Fig. 4. Second stage larva, bulbus esophagi Enlarg., Ö1 Imm. 1.25 mm., Oc. 12, Tubus 140 mm. $\times \frac{2}{3}$.
- Fig. 5. First stage larva, bulbus esophagi, same animal as Fig. 1, Enlarg. as in Fig. 2.
- Fig. 5a, b, c, d, different stages of the valvulae esophagi, while functioning.
- Fig. 6. Second stage larva, head portion of a larva, Enl. as in Fig. 1.

amph. = amphid; *cyl.* = esophaguscylinder; *cut.* = cuticularisation; *int.* = intestine; *m.* = oral cavity; *n.r.* = nerve ring; *o.a.* = oral aperture; *oes.* = esophagus; *oes. b.* = beginning of esophagus; *oes. bu.* = bulbus esophagi; *pap.* = papilla; *sph.* = sphincter; *valv.* = valvula. The arrows in Fig. 5 indicate the direction of the movements; the dotted lines show the shape of the lumen, on the moment the valvulae are open, see also Fig. 5b.

younger larvae of *Necator* only, although I made some observations upon the younger larvae of *A. duodenale*. I will however postpone the publication of the data I collected upon the younger larvae of this *Anchylostoma*-species, till I have had another opportunity to study them more minutely. As a matter of fact my notices upon the younger larvae of *Necator* are also more or less fragmentary, since the first and second stage larva accomplishes its development in a few days. Four or five days after the preparing of the cultures ensheathed third stage larvae usually are present.

The first stage larva of *Necator americanus*, 285 μ long is in different respects less developed than the third stage larva; esophageal glands are not yet visible and no excretory apparatus has to be found, although LOOSS mentions a group of cells, which forms the matrix of this organ. From a diagnostical point of view the head and its organs are the most important. The anterior head end (Pl. II, Fig. 1, 2) is rounded; no lips are present. Amphids (*amph.*) are to be found and the labial papillae (*pap.*) are especially conspicuous when the larva is observed under sufficient magnification. The more or less cylindrical oral cavity (*m.*) is short and opens immediately into the esophageal cylinder (*cyl.*). The cuticularisation of its wall presents no special differentiations but is wholly smooth. Proximally this cylinder ends with a ring. Here the parallel-sided anterior portion of the esophagus sets off. Where one finds the nerve ring the esophagus is constricted over a remarkable distance, afterwards it swells rather suddenly to the bulbus esophagi (*oes. bu.*). In this way the esophagus is divided into 3 well separated regions (Pl. II, Fig. 1). These divisions fade away during growth and after the ecdyses, but never disappear wholly. The sphincter (*sph.*) situated between intestine and esophagus, does not protrude so wart-like into the lumen of the latter as this is the case in the third stage larva.

The intestine (*int.*) possesses a comparatively thick wall, whereas its lumen is narrow and sinuous, till it opens, just in front of the long tapering tail (Pl. III, Fig. 1), through the anus (*a.a.*). The rectum can be closed by the sphincter ani (*sph. a.*), which has a fine boring. The peristaltic movements, which the intestine showed during digestion were easily to follow; the lumen of the intestine alternately widened and narrowed and small particles were expelled by the anus.

It is necessary to pay more attention to the valvular apparatus, situated in the bulbus esophagi (*valv.* Pl. II, Fig. 1). The apical wall of the lumen bulbi is not strengthened by cuticularisations and resembles a mediastine (Pl. II, Fig. 5 *a—d*); it vaults the caudal part, which has a triangular optical section, although in profile two parts only are visible. The proper valvulae form the distal portion of the latter part; their upper borders are strongly thickened at their median angles (Pl. II, Fig. 4, 5, 5a); the walls bordering the lumen in caudal direction are likewise strongly cuticularised (Pl. II, Fig. 1, 4). In optical longitudinal section this part

gets therefore the shape of an Y. At the corners, where the branches of the Y pass into the longitudinal leg, the thickenings are strongest. From here and from the slanting branches muscles radiate (Pl. II, Fig. 4, 5). Their function is to open the valvulae and by doing this to widen the lumen of the esophagus which gets then its sucking power.

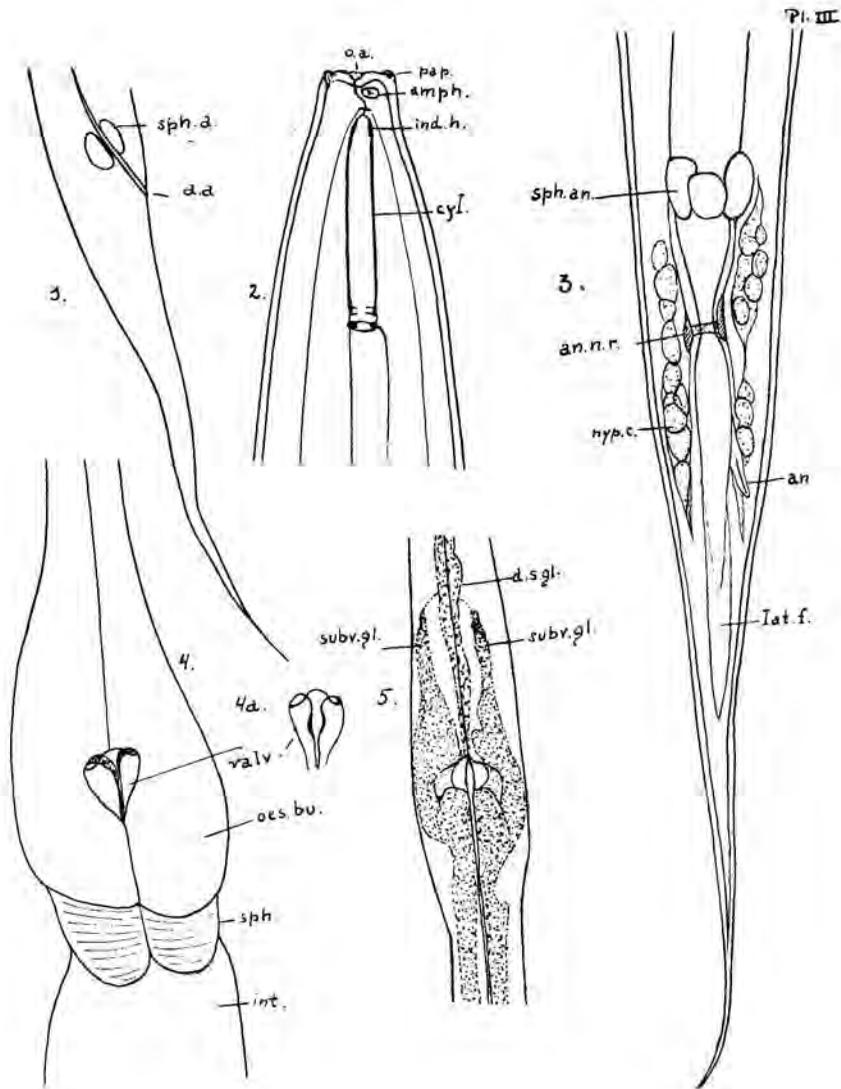
During the ingestion of food particles, the latter are driven forward by means of peristalsis. Then they reach the small space, just in front of the Y which I will call the atrium bulbi. Now the muscles which radiate from the points, where the atrium vault articulates with the Y, contract. I imagine that things happen in this way that the inflow of the fluid in the atrium affords the stimulus to which the muscles answer by contraction. Now the vault of the atrium flattens and functions therefore as the sucker of a pump, which drives the fluid forward in caudal direction and throws it against the valves. The described radiating muscles serve two purposes. The point where the vault articulates with the branches of the Y is torn backwards and at the same time do the branches of the Y turn round these points as fixed axes in outward direction, whereas the leg of the Y placed in the longitudinal axis of the body remains almost in the same position.

Since the thickenings of the wall are not evenly distributed and therefore the elasticity is not the same at all points, a complicate vase-shaped figure is formed (Pl. II, Fig. 5b). As soon as the contraction of the muscles ceases and the vault regains its former position, the valvulae come to rest again, thanks to the elasticity of the cuticularisations (*cut.*); now the upper borders of the thickenings overlap. After the valvulae have closed off the atrium from the rest of the bulbus, the food materials are driven forward through the caudal portion of the bulbus by peristalsis.

Several authors described the valvular apparatus of the bulbus esophagi, most of them for *A. duodenale*, but I think they have not emphasized enough the complexity of its structure.

PERRONCITO mentions "le bulbe pharyngien, muni de dents chitineuses, renfermant dans sa partie centrale un espace triangulaire". LOOSS is in opposition to PERRONCITO, which asserts that the bulbus is provided with teeth. He also disputes with LEICHTENSTERN who considers the bulbus to be "kein Zahn-oder Kau-Apparat, sondern eine Pump- oder Ansaug-Vorrichtung". LOOSS thinks that the "teeth represent the primordia of the future esophageal valves of the adult worm and constitute an apparatus for preventing regurgitation. In order to secure an inflow of fresh nourishment through the mouth during such an extension, the opposite end of the oesophagus must be closed, as will be understood without further explanation. The teeth under discussion provide for this closure and thus represent valves. They are closed in a state of rest and become opened by the contraction of the muscle fibres proceeding from them, while when the muscles relax, they spring back into this former position".

The description I gave makes it sufficient clear, I think, that I join



Necator americanus, first, second and third stage larva.

Fig. 1. Tail of first stage larva, same animal as Pl. II, Fig. 1, Enlarg. same.

Fig. 2. Head of second stage larva. Enlarg. ÖI Imm. 1.25 mm. Oc. 12, Tubus 140 mm. $\times \frac{2}{3}$.

Fig. 3. Tail of second stage larva of *A. duodenale* of *N. americanus*? Enlarg. ÖI Imm. 1,25 mm. Oc. 6, Tubus 140 mm. $\times \frac{2}{3}$.

Fig. 4. Bulbus esophagi second stage larva *Necator*, Enlarg. as 2.

Fig. 4a. Valvulae bulbi esophagi.

Fig. 5. Third stage larva *Necator*, salivary glands. Enlarg. as in 2.

a. a. = anal aperture; an. n. r. = anal nerve ring; an. = anus; amph. = amphid; cyl. = esophageal cylinder; d. s. gl. = dorsal salivary glands; hyp. c. = hypodermis cells; ind. h. = indications of clasps; int. = intestine; lat. f. = lateral cords; m. = oral cavity; oes. bu. = bulbus oesophagi; sph. = sphincter; sph. a. or sph. an. = sphincter ani; subv. gl. = subventral salivary glands; valv. = valvulae.

LEICHTENSTERN's opinion, since the atrial vault (Pl. II, Fig. 5*b-d*) prohibits a regurgitation. Now I will abstain from a discussion over the value which the valvular apparatuses of free living and parasitic nemas have in respect with phylogenetical speculations. I think however that the loss of the valvular apparatus during the lifecycle or even the loss of its function speaks in favour of a secondary reduction, probably in relation with parasitism and yields another argument for STEINER's opinion that parasitic nemas are not primitive forms.

