

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM

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SECTION OF SCIENCES

VOLUME XXIX

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(Translated from: Verslag van de gewone vergaderingen der Afdeeling
Natuurkunde, Vols. XXXIV and XXXV).

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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
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PROCEEDINGS

VOLUME XXIX

N^o. 6

President: Prof. F. A. F. C. WENT

Secretary: Prof. L. BOLK

(Translated from: "Verslag van de gewone vergaderingen der Afdeling
Natuurkunde", Vols. XXXIV and XXXV)

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Serology. — “A Contribution to the knowledge of the nature of the *Tuberculin-reaction.*” By Prof. C. H. H. SPRONCK and Miss W. HAMBURGER.

(Communicated at the meeting of December 19, 1925).

A few months ago the American researchers LONG and SEIBERT¹⁾ recorded that sensitized, normal tissues of tuberculous caviae do not adsorb tuberculin in vitro. They had made experiments with fine-grounded testis-tissue and skin. To 0.8 c.c. of emulsion 0.2 c.c. of tuberculin was added and the mixture was placed in an incubator for 6 hours. Subsequently it was centrifugalized and the supernatant fluid was injected into the testis of a tuberculous cavia and of a healthy one. The healthy animal showed no reaction, but a violent inflammation came forth in the tuberculous animals.

In this way, then, nothing could be observed of tuberculin combination, no more of the generation of a poisonous substance, which according to some was to arise from the tuberculin, when it is combined with its reciprocal antibody and to cause the tuberculin reaction. This induced LONG and SEIBERT to side with the adherents of SELTER's theory, which they consider, however, as “a statement of our ignorance of the true nature of the reaction rather than an explanation of it”. Indeed, this somewhat singular theory runs as follows: tuberculin is a specific irritant for the sensitized cells, which poisons the cells without, however, being bound by them.

The allergy for albumins, proteins, micro-organisms, is generally attributed to antibodies, which originate in cells, especially in endothelium cells and can pass into the blood. Allowance should be made for the possibility that a substance that lacks antigenic properties, acquires them by combination with a protein. This has already been demonstrated for lipid cell-substances.

In the investigation of the tuberculin-sensitivity of the tuberculous organism WASSERMANN and BRUCK's²⁾ antigenic-antibody theory clashed with the fact that tuberculin has no antigenic properties or only very weak ones. The degree of allergy which, through repeated injection of tuberculin is difficult to elicit in healthy animals, is but the faint shadow of the sensitivity readily generated by the living bacillus tuberculosis. Nay, the question has been propounded whether the artificial tuberculin-sensitivity is, indeed, a true tuberculin sensitivity. Not without reason, since the

¹⁾ The Journal of the American med. Association, V. 85, N^o. 9.

²⁾ Deutsche med. Wochenschrift 1906, Nr. 12.

product called tuberculin, is a mixture of all sorts of substances, and investigators have not yet succeeded in acquiring the pure substance that is the etiological factor of the tuberculin-reaction.

Experiments on analysis of the immunity against tuberculosis¹⁾, which are still continued, have elucidated this problem. Filtrates of aqueous extracts of tuberculous cavia-tissue appeared to contain a substance that acts in tuberculous caviae like tuberculin, but possesses antigenic properties. In view of its active sensitization it was termed „tuberculan“ (SPRONCK). Filtrates of cultures of bacilli tuberculosis contained tuberculin, never tuberculan. The reason assumed was that the bacillus tuberculosis produces in the body a complete sensitizing antigen, in vitro, however, a half, incomplete antigen without antigen power, but attracted and bound by the sensitized cells in the same way as the complete antigen. So in LANDSTEINER's nomenclature, tuberculin would be the hapten of tuberculan. In an earlier paper LANDSTEINER had already suggested the idea that tuberculan might probably be a hapten.

Moreover, we had already ascertained through our own inquiries that under the influence of repeated tuberculin stimuli the sensitized cells of the tuberculous cavia produce an antibody, that is transferred to the blood. This imparts to the serum the property to neutralize the action of tuberculin in vitro and to sensitize healthy caviae passively.

Also RUPPEL and RICKMANN's²⁾ tuberculosis-serum possessed the quality to counteract tuberculin in vitro. This serum had been manufactured by sensitizing oxen and mules beforehand through infection with living virulent tubercle bacilli from man, and by obtunding after some interval of time the tuberculin-reaction by an injection of tuberculin, extracts from tubercle bacilli and living bacilli. Such a serum has latterly been applied in MOLLGAARD's sanocrysin-therapy³⁾.

When tuberculin is binding with its antibody, the production of a poison in the serum is out of the question. If a poison were produced, the binding would not have been detected, as this can appear only from the harmlessness for the sensitized organism. Another explanation of the symptoms that characterize the tuberculin-reaction had, therefore, to be looked for, and we assumed with DÖRR that the physical consequences of the process of the intracellular binding produces as such the irritation, injury, necrosis of cells, inflammation, fever, and also tuberculin death.

Two years ago it was generally admitted that only living bacilli tuberculosis could sensitize the organism for tuberculin. It has since been proved that it is possible to evoke allergy for tuberculin also by injecting dead bacilli into animals and men. BESSAU⁴⁾ and others hold the view that not the dead bacilli themselves evoke the sensitization, but the morbid growth

1) Ned. Tijdschr. v. Geneeskunde 1923, II Nr. 10; Medizinische Klinik 1923, Nr. 32.

2) Zeitschr. f. Immunitätsforschung 1910, VI, p. 344.

3) H. MOLLGAARD, Chemotherapy of Tuberculosis. Copenhagen, 1924.

4) Klin. Wochenschr. 19 Febr. 1925.

of tissue, induced by the bacilli; in other words, the tuberculous cells are assumed to produce tuberculan, whereas we think that they form anti-tuberculan. We cannot adhere to this view and will only point to the bacillus tuberculosis of CALMETTE, the so called "bacil bilié", which produces tuberculin and also sensitizes the organism, but by no means induces a morbid growth of tissue. Besides this we believe to have succeeded in obtaining tuberculan from cultures of bacillus tuberculosis also outside the organism.

It is obvious, therefore, that we felt justified in testing the above named results of LONG and SEIBERT. But their method, by which the binding of small quanta of tuberculin could not be demonstrated, has been modified by us in the following way.

We have made experiments with testis-tissue and determined in the first place the smallest quantum of tuberculin (standard tuberculin) that had to be added to an emulsion, consisting of 4 grms of testis-tissue of a healthy cavia and 6 c.c. of a physiological salt solution with which tuberculin could still be demonstrated distinctly in the filtrate (paper) of this emulsion, through intracutaneous injection of 0.1 c.c. into tuberculous caviae. It thereby appeared that the addition of 0.2 c.c. answered our purpose. It should be added that we made use of tuberculous caviae from 700 to 800 grms in weight because older animals will get more susceptible than young ones, and the susceptibility of the older animals persists so to say until death¹⁾. White animals were preferred, anyhow animals with a white spot on the skin of the belly, destined for intracutaneous injections after previous epilation. After these preliminary experiments we passed on to the definitive ones. A tuberculous cavia was bled to death, 4 grms of testis-tissue was weighed out and crushed in a mortar with addition of 0.2 c.c. of tuberculin, and subsequently 6 c.c. of a physiological salt solution = emulsion T (tuberculosis). In the same way we treated testis-tissue of a healthy cavia of approximately the same weight = emulsion N (normal). Both emulsions were stirred off and on, and were allowed to stand for three hours at room-temperature. Subsequently they were filtrated through paper. With these rather clear filtrates the following experiments were carried out:

10. An intracutaneous injection was given to a tuberculous cavia, into the upper part of the belly 0.1 c.c. filtrate T, into the lower part 0.1 c.c. filtrate N.

20. Similar injections were given to a healthy cavia.

30. Another healthy cavia received under the skin of the upper part of the belly 2 c.c. filtrate T and in the lower part 2 c.c. filtrate N.

As to distinctness the result of these experiments left nothing to be desired. The healthy caviae evinced not the slightest of reaction. In the case of the tuberculous cavia, on the contrary, a distinct tuberculin-reaction (a small cockade with infiltration and a trace of necrosis in the center)

¹⁾ R. DEBRÉ et H. BONNET. C.R. de la Société de Biologie, T. 86, p. 485.

manifested itself regularly on the site of inoculation of filtrate N. This reaction persisted for many days. In the upper part of the belly, at the site of inoculation of filtrate T either some red colouring appeared, that faded already after a couple of days, or no reaction followed at all.

It cannot be mistaken, therefore, that *in vitro* normal, sensitized testis-tissue of tuberculous caviae adsorbs tuberculin. But of the hypothetical poison, that was supposed by some to be produced, we could no more than LONG and SEIBERT detect anything. We have not tried to determine how much tuberculin is adsorbed by sensitized testis-tissue, because this quantity may be considered to depend upon the degree of sensitization. But 1 grm of testis-tissue adsorbed, as we think, certainly not more than 50 mgr. of tuberculin, so that the result of the experiments by LONG and SEIBERT is quite intelligible, as they had added more than 200 mgr. per gram of tissue.

The adsorption of tuberculin by sensitized tissue, which is readily demonstrable by our method, is quite in harmony with the antigen-antibody theory and goes against SELTER's theory. Our simple method, which may still be improved, can also be applied to compare the sensitivity of different organs of a tuberculous animal.

A short time ago DIENES and FREUND¹⁾ reported *inter alia* that they could not demonstrate tuberculin in the urine after the injection of an aqueous extract of bacilli tuberculosis into tuberculous caviae apparently on account of binding with sensitized cells; in healthy caviae on the other hand, which had received an equal dosis of the same extract, they found a large quantum of tuberculin in the urine. Clinical experience also lent support to the antigen-antibody theory. In their recently published "Leerboek der Longtuberculose" VOS and LEUSDEN state: „In practice we can attain to the best and persistent results with the treatment of those patients with whom desensitization can be obtained to high doses of tuberculin”²⁾. As regards tuberculin-treatment the tuberculosis-specialists are divided into two groups. The allergists strive for the maintenance of tuberculin sensitivity; the anergists on the contrary try to obtund it. The allergists receive support from SELTER's theory, the anergists from WASSERMANN and BRUCK's antigen-antibody theory, as modified by us. According to the latter theory the favourable action of the obtusion of the sensitivity, is easy to understand. Allergy of the tuberculous patient indicates a lack of antibodies in the blood, and that, owing to this the cells are not protected against tuberculin, which is transferred spontaneously from the tuberculous foci to the blood. Through an appropriate tuberculin treatment the sensitized cells are incited to secrete antibodies, which circulate in the blood and cause a decrease of the tuberculin-allergy, because the tuberculin that

¹⁾ DIENES and FREUND, *American Review of Tuberculosis*, Baltimore, Bd. 12, Sept. 1925.

²⁾ B. H. VOS and J. TH. LEUSDEN, *Leerboek der Longtuberculose*. Utrecht. A. OOSTHOEK, 1925, DI. II, p. 233.

is introduced, has already been saturated with antibodies in the blood, and consequently does not reach the cells. The obtunding treatment transfers the process of the binding which is deleterious to the cells, from the cells to the blood, where the binding is not noxious. So the obtusion promotes the rest of the sufferer, because it promotes the relaxation required for the healing of the tuberculous foci, since the foci can no more stimulate and deteriorate each other through spontaneous inoculation of tuberculan into the blood. Thus the generally admitted favourable effect of the rest-cure is accomplished by obtusion of the tuberculin-allergy.

From the Bacteriotherapeutic Laboratory at Zeist.)

Physiology. — "*On the Transitory Arrest of the Heart's Action*". By Prof. H. ZWAARDEMAKER and W. WITANOWSKI. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated at the meeting of January 30, 1926).

When the conditions under which the heart functionates are of a sudden considerably altered, it often happens that the organ ceases to beat for some time and resumes its pulsations a few moments later. One receives an impression that the heart wants to accommodate itself before getting used to the new condition.

These transitory conditions have often been studied before, also with the isolated heart. NOYONS and LIBBRECHT ¹⁾ were confronted with them during sudden changes of temperature; during a sudden change of pressure every researcher must have observed them; during a sudden change of osmotic pressure and of concentration of hydrogen-ions ODINOT ²⁾ noted them; during sudden changes of potassium-concentration in the circulating fluid they are quite common. (NOYONS, LIBBRECHT, BUSQUET ³⁾).

Lately we could establish them also with abrupt changes in the concentration of Na and Ca, and also in artificial systems with a change in the concentration of NH₄, Rb, Cs, UO₂, Th., so that it is safe to say that nearly all sudden changes are attended with transitory standstills.

Together with the visible movements the action-currents of the heart disappear in all these cases ⁴⁾.

Besides transitory standstills other transition phenomena can appear with abrupt quantitative transitions. (T, p, π , P_H, Conc. Na, Conc. K, Conc. Ca, Conc. Rb, Conc. Cs, Conc. NH₄, Conc. U, Conc. Th.). Among these we reckon, e. g., also the group-formation appearing for some time in the pulsation instead of a transitory arrest. Sudden changes of force or relaxation have also been observed.

We purpose to discuss very briefly the significance of these transition-phenomena.

All of them have one characteristic in common: they are generated by an abrupt quantitative modification in a certain condition and they all have the peculiarity that they appear and disappear critically with extreme rapidity, with a latent period of half a min. We suspect, therefore, that we have to do here with a surface phenomenon, and we

¹⁾ NOYONS and LIBBRECHT, Arch. internat. de Physiol. T. 16, p. 451.

²⁾ Unpublished investigation in the Physiol. Lab. of Utrecht.

³⁾ Only calcium forms an exception. See Mines, Arbeiter, etc.

⁴⁾ An investigation by STEYNS has lately cleared up the Ca-controversy.

naturally feel inclined to correlate this with the so-called antagonisms, resp. pseudo-antagonisms of the ions. The ions of which this antagonism, resp. pseudo-antagonism has been established are Na, K, Ca. The numerical relations are represented schematically:

A. In the case of single antagonism by the fraction $\frac{\text{Na} + \text{K}}{\text{Ca}} =$
 $=$ constant.

B. In the case of threefold antagonism or pseudo-antagonism by
 the triangle



Geometrically the triangle is the base of a pyramid, whose apex is located in the origin of coordinates of a three-dimensional system of coordinates Na, K, Ca. The base of the pyramid may be displaced in a direction parallel to itself, but a displacement in another direction is not consistent with normal functions.

It should not be imagined, however, that the relations in question $\frac{\text{Na}}{\text{Ca}} =$ constant, $\frac{\text{K}}{\text{Ca}} =$ constant, $\frac{\text{Na}}{\text{K}} =$ constant, must be kept invariable. On the contrary, in practice ample variation is admissible as well in J. LOEB's classical researches as in the most recent ones.

The antagonisms are most often correlated with softening or hardening of the surface layer (H. MEYER, SPIRO, HÖBER), and the pseudo-antagonisms are ascribed to the necessity of maintaining a proper permeability (J. LOEB, H. J. HAMBURGER).

We shall not enter into a discussion of these explanations as we prefer to say something about the phenomena as real facts, premising that we deal only with transition-phenomena that arise with abrupt transitions between the several conditions, in which the antagonism as well as the pseudo-antagonism of the ions, of themselves, are fairly well maintained, so that after some time, say for the space of 60 minutes, regular cardiac action is guaranteed.

A. Inotropies and tonotropies as transition-phenomena conforming to the fractional scheme.

For instance sudden changes of lifting-power are met with both in a positive and in a negative sense. They generally reveal themselves with a staircase. The latent period is short, yet in the cold-blooded heart it is mostly not shorter than some seconds.

A very frequent transition-phenomenon is a sudden tonotropy, sometimes in a positive, sometimes in a negative sense. This may be a staircase or a steep reaction. The latent period is longer, it may last several minutes.

From the occurrence of a staircase and from the rather long latent

period we infer that the surface layer in which this occurs should not be looked for too close to the surface of the cell. Considering that in fraction $\frac{\text{Na} + \text{K}}{\text{Ca}}$ increase of the numerator brings on rapid relaxation, and increase of the denominator induces strengthening of the muscle, we localize this transition-phenomenon in the surface-layer of the fibrils, which is probably also the location of the elastic tonus (LANGELAAN, FEENSTRA).

A long series of other ions may also be included in the numerator of our fraction. In the series $\text{Li} : \text{Na} : \text{NH}_4 : \text{K}$ we know the quantitative relation 1 : 3 : 12 : 120 (V. D. BOVENKAMP). Afterwards the series could still be prolonged via Rb to Cs. (SMITS). In the denominator Sr and Ba find a place beside Ca, but Be and Mg appeared to be inactive. Quite foreign ions, viz. U and Th can exert, together with calcium, an influence analogous to that of calcium, not, however, when the calcium is absent.

In the glucose heart of NOYONS¹⁾ the denominator prevails over the numerator as far as tonus is concerned; as regards force the denominator is inferior to the numerator.

B. Chronotropies as transition-phenomenon conforming to the triangle-scheme:

As already stated, the phenomena themselves admit of a division into three groups:

- 1^o. transitory standstills;
- 2^o. group-formations;
- 3^o. transient increase of the frequency.

The latent period of these phenomena is very short, sometimes even no longer than a few seconds, in a coldblooded heart equal to the periodical time of one pulsation.

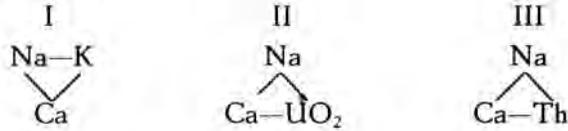
The duration is very variable, sometimes the standstill lasts only a few seconds, then again a few minutes up to 20. The group-formation manifests itself after 10 minutes or less, and perhaps persists for half an hour. The increase of the frequency has again a short latent period (1 min.) and continues only for 1, 2 to 5 minutes (J. B. ZWAARDEMAKER).¹⁾

The phenomena are closely allied, for not only can we let them come and go by the same contrivances, but they can also link up to each other, those with a short latent period, of course, coming first and later on those with a longer period.

Their character is exclusively chronotropic. For instance, instead of Na half the substance can be replaced by Li. The vertex of the triangle is then $\text{Li} + \text{Na}$. Partially K can be replaced by NH_4 , also by Rb. The vertex is then $\text{NH}_4 + \text{K} + \text{Rb}$. In place of Ca Cs can be used. Entirely new triangles may also appear when we introduce the radio-

¹⁾ J. B. ZWAARDEMAKER, Arch. néerl. de physiol. T. 8 p. 414, 1923.

active ions U or Th. Then the following schemata are obtained that are analogous inter se.



Meanwhile this time the transitions are no longer purely quantitative, as the radio-active elements K and U, when acting separately (or all but separately) are, indeed, interchangeable, but when acting simultaneously (which they do of necessity in such transitions), the radio-physiological antagonism asserts itself. The transitions $\text{I} \rightleftharpoons \text{II}$, resp. $\text{I} \rightleftharpoons \text{III}$ will, therefore, acquire a quantitative and a qualitative character. The transitions $\text{I} \rightleftharpoons \text{III}$ on the other hand are of a qualitative nature.

From what has been set forth here it appears, then, that the following phenomena reveal themselves:

1^o. transition-phenomena of a quantitative origin conforming to the fraction scheme;

2^o. transition-phenomena of a quantitative origin conforming to the triangle-scheme;

3^o. transition-phenomena of a qualitative origin.

I must not omit to call attention to the following remarkable peculiarity:

Whereas the transition-phenomena sub 1 and 2 still show some difference in their mode of appearing (gradual development, resp. recession sub 1 sudden coming and going sub 2), not any outward difference is noticeable in the transition-phenomena sub 2 and 3. We there observe the same suddenness in appearing and disappearing, equal latent period (on an average 30 sec.) equal duration (on an average 1 or 2 minutes), mostly complete relaxation of the muscle during the standstill. Yet they originate in a different way. In the one case the standstill is brought about by the sudden modification in the quantity of a condition, in the other by a sudden modification in the nature of a condition (alpha-radiation, i.e. emission of positively charged, very large ions over against beta-radiation, i.e. negatively charged, very small particles). True, there is no denying the fact that between alpha-, and beta-radiation there is besides a qualitative also a quantitative difference (e.g. in the amount of kinetic energy), but it would not do to attach value to this quantitative difference alone, for in doing so the difference in positive and negative charge would be eliminated, and the biological contrast between the two radiations (radio-physiological antagonism) would not be taken into account. The connection will be readily understood, however, when we examine the frequency of the occurrence of the standstills, i.e. when we compare e.g. a frequency with purely quantitative difference with the frequency of the standstills in case of a complex of quantitative difference and

qualitative difference. The frequency of the standstills amounted, for purely quantitative difference, to 33 % of the expressly examined cases (VAN DISHOECK), over against 50 % when qualitative and quantitative differences were combined. Occasionally the two transitions will appear the one after the other. The artificial Na Ca UO₂-system, and the equally artificial Na Ca Th-system have not yet been dealt with systematically. Provisionally we are impressed with the idea that in the main (not entirely) ¹⁾ they are analogously related to the natural Na K Ca-system.

¹⁾ Not entirely, as there is no abrupt increase of frequency in Petromyzon on a sudden removal of the UO₂, resp. Th. from the circulating fluid. In fact also the abrupt transition-standstills and transition-groupings are less frequent in the artificial systems. Meantime there remains a possibility that owing to the slow diffusion of the UO₂- and Th-ions the transitions are not comparable.

Chemistry. — "*The Potentiometric Determination of the Hydrogen Ion Concentration at Higher Temperatures*". By I. M. KOLTHOFF and F. TEKELENBURG. (Communicated by Prof. H. R. KRUYT).

(Communicated at the meeting of January 30, 1926).

1. In an investigation on the variation of the constant of dissociation of organic acids and the p_H of buffer mixtures at higher temperatures we have performed measurements of the hydrogen ion concentration, both with the use of the hydrogen and of the quinhydrone electrode.

The E. M. F. of one of these electrodes is measured by some standard electrode or other, and then the p_H is calculated according to the formula:

$$p_H = \frac{\pi - \pi_0}{\frac{RT}{F}} = \frac{\pi - \pi_0}{0.0001983 T}$$

π is the E. M. F. measured, compared with the standard electrode.

π_0 is the E.M.F. of the standard electrode compared with the electrode which is normal as regards hydrogen ions, T representing the absolute temperature.

The formula holds for different temperatures. If from the measured value of π it is desired to calculate the p_H , we must know the value of π_0 at the temperature T. The variation of π with the temperature depends on the temperature coefficient of the electrode which is normal as regards hydrogen ions, and on that of the standard electrode; the temperature coefficient of π_0 being equal to the algebraic sum of the two electrodes separately.

The value of π_0 at room temperature is not known with great certainty. Internationally with use of 0.1 N calomel electrode the value found by SÖRENSEN ¹⁾ is universally accepted as standard electrode at 18°. This value has been calculated from the E. M. F. of the circuit: H₂ electrode — 0.01 N HCl + 0.09 N KCl / — 0.1 N calomel electrode. Besides SÖRENSEN has calculated the p_H of the hydrochloric mixture from the electric conductivity according to ARRHENIUS' isohydric principle, finding a value of 2.038. We should, however, bear in mind that the electric conductivity does not convey a correct impression of the activity of the hydrogen ions.

In this connection we may especially refer to an important communication by G. N. LEWIS, T. B. BRIGHTON, and R. L. SEBASTIAN ²⁾, who

¹⁾ S. P. L. SÖRENSEN, *Compt. rend. Lab. de Carlsberg* 8, bl. 22 (1909); and also SÖRENSEN and K. LINDERSTRÖM-LANG, *Ibid.* 15, No. 6 (1924).

²⁾ LEWIS, BRIGHTON and SEBASTIAN, *J. Amer. Chem. Soc.* 39, 2245 (1917).

have carried out measurements with diluted hydrochloric acid solutions without chloride of potassium, and have calculated from it a value of π_0 , 2 m. Volt lower than that according to SÖRENSEN.

We hope to return more at length to the correct standard value later on. For the present we have assumed the value π_0 given by SÖRENSEN, to be correct.

For the results which have been mentioned in these communications it is, for the rest, immaterial which standard value is assumed at 18°, as it was chiefly the intention to get to know the temperature coefficient of π . The only investigation on the temperature coefficient of the N-hydrogen electrode has been performed by KOEFOED.

His results have been communicated by SÖRENSEN¹⁾. The measurements have been made against 0.1 N calomel electrode in an air-thermostat. Further particulars have not been mentioned. From the known temperature coefficient of the N-calomel electrode SÖRENSEN calculates that the potential of the normal hydrogen electrode at increasing temperature diminishes 0.85 m. Volt per degree. (SÖRENSEN has made here a mistake about the sign, for the potential of the N-hydrogen electrode increases with the temperature).

E. BILLMANN and J. KRARUP²⁾ have measured the temperature coefficient of the element hydrogenelectrode-quinhydrone electrode. They found that the E. M. F. of this element decreases by 0.74 m. Volt per degree at rising temperature.

As there are so few data in the literature on the temperature coefficient of the hydrogen- and quinhydronelectrode, we have made a thorough investigation of this question.

In the course of this examination we have determined the coefficients of these two electrodes and of the *n*-calomelectrode both separately, as of elements built up of different electrodes. In this way it was easy to check the reliability of the results. It may already be mentioned here that we found a good agreement with the values which have been communicated by SÖRENSEN, BILLMANN and their collaborators.

2. *Apparatus.* In the application of the compensation method we always used a student's potentiometer of LEEDS & NORTHRUP Company, with accessory galvanometer. The readings are very sensitive generally up to 0.05 m. Volt. In all cases the measurements could be made with an accuracy of 0.1 m. Volt. The scalar division of the potentiometer was checked by comparing the readings with those obtained by 2 precision rheostats instead of using the potentiometer.

As electrodes we used the N-calomel-, the hydrogen-, and the quinhydrone electrode. All the determinations were carried out in a water thermostat, in which a layer of paraffin oil prevented too strong evapo-

¹⁾ SÖRENSEN, Compt. rend. Lab. de Carlsberg, No. 9 (1912); cf. SÖRENSEN and LINDERSTRÖM-LANG l. c.

²⁾ E. BILLMANN and J. KRARUP, J. Chem. Soc., 125, 1954 (1924).

risation of the water. As we gradually raised the temperature in every series of experiments, an air thermostat was not suitable for our purpose, because the time that elapses before the different vessels have assumed a constant temperature, is much too long.

In all the cases the temperature was constant up to 0.1° C., a more accurate regulation of the temperature was not necessary for our purpose.

As the forms of the electrode vessels generally used are not very suitable when a water thermostat is worked with, we have used somewhat different models, so that electrode and liquid were entirely in the thermostat. The electrode vessels were made of Jena normal glass, and were cleaned by steam before the filling. The tube that served for the calomel electrode had an inner diameter of 2 cm. and was 7 cm. long. It was closed by a rubber stopper with 2 perforations; through one passed the capillary, in which at the bottom a piece of platinum had been fused which formed contact with the mercury; through the other the connecting siphon could be adjusted, which joined the normal calomel electrode with the other electrode. The filling of the "half-cell" took place with the usual precautions. When the quinhydrone electrode was used, we took an analogous vessel as has been described for the calomel electrode. The best results appeared to be obtained when a piece of platinum gauze was used. The quinhydrone had been prepared in the usual way out of hydroquinone and ferri-ammonium sulphate, and then recrystallized out of water. The vessel in which the measurements with the hydrogen electrode were performed, was wider than those described above. The vessel was also closed with a rubber cork, in which several perforations had been made. Through one passed the inlet tube for the hydrogen. This tube was bent to a U-shape at the bottom, and care was taken that a part of the platinized platinum filament electrode extended into the tube. In this way the electrode was continually in contact with the in-flowing hydrogen. In order to prevent evaporation of the liquid in the electrode vessel at higher temperature, we always led the hydrogen through a water bottle placed in the thermostat, and filled with the same liquid as the electrode vessel. Through another perforation of the cork passed a narrow tube, through which the hydrogen could escape, and through a third perforation the connecting siphon with the standard electrode could be passed.

The measurements with the hydrogen electrode were easily reproducible to a 0.1 m. Volt.

The siphons, which served to connect the electrodes with each other, were U-shaped tubes, with a bore of 1 to 2 mm. The tubes were filled with solutions containing 1.75 resp. 3.5 N. chloride of potassium and 2% agar. The agar was dissolved in the heated salt-solution, then after filtration the heated liquid was sucked up into the tubes. At room-temperature the contents get solid. Also in the measurements at 60° to 70°

these tubes could be used. By always using two siphons, one containing 1.75 N. and the other 3.5 N. chloride of potassium, the potential of diffusion according to N. BJERRUM ¹⁾ could be eliminated.

In the measurement of the temperature coefficient of an element, part of the siphon was in the air, so that the contents had not the same temperature everywhere. As it was possible that in this way errors might arise, we have also carried out measurements, in which this drawback was obviated. The apparatus used is represented in Fig. 1.

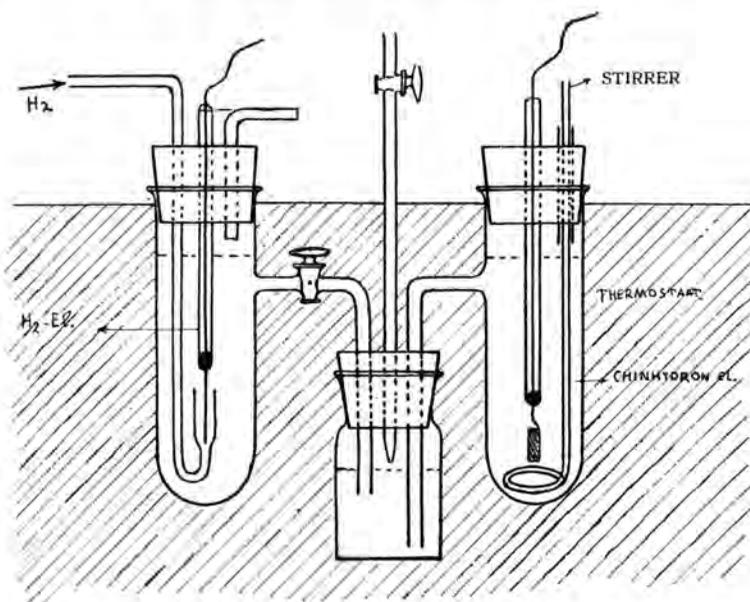


Fig. 1.

One leg served as receptacle for the hydrogen, the other as receptacle for the quinhydrone electrode. The two vessels were filled with the same liquid. To prevent diffusion of quinhydrone into the hydrogen electrode, a connecting vessel was applied, also filled with the same liquid as that with which the electrodes were in contact. Hence a diffusion potential could not appear. In the vessel in which the quinhydrone electrode was immersed, a glass stirrer was placed, so that the liquid could be stirred at the higher temperatures, and could be quickly saturated with quinhydrone in this way. The connecting tap, which was also immersed in the thermostat, was opened during the measurement by the aid of a wooden fork, and closed again after the reading. The whole element could be immersed in the thermostat. It now appeared that we practically obtained the same results with this element as with our usual arrangement.

In the measurement of the temperature coefficient of one electrode

N. BJERRUM, Z. physik. Chem. 53, 428 (1905).

separately, the electrode of comparison was placed in a DEWAR vessel in melting ice, so that the temperature remained constant during the series of measurements.

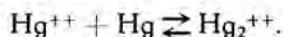
3. *The Reproducibility of the Electrodes at Higher Temperatures.*

In measurements of ion concentrations as a rule some calomel electrode or other is used as standard electrode.

In the measurements of ion concentrations at varying temperatures our experiences with these electrodes have not been favourable. When the normal calomelectrode is heated, without moving or shaking, it takes a pretty long time, before the state of equilibrium is reached. Sometimes we had to wait as much as two hours or more until the potential was constant. Of course the establishment of the equilibrium could be accelerated, by a few minutes' shaking in the thermostat at the higher temperatures. It appeared, however, that after the shaking, the electrode had a too noble potential due to the suspended calomel. To obtain reproducible values we had to wait till all the calomel had settled down the bottom, which may take a pretty long time. If the measurement is made while the liquid is still turbid of the suspended calomel, errors may ensue of 1 to 1.5 m.Volt. A more serious drawback is, that the calomel electrode, after having been at high temperature, no longer shows the original potential after having been cooled down to room temperature. Thus we have repeatedly found that an electrode that had been heated to 60° had a potential at 18° which was 5 m. Volt more positive than originally.

This deviation is a consequence:

1. of the super-saturation of the liquid with calomel after cooling;
2. of the changed equilibrium:



At higher temperature the state of equilibrium changes and it takes a long time after the cooling, before the original state has again been reached. When the liquid is shaken for a long time after the cooling and when it is allowed to settle, or when the electrode liquid is refreshed, the original values are found back.

It appears from the above:

1. that when a calomel electrode is used as standard, the vessel must always be kept at the temperature at which the measurement is made. Only in this way one can be sure that the potential remains constant.
2. that when varying temperatures are worked with, the use of the calomel electrode is attended with many difficulties. For series measurements at different temperatures it is therefore *not* to be recommended as a standard.

For these reasons we have also worked with another standard electrode; we took a quinhydrone electrode in contact with a liquid containing

0.01 N. hydrochloric acid and 0.09 N. potassium chloride. Such an electrode has already been recommended by S. VEIBEL¹⁾ as standard, and he has also already determined its temperaturecoefficient between 0° and 18° in connection to the hydrogen electrode.

With this standard-quinhydrone electrode we have obtained very satisfactory results. Up to a temperature of 40° the potential remains constant for a long time; at higher temperatures, however, changes may be observed when the electrode is left for some time; the electrode becomes less noble. This may be partly a consequence of an evaporation of quinone, but besides a reaction seems to take place between quinone and hydrochloric acid. For after the heating of quinhydrone with the electrode liquid at 60° in a closed vessel it is found that the potential has considerably fallen after cooling.

After the slow heating of a quinhydrone electrode to 50° we have repeatedly observed that after cooling to 18° the potential had fallen from 6 to 8 m. Volt. When the quinhydrone electrode is used at higher temperatures, it is, however, very well possible to obtain reproducible values, when the electrode is refreshed resp. at 50°, 60°, and 70°. Every time fresh quinhydrone is added, the electrode is shaken for about 3 minutes, after which the potential remains constant for a short time.

As the state of equilibrium is reached so quickly, and the manipulation of the quinhydrone electrode is very simple, we can recommend it as standard electrode. In series of measurements which were made on different days under the same conditions, we found results agreeing to at least 0.2 m. Volt.

We may still mention that instead of the hydrochloric acid potassium chloride mixture we have also tried several buffer solutions as electrode liquids. Here, too, we found that at higher temperatures the potential remained constant only for a short time.

In conclusion we may still remark that in principle it is most advisable to use a hydrogen electrode at higher temperatures as standard. L. MICHAELIS²⁾ has already recommended an acetic acid-acetate mixture for this purpose; we, however, prefer SÖRENSEN's hydrochloric acid-potassium chloride mixture (0.01 NHCl + 0.09 NKCl). The presence of the potassium chloride causes the diffusion potential to be exceedingly small.

4. Results.

In the first place we have measured the temperature coefficients of the normal calomel electrode with due observation of the different precautions described under 3. For this purpose we have worked with 2

¹⁾ S. VEIBEL, J. Chem. Soc. 123, 2203 (1923).

²⁾ L. MICHAELIS, Die Wasserstoffionen-konzentration, 1e Aufl. Berlin 1914, p. 148.

different electrodes, a third electrode being kept constant at 0°. It was first ascertained that the electrodes at 18° connected with each other had an E.M.F. of 0.0 m.Volt. At higher temperatures the potential of the calomel electrode becomes more positive. Some results are laid down in the following tables:

Temperature coefficient N-calomel electrode.

| N. C. E. I t | N. C. E. II t | E. M. F. in Volts | Temp. coeff. |
|-----------------|------------------|----------------------|--------------|
| 0° | 18.6 | 0.0114 | 0.00061 |
| 0 | 26.2 | 0.0155 | 0.00060 |
| 0 | 30.3 | 0.0175 | 0.00057 |
| 0 | 40.2 | 0.0231 | 0.00057 |
| 0 | 50.2 | 0.0288 | 0.00057 |
| 0 | 60.2 | 0.0343 | 0.00056 |

mean: 0.00058
(between 0° and 30°: 0.00059)

| N. C. E. III | N. C. E. IV | E. M. F. | Temp. coeff. |
|--------------|-------------|----------|--------------|
| 0° | 26.2 | 0.0161 | 0.00061 |
| 0 | 30.4 | 0.0184 | 0.00060 |
| 0 | 40.2 | 0.0229 | 0.00057 |
| 0 | 50.1 | 0.0291 | 0.00058 |

mean: 0.00059
(between 0° and 30°: 0.00060)

As a mean we find for the normal-calomel electrode a temperature coefficient of 0.00059, which value is in good harmony with the result of the accurate measurements of TH. W. RICHARDS¹⁾, who found a coefficient of 0.00061.

In the next table we state the results obtained in the measurement of the temperature coefficient of the quinhydrone electrode in the standard hydrochloric acid mixture. The measurement was made by comparison with a quinhydrone electrode, which was kept at 0°. Originally the two electrodes had, at 18°, an E.M.F. of 0.0 m.Volt as compared with each other.

¹⁾ TH. W. RICHARDS, Z. physik. Chem. 24. 49 (1897).

Temperature coefficient quinhydrone electrode.

| Quinhydrone electrode I t | Quinhydrone electrode II t | E. M. F. in Volts | Temp. coeff. |
|------------------------------|-------------------------------|----------------------|--------------|
| 0° | 18.0 | -0.0054 | -0.00030 |
| 0 | 25.0 | -0.0074 | -0.00030 |
| 0 | 30.0 | -0.0092 | -0.00031 |
| 0 | 40.0 | -0.0123 | -0.00031 |
| 0 | 50.0 | -0.0152 | -0.00030 |
| 0 | 60.0 | -0.0183 | -0.00030 |

average -0.00030

In another series -0.00031 was found as a mean. We have also carried out a series of measurements against the N-calomel electrode, which latter was kept at 0°. The mean value, which then was found, was -0.00033. From all the observations which were then made the conclusion may be drawn that the quinhydrone electrode has a temperature coefficient of 0.00031.

We then determined the temperature coefficient of the hydrogen electrode, which was in contact with the standard hydrochloric acid mixture. The measurements were carried out against the quinhydrone electrode in the same mixture kept at 0°.

In the third column the directly measured E.M.F. is recorded, in the fourth column the recalculated values for a pressure of the hydrogen of 1 atmosphere (hence corrected for barometer position and vapour pressure of the water).

Temperature coefficient of the hydrogen electrode.

| Quinhydrone- electrode t | Hydrogen- electrode t | E. M. F. in volts | E. M. F. in volts corrected B = 756.8 | Temp. coeff. |
|--------------------------------|-----------------------------|----------------------|--|--------------|
| 0° | 18° | 0.7093 | 0.7096 | |
| 0 | 25 | 0.7060 | 0.7064 | + 0.00046 |
| 0 | 30 | 0.7035 | 0.7041 | + 0.00046 |
| 0 | 40 | 0.6985 | 0.6995 | + 0.00046 |
| 0 | 50 | 0.6930 | 0.6948 | + 0.00047 |
| 0 | 60 | 0.6865 | 0.6896 | + 0.00052 |
| 0 | 70 | 0.6793 | 0.6847 | + 0.00049 |

+ 0.00047

51*

In 3 other series of measurements, the particulars of which will be communicated in the thesis for the doctorate of one of us (T.), the mean values of 0.00047, resp. 0.00046, resp. 0.00045 were found. As total average between 0° and 70° we can therefore assume a value of 0.00046 ± 0.00001 .

As the quinhydrone electrode in the hydrochloric acid-potassium chloride mixture has a temperature coefficient of -0.00031 , the hydrogen electrode of $+0.00046$, the element quinhydrone-hydrogen electrode both in the same liquid must have a temperature coefficient of -0.00077 ± 0.00002 . We now carried out a great number of measurements between 18° and 70°; in this it was necessary to bring the quinhydrone electrode every time in contact with fresh liquid at 50°, 60°, and 70°. If this was not done, irregularities were found.

In TEKELENBURG's thesis for the doctorate all the values will be given: we will mention here only the mean values found in the different series of observations: -0.00074 ; -0.00076 ; -0.00077 ; -0.00077 ; 0.00075 . As total mean we find therefore a temperature coefficient for the element quinhydrone-normal-hydrogen-electrode of -0.00076 ± 0.00001 Volt. Hence this value is in accordance with that which we had calculated from the temperature coefficients of the two electrodes separately. E. BILLMANN and KRARUP (loc. cit.) found, between 0° and 40°, a temperature coefficient of the same element of -0.00074 , which is in satisfactory agreement with the value found by us. In order to ascertain whether there is an error in the determination due to the connecting siphon being only partially immersed in the thermostat, we also made a number of series of observations with the element as represented in Fig. 1. We then found, in perfect harmony with the value given above, a mean temperature coefficient of -0.00076 .

Besides, we also made series of measurements, in which the hydrogen and quinhydrone electrode were in contact with buffer mixtures instead of with the standard hydrochloric acid mixture.

In this way the following values were found:

- liquid in 0.1 molar secondary sodium citrate $p_H = 4.96$;
temperature coefficient element -0.00076
- liquid in 0.1 molar sodium bi-tartrate
temperature coefficient element -0.00075 .

These values are, therefore, in good agreement with those that were found with the standard hydrochloric acid mixture.

We will still state here that the temperature coefficient of the hydrogen electrode in the standard hydrochloric acid mixture is not equal to that of the *normal* hydrogen electrode.

For if the p_H of the standard mixture is 2.031, and remains unchanged

at higher temperatures, the E. M. F. is with regard to the normal hydrogen electrode

$$\pi = -2.038 \times \frac{RT}{F}.$$

From this formula and the temperature coefficient of the hydrogen electrode in the standard mixture, we then find a coefficient for the *normal hydrogen electrode* of $+(0.00047 + 0.0004) = +0.00087$, while SÖRENSEN has derived a coefficient of the temperature of 0.00085 from KOEFOED's observations (loc. cit.). Consequently the agreement is excellent.

Summary.

1. At higher temperatures the use of the calomel electrode as standard electrode is not to be recommended. The state of equilibrium is only slowly reached with change of temperature, calomel having influence on the potential in suspended condition. In particular the equilibrium is established very slowly when an electrode of a higher temperature is brought to a lower.

2. In series measurements the quinhydrone electrode in a liquid containing 0.01 N of hydrochloric acid 0.09 N of potassium chloride (standard hydrochloride mixture) is to be recommended as comparison electrode. At a temperature higher than 40° C. the potential however, changes comparatively quickly with the time, so that at higher temperatures the liquid must be repeatedly refreshed. In principle the hydrogen electrode is, therefore, to be preferred in the standard acid mixture.

3. The temperature coefficient of the N-calomel electrode was found equal to $+0.00059$ (RICHARDS $+0.00061$); of the quinhydrone electrode in the standard acid mixture to -0.00031 ; of the hydrogen electrode in the same mixture to $+0.00046$. From the last value we calculate a temperature coefficient of the normal hydrogen electrode of $+0.00087$ (SÖRENSEN and KOEFOED 0.00085).

4. Of the element quinhydrone electrode — hydrogen electrode a temperature coefficient of -0.00076 was found. (BILLMANN and KRARUP -0.00074).

Pharmaceutical Laboratory of the University.

Utrecht, January 1926.

Mathematics. — “A Representation of the Congruence of the Twisted Cubics which Pass through Four given Points and which Cut a given Straight Line Twice”. By Dr. G. SCHAAKE. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of January 30, 1926).

§ 1. The twisted cubics k^3 which pass through four given points H_1, \dots, H_4 and which cut a given straight line l twice ¹⁾, may be represented on the points of a plane α in the following way. We assume a one-one correspondence between the points P of l and the points Q of a conic k^2 in α . To a curve k^3 which cuts l in the points P_1 and P_2 , we associate the point of intersection K of the lines q_1 and q_2 which touch k^2 at the points Q_1 and Q_2 corresponding resp. to P_1 and P_2 .

§ 2. If P_1 lies in the point of intersection S_1 of l with the plane $\beta_1 \equiv H_2H_3H_4$, the curve k^3 , which in this case has four points in common with β_1 , must degenerate into a conic k_1^2 of the pencil of which H_2, H_3, H_4 , and S_1 are the base points and a straight line k_1 which cuts k_1^2 outside l , passes through H_1 , and cuts l in P_2 . Any conic of the said pencil is a part of one degenerate individual of our system. In this way we find ∞^1 degenerate curves k^3 , which are represented on the points of the tangent t_1 to k^2 at the point T_1 which corresponds to S_1 .

Accordingly the curves of our system which are degenerate in a conic and a straight line, form four systems which are represented on the four tangents t_1, \dots, t_4 to k^2 at the points T_1, \dots, T_4 that are associated to the points of intersection S_1, \dots, S_4 of l with the four sides β_1, \dots, β_4 of the tetrahedron of which H_1, \dots, H_4 are the angular points.

The conic k_1^2 may degenerate into S_1H_2 and H_3H_4 , and is then completed to a curve k^3 by the straight line H_1S_2 in β_2 . Together with H_3H_4 this line forms a degenerate conic k_2^2 of the pencil which has H_3, H_4, H_1 , and S_2 as base points. Consequently we have here a k^3 degenerate in three straight lines, which is represented on the intersection T_{12} of t_1 and t_2 .

Hence our system contains six curves k^3 which are degenerate in three straight lines. They are represented on the six points of intersection T_{ik} of the lines t_1, \dots, t_4 . The curve k^3 corresponding to the point T_{ik} consists of the edge opposite to H_iH_k and of the lines S_iH_k and S_kH_i .

¹⁾ The congruence which is formed by these curves, has been treated by Prof. JAN DE VRIES: A Congruence (1,0) of Twisted Cubics, these Proceedings 26, 126.

§ 3. The tangent at a point Q of k^2 is the image of the system of the ∞^1 curves k^3 which pass through the point P of l corresponding to Q . These curves form the quadratic cone \varkappa which has P as vertex, which passes through the points H_1, \dots, H_4 , and which contains the straight line l , as it must have seven points in common with \varkappa , to wit two points which coincide in P , one point of intersection with l outside P , and the four points H_1, \dots, H_4 .

The system Σ_1 of the ∞^1 curves k^3 which cut a given line m , is represented on a curve of a . As through a given point P of l there pass two curves of Σ_1 , which cut m in the two points of intersection of m with \varkappa , the aforesaid image curve has two points in common with any tangent to k^2 .

Accordingly the system Σ_1 of the ∞^1 curves k^3 which cut a given straight line m , is represented on a conic m^2 of a .

To a curve of Σ_1 which cuts m twice, there would correspond a node of the image curve of Σ_1 . As a rule a conic m^2 is not degenerate, as in this case our congruence would be compound, and consequently it does not generally contain any double point.

There is, therefore, no twisted cubic which passes through four given points and which has two given straight lines as chords ¹⁾.

As the conics m^2 corresponding to two different lines m have four points in common, we find:

There are four twisted cubics which pass through four given points, have a given chord, and cut two given straight lines.

The points of k^2 are the images of the curves k^3 which touch l . Now a conic m^2 has four points in common with k^2 .

There are accordingly four twisted cubics which pass through four given points, touch a given straight line, and cut another given line.

§ 4. The points K of a straight line k of a are the image points of ∞^1 curves k^3 , which form a system S . As the pairs of points (Q_1, Q_2) which correspond to the points K of k , define an involution with carrier k^2 , the pairs of points of intersection of l with the curves of S form an involution on l . The locus of the curves of S is a quadratic surface σ^2 , as k has two points in common with a conic m^2 and there are, therefore, two curves of S which cut a given straight line m . This surface σ passes through the four points H_1, \dots, H_4 and it contains the line l . Each of the ∞^2 quadratic surfaces which satisfy the latter conditions, contains a system of ∞^1 twisted cubics k^3 which pass through H_1, \dots, H_4 and have l as chord, which system is always represented on a straight line of a .

A straight line k of a is the image of a system S of twisted cubics of a quadratic surface σ which passes through H_1, \dots, H_4 and contains l .

¹⁾ For these results and other derived numbers cf. SCHUBERT: *Kalkül der Abzählenden Geometrie*, pp. 176 and 182.

The individuals of S are those twisted cubics of σ which pass through H_1, \dots, H_4 and have l as chord.

The surfaces σ which correspond to the lines k of a , form a net.

The curves of S have one point in common with any straight line of σ which cuts l . Hence a line k of a forms a degenerate conic m^2 with any tangent q to k^2 . For the system of curves k^3 cutting the straight line of σ which passes through the point P of l that corresponds to q and which is different from l , consists of the system S associated to k and of the curves k^3 through P .

To the straight lines of a plane pencil of a there correspond the surfaces σ of the pencil the base curve of which consists of l and the curve k^3 which is associated to the vertex of the pencil. The surfaces σ corresponding to the two tangents of k^2 in the plane pencil, are the quadratic cones which project k^3 out its two points of intersection with l .

A surface σ is the locus of the curves k^3 which cut a straight line through a point H_i outside H_i . This line is the straight line of this surface belonging to the same scroll as l which passes through H_i and it may coincide with any straight line through H_i . The two points of intersection of a straight line k with k^2 or with a conic m^2 are accordingly the images of two curves of the system associated to k^2 or to m^2 which cut a straight line through H_i outside H_i .

Accordingly the biquadratic surfaces of the twisted cubics which pass through four given points and which, besides, have a given chord and cut a given straight line or touch a given straight line, have a double point in each of the four given points. These surfaces have b resp. as nodal line and as cuspidal line.

If the line m is a chord of a curve k^3 , m and l belong to the same scroll of a surface σ , the individual of the pencil containing k^3 which passes through a point of m outside k^3 . In this case m is cut twice by any of the ∞^1 curves k^3 of σ , so that the conic m^2 corresponding to m is degenerate in a double line k . The chords of the curves k^3 , i. e. the straight lines of the surfaces σ which belong to the same scroll as l , form a quadratic complex C^2 . For there is a pencil of surfaces σ which pass through a given point of space. The base curve of this pencil consists of l and the curve k^3 which passes through the chosen point, and the lines of the surfaces σ through this point which belong to the same system of generatrices as l , form the quadratic cone which projects the said curve k^3 out of the point in question. Evidently the complex C^2 has cardinal points H_i and it contains the line l . The complex cone of which the vertex lies in a point of l , contains l and the four points H_i . C^2 is the tetrahedral complex with tetrahedron of singularities $H_1H_2H_3H_4$ to which l belongs ¹⁾.

As a straight line k of a has two points in common with k^2 , a surface

¹⁾ Cf. STURM: Liniengeometrie, I p. 335.

σ contains two curves k^3 which touch l , hence also two which touch a generatrix of the system which contains l . C^2 is particularly the complex of the tangents to the curves of our congruence. Any straight line of C^2 is a tangent to two curves k^3 .

§ 5. The curves of our congruence which touch a given plane β and which, accordingly, form a surface, are represented on the points of a curve of α . The number of points of intersection of this curve with a line k of α is the number of curves k^3 touching β which lie on the surface σ corresponding to k . Now the curves k^3 of σ cut the plane β in the point-triplets of a cubic involution, which has the intersection of σ and β as carrier. As this involution contains three triplets with two coinciding points, there lie on σ four curves k^3 touching β , so that the image curve of the system Σ_2 of the curves k^3 which touch β , is a curve b^4 of the fourth order.

The image point of a curve k^3 which touches β , always lies on a generatrix of a plane pencil chosen at random in α . Consequently any curve of Σ_2 lies on one of an arbitrary pencil of surfaces σ .

This pencil intersects β along a pencil of conics of which one of the base points lies in the point $l\beta$. The curves of Σ_2 touch β in the points of a curve which has four points outside the base points in common with any conic of the latter pencil. The point $l\beta$ is a double point of the locus of the points of contact. For on the cone \varkappa which has $l\beta$ as vertex and which contains l and H_1, \dots, H_4 , there lie two curves k^3 each of which touches one of the common straight lines of \varkappa and β . We see that the locus of the points where the curves of Σ_2 touch the plane β , is a cubic with a double point in $l\beta$.

Consequently the system Σ_2 and the curve b^4 are rational. Any surface σ through a curve k^3 which is associated to a double point of b^4 , to which surface there corresponds a straight line of α through this double point, has the property that two of the four curves of Σ_2 which lie on an arbitrary surface σ , coincide in the curve associated to the double point. Hence two of the double points of the involutory (2, 2)-correspondence on the intersection of σ with β , which we get by associating to each other the points of intersection of the same k^3 with β , coincide in the point of contact of the k^3 which corresponds to the double point of b^4 . This k^3 has, therefore, three points of intersection with β coinciding in its point of contact, so that it osculates β .

Now the rational curve b^4 has three double points.

There are accordingly three twisted cubics which pass through four given points, have a given chord and osculate a given plane.

To the tangent to b^4 at an arbitrary point of this curve there corresponds the surface σ through the curve k^3 associated to this point which touches β in the point of contact of k^3 . For the intersection of this surface σ and β consists of the lines which join the point of contact

with $l\beta$ and with the point of intersection of k^3 and β . The latter straight line is touched by two curves k^3 of σ . The other two curves k^3 of σ which touch β , coincide in the curve which is associated to the point chosen on b^4 .

As a pencil of surfaces σ contains three individuals which touch a plane β , in a plane pencil of a there lie three tangents of b^4 , so that b^4 is of the third class. Hence the three double points of b^4 must be cusps. This may also be seen by observing that through a k^3 which corresponds to a double point of b^4 , there passes only one surface σ touching β . This surface cuts β along the tangent in β of k^3 and the join of the point of contact and $l\beta$. One of the two curves k^3 of σ which touch the former straight line, coincides with the k^3 which corresponds to the double point of b^4 , so that three of the curves k^3 of σ which touch β , coincide with this curve and this surface σ is accordingly associated to the tangent of b^4 in the double point in question.

The cone \varkappa containing l and H_1, \dots, H_4 which has $l\beta$ as vertex, is represented on a tangent to k^2 which is a double tangent of b^4 . For \varkappa contains only two curves k^3 of Σ_2 , those curves k^3 of \varkappa which touch the two common lines of \varkappa and β in the vertex of \varkappa .

The system Σ_2 of the curves k^3 of our congruence which touch a given plane β , is represented on a biquadratic curve with three cusps. The double tangent of this curve is at the same time the tangent to k^2 at the point associated to $l\beta$.

If the points of contact of the double tangent of b^4 are real, b^4 has only one real cusp; if the said points of contact are imaginary b^4 has three real cusps.

Hence:

The three curves of our congruence that osculate a given plane β , are real if the cone \varkappa which contains l and H_1, \dots, H_4 and which has $l\beta$ as vertex, has no real straight line in common with β . If \varkappa cuts the plane β along two real lines, two of the three curves of our congruence which osculate β , are imaginary.

If β touches the cone \varkappa , the image curve of Σ_2 degenerates into the tangent of k^2 which represents the system of curves k^3 of \varkappa , all of which touch β , and into a cubic with one cusp of which the said tangent of k^3 is an inflectional tangent. The point of inflection is the image of the curve k^3 on \varkappa which touches the line common to \varkappa and β in the vertex of \varkappa and which, accordingly, osculates β there. Two of the three curves k^3 which osculate an arbitrary plane, coincide in this curve; the third is associated to the cusp of the said cubic.

The three cuspidal tangents of b^4 pass through one point. The three surfaces σ which pass through the curves k^3 that osculate β and which touch β , belong, therefore, to a pencil and contain the same curve k^3 .

As b^4 has eight points in common with a conic m^2 and with k^2 we find:

There are eight twisted cubics which pass through four given points, have a given chord, cut a given straight line, and touch a given plane.

There are eight twisted cubics which pass through four given points and touch a given straight line and a given plane.

Two curves b^4 have sixteen points of intersection.

Accordingly there are sixteen twisted cubics which pass through four given points, have a given chord, and touch two given planes.

§ 6. The chords of the curves k^3 of a surface σ through a point P form the quadratic complex-cone of the tetrahedral complex C^2 with vertex in P . For these chords belong certainly to C^2 and, inversely, the curve k^3 through a point of intersection of a generatrix of the said complex-cone with σ which cuts this generatrix twice, must lie entirely on σ . This k^3 passes, therefore, through the second point of intersection of the above mentioned generatrix and σ .

The complex-cone of C^2 and the cone of the tangents to σ , both of which have their vertex in P , cut each other along four straight lines. These are tangents through P of curves k^3 of σ . As, accordingly, on a surface σ there lie four curves k^3 of which a tangent passes through an arbitrary point P , the image curve of the system Σ_3 of the ∞^1 curves k^3 of which the developable surfaces pass through a given point P , has four points in common with any straight line of α . Consequently the system Σ_3 is represented on a biquadratic curve p^4 of α .

The numbers of points of intersection of a curve p^4 with a conic m^2 , k^2 , a curve b^4 and another curve p^4 lead resp. to :

There are eight twisted cubics which pass through four given points, have a given chord, have a tangent through a given point, and cut a given straight line.

There are eight twisted cubics which pass through four given points, touch a given straight line, and have a tangent through a given point.

There are sixteen twisted cubics which pass through four given points, have a given chord, have a tangent through a given point, and touch a given plane.

There are sixteen twisted cubics which pass through four given points, have a given chord, and have a tangent through each of two given points.

§ 7. We shall determine the number of curves k^3 of a surface σ of which one of the planes of osculation passes through a given straight line g . If we project the curves k^3 of σ out of one of the points of intersection G of g and σ on an arbitrary plane γ , the projections k'^3 form a pencil of cubics which have a double point in the intersection D of γ with the generatrix d of σ through G that belongs to the same scroll as l and which have single points in the intersection of γ with the other straight line of σ through G and in the projections of H_1, \dots, H_4 out of G on γ . This pencil contains five degenerations. Such

a degeneration consists of the conic through four of the single base points and D and the join of D and the fifth single base point.

The locus of the points where curves of the pencil touch the straight lines through the intersection Q of g and γ , is a curve k^5 of the fifth order as a straight line through Q is touched by four curves and the locus contains Q . As there are two curves which have a cusp in D , the projections of the two curves k^3 of σ which touch d , and as there is at the same time a curve which touches the line QD at D , k^5 has a triple point in D . The branch of k^5 corresponding to the latter curve evidently touches this curve and, therefore, also the line QD at D . Further k^5 passes through the five single base points and the tangent at such a base point to k^5 passes through Q .

As the triple point D counts for three double points, k^5 has the class $5 \times 4 - 3 \times 2 = 14$. Two of the fourteen tangents to k^5 through Q coincide with the tangent at Q , five pass through the single base points of our pencil and one joins Q to the common double point. Accordingly there remain six tangents for each of which two of the curves k'^3 which touch it, coincide. These are inflectional tangents of curves k'^3 through Q . Consequently the inflectional tangents of the curves of our pencil envelop a curve of the sixth class ¹⁾.

Now an inflectional tangent through Q is the intersection with the plane of projection of a plane of osculation through g to a curve k^3 on σ . We see, therefore, that a surface σ contains six curves k^3 of which a plane of osculation passes through a straight line g given at random, and that, consequently, the image curve of the system of the twisted cubics of our congruence which have a plane of osculation through g and of which the developable surfaces, therefore, touch g , is a curve g^6 of the sixth order.

The numbers of points of intersection of g^6 with a conic m^2 , k^2 , a curve b^4 , a curve p^4 , and another curve g^6 lead resp. to:

There are twelve twisted cubics which pass through four given points, have a given chord, cut a given straight line and have a plane of osculation through a given straight line.

There are twelve twisted cubics which pass through four given points, touch a given straight line and have a plane of osculation through a given straight line.

There are 24 twisted cubics which pass through four given points, have a given chord, touch a given plane and have a plane of osculation through a given straight line.

There are 24 twisted cubics which pass through four given points, have a given chord, have a tangent through a given point, and a plane of osculation through a given straight line.

There are 36 twisted cubics which pass through four given points, have

¹⁾ In this way the same class has been determined for a general pencil of cubics by Prof. H. J. VAN VEEN, Nieuw Archief voor Wiskunde, 2e reeks, dl. XII, p. 284.

a given chord and have a plane of osculation through each of two given straight lines.

§ 8. Now the system M of the ∞^4 conics m^2 of § 3 will be investigated more closely. Besides the system Δ of the ∞^3 pairs of lines which we found by combining the straight lines of α with the tangents to k^2 , M contains four more systems of ∞^3 pairs of lines. For if m approaches a straight line m_1 through H_1 and if we consider the curves k^3 which cut m , the curves k^3 of which the points of intersection with m do not approach H_1 , finally form the surface σ_1 which contains m_1 , whereas the curves k^3 of which the points of intersection with m approach H_1 , become those curves which touch at H_1 the limitary position μ_1 of the plane Hm . The locus of the latter curves k^3 is the surface σ'_1 which touches μ_1 at H_1 . Now for μ_1 we may choose any plane through m_1 . We find accordingly that to a surface σ_1 there correspond those surfaces σ'_1 which touch the line m_1 of σ_1 through H_1 belonging to the same scroll as l , at H_1 and which form, therefore, a pencil that contains σ_1 of which the base curve consists besides of l of the curve k^3 which touches m_1 at H_1 .

The ∞^3 pairs (σ_1, σ'_1) are represented on ∞^3 pairs of rays (k_1, k'_1) of α which form degenerate conics of M . To a straight line k_1 there corresponds a plane pencil of lines k'_1 which has a point K_1 on k_1 as vertex. To a pencil of surfaces σ'_1 there corresponds one surface σ_1 , the one which contains the tangent at H_1 to the cubic part of the base curve of the pencil. Consequently to a point K_1 there corresponds one line k_1 . The line elements (k_1, K_1) form, therefore, a *bilinear null system* ¹⁾ N_1 , which we get by associating to any line k_1 of α the image point of the curve k^3 which touches the line m_1 through H_1 of the surface σ_1 corresponding to k_1 , at H_1 . It is easily seen that the lines t_2, t_3 and t_4 and the points T_{34}, T_{24} , and T_{23} are singular for N_1 .

In this way we find a system Δ_1 of ∞^3 degenerations of M consisting of the pairs of rays which are formed by the lines k_1 of α and the straight lines through the null point K_1 of k_1 . In the same way it appears that to the points H_2, H_3 , and H_4 there correspond three systems Δ_2, Δ_3 , and Δ_4 of degenerations of M which may be derived from null systems N_2, N_3 , and N_4 .

M also contains ∞^3 degenerations in pairs of points. The surfaces σ which touch an arbitrary line m , are represented on the tangents to the conic m^2 corresponding to m . If we choose m in the tetrahedral complex C^2 , the surfaces σ touching m form two pencils which have the two curves k^3 that touch m , as base curves so that m^2 degenerates into the pair of points which consists of the image points of the two above mentioned curves k^3 . The carrier of this pair of points is the image of

¹⁾ Cf. Prof. JAN DE VRIES. *Plane Linear Null Systems*, these Proceedings 15, 1165.

the surface σ which contains m . Accordingly an arbitrary line k of a to which the ∞^1 lines m of a scroll of the surface σ corresponding to k are associated, contains ∞^1 of such pairs of points. As a curve k^3 of σ is touched by two generatrices of the said scroll, and, therefore, touches the same generatrix as two other curves k^3 of σ , the pairs of points of a line k of a which give degenerations of M , form an involutory (2, 2)-correspondence on k . Any pair of points of one of the lines t_1, \dots, t_4 is a quadruple pair of M , as two conics in one of the planes β, \dots, β_4 have four common tangents.

Let us now determine the number of the conics m^2 that pass through four given points a . To them correspond the four lines m which cut in different points the four curves k^3 of which the given points are the images.

Let us first consider the surface of the lines which cut a given straight line r and two curves k^3 in different points. As through any point of r there pass five generatrices of this surface and a plane through r contains nine generatrices, this surface is of the order 14. The plane through H_k and r contains four generatrices through H_k two of which cut one curve and two of which cut the other one outside H_k . We find one more generatrix through H_k , if we cut the common plane of contact of the two curves at H_k with r and if we join the point of intersection to H_k . Consequently each point H_k is a five-fold point of the surface which cuts a third curve k^3 outside the points H_k , hence in $3 \times 14 - 4 \times 5 = 22$ points.

The surface of the lines which cut three curves k^3 in different points, is, therefore, of the order 22. The generatrices of this surface through H_k are in the first place three lines each of which cuts two of the curves k^3 in separate points different from H_k . Further any plane of contact of two curves k^3 contains two generatrices which cut the third curve k^3 outside H_k . Accordingly we find that any point H_k is a nine-fold point of the surface and that a fourth curve k^3 cuts this surface outside the points H_k in $3 \times 22 - 4 \times 9 = 30$ points. There are, therefore, thirty lines which cut four curves k^3 in different points. Among them the line l , which cuts each of the curves k^3 twice, counts 16 times. As many conics m^2 passing through the four given points are associated to the other 14 lines.

The result found above may also be derived by the aid of the well known property that the number of conics of a system of ∞^1 individuals which pass through a given point, is equal to $1/3$ of the number of degenerations into two straight lines which are contained in the system if this system does not contain any degenerations into pairs of points. The latter is the case with the system T of the conics of M which pass through the three given points A_1, A_2 , and A_3 of a . To T there belong six pairs of lines of Δ , e. g. those which are formed by the lines A_2A_3 and the two tangents through A_1 to k^2 . If we combine

$A_2 A_3$ and the line which joins A_1 and the null-point K_1 of $A_2 A_3$, we get a pair of lines of Δ_1 which belongs to T . As the null points K_1 of the lines through A_1 form a conic which cuts $A_2 A_3$ in two points, there are two more pairs of lines of Δ_1 belonging to T , which consist of $A_2 A_3$ and a straight line through A_1 . Accordingly 9 pairs of each of the systems $\Delta_1, \dots, \Delta_4$ belong to T so that in all T contains 42 pairs of lines. Hence 14 conics of M pass through four given points.

If we associate to each other the points of intersection of each conic of T with an arbitrary straight line, a (14, 14)- correspondence arises on this straight line, so that, as T does not contain any double lines, M contains 28 conics which pass through three given points and touch a given straight line.

This appears also if we remark that to the conics that satisfy the above mentioned conditions, there correspond lines m which cut three given curves k^3 in different points and which touch a surface σ . The surface of the order 22 of the lines which cut three curves k^3 , has 44 lines in common with the quadratic line-complex of σ . Among them l , which is a double line of the complex of tangents, counts 16 times, so that in fact there remain 28 lines.

The congruence (5, 9) of the lines which intersect two curves k^3 , has 56 lines in common with the congruence (4, 4) of the tangents to two surfaces σ , among which l again counts 16 times. There are, therefore, 40 conics m^2 which touch two given straight lines and pass through two given points.

The scroll of the common tangents to three surfaces σ , which is of the order 16, contains 48 lines which intersect a curve k^3 . If again we set apart l 16 times, it appears that there are 32 conics m^2 which pass through a given point and touch three given straight lines.

Finally we find that four surfaces σ have 32 common tangents, if we count l 16 times. Consequently there are 16 conics m^2 which touch four given straight lines.

Physics. — “*The thermal expansion of copper between $+101^{\circ}$ C. and -253° C.*” By W. H. KEESOM, F. P. G. A. J. VAN AGT and Miss A. F. J. JANSEN. (Communication N^o. 182a from the Physical Laboratory at Leiden).

(Communicated at the meeting of February 27, 1926)

§ 1. *Introduction.* As a continuation of the investigations of the thermal expansion of solid substances at low temperatures, which have been made in this laboratory by KAMERLINGH ONNES and HEUSE (Jena-glass 16^{III} and Thuringian-glass to -182° C.)¹⁾, by KAMERLINGH ONNES and CLAY (Jena-glass 16^{III} and platinum to -182° C.)²⁾ and by VAN AGT and KAMERLINGH ONNES (Jena-glass 16^{III} to -253° C.)³⁾, we have investigated that of copper to the temperature of liquid hydrogen boiling under atmospheric pressure.

§ 2. *Method and apparatus.* We followed the method given by VAN AGT and KAMERLINGH ONNES (l. c.). A vacuumvessel about 1 metre long (see fig. 1) was constructed from electrolytic copper by L. OUWERKERK, “technician 1st class” at the Physical Laboratory. The inner wall of this vacuumvessel was brought to different temperatures by pouring into it successively water at about room temperature, boiling water, and liquefied gases. Bent platinum points *B, C, B', C'* were fixed to the bottom and to the top of the inner and outer tubes, and the variations of the distances between them were measured by the method described in Comm. N^o. 176a. The outer wall was kept at constant temperature by means of a waterjacket. The vacuum was produced and maintained by means of charcoal in liquid air, the charcoaltubes *N₁* and *N₂* being connected in turn through the taps *P₃*, *P₁* and *P₂*. Charcoal was also placed into the vacuum space at *O*. For the measurement with liquid hydrogen tap *P₃* was closed. Owing to the greater conductivity of copper, compared to that of glass, the liquefied gas, brought in it, evaporated appreciably quicker than with the measurements of Comm. N^o. 176a. The vacuumvessel was therefore provided with a widened part *R*, made of new-silver, so that it could hold a larger quantity of liquid.

The kathetometers, provided with micrometer-eyepieces, which served for observing the platinum points, now carried a lamp with reflector to illuminate the field of view. A standard-metre was mounted near the tube, so that the standard-metre and points could be seen sharply with

¹⁾ These Proceedings 7, 674, 1905, Comm. Leiden N^o. 85.

²⁾ These Proceedings 9, 199 1906, Comm. Leiden N^o. 95b.

³⁾ These Proceedings 28, 667, 1925, Comm. Leiden N^o. 176a. Comp. F. P. G. A. J. VAN AGT, Thesis for the Doctorate Leiden, p. 61.

the same adjustment of the telescope. The magnification of the microscopes was determined by measuring, before each measurement of the distance of the platinum points, some mm.'s of the standard-metre.

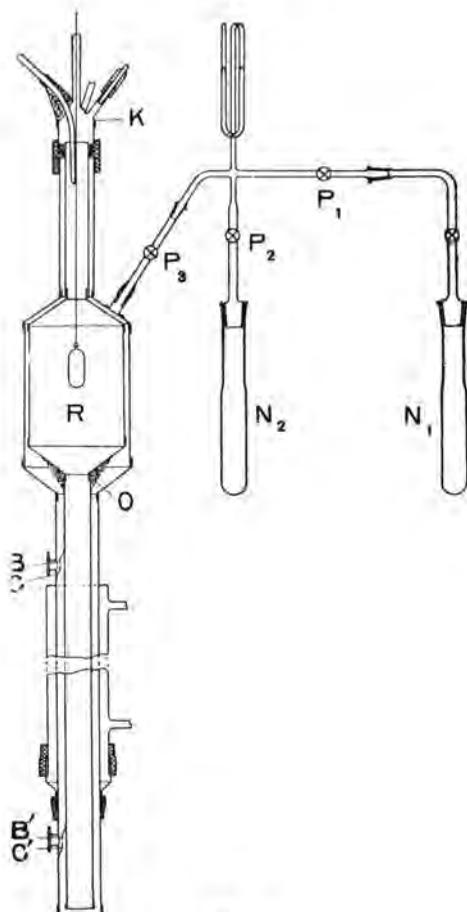


Fig. 1.

The distance of the points was determined repeatedly at room temperature. The temperatures of the inner and outer walls were read off with mercury thermometers, graduated in tenths of a degree. Using the value, found by FIZEAU, for the expansion coefficient at this temperature, this result was reduced to the temperature of 0° C. for both inner and outer walls. The measurements at room temperature were repeated before and after each measurement at low temperatures in order to ascertain that the distance between the points had regained its original value, and that the points had not been distorted; for, during the filling with liquid hydrogen, the tube bent appreciably.

Besides those at room temperature, measurements were made at the boiling-points of water, ethylene, oxygen and hydrogen.

In order to obtain a temperature of about 100° C., boiling distilled

water was poured out into the tube, and then a heating coil introduced to keep the water boiling. The temperature of the water at the top was derived from the reading of the barometer; that of the tube at the bottom by taking into account the hydrostatic pressure of the water column; the average of both temperatures being taken. The temperature of the hydrogen bath was determined in the same way. In this case the complementary pressure in the gasholder was added to the height of the barometer.

The temperatures of the ethylene and of the oxygen baths were determined by means of the resistance thermometer Pt_{32} , which had been calibrated several times with the helium thermometer, and which was placed half-way between the platinum-points.

Before the tube was filled with a liquefied gas, it was cooled by means of liquid air. This appeared to be particularly necessary in the case of hydrogen, otherwise too much of it had to be used to cool the tube. The tube was kept for some moments filled with liquid air, and then the air was quickly siphoned out. After this the new-silver cap K was put on, and connected with the vacuum-lead in order to evacuate the tube. The tube could now be filled with liquid hydrogen. A float was passed through the cap to indicate if the liquid stood in the tube, and its height.

§ 3. *Results.* After some provisional measurements, the results of Table I were obtained.

TABLE I.

| Date | Temperature | | $L_{inner} - L_{outer}$ |
|-------------|------------------|------------------|-------------------------|
| | Inner tube | Outer tube | |
| Dec. 1 '25 | reduced to 0° C. | reduced to 0° C. | 0.129 mm. |
| " 17 '25 | " | " | 0.120 " |
| " 22 '25 | -252.84° C. | " | -2.600 " |
| " 23 '25 | reduced to 0° C. | " | 0.158 " |
| Jan. 5 '26 | " | " | 0.149 " |
| " 6 '26 | " | " | 0.153 " |
| " 18 '26 | " | " | 0.155 " |
| " 19 '26 | " | " | 0.159 " |
| " 21 '26 | -102.87° C. | " | -1.342 " |
| Febr. 3 '26 | reduced to 0° C. | " | 0.157 " |
| " 4 '26 | -185.47° C. | " | -2.286 " |
| " 10 '26 | reduced to 0° C. | " | 1.600 " |

The platinum-points were distorted somewhat before the measurement at 100° C. The following measurements were then made.

TABLE I (Continuation).

| | | | |
|--------------|---------------------------|---------------------------|-----------|
| Febr. 17 '26 | reduced to 0° C. | reduced to 0° C. | 0.782 mm. |
| .. 18 '26 | 100.94° C. | .. | 2.323 .. |
| .. 19 '26 | reduced to 0° C. | .. | 0.772 .. |

From the above the following changes of length were derived:

| | | |
|--|-------------|-----------------------------|
| From 0° C. to -102.87° C. | contraction | $0.155 + 1.342 = 1.497$ mm. |
| .. -102.87° C. to -185.47° C. | .. | 0.944 .. |
| .. -185.47° -252.84° | .. | 0.314 .. |
| .. 0° C. -100.94° .. | expansion | 1.546 .. |

Length of the inner tube: 947.96 mm.

The change of length as a function of the temperature is represented in Fig. 2.

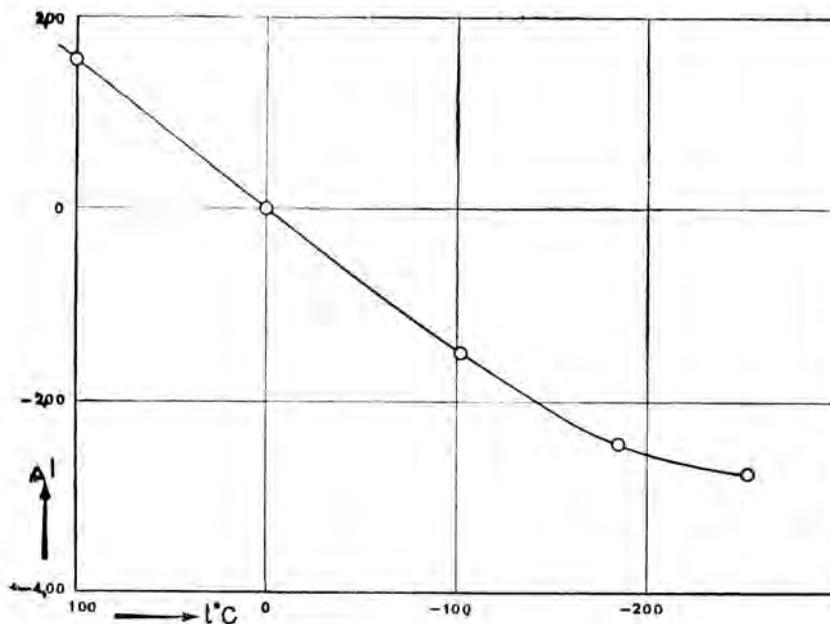


Fig. 2.

§ 4. *The expansion coefficient.* From the results the following mean expansion coefficients were calculated.

| temperature range | \bar{a} |
|------------------------------|------------------------|
| between 100.94° C. and 0° C. | 16.16 $\times 10^{-6}$ |
| 0 ° C. „ — 102.87° C. | 15.35 „ |
| — 102.87° C. „ — 185.47° C. | 12.10 „ |
| — 185.47° C. „ — 252.84° C. | 4.92 „ |

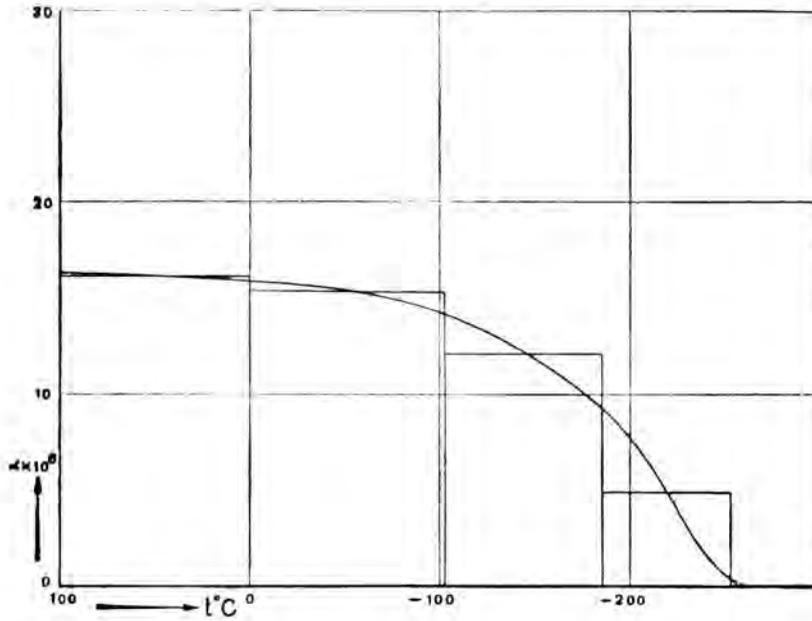


Fig. 3.

In Fig. 3 we derived graphically the expansion coefficient $a = \frac{1}{l} \frac{dl}{dt}$ by drawing a smooth curve, so that, for the temperature ranges mentioned above, $\int_{t_1}^{t_2} a dt = \bar{a}(t_2 - t_1)$. The equality of the areas in question was tested with the aid of a planimeter.

§ 5. Representation of the observations in a formula.

It was not possible to represent the observations over the whole range from + 100.94° C. to — 252.84° C. by a third degree equation. Therefore, in analogy to the formula for the expansion of glass¹⁾, a fourth-degree equation was made of the form:

$$L_t = L_0 \left[1 + \left\{ a \frac{t}{100} + b \left(\frac{t}{100} \right)^2 + c \left(\frac{t}{100} \right)^3 + d \left(\frac{t}{100} \right)^4 \right\} \cdot 10^{-6} \right],$$

¹⁾ F. P. G. A. J. VAN AGT and H. KAMERLINGH ONNES, l.c. p. 786 note 3.

From our observations we calculated the coefficients:

$$\begin{aligned} a &= 1585.338 \\ b &= 18.121 \\ c &= -8.698 \\ d &= 20.662 \end{aligned}$$

§ 6. *Comparison with previous observations.* Previous investigations on copper at low temperatures were made by DORSEY¹⁾, following the method of FIZEAU, in the temperature range from $+10^{\circ}\text{C.}$ to -170°C. ; by HENNING²⁾, who measured between -191°C. and $+16^{\circ}\text{C.}$ the expansion relative to platinum³⁾; by CH. L. LINDEMANN⁴⁾, who measured down to -253°C. the expansion relative to fused quartz, and by BORELIUS and JOHANSSON⁵⁾, who measured the expansion of copper relative to quartz⁶⁾, between 0°C. and -170°C.

The measurements of LINDEMANN do not allow a direct comparison as the measurements are relative to fused quartz, of which the expansion coefficient is not known.⁷⁾

For the other observations we have calculated from their results the mean expansion coefficients for the temperature ranges, mentioned in Table II, and compared these coefficients with ours:

TABLE II.

| Mean expansion coefficient $\times 10^6$ | | | | |
|--|--------|---------|------------------------|--------------------------------|
| Temperature-range | DORSEY | HENNING | BORELIUS and JOHANSSON | KEESOM, v. AGT and Miss JANSEN |
| $0^{\circ}\text{C. to } -100^{\circ}\text{C.}$ | 15.63 | | 15.4 | 15.36 |
| $0^{\circ} \text{ .. } -180^{\circ} \text{ ..}$ | 14.24 | 14.22 | 14.0 | 14.01 |
| $-100^{\circ} \text{ .. } -180^{\circ} \text{ ..}$ | 12.51 | | 12.2 | 12.35 |
| $-180^{\circ} \text{ .. } -250^{\circ} \text{ ..}$ | — | | — | 5.45 |

¹⁾ H. G. DORSEY. Phys. Rev. 25, 88, 1907.

²⁾ F. HENNING. Ann. d. Phys. 22, 631, 1907.

³⁾ The expansion coefficient of platinum in this range was known from measurements made by SCHEEL, Verh. d. D. Physik. Ges. 9, 3, 1907, by the method of FIZEAU.

⁴⁾ CH. L. LINDEMANN. Physik. Z. S. 12, 1197, 1911.

⁵⁾ G. BORELIUS and C. H. JOHANSSON. Ann. d. Phys. (4) 75, 23, 1924.

⁶⁾ For quartz (the authors do not mention what sort of quartz they used) use was made of the measurements of H. G. DORSEY, Phys. Rev. 27, 1, 1908.

⁷⁾ Comp. H. G. DORSEY, note 1.

Physics. — “*Thermionic Emission of the Metals Tungsten, Molybdenum, Thorium, Zircon and Hafnium*”. By C. ZWIKKER. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of February 27, 1926).

1. When the thermionic emission of the metals is considered as an evaporation of the electron gas from the metal, it can be thermodynamically deduced in a simple way that the saturation current per square centimeter may be represented by the following formula¹⁾:

$$i = AT^{\frac{c_p}{R} - \frac{1}{2}} e^{-\frac{\lambda_0}{RT} - \int_0^T \frac{\epsilon}{RT^2} dT} \dots \dots \dots (1)$$

where:

A is a constant.

c_p = heat capacity of 1 gr. mol. of electrons outside the metal.

λ_0 = external heat of evaporation of 1 gr. mol. of electrons at the absolute zero point.

ϵ = energy content of 1 gr. mol. of electrons found inside the metal at the temperature T .

As the “electron gas” may certainly be considered as mon-atomic, we may put $c_p = \frac{5}{2} R$, so that the exponent of T becomes equal to 2 in the above formula.

When also ϵ is neglected, we arrive at RICHARDSON-DUSHMAN's simple formula:

$$i = AT^2 10^{-\frac{b}{T}} \dots \dots \dots (2)$$

in which A and b are constants of material.

The same result is arrived at when ϵ is put independent of the temperature, because it is then included in the b . Consequently the variation of the energy content of the electrons with the temperature, i.e. the THOMSON effect, is neglected in both cases.

By application of SACKUR and TETRODE's theory of the chemical constant, DUSHMAN²⁾ has also been able to calculate the constant A quantitatively. It then appeared that A is independent of the nature of the material, and is equal to:

$$A = \frac{2\pi emk^2}{h^3},$$

¹⁾ Cf. e.g. v. RASCHEVSKY, Z. f. Phys. 32, 1925, p. 746.

²⁾ Phys. Rev. 21, 1923, p. 623.

in which e, m, k , and h have the usual meaning of the fundamental constants. When i is expressed in amperes per cm^2 , then $A=60,2$. v. RASCHEVSKY ¹⁾ arrives at the same value of A by means of a quanta-statistic calculation, however, again if the THOMSON effect is neglected.

Introducing the THOMSON effect, we must, therefore, calculate the integral

$$\int_0^T \frac{\epsilon}{RT^2} dT. \dots \dots \dots (3)$$

A first condition will be that at the absolute zero ϵ be $=0$, and also $\frac{d\epsilon}{dT}$, because otherwise the integral becomes infinite.

When now, in order to find the order of magnitude of the correction of the saturation current ensuing from the THOMSON effect, we put the energy content proportional to the temperature, hence

$$\epsilon = Ne\sigma T,$$

in which, therefore, the THOMSON coefficient σ is taken to be independent of the temperature, and N represents the number of AVOGADRO, we have probably taken ϵ too great, and with even greater probability the integral (3) likewise too great.

This assumption for ϵ gives rise to the appearance of the factor

$$T^{-\frac{Ne\sigma}{R}}$$

in the formula for i .

$\frac{Ne\sigma}{R}$ is of the order $\pm 0,1$, the factor $T^{-\frac{Ne\sigma}{R}}$ is of the order $\frac{1}{2}$ or 2, and almost independent of the temperature. If nevertheless we wish to determine the emission by means of a formula of the form:

$$i = AT^2 10^{-\frac{b}{T}},$$

the quantity A will include the almost constant factor $T^{-\frac{Ne\sigma}{R}}$, i.e. the exact value of A need not be 60,2, but can be a few times larger or a few times smaller. For tungsten for instance, σ is known at high temperatures, viz.:

$$-2.10^{-5} \text{ Volt/degree, } e\sigma = -3,184 \cdot 10^{-24} \text{ JOULE}$$

$$k = \frac{R}{N} = 13,72 \cdot 10^{-24} \text{ JOULE } \text{ hence } \frac{e\sigma}{k} = -0,232.$$

Consequently we expect for tungsten the formula:

$$i = 60,2 T^{2+0,232} \cdot 10^{-\frac{b}{T}}$$

¹⁾ l. c.

which for a mean temperature of 2000° almost coincides with:

$$i = 355 T^2 10^{-\frac{b}{T}}.$$

Tungsten has an exceedingly great THOMSON coefficient. For most other metals A will depart less from the universal value 60.2. Experimentally it can, however, scarcely be decided whether the factor 355 represents the measurements better than the factor 60.2, because a small variation of the quantity b will bring the absolute value of the emission in harmony with the formula.

An attempt to find experimentally something about the value of the quantity A , has been made by DUSHMAN, ROWE, EWALD and KIDNER¹⁾, by measuring the emission of tungsten, molybdenum and tantalum. The result at which they arrived, was that it was not to be proved experimentally that the value 60.2 is wrong.

The writer of this paper has made emission measurements of some metals which enable him to make an investigation of the constant A .

2. Measurements of the very pure tungsten have been published by me in detail elsewhere²⁾.

The mean of the measurements on 5 filaments was:

| T K | i Amp./cm. ² |
|-------|---------------------------|
| 2500 | 0.324 |
| 2400 | 0.119 |
| 2300 | 0.0425 |
| 2200 | 0.0140 |
| 2100 | 0.00398 |
| 2000 | 0.000978 |

For each of these temperatures we can calculate with the formula:

$$i = AT^2 10^{-\frac{b}{T}}$$

what b we must fill in when we put $A = 60.2$. We can do this too for any value of A and will, accordingly prefer that A for which the six values for b are as much as possible equal to each other. I have carried out this calculation for the 5 values of A : 10; 60.2; 100; 355; and 1000, with the following results:

¹⁾ Phys. Rev. 25, 1925, p. 338.

²⁾ Dissertatie Amsterdam 1925. Archives Néerlandaises IX. 1925, p. 301.

Value of b :

| T | $A = 10$ | $A = 60.2$ | $A = 100$ | $A = 355$ | $A = 1000$ |
|------|----------|------------|-----------|-----------|------------|
| 2500 | 20740 | 22670 | 23200 | 24590 | 25730 |
| 2400 | 20850 | 22730 | 23230 | 24600 | 25640 |
| 2300 | 20980 | 22720 | 23230 | 24490 | 25570 |
| 2200 | 20990 | 22730 | 23200 | 24380 | 25420 |
| 2100 | 21120 | 22770 | 23200 | 24390 | 25320 |
| 2000 | 21238 | 22780 | 23230 | 24320 | 25220 |

These calculated values have been drawn in figure 1. It is seen at a glance that the value $A = 100$ gives the most constant value for b . For

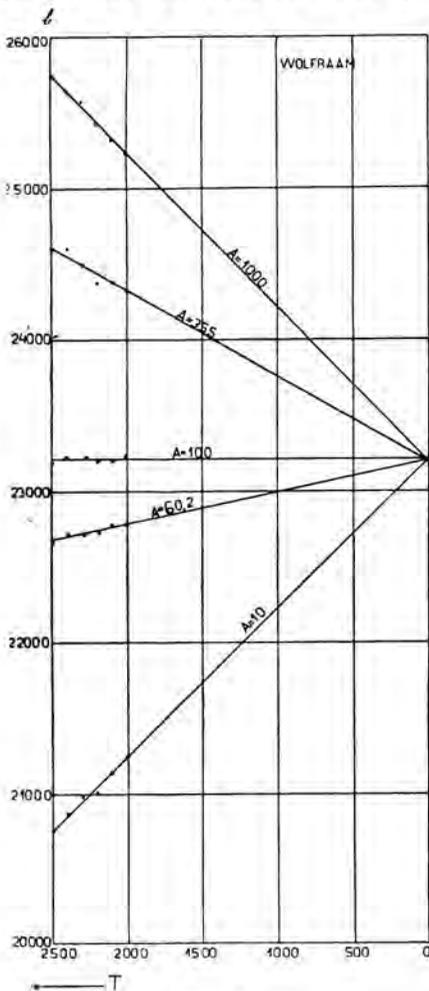


Fig. 1.

$A = 60.2$ the values of b diverge 11 on 2270. In the formula occurs the quotient of b and T . In the temperature a difference of 11° at 2270° can certainly be measured. Estimating the error in the temperature at 5° , that in the emission measurement at 5% , once more corresponding to an error in the temperature of 5° , we see that the experimental error is 10° . Unless we underrate the accuracy of the (already averaged) measurements, it follows from this that it cannot be concluded from the measurements of tungsten that the value 60.2 for A is really wrong. It is, however, certain that the value $A = 355$, which would suppose an error of 27° in the temperature measurement, is inadmissible. This latter is probably caused by our having introduced a too high amount for the integral (3).

3. Emission measurements of molybdenum have been performed in tubes of the model represented in fig. 2 by receiving the electron emission of the central incandescent filament on a spiralized anode. The

course of the measurement was the same as that for tungsten¹⁾ with this difference that for tungsten the right emission is only obtained when the lamp is immersed in liquid air, whereas this appeared to be unnecessary for molybdenum. Part of the glass wall was, however, covered with magnesium to maintain the vacuum. The emission has been measured for 2 different kinds of filaments. The first filament was supplied by PHILIPS, the second was an American filament of the General Electric Company. The incandescent-temperatures have been determined with an optical pyrometer. The brightness-temperature is measured. To reduce this to true temperature the emissive power of molybdenum is required. I have taken for this the values given by WORTHING²⁾.

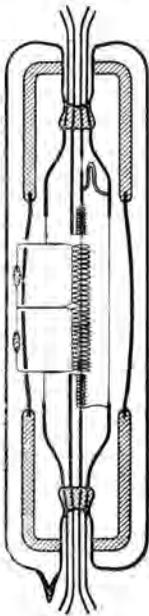


Fig. 2.

The two kinds of filaments gave almost the same results. The PHILIPS filament gave slightly greater emission, about 10%. The deviation inter se of the two filaments was always less than 10° in temperature.

After taking the mean of the two filaments, and after a correction had been applied for the SCHOTTKY effect³⁾ the relation between emission and temperature was:

| T °K | i Amp./cm. ² |
|--------|---------------------------|
| 1900 | 0.000500 |
| 2000 | 0.00234 |
| 2100 | 0.00859 |
| 2200 | 0.0285 |
| 2300 | 0.0835 |
| 2400 | 0.236 |

We now repeat the calculations we have also performed for tungsten, and find the following table for b as function of A and T . (See Table following page).

These points are given in fig. 3. We see from this figure that the most probable value of A is: 65. There remains, however, some licence in the figure as regards the tracing of the lines $A = \text{constant}$. The value 100 for A is not excluded.

4. Dr. DE BOER put some strips of metallic thorium at my disposal, made by decomposition of gaseous thorium compounds at an incandescent

1) l. c.

2) Phys. Rev. 25, 1925, p. 846.

3) By SCHOTTKY effect is understood the phenomenon that the saturation current still increases slightly with the tension. The emissions mentioned here are those which the filament would give in the absence of anode tension.

tungsten filament¹⁾. If this tungsten filament diffuses through the thorium, when the thorium is aged, the thorium strips used can contain about 6‰ of tungsten.

Value of b in molybdenum.

| T | $A = 10$ | $A = 100$ | $A = 1000$ |
|------|----------|-----------|------------|
| 2400 | 20140 | 22530 | 24900 |
| 2300 | 20250 | 22560 | 24860 |
| 2200 | 20300 | 22510 | 24710 |
| 2100 | 20370 | 22430 | 24550 |
| 2000 | 20470 | 22460 | 24460 |
| 1900 | 20630 | 22530 | 24420 |

The electron emission of these strips was very constant and reproducible, when only in the tube a magnesium mirror was adjusted. Definitive measurements have been made with a strip of a cross-section of 0.110×0.0090 cm. and a length of 4.23 cm. According to estimation 1 mm. was lost at each end of the strip, because it had not fully the required temperature. If this estimation is 100% wrong, this gives in the emission per cm^2 an error of no more than 5%.

The temperatures have again been optically determined. The recalculation of brightness temperature to true temperature cannot take place so accurately as in tungsten or molybdenum, because little is known about the emissive power. I have put the emissive power at 0.40.

With these data the relation between temperature and emission corrected for the SCHOTTKY effect becomes:

| T °K | i Amp./ cm^2 |
|--------|-------------------------|
| 1300 | 0.0000133 |
| 1400 | 0.000100 |
| 1500 | 0.000738 |
| 1600 | 0.00430 |
| 1700 | 0.0203 |
| 1800 | 0.0794 |

This emission may be expressed in the formula:

$$i = AT^2 10^{-\frac{b}{T}}$$

¹⁾ Z. f. Anorg. u. Allgem. Chemie 148, 1925, p. 345.

when b is taken as follows:

| T | $A = 10$ | $A = 100$ | $A = 1000$ |
|------|----------|-----------|------------|
| 1300 | 15770 | 17060 | 18350 |
| 1400 | 15800 | 17210 | 18610 |
| 1500 | 15720 | 17220 | 18730 |
| 1600 | 15630 | 17220 | 18820 |
| 1700 | 15570 | 17270 | 18950 |
| 1800 | 15500 | 17310 | 19120 |

Cf. fig. 4. As most probable value of A follows 70. It is, however, the question in how far this result is reliable in connection with the unknown emission coefficient of thorium. I have taken 0.4 for this. If in reality it is 0.5, my temperatures have been taken too high. The error is 20° at 1300° and 32° at 1800° . The relative error is 12° to an average of 1500° . The same relative error occurs in b . Then the most probable value for A becomes 40. It is, however, not probable that the emissive power is so far wrong, as was supposed here.

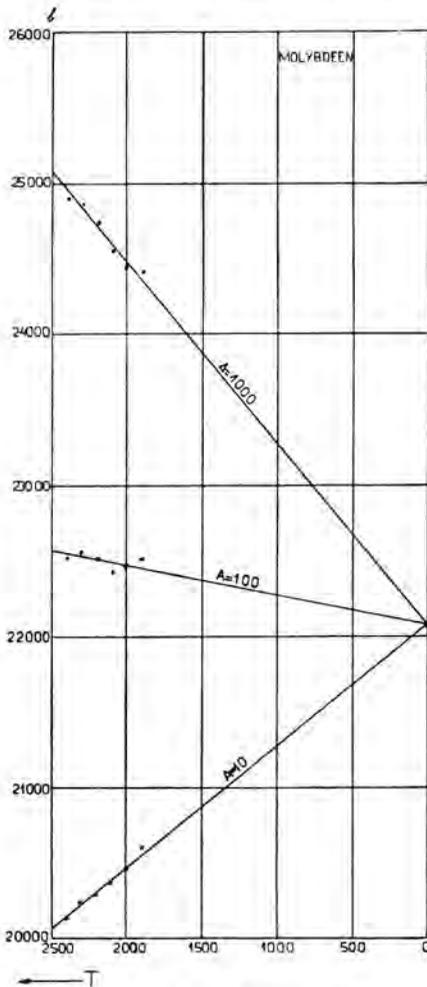


Fig. 3.

5. Also Dr. DE BOER supplied me with a zircon filament which he had prepared on a tungsten nucleus by decomposition. The nucleus was 40 microns, the zircon filament 770 microns, so that the tungsten content was 4.8% in weight. Of this filament I have made three tubes with incandescent filament lengths resp. 8.85, 12.60, and 16.45 cm. By subtracting the emissions of these filaments from each other, I was enabled to eliminate the cooled extremities. In order to convert the measured brightness temperature into true temperature the emissive power has been used, as given by BURGESS

and WALTENBERG ¹⁾, viz. 0.32. The electron emission of zircon appeared to be little sensitive to gas rests, greatly in contrast to that of tungsten. The emission corrected for the SCHOTTKY effect was as function of the temperature:

| T K | i Amp./cm. ² |
|-------|---------------------------|
| 1600 | $5.06 \cdot 10^{-5}$ |
| 1700 | $3.96 \cdot 10^{-4}$ |
| 1800 | $2.29 \cdot 10^{-3}$ |
| 1900 | $1.18 \cdot 10^{-2}$ |
| 2000 | $5.50 \cdot 10^{-2}$ |
| 2100 | $1.97 \cdot 10^{-1}$ |

Calculating the relation between A and b with these data, we get the following table for b :

| T | $A = 10$ | $A = 100$ | $A = 1000$ | $A = 10000$ |
|------|----------|-----------|------------|-------------|
| 1600 | 18720 | 20340 | 21930 | 23550 |
| 1700 | 18460 | 18460 | 21860 | 23570 |
| 1800 | 18270 | 18270 | 21860 | 23670 |
| 1900 | 18030 | 18030 | 21820 | 23690 |
| 2000 | 17720 | 17720 | 21730 | 23720 |
| 2100 | 17530 | 17530 | 21750 | 23820 |

Compare fig. 5. We see that it is out of the question that 60.2 could be the correct value for A . The most probable value for A is according to the figure: $A = 3000$.

6. Finally we can derive the value of A from measurements with a hafnium filament, which has also been made by Dr. DE BOER. These measurements bear a preliminary character, because:

1. the material is not quite pure; it contains 5% of tungsten in consequence of the preparation on a tungsten nucleus, it not being excluded either that it contains a few percentages of zircon, which is difficult to detect by the side of hafnium.

2. there is little known about the losses at the cooled extremity of the filament.

3. the emission coefficient is not known.

¹⁾ Bull. Bur. of St. 11, 1915, p. 591.

Yet the measurements yield a value for A , which departs so much from the theoretical value, that it should certainly be mentioned in this connection.

With regard to the losses at the ends, I have proceeded as if it were zircon. For the emission coefficient I have assumed the estimated value 0.37.

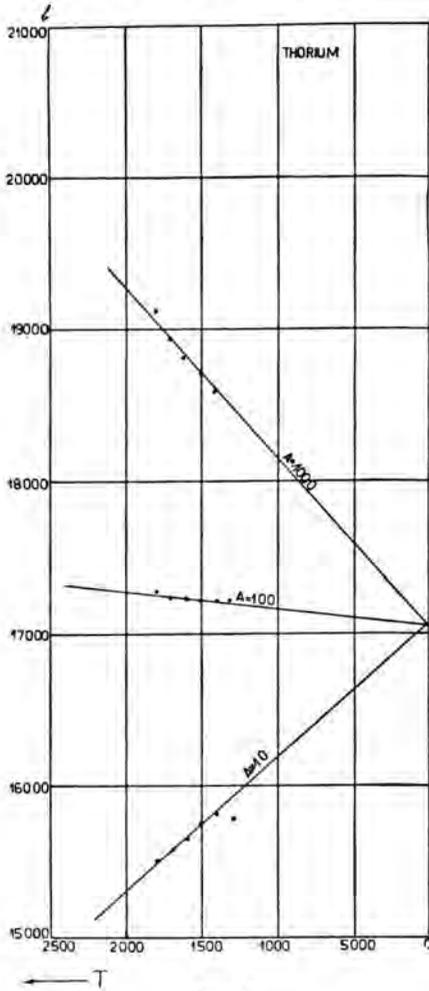


Fig. 4.

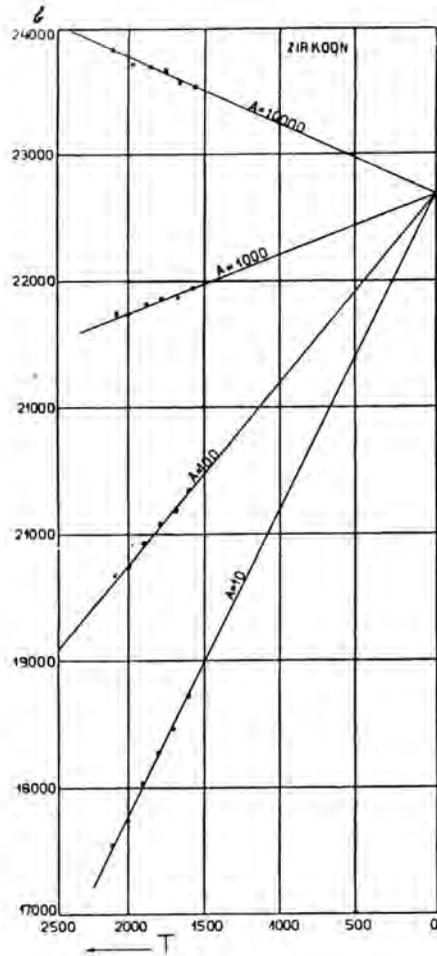


Fig. 5.

The measurements above 1900° (under 1900° not yet explained irregularities occurred) yielded the result.

| T °K | i Amp./cm. ² |
|--------|---------------------------|
| 1900 | $4.80 \cdot 10^{-3}$ |
| 2000 | $2.62 \cdot 10^{-2}$ |
| 2100 | $1.23 \cdot 10^{-1}$ |
| 2200 | $4.85 \cdot 10^{-1}$ |

The corresponding values for A and b are:

| T | $A = 100$ | $A = 1000$ | $A = 10000$ | $A = 100000$ |
|------|-----------|------------|-------------|--------------|
| 1900 | 20620 | 22520 | 24420 | 26340 |
| 2000 | 20350 | 22350 | 24330 | 26350 |
| 2100 | 20070 | 22160 | 24250 | 26390 |
| 2200 | 19810 | 22000 | 24200 | 26410 |

From figure 6, in which these points are indicated, it appears that the most probable value for A is: 55000.

7. Summarizing we may, therefore, say that the metals tungsten, molybdenum, thorium, and according to DUSHMAN c. s. also tantalum yield a

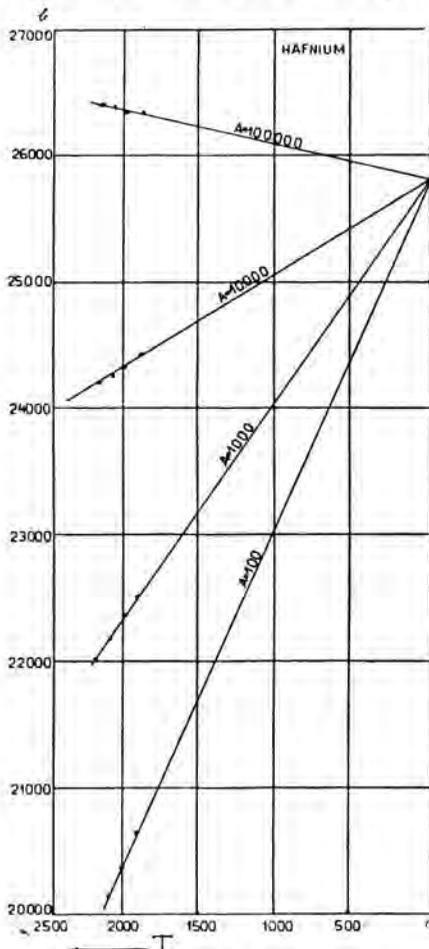


Fig. 6.

value for A which is in accordance with the theoretical value, perhaps slightly modified by the THOMSON effect. The values which zircon and hafnium give, depart however greatly from it.

Also DUSHMAN c. s. observe that greatly divergent values of A do occur. In this case there is, however, always question of a thin (sometimes mono-molecular) layer of emitting substance on a metal filament. For instance in the emission of a tungsten filament covered with cesium A is smaller than 0.003^1), and in the emission of a superficially oxidized tungsten filament A is $\approx 5 \cdot 10^{11}$. Without giving a further explanation of it, DUSHMAN c. s. state that A will always become greater than the theoretical value when the covering layer is electronegative.

When we adopt this view we might account for the high values which zircon and hafnium give for A by assuming that they are covered up to $2200^\circ K$ (melting-point of zircon) by an electronegative layer. In favour of the view that zircon

¹ KINGDON, Phys. Rev. 1924, p. 510.

at low temperatures is covered with an oxide layer, speaks its unsusceptibility to mineral acids, a property that it shares with aluminium and magnesium, of which it is almost certain that they are covered with a coat of oxide. If it is, besides, taken into account that zircon oxide melts higher than zircon itself, and probably the velocity of evaporation of zircon-oxide is smaller than that of zircon, it is very well probable that zircon remains covered up to its melting-point. A same reasoning applies to hafnium. In this case we should, so far, have no example as yet of a pure metal surface the emission of which could not be represented by:

$$i = 60.2 T^2 10^{-b/T} \text{ Amp/cm}^2.$$

Mathematics. — "On the Geometry of the Group-manifold of simple and semi-simple groups". By E. CARTAN and J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 27, 1926).

If the ∞^r transformations of a continuous group with r parameters are represented by points of an X_r , then the group-manifold (variété du groupe, Gruppenmannigfaltigkeit) arises. The parameters play the part of coordinates of the X_r . In the present paper we will prove that there exist in X_r three connexions (connexions, Uebertragungen) two of them being not symmetrical but having zero curvature, the third being symmetrical (à torsion nulle) and in the case of a simple or a semi-simple group Riemannian. The Riemannian connexions, formed in the above manner, are of a very particular type, having, as far as we know, not yet been examined except the case $r=3$, where the curvature becomes constant. The main feature of these geometries is the possession of two absolute parallelisms, who in the case $r=3$ pass into the well known parallelisms of CLIFFORD. In a subsequent article we will prove that this property is characteristic for the geometries of semi-simple groups, but for one exception, which is in close connexion with the non-associative numbersystem with 8 unities of GRAVES-CAYLEY.

§ 1. *The first displacement with zero curvature* (—).

The transformations of the group being given by the equations

$$x^k = x^k(x^1, \dots, x^n, \xi^{a_1}, \dots, \xi^{a_r}); k=1, \dots, n, \dots \dots (1)$$

the ξ^v are coordinates in the group-manifold X_r .

T_ξ corresponding with ξ^v and $T_{\xi+d\xi}$ with $\xi^v + d\xi^v$, the linear element $d\xi^v$ corresponds with the infinitesimal transformation

$$T_{\xi+d\xi} T_\xi^{-1} \quad 1)$$

This infinitesimal transformation has not only significance in the point ξ^v . In any other point η^v it corresponds with the linear element, extending from T_ξ to $T_{\eta+d\xi} T_\xi^{-1} T_\eta$. By this correspondence, which may be characterized by the equation

$$T_{\xi+d\xi} T_\xi^{-1} = T_{\eta+d\xi} T_\eta^{-1} \quad 2) \dots \dots \dots (2)$$

1) This means that T_ξ^{-1} has to be executed first and then $T_{\xi+d\xi}$.
 2) See note page 807.

a connexion in X_r is defined, and this connexion is integrable, since because of (2) with a definite linear element in any point corresponds one and only one linear element in any point of the neighbourhood. If this connexion, which we will represent by $(-)$, were symmetrical, it would be possible to lay in X_r a Cartesian system of coordinates and X_r would be an E_r . We will, however, see that this is not the case in general.

T_u being the general transformation of a subgroup with one parameter, the equation

$$T_\xi = T_u T_{\xi_0} \dots \dots \dots (3)$$

represents a geodesic through ξ^v_0 . Writing (2) in the form

$$T_{\xi'} T_{\xi'}^{-1} = T_{\xi'} T_{\xi'}^{-1} \dots \dots \dots (4)$$

this equation defines the $(-)$ -aequipollence of two finite segments of geodesics.

Besides the system of coordinates ξ^v we introduce in every point of X_r a system of coordinates ε^k , geodesic in that point and we place these systems in such manner that two elements with the same coordinates $d\varepsilon^k$ in different points arise from one another by a $(-)$ -parallel displacement and correspond consequently with the same infinitesimal transformation. Then in every point

$$\frac{\partial' x^k}{\partial \xi^\lambda} = \frac{\partial' x^k}{\partial \varepsilon^j} \frac{\partial \varepsilon^j}{\partial \xi^\lambda} \dots \dots \dots (5)$$

holds good. Now $\frac{\partial' x^k}{\partial \varepsilon^j}$ depends only on $'x^k$ and is independent of the ξ^v because $d\varepsilon^k$ represents in every point the same infinitesimal transformation. $\frac{\partial \varepsilon^j}{\partial \xi^\lambda}$ depends only on the relative situation of the systems ξ^v and ε^k , i.e. therefore on ξ^v and is independent of the $'x^k$. Writing for these functions

$$\frac{\partial' x^k}{\partial \varepsilon^j} = \xi_j^k ('x) \quad ; \quad \frac{\partial \varepsilon^j}{\partial \xi^\lambda} = \psi_\lambda^j (\xi)$$

we obtain

$$\frac{\partial' x^k}{\partial \xi^\lambda} = \xi_j^k \psi_\lambda^j \dots \dots \dots (6)$$

ψ_λ^j being nothing else but the affinor of unity, with regard to ε^k above and to ξ^λ beneath, the rank of ψ_λ^j is r and the equation (5) admits therefore an inversion of the form

$$\xi_i^k = a_i^\lambda \frac{\partial' x^k}{\partial \xi^\lambda} \dots \dots \dots (7)$$

So we have obtained the first part of the first fundamental theorem

1) We omit all signs Σ with regard to two differently placed indices.

of LIE¹⁾ and we see that this theorem is in close connexion with the displacement (—).

Writing $e^v dt$ for $d\xi^v$ and consequently $e^k dt$ for de^k , e^v is a (—)-constant vector in X_r , having consequently constant coordinates with regard to the systems e^k . (5) gives, that de^k corresponds with the infinitesimal transformation

$$d'x^k = \xi_j^k e^j dt$$

or

$$df = \xi_j^k e^j \frac{\partial f}{\partial x^k} = e^j X_j f dt \quad ; \quad X_j f = \xi_j^k (x) \frac{\partial f}{\partial x^k}$$

With e^k corresponds therefore the infinitesimal transformation $e^j X_j f$ or $e^\mu X_\mu f$ in x^k , where

$$X_j f = \xi_j^k(x) \frac{\partial f}{\partial x^k} \quad ; \quad X_\mu f = \psi_\mu^j \xi_j^k(x) \frac{\partial f}{\partial x^k} \quad . \quad . \quad . \quad (8)$$

If v^v_1 and v^v_2 are two vectors which are (—)-constant and correspond therefore with two infinitesimal transformations

$$\begin{aligned} \delta^-_1 v^v &= 0 \quad ; \quad dv^v_1 = -\bar{I}^v_{\lambda\mu} v^\lambda d\xi^v_1 \quad ; \quad dv^k_1 = 0, \\ \delta^-_2 v^v &= 0 \quad ; \quad dv^v_2 = -\bar{I}^v_{\lambda\mu} v^\lambda d\xi^v_2 \quad ; \quad dv^k_2 = 0. \end{aligned}$$

then we have

$$df = v^j_1 X_j f \quad ; \quad df = v^j_2 X_j f$$

Now it is well known that

$$v^i_1 v^j_2 X_i X_j f - v^i_2 v^j_1 X_j X_i f = v^i_1 v^j_2 c_{ij}^k X_k f \quad . \quad . \quad . \quad (9)$$

where $c_{ij}^k = -c_{ji}^k$ are the $1/2 r^2(r-1)$ constants of LIE, which determine the structure of the group. The corresponding linear element is the difference of

$$v^v_2 + v^v_1 + d v^v_1 = v^v_2 + v^v_1 - \bar{I}^v_{\lambda\mu} v^\lambda v^\mu$$

and

$$v^v_1 + v^v_2 + d v^v_2 = v^v_1 + v^v_2 - \bar{I}^v_{\lambda\mu} v^\lambda v^\mu$$

so that

$$v^i_1 v^j_2 c_{ij}^k = -v^\lambda_1 v^\mu_2 (\bar{I}^v_{\lambda\mu} - \bar{I}^v_{\lambda\mu}) \psi^k_v$$

or

$$S^{\lambda\mu}_v = \bar{I}^v_{[\lambda\mu]} = -1/2 c_{\lambda\mu}^v = -1/2 A^{\lambda\mu k} c_{ij}^k \quad . \quad . \quad . \quad (10)$$

¹⁾ Comp. f.i. LIE-SCHEFFERS, p. 376. We have used ξ and α instead of A^j_λ and A^λ_i to make comparison more easy.

$S_{\lambda\mu}^{\nu}$ is the quantity which makes the connexion asymmetrical. The connexion $(-)$ is defined by the equation

$$\delta v^\nu = dv^\nu + I_{\lambda\mu}^{\nu 0} v^\lambda d\xi^\mu + S_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu \dots \dots \dots (11)$$

The c_{ij}^k being constants, $S_{\lambda\mu}^{\nu}$ is a $(-)$ -constant quantity

$$\bar{\nabla}_\nu S_{\lambda\mu}^{\nu} = 0 \quad ; \quad \delta^- S_{ij}^k = dS_{ij}^k = 0 \dots \dots \dots (12)$$

$S_{\lambda\mu}^{\nu}$ is only then equal to zero, if all the expressions $(X_i X_j)$ vanish.

§ 2. *The second connexion with zero curvature (+).*

A second connexion may be defined by means of the equation

$$T_{\xi}^{-1} T_{\xi+d\xi} = T_{\xi}^{-1} T_{\xi+d\xi} \dots \dots \dots (13)$$

This connexion being equally integrable, the curvature is here zero as well. A geodesic passing through ξ^0 is given by the equation

$$T_{\xi} = T_{\xi_0} T_u \dots \dots \dots (14)$$

and is at the same time a geodesic of the $(-)$ -connexion, since

$$T_{\xi_0} T_u = (T_{\xi_0} T_u T_{\xi_0}^{-1}) T_{\xi_0} \dots \dots \dots (15)$$

and $T_{\xi_0} T_u T_{\xi_0}^{-1}$ represents also a group with one parameter. Writing (13) in a finite form

$$T_{\xi}^{-1} T_{\xi'} = T_{\xi}^{-1} T_{\xi'} \dots \dots \dots (16)$$

it follows that the segments $\xi \xi'$ and $\eta \eta'$ are $(+)$ -aequipollent. We will deduce this second connexion in another way and will find a formula corresponding with (11).

Introducing in the equation

$$'x = Tx$$

new variables on both sides,

$$'x' = S'x \quad ; \quad x' = Sx$$

we have

$$'x' = S T S^{-1} x'$$

S and T being transformations of the group, applying the transformation T on x is therefore the same as applying STS^{-1} on Sx . If we take S and T both infinitesimal

$$\begin{aligned} Sx^k &= x^k + s^j X_j x^k dt + \dots \\ Tx^k &= x^k + t^j X_j x^k dt + \dots \end{aligned}$$

in which s^j and t^j are $(-)$ -constant vectors, the transition from T into STS^{-1} corresponds with the linear element $s^j dt$ or, because

$$\begin{aligned}
 S^{-1} &= x^k - s^j X_j x^k dt + \dots \\
 TS^{-1} x^k &= x^k - s^j X_j x^k dt + t^j X_j x^k dt - t^i X_i s^j X_j x^k dt^2 + \dots \\
 STS^{-1} x^k &= x^k + t^j X_j x^k dt - t^i s^j (X_i X_j - X_j X_i) x^k dt^2 + \dots
 \end{aligned}$$

with the transition from t^k into $t^k + s^i t^j c_{ij}^k dt$. So we obtain a new connexion

$$\delta^+ t^k = -s^i t^j c_{ij}^k dt = t^i s^j c_{ij}^k dt = \delta^- t^k + t^i s^j c_{ij}^k dt \quad (17)$$

or in a more general form

$$\left. \begin{aligned}
 \delta^+ v^\nu &= \delta^- v^\nu + c_{\lambda\mu}^{\nu\cdot\cdot} v^\lambda d\xi^\mu \\
 &= dv^\nu + \overset{0}{\Gamma}_{\lambda\mu}^{\nu\cdot\cdot} v^\lambda d\xi^\mu + 1/2 c_{\lambda\mu}^{\nu\cdot\cdot} v^\lambda d\xi^\mu \\
 &= dv^\nu + \overset{0}{\Gamma}_{\lambda\mu}^{\nu\cdot\cdot} v^\lambda d\xi^\mu - S_{\lambda\mu}^{\nu\cdot\cdot} v^\lambda d\xi^\mu
 \end{aligned} \right\} \dots (18)$$

By the (+)-connexion the infinitesimal transformations of the group undergo a transformation and these transformations form the *adjoint group*. Both connexions correspond therefore exactly with the two fundamental relations between the infinitesimal transformations which occur in the classical theory¹⁾.

The quantity $S_{\lambda\mu}^{\nu\cdot\cdot}$ is constant by the (+)-connexion also. Indeed we have

$$\left. \begin{aligned}
 \nabla_{\omega}^+ S_{\lambda\mu}^{\nu\cdot\cdot} &= \bar{\nabla}_{\omega} S_{\lambda\mu}^{\nu\cdot\cdot} - c_{\rho\omega}^{\nu\cdot\cdot} S_{\lambda\mu}^{\rho\cdot\cdot} - c_{\mu\omega}^{\nu\cdot\cdot} S_{\lambda\alpha}^{\rho\cdot\cdot} + c_{\alpha\omega}^{\nu\cdot\cdot} S_{\lambda\mu}^{\rho\cdot\cdot} \\
 &= 0 + \frac{2}{3} c_{[\omega\alpha}^{\nu\cdot\cdot} c_{\mu]}^{\rho\cdot\cdot} = 0
 \end{aligned} \right\} \dots (19)$$

since the JACOBIAN identity gives:

$$c_{[ij}^m c_{l]m}^k = 0 \dots \dots \dots (20)$$

The integrability of the (+)-connexion follows also by applying the general formula for the transformation of the quantity of curvature:

$${}^{\nu}R_{\omega\lambda\mu}^{\nu\cdot\cdot} = R_{\omega\lambda\mu}^{\nu\cdot\cdot} - 2 \nabla_{[\omega} A_{|\lambda|\mu]}^{\nu\cdot\cdot} + 2 \overset{0}{\Gamma}_{[\omega\lambda\mu]}^{\nu\cdot\cdot} A_{\lambda\alpha}^{\nu\cdot\cdot} - 2 A_{\alpha[\omega}^{\nu\cdot\cdot} A_{|\lambda|\mu]}^{\nu\cdot\cdot} \dots (21)$$

valid by the transformation of parameters

$${}^{\nu} \overset{0}{\Gamma}_{\lambda\mu}^{\nu\cdot\cdot} = \overset{0}{\Gamma}_{\lambda\mu}^{\nu\cdot\cdot} + A_{\lambda\mu}^{\nu\cdot\cdot}$$

§ 3. *The symmetrical displacement (0).*

$S_{\lambda\mu}^{\nu\cdot\cdot}$ being an affinor in X_r , the equation

$$\delta^0 v^\nu = dv^\nu + \overset{0}{\Gamma}_{\lambda\mu}^{\nu\cdot\cdot} v^\lambda d\xi^\mu \dots \dots \dots (23)$$

defines a third displacement (0), which is symmetrical.

$S_{\lambda\mu}^{\nu\cdot\cdot}$ is also constant by (0):

$$\overset{0}{\nabla}_{\omega} S_{\lambda\mu}^{\nu\cdot\cdot} = 0 \dots \dots \dots (24)$$

¹⁾ The two expressions (2) and (13) occur for the first time side by side in CARTAN'S paper Bull. d. Sc. Math. 34. 1910, page 250—283.

in consequence of the JACOBIAN identity. By applying (21), we have for the quantity of curvature

$$\left. \begin{aligned} R_{\alpha\mu\lambda}^{0\ \cdot\ \cdot\ \cdot\ \nu} &= -2 S_{\alpha\mu}^{\cdot\ \cdot\ \cdot\ \alpha} S_{\lambda\alpha}^{\cdot\ \cdot\ \cdot\ \nu} - 2 S_{\alpha\lambda}^{\cdot\ \cdot\ \cdot\ \nu} S_{\lambda|\mu}^{\cdot\ \cdot\ \cdot\ \alpha} \\ &= -2 S_{\alpha\mu}^{\cdot\ \cdot\ \cdot\ \alpha} S_{\lambda\alpha}^{\cdot\ \cdot\ \cdot\ \nu} - S_{\lambda\mu}^{\cdot\ \cdot\ \cdot\ \alpha} S_{\alpha\nu}^{\cdot\ \cdot\ \cdot\ \nu} + S_{\lambda\alpha}^{\cdot\ \cdot\ \cdot\ \alpha} S_{\alpha\mu}^{\cdot\ \cdot\ \cdot\ \nu} \\ &= S_{\alpha\mu}^{\cdot\ \cdot\ \cdot\ \alpha} S_{\alpha\lambda}^{\cdot\ \cdot\ \cdot\ \nu} \end{aligned} \right\} \dots (25)$$

from which it follows that $R_{\alpha\mu\lambda}^{0\ \cdot\ \cdot\ \cdot\ \nu}$ is constant for all the three connexions.

The part of the parameters that is symmetrical in $\lambda\mu$ being equal for the three connexions, it follows that they have the same geodesics. The geometrical link between the three connexions is as follows. If $d\xi_\nu$ and $d\xi_\nu$ are two linear elements in ξ_ν , and if $d^0 \xi_\nu$, $d^+ \xi_\nu$ and $d^- \xi_\nu$ arise by (0) -, $(+)$ - and $(-)$ -displacement of $d \xi_\nu$ along $d \xi_\nu$ and $d^0 \xi_\nu$, $d^+ \xi_\nu$ and $d^- \xi_\nu$ by the analogous displacement of $d \xi_\nu$ along $d \xi_\nu$, then $d^0 \xi_\nu$ and $d^0 \xi_\nu$ form together with $d \xi_\nu$ and $d \xi_\nu$ a closed figure, as well as $d^+ \xi_\nu$ and $d^- \xi_\nu$ and $d^- \xi_\nu$ and $d^+ \xi_\nu$, while among the three points opposite to ξ_ν , the point belonging to $d^0 \xi_\nu$ and $d^0 \xi_\nu$ lies in the midst between the other two. If $f^{\lambda\mu} d\sigma$ is an infinitesimal bivector, and if we call generally $f^{\lambda\mu} d\sigma \Gamma_{[\lambda\mu]}^\nu$ the corresponding torsion, it follows that the $(+)$ -torsion and the $(-)$ -torsion are equal and of opposite sign, the (0) -torsion being zero. If the bivector is formed by the two infinitesimal transformations X_i and X_k , then the torsion has the direction of $(X_i X_k)$.

§ 4. The adjoint group.

If we write always e^k for a $(-)$ -constant vector, which corresponds therefore with an infinitesimal transformation $e^k X_k$, the $(+)$ -differential is according to (18) given by the equation

$$\delta^+ e^k = e^i c_{ij}^k d\xi^j \dots \dots \dots (26)$$

Any function f of e^k undergoes by an infinitesimal transformation of the adjoint group a change

$$df = \frac{\partial f}{\partial e^k} e^i c_{ij}^k d\xi^j,$$

so that we get for the symbol of LIE of the infinitesimal transformations the well known equation

$$E_j f = e^i c_{ij}^k \frac{\partial f}{\partial e^k} \dots \dots \dots (27)$$

If we call *constant quantities* such quantities, which are constant by $(+)$, $(-)$ and (0) , then the following propositions hold good:

form volume-integrals in the group-manifold of an arbitrary finite continuous group. If we start from an unit of volume in ξ^v and denote by dt^0 , dt^+ and dt^- the volumes of a space-element, measured with these unit, then generally dt^0 , dt^+ and dt^- are unequal. Moreover, $dt^+ = dt^-$ in the E_{r-1} of the invariant subgroup of g_i passing by ξ^v , whilst always $dt^0 = \sqrt{dt^+ dt^-}$. Only in the case $g_i = 0$ the measured volumina get equal in every point.

Applying the JACOBIAN identity, we have

$$\left. \begin{aligned} S_{ij}^{\dots l} g_{lk} &= \frac{1}{c} S_{ij}^{\dots l} S_{lm}^{\dots n} S_{kn}^{\dots m} \\ &= -\frac{1}{c} S_{jm}^{\dots l} S_{li}^{\dots n} S_{kn}^{\dots m} \\ &\quad -\frac{1}{c} S_{mi}^{\dots l} S_{lj}^{\dots n} S_{kn}^{\dots m} = \\ &= -\frac{1}{c} g_{jlk} + \frac{1}{c} g_{ljk} = -\frac{1}{c} g_{lkj} + \frac{1}{c} g_{ljk} \end{aligned} \right\} \dots (31)$$

whence it follows that $S_{ij}^{\dots l} g_{lk}$ is alternating in jk and therefore is a trivector.

§ 5. Semi-simple and simple groups.

A simple group is a group, which does not possess invariant subgroups. A group is called semi-simple, if it possesses no integrable invariant subgroups. CARTAN has proved¹⁾ that a group is then and only then semi-simple, if the tensor $g_{\lambda\mu}$ has the rank r . There exists therefore in that case a tensor of rank r , which is (—)-constant and it follows that for a semi-simple group the connexion (0) is a Riemannian one with $g_{\lambda\mu}$ as fundamental tensor²⁾. For the constant c it follows from (30):

$$c = \frac{1}{n} K = -(n-1)K_0 ; \quad K = K_{\alpha\beta}^{\dots \gamma} ; \quad K_{\alpha\omega\lambda}^{\dots \nu} = R_{\alpha\mu\lambda}^{\dots \nu} \dots (32)$$

Moreover it follows that g_λ is zero.

If the group is semi-simple, there exist invariant sub-groups. Such a subgroup determines a real p -direction which is constant by the three displacements³⁾. These p -directions determine a system of ∞^{n-p} V_p which are totally geodesic and mutually parallel. The $(n-p)$ -direction orthogonal to these V_p is also constant and determines in the same way ∞^p totally geodesic mutually parallel V_{n-p} . The coordinates may be

¹⁾ Thèse, Paris 1894.

²⁾ L. P. EISENHART, Proc. Nat. Acad. 11, 1925, p. 246—250, has proved that every simple transitive group is connected with a symmetrical connexion and with one or more symmetrical quantities of degree 2.

³⁾ Compare the division of semi-simple groups by CARTAN, Thèse, p. 52, especially for the demonstration that the case where the direction would have in common a p -direction with the orthogonal $(n-p)$ -direction, can not occur.

chosen in such a way that their congruences are entirely in the V_p and V_{n-p} , so that the linear element decomposes into

$$ds^2 = ds_1^2 + ds_2^2$$

where ds_1 represents the linear element of V_p and ds_2 the linear element of V_{n-p} . $S_{i,j}$ and $R_{\alpha,\beta,\gamma}$ decompose into two separately constant parts which lie totally in V_p resp. V_{n-p} . If there are more invariant subgroups, then it is possible to repeat this decomposition, so that V_r finally decomposes into a certain number of V_{p_1}, V_{p_2} etc., which are all geodesic, parallel and mutually orthogonal. Every system belongs to a simple group, which is an invariant subgroup of the given group, and every V_p of this system is groupmanifold of this subgroup. The geometry of the V_r decomposes into the geometries of these simple invariant subgroups. The geometry of a semi-simple group is therefore reduced to the geometries of simple groups. ¹⁾

§ 6. Some geometrical properties.

Starting from an arbitrary geodesic we obtain by means of the connexion (+) a congruence of geodesics which are mutually (+)-parallel, and in the same manner we obtain a congruence of mutually (-)-parallel geodesics. For $r=3$ it follows from (30) that V_3 is an S_3 . The two parallelisms coincide then with the parallelisms of CLIFFORD. ²⁾

If v^ν is a tangential vector in a point of the first geodesic, then the first congruence is given by the equation

$$\overset{+}{\nabla}_\mu v^\nu = \overset{0}{\nabla}_\mu v^\nu + S_{\lambda\mu}^{\nu\sigma} v^\lambda = 0,$$

from which it follows that v_λ is a solution of the equation of KILLING

$$\nabla_{(\mu} v_{\lambda)} = 0. \text{ } ^3)$$

The transformation

$${}' \xi^\nu = \xi^\nu + v^\nu dt$$

is therefore a *motion*, and even a *translation* since the length of the vector v^ν is the same in all points of V_r . The same consideration being

¹⁾ H. LEVY, Rend. Acc. Linc. 3, 1926, p. 65—69 (see 'also p. 124—129) has found that every Riemannian geometry with $\nabla_\xi K_{\alpha\beta\gamma\delta} = 0$ decomposes into Riemannian geometries with zero curvature. This does not agree with our results.

²⁾ A treatment of the geometry in S_3 and of CLIFFORD's parallelisms from the point of view of the different connexions is to be found by CARTAN, Récentes généralisations de la notion d'espace, Bull. Sc. Math. 48, 1924, p. 294—320 and from another point of view by EN. BORTOLOTTI, Parallelismo assoluto e vincolato negli S_3 etc. Atti Veneto 84, 1925, p. 821—858.

³⁾ Der Ricci Kalkül, SPRINGER 1924, p. 212.

valid for the displacement $(-)$, two kinds of translations are possible in V_c .

The first kind may be given also by the equation

$$T_{\xi'} = T_a T_{\xi} \dots \dots \dots (33)$$

where T_a represents a definite transformation, the second by

$$T_{\xi'} = T_{\xi} T_a \dots \dots \dots (34)$$

whence it appears that the translations belong to the first and second parametergroup. The described paths are $(-)$ -aequipollent, while a segment passes into a $(+)$ -aequipollent segment. The same holds m.m. for the second transformation, and for all continuous groups, not only for the semi-simple ones. Transformations of the form

$$T_{\xi'} = T_a T_{\xi} T_b \dots \dots \dots (35)$$

leave invariant all three displacements, while $(+)$ - as good as $(-)$ -parallel segments pass into segments of the same kind. We may ask if there exist yet other pointtransformations of X_r , which leave invariant the connexions $(+)$ and $(-)$. Determining first the transformations which leave invariant the group-structure, i.e. which transform $T_{\xi} T_{\eta} = T_{\zeta}$ into $T_{\xi'} T_{\eta'} = T_{\zeta'}$ ¹⁾, to these transformations only the transformations (33) or (34) have to be added to obtain the desired transformations. The transformations (35) form an invariant subgroup of the so obtained group.

The connexions $(+)$ and $(-)$ determine each the structure of the group entirely. On the contrary, the group being not simple or semi-simple, it may occur that the structure is not entirely determined by (0). In the first place the group of transformations of (0) in itself contains among others the transformations $T_{\xi'} = T_{\xi}^{-1}$ that changes $(+)$ in $(-)$ v.v. The group being not simple or semi-simple there may exist moreover transformation of (0) in itself, conserving not the total of $(+)$ and $(-)$ parallelism. If this case occurs, there exist different absolute parallelisms. A simple example is given by the group

$$(X_1 X_2) = X_3 ; (X_1 X_3) = (X_2 X_3) = 0.$$

The (0)-connexion is here an ordinary affine one. In the corresponding E_3 besides the ordinary parallelisms there can be defined an infinity of absolute parallelisms with the straight lines as geodesics. With the given group is connected the parallelism by which the direction (a, β, γ) in 0 corresponds with the direction $(a, \beta, \gamma + a y - \beta x)$ in (x, y, z) .

We can indicate also in which case a given integrable connexion is

¹⁾ For the semi-simple groups this investigation has been executed by CARTAN, Bull. des Sc. Math. 49, 1925, p. 361-374.

(-)- or (+)-connexion of a group. If ABC is a triangle of geodesics in X_r and if the connexion belongs to a group, then $A'C'$ is aequipollent with AC , if this property holds good for $A'B'$ and AB as well as for $B'C'$ and BC , for every choice of A' .

The reciprocal is also true. If we make correspond with each segment \vec{a} in X_r and its aequipollent segments the transformation, which transforms an arbitrary point X into $'X$, so that $\vec{X}'X = \vec{a}$, then these transformations form a continuous group with r parameters and the given connexion is the (-)-connexion of this group. Infinitesimally the condition is as follows. The connexion belongs then and only then to a group, if the expression $\int^{\lambda\mu} d\sigma S_{\lambda\mu}^{\dots\nu}$ does not change if the surface element is replaced by an aequipollent element, or, which is the same, if $S_{\lambda\mu}^{\dots\nu}$ is a quantity constant by the given connexion,

Starting from two vectors v^ν and w^ν in ξ^ν , we may construct a congruence v^ν by means of the connexion (+) and a congruence w^ν by means of the connexion (-). These two congruences are V_2 -building¹⁾, since

$$v^\mu \overset{0}{\nabla}_\mu w^\nu - w^\mu \overset{0}{\nabla}_\mu v^\nu = v^\mu S_{\lambda\mu}^{\dots\nu} w^\lambda + w^\mu S_{\lambda\mu}^{\dots\nu} v^\lambda = 0.$$

In any of these V_2 the angle between two vectors v^ν and w^ν in every point is equal, since

$$v^\mu \overset{0}{\nabla}_\mu v_\lambda w^\lambda = v^\mu S_{\lambda\mu}^{\dots\nu} v_\nu w^\lambda + v^\mu S_{\mu\lambda}^{\dots\nu} w^\lambda v_\nu = 0$$

$$w^\mu \overset{0}{\nabla}_\mu v_\lambda w^\lambda = 0.$$

In the V_2 obtained in this manner two translations are possible whence it follows that their geometry must be euclidian. For $r=3$ the V_2 are the well known surfaces of CLIFFORD.

We may generalise these considerations. If T_u is the general transformation of a p -fold continuous group, $T_u T_\xi$ is a total geodesic X_p passing through ξ^ν . Comparing this X_p with the X_p passing through η^ν given by $T_u T_\eta$, with every geodesic in one X_p corresponds a (-)-parallel in the other. Indeed, T_u and T_v being two general transformations of the subgroup, the segment $T_v T_\xi (T_u T_\xi)^{-1}$ is (-)-aequipollent with the segment $T_v T_\eta (T_u T_\eta)^{-1}$ since

$$T_v T_\xi T_\xi^{-1} T_u^{-1} = T_v T_u^{-1} = T_v T_\eta T_\eta^{-1} T_u^{-1}.$$

We may therefore say that X_p displaces itself (-)-parallel if ξ^ν changes. Choosing η^ν in the first X_p it follows that, being given a geodesic in that X_p passing through ξ^ν , through every other point of that X_p passes a (-)-parallel geodesic, situated entirely in X_p . In the same way $T_\xi T_\nu$

¹⁾ Ricci Kalkül, p. 107.

gives a totally geodesic X_p through ξ^v , for which the same holds good with respect to the displacement (+). These two X_p coincide if T_ξ is a not singular transformation of the subgroup, since

$$T_\xi T_u = (T_\xi T_u T_\xi^{-1}) T_\xi$$

and $T_\xi T_u T_\xi^{-1}$ represents a general transformation of the subgroup also. The property of parallel motion of geodesics exists also for the (+)-parallelisms.

The reciprocal holds good: if it is possible to make correspond to every geodesic X_p through every point a (-) or (+)-parallel, which lies entirely in X_p , then X_p belongs to a subgroup.

The X_p arising from the X_p through ξ^v by parallel motion are generally different, according as the (+) or (-)-displacement is used.

The invariant subgroups distinguish themselves by the coincidence of the two systems of X_p , which are obtained from the X_p in ξ^v by means of (+) and (-).

In another kind of totally geodesic X_p the (+) or (-)-parallel of a geodesic through a point of the X_p not situated on that line lies not in general in X_p . (A (o)-parallel direction is always situated in X_p). If we construct through a point not situated in X_p all geodesics, which are (+) or (-)-parallel with all geodesics through a definite point of X_p , then there arises a totally geodesic X_p , but this X_p is different according as the point in the first X_p is chosen differently. In an S_3 exist only real geodesical S_2 of the second kind, since the corresponding group of the ordinary rotations in R_3 contains no subgroups with two parameters. The isotropical planes belong however to the first kind.

§ 7. Classification of the geometries belonging to the simple groups.

For a classification of the geometries of simple groups we have to start with the classification of these groups given by CARTAN¹⁾. By this classification a great part is played by a number l that has the following geometrical signification. If v^a is a definite vector, which has not a singular direction, the vectors w^a , satisfying the equation

$$v^a w^a R_{\alpha\beta\gamma} = 0$$

fill an R_l . There exist four normal types, A, B, C and D:

- | | | | |
|----|-------------------|-----------------|---------------------|
| A. | $l = 1, 2, \dots$ | $r = l(l+2) =$ | 3, 8, 15, 24, 35 |
| B. | $l = 3, 4, \dots$ | $r = l(2l+1) =$ | 21, 36, 55, 78, 105 |
| C. | $l = 2, 3, \dots$ | $r = l(2l+1) =$ | 10, 21, 36 |
| D. | $l = 4, 5, \dots$ | $r = l(2l-1) =$ | 28, 45, 66, 91, 120 |

and five abnormal ones:

¹⁾ These 1894.

| | | |
|-----------|-------|----------|
| <i>E.</i> | $l=6$ | $r=78$ |
| <i>E.</i> | $l=7$ | $r=133$ |
| <i>E.</i> | $l=8$ | $r=248$ |
| <i>F.</i> | $l=4$ | $r=52$ |
| <i>G.</i> | $l=2$ | $r=14$. |

For each type the form of $S_{i,uv}$ may be deduced and the corresponding geometry constructed ¹⁾.

We will draw the attention on some important questions. Realising V_r in R_{r+m} it would be important to know the minimum value of m and to have a method to obtain such a realisation in the most simple manner. Also there may be asked for the necessary and sufficient conditions for a given symmetrical connexion being (o)-connexion of a group.

¹⁾ Compare WEYL. Math. Zeitschr. 24, 1925 p. 354 f.

Physiology. "On the reaction course of physiological buffermixtures, examined by direct registration of the P_H changes". By Prof. F. J. J. BUYTENDIJK and R. BRINKMAN. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated at the meeting of February 27, 1926).

In the field of study of the biological neutrality regulation, in spite of the steady increase in detail, the most important problem is still the properties of the buffer-system as formed by the carbonic acid and its salts, the phosphates, the oxy-haemoglobin and perhaps by other weak acids. These systems have been so exhaustively treated in the well-known researches of L. HENDERSON, SPIRO, SÖRENSEN, HASSELBACH, MICHAELIS, WARBURG, VAN SLYKE and many others, and their study has given us such a deep insight into the mechanism of the biological regulation of the acidity, that we have an impression of the comparative completeness of our knowledge. The reaction of the respiratory centrum, of the function of the kidneys and other P_H regulating organs on the change in concentration of the buffer components may still be frequently less clear, but there is evidently little to add to our physico-chemical knowledge of the central buffer-systems, at least of the carbonic acid-bicarbonate and of the phosphate system.

The aim of this communication is to draw attention to the changes which occur in a bicarbonate and phosphate system before an equilibrium is attained.

It was obvious that the chemical examination of the neutrality regulation, as a typical application of the law of mass-action with ion-reactions, showed only the equilibrium conditions but not the course of the reaction. And yet there is, e.g. in the properties of the carbonic acid certainly a reason to examine this course of reaction more in detail, since the non-dissociated H_2CO_3 , according to THIEL and STROHECKER, may be present for a short time after its formation as the strong oxy-formic acid, and in that time it may penetrate into cells by its very strong solubility and permeability. Thus for these reasons, besides by their difference in buffering action, the systems

$$\frac{[H_2CO_3]}{[NaHCO_3]} \text{ and } \frac{2[H_2CO_3]}{2[NaHCO_3]}$$

are by no means equal in their biological effect.

Thus, although the system $HCO'_3 + H \rightarrow H_2CO_3 \rightleftharpoons CO_2 + H_2O$. or perhaps $HCO'_3 \rightleftharpoons CO_2 + OH'$ is exactly known in its state of equilibrium, the time required for the accomplishment of this equilibrium is so long

that, for instance, the strong momentary H_2CO_3 might have a great biological influence. Of the duration of its existence, however, very little is known.

On the other hand, COLLINGWOOD¹⁾ has pointed out the phenomenon that when a current of gaseous CO_2 be passed 2—3 seconds through a "weak alkaline" solution and all the CO_2 above the solution carefully blown off, it may take as long as 30 seconds before the addition of phenol red as indicator will show a change to a more acid reaction. He concluded that if CO_2 , formed in the tissues, were to be transported by the blood as HCO'_3 the formation time for this ion to be produced ($\text{CO}_2 + \text{OH}' = \text{HCO}'_3$) is much too long to ensure a sufficiently rapid transport of CO_2 , so that there must be still unknown factors in this mechanism.

An indication that in the formation of the bicarbonate-carbonic acid equilibrium intermediate reactions may occur, whereby the time factor biologically becomes important in this equilibrium, is found in a simple experiment, part of which is described by MICHAELIS: Add to a solution of 0.1 N NaHCO_3 a little neutral red, which is yellow in alkaline sol.; add enough 0.1 N HCl to turn the colour red after mixing. *After a few seconds* the colour will change back to yellow; if more HCl be added the change to yellow-red-yellow will ensue.

Loss of CO_2 is not the cause of this phenomenon which, according to THIEL and STROHECKER, depends upon the relatively smaller velocity of the reaction $\text{H}_2\text{CO}_3 \rightarrow \text{H}_2\text{O} + \text{CO}_2$.

There is, however, a similar experiment of which we find no mention. If we take again 0.1 N NaHCO_3 , add to it phenol red as indicator and then a little 0.1 N HCl , *the colour change from red to yellow will begin a few seconds after the mixing*, and not suddenly but gradually. And this colour does not change back to the alkaline red tint.

The field of change of phenol red lies between P_{H} 8.4—7.4 of neutral red between P_{H} 8.7.

From this simple experiment it may already be assumed that the increase of the hydrogen ion concentration in a bicarbonate sol. to which acid has been added, would, in the vicinity of the neutral reaction, not take place so regularly as might be expected. That the difference in the manner of change is not a peculiarity of both indicators themselves is evident from the fact that in the phosphate solutions of analogous reaction, buffer capacity and molarity, the change takes place *in both always* immediately and definitely.

A further examination of the process of buffer reactions was only possible by a method which directly indicated the momentary $[\text{H}]$ with as little latency as possible.

¹⁾ Proc. Physiol. Soc. July 5th, 1925.
Journ. Physiol. 59, XXII, 1924—25.

For this, an ordinary electrometric P_H determination with compensation connections as zero-method is not practicable, but we must be able to follow direct the strength of the E. M. F. produced between the solution examined and a standard sol., and by preference be able to register it. A galvano-meter method with a lead-off by common Pt. electrodes is not possible owing to polarisation phenomena.

TREADWELL and WEISS¹⁾ have indeed indicated an H electrode which is nearly non-polarisable and which can sustain a permanent very slight lead-off of current, but this requires a passage of gaseous hydrogen, is very complicated, and acts for our purpose too slowly.

A purely electro-metric method with a binant or quadrant electro-meter as measuring instrument likewise appeared less well adapted, owing to the instrument adjusting slowly, whereby, moreover, it is impossible to register a rapid change of the E. M. F. to be measured in two directions.

Very suitable, however, appeared to us a triode-lamp connection, whereby the E. M. F. as the voltage on the grid influences the discharge current between the filament and the plate, as has been indicated by GOODE²⁾. This system is unpolarisable, and the variation of voltage anode involves a change of current which may be multiplied and registered.

Moreover, in order to trace directly the P_H to be measured, a hydrogen-electrode, which was in equilibrium with the free gas, could not be used as electrode, so we employed the homogenous system with a constant hydrogen pressure by the equilibrium $\text{chinon} + \text{H}_2 = \text{hydrochinon}$ (so-called chinhydronelectrode), and in which the electrode adjusts itself almost instantaneously.

When using the beaker electrode, after MISLOWITZER³⁾, in which the conducting and comparison electrodes are combined, blank platinum electrodes are used, and all connecting pieces are avoided. The electrometric determination of $[\text{H}]$ is by this method very simple, while in the range of the H-ion concentration, which we will examine, the chinhydronelectrode yields results as exact as the classic H-electrode.

As galvanometer the string-galvanometer was the proper instrument, with which every variation in current could be instantaneously registered.

Schematically our method was as follows:

The buffer-system to be examined was in the inmost beaker of the chinhydronelectrode. Very vigorous stirring with the Pt-electrode itself ensures as rapid as possible a mixing of the added acid. The E.M.F. to be measured is conducted through a potentiometer where it can be exactly compensated and measured. From this potentiometer the

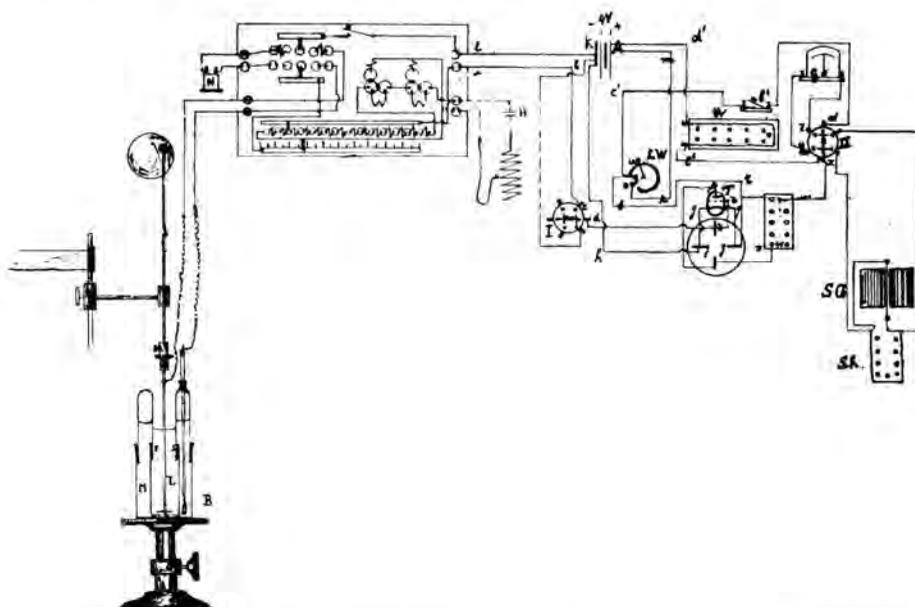
¹⁾ Helv. Chim. Acta 1, 410 (1919).

²⁾ Journ. Amer. Chem. Soc. 1922.

³⁾ Biochem. Zeitschr. 159 58 (1925).

tension arrives at the grid of a triode-lamp, while the variation of the compensated anode-current of this lamp is registered with the string galvanometer.

For the exact description see diagram below:



B is a chinhydron electrode vessel model I, after MISLOWITZER¹⁾, in which a ground surface S wetted with saturated KCL forms the conductive connection between the standard buffer solution in the outermost beaker (M) and the unknown solution in the inner beaker (L). The outer solution (in M) is a mixture of equal parts of saturated KCL and a standard acetate solution (100 cc. N. NaOH + 200 cc. N. CH₃COOH + 700 aq. PH = 4.62). Both liquids are saturated with chinhydron.

For a rapid registration of the effect of an acid addition on the PH vigorous stirring is of great importance. We obtained the best results with a disk electrode measuring $\frac{5}{6}$ diameter of the inner beaker lumen and moving rapidly up and down, over the whole length of the fluid column with a frequency of about 10 per sec.

The E.M.F. to be measured can be compensated by potentiometer, drawn schematically in the figure above. The compensation was accurate up to $\frac{1}{4}$ millivolt. (N standard cell, H additional battery.)

The E.M.F. reaches through the circuit *abcd rghkl* the grid of the triode-lamp.

The lamp is a miniwatt lamp B 406, with a plate potential of 36 V. A small Weston needle galvanometer with a sensibility of 22 micro-amp. serves as zero-instrument. After compensation the discharge current could be directed by the POHL's reverser II to the stringgalvanometer SG (with shunt Sh),²⁾

The sensitivity of this apparatus easily exceeds 0.02 P_H for a displacement of 1 mM of the string shadow, and might even be increased by using more than one triode lamp.

¹⁾ l. c.

²⁾ Our thanks are due to Mr. HUIZING for his valued explanation.

In the first place now we examined how the P_H varied in the time period immediately after the addition of diluted HCl to a 0.02 N. solution of NaHCO_3 and Na_2HPO_4 .

The effect of adding separately 1 c.c. 0.04 N. HCl to 40 c.c. buffer mixture (i. e. each time 0.001 N. HCl) proved to be as follows:

With the phosphate solution the calculated P_H was reached immediately, and this P_H remains constant. The speed of the adjustment is entirely dependent upon the speed of stirring, and the phenomenon remains the same in the entire P_H range examined (8.5—6.5). Only the buffer capacity is naturally greatest near the neutral reaction.

Fig. 1 gives an example.

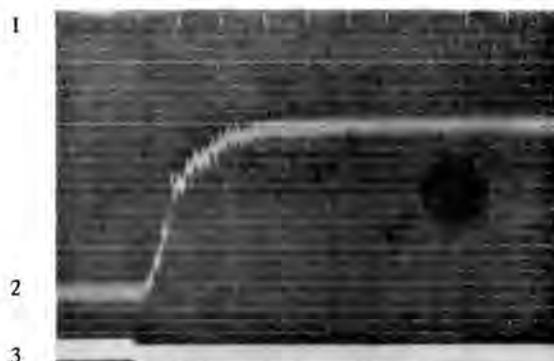


Fig. 1. Addition of diluted HCl to 0.02 N. $\text{Na}_2\text{HPO}_4 + \text{NaH}_2\text{PO}_4$.
 P_H changes from 7.93 to 7.35.

1. Time in secs.
2. String shadow in which stirring is visible.
3. Signal for the acid to be added.

The process in a bicarbonate system was entirely different, as can be seen in Fig. 2.

The variation of P_H thus actually took place as the indicator reactions had led us to suppose; from P_H 8.3 to P_H 7.7 added acid is at once completely buffered, and not until some tenths of seconds later does the reaction reach its equilibrium value.

Between P_H 7.7—7.5 the rise becomes steeper, and with P_H 7.4 a typical „acid summit” is seen, which is very probably caused by the existence of H_2CO_3 during 1—2 seconds.

It is very interesting to note that this variation in the reaction should just occur in the range of blood reaction, where one may expect the one type as well as the other.

We see from this observation thus that the short duration of the strong H_2CO_3 in vivo must certainly be taken into account.

We cannot in this communication go further into this question, but

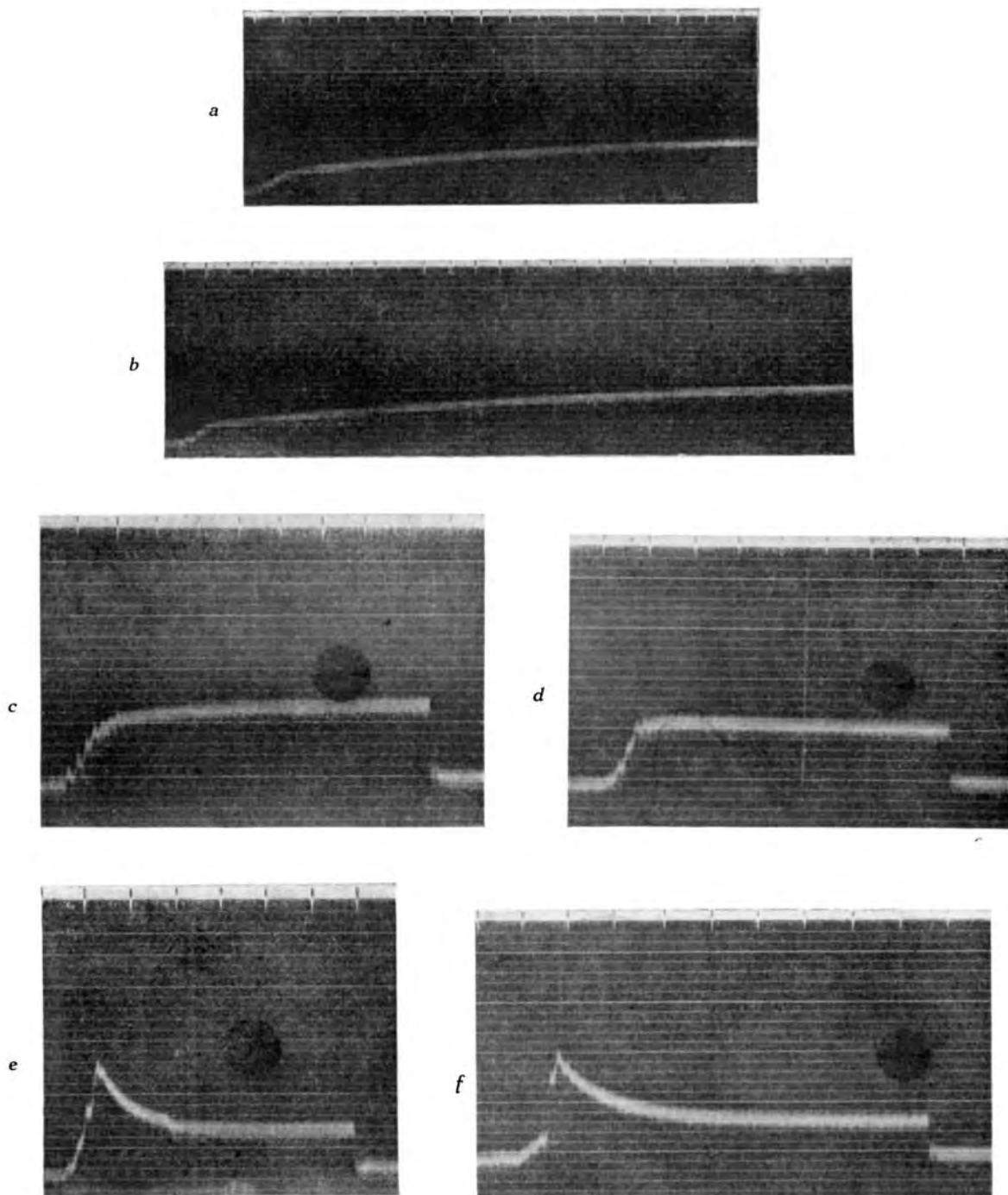


Fig. 2. Successive adding of 0.001 N. HCl to 0.02 N. NaHCO₃
Time in seconds.

d 0.004 N. HCl, P_H of 7.58—7.45

e 0.005 N. HCl, P_H of 7.42—7.36

f 0.006 N. HCl, P_H of 7.36—7.28

a 0.001 N. HCl, P_H of 8.31—8.03

b 0.002 N. HCl, P_H of 8.03—7.77

c 0.003 N. HCl, P_H of 7.77—7.58

wish to point out again that with a continued production of acid the H_2CO_3 may be permanently present in a larger concentration if the rapidity of the formation of this acid is at all comparable to the slowness of the CO_2 decomposition.

The actual P_{H} in an acidotic state can thus not be determined in the usual way, but must be examined in the living blood in the way we have indicated. Then it is possible that a more acid reaction may be found than has hitherto been observed.

Besides in the investigations as to the velocity of the buffer reactions, and in particular as to the significance of free H_2CO_3 , we suppose that the method of registration above described may be useful in the study of different problems to which we hope to return later. We have here in view especially the direct registration of blood and tissue-fluid reactions in vivo, the measurement of biological membrane potentials and their change under nerve stimulation, and the direct registration of the carbonic acid percentage of the air exhaled.

Physiological Laboratorium of the State University.

Groningen, February 1926.

Botany. — “*The heredity of the need of light for germination in tobacco-seeds.*” By J. A. HONING. (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of March 27, 1926).

The researches of CIESLAR (1883), RACIBORSKI (1900), JENSEN (1901 and 1902) and KINZEL (1907) proved, that tobacco-seed needs light to germinate. GASZNER (1915) however counts *Nicotiana Tabacum* among the “Samen mit indifferenter Lichtwirkung” and found that the germination in darkness even occurred a little more rapidly. This made me surmise that in the great group of types, joined in the Linnean species *N. Tabacum*, there must exist considerable differences as to the need of light to germinate. In order to find this out I examined (1916) 51 samples of tobacco-seeds, viz. 8 from Deli, 6 from Western and Central Europe, 14 from the Balkan and Asia Minor, 21 from America, 1 of *N. quadrivalvis* and 1 of *N. rustica*, every time 300 seeds in darkness and 300 in diffuse day-light.

The samples of Deli-tobacco did not germinate at all or badly in the dark, in diffuse day-light very well, most of them for more than 90 % in 3 days. The seeds of types from South-Eastern Europe and Asia Minor however germinated excellently in darkness, for 50 to 94 % in 7 days, those of Western and Central Europe varied greatly, from 2 to 68 % and of the 21 American specimens 10 had not yet started after 7 days. This I considered a satisfactory explanation of the difference in results of CIESLAR a.o. as contrasted with GASZNER's.

I was rather sorry not to get acquainted with GOODSPEED's publication (1919) before 1925, whose experiences, were more in correspondence with GASZNER's and who repeated my experiments with *N. Tabacum* and *rustica* for this reason. GOODSPEED writes: “I am at a loss to explain this result of HONING or a number of the others which he reports.” The explanation may be quite plain: GOODSPEED used in repeating the experiments not a single of the 21 American races I mentioned, but 5 others, so that practically we cannot speak of contradictory results.

Yet GOODSPEED's criticism has led me to resume the research on a larger scale. From the 139 samples now examined, among which there was no progeny of the 51 of 1916, 10 germinated badly, so that they must be left out of account. The remaining 129 belonged to the following species, varieties and races: *N. acuminata* Hook., *affinis*, *alata* L. et O., *alata* L. et O. var. *grandiflora* Comes., *atropurpurea grandiflora*, *auriculata* Bertol., *campanulata* W. K., *cerinthioides* Horn., *chinensis* Fisch., *fragrans* Hort., *fruticosa* L., *glauca* R. Grah., *glutinosa* L., *Langsdorffii* Schrank., *longiflora*

Cav., *macrophylla* Spreng., *macrophylla gigantea*, *paniculata* L., *petiolaris* Schlecht., *Plumbaginifolia* Willd., *rustica* L. (7 samples), *Sanderae* Hort., *silvestris* Spreng., *solanifera* Walp., *solanifolia* Walp., *spososiflora* Dun., *suaveolens* Lehm., *Tabacum* L. (10 samples and further the culture-races: Amersfoort, Cavalla, Cuba, Deli, Hatano, Kedoe, Pajakombo, Samson, Smyrna, Tombak, Trafasund, Vorstenlanden and Xanthi), *virginica* Agard., *viscosum* Lehm.

The seeds originated from: Bordeaux, Bucharest, Cluj, Delft, Dyon, Genoa, Göttingen, Jassy, Kew, Königsberg, Maltha, Montpellier, Naples, Palermo, Wilna, Wageningen, Xanthi, Zürich, Palestina, Soechoem (Trans Caucasia) and the Dutch Indies (Djember, Klaten, Medan, Pajakombo).

During these experiments I had the great advantage of working in the plant-physiological laboratory of Prof. BLAAUW, who was kind enough to put two rooms of a constant temperature at my disposal. In one of them kept at 20° C. stood a thermostat heated to 26° C., in which the Petri-dishes containing the seeds for the experiment in darkness were put in light-proof boxes, every lot consisting of 200 seeds of each sample. As a rule the temperature-oscillations amounted to some tenths of a degree. The seeds which were to be exposed to the light, were put in the other room on four tables each with 5 rows of 5 dishes (likewise 2 either of them containing 100 seeds of each sample) illuminated by a lamp with a parabolic reflector 115 cms. above the centre of each table. As the lamps raise the temperature, forming part of the heating-system, they were initially kept burning constantly. By turns a 12 hours' exposure to light was succeeded by a 12 hours' darkening by means of a green cloth folded double on top of the dishes (without touching them). As a rule the temperature was 24 to 26° C., it never fell below 21.9 nor exceeded 27.9° C. (the thermometer on moist filter-paper in a Petri-dish on one of the tables). Since oscillations could not be prevented (which did not matter in these experiments as they were only meant as a control on the germination in darkness), the light was afterwards put in and out by means of an electric clock of SAUTER every 12 hours. 100 Watt's Argenta lamps appeared to be too strong for this arrangement, e.g. for the *rustica* from Kew, which germinated more rapidly in darkness. So they were exchanged for 40 Watt Argenta.

Controlling the seeds exposed to light occurred in diffuse day-light, that of the seeds kept in the dark in a dark room by a dark brownish-yellow 16 Watt photo-lamp, as far as I could observe without detriment, though theoretically red or blue light would have been preferable (KINZEL 1907).

No difference after an exposure to light or dark, or a difference of at most 5% in 14 days was found in: *N. auriculata*, *chinensis* from Delft, Jassy and Wageningen, *fruticosa*, *glutinosa* from Naples, *macrophylla gigantea*, *petiolaris* from Delft, *Plumbaginifolia* from Bucharest and Wageningen, *rustica* from Cluj, Wageningen and Zürich, *silvestris* from Bucharest, *suaveolens* from Delft and Jassy, *Tabacum* from Cluj, Genoa, Naples

and Zürich (and from the *culture-races* germinated equally good without light: Cavalla, Cuba, Samson, Smyrna, Tombak, Trafasund and Xanthi), *texana* from Delft and Naples, *viscosa* from Cluj.

Speaking only of need of light, if the difference between light and dark amounts to at least 20 % (which is of course arbitrary and purposely exaggerated), we must consider as absolute or partial light-germinators: *N. alata* from Delft and Maltha (difference between light and dark 36 resp. 79 % in 14 days), *fragrans* (diff. 33 %), *glauca* from Cluj, Dyon, Maltha, Montpellier and Naples (d. resp. 50, 31, 35, 21 and 36 %), *glutinosa* from Cluj (d. 69 %), *Langsdorffii* from Cluj (d. 77 %), *longiflora* (d. 51 %), *macrophylla* (d. 75 %), *paniculata* from Cluj and Dyon (d. 39 and 34 %), *rustica* from Bucharest and Dyon (d. 36 and 24 %), *silvestris* from Cluj, Wageningen and Zürich (d. 60, 84 and 32 %), *solanifera* (d. 89 %), *Tabacum* from Bucharest and Dyon (d. 26 and 39 %) and the cultivated races Amersfoorter from Wageningen (d. 40 %) and Pajakombo from Western-Sumatra (d. 90 %), *texana* from Wageningen (d. 87 %), *virginica* from Bucharest and Maltha (d. 30 and 75 %).

In both series the species: *glutinosa*, *rustica*, *silvestris*, *Tabacum* and *texana* occur. Perhaps *chinensis*, *petiolaris* and *suaveolens* might be reckoned among them, but in that case the decisive limit should be reduced from 20 to 10 %.

With respect to *Tabacum* this is a corroboration, made a little less urgent, since with regard to two *Tabacum*-races, Mexico Santiago and Mexico Compostella, BUSSE (1925) stated the need of light, viz. 22 and 73 % with light and 0 and 18 % in darkness. Now it appears, that likewise *rustica*, as to which I initially agreed with GOODSPEED, that it could do without light, be it within narrower bounds, also varies (table I).

Part of the seeds kept in the dark, which remained considerably behind those exposed to light, were subjected to alternating exposure after 14, 21 or 30 days. It usually appeared, that the imbibed seeds had only waited for light and the germination-figure rose suddenly, sometimes even above that of the directly exposed control-seeds. For some lots however the abode in moist air in a constant temperature without germination had proved injurious. The retrogression amounted to 10 to 87 % (e.g. *texana* table I).

Experiments with pure lines.

By far the greater part of the samples discussed concerns seed the genetic purity of which is unknown. An exception are some 8 samples of the 2nd to 5th generations after self-pollination at Wageningen, viz. *Tabacum* Amersfoorter D₅, *rustica* D₁, *atropurpurea grandiflora* D₅, *chinensis* D₃, *macrophylla gigantea* D₁ and D₂, *Plumbaginifolia* D₄, *silvestris* D₄ and *texana* D₃. As to the rest BUSSE's objection to my public-

TABLE I. Germination-percentage of some samples of seed of species and varieties of the genus *Nicotiana*.¹⁾

| Species and variety Origin | light dark | Germination-percentage after .. days | | | | | | | | | |
|---|---------------|--------------------------------------|----|----|----|------|----|------|----|----|--|
| | | 3 | 5 | 7 | 10 | 14 | 18 | 21 | 25 | 30 | |
| <i>N. alata</i> L. et O., Delft | l | 90 | . | . | . | 90 | | | | | |
| | d | 1 | 48 | 54 | . | 54 L | 67 | 67 | . | 67 | |
| " Maltha | l | — | — | 3 | 46 | 79 | . | 84 | . | 84 | |
| | d | — | — | — | — | — L | 64 | 71 | 73 | 74 | |
| <i>N. glauca</i> R. Grah., Cluj | l | — | 33 | 39 | 45 | 50 | 51 | 53 | . | 55 | |
| | d | — | — | — | — | — L | 62 | 64 | 65 | 66 | |
| " Maltha | l | 1 | 87 | 88 | 91 | 92 | | | | | |
| | d | — | — | 4 | 32 | 57 | . | 60 L | 86 | 88 | |
| <i>N. glutinosa</i> L., Cluj | l | 43 | 73 | 77 | 79 | 79 | | | | | |
| | d | — | — | 3 | 6 | 10 L | 91 | 92 | . | 92 | |
| " Naples | l | 60 | . | 61 | 62 | 63 | | 63 | | | |
| | d | 57 | 63 | 65 | . | 65 | | | | | |
| <i>N. paniculata</i> L., Cluj | l | 85 | 87 | 89 | . | 90 | | | | | |
| | d | 1 | 9 | 36 | 49 | 51 L | 89 | 94 | . | 94 | |
| " Montpellier | l | 78 | 84 | . | 86 | 90 | | | | | |
| | d | 44 | 80 | 84 | . | 84 | | | | | |
| <i>N. rustica</i> L., Bucharest | l | 59 | 62 | . | 64 | 64 | | 64 | | | |
| | d | 21 | 26 | 28 | . | 28 | | 29 | | | |
| " Dyon | l | 87 | 91 | 92 | . | 93 | | | | | |
| | d | 34 | 58 | 66 | 68 | 69 L | 79 | 80 | . | 81 | |
| " Zürich | l | 91 | . | 92 | . | 92 | | | | | |
| | d | 89 | 91 | . | . | 91 | | | | | |
| <i>N. silvestris</i> Comes et Speg., Cluj | l | 1 | 26 | 46 | 57 | 60 | . | 61 | | | |
| | d | — | — | — | — | — L | 13 | 79 | 86 | 87 | |
| " Jassy | l | 86 | 97 | 99 | . | 99 | | 99 | | | |
| | d | 3 | 53 | 88 | 91 | 93 | | | | | |
| <i>N. Tabacum</i> L., Dyon | l | 19 | 84 | 88 | 90 | 91 | | | | | |
| | d | — | 4 | 31 | 41 | 42 L | 82 | 90 | 91 | 91 | |
| " Amersfoorter D ₅ | l | 47 | 88 | . | . | 89 | | | | | |
| | d | 1 | 35 | 48 | 49 | 49 L | 67 | 84 | . | 85 | |
| " Pajakombo | l | — | 76 | 87 | 90 | 90 | | | | | |
| | d | — | — | — | — | — L | 79 | 84 | | | |
| " Smyrna | l | 82 | 90 | . | 91 | 91 | | | | | |
| | d | 3 | 85 | 92 | 94 | 94 | | | | | |
| <i>N. texana</i> Hort., Delft | l | 93 | . | . | . | 93 | | | | | |
| | d | 23 | 97 | . | . | 97 | | | | | |
| " D ₃ , Wageningen | l | 89 | 90 | . | . | 90 | | | | | |
| | d | — | 3 | . | . | 3 L | . | . | . | 3 | |

¹⁾ L in column 14 days means: transmitted to light.

ation of 1916 holds good, that it is unknown, whether the material is homozygous or not. Now lines from four races of *N. Tabacum* have been examined: Deli, Vorstenlanden, Kedoe and Hatano and the result of the experiments with the pure lines is perfectly parallel to those of the samples perhaps not pure from a genetic point of view. With regard to the term "pure lines" it should be remembered that the purity was judged before from various morphological characteristics, so that the possibility remained, that those races were not pure as to the need of light for the germination, a fact which had not been taken into account. This, however, did not appear to be the case, for in two and three generations the phenomena remained constant, if only the degree of ripening was identical. The following was observed :

A. Deli. a. Of a half-chlorina-line the seed of two chlorina-individuals was examined, a D_2 from Sumatra and a D_3 from Wageningen at the same time with an extracted green individual D_2 from Sumatra. In darkness the germination-figures for half-chlorina were 11 and 10 %, of the green plant even 20 %, exactly the maximum found for Deli in 1916.

b. Of the Kloempang-line, the heterozygotes of which always produce $\frac{1}{4}$ plants of normal height, $\frac{1}{2}$ halfhigh and $\frac{1}{4}$ dwarfs, the two homozygotes have been examined. The seed of the normal type germinated in darkness in 14 days for $18\frac{1}{2}$ % (in the light 92 % in 7 days) and the seed of the dwarf in darkness for $\frac{1}{2}$ % (in the light 61 %).

c. The deformis-line was very "hard". The *deformis*, *deformis*-heterozygote and the normal type germinated in darkness for $\frac{1}{2}$, 0 and 2 %, in the light in the same order of succession : 98, 91 and 99 %.

In all three lines it is the normal form which germinates least unfavourably without light :

half-chlorina-line: green 20 %, half-chlorina 11 and 10 %,
 Kloempang-line: high $18\frac{1}{2}$, dwarf $\frac{1}{2}$ %,
 deformis-line: normal 2 %, *deformis*-heterozygote 0 %, *deformis* $\frac{1}{2}$ %.

B. Vorstenlanden ("the Principalities"). Of this I possess 3 lines, two homozygous double-flowered and a "doorschieter" ("shooter") or giant-line, apparently heterozygous for the number of leaves. The two double-flowered specimens germinate excellently in darkness, only initially somewhat slower, but within the first week the difference with the seeds exposed to light had disappeared. This is remarkable, as, but for the double flowers and some features connected with them, this tobacco shows much resemblance with the Canari, for which RACIBORSKI stated a need of light. The seeds of the shooter germinated very slowly, also exposed to light. In darkness however nearly half the seeds had shot within a fortnight.

C. Kedoe. Seed of mutant C of ARISZ germinated without light in a fortnight for 65 %. A subsequent exposure had practically no effect, viz.

TABLE II. Germination of some pure lines of *Nicotiana Tabacum* L.
(Deli, Vorstenlanden, Kedoe, Hatano).

| Race, cultivated in year, generation | light dark | Germination-percentage after . . days | | | | | | | | | | |
|---|---------------|---------------------------------------|----|----|----|------|----|----|----|----|----|----|
| | | 3 | 5 | 7 | 10 | 14 | 18 | 21 | 25 | 30 | 35 | 40 |
| A. Deli | | | | | | | | | | | | |
| <i>a. half-chlorina-line</i> | | | | | | | | | | | | |
| half-chlorina D ₂ | l | 31 | 80 | 89 | 92 | 92 | | | | | | |
| Sumatra 1920 | d | — | — | 3 | 11 | 11 L | 91 | 92 | | | | |
| the same D ₃ | l | 19 | 79 | 82 | . | 83 | | | | | | |
| Wageningen 1922 | d | — | 2 | 7 | 9 | 10 L | 82 | 84 | | | | |
| extracted green D ₂ | l | 27 | 82 | 88 | 92 | 92 | | | | | | |
| Sumatra 1920 | d | — | 1 | 10 | 19 | 20 L | 90 | 91 | | | | |
| <i>b. Kloempang-line</i> | | | | | | | | | | | | |
| dwarf D ₂ | l | 34 | 51 | 57 | 60 | 61 | | | | | | |
| Wageningen 1922 | d | — | — | — | — | — L | 64 | 65 | | | | |
| high D ₂ | l | 64 | 91 | 92 | . | 92 | | | | | | |
| Wageningen 1922 | d | — | — | 13 | 18 | 18 L | 86 | 90 | 91 | 91 | | |
| <i>c. deformis-line</i> | | | | | | | | | | | | |
| <i>deformis</i> D ₃ | l | 71 | 97 | 98 | . | 98 | | | | | | |
| Wageningen 1922 | d | — | — | — | — | — L | 96 | 97 | | | | |
| <i>deformis</i> -heterozyg. | l | 84 | 90 | . | . | 91 | | | | | | |
| F ₄ Wageningen 1923 | d | — | — | — | — | — L | 87 | 90 | . | 90 | | |
| extracted normal | l | 45 | 97 | 99 | . | 99 | | | | | | |
| D ₂ Sumatra 1916 | d | — | — | 1 | 2 | L | 96 | 96 | | | | |
| B. Vorstenlanden | | | | | | | | | | | | |
| <i>a. double-flowered line</i> | | | | | | | | | | | | |
| D _x Klaten n ⁰ . 99 1917 | d | — | 3 | 94 | 98 | 98 | | | | | | |
| the same D _{x+1} | l | 41 | 59 | 65 | 72 | 73 | . | 73 | | | | |
| Wageningen 1924 | d | 6 | 52 | 64 | 68 | 72 | . | 73 | | | | |
| <i>b. double-flow. line D_x</i> | | | | | | | | | | | | |
| Klaten n ⁰ . 120 1917 | d | — | — | 87 | 94 | 96 | | | | | | |
| the same D _{x+1} | l | 45 | 66 | 68 | 69 | 71 | 72 | 72 | | | | |
| Wageningen 1924 | d | 26 | 58 | 72 | 74 | 75 | 76 | 76 | | | | |

TABLE II. Germination of some pure lines of *Nicotiana Tabacum* L.
(Deli, Vorstenlanden, Kedoe, Hatano). (Continued).

| Race, cultivated in . . . year . . . generation . . . | light dark | Germination-percentage after .. days | | | | | | | | | | |
|--|---------------|--------------------------------------|----|----|----|------|----|------|----|----|----|----|
| | | 3 | 5 | 7 | 10 | 14 | 18 | 21 | 25 | 30 | 35 | 40 |
| the same D_{x+2} Nov. | l | 35 | 80 | 83 | 84 | 85 | | | | | | |
| Wageningen 1925 .. | d | 12 | 27 | 54 | 62 | 63 L | 80 | 84 | 85 | 85 | | |
| Jan. | d | 7 | 24 | 35 | 58 | 62 | . | 62 L | 86 | 91 | . | 91 |
| c. „Shooter” | l | — | 20 | 46 | 55 | 59 | 61 | 62 | . | 62 | | |
| Klaten 1919 | d | — | — | 23 | 41 | 46 | 48 | 48 L | 53 | 53 | | |
| C. Kedoe | | | | | | | | | | | | |
| mutant C of Arisz | l | 3 | 69 | 79 | 82 | 82 | | | | | | |
| D_x Djember | d | — | — | 52 | 62 | 65 | . | 65 L | 66 | 67 | | |
| the same D_{x+1} Nov. | l | 10 | 59 | 78 | 96 | 96 | | | | | | |
| Wageningen 1925 .. | d | — | — | 1 | 3 | 3 L | 93 | 94 | | | | |
| Jan. | l | 4 | 12 | 17 | 20 | 22 | 23 | 23 | 23 | 76 | 95 | 97 |
| .. | d | 1 | 1 | 2 | 3 | 4 | . | 4 L | 94 | 97 | . | 97 |
| D. Hatano | | | | | | | | | | | | |
| D_x | l | 73 | 97 | . | . | 97 | | | | | | |
| Wageningen 1923 | d | — | 1 | 9 | 35 | 56 | 69 | 70 L | 95 | 97 | | |
| D_{x+1} Nov. | l | 15 | 76 | 87 | 90 | 91 | | | | | | |
| Wageningen 1925 .. | d | — | — | 2 | 3 | 3 L | 89 | 90 | | | | |
| Jan. | l | 4 | 30 | 39 | 50 | 64 | 65 | 65 | 87 | 95 | . | 96 |
| .. | d | — | — | — | 1 | 1 | . | 1 L | 92 | 93 | . | 93 |

but 2 %, though on exposure from the beginning 82 % germinated. This Kedoe-line therefore is not so indifferent to light as the two double-flowered lines of the Vorstenlanden ; its behaviour is more parallel with the shooter's.

D. Hatano. This does not germinate in darkness quite so well as the Kedoe-mutant, viz. 56 % in a fortnight, 70 % in 3 weeks, compared with 97 % in 5 days in the light.

Especially the results with the pure lines (table II) have in my opinion proved, that in the "species" *Tabacum* there are very great differences as to the need of light for germination and that those differences are hereditary.

Influence of the degree of ripeness. The results obtained with seed of the Vorstenlanden, Kedoe and Hatano, gathered in September 1925 and

examined in November next following, deviated in so far from those with seed of 1917, 1923 and 1924, that without light the germination was distinctly less favourable in seeds of Vorstenlanden and almost absent in Kedoe and Hatano. As this phenomenon also occurred in the hybrids to be discussed presently, a further research was necessary. From my practise at experiment-stations I knew, that Deli-seed, still a little fresh, does not germinate so well as seed at least six months old. Now it appeared, that seed not yet quite mature is less capable of germinating without light (Vorstenlanden 22 % less), or has not got this capacity at all (Kedoe and Hatano both 3 % compared with 96 and 91 % with light). This proves that a type indifferent to light has lived through a stage of much greater need of light, in other words that the forms that always need light for germination, do not reach the indifferent stage in after-ripening. And this renders the fact a little more comprehensible that we find all kinds of transitions between the absolute need of light and the absolute doing without. (How the seeds behave directly after they have been gathered has still to be investigated into).

On repeating the experiments in January I found, that on the whole the seeds were even more latent, that also in the light the progress of the germination was much slower. In March they recovered, i.e. with regard to the germination with light, not yet for the germination in darkness.

The crosses.

Crosses between *N. Tabacum* Amersfoorter (half-"light-germinator") and an indifferent *N. rustica* and of Amersfoorter with *silvestris* ("light-germinator") did lead to a flowering F_1 , but in both cases a sterile one, likewise on backcrossing with the parents.

Better results were obtained by crossing some *Tabacum*-races the one with the other. The double-flowered lines of Vorstenlanden (indifferent) were reciprocally crossed with the *deformis*-heterozygote (pronounced "light-germinator"). With Klaten N^o. 99 germination-figures of 0 and 4 % were obtained, with Klaten N^o. 120 on the contrary 84 and 87 %, though we observed a vastly slower progress than with the two parents. In darkness however it was but $1\frac{1}{2}$ and 0 % after 30 days, i.e. the need of light is dominant and indifferent is recessive. The results of the two reciprocal crosses of Vorstenlanden and Deli being identical, the seed-coat has no influence and the need of light is a characteristic of the embryo.

The F_2 behaved differently, viz. 36, 22 and 9 % germinate in darkness in a fortnight to 96, 86 and 63 (after 45 days 87) % in the light. The differences of 27 and 13 % can be the result of an unequal degree of after-ripening and do not prove anything with respect to the ratios, nothing therefore as to the number of factors to which the difference is due. A repetition of the germination-experiment in January yielded 9, 14 and

Likewise with the F₁ of *deformis*-heterozygote and Hatano (half-

TABLE III. Germination of seeds of the F₁ and F₂ after crossing a "light-germinator" with an indifferent race and a "light-germinator" with a half-indifferent race

| Gen. | Parents. Year of harvest | light dark | Percentage of germination . . days | | | | | | | | | | | | |
|----------------|---|---------------|------------------------------------|----|----|----|------|------|------|-----|----|----|----|----|----|
| | | | 3 | 5 | 7 | 10 | 14 | 18 | 21 | 25 | 30 | 35 | 40 | 45 | 50 |
| F ₁ | <i>deformis</i> -heterozygote × Vorstenlanden Klaten No. 120, 1924 | l | — | 6 | 18 | 31 | 32 | . | 32 | 43 | 44 | 64 | 84 | . | 84 |
| | | d | — | — | 1 | 1 | . | 1 | 1 | 1 L | 83 | 86 | . | 86 | |
| F ₂ | F ₁ plant with sessile leaves 1925 Nov. | l | 18 | 94 | 95 | 96 | 96 | . | . | . | . | . | . | . | . |
| | | d | — | 3 | 29 | 36 | 36 L | 93 | 97 | . | 97 | . | . | . | . |
| F ₂ | F ₁ plant with sessile leaves 1925 Nov. | l | — | 1 | 5 | 9 | 9 | . | 9 L | 87 | 94 | . | 95 | . | . |
| | | d | — | — | — | — | — | — | — | — | — | — | — | — | — |
| F ₂ | F ₁ plant with petiolate leaves 1925 Nov. | l | 14 | 81 | 84 | 85 | 86 | . | . | . | . | . | . | . | . |
| | | d | 1 | 5 | 16 | 22 | 22 L | 83 | 90 | 92 | 94 | . | . | . | . |
| F ₁ | Vorstenlanden Klaten No. 120 × <i>deformis</i> -heterozygote 1924 | l | — | 2 | 10 | 13 | 14 | . | 14 L | 72 | 85 | 90 | 90 | . | 87 |
| | | d | — | 3 | . | 4 | 5 | 9 | 32 | 47 | 60 | 85 | 85 | . | 86 |
| F ₂ | F ₁ plant with petiolate leaves 1925 Nov. | l | 8 | 49 | 53 | 61 | 63 | 67 | 77 | 78 | 80 | 85 | 86 | 87 | 87 |
| | | d | 1 | 2 | 7 | 9 | 9 L | 60 | 75 | 78 | 79 | 82 | 82 | 87 | 87 |
| F ₁ | <i>deformis</i> -heterozygote × Hatano 1923 | l | — | — | 1 | . | 1 | . | 1 L | 48 | 55 | 69 | 72 | . | . |
| | | d | 79 | 94 | 97 | . | 97 | . | . | . | . | . | . | . | . |
| F ₂ | F ₁ plant with petiolate leaves 1924 | l | — | — | 9 | 14 | 14 L | 95 | 98 | . | 98 | . | . | . | . |
| | | d | 71 | 92 | . | 94 | 95 | . | . | . | . | . | . | . | . |
| F ₂ | F ₁ plant with petiolate leaves 1924 | l | — | — | 4 | 8 | 8 L | 86 | 94 | 95 | 95 | . | . | . | . |
| | | d | 64 | 94 | 96 | . | 97 | . | . | . | . | . | . | . | . |
| F ₂ | F ₁ plant with winged leaves 1924 | l | — | — | 2 | 5 | 5 L | 95 | 97 | . | 97 | . | . | . | . |
| | | d | 91 | 98 | . | . | 98 | . | . | . | . | . | . | . | . |
| F ₂ | F ₁ plant with winged leaves 1924 | l | — | — | 5 | 55 | 68 | 70 L | 97 | 97 | . | 97 | . | . | . |
| | | d | — | 5 | 55 | 68 | 70 L | 97 | 97 | . | 97 | . | . | . | . |

1% germination in a fortnight and after 21 days exactly the same figures. Yet the germinative-power had not got lost, for when after 21 days the seeds were exposed to light, a total amount of 95, 90 and 72% had germinated after 19 days.

indifferent) it appears, that the need of light is dominant: in darkness 14½ % in a fortnight, in the light 97 % in 7 days. The dominance therefore is not so absolute here as with the crossing with the Vorstenlanden, though the Hatano-line approximates the Deli nearer than the Vorstenlanden-line as to need of light (table III).

The F_2 of *deformis*-heterozygote \times Hatano segregated likewise and as the seed is of the year 1924, the figures ought to be trustworthier. Seed of three, F_1 individuals germinated in 14 days in darkness for 8, 5 and 70 % (in the light 95, 97 and 98 %). The first two F_1 individuals had petiolate leaves (characteristic of heterozygosity for the *deformis*-factor and *deformis* is very "hard" without light), the third had winged petioles (Hatano-characteristic and Hatano germinates fairly well in darkness, 56 % in 14 and 70 % in 21 days). This association of characteristics may be accidental, but it may also point to coupling (though the chance of finding this with a species with haploid 24 chromosomes is not very great). A large number of F_1 individuals will only be decisive. It is true seed of seven F_2 plants has been examined, but that number too is much too slight; moreover this seed of 1925 does not yet give any constant results. For the present it is only a fact, that segregation occurs in the second generation; whether this is Mendelian segregation is an open question.

May be JOST's saying (Pflanzenphysiologie II, p. 131, 1923): "Somit bleibt von dem Begriff "Lichtkeimer" nicht mehr viel übrig", is perfectly correct, though it has still to be proved, if in every case light may be replaced by chemical reagents. Experiments to make my "hardest" seed, the F_4 *deformis*-heterozygote germinate in darkness with diluted acetic acid, nitric acid, potassium nitrate and lime have proved a failure. With the answer to the question whether the idea "light-germinator" is correct or not, the difference existing between the so-called light-germinators and the indifferent tobacco-types does not stand or fall. That difference is a fact, is hereditary.

BUSSE tried to give an explanation for the differences in need of light for the germination stated by him viz. as a result of natural or artificial selection through climate (intensive sun-light in the tropics) or cultural methods (covering the seed-beds with a coat of soil at least 1 cm. thick in South-Eastern Europe and Asia Minor). That sounds natural and plausible. Yet this explanation is presumably incorrect, for all four of my Java types germinated without light either excellently or for nearly 50 % (98, 96, 46 and 65 %), a result totally deviating from what RACIBORSKI and JENSEN found for tobacco of the Principalities, while there is not a single reason for the least doubt about the exactness of their communications. That we should not generalise with regard to this material is obvious.

LITERATURE.

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Botany. — "On new phenomena caused by irritation of roots." By Prof. J. M. JANSE. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of March 27, 1926)

During a recent research of movements effected by excitation of roots, my attention was attracted by some phenomena unknown to me, which therefore I studied more in detail.

The first experiments were made with seedlings of *Sinapis alba*, *Helianthus annuus*, *Pisum sativum* and *Lupinus luteus*, the plants mostly used at the beginning of experiments. Afterwards I examined the radicles of some other plants, in order to test the universality of the phenomena observed. I did not select special plants, but made me give different seeds from the Botanical Garden. Yet all gave fully corresponding results, though there were differences in sensibility and not all were equally adapted for the purpose, be it for accessory reasons. *Sinapis*, *Brassica* and *Onopordon* f.i. turned out to be the most suitable, especially for microscopical examination.

The phenomena observed were the following :

1. If a seedling is pulled up somewhat carelessly out of the soil, the result is sometimes already a certain retardation in growth ; the retardation is more important when the friction is augmented in some way. At an experiment with a radicle of *Helianthus annuus*, growing between two slabs of glass and observed with the horizontal microscope, the speed of growth at the beginning, at about 16° C., amounted to 0.33—0.36 m.m. an hour. After having been very strongly excited, by moving it some 200 times between wet fingers, it showed when it had been brought back between the slabs of glass again, a speed of growth 0 during the first 30 minutes, but in the hours following the rapidity increased successively to 0.08, 0.28, 0.23 and 0.30 m.m. an hour, so that after 5 hours the primitive speed was nearly reached again. So the result is, that a strong excitation provokes a very considerable delay in growth, but without causing a permanent disturbance.

2. After effectual friction, e.g. as in the experiment described above, the root turned out to have become a little thinner and weaker over a certain distance (e.g. from 5—9 m.m. from the top). The diameter of a root of *Helianthus* for instance, decreased from 0.66 to 0.47 m.m.

As the length of the growingzone of such a root amounts to about 10 m.m., the emaciation principally takes place in the zones approaching the full-grown state.

3. Another phenomenon, which always accompanies the former, is the fact that the root becomes more or less transparent. As the opaqueness of the normal root is due to the great quantity of air contained in the numerous intercellular spaces, which are principally arranged in longitudinal direction,

the transparency of the root is founded on the disappearance of this air, or, as we shall call them for convenience sake, the air-filaments, because they present themselves to a certain extent as such.

Besides it was observed that the disappearance of the air could be studied also at a magnification of 100, and in thin roots even of 250 times. On a transverse section, the intercellular spaces are mostly wedge-shaped, so that, when a rootlet is examined lengthwise, they present themselves some times as very fine filaments, sometimes as broad ribbons, according to their being examined from the narrow or from the broad side; they strike the eye especially by the total reflection of light in the cellwalls surrounding them. The unaltered roots are so opaque, that by means of the microscope only the higher situated filaments are perceptible with some distinctness, but during the process the examination becomes more and more easy, as the tissue grows more transparent.

To observe the phenomena, the best way is to excite not too strongly a thin radicle of *Sinapis*, *Brassica* or *Onopordon*, of about 1 to 2 c.m. in length (it is sufficient to move it from 1 to 5 times between the fingers) and to examine it immediately afterwards in water under a cover-glass.

The first thing to be seen then is that often an important number of air-filaments are no longer limited by straight lines, but that they show at one or both sides small depressions; these depressions deepen by degrees and so reach the other side of the filament, which is in this way interrupted in many places. The number of these intermissions increases hand-over-hand in some minutes; at the same time many of them spread in a lateral direction; by this longer or shorter parts of the air-filaments are occasionally pushed aside (sometimes even with a rather considerable speed) or, and far more frequently, two of the intermissions approach one another, so that the air-bubble lying between them decreases gradually in length and disappears wholly at the end.

From this it follows that the infiltration is caused chiefly by the disappearance of the air from intercellular spaces, and not by a movement or displacement of it to the other, unexcited parts of the plant. The phenomenon seems to appear in every cell rather quickly, so it may be that the alterations observed microscopically originate from the cells reacting somewhat slowly. The place of the air is occupied by a liquid emerging in small drops, which make out the mentioned interruptions, from the adjacent parenchymatous cells. This latter fact could not be ascertained directly, when, as usually, the cells contained an untinged cellsap, but another origin of the drops is unconceivable under these circumstances. Yet I have tried to make sure the origin of the liquid in another way; for those new experiments I chose seedlings of *Beta vulgaris*, namely of a very dark coloured variety of the red beet, as I expected that here the cells of the young parenchyma of the root would contain red cellsap. This supposition, however, turned out to be incorrect, the root being untinged over about 20 m.m. from the top; the anthocyan appears only later on, as soon as the cells

are full-grown and so I had to use full-grown cells for these experiments.

After irritation by effectual friction of these full-grown parts the interruption of the filaments and the dissolution of the air could be clearly observed, even when a magnification of 250 times was used. Even scrupulous examination showed not the slightest difference in tinge between the parts of the intercellular spaces where the air had been expelled and the immediately adjacent cells. So we are entitled to conclude, for the present at least, that the liquid leaving the cells had the same colour as the cellsap itself, so that it consisted indeed of a part of the unaltered cellsap, as was presumed.

The explanation of these phenomena follows here later on.

When the root is excited feebly, the quantity of air usually disappearing does not amount to such an extent as to render the root wholly transparent; in this case a strong irritation is necessary to obtain this phenomenon.

This sensibility of a distinct zone of young roots is, as is evident from the accessory phenomena, quite different from the one observed by DARWIN, especially because of the fact that the latter was proved to reside only in the top of the root.

4. When the plants, after having been irritated as described above, are cultivated, e.g. between two slabs of glass, they show after the lapse of a day still another phenomenon, consisting in the fact that the root, which has grown in length more or less, shows a local thickening. This swelling always appears in a corresponding place, i.e. the place occupied the day before by the not yet grown-out cells of the zone of growth. This swollen part occurs always under the part emaciated by the excitation, which at this time is mostly but little restored, and contains, accordingly, only a small quantity of air. The cells that were at the beginning of their development had therefore not increased as much in length, as otherwise would have been the case, but on the contrary their radial dimension had become augmented importantly. This phenomenon can be observed easily after they are brought in alcohol, and then also the fact can be observed that celldivisions have played no part at all in this thickening. Below the thickened part the root slowly resumes its cylindrical form and normal dimensions.

The importance of this modification of the dimensions of the cells in the swollen part is shown in the subjoined table, derived from an experiment where *Sinapis* was used; the dimensions of the cells in the narrowest and the thickest part of the root amounted to:

| | narrowest part. | thickest part. | proportion |
|---------------------|-----------------|----------------|------------|
| longitud. dimens. : | 0.35 m.m. | 0.14 m.m. | 1 : 0.4 |
| radial dimens. : | 0.19 m.m. | 0.28 m.m. | 1 : 1.5 |

When we presume, as we are entitled to, that the tangential dimension of the cells has remained unaltered, we find that the areas and consequently the volumes of these cells bear a proportion of 10 : 6, so that the local

swelling is accompanied by an important staying behind of the volume of the cells.

The degree of swelling is very different according to the plants examined; it depends upon the degree of excitation and may reach a considerable amount as is evident from the subjoined table indicating the diameter of the thicker and of the narrower part (the latter being the part excited the day before) of the roots, one day after the excitation :

| | narrower part | thicker part | difference |
|-------------------------------|---------------|--------------|-------------------------------|
| <i>Sinapis alba</i> | 0.43 mm. | 0.85 mm. | 0.42 mm. or 98 $\frac{0}{10}$ |
| <i>Helianthus annuus</i> | 0.85 .. | 0.95 .. | 0.10 12 .. |
| <i>Pisum sativum</i> | 0.69 .. | 0.95 .. | 0.26 38 .. |
| <i>Lupinus luteus</i> | 0.90 .. | 1.30 .. | 0.40 44 .. |
| <i>Onopordon bracteatum</i> | 0.44 .. | 1.58 .. | 0.14 32 .. |
| <i>Mirabilis Jalapa</i> | 0.57 .. | 1.14 .. | 0.57 100 .. |
| <i>Impatiens glanduligera</i> | 0.57 .. | 0.71 .. | 0.14 24 .. |
| <i>Euphorbia Lagascae</i> | 0.65 .. | 0.90 .. | 0.25 38 .. |
| <i>Nigella aristata</i> | 0.38 .. | 0.52 .. | 0.14 31 .. |
| <i>Beta vulgaris</i> | 0.40 .. | 0.59 .. | 0.19 47 .. |

The reaction comes to an end within 24 hours, as is proved clearly by the fact that after that lapse of time, a renewed friction provokes a second swelling, etc. In this way a root of *Lupinus*, excited on three consecutive days, showed therefore 3 successive swellings at a distance of some millimeters one from the other; the growth amounted during the day after the second excitation to 6.5 m.m. after the third to 4.25 m.m., but the following day, when the rootlet had not been excited previously, the growth amounted to $9\frac{3}{4}$ m.m., showing thus again that the excitation retards the rate of growth.

When, after having been excited, the plants are cultivated in humid air, the fact can be observed that in the narrow part, which is situated immediately below the old root-hairs, no new hairs are produced, but that they reappear quite normally at the beginning of the swollen part. From this might be concluded once more that especially the nearly full-grown cells undergo the influence of the irritation, for the fact that the old root-hairs are all situated above it, proves that the cells situated in this zone were full-grown already when the excitation took place; indeed root-hairs are never formed in the zone of growth.

Now the question arises how to explain the phenomena described.

1. The direct examination showed that the interruption of the

airfilaments was due to small drops of liquid, penetrating into the intercellular spaces. It is evident that those drops can only come from the adjacent parenchyma-cells, so that they must consist of pure water or of cellsap. Theoretically only the last hypothesis is acceptable for reasons, which I developed circumstantionally in a former article ¹⁾ concerning the irritations of *Mimosa*, which are probably identical with those here described. The principal reason was, that in the case of pure water being expressed, the cellsap had necessarily to be divided very quickly into water and a more concentrated solution, which is absolutely unacceptable, among other reasons because of the very strong attractive power, which these liquids exercise on each other.

When indeed a part of the cellsap is expressed this only could have taken place after the wall of the vacuole, having lost its semipermeability, has become permeable for all substances contained in the vacuole; only after this has happened the elastically stretched cellwall will be able to press part of the cellsap into the intercellular spaces. The emission of coloured cellsap as described in the case of *Beta* would have been the best proof that indeed the wall of the vacuole had become entirely permeable, because of the fact that it then must have allowed even the large molecules of anthocyan to pass.

Besides this the last named effect proves that the cells which are just full-grown, may react in the same way as the nearly grown-out elements of the growing-roots, which affords a motive to extent this researches also to full-grown tissues and to those of other plants.

2. A part of the cellsap thus arriving under a certain pressure in the intercellular spaces, will exercise a pressure on the air contained in them. When an air-filament could easily be moved from its place (as is sometimes observed) this would undoubtedly take place everywhere, so that at the end all air would be driven into the unexcited parts of the plant. But, as the resistance in the narrow intercellular spaces is too great for this, this pressure will compress the air contained in them till an equilibrium has been established between this pressure and the rest of the tension of the cellwalls, these cellwalls being then still — be it to a lower degree than before — elastically stretched. The fact that they are still stretched to some extent is evident, as the excited root has kept some turgescence.

The partial or total disappearance of the air from the intercellular spaces, as is observed by means of the microscope, is therefore due to the air being dissolved, as the solubility of gases in water is proportional to their pressure; so this phenomenon, however striking it may be, is not of any theoretical importance.

3. The decrease of the ration of growth of the roots after the irritation is evidently due to the diminishing of the turgor of the still growing cells, as

¹⁾ J. M. JANSE, Ueber die Permeabilität des Protoplasma. Verslagen en Mededeelingen der Kon. Ak. van Wetenschappen, Amsterdam, Afd. Natuurkunde, 1888, Dl. IV, p. 410.

the ration of growth is principally determined by this tension. The fact that the ration of growth increases afterwards again slowly to the primitive value, cannot be explained in so simple a way, as certainly different circumstances, not all entirely known, play a part therein. But we may, however, remark here, that the nearly full-grown cells which had lost a part of their cellsap do not take it in again so easily, as the observation showed that the day after their irritation the radicle had not reassumed its former dimension, nor the air had returned completely into the intercellular spaces.

4. The cause of the modification in growth of the cells of the younger zones of the root consisting in a diminishing in the longitudinal and an increase in the radial direction can not yet be explained. Nevertheless we may remark here that in various other cases already described, a decrease of growing in length taking place for some reason or other was always accompanied by an important development in thickness. Such a case was described already by VÖCHTING in the report of his experiences with *Papaver* ¹⁾. When a nearly full-grown flowerbud (which was still hanging down) of *Papaver Rhoeas* or of other species of this genus was prevented from erection by means of a weight attached to it, it showed soon a diminishing of growth and at the same time a abnormal swelling of the upper part of the stem; VÖCHTING adds that every hindrance of growing in length can cause an abnormal increase in diameter without new celldivisions taking place.

A corresponding phenomenon can often be observed in geotropic curvatures, especially of seedlings; it can appear in the root as well in the hypocotyl; new experiments, with roots of *Pisum* and *Lupinus*, and roots and hypocotyls of *Sinapis* convinced me again. Though these swelling must have been seen often, and by many investigators, so that they can even be recognised in some illustrations, they have apparently always escaped special attention.

When we assume that even the normal vertical growing in length might be considered as a reaction upon the irritation due to gravitation (as to me it seems probable ²⁾) it follows from this, that in the both cases mentioned this irritation by gravity must have decreased, so that we might conclude that the diminishing of this excitation, causing a decrease of the growing in length, could be held responsible also for the stronger growing in thickness. This fact would be of still more importance, if as I can yet but suppose, inversely a swelling, appearing under circumstances of the same kind would indicate a previous decrease of the excitation by gravity, as this would provide us with a direct and permanent indication of a diminishing of the intensity of this excitation having taken place.

BLAAUW has shown by numerous experiences, performed with extremely sensitive instruments, that strong light can slacken also the growth in length;

¹⁾ Die Bewegungen der Blüten und Früchte, Bonn, 1882, p. 120—124.

²⁾ J. M. JANSE. On stimulation in auxotonic movements. These Proceedings 26, p. 171.

it was therefore that I wished to see if the phenomena described above could be observed also in this case: this indeed proved to be the fact. One of the experiments performed in order to solve this question was the following:

Fourteen seedlings of *Sinapis*, of 20—40 m.m. in length, were exposed simultaneously during 65 minutes to the light of an Osramlamp of 1200 candles, placed at a distance of 24 c.m.; a little mirror was placed at the other side of the rootlets. After the exposure all the roots, which contained before the ordinary quantity of air in their intercellular spaces, had become transparent, which was, here too, due to a considerable diminishing of the air in them, as was proved by the microscopical examination.

After this the plants were cultivated in vertical position between two slabs of glass, separated by strips of thick blotting-paper. After 6 hours no growth worth mentioning could be observed, but after 24 hours the length had increased with 2—11 m.m. So the growth had been strongly disturbed, it is true, but the roots had not suffered any permanent injury by the exposure; roothairs too had been developed here, and just in the same way as in the case of irritation by friction; moreover the next day the swellings appeared very plain occurring also at the corresponding places. The result of the measurements was the following (the roots have been tabulated after their decreasing growth in length):

| No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Average |
|-------------------------------------|------|------|------|------|------|------|------|----------|
| Diam. of the narrower part in mm. | 0.54 | 0.54 | 0.54 | 0.50 | 0.57 | 0.63 | 0.60 | 0.56 mm. |
| Diameter of the thicker part in mm. | 0.65 | 0.69 | 0.63 | 0.63 | 0.72 | 0.79 | 0.76 | 0.70 .. |
| Increase of diam. in mm. | 0.11 | 0.15 | 0.09 | 0.13 | 0.15 | 0.16 | 0.16 | 0.14 .. |
| Increase of diameter in % | 20 | 28 | 17 | 26 | 26 | 25 | 27 | 24 % |
| Growth in length in mm. | 11.0 | 11.0 | 10.0 | 10.0 | 10.0 | 10.0 | 7.5 | 10.0 mm. |

| No. | 8 | 9 | 10 | 11 | 12 | 13 | 14 | Average: |
|-----------------------------------|------|------|------|------|------|------|------|----------|
| Diam. of the narrower part in mm. | 0.60 | 0.54 | 0.63 | 0.63 | 0.66 | 0.57 | 0.60 | 0.61 mm. |
| Diam. of the thicker part in mm. | 0.76 | 0.76 | 0.95 | 0.77 | 0.76 | 0.80 | 1.01 | 0.83 .. |
| Increase of diam. in mm. | 0.16 | 0.22 | 0.32 | 0.14 | 0.10 | 0.23 | 0.41 | 0.22 .. |
| Increase of diam. in % | 27 | 41 | 51 | 22 | 15 | 40 | 68 | 38 % |
| Growth in length in mm. | 6.5 | 6.5 | 5.0 | 4.5 | 4.0 | 3.0 | 2.0 | 4.5 mm. |

From this table it is evident that everywhere a plain perceptible swelling was observed, which was by no means less important than in the case of friction. When we compare the 7 plants which had grown most (N^o. 1—7) with those growing more slowly (N^o. 8—14) we see that, on an average, the former show a growth of 10 m.m. and an increase of diameter amounting to 24 percent, while the later grew in average 4.5 m.m. and increased in diameter 38 percent. So swelling was the more important as the slackening in growth was more considerable. Now and again trifling curvatures were observed, owing to an inevitable difference in excitation at the different sides; the curvature was always situated in the swollen part.

From these results it is evident that the reaction of the roots upon friction and upon exposure to intense light were so fully corresponding in all respects, that we may deduce from it that the delay in growth observed by BLAAUW, and which he calls „lightgrowingreaction”, is due just as well to the becoming permeable of the wall of the vacuole of the cells; a microscopical examination of his radicles would certainly have shown him also the more or less complete infiltration of their parenchyma.

This result must give a fully different insight into the importance BLAAUW attributed to his „lightgrowingreaction” as an explanation of the cause of heliotropic movements, as this reaction might be regarded now as a more or less important disturbance of the process of growth, more than as a regulation of this process in behalf of a heliotropic curvature.

From the facts enumerated it has become probable that other stimulations too might provoke the same effects, but this can only be established by further researches.

It seems to me that the temporary permeability of the wall of the vacuole under the influence of a more or less effectual touch is also found in many other plants, for different phenomena have been observed and also described, which are certainly due to such a modification of the properties of the wall of the vacuole.

In first instance this may be the fact with the movements of the leaves of *Mimosa pudica*, as was already remarked before, but besides that there are other leaves able to execute similar though less obvious movements, and furtheron there are many young shoots, etc. that modify temporarily their form after friction or that are bending, also because they loose part of their turgor. I can but mention here the many experiments made by HOFMEISTER¹⁾ and by PRILLIEUX²⁾, but must refer for further expositions to another article, which shall be published elsewhere before long.

These latter phenomena occur exclusively at just full-grown parts, which show rather soon after the friction etc. a certain curvature accompanied by some diminution in turgor; only after a rather long time (some hours) the

¹⁾ W. HOFMEISTER, Ueber die Bewegungen saftreicher Pflanzentheile nach Erschütterung. Jahrbücher für wissenschaftliche Botanik, 1860, Bd. 2, p. 237.

²⁾ Ed. PRILLIEUX, Étude sur les courbures que produisent les secousses sur les jeunes pousses des végétaux. Annales des Sciences Naturelles, 1868, 5me Série, T. 9, p. 248.

organs reassume their primitive state. In these cases we might admit probably also that the cellsap which had been expelled before, was absorbed again by the same cells, thus unlike the phenomenon observed here in the case of still growing rootlets, where only still growing cells were concerned. Here too, however, the decision must be left to a further research.

POSTSCRIPT.

As BLAAUW was not able to provoke the „lightgrowingreaction“ in *Avena sativa* or *Raphanus sativus*, even with very strong light, I examined these species too with regard to their sensitivity to friction. Both proved to be not less sensitive than the other plants used; besides *Raphanus* showed to be very suitable for the microscopic examination of the phenomena described, as e.g. the rubbing between wet fingers, repeated about 20 times, and immediately followed by the microscopical examination gives an excellent example of the direct consequences of excitation.

On the next day all these roots showed the formation of root hairs exactly in the same way as in the other cases described and also the increase in thickness did appear; in the case of *Raphanus* it reached the value of 44 and in one of the cases of *Avena* even that of 153 per cent.

So the roots of *all* kinds examined were sensitive to friction and reacted upon it all exactly in the same way.

Geology. — “Concerning the Occurrence of Quaternary Corals in Angola,”
By F. J. FABER. (Communicated by Prof. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of April 24, 1926).

In an article by Prof. H. GERTH entitled: “De verspreiding en de jonge ouderdom der Koraalriffen in den Atlantischen Oceaan buiten West-Indië”¹⁾ the following passage occurs: „Finally I wish still to observe that, to my knowledge, raised coral-reefs do not occur anywhere along the Atlantic coasts of South America and Africa, which must have been the case if quaternary reefcorals had occurred on these coasts as early as the Tertiary or the Quaternary. True, in some places tertiary deposits prevail, but they do not contain reefcorals. This had already been marked by GERTH in a map: “Vorkommen der Riffkoralen im Känozoikum”²⁾

The occurrence of Cainozoic reefcorals on the Westcoast of Africa is not known beyond a few degrees south latitude, which is pointed out emphatically in the above-quoted publications.

That no corals occur there now, is also owing to the cold ocean-current from the south, which flows along the coast, the so-called Benguelacurrent, of which the influence can still be observed far towards the north³⁾. For their viability the ordinary reef-corals require a temperature of the water of more than 20° C. That corals should not have existed on the Westcoast of Africa throughout the Cainozoicum is *not* the fact.

The sedimentary deposits of the Westcoast of Angola near Mossamedes, which, as a rule, have a horizontal position, are referred to the Tertiary by several researchers⁴⁾; presumably they may even be classed under the young Tertiary. I have no reason to suppose that the rock of Balo do Chapéu Armado, nearly 100 k.m. north of Mossamedes, is of an earlier geological period. Probably a young tertiary coastal fringe extends from

¹⁾ Vakblad voor Biologen, 7e jaargang, No. 5, pp. 68—71 (1926).

²⁾ Gedenkboek-VERBEEK, edited by the Geol. Mijnbouwkundig Genootschap: H. GERTH, „Die Bedeutung der tertiären Riffkorallenfauna des malayischen Archipels für die Entwicklung der lebenden Riff-Fauna im indopazifischen und atlantischen Gebiet”, p. 173—196, Map on p. 191 (1925).

³⁾ According to G. SCHOTT the seawater has a surface-temperature only just to the south of the Congo-mouth, permanently higher than 20° C. (Geogr. des Atl. Ozeans 1926 Tafel IX). Already at a little depth does the temperature of the water fall considerably.

⁴⁾ See, amongst other works, J. B. BEBIANO. Geologia e Riqueza Mineira de Angola. Lisboa 1923. p. 60 sqq.

a point south of Mossámedes to a point north of Chapéu Armado. In this rock, which I had an opportunity to examine, no corals were identified.

The coast supplies an impressive example of decomposition and over nearly its whole length it consists of a normal barrier, high 40 m. and upwards, which is continually excavated by the ocean to the effect that, time and again, large blocks of rock break off and slide down. Even though in this region raised, comparatively recent coralreefs did occur, there would be little probability of their not having fallen a victim to the erosion. Only when extraordinary circumstances might have let a space intact, this space could be detected.

The Balo do Chapéu Armado is situated at about $14^{\circ} 23'$ south-latitude. It does not cut far into the land and is bounded on the north and on the south by the normal, steep sedimentary barrier, on the east by a beach, consisting of sand carried by a rivulet that has silted up half of the bay. Consequently the steep barrier can be traced a few hundreds of meters landwards. Here then the great marine disintegration had been put a stop to for some time.

J. C. SCHAGEN VAN SOELEN as well as the present writer found fragments of coral with a fresh appearance on a small terrace about 8 m. above the sea-level and another 10 m. higher up in the crumbled detritus covering the lower part of the steep barrier. Presumably these fragments were derived from a white speck on the otherwise yellow to reddish-brown rock on considerably greater height. Whether this is so, or is not, we could not make out, no more could we decide whether the coral had originated from the rock or had grown upon it and would consequently be much younger. The latter circumstance would tally with the fresh appearance and is in my opinion the most plausible view. The district being all but uninhabited and uninhabitable we need not think of any other source and a transport by men.

Prof. GERTH of Leyden was so kind as to examine one of the fragments. He referred the coral to the species *Siderastrea siderea*, which still occurs in the Atlantic ocean and is known already from the younger Tertiary of West-Indies. The other fragments have been formed by the same species of corals.

Very likely the land was upheaved at least 200 m. after the formation of the young tertiary deposits; anyhow up to about this height tertiary rocks have been identified. Probably this movement was rather rapid. In complete harmony with this is the markedly revived character displayed by all rivers not far from the shore. The sea attacked the new shore, an erosion-cliff was soon formed and a condition obtained, very similar to the present. The shore lay generally more towards the west, but the bay of Chapéu Armado had already taken origin. Subsequently a subsidence of the land took place (or an upheaval of the level of the sea), while circumstances prevailed that promoted the genesis of corals. It would seem to me that these animals built up their reefs here in the Pleistocene.

It is still a moot point what we have to think of the Benguela-current in this stage. It is certain that the land rose again above the sea, and that the corals came to lie about 30 or at least 20 m. above the sea-level. The immense marine erosion caused the raised reefs to disappear everywhere. Only in a single protected space, where the sea could not undermine the rocks as it did here near Chapéu Armado, a meagre remnant remained.

Chemistry. — “Equilibria in systems, in which phases, separated by a semipermeable membrane” XVI. By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of March 27, 1926).

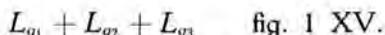
Some general considerations.

We imagine in the diagrams of the previous communications to be drawn a straight line through the point W . If this line is situated totally out of the region of dimixtion, then the $O.W.A.$ of the liquids of this line increases, according to those liquids being situated further from the point W . If this line intersects a region of dimixtion of two or three liquids, then the same is true for the parts of this line situated outside this region. If we take f.i. the line Wg (fig. 2 XIII); its point of intersection with curve m_1a_1 is called s . The $O.W.A.$ of the liquids of this line then increases from W to s and from a_2 to g ; the $O.W.A.$ of liquid a_2 is larger also than that of liquid s . On the line Wm_1m_2 (fig. 2 XIII) the $O.W.A.$ increases from W to m_1 and further from m_2 towards points, situated further from point W ; the $O.W.A.$ of the liquids m_1 and m_2 is the same, however.