3. *The second stage larva of Necator americanus.*

The second stage larva resembles in many respects the first stage larva; from the third stage larva it may be distinguished at once by its much longer tail. The head is more *Necator*-like than the head of the first stage larva of the same species; its anterior end is flattened in the middle (Pl. III, Fig. 2; Pl. III, Fig. 6); lips are absent, the papillae (*pap.*) are to be seen as glistening warts; the amphids (*amph.*) are situated next to the oral opening; oral cavity short, cup-shaped. The covering of the esophageal cylinder (*cyl.*) has undergone slight changes only; thickenings appear as indications of the later hooklets (Pl. III, Fig. 2, *ind. h.*), which resemble in this species the pollensacs of grasses; at the proximal end of the cylinder there is a couple of rings, whereas further on, at the beginning of the caudal portion of the esophagus 2 strongly refractive points are very obvious. The ring at the proximal end of the cylinder seems to be thickened remarkably. The indications of the hooks do not protrude into the oral cavity as is the case in third stage larvae of *Anchylostoma* (Compare LOOSS, page 349). The bulbus esophagi (*oes. bu.*) contains the valvulae (*valv.*) I described for the first stage larva. Pl. III, Fig. 4 shows beautifully how the mentioned muscles radiate from the thickenings of the Y-branches. Salivary glands are faintly visible just now; the bulbus esophagi presents three regions as traces of the later lobes. The nerve ring (Pl. II, Fig. 6 *n. r.*) moved nearer to the bulbus esophagi than in the first stage larva. The excretory apparatus is present but does not yet function. The sphincter (*sph.*) protrudes wartlike into the lumen of the intestine (*int.*).

Finally some fragmentary data may be added. Pl. III, Fig. 3 shows an empty skin, which represents I think the first ecdysis; the amphids, of which the apical portion apparently has been stripped off, are plainly visible.

Further Pl. III, Fig. 3 depicts the tail of a second stage larva probably of *Necator americanus*, but as this larva was isolated from a culture, which contained a mixture of larvae of both human species of hookworm larvae, I could not make out this point. The cited figure depicts the sphincter ani (*sph. an.*), the anal nerve ring (*an. n. r.*), the anus (*an.*) and the cords of hypodermis cells (*hyp. c.*) which divide the cylindrical

body into fields. The bulbus esophagi of the same animal (Pl. III, Fig. 4, 4a) contained valvulae; the cuticularised esophageal cilinder, situated next to the oral cavity, had the same shape as in the aforementioned larvae. Bulbar valvulae I found in all larvae younger than the third stage and so they must be also present in both first and second stage larvae of *A duodenale*.

My description of the third stage larva of *Necator americanus* needs a correction. There I spoke of a rudimentary anterior bulbus and depicted in Textfig. A of my former paper on hookworm larvae a special muscularisation at this point. In the material which served as a base for the present paper I could not find again a similar muscularisation; the lumen of the esophagus was slightly widened only at the level of the ampulla of the subventral salivary glands (Compare Pl. III, Fig. 5).

Utrecht, March 28th, 1927.

POSTSCRIPTUM.

Professor SCHUEFFNER drew my attention to a lecture of VAN THIEL, delivered by the latter author on an assembling of the "Vereeniging voor Tropische Geneeskunde" on March 13th, during which VAN THIEL declared that his communications on the larvae of *A. caninum* were in reality observations on *Uncinaria stenocephala* larvae. In the mean time I examined again the worms, expelled from the same dog, whose stool yielded a large quantity of *A. caninum* larvae when it served for culture purposes, and again I found specimens of *A. caninum* only.

May 24th, 1927.

REFERENCES.

- SCHUURMANS STEKHOVEN, J. H. Jr. en SCHUURMANS STEKHOVEN—MEYER, A. W., Voorbereidende onderzoekingen met betrekking tot de differentiatie van de larven van de mijnwormen van mensch en hond, Tijdschr. v. Vergelijk. Geneesk., Dl. 10, p. 1, 1923.
See for other papers also:
SCHUURMANS STEKHOVEN, J. H. Jr. The Nemas Anchylostoma and Necator I, These Proceedings Vol. 30, No. 1, p. 113, 1927.
STEINER, G. Some Nemas from the alimentary tract of the Carolina Tree Frog (*Hyla-carolinensis* PENNANT) with a discussion on some general problems of Nematology, Journal of Parasitology Vol. 11, p. 1—32, 1924.
STEINER, G. The problem of Host selection and Host specialisation of certain plant-infesting Nemas and its application in the study of nemic pests, Phytopathology, Vol. 15, p. 499—534, 1925.

Histology. — *Concerning an Optic Phenomenon in Tendons. (Part. III).*
by Prof. G. C. HERINGA M. D. and M. MINNAERT SC. D. (Com-
municated by Prof. J. BOEKE).

(Communicated at the meeting of April 30, 1927).

In two preceding numbers of these Proceedings ¹⁾ we have pointed to polarized interference-phenomena, produced in thin longitudinal sections of a cow's tendon. We have demonstrated that these phenomena can be completely explained by assuming that the fibres of the tendon behave like long, quasi-uniaxial crystals, and by conceiving them as not running in a straight line but 1^o, either in the shape of sine curves in a plane, or 2^o, in the shape of helices. Various arguments have been brought forward for adopting the latter hypothesis. We are now in a position to furnish conclusive evidence for this hypothesis: we found namely, *that the cow's tendon rotates the polarized light excessively*. This can only be explained on the basis of the helix-hypothesis. It will appear that the amount of rotation can be computed beforehand, and that the order of magnitude is confirmed by the observation. In order to be able to observe the rotation, we had to look in the direction of the fibres of the tendon, so that we had to make transverse sections. These were taken as thick as possible in order to get a strong rotation; however, the drab colour and the inevitable irregularities inside the tendon did not allow of greater thickness than 3 mm. The preparations were derived from the toe-tensor of the cow. They were cut unfixed by means of the freezing microtome as smoothly as possible, and embedded in levulose-gelatin. It is owing to this means of embedding that the preparations of 3 mm. thickness were still sufficiently translucent.

For our observations we used a simple polarization apparatus of NÖRREMBERG with the parallel light of a mercury lamp, from which the blue and the violet radiations had been removed by means of a filter of potassium-bichromate; the active wavelength was computed as 560 $\mu\mu$.

The tendon-preparations were placed over the opening of a diaphragm, which shut off all the side-light, and they were viewed between crossed polarizers. We shall now first describe the phenomena, which were exhibited by our finest preparation, and which approximated most closely to what we anticipated theoretically.

This tendon-preparation appeared to be extremely dextro-rotatory; the levulose itself, at the side of the preparation, yielded a feebler rota-

¹⁾ These Proceedings 35, p. 939.

tion in the opposite sense. The angle between the two adjustments amounted to about 30° (on an average for two observers), which we shall consider as the particular rotation of the tendon-preparation, on the assumption, therefore, that the levulose permeated the whole tendon, and that its rotation is superadded to the rotation pertaining to the tendon (should this not be the case then the rotation would be somewhat smaller than has been adopted here). The adjustment is somewhat uncertain, the section of the tendon does not get completely dark with any position of the analyzer, and its surface is of course not evenly lucid.

We are now going to show that a bundle of uniaxial crystals, wound along a steep helix, and viewed in the direction of the axis of the helix, must, indeed, display optic rotation. This will be readily understood if we conceive such a preparation as a pile of accumulated disks, so to speak extremely thin transverse sections. In each of these *strongly birefringent* disks the optic axis is *slightly inclined from the vertical* on the plane of section, so that the strong double refraction causes for the observer only small phase-differences; the azimuth of that optic axis alters from one disk to the other and rotates 360° over a distance along the tendon, corresponding with one turn of the helical line.

To simplify our reasoning we imagine this model I substituted by a model II: let the *double refraction be small* in each disk, but the optic axis *be parallel to the plane of section*. The optic axes of the successive disks have successively rotated over a small angle, and we have here an analogue to the well-known experiments of REUSCH with piled up micaplates¹⁾. Just as in REUSCH's experiments we have also here to expect optical rotation. The theory of this phenomenon is furnished by SOHNCKE and MALLARD, A extensive review is to be found in MASCART's Manual²⁾. When the total thickness E of the layers is built up of an infinite number of infinitesimally thin plates, disposed in helical succession with a speed e , the optic rotation R will be :

$$R = \frac{\pi E e}{2 \lambda^2} (n'' - n')^2,$$

in which formula λ represents the wavelength of the light, $n'' - n'$ the double refraction in each of the layers. When applying this formula to the case of the helical lines, and going from the vicarious model II back again to model I, we must bear in mind that in case I we have a stronger double refraction $n_w - n_z$, but that our angle of vision with the optic axis is small, so that that double refraction induces only slight phase-differences.

The ordinary wave, and the extraordinary wave, propagating in an

¹⁾ MULLER—POUILLET, II, 1001.

²⁾ M. E. MASCART, *Traité d'Optique*, II, 324, 1891.

uniaxial crystal at a small angle φ with the optic axis, have according to FRESNEL velocities:

$$V' = V_{\omega} = a.$$

$$\text{and } V'' = \sqrt{a^2 \cos^2 \varphi + c^2 \sin^2 \varphi} = a + \frac{c^2 - a^2}{2a} \varphi^2.$$

From this it follows that the difference of the refractive indices

$$n' - n'' = V \left(\frac{1}{V'} - \frac{1}{V''} \right) = n \varphi^2 \frac{c - a}{a} = (n_{\omega} - n_{\epsilon}) \varphi^2.$$

If the helix is seen in projection, it will present the aspect of a sine curve $y = a \sin \frac{2\pi x}{E}$, with $\operatorname{tg} \varphi = \frac{dy}{dx} = \frac{2\pi a}{E} = \pi k$, k being the relation of the double amplitude to the period of undulation, which may be measured under the microscope.

For steep helices φ is small, and

$$n' - n'' = \pi^2 k^2 (n_{\omega} - n_{\epsilon}).$$

Finally we find for the optic rotation caused by uniaxial crystals, wound in the shape of helical lines:

$$R = 180^{\circ} \times \frac{\pi^4 k^4 E e}{2 \lambda^2} (n_{\omega} - n_{\epsilon})^2.$$

In our case $E = 3 \text{ mm.}$, $e = 0.04 \text{ mm.}$, $\lambda = 0.00056 \text{ mm.}$

Previous measurements of the double refraction in longitudinal sections have already given for our $n_{\omega} - n_{\epsilon}$ values of the order of 0.002. Most difficult of estimation is the magnitude k which varies considerably from the one point of the preparation to the other, and of which an estimation was made in our first communication (Meeting Sept. 25, 1926) that corresponds with $\frac{0.5}{\pi} = 0.16$ in the present notation. When calculating the value to be assigned to k in order to explain the value observed for the rotation R , we find 0.12.

This value is of quite the same order of magnitude as the one previously estimated by us. This is the more satisfactory as the determination of k is attended with some difficulties because of its marked variability. We can say that the amount of rotation has, indeed, been estimated theoretically and confirmed experimentally. That we preferred to calculate a value for k from the observed R , rather than the reverse, is accounted for by the fact that k occurs in our formula in the fourth power, so that every error in the determination of k would tell excessively on the result for R . The high exponent of k explains at the same time why the measured rotation turns out to be larger than the one obtained by calculation; in the preparation different values of k occur, of which the observer chooses an average; for the optic rotation, however, the measure is determined by the average value of k^4 , and that value will be greater.

We have also tried to compare the sense of the optic rotation with the sense in which the helix is wound. It may be estimated by a careful observation of the fibres and by tracing them back by means of the screw of the micrometer. The two directions of the rotation seem to be opposed to each other, which would, indeed, conform with the results of REUSCH's experiments, and with what can easily be explained theoretically.