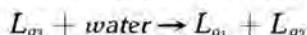
As the liquids of a straight line going through point W can arise from one another by taking in — or losing water, we may say therefore:

of two stable substances, which can be converted into one another by taking in — or losing water, the liquid with the smallest content of water has the greatest $O.W.A.$ and the liquid with the greatest content of water has the smallest $O.W.A.$ If, however, the two liquids can be in equilibrium with one another, then they have the same $O.W.A.$ (f.i. m_1 and m_2 in fig. 2 XIII).

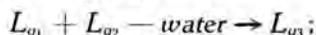
This rule is a special case of a more general rule which we shall discuss. We take an equilibrium, in which, at taking in and losing a little water, a phases-reaction occurs, viz. a reaction, at which the quantities, but not the compositions of the phases change. Let us take f.i. the system



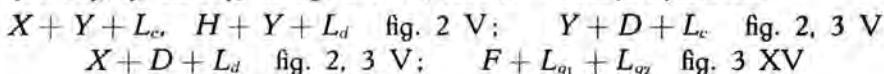
Herein when taking in water occurs the reaction



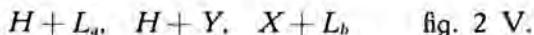
and when losing water the reaction:



the composition of the phases does not change, therefore, but the quantities change. The same is valid also for the equilibrium of the three liquids q_1 q_2 and q_3 in fig. 2 XV; for the ternary equilibria:



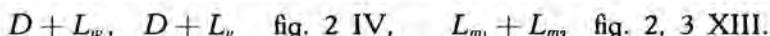
and for the binary equilibria:



In all those equilibria viz. at taking in- and losing a little water a phases-reaction occurs; consequently the *O. W. A.* does not change; we shall call similar systems: "osmotic invariant".

In a system of n components, which consists of n or more phases, at taking in- or losing water always a phases-reaction can occur, unless in special cases. If we take f.i. three phases of a ternary system, which can be represented accidentally by three points, situated on a straight line, which does not go through the point *W*, then between those phases a phases-reaction is possible, but this is not the case when taking in or losing water.

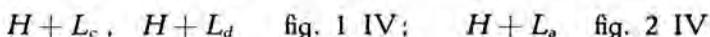
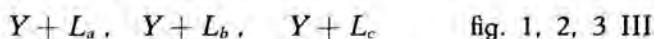
If a system of n components consists of less than n phases, then in this system generally no phases-reaction is possible when taking in or losing water, unless the compositions of the phases satisfy certain conditions. This is f.i. the case in the ternary equilibria:



They are, although they consist of two phases only, yet osmotic invariant.

If in a system by taking in and losing water one or more of the phases change their composition, then the *O. W. A.* changes also; we shall call such a system "osmotic variant".

This is the case f.i. with the systems:



We now take two similar systems of such a composition that the one can arise from the other by taking in or losing water. This is f.i. the case with the systems $Y + L_b$ and $Y + L_c$ in fig. 1 III, if we choose the quantities of the phases in such a way that the points, which represent those two systems, are situated on a straight line going through point *W*.

If both systems, which we shall call E_1 and E_2 , can be accidentally in equilibrium with one another, then they have the same *O. W. A.* If this is not the case, then, if we bring both together, a new system E_3 will be formed. Consequently we have the conversion:



It now depends on the ratio of the quantities of E_1 and E_2 which system E_3 can be formed. If we take a definite ratio and if we consider stable states only, then E_3 is completely defined. If we now assume that E_1 has a greater amount of water than E_2 , then appears from (1) that E_3

has also a greater amount of water than E_2 but smaller than E_1 . Consequently we can have the conversions:

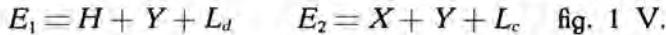


We now form the osmotic system:



As herein by diffusion of water at both sides of the membrane the system E_3 can be formed, and as this system, as we assumed above, also is the most stable state, this will really take place. Therefore, as appears from (2) and (3) water must diffuse from E_1 to E_2 . The *O.W.A.* of system E_2 (viz. of the system which has the smallest amount of water) is, therefore, greater than that of E_1 .

We can apply those considerations also when one of the systems E_1 and E_2 or when both are osmotic invariant. Let us take f.i. the equilibria:



We imagine those equilibria to be represented by the points K_1 and K_2 (not drawn); consequently K_1 is situated within the triangle HYd and K_2 within the triangle XYc of fig. 1 V. As E_2 passes into E_1 by taking in water, the points K_1 , K_2 and W must be situated on a straight line. We now represent by K_3 the equilibrium E_3 which can be formed in accordance with reaction (1); this point K_3 is situated then anywhere on the line WK_1K_2 between K_1 and K_2 ; its place is defined by the ratio's of the quantities of E_1 and E_2 which we bring together. It now depends on the position of K_3 which equilibrium E_3 will arise. In accordance with the position of K_3 within the triangle HYd , the sector Ydc on the triangle XYc , E_3 is one of the equilibria:



The first of those equilibria is the same as E_1 , with this difference, however, that the quantities of the phases are otherwise. The same is true for the last one, which is equal to E_2 .

In similar way as above we now find that in (4) water must diffuse from E_1 towards E_2 , so that the *O.W.A.* of E_2 is greater than that of E_1 .

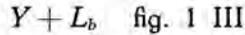
We can apply similar considerations also to systems, in which one or more phases are the same (f.i. system $Y + D + L_c$ and $Y + D + X$ fig. 3 V), to systems which differ infinitely little in composition, etc. We now find:

Of two different stable systems, which can be converted into one another by taking in or losing water, and which are not in equilibrium with one another, the system with the smallest amount of water has the greatest *O.W.A.*

If in a stable equilibrium a phases-reaction occurs at taking in or

losing water, then it is osmotic invariant; its *O.W.A.* viz. remains constant. If no phases-reaction occurs, then it is osmotic variant; its *O.W.A.* becomes greater at losing water, and smaller at taking water in.

We now take the equilibrium



which is represented in the figure by a point of the line *bY* f.i. point *g*. If we take away a little water from this system, then the liquid changes its composition; it shifts, starting from *b* along the saturation-curve in the direction towards *v*. As the *O.W.A.* of the system increases at losing water, the *O.W.A.* in the point *b* of the saturation-curve increases, therefore, in the direction of the arrow. If we apply this to all systems $Y + L$, in which *L* represents each liquid of the saturation-curve, then it appears, which we have already deduced previously in another way, that the *O.W.A.* of the liquids of the saturation-curve increases in the direction of the arrows, viz. from *w* to *v*.

We find the same for the liquids of the saturation-curve in fig. 2 III; here also the *O.W.A.* of the liquids increases in the direction from *w* towards *v*, notwithstanding the content of water of the liquids themselves firstly increases starting from *w*, in order to decrease later of course. It also appears in fig. 3 that the *O.W.A.* increases in the direction of the arrows.

We now take the equilibria



in which *L* represents an arbitrary liquid of the saturation-curve. In the same way as above we now find the property, already previously deduced, that the *O.W.A.* increases from *w* to *v*.

In order to formulate this property in another way than formerly, we shall define the direction in which we move along a saturation-curve, in a corresponding way as in communication XIII for a binodalcurve. The line *Yb* (figs 1, 2 and 3 III) or *Hb* (fig. 1 IV) or *Db* (fig. 2 IV) divides the plane into two parts; in one of those parts the point *W* is situated. We now shall say that we go starting from *b* along the saturation-curve towards *W*, if we go along the saturation-curve towards that part of the plane, in which point *W* is situated. If we go in the other direction, then we shall say that we move away from the point *W*. We may summarize then the deduced above in the following way:

the *O.W.A.* of the liquids of a saturation-curve increases in that direction, in which we move away along this curve from the point *W*.

Of course this rule is valid also, if the diffusing substance is other than water. If f.i. the substance *Y* diffuses, then we have to replace in the above rule *O.W.A.* by *O.Y.A.* and point *W* by point *Y*.

Let us take the figures of the communications XI and XII in which *Y* is the diffusing substance. We then find in accordance with previously

that the *O.W.A.* along the saturation curve *adg* of the substance *D* (fig. 2 XI) must increase in the direction of the arrows. The same is true for the saturation-curve *ab* of the substance *H*, for the saturation-curve *cd* of the substance *X* and for the saturation-curve *fg* of *Ice* in fig. 2 XII.

We can apply also the rule on the change of the *O.W.A.* of an osmotic variant equilibrium to define the direction, in which the *O.W.A.* of the liquids of a binodal-curve increases. Let us take f.i. the equilibrium:

$$L_{a_1} + L_{a_2} \text{ fig. 1-3 XIII.}$$

If we take away from this system a little water, then those liquids shift along the binodal-curve away from the point *W*; consequently they move in the direction of the arrows. Those arrows, therefore, indicate also the direction, in which the *O.W.A.* increases. In accordance with previously we find, therefore:

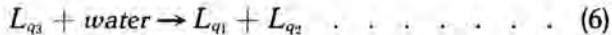
the *O.W.A.* of the liquids of a binodal-curve increases in that direction in which we move away along this curve from the point *W*.

Consequently this rule is the same as that for the saturation-curve of a solid substance; the only difference is, that with the saturation-curve the conjugation-line solid-liquid, and with the binodal-curve the conjugation-line of two coexisting liquids plays a part.

We now take the equilibrium

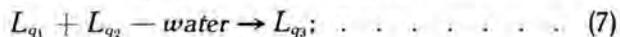
$$E = L_{q_1} + L_{q_2} + L_{q_3} \text{ fig. 1 XV (5)}$$

When taking in water, herein occurs, as appears from the figure, the reaction:



The quantity of the liquid *q*₃ decreases, therefore, that of the liquids *q*₁ and *q*₂ increases; with sufficient taking water in arises, therefore, the equilibrium *L*_{*q*₁} + *L*_{*q*₂}, which passes into an equilibrium *L*(*q*₁*b*₁) + *L*(*q*₂*b*₂) by taking further water in. Herein *L*(*q*₁*b*₁) represents a liquid of curve *q*₁*b*₁ and *L*(*q*₂*b*₂) a liquid of curve *q*₂*b*₂. As this equilibrium arises from (5) by taking water in, it has, therefore, a smaller *O.W.A.* than system (5). This is in accordance with the direction of the arrows on the binodal-curve *q*₁*a**q*₂.

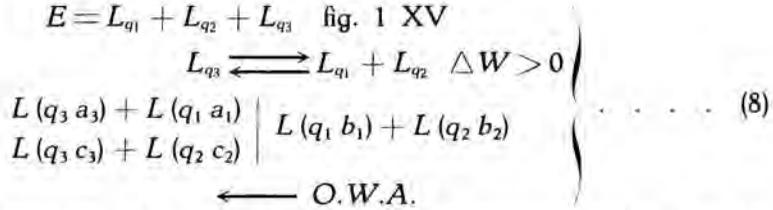
At losing water in the system (5) occurs the reaction:



with sufficient loss of water arises, therefore, one of the systems: *L*_{*q*₁} + *L*_{*q*₃} or *L*_{*q*₂} + *L*_{*q*₃}. It depends on the ratio of the quantities of *L*_{*q*₁} and *L*_{*q*₂} in (5) which of those systems arises in a special case. With further loss of water arises one of the systems: *L*(*q*₁*a*₁) + *L*(*q*₃*a*₃) or *L*(*q*₂*c*₂) + *L*(*q*₃*c*₃) in which are indicated again between parentheses the curves on which the liquids are situated. As those equilibria arise from

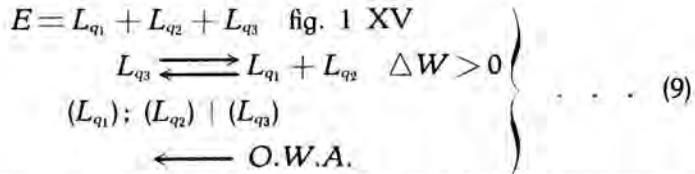
(5) by losing water, they have a greater *O.W.A.* than system (5); this is in accordance with the direction of the arrows in the figure.

We shall summarize the previous in the following way:

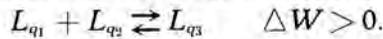


In the first line of (8) we find the osmotic invariant system, in the second the reaction which occurs between the phases at taking in and losing water. On the right side of this reaction we still find $\Delta W > 0$; this means that the reaction takes place from left to right with taking in water. In accordance with this we find at the right side of the vertical line the equilibrium, which is formed with taking in water, at the left side of this line the equilibria which are formed with loss of water. Consequently the equilibria at the left side of this line have a larger *O.W.A.* than the equilibrium *E*, those at the right side of the line have a smaller *O.W.A.* This is indicated by the direction of the lowest arrow.

We shall represent by (L_{q_3}) the equilibrium $L(q_1 b_1) + L(q_2 b_2)$, which is formed from *E*, when herein disappears the phase L_{q_3} ; the equilibrium $L(q_3 a_3) + L(q_1 a_1)$ which is formed from *E*, when herein the phase L_{q_2} disappears, is represented by (L_{q_2}) and the equilibrium $L(q_3 c_3) + L(q_2 c_2)$ which is formed from *E*, when herein the phase L_{q_1} disappears, by (L_{q_1}). Instead of (8) we may write then:

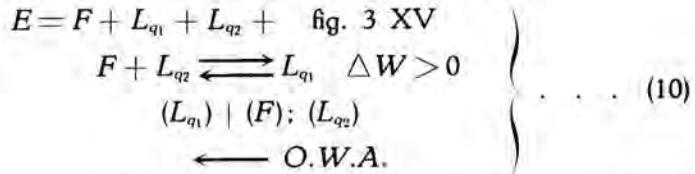


For the equilibrium of the three liquids in fig. 2. XV is the reaction:



In (9) we have, therefore, to replace by one another the systems at the right and at the left of the vertical line. This is also in accordance with the direction of the arrows in this figure.

For the threephases-equilibrium of fig. 3. XV is found:

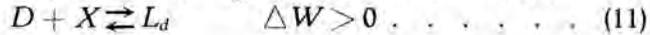


Herein (L_{q_1}) represents the equilibrium, which arises from *E*, if the

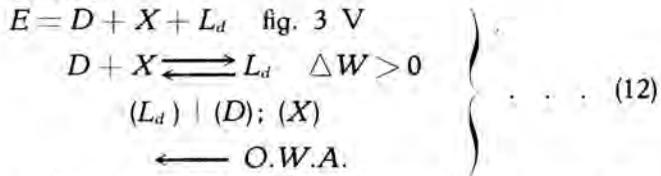
liquid L_{q_1} disappears from this; consequently we have: $(L_{q_1}) = F + L$, in which L is a liquid of curve q_2f . In accordance with (10) the *O.W.A.* of this system must be larger than that of equilibrium E ; this is in accordance with the direction of the arrow on curve d_2f of the figure.

(F) represents the system, which arises from E if the substance F disappears from this, consequently two conjugated liquids of the binodal-curve $q_1 a q_2$; in accordance with (10) the *O.W.A.* of this system is smaller than that of the equilibrium E ; the same is true for the equilibrium $(L_{q_2}) = F + L$, in which L represents a liquid of curve q_1w .

In the equilibrium $D + X + L_d$ of fig. 3. V occurs the reaction:

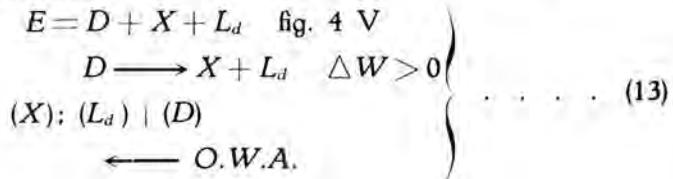


We may write, therefore:



The equilibria $(D) = X + L$ and $(X) = D + L$ have, therefore, a smaller *O.W.A.* than the equilibrium E ; the direction of the arrows on the curves db and ds is in accordance with this. As (L_d) represents the equilibrium which is formed from E by loss of water, we may have, in accordance with the figure: $(L_d) = D + X + Y$. The *O.W.A.* of this system is larger, therefore, than that of the system E . We have already deduced this in another way in communication VI.

We now take the same three-phases-equilibrium as in (12), but now in fig. 4. V. We then find:

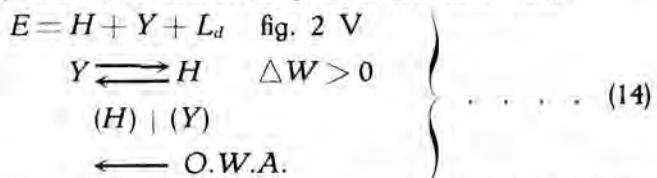


The equilibrium $(X) = D + L$ is situated now on the other side of the vertical line than in (12); therefore, the *O.W.A.* must increase along the saturation-curve of D in fig. 4. V starting from the point d .

In the equilibrium $E = H + Y + L_d$ of fig. 2. V occurs the reaction:



As the liquid does not take part in the reaction we may call E a singular equilibrium¹⁾ and L_d its indifferent phase. We now find:



¹⁾ F. A. H. SCHREINEMAKERS, In-, mono- and plurivariant equilibria XXVII and XXVIII.

In accordance with the arrows in fig. 2, V it follows from this that $(H) = Y + L$ has a greater and $(Y) = H + L$ has a smaller *O. W. A.* than the system E .

If we introduce a semi-permeable membrane in an equilibrium E then it does not change its stability, we then get an osmotic equilibrium:

$$E_1 | E_2 \dots \dots \dots (15)$$

If we represent the thermodynamical potential of the system E by Z , then, if this system contains one or more phases, which change their composition at taking in or losing water, and if E_1 and E_2 each contain the half of the phases of system E , at diffusion of δn quantities of water the thermodynamical potential at the one side of the membrane will become:

$$\frac{1}{2} \left[Z + \frac{\partial Z}{\partial n} \cdot \delta n + \frac{1}{2} \cdot \frac{\partial^2 Z}{\partial n^2} \cdot \delta n^2 + \dots \right]$$

and at the other side:

$$\frac{1}{2} \left[Z - \frac{\partial Z}{\partial n} \cdot \delta n + \frac{1}{2} \cdot \frac{\partial^2 Z}{\partial n^2} \cdot \delta n^2 - \dots \right].$$

Consequently the total thermodynamical potential becomes:

$$Z + \frac{1}{2} \frac{\partial^2 Z}{\partial n^2} \cdot \delta n^2 + \dots \dots \dots (16)$$

If we assume that system E and, therefore, also the osmotic system (15) is stable for small disturbances of equilibrium, then (16) must be greater than Z , so that $\frac{\partial^2 Z}{\partial n^2}$ is positive. As $\frac{\partial Z}{\partial n} = \varphi$, $\frac{\partial \varphi}{\partial n}$ must be positive, therefore. Consequently the φ of system E increases at taking in a little water. Hence follows the rule, already deduced above:

the *O. W. A.* of a stable osmotic variant equilibrium becomes greater when losing water and smaller when taking water in.

This rule needs however to be true no more for systems, which are unstable for small changes of equilibrium. Let us imagine in a diagram f. i. in fig. 1 XIV a straight line to be drawn through point W . We take on this line a point Q and we represent the composition of a liquid L by:

$$x \text{ quant. of } W + y \text{ quant. of } N + (1-x-y) \text{ quant. of } Q$$

in which N is an arbitrary phase. The *O. W. A.* of this liquid L is defined by:

$$\varphi = \zeta + (1-x) \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} \dots \dots \dots (17)$$

For a new liquid L' , which differs infinitely little from L , is true, therefore:

$$d\varphi = [(1-x)r - ys] dx + [(1-x)s - yt] dy \dots \dots (18)$$

If we imagine the liquid L to be in the point Q and the new liquid L' on the line WQ (so that $dy=0$) then (18) passes into:

$$dq = r dx \quad (19)$$

We now go in fig. 1 XIV f.i. along the line Wb from the liquid L towards the new liquid L' , which is situated somewhat further from point W . As then dx is negative and r is positive in all points of this line dq is negative, therefore. The same is true of course for all liquids of the parts of a straight line going through point W , which are situated outside the region of dimixtion. Consequently we find in accordance with previously:

the *O.W.A.* of a stable liquid becomes greater when losing water and smaller when taking water in.

Otherwise it is, however, if we take a liquid which is situated within the spinodalcurve (and, therefore, surely within the region of dimixtion): then r can be negative. If we take f.i. the line Wu of fig. 1 XIV; previously we have assumed that r in the points c and m is zero, and negative between these points. In connection with this an isotonic curve touches this line Wu in c and an other curve (not drawn) touches it in m . It now follows from (19):

the *O.W.A.* of a liquid between c and m (fig. 1 XIV) becomes smaller when losing water and greater when taking water in.

We see this also from the figure, if we imagine to be drawn in this still some isotonic curves, situated between the two, which touch in c and m the line Wu .

Also the rule, deduced above, that the *O.W.A.* of the liquids of a saturation-curve increases in that direction in which we move along this curve from the point W , needs to be true no more when the system is unstable for small changes in equilibrium. Let us take f.i. the saturation-curve $a q_1 q_2 f$ of the substance F in fig. 3 XV. It appears from the direction of the arrows that the rule above mentioned is not valid for the parts lm and pn of this curve, which are situated within the region of dimixtion, but it is valid for the parts $q_1 l$, mn and $p q_2$ which are situated also within the region of dimixtion.

(To be continued.)

Mathematics. — "Intuitionistische Einführung des Dimensionsbegriffes."

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§ 1. *Definition der katalogisiertkompakten Spezies.*

Unter einer *Teilmenge der Menge M* verstehen wir im folgenden eine derweise aus *M* entstehende Menge, dass mittels einer unbeschränkten Folge von Entscheidungen der Reihe nach für jede ungehemmte endliche Wahlfolge von *M* entschieden wird, entweder dass sie ungehemmt bleibt, oder dass sie gehemmt wird. Der Mengenbegriff erleidet hierbei gegenüber der Math. Ann. 93, S. 244 gegebenen Definition insofern eine Erweiterung, dass die Entscheidung zwischen Gehemtheit und Ungehemtheit nicht mehr von vornherein für alle endlichen Wahlfolgen durch ein Gesetz gänzlich festgelegt zu sein braucht.

Unter einer *katalogisierten Folge* verstehen wir eine unbegrenzte Folge p_1, p_2, \dots , in welcher zu je zwei Elementen p_{v_1} und p_{v_2} ein Abstand $\varrho(p_{v_1}, p_{v_2}) \geq 0$ (aber nicht notwendig entweder > 0 oder $= 0$) definiert wird, der folgenden Bedingungen genügt: Es ist stets $\varrho(p_v, p_v) = 0$ und $\varrho(p_{v_1}, p_{v_2}) \leq \varrho(p_{v_1}, p_{v_3}) + \varrho(p_{v_2}, p_{v_3})$. Weiter existiert zu jedem n ein μ_n und ein ν_n , so dass für $\nu > \nu_n$ jedes p_ν in einer Entfernung $\leq a < 4^{-n}$ von $s_{\mu_n} = \mathfrak{S}(p_1, p_2, \dots, p_{\mu_n})$ gelegen ist. Schliesslich ist jedes Element p_{ν_n} von s_{μ_n} entweder ein β_n -Element oder ein α_n -Element (niemals aber beides gleichzeitig); im ersteren Falle ist für $\nu > m(\eta_n)$ jedes p_ν in einer Entfernung $\geq \frac{5}{4} \cdot 4^{-n}$ von p_{ν_n} gelegen, im letzteren Falle gibt es eine unbegrenzte Folge von verschiedenen p_ν in einer Entfernung $\leq b < \frac{3}{2} \cdot 4^{-n}$ ($b > \frac{5}{4} \cdot 4^{-n}$) von p_{ν_n} . (Demzufolge besitzt dann auch ein beliebiges p_ν entweder die für die α_n oder die für die β_n charakteristische Eigenschaft.)

Was wir auf dieser Grundlage unter einer *positiv-konvergenten unbegrenzten Folge* (bzw. unter *zusammengehörigen positiv-konvergenten unbegrenzten Folgen*) von verschiedenen Elementen einer katalogisierten Folge zu verstehen haben, ist ohne weiteres klar.

Die Spezies *S* der Spezies *zusammengehöriger positiv-konvergenter unbegrenzter Folgen von verschiedenen Elementen* einer katalogisierten Folge \mathfrak{r} ist *kompakt*, d.h. wenn wir in auf der Hand liegender Weise die Entfernung je zweier Elemente von *S* definieren, so konvergiert jede

positiv-konvergente Folge von Elementen von S positiv gegen ein Element von S . Weiter ist S in bezug auf F katalogisiert, d.h. die Entfernung zwischen S und einem beliebigen Elemente p_n von F lässt sich mit jedem beliebigen Grade der Genauigkeit approximieren (ohne dass jedoch stets ein Element q von S angegeben werden kann so dass $\varrho(q, p_n) = \varrho(S, p_n)$). Die Spezies S „fällt zusammen“ mit einer kompakten finiten Menge M von positiv-konvergenten unbegrenzten Folgen von (eventuell teilweise identischen) Elementen von F , welche erhalten werden, indem in solcher Weise der Reihe nach ein α_1 -Element, ein α_2 -Element, u.s.w. gewählt werden, dass nach der Wahl eines bestimmten α_n -Elementes α_n^0 jedes α_{n+1} -Element in einer Entfernung $\leq 2 \cdot 4^{-n}$ von α_n^0 wählbar und jedes α_{n+1} -Element in einer Entfernung $\geq 3 \cdot 4^{-n}$ von α_n^0 unwählbar ist. Die Folge F_1 , welche der Reihe nach die α_1 -Elemente, die α_2 -Elemente, u.s.w. enthält, ist wiederum katalogisiert (wobei man diesmal ohne die Analoga der β_n -Elemente auskommen kann) und die Spezies der Spezies zusammengehöriger positiv-konvergenter unbegrenzter Folgen von verschiedenen Elementen von F_1 fällt zusammen mit S . Jede in derselben Weise wie S mittels einer katalogisierten Folge definierte Spezies, und überdies die leere Spezies, heisst eine *katalogisierte-kompakte Spezies*.

Sei S' eine in bezug auf F_1 katalogisierte kompakte Teilspezies von S (also z.B. die „Abschliessung“ einer beliebigen Teilmenge von M ; übrigens werden wir auch die leere Spezies zu den in bezug auf F_1 katalogisierten kompakten Teilspezies von S rechnen). Jedem α_n -Elemente ${}_1\alpha_n$ geben wir entweder das Prädikat eines α'_n -Elementes (in welchem Falle es Elemente von S' in einer Entfernung $\leq c < \frac{3}{2} \cdot 4^{-n}$ ($c > \frac{5}{4} \cdot 4^{-n}$) von ${}_1\alpha_n$ gibt) oder das Prädikat eines β'_n -Elementes (in welchem Falle jedes Element von S' in einer Entfernung $\geq \frac{5}{4} \cdot 4^{-n}$ von ${}_1\alpha_n$ gelegen ist),

mit der Massgabe, dass die beiden Prädikate sich gegenseitig ausschliessen. Sodann bestimmen wir eine mit S' zusammenfallende Teilmenge M' von M , für welche in solcher Weise der Reihe nach ein α'_1 -Element, ein α'_2 -Element, u. s. w. gewählt werden, dass nach der Wahl eines bestimmten α'_n -Elementes ${}_0\alpha'_n$ jedes nach derselben für M unwählbare α'_{n+1} -Element sowie jedes in einer Entfernung $\geq 3 \cdot 4^{-n}$ von ${}_0\alpha'_n$ gelegene α'_{n+1} -Element unwählbar und jedes in einer Entfernung $\leq 2 \cdot 4^{-n}$ von ${}_0\alpha'_n$ gelegene α'_{n+1} -Element wählbar ist. Die Folge F'_1 , welche der Reihe nach die α'_1 -Elemente, die α'_2 -Elemente, u.s.w. enthält, ist wiederum katalogisiert (wobei man wiederum ohne die Analoga der β_n -Elemente auskommen kann) und die Spezies der Spezies zusammengehöriger positiv-konvergenter unbegrenzter Folgen von verschiedenen Elementen von F'_1 fällt zusammen mit S' .

In analoger Weise stellt sich heraus, dass eine in bezug auf F_1 katalogisierte kompakte Teilspezies S'' von S' mit einer Teilmenge M'' von

M' zusammenfällt; demzufolge ist sie ebenfalls in bezug auf r_1 (sowie in bezug auf r und in bezug auf S) katalogisiert.

§ 2. Ein Satz über Erweiterung von Trennungen in katalogisiertkompakten Spezies.

In Anschluss an eine vorgegebene in bezug auf r_1 katalogisierte kompakte Teilspezies S' von S bestimmen wir in folgender Weise eine Teilmenge $M'_n(S')$ von M : Jedem a_{n+2} -Elemente ${}_1a_{n+2}$ sprechen wir entweder das Prädikat eines a''_{n+2} -Elementes zu (in welchen Falle es in einer Entfernung $\geq 4^{-n} + 4^{-n-1}$ von S' gelegen ist) oder das Prädikat eines β''_{n+2} -Elementes (in welchem Falle es einen Abstand $\leq 4^{-n} + 2 \cdot 4^{-n-1}$ von S' besitzt). Alsdann wird $M'_n(S')$ von denjenigen zu M gehörigen Folgen, deren $(n+2)$ -tes Element ein a''_{n+2} -Element ist, gebildet. Jedes Element der Abschliessung $A_n(S')$ von $M'_n(S')$ besitzt einen Abstand $\geq 4^{-n}$ von S' , während jedes in einer Entfernung $\geq 4^{-n} + 2 \cdot 4^{-n-1} + 4^{-n-2}$, a fortiori also jedes in einer Entfernung $\geq 2 \cdot 4^{-n}$ von S' gelegene Element von S zu $A_n(S')$ gehört. Sei $M_n(S')$ eine nach dem Verfahren des § 1 konstruierte mit $A_n(S')$ zusammenfallende Teilmenge von M , und sei jeder Teilmenge τ von $M_n(S')$ eine positive rationale Zahl ϱ_τ zugeordnet. Alsdann bestehen, weil die Spezies der τ eine finite Menge darstellt, nach dem in diesen Proceedings XXVII, S. 192 (vgl. auch S. 646) hergeleiteten Satze, für ϱ_τ nur endlichviele Werte und unter denselben ein positiver, rationaler Minimalwert.

Wir sagen, dass die Teilspezies α von S zwischen den Teilspezies β und γ von S eine Sprungweite b aufweist, wenn in jeder endlichen Folge von Elementen von S , von denen das erste zu β , das letzte zu γ und alle anderen zu α gehören, zwei aufeinanderfolgende Elemente vorkommen, deren Abstand $\geq b$ ist. Im Falle, dass α , β und γ kompakt und in bezug auf r_1 katalogisiert sind, besteht offenbar eine Maximalsprungweite von α zwischen β und γ , welche sich mit jedem beliebigen Grade der Genauigkeit approximieren lässt.

Seien A und B in bezug auf r'_1 katalogisierte kompakte Teilspezies von S' und C und D in bezug auf r_1 katalogisierte kompakte Teilspezies von S , während von den Spezies A , B und C je zwei einen Abstand $\circ > 0$ besitzen, und seien A und B in S' durch C getrennt, womit wir (für Spezies S, S', A, B und C der angegebenen Art) meinen, dass jede in bezug auf r'_1 katalogisierte kompakte Teilspezies S'' von S' , welche eine Entfernung $\circ > 0$ von C besitzt, eine Sprungweite (mithin auch eine rationale Sprungweite) $\circ > 0$ zwischen A und B aufweist¹⁾. Alsdann

¹⁾ Dieser "schwachen Trennung" lassen sich an die Seite stellen:

1. Die „starke Trennung“, welche stattfindet, wenn jedes in einer Entfernung $\circ > 0$ von C gelegene Element von S' entweder mit dem Prädikate eines g_A -Elementes, oder mit

können wir nach der obigen Methode eine mit einer in bezug auf r'_1 katalogisierten kompakten Teilspezies von S' zusammenfallende Teilmenge ${}_S M_n(C)$ von M' bestimmen, so dass jedes Element von ${}_S M_n(C)$ einen Abstand $\geq 4^{-n}$ von C besitzt, während jedes in einer Entfernung $\geq 2 \cdot 4^{-n}$ von C liegende Element von S' mit einem Elemente von ${}_S M_n(C)$ zusammenfällt. Jeder Teilmenge von ${}_S M_n(C)$ ist als Sprungweite ihrer Abschliessung zwischen A und B eine positive rationale Zahl zugeordnet; diese positive rationale Zahl besitzt nach dem ersten Absatze dieses § einen Minimalwert σ_n , welcher als Sprungweite der Abschliessung einer beliebigen Teilmenge von ${}_S M_n(C)$ zwischen A und B , mithin insbesondere als Sprungweite einer beliebigen in einer Entfernung $\geq 2 \cdot 4^{-n}$ von C gelegenen, in bezug auf r'_1 katalogisierten kompakten Teilspezies von S' zwischen A und B auftritt. Wenn n hinreichend gross gewählt wird, dürfen wir, wenn wie immer $\varrho(X, Y)$ den Abstand zwischen X und Y vorstellt, annehmen, dass σ_n , sowie $2 \cdot 4^{-n+1} \cdot \leq \varrho(A, B)$, $\leq \varrho(A, C)$ und $\leq \varrho(B, C)$ sind.

Sei A_1 (bzw. B_1) eine solche gänzlich in einem Abstände $< \frac{1}{4} \sigma_n$ von S' gelegene, in bezug auf r_1 katalogisierte kompakte Teilspezies von S , dass jedes in einer Entfernung $\leq \frac{1}{4} \sigma_n$ von A_1 (bzw. von B_1) gelegene Element von S' zu A (bzw. zu B) gehört, so dass A_1 (bzw. B_1) auch gänzlich in einem Abstände $< \frac{1}{4} \sigma_n$ von A (bzw. von B) gelegen ist.

Sei T eine gänzlich in einer Entfernung $< \frac{1}{4} \sigma_n$ von S' und gänzlich in einer Entfernung $\geq 2 \cdot 4^{-n} + \frac{1}{4} \sigma_n$ von C gelegene, in bezug auf r_1 katalogisierte kompakte Teilspezies von S . *Alsdann weist T zwischen A_1 und B_1 eine Sprungweite $\frac{1}{2} \sigma_n$ auf.* Wenn wir nämlich eine endliche Folge von Elementen von S betrachten, von denen das erste zu A_1 , das letzte zu B_1 und alle anderen zu T gehören, so können wir, indem

dem Prädikate eines g_B -Elementes (welche Prädikate sich gegenseitig ausschliessen) versehen ist, mit der Massgabe, dass jedes Element von A (bzw. von B) ein g_A -Element (bzw. ein g_B -Element) ist, und dass zu jedem $r_1 > 0$ ein solches $r_2 > 0$ bestimmt werden kann, dass jedes in einer Entfernung $\leq r_2$ von einem in einer Entfernung $\geq r_1$ von C gelegenen g_A -Elemente (bzw. g_B -Elemente) liegende Element von S' ebenfalls ein g_A -Element (bzw. g_B -Element) ist.

2. Die "mittlere Trennung", welche stattfindet, wenn jedes in einer Entfernung > 0 von C gelegene Element von S' entweder die t_A -Eigenschaft oder die t_B -Eigenschaft (welche sich übrigens nicht ausschliessen) besitzt. Dabei sagen wir, dass ein Element q' von S' die t_A -Eigenschaft (bzw. die t_B -Eigenschaft) besitzt, wenn ein $r(q') > 0$ existiert, so dass jede in bezug auf r'_1 katalogisierte kompakte Teilspezies von S' , welche gänzlich in einer Entfernung $\leq r(q')$ von q' gelegen ist, in S' durch C von A (bzw. von B) "schwach getrennt" wird.

wir jedes Element der Folge einer passenden Verrückung $< \frac{1}{4} \sigma_n$ unterziehen, eine endliche Folge von Elementen von S erzeugen, von denen das erste zu A , das letzte zu B und alle anderen zu S' gehören und in einer Entfernung $> 2 \cdot 4^{-n}$ von C liegen, in welcher mithin zwei aufeinanderfolgende Elemente vorkommen, deren Abstand $\geq \sigma_n$ ist.

Sei R eine alle Elemente von D in einem Abstände $\leq \frac{1}{8} \sigma_n$ von S' enthaltende und gänzlich in einer Entfernung $< \frac{1}{4} \sigma_n$ von S' gelegene, in bezug auf r_1 katalogisierte kompakte Teilspezies von D ; seien A_1 und B_1 den im vorigen Absatz aufgestellten Forderungen entsprechend, und überdies innerhalb R gewählt; und sei C_1 eine alle Elemente von D in einem Abstände $\leq 2 \cdot 4^{-n} + \frac{1}{4} \sigma_n$ von C enthaltende, in einem Abstände $> \frac{1}{4} \sigma_n$ von A sowie von B gelegene, in bezug auf r_1 katalogisierte kompakte Teilspezies von D . Alsdann besitzen erstens je zwei der Spezies A_1 , B_1 und C_1 einen Abstand > 0 voneinander, zweitens sind nach dem vorigen Absatze A_1 und B_1 in R durch C_1 getrennt.

Mit dem obigen ist folgende Eigenschaft bewiesen: Sei S eine katalogisierte kompakte Spezies; S' und D in bezug auf S katalogisierte kompakte Teilspezies von S ; A und B je eine Relativumgebung von S' enthaltende, in einem Abstände > 0 voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S' ; C eine in einem Abstände > 0 von A sowohl wie von B gelegene, A und B in S' trennende, in bezug auf S katalogisierte kompakte Teilspezies von S ; C_1 eine alle Elemente von D in einem Abstände $\leq a_1$ von C enthaltende, in einem Abstände $> a_2 > 0$ von A sowohl wie von B gelegene, in bezug auf S katalogisierte kompakte Teilspezies von D ; n und ein zugehöriges $\sigma(n)$ (d.h. eine für eine beliebige in einer Entfernung $\geq 2 \cdot 4^{-n}$ von C gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S' bestehende Sprungweite zwischen A und B) so gewählt, dass je zwei der Spezies A , B und C in einer Entfernung $\geq \sigma_n$ und $\geq 2 \cdot 4^{-n+1}$ voneinander gelegen sind und überdies die Ungleichungen $2 \cdot 4^{-n} + \frac{1}{4} \sigma_n \leq a_1$ und $\frac{1}{4} \sigma_n < a_2$ bestehen; R eine alle Elemente von D in einem Abstände $\leq \frac{1}{8} \sigma_n$ von S' enthaltende, gänzlich innerhalb einer Entfernung $< \frac{1}{4} \sigma_n$ von S' gelegene, in bezug auf S katalogisierte kompakte Teilspezies von D ; A_1 (bzw. B_1) eine solche in bezug auf S katalogisierte kompakte Teilspezies von R , dass jedes in einer Entfernung $\leq \frac{1}{4} \sigma_n$ von A_1 (bzw. B_1) gelegene Element von S' zu A (bzw. B) gehört. Alsdann sind A_1 und B_1 in R durch C_1 getrennt.

§ 3. Der Begriff des Dimensionsgrades für katalogisiertkompakte Spezies.

Wir sagen, dass die kompaktkatalogisierte Spezies S den *unteren Dimensionsgrad* n besitzt, wenn in S zwei in bezug auf S katalogisierte kompakte Teilspezies A und B angegeben werden können, welche in einem Abstände $\circ > 0$ voneinander gelegen sind und die Eigenschaft besitzen, dass jede A und B in S trennende²⁾, in bezug auf S katalogisierte kompakte Teilspezies von S den unteren Dimensionsgrad $n-1$ besitzt. Weiter sagen wir, dass S den *unteren Dimensionsgrad Null bzw. ω* besitzt, wenn ein Element von S angegeben werden kann, bzw. wenn S für jede natürliche Zahl n den unteren Dimensionsgrad n besitzt.

Wir sagen, dass die kompaktkatalogisierte Spezies S den *oberen Dimensionsgrad* n besitzt, wenn je zwei in einem Abstände $\circ > 0$ voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S durch eine den oberen Dimensionsgrad $n-1$ besitzende, in bezug auf S katalogisierte kompakte Teilspezies von S , in S getrennt werden können. Weiter sagen wir, dass S den *oberen Dimensionsgrad Null bzw. ω* besitzt, wenn S zwischen je zwei in einem Abstände $\circ > 0$ voneinander gelegenen, in bezug auf S katalogisierten kompakten Teilspezies von S eine Sprungweite $\circ > 0$ aufweist bzw. wenn je zwei in einem Abstände $\circ > 0$ voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S durch eine einen endlichen oberen Dimensionsgrad besitzende, in bezug auf S katalogisierte kompakte Teilspezies von S , in S getrennt werden können.

Wenn eine kompaktkatalogisierte Spezies sowohl den oberen wie den unteren Dimensionsgrad n bzw. 0 bzw. ω besitzt, so sagen wir, dass sie den *allgemeinen Dimensionsgrad* n bzw. 0 bzw. ω besitzt.

Den obigen Definitionen lässt sich leicht eine von der Rekurrenz unabhängige Form geben. Dazu denken wir uns die Spezies S von zwei Personen P und Q der „Dimensionsoperation“ unterzogen, worunter wir folgendes verstehen: P wählt in S zwei in einem Abstände $\circ > 0$ voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies A und B , worauf Q die Spezies A und B in S trennt durch eine in bezug auf S katalogisierte kompakte Teilspezies S_1 von S . Sodann wählt P zwei in einem Abstände $\circ > 0$ voneinander gelegene, in bezug auf S_1 katalogisierte kompakte Teilspezies A_1 und B_1 von S_1 , worauf Q die Spezies A_1 und B_1 in S_1 trennt durch eine in bezug auf S_1 katalogisierte kompakte Teilspezies S_2 von S_1 . Dieser Prozess wird unbeschränkt wiederholt, bis eventuell eine Spezies S_h auftritt, welche zwischen je zwei in einem Abstände $\circ > 0$ voneinander gelegene, in bezug auf S_h katalogisierten kompakten Teilspezies von S_h eine Sprungweite $\circ > 0$ aufweist.

²⁾ Zu je zwei in einem Abstände $\circ > 0$ voneinander gelegenen, in bezug auf S katalogisierten kompakten Teilspezies von S kann *immer eine* in einem Abstände $\circ > 0$ sowohl von A wie von B gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S angegeben werden, welche A und B in S trennt.

Wenn nun Q unabhängig von den Wahlen der A_ν und B_ν (inklusive A und B) dafür sorgen kann, dass eine Spezies S_h auftritt, deren $h \leq n$, so sagen wir, dass S den oberen Dimensionsgrad n besitzt. Wenn dagegen P unabhängig von den Wahlen der S_ν dafür sorgen kann, dass keine Spezies S_h auftritt, deren $h < n$, so sagen wir, dass S den unteren Dimensionsgrad n besitzt³⁾.

§ 4. Die Dimensionsgrade der n -dimensionalen Mannigfaltigkeiten.

SATZ 1. Jedes n -dimensionale Fragment S (worunter wir hier die „vereinigende“ katalogisiertkompakte Punktkernspezies der n -dimensionalen Elemente von S verstehen) besitzt den oberen Dimensionsgrad n .

BEWEIS. Nachdem P in S die Spezies A und B gewählt hat, bestimmt Q eine so dichte simpliziale Zerlegung \mathfrak{J} von S , dass er auf Grund derselben eine Gruppe g von Grundsimplex von \mathfrak{J} und zwei Grössen $a \circ > 0$ und $b \circ > 0$ bestimmen kann mit der Eigenschaft, dass jedes in einem Abstände $\leq a$ von A gelegene Grundsimplex von \mathfrak{J} zu g gehört und jedes zu g gehörige Grundsimplex von \mathfrak{J} einen Abstand $\geq b$ von B besitzt. Die Gruppe g bildet ein n -dimensionales Fragment, deren A und B in S trennende Grenze, welche von Q als S_1 gewählt wird, aus der „vereinigenden“ katalogisiertkompakten Punktkernspezies einer endlichen Anzahl von geschlossenen $(n-1)$ -dimensionalen Pseudomannigfaltigkeiten besteht, mithin ein $(n-1)$ -dimensionales Fragment bildet (in welchem übrigens mehrere Elementseiten, welche als verschieden zu betrachten sind, zusammenfallen können). Falls darauf P die Spezies A_1 und B_1 im selben Teilkontinuum von S_1 wählt, bestimmt Q eine so dichte simpliziale Zerlegung \mathfrak{J}_1 von S_1 , dass er auf Grund derselben eine Gruppe g_1 von Grundsimplex von \mathfrak{J}_1 und zwei Grössen $a_1 \circ > 0$ und $b_1 \circ > 0$ bestimmen kann mit der Eigenschaft dass jedes in einem Abstände $\leq a_1$ von A_1 gelegene Grundsimplex von \mathfrak{J}_1 zu g_1 gehört und jedes zu g_1 gehörige Grundsimplex von \mathfrak{J}_1 einen Abstand $\geq b_1$ von B_1 besitzt. Die Gruppe g_1 bildet ein $(n-1)$ -dimensionales Fragment, deren A_1 und B_1 in S_1 trennende Grenze, welche von Q als S_2 gewählt wird, aus der „vereinigenden“ katalogisiertkompakten Punktkernspezies einer endlichen Anzahl von geschlossenen $(n-2)$ -dimensionalen Pseudomannigfaltigkeiten besteht, mithin ein $(n-2)$ -dimensionales Fragment bildet (in welchem übrigens wieder mehrere Elementseiten, welche als verschieden zu betrachten sind, zusammenfallen können). In dieser Weise fortfahrend, ge-

³⁾ Die im § 3 aufgestellten Definitionen bilden mit der im § 4 enthaltenen Beweisführung eine intuitionistische Ausarbeitung der in diesen Proceedings Bd. XXVII, S. 636–638 gegebenen Variante zur Einführung des „natürlichen Dimensionsbegriffes“. Ich benutze diese Gelegenheit, um darauf hinzuweisen, dass a. a. O. S. 636 Z. 13–14 anstatt: jede zusammenhängende Teilspezies von π , welche sowohl mit p wie mit p' Punkte gemeinsam hat, zu lesen ist: jede zwischen p und p' zusammenhängende Teilspezies von π , welche sowohl mit p wie mit p' Punkte gemeinsam hat.

langt Q schliesslich zu einem aus endlichvielen Punkten bestehenden S_n , es sei denn, dass der Prozess schon früher dadurch beendet wurde, dass P einmal A_ν und B_ν nicht in demselben Teilkontinuum von S_ν wählte.

SATZ 2. Jede kompakte Teilspezies S eines n -dimensionalen Fragmentes F , welche in bezug auf F katalogisiert ist und ein Grundsimplex G einer simplizialen Zerlegung von F enthält, besitzt den unteren Dimensionsgrad n .

BEWEIS. Seien E_1, E_2, \dots, E_{n+1} die Eckpunkte eines Grundsimplexes einer simplizialen Zerlegung von G , das von allen Seiten von G einen Abstand $\circ > 0$ besitzt. Alsdann wählt P für A (bzw. B) eine in bezug auf S katalogisierte kompakte Teilspezies von S , welche für gewisse Grössen ε und $\varepsilon' \circ > 0$ alle in einer Entfernung $\leq \varepsilon$ von E_1 (bzw. vom Simplex $E_2 E_3 \dots E_{n+1}$) gelegenen Punktkerne von S enthält und eine Entfernung $\geq \varepsilon'$ vom Simplex $E_2 E_3 \dots E_{n+1}$ (bzw. von E_1) besitzt, wobei dafür gesorgt wird, dass A und B in einer Entfernung $\circ > 0$ voneinander liegen; für A_1 (bzw. B_1) eine in bezug auf S_1 katalogisierte kompakte Teilspezies von S_1 , welche für gewisse Grössen ε_1 und $\varepsilon'_1 \circ > 0$ alle in einer Entfernung $\leq \varepsilon_1$ vom Segment $E_1 E_2$ (bzw. vom Simplex $E_1 E_3 \dots E_{n+1}$) gelegenen Punktkerne von S_1 enthält und eine Entfernung $\geq \varepsilon'_1$ vom Simplex $E_1 E_3 \dots E_{n+1}$ (bzw. vom Segment $E_1 E_2$) besitzt, wobei dafür gesorgt wird, dass A_1 und B_1 in einer Entfernung $\circ > 0$ voneinander liegen; für A_2 (bzw. B_2) eine in bezug auf S_2 katalogisierte kompakte Teilspezies von S_2 , welche für gewisse Grössen ε_2 und $\varepsilon'_2 \circ > 0$ alle in einer Entfernung $\leq \varepsilon_2$ vom Simplex $E_1 E_2 E_3$ (bzw. vom Simplex $E_1 E_2 E_4 \dots E_{n+1}$) gelegenen Punktkerne von S_2 enthält und eine Entfernung $\geq \varepsilon'_2$ vom Simplex $E_1 E_2 E_4 \dots E_{n+1}$ (bzw. vom Simplex $E_1 E_2 E_3$) besitzt, wobei dafür gesorgt wird, dass A_2 und B_2 in einer Entfernung $\circ > 0$ voneinander liegen; u. s. w. Um zu beweisen, dass von den Spezies S_1, S_2, \dots, S_n keine in Fortfall kommen kann, bezeichnen wir mit T das (abgeschlossene) Simplex $E_1 E_2 E_3 \dots E_{n+1}$ und konstruieren nach irgend einem Gesetz eine Fundamentalreihe $\mathfrak{Z}_1, \mathfrak{Z}_2, \dots$ von simplizialen Zerlegungen von T , deren jede eine Unterteilung der ihr vorangehenden ist, während für unbeschränkt wachsendes ν die Maximalbreite der Grundsimplen von \mathfrak{Z}_ν positiv gegen Null konvergiert. Nehmen wir nun an, dass es ein solches $\nu_1 \leq n$ gibt, dass S_{ν_1-1} noch auftritt, aber S_{ν_1} nicht mehr auftritt. Alsdann bestimmen wir als K_{ν_1-1} eine solche in einer Entfernung $\circ > 0$ von A_{ν_1-2}, B_{ν_1-2} und den Simplexen $E_1 E_2 \dots E_{\nu_1-1}; E_2 E_3 \dots E_{n+1}; E_1 E_3 \dots E_{n+1}; \dots; E_1 E_2 \dots E_{\nu_1-2} E_{\nu_1} \dots E_{n+1}$ gelegene „vereinigende“ katalogisiertkompakte Punktkeimspezies einer Gruppe von Grundsimplen eines passenden $\mathfrak{Z}_{\mu_{\nu_1-1}}$, dass für ein gewisses $e_{\nu_1-1} \circ > 0$ alle in einer Entfernung $\leq e_{\nu_1-1}$ von S_{ν_1-1} gelegenen Grundsimplen von $\mathfrak{Z}_{\mu_{\nu_1-1}}$ zu K_{ν_1-1} gehören, während K_{ν_1-1} zwischen seinen Durchschnitten mit den Simplexen $E_1 E_2 \dots E_{\nu_1}$ und $E_1 E_2 \dots E_{\nu_1-1} E_{\nu_1+1} E_{\nu_1+2} \dots E_{n+1}$ eine Sprungweite $\circ > 0$ aufweist. Sodann wählen wir nach dem Resultate des § 2 als K_{ν_1-2} eine solche in einer Entfernung $\circ > 0$ von A_{ν_1-3}, B_{ν_1-3} und den Simplexen $E_1 E_2 \dots E_{\nu_1-2}; E_2 E_3 \dots E_{n+1}; E_1 E_3 \dots E_{n+1}; \dots; E_1 E_2 \dots E_{\nu_1-3} E_{\nu_1-1} \dots E_{n+1}$ gelegene „vereinigende“

katalogisiertkompakte Punktkernspezies einer Gruppe von Grundsimplen eines passenden $\mathfrak{S}_{\alpha_{\nu-2}}$ ($\alpha_{\nu-2} \geq \alpha_{\nu-1}$), dass für ein gewisses $e_{\nu-2} > 0$ alle in einer Entfernung $\leq e_{\nu-2}$ von $S_{\nu-2}$ gelegenen Grundsimplen von $\mathfrak{S}_{\alpha_{\nu-2}}$ zu $K_{\nu-2}$ gehören, und die Durchschnitte von $K_{\nu-2}$ mit den Simplexen $E_1 E_2 \dots E_{\nu-1}$ und $E_1 E_2 \dots E_{\nu-2} E_{\nu} \dots E_{n+1}$ in $K_{\nu-2}$ durch $K_{\nu-1}$ getrennt werden. Darauf wählen wir, *wiederum nach dem Resultate des § 2*, als $K_{\nu-3}$ eine solche in einer Entfernung > 0 von $A_{\nu-4}, B_{\nu-4}$ und den Simplexen $E_1 E_2 \dots E_{\nu-3}; E_2 E_3 \dots E_{n+1}; E_1 E_3 \dots E_{n+1}; \dots; E_1 E_2 \dots E_{\nu-4} E_{\nu-2} \dots E_{n+1}$ gelegene „vereinigende“ katalogisiertkompakte Punktkernspezies einer Gruppe von Grundsimplen eines passenden $\mathfrak{S}_{\alpha_{\nu-3}}$ ($\alpha_{\nu-3} \geq \alpha_{\nu-2}$), dass für ein gewisses $e_{\nu-3} > 0$ alle in einer Entfernung $\leq e_{\nu-3}$ von $S_{\nu-3}$ gelegenen Grundsimplen von $\mathfrak{S}_{\alpha_{\nu-3}}$ zu $K_{\nu-3}$ gehören, und die Durchschnitte von $K_{\nu-3}$ mit den Simplexen $E_1 E_2 \dots E_{\nu-2}$ und $E_1 E_2 \dots E_{\nu-3} E_{\nu-1} \dots E_{n+1}$ in $K_{\nu-3}$ durch $K_{\nu-2}$ getrennt werden. Indem wir in dieser Weise fortfahren, bestimmen wir schliesslich als K_1 eine solche in einer Entfernung > 0 von A, B , dem Punkte E_1 und dem Simplexe $E_2 E_3 \dots E_{n+1}$ gelegene „vereinigende“ katalogisiertkompakte Punktkernspezies einer Gruppe von Grundsimplen eines passenden \mathfrak{S}_{α_1} ($\alpha_1 \geq \alpha_2$), dass für ein gewisses $e_1 > 0$ alle in einer Entfernung $\leq e_1$ von S_1 gelegenen Grundsimplen von \mathfrak{S}_{α_1} zu K_1 gehören, und die Durchschnitte von K_1 mit dem Segmente $E_1 E_2$ und dem Simplexe $E_1 E_3 \dots E_{n+1}$ in K_1 durch K_2 getrennt werden. (Hierzu ist zu bemerken, dass auch $K_{\nu-1}, K_{\nu-2}, \dots, K_2$ „vereinigende“ katalogisiertkompakte Punktkernspezies von Gruppen von Grundsimplen von \mathfrak{S}_{α_1} darstellen). Bezeichnen wir noch das Grundsimplen $E_1 E_2 \dots E_{n+1}$ mit K , so werden auch die Durchschnitte von K mit dem Punkte E_1 und dem Simplexe $E_2 E_3 \dots E_{n+1}$ in K durch K_1 getrennt.

Mit der obigen Konstruktion ist der Beweis von Satz 2 zurückgeführt auf den S. 150 von Bd. 142 des Journ. f. Math. formulierten Hilfssatz, dessen a.a. O. S. 151 u. 152 gegebene Herleitung auch für die intuitionistische Mathematik in Kraft bleibt⁴⁾.

⁴⁾ Aus den Entwicklungen des § 4 geht gleichzeitig hervor, dass der aus den Sätzen 1 und 2 zusammengesetzte „Satz von der n -dimensionalität des R_n “ bestehen bleibt, wenn anstatt des schwachen der mittlere oder der starke Trennungsbegriff dem Dimensionsbegriffe zu Grunde gelegt wird.

Mathematics. — “On transformations of projective spaces”. By Prof. L. E. J. BROUWER.

(Communicated at the meeting of May 29, 1926).

In my article: “*Ueber Abbildung von Mannigfaltigkeiten*” (Math. Ann. 71, p. 97—115) I have proved the following theorem:

A uniform continuous transformation without fixed point of an n -dimensional sphere into itself, has degree -1 for even n and $+1$ for odd n .

Since it appears that the specialisation of this theorem which gives an analogous property of uniform continuous transformations of n -dimensional projective spaces into themselves, and of which a particular case had already been published in these Proceedings XI, p. 798, is not to that degree, as I then thought permissible to assume, self evident to all readers, it shall be set forth here in full detail.

In the first place let there be given a uniform continuous transformation τ of a $(2n-1)$ -dimensional projective space E into itself. We provide E with a positive indicatrix and with an elliptic metric. Let S be the $(2n-1)$ -dimensional sphere provided with a positive indicatrix and a metric, obtained by duplication of E . Let P be a point of E , P_1 and P_2 the corresponding points of S , P' the image of P in E under τ , and P'_1 and P'_2 the points of S corresponding to P' . Let τ_1 be the uniform continuous transformation of S into itself which brings P_1 into P'_1 , and which becomes τ by the folding of S into E ¹⁾. Then the volume of the image of S under τ_1 (measured by the volume of the image of S under the simplicial approximations of τ_1) is twice as much as the volume of the image of E under τ . Since, however, the volume of S is also twice that of E , the degree of τ_1 appears to be equal to that of τ . Further, since the absence of a fixed point for τ implies the absence of one for τ_1 , the transformation τ , if exhibiting no fixed point must be necessarily of degree $+1$.

On the other hand there exist arbitrarily small congruent transformations of S into itself without fixed point. To these correspond arbitrarily

¹⁾ The existence of τ_1 is due to the fact that to a circuit of S passing through P_1 corresponds a contractible circuit of E passing through P the image of which under τ is a contractible circuit of E passing through P' which corresponds to a circuit of S passing through P'_1 .

The antipodal point-pairs of S become under τ_1 again antipodal point-pairs or simple points, according as uncontractible circuits of E become under τ again uncontractible or contractible.

small congruent transformations of E into itself without fixed point. Thus there exist uniform continuous transformations of E into itself of degree $+1$ and without fixed point.

In the second place let there be given a uniform continuous transformation τ of a $2n$ -dimensional projective space E into itself. We provide E with an elliptic metric. Let S be the $2n$ -dimensional sphere provided with a metric, obtained by duplication of E . We provide S with a positive indicatrix. Let P be a point of E , P_1 and P_2 the corresponding points of S , P' the image of P in E under τ , and P'_1 and P'_2 the points of S corresponding to P' . Let τ_1 and τ_2 be the uniform continuous transformations of S into itself which carry P_1 into P'_1 and P_2 respectively, and which become τ by the folding of S into E . Then corresponding image simplexes of corresponding simplicial approximations of τ_1 and τ_2 have equal volumes of opposite signs; thus τ_1 and τ_2 have equal degrees of opposite signs, and thus either τ_1 or τ_2 has a fixed point. Then, however, τ also must have a fixed point²⁾.

²⁾ Dr. HOPF points out to me that a uniform continuous transformation of a $(2n-1)$ -dimensional projective space E into itself possesses at least two invariant points, if its degree is odd and $\neq +1$. We can add that on the other hand a uniform continuous transformation of a $2n$ -dimensional projective space E into itself has at least two invariant points, if its absolute degree (i.e. the absolute value of the degrees of the correspondent transformations of the sphere S duplicating E) is ≥ 2 .

Mathematics. — “Die intuitionistische Form des HEINE-BORELSchen Theorems”. By Prof. L. E. J. BROUWER.

(Communicated at the meeting of June 26, 1926).

In der klassischen Mathematik gilt für einen kompakten metrischen Raum R folgende Eigenschaft:

HEINE-BORELSCHES THEOREM: Wenn in R jedem Punkte einer kompakten Teilspezies R' von R eine Umgebung dieses Punktes zugeordnet ist, so besteht eine endliche Anzahl dieser Umgebungen, in denen R' vollständig enthalten ist.

Bei einer früheren Gelegenheit¹⁾ habe ich gezeigt, dass diese Eigenschaft in ihrer allgemeinen Form in der intuitionistischen Mathematik unrichtig ist. Hier werde ich eine Spezialisierung des Theorems herleiten, welche auch in der intuitionistischen Mathematik in Kraft bleibt.

Es sei S eine katalogisiertkompakte Spezies²⁾, P ein Element von S , n eine natürliche Zahl. Eine Teilspezies von S , zu der jedes in einer Entfernung $\leq 4^{-n}$ von P gelegene Element von S gehört, nennen wir eine n -Umgebung von P in S . Eine Teilspezies von S , welche für eine passende natürliche Zahl n eine n -Umgebung von P ist, nennen wir eine Umgebung von P in S .

Es sei S' eine in bezug auf S katalogisierte kompakte Teilspezies von S , und es sei mittels eines Gesetzes w jedem Element P' von S' eine Umgebung $\omega(P')$ von P' in S zugeordnet. In der in diesen Proceedings XXIX, S. 856 angegebenen Weise bestimmen wir zu S' für jede natürliche Zahl n die α'_n -Elemente, und sodann die mit S' „zusammenfallende“ finite Menge M' . Alsdann ist jedem Elemente e' von M' infolge w eine Umgebung in S eines mit e' zusammenfallenden Elementes P' von S' , und somit gleichzeitig eine natürliche Zahl $m(e')$, mittels deren die betreffende Umgebung als $m(e')$ -Umgebung charakterisiert wird, zugeordnet. Auf Grund der in diesen Proceedings XXVII, S. 192 (vgl. auch S. 646) hergeleiteten Eigenschaft besteht dann, bei Variierung von e' innerhalb M' , für $m(e')$ nur eine endliche Anzahl von Werten, und darunter ein Maximalwert $m_1 - 1$. Zum Gesetze w gehört also eine solche natürliche Zahl m_1 , dass für jedes Element P' von S' die Umgebung $\omega(P')$ eine $(m_1 - 1)$ -Umgebung ist.

Wir bezeichnen die Anzahl der α'_{m_1} -Elemente mit d_{m_1} und die α'_{m_1} -Elemente selber der Reihe nach mit $\alpha'_{m_1, \nu}$ ($1 \leq \nu \leq d_{m_1}$). Für jedes ν wählen

¹⁾ Vgl. Wis- en Natuurkundig Tijdschrift 2, S. 4; Journ. f. Math. 154, S. 4–5.

²⁾ Vgl. diese Proceedings XXIX, S. 856.

wir als $P(\alpha'_m)$ ein in einer Entfernung $\leq c < \frac{3}{2} \cdot 4^{-m}$ von α'_m gelegenes Element von S' . Alsdann liegt ein beliebiges Element von S' , weil es in einer Entfernung $< 4^{-m}$ von einem der α'_m gelegen ist, in einer Entfernung $< \frac{5}{2} \cdot 4^{-m}$ von einem der $P(\alpha'_m)$, liegt also innerhalb eines der $\omega \{P(\alpha'_m)\}$.

Hiermit ist folgende Eigenschaft bewiesen:

INTUITIONISTISCHES ÜBERDECKUNGSTHEOREM: *Wenn in der katalogisiertkompakten Spezies S durch ein Gesetz w einem beliebigen Elemente e einer in bezug auf S katalogisierten kompakten Teilspezies S' von S eine Umgebung von e in S zugeordnet ist, so kann eine endliche Anzahl dieser Umgebungen angegeben werden, in denen S' vollständig enthalten ist³⁾.*

³⁾ Die Aussage des Satzes bleibt bestehen, wenn durch w nicht einem beliebigen Elemente e von S' , sondern nur einem beliebigen (eine positiv-konvergente unbegrenzte Folge von verschiedenen Elementen einer katalogisierten Folge darstellenden) Elemente eines beliebigen Elementes e von S' eine Umgebung von e in S zugeordnet ist (ohne dass dabei notwendigerweise den verschiedenen Elementen von e die gleiche Umgebung zugeordnet zu sein braucht).

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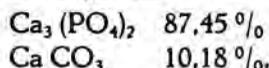
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Mineralogy. — “On the Mineral Component of Bones”, By W. F. DE JONG.
(Communicated by Prof. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of March 27, 1926).

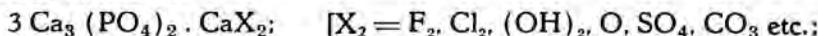
The chemical analysis of recent bones shows that the proportion in which they contain mineral constituents is approximately constant; f. i. the composition of a human thigh-bone is according to CARNOT: ¹⁾



and a varying amount of CaF_2 , CaCl_2 etc. making up only a small percentage of the whole. Fossilisation raises the amount of F gradually and in palaeozoic bones it is about as high as in normal fluor-apatite.

The ratio $\frac{\text{CaCO}_3}{\text{Ca}_3(\text{PO}_4)_2}$ being found about the same everywhere ²⁾ it may readily be suspected that bones are built up chiefly by one compound.

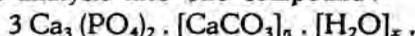
In fact CARNOT has already suspected the occurrence of a compound $3 \text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaCO}_3$, i. e. a mineral of the apatite series, of which the general formula is:



Ca can be substituted by the metals Fe, Mg etc. ³⁾]

The compound $3 \text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaCO}_3$ is not known as a pure, well-crystallized material ⁴⁾, but minerals of this composition are known by the name of podolite and dahllite ⁵⁾; both are poorly studied and often reported to contain water ⁶⁾.

ROGERS ⁷⁾ enlarges the scope of the conception “*collophanite*” and combines the whole analysis into one compound:



(n ranging from 0.55—1.43 and x from 0.28—3.03), and describes it as being amorphous with an index of refraction of 1.573—1.621, a weak anomalous double-refraction and an anomalous pleochroism.

To get to the root of this question exclusively optic investigation is not sufficient. This is why we have endeavoured to approach a solution

¹⁾ A. CARNOT, Recherches sur la composition générale et la teneur en fluor des os modernes et des os fossiles. Ann. des Mines, 1893 III p. 164.

²⁾ Cf. also FRANCHET, Sur la dissolution des os et des dents des sépultures préhistoriques. Revue Anthropologique 1925 p. 34.

³⁾ A. CARNOT. Ann. des Mines. Série 9, Tome X, p. 45.

A. F. ROGERS. Zt. f. Krist. 52, p. 213 (1913).

⁴⁾ C. HINTZE, Handbuch der Mineralogie I p. 515.

⁵⁾ C. DOELTER, Handbuch der Mineralchemie, Bd. III. Abt. I p. 352 and 577.

⁶⁾ GROTH—MIELEITNER, Mineralogische Tabellen, 1921.

⁷⁾ A. F. ROGERS, Mineralogy and petrography of fossil bones. Bull. of the Geol. Soc. of Am. Vol. 35 p. 535, 1924.

by means of röntgen-rays. Photos were taken of a number of pulverized bones with CuK_α -rays after the exposure-method of DEBIJE-SCHERRER¹⁾. The bones were fossil bones derived from the Limburg-Chalk, from the phosphatic deposits in Tunis²⁾, from *Ursus spelaeus* (diluvium in Moravia) and from human bones exhumed from the graves in Hungary and Drenthe; besides some recent bones of a seal, etc.

These photos cannot be distinguished from those of apatite, though the lines may sometimes appear to be more or less hazy and broadened. For further comparison photos have also been taken of the minerals staffelite, osteolite and pyromorphite [$3\text{Pb}_3(\text{PO}_4)_2 \cdot \text{PbCl}_2$], which belong to the apatite series. These pictures are approximately the same, Collophan³⁾, however, presents a totally different picture, as can be seen in the photometer-diagrams of Fig. 1, made with the self-recording contrivance devised by MOLL⁴⁾.

We are, therefore, justified in concluding, that a mineral from the apatite-series preponderates in the composition of the bones. The facts that no lines are found implying the presence of other components⁵⁾ and

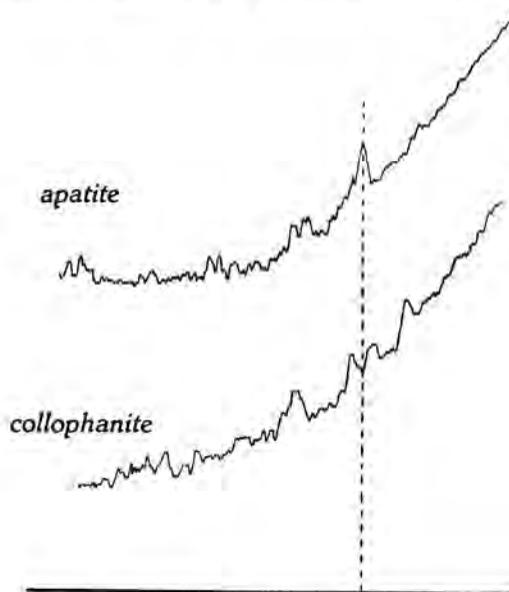


Fig. 1.

The lines on the films appear in the diagram as peaks. The place and the intensity of the lines of the two minerals deviate considerably.

1) DEBIJE and SCHERRER. Kgl. Ges. d. Wiss. Göttingen, Dez. 1915; Phys. Zt. 17, p. 277, 1916.

2) A. CARNOT. Ann. des Mines, Tome X, p. 200.

3) From the island Mona. Dana, The System of Mineralogy. 6 Ed. p. 108.

4) Proc. of the Phys. Soc. of London, 1921, p. 207.

5) Some percentage of an other crystalline substance is very difficult to detect by means of an analysis after the method we used. (Cf. WYCKOFF, The structure of crystals, 1924, p. 419), but it is not likely that a more or less considerable amount of CaCO_3 should not have yielded distinguishable lines.

that the ratio $\frac{\text{CaCO}_3}{\text{Ca}_3(\text{PO}_4)_2}$ in $3 \text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaCO}_3$ coincides with the observation i.e. 9.7 : 90.3 (cf. p. 870), induce us to assume the presence chiefly of $3 \text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaCO}_3$, or perhaps $3 \text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaO}$.

It need not surprise us that no difference is noticeable between the röntgen-pictures of the hypothetical carbonate-apatite and the ordinary apatite. This may be due partly to the haziness and the faintness of various lines, but it will no doubt chiefly be caused by the great resemblance of the two molecules in which the radicals CO_3 etc. exercise but little influence upon the general structure.

The assumption of an apatite-mineral in bones is quite in keeping with the observations of ROGERS and others. The faint double-refraction, the minute pleochroism and the index of refraction, (F-apatite = 1.64—1.65), the solubility in HNO_3 , the S.G. are entirely or approximately similar to those in the familiar terms of the apatite-series.

It is noteworthy that the crystals in fresh bones are so small that the röntgen-lines are hazy; so they do not contain more than, say, a hundred molecules. Most fossil bones also yield these lines. On the other hand while the bones are being burned, crystals accrue so that the lines appear sharply bounded. This perhaps affords a mean of differentiation between prehistorical findings that have been exposed to fire and those that have not.

S U M M A R Y.

Bones contain a mineral of the apatite-series, most probably $3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaCO}_3$. The radical CO_3 can be replaced here — as is often the case in fossilization — by F_2 , Cl_2 , SO_4 and perhaps others, Ca by Mg and perhaps by Fe and Al.

The mineral collophanite cannot be recognized.

The crystals contain some tens or hundreds of molecules, the röntgen-photo showing hazy lines; when the bones are being burned the crystals accrue and the lines become distinct. This may be of some importance for prehistorici.

The great significance of a research by means of röntgen-rays of bones was suggested to me by Dr. P. KRUIZINGA through whose interference I also obtained suitable material from Dr. A. E. VAN GIFFEN. Prof. J. A. GRUTTERINK allowed me free use of the Mineralogical Laboratory and gave me his valuable aid. To these gentlemen I desire to express my great obligations.

*From the Mineralogical Laboratory of the
Technical University of Delft.*

Physiology. — "*Diet and reproduction*" II. By G. GRIJNS and K. DE HAAN.

(Communicated at the meeting of April 24, 1926.)

After our communication at the meeting in October a.p. we continued our investigations on the connection between nutrition and reproduction.

We still observed 4 male rats: 2 on maizfood II and 2 on maizfood III all second generation. All 4 were sterile. Thus we found till now among 14 males on the 3 maizfoods that only one was fertile. We also examined some gonads of these rats. They proved to differ much in development; some ones only weighing $\frac{1}{6}$ of normal testes other ones about $\frac{1}{2}$. In the smaller gonads the epithelial cells of the seminiferous tubules were almost totally vanished, and only a few cells rich in protoplasm remembered the spermatocytes. In the other ones spermatogonia, spermatocytes and spermatozoa were present and did not show many differences with those in normal testes. But in fresh smears they showed little mobility even after addition of a solution of glucose.

In an other experience we used an other ration viz. basal ration with yellow fat. This is a vegetable extract mixed to a vitamin free fat, on which we intend to report afterwards, as our investigations on it are not yet finished. The composition of this diet was: albumen 2, casein 15, hardened fat 12, yellow fat 3, rice starch 52, McCollums saltmixture 5, marmite 5, decitrated lemonsquash 5.

The young rats were put to this diet immediately after weaning; they grew well on it, even a bit faster than rats on normal diet, as may be seen from diagram 1.

The results of matings have been recorded in table 1. It shows that fertility, judged by the number of young rats born was very sufficient. Out of 11 matings 2 were sterile; in one case the course of the weight indicated resorption of young; once we only found blood in the cotton-wool of the case of a female we had separated as pregnant. In this case dead young had probably been eaten.

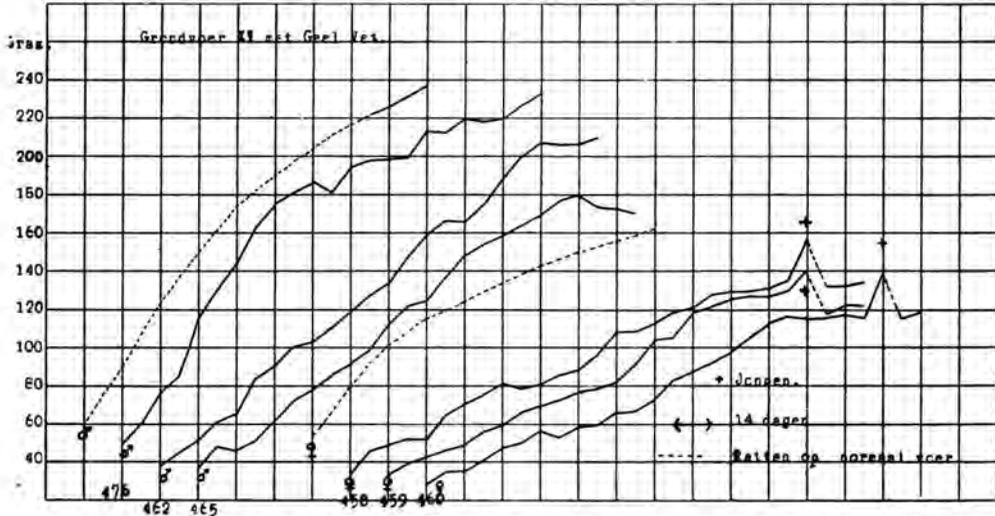
The remaining 6 matings yielded 38 youngs that died all except 2 a few days after birth. The mother of the 2 surviving (young of 460 and 483) had been given normal diet 3 days after whelping.

In a second experiment we used as main food whole wheat meal that had been exhausted with acetone for 10—12 days. According to BARNETT SURE's publications we supposed to be able to remove in this way the fertilising factor. Other vitamins however also being soluble in acetone, we had to supply these.

Therefore we composed a ration XIII as follows: Acetone extracted

whole wheat meal 250, wheat starch 125, hardened fat 60, butterfat 15, casein 25, decitrated lemonsquash 20, saltmixture as in cornfood 15. This

Grondvoer XI met Geel Vet = Basal diet XI with yellow fat.



Jongens = young. 14 dagen = 14 days. Ratten op normaal voer = Rats on normal diet.
Fig. 1.

TABLE 1.
Basal diet XI with yellow fat.

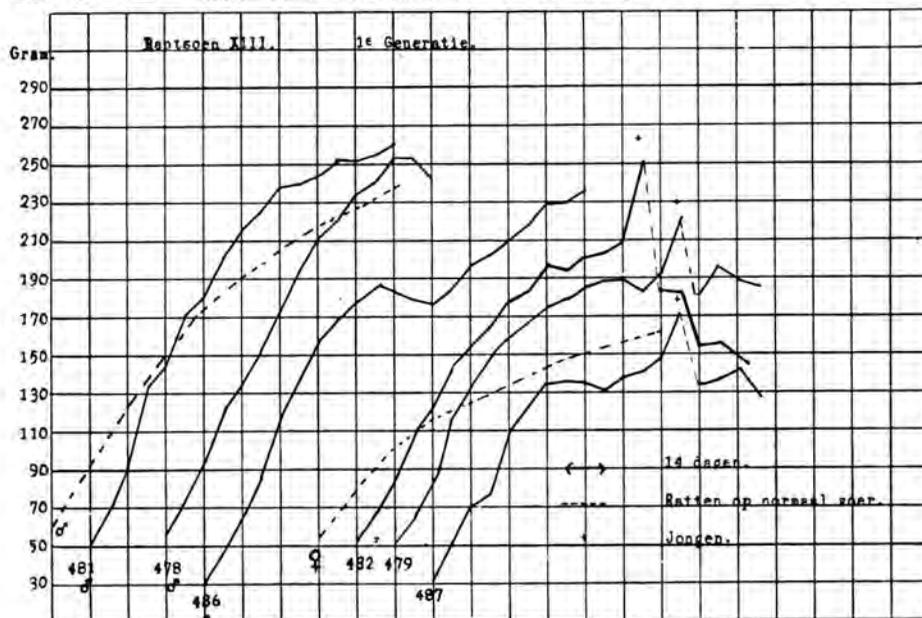
| Females | Males | Number of young | Weaned |
|---------|-------|--------------------------|-----------------|
| 458 | 483 | 7 | 0 |
| 458 | 467 | 6 | 0 |
| 458 | 483 | 6 | 0 |
| 460 | 467 | 7 | 0 |
| 460 | 483 | 5 | 0 |
| 460 | 483 | 2 | 2 ¹⁾ |
| 463 | 477 | abortus dead young | |
| 463 | 477 | | |
| 459 | 477 | 0 | 0 |
| 459 | 471 | 5 | 0 |
| 459 | 476 | 0 | — |

ration was also fed immediately after weaning. The rats grew very well, as may be taken from the curves in chart 2, where the growth of some at

¹⁾ Normal ration after whelping.

random rats is represented. From 8 matings (see table 2.) we got 51 young rats, 30 of which were weaned or 59 %.

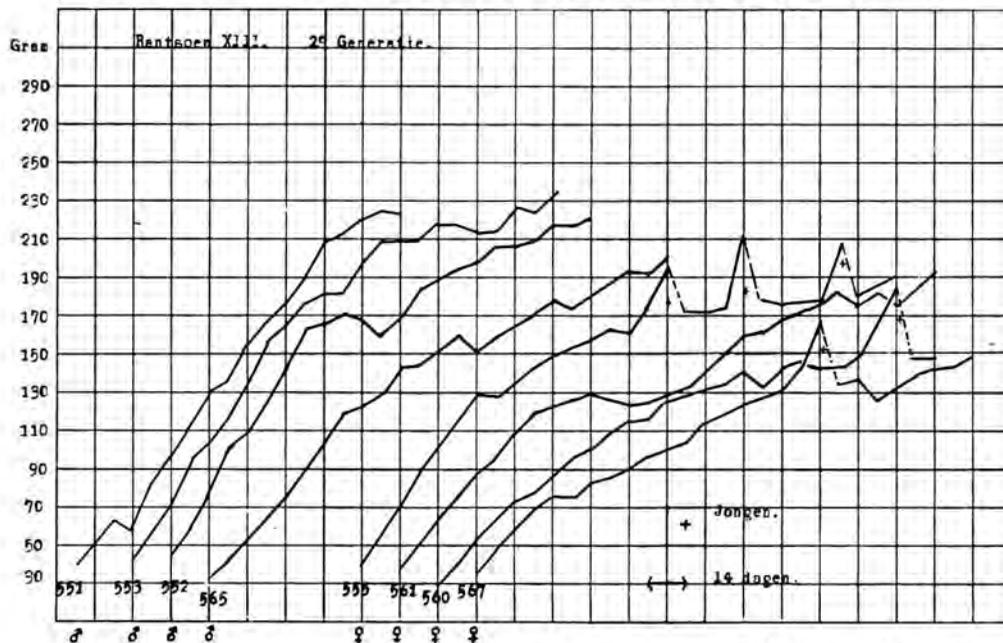
Rantsoen XIII = Ration XIII. 1e Generatie = 1st Generation.



14 dagen = 14 days. Ratten op normaal voer = Rats on normal diet. Jongen = young.

Fig. 2.

Rantsoen XIII = Ration XIII. 2e Generatie = 2nd Generation.



Jongen = young. 14 dagen = 14 days.

Fig. 3.

These were divided in 2 lots, the first getting the same diet as before, while for the second lot the whole wheat meal was not digested with acetone (ration XIIIa).

In both lots growth was normal. Of a number of the first growth curves have been reproduced in figure 3. Of the second lot, that grew as

TABLE 2.
Diet XIII. 1st generation.

| Females | Males | Number of young | Weaned |
|---------|-------|-----------------|--------|
| 479 | 489 | 9 | 3 |
| 482 | 486 | 10 | 9 |
| 484 | 478 | 7 | 0 |
| 484 | 486 | 4 | 3 |
| 487 | 480 | 2 | 0 |
| 487 | 486 | 3 | 0 |
| 488 | 481 | 8 | 7 |
| 497 | 458 | 8 | 8 |
| | | 51 | 30 |

TABLE 3.
Diet XIII. 2nd generation.

| Females | Males | Number of young | Weaned |
|---------|-------|-----------------|--------|
| 550 | 565 | 6 | 0 |
| 555 | 565 | 9 | 0 |
| 555 | 565 | 6 | 0 |
| 555 | 553 | 6 | 0 |
| 560 | 552 | 5 | 0 |
| 567 | 552 | 8 | 3 |
| 568 | 553 | 8 | 0 |
| 568 | 553 | 7 | 0 |
| | | 55 | 3 |

N.B. The 3 young from this litter grew very slowly. They could not be weaned before the seventh week. On normal diet this happens within 4 weeks.

fast, we added no curves to spare space. Table 3 shows the effect of matings in the first, table 4 of those in the second lot.

TABLE 4.
Diet XIIIa. 2nd generation.

| Females | Males | Number of young | Weaned |
|---------|-------|-----------------|--------|
| 549 | 558 | 1 | 0 |
| 554 | 563 | 9 | 0 |
| 554 | 563 | 5 | 0 |
| 564 | 557 | aborted | |
| 566 | 558 | 6 | 0 |
| 566 | 558 | 8 | 0 |
| 566 | 559 | 8 | 0 |

We see two rations, from which the first unables the mothers in the first generation immediately, the other in the second, to rear their young, while fertility in the males remains unchanged and that of the females, as far as to produce living young is considered, is but very little hampered.

That the animals fed on diet XIIIa were unable to rear their young, astonished us, for American investigations, we reported in our former communication, are in favour of a fertilising vitamin in wheatembryo.

In every case it is clear, that we can not speak of a single fertility controlling factor, but that different foodstuffs are wanted for the function of the testis, than for that of the milklands.

In a minute investigation on histological changes in sterile rats KARL E. MASON ¹⁾ used the following ration to make male rats sterile: casein 18, starch 54, lard 15, butterfat 9, saltmixture 4 plus yeast tablets 0.4 gr.

Addition of 40 gr. fresh lettuce prevented sterility, as in the case of BARNETT SURE.

We are continuing our observations with different diets, to see if it is necessary to postulate still other specific substances that are indispensable for the function of other organs playing a role in reproduction.

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¹⁾ K. E. MASON: A histological study of sterility in the albino rat due to a dietary deficiency. Proceedings Nat. Acad. of Sciences July 1925. Vol. 11.

Physiology. — "*On After-images and contrast in coloured light.*" By Prof. G. GRIJNS and K. DE HAAN.

(Communicated at the meeting of May 29, 1926).

It is a remarkable fact, that half the century, that passed since HERING defended his „Gegenfarbentheorie“ against the objections of DONDERS, did not bring a decision between this and YOUNG-HELMHOLTZ-MAXWELL's theory of complementary colours. Both have their partisans among physiologists and oculists, and all the work done on light- and coloursense has not been able to carry one of them to victory. Yet the adversary principles of those two views make one expect the possibility of finding such facts as are only compatible with one of them. HERING supposing an invertable reaction to be the cause of seeing colours (white and black included) one direction being called „Assimilation“, the other „Dissimilation“, the first associated with one sensation (black, green or blue) the other with the antagonistic sensation (white, red or yellow), seeks the explanation for the after-images in processes occurring in the visual substance (Sehsubstanz) involved by the preceding stimulus, while the assimilation-dissimilation rate that was removed from its rest-equilibrium returns to it, the opposite reaction prevails then and provokes the antagonistic sensation (Gegenfarbe). The colours of the complimentary after-image originate therefore in the „Sehsubstanz“ itself, and must prevail, wherever such after-images appear.

The theory of YOUNG-HELMHOLTZ-MAXWELL on the contrary supposes three independent fundamental colours. It explains after-images by the intervention of fatigue after contemplating a colour for some time, wherefrom the other colours of the background prevail. According to this view the colour of the after-image is determined by that of the background.

HERING explains the contrast by induction. A beam striking a part of the „Sehsubstanz“ and causing there assimilation, originates dissimilation in the adjacent parts and the reverse. How the action of irritated parts on not irritated ones occurs, HERING does not indicate. It can not be diffusion of dissociationproducts, as some of his partisans have suggested, because there would not remain matter for the after-images, and the immediate appearance of the contrast makes diffusion, which would require time, unacceptable.

HELMHOLTZ counts contrast to the erroneous judgments for which pleads, that it is most obvious, under circumstances, which suggest a coloured illumination, in which objects that are seen white ought to have the contrasting colour.

According to HERING's theory the contrast as well as the after-images always

must show the "Gegenfarbe", be it not unmixed under certain circumstances. While according to YOUNG-HELMHOLTZ's view colour of after-images and contrast will be destined by the difference of the colour present in the irritating light and the background.

Firstly we will consider those after-images, that originate if we fix for some time a coloured surface on grey paper, and without moving the eyes, remove the colour. But we use coloured light instead of white.

One of us ¹⁾ published observations on after-images in monochromatic light, by which he showed them to be always black. We now extend them to polychromatic light and to contrast.

We used an electrical light of 100 candles, shut up in a light-tight ventilated box, with at one end an opening, in which we screwed a kind of nosepiece carrying the coloured glasses.

This illuminated a circle of about half a meter diameter on a grey paper, in which we placed the coloured paper, of which we wanted to study the after-images. All was placed in our dark room, so that only the filtered light entered the room.

Spectroscopically we found our lightfilters to let pass the following wavelengths:

| | | | |
|----------------|---------|----------------|---------------------|
| 1. Red | 690—640 | 5 Bluish green | 575—515 |
| 2. Orange | 650—595 | 6 Blue | 465—420 |
| 3. Yellow | 690—510 | 7 Violet | 500—420 and 690—660 |
| 4. Yellowgreen | 570—540 | | |

The papers used ordinarily have longer spectra and some white light mixed with the colour, but if we contemplate them in coloured light they only reflect such light as is common to paper and glass.

HELMHOLTZ ²⁾ experimented with coloured paper on coloured ground. He concludes that the after-images in those cases have a colour that is a mixture from that of the background and the complementary one of the paper.

We prefer to use coloured light, because colours that are not present in the ground are avoided and the white light from the background is excluded.

If we use white as image in all those experiments the after-image is black, as we might expect from our former investigations.

We thoroughly have discussed if we ought to use spectral light for composing our coloured light. This, however, would have necessitated a complex and expensive installation, and as we were free in choosing the

¹⁾ G. GRIJNS: L'influence de la lumière colorée sur les images consécutives négatives. Arch. Néerl. de Physiol. VII p. 355, 1922.

²⁾ H. VON HELMHOLTZ: Handbuch der Physiologischen Optik. Vol. II. p. 204. See also C. HESS, Unters. über die nach kurzdauernder Reizung des Sehorgans auftretenden Nachbilder. Pflügers Arch. 49 p. 201, 1891,

combinations of colours we were to experiment with, we took gelatinous lightfilters which we examined spectroscopically.

The results of our observations are given in table I where those on contrast are joined too.

For studying contrast we laid a grey disc on a coloured background or a coloured paper on a grey one, covered this with a thin transparent white paper and illuminated with one of the coloured lights. Contrast is very distinct then, and it makes no difference if the colour or the grey is in the midst. With yellow paper we did not observe contrasts, and ordinarily no coloured after-images, because we could not get a yellow paper or yellow stuff, that did not contain a large part of red and green. So it showed in orange or light green light the colour of this light, the after-image being dark, still no contrast appearing.

From the table you may see, that according to the light used we may see from red a green, a blue or a yellow after-image or contrast, from blue a red, a yellow or a green, and so on. In all cases examined the colour of the after-image and the contrast is that what remains, if we subtract from the colour of the light the colour we see the disc in. We never discovered a trace of a colour, that was not present in the light offered. The result of these experiments is that in not a single case is shown anything of light sensation of antagonistic colour originating in the eye, or more centrally.

We now will discuss another group of after-images, that give rise to more trouble, because they are more difficult to observe and different investigators describe them differently. It is the images we see after exposition of the eye to light, if this is darkened afterwards. As the time of illumination is of great influence on the effect, we firstly will consider what happens after a short exposition. We can experiment like BRÜCKE ¹⁾, by lighting up an object for a moment e.g. by an electric spark or a momentshutter or like PURKINJE ²⁾, who moved a small object through the visual field while a given point was fixed. In the first case, we must attend to the phenomena, that arrive after the illumination, in the other we see the phases, through which the irritationprocesses pass away, behind each other. We must be aware, however, that what we observe with these two methods does not want to be identical, because in the latter the inducing light is still in the visual field, and a priori you cannot know, if this will exert an influence on sensation, or on the judgment about what happens in our visual space.

HAMAKER ³⁾ resumes the results of precedent observators in this way: After a short illumination the following phenomena occur.

¹⁾ F. BRÜCKE: Versuche über subjective Farben. Poggend. Annalen d. Physik und Chemie LXXXIV. 418, 1851.

²⁾ J. PURKINJE. Beobachtungen und Versuche zur Physiologie der Sinnen. Bd. II, p. 110. Quoted from HAMAKER.

³⁾ H. G. HAMAKER. Ueber Nachbilder nach momentaner Helligkeit. Onderz. Physiol. Lab. Utrecht, Reeks V, Deel I, 1899.

TABLE I.

| | | COLOUR OF THE LIGHT. | | | | | | | |
|---------------------|----------------|----------------------------------|-----------------------------------|--|---------------------------------------|---|---------------------------------------|-------------------------------------|---|
| | | Red | Orange | Yellow | Yellowgreen | Blue-green | Blue | Violet | |
| COLOUR OF THE DISC. | Red | image after-image contrast | red grey black grey black | red yellow yellow | red green green | black bright green bright green | black bright green bright green | black bright blue bright blue | red blue blue |
| | Orange | image after-image contrast | red grey black grey black | orange dark black black | orange green no blue green | grey uncertain uncertain | grey bright green green | black bright blue blue | red blue blue |
| | Yellow | image after-image contrast | red grey black grey black | yelloworange grey black grey black | yellow none none | greenish yell. violet black uncertain | yellow blue none | black bright blue bright blue | red blue blue |
| | Light green | image after-image contrast | dull red none none | orange none none | green dark grey dark | green none none | green grey grey | blue none none | black bright violet bright violet |
| | Dark green | image after-image contrast | black bright red bright red | black bright orange bright orange | green orange orange | green none none | yellowgreen grey grey | black bright blue bright blue | black bright violet bright violet |
| | Blue | image after-image contrast | black bright red bright red | black bright orange orange | dark green yellow uncertain | blue bright green bright green | blue bright green uncertain | blue none none | blue red red |
| | Blue a. | image after-image contrast | | | black green uncertain | grey bright green bright green | | | |
| | Blue b. | image after-image contrast | | | dark red bright green uncertain | black green green | | | |

1. Primary short lasting image, that stays a moment after light is removed.
2. Very short dark interval.
3. After-image of PURKINJE (Secondary image of BOSSCHA) brighter than surrounding, of nearly complementary colour; lasts about $\frac{1}{4}$ sec.
4. Dark interval.

5. Faint light image, lasting a few seconds, colour indistinct.

6. A negative complementarily coloured after-image, only observable in special circumstances.

This series has been augmented with two phases by DITTLER and EISENMEIER¹⁾. They described a positive, isochrome, very short lasting image between 1 and 2 separated from 1 by a short dark space. It was observed behind a moving slit in a moderately lighted room, or in dark after short adaptation.

As we intend to publish our investigations in extenso elsewhere we will be concise here.

We investigated in both ways. With the momentshutter, we mounted on a light-tight lamp, we were not able to see the after-images of DITTLER and EISENMEIER. Our shortest exposition was 1/90 second. Then the image of PURKINJE appeared so soon, that the dark interval was not always observable, but often it looked as if suddenly a veil of different colour arose from the figure, thereafter all vanished. A moment later the 5th phase appeared.

We operated with nearly white light (incandescent lamp) and put coloured discs on the illuminated surface. The room was quite dark. Generally the after-image of PURKINJE is described as complementary, but if one thoroughly reads the protocols of several investigators, this proves not true.

HAMAKER writes²⁾ "Seine Färbung war nicht immer eine komplementäre" and in table VIII page 30 it is purple or red for all colours. DITTLER and EISENMEIER³⁾ call it more or less distinctly complementary. "Jedenfalls macht es nie einen mit dem Reizlichte gleichfarbigen Eindruck." HELMHOLTZ describes how the after-image of a shortly seen spectrum projected on a screen shows a reddish white spot, that corresponds as to its situation with the colours from orange to indigo-blue. Also after contemplating momentary different colours a crimson hue develops. Crimson HELMHOLTZ⁴⁾ defines as purple mixed with white.

Most investigators assure that the after-image of PURKINJE is not seen in the fovea centralis and that it is absent after red.

The colour of this after-image to our opinion may be best compared with that of lightning, if we contemplated white objects. It is bright with a little bluish or purplish hue, that some times varies a little more to blue, other times more to reddish. It lasts very short and after a dark interval the under 5 mentioned phase follows. If the primary light was feeble, it is almost imperceptible. It seems like a thin mist of very minute lightdust

¹⁾ R. DITTLER und J. EISENMEIER. Ueber das erste positive Nachbild nach kurzdauernder Reizung der Sehorgans, mittelst beweglicher Lichtquelle. Pflügers Arch. f. die Ges. Physiol. Bd. 126 p. 610, 1909.

²⁾ l. c. p. 8.

³⁾ l. c. p. 643.

⁴⁾ HELMHOLTZ. Physiol. Optik. Bd. II p. 213, 1911.

you scarcely can ascribe a colour to, but composed as it seems of more or less yellowish and greenish small points. HESS ¹⁾ gives a good reproduction. The contours of the image are always very vague and its dimensions vary.

We put coloured discs one time on a black, the other time on a grey background, and with the aid of a momentshutter and a light of about 100 candles we illuminated them. We found the after-image of PURKINJE to be black for red, orange-red and violet. With orange, yellow, green and blue they were always bluish, sometimes a bit purple. On a gray ground red-orange and dark blue also have black after-images with a bluish bright area round them.

In our experiments with moving light, we had a black disc revolving before a uniformly lighted surface. In the disc was a small hole, before which slits of 1 to 5 mM. width and 3 to 4 cM. height could be adapted as well as coloured glasses. Screens prevented light to hit the foremost surface. After green, yellow or blue we saw the satellite moving very distinctly. Also behind a white slit. The colour always varied between purplish and bluish, behind a green slit tending a little more to purple, behind a yellow one to blue. Behind a white slit comes a bluish satellite. Our violet glass, that let pass red and blue had a bluish satellite. Then comes a dark phase, and afterwards an often long tail of lightcloud, that with a bluish green glass has a faint purple hue.

With red and orange glass the after-image of PURKINJE stays out. We paid particular attention to the after-images described by DITTLER and EISENMEIER. As we could give to our slit a velocity of 5 till 60 cM. pro second, we thought they would be easily seen. This was not the case. On rotating fastly different colours made different phenomena. Yellow, green and blue showed an enlargement of the slit, in which the former part of it ordinarily seemed more saturated than the latter. A red that contained rays of wavelength longer than 680 did the same, but less strongly. The other red (690—640) divided in two rectangles parted by a dark space, the first was saturated red, the second vermillion. An orange (640—590) slit in orange and yellow, another one (655—605) in red and yellow. The purple glass showed, at sufficient speed, a foremost purple and an immediately following greyish blue part.

On account of the measured rotationspeed, when the parting line just appeared between the dividing coloured slits, we find a difference with the primary image of about 0.04 seconds. The after-image of PURKINJE is certainly 6 till 8 times later.

We will continue our investigations on the latter phenomenon, which from the time it appears must be connected with that described by DITTLER and EISENMEIER.

¹⁾ C. HESS. Ueber das Abklingen der Erregung im Sehorgan nach kurzdauernder Reizung. Pflügers Arch. 95. p. 1, 1903.

If we contemplate all that has been communicated on the after-image of PURKINJE it is evident, that it is as a rule not complimentary to the original image, but that in experiments with moving light the coloured image has a little influence on the after-image. This may be explained for as well by HERING's as by YOUNG's theory. As for the rest it has a not strictly defined hue, that remembers lightning, it does not appear with red, orange and violet, and evidently neither in the fovea. In HESS' ¹⁾ experiment with white stripes on dark ground the after-image (which I see bluish also) may seem to pass over the fovea with a curve, but it may be as well, that we complete the hiatus in the movement with our imagination, so as we do in the case of moving pictures.

We therefore hold with HAMAKER, that it originates in the rods, and not in the cones, but we do not consider it as an after-image, but as the primary rod-image. We therefore suppose in the rods a longer latent periode than in the cones. If the stimulus is short enough the cones will be returned to rest before the rods answer and the primary image is seen twice: first by the cones in its own colour, then by the rods, that truly have no distinctionability for colours, but whose image however must appear to us in some hue, if we compare it with the sensation of the cones. It would be mere accident, if it was just alike to white.

The lack of coloursense explains for the incertainty of the colourquality, that must be ascribed to it and the variability on different days.

Our conception explains also for HERING's experience ²⁾, in which behind a black stripe, that is moved over a white ground, follows a black after-image, without supposing that a darkening of some spot in the retina gives a dark after-image on the enlightened part of this membrane. Here too the second image is the image of the rods, that discriminate between light and dark. Also what time concerns, as far as I can estimate it by the velocity, with which the black stripe must be moved, this black image corresponds to that of PURKINJE.

If the after-images of PURKINJE are properly the primary sensations of the rods, then the colours we attribute to them cannot be dependent of a reaction in the "Sehsubstanz", but ought to be the results of a judgment not of a direct perception.

¹⁾ C. HESS, l. c

²⁾ E. HERING. Eine Methode zur Beobachtung und Zeitbestimmung des ersten positiven Nachbildes kleiner bewegter Objecten. Pflügers Arch 126, p. 609, 1909.

Chemistry. — "*The hydration of dissolved saccharose and the expression of the concentration in measuring the activity of ions.*" By I. M. KOLTHOFF. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of February 27, 1926)

1. In an investigation on the influence of neutral salts on the concentration of hydrogen ions of dilute hydrochloric acid, of which a later communication will follow, the difficulty was how to express the concentration of the substances dissolved. This concentration is often measured in grams or mols of dissolved substance per liter of solution; others express it in grams or mols of dissolved substance per 1000 gms. of solution. In the measurements of osmotic pressure and related quantities the concentration is expressed in grammes of substance per 100 gms. of solvent, or in mols dissolved in a constant number of mols of solvent. When n mols of substance are dissolved in N mols of solvent the concentration is

$$x = \frac{n}{N}.$$

Finally the concentration can also be expressed in mols of dissolved substance in mols of solution, in other words, the concentration is

$$x = \frac{n}{n + N}.$$

In interpreting the influence of the concentration in measurements of osmotic pressure A. FINDLAY ¹⁾ prefers the latter way of expression, and N. BJERRUM ¹⁾ too uses it in calculating the activities of ions.

When experimenting with dilute solutions, it is immaterial which way of expressing the concentration is preferred. When, however, we experiment with liquids, in which the number of mols of dissolved substance is of importance with respect to the number of mols of solvent, the way how the concentration is expressed is of great importance.

2. In order to find out what way of expression is the most rational in my investigation, I have studied the influence of non-electrolytes on the concentration of hydrogen ions in 0.01 mol of hydrochloric-acid per liter of solution. The influence of saccharose was more particularly investigated.

As will be seen from what follows, saccharose practically does not influence the activity of hydrogen ions at this dilution of acid.

¹⁾ A. FINDLAY, „The osmotic pressure“ cited from N. BJERRUM, *Z. anorgan. allgem. Chem.* **109**, 275 (1920).

In the first place a series of experiments was made in which the concentration of the hydrogen ions is measured in 0.01 N hydrochloric acid, with and without the presence of saccharose. The measurements have been made with the hydrogen-electrode in the Physiological Laboratory of the University. The normal calomel-electrode was used as standard half cell. The results obtained are given in the following table. 1.75 N as well as 3.5 N potassium chloride was used as a salt-bridge between the hydrogen- and calomel-electrode. By applying the extrapolation method of BJERRUM the diffusion potential could in this way be more or less eliminated. It is to be noted that this method could not be applied in solutions which contain more than 1 mol of saccharose per liter. The concentrations are expressed in mols per liter.

Concentration of hydrogen ions in 0.01 N HCl in a solution of saccharose.

| Composition of the liquid | E. M. F. at 18° against N. C. E. | p _H | [H ⁺] |
|--|-------------------------------------|----------------|-----------------------|
| 0.01 mol HCl p. liter | 0.4041 | 2.040 | 0.91×10^{-2} |
| 0.01 mol HCl + 0.5 mol of saccharose p. l. | 0.3924 | 1.838 | $1.45 \times -$ |
| 0.01 + 1 | 0.387 | 1.744 | $1.80 \times -$ |
| 0.01 + 1.5 | 0.380 | 1.623 | $2.4 \times -$ |

These experiments show that the concentration of hydrogen ions of a solution of hydrochloric-acid apparently increases very much when saccharose is present. I have been able to confirm this result also with colourindicators. Of the liquids mentioned in the table above, [H⁺] was also colorimetrically determined with thymolblue and tropaeoline 00 respectively as indicator. The colour was compared with freshly prepared solutions of hydrochloric-acid, which were obtained by diluting 0.1 N HCl.

Colorimetrically determined [H⁺] of 0.01 N HCl in solution of saccharose.

| Composition of the liquid | [H ⁺] on thymol- blue | [H ⁺] on tropaeoline 00 |
|--|--------------------------------------|--|
| 0.01 mol HCl per liter | 1×10^{-2} | 1×10^{-2} |
| 0.01 mol HCl + 0.5 mol of saccharose p. l. | $1.4 \times -$ | $1.32 \times -$ |
| 0.01 + 1 | $1.8 \times -$ | $1.75 \times -$ |
| 0.01 + 1.5 | $2.5 \times -$ | $2.1 \times -$ |
| 0.01 + 2 | $5-5.5 \times -$ | |

As is well known, thymolblue is an indicator which behaves as an acid, tropaeoline 00, on the contrary, as a very weak base. Yet, in the saccharose solution their indication of the concentration of the hydrogen ions is almost

identical. Generally the values found with thymolblue are a little higher than those with the other indicator. This is explained by the fact that the dissociation constant of either indicator is very little diminished by the saccharose. The result is that thymolblue indicates a reaction which is a little too acidic, tropaeoline 00, on the contrary, a reaction slightly less acidic than is actually the case.

We may further observe that the agreement between the electrometric and colorimetric determinations is very satisfactory. It is seen that the concentration of hydrogen ions regularly increases with increasing concentration of saccharose. In a solution, containing 2.5 mols of saccharose and 0.01 N of hydrochloric-acid, I found, with thymolblue, a concentration of hydrogen ions of even $\pm 1.5 \times 10^{-1}$, consequently 15 times greater than might be expected according to the concentration of hydrochloric-acid (expressed per liter).

There is a great element of uncertainty in electrometric determinations. In liquids, containing hydrochloric-acid only, the diffusion potential, against potassium-chloride solutions, is rather large, so that the extrapolated value of the E.M.F. is at least 1 millivolt inaccurate. It is very difficult to find, when repeating the experiments, an agreement of 1 millivolt with the solutions mentioned.

Therefore another series of measurements has been made under the same circumstances; the difference is that all the liquids also contain 0.09 mol of potassium chloride per liter. A liquid containing 0.01 N HCl and 0.09 N potassium chloride is recommended by S. P. L. SÖRENSEN as standard in electrometric determinations of hydrogen ions. In agreement with SÖRENSEN we found that, in such a liquid at 18°, a hydrogen electrode has a potential against the N calomel-electrode of 0.4040 ± 0.0001 Volt.

I have made the measurements not only with the hydrogen electrode but also with the quinhydrone electrode at 18°, using the quinhydrone electrode in the standard hydrochloric-acid mixture as reference electrode. In this case the measurements could be reproduced to 0.0001 Volt.

When we refer the E.M.F. to the standard hydrochloric mixture the calculation of the change of p_H and $[H^+]$ is very simple, for the saccharose solutions contain the same quantity of hydrochloric acid and potassium chloride as the standard mixture. When π is the E.M.F. measured, the change of p_H is equal to

$$\Delta p_H = \frac{\pi}{0,0577} (18^\circ).$$

Moreover I have colorimetrically determined $[H^+]$ in the mixtures, by comparing them with solutions of hydrochloric-acid in 0.09 N potassium-chloride. The addition of the latter salt is absolutely necessary, as otherwise greatly divergent values are found. The presence of a minute quantity of salt has a rather great influence on the intensity and the tint of the indicators. I hope to return to this part of the subject in a future paper.

Measurements with the hydrogenelectrode in 0.01 N HCl, 0.09 N KCl and saccharose.

| Composition of the liquid | π with resp. to 0.01 N HCl + 0.09 N KCl | Decrease of p_H by sacch. | Increase of the [H ⁺] by sacch. |
|---|---|--------------------------------|--|
| 0.01 N HCl + 0.09 N KCl + 0.5 mol of sacch. | 6.1 ± 0.4 m. Volt | 0.104 ± 0.01 | $1.27 \times \pm 0.03$ |
| 0.01 " " + 0.09 " " + 1 " " " | 11.6 ± 0.4 " " | 0.201 | $1.59 \times \pm 0.03$ |
| 0.01 " " + 0.09 " " + 1.5 " " " | 18.4 ± 0.4 " " | 0.319 | $2.08 \times \pm 0.03$ |

The same with the quinhydronelectrode.

| | | | |
|---|-----------------------|-------|------------|
| 0.01 N HCl + 0.09 N KCl + 0.5 mol of sacch. | 5.8 ± 0.1 m. Volt | 0.101 | 1.26 times |
| 0.01 " " + 0.09 " " + 1 " " " | 12.5 ± 0.1 " " | 0.217 | 1.65 .. |
| 0.01 " " + 0.09 " " + 1.5 " " " | 19.2 ± 0.1 " " | 0.334 | 2.16 .. |

Below are given the values which are found colorimetrically.

| Compositon of the liquid | Increase of the concentration of the hydrogen ions with | |
|---|--|---------------------------------------|
| | Thymolblue | Tropaeoline 00 |
| 0.01 N HCl + 0.09 N KCl + 0.5 mol of sacch. | 1.29 times | 1.25 times (not sharp) |
| 0.01 " " + 0.09 " " + 1 " " " | 1.65 .. | ± 1.5 " " " |
| 0.01 " " + 0.09 " " + 1.5 " " " | 2.4 .. | no accurate determination possible |

The values found with thymolblue are probably a little too high. For the rest the agreement between the results is very satisfactory. Summarizing we find that by the presence of the saccharose the activity of the hydrogen ions apparently is increased; on an average we find:

in a solution with 0.5 mol of saccharose per liter an increase of $1.27 \times$
 " " " " 1 " " " " " " " " " " " " $1.60 \times$
 " " " " 1.5 " " " " " " " " " " " " $2.1 \times$

Some objections may be made to these results; these we shall discuss first:

10. It is possible that the constants of the hydrogen- and quinhydronelectrode are changed by the saccharose so that the method of calculation is inaccurate. This, however, is beside the truth, for I have also made measurements in buffer-solutions of different composition. In these the concentration of hydrogen ions is fixed, and is controlled by the relations of the undissociated acid and its anion. The total concentration of the two components is of subordinate value. Now I observed that the saccharose in the buffer-mixtures practically has no influence on the concentration of the

hydrogen ions; the very small effect that was found points to a minute decrease of the dissociation constant of the acid, which occurs in the buffermixture.

2^o. It is possible that the sensitivity for hydrogen ions of the indicators used is changed by the saccharose. Indeed, the saccharose has a very slight influence on the dissociation constant of the indicators. The change which might be due to this influence, however, is much smaller than the effect we have observed. Moreover, the indicator tropaeoline 00 becomes a little less sensitive, so that in the saccharose mixtures it indicates a $[H^+]$ which is a little too low.

3^o. Saccharose itself is a very weak acid. The dissociation constant, however, is very small, about 2×10^{-13} , and the concentration of hydrogen ions of 1 or 1.5 N saccharose is of the order of only 10^{-6} .

On account of its acid nature the saccharose cannot increase the concentration of hydrogen ions of 0.01 N hydrochloric-acid.

The increase of the activity of the hydrogen ions in the presence of saccharose really exists. In the first instance we can explain this influence in two ways:

1^o. It is admitted that hydrogen ions in aqueous solution are hydrated and that the electrometric determinations give us only the concentration of anhydrous ions.

It would therefore be possible to assume that the dissolved saccharose has a dehydrating influence on the hydrogen-ions, so that more anhydrous ions appear. Then we should also have to admit that the colour indicators are sensitive only for anhydrous hydrogen-ions.

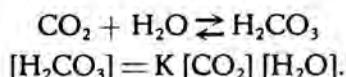
If this explanation were correct we should have to find that the saccharose would increase the concentration of hydrogen-ions in buffer-mixtures as much as in hydrochloric-acid. As is mentioned already this is not the case.

In solutions of organic acids saccharose has another effect than in hydrochloric-acid. We have found that 1 mol saccharose per liter increases the concentration of hydrogen-ions of hydrochloric acid $1.6 \times$. In solutions of organic-acids $[H^+]$ also increases, and this not $1.6 \times$, but less, namely so much as corresponds with an increase of the total concentration of 1.6 times.

2^o. The other explanation which remains is that the saccharose displaces the water in the solution; the concentration of the water, or, the free volume of the water, diminishes, and the concentration of the solute increases. This would lead us to deduce that the concentration of the acid must be expressed in mols of solute, dissolved in mols of solution or mols of solvent. In our case there is little difference between the two modes of expression; experimentally we cannot decide which is the correct one. It is obvious to express the concentration in mols of solution.

I could also prove another way that one of the two modes of expression is correct.

As is said above, saccharose increases the concentration of hydrogen-ions in weak acids. Carbonic-acid, however, behaves differently; saccharose slightly diminishes the concentration of H-ions. The reason is, that in a carbonic-acid solution the greater part of the acid is present in the form of CO_2 (about 99 %), and that a small part is in the acid form as H_2CO_3 . The concentration of the latter is also dependent of the concentration of the water :



If the concentration of the water becomes a \times smaller, $[\text{H}_2\text{CO}_3]$ also becomes a \times smaller. On the other hand the total concentration of the acid becomes as many times greater by the displacement of the water by the saccharose. If therefore the saccharose has no influence on the dissociation constant of carbonic-acid, we should find that the reaction is not changed by the saccharose. We indeed find a slight decrease of the concentration of the hydrogen-ions, corresponding with a slight fall of the constant.

The difference between saccharose and a substance as aethylalcohol is very peculiar as regards their influence on the dissociation constant of weak acids. The latter is strongly diminished by aethylalcohol, but hardly by saccharose. The explanation probably is that solutions of saccharose in water have a rather great dielectric constant¹⁾. In this respect saccharose, as a solvent, may be compared with water.

30. If, from what precedes, we conclude that the concentration of the solute must be expressed as $\frac{n}{N}$ (or as $\frac{n}{n+N}$) in which n is the number of mols of solute and N the number of mols of solution, we can calculate from the increase of the concentration of the hydrogen-ions, how great the free volume is of the remaining water.

For, if we find that in 1 liter of solution, containing 0.01 mol of saccharose, the increase is 1.28 \times , this means we have 0.01 mol of HCl in $\frac{1000}{1.28} = 780$ C. of water. According to the specific gravity of a saccharose solution of 0.5 molal (WINDISCH's table) the liquid contains 895 cc of water per liter. From this we may conclude that 0.5 mol of saccharose binds $895 - 780 = 115$ cC of water.

If we consider this water as hydration water, it follows, from what is said above, that in 0.5 molal solution of saccharose 230 gms. of water are bound by 1 mol saccharose, corresponding to a hydration of 12.8 mols of water for 1 mol of saccharose.

On account of the uncertainty in the measurements, mentioned in sub. 2, this last figure is not very certain, and changes between 10 and 14 mols.

¹⁾ Compare i. a. THWING, Z. Physik. Chem. **14**, 292 (1894).

DRUDE, Z. Physik. Chem. **23**, 305 (1897).

From the results of the measurements with the other saccharose solutions we can also calculate the hydration, and then we find :

Hydration of saccharose in solution.

| Concentration of saccharose per liter | Mols of water per mol. of saccharose |
|---------------------------------------|--------------------------------------|
| 0.5 molal | 10 to 11 |
| 1 .. | 6.6 .. 11.2 |
| (1.5 .. | 9.2 .. 10.8) |

As a slight error in the determinations of the concentration of the hydrogen-ions causes a great change in the hydration calculated, the latter cannot be deduced from the measurements with great certainty. We can only give the order of magnitude approximately (about 10 mols of water per mol of saccharose), which of course will diminish in concentrated solutions.

4. If saccharose apparently increases the activity of hydrogen-ions in solutions of hydrochloric acid, this must also manifest itself when the inversion constant in concentrated solutions of saccharose is determined.

Data in the literature prove indeed that the constant increases with increasing saccharose content. J. SPOHR¹⁾ already gave an explanation for this anomaly, by pointing out that the concentration should not be expressed in gms. of saccharose per liter of acid solution, but per gms. of solution. If this is done we get, according to SPOHR, a constant which is independent of the concentration of the saccharose dissolved.

ERNST COHEN²⁾ formulated the deviation found more clearly. According to him, the volume of the saccharose in solution should be deducted, in order to calculate the exact concentration of the acid. For, considered kinetically, in a stronger saccharose solution the space, in which hydrochloric-acid and saccharose can collide, is smaller than when the volume of saccharose is small. Therefore the constant found in a stronger saccharose solution will be greater than in a more diluted solution.

M. A. ROSANOFF, R. H. CLARK, and R. L. SIBLEY³⁾ demonstrated that the inversion constant really becomes independent of the concentration of the saccharose dissolved, when the concentration of the acid is expressed in the quantity of water present.

ERNST COHEN assumes that the volume of the saccharose dissolved is independent of the solution. In this way he gets the equation :

$$k_{40} : k_{20} = \frac{1}{100 - b_{40}} : \frac{1}{100 - b_{20}}$$

Here k_{20} and k_{40} represent the reaction constants in a saccharose solution

¹⁾ J. SPOHR, J. prakt. Chem. **33**, 266 (1886).

²⁾ ERNST COHEN, Z. Physik. Chem. **23**, 442 (1897).

³⁾ ROSANOFF, CLARK and SIBLEY, J. Amer. Chem. Soc. **33**, 1911 (1911).

of 40 %, b_{40} is the volume of the dissolved saccharose in a solution of 40 %, b_{20} in a solution of 20 %.

In this way he calculates that 200 gms. of dissolved saccharose occupy a volume of 177 cc. According to the specific gravity this figure is 123 cc. Hence 200 gms. of saccharose bind 54 gms. of water, or 1 mol of saccharose 5.1 mols of water.

I have compared the values of the inversion constant found by W. OSTWALD ¹⁾ and by J. SPOHR (l.c.) and moreover graphically deduced the value for a solution containing 0 % saccharose.

The figures are found in the following table :

| Concentration of sacch. p. 100 Cc | Constant of OSTWALD | of SPOHR |
|-----------------------------------|---------------------|----------|
| 0 % | 18.25 | 18.25 |
| 2 .. | — | (20.47) |
| 4 .. | 19.19 | — |
| 10 .. | 20.69 | 21.00 |
| 20 .. | 22.92 | 24.21 |
| 30 .. | — | 27.21 |
| 40 | 29.20 | — |

The value found by SPOHR in a solution of 2 % cannot possibly be good. For the rest the agreement between his values and OSTWALD's is not very fine. From the two series of observations I have calculated the inversion constant in a 0.5-, respectively 1 molal saccharose solution. At the same time is indicated how many times the inversion constant is increased, compared with the constant in a solution with 0 % of saccharose. In the two last columns the hydration of the saccharose has been calculated in the way as described above.

Hydration of saccharose according to inversion constant.

| Concentration of saccharose | Inversion constant of | | Increase of constant of | | Hydration of sacchar. according to | |
|-----------------------------|-----------------------|-------|-------------------------|--------|------------------------------------|-------|
| | OSTWALD | SPOHR | OSTWALD | SPOHR | OSTWALD | SPOHR |
| 0.5 molal | 22.25 | 23.4 | 1.22 × | 1.28 × | 8.3 | 12 |
| 1 .. | 26.90 | 28.4 | 1.47 × | 1.55 × | 6.2 | 8 |

In this way we therefore find for the hydration of the saccharose figures of the same order of magnitude as have been deduced from the increase of the concentration of the hydrogen-ions. Further we see from the last table

¹⁾ W. OSTWALD, J. prakt. Chem. 31, 316 (1885).

that the hydration decreases when the concentration of the saccharose increases.

5⁰. *Deduction of the hydration from other physical properties of solutions of saccharose.*

The relative decrease of the osmotic pressure by a dissolved substance is expressed by the equation :

$$\frac{p-p'}{p'} = \frac{n}{N} \left(\text{or} = \frac{n}{n+N} \right),$$

in which p and p' represent the osmotic pressure of the solution, respectively the solvent. From the measurements of the osmotic pressure of solutions of saccharose which have been made by H. N. MORSE and W. W. HOLLAND ¹) we cannot calculate the hydration with sufficient accuracy, as there is some irregularity in their figures. As this is the most accurate investigation on osmotic pressure of solutions of saccharose, we can draw from their figures this qualitative conclusion only, that the saccharose is hydrated.

A better result is reached with the data on the lowering of the vapour pressure, respectively the freezing-point of water by saccharose. Direct measurements of the vapour pressure of saccharose solutions have been made by A. SMITS ²), who found the same figures as DIETERICI ³).

Whereas the molecular lowering of the vapour pressure of saccharose in a very dilute solution is 0.084, it is 0.0902 in a solution of 1 mol of saccharose in 1000 gms. of water.

Hence the hydration of saccharose in the latter solution corresponds with 6 mols of water.

The most accurate measurements on the lowering of the freezing point of water by saccharose have been made by F. M. RAOULT ⁴).

He found :

| Grammes of sacch. in 100 g. of water | Molec. freezing-point depression |
|--------------------------------------|----------------------------------|
| 0 | 18.72 |
| 8.55 | 19.22 |
| 17.292 | 19.59 |
| 34.565 | 20.79 |

The molecular freezing-point depression increases with increasing concentration of saccharose ; we therefore get the impression that the

¹) H. N. MORSE and W. W. HOLLAND, Amer. Chem. J. **41**, 1 (1909).

²) A. SMITS, Z. Physik. Chem. **39**, 385 (1902).

³) DIETERICI, Wied. Ann. **62**, 616 (1897).

⁴) F. M. RAOULT, Z. Physik. Chem. **27**, 617 (1898).

concentration of the saccharose increases more strongly than seems to correspond with the figures in the first column. If we assume that this is caused by the hydration of the saccharose, we find in the solution, which contains 34.565 gms. of saccharose, not 100 gms. of free water, but only

$$100 \times \frac{18,72}{20,79} = 90 \text{ gms.}$$

Hence we deduce a hydration of 5.5 mols of water, and by analogy 5.3 for a solution containing 0.5 mol of saccharose per 1000 gms. of water.

If we express the concentration as $\frac{n}{n+N}$ instead of $\frac{n}{N}$, we find a hydration of 6.4 for a solution of 1 mol of saccharose in 1000 gms. of water, and of 6.0 for a solution of 0.5 mol of saccharose in a similar amount of water.

Viscosity: According to A. EINSTEIN ¹⁾ the relative viscosity of a very dilute solution is:

$$\mu = 1 + 2,5 v.$$

Here v represents the volume of the solute. BURKHARD ²⁾ has determined the relative viscosity of saccharose at 20° with great accuracy, and has found a value of 1.0245 for a solution of 1%. By means of EINSTEIN'S equation and the specific gravity we calculate a hydration of 6.8 mols of water per 1 mol of saccharose. In this way we find a hydration of 8 for a solution of 2% ($\mu = 1.0521$).

Lowering of the solubility of substances in water in the presence of saccharose.

According to NERNST ³⁾ the relative lowering of solubility on account of the presence of a foreign substance is equal to the number of molecules of that substance, divided by the number of molecules of solvent, therefore:

$$\frac{L-L'}{L'} = \frac{n}{N}$$

So we have here a similar equation as for the lowering of the freezing-point and analogous magnitudes.

However we must apply NERNST'S rule with the greatest circumspection, for it is based on the fact that the foreign substance — in our case saccharose — does not function as a solvent.

I have determined at 18° the solubility of different substances in water, and in saccharose solutions, and it appeared indeed, that the rule usually cannot be applied here.

¹⁾ A. EINSTEIN, Ann. d. Physik. **24**, 34 (1911); Koll. Z. **27**, 137 (1912).

Originally EINSTEIN (Ann. de Physik. **19**, 301 (1906)) had deduced the equation: $\mu = 1 + v$, which, however, he has corrected later on.

²⁾ BURKHARD, Z. des Vereins der Deutschen Zuckerindustrie **24**, 199.

³⁾ W. NERNST, Z. Physik. Chem. **6**, 19 (1890).

Here are a few examples :

| Solubility of silveracetate. | | | |
|--|--|--------------------------------------|------------------------------|
| Concentration of saccharose per liter | Concentration of silveracetate per liter | Relative decrease of solubility in % | |
| 0 | 0.0745 molal | | |
| 0.5 molal | 0.0745 .. | 0 | 0% |
| 1 .. | 0.0740 .. | 0.7 | .. |
| Solubility of sulfanilic-acid. | | | |
| 0 .. | 0.0614 molal | | |
| 0.5 .. | 0.0591 .. | 3.5 | .. |
| 1 .. | 0.0524 .. | 15 | .. |
| Solubility of salicylic-acid. | | | |
| 0 .. | 0.0125 molal | | |
| 0.5 .. | 0.0131 .. | - 4.5 | .. |
| 1 .. | 0.0134 .. | - 7 | .. |
| Solubility of bitartrate of potassium. | | | |
| 0 .. | 0.0269 molal | | (S. ARRHENIUS) ¹⁾ |
| 0.5 .. | 0.0243 .. | 10 | .. (15 %) |
| 1 .. | 0.0208 .. | 22 | .. (20 ..) |

In the last table I have also given the figures for the lowering of solubility in potassium-acid-tartrate, which have been deduced from the observations of SV. ARRHENIUS. His results are not in agreement with mine.

From the figures found for the different substances we see that the relative lowering of the solubility on account of the presence of saccharose is not constant, but varies according to the nature of the substance. There is no doubt that the saccharose exercises a specific influence as a solvent.

It is to be expected that saccharose will have a much smaller specific influence on the solubility of gases, as the gases have a much smaller polar character than the substances mentioned in the table above.

In the following table I give a few results on the solubility of hydrogen (deduced from the determinations of P. STEINER²⁾), and carbonic-acid (deduced from the determinations of A. CHRISTOFF³⁾).

¹⁾ SV. ARRHENIUS, Z. Physik. Chem. **31**, 226 (1899).

²⁾ P. STEINER, Wied. Ann. **52**, 275 (1894).

³⁾ A. CHRISTOFF, Z. Physik. Chem. **53**, 329 (1905).

In the first column the concentration of saccharose is given in mols per liter, in the second column the calculated molecular depression of the solubility by saccharose, and in the third column the hydration derived from it.

Solubility of hydrogen in saccharose solutions (STEINER).

| Concentration of sacch. in mols per liter | Molecular depression of the solubility in % | Calculated hydration of saccharose |
|---|---|------------------------------------|
| 1 molal | 32 % | 7 mols of water |

Solubility of carbonic-acid in sacch. solutions (CHRISTOFF).

| | | |
|-----------|------|------------------|
| 0.5 molal | 42 % | 12 mols of water |
| 1 molal | 33 % | 7 mols of water |

So we find here in 1 molal saccharose solution from the solubility of hydrogen, and of carbonic acid the same value for the hydration of saccharose.

6. Discussion of the results.

From the figures given for the hydration of saccharose we might suppose that the dissolved cane-sugar binds a certain number of molecules of water. This seems to me to be not the case. It is much better to assume that the saccharose possesses an affinity with respect to the molecules of water, which causes the latter to be "directed" in a certain position, by means of which the activity of the water, or the quantity of free water which remains, is lowered. It is not probable that all the molecules of the hydrate are in the same position; it is rather to be expected that the molecules of water, nearest to the saccharose, have lost most of their free mobility. The farther they are distant from the molecule of saccharose, the less they have lost of their activity, so that we get a gradual transition of the decrease of the activity of the directed molecules of water, nearest to the molecule of saccharose, to the free water in the solution. The effect which we observe, and which we call hydration, is therefore an expression for the total decrease of the free mobility of the molecules of water. In the preceding paragraphs we have calculated the hydration of the saccharose from different sorts of experiments. A summary has been given in the following table; here the hydration is given for a solution containing 1 mol of saccharose in 1 liter of solution.

Summary of the computed values of the hydration of saccharose (1 mol per liter)

| Increase of [H ⁺] | Increase of constant of inversion | Lowering of vapour pressure | Lowering of freezing-point | Viscosity | Lowering of solubility of gases |
|-------------------------------|-----------------------------------|-----------------------------|----------------------------|-----------|---------------------------------|
| 7-11 | 6-10 | 6 | 6.4 | 6.4 | 7 |

This table gives rise to the following observations :

a. The value calculated from the increase of the concentration of the hydrogen-ions of a dilute solution of hydrochloric-acid by saccharose is very uncertain. A small error in the measurement of the E.M.F. causes a great change in the value of the hydration. Moreover we have assumed in the calculation that the saccharose does not change the activity of the hydrogen-ions at all. If this activity should decrease by the influence of the saccharose, the values for the hydration would be too small. The average value of 9 mols of water per mol of saccharose is therefore to be considered as a minimum.

b. In the derivation from the increase of the inversion constant we have assumed that the viscosity of the medium has no influence on the reaction velocity. It may be expected, however, that the greater viscosity diminishes the chance of collision, and that therefore also the reaction velocity is somewhat decreased.

In this connexion I wish to point out that ERNST COHEN ¹⁾ has made very accurate determinations on the inversion constant in solutions of hydrochloric-acid, in 20 % of alcohol. Here the alcohol has no influence as yet on the activity of the hydrogen-ions. Yet COHEN found a decrease of the constant of 4%, with respect to solutions free of alcohol, which perhaps must be attributed to an increase of the viscosity of the medium. However, too little is known of the influence of the viscosity on the reaction velocity to be very certain on this point. The mean value of the hydration of 8 is in any case to be considered as a minimum.

c. As was to be expected, the measurements of the lowering of the vapour pressure, respectively of the freezing-point of water, give the same value for the hydration of saccharose.

The calculation has been made on the assumption that the hydrate water has a vapour tension of 0. This seems improbable to me; for even in crystallized salts, with water of crystallization, the water has still a vapour pressure. This may also explain why, according to these two methods, we find too low a value for the hydration.

d. Neither is the value for the hydration, calculated from the viscosity, trustworthy. For EINSTEIN's equation, which is true for very dilute solutions only, may be applied only when the dissolved molecules are spheres, which have a large volume with respect to the molecules of the solvent among which they move.

Here too, therefore, we may consider as correct only the order of magnitude of the hydration calculated from the viscosity.

e. It has been observed already, that we have to be very careful with the explanation of the lowering of the solubility, in consequence of the presence of saccharose. Like glycerine, saccharose is also to be considered

¹⁾ ERNST COHEN, Z. Physik. Chem. 28, 144 (1899), where other literature is also referred to.

as a solvent, and from the examples, given sub 5, we see that some substances dissolve better in the presence of saccharose than in pure water (salicylic acid; also benzoic acid, which is not mentioned in the table). Further it is also possible that the water of hydration has a dissolving action.

Determinations of solubility of substances that have little or no polarity, as carbon dioxide, and hydrogen, are more reliable for this computation.

With the data given on this subject in the literature, we find a value for the hydration which agrees with the one found by other methods.

Summarizing we may say, that it is not possible to calculate the hydration accurately from physical data of saccharose solutions. Only the order of magnitude can be approximated, and then we come to the conclusion that a solution of 1 mol of saccharose per liter, contains 8 ± 2 mols of water as water of hydration.

Summary.

1. Sugar increases the concentration of hydrogen-ions of a solution of hydrochloric-acid.

2. This effect is explained by assuming that, when measuring the activity of ions, we must not express the concentration of the solute in mols per liter, but, as BJERRUM does, in mols of solute n per mols of solution $(n + N)$, therefore

$$x = \frac{n}{n + N}.$$

3. If we express the concentration in the way mentioned above, we find a slight increase of the activity of the hydrogen-ions by the presence of saccharose. This is to be attributed to the hydration of the dissolved saccharose.

4. The hydration of 1 mol of saccharose in 1 liter of solution has been calculated in different ways. We find an agreement in the order of magnitude, namely of 8 ± 2 mols of water per mol of saccharose.

5. The hydrate is not to be considered as a stoichiometric component of water and saccharose. It is probable that the molecules of water are directed by the saccharose in a certain position, so that the free mobility decreases. Hence the value which we calculate as hydration is in fact a measure for the decrease of the activity of the water by the dissolved saccharose.

Pharmaceutic Laboratory of the University.

Utrecht, December 1925.

Physics. — “*Prolegomena to a Theoretical Atomism*”. By Prof. J. D. VAN DER WAALS JR.

(Communicated at the meeting of May 29, 1926).

Of late considerations have been developed by LOUIS DE BROGLIE ¹⁾, EINSTEIN ²⁾ and SCHRÖDINGER ³⁾, in which it was tried to combine the existence of electrons, protons, and light quanta (which three quantities I will comprise under the name of “atoms”) under one theoretic point of view. They assign to these atoms a pulsating phenomenon with the frequency $\nu = \frac{mc^2}{h}$. In this way the time-dependence of the phenomenon is determined, which dependence is more closely defined by DE BROGLIE, who assumes that for the electron that is not subjected to external forces, isophasia will prevail in space. But for the rest the extension in space is left undetermined. Now it seems to me that it is possible to get, in a natural way, a limitation in space for the isolated atom. For this purpose we describe the pulsating phenomenon as it presents itself to an observer who moves with it, by the aid of a function $e^{-\nu s}$, in which however we do not take for s the time, but the scalar quantity $\int \sqrt{dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)}$. This quantity is independent of the system of coordinates chosen, but dependent on the integration path chosen. In a ray of light which is emitted at a moment t_0 by an electron, it is this quantity that actually occurs under the sinus sign, in which as integration path is taken the “world-line” of the electron (hence also of the quantity of energy considered) up to the moment t_0 , and further the world-line of the ray of light for which $\int ds = 0$.

This is valid for energy emitted by the electron. For energy belonging to the electron itself I would propose an integration path such that:

$$\int ds = t + \nu r.$$

It is easy to read from this formula what path is meant.

¹⁾ LOUIS DE BROGLIE. Thèses etc. Paris. Masson et Cie 1924; also Ann. d. Physique X serie 3, 22, 1925.

²⁾ A. EINSTEIN: Berl. Sitzungsberichte 1925. pag. 3.

³⁾ E. SCHRÖDINGER. Ann. d. Physik. 79, 361 and 489, 1926. Phys. Zeitschr. 27 Jahrg. p. 95, 1926.

The pulsating phenomenon is then described by the function

$$e^{-\nu r} (\cos \nu t + i \sin \nu t)$$

and the same quantum hypothesis that determines the periodicity in time, determines at the same time the limited extension in space; and the reason that the atoms have a limited extension in spatial direction, but (at least on an average) an unlimited extension in time direction then lies in this, that the space-time continuum is not four dimensional, but one + three dimensional.

The above solution is not entirely satisfactory. The isophasia for a resting and non-accelerated atom is maintained. And it seems to me that serious objections may be advanced to this. DE BROGLIE discusses a mechanic analogon, in which a number of weights hanging on springs executes isophasic vibrations. But in this it is clear that each of the weights performs its vibrations independent of the existence of the others. It behaves as a "windowless monade" of LEIBNITZ, and the isophasia is only possible through a "harmonia praestabilita". If a dynamic relation is supposed to exist between the different points participating in the pulsating phenomenon, progressive or stationary waves will sooner be assumed between them.

And for this reason I should prefer to determine the pulsating phenomenon by another function or choose another integration path for ds , if it appears to be possible to preserve at the same time the advantage of the natural restriction of the extension in space of the phenomenon through the quantum condition.

Mathematics. — “A Quadruple Involution in Space”. By G. SCHAAKE.
(Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of February 27, 1926).

§ 1. On p. 295 of Vol. XXI of these Proceedings Prof. JAN DE VRIES observed that the triple involution of REIJE in the plane may be produced by the aid of a net of cubics which has the angular points of a complete quadrilateral as base points. The triplets of the involution which may be derived from this net, are the triplets of the base points different from the said angular points of the pencils which are contained in the net.

In this communication we shall investigate a quadruple involution in space which may be produced in a similar way and which, also as regards its properties, forms an analogy to the triple involution of REIJE.

§ 2. We choose five planes a_1, a_2, \dots, a_5 , and we indicate the first member of the equation of the plane a_i , with second member zero by a_i . Then

$$\sum \frac{\lambda_i}{a_i} = 0$$

where $\lambda_1, \dots, \lambda_5$ are arbitrary constants, is the general equation of the biquadratic surfaces ω^4 which pass through the ten lines of intersection $a_i a_k$ of the planes a and which have a conical point in each of the ten points of intersection $a_i a_k a_l$ of three of these planes.

Besides the ten lines $a_i a_k$ two surfaces ω^4 have a twisted curve k^6 of the sixth order in common, which passes through the ten points $a_i a_k a_l$, as the complete intersection has a quadruple point in a point $a_i a_k a_l$ and three lines $a_i a_k$ pass through such a point.

Besides along this straight line a plane through $a_i a_k$ cuts the two surfaces ω^4 along two cubics. These curves cut each other in the first place in the three points of intersection of the plane with the three lines of intersection of the planes a different from a_i and a_k . The remaining six points of intersection belong to k^6 . Three of these six points lie on $a_i a_k$ in the three conical points of the surfaces ω^4 on this line, and three lie outside $a_i a_k$. Hence the curve k^6 does not cut a line $a_i a_k$ outside the three conical points of the surfaces ω^4 which lie on such a straight line.

A third surface ω^4 cuts k^6 in four points outside the ten points $a_i a_k a_l$, in each of which two points of intersection of ω^4 and k^6 coincide. These four points are the base points of a net of surfaces ω^4 .

Let C be a linear complex of surfaces ω^4 . C contains ∞^3 nets of surfaces ω^4 . The quadruple involution in question I^4 consists of the ∞^3 quadruplets of the base points of the said nets.

I^4 contains one quadruplet of which one of the points lies in a given point of space P . This is formed by the base points of the net of the surfaces of C which pass through P .

§ 3. Let us consider a four-dimensional space R_4 which contains the considered space R_3 . Through each of the planes a_i we pass a three-dimensional space A_i lying in R_4 . Any four of the spaces A_i cut each other in the points $H_1 \equiv A_2 A_3 A_4 A_5, \dots, H_5 \equiv A_1 A_2 A_3 A_4$. If we choose the angular points of the pentahedron of coordinates in H_1, \dots, H_5 , the equation:

$$\sum \frac{\lambda_i}{x_i} = 0$$

is the equation of the biquadratic varieties Ω^4 which have triple points in H_1, \dots, H_5 . These varieties contain the ten planes $A_i A_k \equiv H_i H_m H_n$ and have double lines in the ten lines $A_i A_k A_l \equiv H_m H_n$. They cut R_3 along the surfaces ω^4 of § 2.

Besides the ten planes $A_i A_k$ two varieties Ω^4 have a surface ω^6 of the sixth degree in common. This surface has a triple point in any point H_i , as the complete intersection must have a nine-fold point in a point H_i and six planes belonging to this intersection pass through a point H_i . ω^6 passes through each edge $H_i H_k$, as this is quadruple for the complete intersection and contains three planes belonging to this intersection. The surface ω^6 cuts R_3 along a curve k^6 of § 2.

A third variety Ω^4 has in common with ω^6 besides the ten edges $H_i H_k$, each counted twice, a rational biquadratic curve k^4 , which passes through the points H_i , because of the nine branches of the complete intersection which pass through a point H_i , each of the four straight lines $H_i H_k$ through this point splits off twice.

The linear complex C of surfaces ω^4 that has been assumed in § 2, may be considered as the intersection of R_3 and a linear complex I' of varieties Ω^4 which, besides the points H_1, \dots, H_5 have another point, H_6 , in common. A quadruplet of I' consisting of the four common points of three surfaces ω^4 not belonging to a pencil, which are different from the ten points $a_i a_k a_l$, is the intersection of R_3 with a base curve k^4 of a net in I' , which, therefore, passes through H_6 . Now any curve k^4 is a base curve of a net of varieties Ω^4 , as k^4 must have seventeen points of intersection with a variety Ω^4 through two points of k^4 different from H_1, \dots, H_5 , and, therefore, lies on this variety.

Accordingly the quadruple involution I^4 consists of the quadruplets of points of intersection of R_3 with the biquadratic curves which pass through the six points H_i in R_4 .

§ 4. As R_4 contains ∞^{14} quadratic varieties, those which pass through the six points H_i form a linear system of ∞^8 individuals. The condition for a quadratic variety to contain a plane is six-fold because it is equivalent to the condition that the variety contains six points of this plane which do not lie on a conic. Consequently there are ∞^2 quadratic varieties which pass through the six points H_i and contain a given plane φ of R_3 .

The intersection of two of these varieties consists of φ and a cubic surface which passes through the six points H_i and has a conic in common with φ as, besides φ , a space through φ has a plane in common with each of the two varieties, so that this space contains a straight line outside φ of the intersection of the varieties.

Besides the conic in φ the cubic surface has in common with a third variety containing φ which passes through the six points H_i , a biquadratic curve k^4 through the points H_i that cuts φ three times, because a space through φ which has also a straight line in common with the surface and a plane in common with the variety, contains one point of k^4 outside φ . Through this curve there pass all the quadratic varieties of the net which contain the six points H_i and φ .

The planes through a given point which cut the rational curve k^4 three times, form a quadratic hypercone K . For this hypercone intersects an arbitrary three-dimensional space in the surface of the trisecants of the projection of k^4 on this space out of the given point. Consequently the system of the ∞^2 quadratic varieties through the points H_i which contain φ , consists of hypercones K with vertices in φ . There is, therefore, only one curve k^4 which cuts φ three times.

Accordingly the quadruple involution I^4 contains one quadruplet of which three points lie in a given plane.

The surfaces ω^4 of the complex C cut a plane φ along a complex of biquadratic curves which pass through the angular points of a complete pentilateral. The triplet of I^4 in φ forms with the said angular points the system of base points of a net of biquadratic curves. Hence:

A linear complex of biquadratic curves all of which pass through the angular points of a complete pentilateral, contains one net with thirteen base points¹⁾.

§ 5. The system of the ∞^8 quadratic varieties which pass through the six points H_i , cuts R_3 in a linear system of ∞^8 quadratic surfaces. The ten coefficients a_{ik} of the equations of these surfaces always satisfy the relation:

$$\sum \lambda_{ik} a_{ik} = 0.$$

¹⁾ Prof. JAN DE VRIES drew my attention to the fact that these biquadratic curves are involution curves of involutions I^5 of the tangents to the conic which may be inscribed in the pentilateral and of which one group consists of the pentilateral.

Let us choose one variety out of this system which is degenerate in the planes $(\xi'_1, \xi'_2, \xi'_3, \xi'_4)$ and $(\xi''_1, \xi''_2, \xi''_3, \xi''_4)$. The coordinates of these planes satisfy the equation :

$$\sum \lambda_{ik} \cdot \frac{1}{2} (\xi'_i \xi'_k + \xi''_i \xi''_k) = 0.$$

Hence these planes are associated to each other relative to a quadratic surface ω^2 which is represented by the equation in plane-coordinates :

$$\sum \lambda_{ik} \xi_i \xi_k = 0.$$

Accordingly those quadratic varieties of the net which contain the six points H_i and the plane φ of R_3 , cut R_3 also in the planes of a sheaf that has the pole F of φ relative to ω^2 as vertex. An arbitrary plane φ' through F belongs entirely to that variety of the said net that contains two points of φ' which do not lie on a straight line with F . For besides the line $\varphi\varphi'$ this variety has three points in common with φ which do not lie on a straight line.

The point F is the intersection outside φ of the curve k^4 which is common to all the individuals of the net, and R_3 . Consequently a point of intersection of a curve k^4 through the six points H_i and R_3 is the pole relative to ω^2 of the plane through the other three points of intersection of the same curve and R_3 .

The quadruplets of the quadruple involution I^4 are, therefore, the systems of angular points of ∞^3 polar tetrahedrons of a quadratic surface ω^2 .

§ 6. The joins $H_i H_k$ cut R_3 in the fifteen points D_{ik} , the planes $H_i H_k H_l$ along the twenty lines d_{ikl} , and the spaces $H_i H_k H_l H_m$ in the fifteen planes δ_{iklm} . The said points, lines and planes of R_3 form a configuration $\Delta(15_6, 20_3)$ of DESARGUES. The tetrahedrons $D_{13} D_{14} D_{15} D_{16}$ and $D_{23} D_{24} D_{25} D_{26}$ for instance are resp. the projections out of H_1 and H_2 of the tetrahedron $H_3 H_4 H_5 H_6$ on R_3 , so that the lines $D_{13} D_{23}, D_{14} D_{24}$ etc. all cut $H_1 H_2$ and, therefore, pass through D_{12} . Further the points of intersection $D_{34}, D_{35}, D_{36}, D_{45}, D_{46}, D_{56}$ of the corresponding edges lie in the plane δ_{3456} as well as the lines of intersection $d_{456}, d_{356}, d_{346}, d_{345}$ of the corresponding sides of the two above mentioned tetrahedrons of R_3 . The planes $\alpha_1, \alpha_2, \dots, \alpha_5$ of § 2 are resp. identical with the planes $\delta_{2345}, \delta_{1345}, \delta_{1245}, \delta_{1235}$ and δ_{1234} , i.e. the five planes without index 6. As we may exclude another index as well we find :

The quadruple involution I^4 may be produced in six ways as indicated in § 2.

If we choose one of the points P_1 of a quadruplet of I^4 in D_{12} , k^4 degenerates into the line $H_1 H_2$ and one of the ∞^2 twisted cubics which lie in the three-dimensional space $H_3 H_4 H_5 H_6$ and which pass through the points H_3, H_4, H_5, H_6 and the point of intersection S_{12} of the said

space with H_1H_2 . These twisted cubics form a congruence of REYE¹⁾ and cut the plane δ_{3456} in a triple involution of REYE of which any triplet, together with D_{12} , forms a quadruplet of I^4 .

The singular points of this triple involution, the points of a configuration (10_3) of DESARGUES, are the six points of Δ in δ_{3456} and the four points of intersection of δ_{3456} with the straight lines of Δ through D_{12} .

Accordingly the quadruple involution I^4 has cardinal points in the points D_{ik} of the configuration Δ . To the point D_{ik} there correspond the point triplets of a triple involution of REYE in the plane δ_{lmno} .

As any quadruplet of I^4 forms the system of angular points of a polar tetrahedron of ω^2 , the point D_{ik} must be the pole of the associated plane δ_{lmno} relative to ω^2 . Hence the perspective tetrahedrons $D_{13}D_{14}D_{15}D_{16}$ and $D_{23}D_{24}D_{25}D_{26}$ are in a polar correspondence to each other relative to ω^2 and the center of perspectivity is the pole of the axial plane relative to ω^2 . This holds good for all fifteen pairs of perspective tetrahedrons by the aid of which Δ may be produced.

If P_1 lies on the line d_{123} , k^4 degenerates into the conic through P_1 of the pencil of conics in the plane $H_1H_2H_3$ which has the points H_1, H_2, H_3 , and the point of intersection S_{123} of the planes $H_1H_2H_3$ and $H_4H_5H_6$ as base points and into an arbitrary conic of the pencil of conics in the plane $H_4H_5H_6$ of which the points H_4, H_5, H_6 and S_{123} are the base points. The said pencils cut d_{123} and d_{456} resp. in the pairs of points of the involutions i_{123} and i_{456} . Accordingly P_1 belongs to ∞^1 quadruplets of I^4 . Besides P_1 these quadruplets have one more fixed point, P_2 , which is associated to P_1 in i_{123} . The other two points are variable and always form a pair of i_{456} .

Consequently the twenty straight lines d_{ikl} of Δ consist of points that are singular for I^4 . The quadruplets of I^4 which contain a given point P_1 of d_{ikl} , consist of the point associated to P_1 in i_{ikl} and of a pair of the involution i_{mno} on d_{mno} .

As any quadruplet of I^4 consists of the angular points of a polar tetrahedron of ω^2 , the two straight lines of each of the ten pairs (d_{ikl}, d_{mno}) are associated to each other relative to ω^2 and the twenty involutions i_{ikl} consist of the pairs of points of the corresponding straight lines d_{ikl} that are associated relative to ω^2 .

§ 7. A curve k^4 is projected out of H_1 by a cubic cone \varkappa_1^3 which cuts R_3 along a cubic k^3 through the five points $D_{12}, D_{13}, D_{14}, D_{15}$, and D_{16} . Any cubic cone \varkappa_1^3 contains ∞^1 curves k^4 as the k^4 which passes through an arbitrary point of \varkappa_1^3 , is projected on R_3 in a cubic which must have six points in common with the intersection k_1^3 of \varkappa_1^3 and R_3 and which, therefore, coincides with k_1^3 .

The quadruplet of I^4 of which one point lies in a given point of a

¹⁾ Cf. the thesis for the doctorate of Dr. J. DE VRIES: *Bilinéaire congruenties van kubische ruimtekrommen*, Utrecht 1917, cap. I.

curve k_1^3 , belongs entirely to this curve as the k^4 through the given point lies on the cone which projects k^4 out of H_1 , and which, therefore only cuts R_3 on k_1^3 .

The curves in question k_1^3 form a congruence of REYE C_1 , of which the base points are the angular points of a polar pentagon of ω^2 , i.e. a skew pentagon so that each plane through three of the angular points is associated relative to ω^2 to the join of the other two angular points. The straight line for instance which is associated relative to ω^2 to $D_{12}D_{13} \equiv d_{123}$, i.e. the line d_{456} , lies in the plane $D_{14}D_{15}D_{16}$. And the pole of the plane $D_{14}D_{15}D_{16} \equiv \delta_{1456}$, i.e. D_{23} , also lies on d_{123} .

As we might as well have projected the curves k_1^4 out of H_2, \dots, H_6 there are six such congruences of REYE, C_1, \dots, C_6 . Each of the congruences consists of the cubics containing ∞^1 inscribed polar tetrahedrons of ω^2 and I^4 is formed by the quadruplets of angular points of the ∞^3 inscribed polar tetrahedrons of the curves of one of these congruences.

Consequently the points P_2, P_3 and P_4 corresponding in I^4 to a given point P_1 , may be found by determining the intersection of the polar plane of P_1 relative to ω^2 and the curve through P_1 of a congruence of REYE of which the base points form a polar pentagon of ω^2 . Thus I^4 may be produced in six ways.

§ 8. The planes through the straight line H_2H_3 which project a curve k^4 , form a quadratic bicone K_{23}^2 which cuts R_3 in a cone \varkappa_{23}^2 with vertex in D_{23} . This cone contains the four lines of Δ through D_{23} , i.e. $d_{123}, d_{234}, d_{235}$ and d_{236} . Any bicone K_{23}^2 contains ∞^2 curves k^4 as a k^4 through an arbitrary point of K_{23}^2 has two points of intersection with K_{23}^2 each counted double, i.e. the points H_2 and H_3 , and five more single points of intersection, to wit the arbitrary point and the other four points H . A quadruplet of I^4 of which one point lies on the cone \varkappa_{23}^2 , must entirely belong to \varkappa_{23}^2 because the k^4 through this point lies on K_{23}^2 .

As this quadruplet also quite belongs to the curve k_1^3 through the same point, it consists of the four points of intersection outside D_{12} and D_{13} of this curve k_1^3 and \varkappa_{23}^2 . For this reason the intersection of a cone \varkappa_1^3 and a bicone K_{23}^2 consists of the lines H_1H_2 and H_1H_3 and of a curve k^4 .

Accordingly to each of the congruences of REYE mentioned in § 7 there correspond ten pencils of quadratic cones which have their vertices in a point of Δ and of which the four straight lines through this point are generatrices so that I^4 consists of the quadruplets of the variable points of intersection of the curves of the congruence with the cones of each of the corresponding pencils. We have, therefore, found six congruences of REYE and fifteen pencils of cones by the aid of which I^4 may be produced in sixty ways.

§ 9. The twisted cubics of the six congruences of REYE found in § 7, are invariant for I^4 . The same holds good for the ∞^4 curves k^6 which are the base curves of the pencils of the complex C of surfaces ω^4 indicated in § 2. The surface of such a pencil which passes through a point of a_1 , is degenerate in a_1 and a cubic surface that contains the lines of intersection of the planes a_2, \dots, a_5 and has conical points in the points of intersection of any three of these planes. This surface has in common with the cubic surface which is a part of the surface of the pencil through an arbitrary point of a_2 , the three straight lines $a_3 a_4$, $a_4 a_5$ and $a_3 a_5$ passing through the point $a_3 a_4 a_5$ and a curve k^6 of which the said three lines are the only trisecants through $a_3 a_4 a_5$. For both cubic surfaces have a conical point in $a_3 a_4 a_5$; accordingly any trisecant of k^6 through $a_3 a_4 a_5$ has four points in common with these surfaces and, therefore, belongs to them. Hence the projection of k^6 out of $a_3 a_4 a_5$ on an arbitrary plane is a quintic with three double points; k^6 is, therefore, of the genus three.

As I^4 may be produced in six ways, as has been indicated in § 2, there are, accordingly, six systems of ∞^4 twisted curves of the order six and the genus three which are invariant for I^4 .

§ 10. The curves k^4 which cut a given line l of R_3 , form a surface. In the first place this surface has in common with the space $H_1 H_2 H_3 H_4$ the cubic through the points H_1, \dots, H_4 and the points of intersection of l and $H_5 H_6$ with the said space, as together with $H_5 H_6$ this curve forms a k^4 cutting l . The edges of the tetrahedron $H_1 H_2 H_3 H_4$ also belong to the surface because just as $H_5 H_6$ any of these edges is completed to a k^4 cutting l by a cubic.

Consequently the surface is of the degree nine; besides along l it cuts R_3 along a curve k^8 of the order eight, the locus of the point triplets which, together with the points of l , form quadruplets of I^4 .

This ensues also from the way of producing I^4 of § 7. If P_1 describes the line l , the plane π_1 through the three points corresponding to P_1 in I^4 , describes a pencil which has as axis the line m associated to l relative to ω^2 . Any plane π_1 of this pencil contains three points outside m of the curve associated to l , i.e. the points of intersection of π_1 with the curve through P_1 of one of the six congruences of REYE found in § 7. As the curves of these congruences which cut l form a surface of the fifth degree, the curve corresponding to l in I^4 has five points in common with m , which, therefore, is indeed a curve of the order eight, k^8 . The line l is a chord of k^8 . For this curve cuts l in the two points of intersection of l and ω^2 , because one of the points associated to such a point in I^4 , coincides with this point of intersection. In each of the remaining fourteen points of intersection of k^8 with ω^2 there coincide two points associated to a point of l .

The projection k'^8 of k^8 out of a point of m on an arbitrary plane is

a curve of the order eight with a five-fold point M , through which point there pass fourteen tangents. If k'^8 had no multiple points besides M , this number of tangents would be 26. Hence the curve k'^8 has six more double points and it is, therefore, of the genus five. Through a point of m there pass six chords of k'^8 .

One of the three points which in I^4 are associated to the points of intersection of l with a plane δ of the configuration Δ , lies in a point of Δ , to wit in the pole of this plane relative to ω^2 . The other two with $l\delta$ form a triplet of an involution of REYE in δ ; k'^8 cuts δ in these points and in the six points of Δ in δ .

Consequently the point triplets which in I^4 are associated to the points of a line l , form a curve of the order eight and the genus five, k'^8 , which cuts l twice, the line m , associated to l relative to ω^2 , five times, and which passes through the points of the configuration Δ .

The five points of intersection of k'^8 with m are parts of quadruplets of I^4 of which one point belongs to l while the plane through this point and two of the other points of the same quadruplet, passes through l .

There are five quadruplets of I^4 which define tetrahedrons of which one side passes through a given straight line while one of the angular points in this side lies on the given line.

It appears also in the following way that through a point of m there pass six chords of k'^8 . A chord through a point P of m cuts k'^8 in a pair of points which is a part of a quadruplet of I^4 of which one point P_1 lies on l and another point P_2 is a point of intersection outside l of k'^8 with the polar plane π of P relative to ω^2 , as $P_1 P_2$ must be associated to this chord relative to ω^2 . π cuts the curve k'^8 in six points outside l each of which, together with a point of l , forms a pair of points of I^4 .

Consequently the locus of the pairs of I^4 in π cuts l six times. A point of intersection of π with any of the twenty straight lines of Δ together with the point in which the line associated to this line relative to ω^2 cuts π , forms a pair of I^4 . Each of the points of the triplet of I^4 which belongs to π , together with the other two points of this triplet, forms two pairs of I^4 .

The locus of the pairs of points of I^4 belonging to a given plane, is, therefore, a curve of the sixth order which cuts the twenty straight lines of Δ and which has double points in the three points of the triplet of I^4 lying in this plane.

§ 11. If, for instance, l cuts the line d_{123} of Δ , d_{456} splits off from k'^8 and there remains, accordingly, a curve k'^7 of the order seven which passes through the twelve points of Δ outside d_{456} and which cuts m four times. Outside l this curve has twelve points in common with ω^2 . Hence k'^7 is of the genus four. In this way we may also examine the

curve that is associated to a straight line which cuts more than one line of Δ .

Let us now consider the case that l passes, for instance, through the point D_{12} of Δ . We use the way of producing I^4 of § 7 and that by the aid of a congruence of REYE which has a base point in D_{12} . Again a plane through m cuts the curve associated to l in three points outside m and this curve cuts m in the points of intersection of m and the quadratic cone which is formed by the curves of the congruence which cut l . To l there corresponds, therefore, a curve k^5 of the fifth order, which has l and m as chords. This curve passes through the points of Δ outside δ_{3456} , cuts ω^2 outside l in eight points and is, therefore, of the genus two.

In § 10 we found that there are fourteen quadruplets of I^4 which have one point on l and of which two other points coincide. Accordingly the surface of the bifurcation-points of I^4 is of the degree fourteen. We shall call it ω^{14} . The quadruplets of which one point lies in a given point of a straight line of Δ , consist of two fixed points of the straight line and the pairs of an involution on the line associated to it relative to ω^2 , so that among the said quadruplets there are two with two coinciding points. Consequently any point of a straight line of Δ is a double point of ω^{14} . As further, as appeared above, a straight line through a point of Δ contains eight bifurcation-points of I^4 besides this point, any point of Δ is a six-fold point of ω^{14} . A plane of Δ cuts ω^{14} along the four straight lines of Δ in this plane, each counted twice, and along the bifurcation-curve of the involution of REYE which is associated to the pole of this plane relative to ω^2 . This bifurcation-curve is of the sixth order and has a double point in each point of Δ in this plane.

Accordingly the locus of the bifurcation-points of I^4 is a surface of the fourteenth degree, ω^{14} , which has the straight lines of Δ as double lines and which has sixfold points in the points of Δ .

§ 12. The ∞^3 carriers of the pairs of points of I^4 form a complex of which we shall determine the order by investigating which lines this complex has in common with the congruence of the chords of a twisted cubic k^3 of one of the congruences of REYE by the aid of which I^4 may be produced according to § 7. The common lines are in the first place the sextuplets of the rays joining any two points of a quadruplet of I^4 on k^3 . Through any point of k^3 there pass three such lines so that a chord of k^3 cuts six generatrices of the surface formed by these rays, which surface is, therefore, of the sixth degree. As a point of Δ forms a pair of I^4 with any point of its polar plane relative to ω^2 , all the points of Δ are cardinal points of the complex under consideration. This has, therefore, besides, five quadratic cones in common with the congruence of the chords of k^3 ; together with the found surface of the sixth degree, these form a scroll of the degree sixteen. There are no

other carriers of pairs of I^4 which are, at the same time, chords of k^3 , because such a carrier would be a chord of two curves of the congruence of REYE of which k^3 is a part, hence a singular chord, and singular chords are only the lines through the base points of the congruence, which have already been counted. As the congruence of the chords of k^3 is a congruence (1,3), we find the order four for the complex of the carriers of the pairs of I^4 . Any line of a plane of Δ as well as any line cutting two lines of Δ associated to each other relative to ω^2 , is a carrier of a pair of I^4 .

The carriers of the pairs of I^4 form a biquadratic complex C^4 . The fifteen points of Δ are cardinal points, the fifteen planes of Δ cardinal planes of C^4 . This complex contains the ten bilinear congruences each of which has as directrices a pair of lines of Δ associated to each other relative to ω^2 .

The complex curve of C^4 belonging to a given plane u , has as tangents the fifteen lines of intersection of u with the cardinal planes of Δ , the ten joins of the points of intersection of u with the pairs of directrices of the bilinear congruences belonging to C^4 , and the three carriers of the pairs of the triplet of C^4 belonging to u . The complex cone of C^4 with vertex T has as generatrices the fifteen joins of T with the points of Δ , the ten transversals through T of the pairs of directrices of the bilinear congruences belonging to C^4 , and the lines which join T to the three points which, together with T , form a quadruplet of I^4 .

§ 13. We shall now examine the surface which is the locus of the triplets which form quadruplets of I^4 with the points of a given plane η . To each point of intersection of this surface with a given line l there corresponds in I^4 a point of η lying in a point of intersection with η of the curve k^8 associated to l . Hence the surface corresponding to η in I^4 is of the eighth degree. We shall call it ω^8 .

To the point of intersection of η with a straight line of Δ there correspond the pairs of points of an involution on the line associated to it relative to ω^2 and one point of the same straight line. Hence the surface ω^8 contains all the lines of Δ and it has a double point in any point which is associated relative to ω^2 to the intersection of η with a line of Δ and which belongs to the same line.

As to a straight line through a point of Δ there corresponds a curve of the fifth order, such a line has five points in common with ω^8 besides the said cardinal point of I^4 , so that any point of Δ is a triple point of ω^8 .

The surface ω^8 cuts a plane of Δ along the four lines of Δ in this plane and the quartic corresponding to the intersection of η and the above mentioned cardinal plane of I^4 in the triple involution of REYE formed by the triplets of I^4 in this cardinal plane. The intersection of

ω^8 with φ consists of the curve of the sixth order c^6 which is the locus of pairs of points of I^4 in φ , and of the intersection of φ with ω^2 .

Besides in the points of Δ , ω^8 has a triple point in the pole F of φ relative to ω^2 , as this point is associated to the three points of the triplet of I^4 which lies in φ .

A pair of points (P_3, P_4) which forms a quadruplet of I^4 together with a pair (P_1, P_2) of this involution which lies in φ and which, therefore, belongs to c^6 , always lies on a straight line through F . Inversely to any pair (P_3, P_4) on a carrier through F there corresponds a pair (P_1, P_2) of c^6 . The locus of the pairs (P_3, P_4) lies on the complex cone κ^4 of C^4 with vertex in F . It has a triple point in F as F forms a pair (P_3, P_4) with any point of the triplet of I^4 in φ ; it cuts any plane through F in eight points outside F and is, therefore, of the eleventh order. This curve c^{11} is a double curve of ω^8 as the points P_3 and P_4 are always associated in I^4 to each of the corresponding points P_1 and P_2 of φ .

The points where a transversal through F of the straight lines of Δ associated to each other relative to ω^2 cuts these lines, form a quadruplet of I^4 with the points of intersection of these lines and φ . For this reason c^{11} passes through the former two points. In accordance with this we found already that ω^8 has a double point outside the points of Δ on any line of Δ .

The line which joins F to a point of Δ , cuts the polar plane relative to ω^2 of the latter point in a point which forms a quadruplet of I^4 with a pair of points of the line of intersection of this polar plane with φ and the chosen point of Δ . Consequently the curve c^{11} passes through all the points of Δ and through any point of intersection of the join of F and a point of Δ with the polar plane relative to ω^2 of the latter point.

Accordingly six of the points of intersection of c^{11} with a plane of Δ lie in the points of Δ in this plane, four lie on the straight lines of Δ in this plane and the eleventh point of intersection lies in the point where the plane is cut by the line which joins the pole relative to ω^2 of this plane to F .

The locus of the point triplets which form quadruplets of I^4 with the points of a plane φ , is a surface of the eighth degree which contains the lines of Δ , has triple points in the points of Δ and in the pole F of φ relative to ω^2 , and which has a double curve of the eleventh order.

§ 14. The double curve c^{11} of ω^8 cuts φ in the first place in the three points of the triplet of I^4 belonging to φ . In the other eight points of intersection a point P_2 of a pair of I^4 in φ must coincide with a point P_3 of the corresponding pair (P_3, P_4) . These eight points of intersection lie, therefore, on the intersection c^2 of φ with ω^2 . They also belong to c^6 as well as to c^{11} .

In the other four points of intersection of c^6 and c^2 the two points of a pair (P_1, P_2) in φ coincide. Hence the congruence K of the carriers of the coincidences among the pairs of I^4 has the class four. In the same way the two points of a pair (P_3, P_4) on a carrier through F coincide in the fourteen points of intersection of c^{11} with ω^2 which are different from the eight points already found. Accordingly the order of K is fourteen.

The carriers of the coincidences among the pairs of I^4 form a congruence $K(14,4)$.

The pairs of I^4 which belong to a straight line of Δ , form an involution. As among these pairs there are two coincidences, any line of Δ is a double line of K . The same holds good e.g. for the line of intersection of the two planes δ_{1234} and δ_{1256} , as this line with $D_{13}D_{14}$ and $D_{15}D_{16}$ forms a degenerate curve of the congruence of REYE with D_{12}, \dots, D_{16} as base points, by the aid of which I^4 may be produced according to § 7.

Hence seven double rays of K pass through D_{12} , to wit the four lines of Δ through D_{12} and the lines $(\delta_{1234}, \delta_{1256}), (\delta_{1235}, \delta_{1246})$ and $(\delta_{1236}, \delta_{1245})$.

The planes of Δ are singular for K . The lines of K in a plane of Δ , i. e. the bifurcation-lines of the involution of REYE in this plane of point triplets of I^4 , envelop a curve of the sixth class. This curve has as double tangents the four lines of Δ in the plane and the six lines of intersection with the planes of Δ which have only two indices in common with the plane.

§ 15. The curve k^8 and the surface ω^8 corresponding resp. to the line l and the plane φ , have 64 points in common. Three of them lie in each of the fifteen points of Δ . Among the other nineteen there are three which form a quadruplet of I^4 with the point $l\varphi$. There are eight points P_1 of l that belong to the same quadruplet of I^4 with another point P_2 of φ which always lies in a point of intersection of k^8 and φ . The eight pairs of points which complete these pairs (P_1, P_2) to quadruplets of I^4 , give sixteen more points of intersection of k^8 and ω^8 .

The surfaces ω^8 and ω'^8 corresponding to the planes φ and φ' have an intersection of the order 64. To this intersection belong the twenty lines of Δ and the curve k^8 which is associated to the line $\varphi\varphi'$. There remains a curve of the order 36, the locus of the pairs of points which complete a pair (P_1, P_2) of which P_1 lies in φ , P_2 in φ' , to a quadruplet of I^4 . This curve k^{36} has a quadruple point in each point of Δ as the corresponding triple involution of REYE contains four triplets of which one of the points lies in φ and another point in φ' . If d and d' are two lines of Δ associated relative to ω^2 , φd , $\varphi' d'$, and the points of d and d' associated relative to ω^2 resp. to φd and $\varphi' d'$, form a quadruplet of I^4 . In the same way a quadruplet of I^4 may be derived from the

points $\varphi'd$ and $\varphi d'$. Accordingly k^{36} cuts each straight line of Δ besides in three points of Δ in two more points which are associated relative to ω^2 to the points of intersection of this line with φ and φ' .

As k^{36} cuts a third plane φ'' in 36 points, we find:

There are 36 quadruplets of I^4 which have a point in each of three given planes.

The curve k^{36} cuts a third surface ω''^8 in $36 \times 8 - 15 \times 12 - 20 \times 2 = 68$ points which are not singular for I^4 . Hence three surfaces ω^8 , ω'^8 , and ω''^8 corresponding resp. to the planes φ , φ' , and φ'' , have $68 + 19 = 87$ isolated points of intersection. Among them are the three points which form a quadruplet of I^4 with the point $\varphi\varphi'\varphi''$ and the 48 points of the $3 \times 8 = 24$ pairs which complete a point of the intersection of two of the planes φ , φ' and φ'' and a point of the third plane to a quadruplet of I^4 . The remaining 36 points are those which form quadruplets of I^4 with three different points of which the first lies in φ , the second in φ' , and the third in φ'' .

Physics. — "*Measurements on the surface tension of liquid neon*".
(Communication No. 182*b* from the Physical Laboratory, Leiden.)
By A. TH. VAN URK, W. H. KEESOM and G. P. NIJHOFF.

(Communicated at the meeting of April 24, 1926).

§ 1. With the same apparatus, which was used for the measurement of the surface tension of liquid helium¹⁾, we determined the surface

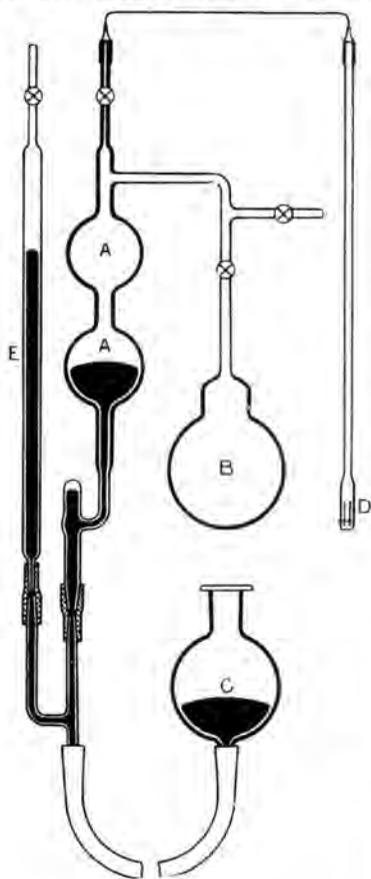


Fig. 1.

tension of liquid neon. The method used was just the same as in the case of helium. In order to get pure liquid neon in the reservoir with the two capillary tubes the apparatus was joined by means of a steel capillary to a big glass retainer formed by two glass bulbs (see fig. 1, A). These bulbs were previously exhausted and the vacuum controlled during some days. The neon was purified by freezing it in liquid hydrogen, after which the first part of it was pumped away, and only the middle part was used for the filling of bulb *B*, which then was blown to the apparatus.

At the beginning of the measurements the gas was liquified in the reservoir *D*, which was held at the triple point temperature of neon, by raising the mercury reservoir *C*. For these measurements the reservoir *D* was put into the hydrogen vapour cryostat.²⁾

By regulating the height of the mercury reservoir *C*, we made certain that a large amount of liquid was first formed. Then some of it was allowed to evaporate in order to be sure that the walls of the capillary tubes were well wetted.

¹⁾ A. TH. VAN URK, W. H. KEESOM and H. KAMERLINGH ONNES. *These Proc.* 28 958, 1925; *Comm. Leiden* No. 179*a*.

²⁾ *These Proc.* 19, 1049, 1917 and 23, 1185, 1921; *Comm. Leiden* Nos. 151*a* and 154*c*.

§ 2. The results of the measurements were calculated in the same way as given in the communication about helium. ¹⁾ They are given in the table. ²⁾ In fig. 2 the values of ψ_M are plotted against T , the \circ and Δ referring to the two capillaries placed alongside each other.

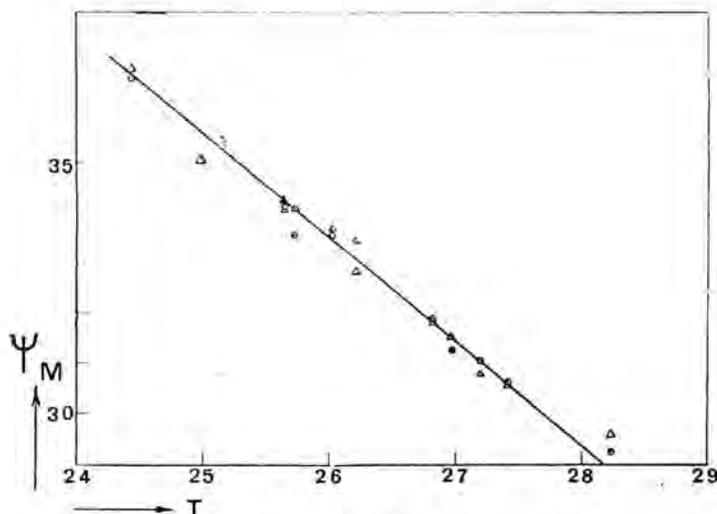


Fig. 2.

The constants for v. d. WAALS' formula

$$\psi_s = A (1 - t)^B$$

become $A = 14.73$ and $B = 1.202$.

The molecular surface tension is given by

$$\psi_M = 2.1 (41.78 - T)$$

in which 2.1 is EÖTVÖS' constant.

By extrapolating down to $T = 0$ one gets for the surface tension at that temperature 1.18×10^{-14} .

For the heat of evaporation one finds at $T = 0$ if one extrapolates linearly the known values ³⁾ between 33° and 25° K.

$$\frac{20.2 \times 30 \times 4.19 \times 10^7}{6.06 \times 10^{23}} = 4.2 \times 10^{-14}$$

and so we find for the ratio $\frac{\sigma_0}{L_0}$ ⁴⁾ 0.28, a value in very good agreement with the calculations in Comm. Leiden, Suppl. N^o. 54.

§ 3. Measurements with oxygen, nitrogen, argon and hydrogen have

¹⁾ See note 1, p. 914.

²⁾ The densities have been taken from Comm. Leiden, No. 162b.

³⁾ Comm. Leiden, No. 162b.

⁴⁾ These Proc. 28, 356, 1925; Comm. Leiden, Suppl. No. 54.

also been done by us. As the results of those measurements agree within the limits of accuracy with the existing measurements¹⁾, there is no need to give them here.

Surface tension of Neon.

| T | First capillary | | | Second capillary | | |
|-------|-----------------|-----------------|----------|------------------|-----------------|----------|
| | hc | Ψ_{σ} | Ψ_M | hc | Ψ_{σ} | Ψ_M |
| 24.44 | 0.00939 | 5.75 | 36.7 | 0.00941 | 5.76 | 36.8 |
| 24.98 | 900 | 5.46 | 35.1 | 900 | 5.46 | 35.1 |
| 25.15 | 908 | 5.50 | 35.3 | 911 | 5.52 | 35.5 |
| 25.65 | 880 | 5.29 | 34.2 | 878 | 5.28 | 34.1 |
| 25.65 | 882 | 5.30 | 34.2 | 882 | 5.30 | 34.2 |
| 25.73 | 865 | 5.19 | 33.6 | 869 | 5.28 | 34.1 |
| 26.03 | 866 | 5.18 | 33.5 | 871 | 5.21 | 33.7 |
| 26.21 | 865 | 5.16 | 33.5 | 849 | 5.06 | 32.8 |
| 26.82 | 829 | 4.88 | 31.9 | 827 | 4.87 | 31.8 |
| 26.97 | 813 | 4.78 | 31.3 | 820 | 4.82 | 31.5 |
| 27.20 | 810 | 4.74 | 31.1 | 803 | 4.70 | 30.8 |
| 27.42 | 800 | 4.67 | 30.6 | 799 | 4.66 | 30.6 |
| 28.23 | 768 | 4.42 | 29.2 | 777 | 4.47 | 29.6 |

¹⁾ E. C. BALY and F. G. DONNAN, Journ. Chem. Soc. London, Transactions **81**, 907, 1902. H. KAMERLINGH ONNES and H. A. KUYPERS, These Proc. **17**, 525, 1914; Comm. Leiden No. 142d. See for the results J. E. VERSCHAFFELT, Mededeeling uit het Nat. Lab. Gent, No. 2, 1925.

Botany. — "*Vegetative Cultivation of Hippeastrum.*" By Miss IDA LUYTEN. First Part. (Communication N^o. 20, Laboratory for Plant-physiological Research, Wageningen.) (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of April 24, 1926).

§ 1. *Introductory.*

The bulbs of the *Hippeastrum*-hybrids (*Amaryllidaceae*) have already been offered to the trade for several decades; usually however they have been offered under the incorrect name of "*Amaryllis Hippeastrum*" or "*Amaryllis-hybrids*".

The original species come from South-America. According to inquiries made the species of this genus were imported into Holland in the order of succession subjoined:

| | | | |
|------------|--------------------|------------------------------------|-----------|
| about 1700 | <i>Hippeastrum</i> | <i>equestre</i> | Herb. |
| 1728 | " | <i>Reginae</i> | Herb. |
| 1769 | " | <i>vittatum</i> | Herb. |
| 1777 | " | <i>reticulatum</i> | Herb. |
| 1810 | " | <i>rutilum</i> (<i>fulgidum</i>) | Herb.) |
| 1814 | " | <i>psittacinum</i> | Herb. |
| 1819 | " | <i>aulicum</i> | Herb. |
| 1820 | " | <i>solandriflorum</i> | Herb. |
| 1863 | " | <i>procerum</i> | Lemaire. |
| 1867 | " | <i>pardinum</i> | Dombrain. |
| 1869 | " | <i>Leopoldi</i> | Dombrain. |

According to BAKER ¹⁾ (1888) the first hybrid was obtained in 1799 by a watchmaker JOHNSON of Prescott (Lancashire), who crossed the species *Reginae* with the species *vittatum*. It appeared that on crossing the broad-leaved species were easily to yield good seeds and that the hybrids first obtained were useful for crossing. About 1830 the chief species usually crossed, were: *Reginae*, *reticulatum*, *vittatum*, *aulicum*, and *solandriflorum*; more rarely used were: *equestre*, *fulgidum* and *stylosum*.

After 1870 the species imported by the English firm VEITCH: *pardinum* and *Leopoldi* were frequently used for crossing. The former yielded the spotted types in the hybrids of VEITCH.

¹⁾ BAKER, J. G. Handbook of the Amaryllideae.

We were told by growers of much experience, that especially the import of *pardinum* (1867) made the hybrids much more beautiful, but according to their experience their flowering-ability was reduced. As to this we deem it not impossible, that the wealth of flowers may be improved by certain conditions. On this important point experiments will be made in our laboratory to find the best conditions for flowering-ability.

Accordingly these bulbs are propagated by crossing fine, selected varieties, the flowers of which excel in colour, size and position. From this seed bulbs may be obtained, flowering as a rule after 3 or 4 years. This method of propagation has a disadvantage, viz. that it is uncertain, how the seedlings will flower, whether they will really meet the requirements put to the flowers. As all *seedlings* may vary slightly, but sometimes even greatly, "named" bulbs cannot be supplied. The buyer therefore can never get at the same time or a couple of years at a stretch a number of bulbs *exactly* identical to each other. Growers do sell named grown seedlings of a fine prize plant; these are the descendants of an individual prized at the time. They resemble the original, also in consequence of good selection, but they are not the same being as the former prize-generation. Virtually named specimens must not be supplied, though it is customary and hitherto the only practical method.

Now some years ago we put ourselves the question whether it would not be possible to propagate these *Hippeastrum*-bulbs *vegetatively*, for instance by regeneration of the bulbscales cut away, in the way it is done with *Hyacinthus* on a large scale. As far as could be ascertained, growers had not yet succeeded in propagating the *Hippeastrum*-bulbs in this way. Could this really be done, the fine varieties once obtained could be preserved, their number increased and the bulbs supplied "named".

On inquiry it appeared but recently, that various growers were still experimenting on vegetative reproduction and that owing to the slight results some had not expected much from scooping or crossing these bulbs.

The bulbs yield but rarely or most rarely new bulbs originated from axillary buds. Had these been more frequent, they might have been oftener used to increase the number of a variety. From an old catalogue of the year 1862 of E. H. KRELAGE and SON of Haarlem, it appears, that the number could but be increased by a couple from such vegetative shoots. Whether the bulbs will form these offsets is uncertain; we must abide passively.

In practice some people are of opinion, that such bulbs vegetatively reproduced, will never flower.

§ 2. *Arrangement of the experiments.*

As the reports we received about vegetative reproduction were not very hopeful and there was little chance of success, the research in the laboratory

was arranged in such a way, that small quantities would be treated according to various methods. Thus we could trace whether there was any indication of regeneration and find the direction in which the research should be continued.

For starting the experiments the month was chosen in which the bulbs are put dry in culture to undergo a so-called period of rest in a lower temperature. This was done in accordance with the scooping and cutting of the Hyacinth; these bulbs too are used to be scooped and crossed some time after the lifting.

After receiving the bulbs on November 20, we put them in a room of 17° for some 10 days in order to be dried; in consequence of the pot-culture they were still slightly moist.

Next 48 combinations of 5 different external conditions, viz. species of soil, moisture, depth of planting, light and temperature were chosen, in which the different parts of the bulbs were to be treated, viz :

| | | | | |
|---------------------|-------|-------|-------|-----------------------|
| Peatdust | moist | below | | } in 13°, 20° and 27° |
| " | " | atop | light | |
| " | " | " | dark | |
| " | dry | below | | |
| " | " | atop | light | |
| " | " | " | dark | |
| River sand | moist | below | | } in 13°, 20° and 27° |
| " | " | atop | light | |
| " | " | " | dark | |
| " | dry | below | | |
| " | " | atop | light | |
| " | " | " | dark | |
| Calcareous dunesoil | moist | below | | } in 13°, 20° and 27° |
| " | " | atop | light | |
| " | dry | below | | |
| " | " | atop | light | |

On Nov. 29, 1922 we scooped 48 bulbs, i.e. with a scooping-knife we cut the disk from the bulb, next the bulb was cut lengthwise into halves, so that the section ran parallel with the rest of the leaf-blades. This halving was necessary, because otherwise the bulbs could not be peeled without injury, the scales being closed; and in contradistinction to the scooped hyacinths where the scales are left together, we use them here separately. Every half bulb was carefully taken to pieces, so that the least damage possible was done. All parts were used, while the offsets sometimes present, were cleared off to prevent future mistakes. Next on all scales and leaves, coming from one half of the bulb, a diagonal crucial incision was made on the upper- and under-side at some distance from the base. It might be that near or on this crucial incision regeneration would be more likely to occur than near the wound at the base. The parts of a half bulb with and

without crucial incisions were brought into the 48 combinations, so that in this way 96 combinations were formed. The number of possibilities was increased by giving a certain position to the scales and leaves when they were put on or dug into the ground. It might be that the way in which gravitation effected the parts of the bulb, had some influence.

For this purpose those parts which were to be put "below" and bore no crucial incision were planted upright, the top parts of leaves or scales showing just above ground. One half of those bearing a crucial incision were as to the combination "below" put the convex side upwards, the other half the concave side upwards; next they were covered with a layer of soil. In putting them above ground this system of putting the concave side of half the number upwards the other half downwards was likewise applied.

The different pots were brought to three greenhouses; resp. kept at a constant temperature of 13°, 20° and 27°, so that there was plenty of light for that combination that was intended to get light. Those pots in which the bulb-parts lay on top, but needed dark, were covered with black paper. As the moist combinations dried rapidly without a black covering, these pots were covered with a wet piece of white muslin, transmitting much light and preventing a too rapid evaporation.

Besides in 13°, 20° and 27° a bulb was added, cut in two halves, one half of which was put on dry dune-soil, plane of section downwards, whereas in the other half the planes of section were exposed to the air.

The bulbs were supplied by Messrs VAN TUBERGEN, named "*Hippeastrum*"-hybrids. Dark red varieties were selected, since in case the experiments might prove a success, we should prefer this colour for further researches.

§ 3. *The origin of the vegetative bulbs.*

After some weeks (Dec. 18, 1922) it appeared, that the scales exposed to light, formed chlorophyll, especially in 20°, but also in 27° and hardly any in 13°. There likewise occurred a strong anthocyanin-formation. Besides the fact that bulb-parts were coloured entirely red, the planes of section were also coloured red. The strongest anthocyanin-formation occurred in 13°. It was a remarkable fact that in that case anthocyanin was sometimes formed about 3 mms. from the wound of the crucial incision, filling in the space between the crucial incisions. Something similar we saw in the longitudinal sections of the scales. The plane of section was white and at 1—2 mms. distance behind it a stripe of anthocyanin was found.

At the beginning of January 1923 many experiments were stopped, because so many bulb-parts had rotten away. This was so bad in 13°, that but two combinations could be preserved. In 20° and 27° the rotting was less serious. Besides it appeared that the combination kept in dry soil, led to total desiccation and decay of the scales. The unhealthy material

IDA LUYTEN: "VEGETATIVE CULTIVATION OF HIPPEASTRUM".
(First Part).

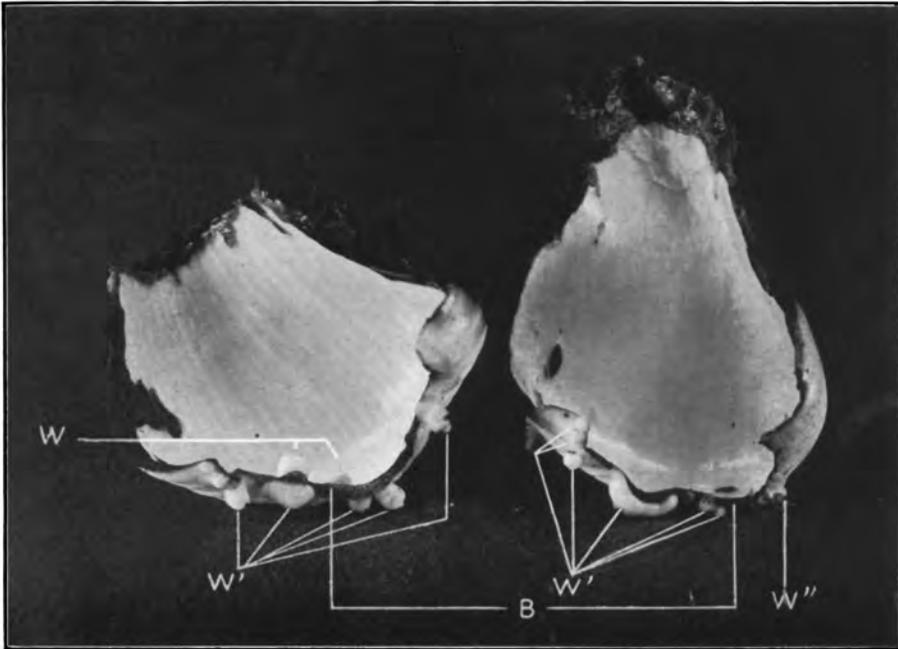


Fig. 1.

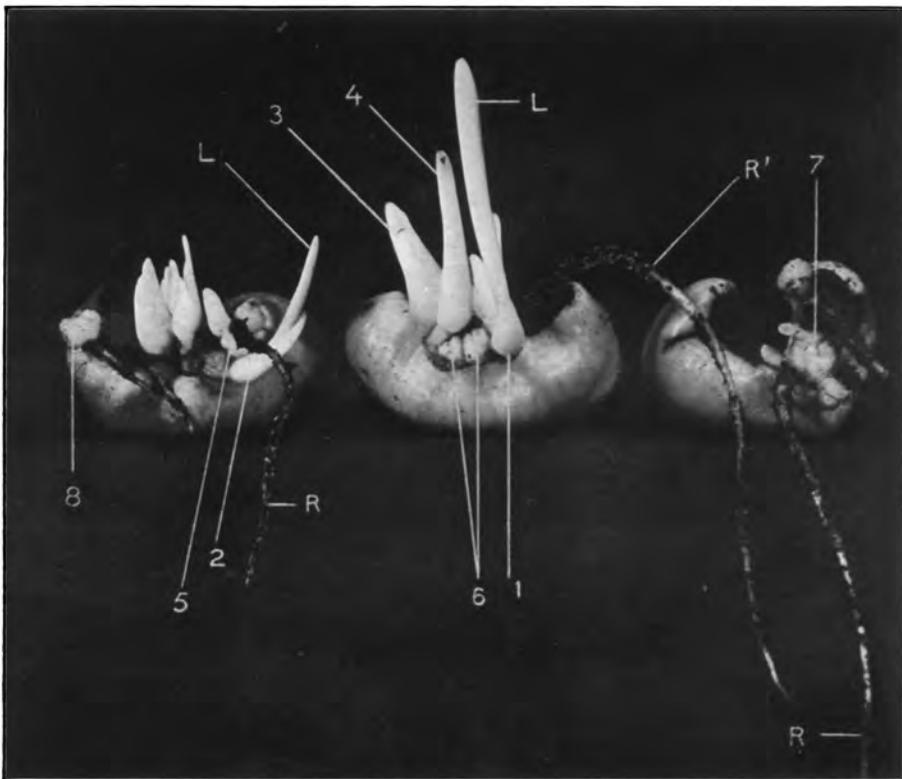


Fig. 2.

was cleared off, *all* the remaining combinations kept moist. Nothing however could still be observed, when here and there an individual was lifted.

On January 24, 1923, i.e. nearly 2 months after the experiments had been started, swellings were found along the planes of section of the bases of some scales in 27°. Fearing to injure something, we left the other experiments untouched and photographed only these scales. Fig. 1 shows the inside of a couple of half scales. Along the planes of section at the base (*B*) we notice smaller and larger protuberances (*W*, *W'*, *W''*). We also noticed a scale bearing a small protuberance, already provided with a root. We point out that the scales after these two months look quite healthy. Further we notice that this regeneration may occur along the base at the inside of the scales (*W*), along the base at the outside (*W'*), and also that it need not arise close to the base, but may begin at some distance from the base (*W''*).

Now the question remained whether those protuberances and swellings would really produce healthy *Hippeastrum*-bulbs.

After a month (Febr. 23, 1923) little bulbs had grown from these tissue-growths, sometimes even with leaflets; they were potted up in the greenhouse in a soil-temperature of 25°—28° under glass cover and an air-temperature of 23°—26° C.

For an illustration of the young *Hippeastrum*-bulbs in a somewhat older stage than given in fig. 1, we refer to fig. 2. This is a photograph taken of some scales of *Hippeastrum*-bulbs, scooped a year later, Dec. 5, 1923. We are shown the lower surface of the bases of the scales.

In this photograph all kinds of forms may be found, viz. various stages of development and various modes of origin. 1 and 2 have already formed foliage-leaves (*L*), whilst 3 and 4 show a somewhat longer outer scale. From the opening of the scale at the top the first leaf will soon appear. Most of the little bulbs have not yet any roots; if the scales lie a little longer, the roots shoot either near the base of the bulbs or they appear later after the bulbs have been potted. N^o. 5 has already formed a root (*R*). As stated above the place of origin of the tissue-growths may vary, but the primary development also appears to be different. Bulbs are namely found, grown from the first tissue-growth. In that case the base of the bulb is directly attached to the scale (1 and 3), others arise on a tissue-protuberance previously formed and stand more or less on a pedestal (5). N^o. 6 apparently starts forming bulbs on the tissue-formations. At (7) various bulbs are developing on an older tissue-protuberance; a root (*R*) is already present. It sometimes happens, that the wound at the base of the scale gives a root (*R'*), without ever getting to regeneration of any other tissue on that spot. It may also occur, that instead of a fine, smooth, glossy-white tissue-growth, a more glassy rough-looking protuberance arises (Fig. 2 at 8). Such growths usually spread without yielding any bulbs. Yet they are sometimes found to have formed a bulb (2).

The above is a comparatively superficial description of what is noticed on examining the scales. A special research may trace the origin of this regeneration, i.e. the first cell-division, the further course of the origin of the various tissues and the various modes of formation of the bulbs. Next it may be accurately determined, at what distance from the base regeneration can occur. It may be pointed out already now, that it is not less than 3 mms. After the base had produced various bulbs, a strip of a scale, 3 mms. wide was cut off along the base. On this new wound new bulbs originated.

§ 4. *Result of the Experiments.*

When regenerative phenomena indicated, that experiments should be made in a comparatively high temperature, the pots from 13° were transmitted to 27°, the more so as the low temperature did not effect the scales favourably. On our continuing the experiments regeneration appeared to occur also at 20°, but much later and to a less degree. This latter is shown in table 1. Besides the bulbs took a longer time to attain a certain size.

TABLE 1. Number of bulbs on May 23, 1923.

| | In 20° | In 27° | Light 20° | Dark 20° | Light 27° | Dark 27° | Without cruc. incisions 20° | With cruc. incisions 20° | Without cruc. incisions 27° | With cruc. incisions 27° |
|--------------|-----------|------------|--------------|-------------|--------------|-------------|--------------------------------------|-----------------------------------|--------------------------------------|-----------------------------------|
| Peatdust | 22 | 39 | — | 22 | — | 39 | 17 | 5 | 26 | 13 |
| Riversand | 20 | 36 | 4 | 16 | 22 | 14 | 14 | 6 | 21 | 15 |
| Dunesoil | 15 | 28 | 14 | 1 | 25 | 3 | 8 | 7 | 9 | 19 |
| Total | 57 | 103 | | | | | | | | |

From table 1 various conclusions may be drawn, yet we should be careful in attaching too much value to the great difference in number of bulbs obtained in some experiments. The following items namely have influenced the result too much.

I. When the bulbs had reached a size of ± 8 mms., or even before that, when they bore a foliage-leaf very early, they were potted up. We did not risk separating these bulbs from the scale on which they had originated, because we did not yet know the strength of this new generation. Accordingly the young bulbs were potted up together with the whole scale or a large part of it. On account of this a great quantity of material which might have borne new bulbs, got lost. For our experiments in later years the bulbs were removed with a sharp knife together with a piece of scale as small as possible, so that the scale could continue yielding new bulbs.

II. In this first year we did not yet know in what temperature and what

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Fig. 3. Vegetatively cultivated *Hippeastrum* after about one year.



Fig. 4. Vegetatively cultivated *Hippeastrum* after about two years.

combination the bulbs were which should give a small or a large crop. When later on the bulbs were numbered, it appeared, that between the individual bulbs there was a great difference in yield of young bulbs.

III. By keeping the material initially dry, much of it had become useless.

IV. The number of scales in 27° exceeded the number in 20°, because combinations from 13° had been transferred to 27° (see above).

V. The experiments were stopped on May 23, 1923, while even more bulbs could have been formed, as became evident in following years.

Yet the following conclusions may be drawn in combination with the observations made during the care of the experiments :

The higher temperatures, as stated above, had a more favourable influence than the lower.

Keeping the scales dry appeared to be injurious.

Light or dark appears to be of no importance. As with respect to keeping the scales moist digging them in a little is much more favourable, the latter was recorded as better treatment. Whether the concave or convex side faced downward has no noticeable influence on the origin of the young bulbs. Later experiences taught us, that practically it is best to put the scales in a slanting position, the concave side upwards.

Near or on the crucial incisions there never occurred any regeneration ; neither had it any favourable influence on the origin of bulbs along the base. It was even a disadvantage, that in consequence of the crucial incision, the scales were sooner subject to rotting.

As to the choice of soil, neither did this give rise to great differences ; we got the impression however that on account of its sourness the moist peatdust made the scales rot sooner.

Never were regenerative-phenomena met on the foliage-leaves, flower-stalks or the disk. These parts of the bulb therefore were removed on scooping in later years when the experiments were started. Potting up the disks may be worth while however. If namely the vegetation-point has remained intact, it may continue splitting and by doing so restore the old bulb. In our case 5 bulbs, treated in this way were preserved ; 2 of them flowered again after 3 years.

It should be mentioned, that in the halved bulbs months later there appeared minute swellings at the base of the halved scales. They were so small that compared to the other results, this treatment evidently offered no advantages.

It will be investigated into whether another month than Nov.-Dec. is more favourable with regard to scooping.

§ 5. *Cultivation and bringing into bloom of the bulbs vegetatively obtained.*

Now it still had to be proved whether these little bulbs really had the

same vitality as the seedlings. In the previous § we already pointed out, that when the bulbs on the scales had attained a certain size, they were potted up. In the first year of the experiments the whole scale was potted up with them, or when each half bore a bulb, half the scale. If a couple of bulbs grew close together, they were put in the same pot. In later years it appeared that such bulbs could be separated. Even when they are growing on one protuberance, we may halve the pedestal without causing a lasting injury.

As soil we chose equal parts of sieved peatdust and dunesoil, the pots being dug in moist peatdust in a plant-box in the greenhouse. The temperature in this greenhouse amounted to 21°—23°, while the temperature of the soil in the plant-box was 25°—28°. To keep the space moist the plant-box was covered with glass, which kept the temperature under the glass at 23°—26°. To the right of figure 3 we see part of such a closed box. When the plants had struck root and become vigorous, they were removed to a box with the same temperature of soil, but without glass-cover.

Part of the material of the 27° experiments was potted up on *Febr.* 23, 1923 and consisted of 26 individuals. From 20° the first 8 individuals were put under glass on *March* 5. The harvest continued till *May* 23, the point of time, when the experiments were stopped in the first year. In later years the experiments were continued much longer.

On that date 103 bulbs had been obtained in 27°, 57 in 20°. When such young plants have attained an age of 7—10 month, they look as represented in fig. 3. On *Oct.* 23, 1923, the bulbs were put in larger pots if necessary, and the soil was made a little heavier with clay. The bulbs were kept growing.

Fig. 4 gives the size of the plants, when they are a year older again (*Nov.* 30, 1924). On the previous *Sept.* 27, they had again been put in larger pots and the soil had been made heavier. So they passed the winter and summer of 1925.

The vigour of the plants had already proved, that bulbs vegetatively obtained, remain healthy and develop as well as seedlings.

Now it had to be proved whether the bulbs would really yield flowers, which was doubted by some people. For this purpose all plants were put dry at the beginning of *Oct.* 1925 and in the middle of *October* (the leaves are decaying by that time) transmitted to a lower temperature (16°). On *Dec.* 17 the first inflorescence shows in a bulb potted up on *April* 6, 1923 as scale-bulb from a combination 27°, dunesoil. On *Dec.* 30 the bulb was put in \pm 23° and kept moist, so that the first flower opens on *Jan.* 23, 1926. Fig. 5 is an illustration of this first flowering *Hippeastrum* vegetatively obtained. At the bottom near the bulb a second inflorescence may be discovered (B 2).

That these plants equal the seedlings in strength follows from this figure, but is moreover proved by fig. 6 (photographed *Febr.* 4, 1926), where the two vigorous stems simultaneously bear flowers. One stem

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Fig. 5. The first flowering *Hippeastrum* cultivated from scales.

bears three, the other 4 flowers. This bulb obtained from a bulb scooped on Nov. 29, 1922 was potted up on April 6, 1923 and flowers¹ \pm Febr. 4 1926. Within three years it has grown into a flowering plant.

It might be expected that not all bulbs should flower in 1926, no more as is the case with the seedlings. In the course of the 3 years 6 bulbs got lost, three of which died. 154 bulbs were left, which may be classed as follows :

TABLE 2. Flowering after \pm 3 years.

| | Peat dust 27° | River- sand 27° | Dune- soil 27° | Peat dust 20° | River- sand 20° | Dune- soil 20° | Total | |
|----------------|---------------------|-----------------------|----------------------|---------------------|-----------------------|----------------------|-----------|-----------|
| | | | | | | | in 27° | in 20° |
| 2 flowerstalks | 6 | 4 | 5 | — | — | 1 | } 43 | } 6 |
| 1 flowerstalk | 17 | 3 | 8 | 1 | 1 | 3 | | |
| No flowerstalk | 15 | 25 | 14 | 23 | 19 | 9 | 54 | 51 |

We read from this table, that nearly half the bulbs, originally formed in 27°, flowered within three years ; viz. 43 of the 97 individuals flowered, 15 of them with 2 flower-stalks. From the bulbs obtained in 20° only 6 of the 57 individuals flowered. We may conclude with certainty, *that the bulbs from 27° are more vigorous, than the bulbs from 20°.*

As in the case of seedlings no more than 40 to 50 % flowers after 3 years, we may call the ratio obtained from 27°, really favourable, especially as of the various treatments applied some did not prove favourable, so that the cultivation of many weak bulbs was also continued. Later experiments will have to prove whether this percentage of bulbs producing flowers after 3 years cannot be increased, when namely after scooping the correct treatment is directly applied. Besides the continuation of our *Hippeastrum*-culture will prove whether the yield of flowers increases in the fourth year, which may surely be expected.

From the above it appears, that *in future* the grower can cultivate the *Hippeastrum* pure in shape and colour, i.e. as varieties — he may therefore in future supply the *Hippeastrum*-bulbs named. He may continue trying to gain new varieties from seed by crossing, while moreover he will be able to preserve pure varieties and to increase their number by propagation from scales.

Meanwhile the experiments have been continued every year since 1922. Since the first experiments indicated 27° as most favourable temperature and it proved best to put the scales in moist, not too sour soil, these results were used as a base. The young scale-bulbs, obtained from the same bulb,

were kept together under the same number, so that already now we possess lots of pure vegetative descent.

As soon as they flowered, we shall publish further data on the best mode of cultivation to our experience in a second paper. We can state already now, that 29°—30° C. and seed-pans with *glass cover* yield the most favourable results.

Our grateful acknowledgement is due to the *Hollandsche Cultuurmaatschappij* for their quite disinterested support in the extra-expenses of this and other publications of our laboratory.

We want to express our thanks to Messrs: C. G. VAN TUBERGEN of *Haarlem*, Messrs: DEN OLDER BROS. of *Leiden*, E. H. KRELAGE of *Haarlem* and Messrs: WARMENHOVEN of *Hillegom*, for the various informations on *Hippeastrum*-culture we might receive.

*Laboratorium voor Plantenphysiologisch Onderzoek,
Wageningen.*

February 1926.

EXPLANATION OF THE FIGURES.

Fig. 1 and Fig. 2 see text.

Fig. 3. Scooped on Dec. 4, 1923; the first potted up on Jan. 23, 1924, the last on Oct. 21, 1924. Photo Nov. 30, 1924.

Fig. 4. Scooped on Nov. 29, 1922; potted up from Febr. to the close of May 1922. Photo Nov. 30, 1924, i.e. a year older than in fig. 3.

Fig. 5. First flowering *Hippeastrum* cultivated from scales. Scooped Nov. 29, 1922, potted April 6, 1923; flowering Jan. 23, 1925.

Fig. 6. *Hippeastrum* cultivated from scales, 2 stems flowering simultaneously.

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Fig. 6. *Hippeastrum* cultivated from scales, flowering with two stalks at the same time.

Neurology. — *"On Recovery of Function of the Facial Muscles after Hypoglossus-facialisanastomosis."* By Prof. E. D. WIERSMA.

(Communicated at the meeting of April 24, 1926).

A boy of 4 years was run over by a motorcar in August 1924. It is not quite sure whether he lost consciousness. An hour after the accident he vomited repeatedly. Directly after the trauma complete paralysis of the right n. facialis was established. The right corner of the mouth did not move when the boy was crying and the right eye could not be shut. The doctor who saw him an hour later stated deafness of the right ear. No further anomalies were found. The other cranial nerves and the extremities functioned normally. When in November 1924 the facialis-paralysis had not taken a turn for the better the patient was admitted to the neurological clinic of Groningen. There the verdict was given: complete deafness of the right ear and complete facialis-paralysis with reaction of degeneration on the right. For the rest there were no anomalies. After consultation with the surgeon a nerve-plastic was decided on. For the facialis-anastomosis only two nerves can receive consideration, viz. the hypoglossus and the accessorius. Although our considerations for choosing the hypoglossus are more appropriate to a surgical and neurological exposition, it is perhaps well to point out that, according to the experience of F. KÖNIG, E. LEXER and L. WREDE¹⁾ the hypoglossus-anastomosis is preferable to the accessorius-anastomosis, as well with regard to the secondary paralyse, as to the associated movements. My object in writing this article is to give an insight into the way in which the recovery of function of the paralyzed right half of the face was effected after the plastic. The hypoglossus was cut through, and the central end was attached to the peripheral end of the facialis, which also had been cut through. Before the cutting it had been ascertained that the exposed nerve did not respond anymore to electric stimuli. This operation was performed in the latter part of November. Of course it occasioned a complete paralysis of the right half of the tongue, which was soon followed by atrophy and reaction of degeneration.

This status persisted some time. The facialis muscles remained unchanged. Not before May 1925 did some spontaneous twitches of the right eyelids appear, which on closer examination had to be considered as associated movements. Whenever the tongue was moved while the boy was speaking or eating, contractions were perceivable of the right half of the face. At first they were still faint, but gradually they increased in

¹⁾ BIER, BRAUN, KÜMMELL, *Chirurgische Operationslehre*, 1917. p. 456—459.

intensity and extent, while after some weeks the whole right half of the face was moving continually, when the boy ate or spoke. These associated movements were so strong that at times we doubted whether the operation had yielded any profit.

Other concomitant movements revealed themselves about the same time. Whenever the left eye was shut, the right eye was also closed; when the boy was laughing or weeping the right corner of the mouth moved together with the left one. These associated movements were hardly visible at first, but grew stronger every week. In this way a simultaneous cooperation of the two halves of the face was brought about.

Little by little the associated movements were further modified, for in January 1926 it became clear that when the patient was speaking or eating i.e. together with the movements of the tongue, the associated movements of the facial muscles were much weaker. The tongue could be put out without large contractions of the facial muscles making their appearance on the right side, and at length hardly any contractions were visible. In addition weak, voluntary movements could be made with the right corner of the mouth. This condition persisted up to now (April 1926). These associated movements recur only when the child gets very excited: when he gets angry or also when he is merry.

These three phenomena, i.e. the movements of the facial muscles on the right, accompanying the movements of the tongue, the recurrence of the simultaneous movement of either half of the face and the slow disappearance of the associated movements, accompanying the movements of the tongue (so that an increased dissociation of the movement of the tongue and the face is brought about as in normal cases) and the voluntary contraction of the right corner of the mouth, require further explanation.

It is easy to see that after regeneration of the hypoglossus that is attached to the facialis, the facialis-muscles will contract at every impulse originating from the hypoglossus-centrum. This will also be the case with movement of the non-paralyzed half of the tongue, because the two halves of the tongue will always cooperate. The action of the one hypoglossus-centrum will of necessity activate the other.

It is not so easy to explain the recurrence of the simultaneous cooperation of the facial muscles on the left and on the right side. As a rule the two halves cooperate. There is a regular association between the motor impulses on the right and on the left. But in our case the impulses for the left half of the face cannot excite simultaneous movements for the right half, because the nerve that is to transmit those impulses to the right half, is paralyzed. If in spite of this there is really cooperation between the two halves it must be accomplished along other paths. To render this conceivable I point to the marked association that always exists between

movements of the tongue and those of the face. When the tongue is put out, the mouth-aperture is widened, the naso-labial folds are getting more conspicuous, sometimes the forehead is wrinkled, frequently the eyes are closed or opened wide. The converse association also exists. Contractions of the muscles of the face are accompanied by movements of the tongue. When the eyes are fast shut, when the forehead is extremely wrinkled, or when the mouth is forcibly widened, movements of the tongue can distinctly be observed in the open mouth. This is noticeable above all in children, and in grown-ups with a lively face-play. Our knowledge of these associated relations affords an explanation of the cooperation of the two halves of the face of our little patient. The motorial impulses, arising from the right facialis-centrum, and moving the left half of the face activate the right hypoglossus-centrum, by which, as stated above, also the activity of the left one is brought about. The impulses that arise here, will produce a contraction of the right facial muscles. This then explains that a bilateral cooperation of the muscles of the face occurs. This instance shows that under some conditions the neurologist can turn to account psychological ideas. If it were not known that anatomical connections exist between the hypoglossus-centra inter se, and between the facialis- and the hypoglossus-centra, the above psychological speculations would entitle us to say with certainty that they must exist and that an elaborate experimental procedure will most probably bring them to light.

The third phenomenon, viz. the slow disappearance of the associated movements of the facial muscles accompanying the movements of the tongue, and the voluntary contraction of the right half of the face, also requires further explanation. A dissociation occurs, the contractions of the face become more dissociated from the movement of the tongue. The impulses from the right hypoglossus-centrum are utilized for the muscles of the face.

Analogous phenomena occur in the recovery of function, which is observed in some physiological experiments.

KENNEDY attached the nerve that innervates the extensors in a dog's foreleg to the nerve supplying the flexors, and the reverse. After regeneration recovery of function was effected by some exercise. When after trepanation the extensor-centrum in the cortex was stimulated the leg was flexed, and the reverse. Here, then, a transformation of the function of the centra takes place, for the animal has learned by practice that an impulse towards the extensors is needed for flexion of the leg. This is quite in keeping with our case; the child has learned to move the muscles of the face by a stimulus towards the tongue.

OSBORNE and KELVINGTON¹⁾ have repeated these experiments in a slightly different way. After severing one of the main cord of the left plexus brachialis in a dog they attached the central end to the distal end

¹⁾ W. A. OSBORNE and BASIL KELVINGTON. *Journal of Physiology*, 1908

of the corresponding severed cord of the right plexus with due precaution to render only regeneration of the attached cords possible. After regeneration the animal learned by practising to use his foreleg again. When after trepanation the right hemisphere was stimulated the muscles of the left leg, which were innervated by the intact cords of the left plexus and also a movement of those muscles of the right leg which were innervated by the transplanted cord of the left plexus, could be seen. This goes to show that those movements are induced in the right leg by a motorial impulse towards the left leg. The dog has gradually learned this from experience, by observing sensations of feeling and here also by eye-sensation, just as in our case exercise has brought about the transformation of the hypoglossus-centrum.

The results of these physiological experiments and those of hypoglossus-anastomosis are to a great degree indicative of the development of the voluntary movements of the child. The neonatus shows only spontaneous movements and reflex-movements, from which voluntary movements gradually develop. The child is aware of every muscle-sensation, and these experiences will vary with the difference in duration, intensity and extent of the movement. Of every sensitive impression some trace is left behind, which belongs to a definite impulse. By way of those sensations the child will learn to find through exertion the right impulse for a wished for movement.

The great influence of sensitive impressions upon movements appears first of all from the fact that embryologically the sensitive paths are sooner developed than the motor ones, and secondly from the excellent comparative anatomical research by ARIENS KAPPERS¹⁾, which proves that the location of the sensitive paths and nuclei is constant, whereas that of the motor ones is variable and determined by the corresponding sensitive paths. Furthermore we must assume that an association appears between the sensations of the contraction of cooperating muscles and those of symmetrical groups of muscles. This enables us to imitate at will the attitude of the arm or leg on the opposite side. In the same way when the muscles of the face are in function a contraction will arise of equal intensity on both sides.

Just as in the child the voluntary movements will originate from the spontaneous movements and the reflex-movements, the animals of the above physiological experiments will learn through exercise and apprehension to alter undesigned movements into intentional movements. Our little patient thus apprehends that for every movement of the face a definite impulse starting from the hypoglossus-centrum is required. By dint of exercise harmony is established in the movement of the two halves of the face.

There is still another similarity with the development of voluntary

¹⁾ ARIENS KAPPERS. *Folia Neurobiologica*. Bnd. I.

movements. The training for new intentional movements is first accompanied by a number of superfluous associated movements, e. g. for cycling, writing etc. These associated movements are after some time inhibited by the concentration of the energy upon the voluntary movements. They are not annulled, however, since they will recur when the said concentration is abated in one way or other. This becomes evident from the awkwardness in emotional conditions, such as fear or anger, or from atactic writing, and from the accompanying movements of lips and face of some demented patients. In the same way the associated movements of the muscles of the face are in our patients not annulled but only inhibited, for they will recur directly with full vigour when the boy gets excited or angry or merry.

Now, how do the sensations of feeling arise that direct the motor impulses and are on that account of great significance for the voluntary movements?

Signal data on this point have been afforded by the very minute histological investigation by J. BOEKE ¹⁾. After SHERRINGTON had demonstrated already some years ago that severing of the motor-nerves of the eye-muscles resulted in degeneration of sensitive end-bodies in the tendons, BOEKE established that the trochlearis and the abducens possess also centripetal fibres. Severing of these nerves brought about a degeneration of the sensitive end-bodies in the muscles and in the tendons. In the loose connective tissue in and round about the muscle a few sensitive end-bodies could still be found, which do not degenerate after the nerves of the eye-muscles are cut through, and whose impulses are probably directed to the centrum along the trigeminus. A similar condition was found by BOEKE for the hypoglossus. Most likely that of the facialis, with which we have to do here, does not differ much from it. Furthermore it is known that also in the joints sensitive nerves are lodged, which are stimulated when active or passive movements take place, and that the skin round about a point is more or less strained during a movement by which also stimuli run to the centrum. We are, therefore, safe to assume that sensation of movement can originate in different ways, viz. along the motor nerves and the sensitive nerves of the muscle and along the sensitive nerves of skin and joints. Also visual impressions can promote an accurate cooperation of the muscles. So there is a complex of causes for tactile sensation which is hard to disentangle. Most probably there are associations between those sensitive sensations, so that the one can compensate the other.

Little is known of each of these sensations. It would seem to me, that the clinic is competent to throw more light upon it. Sensation of movement can be established e.g. by moving the extremities in the joints passively and by recording the slighted movement observable. Another method

¹⁾ J. BOEKE. Studien zur Nervenregeneration. Verhandelingen der Kon. Akad. v. Wetensch. Deel XIX, N^o. 5.

might be employed which has been suggested by CURSCHMANN¹⁾. First the threshold value of the cathodal closure contraction is determined in m.A: and subsequently the threshold value at which the contraction is observed. These two values are under normal conditions nearly equal. In the same way the threshold value of the movement and of the sensation of it in the joints can be estimated.

A number of experiments on peripheral facialisparalysis showed me that the sensation of contraction appears a little later here than the minimal movement. It is probable, therefore, that the facialis conducts also centripetal paths.

However, the awareness of movement is also experienced along the trigeminus. This view is corroborated by a case in which the ganglion Gasseri was extirpated because of violent trigeminus neuralgia. This brought on complete insensibility of the right half of the face and of the right half of the tongue. On examining the muscular sense there appear to be considerable disturbances. On showing the teeth the right half of the face is moved less than the left half. It is evident that the harmony between the movement of the muscles of the face on the right and on the left is diminished. This is also borne out by the following experiment. When I ask patient to raise a little first the left corner of the mouth and then do the same with the right corner, the latter movement is much stronger. Are we perhaps to ascribe this to the fact that the same muscular sense is aroused on the left, only with a much stronger contraction? On applying galvanic stimuli it also appeared that the muscular contraction was felt on the right only when it was much stronger than on the left. Likewise on the right only a stronger electric contraction of the tongue is observed, whereas on the left a very weak, hardly perceptible movement is already apprehended.

¹⁾ CURSCHMANN. Neurol. Centralblatt, 1915.

Mathematics. — "On RIEMANNIAN Geometries admitting an absolute parallelism." By E. CARTAN and J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of April 24, 1926).