Besides this fine, well-measurable preparation we have examined others with which the measurement of the optic rotation appeared to be well-nigh impossible. This need not surprise us. With the slightest difference in the values of k (which occurs in most preparations) every part of the tendon will cause a different rotation, and there will be nowhere any total extinction. That the examined preparations, however, really consist of optically strongly rotating parts, becomes evident from the fact *that they are bright between crossed nicols, and that they remain so with any rotation of the preparation or the analyzer.*

It still remains for us to say, that we have tried to find the optic phenomena observed by us in tendons, also in striated muscle-fibres.

a. Longitudinal sections of muscles display magnificent diffraction spectra fairly agreeing in dispersion with those of ROWLAND's gratings.

b. But these spectra do not show polarization.

c. In transverse sections no optic rotation was distinguishable. The field remained dark.

From the facts under *b* and *c*, which fairly agree *inter se*, it might be concluded that fundamentally the way in which the diffraction spectra (*a*) arise in muscles, differs from that in tendons.

*From the Histological Laboratory of Amsterdam
and the Physical Laboratory of Utrecht.*

September 1926.

Physics. — *Rigidity of Super-conductive Metals.* By Prof. W. J. DE HAAS and M. KINOSHITA. (Communication N^o. 187*b* from the Physical Laboratory at Leyden).

(Communicated at the meeting of May 28, 1927).

§ 1. Very little is known of the physical properties of super-conducting metals in their super-conducting states. The object of the following experiments was to investigate whether there is any change, sudden or otherwise, in the material affecting its elastic constants in transition from non-super-conductive to super-conductive state. At the same time, it was thought instructive to see how the constants varied at very low temperatures.

§ 2. *Method.* For the sake of simplicity and ease of experiment, the rigidity — or the torsion modulus — of wires of different metals was measured.

Two metals were chosen namely tin and mercury, both undergoing the change of state into super-conductivity at conveniently attainable temperatures.

When the measurement is made by a static method of twisting a wire by a given torque, any change in the angle so twisted may be caused either by a change in the rigidity of the metal itself or by that in its density affecting the radius, or by a combined effect of both. As the coefficient of thermal expansion of metals at very low temperatures is yet to be found, it was not possible to eliminate this effect from the results obtained. Numerical figures of the rigidity given later are, therefore, without this temperature correction.

As a matter of comparison, the rigidity thus defined was measured within a reasonably wide range of temperatures; in the case of tin, from below the super-conductive state to room temperatures; in the case of mercury, to the temperature of liquid air.

A piece of wire under test was placed vertically at the bottom of a cryostat. It was twisted through a long piece of glass tube and a short length of thin phosphor bronze wire, the latter being well out of and above the cryostat so that it was at a room temperature. Angles of twist of both wires were measured simultaneously by reading through a telescope a scale reflected by two mirrors attached to each end of the wires. With a suitable constructive arrangement, these mirrors were brought close together and so adjusted that two images of the scale could be seen just one above the other.

At a scale distance of 10 metres the accuracy of readings was $1/20$ mm. As the angle of twist of a wire by a given torque is inversely proportional to the fourth power of its radius, if only the volume of the metal changed, a change of the order of 5×10^{-7} cm in the radius of a wire of 0.5 mm. diameter could be detected with a scale deflection of 50 cm. ¹⁾

Let φ_p , α_p , r_p , l_p , φ , α , r and l be the rigidity, the angle of twist, the radius and the length of the phosphor bronze wire and the test wire respectively. If D is the torque applied,

$$D = \frac{H}{2} \varphi_p \alpha_p r_p^4 \frac{1}{l_p} = \frac{\pi}{2} \varphi \alpha r^4 \frac{1}{l}.$$

Denote

$$C = \varphi_p r_p^4 \frac{1}{l_p}.$$

C can then be considered as a torsion constant of the phosphor bronze wire, the numerical value of which is found by the oscillation method with a relation

$$C = 2\pi \frac{K}{t^2}$$

where K is the moment of inertia, t the period of oscillation.

In some of the experiments, a spiral spring of phosphor bronze wire was used in place of the straight wire. The torsion constant C of such a spring can also be found in a similar way.

The rigidity φ of the wire is then given by

$$\varphi = C \frac{l}{r^4} \cdot \frac{\alpha_p}{\alpha}.$$

When the temperature correction to r is not considered, the accuracy of the measurement of the variation of φ rests on that of the scale readings corresponding to α_p and α .

By taking a mean of number of readings in twisting the wire to the right and to the left at one temperature for an angle of 100 mm scale reading both sides of the zero point, it was possible to estimate the relative value of φ to five figures.

During the whole series of experiments, great care was taken not to

¹⁾ If n is the scale reading, we have

$$\frac{(r + \delta r)^4}{r^4} = \frac{n}{n + \delta n} \quad \text{or} \quad \delta r = \frac{r \delta n}{4n}$$

For $n = 50$ cm. $\delta n = 1/20$ mm and $r = 1/4$ mm,

$$\delta r = \frac{1}{40} \cdot \frac{1}{200} \cdot \frac{1}{4} \cdot \frac{1}{50} = 6 \cdot 10^{-7} \text{ cm.}$$

exceed the elastic limit of torsion so that the original condition of the wires might be preserved. The amount of twist was kept within $10''$ per diameter length at liquid helium and liquid hydrogen temperatures. Above liquid air temperatures it did not exceed $5''$. There was practically no creep in bringing back the zero point.

Apart from the questions of elastic limit and creep, however, there was a kind of hysteresis effect in raising and lowering the temperature.

When two or more different cooling agents were used to cover a wide range of temperature, it was found necessary, therefore, to vary it only in one direction. When, for instance, the range was required from liquid helium to room temperatures, using liquid hydrogen and liquid air as intermediate media, liquid hydrogen was introduced into the cryostat as soon as liquid helium was evaporated out of it. In this procedure hydrogen snow was generally observed inside the cryostat which showed that the temperature did not rise above that of liquid hydrogen. In similar manner the temperature was raised from liquid hydrogen to liquid air and then to room air temperatures.

Determination of the absolute value of the rigidity was not within the scope of the present experiments. Although its numerical values are given in some of the results, they have little bearing on their true values except that they show the order of magnitude at different temperatures.

§ 3. Results.

1. Tin.

Extruded wires of KAHLBAUM tin of 0.53 mm. diameter were used.

As an arbitrary fixed point on the temperature scale, 0°C . was taken and numerical values of the rigidity measured at other temperatures were divided by those at 0°C . to give a clear view on the variation.

To obtain the value at 0°C . ($\varphi_{0^{\circ}\text{C}}$), an independent experiment was made by using alcohol as a cooling medium. From the result shown in Table 1, it was found by interpolation

$$\frac{\varphi_t}{\varphi_{0^{\circ}\text{C}}} = \left(\frac{\alpha_p}{\alpha}\right)_t \bigg/ \left(\frac{\alpha_p}{\alpha}\right)_{0^{\circ}\text{C}} = 1 - 0.0043 t$$

where φ_t is the value obtained at $t^{\circ}\text{C}$. This gave $\varphi_{0^{\circ}\text{C}}$ with a sufficient accuracy between -10°C . and $+20^{\circ}\text{C}$.

The state of super-conductivity is rendered non-super-conductive by an action of a magnetic field. Examinations were repeated several times with different pieces of the wire at the super-conductive temperatures to see if a magnetic field affected the scale readings. The whole cryostat was encased in a long solenoid which produced a reasonably uniform field of 21 Gauss per Ampère in its centre where the tin wire was coaxially placed. Different strengths of the field between 40 and 350 Gauss

TABLE 1. Tin wire n^o. 1. Diameter 0.53 mm. Length 120 mm.

Θ	α_p/α
32.0C.	0.455
18.0	0.476
7.0	0.500
- 5.0	0.526
-17.0	0.555

which were more than sufficient to destroy the super-conductivity, were tried at all temperatures above and below the threshold value without any effect which shows that the rigidity remained apparently undisturbed in the transition from non-super-conductive to super-conductive state and vice versa.

In view of the threshold value of super-conductive temperature of tin changing under a tensile stress¹⁾, it was also tried to find if a similar effect is obtained by twisting. For this purpose, a special arrangement was made to measure the electrical resistance of the wire. An electric current of 250 m. Amp. was allowed to pass through the whole system of suspension from the phosphor bronze wire to the tin wire. The resistance was then measured by the potential difference between the terminals of the latter. A potential lead from the upper end of the tin wire was taken through a mercury cup to avoid a torque. At the critical temperatures, at which the resistance was disappearing, the wire was twisted and untwisted, right and left, up to 3.5' per diameter length which was far beyond its elastic limit. The maximum shearing stress calculated on the surface of the wire was 1.3 Kgm per sq. mm.²⁾ No deviation of the resistance was, however, detected. It has to be taken in mind that the shearing stress is not a homogeneous force as the tensile stress.

The results of the measurements are given in Tables 2, 3 and 4, and Figs. 1, 2 and 3.

Incidentally, it can be remarked that the rigidity varies little, if not increasing, from the temperatures of liquid hydrogen to those at which the metal is super-conductive.

There was a marked difference in the value taken at the intermediate temperatures of liquid air in raising and lowering the temperature as seen in Fig. 2c and Fig. 3c. Similar effect was also noticeable at liquid hydrogen temperatures. At a given temperature, the metal appeared to be softer when the temperature was rising which is just the reverse of the usual hysteresis phenomena.

¹⁾ G. J. SIZOO and H. KAMERLINGH ONNES, Leiden, Comm. 180b, p. 16.

²⁾ The tensile stress used by SIZOO was about 2.5 Kgm. per sq. mm.

There seems to be a change in the rate of variation between 200° Abs. and 250° Abs. This is analogous to YOUNG's modulus of metals in which a similar change occurs at about half the absolute temperature of fusion ¹⁾).

TABLE 2. Tin wire n^o. 2. Diameter 0.53 mm. Length 44.4 mm.

Order of observation	Date	Bath	T	α_p/α	$\varphi/\varphi_{0^\circ\text{C.}}$
1	Nov. 11	Liquid Ethylene	169.5	0.672	1.238
2	"		160	0.677	1.246
3	"		150	0.682	1.256
4	"		141	0.694	1.278
5	"		154.5	0.677	1.246
6	"		170	0.672	1.238
7*)	Nov. 13	Room air	287	0.510	0.940
8	Nov. 25	Liquid methyl chloride	190.5	0.649	1.189
9	"		214	0.625	1.145
10	"		231	0.619	1.134
11	"		237	0.609	1.115
12	"		245	0.597	1.094
13	"		249	0.585	1.071
14	"	Water	237	0.602	1.102
15	"		216.5	0.625	1.145
16**)	Nov. 26		285	0.521	0.955
17	"	Room air	290	0.506	0.927

*) From this observation
 $\varphi_{0^\circ\text{C.}} = (\alpha_p/\alpha)_{0^\circ\text{C.}} = 0.543.$

***) From this observation: $\varphi_{0^\circ\text{C.}} = 0.546.$

2. Single crystal tin.

The wire used was made in the laboratory and measured 121 mm. in length, 0.94 mm. in average diameter. The orientation of the crystal faces was 001, 010 and 100 planes making angles of 94°, 122° and 32° respectively to the axis of the wire.

The result is shown in Table 5 and Fig. 4.

¹⁾ J. P. ANDREWS, Phil. Mag., 50, p. 668, 1925.