We will say that a RIEMANNIAN geometry admits an absolute parallelism, if it is possible to define the parallelism of two directions in two different points in a manner, which is absolute (viz independent of the choice of the coordinates) and satisfies the following conditions:

1. A geodesic is in all her points self-parallel;
2. The angle between two different directions in an arbitrary point P is equal to the angle between the two parallel directions in another arbitrary point Q .

By the thus defined parallelism a connexion arises, which

1. leaves invariant the tensor $g_{\lambda\mu}$
2. possesses the same geodesics as the given RIEMANNIAN connexion
3. has zero curvature.

This new connexion is not necessary symmetrical. In such a RIEMANNIAN geometry evidently through every non-singular point not situated on a given geodesic one and only one geodesic may be drawn, which is in each of her points parallel to the given one.

In a previous paper we proved, that with every simple or semi-simple group such a RIEMANNIAN geometry corresponds and that the geometries corresponding with simple groups admit two different absolute parallelisms. The most simple case is the geometry of the elliptical S_3 , the two parallelisms being those of CLIFFORD.

We will prove presently that, supposing the fundamental form definite, there exists besides these geometries corresponding with the mentioned groups only one other geometry with the designed property and that this geometry is in close connexion with the non-associative number-system of GRAVES—CAYLEY.

§ 1. *Fundamental relations.*

A connexion with the same geodesics as the given RIEMANNIAN geometry, has parameters of the form

$$\bar{I}_{\lambda\mu}^{\nu} = I_{\lambda\mu}^{\nu 0} + p_{\lambda} A_{\mu}^{\nu} + p_{\mu} A_{\lambda}^{\nu} + S_{\lambda\mu}^{\nu} \dots \dots \dots (1)$$

where p_{λ} is an arbitrary vector and $S_{\lambda\mu}^{\nu}$ an arbitrary in $\lambda\mu$ alternating affiner and $I_{\lambda\mu}^{\nu 0}$ are the parameters of the RIEMANNIAN geometry.

If we postulate

$$\bar{\nabla}_{\omega} g_{j\mu} = 0 \dots \dots \dots (2)$$

it follows immediately

$$p_{\lambda} = 0 \dots \dots \dots (3)$$

$$S_{j\mu\nu} = S_{j\mu\nu} \dots \dots \dots (4)$$

$S_{j\mu\nu}$ is therefore a *trivector*. For the quantity of curvature of the new connexion it follows from (1)

$$0 = R_{\omega\mu\lambda}^{\nu} = K_{\omega\mu\lambda}^{\nu} + 2 \nabla_{[\omega} S_{\mu] \lambda}^{\nu} - 2 S_{\alpha [\omega}^{\nu} S_{\mu] \lambda}^{\alpha} \dots \dots (5)$$

and from (5) after some calculations

$$K_{\omega\mu\lambda}^{\nu} = 1/3 S_{\alpha\mu\lambda} S^{\alpha}_{\nu\omega} - 2/3 S_{\alpha\lambda\nu} S^{\alpha}_{\omega\mu} + 1/3 S_{\alpha\nu\omega} S^{\alpha}_{\mu\lambda} \dots \dots (6)$$

$$S_{\omega\mu\lambda}^{\nu} = \overset{0}{\nabla}_{\omega} S_{\mu\lambda}^{\nu} = 1/3 S_{\alpha\mu\lambda} S^{\alpha}_{\nu\omega} + 1/3 S_{\alpha\lambda\nu} S^{\alpha}_{\omega\mu} + 1/3 S_{\alpha\nu\omega} S^{\alpha}_{\mu\lambda} \dots (7)$$

From this latter equation it follows that $S_{\omega\mu\lambda}^{\nu}$ is a *quadrivector*. By differentiation we deduce from (6) and (7), using the identity

$$\overset{0}{\nabla}_{\nu} K_{\omega\mu\lambda}^{\nu} = 2 \nabla_{[\omega} K_{\mu] \lambda}^{\nu)} \dots \dots \dots (8)$$

the equation

$$\overset{0}{\nabla}_{\omega} K_{\mu\lambda} = 0, \dots \dots \dots (9)$$

§ 2. *The case of constant real p-directions.*

If a real p -direction is constant by the (0)-connexion, then the orthogonal $(n-p)$ -direction is constant also and it is well-known that V_n is composed by $\infty^{n-p} V_p$ which are totally geodesic and mutually (0)-parallel and also in $\infty^p V_{n-p}$ with the same properties orthogonal to the V_p . The trivector is the sum of two trivectors $S'_{j\mu\nu}$ and $S''_{j\mu\nu}$ which are entirely situated in the V_p and V_{n-p} resp. and the transformation (1) is composed by a geodesic transformation (viz. a transformation which leaves the geodesics invariant) of every V_p in itself, given by $S'_{j\mu\nu}$ and an analogous transformation of every V_{n-p} in itself, given by $S''_{j\mu\nu}$. Indeed, $K_{\omega\mu\lambda}^{\nu}$ is divided, as is well-known, into two parts, which are situated entirely in V_p and V_{n-p} resp. If therefore v^{ν} and w^{ν} are two vectors lying entirely in V_p and V_{n-p} resp., then from (6) it follows:

$$0 = v^{\omega} w^{\mu} v^{\lambda} w^{\nu} K_{\omega\mu\lambda}^{\nu} = w^{\mu} v^{\lambda} S_{\alpha\mu\lambda}^{\nu} w^{\nu} v^{\omega} S_{\nu\omega}^{\alpha} \dots \dots (10)$$

The real vector $w^{\mu} v^{\lambda} S_{\alpha\mu\lambda}^{\nu}$ is therefore zero, whence it follows that $S_{\alpha\mu\lambda}^{\nu}$ is divided in the described manner.

1) R. K. p. 168.

In consequence of this proposition the case, where V_p admits constant real p -directions is reduced to the case, where these p -directions do not exist. We will therefore suppose in the following, that there exist no constant real p -directions.

§ 3. $K_{\mu i} = cg_{\mu i}$.

From this supposition it follows that $K_{\mu i}$ must be equal to $g_{\mu i}$, but for a constant factor. Indeed, if this were not true, the principal regions of $K_{\mu i}$ would define constant p -directions in consequence of (9). The case $K_{\mu i} = 0$ is to be excluded immediately, for in consequence of (6) we would have $S_{\alpha\beta\gamma} S^{\alpha\beta\gamma} = 0$ which is by a real trivector only possible for $S_{i\alpha\beta} = 0$, giving the trivial case $K_{\alpha\mu i\alpha} = 0$. We have therefore

$$K_{\mu\lambda} = cg_{\mu\lambda} ; \quad c = \text{constant} \neq 0. \quad \dots \dots \dots (11)$$

or

$$S_{\alpha\beta\lambda} S^{\alpha\beta\mu} = -cg_{\lambda\mu}. \quad \dots \dots \dots (12)$$

§ 4. BIANCHI's identity.

Applying BIANCHI's identity on (6), we obtain

$$0 = \nabla_{[\xi} K_{\alpha\mu|\beta\gamma]} = S_{\beta\alpha\gamma|\mu} S^{\alpha\gamma|\xi]} S^{\beta}_{\nu} + S_{\beta\alpha\gamma} S^{\alpha}_{\nu|\mu} S^{\beta\xi}_{\nu]} - S_{\beta\alpha\nu} S^{\alpha}_{\gamma|\mu} S^{\beta\xi}_{\nu]} \quad (13)$$

From $S_{i\alpha\beta}$ we derive the covariants

$$\left. \begin{aligned} a) \quad cg_{\lambda\mu} &= S_{\alpha\lambda}^{\beta} S_{\beta\mu}^{\alpha} \\ b) \quad g_{\lambda\mu\nu} &= S_{\alpha\lambda}^{\beta} S_{\beta\mu}^{\gamma} S_{\gamma\nu}^{\alpha} \end{aligned} \right\} \dots \dots \dots (14)$$

etc.

and we remark that all these covariants admit cyclical permutation of the indices. From (13) it follows

$$c S_{i\alpha|\mu} S^{\alpha}_{\nu|\xi]} = -2 g_{i\alpha|\mu} S^{\alpha}_{\nu|\xi]} \quad \dots \dots \dots (15)$$

By transvection of this equation with $S^{\xi\lambda\gamma}$ and $S^{\alpha\mu\xi}$ arises

$$2 cg_{\mu\nu\alpha} - c^2 S_{\mu\nu\alpha} = 4g_{|\alpha i|\beta|}^{\delta} + 2g_{i\alpha\beta} S^{\alpha\beta\gamma} S_{\gamma\mu\nu} \quad \dots \dots \dots (16)$$

and

$$2 cg_{\mu\nu\alpha} - c^2 S_{\mu\nu\alpha} = 4g_{|\alpha i|\beta|}^{\delta} + 2 cg_{\mu\nu\alpha} \quad \dots \dots \dots (17)$$

hence

$$g_{i\alpha\beta} S^{\alpha\beta\gamma} S_{\gamma\mu\nu} = -cg_{i\mu\nu} \quad \dots \dots \dots (18)$$

Now it follows from (14b) by differentiation

$$\nabla_{\alpha} g_{i\mu\nu} = -\frac{c}{2} \nabla_{\alpha} S_{i\mu\nu} \quad \dots \dots \dots (19)$$

and therefore

$$\nabla_{\alpha} g_{i\alpha\beta} S_{\mu}^{\alpha\beta} = \frac{c^2}{2} \nabla_{\alpha} g_{i\mu} = 0 \quad \dots \dots \dots (20)$$

Consequently $g_{i\alpha\beta} S_{\rho}^{\alpha\beta}$ is a tensor, which can differ from $g_{i\rho}$ only by a constant factor. Writing for this factor $-c\varrho$, $\varrho = \text{constant}$, then it follows by substitution in (18)

$$g_{i\mu\nu} = \varrho S_{i\mu\nu} \dots \dots \dots (21)$$

Substituting this value in (15), it appears, that two cases are possible, either

$$S_{i\alpha\beta} S_{\rho}^{\alpha\beta} = 0 \dots \dots \dots (22)$$

or

$$c = -2\varrho \dots \dots \dots (23)$$

From (7) in relation with (22) we have

$$\nabla_{\alpha} S_{\lambda\mu\nu} = \overset{0}{\nabla_{\alpha}} S_{\lambda\mu\nu} = 0, \dots \dots \dots (24)$$

hence the first case gives again the geometries for which $c = +2\varrho$ mentioned above and treated in our previous paper.

For both cases, $c = \pm 2\varrho$, the following relations hold good:

$$\left. \begin{aligned} a) \quad \nabla_{\xi} K_{\alpha\mu\nu} &= 0 \\ b) \quad \nabla_{\xi} S_{\alpha(\lambda|\nu)} S_{\rho)^{\alpha}} &= 0 \\ c) \quad (c - \varrho) K_{\alpha\mu\nu} &= g_{\alpha\mu\nu} - g_{\nu\mu\alpha} \\ d) \quad 9 S_{\alpha\mu\nu} S_{\rho}^{\alpha\beta} &= -(c - 4\varrho) S_{\alpha\mu\nu} S_{\rho}^{\alpha\beta} - 4 g_{\alpha\mu\nu} S_{\rho}^{\alpha\beta} \end{aligned} \right\} \dots (25)$$

which are all a consequence of BIANCHI's identity.

We have therefore as yet to prove, that the case $c = -2\varrho$ leads to the elliptic geometry in S_7 . For this it is necessary to use some propositions of the theory of groups.

§ 5. *The group of the RIEMANN-CHRISTOFFEL affinor.*

Just as in the previous communication we make use of a system of measure-vectors $e^k, k = 1, \dots, n$, that is constant by the connexion (-). To two surface-elements in two different points, each defined by two directions, which have in both points the same coordinates x^k, y^k with reference to the system e^k , corresponds the same RIEMANNIAN curvature, defined analytically by the form

$$R = K_{ijkl} x^i y^j x^k y^l.$$

V_n admits a translation, whereby every point moves along a linear element, given by a (-)-constant vector¹⁾. By this translation the change of the vector x^k is given by

$$dx^k = 2 S_{ij}^k x^i d\xi^j \dots \dots \dots (26)$$

¹⁾ It may easily be proved that the problem treated above is equivalent to the determination of the V_n , which admit n infinitesimal isogonal translations, not situated in one $(n-1)$ -direction.

Since the RIEMANNIAN curvature of a surface-element does not change by this translation, it follows easily, that the form is invariant by all infinitesimal transformations

$$X_i f = S_{ij}^{i,k} x^j \frac{\partial f}{\partial x^k}; Y_i f = S_{ij}^{i,k} y^j \frac{\partial f}{\partial y^k} \dots \dots \dots (27)$$

applied simultaneously on the vectors x^k and y^k .

The transformations (27), whose coefficients because of $c = -2\varrho$ change by the transition from one point of V_n to another, form no group. But they are contained in the group I' of all rotations which leave the form R invariant. Especially the alternated combinations $(X_i X_j)$ belong to this group¹⁾:

$$(X_i X_j) = (S_{ik}^{i,\lambda} S_{j\lambda}^{i,l} - S_{jk}^{i,\lambda} S_{i\lambda}^{j,l}) x^k \frac{\partial f}{\partial x^l} \dots \dots \dots (28)$$

The differential

$$dx^k = S_{ij}^{i,k} x^i d\xi^j$$

obtained by displacing the vector x^k (0)-parallel, is half the differential (26), whence it follows, that the RIEMANNIAN curvature is invariant also by this displacement. Hence the V_n belongs to an important class of V_n , viz those in which the curvature is invariant by pseudo-parallel displacements.

Particularly the rotation, corresponding with a surface-element, will belong to the group I' . This is confirmed by calculation. With a surface-element with the coordinates p^{ij} corresponds the rotation

$$\left. \begin{aligned} p^{ij} K_{ij}^{i,l} x^k \frac{\partial f}{\partial x^l} &= p^{ij} (1/3 S_{kj}^{i,\lambda} S_{i\lambda}^{j,l} - 2/3 S_{ji}^{i,\lambda} S_{k\lambda}^{j,l} + \\ &+ 1/3 S_{ik}^{i,\lambda} S_{j\lambda}^{i,l}) x^k \frac{\partial f}{\partial x^l} = 2/3 p^{ij} S_{ij}^{i,k} X_k f - 1/3 p^{ij} (X_i X_j). \end{aligned} \right\} (29)$$

Hence it follows that the group of holonomy of V_n is the group I' or one of his subgroups²⁾.

§ 6. *The group I' leaves invariant no p -direction.*

It has already been proved in § 2 that there cannot exist a real by I' invariant p -direction. If there were an invariant imaginary p -direction, the conjugate imaginary p -direction and the orthogonal $(n-p)$ -direction would also be invariant. Hence it would follow, that there were an invariant real q -direction, except in the case $n = 2p$ and that the p -direction were totally isotropical. In this latter supposition we choose the system in such a way, that

$$2 x^1 x^2 + \dots + 2 x^{n-1} x^n$$

¹⁾ It may be verified easily that X_i and $(X_i X_j)$ does form a group, but we will not use this property.

²⁾ E. CARTAN, Ann. Ec. Norm. 3,42 (25) p. 21. It may be proved, that I' is the group of holonomy itself.

is the fundamental form, the given p -direction is defined by

$$x^1 = x^3 = \dots = x^{n-1} = 0$$

and the conjugate p -direction by

$$x^2 = x^4 = \dots = x^n = 0.$$

Every infinitesimal transformation of I' contains then only coefficients a_i^{j+1}), whereby $i+j$ is even and therefore only coefficients a_{ij} whereby $i+j$ is odd. One of the S_{ijk} can therefore only then be equal to zero, if not only $i+j$, but also $j+k$ and $k+i$ are odd.

We find therefore $S_{ijk} = 0$.

§ 7. *The group I' is simple.*

Every linear group which leaves no p -direction invariant is either simple or semi-simple²⁾.

In the latter case I' may be obtained from two simple or semi-simple groups γ_1 and γ_2 in q variables x^1, \dots, x^q , resp. s variables y^1, \dots, y^s , which leave invariant no p -direction. The group is then the group of transformations of the $n = qs$ products $z^{(i\alpha)} = x^i y^\alpha$, $i = 1, \dots, q$, $\alpha = 1, \dots, s$ by the transformations of γ_1 and γ_2 . Every transformation of I' is of the form

$$a_i^j z^{(i\lambda)} \frac{\partial f}{\partial z^{(j\lambda)}} + b_x^{\beta} z^{(k\alpha)} \frac{\partial f}{\partial z^{(k\beta)}}$$

Consequently in every vanishing coefficient $A_{(i\alpha)}^{(j\beta)}$ of an infinitesimal transformation of I' either $i=j$ or $\alpha = \beta$, moreover, if $i \neq j$ then we have

$$A_{(i\alpha)}^{(j\alpha)} = A_{(i\beta)}^{(j\beta)}$$

for every value of λ .

The group I' being a group of rotations, it leaves invariant a non-degenerate quadratic form $\Phi(z^{(i\alpha)})$. Two cases are possible, either Φ vanishes identically or Φ don't vanish. In the first case by giving to the y^α fixed values, we obtain a quadratic form $f(x^i)$, invariant by γ_1 , which is determined uniquely but for a constant factor and is not degenerate, since otherwise γ_1 would leave invariant a p -direction. In the same manner a quadratic form $\eta(y^\alpha)$ may be obtained, so that $\Phi(x^i y^\alpha) = f(x^i) \eta(y^\alpha)$. We may therefore suppose, that Φ has the form

$$\Phi = \sum_{i, \alpha} (z^{(i\alpha)})^2 \dots \dots \dots (30)$$

1) By coefficients of the infinitesimal transformation $e^k X_k f$ we mean the n^2 expressions $e^k S_k^{i,j}$, $i, j = 1, \dots, n$.

2) To be compared by this and the following §§. E. CARTAN, Les groupes projectifs, qui ne laissent invariante aucune multiplicité plane. Bull. Soc. Math. 41 (13) p. 53—96.

In the second case, setting

$$\Phi(z^{(i\alpha)}) = g_{(i\alpha)(j\beta)} z^{(i\alpha)} z^{(j\beta)}$$

it follows

$$g_{(i\alpha)(j\beta)} + g_{(i\beta)(j\alpha)} = 0.$$

The bivector $a_{ij} = g_{(i\alpha)(j\beta)}$, in which expression to α and β fixed values are to be given, is therefore invariant by γ_1 and so is by γ_2 the bivector $b_{\alpha\beta} = g_{(i\alpha)(j\beta)}$, in which i and j have fixed values. These two bivectors are uniquely determined but for constant factors and non-degenerate, for otherwise γ_1 resp. γ_2 would leave invariant a p -direction. The constant factors may be chosen so that for Φ holds good

$$g_{(i\alpha)(j\beta)} = \bar{a}_{ij} b_{\alpha\beta}.$$

Hence it may be supposed that Φ has the form:

$$\left. \begin{aligned} \Phi &= \sum_{i, \alpha} z^{(2i-1, 2\alpha-1)} z^{(2i, 2\alpha)} - z^{(2i-1, 2\alpha)} z^{(2i, 2\alpha-1)}, \\ i &= 1, \dots, \frac{q}{2}; \quad \alpha = 1, \dots, \frac{s}{2}. \end{aligned} \right\} \dots \dots (31)$$

Now we will prove that the cases (30) and (31) cannot occur. In the case (30) we have

$$S_{(i\alpha)(j\beta)(k\gamma)} = S_{(i\alpha)(j\beta)}^{(k\gamma)}.$$

If the coefficient $S_{(i\alpha)(j\beta)(k\gamma)}$ is unequal to zero, then either $j = k$ or $\beta = \gamma$, likewise either $i = j$ or $\alpha = \beta$ and either $i = k$ or $\alpha = \gamma$, therefore either $i = j = k$ or $\alpha = \beta = \gamma$. Now supposing f.i. $i = j = k$ and $\alpha \neq \beta$, then we have for every value of k

$$S_{(i\alpha)(i\beta)(i\gamma)} = S_{(k\alpha)(k\beta)(i\gamma)}.$$

For $i \neq k$ the condition mentioned above is not satisfied on the right side of this equation and the trivector S must therefore be equal to zero.

In the case (31) we have

$$S_{(i\alpha)(j\beta)(k\gamma)} = (-1)^{k+\gamma} S_{(i\alpha)(j\beta)}^{(k'\gamma')}$$

where

$$k' = \begin{cases} k + 1, & k \text{ odd} \\ k - 1, & k \text{ even} \end{cases}; \quad \gamma' = \begin{cases} \gamma + 1, & \gamma \text{ odd} \\ \gamma - 1, & \gamma \text{ even} \end{cases}.$$

Hence if the coefficient $S_{(i\alpha)(j\beta)(k\gamma)} \neq 0$ we have

$$j = k' \text{ or } \beta = \gamma'; \quad i = k' \text{ or } \alpha = \gamma'; \quad i = j' \text{ or } \alpha = \beta'.$$

Now supposing f.i. $i = j = k'$ with $\alpha = \beta'$, then we have for every value of λ

$$S_{(i\alpha)(i\alpha')(i'\gamma)} = (-1)^{\alpha+\lambda} S_{(i\lambda)(i\lambda')(i'\gamma)}.$$

For $\lambda \neq \gamma$ we have for every value of k

$$S_{(i\lambda)(i\lambda')(i'\gamma)} = (-1)^{i+k} S_{(i\lambda)(k\lambda')(k'\gamma)}.$$

Taking $k \neq i$, we must have $\lambda = \gamma'$, i.e. every index λ different from

γ , must be equal to γ' . This however is only possible if $s=2$. If $\lambda=\gamma'$ then we have, supposing $k \neq i$,

$$S_{(i\gamma') (k\gamma) (k'\gamma')} = -S_{(i\gamma) (k\gamma) (k'\gamma')}$$

from which it follows $k=i'$. From this we derive that $q=2$ also.

The only possible semi-simple group is therefore the orthogonal group in four variables. The following developments suppose Γ to be simple. They would however be solid also for the orthogonal group in 4 variables. We may however easily verify, that the case $n=4$ cannot occur. Indeed for $n=4$ it follows from (7) that $S_{\gamma_1\gamma_2\gamma_3}=0$, by which we return to the case $c=+2\varrho$.

§ 8. *The form R derived from the infinitesimal transformations of the group.*

Be

$$Z_l f = a_{ij}^k x^j \frac{\partial f}{\partial x^k} ; \quad l, j, k, l = 1, \dots, n$$

a system of r infinitesimal transformations with real coefficients of Γ . To each of these transformations we adjoin the bilinear form

$$\zeta_l = a_{jk} x^j y^k = 1/2 a_{ljk} p^{jk}$$

From the equation

$$(Z_l Z_j) = c_{ij}^k Z_k f \dots \dots \dots (32)$$

it follows that

$$Z_l \zeta_j = c_{ij}^k \zeta_k.$$

Now we introduce the fundamental tensor

$$G_{ij} = 1/2 a_{ijk} a_j^{jk} \dots \dots \dots (33)$$

and by using this tensor we obtain from c_{ij}^k the covariant components c_{ijk} . From (32) it follows that

$$a_{i\dot{i}}^k a_{jk\dot{i}} - a_{j\dot{i}}^k a_{ik\dot{j}} = c_{ij}^k a_{kij}$$

from which we obtain by transvection with $a_{\dot{l}}^{ij}$

$$-a_{i\dot{i}}^j a_{j\dot{j}}^k a_{k\dot{k}}^i = c_{ijk} \dots \dots \dots (34)$$

The c_{ijk} are therefore the components of a trivector.

The form

$$R = K_{ijkl} x^i y^j x^k y^l = 1/4 K_{ijkl} p^{ij} p^{kl}$$

is a quadratic form in ζ_1, \dots, ζ_r . Indeed it follows from (29)

$$\frac{1}{2} \frac{\partial R}{\partial p^j} = \frac{2}{3} S_{ij}^k \xi_k - 1/3 \xi_{ij} \dots \dots \dots (35)$$

where ξ_k and ξ_{ij} are forms which correspond to the infinitesimal trans-

1) For this introduction it is evidently necessary that g_{ij} be known.

formations $X_i f$ (27) and $(X_i X_j)$ (28). Now we deduce the general form of a quadratic form $R(\zeta_j)$, invariant by the group I' . If

$$R(\zeta_j) = A^{IJ} \zeta_I \zeta_J$$

then we have

$$Z_I(R) = A^{JK} \zeta_K c_{IJ}{}^L \zeta_L$$

and therefore

$$C_{IJ}{}^L A^{JK} + C_{IJ}{}^K A^{JL} = 0.$$

These relations express that the adjoint group of I' , which is generated by (est engendré par, erzeugt wird durch) the transformations

$$E_I f = c_{ij}{}^k e^j \frac{\partial f}{\partial e^k},$$

leaves invariant the form $A_{IJ} e^I e^J$. Consequently, I' being simple and the adjoint group leaving invariant the form $G_{IJ} e^I e^J$, we have

$$A_{IJ} = h G_{IJ}$$

therefore

$$R(\zeta_j) = h G^{IJ} \zeta_I \zeta_J = h \zeta^I \zeta_I. \dots \dots \dots (36)$$

From this equation it follows that

$$K_{ij}{}^{ij} = 2hr = -nc; \quad h = -\frac{nc}{2r} \dots \dots \dots (37)$$

§ 9. The order r of I' is equal to $3n$.

Taking for the infinitesimal transformations of I' the n transformations $X_i f$ and $r-n$ other independent ones, then we have for $i, j, k \leq n$ in consequence of (12)

$$G_{ij} = 1/2 S_{ik3} S_j{}^{3k} = -1/2 c g_{ij} \dots \dots \dots (38)$$

On the other hand it follows from (35) and (36) that

$$\frac{1}{2} \frac{\partial R}{\partial p^{ij}} = h \zeta_I \frac{\partial \zeta^I}{\partial p^{ij}} = \frac{2}{3} S_{ij}{}^k \zeta_k - 1/3 c_{ij}{}^l \zeta_l.$$

Hence

$$h \frac{\partial \zeta^I}{\partial p^{ij}} = \frac{2}{3} \epsilon_I S_{ij}{}^I - 1/3 c_{ij}{}^I \zeta_I \quad \epsilon_I = \begin{cases} 1, I \leq n \\ 0, I > n \end{cases}$$

$$h \frac{\partial \zeta_j}{\partial p^{ij}} = \frac{2}{3} G_{ik} S_{ij}{}^k - 1/3 G_{ij} c_{ij}{}^I \zeta_I.$$

By (38) we have

$$h \frac{\partial \zeta_k}{\partial p^{ij}} = h S_{kij} = -1/3 c S_{ijk} - 1/3 c_{ijk}.$$

hence

$$c_{ijk} = -(c + 3h) S_{ijk} \dots \dots \dots (39)$$

Now it follows from (34) that $c_{ijk}, i, j, k \leq n$, is identical with the quantity g_{ijk} defined by (14) so that we have derived in another way once more the identity (21)

$$g_{ijk} = \varrho S_{ijk} \dots \dots \dots (21)$$

obtaining at the same time as a new result

$$\varrho = -(c + 3h).$$

Now it was proved in § 4 that $c = \pm 2\varrho$. Hence from (37) follows that the only possibilities are

$$c = +2\varrho, \quad h = -\frac{c}{2}, \quad r = n$$

giving the V_n of the simple groups, and

$$c = -2\varrho, \quad h = -\frac{c}{6}, \quad r = 3n.$$

§ 10. *The group I' is for $c = -2\varrho$ the orthogonal group in 7 variables.*

Now rests only to examine for which types of simple groups, leaving invariant a non-degenerate quadratic form but no linear manifold, r may be equal to $3n$. Previously we remark that the roots of the characteristic equation of an orthogonal group are in pairs equal and opposite. These roots are called by CARTAN the *weights* of the group¹⁾. It may be remembered that every simple group, which leaves invariant no linear manifold, is entirely determined by her principal weight (poids dominant).

Type A. If the rank of the group is l , then the order is $r = l(l+2)$ and every weight is of the form

$$m_1 \omega_1 + \dots + m_{l+1} \omega_{l+1}$$

in which the sum of the rational coefficients m_i is zero. The difference between two coefficients, corresponding with the same weight or with two different weights is a whole number. Since with every weight corresponds an equal and opposite weight, the coefficients m_i are either all whole numbers or all half odd numbers. If the m_i are all whole numbers, the group contains all weights $\omega_i - \omega_j$ and 0 and the number of the variables is therefore at least equal to r , which is not possible if $r = 3n$. If the m_i are fractions, then l is odd and all weights of the form

$$\frac{1}{2}(\omega_1 + \dots + \omega_{\frac{l+1}{2}} - \omega_{\frac{l+1}{2}+1} - \dots - \omega_{l+1})$$

exist. Their number is therefore

$$\frac{(l+1)!}{\left(\frac{l+1}{2}\right)! \left(\frac{l+1}{2}\right)!} > \frac{l(l+2)}{3}$$

from which it would follow $n > \frac{r}{3}$.

¹⁾ Loc. cit. § 1, No. 3. We make use of the notations used there.

Type B. Here is $l=2$, $r=l(2l+1)$ and the weights are of the form

$$m_1 \omega_1 + \dots + m_l \omega_l$$

in which the m_i are whole numbers or half whole numbers. In the latter case is $n \equiv 2l > \frac{l(2l+1)}{3} = \frac{r}{3}$. In the first case the existence of a weight, which is not of the form $\pm \omega_i$, has as a consequence that all weights of the form $\pm \omega_i \pm \omega_j$ exist, whence it follows that $n \equiv 2l(l-1) > \frac{l(2l+1)}{2} = \frac{r}{3}$. There remains therefore only the case where the weights have the form $\pm \omega_i$. This case corresponds with the orthogonal group in $2l+1$ variables. From the identity $2l+1 = \frac{l(2l+1)}{3}$ it follows that $l=3$, so that the orthogonal group in 7 variables appears to be possible.

Type C. Here is $l \equiv 3$, $r=l(2l+1)$ and the weights have the form

$$m_1 \omega_1 + \dots + m_l \omega_l$$

in which the m_i are whole numbers. The group, for which the weights are $\pm \omega_i$, has $2l$ variables and $2l \neq \frac{l(2l+1)}{3} = \frac{r}{3}$; the other groups have at least r variables.

Type D. Here is $l \equiv 4$, $r=l(2l-1)$ and the weights have the form $m_1 \omega_1 + \dots + m_l \omega_l$. Is $l \equiv 5$, then the m_i may be half odd numbers, and we have $n \equiv 2^{l-1} > \frac{l(2l-1)}{3} = \frac{r}{3}$. If the m_i are whole numbers, then we have in the first place the system of weights that corresponds with $n = 2l \neq \frac{l(2l-1)}{3} = \frac{r}{3}$ and further other groups, for which $n \equiv r$.

The types *E, F, G* give groups for which $n > \frac{r}{3}$.

There is consequently but one possibility, I' is the group of all orthogonal transformations in seven variables. From the expression for R it follows that V_7 is an elliptical S_7 .

§ 11. *The absolute parallelism in S_7 .*

It is indeed very easy to indicate in S_7 an infinity of parallelisms which satisfy the prescribed conditions. In the projective space of 7 dimensions the absolute be defined by

$$x_0^2 + x_1^2 + \dots + x_7^2 = 0.$$

A point be given by 8 coordinates whose squares have the sum 1. The distributive, but not associative numbersystem of GRAVES-CAYLEY with the unities $1, e_1, \dots, e_7$ is given by the rules of multiplication

$$\begin{aligned}
 e_i^2 = -1; \quad e_i = e_{i+1} \quad e_{i+3} = -e_{i+3} \quad e_{i+1} = e_{i+2} \quad e_{i+6} = \\
 = -e_{i+6} \quad e_{i+2} = e_{i+4} \quad e_{i+5} = -e_{i+5} \quad e_{i+4} \quad ; \quad \left. \begin{aligned} e_i = e_{i+7} \\ i = 1, \dots, 7 \end{aligned} \right\} \quad (40)
 \end{aligned}$$

With the point x_0, \dots, x_7 corresponds the number

$$X = x_0 + \sum_i x_i e_i$$

with the module $\sqrt{x_0^2 + x_1^2 + \dots + x_7^2} = 1$.

\vec{XY} being an arbitrary segment of a geodesic and X' an arbitrary point, we will call the segment \vec{XY} and $\vec{X'Y'}$ aequipollent, if

$$Y' X'^{-1} = Y X^{-1}; (X^{-1} = x_0 - \sum x_i e_i) \dots \dots \dots (41)$$

Equalising the scalar parts of both members of (41) we obtain

$$x_0' y_0' + \dots + x_7' y_7' = x_0 y_0 + \dots + x_7 y_7$$

from which follows the equality of length of the segments \vec{XY} and $\vec{X'Y'}$. If X and X' are given, the relation between the y_i and the y_i' is linear and such, that with a vector in X corresponds a vector in X' with the same length.

The aequipollence is therefore conformal. The geodesics of S_7 are selfparallel. Indeed, if we put

$$Y X^{-1} = Z, \text{ whence } Y = Z X^{-1}$$

and if X' is situated on the geodesic \vec{XY} :

$$X' = \lambda X + \mu Y = \lambda X + \mu Z X$$

then we have

$$Y' = Z X' = \lambda Z X + \mu Z (Z X)$$

so that

$$\begin{aligned}
 &= \lambda Y + \mu (2z_0 - Z^{-1}) Z X \\
 &= (\lambda + 2\mu z_0) Y - \mu X.
 \end{aligned}$$

We get another absolute parallelism by putting

$$X'^{-1} Y' = X^{-1} Y \dots \dots \dots (42)$$

If generally A is an arbitrary fixed number of the numbersystem, then we have two families of absolute parallelisms, each depending from 7 parameters, by the equations

$$Y' (X'^{-1} A) = Y (X^{-1} A) \dots \dots \dots (43)$$

$$(A X'^{-1}) Y' = (A X^{-1}) Y \dots \dots \dots (44)$$

It may be foreseen that there exists an infinity of absolute parallelisms.

1) This result remains valid, although the multiplication is no longer associative.

Indeed, by the infinitesimal translation (26) two (—)-parallel vectors in two different points are transferred into two vectors which are no more (—)-parallel. The metrical properties of S_7 being invariant by translation, the (—)-parallelism is transformed into a (—)-parallelism different from the first, so that there exists a continuous family of (—)-parallelisms.

In S_7 there do not exist other absolute parallelisms than those defined by (43) and (44). Let us consider a determined absolute parallelism and a congruence of geodesics invariant by this absolute parallelism. We get a translation by moving every point on the corresponding geodesic over a distance equal for all points. In an S_n , wherein the equation of the absolute quadric is $x_1^2 + \dots + x_n^2 = 0$, every infinitesimal translation may be reduced to the form

$$x_1 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_1} + x_3 \frac{\partial f}{\partial x_4} - x_4 \frac{\partial f}{\partial x_3} + \dots$$

and the trajectories of the translation all intersect the two plane manifolds in the absolute

$$\begin{aligned} x_1 + ix_2 = x_3 + ix_4 = \dots = 0, \\ x_1 - ix_2 = x_3 - ix_4 = \dots = 0. \end{aligned}$$

For $n=7$ these two manifolds P_3 are situated on the absolute quadric, there exist two different families and two conjugate imaginary manifolds belong to the same family. The P_3 of the first family (P_3^+) are characterised by the property that their equations being written in the form

$$y_0 + iy_1 = y_2 + iy_3 = \dots = y_6 + iy_7 = 0$$

the determinant of the coefficients of x_0, \dots, x_7 in y_0, \dots, y_7 is positive. By the other family (P_3^-) this determinant is negative.

Reciprocally to every P_3 and his conjugate corresponds a determined congruence of geodesic lines, to which belongs a group of translations with one parameter.

An absolute parallelism may be obtained by choosing ∞^7 manifolds P_3 such that the corresponding congruences are isogonal. It may be proved, that in this manner no other absolute parallelisms are obtained than those which are defined by (43) and (44). The (+)- and (—)-parallelisms are obtained by means of manifolds from (P_3^+) resp. (P_3^-).

The points of S_7 , the (+)- and (—)-parallelisms may be considered as elements of S_7 . In the same way as we extend in the ordinary projective space the group of projective point-transformations by adjunction of correlations, the group of motions and reflexions in S_7 may be extended by adjunction of the four continuous families of transformations, which transform points in (+)- and (—)-parallelisms. In this manner we have in S_7 a *triatlity*¹⁾ by which it is possible to define the distance of two (+)- or two (—)-parallelisms, etc.

¹⁾ E. CARTAN, Bull. Sc. Math. 2, 49 (25) p. 361—371.

§ 11. *General conclusion.*

In a RIEMANNIAN geometry, in which the linear element is the sum of h linear elements, corresponding with geometries of finite groups and k linear elements, corresponding with the geometry of S_7 , there exist, for $k=0$, 2^h absolute parallelisms and, for $k > 0$, 2^{h+k} continuous families of ∞^{7k} absolute parallelisms.

This result remains valid, if a euclidian linear element of an arbitrary number of dimensions is added.

P.S. We remark, that an error has been made in the deduction of (10) in the first note, the linear element having really the opposite sign, hence

$$S_{ij}^{\prime\prime} = + \frac{1}{2} c_{ij}^{\prime\prime}$$

instead of $-\frac{1}{2} c_{ij}^{\prime\prime}$. Also on page 807 the linear element $s' dt$ corresponds with the transition from t^k into $t^k - s' t' c_{ij}^{\prime\prime k} dt$, hence in (18) $c_{ij}^{\prime\prime}$ ought to be substituted by $-c_{ij}^{\prime\prime}$. The error has had no serious consequences.

Physics. — “*Further measurements on the magnetic disturbance of the supraconductivity with tin and mercury*”. (Comm. N^o. 180 from the Physical Laboratory at Leiden). By W. J. DE HAAS and G. J. SIZOO.

(Communicated at the meeting of May 26, 1926)

§ 1. *Introduction.*

The measurements on the magnetic disturbance of the supraconductivity with tin ¹⁾ and mercury ²⁾, which have already been published in these Proceedings, had shown, that the magnetic transition lines, measured with decreasing and increasing magnetic fields, do not coincide, as formerly had been assumed, but do form a hysteresis figure.

Besides, it was observed that the transition lines with *mercury*, especially the descending one, show sharp discontinuities, whilst those with *tin* also showed indications of the existence of such discontinuities. The whole of the results led us to assume, that these discontinuities did not correspond to sudden changes in the specific resistance of the metal, but that every jump was due to the disappearance of the resistance of a definite part of the wire. More precisely we supposed, that because of the very slow cooling, in the mercury thread pieces of single crystals would be formed and that a jump would be due to the appearance of supraconductivity in such a single crystal.

We wish to communicate here some new measurements, which partly serve as a completion of those already published and partly may be considered as a confirmation of the given explanation of the discontinuities.

A. Tin.

§ 2. *The hysteresis figure in a longitudinal field.*

Our former measurements on the magnetic disturbance of the supraconductivity with tin, were made with extruded tin wires, which were wound in many turns on a glass tube. Here the direction of the magnetic field was perpendicular to the plane of the turns, and therefore was mainly transverse, that is perpendicular to the direction of the current. It was therefore desirable to measure also the hysteresis figure, for the case of a field, quite parallel to the direction of the current. This was done with the measurement of 17 March 1926.

¹⁾ These Proceedings 39, 221, 1926.

²⁾ These Proceedings 39, 233, 1926.

The resistance used *Sn-1926-1*, was made from a tin wire, extruded from "KAHLBAUM" tin, with a diameter of 0.17 mm. The wire was non-inductively wound along a mica strip of 5 cm length. The number of turns, which were separated by silk wire, amounted to 4. Therefore the part of the thread, for which the field was not quite longitudinal, was very small compared to the whole length.

At a temperature of $3^{\circ}.303$ K. the hysteresis figure was measured¹⁾. The results follow in table I and are represented in fig. 1. The measurement consisted of two series. The points of the two series combine

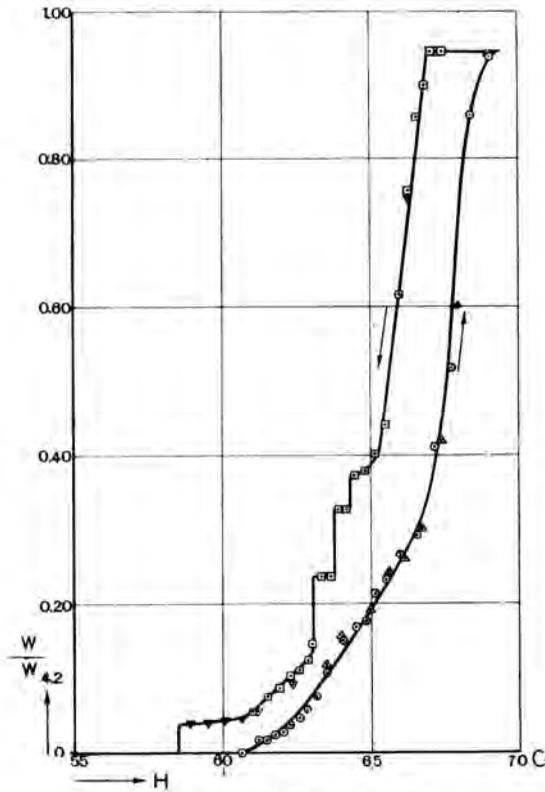


Fig. 1. *Sn-1926-1* (Table I).

very well to one figure. The figure has a peculiar form, recalling the figures obtained with a transverse field, but still a little different. The descending and the ascending line do not touch each other. Discontinuities are present, especially in the descending line. They are much more distinct than in the transverse field. This is in agreement with the supposition that they are due to the presence of single-crystals. When in an extruded tin wire such single crystals are present, they may easily be destroyed, by winding the wire on a thin glass tube, as was done

¹⁾ For the method of the measurements, reference may be made to former communications.

TABLE I. S_n -1926-I. Longitudinal. $P_{\text{helium}} = 264.6 \text{ mm Hg.}$ $T = 3^\circ.303 \text{ K.}$ Coil A. ¹⁾ $A_{\text{max.}} = 0.992.$ $A_{\text{min.}} = 0.990.$ $\frac{A_{\text{min.}}}{A_{\text{max.}}} = 0.991.$ $A_{\text{avg.}} = 0.991.$ $H_{\text{avg.}} \text{ (in gauss)} = 113.1 i \text{ (} i \text{ in amp.)}$

| Date | Current | $H_{\text{avg.}}$ in gauss | $W_{S_n-1926-I}$ | $\frac{W}{W_{4.2}}$ |
|---------------|---------|-------------------------------|------------------|---------------------|
| 17 March 1916 | 8 mA | 63.50 | 0.00021 | 0.119 |
| | | 64.01 | 0.00028 | 0.158 |
| | | 64.98 | 0.00035 | 0.193 |
| | | 65.60 | 0.00044 | 0.242 |
| | | 66.11 | 0.00048 | 0.266 |
| | | 66.73 | 0.00057 | 0.315 |
| | | 67.41 | 0.00076 | 0.421 |
| | | 67.75 | 0.00132 | 0.732 |
| | | 68.60 | 0.00163 | 0.904 |
| | | 69.06 | 0.00170 | 0.942 |
| | | 69.68 | 0.00170 | 0.942 |
| | | 68.09 | 0.00170 | 0.942 |
| | | 67.75 | 0.00170 | 0.942 |
| | | 67.41 | 0.00170 | 0.942 |
| | | 67.13 | 0.00170 | 0.942 |
| | | 66.85 | 0.00160 | 0.900 |
| | | 66.56 | 0.00155 | 0.858 |
| | | 66.28 | 0.00136 | 0.753 |
| | | 66.00 | 0.00111 | 0.617 |
| | | 65.54 | 0.00079 | 0.441 |
| | | 65.15 | 0.00073 | 0.403 |
| | | 64.81 | 0.00068 | 0.378 |
| | | 64.75 | 0.00068 | 0.375 |
| | | 64.13 | 0.00059 | 0.329 |
| 64.01 | 0.00059 | 0.329 | | |
| 63.67 | 0.00043 | 0.238 | | |
| 63.45 | 0.00043 | 0.238 | | |
| 63.05 | 0.00027 | 0.147 | | |
| 62.88 | 0.00023 | 0.126 | | |
| 62.60 | 0.0020 | 0.112 | | |

¹⁾ These Proceedings 39, 233, 1926.

TABLE I. *Sn-1926-I*. Longitudinal.
(Continued).

| Date | Current | $H_{avg.}$ in gauss | $W_{Sn-1926-I}$ | $\frac{W}{W_{4.2}}$ |
|---------------|---------|------------------------|-----------------|---------------------|
| 17 March 1926 | 6.7 mA | 63.32 | 0.00019 | 0.105 |
| | | 61.92 | 0.00016 | 0.088 |
| | | 61.52 | 0.00014 | 0.077 |
| | | 61.01 | 0.00010 | 0.056 |
| | | 48.55 | 0.00000 | 0.000 |
| | | 60.62 | 0.000000 | 0.000 |
| | | 61.23 | 0.00003 | 0.018 |
| | | 61.47 | 0.00003 | 0.018 |
| | | 61.75 | 0.00004 | 0.024 |
| | | 62.03 | 0.00005 | 0.028 |
| | | 62.32 | 0.00007 | 0.038 |
| | | 62.60 | 0.00009 | 0.049 |
| | | 62.88 | 0.00011 | 0.059 |
| | | 63.16 | 0.00014 | 0.077 |
| | | 63.50 | 0.00029 | 0.108 |
| | | 64.07 | 0.00027 | 0.151 |
| | | 64.52 | 0.00030 | 0.168 |
| | | 64.86 | 0.00032 | 0.175 |
| | | 65.15 | 0.00039 | 0.214 |
| | | 65.54 | 0.00042 | 0.231 |
| | | 66.00 | 0.00048 | 0.266 |
| | | 66.62 | 0.00053 | 0.294 |
| | | 67.24 | 0.00074 | 0.413 |
| | | 67.81 | 0.00093 | 0.518 |
| | | 68.43 | 0.00155 | 0.858 |
| | | 69.11 | 0.00165 | 0.918 |
| | | 69.68 | 0.00165 | 0.918 |
| | | 66.28 | 0.00135 | 0.750 |
| | | 62.32 | 0.00016 | 0.091 |
| | | 61.18 | 0.00011 | 0.059 |
| | | 60.62 | 0.00008 | 0.044 |
| | | 60.05 | 0.00008 | 0.044 |
| | | 59.48 | 0.00008 | 0.044 |
| 58.92 | 0.00008 | 0.044 | | |
| 56.65 | 0.00000 | 0.000 | | |

with the resistances formerly used¹⁾. When the wire is only stretched along the length of a mica strip, there is much more chance that the crystals remain intact.

§ 3. *Preliminary measurements on the hysteresis figures with single crystals of tin.*

a. A first preliminary measurement with a single crystal wire of tin has already been given in a preceding communication²⁾. This wire showed an hysteresis figure, which recalled the figures obtained with extruded tin wires, but differed from those by its greater simplicity. The descending, as well as the ascending line seemed only to consist of two rectilinear pieces.

b. The results obtained with mercury resistances caused us to investigate a tin wire, which first was heated till above its melting point and afterwards very slowly cooled. It might be expected, that with this slow cooling large crystals would be formed, which might give rise to similar

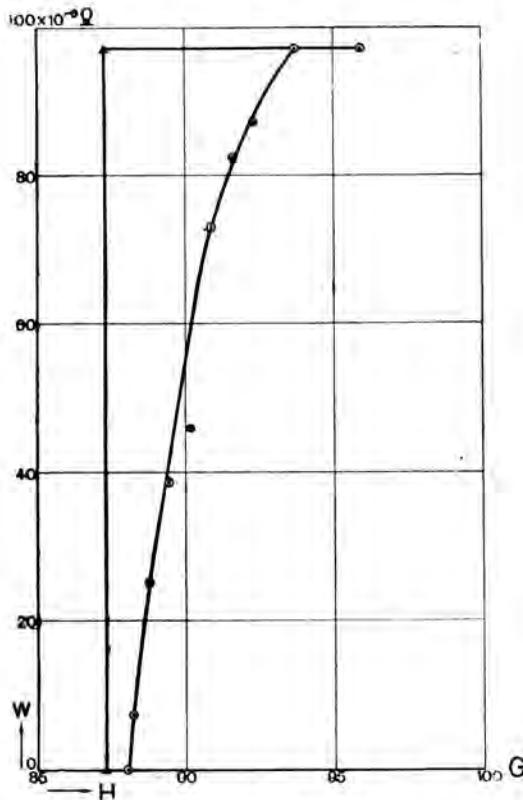


Fig. 2. Sn-1926-K₂ (Table II).

¹⁾ These Proceedings 39, 221, 1926.

²⁾ These Proceedings 39, 253, 1926, fig. 8.

discontinuities as were found with mercury, and, though less distinctly, with tin.

A tin wire, 3.7 cm long, 0.24 mm thick, was fixed between two mica strips, placed in an evacuated glass tube, heated till above its melting

TABLE II. $S_n-1926-K_2$. Longitudinal.

$$\rho_{\text{helium}} = 200 \text{ mm Hg.} \quad T = 3^\circ.120 \text{ K.}$$

Coil W .¹⁾

$$A_{\text{max.}} = 0.783. \quad A_{\text{min.}} = 0.776. \quad \frac{A_{\text{min.}}}{A_{\text{max.}}} = 0.991.$$

$$A_{\text{avg.}} = 0.780. \quad H_{\text{avg.}} \text{ (in gauss)} = 35.87 i \text{ (} i \text{ in amp.)}$$

| Date | Current | $H_{\text{avg.}}$ in gauss | $W_{S_n-1926-K_2}$ | Remarks |
|---------------|---------|-------------------------------|--------------------|---------|
| 17 March 1926 | 200 mA | 104.05 | 0.0000978 Ω | jump |
| | | ↓ | | |
| | | 87.09 | 0.0000000 | |
| | | 88.02 | 0.0000000 | |
| | | 88.23 | 0.0000073 | |
| | | 88.73 | 0.0000251 | |
| | | 89.45 | 0.0000387 | |
| | | 90.17 | 0.0000459 | |
| | | 90.88 | 0.0000730 | |
| | | 91.60 | 0.0000823 | |
| | | 92.31 | 0.0000872 | |
| | | 93.67 | 0.0000971 | |
| | | 95.89 | 0.0000971 | |
| | | ↓ | | |
| | | 87.30 | 0.0000000 | |
| 100 mA | ↓ | 95.89 | 0.0000971 | jump |
| | | ↓ | | |
| | | 87.0 | 0.0000000 | |
| 40 mA | ↓ | 95.89 | 0.0000971 | jump |
| | | ↓ | | |
| | | 87.3 | 0.0000971 | |

¹⁾ These Proceedings, 39, 222, 1926.

point and than very slowly cooled. The hysteresis figure was measured on 17 March 1926. See table II and fig. 2.

The result was more surprising than was expected. Namely, whilst the return of the resistance was continuous, the disappearance took place discontinuously, and in fact in one single jump.

We probably may conclude from this, that the whole wire, by the slow cooling, had become a single crystal wire.

The resistance at 0° C. was 0.0798Ω . The resistance at $4^{\circ}.2$ K. was $98 \times 10^{-6} \Omega$. The ratio between these values amounts to 0.00122. With the wire, mentioned under *a*, this ratio was 0.00027, whilst for extruded tin wires it amounts to about 0.0008. It is evident that this ratio depends largely on the direction of the crystal axis in reference to the length of the wire.

The situation of the jump, namely the value of the field, at which the resistance disappears, was repeatedly determined with different current strengths (200, 100 and 40 mA.). The found differences fall within the accuracy of the measurement. Also with mercury resistances it was always found that the situation of the jumps was independent of the current.

We found no explanation for the difference between the hysteresis figures measured with the two single crystal wires (especially with respect to the ascending line).

We hope to be able to clear up the peculiarities of the magnetic disturbance of the supraconductivity with single crystal wires, by means of new measurements, which are already in progress.

B. Mercury.

§ 4. *The explanation of the discontinuities.*

In a preceding communication ¹⁾ we have already mentioned a measurement, which served to prove the given explanation of the discontinuities by demonstrating that the discontinuous changes of the resistance took place in definite parts of the threads.



Fig. 3.

This measurement was carried out with a resistance of the type, shown in fig. 3. It consists of two capillaries (*a* and *b*) connected by a mercury reservoir, provided with platinum wires, so that the resistance of the capillaries might be measured separately (*a* and *b*) and together (*c*). As soon as a sudden change in the resistance was found by decreasing the magnetic field, the resistance of *c* as well as of *a* and *b* was determined. Only with the first jump, *a* as well as *b* appeared to have lost a part of their resistance. With all following jumps, the change occurred either in *a* or in *b*. The decrease of the resistances in

¹⁾ These Proceedings 39, 250, 1926.

a and b with the first jump were not all proportional to the resistances before the jump. There is therefore no question, that the changes in the resistance should be continuously spread over the whole length of both capillaries. By this experiment the local character of the discontinuities might be considered as proved.

TABLE III. $Hg-1926-D$.

| | | | |
|-------------------------|--|--|------------------------------|
| Type III. ¹⁾ | capillary a : $l = 11$ mm. ²⁾ | capillary b : $l = 11$ mm. | $d = 0.10$ mm. ²⁾ |
| | $W_{\text{roomtemp.}}$ not measured. ³⁾ | $W_{4.02\text{ K.}}$ not measured. ³⁾ | |
| | 1) $p_{\text{helium}} = 624$ mm Hg. | $T = 4.001$ K. | |
| | 2) $p_{\text{helium}} = 347$ mm Hg. | $T = 3.499$ K. | |
| Current = 40 mA. | | | |
| Coil A. | H (in gauss) = 113.1 i (i in amp.). | | |

| T | H _{avg.} in gauss | W _{Hg-1926-D} a | W _{Hg-1926-D} b | W _{Hg-1926-D} c | Remarks |
|-----------|-------------------------------|-----------------------------|-----------------------------|-----------------------------|--------------------------|
| 3°.001 K. | 33.85 | 0.000031 Ω | 0.000051 Ω | 0.000085 Ω | |
| | 40.27 | 0.000064 | 0.000063 | 0.000130 | |
| | 48.15 | 0.000064 | 0.000064 | 0.000130 | |
| | ↓ 30.48 | 0.000059 | 0.000051 | 0.000110 | jump in a and b . |
| | ↓ 27.87 | 0.000059 | 0.000000 | 0.000059 | jump in b . |
| | ↓ 26.85 | 0.000044 | 0.000000 | 0.000044 | jump in a . |
| | ↓ 26.40 | 0.000000 | 0.000000 | 2.000000 | jump in a . |
| 3°.499 K. | 138.5 | 0.000044 | 0.000042 | 0.000086 | |
| | ↓ 123.5 | 0.000041 | 0.000000 | 0.000041 | jump in a and b . |
| | ↓ 119.0 | 0.000000 | 0.000000 | 0.000000 | jump in a . |

To see if the combining of the first jumps was a matter of chance or not, a new measurement was made with $Hg-1926-D$, of the same type. See table III and fig. 4. The result was the same. At both temperatures at which was measured the first jump occurs in both capillaries at the same time, the other either in the first or in the second. Also here the

¹⁾ See fig. 3.

²⁾ l and d represent the length and the diameter of the mercury thread.

³⁾ $W_{\text{roomtemp.}}$ and $W_{4.02\text{ K.}}$ represent the resistances of the thread measured at room-temperature and at 4°.2 K, respectively.

first changes in the resistances of the capillaries are not proportional to the resistances before the jump.

That there exists a jump, with which both capillaries loose a part of

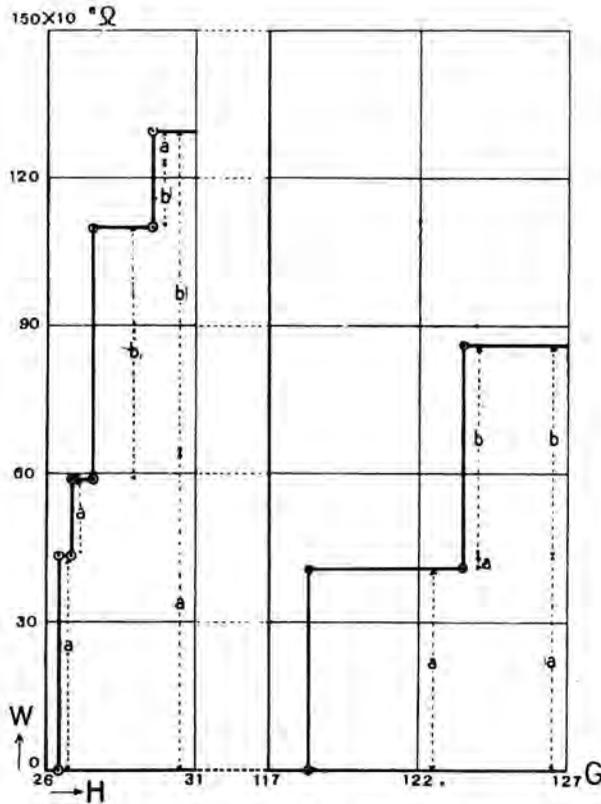


Fig. 4. Hg-1926-D (Table III).

their resistance, is not to be wondered at. There may very well be formed a single crystal which extends over a part of both capillaries and over the middle reservoir. Why this part prefers to loose its resistance with the first jump, remains however unexplained.

TABLE IV. Hg-1926-A.

| | | |
|---|--|------------------------|
| Type II. ¹⁾ | $l = 10 \text{ mm.}$ | $d = 0.10 \text{ mm.}$ |
| $W_{\text{roomtemp.}} = 0.926 \Omega.$ | $W_{4^{\circ}2 \text{ K.}} = 0.000458 \Omega.$ | |
| $p_{\text{helium}} = 9.7 - 10.2 \text{ mm Hg.}$ | $T = 1^{\circ}.79 - 1^{\circ}.84 \text{ K.}$ | |
| Current = 28.5 mA. | | |

Coil W.

¹⁾ These Proceedings 39, 233, 1926.

$A_{max} = 0.783$

$A_{min.} = 0.781.$

$\frac{A_{min.}}{A_{max.}} = 0.99$

$A_{avg.} = 0.782.$

$H_{avg.}$ (in gauss) = 35.86 i (i in amp.).

| $H_{avg.}$ in gauss | $W_{Hg-1926-A}$ | $\frac{W}{W_{4.2}}$ | Remarks |
|------------------------|--------------------|---------------------|---------|
| 340.5 | 0.0000192 Ω | 0.0419 | |
| 341.9 | 0.0000258 | 0.0563 | |
| 344.8 | 0.0000397 | 0.0867 | |
| 345.6 | 0.0000568 | 0.1249 | |
| 352.7 | 0.0000568 | 0.1249 | |
| ↓ | | | |
| 339.1 | 0.0000000 | 0.0000 | jump |

TABLE V. $Hg-1926-A$.

See Table IV.

$p_{helium} = 615.0$ mm Hg

$T = 2^{\circ}.493$ K.

Current = 28.5 mA.

| $H_{avg.}$ in gauss | $W_{Hg-1925-A}$ | $\frac{W}{W_{4.2}}$ | Remarks |
|------------------------|-------------------|---------------------|---------|
| 278.0 | 0.000000 Ω | 0.0000 | |
| 278.3 | 0.000043 | 0.095 | |
| 293.5 | 0.000089 | 0.195 | |
| 302.1 | 0.000090 | 0.197 | |
| 359.2 | 0.000090 | 0.197 | |
| ↓ | | | |
| 284.5 | 0.000026 | 0.056 | jump |
| ↓ | | | |
| 283.7 | 0.000007 | 0.016 | .. |
| ↓ | | | |
| 282.7 | 0.000000 | 0.000 | .. |
| 283.8 | 0.000002 | 0.004 | |
| 284.8 | 0.000034 | 0.074 | |
| 287.4 | 0.000049 | 0.107 | |
| 288.4 | 0.000061 | 0.133 | |
| 291.0 | 0.000090 | 0.197 | |

§ 5. *The hysteresis figure at different temperatures.*

To find out how the hysteresis figure depends on the temperature, on 11 Febr. 1926, we measured the hysteresis figure of the resistance *Hg-1926-A* at four temperatures. The results are contained in the tables IV, V, VI and VII and are represented in fig. 5. As there was

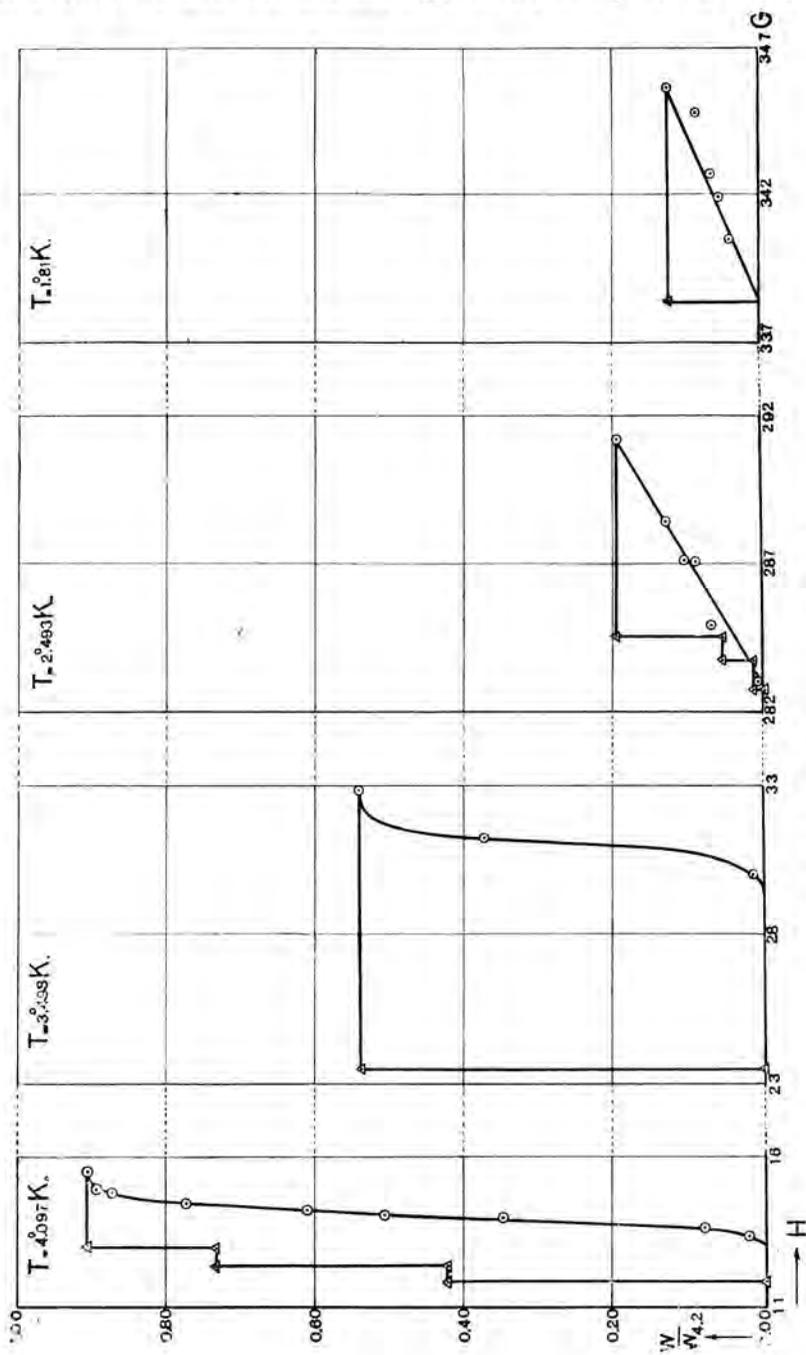


Fig. 5. *Hg-1926-A* (Tables IV, V, VI, VII).

TABLE VI. Hg-1926-A.

See Table IV.

$$p_{\text{helium}} = 346.2 \text{ mm Hg.}$$

$$T = 3^{\circ}.498 \text{ K.}$$

$$\text{Current} = 20 \text{ mA.}$$

| $H_{\text{avg.}}$ in gauss | $W_{\text{Hg-1925-A}}$ | $\frac{W}{W_{4.2}}$ | Remarks |
|-------------------------------|------------------------|---------------------|---------|
| 124.3 | 0.000000 Ω | 0.000 | |
| 130.0 | 0.000015 | 0.000 | |
| 131.5 | 0.000173 | 0.377 | |
| 132.9 | 0.000249 | 0.544 | |
| 136.1 | 0.000248 | 0.541 | |
| ↓ | | | |
| 123.5 | 0.000000 | 0.000 | jump |

TABLE VII. Hg-1926-A.

See Table IV.

$$p_{\text{helium}} = 687 \text{ mm Hg.}$$

$$T = 4^{\circ}.097 \text{ K.}$$

$$\text{Current} = 20 \text{ mA.}$$

| $H_{\text{avg.}}$ in gauss | $W_{\text{Hg-1925-A}}$ | $\frac{W}{W_{4.2}}$ | Remarks |
|-------------------------------|------------------------|---------------------|---------|
| 14.37 | 0.000358 Ω | 0.781 | |
| 14.91 | 0.000413 | 0.902 | |
| 15.45 | 0.000418 | 0.912 | |
| 18.03 | 0.000418 | 0.912 | |
| ↓ | | | |
| 12.90 | 0.000340 | 0.742 | jump |
| ↓ | | | |
| 12.36 | 0.000196 | 0.428 | .. |
| ↓ | | | |
| 11.85 | 0.000000 | 0.000 | .. |
| 13.34 | 0.000010 | 0.023 | |
| 13.63 | 0.000038 | 0.082 | |
| 13.95 | 0.000162 | 0.354 | |
| 14.04 | 0.000235 | 0.513 | |
| 14.22 | 0.000283 | 0.618 | |
| 14.76 | 0.000403 | 0.880 | |
| 17.96 | 0.000418 | 0.912 | |

little time available we had to measure quickly whence the ascending lines, especially at the lower temperatures, were not so thoroughly determined as was desirable. However, from fig. 5, it is evident that at lower temperatures they are much more inclined to the axis of the field than at the higher temperatures. The distance of the ascending and descending lines measured along the axis of the fields seems to become zero at the high, as well as at the lower temperatures.

The descending line, as is clear from fig. 5, does not always show the same jumps, though this might be expected from the given explanation. However, we have more than once drawn the attention to the fact that two jumps may be combined, by which the form of the descending line may change. The different fields strengths at which the jumps occur, seem to depend on the temperature to a different way, by which also the character of the line may change (see for example fig. 4).

The measurements of the resistance were this time made by comparing the potentials at the ends of the unknown and a known resistance by means of the deflections of a ZERNIKE galvanometer. The proportionality of the deflections to the potentials was secured by the great resistance of the galvanometer compared to the resistances, which were to be measured. Besides it was controlled with the aid of known resistances.

Of the resistance *Hg-1925-R* we measured the hysteresis figure at two temperatures ($4^{\circ}.00$ and $3^{\circ}.14$ K.). Herewith we found the peculiarity, that a part of the resistance (about 6%) returned at a much lower value of the field than the rest. Also the disappearance of this part of the resistance look place at an abnormally low value of the field. Besides, in the lower part of the ascending line, there seemed to exist some small discontinuities. For sake of completeness we give the results in tables VIII and IX, but do not wish to attribute a high value to these measurements.

§ 6. *Dependency of the threshold value ($H_{\frac{1}{2}}$) on the temperature.*

The threshold value of the magnetic field was defined by TUYN and KAMERLINGH ONNES¹⁾ as the value of the field, at which the resistance had half returned. This value was called $H_{\frac{1}{2}}$. In accordance to this definition, we now wish to understand by $(H_{\frac{1}{2}})_{asc.}$: the value of the field, at which the resistance, with increasing magnetic field, has reached half of its maximum value at the given temperature, and by $(H_{\frac{1}{2}})_{desc.}$ the value of the field at which the resistance, with decreasing field, has fallen to the same amount. We only give this definition to be able to compare our results with those of previous measurements but do not wish to ascribe any physical meaning to this definition. The peculiar form of hysteresis figure and the discontinuous character of the descending lines make this impossible.

¹⁾ Leiden, Comm. No. 174a.

TABLE VIII. *Hg*-1925-R.Type II. $l = 10$ mm. $d = 0.05$ mm.

$$W_{\text{roomtemp.}} = 3.96 \Omega. \quad W_{4^{\circ}2 \text{ K.}} = 0.00242 \Omega.$$

$$p_{\text{helium}} = 625 \text{ mmHg.} \quad T = 4^{\circ}.001 \text{ K.}$$

$$\text{Current} = 4 \text{ mA.}$$

Coil *W*.

$$A_{\text{max.}} = 0.783. \quad A_{\text{min.}} = 0.781. \quad \frac{A_{\text{min.}}}{A_{\text{max.}}} = 0.997.$$

$$A_{\text{avg.}} = 0.782. \quad H_{\text{avg.}} \text{ (in gauss)} = 35.86 i \text{ (} i \text{ in amp.)}$$

| $H_{\text{avg.}}$ in gauss | $W_{\text{Hg-1925-R}}$ | $\frac{W}{W_{4.2}}$ | Remarks | |
|-------------------------------|------------------------|---------------------|---|------|
| 7.2 | 0.00000 Ω | 0.000 | } jump ? | |
| 19.32 | 0.00016 | 0.066 | | |
| 19.39 | 0.00016 | 0.066 | | |
| 19.46 | 0.00016 | 0.066 | | |
| 19.61 | 0.00016 | 0.066 | | |
| 19.75 | 0.00016 | 0.066 | | |
| 19.93 | 0.00016 | 0.066 | } jump ? | |
| 23.81 | 0.00024 | 0.099 | | |
| 24.06 | 0.00020 | 0.083 | } continuous or very small discontinuities? | |
| 26.93 | 0.00025 | 0.103 | | |
| 29.05 | 0.00030 | 0.124 | | |
| 32.68 | 0.00042 | 0.174 | | |
| 33.14 | 0.00057 | 0.236 | | |
| 33.72 | 0.00113 | 0.467 | | |
| 34.04 | 0.00153 | 0.632 | | |
| 35.98 | 0.00210 | 0.868 | | |
| 36.52 | 0.00208 | 0.859 | | |
| ↓ 26.79 | 0.00168 | 0.694 | | jump |
| ↓ 24.32 | 0.00089 | 0.368 | | .. |
| ↓ 24.63 | 0.00027 | 0.112 | | .. |
| ↓ 14.38 | 0.00016 | 0.066 | | |
| 13.65 | 0.00014 | 0.058 | | |
| 11.96 | 0.00010 | 0.041 | | |
| 10.77 | 0.00009 | 0.036 | | |
| 8.98 | 0.00009 | 0.036 | | |
| 7.79 | 0.00005 | 0.021 | | |
| 5.74 | 0.00000 | 0.000 | | |

TABLE IX. *Hg*-1925-*R*.

See Table VIII.

$$p_{\text{helium}} = 170 \text{ mm Hg.}$$

$$T = 3^{\circ}.14 \text{ K.}$$

$$\text{Current} = 4 \text{ mA.}$$

Coil *W* (see Table VIII).

| $H_{\text{avg.}}$ in gauss | $W_{\text{Hg-1925-R}}$ | $\frac{W}{W_{4.2}}$ | Remarks |
|-------------------------------|------------------------|---------------------|---|
| 265.7 | 0.00098 Ω | 0.405 | |
| ↓ | | | |
| 195.7 | 0.00057 | 0.236 | jump |
| ↓ | | | |
| 193.9 | 0.00014 | 0.056 | jump |
| 175.2 | 0.00006 | 0.026 | } continuous or very small discontinuities? |
| 168.1 | 0.00000 | 0.000 | |
| 190.7 | 0.00000 | 0.000 | } jump? |
| 194.3 | 0.00012 | 0.051 | |
| 197.5 | 0.00014 | 0.062 | } horizontal? |
| 204.7 | 0.00013 | 0.055 | |
| 209.9 | 0.00060 | 0.246 | jump? |
| 211.9 | 0.00100 | 0.414 | |

In fig. 6 the found values for $(H_{\frac{1}{2}})_{\text{asc.}}$ and $(H_{\frac{1}{2}})_{\text{desc.}}$ are plotted as functions of the temperature, together with the older measurements on tin, lead and indium. With those measurements the hysteresis phenomena had not yet been found. As it was to be expected, both lines which represent the connection between $(H_{\frac{1}{2}})_{\text{asc.}}$ and $(H_{\frac{1}{2}})_{\text{desc.}}$ respectively, with the temperature, meet in the same point of the axis of the fields. This point corresponds to the vanishing-point of mercury, namely $4^{\circ}.17 \text{ K}$. In comparing these two curves, with the three others, it is to be remembered that our measurements refer to a longitudinal field, whilst the former refer to a transverse field. From some measurements on the threshold values of tin in a longitudinal field we know that the curve $H_{\frac{1}{2}}, T$ in a longitudinal field is somewhat steeper than in a transverse field.

According to TUYN and KAMERLINGH ONNES, the curve for tin may be represented by a formula of the form $H_{\frac{1}{2}} = h(T_S^2 - T^2)$, where $h = 20.1$ and T_S is the vanishing point of tin ($3^{\circ}.74 \text{ K}$). For mercury a formula of the same form did not appear to hold so well. In fig. 7 the values for $(H_{\frac{1}{2}})_{\text{asc.}}$ are plotted against the corresponding values of T^2 .

where $h = 25.16$ and $T_S = 4^{\circ}.17$ K. In table X, for all measured resistances, are given the values of $(H_{\frac{1}{2}})_{asc.}$ calculated with this formula and also the measured values of $(H_{\frac{1}{2}})_{asc.}$ and $(H_{\frac{1}{2}})_{desc.}$. The differences between

TABLE X.

| Resistance | T | T^2 | $(H_{\frac{1}{2}})_{asc.}$ calc. | $(H_{\frac{1}{2}})_{desc.}$ obs. | $(H_{\frac{1}{2}})_{desc.}$ obs. | In fig. 7 indicated by |
|-----------------------------|----------|-------|-------------------------------------|-------------------------------------|-------------------------------------|------------------------------|
| <i>Hg</i> -1926- <i>A</i> | 4.097 K. | 16.79 | 15 | 14 | 12 | ⊙ |
| <i>Hg</i> -1925- <i>C</i> | 4.036 | 16.29 | 27 | 25 | 23 | } ∇ |
| <i>Hg</i> -1925- <i>R</i> | 4.002 | 16.02 | 34 | 34 | 26 | |
| <i>Hg</i> -1926- <i>D</i> | 4.001 | 16.01 | 34 | 34 | 28 | |
| <i>Hg</i> -1925- <i>P</i> | 3.963 | 15.71 | 42 | 42 | 35 | } + |
| <i>Hg</i> -1925- <i>G</i> | 3.962 | 15.70 | 42 | 42 | 35 | |
| <i>Hg</i> -1925- <i>D</i> | 3.924 | 15.40 | 50 | — | 45 | △ |
| <i>Hg</i> -1925- <i>Z</i> | 3.883 | 15.08 | 58 | 59 | 51 | } □ |
| <i>Hg</i> -1925- <i>G'</i> | 3.797 | 14.42 | 75 | 75 | 67 | |
| <i>Hg</i> -1925- <i>O</i> | 3.797 | 14.42 | 75 | 77.5 | 64 | |
| <i>Hg</i> -1925- <i>E</i> | 3.797 | 14.42 | 75 | 77 | 72 | |
| <i>Hg</i> -1925- <i>G</i> | 3.796 | 14.41 | 75 | 75.5 | 67 | |
| <i>Hg</i> -1925- <i>L</i> | 3.792 | 14.38 | 76 | 77 | 65 | |
| <i>Hg</i> -1925- <i>K</i> | 3.792 | 14.38 | 76 | 72 | 62 | |
| <i>Hg</i> -1925- <i>A</i> | 3.498 | 12.24 | 130 | 131 | 124 | ⊙ |
| [<i>Hg</i> -1925- <i>R</i> | 3.14 | 9.86 | 190 | 210 | 192] ¹⁾ | [X] |
| <i>Hg</i> -1926- <i>A</i> | 2.493 | 6.22 | 281 | 288 | 285 | ⊙ |
| <i>Hg</i> -1926- <i>A</i> | 1.813 | 3.29 | 355 | 342 | 338 | ⊙ |

the measured and the calculated values exceed the accuracy of the measurement by some percents. This is especially evident at the two lowest temperatures at which we measured, with the resistance *Hg*-1926-*A*. For mercury therefore the formula $(H_{\frac{1}{2}})_{asc.} = h(T_S^2 - T^2)$ can only serve as an approximation formula.

¹⁾ Put between [] because of the strange form of the hysteresis-figure (see p. 959).

Chemistry. — “Equilibria in systems in which phases, separated by a semi-permeable membrane” XVII. By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of May 29, 1926).

Ternary systems with vapour-phases.

We now consider an osmotic system:

$$G \mid G_1 \dots \dots \dots (1)$$

in which a vapour occurs on both sides of the membrane. We represent the composition of the vapour G by:

$$x \text{ Mol } X + y \text{ Mol } Y + (1-x-y) \text{ Mol } W$$

and that of the vapour G_1 by:

$$x_1 \text{ Mol } X + y_1 \text{ Mol } Y + (1-x_1-y_1) \text{ Mol } W.$$

If we assume that the membrane allows the substance W only to pass through, then we find that system (1) is in osmotic equilibrium, when:

$$\zeta - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} = \zeta_1 - x_1 \frac{\partial \zeta_1}{\partial x_1} - y_1 \frac{\partial \zeta_1}{\partial y_1} \dots \dots (2)$$

As we find, therefore, for the osmotic equilibrium of two vapours the same equations as for the osmotic equilibrium of two liquids, we can say, therefore, that the *O.W.A.* of a vapour is defined by:

$$\eta = \zeta - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} \dots \dots (3)$$

and consequently we can apply to those osmotic vapour-systems the same considerations as formerly to the osmotic liquid-systems.

Let us take f.i. fig. 1 Comm. II and let us assume that at given temperature and pressure all binary and ternary mixtures are gaseous, then the curves ad , eh etc. can represent isotonic vapour-curves. All vapours of curve ad then are mutually isotonic, also those of eh etc.; of course vapours of different curves are not mutually isotonic, as the *O.W.A.* of the vapours of an isotonic curve is larger, the more this curve is situated farther from the point W .

Previously (Communication II) we have deduced that the isotonic liquid curves are in the vicinity of point w (fig. 1. II) straight lines which cut equal parts of the sides WX and WY ; on greater distance of point W , however, they are curved, as is drawn in the figure. If we assume, however, as we shall do always in the following, that the vapours follow the law of BOYLE GAY-LUSSAC, then the isotonic vapour-curves

are not only straight lines in the vicinity of the point W , but everywhere. For a vapour, which follows the law of BOYLE GAY-LUSSAC, we have viz.

$$\zeta = x C_x + y C_y + (1-x-y) C_w + \left. \begin{array}{l} \\ + RT [x \log x + y \log y + (1-x-y) \log (1-x-y)] \end{array} \right\} \quad (4)$$

Hence follows for the osmotic equilibrium (1):

$$\varphi = C_w + RT \log (1-x-y) \quad \varphi_1 = C_w + RT \log (1-x_1-y_1) \quad (4^a)$$

consequently:

$$x + y = x_1 + y_1 \quad (5)$$

If we give a definite composition to the vapour G_1 , then it appears from (5) that the vapour G is represented by the points of a straight line, which goes through the point which represents the point G_1 . The isotonic vapour-curves, therefore, are straight lines, parallel to the side XY .

As $1-x-y$ is the concentration of the water in the vapour G and $1-x_1-y_1$ that in the vapour G_1 , it follows, therefore:

the vapours G and G_1 have the same *O.W.A.* if the concentration of the water and, therefore, also the partial vapour-pressure of the water is equal in both vapours.

It now is evident that properties and phenomena, which we have previously deduced, with assumption that the isotonic liquid-curves are curved, are no more valid now. Previously we have seen a.o. that the *O.W.A.* of the liquid t (fig. 1. II) does not increase immediately with addition of the substance Y ; its *O.W.A.* viz, firstly becomes smaller, till it becomes a minimum in g and only at further addition of Y the *O.W.A.* increases. Of course a similar phenomenon cannot occur with vapours.

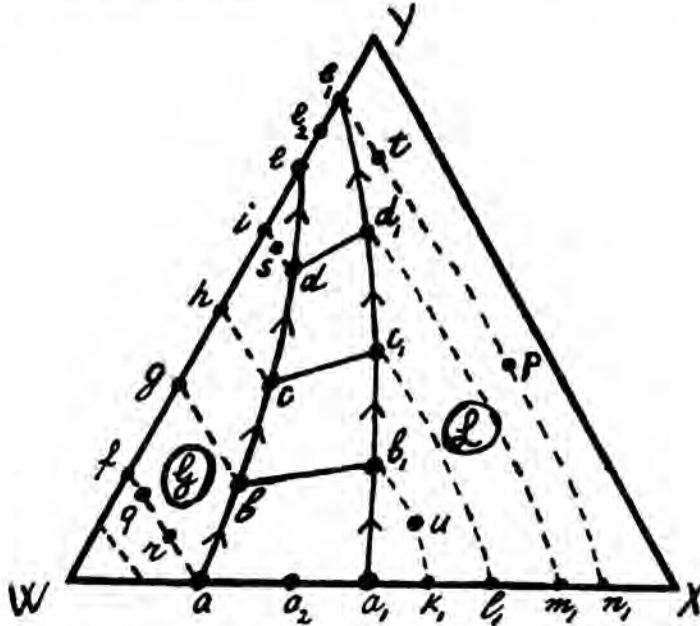
Now we shall assume that with a given temperature and pressure a part of the mixtures is liquid and another part gaseous, so that also may occur equilibria: $G + L$ viz. vapour + liquid. Of the many cases which can occur with this ¹⁾ we shall discuss only some.

If we assume that at the given T and P the substance W is gaseous, but the substances X and Y are liquid, then we may get a diagram as fig. 1, in which Wae represents the vapour-region and $a_1 e_1 YX$ the liquid-region; the first one is indicated by an encircled G , the latter one by an encircled L . Consequently at the given T and P all mixtures of the region Wae are gaseous and those of the region $a_1 e_1 YX$ are liquid. With every vapour of curve $a_1 e_1$ a definite liquid of curve $a_1 e_1$ can be in equilibrium, f.i. vapour a with the liquid a_1 ; vapour b with

¹⁾ For considerations more in detail on ternary systems, in which vapour occurs, compare: F. A. H. SCHREINEMAKERS, Zeitschrift f. Phys. Chem. 36 257, 413, 710 (1901); 37 129 (1901); 38 227 (1901); 43 671 (1903) Archives Néerlandaises. These Proceedings. Equilibria in ternary systems. Comm. I—XVIII (1912—1915).

the liquid b_1 ; etc.; some of the vapours and liquids which are in equilibrium with one another are united with one another in the figure by straight lines (conjugation-lines: vapour-liquid).

The field $a a_1 e_1 e$ situated between the curves $a e$ and $a_1 e_1$ is a heterogeneous field; each point of this field represents viz. a complex of a vapour of curve $a e$ and a liquid of curve $a_1 e_1$; f.i. a point of the line $b b_1$ represents an equilibrium $G_b + L_{b_1}$; a point of the line $c c_1$ an equilibrium $G_c + L_{c_1}$; etc.



The two binary conjugation-lines $a a_1$ and $e e_1$ of course go through the point W ; referring to the position of the ternary conjugation lines we may distinguish two cases, viz.:

- a) one (or some) of the conjugation-lines goes through the point W .
- b) none of the conjugation-lines goes through the point W .

In fig. 1 the latter case is drawn; with this we have assumed that all conjugation-lines intersect the side WY in points between W and Y . In connection with this assumption we may say, as in the previous communications, that we move away from the point W along the curves $a e$ and $a_1 e_1$ if we proceed along these curves in the direction of the arrows viz. from a towards e and from a_1 towards e_1 .

We now take an equilibrium:

$$G + L \dots \dots \dots (6)$$

of which G and L are represented, therefore, in fig. 1 by two conjugated points, the one of which is situated on curve $a e$ and the other on curve $a_1 e_1$. As G and L are in equilibrium with one another, they are isotonic with respect to all components and, therefore, also with respect to the

diffusing component *W*. Consequently the vapour *a* has the same *O.W.A.* as the liquid *a*₁; the vapour *b* the same as the liquid *b*₁; etc. Consequently we have a.o. the following osmotic equilibria:

$$G_a \mid L_{a_1} \quad G_b \mid L_{b_1} \quad G_c \mid L_{c_1} \dots \dots \dots (7)$$

in which, however, we can also omit the membrane.

In the previous communication we have deduced:

the *O.W.A.* of a stable system becomes smaller at taking in of water and greater at losing of water.

If we apply this rule to the equilibria (6) above-mentioned then we find:

the *O.W.A.* of the vapours of curve *ae* and that of the liquids of curve *a₁e₁* increases in the direction of the arrows, viz. from *a* towards *e* and from *a₁* towards *e₁*.

In the osmotic system:

$$G_b + L_{b_1} \xrightarrow{\quad} G_c + L_{c_1} \dots \dots \dots (8)$$

the system at the right of the membrane has, therefore, a greater *O.W.A.* than the left system; consequently the water will diffuse in the direction of the arrow, viz. from left to right.

The curves dotted in fig. 1 represent isotonic curves; if we limit ourselves to stable states then we find in the vapour-region *Wae* only isotonic vapour-curves and in the liquid-region *a₁XYe₁* only isotonic liquid-curves. In the heterogeneous region *a₁e₁e* all isotonic curves are metastable. The isotonic curves situated in the vapour-region are straight lines parallel to the side *XY*; the isotonic curves of the liquid-region are more or less curved.

Isotonic curves with a smaller *O.W.A.* than that of the vapour *a* are situated within the triangle *Waf*; the isotonic curve with the same *O.W.A.* as that of the vapour *a* consists of curve *af* and the isolated point *a₁*. Consequently we can have the osmotic equilibrium:

$$G(af) \mid L_{a_1} \dots \dots \dots (9)$$

in which *G(af)* represents an arbitrary vapour of the curve *af*. Consequently we may have a.o. the following osmotic equilibria:

$$G_f \mid G_a ; G_q \mid G_r ; G_q \mid L_{a_1} ; G_f \mid L_{a_1} \dots \dots (10)$$

In the last of those systems a binary vapour, consisting of *W + Y*, is in osmotic equilibrium with a binary liquid consisting of *W + X*.

The isotonic curve with a same *O.W.A.* as the vapour *b* consists of two branches viz. of the vapour-branch *gb* and the liquid-branch *b₁k₁*, which are united with one another by the conjugation-line *bb₁*. Consequently all vapours of branch *gb* have the same *O.W.A.*, also all

in which G'_q is represented by the point of intersection of Wq and hc and L'_p by the point of intersection of Wp and $c_1 l_1$.

If the equilibrium is formed on the isotonic curve im_1 , then (11) passes into the osmotic equilibrium:

$$G_d + L_{d_1} \mid L'_p \dots \dots \dots (14)$$

in which L'_p is represented by the point of intersection of Wp and $d_1 m_1$.

In the osmotic system:

$$G_r \overset{\downarrow}{\rightarrow} L_p \dots \dots \dots (15)$$

as is apparent from the figure, the liquid has a greater *O.W.A.* than the vapour, so that water must diffuse in the direction of the arrow. If we assume that the line Wr intersects the conjugation-line bb_1 also, then (15) may pass into the osmotic equilibrium:

$$G_b + L_{b_1} \mid G_b + L_{b_1} \dots \dots \dots (16)$$

If the equilibrium is formed on curve $c_1 l_1$ then (15) passes into the osmotic equilibrium:

$$L'_r \mid L'_p \dots \dots \dots (17)$$

consequently into an equilibrium of two liquids which are represented by the points of intersection of the lines Wr and Wp with curve $c_1 l_1$. Therefore, the vapour G_r of system (15) is totally condensed with loss of water.

In the osmotic system:

$$G_r \overset{\downarrow}{\rightarrow} L_t \dots \dots \dots (18)$$

as is apparent from the figure, the liquid has a greater *O.W.A.* than the vapour, so that the water diffuses in the direction of the arrow. If we choose the ratio of the quantities of both phases in such a way that the equilibrium is formed on the isotonic curve hl_1 , then (18) passes into the osmotic equilibrium:

$$L'_r \mid G'_t \dots \dots \dots (19)$$

in which L'_r is represented by the point of intersection of Wr and $c_1 l_1$ and G'_t by the point of intersection of Wt and hc . The result of the diffusion is, therefore, that in (18) the vapour (at the left side of the membrane) passes into a liquid and that the liquid (at the right side of the membrane) passes into a vapour.

We now take an osmotic system of two vapours f.i.

$$G_r \overset{\downarrow}{\rightarrow} G_s \dots \dots \dots (20)$$

in which, as is apparent from the figure, the vapour s has a greater

O.W.A. than the vapour r . If the equilibrium is formed on the isotonic curve gk_1 then (20) passes into the osmotic equilibrium:

$$G_b + L_{b_1} \mid G'_s \dots \dots \dots (21)$$

in which G'_s is represented by the point of intersection of Ws and gb . The vapour G_r of system (20) is partly condensed, therefore, with loss of water.

The vapour G_r of system (20) can also condense totally; this is the case f.i. if we choose the ratio's of the quantities of the phases in (20) in such a way that the equilibrium is formed on the isotonic curve hl_1 . Then system (20) passes into the osmotic equilibrium:

$$L'_r \mid G'_s \dots \dots \dots (22)$$

in which L'_r is represented by the point of intersection of Wr and $c_1 l_1$ and G'_s by the point of intersection of Ws and hc . The vapour G_r of system (20) is totally converted into liquid, therefore.

At last we still consider an osmotic system of two liquids, f.i.:

$$L_u \xrightarrow{\mid} L_t \dots \dots \dots (23)$$

in which, as appears from the figure, the water must diffuse from left to right. If we choose the ratio of the quantities of both phases in such a way that the equilibrium is formed on the isotonic curve hl_1 , then (23) is converted into the osmotic system:

$$L'_u \mid G'_t \dots \dots \dots (24)$$

in which L'_u is represented by the point of intersection of Wu and $c_1 l_1$ and G'_t by the point of intersection of Wt and hc . Consequently the liquid at the right side of the membrane in (23) is totally passed into vapour with taking up water.

In fig. 1 we have assumed that at the given T and P the vapour-region is represented by Wae and the liquid-region by $a_1 e_1 YX$. We now shall suppose, however, the reverse, viz. that Wae represents a liquid-region and $a_1 e_1 YX$ a vapour-region; we have to interchange, therefore, in fig. 1 the encircled letters L and G ; also we have to imagine the curves $b_1 k_1$, $c_1 l_1$, etc. to be replaced by straight lines, parallel to the side XY .

The previous considerations remain all valid now, but of course we have to interchange everywhere vapour and liquid and in the osmotic systems and equilibria (9)–(24) we have to replace the G by L and reversally. If we take f.i. the osmotic system (20) then this now becomes an osmotic system of two liquids viz.:

$$L_c \xrightarrow{\mid} L_s \dots \dots \dots (25)$$

which, if the equilibrium is formed on the isotonic curve $h l_1$, passes into the osmotic equilibrium

$$G_r \mid L_s \dots \dots \dots (26)$$

instead of into (22).

One of the liquids of system (25) viz. L_r now passes with loss of water into the unsaturated vapour G_r' , which is represented in the figure by the point of intersection of $W r$ and $c_1 l_1$.

The ternary conjugation-lines are drawn in fig. 1 all in such a way that none of them passes through the point W ; if we assume, however, that one of those conjugation-lines f.i. the line $c c_1$ goes through the point W , then, if we proceed along the curves $a e$ and $a_1 e_1$, the *O.W.A.* in c and c_1 is maximum or minimum. We can deduce this in the same way as in communication XIV the corresponding property for the contact of an isotonic curve with the binodal-curve. In previous communications on vapour-tensions in ternary systems¹⁾ we find many diagrams in which conjugation-lines: vapour-liquid go through the point W .

As in fig. 1 stable states only are indicated, we find the isotonic vapour-curves only in the region $W a e$ and the isotonic liquid-curves only in the region $a_1 e_1 Y X$. The first is divided by curve $a f$, the second by $e_1 n_1$ into two parts, which behave themselves differently with respect to osmotic equilibria. A vapour of part $W a f$ viz. can be isotonic only with vapours, and a liquid of part $e_1 n_1 X Y$ only with liquids; however, a vapour of part $a f e$ can be isotonic as well with vapours as with liquids; the same is valid for a liquid of part $a_1 e_1 n_1$.

Otherwise it is, however, if we consider metastable states also; then the vapour-region does not terminate in curve $a e$ and the liquid-region not in curve $a_1 e_1$; it depends on the T and P for which the figure is valid, how far both are prolonged. If we choose, as we shall assume further, the T and P in such a way that the components and all their mixtures may be as well liquid as gaseous, then each of both the regions covers the total triangle. Within the region $a e e_1 a_1$ then is situated a curve $a_2 e_2$ of which only the terminating-points a_2 and e_2 are drawn; its interpretation appears from the following. If we leave out of consideration the heterogeneous region $a e e_1 a_1$, then all vapours of the region $W a_2 e_2$ and all liquids of the region $a_2 e_2 Y X$ are stable and consequently all liquids of $W a_2 e_2$ and all vapours of $a_2 e_2 Y X$ are metastable. If we take into consideration, however, also the heterogeneous region, then the vapours of $a e e_2 a_2$ and the liquids of $a_1 e_1 e_2 a_2$ are metastable with respect to the equilibria $L + G$.

Of the many cases, which may occur, if we consider metastable phases

¹⁾ l. c.

also, we shall discuss a few only. We represent the composition of a vapour G by:

$$x \text{ Mol } X + y \text{ Mol } Y + (1 - x - y) \text{ Mol } W$$

and that of a liquid L by:

$$x_1 \text{ Mol } X + y_1 \text{ Mol } Y + (1 - x_1 - y_1) \text{ Mol } W.$$

The thermodynamical potential ζ of the vapour is then defined by (4); in order to represent the thermodynamical potential ζ_1 of the liquid we have to add to (4) still a term which we shall write in the form $RT\mu_1$; further of course we have to replace in (4) x and y by x_1 and y_1 . If the vapour is stable with respect to a liquid of the same composition, then is $\zeta < \zeta_1$ and μ_1 is positive, therefore; in the opposite case μ_1 is negative. We now find for the vapour G :

$$\varphi = C_w + RT \log (1 - x - y) \dots \dots \dots (27)$$

and for the liquid L :

$$\varphi_1 = C_w + RT \log (1 - x_1 - y_1) + RTm \dots \dots \dots (28)$$

in which:

$$m = \mu_1 - x_1 \frac{\partial \mu_1}{\partial x_1} - y_1 \frac{\partial \mu_1}{\partial y_1} \dots \dots \dots (29)$$

We now take a vapour G_0 of the composition $x_0 y_0$ and we consider the osmotic systems:

$$G_0 \mid G \dots (30^a) \quad G_0 \mid L \dots \dots \dots (30^b)$$

For (30^a) follows from (27) the equation:

$$x_0 + y_0 = x + y \dots \dots \dots (31)$$

for (30^b) we find with the aid of (27) and (28):

$$1 - x_0 - y_0 = (1 - x_1 - y_1) e^m \dots \dots \dots (32)$$

As we can always satisfy (31) also always an osmotic equilibrium (30^a) exists, therefore; that is to say: always a series of vapours exists (totally or partly metastable) which are isotonic with G_0 .

However, as we shall see further, it is not always possible to satisfy (32) [viz. by values of x_1 and y_1 which represent points within the triangle]. If we represent the second part of (32) by K then K is a function of x_1 and y_1 which has the value e^{μ_1} in the point W ($x_1 = 0$ and $y_1 = 0$). If we give to x_1 and y_1 such values that the liquid moves away from point W along a straight line going through W , then, as we shall show further, the value of K decreases from e^{μ_1} to zero. (In accordance with fig. 1 we leave here out of consideration dimixtion into 2 of 3 liquids).

We now take the osmotic system (30^b) and we assume, in accordance with fig. 1, that T and P have been chosen in such a way that the watervapour is stable and, therefore, the liquid water is metastable. In point W , therefore, μ_1 is positive and consequently $K = e^{\mu_1} > 1$. As the

first part of (32) is smaller than 1 and as K decreases, when the liquid moves away from W along a line going through point W , consequently on each straight line going through point W is situated a liquid L , which is isotonic with the vapour G_0 . Consequently we find:

if at given T and P water-vapour is the stable state of the water, then with every vapour G_0 a series of vapours and a series of liquids can be isotonic.

We now imagine in fig. 1 vapour- and liquid-region to be interchanged then the water is in liquid-state more stable than in vapour-state. In point W , therefore, μ_1 is negative and consequently $K = e^{\mu_1} < 1$. It now depends on the values of x_0 and y_0 which of the two cases

$$1 - x_0 - y_0 < e^{\mu_1} \qquad 1 - x_0 - y_0 > e^{\mu_1}$$

will occur. In the first case we can satisfy again (32), in the second case, however, not. Hence follows:

if at given T and P the water in stable state occurs as liquid, then with every vapour G_0 a series of vapours can be in equilibrium; however, it depends on the composition of the vapour G_0 whether there is a series of liquids, isotonic with this vapour, or not.

If the liquid L of the osmotic system (30^b) consists of water only, then (32) passes into:

$$1 - x_0 - y_0 = e^{\mu_1} \dots \dots \dots (33)$$

If the liquid water is stable, then is $\mu_1 < 0$ therefore $e^{\mu_1} < 1$; if, however, the liquid water is metastable then is $\mu_1 > 0$ and consequently $e^{\mu_1} > 1$. In the first case, therefore, we can satisfy (33) by positive values of x_0 and y_0 , in the second case we cannot do so. Hence follows:

if water is stable in liquid state, then a series of vapours exists, which have the same *O.W.A.* as this liquid water. (Those vapours, however, are all metastable).

If the vapour G_0 in (30^b) consists of water-vapour only, then (32) passes into:

$$1 = (1 - x_1 - y_1) e^m \dots \dots \dots (34)$$

in which the second part has the value $K = e^{\mu_1}$ for $x_1 = 0$ and $y_1 = 0$. In a similar way as above, it now appears:

if water-vapour is the stable state of the water, then a series of liquids exists, which have the same *O.W.A.* as this water-vapour. (Those liquids, however, are all metastable).

The property, used above, on the change of K viz. the second part of (32) can be deduced on the following way. It follows viz. from the value of K :

$$dK = - \left[1 + (1 - x_1 - y_1) \left(x \frac{\partial^2 \mu}{\partial x^2} + y \frac{\partial^2 \mu}{\partial x \partial y} \right)_1 \right] e^m dx - \left[1 + (1 - x_1 - y_1) \left(x \frac{\partial^2 \mu}{\partial x \partial y} + y \frac{\partial^2 \mu}{\partial y^2} \right)_1 \right] e^m dy \dots \dots \dots (35)$$

If we deduce from ζ_1 the values of r_1, s_1 and t_1 then it appears that we may write also for (35):

$$RT \cdot dK = - (1 - x_1 - y_1) [(rx + sy)_1 dx_1 + (sx + ty)_1 dy_1] e^m \quad (36)$$

If we choose dx_1 and dy_1 in such a way that the liquid moves away from W along a line going through point W then is:

$$dx_1 : x_1 = dy_1 : y_1 = d\lambda$$

in which $d\lambda$ is positive; (36) now passes into:

$$RT \cdot dK = - (1 - x_1 - y_1) (rx^2 + 2sxy + ty^2)_1 e^m \cdot d\lambda.$$

If we exclude dimixtion into two or more liquids, then follows that dK is negative, from which follows the above-mentioned property.

We may easily deduce the results obtained in this and previous communications and many other also with the aid of the ζ -surface of the vapours and liquids.

(To be continued).

Histology. — "*Reversible Gelation and Fixation of Tissues.*" By Miss M. A. VAN HERWERDEN. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of May 29, 1926)

The process of fixation is technically known to the histologist in details, but the physical-chemical event underlying this sudden death of the protoplasm is as yet perfectly unknown.

It seems to me that researches on reversible gelation in the living protoplasm enable us to get an insight into the course of the process of fixation. Experiments in this direction have convinced me that reversible gelation lies on the road to irreversible coagulation, therefore, as far as the tissues are concerned, to *fixation*.

Although there is not yet a constant definition in colloid chemistry of a gel, the most current idea is that the strongly hydrated micellae are united into a threedimensional structure, while inbetween one finds the disperse phase in ultramicroscopical canaliculi. In favour of such a structure (which is easily broken down) is e.g. the possibility — noticed as well in inorganic gels (ironoxide gel) as in living protoplasm — to change suddenly a gel through a simple shaking movement into a less viscid phase (temporary disappearance of mitotic figures a.s.o.).

Having gone beyond the limit of reversibility of such a gel, one has to deal with irreversible coagulation. It is even to be questioned whether there is a sharp limit between gelation and coagulation, whether the change is not a very gradual one. A gel may be regarded as the state of a colloid of which the equilibrium is near coagulation.

Where there is coagulation there is also a tendency to flocculation, as a result of local discharges and dehydration, and a tendency to the formation of filaments (threads of fibrin, tissue fibrils, etc.) Thus one gets a gradual transition of reversible protoplasm gelation to the histological fixation.

In my experiments here following ¹⁾ pointing in this direction, purposely poisons have been chosen which are used in the technique of fixation in histology:

If *Paramaecium aurelia* is put into 1% formol, death results. The animals extrude their trichocysts, small rods at the periphery of their body, which, on getting outside, harden into a network of long interwoven stiff threads, a process which often precedes the death of these animals and which reminds one at first of the sudden formation of threads of fibrin. This

¹⁾ These form part of a more extensive research on reversible gelation. (These Proceedings 27, p. 867 and Archiv. f. exp. Zellforschung, Bd. I, 1925, p. 145).

is doubtlessly a process worth studying from a colloid-chemical point of view, but on which I cannot dwell here.

Thus in 1 % formol an irreversible coagulation takes place, which already may be called fixation although this fixation is better if one increases the percentage, so that death occurs before the trichocysts have been extruded. If, however, *Paramecia* of the same culture are put into 0.01—0.1 % formol, a solidification takes place also, but this is no fixation, but a reversible gelation, with which — if only one washes out in time — life is not at stake. The condition returns to normal after washing the animals in ditchwater.

But this is only the case if the formol is soon eliminated, time factor being of considerable influence. It is not without importance that with a solution of gelatine one can do a very similar experiment. If one adds to 9 parts of a 10 % warm solution of gelatine one part of formol, there is no visible change. The gelatine remains transparent even after solidification on cooling the warmed solution has taken place. That the formol, however, has caused a real change in the structure of the gelatine, appears from the fact that on heating the gelatine does not dissolve as before. *The reversible gel has been changed into an irreversible coagulum.*

If, however, heating takes place very soon after adding formol, (e.g. within half an hour) the process is still reversible.

This shows the analogy with what occurs in protoplasm during the process of gelation and coagulation — with this difference, that in the polyphasic protoplasm the relations are of course much more complicated.

The change which takes place in the infusorium may be studied easily — in the dark field as well as with the light field illumination — on adding 0.01—0.1 % formol under the coverglass. The initially optically empty macronucleus becomes quite opaque while the animal moves slowly. The rigidity of the body appears already in its collisions against companions. The ordinary supple, pliable movements are lost.

The foundation of this process on a gelation was even more convincingly demonstrated in another protozoan, *Actinophrys sol*, which stopped its Brownian molecular movement. I have used this criterion before in another research on leucocytes from the buccal cavity treated with acetic acid¹⁾. *Actinophrys* has a round body, surrounded by a halo of fine protoplasmic threads. The type of the different animals changes according to their age and culture medium. In an old culture the animals with coarse opaque granules are preponderant; in younger cultures there are many of which the protoplasm shows a clear Brownian movement. The latter may be stopped by adding a 0.01 % formol solution and restored completely after rinsing out — an example of a reversible change of phase. Simultaneous with the cessation of the Brownian movement *Actinophrys* detaches from the slide and tends to roll like a ball through the microscopical field.

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a phenomenon which may also be noticed in Amoebae which have been solidificated artificially.

Besides with formol one can also show with another fixative, having a toxic action on the cell, that reversible gelation leads to fixation. With 0.001 % sublimate in water it is namely possible — if quickly rinsed out — to get reversible gelation. With a higher concentration fixation takes place. Just as with formol one can stop with sublimate in Actinophrys the Brownian molecular movement reversibly.

Another fixative commonly used, picric acid, also gives a reversible reaction with a concentration of about 0.01 %, but less easily than formol or sublimate. The time limits between gelation and coagulation approach each other nearer, making it more difficult to observe the phenomenon.

It seems to me that our insight into the nature of fixation is increased by the knowledge that coagulation with a weaker concentration of the fixative goes through a reversible phase. The limits of reversibility are probably fixed by a number of factors among which however — if one experiments with the same substance — concentration is the chief thing. In 0.01 % sublimate the change of gel to coagulum is so rapid that the time limits are too near to intervene experimentally. In 0.001 %, however, the extension of these limits permits to reverse the course of the process and to prolong life, preventing that the change of electric charge of the micellae causes a dehydration and a lasting increase of aggregates, together with a decrease in surface, which is not compatible with the continuation of life.

There are examples in nature in which time, without noticeable outside influence, accomplishes the same change of reversible gel to irreversible coagulum, e.g. the change of silicic acid gels which coagulate in the course of time. Analogies, although of a more complicated nature — owing to the polyphasic nature of the protoplasm — are to be found in living nature in phenomena of senescence.

An important question, the solution of which depends only on the suitability of the material to be tested, is the degree of reversibility in different phases of life. There is every reason to expect that, as life progresses, the reversibility of the started gelation diminishes. Is e.g. the time elapsing between the moment of the still reversible gelation and coagulation different in old and in young cells? — a question which I hope to take in consideration soon. Does this also hold good for processes which take place in the cell normally? It may be expected that also there in the future a similar change has to be tested.

Is not e.g. the increasing cloudiness shown by transparent organisms, which are old or in a state of depression, the manifestation of such an increasing tendency to coagulation, gradually leading to death? Is in such animals — as I described before in fresh water Daphnia — the less transparent diseased organism not to be distinguished at once from the normal, even without this being caused — as shown by measurements — by a general loss of water? One glance at a film with kinematographically

reproduced infusoria — if the eye is accustomed to such matters — is often sufficient to distinguish the animals which have suffered from the ultraviolet light of the arc lamp (perhaps there are some decrepit ones among them) through their cloudy protoplasm and very opaque macronucleus from the normal ones, even if they are still moving about.