TABLE 3. Tin wire n^o. 3. Diameter 0.533 mm. Length 111.2 mm.
Phosphor bronze spring torsion const. $C = 19800$.

Order of observation	Date	T	α_p/α	$\varphi/\varphi_{0^\circ\text{C}}$	φ in dyne/cm ²
1	May 21	4.21 K.	0.550	1.368	$2.40 \cdot 10^{11}$
2	"	3.41	0.550	1.368	2.40
3	"	3.79	0.548	1.363	2.40
4	"	3.92	0.549	1.366	2.40
5	"	2.76	0.551	1.371	2.41
6	"	20.4	0.549	1.366	2.40
7	"	14.3	0.551	1.370	2.40
8	"	20.4	0.548	1.363	2.40
9	"	79	0.519	1.291	2.27
10 *)	May 22	287	0.378	0.940	1.65
11	"	82	0.531	1.320	2.32

*) From this observation: $\varphi_{0^\circ\text{C}} = (\alpha_p/\alpha)_{0^\circ\text{C}} = 0.402$.

TABLE 4. Tin wire n^o. 4. Diameter 0.533 mm. Length 85.0 mm.
Phosphor bronze spring torsion const. $C = 4620$.

Order of observation	T	α_p/α	$\varphi/\varphi_{0^\circ\text{C}}$	φ in dyne/cm ²
1	4.21 K.	3.2268	1.3529	$2.51 \cdot 10^{11}$
2	3.90	3.2273	1.3532	2.51
3	3.70	3.2278	1.3534	2.51
4	3.20	3.2273	1.3532	2.51
5	2.34	3.2278	1.3534	2.51
6	20.4	3.2268	1.3529	2.51
7	14.2	3.2373?	1.3573?	2.52?
8	20.3	3.2213	1.3505	2.51
9	73	3.040	1.2746	2.37
10 *)	288.5	2.220	0.9308	1.73
11	80	3.122	1.3090	2.43
12 *)	290	2.218	0.9300	1.73

*) From these two observations $\varphi_{0^\circ\text{C}} = (\alpha_p/\alpha)_{0^\circ\text{C}} = 2.385$.

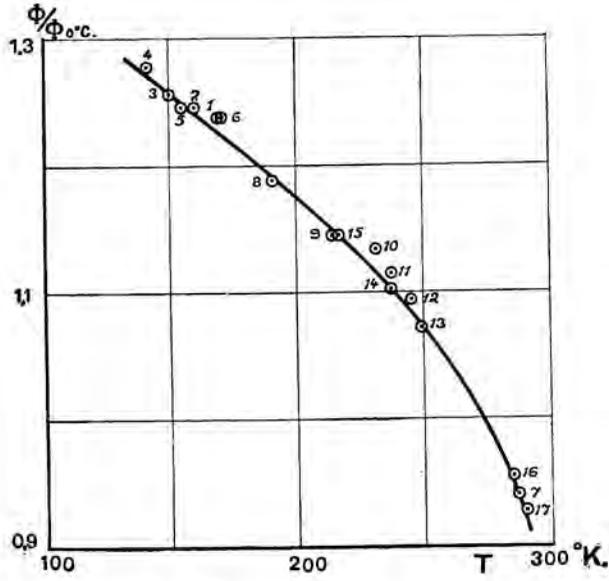


Fig. 1. Tin n⁰. 2.

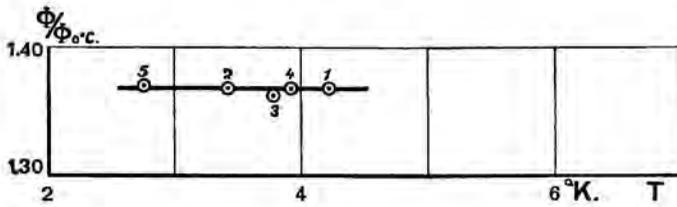


Fig. 2a. Tin n⁰. 3.

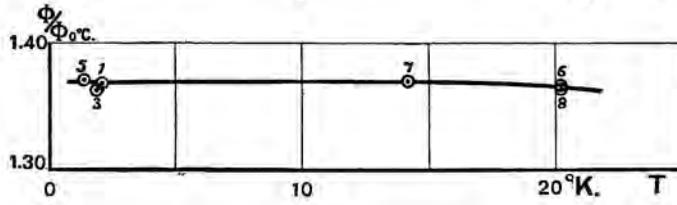


Fig. 2b. Tin n⁰. 3.

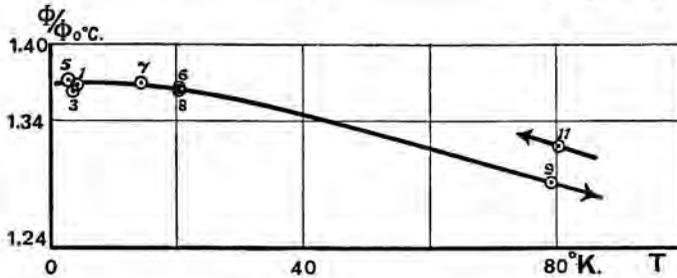


Fig. 2c. Tin n⁰. 3.
Point 11 from the chamber temperat.
Point 9 to the chamber temperat.

Fig. 2.

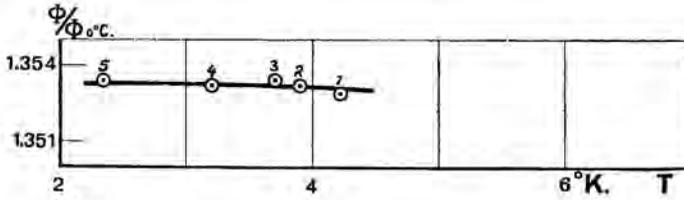


Fig. 3a. Tin n°. 4.

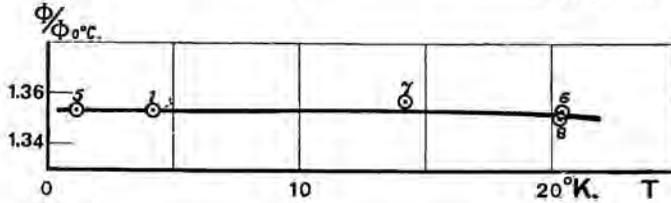


Fig. 3b. Tin n°. 4.

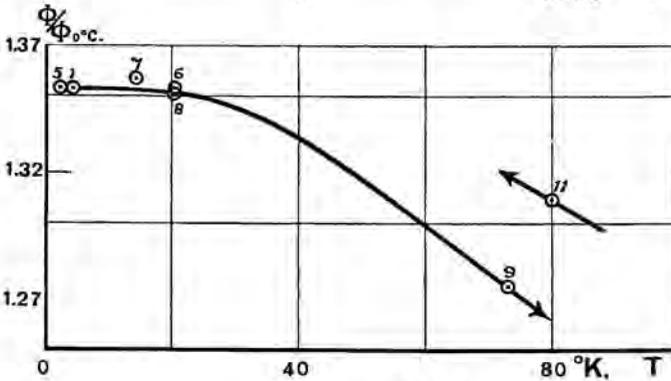


Fig. 3c. Tin n°. 4.
Point 11 from the chamber temperat.
Point 9 to the chamber temperat.

Fig. 3.

On account of the cross section of the wire being more or less irregular, only the relative values are given.

The general feature of the variation was very much the same as that of ordinary tin. There was no change in the rigidity in transition from non-super-conductive to super-conductive state.

3. Mercury.

A thin thread of mercury in a paraffin cast was dipped slowly into liquid air. The whole mass of paraffin was then broken away thus leaving a straight piece of mercury wire. Due probably to the process of cooling, a wire so made is apparently of single crystal judging by the way it stretched when pulled. It showed practically no sign of creep at low temperatures and remained very soft even at those of liquid helium.

The result of the experiments, which is given in Tables 6 and 7 and Figs. 5 and 6, shows that the rigidity increases steadily as the temperature drops, even in its super-conductive state¹⁾.

¹⁾ See W. J. DE HAAS, G. J. SIZOO and H. KAMERLINGH ONNES: Leiden Communication 180d, pp. 58, 69 and 75.

At super-conductive temperatures, if rendered non-super-conductive, the electrical resistance of mercury decreases steadily as the temperature is lowered.

TABLE 5. Single crystal tin wire. Diameter (average), 0.94 mm. Length 121 mm.

Order of observation	T	x_p/x	$\varphi/\varphi_{16^\circ\text{C.}}$
1	4.2 K.	5.1324?	1.2197?
2	3.9	5.1304	1.2194
3	3.7	5.1292	1.2189
4	3.3	5.1298	1.2191
5	2.7	5.1282	1.2187
6	2.2	5.1296	1.2190
7	3.0	5.1300	1.2191
8	4.2	5.1286	1.2188
9	20.4	5.1240	1.2177
10	14.4	5.1268	1.2183
11	20.4	5.1208	1.2169
12	77.0	5.0043	1.1893
13 *)	289.5	4.198	} 1.00
14 *)	288.5	4.218	
15	83.0	5.0040	1.1892

*) Interpolated from these two observations (at 16°C.)
 $\varphi_{16^\circ\text{C.}} = (x_p/x)_{16^\circ\text{C.}} = 4.208.$

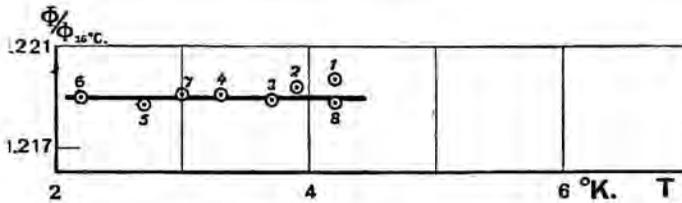


Fig. 4a. A crystal tin wire.

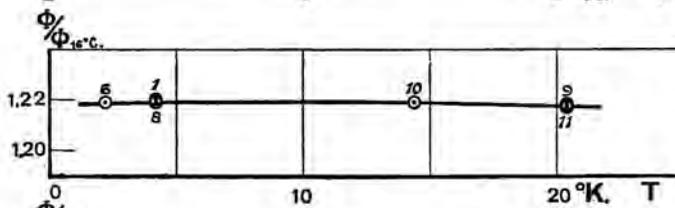


Fig. 4b. A crystal tin wire.

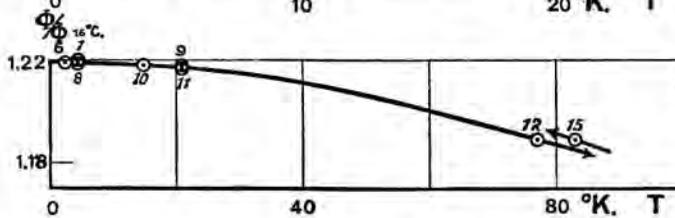


Fig. 4c. A crystal tin wire.

Point 15 from the chamber temperat.
 Point 12 to the chamber temperat.

Fig. 4.

Again there was no apparent effect of the super-conductive state on the rigidity.