That pathological cells show irreversible changes of phase is also known from the aspect of the cloudy swelling. Here also there may be expected that a reversible gelation precedes coagulation. For physiologic-histologic as well as for pathologic research there is an important area, as yet for the greater part unknown, which the biologist may approach in the dawning light on the colloidal properties of protoplasm.

Histology. — "*Day and night period in nuclear divisions.*" By C. E. DROOGLEEVER FORTUIN—VAN LEYDEN. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of May 29, 1926).

In 1916 I tried to demonstrate that in different tissues of young cats, i.e. in the mesenterium, the epithelium of the cornea, and the epithelium cells of the crypts of Lieberkühn, the number of karyokineses is higher at night than in day-time, consequently there exists a certain periodicity in the number of karyokineses on different hours of the day. Further research showed that also other tissues of the cat, i.e. of the blood producing organs (thymus, lymphgland, spleen, and bone-marrow) showed the same phenomenon with two exceptions. In all cases, the bone-marrow excepted, I found a minimum of nuclear divisions at 10½ a.m. In the bone-marrow too I found at that time a low percentage of dividing nuclei, but the minimum was at 2½ p.m. The maximum of divisions in all cases was at 10½ p.m. or at 2½ a.m. with the exception of the spleen and the thymus which showed a great irregularity at 2½ p.m. These two cases, however, are the only exceptions. Different tissues of tadpoles did not show such a regularity in the nuclear division at different times of the day. Yet I would not be surprised if a more systematic research with short intervals on several days would demonstrate a certain rhythm.

There are several opinions on the periodicity of the nuclear division. I should like to discriminate the periodical and rhythmical nuclear division. The periodical nuclear division then would be the regular alternation of maxima and minima in the number of karyokineses at the same hours of a day. In that case we might speak about a daily period. The rhythmical division, however, should be the occurrence of the division in waves, maxima alternating with minima, independent of day and night.

In different lower plants a periodical nuclear division has been demonstrated. Very well known is the case of *Spizogyra* in which alga nuclei and cells divide at night. In higher plants less examples are known. ALEXANDER GURWITCH denies the existence of periodicity in the nuclear division. Already in 1910 GURWITCH has ascribed to chance whether in a certain moment a cell would divide or not. The number of causes inducing a cell to divide would be so large that one gets the impression as if the occurrence of the nuclear division is accidental. In tissues always an accidental number of nuclei would be dividing depending upon temperature, light, and several other factors but independent of day and night. In 1924 GURWITCH supposes rays to exist, radiating from the root-

tips and causing the mitotic division. He found those cells to which the rays were directed to be more dividing than others. Also in animal tissues GURWITCH has found those rays, e.g. in the wounded cornea of frogs. Those rays would cause the repair of the wounded tissue. KARSTEN investigated root-tips of *Vicia Faba* and *Zea Mais*, cultivated in dark, and vegetation-points of *Pisum sativum*, *Zea Mais*, and *Pinus Austrica*, cultivated in alternating light and dark. He denies an occurrence of a periodicity in the root-tips. In the vegetation-points, however, he demonstrates a maximum of nuclear divisions at night but no distinct minimum as much as I can see from his tables. This period is caused by the regular alternation of day and night. Enlightening during the night and darkening during the day causes two maxima with a difference in time of 12 hours. KARSTEN attributes this phenomenon to the continuation of the inherited old period in some vegetation-points and to direct adaption to the new time of enlightening of other ones. Continuous enlightening eliminates the period and inhibits the growth.

BERINSOHN has investigated the influence of light and dark on the root-tips of *Allium Cepa*. He found less divisions in daylight than in the dark, if he kept the same onion first in light and afterwards during several hours in the dark. But as he never fixed his material during the night, one cannot draw any conclusion from his tables about the existence of a day and night period. BERINSOHN concludes that in root-tips of *Allium Cepa* the light inhibits the karyokinesis, in which point he agrees with KARSTEN, when speaking of vegetationpoints of different plants.

A. FISCHER proved photographically that the fibroblasts of an embryonic chicken-heart cultivated in vitro at a temperature of 39° C., are dividing rhythmically. Times of active cell divisions alternate with times of rare divisions or without any division. FISCHER calculates that for his culture the time between two waves of division is about 10 hours. This rhythmic division only occurs in cells of a tissue, not in single cells.

Being of the opinion that the results of my first research indicated the existence of a periodical nuclear division, I decided to continue my research and to investigate, whether still another mammal besides the cat shows a day and night period and which factors have influence upon the process of nuclear division in tissues. For the first research I chose the mouse, for the second I decided to investigate the influence of light as BERINSOHN has done and besides the influence of changes in the permeability in the cell, the results of which I hope to publish later. As a subject for the latter researches I chose the onion, *Allium Cepa*, because, as BERINSOHN correctly remarks, experimenting with animals has this drawback that we always have to do with different individuals showing many individual differences whereas the root-tips of one onion belong to one individual.

Before discussing my researches, I wish to point out that the times indicated in this paper differ about half an hour from the real time of the sun, because Peking has the time of the meridian of Shanghai which city is

situated farther to the east than Peking. So in Peking it is in reality always half an hour earlier than the clock indicates.

My method of research was the same as before. Every four hours, i.e. at 11 a.m., 3 p.m., 7 p.m., 11 p.m., 3 a.m., 7 a.m., the different tissues were fixed. Afterwards they were cut, stained, and in one out of 5 sections the total number of nuclei was counted and the number of nuclei in division. The totals of the counted nuclei and of the nuclei in division were taken and the percentage of karyokinesis calculated. If necessary, the probable error was calculated with the formula :

$$\pm \sqrt{\frac{\text{percentage of dividing nuclei} \times \text{percentage of non-dividing nuclei}}{\text{total number of nuclei}}}$$

On the 1st and 2nd of December 1925, I fixed the small intestine and the cornea of mice, about two weeks old, in the solution of Carnoy. The sections were stained with Heidenhain iron-hematoxylin.

The number of karyokineses in the crypts of Lieberkühn was as follow :

TABLE I.

| Time | Percentage of kar. | Number of counted nuclei |
|---------|--------------------|--------------------------|
| 3 p.m. | 2.008 | 5030 |
| 7 p.m. | 2.809 | 5518 |
| 11 p.m. | 2.245 | 4985 |
| 3 a.m. | 1.508 | 5105 |
| 7 a.m. | 3.449 | 5102 |
| 11 a.m. | 3.763 | 5447 |

Table I shows a series of percentages with the exception of the percentage at 3 p.m. The probable error calculated with the formula given above is

$$\sqrt{\frac{2,008 \times 97,992}{5030}} = \pm 0,19.$$

The number 2.008 — 3 times its probable error could still be considered to be the same number. But $2.008 + 3 \times 0.19 = 2.578$, and this number is lower than 2.809. Therefore the percentage of 3 p.m. certainly does not belong to the series.

So we see that the most active nuclear division is in the morning. There is a decrease in the afternoon and a minimum at night. In the cornea there was not a single division. The mice of two weeks were probably too old for this. These numbers in themselves are not very conclusive. Individual differences may be of much influence. Besides it was not certain that the mice were children from one mother as it was in the case of cats, which fact

makes it still more probable that there are individual differences. Yet I believe, if we always get such series with unimportant exceptions, there is an indication of the existence of a periodic nuclear division.

In plants we have the factors more in our own power. Here we can compare the tissues of one individual. Therefore, I put two bulbs of the onion, *Allium Cepa*, in glasses with water, close to each other in a room not heated, so that they had the same changes of temperatures. One of the onions was kept in full daylight, the other covered by a tin-box so that it was in the dark. When the onions had good strong root-tips, I started my experiment on Dec. 1—2. The weather was cold but the sun was shining. Every 4 hours the root-tips were fixed in corrosive sublimate and sodium chloride and the longitudinal sections later stained with iron-hematoxylin. The counting occurred in the manner described above. I confined myself to the region of the most active growth and the adjacent zone where not so many karyokineses were found but where still an active growth took place. I thought the counting of one half of the root-tip to be sufficient because straight root-tips, which I always choose, have nearly the same number of karyokineses in both halves as GURWITCH has found.

TABLE II. Onion in the dark.

| Time | Perc. kar. | Tot. numb. count. nucl. |
|-----------------|------------|-------------------------|
| Dec. 2, 7 a.m. | — | — |
| Dec. 1, 11 a.m. | 2.865 | 2513 |
| Dec. 1, 3 p.m. | — | — |
| Dec. 1, 7 p.m. | 3.567 | 1738 |
| Dec. 1, 11 p.m. | 6.315 | 2866 |
| Dec. 2, 3 a.m. | 3.779 | 3096 |

TABLE III. Onion in daylight.

| Perc. kar. | Tot. numb. count. nucl. | Time |
|------------|-------------------------|----------------|
| 3.453 | 1245 | 7 a.m. Dec. 2 |
| 0.887 | 789 ¹⁾ | 11 a.m. Dec. 1 |
| 2.486 | 2292 | 3 p.m. Dec. 1 |
| — | — | 7 p.m. Dec. 1 |
| 12.348 | 1563 | 11 p.m. Dec. 1 |
| 5.069 | 1634 | 3 a.m. Dec. 2 |

Table II shows the observed percentage of karyokineses of the onion in the dark, table III that of the onion in daylight. In both cases we find a striking maximum at 11 p.m. and a minimum at 11 a.m.; at the intermediate times a rise and fall from minimum and the reverse. The differences shown by the onion in daylight are much greater than those of the onion in the dark. I made a second experiment on January 16th—17th, 1926. The onions stood before a window in a room with central heating which caused the differences of temperature not to be great. The weather was bright and the sun shone so that one onion had plenty of light, the other onion again

¹⁾ This preparation was not very good so that I could only count the karyokineses in some sections. Out of 789 counted nuclei 7 were in division.

was covered with the tin-box. Root-tips were fixed again every 4 hours, now in the solution of FLEMMING, this fluid being more satisfactory than corrosive sublimate and sodium chloride. Staining occurred as before. This time also the different stages of karyokineses were counted. I reckoned all stages before the chromosomes in the aequatorial plate to the prophase, monaster and diaster to the metaphase and the stage of the chromosomes at the poles and two young nuclei to the ana- and telophase.

TABLE IV. Onion in light.

| Time | Tot. perc. kar. | Proph. | Metaph. | Ana. teloph. | Tot. numb. count. nucl. | Aver. perc. kar. |
|------------------|-----------------|--------|---------|--------------|-------------------------|------------------|
| Jan. 17, 7 a.m. | 11.11 | 6.28 | 2.15 | 2.31 | 4410 | 6.93 |
| Jan. 16, 11 a.m. | 8.61 | 4.27 | 2.32 | 2.01 | 3927 | |
| Jan. 16, 3 p.m. | 7.57 | 4.20 | 1.65 | 1.72 | 5933 | |
| Jan. 16, 7 p.m. | 6.91 | 3.33 | 1.80 | 1.78 | 4445 | |
| Jan. 16, 11 p.m. | 2.79 | 1.30 | 0.76 | 0.72 | 5525 | |
| Jan. 17, 3 a.m. | 4.58 | 2.78 | 0.92 | 0.88 | 2837 | |

In two root-tips of the same onion fixed 5 days at 11 a.m. and 11 p.m., the number of karyokineses was counted in the same manner as before, and I had the surprising result that I found nearly the same numbers as on the corresponding times 5 days earlier as may be seen in table V.

TABLE V. Onion in light.

| Time | Tot. perc. of kar. | Proph. | Metaph. | Ana. teloph. |
|------------------|--------------------|--------|---------|--------------|
| Jan. 21, 11 a.m. | 8.85 | 4.57 | 1.79 | 2.48 |
| Jan. 21, 11 p.m. | 2.36 | 1.86 | 0.11 | 0.30 |

Comparing these numbers 8.85 and 2.36 with the corresponding numbers of table IV, 8.61 and 2.79, we see that there is indeed not much difference between the number of karyokineses of the root-tips of the one onion on different days but at the same times.

Two root-tips of the same onion as the one used in table IV fixed at the same time, i.e. 7 a.m., nearly also had the same percentage of karyokineses, i.e. 12.48 and 12.92. The reason why I now found a higher percentage than in table IV at 7 a.m. is that this time I only counted the region just below the top where most of the karyokineses are found and not the adjacent region where the number of karyokineses already decreases. So I counted the nuclei of

the root-tip of table IV at 7 a.m. and found in this manner the percentage to be 12.48 instead of 11.11 in table IV. Therefore we must compare this number with 12.92, the number 11.11 being comparable with the other numbers of table IV. The latter case indicates a synchronic nuclear division in different root-tips of the same plant, whereas table IV and V indicate a regular returning of a daily period.

Table VI shows the percentages of the cells in division of the onion in darkness.

TABLE VI. Onion in darkness.

| Time | Tot. perc. kar. | Proph. | Metaph. | Ana. teloph. | Tot. numb. nucl. | Count. aver. perc. |
|------------------|-----------------|--------|---------|--------------|------------------|--------------------|
| Jan. 17, 7 a.m. | 7.78 | 2.48 | 1.11 | 4.19 | 2170 | 7.4 |
| Jan. 16, 11 a.m. | 8.00 | 4.97 | 1.18 | 1.85 | 2976 | |
| Jan. 16, 3 p.m. | — | — | — | — | — | |
| Jan. 16, 7 p.m. | 6.70 | 4.00 | 1.27 | 1.44 | 4728 | |
| Jan. 16, 11 p.m. | 7.00 | 3.37 | 1.48 | 2.16 | 3187 | |
| Jan. 17, 3 a.m. | 7.58 | 5.18 | 1.07 | 1.32 | 3088 | |

Careful examination of table IV and VI shows for both tables a series, table IV with a maximum at 7 a.m. and a minimum at 11 p.m., the intermediate numbers giving a regular transition between minima and maxima. Table VI gives a maximum at 11 a.m., a minimum at 7 p.m., and between these again a regular transition; here, however, the differences are very small as compared with the differences of table IV of the onion in light. The maximum is 8.00 % and the minimum 6.70 %, whereas 11.11 % and 2.79 % are the maxima and minima of the onion in light.

Table II and III also show these differences. In the onion in light the numbers are varying from 12.34 % to 0.88 %, in the onion in darkness from 6.3 % to 2.86 %.

The times of the maxima and minima do not agree in both cases. In table II and III is a maximum at 11 p.m., in table IV on the contrary a minimum at this time is found and table VI also shows a low percentage then. Table IV shows a maximum at 7 a.m. for the onion in light, table VI a maximum at 11 a.m. for the onion in darkness, at which time tables II and III on the contrary show a minimum.

A third experiment was made on March 18th and 19th at 11 p.m. and 7 a.m. The onions again stood in a room with central heating. On March 19th at 7 a.m. it was almost dark because early in the morning the sky was covered with heavy snow-clouds.

Table VII shows again that the difference of the numbers of karyokineses at different times is larger in the onion in light than in the onion in darkness.

| Time | Tot. perc. kar. | Tot. perc. kar. |
|--------------------|-----------------|-----------------|
| March 19, 7 a. m. | 11,38 | 8,35 |
| March 18, 11 p. m. | 15,03 | 7,59 |

In the latter case the numbers agree with the corresponding numbers of table VI. The numbers of the onion in light are very high. I am sorry not to have fixed root-tips at other times of the day.

Comparing my results with those of BERINSOHN whose tables are copied below (table VIII and IX), a great conformity may be seen.

TABLE VIII. According to BERINSOHN — in light.

| Time | Tot. numb. count. nucl. | Tot. numb. mitoses | Spir. loose chromos. | Mon. | Diast. | 2 young nuclei |
|----------|-------------------------|--------------------|----------------------|------|--------|----------------|
| 8 a. m. | 4000 | 0,00 | 0,00 | 0,00 | 0,00 | 0,00 |
| 11 a. m. | 4345 | 3,19 | 1,01 | 1,01 | 0,38 | 0,37 |
| 3 p. m. | 2290 | 2,05 | 1,13 | 0,61 | 0,21 | 0,38 |

TABLE IX. According to BERINSOHN — in darkness.

| Time | Tot. numb. count. nucl. | Tot. numb. mitoses | Spir. loose chromos. | Mon. | Diast. | 2 young nuclei |
|----------|-------------------------|--------------------|----------------------|------|--------|----------------|
| 6 a. m. | 4702 | 4,46 | 3,19 | 1,4 | 0,4 | 0,00 |
| 12 m. | 4204 | 4,28 | 3,49 | 0,49 | 0,48 | 0,09 |
| 6½ p. m. | 4034 | 3,06 | 1,60 | 0,71 | 0,24 | 0,49 |

Table VIII shows that the difference between maximum and minimum for the onion in light is $3.19\% - 0.00\% = 3.19\%$, whereas in table IX it is $4.46\% - 3.06\% = 1.40\%$ for the onion in darkness.

BERINSOHN concludes from his tables that in the dark the number of karyokineses in the root-tips of *Allium Cepa* increases, agreeing in this regard with KARSTEN, who says that in the vegetation-points of the stalk of *Pisum Sativum* and *Zea Mais* the maximum of embryonic growth is in the darkness of the night. In this regard I agree with both investigators. So

my tables show that the high percentages are at night or early in the morning when in winter there is still twilight, i.e. at 11 p.m. or at 7 a.m. It is possible that the true maximum of division lies earlier in complete darkness and that at 7 a.m. we have already passed this point. KARSTEN, who examined it every two hours, found the highest numbers varying between $9\frac{1}{2}$ p.m. and 6 a.m., whereas at 8 a.m. the number of karyokineses is still high but already lower than the maximum.

In contradistinction to KARSTEN and BERINSOHN, however, I would not conclude that the light does nothing else but inhibit the nuclear division. It is obvious that in general the maxima in the root-tips in daylight are higher and the minima lower than those of the root-tips always kept in dark. Comparing the average percentage of the karyokineses of the root-tips in light with that of the root-tips in dark, we see, that it is 6.93 in table IV against 7.40 in table VI. The average percentage in dark is therefore a little higher than in light. Table VII shows a much higher percentage for the onion in light but this case is not reliable, because we have only regarded the times with the largest numbers of karyokineses for onions in light and not the other times, whereas the percentages of plants in the dark nearly agree with the average at all times. The tables of BERINSOHN contain the same mistake. He only counted the number of karyokineses of the onion in light during day-time and did not regard the hours of the night and early morning, when KARSTEN and I found the maxima.

I conclude that the intermittent daylight and the darkness of the night cause a period in the nuclear division with maxima at night and in the early morning and minima at different times of the day or night. For different individuals the times of maxima and minima are different, for the same individual those times correspond on different days (compare table IV with table V). Probably, ordinarily in tissues, times of active nuclear division alternate with times of spare division and an inclination to a certain rhythm in the division is very likely.

When the light has inhibited the nuclear division for a certain tissue, there will be after the cessation of this stimulus an inclination to an increased division causing more karyokineses in this than there would be, if there had not been an inhibitive influence. We might say that the light inhibits the nuclear division but increases its potency.

Thus, table VI of the onion in the dark also shows a certain rhythm in the nuclear division. I have already drawn the attention to a series with small differences in the number of karyokineses. But calculating the probable error, we see, that the limits of the numbers cover each other. The limits of the highest percentage (8.00) for instance, are

$$8 \pm 3 \times \sqrt{\frac{8 \times 92}{2976}} = 8 \pm 1.47.$$

that is 6.53 and 9.47. 6.53 is smaller than 6.70, being the lowest percentage of table VI. So we are not justified in saying that there is a real difference

between maximum and minimum of this table as were in table IV. We no more have a right to conclude to the existence of a daily period.

Yet I will point out a certain rhythm in the different stages of the karyokineses. In table VI we observe at 3 a.m. a large percentage of the cells in the prophase, that is the beginning of the karyokineses (5.18=), followed by a low percentage in the prophase at 7 a.m. (2.48). At 11 a.m. this percentage has increased to 4.97. I regret, that the next stage which should be low, is lacking. At 7 p.m. the number is rather high (4.00), whereas it decreases a little at 11 p.m. (3.37), and increases at 3 a.m. (5.18). We are justified to say that there is a real difference between all these numbers except between 4.00 and 3.77, the limits of the highest being

$$5.18 \pm 3 \times \sqrt{\frac{5.18 \times 94.82}{3088}} = 5.18 \pm 1.17$$

and of the lowest

$$2.48 \pm 3 \sqrt{\frac{2.48 \times 97.52}{2170}} = 2.48 \pm 0.99.$$

We cannot deny a certain rhythm for the prophase. In the same way there exists a rhythm in the telophase with the difference, that where a maximum is found in the prophase, a minimum in the telophase occurs and the reverse. One gets the impression that nuclei which are at a certain time in the prophase are four hours later in the ana- or telophase. The high percentage of 3 a.m. in the prophase and of 7 a.m. in the telophase for instance, is very remarkable. Tables IV and V show that the percentage of the prophase in root-tips in light follows the same period as that of the total number of karyokineses.

Although it would be very difficult to estimate the duration of one nuclear division, especially because it depends upon temperature and perhaps also upon light, I believe, that table VI shows that here the karyokinesis lasts longer than four hours because we meet a high percentage in the prophase, after four hours again in the telophase. BERINSOHN estimates the time of the division in his root-tips at three to four hours. From my table IV of the root-tips in light, I should also conclude to a time of less than four hours. Differences of temperature and light may be of great influence here.

I agree with BERINSOHN regarding the space of time in which the different stages remain in so far that the time of the prophase must be the longest one. In my own tables and in those of BERINSOHN always the highest numbers are found in the prophase, those of the meta- and ana-telophase being more or less the same, at least in my tables. This is not in accordance with the remark of WILSON (p. 131) "There is reason to conclude that the metaphase is a condition of relative stability in which the mitotic figure often remains for a considerable time."

Conclusions.

I. In the crypts of LIEBERKÜHN of the small intestine of young mice there has been found, with one exception, a daily period for the nuclear division.

II. In the root-tips of *Allium Cepa* there has been found :

a. Synchronic nuclear division for root-tips of one specimen.

b. In constant darkness a rhythmical division with alternating maxima and minima.

c. Enlightening during day-time changes this rhythm into a daily period with the largest numbers of division in darkness.

d. Although the maxima are higher and the minima are lower in the onion kept in light compared with the one kept in dark, their average percentage of nuclear division is nearly equal.

Finally I wish to thank Dr. DAVIDSON BLACK, Head of the Department of Anatomy of the Peking Union Medical College, for having given me the opportunity to carry out this research.

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Mathematics. — “On a Function which in any Interval Assumes any Value a Non-Enumerable Number of Times, and on a Function Representing a Rectifiable Curve which in any Interval is Non-Differentiable a Non-Enumerable Number of Times”. By Prof. J. A. BARRAU. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 27, 1926).

The communication of J. WOLFF; “On a Function which in any Interval Assumes any Value on a Non-Enumerable Set of Points”,¹⁾ led me to observe that such a function may also be defined in a more elementary way; also M functions y of N variables x so that in the representation of the x space on the y space defined in this way, the image of any N -dimensional region of the former space covers the whole y space a non-enumerable number of times.

The former is done by the aid of the theorem:²⁾

“Any number x ($0 \leq x \leq 1$) is developed into a binary fraction p (if two developments are possible we choose the one which ends in a repeating zero, not the one with repeating one). If u_n is the arithmetic mean of the first n figures behind the comma in p , for any x the sequence u_n ($n = 1, 2, 3, \dots$ etc.) has an upper limit y ; y is considered as a function of x .

On any sub-interval of the interval $0 \leq x \leq 1$ this function assumes all values from 0 to 1 on a set of values of x which has the same power as the continuum”³⁾.

For u_n we may choose as well for instance the mean of the first n figures with *odd* order numbers; in this case an arbitrary choice of the figures on the *even* places after a definite order number (hence in a definite interval), has no influence on the value of y (which by a proper choice of the odd places may be made equal to any number from 0 to 1); in this way it is easily seen that the x -values corresponding to the same y , are non-enumerable in the interval in question.

In order to define in a similar way M functions y_i of N variables x_j , we understand by $(u_i)_n$ the mean of the first Nn of those figures in the N developments p_j (for each j the first n) of a point of the x -space of which the order number

$$\rho \equiv i \pmod{M+1} \quad 4)$$

¹⁾ These Proceedings 29, p. 127.

²⁾ Published as N^o. 44, deel XIV, Wiskundige opgaven, Amsterdam, where a proof will be given afterwards.

³⁾ This function has the property that it has any period $\omega = 2^{-N}$ (N a positive integer).

⁴⁾ We may also assume $\rho \equiv i \pmod{M}$.

and we put

$$y_i = \overline{\lim}_{n \rightarrow \infty} (u_i)_n .$$

After a certain order number (hence in a certain sub-hyper cube of the hyper cube $0 \leq x_j \leq 1$), we may choose the figures with order numbers $\varrho \equiv i \neq 0$ in such a way that any y_i assumes any desired value from 0 to 1; the choice of the figures with order numbers $\varrho \equiv i = 0$ has no influence on the y_i ; hence the power of the set of points x in the sub-hyper cube (and a fortiori in a region containing this), to which there corresponds a given point y , is the same as that of the continuum.

The restriction of x_j and y_i to the values from 0 to 1 is immaterial and is e.g. annulled by repeating the unit-hyper cube of x -space in the directions of the edges and by taking the function

$$z_i = \frac{1-y_i}{y_i} = \frac{y_i}{1-y_i}$$

in stead of any y_i .

A function $y = f(x)$ which represents a rectifiable curve of given length $\lambda > 1$ between the points A and B , (the points 0 and 1 of the X -axis), and which above any sub-interval of AB contains a non-enumerable number of points without tangent, may be defined in the following way:

Between A and B we construct a series of broken lines P_n ($n = 0, 1, 2, \dots$ etc.), with lengths

$$l_n = \lambda^{1-2^{-n}}, \text{ so that } k_n = \frac{l_n}{l_{n-1}} = \lambda^{2^{-n}}$$

and:

$$\lim_{n \rightarrow \infty} l_n = \lambda \quad ; \quad \lim_{n \rightarrow \infty} k_n = 1.$$

P_0 is the line AB .

The odd angular points of P_1 lie on the X -axis in the points $0, \frac{1}{4}, \frac{3}{4}, \dots, 1-2^{-m}, \dots$

The even angular points lie above the middles of these successive segments, so that P_1 consists of the legs of a series of isosceles triangles. The vertices of these triangles are chosen so that the sum of the legs is successively $a_1, a_1 x_1, a_1 x_1^2, \dots, a_1 x_1^m, \dots$ times their base. Here a_1 is an arbitrary number so that $1 < a_1 < k_1 = l_1$ and x_1 is defined by the condition that P_1 has the length l_1 ; hence:

$$\begin{aligned} \frac{a_1}{2} + \frac{a_1 x_1}{4} + \frac{a_1 x_1^2}{8} + \dots &= k_1 = l_1 \\ 2 > x_1 &= \frac{2k_1 - a_1}{k_1} = \frac{2l_1 - a_1}{l_1} > 1. \end{aligned}$$

If φ_m and ψ_m are the angles which (from left to right) resp. the

ascending and the descending leg of the m^{th} triangle make with the X -axis,

$$\lim_{m \rightarrow \infty} \varphi_m = \frac{\pi}{2} \quad ; \quad \lim_{m \rightarrow \infty} \psi_m = \frac{\pi}{2}.$$

There is, therefore, a value M_1 of m for which and above which $\varphi > \alpha$, $\psi > \alpha$, if α represents an arbitrary angle in the first quadrant, e.g. $\frac{\pi}{4}$.

On each side of P_1 , now from right to left, we choose again the dividing points on $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots$ of this side as odd angular points of a branch of P_2 (with the right extremity of this side as first angular point). The even angular points of this branch of P_2 lie again perpendicularly above the middles of the segments between the odd ones, so that P_2 is formed by the legs of a series of triangles which, in this case, are not isosceles. The vertices of these triangles are chosen so that the proportion of the sum of the legs to the base is successively (from right to left)

$$a_2, \quad a_2x_2, \quad a_2x_2^2, \dots, \quad a_2x_2^m, \dots$$

Again

$$1 < a_2 < k_2$$

and

$$2 > x_2 = \frac{2k_2 - a_2}{k_2} > 1,$$

so that the length of P_2 is indeed l_2 .

For the angles ψ_m and φ_m which, counted from a sufficiently large m , are surely positive, we have again as above:

$$\lim \psi_m = \frac{\pi}{2} \quad ; \quad \lim \varphi_m = \frac{\pi}{2},$$

hence for $m \geq M_2$,

$$\psi > \alpha \quad ; \quad \varphi > \alpha.$$

Thus we go on, starting alternately from the left and from the right and always choosing $1 < a_n < k_n$ and $x_n = \frac{2k_n - a_n}{k_n}$: on each side of P_{n-1} we construct a broken line which is a branch of P_n , and always $l_n = \lambda^{1-2^{-n}}$.

The ordinate of a point x is cut by the lines $P_1, P_2, \dots, P_n, \dots$ in points with ordinates $y_1, y_2, \dots, y_n, \dots$; these values form a limited sequence which, if x belongs to the binary scale, consists of terms which remain equal from a definite n , but, if x is not a term of the binary scale, increases monotonely. Hence we may put

$$y = \lim y_n \equiv f(x).$$

It is clear that the curve represented by this function, is continuous and rectifiable and has the length λ .

In the points of the binary scale it is neither differentiable on the right nor on the left, for from both sides there approach points as well in a fixed non-perpendicular direction (odd points of a polygonal branch) as in a variable direction, approaching to a perpendicular (even points of a higher branch); the set of these binary points is, however, enumerable.

Any interval β contains also non-binary points which have no differential coefficient. Let us consider a part γ of this interval above which there lies exactly a complete side P_n ; such a part may always be found if we choose n sufficiently large.

P_{n+1} has above γ a part consisting of legs of an infinite number of triangles of the same series for which $\varphi > a$; we choose two ascending sides of them (now and in the future always counted from left to right), and call them $\delta_{0,0}$ and $\delta_{0,1}$.

Above each of these sides P_{n+1} has a part for which $\psi > a$; in it we choose two descending sides above each δ : $\varepsilon_{0,00}$ and $\varepsilon_{0,01}$ (above $\delta_{0,0}$), $\varepsilon_{0,10}$ and $\varepsilon_{0,11}$ (above $\delta_{0,1}$). If we go on like this a point x of the interval γ corresponds to the binary development of any number from 0 to 1, around which we may contract an interval so that the chord of the curve above that interval successively ascends more steeply than a and descends more steeply than a ; $f(x)$ is, therefore, non-differentiable in that point.

Hence it is evident that the set of these points x on γ (hence a fortiori the set of all the points on β where $f(x)$ is non-differentiable) has the same power as the continuum.

Zoology. — "*On the food of Reef-corals.*" By H. BOSCHMA. (Communicated by Prof. C. PH. SLUITER.)

(Communicated at the meeting of June 26, 1926).

Fifty years ago it was generally believed that the food of reef-corals consisted of the plankton organisms floating in the surface water of the sea. The first note on investigations of the food of reef-corals, as far as I am aware, is found in a paper by Sir JOHN MURRAY (1889), based on observations during the Challenger Expedition. He mentions shortly that during this Expedition actual observations on the feeding of corals were made and also investigations on the contents of the gastric cavity. These researches convinced him that the food of the reef-corals consisted of the organisms floating in the sea water.

In the later part of the 19th century KRÄMER (1897) studied the plankton of Samoa and the other regions, as well in tropical as in colder seas. According to him in the Baltic Sea the quantity of plankton is about four times as large as that in the tropics, but he still is convinced that this quantity is sufficient for the feeding of reef-corals.

Now it is well known that reef-corals contain in their entoderm numerous unicellular algae, zooxanthellae. It is generally understood that the association of the two organisms, polyp and algae, is a kind of *symbiosis*, which may be defined as "*a condition of conjoint life existing between different organisms that in varying degree are benefited by the partnership*" (NUTTALL, 1924, p. 213). The nature of this symbiosis has been studied in a number of actinians. BRANDT (1883) made various experiments with *Anthea cereus* and *Aiptasia diaphana*, actinians which harbour zooxanthellae in their entoderm. Some of these were kept without food in daylight, others without food in darkness. After some months the actinians in daylight were as healthy as at the beginning of the experiment and had but slightly diminished in size. Those which were kept in darkness had after two months lost their zooxanthellae and had died from starvation or were in a decidedly weakened condition. BRANDT concludes from those experiments that the algae in light contributed food in a dissolved form to the actinians. He did not find any zooxanthellae which were being digested (at this time it had recently been discovered where the food of actinians is digested and BRANDT probably did not know this.)

TRENDELENBURG (1909) made accurate gas analyses of the water in which actinians with zooxanthellae in their tissues (*Anemonia sulcata*) lived, as well in darkness as in light. He found that the algae by day furnish oxygen to the polyp and at night utilize oxygen, whilst they derive

their carbon dioxide partly from the animal and partly from the surrounding water. PÜTTER (1911) found that in *Aiptasia* the algae use the nitrogenous waste products of the polyps (ammonia) for the synthesis of their proteins. These researches together with the results obtained by BRANDT show that the association of the actinians with the zooxanthellae has to be regarded as a true symbiosis.

After the study of the contents of the gastric cavity of coral polyps GARDINER (1903) came to the conclusion that the food of reef-corals consists chiefly of their zooxanthellae. In preserved polyps of *Pocillopora* and "*Astraea*" only in 1 or 2 per cent remains of foreign organisms were found. GRAVIER (1908) also was convinced that the food of reef-corals consists for the greater part of the symbiotic algae; his opinion was based on GARDINER's and his own investigations on the contents of the gastric cavity of the polyps and KRÄMER's statement on the scarcity of the plankton in warmer seas. I came to a similar conclusion (BOSCHMA, 1924) after the study of the contents of the gastric cavity of a great number of East Indian coral polyps. The polyps of smaller size (*Porites*, *Acropora*) very seldom contained other matter than mucus with zooxanthellae, those of larger size (*Fungia*, *Favia*) had a mass of mucous matter in their gastric cavity which besides many zooxanthellae consisted of remains of different plankton organisms. I concluded that the food of reef-corals for a considerable part is derived from the zooxanthellae which are digested by the polyps.

These investigations, however, do not prove that the coral polyps derive a part of their nourishment from their zooxanthellae. Coelenterates have no separate system for ingestion and for excretion. These contents of the gastric cavity were for a part certainly food-remnants (the animal remains) but the zooxanthellae might as well have been the products of excretion: the surplus of the quantity needed by the polyps in the tissues. The zooxanthellae are constantly multiplying in the entoderm; when the cells contain too many algae these are extruded into the gastric cavity. The fact that these algae are found in the gastric cavity does not prove that they are being digested here.

Besides the study of the contents of the gastric cavity feeding experiments have been made on reef-corals. The first who made more or less elaborate investigations of this kind was DUERDEN (1906). He saw that small particles, also plankton organisms, became entangled in the mucus which is secreted on the surface of the polyps of *Fungia* and *Favia*. This mucous layer is afterwards ingested and the food-particles it contains are digested. Much more extensive feeding experiments were made by VAUGHAN (1912) with Floridian and Bahamian reef-coral polyps. Meat of crabs, molluscs or fish, or meat juice was given to the polyps and these substances always brought about the feeding reactions. Also animal plankton (copepods) was eagerly ingested by the polyps. In fact the coral-polyps would feed very readily upon all kinds of animal matter. VAUGHAN also tried vegetable matter: pieces of seaweed and mats of diatoms. This

was invariably refused. Only when the vegetable objects were coated by animal matter it was taken, but soon afterwards the seaweed or the diatoms were removed in an undigested state from the gastric cavity. On the reef also observations were made on corals which had captured animals from the sea-water. VAUGHAN concluded from his experiments: "*The food of reef-corals consists solely of animal matter*" (VAUGHAN, 1912, p. 161). This conclusion in my opinion should be put in a slightly modified form: "*The food of reef-corals as far as it is taken from the outside, in all probability consists solely of animal matter.*"

The investigations of METSCHNIKOFF (1880), KRUKENBERG (1880) and WILLEM (1892) showed that the food of actinians is digested in the mesenterial filaments, the free edges of the mesenteries. Next to the border with its numerous nematocysts there is a region the entoderm cells of which ingest small particles in an amoeboid way and digest these particles there. Further particulars on the digestion of actinians are found in papers by MESNIL (1901) and JORDAN (1907). Besides intracellular digestion in the mesenterial filaments there is much evidence for the secretion of a digestive fluid on the larger objects which come into contact with the mesenterial filaments. This fluid causes the disintegration of these objects into small particles, fit to be ingested by the entoderm cells (cf. also BIEDERMANN, 1911).

In 1924 I made some experiments on the feeding reactions and the digestion in the coral *Astrangia danae* in the Marine Biological Laboratory at Woods Hole, Mass. (BOSCHMA, 1925a). As in actinians the food is digested in the mesenterial filaments in the region next to the border with its nematocysts. This can be proved by mixing the food with some colouring matter (e.g. India ink, litmus or ammonium carminate). After ingestion of the food the food-vacuoles then contain coloured particles. Most of the colonies of *Astrangia danae* have polyps without any zooxanthellae; the food of these polyps consists exclusively of plankton. On the other hand there are other colonies the polyps of which harbour a great many zooxanthellae in their tissues. In the contents of the gastric cavity of these polyps when freshly collected, besides remains of plankton organisms zooxanthellae are always present. Moreover in the mesenterial filaments of such polyps with zooxanthellae a number of these algae are found in the exact place where after feeding experiments the food is ingested. These zooxanthellae in the mesenterial filaments have lost their natural appearance. Whilst the algae from the entoderm cells of the tentacles and oral disk have a more or less uniform yellow colour, those in the mesenterial filaments contain some brown patches and parts of the cells are completely discoloured. They have undergone here similar changes as occur when zooxanthellae die. It seems to me safe to conclude from this that the disintegration of the zooxanthellae in the mesenterial filaments is due to their being digested here by the polyps.

When polyps containing zooxanthellae are abundantly fed with animal

matter no more zooxanthellae are ingested in the mesenterial filaments. Those already present gradually diminish in size and disappear, evidently by being completely digested, and in the course of a few days during abundant feeding the mesenterial filaments of these polyps become completely devoid of zooxanthellae.

Experiments conducted in the Bermuda Biological Station, (cf. BOSCHMA, 1925*b*) and in the Laboratory of the Carnegie Institution of Washington at Tortugas with various Coelenterates which contain zooxanthellae in their entoderm, viz. reef-corals, actinians, Zoanthids, and Gorgoniids, proved that the mesenterial filaments of these organisms in the natural state always contain a number of zooxanthellae in various stages of disintegration. The symbiotic algae therefore constitute a part of the food of the polyps. As for reef-corals it was possible to make the mesenterial filaments devoid of zooxanthellae by feeding the polyps abundantly with animal food.

To study the role of the zooxanthellae in the feeding of actinians I made some experiments with *Cribrina xanthogrammica* at the Scripps Institution for Oceanography in La Jolla, Calif. The feeding reactions of this species have been studied by GEE (1913). The study of the mesenterial filaments in my experiments gave the following results: Hungry actinians kept in full light ingested many zooxanthellae in the mesenterial filaments and extruded a great number of zooxanthellae in masses of mucus through the mouth. For a month they did not diminish noticeable in size. Though a great number of zooxanthellae were extruded through the mouth the actinians kept the same colour. In diffuse daylight the results were quite similar to those in full sunlight. In general the quantity of zooxanthellae ingested in the mesenterial filaments of hungry polyps in diffuse daylight exceeded that of those in full sunlight. In darkness hungry polyps lost the greater part of their zooxanthellae, thereby assuming a much lighter colour; on the whole these actinians were in an ill state of health, probably owing to the lack of oxygen: as the zooxanthellae could not assimilate in the darkness they could not act as source of oxygen to the tissues of the polyp. The size of these actinians after a month in darkness was about one half of what it originally was. Well fed actinians in all different intensities of light increased in size, though those in darkness were more or less ill, as often bladdery lobes of the stomodaeum were extruded through the mouth, probably due to the lack of oxygen. A result of abundant feeding of the polyp was that as well in light as in darkness no zooxanthellae were ingested in the mesenterial filaments. Great quantities of these algae were removed from the gastric cavity through the mouth. After a month the well fed actinians were often more than twice as large as at the beginning of the experiment.

From these experiments we may conclude the following statement: *The food of reef-corals and of actinians which live in association with zooxanthellae consists for a part of these zooxanthellae and for another part of animal matter. The polyps try to get as much animal matter as*

possible, but in case of starvation they depend chiefly upon the zooxanthellae. The surplus of the rapidly multiplying zooxanthellae in the tissues is removed from the entoderm cells to the gastric cavity, and, as far as needs may be, these algae are digested or removed through the mouth.

The experiments prove that the zooxanthellae are not ingested in the mesenterial filaments because they happen to be in the gastric cavity. The algae are ingested as a source of food, but when sufficient other food is available, they are simply removed through the mouth.

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Chemistry. — "*Optical resolution of chlorobromoacetic acid.*" By H. J. BACKER and H. W. MOOK. (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of June 26, 1926).

Chlorobromo- and chloriodomethanesulphonic acids show a remarkable difference.

POPE and READ ¹⁾ resolved the latter acid into its enantiomorphous components, which were completely stable. They did not, however, succeed in resolving the chlorobromo-compound.

After it had been shown, that in the analogous case of chlorosulphoacetic acid the failure was not due to partial racemism of the alkaloidal salts as had been supposed before ²⁾, but that it was caused by a strong tendency to racemisation ³⁾, this same cause was accepted by READ and MCMATH ⁴⁾ for the chlorobromomethanesulphonic acid. They stated, that under special conditions salts of different rotation were formed, but that in aqueous solutions only inactive mixtures could be obtained.

This striking difference between the chlorobromo- and chloriodo-compounds raises the question, whether perhaps the close chemical analogy of chlorine and bromine might prevent resolution or promote racemisation.

In the literature we only found two examples of resolvable chlorobromo-carboxylic acids, the asymmetry of which is due to the carbon atom bearing chlorine and bromine, namely fluorochlorobromoacetic acid $\text{CFCIBr.CO}_2\text{H}$ ⁵⁾ and methylchlorocyclohexyl-chlorobromoacetic acid $\text{C}_7\text{H}_{12}\text{Cl.CClBr.CO}_2\text{H}$ ⁶⁾.

In both acids the asymmetric carbon atom has no hydrogen atom and yet in the first case the tendency to racemisation is so strong, that the active acids could not be separated from the alkaloid salts of different rotation.

To test the influence of chlorine and bromine attached to an asymmetric carbon atom, we have now examined one of the simplest examples namely chlorobromoacetic acid, $\text{ClBrCH.CO}_2\text{H}$.

This compound, which is closely analogous in structure with chlorobro-

¹⁾ Journ. chem. Soc. **105**, 811 (1914).

²⁾ POPE and READ, Journ. chem. Soc. **93**, 796 (1908).

³⁾ BACKER and BURGERS, These Proc. **28**, 64 (1925).

⁴⁾ Journ. chem. Soc. **127**, 1572 (1925).

⁵⁾ SWARTS, Bull. Ac. r. Belg. [3], **31**, 28 (1896); Mém. cour. Ac. Belg. **54** (1896).

⁶⁾ PERKIN and POPE, Journ. chem. Soc. **99**, 1527 (1911).

momethanesulphonic acid $\text{ClBrCH.SO}_3\text{H}$, may be prepared from commercial trichloroethylene ¹⁾).

To effect its resolution the method of „cold crystallisation“ ²⁾ was applied to various alkaloidal salts.

With brucine a laevorotatory acid was obtained, whilst quinine gave a dextrorotatory compound.

The latter alkaloid gave the best results.

Since neutralisation of the carboxyl group in the sulphocarboxylic acids causes a considerable change in molecular rotation, it was important to determine the influence of neutralisation in this case.

It was found, that the salts show a rotation of about the same value as that of the acid, but in the opposite direction.

The highest molecular rotation for the D-line, so far observed, is $+8^\circ$ for the free acid and -8° for the ammonium salt.

Further experiments on the dispersion are in progress, which may perhaps give a greater rotation.

In the first place we were interested in the tendency to racemisation; it proved to be not so great as might have been expected.

A solution of the ammonium salt did not racemise in 24 hours, even in the presence of a molecule of sodium hydroxide.

After heating for half an hour on a water bath the alkaline solution was half racemised.

For a solution of the ammonium salt, containing 0.089 G.mol. per litre, kept at ordinary temperature from October 1925 until June 1926, the rotation had diminished to half the original value.

Thus a compound with chlorine and bromine, attached to an asymmetric carbon atom, has now been resolved into its enantiomorphs; further in the case examined racemisation of the active components was slow.

Org. Chem. Laboratory of the State University.

Groningen, June 1926.

¹⁾ Method of CROMPTON, Compare W. H. VAN MELS, Dissertation Groningen, 26 June 1926, p. 48.

²⁾ These Proc. 28, 64 (1925).

Physiology. — “*On the permeability of the gut in seacucumbers.*” By H. A. P. C. OOMEN. (From the Physiological Department of the Zoological Station at Naples.) (Communicated by Prof. R. MAGNUS.)

(Communicated at the meeting of October 31, 1925).

In 1901 COHNHEIM who previously had demonstrated *active* resorption to occur in the surviving gut of mammals (1) published a paper on “Resorption, Verdauung und Stoffwechsel der Echinodermen” (2). Among other things he states that the Holothurian gut, though, with regard to solved substances, it behaves in general like a simple diffusion membrane, is capable to resorb isotonic solutions from its lumen. His methods of experimentation and results obtained thereby have been severely criticized by ENRIQUES (3). Nevertheless they were accepted and in the literature concerning cellpermeability his observations are cited as remarkable examples (e.g. HÖBER 1924). The results of ENRIQUES’ did not seem to agree with the main function of an intestinal tract. He found the intestinal wall to be absolutely semi-permeable in Holothurians: it allows water to pass, but no solved substances. He denied the existence of active resorption from the isolated surviving gut. These two paradoxical properties render the answer to the question as to how nutritive substances to be resorbed, pass through the intestinal wall, rather difficult.

Neither do the results obtained by ENRIQUES seem to agree with the data furnished by other Evertbrates. In general we may distinguish two different ways in which substances present in the intestinal tract may reach the blood and tissues. The first of these is the above mentioned true resorption, a polarised process by which solved and solving substances may disappear from the intestinal lumen, without or against a difference in osmotic pressure. In other cases diffusion occurs; the intestinal contents pass through the wall as they would through a membrane of parchment paper, according to the laws of diffusion.

The difference between the two is very evident in experiments by H. J. JORDAN and H. BEGEMANN (4) on frogs and snails, who in a living frog filled the gut with solutions, hypo- or isotonic with regard to the blood, and then ligated them at both ends. Two or three days afterwards the contents had disappeared completely. The gut of the snail, however, behaves quite differently. During the same lapse of time the volume of the intestinal contents remains practically the same, but part of the solved glucose has disappeared, evidently by diffusion.

In experiments on the physiology of digestion in sea-cucumbers, made

at Naples, I tried to decide whether COHNHEIM or ENRIQUES were right and furthermore to find out in which way solved substances pass the intestinal wall in these animals.

It proved to be practically impossible to fill and ligate the gut in the living animals; the animals autotomise the whole intestinal tract and its appendages on nearly every strong stimulus. I therefore was obliged to work on isolated surviving guts¹⁾. The experiments were carried out as follows: the animals (*Holothuria tubulosa* and *Stellati*) were carefully dissected and their guts procured. With fine scissors I then removed the wonder-nets and other vessels from the intestinal wall. I then removed the contents of stomach and gut, rinsed them if necessary, ligated one end with a strong thread and filled them with the test-solution from a buret. Then the other end was ligated and the preparation was immersed very carefully in a small container, the contents of which were ventilated and early replaced by running water if the experiment allowed to act thus. At the end of the experiment I lifted the organ by means of the two threads, dried it with filter-paper (not too carefully, in order to be able to notice weak spots) and returned the contents to a buret, set up previously. Control experiments have shown that one must not continue the experiments for longer than 18 hours, if one wishes to be sure to experiment on a surviving and not on a dying gut. After this lapse of time and when the ventilation had not well been taken care of, the results soon become very irregular.

In the first place I had to investigate whether in our animals the surviving gut is capable of resorption. I therefore took a number of stomachs and guts, filled them in the above mentioned way with seawater and suspended them in running sea-water. (The fluid of Holothurians is nearly always isotonic with the medium in which they live.) The following table shows the status at the end of an experiment of six hours.

| Object | Length in cm. | Volume in cc | | Total Cl expressed in cc 0.2 N. AgNO ₃ | |
|-----------|---------------|--------------|-------|--|-------|
| | | before | after | before | after |
| intestine | 25 | 1.7 | 1.8 | 5.5 | 5.6 |
| stomach | 25 | 7.1 | 7.1 | 22.9 | 23.3 |
| intestine | 35 | 5.8 | 5.6 | 18.7 | 18.3 |
| stomach | 30 | 3.0 | 3.4 | 9.7 | 10.3 |
| stomach | 40 | 9.1 | 9.2 | 29.3 | 30.0 |

The initial volume was determined by the readings of the filling buret, the final volume is the quantity found at the end of the experiment. The

¹⁾ For convenience' sake, although in the intestinal tract of Holothurians one may distinguish a distinct stomach and (end-) gut, I use the word intestinal tract or gut, where there is no need of discrimination.

percentage of chlorids was determined by MOHR'S titration with AgNO_3 . 1 cc 0.2 N. $\text{AgNO}_3 = 7.092$ mg Cl. Limit of error with regard to volume ± 0.15 cc, with regard to the Cl percentage ± 0.3 cc, 0.2 N. AgNO_3 .

Neither the volume, nor the percentage of chlorids in this series of experiments show any variation outside the limit of error. I therefore conclude that under proper precautions there is no permeability in either direction and no resorption. If, however, one neglects the limit of duration on the oxygen provision or if one fills the organs to the limit of their capacity one finds a diminution of volume and total chlorides. With my experimental set-up, one hardly obtains positive results with such abnormal organs. They become soft and fragile, the epithelium dies and disappears and they nearly always break when lifted from the water and put on the filter paper.

The next thing to find out was whether and how solved substances are influenced by the intestinal wall, separating two different solutions. For this purpose I chose in the first place different concentrations of NaCl.

The following solutions were used: 1^o. sea-water; 2^o. four parts of sea-water, one part of distilled water; 3^o. sea-water and 0.8% NaCl.

1. Seawater outside, diluted seawater inside. Duration of the experiments 6 hours.

| Object | Length in cm | Volume in cc | | Total Cl in cc 0.2 N. AgNO_3 | |
|-----------|--------------|--------------|-------|--|-------|
| | | before | after | before | after |
| intestine | 12 | 0.8 | 0.65 | 2.0 | 2.0 |
| stomach | 8 | 0.4 | 0.3 | 1.0 | 1.2 |
| intestine | 10 | 1.3 | 1.05 | 3.3 | 3.5 |
| intestine | 12 | 4.0 | (2.2) | 10.1 | 10.6 |
| intestine | 12 | 1.0 | 0.8 | 2.5 | 2.5 |

2. Seawater outside, seawater + 0.8% NaCl inside. Duration of the experiments 6 hours.

| Object | Length in cm | Volume in cc | | Total Cl in cc 0.2 N. AgNO_3 | |
|-----------|--------------|--------------|-------|--|-------|
| | | before | after | before | after |
| stomach | 16 | 1.9 | 2.4 | 7.3 | 7.8 |
| stomach | 35 | 4.6 | 5.7 | 17.7 | 18.5 |
| intestine | 40 | 3.9 | 4.7 | 15.0 | 15.1 |
| stomach | 35 | 6.1 | 7.2 | 23.4 | 23.0 |
| stomach | 50 | 15.1 | 17.8 | — | — |
| intestine | 18 | 1.5 | 1.9 | 5.8 | 5.8 |

3. Seawater + 0.8% NaCl outside, seawater inside. Duration of the experiment 7 hours.

| Object | Length in cm | Volume in cc | | Total Cl in cc 0.2 N. AgNO ₃ | |
|-----------|--------------|--------------|-------|--|-------|
| | | before | after | before | after |
| stomach | 30 | 8.5 | 7.0 | 27.4 | 28.0 |
| intestine | 12 | 1.0 | 0.8 | 3.2 | 3.1 |
| intestine | 20 | 3.9 | 3.2 | 12.6 | 10.5 |
| intestine | 14 | 1.1 | 0.9 | 3.5 | 3.6 |

4. Diluted seawater outside, seawater inside. Duration of the experiments 5 hours.

| Object | Length in cm | Volume in cc | | Total Cl in cc 0.2 N. AgNO ₃ | |
|-----------|--------------|--------------|-------|--|-------|
| | | before | after | before | after |
| stomach | 12 | 4.9 | 5.2 | 15.8 | 15.3 |
| intestine | 38 | 4.9 | 5.7 | 15.8 | 16.5 |
| stomach | 17 | 4.0 | 4.6 | 12.9 | 13.2 |

Alternatively I put them inside or outside in the following combinations :

1. sea-water outside, diluted sea-water inside ;
2. sea-water outside, sea-water + 0.8 % NaCl inside ;
3. sea-water + 0.8 % NaCl outside, sea-water inside ;
4. diluted sea-water outside, sea-water inside.

The next tables demonstrate the change in volume and salt-concentration, occurring by the action of the living intestinal wall.

We notice the volume to change as might be expected from the difference in osmotic pressure. The experiments show the water to move in one direction, the salt percentage remaining nearly the same. From these data I conclude that under these experimental conditions, the living wall of the stomach or gut behaves like a membrane impermeable for *chlorides* but not for *water*.

With regard to salt, we have found absolute semi-permeability. The question now arises whether the same impermeability also exists for other substances. I therefore made analogous experiments with the following substances, dissolved in sea-water ; corrections having been made for changes in osmotic pressure : glucose, urea, methylen-blue and trypan-blue.

I. Experiments on glucose.

In order to avoid pathological conditions and to keep the experiments as "natural" as possible it seemed advisable to use as low concentrations as possible of the "abnormal" substances.

Solutions of glucose in distilled water, isotonic with sea-water, penetrate

the gut wall almost at once (they contain 21—22 %). In the case of a 1 % solution obtained by mixing isotonic glucose in distilled water with sea-water, I hardly was able to demonstrate a reducing action of the surrounding fluid on FEHLING. I also made some experiments using a percentage of 0.4 % in the gut contents and tried to study quantitatively the permeation if there were such. However, it proved to be difficult to estimate with any degree of accuracy the very small quantities passing from inside to outside. I could demonstrate no permeation lying outside the limit of error. In some experiments in which I only quantitatively controlled the permeation of small percentages, the results were always negative too, as long as the organ might be supposed to be living.

If under physiological conditions a permeability for glucose exists, it certainly does not amount much.

II. *Experiments on urea.*

I now filled the guts with solutions of urea in sea-water (0.4 and 0.8 %). The increase in osmotic pressure with regard to seawater is negligible. At the end of the experiment I determined the concentration of urea in the outside fluid by means of the xanthidrol method of FOSSE (5). The results are shown in the next table.

| Object | Length in cm | Volume of filling in cc | Total urea | | Time in min. |
|-----------|--------------|-------------------------|---------------------------|---------------------------|--------------|
| | | | Before exp. inside in mg. | After exp. outside in mg. | |
| stomach | 20 | 2.5 | 10.0 | 0 | 170 |
| stomach | 12 | 2.6 | 10.4 | 0 | 120 |
| stomach | 10 | 2.3 | 9.0 | 0 | 110 |
| stomach | — | 4.1 | 32.8 | 0 | 210 |
| intestine | — | 2.9 | 23.2 | 0 | 210 |

The quantities of urea disappearing during 3½ hours from a 0.8 % solution under perfectly physiological conditions cannot be measured, even by this accurate method. In other words, the gut is impermeable for this substance.

III. *Experiments with dyes.*

Dyes have the advantage of a very slight toxicity, furthermore they can be demonstrated in extremely small concentrations: I used trypane-blue and methylen-blue (the former in colloidal solution).

In some of these experiments I did not remove the vessels from the intestinal tract, in order to keep conditions as natural as possible. The percentage of the sea-water solution used amounted to 0.1 %.

Under the conditions of previous experiments I never saw a permeation

of the dye into the surrounding sea water. As long as the pieces of gut might be supposed to be living (judging from their appearance and from the presence of peristalsis) the dye remained in their interior. When, however, the experiment was continued for too long a time e.g. 24 hours, the surrounding fluid soon became blue. I, therefore, conclude that the gut epithelium as long as it is living does not allow these two dyes to pass.

For all substances investigated we have demonstrated the impermeability of the living gut wall. One might ask, whether this is also a property of the dead gut wall or the gut wall in pathological condition.

Experiments on poisoned gut walls are hardly possible without changing the structure of its epithelium. If, in order to kill it, one uses sublimate, alcohol, formaldehyde, or sodium fluoride, it soon becomes very brittle and detached from the rest of the gut wall. Experiments on such organs are practically useless, because now this epithelium forms the limit between lumen and surrounding fluid; the lacunes behind it communicating with the muscular system on various places where the vessels have been torn away.

I therefore preferred to injure the epithelium by other substances, e.g. by high concentrations of glucose. Some parts of the intestinal tract were filled with: 1. a glucose solution of 18%; 2. equal parts of 18% glucose and sea-water; 3. a mixture of one part 18% glucose and three parts sea-water. The results are summarized in the next table:

| Object | Time in min. | Volume of filling | | Total glucose | | Total Cl in cc 0.2 N. AgNO ₃ | |
|-----------|--------------|-------------------|----------|---------------|----------|---|-------|
| | | before cc | after cc | before mg | after mg | before | after |
| intestine | 140 | 5.0 | 8.8 | 900 | 557 | 0.0 | 21.0 |
| stomach | 135 | 5.0 | 7.8 | 900 | 670 | 0.0 | 17.0 |
| intestine | 125 | 5.0 | — | 450 | 298 | 8.0 | 18.5 |
| stomach | 105 | 3.6 | 4.1 | 160 | 81 | 8.6 | 12.0 |

It is evident that these solutions, although nearly isotonic, but very differently composed destroy the vital conditions of the epithelium of stomach and intestines, and thereby abolish the existing impermeability. By the non-physiological differences in concentration between the glucose inside and the NaCl outside, the gut is changed into a dialysing tube. The glucose now permeates very easily as do the chlorides.

From the preceding experiments I conclude that the remarkable properties attributed by COHNHEIM to the gut wall (he supposed it to be capable of active resorption of isotonic solutions and to behave otherwise with regard to solved substances as a diffusion membrane) are to be explained by pathological changes. I believe that he ran his experiments during too long a lapse of time, and that he chose too high concentrations of the solved

substances. The same reproach has been made by ENRIQUES and never an answer has been given, either by COHNHEIM or by other investigators.

Perhaps either the language in which ENRIQUES wrote his paper, or his really severe criticism, perhaps also his paradoxical results, have prevented that his conclusions were accepted. Nevertheless the results of my more extensive experiments plead entirely in favour of his view.

Finally the question arises as to how nutritive substances are utilised in the animal's metabolism, if in fact the living gut wall neither actively resorbs, nor passively allows solved substances to permeate.

The experimental answer to this question is very difficult, owing to the fact, that the animals, 1. if kept in aquaria never take any food, and therefore surely not indicator-solutions; 2. autotomise their intestinal tract as soon as one tries to handle them.

Nevertheless I have been able to get some information with regard to this question by experiments on surviving guts. My experimental procedure was as follows:

From some animals which might be assumed to be in full digestion and resorption, I took the whole intestinal tract, ligated it distally and without any further operation put it into its own body-fluid, previously procured. During the experiment a fine current of O_2 was led through. With a fine pipet I then added a few ccm of a concentrated solution of trypane-blue or methylen-blue in sea-water to the digestive fluid which is always present in the stomach; then I ligated the proximal end. In these experiments I saw the dye mix slowly with the contents by the peristalsis of the stomach. After some hours the contents were collected in order to study changes in intensity of the coloration, if such there were. By means of a binocular microscope I studied the wall and its surroundings, while still showing peristaltical movements.

In two such experiments with trypane-blue I saw no changes; in the two other experiments with methylen-blue I saw a remarkable migration of the dye. In these cases the contents were either completely or almost colourless; the dye had accumulated in some aggregates of amoebocytes, which had become dark blue. But also the stomach wall (not the wall of the intestine *sensu stricto*) had assumed a bluish shade. Microscopic observation showed that this colour was caused by large quantities of blue amoebocytes, which were found in the lacunes, behind the epithelium itself. The epithelium itself, however, showed hardly any coloration. At some places where the lacunes communicated with the ventral vessel the blue amoebocytes were visible in both. I never saw a diffuse blue coloration. The dye had accumulated everywhere in the phagocytes. Just as in the previous experiments I saw no coloration in the surrounding fluid. Trials to fixate the organs in this condition were failures. The dye cannot be fixated and withdrawal of water from these highly watery tissues causes a contraction and disappearance of the lacunes in the stomach wall.

Very incomplete and full of new problems as my results may be, I

nevertheless believe that they give us definite information with regard to the way in which solved substances may leave the stomach lumen and arrive in the vascular system. They give reason for the believe that other solved substances may also reach the phagocytes behind the epithelium and then be transported by them. In this way the Holothurian gut would be the realisation of a remarkable possibility in the comparative physiology of digestion and show once more how far removed the Echinoderms are from the other Invertebrates, with regard to structure and functions.

In a paper published elsewhere (6) on the physiology of digestion in Holothurians I dealt more extensively with these and related problems in connection with the utilisation of the food; I hope to be able to collect as soon as possible further data on this remarkable mode of resorption.

The results of the experiments here described may be summarised in the following

CONCLUSIONS.

1^o. In the surviving gut of sea-cucumbers no active resorption of isotonic contents takes place.

2^o. The living wall of stomach and gut possesses an absolute semi-permeability. Water passes readily, whereas a highly diminished permeation can be demonstrated in the case of other substances, studied in physiological concentrations.

3^o. The dead or pathological gut-wall is a simple diffusion membrane.

4^o. The omnipresent amoebocytes probably play an important role in the resorption of solved substances.

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Mathematics. — "Ueber den höheren Zusammenhang von kompakten Räumen und eine Klasse von Abbildungen, welche ihn ungeändert lässt." By Dr. L. VIETORIS. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of May 29, 1926).

Wir wollen im Folgenden den Begriff des mehrfachen, beliebig viel-dimensionalen Zusammenhangs der kombinatorischen Topologie auf beliebige kompakte metrische Räume übertragen und zeigen, dass die Zusammenhangsverhältnisse kompakter abgeschlossener Mengen gegenüber eindeutigen stetigen Abbildungen, in welchen jeder Bildpunkt eine Urbildmenge von gewissen einfachsten Zusammenhangsverhältnissen hat, eine gewisse Invarianz besitzen.¹⁾

I. *Kombinatorische Grundlage.* Wir verstehen unter einem *n*-dimensionalen Simplex S^n eine Menge von $(n+1)$ Punkten, $\binom{n+1}{2}$ Punktepaaren, $\binom{n+1}{3}$ Punktetripeln, ... und dem einen Punkte- $(n+1)$ -tupel, welche man aus ihnen bilden kann. Jedes dieser Punkte- k -tupel heisse eine $(k-1)$ -dimensionale Seite von S^n . Für $k=1,2$ mögen sie Ecken, bzw. Kanten heissen.

Unter einem (*simplizialen*) Komplex C verstehen wir eine *endliche* Menge von Simplexen (Teilsimplexen), von denen keines eine Seite des anderen ist, im Verein mit allen Seiten dieser Simplexe. C heisst *n*-dimensional, wenn n die Dimension seines höchstdimensionalen Teilsimplexes ist; C heisst *homogen n*-dimensional, wenn alle Teilsimplexe die Dimension n haben.

Dabei lassen wir auch zu, dass ein Komplex ein Teilsimplex mit einer endlichen positiven oder negativen Vielfachheit enthält.

Der Komplex aller $(n-1)$ -dimensionalen Seiten eines *n*-dimensionalen Simplexes heisse der *Rand* desselben. Der Komplex aller jener $(k-1)$ -dimensionalen Simplexe eines homogenen k -dimensionalen Komplexes C , welche Seiten einer ungeraden Anzahl von Teilsimplexen von C sind, heisse der *Rand* von C . Ein homogen *n*-dimensionaler Komplex ohne Rand heisse ein *n*-dimensionaler Zyklus.

Sind C_1, C_2, \dots, C_k homogen dimensionale Komplexe derselben Dimension, so verstehen wir unter $C_1 + C_2 + \dots + C_k$ den Komplex aller in mindestens einem der C_i enthaltenen Teilsimplexe, jedes so oft gezählt

¹⁾ Diese Untersuchungen gehen von einer mündlichen Bemerkung BROUWERS aus (Vgl. diese Proceedings 29 (1926), S. 445, Anm. 14).

als er in allen C_i zusammengenommen vorkommt. Wir schreiben für zwei homogen n -dimensionale Komplexe K_1, K_2 derselben Dimension $K_1 = K_2 \pmod{2}$, wenn jedes in K_1 mit einer ungeraden Vielfachheit vorkommende n -dimensionale Simplex auch in K_2 mit ungerader Vielfachheit vorkommt und umgekehrt.

Ist $R^{(k-1)}$ der Rand eines in einem Komplex $C^{(n)}$ enthaltenen Simplexes $S^{(k)}$, so schreiben wir die Homologie $R^{(k-1)} = 0$ in $C^{(n)}$. Ferner setzen wir fest:

Gilt für zwei Zyklen $T_1^{(k)}, T_2^{(k)}$, $T_1^{(k)} = 0$ und $T_2^{(k)} = 0$ in $C^{(n)}$ und ist $T_1^{(k)} + T_2^{(k)} = T^{(k)} \pmod{2}$, dann soll auch $T^{(k)} = 0$ in $C^{(n)}$ gelten. Für $T_1^{(k)} + T_2^{(k)} = 0$ schreiben wir auch $T_1^{(k)} = T_2^{(k)}$.

Sehen wir für $T_1^{(k)} = T_2^{(k)}$ in $C^{(n)}$ die Operationen „ $+T_1^{(k)}$ “ und „ $+T_2^{(k)}$ “ als dieselbe Operation an, so erhalten wir eine endliche kommutative Gruppe, welche wir die k -te *Zusammenhangsgruppe* von $C^{(n)}$ nennen.

Die Maximalzahl von in $C^{(n)}$ liegenden k -dimensionalen Zykeln T_1, T_2, \dots, T_s , zwischen denen keine Homologie $a_1 T_1 + a_2 T_2 + \dots + a_s T_s = 0$ ($a_i = 0, 1$) besteht, heiße die k -te *Zusammenhangszahl* von $C^{(n)}$ ²⁾.

Als 0-te Zusammenhangszahl ergibt diese Definition die um 1 verminderte Komponentenzahl.

Streichen wir in der Definition der obigen Homologien die Festsetzung, dass Zyklen, die *modulo 2* übereinstimmen, homolog sind, so erhalten wir die Homologien POINCARÉ's. Nun bilden die Additionen *orientierter* in $C^{(n)}$ liegender k -dimensionaler Zyklen eine Gruppe. Wir nennen sie die k -te *Homologiegruppe* von $C^{(n)}$. Sie ist kommutativ und im allgemeinen unendlich, aber diskret. Die Maximalzahl von in $C^{(n)}$ liegenden *orientierten* k -dimensionalen Zykeln, zwischen denen keine Homologie besteht, heiße die k -te *orientierte Zusammenhangszahl* ³⁾.

II. *Übertragung auf beliebige kompakte Räume.* Wir sagen, ein Komplex C liege in einer Menge M , wenn seine Ecken in M liegen. Ob in M . z. B. zu jeder Kante $[a, b]$ von C , die ja nur als Punktepaar erklärt ist, eine stetige Verbindung zwischen a und b besteht oder nicht, ist uns dabei gleichgültig.

Ist S der Rand eines Simplexes beliebiger Dimension, dessen Kanten $< \varepsilon$ sind, dann schreiben wir $S \approx 0$ (S ε -homolog 0). Ist $K = S_1 + S_2 + \dots + S_r$ und $S_i \approx 0$, so sei auch $K \approx 0$, ebenso für orientierte, wie für nicht orientierte Zyklen. Für $K_1 = K_2 \approx 0$ schreiben wir auch $K_1 \approx K_2$.

²⁾ Diese Zusammenhangszahlen sind um 1 kleiner als die „Connectivities“ im Sinn von O. VEBLEN, *Analysis situs* (Cambridge Colloquium 1916), S. 77 ff. Wir haben hier nur die Definition von der Darstellung der Komplexe durch Matrizen losgelöst.

³⁾ Sie ist um 1 kleiner als POINCARÉ's k -te BETTISCHE Zahl (Journ. Ex. Fol. (2), 1 cah. (1895). Wir folgen in der Zählung dem Vorgang von SCHLÄFLI, KLEIN, DYCK (Siehe die Zitate bei DYCK, *Math. Ann.* 32 (1888), S. 483) und MANNOURY (Nieuw Archief v. Wis-kunde (2) 3 (1898) S. 126-152).

Die unendliche Folge F von k -dimensionalen (nicht orientierten, bzw. orientierten) Zykeln C_1, C_2, \dots heisse eine *Fundamentalfolge* in M , wenn die Kantenlänge von C_m mit wachsendem m gegen 0 konvergiert und zu jedem $\varepsilon > 0$ ein n_ε besteht, sodass $C_{n_1} \varepsilon C_{n_2}$ gilt, sobald $n_1 > n_\varepsilon$ und $n_2 > n_\varepsilon$ sind. F heisse ε -homolog 0, wenn es ein n_ε gibt, sodass für $n > n_\varepsilon$ $C_n \varepsilon 0$ gilt. F heisse eine *Nullfolge* $F = 0$, wenn für jedes $\varepsilon > 0$ $F \varepsilon 0$ gilt.

Sind $\{C_i\}$ und $\{D_j\}$ Fundamentalfolgen (Nullfolgen), so ist auch $\{C_i + D_j\}$ eine Fundamentalfolge (bzw. Nullfolge). $\{C_i\} \varepsilon \{D_j\}$ und $\{C_i\} - \{D_j\}$ bedeute soviel, wie $\{C_i\} - \{D_j\} \varepsilon 0$, ber. $\{C_i\} - \{D_j\} = 0$.

Die Gruppe der Additionen von Fundamentalfolgen heisse die k^{te} *Zusammenhangs-*, bzw. k^{te} *Homologiegruppe* von M .

(1) Die Zusammenhangsgruppen und Homologiegruppen einer beliebigen Menge eines metrischen Raumes können als metrische, und zwar vollständige⁴⁾ Räume aufgefasst werden.

Man hat dazu nur als Abstand zweier Fundamentalfolgen F_1, F_2 die untere Grenze aller Zahlen $\varrho > 0$ zu erklären, für welche $F_1 \varepsilon F_2$ ist.

Entsprechend der Endlichkeit der Zusammenhangsgruppe in Komplexen gilt hier:

(2) Die Zusammenhangsgruppe einer kompakten abgeschlossenen Menge M ist kompakt und abgeschlossen.

Denn es gilt zunächst:

(3) In einer kompakten abgeschlossenen Menge M gibt es für festes n und vorgegebenes ε nur endlich viele paarweise nicht ε -homologe nicht orientierte n -dimensionale $\frac{\varepsilon}{3}$ -Zykel.

Zum Beweis⁵⁾ wird in M eine endliche Menge A so angenommen, dass jeder Punkt von M von mindestens einem Punkt von A einen Abstand $< \frac{\varepsilon}{3}$ hat. Dann lässt sich zeigen, dass jeder in M liegende $\frac{\varepsilon}{3}$ -Zykel C durch endlich viele ε -Abänderungen (Additionen von Simplexrändern $R_i^{(n)} \varepsilon 0$) in einen in A liegenden Zyklus C^* übergeführt werden kann. Da in A überhaupt nur endlich viele *modulo 2* verschiedene Zykeln liegen, ist damit (3) bewiesen.

Aus (3) folgt fast unmittelbar:

(4) In einer kompakten abgeschlossenen Menge M gibt es zu einem festen n und vorgegebenen $\varepsilon > 0$ nur endlich viele, paarweise nicht ε -homologe Fundamentalfolgen von nicht orientierten, n -dimensionalen Zykeln.

Mit anderen Worten: Die Zusammenhangsgruppe einer kompakten

⁴⁾ Hausdorff, Grundzüge, S. 315.

⁵⁾ Bezüglich der genauen Ausführung dieses und der folgenden Beweise, verweisen wir auf eine demnächst in den Math. Ann. erscheinende Abhandlung gleichen Titels.

abgeschlossenen Menge ist *total beschränkt*⁶⁾. Da ein total beschränkter vollständiger Raum kompakt ist,⁷⁾ folgt aus (1) und (4) unmittelbar (2).

Für die Homologiegruppe können wir bloss (3') und (4') behaupten. Die Beweise sind analog zu denen von (3) und (4).

(3') Ist M eine kompakte abgeschlossene Menge, so gibt es zu einem festen n und vorgegebenen $\varepsilon > 0$ eine natürliche Zahl s , sodass zwischen irgend s in M liegenden orientierten $\frac{\varepsilon}{3}$ -Zykeln C_1, C_2, \dots, C_s eine ε -Homologie $a_1 C_1 + a_2 C_2 + \dots + a_s C_s = 0$ besteht.

(4') Ist M eine kompakte abgeschlossene Menge, so gibt es zu einem festen n und vorgegebenen $\varepsilon > 0$ eine natürliche Zahl s , sodass zwischen irgend s Fundamentalfolgen von orientierten in M liegenden n -dimensionalen Zykeln eine ε -Homologie besteht.

Für nicht orientierte Zykel gilt (4') infolge von (4) erst recht.

Bleibt s , wenn ε gegen 0 abnimmt, beschränkt, so kann man \limsup als n^{te} (nicht orientierte, bzw. orientierte) Zusammenhangszahl erklären. Die *Ordnung* einer Zusammenhangs-, oder einer Homologiegruppe kann aber endlich, abzählbar unendlich und von der Mächtigkeit des Kontinuums sein.

Wir sagen, die kompakte abgeschlossene Menge M habe eine p -fache k -dimensionale (nicht orientierte, bzw. orientierte) *Zyklose*, wenn es zu jedem hinreichend kleinen $\varepsilon > 0$ ein $\delta_\varepsilon > 0$ gibt, sodass in M genau p linear ε -unabhängige k -dimensionale (nicht orientierte, bzw. orientierte) δ -Zykel vorkommen, sobald $\delta < \delta_\varepsilon$ ist.⁸⁾

Die Vielfachheit der nicht orientierten *Zyklose* stimmt, soweit sie endlich ist, mit der Anzahl linear unabhängiger Elemente der Zusammenhangsgruppe überein. Für die orientierte *Zyklose* lässt sich durch ein Beispiel das Gegenteil zeigen.

III. *Abbildungssätze*. Wir stützen uns im Folgenden auf den Doppelsatz:

(5), (5') Ist die kompakte abgeschlossene Menge K auf eine andere \mathfrak{R} eindeutig stetig so abgebildet, dass die Menge Y aller Urbilder jedes Punktes y von \mathfrak{R} für $k=0, 1, 2, \dots, n-1$ eine 0-fache k -dimensionale nicht orientierte, bzw. orientierte *Zyklose* hat, dann gibt es zu jedem $\varepsilon > 0$ und $\varepsilon' > 0$ ein $\delta > 0$, sodass jeder in \mathfrak{R} liegende n -dimensionale nicht orientierte, bzw. orientierte δ -Zyklus durch ε -Abänderun-

⁶⁾ Hausdorff, a. a. O., S. 311.

⁷⁾ Hausdorff, a. a. O., S. 314 V.

⁸⁾ Dieser Begriff ist in Analogie zur Vielfachheit der Basis der (eindimensionalen) *Zyklosis* (BROUWER, Math. Ann. 72, S. 422-425) gebildet. Die Analogie ist insofern nicht vollständig, als sich BROUWERs Basis auf die Fundamentalgruppe, der obige Begriff auf die Zusammenhangs-, bzw. Homologiegruppen bezieht.

gen innerhalb \mathfrak{R} in einen n -dimensionalen ε -Zyklus übergeführt werden kann, der das Bild (mindestens) eines in K liegenden nicht orientierten, bzw. orientierten n -dimensionalen ε' -Zyklus ist.

Der Beweis verläuft für nicht orientierte und orientierte Zykel fast gleich und stützt sich auf den Hilfssatz:

Ist die kompakte abgeschlossene Menge K eindeutig stetig auf \mathfrak{R} abgebildet, so gibt es zu jedem $\varepsilon > 0$ und $\varepsilon' > 0$ ein $\delta > 0$, derart, dass zu jeder Menge $\mathfrak{R}_1 \subset \mathfrak{R}$ von einem Durchmesser $< \delta$ ein Punkt y_0 besteht, sodass jeder Punkt x von K , dessen Bild in \mathfrak{R}_1 liegt, von der Urbildmenge Y_0 von y_0 einen Abstand $< \varepsilon$ hat und ausserdem y_0 von jedem Punkt von \mathfrak{R}_1 weniger als ε' entfernt ist.

Zum Beweis von (5) wird, wenn \mathfrak{Z} der gegebene δ -Zyklus ist, zunächst zu jeder Ecke y_i von \mathfrak{Z} ein Punkt x_i in Y_i beliebig gewählt, dann zu jeder Kante $[y_i, y_j]$ von \mathfrak{Z} zufolge (6) ein Punkt y_{ij} von \mathfrak{R} der von y_i und y_j „mit δ kleine“ Abstände hat und für welchen auch jeder Punkt von $Y_i + Y_j$ von Y_{ij} einen mit δ kleinen Abstand hat. Weil Y_{ij} eine 0-fache 0-dimensionale Zyklöse hat, d.h. zusammenhängend ist, kann also x_i mit x_j durch eine sonst ganz in Y_{ij} liegende „mit δ feine“ Kette K_{ij} von endlich vielen Punkten verbunden werden. Ist $[y_i, y_j, y_k]$ eine zweidimensionale Seite von \mathfrak{Z} , so ist der Durchmesser des Sechsecks $[y_i, y_{ij}, y_j, y_{jk}, y_k, y_{ki}]$ mit δ klein. Nach (6) kann ich in \mathfrak{R} einen Punkt y_{ijk} wählen, der von allen Punkten des Sechsecks mit δ kleine Abstände hat, derart, dass jeder Punkt der Menge $Y_i + Y_{ij} + Y_j + Y_{jk} + Y_k + Y_{ki}$ von Y_{ijk} einen mit δ kleinen Abstand hat. Ich kann also zu jedem Punkt x des eindimensionalen Zyklus $Z_{ijk} = K_{ij} + K_{jk} + K_{ki}$ in Y_{ijk} einen Punkt x' wählen, dessen Abstand von x mit δ klein ist, d.h. zu Z_{ijk} einen eindimensionalen Zyklus Z'_{ijk} . Weil nun Y_{ijk} eine 0-fache eindimensionale Zyklöse hat, ist Z'_{ijk} Rand eines in Y_{ijk} liegenden homogen 2-dimensionalen Komplexes, der mit δ fein ist. Zusammen mit dem „zylindrischen Streifen“ zwischen Z_{ijk} und Z'_{ijk} , den wir uns simplizial zerlegt denken, gibt er einen mit δ feinen Komplex K_{ijk} , dessen Rand Z_{ijk} ist. So fahren wir fort, bis wir schliesslich mithilfe der 0-fachen $(n-1)$ -dimensionalen Zyklöse der Y alle jene $(n-1)$ -dimensionalen Zykel, ausgefüllt haben, welche beim Schritt vorher konstruiert worden sind.

Aus (5) lässt sich unter Verwendung der Kompaktheit der Zusammenhangsgruppen ableiten:

(7) Unter den Voraussetzungen von (5) ist die n^{te} Zusammenhangsgruppe $\mathfrak{Z}^{(n)}$ von \mathfrak{R} das Bild der n^{ten} Zusammenhangsgruppe $Z^{(n)}$ von K (bezüglich der zugrunde gelegten Abbildung), und die n^{te} Zusammenhangszahl von \mathfrak{R} ist nicht grösser als die von K^9 .

⁹⁾ Der zweite Teil von (7) ist eine Verallgemeinerung eines von BROUWER herrührenden, in meiner in Anm. 1 zitierten Abhandlung wiedergegebenen Satzes.

Ob (7) auch für orientierte Zykel gilt, bleibe offen. Dagegen können wir auch für diese behaupten:

(7b) Unter den Voraussetzungen von (5') ist die k^{te} Homologiegruppe von \mathfrak{R} für $k < n$ das Bild der k^{ten} Homologiegruppe von K .

Der Beweis stützt sich auf folgende Sätze (5b) und (8), die ähnlich wie (5) und (5') bewiesen werden.

(5b) Unter den Voraussetzungen von (5) bzw. (5') gibt es für $k < n$ zu jedem $\varepsilon > 0$ und $\varepsilon' > 0$ ein $\delta > 0$, sodass jeder in \mathfrak{R} liegende mit 0 δ -homologe k -dimensionale Zyklus mit einem k -dimensionalen Zyklus in \mathfrak{R} ε -homolog ist, der das Bild eines in K mit 0 ε' -homologen k -dimensionalen Zyklus ist.

(8) Unter den Voraussetzungen von (5), bzw. (5') ist jede Folge F von in K liegenden k -dimensionalen (nicht orientierten, bzw. orientierten) Zykeln mit gegen 0 konvergierenden Kantenlängen und $k < n$, deren Bild in \mathfrak{R} eine Nullfolge ist, selbst eine Nullfolge.

Auf Grund von (5b) und (8) kann man nämlich zu jeder in \mathfrak{R} liegenden Fundamentalfolge von k -dimensionalen Zykeln eine mit ihr homologe Fundamentalfolge finden, welche Bild einer in K liegenden Fundamentalfolge ist. Da nun ausserdem wegen (8) zwei Fundamentalfolgen F_1, F_2 von k -dimensionalen Zykeln in K , deren Bilder f_1, f_2 mit einander homolog in \mathfrak{R} sind, homolog in K sind, gilt schliesslich:

(7c) Unter den Voraussetzungen von (5), bzw. (5') ist für $k < n$ die k^{te} Zusammenhangsgruppe, bzw. die k^{te} Homologiegruppe von \mathfrak{R} mit der von K identisch¹⁰⁾.

Für die Vielfachheit der Zyklosis lässt sich mit ähnlichen Mitteln und zwar in gleicher Weise für orientierte, wie für nicht orientierte Zykel zeigen:

(7d) Unter den Voraussetzungen von (5), bzw. (5'), ist für $k < n$ die Vielfachheit der k -dimensionalen nicht orientierten, bzw. orientierten Zyklosis von \mathfrak{R} genau gleich der von K .

¹⁰⁾ Bezüglich der analogen Behandlung der *Fundamentalgruppe* verweisen wir auf die in Anm. 5 angekündigte Abhandlung.

Mathematics. — "*Ueber stetige Bilder von Punktmengen.*" By Dr. W. HUREWICZ. (Communicated by Prof. L. É. J. BROUWER).

(Communicated at the meeting of May 29, 1926).

Im Folgenden untersuchen wir die stetigen Bilder von Punktmengen vom Standpunkt der allgemeinen Dimensionstheorie.¹⁾ Es gilt diesbezüglich vor allem eine Charakterisierung der n -dimensionalen kompakten Räume: Jeder kompakte n -dimensionale metrische Raum K lässt sich darstellen als eindeutiges stetiges Bild einer abgeschlossenen nulldimensionalen Menge (insbesondere also einer abgeschlossenen nirgends dichten linearen Menge²⁾, derart dass jeder Punkt von K Bild von höchstens $n + 1$ Punkten ist, — und umgekehrt ist ein eindeutiges stetiges Bild einer kompakten nulldimensionalen Menge, das höchstens $(n + 1)$ -fache Punkte enthält, höchstens n -dimensional³⁾.

Dieser Satz ist ein Spezialfall einer analogen Charakterisierung für allgemeine separable Räume, insbesondere also für Teilmengen kompakter Räume. Zwar können schon eineindeutige stetige Bilder nulldimensionaler Mengen eindimensional sein⁴⁾, ja beliebig hohe Dimension besitzen. Die Uebertragung der angeführten Charakterisierung auf nicht kompakte Räume gelingt indes durch Einführung folgender Begriffe: Wir nennen zunächst die *mehrdeutige Abbildung* A der Menge M auf die Menge M^* *stetig*, falls zu jedem Punkt p von M und zu jeder Umgebung V der Bildmenge $A(p)$ eine Umgebung $U(p)$ von p in M existiert, so dass alle Bilder von Punkten aus $U(p)$ in V liegen, oder, was gleichbedeutend ist, falls jede in M^* abgeschlossene Menge ein in M abgeschlossene Urbildmenge⁵⁾ hat.

¹⁾ Vgl. Menger. Bericht über die Dimensionstheorie. Jahresber. d. deutschen Mathem. Vereinig. 35, 1926 und URYSOHN. Mémoire sur les multiplicités Cantorienes, Fund. Math. VII u. VIII.

²⁾ Mit Rücksicht auf den SIERPINSKISCHEN Satz von der Homöomorphie der nulldimensionalen mit linearen Mengen, Fund. Math. II, 1921, S. 89.

³⁾ LEBESGUE hat nachgewiesen (Fund. Math. II, 1921, S. 283), dass jede Begrenzung einer offenen Menge des R_{n+1} (variété frontière à n dimensions) sich darstellen lässt als eindeutiges stetiges Bild einer linearen perfekten Menge mit höchstens $(n+1)$ -fachen Punkten; da die Begrenzungen offener Mengen des R_{n+1} höchstens n -dimensional sind, ist dies ein Spezialfall des obigen Satzes. Jeder perfekte n -dimensionale Raum kann durch eindeutige stetige Abbildung mit höchstens $(n+1)$ -fachen Punkten aus der nirgends dichten perfekten CANTOR'schen Menge gewonnen werden. Dass überhaupt jeder kompakte metrische Raum stetiges Bild der CANTOR'schen Menge ist, wurde von ALEXANDROFF (diese Proceedings 28, 1925, S. 997) ausgesprochen; vgl. übrigens eine Bemerkung von MAZURKIEWICZ (Fund. Math. I, 1920, S. 179—180).

⁴⁾ Vgl. das einfache Beispiel von Menger, Ueber die Dimension von Punktmengen II, Monatshefte f. Math. u. Phys. 34, 1924, S. 141.

⁵⁾ Unter der Urbildmenge eines Teiles N von M^* verstehen wir die Gesamtheit aller Punkte von M , die mindestens einen in N liegenden Bildpunkt besitzen.

Wir nennen eine eindeutige Abbildung beiderseits stetig, wenn sie stetig ist und ihre (im Allgemeinen mehrdeutige) Umkehrung im definierten Sinn ebenfalls stetig ist.¹⁾ Es gilt nun das folgende

Theorem. *Unter den separablen metrischen Räumen sind die n -dimensionalen dadurch charakterisiert, dass sie sich erstens als eindeutige beiderseits stetige Bilder einer nulldimensionalen Menge (also auch einer linearen Menge ohne Teilintervall) mit höchstens $(n+1)$ -fachen Punkten darstellen lassen, während sie sich zweitens nicht als eindeutige beiderseits stetige Bilder einer nulldimensionalen Menge darstellen lassen, so dass jeder Punkt weniger als $n+1$ Urbildern zugeordnet ist.*

Den Beweis der ersten Hälfte des Theorems stützen wir darauf, dass in jedem n -dimensionalen separablen Raum R ein System von abgeschlossenen Mengen²⁾ mit folgenden Eigenschaften existiert:

1. Es gibt endlich viele, etwa ν_1 , in R abgeschlossene Mengen $R_1, R_2, \dots, R_{\nu_1}$ "des ersten Schrittes", und $R = \sum_{k=1}^{\nu_1} R_k$.
2. Wenn R_{i_1, i_2, \dots, i_n} eine bereits definierte Menge des Systems ist, dann gibt es endlich viele, etwa $\nu_{i_1, i_2, \dots, i_n}$ abgeschlossene Mengen des $(n+1)$ -ten Schrittes, $R_{i_1, i_2, \dots, i_n, k}$, und $R_{i_1, i_2, \dots, i_n} = \sum_{k=1}^{\nu_{i_1, i_2, \dots, i_n}} R_{i_1, i_2, \dots, i_n, k}$.
3. Jede Folge von in einander geschachtelten Mengen $R_{i_1}, R_{i_1, i_2}, \dots, \dots, R_{i_1, i_2, \dots, i_n}, \dots$ zieht sich entweder auf einen Punkt zusammen oder hat einen leeren Durchschnitt.

4. Für jedes k sind je $n+2$ Mengen des k -ten Schrittes fremd.

(Dieser Satz ist eine Verallgemeinerung des von MENGER³⁾ und URYSOHN⁴⁾ bewiesenen Satzes von der Zerlegbarkeit kompakter n -dimensionaler Räume in endlich viele beliebig kleine abgeschlossene Teile, die zu je $n+2$ fremd sind. Wie MENGER bemerkt hat⁵⁾, gilt dieser letztere Satz auf Grund eines Ergebnisses von mir⁶⁾ für beliebige Teilmengen kompakter Räume.)

Wir ordnen nun jeder Menge R_{i_1, i_2, \dots, i_k} des k -ten Schrittes ein abgeschlossenes Intervall I_{i_1, i_2, \dots, i_k} von einer Länge $< \frac{1}{k}$ im Innern des Intervalls $[0,1]$ zu, so dass *erstens* je zwei Intervalle mit gleich vielen Indizes

¹⁾ Jede eindeutige stetige Abbildung eines kompakten Raumes ist beiderseits stetig.

²⁾ Vgl. den verwandten Begriff des finiten Umgebungssystems bei MENGER, diese Proceedings 29.

³⁾ Monatshefte f. Math. u. Phys. 34, 1924, S. 153.

⁴⁾ Fund. Math. VIII.

⁵⁾ Vgl. MENGER, Bericht.

⁶⁾ Nämlich, auf Grund des Satzes, dass n -dimensionale Teilmengen kompakter Räume in endlich viele beliebig kleine relativ abgeschlossene Teile zerlegbar sind, die zu je zwei höchstens $(n-1)$ -dimensionale Durchschnitte haben. Vgl. meine Arbeit „Normalbereiche und Dimensionstheorie“, Mathem. Annalen 96.

fremd sind und zweitens stets $I_{i_1, i_2, \dots, i_{k-1}, i_k}$ im offenen Intervall $I_{i_1, i_2, \dots, i_{k-1}}$ liegt. Sodann betrachten wir auf der Strecke die (nulldimensionale) Menge aller Punkte p , auf die sich eine Folge von Intervallen $I_{i_1}, I_{i_1, i_2}, \dots, I_{i_1, i_2, \dots, i_n}, \dots$ zusammenzieht, so dass die Folge der entsprechenden Mengen $R_{i_1}, R_{i_1, i_2}, \dots, R_{i_1, i_2, \dots, i_n}, \dots$ einen nicht leeren Durchschnitt hat und sich mithin auf einen Punkt p^* von R zusammenzieht. Wir ordnen nunmehr p^* dem Punkt p als Bildpunkt zu. Dadurch ist, wie man leicht einsieht, eine eindeutige beiderseits stetige Abbildung einer linearen nulldimensionalen Menge auf den vorgegebenen Raum R definiert, wobei jeder Punkt von R höchstens $n+1$ Urbilder besitzt.

Den Beweis der zweiten Hälfte unseres Theorems stützen wir auf den folgenden vielleicht auch an sich interessanten Satz:

Ist der Raum R^ eindeutiges beiderseits stetiges Bild des separablen n -dimensionalen metrischen Raumes R , so zwar dass alle Punkte von R eine gleiche endliche Anzahl von Urbildern besitzen, dann ist auch R n -dimensional.*

An dieser Stelle zeigen wir, da es für das Folgende ausreicht, bloss, dass R^* höchstens n -dimensional ist¹⁾. Zum Beweise betrachten wir ein abzählbares System S von Umgebungen in R , so dass sich auf jeden Punkt von R eine Umgebungsfolge aus S zusammenzieht. Bezeichnet k die für alle Punkte von R^* gemeinsame Anzahl von Urbildern in R , so ordnen wir alle Systeme von je k Umgebungen aus S , die samt ihren Begrenzungen paarweise fremd sind, in eine abzählbare Folge $K_1, K_2, \dots, K_m, \dots$. Mit R_m^* bezeichnen wir die in R^* abgeschlossene Menge, welche Durchschnitt ist von den Bildern der k Umgebungen aus S , die in K_m zusammengefasst sind. Man überzeugt sich sofort davon, dass $R^* = \sum_{m=1}^{\infty} R_m^*$ gilt. Ferner ist jede Menge R_m^* höchstens n -dimensional. Denn betrachten wir irgend eine bestimmte von den k Umgebungen, aus denen K_m zusammengesetzt ist, und ordnen wir jedem Punkt von R_m^* dasjenige seiner Urbilder zu, welches in dieser Umgebung liegt, — so haben wir, wie man leicht zeigt, eine umkehrbar eindeutige beiderseits stetige (also topologische) Abbildung zwischen R_m^* und einem Teil der n -dimensionalen Menge R . — Demnach ist R^* Summe von abzählbar vielen in R^* abgeschlossenen höchstens n -dimensionalen Mengen, also²⁾ auch selbst höchstens n -dimensional.

Eine unmittelbare Folgerung hieraus lautet:

Ist R^ eindeutiges beiderseits stetiges Bild des separablen nulldimensionalen Raumes R , dann ist für jede natürliche Zahl k die Menge aller Punkte*

¹⁾ Eine Erniedrigung der Dimension kann, nebenbei bemerkt, durch eine eindeutige beiderseits stetige Abbildung schon dann nicht hervorgerufen werden, wenn jeder Punkt der Bildmenge eine nulldimensionale Menge von Urbildern besitzt.

²⁾ Vgl. meine oben zitierte Arbeit und TUMARKIN, diese Proceedings 28, S. 994.

von R^* , die genau k Urbilder in R besitzen, entweder leer oder null-dimensional¹⁾).

Da die Summe von $n + 1$ nulldimensionalen Mengen höchstens n -dimensional ist²⁾, folgt aus dem Bewiesenen:

Wenn der n -dimensionale Raum R^ eindeutiges beiderseits stetiges Bild des separablen nulldimensionalen Raumes R ist, so dass jeder Punkt von R^* von endlicher Vielfachheit ist, d.h. nur endlich viele Urbilder in R besitzt, dann kommen in R^* Punkte von mindestens $n + 1$ verschiedenen Vielfachheiten vor.*

In dieser Behauptung ist die zweite Hälfte unseres Theorems enthalten.

Die bisherigen Resultate lassen sich teilweise auf die eindeutigen stetigen Bilder der n -dimensionalen Mengen, insbesondere auf die stetigen Bilder der Strecke (die Jordanschen Kurven) übertragen. Zwar gilt, wie man sich durch einfache Beispiele klar macht, im allgemeinen nicht, dass sich ein n -dimensionales im kleinen zusammenhängendes Kontinuum als eindeutiges stetiges Bild der Strecke mit höchstens $(n + 1)$ -fachen Punkten darstellen lässt.³⁾ Es gilt aber, dass bei jeder eindeutigen stetigen Abbildung der Strecke auf einen Raum R^* die Menge aller Punkte von einer gegebenen endlichen Vielfachheit null-dimensional ist, sofern es in R^* keinen einfachen Bogen gibt, der einen offenen Teil von R^* vollständig ausfüllt. Ist also der Bildraum n -dimensional und jeder seiner Punkte von endlicher Vielfachheit, so müssen Punkte von mindestens $n + 1$ verschiedenen Vielfachheiten auftreten, insbesondere also auch Punkte von mindestens $(n + 1)$ -facher Vielfachheit.⁴⁾ Weiter ergibt sich, dass bei jeder eindeutigen stetigen Abbildung der Strecke⁵⁾ auf ein Intervall des R_n die Menge aller Punkte von einer Vielfachheit $\geq k$ ($k = 1, 2, \dots, n$) zusammenhängend ist⁶⁾.

Eine ausführliche Darstellung der Resultate dieser Arbeit erscheint demnächst in den Mathematischen Annalen.

¹⁾ Es gilt allgemein, dass in einem eindeutigen beiderseits stetigen Bild eines separablen n -dimensionalen Raumes für jedes k die Menge aller k -fachen Punkte höchstens n -dimensional ist. Die Menge aller höchstens k -fachen Punkte ist stets ein G_j .

²⁾ Vgl. URYSOHN, Fund. Math. VIII.

³⁾ Für Intervalle des R_n gilt dies wohl, wie LEBESGUE, Fund. Math. II. 1921, S. 280 gezeigt hat.

⁴⁾ Die Existenz mindestens $(n + 1)$ -facher Punkte bei der Abbildung der Strecke auf ein Intervall des R_n wurde von LEBESGUE a.O. S. 279 bewiesen.

⁵⁾ Der Satz gilt auch für die Abbildungen von nulldimensionalen Mengen.

⁶⁾ Von dieser Menge kann man ferner zeigen, dass sie (und zwar in jedem ihrer relativ offenen Teile) ein $(n + 1 - k)$ -dimensionales Teilkontinuum enthält. Dass jede Jordankurve, die ein Quadrat ausfüllt, ein Kontinuum von Doppelpunkten enthält, wurde bereits von HAHN bewiesen. Vgl. Annali di Mathem., 1913, S. 48.

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Mathematics. — *“The Congruence of the Twisted Cubics which Pass through Four Given Points and which Cut Two Given Straight Lines.”* By Dr. G. SCHAAKE. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 27, 1926).

§ 1. The twisted cubics k^3 which pass through four given points H_1, \dots, H_4 and which cut two given lines l and m , may be represented on the points of a plane a in the following way. We choose two points L and M in a . We consider a projective correspondence between the points P of l and the lines p of the plane pencil (L, a) and a similar correspondence between the points Q of m and the lines q of the plane pencil (M, a) . To a curve k^3 which cuts l in P and m in Q , we associate the point of intersection K of the lines p and q corresponding resp. to P and Q .

§ 2. To the line LM , which belongs to the plane pencil (L, a) as well as to the plane pencil (M, a) , there correspond on l and m resp. the points A and B . The curve k^3 through A and B has all the points of LM as image points.

Accordingly the curve k^3 through A and B is singular for our representation. To this curve all the points of LM correspond as image points K .

If we choose K in L , the line p is indefinite and q coincides with LM . The point L is, therefore, the image point of the ∞^1 curves k^3 through B . These curves form a surface which cuts the plane $H_2 H_3 H_4$ along the lines $H_2 H_3, H_3 H_4, H_2 H_4$, and the conic through H_2, H_3, H_4 , and the points of intersection of BH_1 and l with the said plane. For this conic, together with BH_1 , forms a degenerate k^3 through B and $H_3 H_4$ forms a similar k^3 with the conic in the plane $BH_1 H_2$ which cuts $H_3 H_4$ and which passes through B, H_1, H_2 and the point of intersection of l with this plane. The surface has such an intersection with any plane through three of the points B and H_1, \dots, H_4 . The surface in question, i. e. the locus of the twisted cubics which pass through five given points and which cut a given line, is, accordingly, a surface of the fifth degree β^5 which has triple points in the points B and H_1, \dots, H_4 .

In the same way it appears that to the point M ∞^1 curves k^3 are associated, the curves k^3 through A , which also form a surface of the fifth degree, α^5 .

Consequently the points L and M are singular for our representation. To L and M there correspond resp. the surfaces of the fifth degree β^5 and α^5 of the curves k^3 through B and A .

The conic k^2 passing through H_2, H_3 and H_4 and cutting l and m , which, therefore, contains the points of intersection P_1 and Q_1 of l and m with the plane $H_2 H_3 H_4$, together with the generatrices of the cone α_1 which projects k_1^2 out of H_1 , forms ∞^1 curves k^3 of our system. All these curves are represented in the point of intersection S_1 of the lines p_1 and q_1 corresponding resp. to P_1 and Q_1 .

Accordingly we find four singular image points S_1, S_2, S_3 , and S_4 . To S_i there correspond the ∞^1 curves k^3 that are formed by the conic k_i^2 cutting l and m resp. in P_i and Q_i , which passes through H_k, H_l , and H_m , and the generatrices of the cone α_i which projects k_i^2 out of H_i .

The transversal t_1 of l and m through H_1 which cuts l and m resp. in P_1' and Q_1' , forms ∞^1 curves k^3 with the conics of the pencil β_1 of conics which cut t_1 and which pass through H_2, H_3 , and H_4 . All these curves k^3 are represented on the point of intersection S_1' of the lines p_1' and q_1' corresponding resp. to P_1' and Q_1' .

Consequently there are four more singular image points S_1', S_2', S_3' , and S_4' . To S_i' there correspond the ∞^1 curves k^3 that are formed by the transversal t_i through H_i' of l and m , which cuts l and m resp. in P_i' and Q_i' , with the conics of the pencil β_i of the conics that pass through H_k, H_l , and H_m and cut t_i .

The lines $H_1 Q_2$ and $H_2 P_1$ both cut the line $H_3 H_4$ and form a k^3 of our congruence together with this line. It is represented in the point of intersection K_{12} of p_1 and q_2 . The points $K_{i,k} \equiv p_i q_k$ ($i \neq k$) are, therefore, the image points of twelve curves k^3 which are degenerate in a join of two of the points H_i and two lines which cut this line.

The degenerate curves k^3 consisting of a conic through H_2, H_3, H_4, P_1 or Q_1 , and a transversal through H_1 of such a conic and m or l , are represented in the points of p_1 or of q_1 . Besides the curves that are represented in the points $p_1 q_2, p_1 q_3$, and $p_1 q_4$, the system corresponding to p_1 contains three more curves k^3 degenerate in three lines e.g. the curve which consists of $H_3 H_4, H_2 P_1$, and the transversal through H_1 of $H_2 P_1$ and m . In this way we find $8 \times 3 = 24$ more curves k^3 degenerate in three lines.

There are twelve more of such curves, those which consist of the transversal t_i and the two lines of a degeneration in β_i , of which the image points, therefore, coincide in the points S_i' in groups of three.

Finally we find six curves k^3 degenerate in three lines, in the following way. Let us consider e.g. the curves k^3 consisting of $h_{12} \equiv H_1 H_2$ and a conic k^2 through H_3 and H_4 which rests on h_{12}, l and m . These conics k^2 form a surface of the fourth degree, a dimonoid with H_3 and H_4 as triple points and $h_{34} \equiv H_3 H_4$ as double torsal line. This arises from the pair of lines formed by h_{34} with one of the two transversals

of h_{34} , h_{12} , l and m . Each of these two transversals forms, therefore, with h_{12} and h_{34} a curve k^3 degenerate in three lines, so that in all our congruence contains six more figures k^3 , each consisting of three lines.

The system of the curves k^3 which form the above mentioned monoid, is represented on a conic through L and M , for the point ranges (P) and (Q) of these curves are projective. This conic contains the points S_1 and S_2 . Analogously to the dimonoid which is formed by the conics passing through H_1 and H_2 and cutting l , m and h_{34} , there corresponds a conic through L , M , S_3 and S_4 .

Outside l the two conics have two points in common, the image points of the curves k^3 which consist of a transversal of l , m , h_{12} and h_{34} and the latter two lines.

The contents of the preceding three paragraphs have been communicated to me by Prof. JAN DE VRIES.

Consequently our congruence contains in all 54 curves degenerate in three lines.

§ 3. A curve k^3 which cuts l twice, has two image points, which lie on the same line q through M . The locus of the image points of these curves cuts any line q twice outside M in the image points of the curve k^3 that cuts l twice and that passes through the corresponding point Q . As l cuts the surface a^5 twice outside the triple point A , two of the curves k^3 through A cut l once more. Hence the locus of the image points of the curves k^3 which cut l twice, is a biquadratic curve λ^4 that has a double point in M .

λ^4 has also a double point in L . The tangents to λ^4 at this point are the two lines p corresponding to the points of intersection P with l of the curve k^3 through B which cuts l twice. Accordingly a line p cuts λ^4 twice outside L . Through a point of l there pass, therefore, just as through A , two curves k^3 which cut l once more.

The curve k^3 consisting of the conic k_i^2 and the line through H_i which cuts k_i^2 and l in different points and which is, accordingly, represented in S_i , cuts l twice, the same as the curve k^3 formed by t_i and the conic cutting l of the pencil β_i , of which the image point lies in S_i' . Consequently λ^4 passes through the points S_i and S_i' .

The points of contact of the four tangents to λ^4 through M are the image points of the four twisted cubics which pass through the four points H_i , touch l , and cut m .

The locus of the image points of the curves k^3 which cut m twice, is a biquadratic curve μ^4 which has double points in L and M and passes through the points S_i and S_i' . Each of the said curves k^3 has two image points, the two points of intersection outside L of μ^4 and the line p associated to its point of intersection with l .

Four of the sixteen points of intersection of λ^4 and μ^4 lie in each of the points L and M ; the remaining eight are the points S_i and S_i' .

Hence no twisted cubic which cuts the lines l and m twice, passes through the points H_i .

§ 4. We shall now examine the representation of the system Σ_1 of the curves k^3 which cut an arbitrary line n . The curves k^3 passing through a point P of l or a point Q of m , form a surface of the fifth degree, which has been proved for the point B in § 2. Hence through any point of l or m there pass five curves of Σ_1 . Consequently the image curve of Σ_1 has five points, resp. outside L and M , in common with each line p and q . As each of the systems associated to L and M contains five curves which cut n , the said image curve has fivefold points in L and M . Accordingly Σ_1 is represented in a curve of the tenth order, k^{10} , which has fivefold points in L and M .

Of the curves k^3 associated to S_i , two belong to Σ_1 , those which are formed by k_i^2 and the two transversals of k_i^2 and n through H_i . Among the curves k^3 corresponding to S_i there is only one which belongs to Σ_1 , the one which consists of t_i and the conic of the pencil β_i which cuts n . Hence the curve k^{10} has a double point in each of the points S_i and passes through all the points S_i' .

Ten of the forty points of intersection of the curves k^{10} and λ^4 coincide in each of the points L and M and two in each of the points S_i , and also the points S_i' belong to them. The remaining eight points of intersection are yielded by the four pairs of image points of the curves k^3 which cut l twice and n once.

As, accordingly, through four points there pass four twisted cubics which cut a given line twice and two other given lines once, Σ_1 contains four curves which cut n twice. The image points of these curves are double points of k^{10} .

Consequently the image curve of the system Σ_1 of the curves k^3 which cut a given line n , is a curve k^{10} of the order ten and the genus eight which has the points L and M as fivefold points, has double points in the points S_i , passes through the points S_i' , and has four more free double points.

25 of the hundred points of intersection of two curves k^{10} lie in each of the points L and M , four in each of the points S_i , and also the points S_i' belong to the points of intersection. There remain, therefore, thirty points of intersection that are not singular for the representation.

Accordingly there are thirty twisted cubics which pass through four given points and cut four given lines.¹⁾

If n cuts the line l , the line p corresponding to the point of intersection of l and n splits off from k^{10} and there remains a curve of the ninth order which has a quadruple point in L and a fivefold point in M . If n is a transversal of l and m , the image curve is of the eighth order and has quadruple points in L and M .

¹⁾ Cf. SCHUBERT, *Kalkül der abzählenden Geometrie*, p. 172.

If n passes through H_1 , the image points of the curves k^3 which cut n outside H_1 , form a curve which cuts any line p or q resp. outside L or M in two points. For n cuts two of the curves k^3 through a point of l or m , which form a surface of the fifth degree with triple point in H_1 , outside H_1 .

If we choose the point of l or m in A or B , it appears that the image curve has double points in L and M and is, therefore, of the fourth order. Among the curves k^3 corresponding to S_1 there is none which cuts n outside H_1 . Of the curves k^3 associated to S_2 one cuts n outside H_1 , to wit the curve which consists of k_2^2 and the line through H_2 which cuts n and k_2^2 in different points. As an analogous remark may be made for S_3 and S_4 , the image curve k_1^4 does not pass through S_1 , but it passes through S_2 , S_3 and S_4 . The curve k^3 consisting of t_1 and the conic of the pencil β_1 cutting n , which, accordingly, cuts n outside H_1 , corresponds to S_1' , and the curves k^3 associated to S_2' , S_3' , and S_4' do not contain any curve which cuts n outside H_1 . Consequently the image curve k_1^4 passes through S_1' , but not through S_2' , S_3' , and S_4' .

The curves k^3 which cut a line through H_i outside H_i , are represented on the points of a biquadratic curve k_i^4 with double points in L and M which passes through the points S different from S_i and through S_i' .

If n approaches to a line through H_1 , the system of the curves k^3 which cut n , ends by degenerating into the system of the curves k^3 which cut the limitative position of n outside H_1 and the system of the curves k^3 which touch the limitative position of the plane $H_1 n$ at H_1 .

Hence the curves k^3 which touch a plane through H_i at H_i , are represented on the points of a curve of the sixth order k_i^6 with triple points in L and M , which has a double point in S_i and passes through the points S and S' differing resp. from S_i and S_i' .

This may also be derived directly.

The points of intersection of a curve k_i^4 and a curve k^{10} are the points L and M , each counted ten times, the three points S different from S_i , each counted twice, the point S_i' , and thirteen points that are not singular for the representation. Hence a straight line through H_i cuts thirteen of the curves k^3 which intersect an arbitrary line outside H_i .

Consequently the surface of the curves k^3 which cut an arbitrary straight line n , has a seventeenfold point in each point H_i . This surface, which is of the degree thirty, has l , m , and n as fivefold straight lines.

The sixteen points of intersection of two curves k_i^4 contain the points L and M , each counted four times, the three points S different from S_i , and the point S_i' . The other four points of intersection are not singular for the representation. A curve k_i^4 has six non-singular points in common with a curve k_k^4 ; for the singular points of intersection are the points L and M , each counted four times, and the two points S different from S_i and S_k .

Hence four of the curves k^3 cutting a line n through H_i outside H_i also cut another line through H_i outside H_i , and six cut a line through H_k outside H_k .

The surface of the thirteenth degree which is formed by the curves k^3 that cut a line through H_i outside H_i , has, therefore, a ninefold point in H_i and sevenfold points in the other points H_i . This surface has the lines l and m as double lines and n as a fivefold line.

By considering the representation of the system of the curves k^3 which touch a plane through H_i at H_i , or by observing that the surface formed by these curves completes the surface of the curves k^3 that cut a line through H_i in the said plane outside H_i , to an individual of the system of surfaces, each consisting of the curves k^3 which cut an arbitrary line, we find:

The curves k^3 which touch a plane through H_i at H_i , form a surface of the degree seventeen which has an eightfold point in H_i and tenfold points in the other points H_i . l and m are triple lines of this surface. For the tangents at H_i to the curves k^3 which pass through a point of l or m , form a cubic cone.

Four of the sixteen points of intersection of a curve λ^4 or μ^4 and a curve k_i^4 lie in each of the points L and M and also the points S different from S_i and the point S_i' belong to them. The other four non-singular points of intersection form the pairs of image points of two curves k^3 that cut l or m twice and that cut the line through H_i corresponding to k_i^4 outside H_i . Accordingly the biquadratic surface of the curves k^3 which cut l or m twice, have double points in the points H_i .

§ 5. A straight line a of a is the image of a system of curves k^3 which form a surface of the tenth degree Ω^{10} , for to the ten points of intersection of a and a curve k^{10} there correspond as many curves of the said system, which cut the line n associated to k^{10} . To the point of intersection of a and LM there corresponds the curve k^3 through A and B , which, therefore, lies on Ω^{10} . As the pairs of rays (p, q) with points of intersection on a define a projective correspondence between the plane pencils L and M , Ω^{10} is the locus of the curves k^3 which contain the pairs of points of a projective correspondence between the lines l and m .

As a has four points in common with each of the curves λ^4 and μ^4 , Ω^{10} contains four curves k^3 which cut l twice and as many curves k^3 which cut m twice. To the six points of intersection of a with a curve k_i^6 there correspond as many curves k^3 of Ω^{10} , which touch the plane through H_i corresponding to k_i^6 at H_i . Accordingly Ω^{10} has a sextuple point in each point H_i . Besides l and m the surface contains eight more lines, the four lines which each join a point H_i to the point on l associated to the point of intersection of m with the plane through the other three points H_i , and the four lines which each pass through a

point H_i and through the point on m corresponding to the point of intersection of l with the plane through the other three points H . For each of these lines, together with a conic, forms a curve k^3 of Ω^{10} (cf. § 2).

A surface which is formed by the curves k^3 containing the pairs of a projective correspondence between l and m to which the pair (A, B) does not belong, is represented on a conic through L and M .

§ 6. The image curve of the system Σ_2 of the curves k^3 which touch a given plane φ , has as many points outside L in common with a straight line through L as curves k^3 touching φ pass through a point P of l . Through a cubic involutory transformation¹⁾ with cardinal points in H_1, \dots, H_4 P is transformed into the point P' , m into a cubic m'^3 through H_1, \dots, H_4 and φ into a cubic surface φ'^3 which has conical points in H_1, \dots, H_4 . To the curves k^3 through P which touch φ , there correspond the tangents of φ'^3 through P which cut m'^3 outside H_1, \dots, H_4 . The tangents of φ'^3 through P' form a cone of the sixth degree which has the lines $P'H_1, \dots, P'H_4$ as double generatrices. There are, accordingly, $6 \times 3 - 4 \times 2 = 10$ lines of the cone which cut m'^3 outside the points H_1, \dots, H_4 .

Consequently the image curve of Σ_2 has ten points outside L in common with any line through L . Through each of the points A and B there pass ten curves k^3 which touch φ . Hence the image curve of Σ_2 is a curve of the order twenty, k^{20} , with tenfold points in L and M .

As a pencil β_i contains two conics which touch φ and which with t_i form two curves k^3 associated to S_i' , k^{20} has a double point in each of the points S_i' .

Among the curves k^3 through P_1 , the point of intersection of l with the plane $H_2 H_3 H_4$, which touch φ , there are in the first place four curves which are formed by the two conics touching φ of the pencil with base points H_2, H_3, H_4 , and P_1 , each combined with the two transversals through H_1 of such a conic and the line m . Further we have the curve k^3 consisting of the conic through H_2, H_3, H_4, P_1 and the point of intersection of the planes $H_2 H_3 H_4$, φ and $H_1 m$, and the join of the latter point of intersection with H_1 . Finally we find the two curves k^3 formed by k_1^2 and the lines which project the points of intersection of k_1^2 and φ out of H_1 . The latter two curves are represented on the point S_1 . Now we have ten curves k^3 through P_1 which touch φ , if we count the latter three curves twice.

In fact k^{20} has a quadruple point in each of the points S_i , so that four of the points of intersection of LS_i with k^{20} lie in S_i , and the latter two curves count, therefore, for four of the curves k^3 through P_1 which touch φ ; hence each of these curves must be counted twice. In order to

¹⁾ Cf. Prof. JAN DE VRIES, these Proceedings, II, 84.

show this we make use of the fact that there are eight curves k^3 which pass through four given points, cut a straight line, cut another straight line twice, and touch a plane ¹⁾. Among the 80 points of intersection of λ^4 and k^{20} the points L and M must each be counted 20 times. Further they contain the four points S_i' , each counted twice, and the eight pairs of image points of the curves k^3 which touch η and which cut l twice. There remain 16 points of intersection, so that four of the points of intersection of λ^4 , and also of μ^4 , with k^{20} coincide in each point S_i . And the curves k^{20} have no definite tangent in S_i as any k^3 may belong to Σ_2 .

Accordingly the system Σ_2 of the curves k^3 which touch a plane η , is represented on a curve of the order twenty, k^{20} , which has tenfold points in L and M , quadruple points in the points S_i and double points in the points S_i' .

A curve k^{10} and a curve k^{20} have sixty points, that are non-singular for the representation in common.

There are, therefore, sixty twisted cubics which pass through four given points, cut three given straight lines, and touch a given plane.

Two curves k^{20} have 120 points of intersection that are not singular for the representation.

Consequently there are 120 twisted cubics which pass through four given points, cut two given straight lines, and touch two given planes.

The sixteen non-singular points of intersection of λ^4 and k^{20} are the eight pairs of image points of the curves k^3 which cut l twice and which touch η .

A curve k_i^4 has 26 points in common with k^{20} which are not singular for the representation.

The curves k^3 which touch a given plane, form a surface of the degree sixty, which has l and m as tenfold lines and has 34-fold points in the points H_i .

If η passes through H_i , the surface of the curves k^3 which touch η at H_i , discussed in § 4, splits off twice.

Accordingly the surface of the curves k^3 which touch a plane through H_i outside H_i , is of the degree 26. It has quadruple lines in l and m , an eighteenfold point in H_i , and fourteenfold points in the other points H .

¹⁾ Cf. e. g. these Proceedings, 29, 781.

Mathematics. — “*Determination of the System (2, 1, 1) of ∞^3 Line Elements of Space*”. By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of April 24, 1926).

§ 1. In my paper: “*Determination of the Bilinear System of ∞^3 Line Elements of Space*”¹⁾ the system (1, 1, 2) of ∞^3 line elements of space has been examined. In this paper we shall treat the system (2, 1, 1) of ∞^3 line elements (P, l) of space. For this system S_3 the order of the complex of the lines l is two; it contains one line element for which P lies in an arbitrary given point of space and the points P of the line elements of S_3 which belong to an arbitrary given plane, form a straight line r .

§ 2. The lines l which define line elements of S_3 with the points P of a given line m , form a scroll which has m as a single directrix. A plane through m contains one generatrix of this scroll, the line which, as line l , corresponds to the point P of m which lies in the point of intersection of the line r of the said plane with m .

The lines l which correspond to the points P of a line m , form, therefore, a quadratic scroll μ^2 .

The locus of the points P of the lines of a complex cone \varkappa of the quadratic complex C^2 of the lines l of S_3 , is a curve which cuts a plane through the vertex T of \varkappa in two points outside T , the points P of the two generatrices l of \varkappa in this plane. Besides this curve passes through T where it touches the line l which forms a line element of S_3 with T .

Accordingly the locus of the points P of the generatrices l of a complex cone \varkappa of C^2 is a twisted cubic which passes through the vertex of \varkappa .

The lines l of C^2 which cut two given lines m and n , form a biquadratic scroll λ^4 of the genus one, which has m and n as double directrices. The locus of the points P of the lines l of λ^4 cuts a plane through m outside m in the points P of the two generatrices l of λ^4 in this plane. There are two lines l of λ^4 for which the corresponding point P lies in m ; they are the two generatrices of the scroll μ^2 corresponding to m which cut n .

Consequently the points P of the lines l of C^2 which cut the lines m and n , form a biquadratic twisted curve k^4 of the genus one which has m and n as chords.

The lines l of C^2 which cut a line m , form a congruence $M(2, 2)$. The locus of the points P of the lines l of a congruence $M(2, 2)$ is a

¹⁾ These Proceedings, 27, 2.

surface which contains m , because among the lines which pass through a point of m , there is one which has its point P in the said point. To the points P where this surface cuts an arbitrary line m' , there correspond as lines l the generatrices which cut m of the scroll μ'^2 corresponding to m' .

Accordingly to a congruence $M(2, 2)$ of lines of C^2 which cut a given line m , there corresponds a quadratic surface Ω^2 of points P which contains m .

§ 3. A curve of singular points P would lie on all surfaces Ω^2 , as one or more of the ∞^1 lines l associated to a point of this curve, would cut an arbitrary line m . And the intersection of two surfaces Ω^2 is formed by the curve k^4 of the points P of the lines l of C^2 which cut the two lines to which the said surfaces Ω^2 correspond.

Accordingly the system S_3 does not contain any curve of singular points P .

Two of the six points of intersection of the curve k^3 associated to a complex cone \varkappa and the quadratic surface corresponding to a line m , lie in the points P of the lines l of \varkappa which cut m . The other four, to which there must correspond a straight line of \varkappa as well as a line outside \varkappa cutting m , are necessarily singular. As S_3 does not contain any singular curve, any curve k^3 and any surface Ω^2 have the same four singular points in common. Through these four points H_1, \dots, H_4 there pass any surface Ω^2 , any curve k^3 and consequently any complex cone \varkappa . Accordingly the points H_1, \dots, H_4 are cardinal points of the complex C^2 and also of S_3 , as each of these points corresponds as point P to any line l through it.

In a plane through three of the points H there lie three pencils of lines l of C^2 , which, therefore, form a curve of the third class. As the lines of C^2 in an arbitrary plane envelop a conic, all the lines of a plane through three points H must belong to C^2 so that the sides of the tetrahedron $H_1 H_2 H_3 H_4$ are cardinal planes of C^2 . The quadratic complexes through the congruence (4, 4) consisting of the four sheaves H_1, \dots, H_4 and the four fields in the sides of the tetrahedron $H_1 H_2 H_3 H_4$, form a pencil of which one individual also contains a line l given at random. Consequently C^2 must be the tetrahedral complex of which $H_1 H_2 H_3 H_4$ is the tetrahedron of singularities and which contains a line of C^2 that does not pass through any of the angular points of the said tetrahedron and does not lie in any of the sides either.

Accordingly the complex C^2 of the lines of S_3 is a tetrahedral complex. The four cardinal points of this complex are also cardinal points of S_3 .

§ 4. Of a complex cone \varkappa of C^2 with vertex T we consider two generatrices l_1 and l_2 to which resp. the points P_1 and P_2 correspond in S_3 . The curve k^3 of the points P of the lines of \varkappa passes through

$T, P_1, P_2, H_1, H_2, H_3$ and H_4 . If we consider the projective correspondence between the sheaves P_1 and P_2 in which to the lines $P_1 H_1, \dots, P_1 H_4$ there correspond resp. the lines $P_2 H_1, \dots, P_2 H_4$, k^3 is the locus of the points of intersection of lines associated to each other in this correspondence. To the line l_1 through P_1 the line l_2 through P_2 is conjugated as these cut each other in the point T of k^3 . This also holds good if P_2 lies for instance in T . In this case to the line $l_1 \equiv P_1 T$, the tangent to k^3 at T is associated which line l_2 corresponds to the point P_2 in T .

Let us now choose two points P_1 and P_2 so that the lines l_1 and l_2 do not cut each other. If P describes the line l_1 , l describes a scroll with l_1 not only as directrix but also as generatrix, i. e. the system of tangents to a complex conic of C^2 in a plane through l_1 , to which plane, accordingly, l_1 is associated as line r . Through the point of intersection of l_2 and this plane we draw a tangent l_3 to the said complex conic. To l_3 there corresponds in S_3 the point of intersection P_3 of l_3 and l_1 .

In the projective correspondence between the sheaves P_1 and P_3 which is defined by the four pairs of rays $(P_1 H_i, P_3 H_i)$, the line l_3 corresponds to l_1 according to the above. In the same way the line l_2 corresponds to l_3 in the projective correspondence between the sheaves P_3 and P_2 which is defined by the four pairs $(P_3 H_i, P_2 H_i)$. Hence in the projective correspondence between the sheaves P_1 and P_2 which is defined by the four pairs $(P_1 H_i, P_2 H_i)$, the line l_2 corresponds to l_1 .

Accordingly a system $S_3(2, 1, 1)$ of ∞^3 line elements may always be produced by the aid of four given points H_1, \dots, H_4 and a given line element (P_1, l_1) by associating to each point P of space the line l through P which in the projective correspondence between the sheaves P_1 and P defined by the four pairs $(P_1 H_i, P H_i)$, corresponds to l_1 .

Instead of a line element (P_1, l_1) of S_3 we may also choose five rays s_1, \dots, s_5 which pass through an arbitrary point S , and we may associate to any point P the line l which in the projective correspondence between the sheaves S and P defined by the four pairs $(s_i, P H_i)$ corresponds to s_5 .

§ 5. On the line l_1 of the line element (P_1, l_1) belonging to S_3 , we choose a point Q_1 different from P_1 . We consider the collinear transformation (P, Q) which has its coincidences in the points H_1, \dots, H_4 , and in which the point Q_1 corresponds to P_1 . The join l of the points P and Q corresponding to each other in this transformation, form a tetrahedral complex which has its cardinal points in H_1, \dots, H_4 and contains l_1 , and which is, therefore, identical with C^2 . If to any line l of C^2 we associate the point P of the pair (P, Q) on l , there arises a system $(2, 1, 1)$ of ∞^3 line elements which has the cardinal points H_1, \dots, H_4 and the line element (P_1, l_1) in common with S_3 and which is, therefore, identical with S_3 .

A system $S_3(2, 1, 1)$ of ∞^3 line elements can always be derived from a collinear transformation (P, Q) of space if to any point P the join l of P and the point Q corresponding to P in the transformation, is associated.

This leads to the well known theorem ¹⁾, that the quintets of rays which join the points P of space with the coincidences of a collinear correspondence and with the points Q corresponding to P , can be transformed into each other by collinear transformations.

¹⁾ STURM, Liniengeometrie, I. p. 354.

Mathematics. — "*The Involution in the Rays of Space with Rank Numbers One, One, and Zero*". By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES.)

(Communicated at the meeting of May 29, 1926).

§ 1. On p. 19 of Vol. 28. of these Proceedings we have indicated how the involution of rays with rank numbers *one, one, and zero* is defined through two given straight lines m and n and a bilinear congruence K . For by the aid of these data we can construct the line l' associated to a line l by determining the line different from l which the scroll (l, m, n) has in common with the linear complex containing l and K .

By means of a representation of the rays of space on the points of a linear four-dimensional space this involution has been treated by the author on p. 84—92 of his thesis for the doctorate ¹⁾).

In this paper we shall first investigate the said involution of rays without using more-dimensional considerations. Further we shall indicate the special involutions of rays with rank numbers *one, one, and zero* for which the scroll associated to a plane pencil and the congruence corresponding to a star and a field, have lower degree numbers than those for the general involutions of rays with the same rank numbers.

§ 2. If l is chosen in m , the linear complex through l and K , i.e. the linear complex C_m through m and K , is entirely defined; for the scroll (l, m, n) , however, we may choose any scroll which contains m and n , so that any line of C_m corresponds as line l' to the line l coinciding with m . In the same way any line of the linear complex C_n through n and K corresponds to n .

Accordingly the lines m and n are singular rays of the third rank ²⁾. To each of these lines there corresponds a linear complex which contains K .

If, however, we choose l in K , there is one scroll (l, m, n) whereas there are an infinite number of complexes (l, K) . In this case, therefore, any line of the scroll (l, m, n) corresponds to l as line l' . The ∞^2 scrolls (l, m, n) through the rays l of K form a complex. We shall try to find the number of lines of this complex which lie in a given plane pencil w . $m, n,$ and w define a bilinear congruence of which the directrices are

¹⁾ „*Afbeeldingen van figuren op de punten eener lineaire ruimte*”, Groningen 1922.

²⁾ This means that to m as well as to n there correspond ∞^3 rays.

the transversals of m and n which lie in the plane of w and pass through the vertex of w . Any scroll (l, m, n) which contains a line of w , must lie entirely in the bilinear congruence (m, n, w) , as m, n and the generatrix in w cut the two directrices of the said congruence. Now the bilinear congruence (m, n, w) has two lines l in common with K and each of the corresponding scrolls (l, m, n) has a generatrix in w . Accordingly the scrolls (l, m, n) corresponding to the lines l of K , form a quadratic complex C^2 which contains K .

A straight line of C^2 through a point P of m must belong to a scroll (l, m, n) which is degenerate and which, accordingly, corresponds to a line l of K which cuts m or n . The lines of C^2 through a point P of m form, therefore, the plane pencil defined by m and the line of K through P , and another plane pencil that has P as vertex and of which the plane passes through m and the point of n where the line of K in the plane (P, n) cuts the line n . Hence m and n are double lines of C^2 .

The lines of K are, therefore, singular. To any ray l of K there correspond the generatrices of the scroll (l, m, n) . These scrolls form a quadratic complex C^2 which contains K and of which m and n are double lines.

A line l is also singular if the corresponding scroll (l, m, n) has an infinite number of rays in common with the complex (K, l) . As there exists no linear complex which contains K, m and n , for such a line l the scroll (l, m, n) must degenerate so that l must cut m or n .

If we choose a line l which cuts m , the scroll (l, m, n) degenerates into the plane pencil (l, m) and the plane pencil which contains n and has a ray in common with (l, m) . Now, if l cuts m but does not cut n , the plane pencil which contains n , cannot lie in (K, l) . For if this were the case (K, l) would contain two rays of the plane pencil (l, m) , hence also m , so that m, n and K would lie in a linear complex. The entire plane pencil (l, m) lies in (K, l) , if l is a generatrix of one of the plane pencils of C_m which contain m , i.e. the plane pencils defined by m and a line of K cutting m . These plane pencils form a special bilinear congruence K_m which belongs to the complex C^2 .

There are two special bilinear congruences K_m and K_n of singular rays, which are composed of singular plane pencils, i.e. plane pencils of which any two generatrices are always associated to each other in the involution under consideration.

If a straight line l describes a pencil (F, φ) with vertex F in a plane φ , the scroll (l, m, n) always belongs to the bilinear congruence of which the transversal f_1 through F and the transversal f_2 in φ of m and n are directrices. If we associate to each other those generatrices of a plane pencil of this congruence through which there pass a scroll (l, m, n) and a complex (K, l) corresponding to the same line l of the plane pencil (F, φ) , there arises a (1,1) correspondence in the former plane pencil. Accordingly the scroll consisting of the rays each of which is common

to a scroll (l, m, n) and a complex (K, l) which contain the same line l of (F, φ) , has f_1 and f_2 as double directrices and is, therefore, of the fourth degree. The plane pencil (F, φ) belongs to this surface.

Consequently the lines l' which are associated to the generatrices l of (F, φ) , form a cubic scroll ϱ^3 which has f_1 as a double directrix, f_2 as a single directrix. To the generatrices of (F, φ) in C^2 there correspond two lines of K , to the generatrices which lie in C_m and C_n are resp. associated m and n .

Accordingly to a plane pencil there corresponds a cubic scroll which contains m and n and which has two lines in common with the bilinear congruence K .

As F is a point of the double directrix f_1 of ϱ^3 , two generatrices of this surface that lie in the plane pencil (F, φ) , pass through F . These are double rays of our involution.

The double lines form a quadratic complex. This complex contains the congruences K , K_m and K_n , hence also the lines m and n . The lines of the complex through a point of m or of n form the plane pencil of K_m or K_n which has this point as vertex, and a pencil in a plane through resp. n or m . Hence the bilinear congruence of which m and n are the directrices, belongs to the complex of the double lines.

If for l we choose a directrix of K , (K, l) becomes the special linear complex that has l as axis, and l' coincides, therefore, with l . The double lines of the involution through a point of a directrix of K form two plane pencils. The plane of one of them passes through the other directrix of K , the plane of the other one touches the quadratic surface which is defined by m , n , and the chosen directrix. For if we choose l in the indicated second plane pencil, (K, l) is the special linear complex which has the chosen directrix as axis, and this directrix touches the surface (l, m, n) so that l' coincides with l . In the same way it is evident that the double lines in a plane through a directrix of K form two pencils one of which contains this directrix. Hence the two directrices of K are single lines of the complex of the double rays. Just as m and n they are lines of the surface of KUMMER of this complex.

If (F, φ) contains one of the lines m and n , for any generatrix l of (F, φ) the scroll (l, m, n) degenerates into (F, φ) and the plane pencil through n or m which has a line in common with (F, φ) . In this case the latter plane pencil is associated to (F, φ) .

If the plane pencil (F, φ) has a line l in common with K , the scroll (l, m, n) corresponding to this line l splits off from the associated ϱ^3 . In this case, accordingly, there corresponds to (F, φ) a plane pencil which has a line in common with K , of which the plane passes through f_1 , and of which the vertex lies on f_2 . This plane pencil, associated to (F, φ) , lies with (F, φ) in the linear complex through K that contains (F, φ) , and it contains the lines which, together with the generatrices of (F, φ) , form pairs of associated directrices of that individual of the pencil of complexes

defined by the bilinear congruence (m, n) which is involutory with the complex through K and (F, φ) .

If (F, φ) contains a generatrix of K_m or of K_n , a plane pencil of K_m or of K_n splits off from ϱ^3 and there remains a quadratic scroll which contains n or m and which has a line in common with K and with K_m or K_n . If (F, φ) contains a line of each of the congruences K_m and K_n , a similar plane pencil is associated to (F, φ) . Especially a pencil with vertex on m or n and plane through n or m is transformed into itself.

The order of the congruence Σ associated to a star S of rays l , is equal to the number of pairs of rays (l, l') of which one line passes through S , the other one through a point T chosen arbitrarily. Such a pair of rays must belong to one of the scrolls containing m and n which pass through S and T and of which the transversals s and t of m and n through S and T are, therefore, directrices. Hence the line l through S of such a pair of rays must belong to the pencil which has S as vertex and which lies in a plane through t . t is a single directrix of the ϱ^3 corresponding to this plane pencil, so that one directrix of this scroll passes through T . Accordingly the order of Σ is one.

The pairs (l, l') of which l passes through S and l' lies in a given plane π , must belong to the bilinear congruence which has s and the transversal p of m and n in π as directrices. To the pencil that has S as vertex and of which the plane passes through p , there corresponds a ϱ^3 that has p as single directrix so that two generatrices of this surface lie in π . The class of Σ is, therefore, two.

The singular line of Σ is the transversal s of m and n through S . The congruence Σ contains m and n , which lines correspond to the rays of C_m and C_n through S , and with each of the congruences K_m and K_n it has the plane pencil in common to which the generatrix through S of K_m or K_n belongs. To the quadratic complex cone of C^2 that has S as vertex, there corresponds the scroll which is formed by the lines of K that cut s .

To a star of rays l a congruence $\Sigma(1,2)$ is associated which contains m and n , which has a scroll in common with K and a plane pencil in common with K_m and K_n .

The rays of Σ through S form the quadratic cone of the double rays through S . Accordingly Σ has a singular conic. This is one of the conics that lie on the quadratic surface which is formed by the generatrices of K that cut s and that pass through the points of intersection outside s of this quadratic surface with m and n , i.e. the vertices of the plane pencils of K_m and K_n in Σ .

If S lies on m or n , for any line l of the star the scroll (l, m, n) degenerates into the plane pencil (l, m) or (l, n) and a pencil containing n or m which lies in the plane (n, S) or (m, S) . Accordingly to a star with vertex S on m or n the fields of rays (n, S) or (m, S) are resp. associated.

If S lies on one of the directrices of K , the complex (K, l) for a line l outside K of the star is always the special linear complex which has the directrix containing S as axis. The line l' corresponding to l always lies in the plane through this directrix and the transversal through S of m and n . Accordingly also in this case a field of rays corresponds to S . Therefore, from the congruence (1,2) which generally corresponds to S , the bilinear congruence has split off which is formed by the scrolls (l, m, n) of the generatrices l of the plane pencil which S has in common with K .

As there are two pairs (l, l') of which one line lies in a given plane φ , the other one passes through a given point T , the congruence Φ associated to a field of rays φ has the order *two*. If we have a line l' in a plane π corresponding to a line l of φ , l and l' both cut the transversals f and p of m and n in φ and π . In this case the line l belongs to the pencil in φ which has the point of intersection of p with φ as vertex. p is a double directrix of the cubic scroll ϱ^3 associated to this plane pencil, so that one generatrix of ϱ^3 lies in π . The class of Φ is, therefore, one. With respect to m, n, K, K_m and K_n , apparently Φ entirely behaves as a congruence Σ . The scroll which Φ has in common with K , consists of the lines of K which cut the singular line of Φ , i.e. f .

To a field of rays there corresponds a congruence (2,1) which contains m and n , which has a scroll in common with K , and a plane pencil in common with K_m and K_n .

To a field of rays φ containing m or n the star is associated which has the point of intersection of φ with n or m as vertex. If one of the directrices of K lies in φ , there corresponds to φ the star of which the vertex lies in the intersection of the transversal of m and n in φ with the directrix of K in φ . To a bilinear congruence, which, according to the theorem of HALPHEN, has three rays in common with a congruence Σ as well as with a congruence Φ , a congruence Ψ (3,3) is, therefore, associated. A star and a field of rays which contain m , to which resp. a field and a star containing n are associated, each contain, therefore, only one line of Ψ which is non-singular for the involution of rays, so that m , and also n , is a double line of Ψ . Any plane pencil of K contains two lines of Ψ , to wit those which are associated to the lines which the bilinear congruence corresponding to such a plane pencil, has in common with the chosen bilinear congruence.

Accordingly to a bilinear congruence there corresponds a congruence (3,3) which has double lines in m and n and which has a scroll of the fourth degree in common with K . This congruence (3,3) has two plane pencils in common with K_m as well as with K_n .

If the bilinear congruence B contains one of the lines m and n , for instance m , it has two lines, non-singular for the involution, in common with each of the congruences which correspond to a sheaf and to a field. In this case B corresponds, therefore, to a congruence $\Psi(2,2)$, which has a single line in m , a double line in n , as a star and

a field containing n have one non-singular ray in common with B , a star and a field containing m none. The bilinear congruence corresponding to a plane pencil of K , contains only one non-singular line of B , so that $\Psi(2,2)$ has a quadratic scroll in common with K .

In the same way it appears that a bilinear congruence containing m and n corresponds to a similar bilinear congruence. The same holds good for a bilinear congruence of which one of the directrices coincides with a directrix of K .

As a linear complex Γ has three rays in common with the ϱ^3 which is associated to a plane pencil, a cubic complex Γ' corresponds to Γ . This complex contains K_m and K_n , because Γ has one line in each singular plane pencil of the involution. The lines of Γ' through a point of m form in the first place the plane pencil of K_m that contains m and that has this point as vertex. The other lines correspond to the generatrices of the pencil of Γ in a plane through n . This plane pencil has a line in common with K_n so that a quadratic cone containing m corresponds to it. Hence m and also n is a double line of Γ' . As $\overline{\Gamma}$ has two lines in common with each scroll (l, m, n) corresponding to a line l of K , any ray of K is a double line of Γ' .

To a linear complex there corresponds a cubic complex which has m and n as double lines just as the generatrices of K , and which contains K_m and K_n .

If m is a line of Γ , the complex C_m splits off from Γ' and there remains, therefore, a quadratic complex which has n as double line and of which m and the generatrices of K and K_n are single lines. The quadratic complex which corresponds to a linear complex through n , has m as double line and contains n and the generatrices of K and K_m as single lines. A linear complex containing m and n , passes into a similar complex. The special linear complexes that have m and n as axes, are associated to each other. A linear complex which contains K , is transformed into itself. ¹⁾

§ 3. We shall now investigate the special cases of the involution of rays which arise for special relative situations of m , n , and K .

In the first place we shall suppose that m , n , and K belong to a linear complex A . In this case the directrices of K also lie in a linear complex with the bilinear congruence (m, n) which has m and n as directrices. For the linear complexes which contain the congruence (m, n) and form a pencil, are all involutory relative to A , and the complex B of this pencil which contains one directrix of K , must also contain the other directrix associated to it relative to A .

If we choose a line l outside A , the line which is associated to l relative to B must cut all the generatrices of the scroll of which l , m ,

¹⁾ For further properties of the involution of rays cf. the paper mentioned in § 1.

and n are directrices and must, therefore, belong to the scroll (l, m, n) but also to the complex (K, l) , as (K, l) and B are involutory. In this case l' is the line associated to l relative to B . If l lies in A , (l, m, n) belongs entirely to A and (K, l) coincides with A .

Accordingly if m, n and K lie in one linear complex, the involution of rays degenerates into that of the pairs of associated directrices relative to a linear complex and into the system of ∞^4 pairs of rays of the scrolls of A which contain m and n , hence into an involution with rank numbers one, zero, and zero and an involution with rank numbers zero, one, and zero.

We shall now consider the case that m coincides with one of the directrices of K . For a line l which cuts m , the complex (K, l) is the special linear complex that has m as axis, and the scroll (l, m, n) degenerates into two plane pencils one of which, (l, m) , belongs entirely to (K, l) , so that to a line which cuts m , there corresponds a plane pencil. Hence from the involution of rays there splits off the system of the ∞^4 pairs of rays which lie in plane pencils containing m . In the remaining involution of rays a scroll ϱ^2 corresponds to a plane pencil w , as from the ϱ^3 which in the general case corresponds to a plane pencil, the plane pencil containing m splits off which has a generatrix in common with w .

Now from the quadratic complex of the double rays which contains m, n and K , there splits off the special linear complex that has m as axis, so that there remains a linear complex C . This complex is defined by K and n .

The pairs of rays which the linear complexes through K have in common with a definite scroll that contains m and n , form an involution which has as double rays the generatrix o of this scroll different from n and belonging to C , and the line m . For if we choose l in m , (K, l) is the special linear complex which has m as axis, and the common rays of (K, l) and the scroll which contains m and n , coincide in m .

As a special case of the involution of rays with rank numbers one, one, and zero we get, therefore, the involution for which the line l' corresponding to l , is constructed in the following way. Let a linear complex C be given, a line m which does not belong to C , and a line n in C . The scroll ϱ defined by l, m , and n has a line o in common with C besides n . Now to l the generatrix l' of ϱ is associated which is harmonically separated from l by m and o .

This involution of rays has been treated by the author in these Proceedings, Vol. 28, p. 556.

If m and n coincide with the directrices of K , we get an involution of rays in which a plane pencil w' is associated to a plane pencil w . In this case the involution of the pairs of rays which the linear complexes through K have in common with a scroll containing m and n , has m and n as double lines.

Consequently to the involutions of rays with rank numbers one, one, and zero, the involution of the pairs of lines is associated which correspond to each other in a scrolled involutory collineation of space.

It is easily seen that these two involutions of rays satisfy the formulas of § 3 of the paper mentioned in § 1 of this communication.

§ 4. We have another special case if K is degenerate in a star T and a field τ which have a plane pencil (T, τ) in common, so that the vertex T of the star lies in the plane τ . In this case for any line l the complex (K, l) is the special linear complex which has as directrix the generatrix of the pencil (T, τ) which cuts l .

We have, accordingly, an involution of rays for which the line l' corresponding to a line l , is defined in the following way. The line l cuts one generatrix t of the plane pencil (T, τ) . The line l' corresponding to l is the generatrix of the scroll (l, m, n) different from l which cuts t .

The scrolls (l, m, n) which correspond to the lines l through T , form the special linear complex R_1 which has the transversal r_1 of m and n through T as directrix. The scrolls (l, m, n) associated to the lines l of τ , form the special linear complex R_2 for which the directrix lies in the transversal r_2 of m and n in τ . The quadratic complex associated to K , has degenerated in this case into R_1 and R_2 .

Now the complexes C_m and C_n are also special and have resp. the generatrix t_m cutting m and the line t_n of (T, τ) cutting n as directrices. The special bilinear congruence K_m has degenerated into the field (t_m, m) and the star (t_m, m) , K_n into the field (t_n, n) and the star (t_n, n) .

The cubic scroll ρ^3 associated to a plane pencil, contains m, n , a generatrix in τ , and a line through T . To a plane pencil with vertex in the plane (t_m, m) there corresponds a quadratic scroll through n which has a generatrix in each of the planes (t_m, m) and τ . If the plane of the pencil passes through the point (t_m, m) , there corresponds to it a quadratic scroll through n which passes through the points (t_m, m) and T .

The congruence $\Sigma(1, 2)$ associated to a star S , has plane pencils in the star T and in the field τ which cut the transversal s of m and n through S just as all the other generatrices of Σ . Σ has also a plane pencil containing m in common with the star (t_m, m) and a plane pencil containing n with the star (t_n, n) . The singular conic of $\Sigma(1, 2)$ lies in τ and belongs to the pencil that has T and the points of intersection of τ with m, n , and s as base points. To the rays of a star with vertex S in τ the lines of the plane (r_1, s) through T are associated.

The congruence $\Phi(2, 1)$ corresponding to a field φ , has a plane pencil in the star T and in the field τ , and a plane pencil containing m or n in each of the fields (t_m, m) and (t_n, n) . If the plane φ passes through T , a star with vertex in τ is associated to the field.

The congruence $(3, 3)$, which corresponds to a bilinear congruence and has double lines in m and n , has a singular point in T and a

singular plane in τ . Its generatrices through T form a quadratic cone, its lines in τ envelop a conic. The tangents to this conic in the plane pencil (T, τ) lie on the cone that has T as vertex. This congruence has also a plane pencil containing m in the star and in the field (t_m, m) and a plane pencil containing n in the star and also in the field (t_n, n) .

The cubic complex associated to a linear complex which has m, n , the lines of the star T and those of the field τ as double lines, also contains the stars (t_m, m) and (t_n, n) and the fields (t_m, m) and (t_n, n) .

Mathematics. — “On the Bilinear System of ∞^3 Line Elements of Space”. By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of June 26, 1926).

§ 1. We consider two bilinear systems S_3 and S'_3 of ∞^3 line elements of space, i.e. systems of line elements (P, l) of which the carriers l form resp. the linear complex C and the linear complex C' ; S_3 as well as S'_3 contain one line element of which the point P is given¹⁾.

The line elements (P, l) of which the carriers l lie in one plane pencil with the carriers of the line elements of S_3 and S'_3 of which the points lie in P , form a system Σ_4 of ∞^4 individuals.

The characteristic numbers μ and ν of such a system indicate resp. the degree of the cone of the carriers l of the line elements of the system for which P is a given point, and the number of points P which correspond to a given line l .

As the carriers l of the line elements of Σ_4 with common point P form a plane pencil, for this system $\mu = 1$. The carriers of the line elements of S_3 with point P on a given line l_1 , form a cubic scroll ϱ^3 which has l_1 as single directrix, and the carriers of the line elements of S'_3 with point P on l_1 form a similar scroll ϱ'^3 . If to a point Q of l_1 we associate the two points where l_1 is cut by the generatrices of ϱ^3 in the plane through l_1 and the line of ϱ^3 through Q , there arises a (2,2)-correspondence on l_1 . The four coincidences of this correspondence are the points P of l_1 to which there correspond in S_3 and S'_3 lines which belong to the same plane pencil as l_1 . Hence the second characteristic number ν of Σ_4 is equal to *four*.

§ 2. In the communication indicated in the footnote we have derived that the singular points P of S_3 , to each of which there corresponds a plane pencil of lines l , form a twisted curve k^5 of the order five and the genus one. The locus of the points P of the lines l of a bilinear congruence Γ of C consists of a cubic surface Ω^3 which contains the directrices r and r' of Γ and k^5 .

The cubic surfaces Ω^3 through k^5 form a linear system of ∞^4 individuals. A surface Ω^3 contains two lines r and r' which do not cut k^5 . For if we represent Ω^3 as simply as possible on a plane V , a cubic k^3 in V

¹⁾ Cf. my communication: *Determination of the Bilinear System of ∞^3 Line Elements of Space*, these Proceedings, 27, 2.

which passes through four of the six singular points, is the image of a quintic k^5 on Ω^3 of the genus one, so that only those two lines of Ω^3 do not cut the curve k^5 which are associated to the two singular points through which k^3 does not pass. These lines are the directrices of the bilinear congruence I' of C of which the lines l have their image points on Ω^3 . The point P which in S_3 corresponds to a line l of I' , is the point of intersection outside r and r' of l with Ω^3 .

From this we can derive a way to produce S_3 . Let us consider a pencil Φ of surfaces Ω^3 through k^5 . Besides k^5 this pencil has a base curve k^4 which is represented on a curve of the sixth order in V with two triple points and four double points and which is, accordingly, rational. The lines r and r' of the surface Ω^3 of Φ are always trisecants of k^4 and form, therefore, a scroll ϱ^2 . The bilinear congruences of which the pairs (r, r') which define an involution on ϱ^2 are the pairs of directrices, form a linear complex, namely C .

The point P associated in S_3 to a line l of C , may be determined in the following way. We consider the surface Ω^3 of Φ which contains the two generatrices of ϱ^2 that cut l . Now the point in question P is the point of intersection outside these generatrices of l with the said surface Ω^3 .

§ 3. Let us choose another quintic k'^5 of the genus one on an arbitrary surface Ω^3 through k^5 which, the same as k^5 , does not cut the lines r and r' of Ω^3 . In the way described in § 5 of the above mentioned communication or in the way just indicated we derive from k'^5 the bilinear system S'_3 of ∞^3 line elements. As a rule the linear complex C' which contains the lines l of S'_3 , differs from C . We can make sure of this by choosing a trisecant of k'^5 outside C , as the surface of the trisecants of k'^5 belongs entirely to C' , and in this case C' contains, accordingly, a line which does not lie in C .

For S_3 as well as for S'_3 Ω^3 is the locus of the points P of the lines of the bilinear congruence I' which has the two lines r and r' as directrices. The scrolls ϱ^3 and ϱ'^3 that are formed by the carriers of the line elements of S_3 and S'_3 of which the points P lie on a given line l_1 , have, therefore, always three common generatrices belonging to I' . These generatrices cut l_1 in three of the four coincidences of the $(2, 2)$ -correspondence on l_1 constructed in § 1. Consequently l_1 contains only one point P to which there correspond lines in S_3 and S'_3 that lie in a plane pencil with l_1 .

Accordingly a bilinear system of ∞^3 line elements always consists of those line elements of a bilinear system of ∞^4 individuals of which the carriers lie in a linear complex.

§ 4. The loci of the points P of the line elements of S_3 and S'_3 that belong to a plane α , are resp. the conics k^2 and k'^2 . One of the four

common points of these conics is the point which in S_3 as well as in S'_3 is associated to the line l of Γ which joins the two null points A and A' that correspond to a in C and C' . The other three points of intersection of k^2 and k'^2 are the vertices of the pencils in a that consist of lines l of the bilinear system Σ_4 of ∞^4 line elements which contains S_3 and S'_3 . The joins of these three points of intersection, to each of which there correspond more than one point P in Σ_4 , are, therefore, singular lines of Σ_4 .

An arbitrary point P of space, to which in Σ_4 there corresponds a plane pencil (P, a) of lines l , must be one of the points of intersection of the conics k^2 and k'^2 in a which are different from the point P of the line AA' . Consequently two singular lines in a pass through P .

The lines l associated in Σ_4 to a point P of space, are, therefore, those lines which belong to one plane pencil with the two rays through P of a congruence $K(2, 3)$.

The line elements of Σ_4 which belong to a plane a , form a *bilinear null system*¹⁾. The singular lines of this null system are the rays of $K(2, 3)$ in a , the singular points are the angular points of the triangle which is formed by the singular lines.

As a congruence $K(2, 3)$ without singular curve has the rank one, and, accordingly, any line of space belongs to the same plane pencil with one pair of rays of K , inversely a bilinear system Σ_4 may always be derived from such a congruence.²⁾ The singular lines of Σ_4 are the rays of $K(2, 3)$.

The five singular points of the second order of $K(2, 3)$ are evidently cardinal points of Σ_4 and singular points for S_3 and S'_3 . They are, therefore, the five points of intersection of the curves k^5 and k'^5 from which resp. S_3 and S'_3 may be derived.

§ 5. The congruence $K(2, 2)$ of the lines which touch a quadratic cone κ on a give plane V , has the rank one. For if a point P describes a line r , the two lines of $K(2, 2)$ through P lie in one plane with r if the line of intersection of the polar plane of P relative to κ with V cuts the line r . This is only the case if P is the pole on r of the plane through the line associated to r relative to κ and the point of intersection of r with V .

The bilinear system Σ_4 of the line elements (P, l) of which the carriers l lie in one plane pencil with the rays through P of $K(2, 2)$, has evidently singular rays in the lines of V besides in the lines of $K(2, 2)$ so that also in this case the singular lines form a congruence $(2, 3)$.

If for V we choose the plane at infinity and for κ an isotropous cone, in the bilinear system Σ_4 there corresponds to any line l the point of

1) Prof. JAN DE VRIES: *Plane Linear Null Systems*, these Proceedings, 15, 1165.

2) Cf. STURM, *Liniengeometrie* II, p. 244.

intersection with the perpendicular which may be let down on l out of the vertex of the isotropous cone.

The system S_3 which consists of the line elements of Σ_4 of which the carriers belong to a given linear complex, has two curves of singular points, to wit the imaginary circle and a twisted cubic which cuts the imaginary circle twice and passes through the null point of the plane at infinity.

Another system Σ_4 associates to any line l the point of intersection P of l with the line which cuts l and a given line a at right angles. For this system the congruence (2, 3) of the singular rays consists of the lines of the plane at infinity, of the two isotropous planes through a , and of the star with vertex in the point at infinity of a counted twice. The star of rays must be counted twice because in this case the plane pencil of lines l associated in Σ_4 to an arbitrary point P , contains only one singular line which belongs to this star.

Mathematics. — “A Special Congruence of Twisted Cubics”. By
Prof. JAN DE VRIES.

(Communicated at the meeting of May 29, 1926).

1. The twisted cubics ϱ^3 through the points A_1, A_2 and A_3 which have the straight line b as bisecant and the straight lines c_1, c_2 as secants, form a congruence $[\varrho^3]$. Let the plane of the points A be indicated by α , the transits of b, c_1, c_2 through α by B, C_1, C_2 .

To this congruence there belongs the **pencil of nodal curves** k^3 which have a node in B and which pass through A_1, A_2, A_3, C_1, C_2 . This pencil contains two cuspidal curves with cusp B and five figures consisting of a k^2 and a straight line.

2. In order to arrive at a *representation* of $[\varrho^3]$, I assume two pencils (p) and (q) in one plane with vertices L and M , and I establish a projective correspondence between the point range (P) of c_1 and (p) , and likewise between (Q) of c_2 and (q) . As the image of the curve through P and Q I consider the point $R \equiv p q$.

If P^* and Q^* are associated to the rays LM and ML , the ϱ^3 defined by them is represented on the point range of LM ; this curve is, therefore, *singular*.

All the curves through P^* have their image in M , all the curves through Q^* are represented in L . Hence L and M are *singular points* for the representation.

If p_0 and q_0 are associated to the points C_1 and C_2 , their point of intersection S_0 is the image of any curve k^3 (§ 1). Accordingly also S_0 is *singular*.

Let α_1 be the plane through A_1 and b , A' the transit of a_1 ($A_2 A_3$), $C'_1, (C'_2)$ the transit of c_1 (c_2). Each conic ϱ^2 through A_1, A', C'_1, C'_2 forms with a_1 a configuration belonging to $[\varrho^3]$; all these ϱ^3 have their images in the point of intersection S_1 of the rays p', q' corresponding to C'_1, C'_2 . The planes α_2 and α_3 contain analogous systems. Also the points S_1, S_2 and S_3 are *singular*.

3. The surface Γ formed by the ϱ^3 which rest on a line c_3 , is represented on a curve γ of which we shall determine the order.

Let P be a point of c_1 . Through P and the points A there pass four ϱ^3 which cut b twice and which rest on c_2 and c_3 .¹⁾

¹⁾ This well known number may be found in the following way by applying the principle of the conservation of the number. If the four points lie in a plane φ , in the first place the curve k^3 satisfies the conditions which has the transit of b as double point and which rests on the two secants. Further the 3 given lines define a hyperboloid which has three more points in common with the conic in φ passing through the given points and resting on b . With the straight lines of the scroll through the said points of intersection this k^2 forms three configurations ρ^3 . Hence in the notation of SCHUBERT $P^4 B \gamma^2 = 4$.

Accordingly by I' four points Q of c_2 are associated to P ; hence a ray p cuts γ in four points outside L . In the same way four rays p are associated to the ray ML ; hence γ has a quadruple point in L . It may, therefore, be indicated by the symbol $\gamma^8 (L^4, M^4)$.

If we replace c_3 by a line c_3^* , hence γ by γ^* , the points of intersection of the two curves outside the singular points are the images of the curves ϱ^3 which rest on c_3 and on c_3^* . Accordingly through A_1, A_2, A_3 there pass 28 curves which have b as bisecant and which cut four given straight lines; $P^3Bv^4 = 28$. Consequently the surface I' is of the degree 28.

4. The congruence contains still other composite configurations:

a. Any conic β^2 in u through A_1, A_2, A_3, B forms a ϱ^3 with any of the three lines r (PQ) which rest on β^2, b, c_1 and c_2 . The lines r form a quadratic scroll $(r)^2$; to each r there corresponds one β^2 . As (P) and (Q) are projective this system has as *image* the conic through the singular points L, M, S_1, S_2 and S_3 .

b. The conic $\gamma_1^2, (A_1 A_2 A_3 B C_1)$ is completed to configurations ϱ^3 by any transversal r of b and c_2 which rests on it. These lines form a cubic scroll $(r)^3$ of which b is the double directrix. The system (γ_1^2, r) has as *image* the point range on p_0 (§ 2).

Analogously q_0 is the *image* of the system (γ_2^2, r) .

c. The conics $\delta_{1,1}^2$ through A_2 and A_3 which rest on b , on c_1 , and on the transversal $t_{1,2}$ of b and c_2 through A_1 , form a cubic dimonoid with double points A_2 and A_3 . The line $t_{1,2}$ forms a ϱ^3 with any $\delta_{1,1}^2$. This system has as *image* the point range on the ray $q_{1,2} \equiv MS_1$, which corresponds to the point of intersection of c_2 and $t_{1,2}$.

There are five analogous systems; for we may interchange c_1 and c_2 and replace A_1 by A_2 or A_3 .

The *images* of these systems are the point ranges on rays which may be indicated by $p_{1,1}, p_{2,1}, p_{3,1}$ and $q_{2,2}, q_{3,2}$.

d. The conics ε_1^2 through A_2 and A_3 which cut b, c_1 and c_2 , form a dimonoid of the fourth degree with triple points A_2, A_3 . To each ε_1^2 the transversal t_1 through A_1 of b and ε_1^2 is associated. Any ray of the plane pencil (t_1) corresponds to three conics. As again (P) and (Q) are projective, the system (ε_1^2, t_1) has as *image* a conic through L, M and S_0 .

Analogously there are the systems (ε_2^2, t_2) and (ε_3^2, t_3) .

5. The degree of the surface I' (§ 3) may also be determined by the aid of the intersection of I' with u . This contains in the first place the curve k^3 which rests on c_3 (§ 1). Further the lines a_1, a_2, a_3 (§ 2), a conic β^2 (§ 4, a), which is triple, a γ_1^2 and a γ_2^2 (§ 4, b), which are likewise triple, and finally two conics β^2 of which the completing lines r rest on c_3 . Apparently the complete intersection is of the order 28.

On I' there also lie 18 conics δ^2 (§ 4, c), 21 conics ε^2 (§ 4, d) and 3 conics ϱ^2 , which are completed by a_1, a_2, a_3 (§ 2). Further I' contains the 9 straight lines corresponding to the above mentioned conics $\beta^2, \gamma_1^2, \gamma_2^2$

and the 2 lines r resting on c_3 (§ 4, a); also the 6 lines $t_{k,l}$ (§ 4, c), which are triple, 3 lines t_k (§ 4, d), likewise triple, and 12 single lines t_k .

The curves ϱ^3 which have c_1 as bisecant, lie on the quadratic surface defined by A_1, A_2, A_3, b and c_1 . Each of the two points of intersection of this scroll with c_2 , with each of the two points of intersection with c_3 , defines a ϱ^3 which belongs to the congruence and which cuts c_1 twice. Hence Γ contains twelve cubic *nodal curves*.

The lines c are *quadruple* on Γ (§ 3).

The intersection of Γ with the plane a_1 (through A_1 and b) consists of a ϱ^2 (§ 2), the triple lines $t_{1,1}$ and $t_{1,2}$ (§ 4, c), the triple line t_1 , which rests on c_3 (§ 5, d), four rays t_1 , which are single, and the line b . Consequently the line b is *thirteen-fold* on Γ .

The consideration of the intersection of Γ with the plane a readily shows that A_1, A_2 and A_3 are *fourteen-fold* points.

6. Let A_4 be a point of c_1 . The curves ϱ^3 through the four points A_k which cut b twice and which rest on c_2 , form a surface of the 4th degree (§ 3). The intersection of this surface with the plane through three of the points A_k consists of two conics; hence the points A_k are double points.

Consequently through A_4 there pass two curves of the $[\varrho^3]$ which cut a line g through A_1 outside A_1 . If A_4 describes the line c_1 , these two curves describe a surface which has a curve $\gamma^4(L^2, M^2)$ as image. For any ray p contains two image points besides L , and the images of the ϱ^3 through Q^* (§ 2) lie in L . This surface contains two conics, which are represented in S_2 and S_3 (§ 2). $\gamma^8(L^4, M^4)$ and $\gamma^4(L^2, M^2)$ have 14 points in common besides the singular points L, M, S_2, S_3 ; accordingly the line g cuts Γ in 14 points besides A_1 . This again shows that A_1, A_2 and A_3 are *fourteen-fold* points.

Botany. — “Concerning the Function of Caffein in the metabolism of *Paullinia cupana*.” By Prof. TH. WEEVERS.

(Communicated at the meeting of February 27, 1926).

Among the various plants containing caffein *Paullinia cupana* is the one that has received least attention. This is due to the fact that the plant is little known, as it is found only in the basin of the Amazon and in large amount only at Mauès, miles and miles from the coast. On a study-tour, which a subsidy from the Langerhuizen-Fund enabled me to undertake, I found at Para a number of well-developed plants.

Before entering upon a description of my experiments on this plant, I wish to acknowledge gratefully the helpful services received from the Brazilian authorities and the Dutch Embassy ¹⁾.

The present paper deals only with the microchemical examination of the plant under consideration; the results of the quantitative macrochemical investigations will be made known in a future communication, together with those concerning *Ilex paraguensis* St. Hil.

PECKOLT's ²⁾ experiments have demonstrated that the seeds of *Paullinia cupana* are rich in caffein (4½ % dry weight), while he maintains further that the leaves do not contain any caffein. In connection with my previous observations upon the plants containing xanthin-derivatives, viz. *Thea* and *Coffea spec.*, *Theobroma Cacao* and *Cola acuminata* ³⁾ this statement seemed so strange to me, that it alone gave sufficient reasons to study also *Paullinia*. The method used has been described by H. BEHRENS in his “Anleitung zur mikrochemischen Analyse”, and has also been applied by my wife and myself in the above investigation.

The parts of the plant are pounded with quick-lime and the dry powder is extracted with 96 % alcohol. After evaporating the alcoholic extract to dryness a portion of the residue is sublimated and from the form of the crystals and their extinction the presence of caffein in the cooled sublimate is determined. The quantity of the sublimate moistened by breathing affords an index to ascertain whether there is much or little caffein in the extracted parts of the plant.

Paullinia cupana ⁴⁾ belonging to the Sapindaceae is a liane, a tendril climber with uneven-pinnate leaves. Like many other plants of the tropical

¹⁾ I also wish to express my thanks to Prof. Dr. E. J. LOHMAN, Sc. D., Dr. DUK, and KUHLMANN of Rio de Janeiro, as well as to Dr. P. LE COMTE, Sc. D., Para.

²⁾ PECKOLT, Sitz. d. Wien. Akad. 1866; Ber. d. pharm. Ges. 1901.

³⁾ TH. WEEVERS and C. J. WEEVERS—DE GRAAFF, Zittingsverslag Kon. Akad. v. Wet. Amsterdam 11, 342, 1902; 12, 369, 1903.

TH. WEEVERS, Ann. Jard. bot. Buitenzorg 2e Ser. Vol. VI.

⁴⁾ MARTIUS (Kast. Archiv. 1826, Lieb. Ann. 1840) found guaranin in *Paullinia*; BERTHEMOT and DECHASTELE Journ. Pharm. 26, 1840 identified it with caffein.

rain-forest *Paullinia* always possesses at the same time leaves of various generations of buds that have developed with longer or shorter intervals of rest. The leaves overgrown with epiphytic lichens are old ones; those without lichens are recently full-grown.

Leaves.

- a. very young leaves, coloured red by anthocyan, contain rather much caffein.
- b. young, not quite developed green leaves: much caffein.
- c. barely full-grown, green leaves: rather much caffein.
- d. old, green leaves overgrown with lichen: no caffein.
- e. yellow leaves, just falling off: no caffein.

In all cases the amount of caffein was compared per 1.5 grm fresh weight, not per equal number of leaves, so it cannot be deduced from these experiments when the increase of caffein stops and the decrease begins. Macro-quantitative examinations are needed to solve this question.

These experiments warrant the conclusion that the caffein first contained in the leaves disappears being either transported or taken up in the metabolism.

Petioles.

- a. in very young petioles: rather much caffein;
- b. in stalks of full grown and old leaves no caffein. Transport of caffein is therefore not very probable.

Shoots.

- a. young, growing shoots: rather much caffein;
- b. branches with incipient lignification and barely developed leaves: very little caffein;
- c. in lignified branches with old leaves: wood and marrow appeared not to contain caffein; inner-, and outer-bark contained a small quantity: older ones traces in the cambium;
- d. branches of 2 and 3 years: no caffein.

Roots.

In young seedlings caffein occurs in inner, and outer-bark, not in wood and marrow. Old roots were not examined.

Flowers.

Male flowers much caffein, perhaps traces of theobromin. (The flowers are too small to ascertain whether here, just as in *Kola acuminata*, the greatest amount occurs in the stamens.)

Female flowers much caffein, perhaps a little theobromin¹⁾.

¹⁾ TSCHIRCH and GROGG, *Handb. der Pharmakognosie*, found theobromin in the seed-coat of the ripe seed.

The greatest amount of caffeine is contained in the ovaries, especially in the ovulae; the wall of the ovary contains only very little of it. The growing fruits hold it as well in the ovula as in the arillus. With these young ovulae it was possible to perform a simple quantitative estimation of the caffeine by evaporating to dryness the colourless alcoholic extract of 50 ovulae and by dissolving the rest in some c.c. of warm water. The aqueous solution was filtered off and finally shaken out. Chloroform was added to the solution. Then the mixture was poured into a separating funnel. All the caffeine passed into solution in the chloroform, if this process was repeated some times. By evaporating the chloroform, to dryness a crystallizing residue was left, which was nearly pure caffeine and was determined by weighing;

per 50 very young ovulae \pm 50 mgrms, of which \pm 10 mgrs in the arillus;

per 50 very young seeds (dry weight 4 grms) 255 mgrs, of caffeine, that is 5.6 % or 4.5 mgs each.

The dry weight of one adult seed is \pm 0.7 grms, the caffeine \pm 30 mgrs.

We see, then, that caffeine occurs in all young, developing parts of *Paullinia*, from which it disappears after the parts have attained full growth. It is only the seeds that behave differently, as the increase of the amount of caffeine continues until the ripening. So this plant presents completely the same aspect of the occurrence of caffeine as has been set forth previously for *Coffea liberica*. Whether the caffeine is used during the germination has to be made out by further experimentation.

In Para, *Paullinia cupana* has periods of rest, in which no young shoots and sometimes no ripening fruits are to be seen; then, as appears from the above, no caffeine occurs either in the leaves or in the branches, whereas shortly before a rather large quantity was present during the opening of the buds. With the falling fruits and seeds the plant loses caffeine, the falling leaves however do not contain any caffeine.

The conclusion at which I arrived previously by the study of *Theobroma Cacao* and *Coffea liberica*, viz that the xanthin derivatives can enter again into the metabolism, holds therefore with perfect justice for *Paullinia cupana*. Here too caffeine is an intermediate product, not an end-product of the metabolism, a conclusion that may be confined more precisely by future quantitative investigation, but does not require this investigation to prove its correctness.

The question whether also in *Paullinia* the caffeine is utilized for the generation of protein, and whether the final conclusion of previous investigations with tea, coffee, cacao and kolaplants is completely confirmed by *Paullinia* also in this respect, must be solved by the quantitative examinations.

Physics. — “*Further experiments with liquid helium. BA. Preliminary determinations of the latent heat of vaporization of liquid helium.*”
By L. I. DANA¹⁾ and H. KAMERLINGH ONNES. (Communication N^o. 179c from the Physical Laboratory at Leiden).²⁾

(Communicated at the meetings of June 27, and December 19, 1925).

The object of these experiments was to measure the latent heat of vaporization of liquid helium for pressures at and below one atmosphere by a simple method. Measurements of this thermal constant have not previously been made. They are of especial importance in the problems which present themselves at the lowest temperatures with liquid helium, as a check on the helium temperature scale, and for the purpose of correlation with the vapor pressure and density data by means of the CLAUSIUS-CLAPEYRON equation.

Description of the Method and Apparatus.

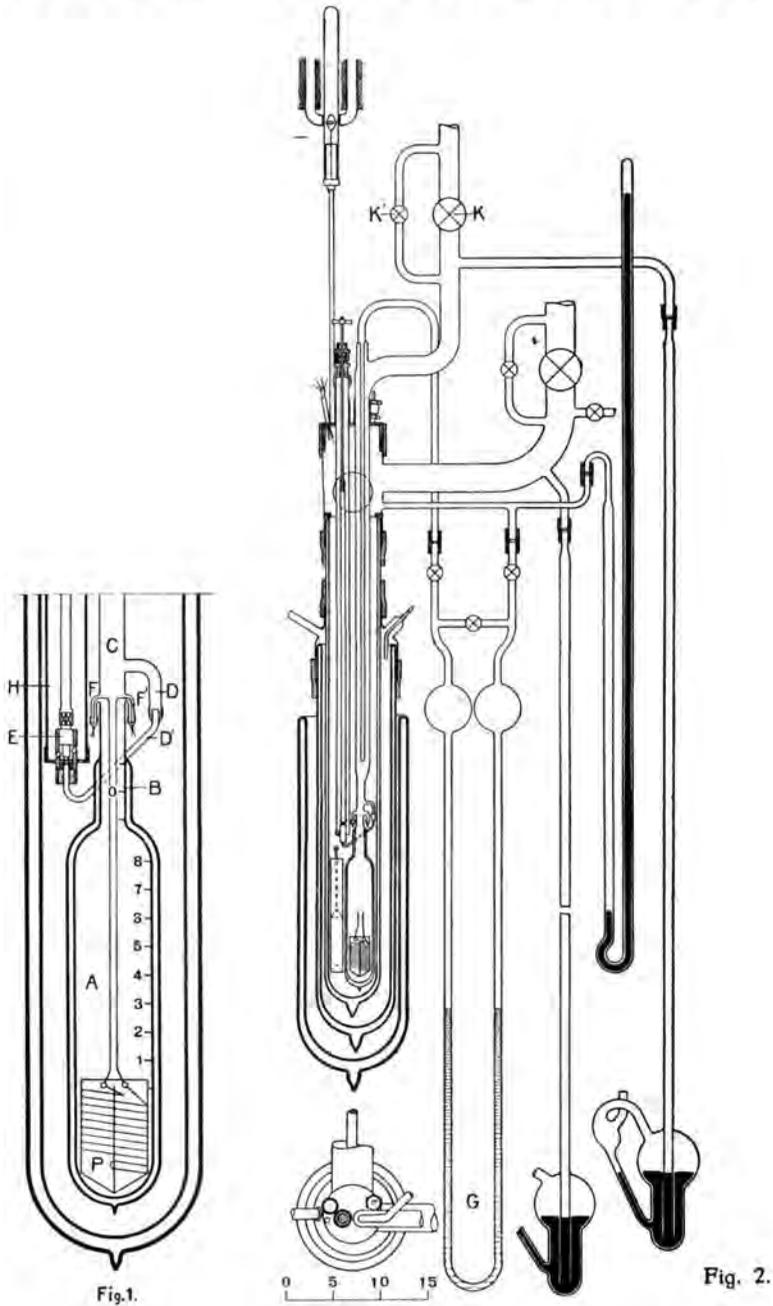
It was decided to adopt as simple a method as possible because the operation of the helium cryostat involves a sufficient number of difficulties without introducing extra ones. Fig. 2 shows the calorimeter set up in the helium cryostat, the latter being the same as that described previously in Comm. N^o. 159 with the exception that in these experiments the helium flask was longer than usual. Fig. 1 shows the calorimeter itself drawn on a larger scale. The latter consists of a small DEWAR flask supported by an outlet tube *C* and containing an electric heating coil *P*. Through a small valve *E* the flask could be filled with liquid helium from the surrounding bath. The initial level of the helium was read off by a cathetometer from the millimeter marks engraved on the inner tube. After a current was passed through and the power measured, the final level was ascertained. From the volume and density of the liquid evaporated and the power input, the latent heat could be readily computed.

Before the sealing of the lower part of the calorimeter, the inner tube was carefully calibrated with mercury. Then followed the insertion of the heating coil and the sealing and evacuation of the flask. The heating coil, of 0.1 mm. constantan wire and of 100 ohms resistance, was wound in bifilar fashion on a mica frame. The current leads, of 0.2 mm. copper wire, led out through two side tubes *F* and *F'*, Fig. 1, while the openings at the ends of the tube were closed with small brass caps soldered to

¹⁾ SHELDON fellow in physics, Harvard University, 1922—1923.

²⁾ This paper was printed in an abridged form in this Academy's „Zittingsverslagen”, 34, 1335, (1925).

the glass. It was permissible to connect the current leads outside the calorimeter since the resistances of those within were negligible in com-



parison to that of the heating coil; this procedure dispensed with two extra outlet tubes.

The helium could have been condensed into the calorimeter under pressure, but as it was not allowed to have the level of the helium in the

cryostat above the part protected by the vacuum the time of filling would have been excessive. The use of the small valve *E* made it possible to fill the calorimeter quickly and thus to make a number of determinations on one day. A copper capillary *D'* made connection from the valve to the calorimeter, one end being soldered to the bottom of the valve and the other to the inlet tube *D*. To prevent the valve from rotating when the valve stem was turned, and the consequent breaking off of the glass tube *D*, the outer shell of the valve was soldered to the glass tube *H*, which reached to the head of the cryostat, and was there cemented in place. Steel is so hard at low temperatures that a steel valve stem would not make a tight fit with a steel seat; hence the seat was made of brass, while the stem consisted of hardened steel. A long handle, made of german silver tubing to reduce conduction of heat, fitting through a packing gland at the head of the cryostat, actuated the valve stem.

Initially, the diameter of the outlet tube was about 7 or 8 mm. It developed on trial that at the lower pressures of evaporation, below 10 mm. Hg or so, the frictional resistance to the flow of the helium at the rates of evaporation maintained was so great that with the lowest pressures which could easily be maintained with the pump at the top of the outlet, the pressure in the calorimeter was considerably higher. This necessitated a larger outlet tube (see Fig. 2) and, in addition, another concentric tube, leading to the top of the calorimeter, was installed as a static pressure measuring tube; thus no error as a result of a pressure drop could be introduced. The impact pressure of the helium flowing against the central tube was found to be negligible.

Obviously, to fill the calorimeter with helium a sufficient quantity had to be formed to cover the valve; further, it was desirable at all times to have a considerable height of the liquid to cut down heat leaks. It is evident that if the vapor pressure of the liquid inside the calorimeter is somewhat higher than that outside along the outlet tube, or what amounts to the same thing, if the temperature inside is slightly higher, then, while the vapors resulting from evaporation passed through the outlet tube, they would come in contact with the glass wall cooled by the colder liquid and would partly condense, the result being a large value of the observed latent heat. Thus previous to and during evaporation it was essential to keep the vapor pressure of the helium in the outside bath somewhat higher than inside the calorimeter in order that the vapors should be allowed to come in contact only with warmer spots than itself. This pressure difference was maintained by a valve *K*, Fig. 2, connected to the calorimeter and the vacuum line, and observed on a differential oil manometer *G*, one leg of which led from the outer bath and the other to the calorimeter. The differential height of the oil was so regulated as to give at the various pressures of evaporation a temperature difference of 0.05°C. , — one which experiment showed to

suffice. To help maintain uniformity in temperature in the outer bath, a pump stirrer, shown in Fig. 2, was situated by the side of the calorimeter.

To read the levels of the helium meniscus, the lighting of the bath had to be arranged with care. The line of sight was tangent to the millimeter marks. Slits in the silvering of the liquid air and liquid hydrogen flasks, which surrounded the helium flask, limited the light passage; the lower part of the helium flask was unsilvered. Although the rays passed through eight glass walls, the distortion appeared to be small. A low wattage electric lamp behind the outer flask furnished the illumination and an alum cell between the two absorbed the infra-red rays. By inserting a sheet of black paper between the cell and slit, and by raising it till the edge stood just below the meniscus, the latter could be made to stand out sharply. This screen also served to protect most of the liquid in the calorimeter from the lamp rays, with the result that evaporation due to external radiation remained very small.

A WOLFF potentiometer of the DIESELHORST type served to measure the current by determining the potential drop across a standard in series with the heating coil, and to measure the voltage across the heating coil by ascertaining the potential drop across a part of a standard resistance shunted across the heating coil. Proper corrections were always made for the current shunted through the standard resistance. The time of power input averaged 1000 seconds and to measure it a stop watch was employed.

Heat Leaks and other Sources of Error.

Perhaps the greatest difficulty to overcome in the study of the calorimetric properties of helium is the leakage of heat, because of the low density and the consequent small mass of liquid that can be put into the calorimeter, and principally because of the very low value of the latent heat, which is the smallest of any substance. The enclosure of the liquid to be measured in a small DEWAR flask helps greatly in the limiting of the leakage from the surroundings, but the necessity of having current leads and of maintaining a slightly higher temperature in the liquid outside introduces an unavoidable leak. The copper leads were chosen as small as consistent with mechanical strength. Further, other avenues of leakage are along the inner glass wall of the neck, and down the column of vapor existing in the latter. The amount of heat leaking along these two paths can be shown by calculation to be very small.

At any rate, the method of experiment was such as to take account of the leakage from the surroundings. Observations of level were taken for at least five minutes with no power input after the calorimeter had been filled and pressure equilibrium attained. When the evaporation of liquid by the current had ceased, readings of the leak were again taken. As was to be expected, the final leak resulted in general smaller than the initial one because of the longer path of conduction down the copper leads to the liquid. The correction to the volume evaporated by the

leak was assumed to be the mean of the initial and final values. It could be maintained that during evaporation the leakage was less than either before or after because of the absorption of some of it by the helium vapors flowing by the wires and the neck of the flask. However, the facts that the velocity of vapor did not reach high values, that the latent heat did not appear to be affected by the rate of evaporation, and that the leak on the average amounted to not more than 1% of the total volume evaporated, appear to show that any error due to the assumption of the mean leak could not be greater than the error in any individual experimental factor.

Since the boiling of the helium under power input is accompanied by the formation of bubbles of vapor, it may be thought that the bursting of these bubbles results in the formation of a considerable quantity of mist and spray which is carried along with the evaporated vapor, the net result being a quantity of liquid removed without any corresponding heat input and a consequent low value of the observed latent heat. Nevertheless, the rate of evaporation was not so great as to cause violent ebullition, and if such effects did occur, it might be expected that they would vary greatly with the rate of evaporation; the results of the latent heat for various rates would appear to show that these effects are negligible.

Since the heating took place at the bottom of the liquid, the formation of bubbles was beneficial in so far as it produced stirring and better uniformity of temperature throughout the liquid. Further, any colder liquid at the top would give rise to convection currents, at least for temperatures higher than that of the maximum density of the liquid.

A height of 6 to 6.5 cm. of liquid usually was evaporated off. Thus the hydrostatic pressure on the lower levels of the liquid varied during evaporation. Because liquid helium is so light, the maximum effect due to this cause at the lowest evaporation pressure, 4 mm. Hg, is 0.3 mm. which is equivalent to about 0.03° K. and which is nearly negligible.

In ascertaining the levels of the liquid, the micrometer eyepiece of the cathetometer telescope served to determine the difference in height between the meniscus and the nearest millimeter mark on the inner wall of the calorimeter. Under the conditions of the experiments, with slight tremors of the helium meniscus due to vibration of compressors, with slight distortion of the meniscus, and with the necessity of taking readings rapidly, it is possible that there may be an error of 0.05 mm. in each meniscus reading, or say, an error of 0.1 mm. for both. The error in volume due to an error in meniscus height may be about 0.5%; and this is probably the largest single experimental error. Both the power input and time measurement could readily be ascertained with an accuracy of 0.1%.

Method of Operation.

When some liquid helium had been formed in the outer helium flask,

a slight vacuum was produced in the calorimeter, the valve opened, and a current of cold helium vapor passed through to cool off the inner walls of the calorimeter. When liquid covered the valve, it was allowed to flow in. The liquid stood for a few minutes with the proper pressure difference between the inside and outside in order to attain equilibrium and to make sure that any liquid remaining in the copper capillary D' had boiled off. It required but a short time for the latter to occur for the capillary rested in direct thermal contact with the warmer liquid outside. Then the heat leak observations were begun. Between the last reading of the liquid level and the beginning of the current a short time necessarily elapsed during which the level fell slightly. By noting this time interval with the stop watch the exact level at the time of the beginning of the power input was readily ascertained. At the end of the experiment the same procedure held but the correction was somewhat larger and of opposite sign; as a rule, it amounted to less than 0.1 % of the volume evaporated.

Results.

The results of the experiments are summarized in the table below together

| Series | N ^o . | Pressure | Temp. | Latent heat of vaporization |
|--------|------------------|----------|-----------------------|-----------------------------|
| A | I | 764 mm. | 4 ^o .21 K. | 4.95 |
| | II | 496 | 3.79 | 5.37 |
| | III | 153 | 2.96 | 5.64 |
| B | I | 768 | 4.21 | 4.89 |
| | II | 338 | 3.48 | 5.56 |
| | III | 10.4 | 1.81 | 5.55 |
| | IV | 25.9 | 2.14 | 5.58 |
| | V | 74.3 | 2.58 | 5.58 |
| C | I | 211 | 3.15 | 5.66 |
| | II | 108 | 2.77 | 5.60 |
| | III | 46.2 | 2.36 | 5.47 |
| D | I | 31.7 | 2.22 | 5.53 |
| | II | 4.26 | 1.49 | 5.45 |
| | III | 14.5 | 1.93 | 5.57 |
| | IV | 58.1 | 2.47 | 5.59 |
| | V | 9.04 | 1.76 | 5.52 |

with the important factors entering into the latent heat. The temperatures of the liquid, corresponding to the vapor pressures, were derived from the formula given in Comm. N^o. 147*b*. Densities of the liquid are derived from recent measurements by KAMERLINGH ONNES and BOKS¹⁾. Because the same kind of glass was used in the construction of the calorimeter as in those experiments, the latent heat results are independent of the value assumed for the coefficient of expansion of the glass in the density measurements. The latent heats are given in 15^o calories.

These results are plotted in Fig. 3.

Discussion of Results.

From the observations indicated by \odot it follows that the curve, representing the heat of vaporization as a function of the temperature shows a maximum near 3^o K. and descends again at lower temperatures²⁾. This should support the calculations of VERSCHAFFELT (c.f. Comm. Suppl. N^o. 49) who arrives at the conclusion, that at $T = 0^{\circ}$ K. the heat of vaporization

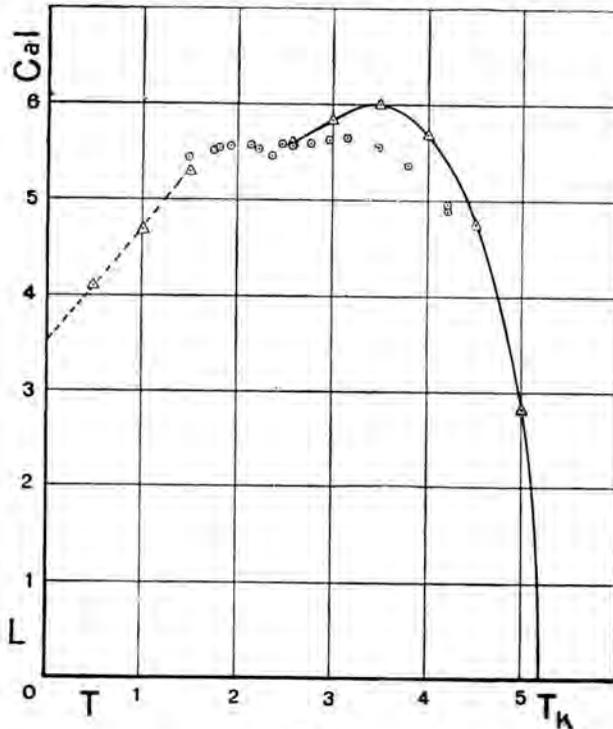


Fig. 3.

¹⁾ Leiden, Comm. No. 170, p. 22.

²⁾ Further there is a noteworthy anomaly at the temperature of maximum density. It is possible that only the variation of the state of convection at surpassing this temperature causes a change in a systematic error. The accuracy of the results, however, being sufficient with regard to the deviation with near temperatures, it is remarkable that these results would indicate, that near the maximum density something happens to the helium, which within a small temperature range is perhaps even discontinuous. The change of density of the liquid indicates also something of the same kind.

of helium (solid or liquid) would be about 3.5 cal., and that near the absolute zero the heat of vaporization would change with temperature according to the formula:

$$L = 3.5 + 1.25 T.$$

The curve, which represents graphically the formula (b) pg. 1059, has been drawn as a dotted line. This formula (b) agrees with the formula given above for small values of T . The joining of the curve of VERSCHAFFELT with the curve of the observations is still giving difficulties¹⁾.

Our observations, considering the small accuracy of the data for the vapor pressure, agree moderately well with the values L , which can be calculated from the data of MATHIAS, CROMMELIN, KAMERLINGH ONNES and SWALLOW by making use of the formula of CLAPEYRON-CLAUSIUS. One obtains then:

| T | $v_{vap.}$ | $v_{liq.}$ | $\frac{dp}{dT}$ mm. Hg/degree | L |
|-------------------|-----------------|------------|----------------------------------|-----------|
| 3 ^o K. | 233 | 6.9 | 270 | ± 5.8 |
| 3.5 | 127 | 7.3 | 450 | 6.0 |
| 4 | 77 ⁵ | 7.7 | 640 | 5.7 |
| 4.5 | 46 ⁵ | 8.4 | 870 | 4.0 |
| 5 | 26 | 10 | 1110 | 2.8 |

These points are also represented in the figure by Δ .

The values of $\frac{dp}{dT}$ were calculated from the formula

$$\log_{10} p = 2.19 - \frac{3.1}{T} + 2.5 \log_{10} T - 0.027 T, \dots \dots (1)$$

which was derived from the following reduced formula

$$\log_{10} \pi = -\frac{0.60}{\tau} + 2.5 \log_{10} \tau + 0.74 - 0.14 \tau.$$

by using $T_k = 5.19^{\circ}$ K. and $p_k = 2.26$ int. atm. or 1720 mm. Hg.

The reduced formula was obtained by adding to the formula of VERSCHAFFELT (loc. cit.) a correction term, so that for $T = T_k$ also $p = p_k$, thus for $\tau = 1$, $\pi = 1$. The formula thus obtained agreed only little less well with the observations than the formula in Leiden Comm. N^o. 147b; one obtains namely:

¹⁾ Within the limits considered indeed the temperatures, which can be calculated from this formula and from the formula of Comm. N^o. 147b for a definite value of p , deviate from each other by several hundredths of a degree.

| T | p mm. Hg. | p atm. | $\frac{dp}{dT}$ atm./degree |
|-----|----------------|-------------|--------------------------------|
| 1.5 | 3.3 | — | — |
| 2.0 | 22 | 0.029 | 0.085 |
| 2.5 | 75 | 0.099 | 0.210 |
| 3.0 | 180 | 0.237 | 0.35 |
| 3.5 | 375 | 0.50 | 0.59 |
| 4.0 | 630 | 0.82 | 0.84 |
| 4.5 | 1030 | 1.32 | 1.15 |
| 5.0 | 1520 | 2.00 | 1.46 |

By using the formula :

$$\frac{d \log p}{dT} = \frac{ML}{RT^2} = \frac{4L}{2T^2} \dots \dots \dots (a)$$

one derives from the vapor pressure formula in the neighbourhood of $T=0^\circ$ K. :

$$L = 3.5 + 1.25 T - 0.03 T^2 \dots \dots \dots (b)$$

so that for

| T | L |
|-----|-----|
| 0.5 | 4.1 |
| 1.0 | 4.7 |
| 1.5 | 5.3 |

These points are also represented in the figure by \blacktriangle .

At higher temperatures the simple formula (a) becomes too inaccurate because of the deviations from the gas laws, which the vapor shows. Even at $\pm 2.5^\circ$ K. these deviations must still be considerable after the isotherm determinations of KAMERLINGH ONNES and BOKS. One has to apply then the correction, following from the observations of KAMERLINGH ONNES and BOKS, or when this is no longer satisfactory, one has to calculate at once with the formula of CLAPEYRON. The specific volume of the saturated vapor, as far as it has not been observed, can then approximately be calculated with the observations in question for some temperatures.

From the values of L obtained by making use of the formula of CLAPEYRON-CLAUSIUS, just as from our own observations it seems to follow that the heat of vaporization of helium reaches a maximum at

3.5° K., by which our previous surmise¹⁾ is established. Moreover the course of L , so calculated, seems to indicate that also at temperatures lower than 2.5° K. the simple formula (a) is not yet applicable, and that this formula is not valid until immediately near $T=0$; this should indicate that, also to lower than 2° K. thus at a reduced temperature lower than 0.4, the behaviour of the saturated vapor of helium deviates still appreciably from the ideal gas laws. However, it could also be possible, that the value of L_0 is another one than was adopted by VERSCHAFFELT.

¹⁾ H. KAMERLINGH ONNES. Rec. Trav. Chim. Pays-Bas (4) 3, 535, 1923

Physics. — “*Further experiments with liquid helium. BB. Preliminary determinations of the specific heat of liquid helium.*” By L. I. DANA ¹⁾ and H. KAMERLINGH ONNES. (Communication No. 179d from the Physical Laboratory at Leiden). ²⁾

(Communicated at the meetings of June 27, and December 19, 1925).

Measurements on the specific heat of liquid helium have not previously been made. They are of great importance in the study of the liquid state near the absolute zero. The object of these experiments has been, not so much to measure the specific heat with high precision as to get a reasonably good idea of the values and their trend with the temperature for saturation pressures at and below one atmosphere as far as could be conveniently gone.

Description of the Method and Apparatus.

In Fig. 1 is shown the general set-up of the calorimeter in the helium cryostat. In some ways, the arrangement is similar to that of the latent heat apparatus. (Comm. N^o. 179c). The specific heat calorimeter is drawn on a larger scale in Fig. 2. The body is a small DEWAR flask supported by the tube *B* which leads to the vapor pressure manometer. The calorimeter was filled with liquid helium from the outer bath through the small valve *A*.

To measure the specific heat arrangements had to be provided to (a) heat the liquid (b) stir the liquid (c) measure the rise in temperature.

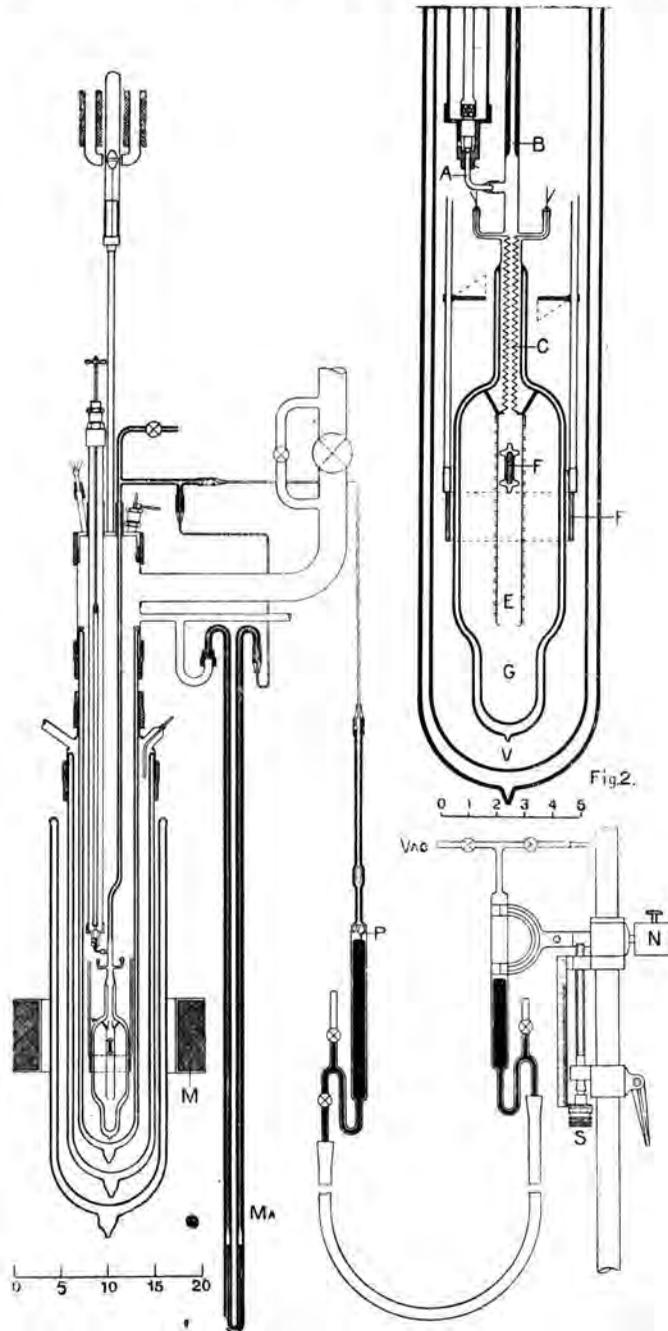
The heating coil of manganin wire was wound on a grooved glass core *E* which had its support at the top of the inner wall by two glass stems. To stir the liquid and thus to maintain approximate uniformity of temperature especially during heating, a small glass plunger *F*, containing an iron wire was moved up and down by moving the iron ring *R* in the magnetic field produced by the magnet *M*, Fig. 1. This scheme for magnetic stirring is described in Leiden Comm. N^o. 161.

Since the means of temperature measurement at these very low temperatures are limited, the simplest procedure to follow in measuring the rise in temperature of the liquid helium produced by heating appeared to be to determine the rise in vapor pressure; that is, to use the whole calorimeter as a vapor pressure thermometer. Especially with this method did the liquid have to be stirred, for the vapor pressure reading would be principally governed by the temperature of the meniscus. The mercury

¹⁾ SHELDON fellow in physics, Harvard University, 1922–1923.

²⁾ This paper was printed in an abridged form in this Academy's „Zittingsverslagen” 34, 1340, (1925).

manometer adopted to measure the rise in pressure is shown in Fig. 1. This rise in pressure, before, during, and after the power input, had to



be determined quickly so that the arrangement shown in Fig. 1 of adjusting one meniscus to a fixed position and reading the position of the

other was used. A flexible rubber tube connection between the top and bottom of the manometer allowed freedom of motion of the upper part which was attached to a moveable nut N , actuated by a micrometer screw S . As the pressure rose, one observer maintained the lower meniscus at the fixed point P by sighting through a telescope and rotating the screw S while another observer recorded the readings of the screw. The actual pressure was read with a cathetometer while the rise in pressure was given by the screw reading; the latter being 11 cm. long, limited the rise in pressure for one experiment to about 10 cm. Hg.

The time of power input, measured with a stopwatch, averaged about 500 seconds. The current and voltage were measured with a calibrated standard WESTON voltmeter and ammeter; the power was of the order of 0.01 watt.

Before an experiment began, the liquid helium was brought to a level somewhat below the bottom of the neck of the calorimeter so that at the end of the power input the level should still be below. If the meniscus rose up high into the neck the temperature of the meniscus was very much influenced by the heat leaking in and false pressure readings would be obtained. After the experiments with a particular filling of the liquid were over, the liquid was warmed up until the meniscus rose into the neck. Since the calorimeter volume for various points in the neck had previously been determined, and the density of the liquid was known, the mass of the liquid filling could readily be ascertained. The bottom of the calorimeter was drawn down to facilitate the insertion of extra glass to determine the heat capacity of the calorimeter itself. But previous measurements on the specific heat of glass have indicated that the values are so small at helium temperatures that the correction for the heat capacity of the calorimeter would be very small and thus an extrapolated value could be assumed.

Heat Leaks.

It was demonstrated during the beginnings of the experiments, even though the specific heat determination did not involve the flow of matter out through the capillary, that if the outer bath were colder than the inner one, the possibility of condensation in the capillary leading to the manometer still existed. In the latter case, the vapor would condense, fall below, and new vapor would arise to go through the same process; this was a so-called convective heat leak, and when it did occur it produced a very large effect on the observed specific heats. To avoid this the same scheme as used in the latent heat experiments was adopted, namely, the maintenance of the outer helium bath at a somewhat higher temperature than the liquid in the calorimeter. In doing this with a differential manometer the essential condition of maintaining the volume of the dead vapor space as small as possible had to be observed, for toherwise the correction for evaporation of gas into this space would

have been too large. Thus, instead of an oil manometer, the mercury manometer *Ma*, Fig. 1, was employed. The difference in pressure had to be kept small to reduce the heat leak, and it was measured by a cathetometer. Due to the sudden input of current at the beginning of heating, it was difficult to maintain the desired pressure difference from the beginning, and it could happen that for a moment or so the surface of the outer bath became colder than the gas inside the capillary and condensation might result. A very small amount of condensation produces a very large error in the observed specific heat, for the heat input in the specific heat experiments is so much smaller than in the case of the latent heats. The effect was further increased because mercury instead of oil was used as the manometer fluid. As a result of this condensation a large number of points were erroneous. As was to be expected these points fell very irregularly off the curve obtained when condensation did not occur and always resulted in large values of the observed specific heat. Since the cause of this effect was well known from the latent heat experiments, where it could more easily be avoided and since the effect could be produced at will in the specific heat experiments, we felt justified in discarding the irregular points. The frequency of occurrence of condensation could be minimized by working with the level of the liquid in the outer bath somewhat below the top of the neck of the calorimeter. Measurements in this case could not proceed for a long period because the level would quickly fall to so low a value as to entail a large heat leak.

Vapor Space Correction.

It is evident that the exact volume of the vapor to which the measured heat input is applied is somewhat indefinite because the boundary of the calorimeter at the top is somewhat indefinite. Nevertheless, if we make the assumption that the saturated vapor space terminated at the bottom of the capillary tube, and assume that the vapor as well as the liquid took part in the rise in temperature, then it turns out that the volume of this vapor space need not be accurately known in determining the correction to be applied to the specific heat of the liquid. This correction is more dependent on the variation of the latent heat and specific volumes with the temperature; it is given by the following formula:

$$-\frac{L_2 x_2 - L_1 x_1}{\Delta T} + \left(\frac{Lx}{T}\right)_m$$

where L , x , T are respectively the latent heat, dryness factor, and absolute temperature, the subscripts 1, 2, and m referring to the lower, upper, and mean temperatures respectively.

The dryness factor is given by

$$x = \frac{V}{M} \frac{-u}{u' - u}, \text{ in which}$$

V is the total volume of the calorimeter,
 M is the mass of liquid in the calorimeter,
 u' is the specific volume of the vapor, and
 u is the specific volume of the liquid.

This first correction as calculated for five points is given in the following table, from which the general manner of variation can be seen.

| Mean T °K. | Correction Cal./g. |
|---------------|-----------------------|
| 4.00 | + 0.057 |
| 3.79 | + 0.020 |
| 3.48 | + 0.007 |
| 2.99 | - 0.003 |
| 2.62 | - 0.014 |

These values are probably good to 10%, entailing an error less than 1% in the specific heat.

Finally, two other corrections are involved, that due to the heat capacity of the calorimeter, which amounts to about 1% of the specific heat, and that due to evaporation into the superheated gas space, amounting to about 0.05%.

Thus the equation giving the specific heat of the liquid in terms of the measured quantities is as follows:

$$C_{He} = \frac{Wxt}{4.184 m_{He} \Delta T} - V. S. - \frac{c_g m_g}{m_{He}} - \frac{m_v L_m}{m_{He} \Delta T}$$

in which

W is the power input in watts,

t is the time of heating in seconds,

c_g and C_{He} are the specific heats of the glass and liquid helium respectively,

$V. S.$ is the vapor space correction given above,

m_{He} , m_g , m_v are the masses of the liquid helium, glass, and evaporated super-heated vapor, respectively,

L_m is the mean latent heat of the liquid helium, and

ΔT is the rise in temperature due to electric heating.

Procedure.

After filling the calorimeter with liquid and adjusting the temperatures of the outer and inner baths to the proper difference, readings were taken of the rise in pressure for about five minutes, and when plotted against the time gave a straight line. During the application of the heat,

the pressure was also read, and finally, the readings were continued for five minutes after the current ceased. For about 30 seconds after the latter ceased the pressure fell slightly and then proceeded to rise slowly. Evidently the rate of stirring was not sufficiently great to produce instantaneous uniformity of temperature during heating, but this lag was small and correction for it could readily be made. To get the initial pressure the leakage line was extrapolated to the instant of beginning of the current, while the final pressure was obtained by extrapolation to the instant of ending of the current.

Due to a variety of causes, the initial and final rates of leakage were not equal; but the change was very probably gradual. On the average, the temperature rise due to leakage was about 4% of the total, and by subtracting the mean value from the total rise during power input, the correct rise was obtained.

To avoid errors due to calculations, to make the values conform to the temperature scale as given by the vapor pressure curve formula in Leiden Comm. N^o. 147*b*, and to facilitate the reduction of the data, the temperature rise for each experiment was determined as follows. The values of $\frac{dp}{dT}$ as calculated from the above formula were plotted against the pressure p . The value of $\frac{dp}{dT}$ for the mean pressure of the experiment was read off, the reciprocal $\frac{dT}{dp}$ found, and this was multiplied by the corrected pressure rise to get the actual rise in temperature. It is proper to take the value of $\frac{dT}{dp}$ at the mean pressure provided it varies practically linearly with the pressure in the range of pressure under consideration. Over most of the range this is the case, but between about 20 and 150 mm. Hg, $\frac{dT}{dp}$ varies rapidly with the pressure. For this range $\frac{dT}{dp}$ was plotted against p and for any particular range

$$\Delta T = \int_{p_1}^{p_2} \frac{dT}{dp} dp \text{ was determined by graphical integration.}$$

Results.

In the following table is given a summary of the results and the factors involved. It will be noted that the temperature scale used is that given by the vapor pressure formula in Leiden Comm. N^o. 147*b*. The specific heats C are those corresponding to saturation conditions. Considering the deviations of the results, the experimental conditions, and the corrections we should say that the order of accuracy of the results should be about 3%.

The specific heat of liquid helium.

| Series N ^o . | Run N ^o . | Temp. Rise, °K. degr. | Pressure Rise mm. Hg | Power Input Watts | Mass of Liquid | Mean Pressure mm. Hg. | Mean Temp. ° K. | Specific Heat Cal./g. |
|-------------------------|----------------------|-----------------------|----------------------|-------------------|----------------|-----------------------|-----------------|-----------------------|
| I June 29 1923 | I | 0.103 | 69.3 | 0.00644 | 7.768 | 621 | 4.00 | 0.963 |
| | II | 0.121 | 74.2 | 0.00665 | .. | 523 | 3.84 | 0.919 |
| | III | 0.1065 | 70.9 | 0.00604 | .. | 603 | 3.97 | 0.974 |
| | IV | 0.205 | 48.2 | 0.00604 | 8.076 | 99 | 2.72 | 0.602 |
| | V | 0.221 | 72.0 | 0.00927 | .. | 180 | 3.05 | 0.717 |
| | VI | 0.455 | 70.5 | 0.02862 | .. | 78 | 2.61 | 0.644 |
| | VII | 0.195 | 66.9 | 0.01532 | .. | 214 | 3.16 | 0.710 |
| II July 6 | I | 0.339 | 94.5 | 0.01280 | 8.473 | 162 | 2.99 | 0.631 |
| | II | 0.235 | 92.2 | 0.01173 | 8.302 | 263 | 3.30 | 0.707 |
| | III | 0.176 | 90.8 | 0.01149 | 7.966 | 386 | 3.58 | 0.863 |
| | IV | 0.127 | 89.1 | 0.00822 | 7.656 | 661 | 4.06 | 1.024 |
| III July 18 | I | 0.172 | 88.6 | 0.01041 | 8.037 | 406 | 3.63 | 0.833 |
| | II | 0.218 | 93.6 | 0.01095 | .. | 298 | 3.39 | 0.758 |
| | III | 0.206 | 96.5 | 0.01035 | .. | 337 | 3.48 | 0.783 |
| | IV | 0.172 | 94.7 | 0.00982 | .. | 439 | 3.69 | 0.859 |
| | V | 0.0926 | 64.8 | 0.00768 | 7.786 | 659 | 4.05 | 0.982 |

Discussion of Results.

It is seen that the deviation of the points at the lower temperatures is greater than at the higher ones. This probably comes from the greater difficulty in handling the calorimeter and in measuring at the lower pressures.

In a graphical representation our observations give points, which tend to lie on a straight line, which passes through the origin; between 2^o.5 and 4^o.0 K. one has about $C_{sat} = 0.222 T$. Within our range of temperature we may with reference to the accuracy attained put $C_{sat} = C_p$.

There would be no sense in applying this relation between larger limits. At first the value of C_{sat} must rise much more rapidly with the temperature above 4^o K., because it becomes infinite at $T = 5^o.19$ K. (critical temperature); and moreover it is improbable that C_p will approach to zero in proportion to T .

If one considers the deficiency in percentages of the specific heat C_v of a monatomic liquefied gas according to the law of DULONG and

PETIT as a measure of the degeneration of the liquid and if 2 is taken as the atomic weight of liquid hydrogen, then, neglecting the difference between C_p and C_v , an analogous degeneration seems to take place with liquid helium and liquid hydrogen; further it appears that as a function of the reduced temperature the degeneration for helium is greater than that for hydrogen. Examining this difference more closely, it must be kept in view, that $C_p - C_v$ is the same for different substances at the same reduced state, if the law of corresponding states holds, but that this difference by *deviations*¹⁾ will have different values at the same reduced state for helium and hydrogen; as $C_p - C_v$ at the examined reduced states is rather large, these deviations must be taken into account rather strongly.

¹⁾ Leiden Comm. Suppl. N^o. 23, § 38.

Physics. — "On crystallized Nitrogen". By D. VORLÄNDER and W. H. KEESOM. (Communication N^o. 182c from the Physical Laboratory at Leiden).

(Communicated at the meeting of June 26, 1926).

§ 1. *Introduction.* According to WAHL ¹⁾ nitrogen when solidifying out of the liquid state at -210° crystallizes in the cubic system and is optically completely isotropic. On the other hand DE SMEDT and KEESOM ²⁾ have examined the structure of crystallized nitrogen by Röntgen analysis at -253° and found that in all probability at this lower temperature the nitrogen is not cubic. EUCKEN ³⁾, as well as KEESOM and KAMERLINGH ONNES ⁴⁾ had already established the fact that at $35^{\circ}.5$ K. a transformation takes place.

According to these results nitrogen would belong to that important class of dimorphic substances, in which the cubic form which arises at higher temperature passes at falling temperature enantiotropically into an optically anisotropic form. Nitrogen would therefore behave in the same way as white phosphorus ⁵⁾.

§ 2. *Micro-observations at temperatures down to -253° C.* We investigated whether by means of a microscope in parallel polarised light a characteristic transformation isotropic \rightleftharpoons anisotropic could be ascertained in nitrogen and in some other substances crystallizing in the cubic system when cooled to -253° . After overcoming the initial difficulties we were able to observe the substances, cooled in a stream of cold hydrogen or plunged in liquid hydrogen, between crossed nicols. In the mean time experiments with cubic substances ⁶⁾ of very various nature, and further observations on a few anisotropic substances had shown that at these low temperatures a large harvest of new forms was not to be anticipated; the transformation isotropic \rightarrow anisotropic in particular, did not take place. Ammoniumchloride for which SIMON ⁷⁾ thought to have found a trans-

¹⁾ W. WAHL, Proc. Roy. Soc. A **87**, 371, 1912; A **88**, 61, 1913; Z.S. physik. Chem. **84**, 106, 1913.

²⁾ Physica **5**, 344, 1925; Leiden Comm. N^o. 178b.

³⁾ A. EUCKEN, Verh. d. D. physik. Ges. **18**, 4, 1916.

⁴⁾ These Proc. **18**, 1247, 1916; Leiden Comm. N^o. 149a, p. 8, note 2.

⁵⁾ D. VORLÄNDER, Ber. d. D. chem. Ges. **58**, 1802, 1925.

⁶⁾ Gold, iron, silver, diamond, KCl, KBr, KJ, NaCl, NH₄Cl, TiCl, TlBr, CaF₂, K₂PtCl₆, As₂O₃, (NH₄)₂SnCl₆, (NH₄)₂TeCl₆, ferric-ammonium-, thallium- and potassium-alum, spinel, corundum, ruby, zinblend, analcym, pyrope.

⁷⁾ F. SIMON, Ann. d. Phys. (4) **65**, 264, 1922; c.f. W. EITEL, Fortschr. d. Mineralogie **8**, 34, 1923, according to whom the transformation of ammonium chloride "extends over a large temperature range and cannot be demonstrated dilatometrically".

formation point at -243° while measuring the molecular heat, remains down to -253° optically isotropic; common and blue rock salt remain unchanged; pleochroic solid crystals retain their pleochroism¹⁾; ice remains hexagonal between 0° and -253° .

What we observed in the anisotropic substances²⁾ through the microscope was principally some changes which gradually took place in the volume and the double refraction with rising and falling temperatures and the formation of cracks in the material, but no ordinary dimorphic transformations, which should take place with a complete change in the crystalline structure with a distinct, even if slow, growth, from the one to the other form. In this way we observed under the microscope for the space of $1/2$ to $3/4$ hour several preparations plunged in liquid hydrogen. This does not entirely exclude the possibility of transformations still taking place, but we regard it as highly improbable, when we consider further that during a very slow re-heating of the substances no transformations were observable.

§ 3. *The preparation of pure nitrogen.* For these experiments with nitrogen the vital thing was to procure the gas absolutely pure. This was done from ammonium nitrite, by a gradual introduction of a concentrated aqueous solution of sodium nitrite (1 equivalent weight) through a dropping funnel into an aqueous solution of ammonium sulphate (2.5 equivalent weight) and yellow potassium chromate³⁾ (1 equivalent weight) heated on a water bath to 80° – 95° ⁴⁾. To expel the air from the apparatus (the solutions having been previously boiled) two of these nitrogen generators were connected in series; they were set working one after the other, after having been washed through with commercial nitrogen and the whole of the rest of the apparatus evacuated. This consisted of glass tubes sealed together, with the exception of the taps to the oxidation tube, which were closed by mercury. The nitrite nitrogen from the generators ran through the following apparatus:

1. two vessels cooled in icewater, containing 5 volumes saturated solution of potassium bichromate and 1 volume concentrated sulfuric acid, to oxidise oxides of nitrogen;
2. two tubes filled with pieces of sodium- or potassium hydroxide;
3. two drying vessels cooled with liquid air, to freeze out water vapour, NO, etc.

¹⁾ Undercooled pleochroitic mono-axial layers of liquid crystals (cyano benzalamino-cinnamic active amyl ester) changed into a fine net-work with very irregular boundaries, while loosing their pleochroism.

²⁾ Sulphur, phosphorus, bromine, iodine, $Pb(NO_3)_2$, CdJ_2 , SbJ_3 , TlJ , Hg_2Cl_2 , K_2PtCl_4 , $AsBr_3$, H_3BO_3 , ZnO , gypsum, arragonite, Iceland spar, quartz, beryl, tourmaline, dichroïte, JCN , $(CH_3)_4N]$, $Hg(CN)_2$, CCl_4 , $CHBr_3$, normal heneicosane, hexadecylene, norm. valeric acid, stearic acid, elaidic acid, tristearine, norm. hexylalcohol, cyclohexanone, methyloxalate.

³⁾ C.f. VON KNORRE, Chem. Industrie 25, 531 and 550, 1902.

⁴⁾ C.f. H. KAMERLINGH ONNES and A. TH. VAN URK, Leiden Comm. N^o. 169d and A. TH. VAN URK, Diss. Leiden, 1924, p. 21.

4. an oxidation tube heated in a gas oven to low red heat, provided for $\frac{2}{5}$ of its length with copper gauze and for $\frac{1}{5}$ with CuO;
5. two drying vessels cooled outside with liquid air (one of them was a spiral glass tube);
6. two condensation vessels cooled outside by liquid air, boiling under reduced pressure; the nitrogen was liquefied in the first of these vessels and then distilled over into the second vessel (after the surrounding liquid air had been brought back to atmospheric pressure).