§ 4. *Discussions.* The rigidity of tin and mercury remained apparently undisturbed by the change of state into super-conductivity. It is improbable

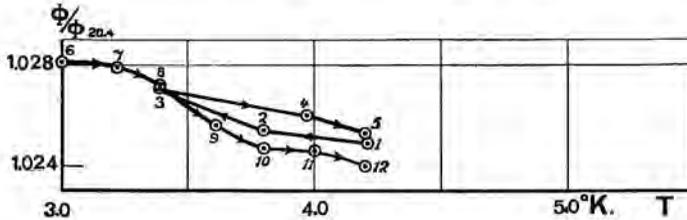


Fig. 5a.
Mercury $n^0. 1.$

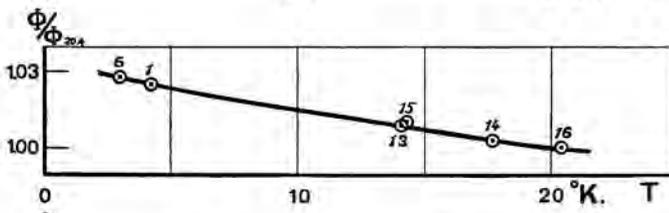


Fig. 5b.
Mercury $n^0. 1.$

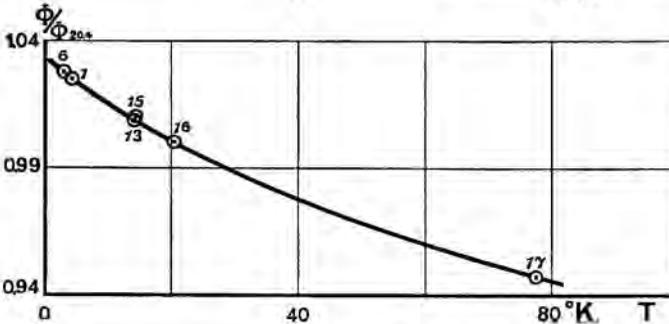


Fig. 5c.
Mercury $n^0. 1.$

Fig. 5.

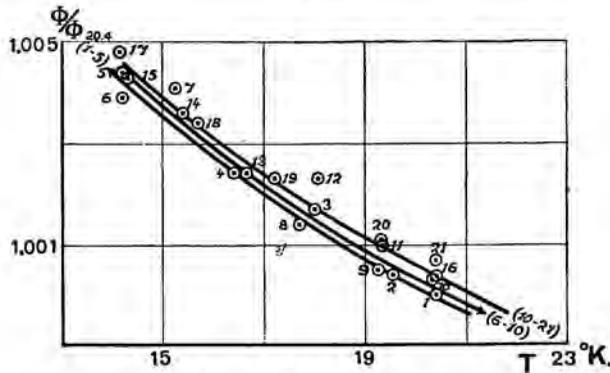


Fig. 6. Mercury $n^0. 2.$

that there occurred some such change in the volume of the metals in the transition that it just counterpoised a change in the rigidity thus leaving its apparent value — uncorrected to volume expansion — unchanged

within the accuracy of the measurement. Until more is known about the volume contraction at these low temperatures, no definite statement can be made. At least it can be said that by the arrangement with which a change of the order of 5×10^{-7} cm. in the radius could be detected, no

TABLE 6. Mercury wire n^o. 1. Diameter ca. 0.85 mm. Length 107 mm.

Order of observation	T	α_p/α	$\varphi/\varphi_{20^{\circ}.4 \text{ K.}}$
1	4.21 K.	3.2407	1.0249
2	3.80	3.2422	1.0254
3	3.39	3.2476	1.0271
4	3.97	3.2444	1.0260
5	4.20	3.2421	1.0253
6	3.00	3.2508	1.0281
7	3.22	3.2503	1.0279
8	3.39	3.2480	1.0272
9	3.61	3.2430	1.0256
10	3.80	3.2399	1.0247
11	4.00	3.2398	1.0246
12	4.20	3.2378	1.0240
13	14.1	3.1888	1.0085
14	17.7	3.1716	1.0031
15	14.3	3.1945	1.0103
16	20.4	3.1620	1.0000
17	77.5	2.944	0.9469

positive result was obtained. This accuracy is about the same as that of the method of optical interference.

It is also inconceivable that a change in the rigidity was counteracted by a change of volume caused by the magnetic field which was applied to destroy the state of super-conductivity (magnetostriction).

The effect of a tensile stress is to render a super-conductive metal more favourable to super-conductivity ¹⁾. The resistance begins to disappear at a higher temperature. By twisting a wire, the shearing stress and also the tensile stress are the greatest on the surface. If the mechanism

¹⁾ G. J. SZOO and H. KAMERLINGH ONNES, Leiden Comm. 180b.

of a super-conductor lies only on its thin surface layer, a twist should be expected to affect the threshold temperature more easily than a pull. Up to the amount of torque applied, by which the elastic limit of torsion was exceeded by far, no change could be noticed.

TABLE 7. Mercury wire n^o. 2. Diameter ca 0.85 mm. Length 107.5 mm.

Order of observation	T	α_p/α	$\varphi/\varphi_{20^{\circ}.4 \text{ K.}}$
1	20.40 K.	2.3920	1.0000
2	19.53	2.3930	1.0004
3	18.00	2.3962	1.0017
4	16.41	2.3978	1.0024
5	14.18	2.4025	1.0044
6	14.22	2.4014	1.0039
7	15.26	2.4019	1.0041
8	17.71	2.3952	1.0014
9	19.25	2.3932	1.0005
10	20.33	2.3927	1.0003
11	19.31	2.3946	1.0010
12	18.07	2.3975	1.0023
13	16.64	2.3978	1.0024
14	15.40	2.4006	1.0036
15	14.30	2.4023	1.0043
16	20.39	2.3930	1.0004
17	14.14	2.4036	1.0048
18	15.69	2.4000	1.0034
19	17.19	2.3975	1.0023
20	19.28	2.3947	1.0011
21	20.39	2.3937	1.0007

Similar results are expected from the other super-conductive metals.

It is likely that other elastic constants will also remain unchanged in transition from non-super-conductive to super-conductive state if measured within the same order of accuracy as obtained in the present experiment.

The authors wish to express their sincere thanks to Mr. J. VOOGD and Dr. H. R. WOLTJER for their valuable assistance.

Chemistry. — *Equilibria in systems, in which phases, separated by a semi-permeable membrane XXII.* By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of May 28, 1927).

Influence of the temperature on osmotic systems.

In the communications XVIII—XXI we have discussed the influence of a change of pressure on osmotic equilibria and on the osmotic pressure; now we shall discuss briefly the influence of a change of temperature.

In communication XVIII we have a.o. deduced the following. If we have a system $E(P)$ viz. a system (of one or more phases) under the pressure P and we bring this, while the temperature and the total composition remain constant, under the pressure $P + dP$, then arises a new system $E'(P + dP)$ the phases of which can differ a little in composition from those of the first system. If we represent the *O.W.A.* and the total thermodynamical potential of the first system by ξ and Z , then we have found:

$$\xi = - \frac{\partial Z}{\partial w} \quad . \quad . \quad (1^a) \qquad d\xi = - \Delta V_E \cdot dP \quad . \quad . \quad (1^b)$$

The *O.W.A.* of the system $E(P)$ is defined by (1^a); the change $d\xi$ which the *O.W.A.* of this system endures, when it passes into the new system $E'(P + dP)$, is defined by (1^b). We found for the meaning of ΔV_E that $\Delta V_E \cdot \delta w$ represents the change, which gets the total volume of the system E , of this takes in δw quantities of water. If we call ΔV_E the osmotic increase of volume of system E , then follows from (1^b):

with increase of pressure the *O.W.A.* of a system becomes smaller, when its osmotic increase of volume is positive and greater, when this is negative.

A corresponding property is valid for the change in temperature of a system. If we bring the system $E(T)$ viz. a system of the temperature T , while the pressure and total composition remain constant, to the temperature $T + dT$, then we find:

$$\xi = - \frac{\partial Z}{\partial w} \quad . \quad . \quad (2^a) \qquad d\xi = \Delta H_E \cdot dT \quad . \quad . \quad (2)$$

The change of the *O.W.A.* which the system endures, if we bring its temperature from T to $T + dT$, is defined by (2^b). For the meaning of ΔH_E we find, that $\Delta H_E \cdot \delta w$ is the change, which the total entropy of the system endures, if it takes in δw quantities of water. We shall call ΔH_E the osmotic increase of entropy of the system E .

When the systems E_1 and E_2 have the same *O.W.A.* at the temperature T , then exists the osmotic equilibrium:

$$(E_1 | E_2)_T \dots \dots \dots (3)$$

If we bring the temperature to $T + dT$ then these systems change their *O.W.A.* with:

$$d\xi_1 = \Delta H_{E_1} \cdot dT \dots (4^a) \qquad d\xi_2 = \Delta H_{E_2} \cdot dT \dots (4^b)$$

If we take $\Delta H_{E_1} > \Delta H_{E_2}$ then, with increase of temperature, the *O.W.A.* of the left system becomes greater than that of the right system; consequently water must diffuse from right to left. We shall represent this by:

$$\Delta H_{E_1} > \Delta H_{E_2} \quad [E_1 \leftarrow E_2]_{T+dT} \dots \dots \dots (5)$$

in which dT is positive. Consequently we may say:

with increase of temperature water diffuses towards the system, which has the greatest increase of osmotic entropy;

or also: with increase of temperature water diffuses in such direction that the total entropy of the whole system increases.

We shall represent by $Q_{1,2} \delta w$ the quantity of heat, which is taken up, when δw quantities of water diffuse from a system E_1 towards a system E_2 ; we call $Q_{1,2}$ the heat of diffusion of water from system E_1 towards system E_2 ; the heat of diffusion of water from E_2 towards E_1 is then $Q_{2,1} = -Q_{1,2}$. We now have:

$$\Delta H_{E_1} - \Delta H_{E_2} = \frac{Q_{2,1}}{T} = -\frac{Q_{1,2}}{T} \dots \dots \dots (6)$$

Therefore we can say also: with increase of temperature water diffuses in an osmotic equilibrium in that direction, in which the heat of diffusion is positive or in other words: that heat is absorbed.

We now can deduce the influence of a change of temperature in a corresponding way as in the communications XVIII—XXI, in which is discussed the influence of the pressure on osmotic systems; we shall discuss only some cases.

We take at the temperature T the liquid c of fig. 1 Comm. XVIII; all liquids, which have at this temperature T the same *O.W.A.* as this liquid c are situated on the isotonic curve acb going through point c . As the liquid q is, therefore, at the temperature T isotonic with the liquid c , we have the osmotic equilibrium:

$$[L_c | L_q]_T \quad \text{fig. 1. XVIII} \dots \dots \dots (7)$$

If we represent the osmotic increases of entropy by ΔH_c and ΔH_q , then, if we bring the temperature to $T + dT$, the *O.W.A.* at the left and at the right of the membrane, changes with:

$$d\xi_c = \Delta H_c \cdot dT \dots (8^a) \qquad d\xi_q = \Delta H_q \cdot dT \dots (8^b)$$

so that in general the two liquids are no more in osmotic equilibrium with one another. If we put $\Delta H_c > \Delta H_q$ then consequently water diffuses from q towards liquid c . We can represent this by:

$$\Delta H_c > \Delta H_q \quad [L_c \leftarrow L_q]_{T+dT} \quad \text{fig. 1. XVIII} \quad \dots (9)$$

in which dT is positive.

As in the osmotic system (9) the *O.W.A.* of the liquid c is greater than that of the liquid q , we can get an osmotic equilibrium of the temperature $T + dT$ by replacing the liquid q by a liquid q' with an *O.W.A.* which is somewhat greater, so that the *O.W.A.* on both sides of the membrane becomes equal again. Amongst others we can get such a liquid q' , as we have seen in previous communications, by withdrawing a little water from the liquid q . We then have the osmotic equilibrium:

$$[L_c \mid L_{q'}]_{T+dT} \quad \text{fig. 1. XVIII} \quad \dots (10)$$

in which the liquid q' can be represented by the point q' of fig. 1 XVIII, situated on the line Wq .

Also the following appears a.o. from this. If the liquids, which are isotonic with the liquid c at the temperature T , are represented by curve acb , then the liquids, which are isotonic with liquid c at the temperature $T + dT$, will be represented by a curve $a_1c b_1$ (fig. 1 XVIII) which does not coincide with acb .

We are able to deduce the above yet also in another way. For the osmotic equilibrium (7) is valid the relation:

$$\left(\zeta - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} \right)_c = \left(\zeta - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} \right)_q \quad \dots (11)$$

For the osmotic equilibrium (10), of which we assume that the liquid q' differs infinitely little (*viz.* dx and dy) from the liquid q , then is valid, as follows from (11).

$$\left(\eta - x \frac{\partial \eta}{\partial x} - y \frac{\partial \eta}{\partial y} \right)_c dT = \left(\eta - x \frac{\partial \eta}{\partial x} - y \frac{\partial \eta}{\partial y} \right)_q dT + (rx + sy) dx + \left. \begin{array}{l} \\ + (sx + ty) dy \end{array} \right\} (12)$$

in which, as we deduce easily, the coefficients of dT are the osmotic increases of entropy of the liquids c and q . As we can satisfy (12) for a definite value of dT still by an infinite number of values of dx and dy , there exists, therefore, in the vicinity of liquid q (fig. 1 XVIII) an infinite number of liquids q' , which are all isotonic with the liquid c at the temperature $T + dT$. Of all those liquids we take a liquid q' , which can arise from q by taking in or giving a little water, so that it is situated in fig. 1 XVIII anywhere on the line Wq ; then dx and dy must satisfy:

$$dx : x = dy : y = d\lambda \quad \dots (13)$$

We now can write for (12):

$$(\Delta H_c - \Delta H_q) dT = (rx^2 + 2sxy + ty^2) d\lambda \dots \dots (14)$$

by which the value of $d\lambda$ is defined. If we assume, in accordance with above, that ΔH_c is greater than ΔH_q and if we take dT positive, then it follows from (14) that $d\lambda$ is positive also. As it follows from (13) that dx and dy are positive also now, it follows that this liquid q' must be situated, as is drawn in fig. 1 XVIII, viz. further from the point W than the point q .

The above is valid no more when point c and consequently also point q are situated in the vicinity of point W , so that the isotonic curves are straight lines, which cut off equal parts of the sides WX and WY ; the isotonic curves of the temperatures T and $T + dT$ going through point c , will coincide then. This follows a.o. also from (12); for infinitely small values of x and y the coefficients of dT in (12) approach both the entropy of the pure water, so that their difference approaches zero. As s remains finite, sx and sy approach zero; rx and ty , however, approach both RT . Then follows from (12):

$$0 = dx + dy \dots \dots \dots (15)$$

from which the above follows at once. We are able to find this also by substituting in (11):

$$\zeta = \varphi + RT(\log x + \log y) \dots \dots \dots (16)$$

φ and its derivations with respect to x and y then remain finite for infinitely small values of x and y . We then find:

$$\left[\varphi - x \frac{\partial \varphi}{\partial x} - y \frac{\partial \varphi}{\partial y} - RT(x + y) \right]_c = \left[\varphi - x \frac{\partial \varphi}{\partial x} - y \frac{\partial \varphi}{\partial y} - RT(x + y) \right]_q$$

For infinitely small values of x and y follows again from this the equation (15).

In order that two systems E_1 and E_2 can be in osmotic equilibrium with one another, it is not sufficient that they have the same *O.W.A.*, but also their temperature must be the same; the pressure of the two systems, however, can be as well equal as different. Consequently, provided that the temperature of the separate systems is the same, an osmotic equilibrium:

$$(E_1)_{P_1} \mid (E_2)_{P_2} \dots \dots \dots (17)$$

may exist, in which P_1 and P_2 can represent the same pressures or not. Let us take f.i. the osmotic equilibrium:

$$L_P \mid L_P \dots \dots \dots (18)$$

in which on both sides of the membrane the same liquid and the same pressure. If we bring the pressure on the right side of the membrane to $P + dP$, then we can compare the *O.W.A.* on both sides of the membrane

with one another; at the right side it increases with respect to the left with $-\Delta V_L \cdot dP$; if we take dP positive, then the *O.W.A.* at the right side of the membrane decreases, therefore; consequently in the osmotic system:

$$L_P \leftarrow L_{P+dP} \dots \dots \dots (19)$$

a little water diffuses from the right towards the left. By changing a little the composition of one or of both liquids, so that the *O.W.A.* becomes equal again on both sides of the membrane, we are able to prevent the diffusion of water; we then get, instead of (19) one of the osmotic equilibria:

$$L_P^{(1)} | L_{P+dP} \quad L_P | L_{P+dP}^{(2)} \quad L_P^{(3)} | L_{P+dP}^{(4)} \dots \dots \dots (20)$$

in which the liquids $L^{(1)}$ $L^{(2)}$ $L^{(3)}$ and $L^{(4)}$ differ a little in composition with the liquid L .

If we take, however, the osmotic equilibrium:

$$(E_1)_T | (E_2)_T \dots \dots \dots (21)$$

in which the pressure on both sides of the membrane is equal or not and if we bring the temperature on the right side of the membrane to $T + dT$ then we get:

$$(E_1)_T | (E_2)_{T+dT} \dots \dots \dots (22)$$

which is not an osmotic system, however, and which never can pass into an osmotic equilibrium, in whatever way we change the composition and the pressure of the separate systems. Consequently we cannot compare the *O.W.A.* of both the separate systems with one another. Of course this is really the case, when we bring the temperature on both sides of the membrane to $T + dT$, so that we get the osmotic system:

$$(E_1)_{T+dT} | (E_2)_{T+dT} \dots \dots \dots (23)$$

While at change of pressure the osmotic changes of volume ΔV_{E_1} and ΔV_{E_2} may assume a part separately, this is not the case at a change of temperature with the osmotic changes of entropy ΔH_{E_1} and ΔH_{E_2} ; as viz. the temperature must be always the same on both sides of the membrane, and must change therefore, also always with a same amount, here always occurs the difference $\Delta H_{E_1} - \Delta H_{E_2}$ and also, therefore, the diffusion heat $Q_{2,1}$.

In communication XIX we have deduced the osmotic change of volume ΔV_E for different systems E ; in a corresponding way we are able to find the osmotic change of entropy ΔH_E . If we take f.i. the system:

$$E = (Y + L_w)_T \quad \text{fig. 1. III} \dots \dots \dots (24)$$

then we find for the osmotic change in entropy:

$$\Delta H_E = \frac{\eta - y \eta_y}{1 - y} \dots \dots \dots (25)$$

in which η represents the entropy of the liquid and η_y that of the solid substance Y . If we take one of the systems:

$$E_1 = (Y + L_a)_T \quad \text{fig. 1. III} \qquad E_2 = (X + Y + L_c)_T \quad \text{fig. 1. V}$$

$$E_3 = (Y + H + L_d)_T \quad \text{fig. 2. V}$$

then the osmotic changes in entropy are defined by:

$$\Delta H_{E_1} = \eta + \frac{\partial \eta}{\partial x} + \frac{s}{t} \left(\eta_y - \eta - \frac{\partial \eta}{\partial y} \right)$$

$$\Delta H_{E_2} = \frac{\eta - x \eta_x - y \eta_y}{1 - x - y} \qquad \Delta H_{E_3} = \frac{\eta_H - \beta \eta_y}{1 - \beta}$$

in which H represents a hydrate with the composition β Mol. $Y + (1 - \beta)$ Mol. W ; the entropies of the solid substances X, Y and H are indicated by η_x, η_y and η_H .

We assume that fig. 1. III is valid for a definite temperature T and pressure P ; the *O.W.A.* of the solid substance Y at this T and P then is the same as that of the liquid w saturated with solid Y and also therefore, equal to the *O.W.A.* of every liquid of the isotonic curve wm . If we take on this curve an arbitrary liquid e , then we have, therefore, the osmotic equilibrium:

$$(Y + L_w)_T \parallel (L_e)_T \quad \text{fig. 1. III} \quad (26)$$

with which the index P is omitted, as further we shall keep the pressure constant. We now bring the pressure on both sides of the membrane to $T + dT$, while we keep constant the composition of the both separate systems; we then get the osmotic system:

$$(Y + L_w)_{T+dT} \parallel (L_e)_{T+dT} \quad (27)$$

Although the total composition at the left side of the membrane remains constant, yet the liquid w will change a little its composition by dissolving or depositing solid Y ; we can imagine this new liquid w' to be represented in fig. 1. III by a point w' (on the side WY) in the vicinity of point w .

If we represent the *O.W.A.* and the osmotic increase of entropy of the left system by ξ_E and ΔH_E and those of the right system by ξ_e and ΔH_e , then the *O.W.A.* on the left and the right side of the membrane increases with:

$$d\xi_E = \Delta H_E dT \qquad d\xi_e = \Delta H_e . dT \quad (28)$$

in general, therefore, (27) forms no more an osmotic equilibrium and a little water must diffuse through the membrane. The direction of this diffusion is defined by the sign of:

$$d\xi_E - d\xi_e = (\Delta H_E - \Delta H_e) dT = \frac{Q_{e.E}}{T} . dT \quad (29)$$

in which $Q_{e.E}$ represents the diffusion-heat of water from the liquid e towards the system $E = Y + L_w$.

We shall divide into two parts the reaction, which occurs when δw quantities of water are taken in by the system $Y + L_w$; at first viz. we shall mix this water with the liquid L_w and afterwards we shall dissolve $y \delta w : (1 - Y)$ quantities of solid Y in this liquid. If we represent the mixtion-heat of water with the liquids L_w and L_e by Q_w and Q_e and the dissolving-heat of the solid substance Y in the liquid L_w by Q_y , then we have:

$$Q_{e.E} = Q_w - Q_e + \frac{y}{1-y} \cdot Q_y \dots \dots \dots (30)$$

Hence it appears that the diffusion-heat can be as well positive as negative; if the liquid w , however, contains still little water only, so that y approaches the unity, then this diffusion-heat is in general positive. If the liquid w viz. contains still little water only, then the dissolving heat Q_y approaches the melting-heat of the solid substance Y and is positive, therefore; as $1 - y$ then is small also, the last term of (30) shall get such a great positive value that the total value of (30) is positive also.

We now take $Q_{e.E}$ positive. Then follows from (29) that in the osmotic system (27) the *O.W.A.* at the left side of the membrane is greater than that at the right side; consequently a little water must diffuse from right to left. The osmotic system (27) passes by this into the osmotic equilibrium:

$$(Y + L_w)_{T+dT} | (L_{e'})_{T+dT} \dots \dots \dots (31)$$

The liquid w' at the left of the membrane keeps with this (by dissolving or depositing a little Y) its composition. On account of its giving of a little water the right liquid e passes into a liquid e' , which we can imagine to be represented in fig. 1. III by a point e' on the line We and somewhat further from W than point e .

We are able to deduce the previous in another way also. For the osmotic equilibrium (26) viz. is valid the equation:

$$\left(\zeta - y \frac{\partial \zeta}{\partial y} \right)_w = \left(\zeta - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} \right)_e \dots \dots \dots (32)$$

which expresses that the *O.W.A.* is equal on both sides of the membrane. As however the liquid w on the left side of the membrane is still also in equilibrium with solid Y we have:

$$\left[\zeta + (1 - y) \frac{\partial \zeta}{\partial y} \right]_w = \zeta_y \dots \dots \dots (33)$$

If we represent the difference in composition of the liquids e' and e by dx and dy and that of the liquids w' and w by dy' then the osmotic equilibrium (31) is defined by:

$$\left(\eta - y \frac{\partial \eta}{\partial y} \right)_w dT + (y t)_w \cdot dy' = \left(\eta - x \frac{\partial \eta}{\partial x} - y \frac{\partial \eta}{\partial y} \right) dT + \left. \begin{aligned} &+ (rx + sy)_e \cdot dx + (sx + ty)_e \cdot dy \end{aligned} \right\} \dots (32^a)$$

$$\left[\eta + (1 - y) \frac{\partial \eta}{\partial y} \right] dT - [(1 - y) t]_w \cdot dy' = \eta_y \cdot dT \dots \dots (33^a)$$

of which (32^a) follows from (32) and (33^a) from (33). By elimination on dy' follows:

$$A \cdot dT = (rx + sy)_e dx + (sx + ty) dy \dots (34)$$

in which:

$$A = \left(\frac{\eta - y \eta_y}{1 - y} \right)_w - \left(\eta - x \frac{\partial \eta}{\partial x} - y \frac{\partial \eta}{\partial y} \right)_e \dots (35)$$

It appears from (24) and (25) that the term with the index w in (35) represents the osmotic increase of entropy of the system $Y + L_w$; the term with the index e represents the osmotic increase of entropy of the liquid L_e . Consequently we have:

$$A = \Delta H_E - \Delta H_e = \frac{Q_{e,E}}{T} \dots (36)$$

If we substitute this value of A in (34) and if we use (13), then (34) passes into:

$$\frac{Q_{e,E}}{T} \cdot dT = (rx^2 + 2sxy + ty^2) d\lambda \dots (37)$$

If we take, as above, $Q_{e,E}$ positive, then follows that $d\lambda$ is positive also; hence follows for the position of liquid L_e' of (31) the same as we have deduced already above.

In fig. 1 are represented by curve $w_1 v_1$ the solutions saturated with the solid substance Y at the temperature T_1 ; the isotonic curve of the same temperature going through point w_1 is indicated by $w_1 m_1$. For

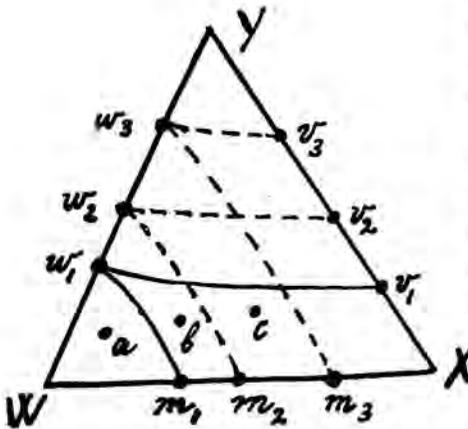


Fig. 1.

fixing the ideas we shall assume now that the solubility of the solid substance Y increases with increase of temperature; if we put $T_1 < T_2 < T_3$ then $w_2 v_2$ and $w_3 v_3$ can represent the saturation-curves of the temperatures T_2 and T_3 ; further we assume that $w_2 m_2$ and $w_3 m_3$ represent the isotonic curves of those temperatures.

The O. W. A. of the solid substance Y is equal to that of the liquid w_1 (at the temperature T_1) and, therefore, also equal to that of every liquid of the curve

$w_1 m_1$. Consequently the solid substance Y has at this temperature a greater O.W.A. than the liquid a , but a smaller O.W.A. than the liquids b and c . In the first of the three systems:

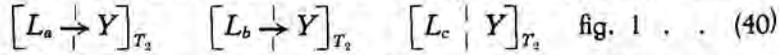
$$\left[L_a \begin{array}{c} \rightarrow \\ \uparrow \end{array} Y \right]_{T_1} \quad \left[L_b \begin{array}{c} | \\ \uparrow \end{array} Y \right]_{T_1} \quad \left[L_c \begin{array}{c} | \\ \uparrow \end{array} Y \right]_{T_1} \quad \text{fig. 1.} \dots (38)$$

water diffuses, therefore, in the direction of the arrow; if there is sufficient solid Y then is formed the osmotic equilibrium:

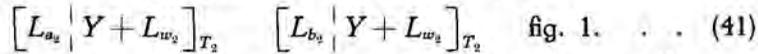
$$\left[L_{a_1} \begin{array}{c} | \\ \uparrow \end{array} Y + L_{w_1} \right]_{T_1} \quad \text{fig. 1.} \dots (39)$$

in which a_1 is the point of intersection of the line Wa with the isotonic curve w_1m_1 . The second and the third system are osmotic complexes, in which nothing happens.

We now bring the three systems (38) to the temperature T_2 ; water will diffuse in the two first systems from left to right; consequently we have:



In the third of those systems nothing happens. If a sufficient quantity of solid Y is present, then both the first pass into:



in which a_2 and b_2 are the points of intersection of the lines Wa and Wb with the curve w_2m_2 .

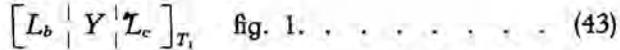
If we raise the temperature to T_3 then we get:



viz. a diffusion of water in each of the three systems. If there is a sufficient quantity of solid Y then arises on the right side of the membrane $Y + L_{w_3}$.

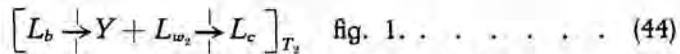
It is apparent from those examples that the diffusing or not of water through the membrane towards a solid substance, also is dependent on the temperature.

We now take the osmotic system:

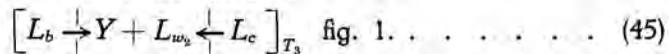


with a double-membrane. (Compare a.o. also Communication X). Although the *O.W.A.* of both liquids is different, herein nothing happens, because the *O.W.A.* of the solid substance Y is smaller than that of each of the liquids. Consequently the double-membrane is impermeable for water.

If we raise, however, the temperature to T_2 then the *O.W.A.* of the solid substance Y becomes greater than that of the liquid b , while it is smaller than that of the liquid c . At the beginning of the diffusion we then have the system:



If we bring the temperature to T_3 then the *O.W.A.* of the solid substance Y is greater than that of each of the liquids; at the beginning of the diffusion we then have the system:



As we have discussed already in Communication X it depends on the ratio of the quantities of the phases, etc., which osmotic equilibria will be formed at last from (44) and (45).

It appears from those examples that, therefore, it depends also on the temperature whether a double-membrane is permeable or not.

(To be continued).

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Microbiology. — *On Digestion in Protozoa.* By S. L. SCHOUTEN Sc. D.
(Communicated by Prof. C. EIJKMAN.)

(Communicated at the meeting of May 28, 1927)

Our knowledge of the physiology of digestion in Protozoa is comparatively limited. RUD. OEHLER, one of the best investigators during late years, even asserts summarily¹⁾ that there is no physiology of the unicellular organisms. He ascribes this to the lack of pure cultures, so that we do not know e.g. what is digested and what is not; nor what bacteria, when ingested, exert a deleterious effect through their metabolic products. Yet, for our knowledge of the self-purification of water, for instance, a close study of these problems is highly necessary.

Up to now researchers have tried to obtain pure cultures with one accompanying strain of bacteria by washing out repeatedly, centrifugalizing and planting out on solid culture media; by starting from cysts that are purified from adhering bacteria with water or soda-solution; by means of an electric current, whereby the protozoa are allured to the cathode; by inoculation in the centre of an agar-plate, previously planted entirely with a pure culture of a bacterium, in order to get finally, after a repetition of this inoculation, the protozoa that are moving rapidly towards the periphery, in company with that one bacterium.

But the results of all these endeavours are not brilliant. OEHLER, who has particularly applied himself to a solution of this problem, arrived at the conclusion: „Gewiss müssen andere, bessere Verfahren ausgearbeitet werden um Sterilzuchten zu erreichen“²⁾.

I thought fit to employ my apparatus for individual treatment of micro-organisms³⁾.

A drop of the material, from which one wishes to isolate, is placed on the coverslip, as well as a number of drops of one sterile culture medium or other. A confluence is now cautiously effected between the "material drop" and the sterile drop immediately contiguous to it, so that protozoa can swim across to the new, limpid portion of the medium.

With a glass needle at one end bent into a loop (Fig. 1; a rather large loop, in which the protozoa can swim about unscathed) and at low-power magnification they are now fished off from this limpid portion, and transferred to the second drop, then to the third etc. In each of these drops they

¹⁾ Arch. f. Protistenkunde, 1924, p. 288.

²⁾ Ibid. 1919 p. 19.

³⁾ See Kon. Akad. van Wetensch. Verslag van de Wis- en Natuurk. Afd. 24 Dec. 1910.

keep swimming about for some time to be deprived of the accompanying bacteria. Experience will show how many drops they have to pass through. Ultimately they are allowed to multiply in the last drop, after having added at will dead bacteria or living ones in pure culture (initially deposited on the cover-glass). Generally after one or two days it will be possible to transfer a subculture from that drop to a test-tube.

After having thus obtained several pure cultures, I selected one of *Colpidium campylum* to experiment on the rate of digestion. I must postpone a more detailed systematic enquiry until later; we can now only make a dip into the subject here and there, which, however admits of forming an idea about the usefulness of the method.

The protozoa were placed on a coverslip, together with a number of washing-drops, as described above, and also a few drops of ordinary nutrient-gelatin-agar ¹⁾ (a mixture of equal parts of nutrient-gelatin and nutrient-agar). Likewise a small lump of pure culture of bacilli whose digestibility I wished to establish, was deposited among the drops.



Fig. 1. (× 240)



Fig. 2. (× 240)



Fig. 3. (× 240)



Fig. 4. (× 240)

¹⁾ PRALL, Beitrag zur Kenntniss der Nährböden. Arb. a. d. Kaiserl. Gesundh. 1902, p. 436

After the protozoa had gone through different drops, I finally left them for some hours in a drop, without food. If I found by means of high-power magnification that they did not contain filled gastric vacuoles any more (with a view to this they were just for a short period removed from the large drop and examined in a little one: see fig. 2), they were fed again. At different intervals the protozoa were opened with the dissecting needle (fig. 3), so that the filled vacuoles were isolated (fig. 4)¹⁾.

These were crushed, and after being washed out the bacteria were deposited at the periphery of the gelatin-, or agar-gelatin drops. Whether they were dead, or alive could be seen from the multiplication.

From these experiments it first of all became evident that when *Colpidium* meets only few bacteria, it ignores them, but it devours them with great zest when it comes across a large mass, by preference a big lump, into which it bores its way. Its rapidly rotating movements give one the impression that it is revelling in food.

In a few minutes three or more vacuoles, filled with bacteria, are distinguishable.

In a special examination of the vacuoles it was not difficult to distribute the bacteria over various drops, and to count them. So I counted 127 spores of *B. subtilis* and in a small gastric vacuole 64 typhus bacilli; so at the same time 500 bacteria could easily be devoured.

Opinions differ about the wall of the gastric vacuoles. It is mostly conceived as a differentiation of the entoplasm, originated from a contact of plasm and water, that is ingested together with the food; all this occurs under the influence of surface-tension and colloid-chemical factors²⁾. By means of the pointed needle it could easily be found out that that wall is rather tough; when on the one side I had made a puncture, a slight push at the opposite side would keep the wall intact, but made the bacteria slip through the opening, the one after the other.

Some idea about the rate of digestion was afforded by the following experiments. Our *Colpidium* was very fond of a non-determined subtilis-like bacterium. The feeding itself lasted five minutes. When 10 minutes afterwards the bacteria were deposited in the way described above, they did not grow. When this was done 5 minutes after the feeding, they appeared to be alive yet. As could be anticipated, spores of these bacteria held out much longer in the vacuoles. When I allowed *Colpidium* to live still 30 minutes after the feeding, then excised the vacuoles and left them alone for 15 minutes, and subsequently inoculated the spores, they all of them appeared still to possess germinative power. I even experienced that germination was possible when these periods of 30 and 15 minutes were lengthened resp. to 90 and 60 minutes. Under certain conditions completely

¹⁾ Fig. 4 is somewhat indistinct, especially as regards the gastric vacuoles, owing to the quality of paper on which it is printed.

²⁾ See also: BOZLER, Arch. f. Protistenkunde, 1924, p. 163.

filled vacuoles are extruded ; when next day I examined their spores, they appeared to be still viable.

I also experimented with Anthrax, Coli, and Typhus.

As regards Anthrax let it be mentioned only that the bacteria were not devoured, apparently because they are too large, whereas on the other hand the spores were ingested.

Coli, and Typhus appeared to be much more resistant than the subtilis-like bacterium. I once left *B. coli communis* for 40 minutes after the feeding in the living *Colpidia* ; after this the animals were killed off, so that the gastric vacuoles were again at disposal. Untoward circumstances prevented me from finishing the experiment the same evening ; so the preparation was placed in the refrigerator. After 19 hours the wall of the vacuole was broken and the bacteria appeared to have survived. Typhus bacilli were not killed off after a sojourn of 40 minutes in the living *Colpidia*.

I may be allowed to add a few purely technical remarks :

For successful isolation experiments the surface of the coverslip is of prime importance. I intimated already in my first publication (1899), that the coverslip should be made fatty, so that the drops do not coalesce. It is not immaterial, though, what sort of fat is used for this purpose. In the beginning I used vaselin, though it is no fat at all ; afterwards I took suet, which satisfied me much better. Nonetheless also with this sort of fat I was more or less troubled by coalescence of the drops, especially with high summer temperatures and when a preparation had to serve a rather long time. At last I found out that we wanted suet, still covered by the peritoneum. Probably on its outside there are only the solid fats, while the liquid fats mixed with the solid ones form the interior. The underside of the coverslip is rubbed over the peritoneum, then nearly all the fat is removed by rubbing with a piece of soft linen, and the coverslip is flamed.

Another experience I often made with cultures in moist chambers was that several micro-organisms are rather sensitive to a rise of vapourpressure, that necessarily must originate in these moist chambers in the incubator from their atmosphere being saturated with water-vapour. This has been pointed out before by other investigators ; CLARK¹), therefore, suggests not to cover the chamber entirely with the coverglass, when it is placed in the incubator, but to leave a slit open, which is closed afterwards. To this end, however, the moist chamber will perhaps have to be taken out of the incubator, so that it will cool down rapidly ; after being closed its temperature will increase again. Moreover through the slit evaporation of the drop can take place. Fig. 5 shows a cross-section of a very simple isobarometric moist chamber.

A square frame of nickle-plated copper with interior dimensions of 14 × 14 mm., the breadth and the height of the border measuring resp. 4 and 3.5 mm., has been stuck on an objectglass with ordinary sealing-wax

¹) CLARK, On the toxic effect etc. Bot. Gazette, 1899, p. 289.

(fig. 5 to the left, from above). The right sidewall has been filed down obliquely over a distance of 6 mm. on either side, so that its lower part is narrowed. In the middle of this narrowed part a fine slit has been filed, so that the interior of the moist chamber is connected with the open air.

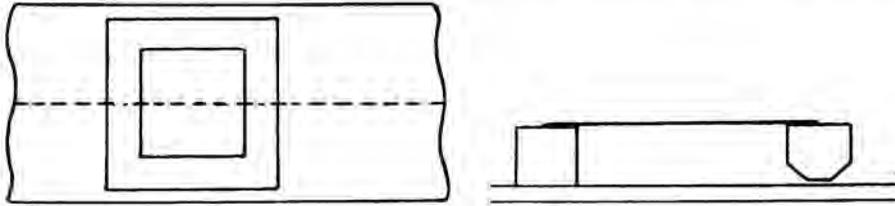


Fig. 5.

Cross-section along the dotted line is seen at the right of the figure, magnified $2\times$. Before closing the chamber with the coverglass a drop of thick oil ($2\frac{1}{2}$ gr. of vaselin to 10 gr. of paraffine liquidum, well mixed on a warm water bath) is placed in the obliquely filed part on the inside. On a rise of the vapour-pressure air-bubbles are seen escaping at the outside of the slit, when the pressure is lessened on the inside. The oil keeps the slit closed.

Utrecht.

*From the Hygienic Institute.
(Director Prof. C. EIJKMAN.)*

Physiology. — *On the Origin of the Radiation substances, and their Action on the Heart.* By Prof. H. ZWAARDEMAKER M. D.

(Communicated at the meeting of May 28, 1927).

In an earlier publication ¹⁾ we reported that in an isolated heart, that has discontinued its pulsation through removal of potassium, radium rays of weak penetrating power, or polonium radiation can generate substances, which, when brought into a small circulation, are able to restore the function of another heart, that has been prepared in the same way.

These substances may appropriately be called *automatins*; if we bear in mind that in the human body alpha-radiation is lacking, and consequently alpha-automatin does not normally occur, the natural automatin will be beta-automatin. It arises from a mother-substance, that is best designated by the name of "automatinogen". It is present throughout the organ that is activated by radiation.

Now it is just the activated substance that can set a heart beating spontaneously for a considerable time, after it has ceased to beat through a paucity of automatin and potassium.

The presence of a mother-substance was suggested to me by the fact that a fluid, that has been perfusing a non-pulsating heart for hours, acquires, after being radiated in a beaker-glass, the same properties as the fluid from a radiated heart.

It may be assumed, that the same occurs in a natural condition i.e. that throughout the organ mother-substance occurs that is activated by the radiation of the potassium contained in the muscle (2—4 β -rays per sec. and per gramme of substance), and that proceeds to the nodal tissue partly through direct diffusion, partly via the circulation (of the blood), where the automatin is eagerly absorbed and sustains the normal automatism.

This state of things, viz. that the mother-substance is contained in the heart-muscle, but that for automatism the action of activated mother-substance is required, is rendered plausible by the following experiments, which were made again in collaboration with Dr. PH. ARONS.

A frog's heart is attached to a Kronecker-canula and the ligature is laid slightly higher than the middle of the ventricle, all that is above the ligature being cut away, so that we have to do only with the apex of the heart without the atrioventricular node. Now, when we radiate this apex, in circulation, with radium, the heart will not pulsate, but the fluid can make a non-pulsating eel's heart resume its beats again.

¹⁾ H. ZWAARDEMAKER, These Proceedings, 30 p. 184 and 420.

If we do not radiate the apex of the heart, but for the rest arrange the experiment in the same way, then the liquid will be inactive.

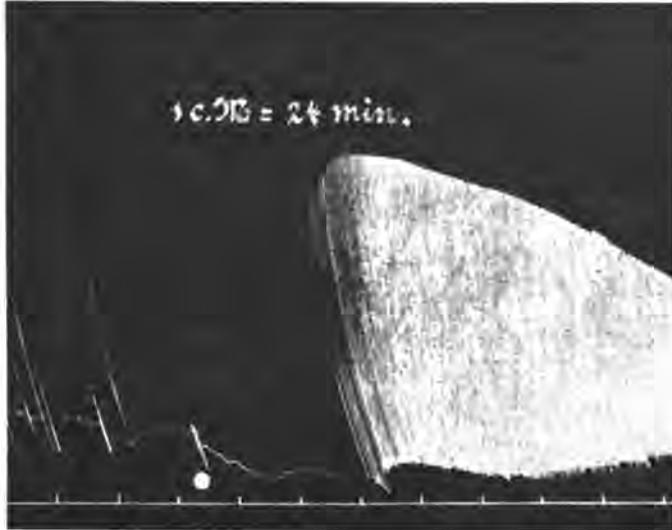


Fig. 1.

Recovery of an eel's heart in circulation, due to a circulating fluid that has been derived from a non-pulsating, but radiated portion of a frog's ventricle (lower two thirds). Commencement of the action at the white speck. One scale-mark of the time-line = 24 min.

So, with adequate radiation, molecules of the mother-substances as well as activated molecules, whose effect on a second heart is illustrated in Fig. 1., pass from the lower two thirds of a motionless frog's heart into the perfusing fluid.

Now that it appears that the mother-substance of the normal automatin is present throughout the heart-muscle, the question arises, whether it also occurs in the skeletal muscle.

This question is set at rest by the following experiment :

Three frogs are deprived of their blood as much as possible through perfusion for three hours with potassium free fluid. Then the muscles of the hindlegs are cut loose from their environment ; the nerves, the vessels and the tendons are removed as much as possible. The muscles are minced finely and shaken in alcohol for half an hour ; after they have been pressed out in a tissue-press, as much as possible, the fluid acquired is filtered. We allow the alcohol to evaporate at a moderate temperature, and wash the residue with ether. Then all is dissolved in potassium-free fluid. If we shake this solution with talc, the talc will absorb mother-substance, as well as the activated substance occasionally present. From the talc we can extract the absorbed substances with alcohol. Again the alcohol is allowed to evaporate and we dissolve the substance in potassium-free fluid.

Half of the extract contained in the beaker-glass is radiated in a thin layer, the other half is not radiated. Both halves are tested separately on a heart that has been brought to a standstill through deprivation of potassium and automatin. The radiated, as well as the non-radiated muscle-extract appears to be active, but the former is always the most powerful.



Fig. 2.

Recovery of an eel's heart in circulation owing to radiated muscle-extract (alcohol, absorption, alcohol, Ringer-solution). Commencement of the action at the white speck. One scale-mark of the time-line = 16 minutes.

This figure instances such a recovery produced by a radiated muscle-extract. The pulsation persisted regularly without flagging for 20 hours, while the heart perfused with the non-radiated half pulsated only 8 hours. Thus, through extraction with alcohol we can obtain from the skeletal muscles of the frog, previously deprived of potassium and automatin, both mother-substance molecules and activated molecules, which, when taken up in circulation restore the pulsations of a non-pulsating heart.

Very probably extraction will also liberate from the skeletal muscles season sensitizers¹⁾, respectively antibodies, whose effect is superadded to that of automatin. The potassium that was also extracted, thanks to the absorption-process, caused no disturbance (at most 5 mgrms of K. per L. of circulating fluid).

The next thing we shall have to try is to follow the automatin on its way from the skeletal muscles via the blood to the heart.

¹⁾ H. ZWAARDEMAKER, K. Akad. van Wetenschappen, Amsterdam, Verslagen van 25 Sept. 1920 en 29 Jan. 1921.

ERRATUM

Page 382, line 5 from top: for 18 read 28.