The nitrogen was then distilled over, into a previously evacuated glass bulb, for storing, or directly into a 25 to 30 cm³ pipette *p* (fig. 1) with a tap connected to the apparatus, to which was sealed an observation chamber *c* made of plane parallel glass plates of about 1 cm diameter¹⁾. For the final filling the pipette and chamber were highly evacuated (6 times) and filled with nitrogen. The nitrogen condensed in the first condensation vessel (No. 6) formed a clear, colourless liquid which showed no trace of turbidity and when distilled off left no trace of foreign matter. After washing, the nitrogen was fractionally distilled over from the first condensation vessel into the second (No. 6); then the pipette with observation chamber was filled with it, up to about 40 cm pressure. This nitrogen contained no oxygen or oxide of nitrogen (tested with alkaline solution of pyrogallol and when admitting air, with acidified potassium iodide). By sealing on in front of the pipette a glass bulb (*b*) filled with nitrogen under increased pressure (80 cm) the possibility of oxygen from the outer atmosphere penetrating through the tap of the pipette was prevented. In the observations with the polarisation microscope the chamber was placed in a cryostat glass, of which the inner wall was cylindrical, while on the outer wall at the height of the chamber two plane parallel glass windows were made. The nitrogen crystallized on being cooled with hydrogen, to a layer of 0.2 to 0.3 mm thickness in the observation chamber²⁾.

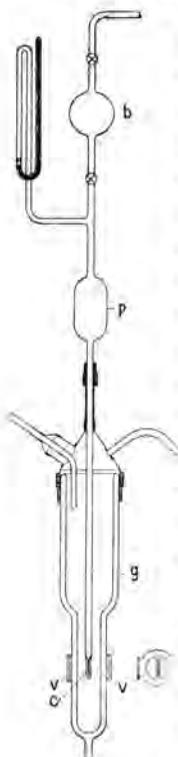


Fig. 1.

§ 4. *Observations of the crystallized nitrogen in the polarisation microscope.* As the nitrogen crystallizes out from the liquid by cooling, the first particles of solid nitrogen still moving in the liquid appear to be double refracting. Shortly after the whole mass has become doubly refracting and solid, a marked contraction, of roughly from $\frac{3}{4}$ to $\frac{4}{5}$ of the original

¹⁾ Simple mercury manometers, acting at the same time as safety valves, were placed: a. between the generator and N^o. 1; b. between N^o. 5 and N^o. 6; c. between the first and the second vessel of N^o. 6; d. behind N^o. 6; e. at the pipette.

²⁾ We wish to express our thanks to Mr. W. PLOEG for his help in preparing the pure nitrogen.

volume of the fluid takes place. The solid optically anisotropic mass is strongly deformed, while forming tubular and vermiform holes, from the surface of which often innumerable very minute and branched cracks and holes spread out, which between crossed nicols appear black. Pleochroïsm could not be observed at the transparent surfaces when rotating the polarisor, the other nicol being removed. The mass which seems to be principally fibrous and in some places tabular, shows no perceivable definite structure, so that the existence of definite angles or extinction directions could not be observed. On further cooling the double refraction changed. At the introduction of a gypsum plate 1st order, red polarisation colours changed into yellow, violet into blue. This change takes place gradually; at the same time a few fairly rapid changes were seen, in which a fine seam appeared to run over the mass. A distinct transformation from one optically anisotropic form into another could not be ascertained. A transformation from an isotropic to an anisotropic phase from the first cristallization at -210° to -253° , is excluded.

With rising temperature first the changes of the polarisation colours in the opposite sense are seen; after that, on approaching the melting point, the holes partly fill up, so that a clear tabular crystalline mass forms, in which no special direction of growth can be ascertained, as is the case with falling temperature, as apparently the crystalline particles are orientated and grown together in every possible direction. The pattern that is formed is very peculiar, we do not remember ever to have seen one like it in ordinary crystalline masses of other substances. Now the melting begins very speedily, during which the solid "tables" remain optically anisotropic to the last particle, floating in liquid nitrogen. It thus appears that here too there is no isotropic phase. These observations make it quite clear that the double refraction which was observed could not be only due to deformation, caused by the contraction of an isotropic form.

§ 5. By the observations described in § 4 the cooling in the stream of cold hydrogen and in liquid hydrogen respectively took place fairly quickly. As Dr. TERPSTRA pointed out to us it was possible that with rapid cooling anomalous forms would arise. We have therefore repeated the observations to about -218° in a bath of oxygen, boiling under reduced pressure. By regulating the pressure of this we could regulate the rapidity of the cristallization as well as that of the re-heating.

In these experiments, in which Dr. TERPSTRA was so kind as to take part, the nitrogen was observed: 1. with an ordinary microscope, 2. with crossed nicols in // light, 3. with crossed nicols and a gypsum plate red 1st order in // light, 4. in a conoscope with convergent light.

The following observations were made:

1. With the microscope the coagulation in the field of vision could

be seen gradually proceeding, mostly in a few // fields with progressing boundaries about \perp to the meniscus of the liquid. The fields were often directly distinguishable on account of the difference in intensity of the light let through.

2. Between crossed nicols the differences in intensity of the fields could not be fixed with certainty. Neither could differences during synchronic rotation of the nicols be observed. The thickness of the crystal layer was about 0.3 mm, the visible field formed a circle of 5 mm diameter.

3. With the gypsum plate changes of colour usually occurred, which indicated a very weak double refraction. Occasionally a field was observed without perceptible double refraction, between others which were distinctly doubly refracting.

4. Conoscopic investigation yielded only negative results. Neither cross, rings or hyperboles were observable. Rotation of the analysor showed that optical activity played no part.

A repetition of observations 1—3 by Messrs ESBACH and PLOEG with nitrogen prepared from sodium azide led to the same conclusions.

§ 6. *Conclusion.* The observations given in §§ 4 and 5 lead to the conclusion that nitrogen immediately at the melting point is doubly refracting. The double refraction is extremely small.

The fact that WAHL did not find any double refraction may be due to his having used a layer only $\frac{1}{20}$ mm in thickness, and making his observations only with crossed nicols, and not with the more sensitive method of the gypsum plate.

Crystallized nitrogen cannot therefore be compared to the dimorphic white phosphorus. Its anisotropic structure corresponds better with the multivarious chemical properties of nitrogen than would be the case with the regular structure so far assumed.

§ 7. *Observations on crystallized argon.* In agreement with WAHL's results ¹⁾ and with the Röntgen analysis of F. SIMON and CLARA VON SIMSON ²⁾ and of DE SMEDT and KEESOM ³⁾ argon according to our experiments is regular to -253° . The manner of crystallizing is very different from that of nitrogen. Argon freezes to an optically isotropic homogeneous mass, which with more intense cooling shrinks as a whole and often springs off from the walls of the observation chamber without becoming doubly refracting.

On being gradually heated the crystallized argon of our experiments began to sublime strongly. The glass wall visible above the crystalline

¹⁾ See p. 1069, note 1.

²⁾ F. SIMON and CLARA VON SIMSON, *Z.s. f. Physik* 25, 160, 1924.

³⁾ See p. 1069, note 2.

mass was covered with small isotropic crystals, as well as with needles which were grown together at right angles to each other. After that the argon melted.

To conclude we are glad to express our thanks to Mr. V. ESBACH for the great help that he has given us, and to H. VORLÄNDER for his help in part of these experiments.

Physiology. — “*The method for graphic determination of the total gaseous exchange in man during muscular activity.*” By J. G. DUSSEY DE BARENNE and G. C. E. BURGER. (Communicated by Prof. H. ZWAARDEMAKER.)

(Communicated at the meeting of May 29, 1926)

In previous communications ¹⁾, ²⁾, ³⁾, ⁴⁾, ⁵⁾ we have described a method for the graphic determination of the total gaseous exchange in man and animal at rest, i.e. the consumption of O₂, the output of CO₂, the respiratory quotient and ventilation. The advantages of this method are the following: 1. gas analyses are not necessary; 2. important saving in time; 3. the experimental data are and remain available in the form of curves for control and reproduction. On comparing this method with different gas analytical methods, e.g. with the most accurate one known so far, that of ZUNTZ—GEPPERT, it appeared to give very well corresponding results; our method is therefore reliable and accurate.

On having determined this, we tried to use this method also for the determination of the gaseous exchange of man during muscular exercise. We believe to have succeeded; with the following method it is possible to determine the gaseous exchange of a subject graphically before, during and after muscular activity.

The principle of this extension of our method is that through the respiration of the subject with the aid of our „trap apparatus” and its volume recorder and a second volume recorder filled with soda-lime, „CO₂-curves” and „O₂-curves” are written alternately below each other on a kymograph. From the descent of the O₂-curves and the descent, resp. rise of the CO₂-curves within a definite period of time, one can obtain the data enabling to determine with a very simple calculation the desired figures on the gaseous exchange during the separate experimental periods.

This method will be best described by referring to the subjoined diagram. (Fig. 1).

M is the mouthpiece which the subject takes in the mouth, causing him to breath (after blocking the nose with a nose clip) through the valves *I* and *E*. These communicate through wide rubber tubes *a* and *b* with the big three-way taps *II* and *III*. In the position

1) *Klinische Wochenschrift* 1924, N^o. 10.

2) *Journal of Physiology* 59, 1924, p. 17.

3) *These Proceedings* 27, p. 382.

4) *Klinische Wochenschrift* 1925, N^o. 2.

5) *Zeitschrift f. d. gesammte experiment. Medizin*, 59, 1926, p. 130.

as given in figure 1, the subject inspires along valve *I*, tube *a*, tap *II* and pipe *e* from volume recorder 1 (pattern of KROGH) and expires along valve *E*, tube *b*, tap *III* and pipe *c* into spirometer *A* of our apparatus, the latter having been indicated diagrammatically by the concentric circles *A* and *B*. With every quantity of expired air, going along the way mentioned above, into spirometer *A* and causing it to rise, an equal volume of air

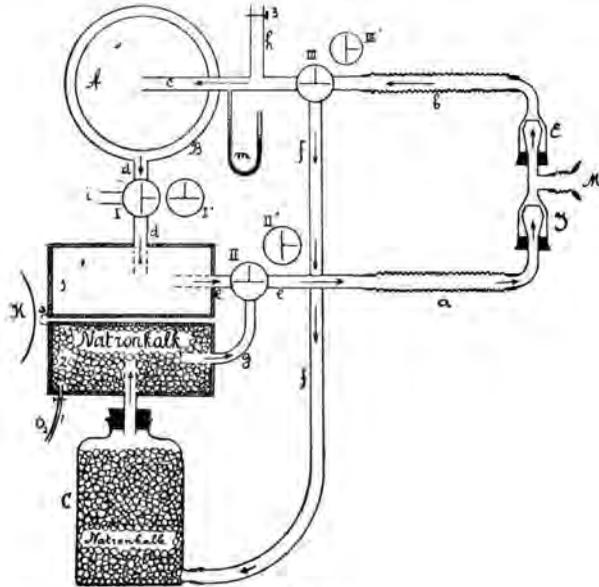


Fig. 1.

from cylinder *B*, (closing off spirometer *A* hermetically) is expelled into volume recorder 1 along pipe *d* and three-way tap *I*. For every one conversant with the working of our respiratory apparatus it will be clear that when taps *II* and *III* are in the position as shown on the fig. 1 the respiration of the subject writes a CO_2 -curve on kymograph *K*. If the three-way taps *II* and *III* are turned 90° to the right, bringing them into position *II'* and *III'*, the trapapparatus is closed and the subject inspires from volumerecorder 2 along pipe *g*, tap *II*, tube *a* and valve *I* and expires into this volumerecorder along valve *E*, tube *b*, tap *III*, pipe *f* and flask *C*. As the tray of volumerecorder *II* and flask *C* are filled with soda-lime, the CO_2 from the expired air will be absorbed if the subject communicates with volumerecorder 2 and through the respiration an ordinary O_2 -curve will be written by its writing point. During this O_2 -period the trapapparatus is closed and the writing point of volume-recorder 1 makes a control line. Reversibly, if the subject communicates with the trapapparatus and volumerecorder 2 a CO_2 -curve is written on the kymograph through his respiration; during this period the pointer of volumerecorder 2 writes a control line.

From the above it will be clear that through turning simultaneously the three-way taps *II* and *III*, then to the left and then to the right, the subject can be made to "write" alternatively then a CO_2 -curve and then an O_2 -curve. In this way we get two curves, which schematically look like figure 2.

From the descent of these O_2 -curves and the descent resp. rise of the CO_2 -curves within a definite period of time, we get all data to calculate

the desired figures on the gaseous exchange during these periods. It is clear that it is thus also possible to determine the gaseous exchange of man at rest, but we wished to use this method in the first place for experiments on work.

The capacity of spirometer *A* of our apparatus for the human subject is about 130 litres, which is amply sufficient during rest to allow for periods

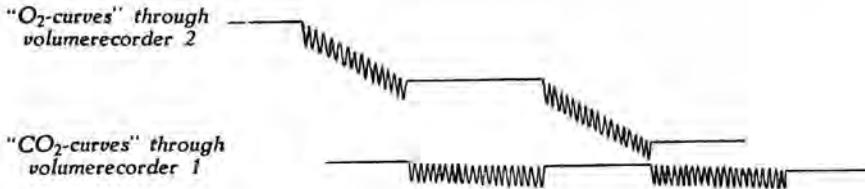


Fig. 2.

of 15—20 minutes. But during work, when the pulmonary ventilation is often 50—80 litres a minute, spirometer *A* is filled in 2—3 minutes and with maximal muscular exercise, still quicker. To render this method useful for experiments on work, spirometer *A* must be emptied quickly when filled with expired air during the registration of an O_2 -curve, so as to be empty before the next CO_2 -curve begins.

This condition we have been able to satisfy in a simple way. As soon as the moment has arrived on which the spirometer must be emptied, 3-way tap *I* is turned 90° to the left during the tracing of an O_2 -curve (position *I'*), causing volume recorder *I* to be closed off and cylinder *B* to communicate with the outside air along the central piece of tube *d*, top *I* and tube *i*. Now through tube *h*, a side tube of *c*, which otherwise is closed at 3 through a tap or stop, we empty spirometer *A* by suction. Spirometer *A* is thus emptied which is possible, because during its descent a corresponding quantity of air is sucked into cylinder *B* along tube *i*, tap *I* and tube *d*.

We use a Phön apparatus for this suction. One of its two perforated sides is hermetically closed, the other one has a tight fitting funnel, on which fits the cork at the end of tube *h*. When the Phön apparatus is put into action through turning the button, it will obtain its air, not from the outside, but from spirometer *A*. The Phön apparatus used by us empties our spirometer in about one minute.

Where we generally take records of periods of from 1.5 to 3 minutes, it is thus possible, during the writing of an O_2 -curve, to empty spirometer *A* (filled with expired air) so that it becomes available for the next CO_2 -curve.

We fill the volume-recorder 2, with which the O_2 curves are written, with O_2 from an O_2 -cylinder, causing enough O_2 to be available for O_2 -periods of 2—3 minutes, even during heavy exercise. After an O_2 -period, and during the writing of a CO_2 -curve, volume recorder 2 may be filled along tube O_2 making enough O_2 available again for the next O_2 -curves.

The results and curves obtained from similar experiments on work, appear from figure 3, which shows two sets of curves in a complete experiment on the gaseous exchange, obtained by the above method (see fig 3).

In this preliminary communication we will have to be satisfied with the discussion of a single item. During the registration of the O_2 -curves, the subject inspires O_2 , during the CO_2 -curves, ordinary atmospheric air. One might suppose that the alternate breathing of gas mixtures of different composition, could lead to inaccuracy. For the most obvious thing to think is that during inspiration of the nearly pure O_2 from volume-recorder 2, a certain extra quantity of O_2 will be dissolved physically in the body and a certain quantity of N_2 will be expelled, in accordance with the smaller partial tension of the N_2 in the O_2 mixture in relation to air; on inspiring ordinary air during registration of a CO_2 -curve, one would expect the opposite. Where the absorption coefficients of O_2 and N_2 in blood and water are approximately as 1 to 2, one might easily suppose that through this factor the figures for the quantities of O_2 used during the respective periods of the experiment, are not accurately corresponding with the quantities used by the organism.

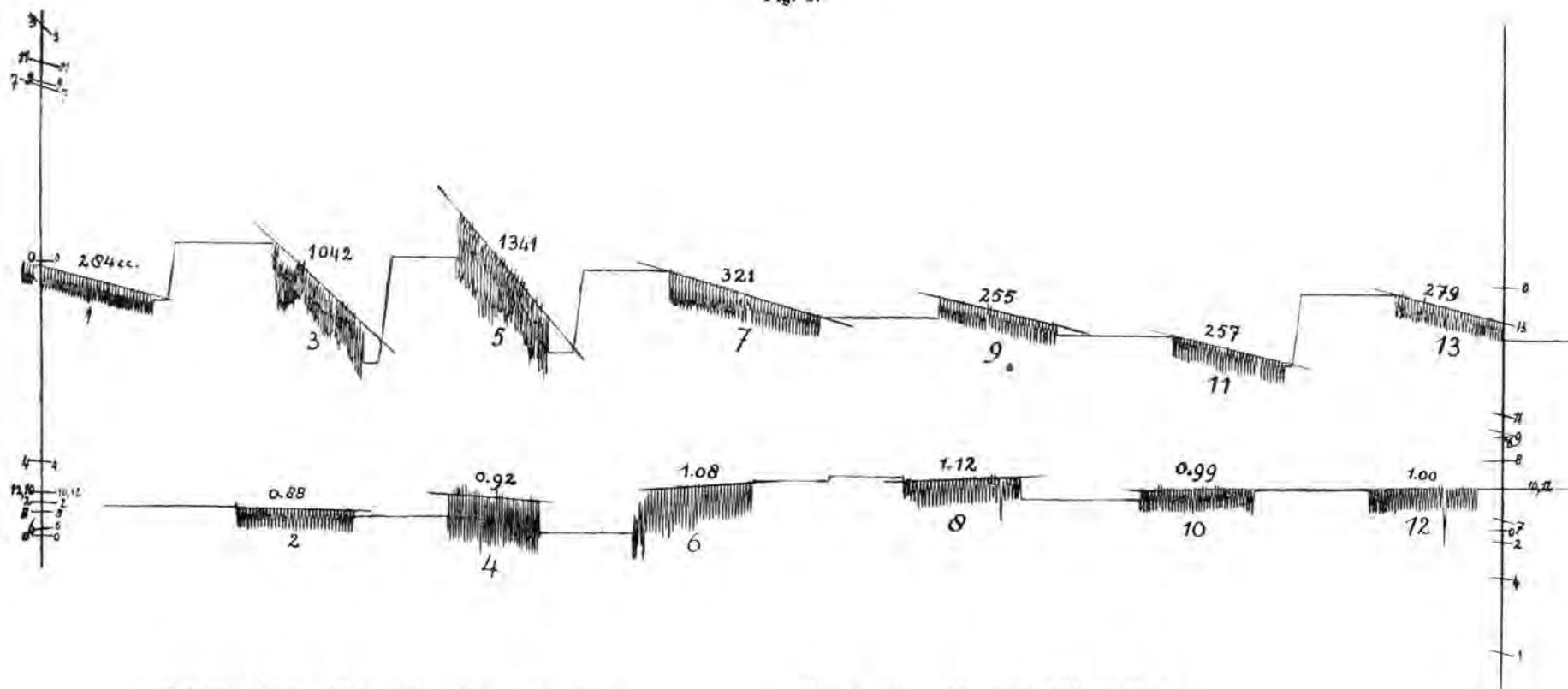
To examine this point we have determined several times graphically in different subjects at rest the O_2 -intake during periods of two minutes with alternate breathing of O_2 and ordinary air, now after a long foreperiod on ordinary air, then on O_2 . The result of this control has been that in our method in inspiring alternately O_2 and air during periods of 1.5—3 minutes, this factor, if present, does not interfere. This is shown by the following table.

The average of these two groups of control experiments is the same. Moreover, the differences which have been found, and which are very small for experiments on gaseous exchange, are in several cases in opposite direction; a systematic deviation therefore not observed. The beautiful

Consumption of O_2 in cc. per minute (reduced to 0° and 760 mm. Hg.)

| After adaptation to O_2 | After adaptation to air |
|---------------------------|-------------------------|
| 280.1 | 274.6 |
| 279.2 | 279.2 |
| 297.7 | 290.1 |
| 295.6 | 296.8 |
| 299.6 | 309.7 |
| Av. 290.5 | Av. 290.1 |

Fig. 3.



23.4'.26. H. Sch. 11³⁰ lunch (rich in carbohydrates).
 Fore period: sitting quietly 35 min. in easy chair.
 6 min. adaptation on O₂.
 curves 1—13: 3³⁷—4⁰² (25 min.).
 exercise during curves 3, 4 en 5 (4½ min. standing
 running 164 steps per min.)

Temp. volumerecorders = 12,5°—12,7° C.
 Corrected bar. = 758 mm. Hg.
 Factor = 0,939.
 1000 cc. of each volumerecorder = 17 mm. on the
 original curves.

Enlargement of this fig. in comparison with the original curves is 0.43.
 The R. Q. given above CO₂-curve 6 must be 1.17 instead of 1.08.

analogy between the two corresponding numbers in each experiment also shows the usefulness and accuracy of the method.

Our method is so simple in its execution, that one can do the whole experiment with two people, the subject and the experimentalist. Further, it entails a very valuable saving in time. The calculation of the curves given in fig. 3, has not taken 35 minutes. Therefore, in about 35 minutes, the different figures of the O_2 -consumption and the 6 given respiratory quotients were known. If we estimate the time necessary for the determination of these data in using one of the gas analytic methods, there would have been necessary for the six gas analyses (taking it of course they were done without interval one after another by one analyst) 6×20 minutes or 2 hours. Adding to this the time necessary for the measurement of the ventilations during the 6 periods and the calculations of the respiratory quotients from these gas analyses and ventilations, these calculations taking much more time than in our method, one would have to reckon $2\frac{3}{4}$ hours at least for obtaining the same data by gas analysis.

Briefly, we may thus summarize this preliminary communication: The above described method is an extension of the method given in previous communications for the graphic determination of the total gaseous exchange. It permits to determine this in man also in the so-called exercise-experiment, i.e. before, during and after muscular activity. The experiment may be continued as long as the experimentalist desires. The method is accurate and offers several advantages against the gas analytic methods hitherto used in this kind of research.

(Physiology Laboratory of the State University, Utrecht.)

(May 1926.)

Histology. — "*An Inquiry into the Physico-chemical Structure of the Collagenic Substance. I. On the Spiral Arrangement and on the so-called Hygroscopic Torsion of Collagenic Bundles of Tendons.*"
By Prof. G. C. HERINGA and Miss H. A. LOHR. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of June 26, 1926).

Being struck by the peculiar sinuous course of the collagenic fibres that we found again and again in all our preparations (as well in the broad tape-shaped bundles as in the most delicate, in dark field optically homogeneous filaments) we have made a close inspection of the arrangement of the bundles in the tendon. Our endeavour resulted in a complete confirmation of WOERDEMAN's ¹⁾ description. That collagenic bundles are intertwined in the tendon in groups, became evident from the following observations:

1⁰. In longitudinal sections of a tendon with purely parallel fibres (mouse-tail) the "azimuth blende" ²⁾ showed us in dark field the interlaced course on a different level of the fibres.

2⁰. In transverse sections (mouse-tail tendon, achilles tendon human embryo) it is seen on turning the screw of the micrometer that bundles are coiling in groups round each other, as described by WOERDEMAN. It is remarkable that the last-named object displays the peculiarity that all over the transverse section complexes occur side by side twisting to the right and to the left.

3⁰. The spiral interlacement of the bundles is demonstrated by the following experiment even more conclusively than by the two preceding facts:

A tendon with parallel fibres (extensor of the toe of a cow) is firmly clamped by a screw. The other end is held by the hand or simply hangs loose. A slender, pointed lancet is inserted superficially into the tendon with the blunt edge of the blade towards the experimenter, and the flat side parallel to the long axis of the tendon. By pushing the knife down so far that the point reappears on the other side a slender bundle of fibres is split off from the tendon. Now the knife is drawn back, so that the split bundle is detached from the tissue. It then appears that under this operation the free extremity of the tendon twirls round as a rope made of twisted fibres does when used for a similar experiment.

A repetition of the experiment, not with the entire tendon, but with a detached bundle showed that the phenomenon became still more apparent according as the

¹⁾ H. W. WOERDEMAN, Histologisch onderzoek naar den fibrillairen bouw van eenige cellen en weefsels. Diss. A'dam, 1921.

²⁾ Cf. G. C. HERINGA and H. A. LOHR. These Proceedings, Vol. 28, p. 509.

bundle is thinner. From this we conclude that intertwining of the bundles, down to increasingly finer texture, is repeated over and over again, which in fact tallies completely with WOERDEMAN's observation, as well as with ours of the microscopic preparation (transverse section).

There is no doubt, then, but that in the tendon there exists a spiral intertwining of the bundles. A similar condition of the bundles is found in microscopic preparations of the skin and of the loose connective tissue (umbilical cord, mesenterium etc.), in which the sinuous structure of the bundles can be watched in one field of vision as the spindle is so much smaller.

Seemingly conflicting with all these observations was the fact that in a great number of largely magnified photos of mouse-tail tendons swollen in acids a very distinct parallel striation was noticeable. While occupying ourselves with the question in how far a detorsion under the influence of the acid-swelling might come into play here, HATSCHEK's publication in the *Kolloid Zeitschrift* 1925 was brought under our notice by Dr. BUNGENBERG DE JONG.

HATSCHEK has bent rectangularly moulded gelatin rods and held them in fixation so long that deformation became permanent. Through desiccation the curvature was then rendered firmer. Swelling caused its detorsion. HATSCHEK also twisted similar rods into spirals; desiccation induced further torsion, swelling brought detorsion. Control-experiments with nondeformed gelatin-models of a definite mould exhibited no morphological change, but a homomorphic change of volume excepting phenomena of "case-hardening".

To put it shortly: HATSCHEK's experiments go to show that desiccation gives permanence to a previously effected deformation of the gelatin, and that swelling causes it to disappear.

We have repeated these experiments with quite similar results in so far as a simple curvature is concerned. Only we feel urged to add two observations to HATSCHEK's description.

1^o. that the morphological change is introduced by a shortening on the concave side without an increase, nay rather with a decrease of the concavity. At the same time the short terminal surfaces are converging towards the concave side. This clearly proves, what in strictness could be anticipated, viz. that the warping of the rod rests on traction emanating from the side compressed during the deformation process. This lends support to the conception that the phenomenon originates entirely from the loss of water through the micellae, which loss asserts itself most where the micellae are most accumulated. This conception also makes the second fact intelligible, which we will mention as an addition to HATSCHEK's description, viz. that curvature caused by desiccation begins only after a considerable, fairly homomorphic diminution: the influence of the disposition of the micellae is more distinctly visible according as the latter are packed

more closely (had we rather say: are more closely accumulated?) through evaporation of intermicellae water.

As we endeavoured to find a parallel between the twisted gelatin and our tendons, we have started experiments on the behaviour of tendons during swelling and desiccation.

Again we took a piece of a cow's tendon (toe-extensor), 10 c.m. in length. Judging from its outward appearance its fibres ran parallel. At the top it was clamped, at the lower end it was provided with a pointer and dipped vertically in a weak acetic-acid solution. The first experiment yielded directly positive results. After two or three hours the pointer indicated a convolution of more than 90° . This experiment was repeated some times, modified in such a sense that lateral curvatures, which greatly vitiated the readings, were counteracted by a balance at the lower extremity. The same torsion phenomenon was noted every time. But the deflections were seldom higher than 15° — 30° , while, moreover an insight into the nature of the problem was impeded by the fact that swelling caused pieces of the same tendon to wind alternately to the right or to the left. This fact invalidated our estimation of the direction of the torsion. Besides this there was, however, the fact that convolution also took place with desiccation (the tendon, mounted in the same way being suspended in the air).

The idea that in such a complex as an entire tendon the combination of elements with a tendency to opposite torsions might play an influence upon the result of the experiment, prompted us on Dr. KOLKMEYER's suggestion to repeat the experiment, with small fibre-bundles, isolated from a tendon in the way above described. Now all at once convolutions made their appearance, not seldom with deflections of 360° or 2×360 degrees. That nevertheless no regularity was found in the size of the angle of torsion, so that a quantitative analysis of the phenomenon is not possible for the present, is quite conceivable, if we reflect that our splitting-method is still a rough-and-ready method, and how much the result is affected, as may be seen during the experiment, by adhering rough particles and probably also by mechanical lesions to the bundles. With these delicate filaments in the same object the same torsions to the right and to the left manifested themselves either side by side or sometimes in succession. It was evident, however, that the torsion caused by swelling leads in the first instance to a winding down of a torsion that always appears at desiccation. This torsion is the fundamentally equally important counterpart of the detorsion caused by swelling. It is very conspicuous at the tapes shaped and slightly crinkled fibre-bundles.

Thus we establish the fact that the collagenic fibres present with desiccation a torsion opposite to that presented with swelling; they behave like the gelatin in HATSCHEK's experiments. Besides the observations described by HATSCHEK we know already for a long time a similar phenomenon from quite another quarter. This may the more readily be correlated with our torsion of collagen, as both relate in the same degree to fibrous substances furnished by nature. We now think of the phenomenon designated by botanists as "hygroscopic torsion".

It had already long been known that many plant-parts flex or twist on (physiological) desiccation, when DARWIN¹⁾ pointed out that the origin of the movements

¹⁾ F. DARWIN, Transactions Lin. Soc., 2nd Series I, 149, 1876.

was to be looked for in the properties of the cells. ZIMMERMANN¹⁾, coinciding with NÄGELI's hypothesis, established the relation between the disposition of the micellae and the location of the „shrivelling-ellipsoid“, and lastly STEINBRINCK²⁾ found an explanation by means of extremely ingenious models for the various morphological changes (flexure, spiralization, torsion) originating from the disposition of the micellae. Without entering into details we wish to remind the reader that STEINBRINCK has demonstrated that, when different micellar systems are united into one system in such a way that their several shrivelling-ellipsoids do not coincide, the deformation agrees on all points with the resultant of the tensions appearing in situ.

When applying STEINBRINCK's reasoning to HATSCHEK's flexed gelatin-rod, it will be seen directly that desiccation, indeed, induces reinforcement of the curvature, anyhow if we assume provisionally, that the gelatin-micellae, like those of plant-fibres, are elongated anisotropic rods, and that these micellae, without changing their position as a whole, are placed by the deformation in such a way that their long sides are turned outward on the convex side, and on the concave side one of their short ones, assuming also that, again as in the case of the plant-fibre, the long axis of the shrivelling ellipsoid coincides with that of the micellae.

For when we examine a slice defined by two parallel planes, taken from the middle of the gelatin-rod *a b c d*, in which there are on the convex side *a b* two micellae with their long axis parallel with the axis of the rod, and on the concave side *c d* two micellae normal to it, then it will follow directly from the position of the shrivelling-ellipsoids that *c d* is shortened more than *a b*, which when totalized must bring on a flexion of the whole rod.

A similar reasoning may afford an explanation of the desiccation-torsion, viz. when we imagine the twisted object, from which we started, divided by parallel planes into a number of thin slices. STEINBRINCK's scheme, which is in every way admissible for botanical objects, tested by polarization-microscopic investigations, is, therefore, also adapted to HATSCHEK's gelatin-model if the gelatin-micellae are anisotropic. Not only in this case, however. When suggesting the above mode of explanation we started from the ascertained fact, (without taking into account the anisotropism of the separate micellae), that on the compressed side the micellae are packed more closely. This, indeed, could readily be verified by mixing the liquid gelatin before the moulding with gold-sol, and by counting the particles by means of an Ehrlich-blende. Then it should only be remembered that for the explanation it is immaterial whether a single particle is answerable for a definite shrivelling or whether it is the sum of the mean decrease of many particles, in order to see that the two explanations coincide completely. From either starting-point we arrive at the conclusion that with dehydration the micellar-anisotropic structure of any object brings about a change in

¹⁾ A. ZIMMERMANN, *Mechan. Einrichtungen zur Verbreitung der Sauren und Früchte*, etc. — *Jahrb. f. Wiss. Botanik* 12, 1881.

²⁾ C. STEINBRINCK, *Ueber Schrumpfunen und Kohäsionsmechanisme der Pflanzen*. — *Biol. C. Blatt*, 26, 1906.

the outward form, agreeing with an accentuation of the micellar arrangement.

Lastly we have still to observe that this formula is reversible in the sense that with evenly arranged micellae loss of water can bring about deformation only when the micellae are anisotropic.

After this lengthy expatiation let us revert to our own object. It is difficult now to interpret the swelling- and the desiccation-torsion of the tendon, respectively the tendon-fibrils, otherwise than by assuming either that the micellae are arranged as a spiral or every micella possesses itself a spiralshaped anisotropism, or finally both possibilities together.

Thanks to the kindness of our co-workers we are so fortunate to be able to verify this hypothesis by two investigations respectively by HERINGA and MINNAERT and by HERINGA and KOLKMEIJER. The firstnamed authors ¹⁾ starting from an optic phenomenon observed in tendon-sections with transmitted light, came to the conclusion that the micellae of AMBRONN are arranged spirally. The great regularity also, with which definite convolutions revealed themselves in the preparation, was a strong argument for the conclusion that we had not to do here with an orientation of the fibrils incited from without (i. e. mechanical force in connection with development): the phenomenon in fact being evenly superposed on all histological arrangement. It was evidently bound to the properties of the collagenic substance itself.

Once so far we could not omit going farther into the real colloid-chemical-micellar formation of the collagen, as had already been done before us by HERTZOG and KATZ and others. Dr. KOLKMEIJER ²⁾ was so kind as to assist us in our investigation. To our great satisfaction we could derive conclusions from the röntgenogram that really favour our hypothesis, as it appeared ^{1°}. that presumably plate-shaped crystallites are arranged invariably at a definite angle with the axis of the fibre; ^{2°}. that, moreover, a periodicity exists in the long axis of the fibre. These facts, especially when combined, are quite compatible with the conception of a spiral structure. Indeed, allied facts can be pointed out in the crystallography of inorganic nature, as according to KOLKMEIJER, BIJVOET and KARSEN's ³⁾ inquiries a similar spiral structure also occurs in sodium-chlorate, and sodium-bromide.

This is a pregnant conclusion, as it leads to many consequences, which, especially when compared with the most recent observations by KATZ ⁴⁾ on the splitting capacity and the röntgenspectrum of elongated gelatin, will not fail to influence also our view of the gelatinizing process. This, however, will require closer inspection.

As to collagen itself, we wish to point out in connection with the foregoing that, if the probable structure of the elementary cell of the collagenic substance, rendered probable by this röntgen-treatment, should be confirmed,

¹⁾ G. C. HERINGA and M. MINNAERT, These Proceedings, p. 1087.

²⁾ G. C. HERINGA and M. H. KOLKMEIJER, These Proceedings, p. 1092.

³⁾ These Proceedings, 23, 644, 1920.

⁴⁾ l. c.

then, in accordance with STEINBRINCK's schema, the arrangement of the crystallites inside this cell (to be imagined as anisotropic rods or plates) must, when dried, naturally lead to torsion of this units. Although the spatical composition of these elements into a crystalline whole, has not yet been accomplished, it may be deemed probable that the composition of all these elemental torsions will induce torsion of the whole fibre, in the same way as DARWIN (l.c.) transfers the torsion of the separate sklerenchymatic cells to the torsion of the plant-part.

From this it would again follow that the torsion of the collagen is owing to the arrangement of the crystallites as well as to the typical lyophil-colloid character of the collagenic substance.

Taking all in all the result of this inquiry on the one hand confirms the relations between the collagenic substance and colloid-chemistry, especially the thread-sols¹⁾, on the other hand it confirms the relation we looked for, between collagen and the mesomorphic substances.

So the röntgendiagram corroborates our previously formulated conception²⁾ of the origin of the collagen substance. Moreover, it explains a fact that had hitherto puzzled us. As the reader will remember we described how, in the first instance, the collagenic fibres presented themselves as thin filaments singly defined in dark field, how they subsequently grow into rather broad, doubly contoured, but still optically homogeneous permanent tapes, and lastly how after this secondary optic inhomogeneities appeared, that, while ever multiplying, gave the final, very closely striated, typical, fibre-bundlelike aspect to the collagenic mass. Well then, it is quite conceivable in addition to what we now know of the crystallography of collagenic substance :

10. that like asbestos this must have numberless longitudinal planes of cleavage ;

20. that this cleavage (cf. splitting of dried wood) will appear spontaneously with the apparently physiological, gradual dehydration of the micellae, so that in connection with the accompanying torsion the planes of cleavage also assume a twisted course.

and 30. that consequently, when micellae arranged round one crystallization-centrum are getting older, the whole collagen mass built up by them will be split up in a number of fibres twisted round each other.

*From the Laboratory for Embryology and Histology
of the State-University at Utrecht.*

Utrecht, 26 June '26.

1) "Stäbchen-Sole" after SZEGVARI.

2) G. C. HERINGA and H. A. LOHR, These Proceedings.

G. C. HERINGA and H. A. LOHR, Bulletin d'Histologie, 1926.

Histology. — “An Inquiry into the Physico-chemical Structure of the Collagenic Substance. II. On an Optic Phenomenon observed in sections of the Tendon”. By Prof. G. C. HERINGA and M. MINNAERT. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of June 26, 1926).

The structure described in this paper was found in longitudinal sections of a cow's tendon (toe-extensor) obtained by means of a microtome after previous freezing without embedding, and enclosed in a levulose-gelatine mixture. The preparations become completely transparent and look beautifully homogeneous.

We hold such a preparation close before our eyes, and look at a point-shaped lightsource. A “nitra-lamp” arranged at some metres' distance suits our purpose very well. Then we see a remarkable diffraction figure, which shows most distinctly when the room is darkened and the background is black. Let the direction of the fibres be horizontal. We see (Fig. 1) round the light source a cross of light with four

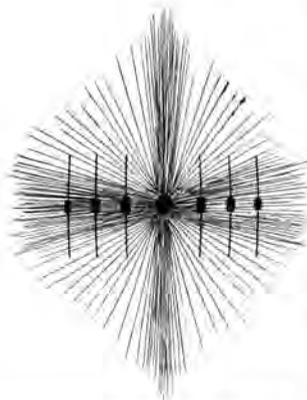


Fig. 1.

somewhat pencil-shaped arms. The vertical arms are rather narrow and long, the horizontal ones broader and shorter. In the horizontal lines of light, on either side of the lightsource there are to be seen three or four intensely bright, beautifully coloured diffraction-spectra. Moreover, the whole field is filled with a weak light radiating from the central light-point. The diffraction-spectra are disposed on a horizontal line, parallel to the longitudinal direction of the fibres.

In the best preparations the vertical arm is almost invisible. This part of the phenomenon must, therefore, be referred to fortuitous small fissures in the preparations in the direction of the fibres.

In order to determine the period of the structure, the light-source was replaced by a screen with a narrow slit, illuminated by sodium-light, and the angle was determined at which the diffraction-spectra could be seen. We found: 0.015 rad., which corresponds with a period of $\frac{0.589}{0.015} \mu = 40 \mu$.

This value is, of course, an average. Probably periodic structures will appear in the preparation of differing λ . The optic examination gives

an average value over an area of the preparation equal to that of the pupil. The marked distinctness and the pure colour of the diffraction-spectra indicates, however, that the period can differ only little from one point to the other, or that portions with considerably deviating periods occur comparatively seldom.

This great regularity and the accurate orientation of the periodicity with respect to the longitudinal direction of the tendon renders it already improbable that some fortuitous ridging should have originated during the microtomy. Moreover, the direction of the cutting microtome was inclined at a large angle to the longitudinal direction of the fibres.

Now it was our desire to make out in a general sense whether the optic phenomenon was due to a ridging of the surface or to periodic alterations in the index of refraction of the tendon. Our reasoning can be outlined as follows:

1. if there were a periodic structure in the surface, it must be just as visible in the reflected-, as in the refracted light.
2. in the reflected light no diffraction-spectra are visible.
3. therefore, the looked-for periodicity is not due to a ridging of the surface.

Ad 1. Let us consider a grating, consisting of a homogeneous medium of an index of refraction n_1 bordering on a medium of an index of refraction n (Fig. 2). We confine ourselves to the

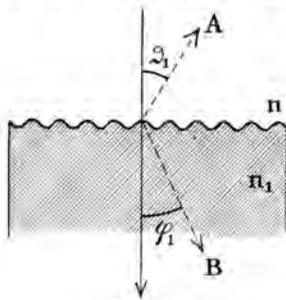


Fig. 2.

simplest grating-profile $\zeta = c_1 \cos px + s_1 \sin px$, and we suppose that the incident pencil of rays is normal to the boundary plane. Now we compare the amplitudes of the flexure-spectra of the first order A_1 , originating from reflection, with the amplitudes of the spectra B_1 , occurring in the refracted light.¹⁾

- a. For polarized light, of which the electric vector is parallel to the grooves, $B_1 = A_1$.
- b. For polarized light, of which the electric vector is perpendicular to the grooves, $B_1 = -A_1 \frac{\cos \vartheta_1 n}{\cos \varphi_1 n_1}$, in which ϑ_1 and φ_1 are the diffraction-angles for the reflected and for the refracted rays. In the preparations studied by us n and n_1 differ little, and the grating-period is so large that we can put $\vartheta_1 = \varphi_1$; so that B_1 is about A_1 . So the diffraction-spectra are about as strong in the reflected as in the refracted light.

Ad 2. Now we try to see the diffraction bands in the reflected light. We look at our point-shaped light-source reflected in the preparation, while holding the object-glass towards the light-source, and the cover-glass away from the light-source. Then we see two images of the light-

¹⁾ Cf. RAYLEIGH, On the dynamical theory of Gratings, Papers V, 403; Proc. R. Soc. 1907, A 79, 322.

source at a short distance apart when we look through a part of the object-glass that is slightly prismatic. The brightest image *A* is reflected by the anterior glass-surface. The less bright image *B* is reflected by the surface of the cover-glass, and has gone twice through the preparation. An image *C* reflected by the preparation itself, would nearly coincide with *B*. Now in order to isolate the image *C*, we press a piece of black paper on the surface of the cover-glass and take care to preserve optic contact by means of a drop of cedar-oil between the paper and the glass. The image *B* reflected by the posterior plane of the cover-glass must disappear now and only an image *C* could remain. Now from observation we learn that no trace is left either of a reflected image or of a diffraction-spectrum. This shows that the index of refraction of the preparation is so small that the reflected spectra have no appreciable intensity.

Ad. 3. Thus a periodic ridging of the surface cannot have contributed appreciably to the formation of diffraction-spectra observed in transmitted light.

On the other hand it is quite conceivable that these diffraction-spectra could originate from periodic differences of the index of refraction, since they exert an influence over the whole thickness of the preparation. Generally the diffraction-spectra must then be the more distinct, according as the preparation is thicker. This generally came true.

Now we still intend to demonstrate that the diffraction-spectra observed, are not at all to be ascribed to periodic changes in the index of refraction of an isotropic medium, i.e. to periodic variations in double refraction.

Let a polarizer be placed before the preparation. The diffraction-spectra will then be seen to retain their intensity whatever the position of this polarizer may be. Now if we arrange it parallel to the direction of the fibres, and if we place between the preparation and the eye an analyzer, which we rotate so that the light-source becomes lighter and darker every 90° , then the diffraction-spectra of the first order appear to exhibit simultaneous variations of intensity, but in an opposite sense. The spectra of the 2^d order (this observation is difficult of execution) seem to become clearer or darker as the light-source becomes clearer or darker.

Now something remarkable can be witnessed, viz. that the light-source itself, which produces the diffraction-spectra is almost extinguished, while the first diffraction-spectra of that light-source still display an almost undiminished intensity. Such a phenomenon can never be explained by a periodic variation in the usual index of refraction, where as it is plausible that it is caused by variation in the double refraction.

We also examined our preparation under the microscope. It showed most distinctly with incident light, but also with transmitted light, a sinuous course of the tendon-bundles, as is illustrated, e.g. in Fig. 143 by SCHMIDT "Die Bausteine des Tierkörpers" (Bonn. 1924). What struck us here, is that the phases of undulation of the various contiguous bundles

are in complete agreement with each other. From this results a striation of the whole preparation. On the face of it this somewhat reminds us of the striation of the fibres of voluntary muscles. Direct measuring of the period of a group of clearly defined wavelets invariably resulted in the optically determined wavelength: 40μ .

We now asked the question whether the phenomenon is of common occurrence. Nine preparations of the same cow's tendon, of thicknesses between 20μ and 200μ all displayed the same phenomenon, and the values of the periods varied from 38μ to 41μ . Afterwards we found the same in cross-sections of other cow's tendons and also of pig's tendons, and always with periods of about the same magnitude.

Between the optic diffraction-phenomenon and the periodic structure in the microscopic preparation there is no doubt a close relation. What this relation is we shall discuss in a later paper.

Further data were furnished by an inquiry under the polarization microscope.

Between crossed Nicol prisms the preparation is rather evenly clear, when the direction of the fibres makes an angle of 45° with the direction of the vibrations of the nicols. If the fibres are made to run parallel to the vibrations, we see alternately dark and light bands, normal to the fibres, with a distance of 20μ between two dark (resp. clear) bands. Wherever the spiral of the bundles is seen to run parallel to the direction of the fibres, the transverse bands are dark, wherever the inclined pieces of the spiral are seen, they are light (fig. 3a).

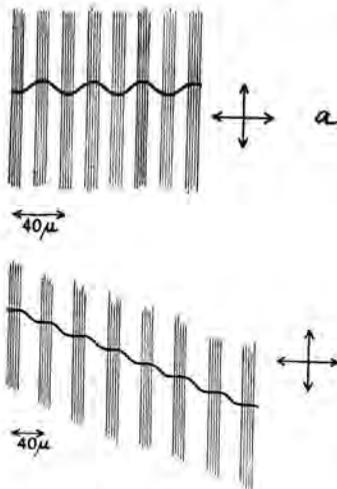


Fig. 3 a and b.

subsequently made to incline at a small angle, the dark bands are seen to merge into each other (fig. 3b), and the period of the dark bands is doubled (40μ). It again comes true here that the dark bands are precisely where the direction of the spiral is parallel to that of the nicol.

All these phenomena prove clearly that the bundles behave completely like uniaxial crystals, of which the axes are disposed in a regularly waving line, or on a spiral which we see in the projection. In connection with the polarization-microscopic investigations of AMBRONN, c.s., according to which the "micellae" that build up our tendon-

substance, are to be regarded as unidirectional, uniaxial crystals, we can conclude that the micellae are also disposed either on a waving line, or on a spiral.

In order to choose between these two suppositions we made a number of cross-sections of one and the same cow's tendon, all parallel to the axis, but in two mutually vertical planes. The microscopic examination showed that in both preparations the bundles exhibited the same periodicity, and that in either of them the same flexure-spectra appear. It can, therefore, hardly be imagined that the preparation should consist of a large number of plane sinecurves, orientated in all sorts of azimuths, considering the complete contiguity of the bundles, observed in the preparation. Moreover, we see distinctly the spiral form, when tracing the boundary-line of a bundle with the screw of the micrometer. On this basis, then, the conclusion is warrantable that the observed sinecurves are, indeed, the projection of spirals, and that consequently the micellae are arranged in such spirals.

*(From the Laboratory for Embryology and Histology and from
the Physical Laboratory of the State-University at Utrecht.)*

Utrecht, June 23, 1926.

Histology. — “*An Inquiry into the Physico-chemical Structure of the Collagenic Substance. III. Report on an Investigation with Röntgenrays of the structure of the Collagenic Substance*”. By Prof. G. C. HERINGA and N. H. KOLKMEIJER. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of June 26, 1926.)

1. Already a few years ago we took a photo of a tendon from a mouse-tail. The film was stretched along the wall of a cylindric camera with a radius of 2.72 cm.; the tendon was placed in the axis of the cylinder in paraffinoil, inside a thin-walled glass-tube, and the parallel bundle of monochromatic (CuK α) Röntgenrays was normal to the axis. The exposure lasted about 20 hours at about 10 mA and 40.000 Volts. Circumstances prevented us from prosecuting the examination until very recently. We have now taken a similar exposure, this time, however, without paraffin-oil in order to avoid the influence of this complication. The tendon, approximately $\frac{1}{2}$ mm. thick, was now dried, consequently stiff, and was thus placed in the thin-walled tube. During the exposure of six hours at 40000 volts, 15 mA, the preparation was rotated round the axis. Although the film is less blackened, this photo showed no appreciable difference with the previous one.

2. It is visible at once that the photo is of the nature of a four-point-diagram. This indicates a fibrous structure such as HERZOG¹⁾, and POLANYI²⁾ and their co-workers found in fibrous substances. This has also been observed by KATZ³⁾. On closer inspection it will be seen that the four points, which belong together, enlarge every time so as to form very broad and vague DEBYE-SCHERRER-lines. Still, two or perhaps four broad serial lines (Schichtlinien) remain visible, also fairly uninterrupted. We have also noticed a series of finer lines, parallel to the serial lines, much more difficult of observation. In order to ascertain the existence of these lines, we thought it necessary to make a photometric measurement of the film along some lines, normal to the direction of the striation. We are greatly indebted to Dr MINNAERT for performing this measurement for us. From the

1) R. O. HERZOG and W. JANCKE, Z. f. Phys. 3, 196, 1921.

2) M. POLANYI, Z. f. Phys. 7, 149, 1921; M. POLANYI and K. WEISSENBERG, *ibid* 9, 123, 1922 and following communications.

3) I. R. KATZ, Erg. der exact. Naturw. 3, 316, 1924 and 4, 154, 1925.

photometric curves it can be concluded that the above mentioned fine, alternately dark and light striation runs approximately parallel and straight, with equal breadths, over a large portion of the film. It may further be observed that this striation can be noted on the reproduction of the centre of an exposure of the tendon published by KATZ in *Kolloidzeitschrift* June 1926.

3. The indistinctness of all the lines on both films is such that exact measurements are impossible. However, in order to verify a theory ¹⁾ advanced by one of us (*H*) concerning the structure of collagenic tissue, it is very desirable to collect quantitative data. Therefore, we publish our results, be it provisionally and with some reserve, hoping that other observers will also publish theirs for the sake of comparison.

The median lines of some broad DEBIJE-SCHERRER-lines are to be seen at distances from the middle of the film, of 1.0, 2.0, 3.65, and 6.2 cm. from which we computed for $\sin^2 \frac{1}{2} \vartheta$ the values 0.033, 0.13, 0.39, and 0.825, in the roughly calculated ratios of 1:4:12:25. It should here be observed that the line 3.65 is very broad and probably consists of two lines. A more accurate estimation suggests that the number 12 of the last series should be replaced by 9 and 16. So we think that only one face of the crystal has reflected, which might point to a plate-shaped rather than to an elongated form of the crystallites. In case this hypothesis is correct $a = 4 \text{ \AA}$ is to be considered as a possible parameter of the net.

From the formula given by DEBIJE and SCHERRER ²⁾

$$B = 2 \sqrt{\frac{\log e}{\pi}} \frac{\lambda}{D \cos \vartheta}$$

we compute some tens of \AA for the dimension of the crystallites with aid of the estimated breadth (to half the intensity of the middle) of the lines, viz. 2 mm for the line of the second order. For the lowest limit we found 14 \AA .

From the distance of the distinct serial lines 5.9 mm we compute 87 degrees for the most frequently occurring angle, made by the one reflecting plane with the axis of the fibre. We believe that no conclusion may as yet be deduced from our exposures for the nature of the special fibre-structure in the sense K. WEISSENBERG takes it ³⁾. It seems to us that our measurements of the distances of two light lines in the fine striation by means of a photometric curve, is more accurate, viz. 2.2 cm. We think that this striation is due to the occurrence of a linear

1) G. C. HERINGA and H. A. LOHR. These Proceedings p. 1081.

2) Nachr. Ges. Wiss., Göttingen, 91. 1918, Assuming the net to be cubic.

3) Zs. f. Physik, 8, 20, 1922.

succession of equal elements in the long axis of the fibres. If this is the case let the distance of the elements be x and let ϑ be the angle at which the distance between two light lines is seen from the preparation, then $\sin \vartheta = \frac{\lambda}{x}$. In working this out the result will be $x = 0.004 \mu$.

*(From the Laboratory for Embryology and Histology and the
VAN 'T HOFF-Laboratory of the State University at Utrecht).*

Utrecht, 27 June 1926.

Petrology. — "*On a Young-Tertiary Limestone of the Isle of Rotti with Coccoliths, Calci- and Manganese-peroxide-Spherulites*".
By TAN SIN HOK. (Communicated by Prof. H. A. BROUWER).

(Communicated at the meeting of June 26, 1926)

Introduction. The in many respects remarkable collection of young-tertiary limestones, collected by Professor Dr. H. A. BROUWER on the occasion of the Dutch Timor-Expedition, was kindly placed at my disposal for sedimentary-petrographical research.

In preparing this paper the author has been put under special obligation to Professor BROUWER for his extremely valuable directions and stimulative interest.

These limestones which are believed to be of pliocene age, are very frequent in the South of Rotti; they are found with reefs ¹⁾. The groups diagenetically not much altered, were examined, all being white and soft. The following division could be made:

1. Radiolarian-limestones.
2. Foraminifera-limestones.

The groundmass of these limestones is very fine-grained; in group 2 aragonite-asterisks occur very numerously, whereas they are not yet met with in group 1. This paper is confined to an abnormal type out of the group of the foraminifera-limestones.

Method of Examination. Thin slices as well as preparations made of the disintegrated limestone, magnified about 375—750, were examined in monochromatic yellow light.

The monochromatic light was used to simplify the recognition of spherulitic and fibro-radiate forms. As is known fibro-radiate aggregates of optically uni-axial crystals exhibit in parallel-polarized light the interferencefigure of homogeneous uni-axial crystals in convergent-polarized light ²⁾. Monochromatic light makes the interference-cross more conspicuous, so that those aggregates cannot be overlooked so easily.

Making of the preparations. A fragment of the stone is boiled in distilled water. By this treatment it was completely disintegrated without any damage to its components.

¹⁾ H. A. BROUWER 3, p. 72.

²⁾ H. ROSENBUSCH 16, p. 398. G. LINCK 9, p. 281.

Aggregates of rhombic crystals of analogous structure form interference-figures of little difference.

A drop of the resulting milk of lime was evaporated on a slide, and on the residue some diluted canada-balsam was brought (as a solvent xylol was used).

Professor S. QUINABOL of Turin recommended this diluted balsam for the mounting of preparations of Radiolaria. In this soft medium they can be brought to rotation by a gentle pressure on the cover-slip, so that now they may be studied in every position.

This diluted balsam can be generally recommended for sedimentary-petrographical researches.

For the concentration of heavier or coarser ingredients the method which is already known, was followed: the beaker with the milk of lime was brought in a rotatory motion, the desired concentrate accumulates in the least agitated parts of the fluid and can be transferred by a pipette.

Description of specimen: $168\times$, from Bebalain. Rotti.

Macroscopically a gray, dirty-white rock, soft; coming-off, with many spots of MnO_2 , often surrounded by a brown envelope.

Microscopically a compact limestone, fine-grained;

A. the coarsest constituents of which are:

- a. Foraminifera which reach a maximum dimension of $100\ \mu$. Globigerinidae and fragments of larger globigerina. Textularidae.

The planktonic foraminifera predominate, very curious is *that larger individuals are absent*.

- b. Terrigenous fragments of lime: they are of the same order of size as the foraminifera, usually with sharp edges, therefore not distinctly marked off against the groundmass, many of them do not extinct homogeneously.

Minerologically they consist of aragonite or calcite, (discernment by means of MOHR's salt: $Fe\ SO_4\ (NH_4)_2\ SO_4\ 6\ H_2O$ ¹⁾).

- c. volcanic glass, subordinate, occurring in transparent pieces with many opaque spots; between crossed nicols, minerals with gray polarization-colours, probable feldspars, are visible.

- d. concretions of manganese-peroxide, numerous, as little spots or cloudy accumulations of irregular shape. The largest accumulation in the sections is $240 \times 135\ \mu$. Where the grains are not closely accumulated a brown colour is often visible. The manganese-peroxide encloses the ground-mass and fills also the chambers of the globigerina.

Qualitatively the manganese was analysed by dissolving the grains in a bead of salt of phosphorus and then performing the colourreaction by oxidation with KNO_3 .

¹⁾ K. KEILHACK, 7, p. 397. For the examination of preparations it is easier to apply this reaction than MEIGEN's, and moreover, it produces still distinct colours with objects of minute dimensions.

The reactions from which the conclusion was drawn that the manganese occurs as superoxide were:

1. evolution of Cl_2 from concentrated HCl .
2. evolution of O_2 from H_2O_2 .

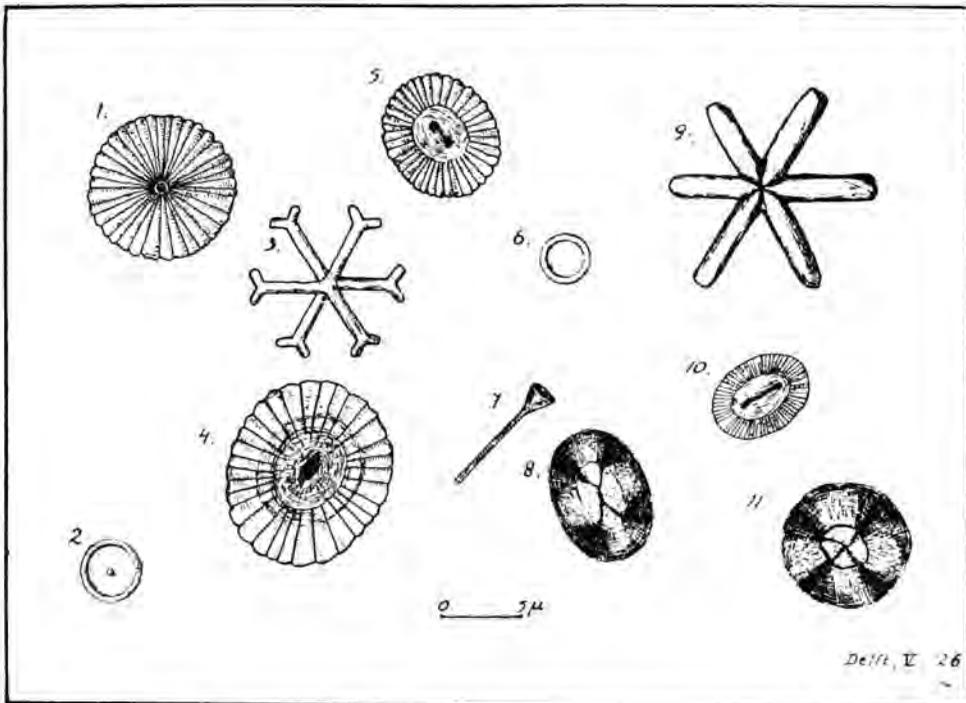
These reactions were performed in the laboratory of microbiology by Ir. C. B. VAN NIEL t. to whom the author feels greatly obliged.

B. The finer components of the ground-mass are:

- e. Coccoliths;
- f. Asterisks ("Disco-asters");
- g. Calci-spherulites;
- h. Grains of calcite;
- i. Grains of aragonite.

e. Coccolithophoridae Lohmann ¹⁾

circular, oval to elliptical disks, diameter 2—20 μ , most of them are about 10 μ and less. Thickness 3 μ and less.



Discoliths: fig. 2, 6.

Placoliths: fig. 1, 4, 5, 8, 10, 11 (8 and 11 are respectively 5 and 1 between crossed nicols).

Rhabdolith: fig. 7.

Disco-asters: fig. 3, 9.

Magnified 2000 \times .

¹⁾ H. LOHMANN 10, p. 147; J. SCHILLER 17, p. 282.

Discoliths fig. 2, 6.

not perforated, diameter till $4\ \mu$, thickness about $1\ \mu$, margin thickened. Sometimes with a button in the center fig. 2.

They are easily to be brought to the system of LOHMANN ¹⁾,

The most frequent of these have been pictured:

Pontosphaera huxleyi Lohmann fig. 6.

Pontosphaera pellucida Lohmann fig. 2.

Placoliths (Lohmann).

with singular perforation, fig. 1, 4.

with double perforation, fig. 5.

with irregular perforation, fig. 10.

The surface is striated radially ²⁾. LOHMANN's conception (loc. cit. page 114) that the margin of the placoliths is thinner than the center, was confirmed.

By rotating of the placoliths, the differentiation in basal and distal disk could be observed, these disks appear to differ in size, the plane of the disks is convex-concave.

The figures given in the sketch belong to the *Coccolithophora* Lohmann and are identified as:

Coccolithophora leptophora Murray and Blackman fig. 1.

Coccolithophora pelagica (Wallich) Lohmann fig. 4, 5, 10.

Rhabdoliths: It is remarkable that they occur in a very subordinate way. Fig. 7: *Discosphaera Thomisonii* Ostenfeld with a distinct axial perforation.

Structure of the Coccoliths: They consist of radially placed needles which diverge in the placoliths from the perforation or perforations; these fibres cause the radial striations on the disk, an adoption of radiating channels within the disk for their explication is superfluous ³⁾.

The optical orientation of the fibres is negative. Which of the other axes of elasticity lies in the disk-plane could not be determined. In the disk-plane the fibres are not placed perfectly radially, eventually caused by ramification of the fibres. This appears from the behaviour of the interference-figure when revolving the microscopic stage. Then the arms of the cross do not keep the same orientation. Fig. 11 is a *Coccolithophora leptophora* Murr. and Blackm., fig. 8 is a *C. pelagica* (Wallich) Lohm. between crossed nicols. In consequence of the structure of 2 disks the interference-figure of the disk-center often differs from that of the margin.

It is very probable that in space the aragonite-fibres do not diverge from one single center, as the spherulites, but from various centers of the disk-axis. In vertical section the extinction is not homogeneous ⁴⁾. By

¹⁾ H. LOHMANN 10, p. 147; J. SCHILLER 17, p. 282.

²⁾ These striations are only visible on the largest disks.

³⁾ A. VOELTZKOW 21, p. 483.

⁴⁾ This observation could only be made on the bigger placoliths.

optical way with the method of SCHROEDER VAN DER KOLK the mineralogical modification of CaCO_3 which form the coccoliths, could not be determined. The colouring with MOHR's salt, however, was positive for aragonite.

Nor do the rhaboliths extinct homogeneously in profile, so that probably they too are built up of fibres which in space are arranged analogically as in the coccoliths.

f. Asterisks. ("Disco-asters") fig. 3, 9.

These too consist of aragonite; to which organisms they belonged, cannot yet be decided. They will be treated in another paper.

g. Calci-spherulites.

Radiated in structure, irregular in external form, size 2–5 μ . First they were thought to be coccoliths, but they differ from these in:

1. compared with coccoliths of the same size they exhibit a higher interference-colour;

2. when rotated round an axis in a plane parallel to the stage, they appear to be no disks at all, the interference-cross *remains visible in all positions*;

3. in structure they are perfectly spherulitic, the interference-figure persist with the same orientation of the cross-arms as well during the revolution of the stage as on being revolved on an axis in the plane of the stage.

In general one can expect that the structure of crystalline products of chemical or biochemical extraplasmatic origin, especially, in the beginning of their formation, follows as accurately as possible the laws of crystallography, whereas the products of physiological-intraplasmatic origin in the first place aims at reaching a definite morphological ultimate shape.

The discernment between spherulites and coccoliths does not always run easily, they also consist of aragonite-needles, with negative orientation; perhaps here we have to do with the modification Vaterite¹⁾.

h. Grains of calcite.

By means of colouring with MOHR's salt they could be recognised and distinguished from:

i. Grains of aragonite,

also occurring in irregular forms, which often do not extinct homogeneously.

Remark: After dissolving the rock in respectively HNO_3 , HCl , acetic acid a residue was obtained, which absorbs organic colouring matters, obviously the lime contains some clay.

¹⁾ F. KLOCKMANN 8, p. 455 and p. 457. Below 29° C., from solutions of calcium-carbonate, calcite will always precipitate. Above this temperature aragonite, and in the presence of free bases as ammonia the spherulitic Vaterite. Aragonite, however, may also be formed below 29° C. especially in the presence of Magnesia-salts, e.g. in seawater.

Views on the various components of the rock.

To avoid repetition in the following statements the premise is made that in this rock diagenetic alteration, if it has taken place, must be very subordinate.

Phenomena of solution were not observed on the aragonite-components, whereas their minute dimensions and the porosity of the rock had made them highly accessible for chemical attack; if solutions had circulated in this rock those particals certainly would have acted as crystallization-centers, and so would have furthered a precipitation, but the fact that the disintegration of the sample was so easily done, as was previously stated, makes it evident that any posterior impregnation by solutions, saturated with calcium- or manganese-carbonates must be excluded.

While the tropical climate favours the chemical weathering the elevation of this rock must be of very recent date.

The occurrence with coralreefs points at a sedimentation in a shallow sea, notwithstanding the pelagic habit of the rock.

Concretions of manganese: MOLENGRAAFF¹⁾ in 1915 came to the following conclusion: "Nodules and concretions of manganese in general (therefore) are not characteristic of abysmal deposits in this way that from the occurrence of such concretions in a certain deposit, one would be justified in concluding that the deposit could be nothing else than an abysmal deposit . . ."

The chemical process of their growth is a very slow one, and the chance of finding them is inversely proportional to the rate of accumulation. About the velocity of the sedimentation of our rock, no data can be obtained of course, but it is certain that it was greater than of abysmal sediments.

In the discussion following the communication of his paper, MOLENGRAAFF on WICHMANN's remark, admits the possibility of the formation of manganese-nodules by biochemical processes, but at the same time he points out that until now the existence of bacterial life in abysmal depths has not been proved (pag. 428).

At this moment manganese-bacteria from the sea or from sea-mud, not to mention those from abysmal depths have not been described.

BEIJERINCK²⁾ demonstrated them in garden-mould, VON WOLZOGEN KÜHR³⁾ found them as hydrobios in dune-water and it is not improbable that they are represented in the halobios, a supposition which Prof. Dr. A. J. KLUYVER thinks very probable (verbal communication).

In this connection the attention is drawn to the following quotation:

1) G. A. F. MOLENGRAAFF 12, p. 418.

2) M. W. BEIJERINCK 2, p. 123.

3) C. A. H. VON WOLZOGEN KÜHR 21, No. 3, 4, 5.

“Vielleicht wird man aber annehmen, dass die Manganknollen ähnlich wie das Sumpferz durch Vermittelung von Bakterien ausgeschieden wurden, und dass sie sich dort in grösseren Mengen anhaufen konnten, wo die Lebensbedingungen für diese besonders günstig waren”¹⁾.

Without further commentary BEIJERINCK (on page 128) calls the grains of manganese-superoxide formed by bacteria: spherulites, they are perfectly round and have a rough surface, and, besides, they should include organic matter. They reach sizes to $350 \mu^2$.

Another possible way of formations is chemical oxidation of the mangano-ion dissolved in the seawater, especially as it is known that seawater reacts alkaline and that the oxidation of mangano-ions depends on the concentration of the hydrogen-ions in the medium.

But, if this process had taken place, the Mn-spherulites would have been arranged more or less in layers, and would not occur as dispersed spots. Post-genetical formation must be excluded for reasons mentioned above. The only possible origin of the accumulations in this rock is that they are a product of bacteria, which lived in the superior parts of the sedimentated mud.

Then the accumulations (“star-accumulations”) might be considered as the products of secretion of the former colonies.

MOLENGRAAFF's pronouncement: “Consequently concretions of manganese are in this manner characteristic of abysmal deposits that they may form an important percentage in proportion to other constituents exclusively in such deposits”, now can be stated precisely in this manner that the manganese-superoxide as: “important percentage in proportion to the other constituents” is only in this case an indication of an abysmal sediment, when it occurs in big grains: *non numerus granulorum sed magnitudo grani*. For, the quantity is as well a function of the favourable-ness of biological factors, whereas the magnitude, it being of no consequence whether the precipitation was caused by chemical or biochemical processes, in the first place, is ruled by the time available for *undisturbed* growth, thus is connected with the velocity of sedimentation of the other constituents of the rock; circumstances which, without any doubt, are as favourable as possible in abysmal depths.

Calci-spherulites. STEINMANN³⁾ asserts that the coccoliths occurring in biancone, lay in a ground-mass consisting of grains which should have a fibrous-crystalline structure. He asserts them to be “Verwesungsfällungskalk”. Besides from the decay of organic matter, calci-spherulites can be obtained in the laboratory in a purely anorganic way. (HARTING⁴⁾)

¹⁾ J. MURRAY and E. PHILIPPI 14, p. 190.

²⁾ On the analogy of BEIJERINCK's nomenclature the MnO_2 -grains in this rock, ar called spherulites.

³⁾ G. STEINMANN 18, p. 444–445.

⁴⁾ P. HARTING 5, p. 1–84.

and see quotation on page 1099) Physical-chemical precipitations of lime are not impossible in nature¹⁾.

It cannot be decided in which of these ways the calci-spherulites in this rock have been formed. A sure indication of the biochemical formation is the occurrence of organisms, whereas there are no terms to consider a co-existing physical-chemical "planktogeneus" formation as impossible.

ANDREE²⁾ makes a difference between "Verwesungsfällungskalk" viz. lime issued from precipitation in sea-water by the action of "decay"-products and "physiologische Fällungskalk" viz. lime, which is formed in direct connection with the physiology of living organism, in the first place of bacteria.

Other geologists too, accept on DREW's authority that there exist bacteria which may be brought to a physiological group of: "Calcium-carbonate-bacteria", on the ground of the fact that the production of calcium-carbonate is to be reckoned among the direct physiological processes of those bacteria.

However, it ought to be mentioned, that evidently this conception is not agreed with in biological circles. So the well known bacteriologist MOLISCH gives the following definition of "calcium-carbonate-bacteria": "Die Bacterien erzeugen entweder aus Eiweisz, ihren Derivaten oder Nitraten Ammoniak, dieses verbindet sich mit der im Wasser gelösten Kalksalzen zu Kohlensäuren Kalk, der entweder für sich allein oder mit Phosphorsäure zusammen in Form der Sphärite erscheint³⁾.

It is evident that the difference between: "Verwesungsfällungskalk" and "physiologische Fällungskalk" mentioned above, *has no reason to be kept*, a conclusion backed by Professor Dr. A. J. KLUYVER.

Bacteria which produce lime were met with by DREW in great numbers in the surface-water and in the mud of the sea of Florida, they only prosper in warm seas and in the most florid way in depths less than 100—200 fathoms⁴⁾.

In the mud in loco VAUGHAN⁵⁾ found calci-spherulites of 4—6 μ . HARTING's and STEINMANN's⁶⁾ experiments might cause the supposition that the formation of spherulites, is only possible in a viscous medium. Professor Dr. A. J. KLUYVER kindly communicated to me: "A viscous medium is *certainly* not necessary for the formation of calci-spherulites, *perhaps not* for the manganese-spherulites.

A further indication of the conditions during the sedimentation of this rock gives the following quotation from MOLISCH's paper (loc. cit. page 135): "Die Menge des Ammoniaks muss eine gewisse Höhe erreichen,

¹⁾ ARN. HEIM 6, p. 30—32.

²⁾ K. ANDREE 1, p. 185—191.

³⁾ H. MOLISCH 11, p. 130—139.

⁴⁾ H. DREW 4, p. 35.

⁵⁾ F. W. VAUGHAN 20, p. 53.

⁶⁾ G. STEINMANN 19 p. 40—45.

wenn die Fällung des CO_2 Ca erfolgen soll". Strong oceanic currents are consequently not to be expected.

HARTING, STEINMANN ¹⁾ and BEIJERINCK (loc. cit.) found in the spherulites from their experiments an organic matter with properties of conchyoline. It is possible that part of the residue, obtained by dissolving the rock, and which was considered as clay, may be conchyoline.

Coccolithophoridae ²⁾ are uni-cellular Flagellata which belong to the nanno-phyto-plankton. They avoid sweetened sea-water and are met with in tropical and temperate seas. According to MURRAY and RENARD *Rhabdosphaera* and *Discosphaera* Lohm. would have their greatest development in equatorial seas. *Coccolithophora* Lohm. in the seas of the temperate zones.

Curious is that in this sediment, though certainly formed in the tropics *Rhabdosphaera* and *Discosphaera* occur very subordinately.

The origin of the rock: ³⁾ About the young-tertiary Globigerina-rocks of Rotti, BROUWER (loc. cit. page 76) writes: . . . "achten wij het waarschijnlijk dat ze — hoewel in samenstelling veel overeenkomst vertoonend met sommige afzettingen van recent globigerinenslib in diepzee — niet op zeer groote diepte, maar in een zee, waarin misschien koraaleilanden aanwezig waren, zijn afgezet. Wij kunnen b.v. denken aan soortgelijke afzettingen als de kalkslibvormingen in lagunen van koraalriffen en die welke ontstaan uit met koraalmelk beladen, „witte water", dat na stormen tot verscheidene kilometers afstand van koraalriffen voorkomt, waaruit het kalkslib gelijktijdig met de globigerinenschalen is bezonken.

Of, de globigerinengesteenten kunnen zijn afgezet op een dicht onder de zeeoppervlakte gelegen rug, die door diepere zeeën met gunstige levensvoorwaarden voor planktonische foraminiferen was omgeven" ⁴⁾

In literature a recent mud of the composition of this rock is not known. Of the fossil limestones the limes of Albrada, a raised atoll in the Indian Ocean ⁵⁾ are to be compared with this rock. But VOELTZKOW

¹⁾ G. STEINMANN 19, p. 40—45.

²⁾ H. LOHMANN loc. cit. J. SCHILLER loc. cit.

³⁾ In the microbiological parts the author's thanks are due to Prof. Dr. A. J. KLUYVER and Ir. C. B. VAN NIEL, to whose better judgment the questions were submitted.

⁴⁾ " . . . we think it probable that — in spite of the great resemblance of their composition with some sediments of recent globigerina-ooze in the deepsea — they were not sedimentated in very great depth, but in a sea, where perhaps coral-islands were present.

For instance, we can think of analogous sediments as the formations of calcareous-mud in lagoons of coral-reefs and those which issue from "white-water" loaded with coral-milk that after storms occurs up to several kilometers' distance from coral-reefs, from which the calcareous-mud simultaneously with the shells of the globigerina were sedimentated.

Either, the globigerina-rocks were formed on a ridge situated close to the sea-surface and surrounded by deeper seas with favourable conditions of life for planktonic foraminifera".

⁵⁾ A. VOELTZKOW 21.

still owes an explanation for the occurrence of the coccoliths in "pure culture".

BROUWER's first explication is most probable for this rock, but under very circumstances.

The geological occurrence in connection with coral-reefs is an indication of a formation in shallow water: the fine-granularity, the occurrence of calci-spherulites for a quiet medium, circumstances which are to be found in the *most quiet* parts of the lagoon of an atoll, far from the entrances to the open ocean and which in this special case were met with far from the shores of the lagoon. So the rock represents the finest "washings" of coral-mud, brought by weak currents, stronger sea-currents which could transport coarser material do not occur or are very scarce. The accumulated coral-mud consisted of fine detrital lime issued from the atoll, of coccospheres and little foraminifera. They could settle in this part of the lagoon, the organisms died, the dead albumen-matter were putrefied by bacteria, the issuing ammonia reached a concentration necessary for the precipitation of calcium-carbonate, besides, the manganese-compounds in the volcanic ash were transformed in manganous-carbonate. In this part of the lagoon probably only bacterial life was possible, all organisms occurring in this rock must be originated from individuals not living in situ, a conclusion drawn from the absence of greater foraminifera. ¹⁾

The organisms, bearers of the "Disco-asters" must not be considered as a form adapted to these special conditions, they are common in the group of the soft globigerina-rocks, which represent more normal coral-mud: (e.g. 178 \times from Bebalain).

This rock-abnormity can be nothing but a local facies.

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¹⁾ For the "azoism" the denitrifying bacteria could be hold responsible (BRANDT's hypothesis). The flora and consequently the fauna too, due to the resulting scanty percentage, eventually, the total absence of nitrates.

The circumstance that the waterrefreshment is very defective makes comprehensible that the bacteria consume the nitrates almost completely.

It is possible too that this part of the lagoon runs dry when the tide is low, but for this I have no indication. (e.g. mud-cracks).

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Laboratory of Stratigraphical Geology and Palaeontology.

Delft, June 1926.

Physics. — *"Influence of the Pressure on the Electric Conductivity of Platinum."* (15th communication of results obtained by the aid of the VAN DER WAALS Fund.) By A. MICHELS and P. GEELS. (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of May 29, 1926).

The desire to improve the temperature measurements in the determinations of isotherms induced us to make new measurements of the influence of the pressure on the electric conductivity of platinum.

In the earlier measurements of isotherms a doubt had risen on the question whether the temperature within the piezometer in which the gas to be examined was compressed, was really the same as that of the thermostat surrounding the piezometer. This doubt was founded on some small deviations in the results of the measurements. A motive for the suspicion that we should have to do here with a difference of the two temperatures, was obvious.

In the earlier arrangement, only that part of the piezometer and its protecting casing jacket reached into the thermostat, where the capillary form of the piezometer began, and the measurements actually were made. The remaining part, which represents a considerable weight of steel and mercury, remains outside this thermostat. As the mercury in this reservoir forms a continuous column with the mercury in the capillary and the much greater quantum of the reservoir is permanently in contact with the outer air, it was quite possible that a continuous transmission of heat took place along this mercury column. Moreover along the inner side of the steel protecting jacket a conduction of heat would be able to prevent a heat equilibrium from ever being established.

A preliminary determination, made in the most favourable case for reaching equality of temperature between compressed gas and thermostat with difference of temperature with the outer air of about 40 degrees gave deviations of $\frac{1}{50}$ degree already.

These observations were arranged as follows (fig. 1) : the head of an old measuring capillary was blown into a bulb and then opened at the top (*a*), after which the capillary was placed in the usual way in its protecting jacket, with this difference, however, that the upper closure of this jacket had been removed. Then the reservoir of a BECKMANN thermometer was stuck into the bulb. Now the mercury in the capillary is just made to rise so as not to touch the thermometer. The thermostat, which surrounds the protecting tube in the usual way (*b*) is brought to the desired temperature, and the difference between this temperature and that of the gas in the bulb is measured. The results were as given above, and this, too, in the

case in which the mercury stood as high as possible in the capillary. At a lower level we should certainly have to expect greater differences.

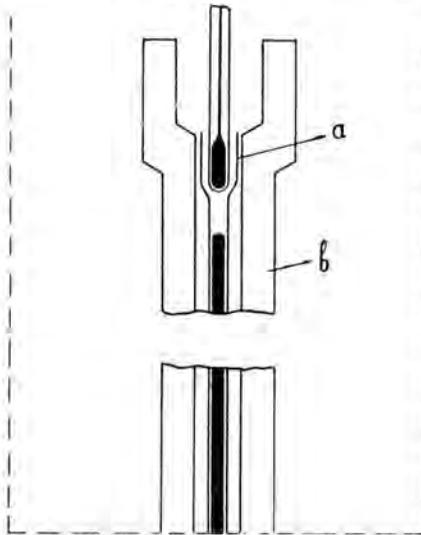


Fig. 1.

Though it is obvious that the above-mentioned measurements can give an insight into the existence and the order of magnitude of the errors made in the isotherm determinations, no accurate determination of errors could be expected in this way. For this purpose it would be necessary to determine the temperature within the piezometer *during* the pressure measurements. If this is possible, there are still two ways open for isotherm measurements. Either keeping the same instruments and using the internal temperature as measured in the calculations, or examining by the aid of an interior thermometer in order the instruments have to be modified to reach a sufficient

equality of the inner and outer temperature. Which solution is to be preferred does not enter into the scope of this communication.

A second motive for the construction of an interior thermometer was the desire to know how much time adiabatic variation of temperature (consequence of variations of pressure) required to be absorbed in the thermostat and to know how rapidly the inner temperature follows the fluctuations of the temperature of the thermostat, which fluctuations can never be entirely avoided.

For the measurement of the inner temperature it seemed to us the most practical course to make use of a *Pt*-resistance thermometer. In this we are, however, at once up against the difficulty that the pressure also exercises an influence on the resistance. As regards the extent of this influence there exist only a few, and what is worse discrepant data. The authors who furnish numerical data are E. LISELL ¹⁾, M. A. LAFAY ²⁾ and P. W. BRIDGMAN ³⁾. The first states that he has found

$$\alpha = 1,77 \times 10^{-6} + 3,85 \times 10^{-11} p$$

(translated by us into KG. per cm³. from atm.), whereas LAFAY gives

$$\alpha = 1,80 \times 10^{-6} \text{ in KG. per cm}^2$$

and BRIDGMAN :

$$\alpha = 1,90 \times 10^{-6} \text{ to } 1000 \text{ atm. and at } 12000 \text{ atm. } 1,77 \times 10^{-6}.$$

The last observer, who draws attention to the lack of harmony between

¹⁾ Thesis Upsala 1902.

²⁾ Ann. de Chim. et de Phys. 19 (1910) 289.

³⁾ Proc. Am. Acad. 52 (1917) 573.

the two first, reduces LISELL's values from atm. to kg. cm²., whereas he does not do the same with LAFAY's figures. Thus he overrates the difference considerably. It must be admitted that LAFAY's communication on this subject leaves something to be desired as regards clearness.

It does not seem desirable to base a thermometer on the above-mentioned data. Besides, since the improvements in the method of the measuring of pressure we were able to work with more accuracy and to express our data in absolute units. These two considerations led us to a new determination of the influence of the pressure on the *Pt*-resistance, at first, however, only to 250 atm., as we required these results first of all.

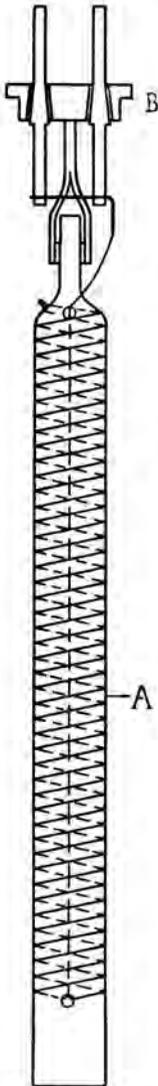


Fig. 2.

Method of Measurement.

A preliminary calculation¹⁾ gave as most favourable dimension of the wire one of about 0.2 mm. with a length of from 5 to 6 m. This wire may be charged with a current of 0.004 amp. without danger of the temperature rising higher than 0.001 degree. The current as used in the experiment always remained below this value.

First this wire was heated for 10 min. at about 400°, and then wound bifilarly round a glass insulator (fig. 2) and suspended from a steel plate *B*. Through this steel plate four insulated transits were made. In our previous measurements the insulations were always made of ivory; here, however, ivory appeared not to possess a sufficient insulating power, especially at somewhat higher temperature. After having tried other insulating substances, amber at last proved successful. The insulation resistance lay mostly between 100 and 2000 mega ohm. In order to render a potentiometric method of measurement possible, four transits were applied. The two pairs of copper cores are connected by a *pt* wire 1 mm. thick at the lower side of the plate. The wire which has to be examined, is welded on to this *pt* wire. By this arrangement all the contacts are brought into the pressure body and close together. This diminishes the chance of thermophenomena and of the influence of a onesided compression strain at the places of contact.

After the thread has been wound, it is once more gently heated in an oven to about 150 or 200° in order to eliminate possible tensions that have arisen through the winding and to prevent later thermal after effects.

The glass insulator with the *pt*-wire is placed in a tube-shaped pressure reservoir, which is closed at the top by plate *b*. A swivel presses *b* on the upper side of the

¹⁾ A. MICHELS. Thesis, Amsterdam, 1924.

reservoir and this, together with a packing ring, brings about a closure free from leakage. The whole reservoir is further filled with oil, which is carefully examined as regards its insulating power.

As it was our intention to measure the influences of the pressure to an accuracy corresponding to a temperature determination accurate to a 0.001 degree, it was necessary to take certain precautions, which exclude differences of temperature of this order. For this purpose a second platinum wire as far as possible of the same thickness, length, and suspension was put in to the thermostat in which the pressure reservoir was placed. This wire was, however, never exposed to pressure.

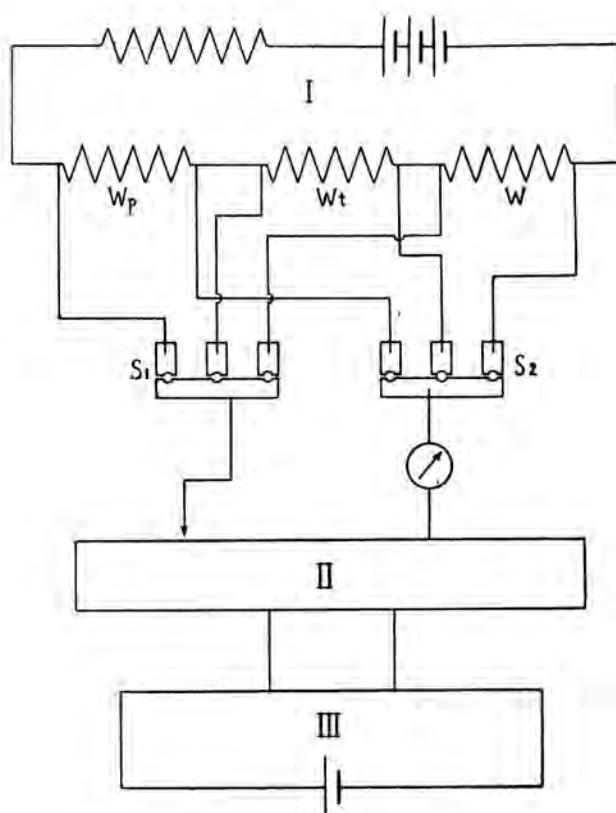


Fig. 3.

Before and after each measurement of the wire under pressure the second wire was measured. (Each series of measurements consisted of 3 such series and extends, therefore, in all over 9 measurements. These measurements took about 10 minutes.) It is possible to derive the required temperature correction for the first resistance from the values obtained from the second. In this of course should be supposed that the two thermometers possess the same inertia. This can, however, be accomplished by also placing the second wire in a casing of about the same thickness.

The electric method measurement consisted in a potentiometric replacement method. The arrangement is schematically represented in fig. 3. Here Wp is the first, Wt the second wire, W being an accurately gauged resistance box. S_1 and S_2 are switches, which enable us to connect Wp , Wt , and W successively to the potentiometer circuit II. The current for this circuit is provided by the shunt circuit III.

In the measurement of e.g. Wp the switches are adjusted in such a way that this resistance is switched on to the potentiometer, and the latter is adjusted to a minimum deviation of the galvanometer. Then W is coupled on the potentiometer, and the resistance, which is equivalent to Wp is determined. If W is not adjusted beforehand at about the right value, so that we have to modify it, the current in the resistance circuit changes a little in consequence: hence a second measurement becomes necessary for Wp .

By a suitable choice of the different resistances the whole operation can, however, be made exceedingly simple, and when e.g. beforehand W has been fixed at 0.01 Ohm, the whole measurement can be confined to a reading of the small galvanometer deviations, the fixing of the scalar value, and the changes in the switches S_1 and S_2 .

To lessen the thermo-couples the whole complex of resistances was placed in a separate isolated space, and all the supply leads and contact knobs were made of the same copper.

The results obtained are given in the subjoined table. The pressure is given in kg. per cm^2 . Under each pressure is found the total decrease of resistance per Ohm of initial resistance at the temperature in question. A second table states the mean decrease of resistance per kg. between every two successive observations. In addition figure 4 gives the graphical course.

All the results show a great difference from the data published up to now, in so far as the pressure coefficients vary more considerably with the pressure.

$$\frac{\Delta w \times 10^6}{w}$$

| Pressure in kg. per cm^2 . | 51.5 | 101.5 | 151.5 | 201.5 | 251.5 |
|-------------------------------------|------|------------------|------------------|-------|------------------|
| 15.57° | 185 | 340 | 465 | 568 | 661 |
| 21.95° | 191 | 351 | 484 ⁵ | 602 | 708 ⁵ |
| 34.75° | 196 | 364 ⁵ | 510 | 629 | 739 |

Besides, at the lower pressures the coefficient is to a much greater extent variable with the temperature.

These differences can be accounted for. All earlier observers have worked

$$\frac{\Delta w \times 10^6}{w \Delta p}$$

| Pressure range in kg. per cm ² . | 0—51.5 | 51.5—101.5 | 101.5—151.5 | 151.5—201.5 | 201.5—251.5 |
|--|-------------------|------------|-------------|-------------|-------------|
| 15.57° | 3.68 | 3.11 | 2.50 | 2.05 | 1.86 |
| 21.95° | 3.70 ⁵ | 3.20 | 2.67 | 2.25 | 2.03 |
| 34.75° | 3.80 ⁵ | 3.37 | 2.92 | 2.37 | 2.20 |

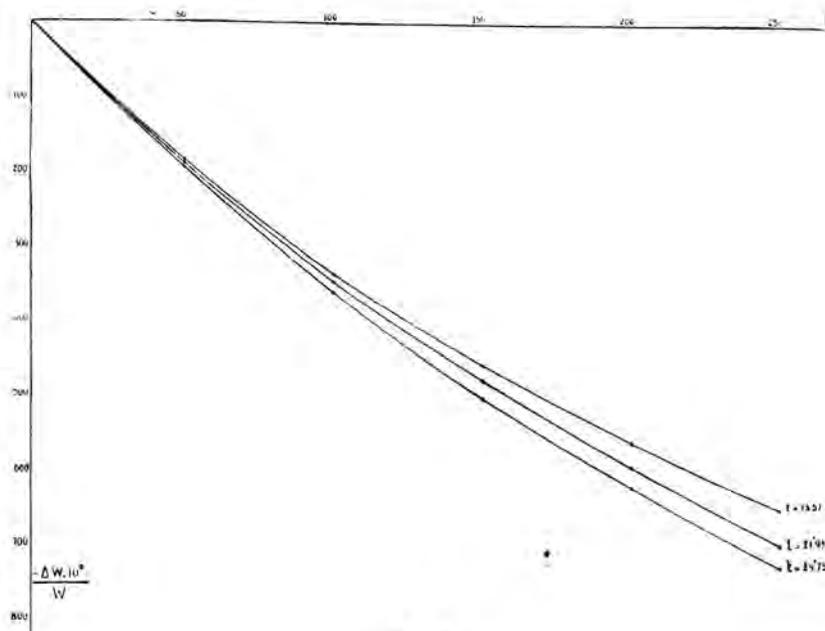


Fig. 4.

with much greater pressure intervals. Thus BRIDGMAN's first interval is 1000 atm. Nor does LISELL mention any observations below this pressure. Of LAFAY this cannot be ascertained, since he mentions only interpolation values ascending with 500 atm.

Now it may be very well possible that if these observers had worked with lower pressures, they would have found values more in harmony with ours. At higher pressures our values show a tendency to agree more with the coefficients known. In how far they are really in harmony with them, cannot yet be decided, because on the data given no extrapolation is possible up to e.g. 1000 atm.

If they actually agreed, it would also appear that a development of the resistance into a function

$$W = W_0 + ap + \beta p^2$$

as has been tried among others by LISELL, is inadmissible. For a higher pressure the resistance shows a tendency towards an asymptotic course, whereas according to the equation the quadratic term would begin to dominate. This is the ground, why we have not tried to draw up a function on the values that we have now at our disposal. Small differences might still be accounted for by the greater accuracy with which we might measure our pressures, while a third cause may be found in a possible difference in the purity of the metals, the *Pt* used by BRIDGMAN and that employed by us not being supplied by the same purveyor, BRIDGMAN, too, was of opinion that the difference between his values and those of LISELL was due to a difference in purity. He supposed LISELL's material to contain 0.28 % of Iridium.

This, however, gives differences of only a small percentage, so that here too not so much influence may be assigned to this as to account for the whole difference. At first, when we did not yet suspect such a great difference in the observations, it was only our intention to construct a thermometer which was able to resist pressure. Hence no attempt was made to obtain material from the same source from which the other observers obtained theirs. Our observations enable us fully to fulfil our first intention. It has, however, also proved necessary to determine anew the influence of pressure on the conductivity of metals with smaller pressure intervals than those with which earlier observers worked. The preparations herefore are already in a pretty advanced stage. Measures have also been taken which we hope will allow us to attain greater accuracy in our electric measurements, in order to be also able to profit by the greater accuracy of our pressure measurements. Of course the greatest care shall be taken to obtain material of purity as high as possible.

Botany. — “Concerning the sensibility of decapitated coleoptiles of *Avena sativa* for light and gravitation.” By H. E. DOLK. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of May 29, 1926).

As the conceptions of the various investigators about the influence of the tip of the coleoptiles of *Avena* on the phototropical and geotropical curvatures differ in some respects (see the historical review of NIELSEN (3)) I thought, it would be interesting to investigate this question further.

For this purpose I used the observations of ROTHERT (5) about the regeneration of “a new physiological tip” in decapitated seedlings. In this paper I purpose to give an account of the results of my experiments.

By the removal of the tip the uppermost zone assumes the properties of the tip, which it did not possess before. ROTHERT has shown this for phototropical sensibility. These experiments were repeated and even extended to geotropism.

The question is, whether the transmission of the stimulus is generated by the formation of specific “stimulus substances” (STARK 7) or by a change of the concentration of the growth accelerating substances already present, (PAAL (4) and others). SÖDING (6) has observed that some time after the decapitation the uppermost zone of the coleoptile again produces growth accelerating substances. If there is an immediate connection between the transmission of the phototropical stimulus and the production of the growth accelerating substances, then the phototropical sensibility has to return at exactly the same time as the growth acceleration in decapitated coleoptiles.

In order to ascertain this one ought to examine accurately :

1. The moment when the growth increases after the decapitation.
2. Whether the growth-accelerating is caused by the growth-accelerating substances or attributable to a temporary influence of the wound.
3. The moment when the phototropical sensibility returns.

Before measuring the growth I made some experiments in order to determine the influence of a regenerated tip on the growth of a stump. For this purpose the seedlings were decapitated (± 4 mm). After twelve hours the new physiological tip is regenerated. Two millimeter of this tip were then cut off and placed unilaterally on the coleoptile stump. Seventeen out of 23 plants curved negatively, 2 positively and 4 remained straight. (The curvature is positive when the plant curves towards the side on which the ring is placed.) This experiment shows that the new regenerated tip produces growth accelerating substances. If the uppermost

coleoptile ring was cut off immediately after the decapitation and placed unilaterally on the stump, then the stumps only curved slightly. NIELSEN (3) also observed slight curvatures in this case.

The growth of decapitated plants was measured and compared with the growth of the plants on which the tip was replaced. It was therefore necessary to examine whether the tip retains its normal properties under these circumstances. The plants were decapitated and the tips replaced on the stumps. After twelve hours the tips were removed and placed unilaterally on the same stumps. Out of 52 plants 5 curved positively, 5 negatively, while 42 remained straight. Some control experiments were made to examine whether the stumps were still able to curve. All the stumps curved very strongly, when tips, that were just cut off, were placed unilaterally on them.

Therefore we may conclude that a tip, placed on a stump, loses its activity after some time.

Is there any regeneration of the tip in a stump on which a tip is replaced? When the tip was placed on the stump for 6 hours no regeneration took place. The stumps did not curve at all, when the uppermost ring was cut off and placed unilaterally on them. This experiment was repeated after the tips had remained 9 hours on the stumps and then a slight regeneration did take place. When the coleoptile rings were placed unilaterally on the stumps these curved away from the ring. Of 40 plants 18 curved negatively, 2 positively and 20 remained straight. If we compare these curvatures with those of the first experiment it is obvious that under these circumstances the regenerated tip produces far less growth-accelerating substances than if the tip had not been replaced on the stump.

In order to determine the exact moment at which the new physiological tip appears, it was necessary to measure the growth of coleoptiles after decapitation.

The increase in length was measured every hour by means of a cathetometer. Fig. 1 demonstrates the average growth of 18 decapitated plants.



Fig. 1. Growthresponse after decapitation. At ↑ the plants were decapitated. The abscissa represents time in hours, the ordinate growth in 10μ per hour. Temp. 21° C.
The average growth of 18 coleoptiles.

This curve shows that just after the decapitation the growth decreases slowly, then more rapidly and after 120—180 minutes there is a marked

minimum. The growth then increases steadily, but never reaches its former intensity. The growth of some decapitated coleoptiles was registered automatically by the auxanometer of KONINGSBERGER (2) in order to get exact data for the minimum. It turned out that the minimum was reached 150 minutes after the decapitation.

The growth of the plants on which the tips were replaced was also measured by means of the cathetometer.

It was found that the growth decreased gradually and in about 9 hours a minimum was reached.

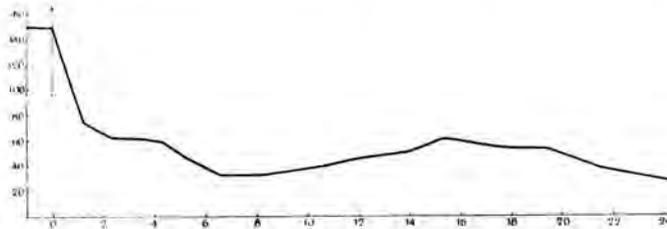


Fig. 2. Growthresponse of coleoptiles, upon which the tip was replaced. At ↑ the tip was cut off and placed on the stump. The abscissa represents time in hours, the ordinate rate of growth in 10μ per hour. Temperature 21° C. Average growth of 15 coleoptiles.

If we compare this curve with that of the decapitated plants, then we find that the minimum, appearing 150 minutes after decapitation, is not due to a transitory effect of the wound.

The following experiment shows that 180 minutes after decapitation the uppermost zone of the coleoptile produces growth-accelerating substances which it did not do before.

TABLE I.

| Number of coleoptiles | Decapitated at: | Upper most part: placed unilaterally | Result at: | + | - | Straight |
|-----------------------|--------------------------|--------------------------------------|------------|---|--------|----------|
| 3 | 0 h. p.m. | 5. ³⁰ p. m. | 9 h. p.m. | 0 | 2 | 1 |
| 4 | 2. ³⁰ h. p.m. | " | " | 0 | 3 | 1 |
| 3 | 4 h. p.m. | " | " | 0 | 0 | 3 |
| 3 | 5. ³⁰ h. p.m. | " | " | 0 | 1 (st) | 2 |

In order to examine when the geotropical sensibility returns a number of coleoptiles were decapitated and then placed horizontally. It turned out that the curvatures showed quite a different progress from the normal geotropical curvatures.

The curvatures appeared first in the base of the coleoptile and then moved to the tip. They were probably caused by the slight geotropical sensibility of the base.

On the other hand the coleoptiles curved quite normally when they were placed horizontally six hours after the decapitation. First the tip of the coleoptile curved, then the base while the upperpart straightened. It is apparent from the above described experiments that regeneration takes place here.

Some of these coleoptiles were decapitated for the second time before placing them horizontally. These coleoptiles curved in the base only, just as those placed horizontally immediately after the first decapitation. It was impossible to determine exactly the moment at which the geotropical sensibility returns, because the perception of the base could not be excluded. It is not difficult to exclude this perception with unilateral illumination. I illuminated only 2 mm of the tip of the coleoptile by using small screens of tin.

So it is much easier to determine the moment at which the phototropical sensibility returns. According to ROTHERT (5) the phototropical sensibility should return 3—6 hours after decapitation. These data are not exact as ROTHERT has not excluded the perception of the base. It is obvious that if a new tip regenerated in the uppermost part of a decapitated coleoptile this zone will become more sensitive to light. It was possible to illuminate exactly the same zone (3—4 mm below the tip) on intact coleoptiles by means of small tubes in which a narrow slit was made.

According to ARISZ (1) the maximum for the first positive curvature can be obtained by a quantity of light of about 250 Lux. I illuminated unilaterally decapitated and intact coleoptiles during 20 sec. with an intensity of 12.5 M.C. No curvatures were visible. The new physiological tip produces comparatively little growth-accelerating substances, therefore a reaction is obtained sooner by a long illumination with a small intensity, than by a short one with a high intensity.

The intact coleoptiles did not curve at all after illuminating the zone 3—5 mm below the tip during 60 min. with 3 M.C. This agrees with the results of WILSCHKE (8) who determined the threshold at 20000 Lux. A number of coleoptiles were decapitated and a few hours afterwards the

TABLE II.
Temperature 21°. Relative moisture 60—70 %.

| Number of coleoptiles | Decapitated at: | Illuminated at (3 M.C.) | Result at: | + | — | Straight | Positively |
|-----------------------|-----------------|-------------------------|---------------|----|---|----------|------------|
| 37 | 11.50 h. a. m. | 0.20-1.20 h. p. m. | 3.10 h. p. m. | 3 | 0 | 34 | 8 % |
| 36 | 2.15 h. p. m. | 3.15-4.15 h. p. m. | 5.50 h. p. m. | 2 | 0 | 34 | 6 % |
| 47 | 10.50 h. a. m. | 0.20-1.20 h. p. m. | 3.10 h. p. m. | 9 | 0 | 38 | 18 % |
| 58 | 11.45 h. a. m. | 2.15-3.15 h. p. m. | 5.15 h. p. m. | 41 | 0 | 17 | 71 % |
| 83 | 3 h. p. m. | 7.15-8.15 h. p. m. | 9.15 h. p. m. | 67 | 0 | 16 | 81 % |

uppermost zone (2 mm) was illuminated unilaterally with the same quantity of light. One hour after illuminating all the coleoptiles curved towards the illuminated side. This experiment shows that after decapitation the uppermost zone of the coleoptile is more sensitive to light than it was before. In order to determine at which moment this sensibility appears, the coleoptiles were illuminated at different intervals after decapitation.

It is apparent from Table II that 150 min. after decapitation the uppermost zone has become sensitive to a quantity of light, which is not the case with the same zone of the intact coleoptiles.

Comparing this with the result of the growth measurements we see that there is a complete agreement between them. The growth acceleration also appears after 150 minutes. Then growth accelerating substances are produced again by the uppermost zone whereas the same zone of the intact coleoptile does not do so. These experiments very strongly favour the conception that light immediately acts upon the production of the growth accelerating substances.

Therefore we have to assume that photochemically light increases or retards a process which may also take place in the dark.

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Botanical Laboratory.

Utrecht, May 1926.

Botany. — "*Phototropical curvatures of seedlings of Avena which appear when reaction of the distal side is excluded.*" By H. RAMAER.
(Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of June 26, 1926).

The phototropical curvatures which appear when seedlings of *Avena* are illuminated unilaterally with different quantities of light, have been carefully studied by ARISZ (1). He found that a positive curvature appears with 1.4—4000 M.C.S., a negative curvature with 4000—70000 to 120000 M.C.S. and a second positive curvature with still greater quantities of light.

It has been tried in several different ways to explain the phototropical curvatures by a difference in the rate of growth between the illuminated and unilluminated sides of the coleoptile.

BOYSEN JENSEN (2) thought that he could conclude from his investigations about the locality of the conduction of the phototropical stimulus that the positive curvature is the result of an increase in the rate of growth of the unilluminated side. This increase would be induced by substances, formed, owing to the unilateral illumination, in the unilluminated side of the tip and diffusing through the distal side to the growing zone.

PAAL (3) showed that *Coix* under normal circumstances always forms growth-accelerating substances in the tip and he explained the positive curvature by a decrease in the rate of growth of the illuminated side as a result of a onesided decrease in the production of growth-accelerating substances.

BLAAUW (4) sought connection between the phototropical curvatures and the lightgrowth-responses studied by him.

According to his theory, every part of a phototropically sensitive organ is able to show a growth-response to the quantity of light it receives, so that a phototropical curvature may be the result of the combinations of these growth-responses.

Bij means of this theory VAN DE SANDE BAKHUYZEN (5), later VAN DILLEWIJN (6) more accurately, could wholly explain the curvatures determined by ARISZ.

In consequence of a recent publication of BOYSEN JENSEN and NIELSEN (7), in which they again bring forward arguments in favour of the theory of BOYSEN JENSEN, I want to compare the three theories with each other and discuss them in connection with some recent investigations of C. VAN DILLEWIJN and F. W. WENT and with the results I obtained by the method of BOYSEN JENSEN and NIELSEN.

In their first experiment they made a median section in the tip of a coleoptile and placed a little screen in the cut. Round the base they placed

a small tube and then illuminated one half of the tip with an intensity of 25 M.C. for an indefinite period (2—3 hours). They found no curvature, but when the light fell in the direction of the screen a curvature appeared. Evidently the wound had no effect on the sensibility of the tip.

This result was an argument in favour of the theory of BOYSEN JENSEN. The same can be said of the following. The tip of a coleoptile was cut off and replaced by two sidely placed tips, between which a small screen was put. When one of the tips was illuminated unilaterally ($2\frac{1}{2}$ hours \times 25 M.C.) the base showed a negative curvature which necessarily was the result of an increase in growth of the proximal side induced by the distal side of the illuminated tip.

I repeated both experiments, although chiefly the first, because the second can afford too many mistakes, for instance individual differences in the production of growth-accelerating substances, the one tip perhaps drying up more rapidly than the other, the contact with the base being unequal etc.

The tip was cut into two by putting a needle through the plant a few m.m. below the tip and then moving it upwards. With a little practice the tip can be accurately split into two in this way. In the cut a little black screen was placed.

The results with on-sided illumination were as follows :

1. After on-sided continuous illumination of the one half of the tip with 25 M.C. for four hours, the base shows a positive curvature.

This curvature is very slight which may be due to the action of the wound, because plants with a cut but without a screen and illuminated for the same time show the slight curvature while intact plants, as a control experiment, show a maximal curvature in the direction of the light.

As working with continuous illumination appeared to me to be an objection, because so little is known about the all-sided continuous illumination, I decided to illuminate with certain definite quantities of light.

2. An all-sided illumination with 800 M.C.S. causes a decrease in growth.

If I illuminated the plants unilaterally with 800 M.C.S. after they had been kept in the dark for five hours after the operation (in order to loose the effect of the wound) a positive curvature appeared which reached its maximum after two hours (fig. 1).



Fig. 1. Curvature after illuminating with 800 M.C.S. (10 sec. \times 80 M.C.) from the right hand side. By means of a little screen light is prevented from reaching the base.

This curvature was shown by 80 % of the plants while 20 % remained straight.

From this result a few very important conclusions may be drawn.

First, that one half of the coleoptile is in itself able to show a light-growth-response, which corresponds with the decrease in growth, found by VAN DILLEWIJN, after allsided illumination with 800 M.C.S. and therefore wholly in accordance with the theory of BLAAUW.

Further, that PAAL was right when he assumed this positive curvature to be the result of a decrease in growth of the proximal side. I must draw attention here to a recent research by WENT (8) who showed convincingly that the production of growth-accelerating substances in the tip decreases considerably with quantities of light of 800 M.C.S.

Finally the theory of BOYSEN JENSEN has been shown to fail in the area of the first positive curvature.

Besides, how could it be possible, as the latter assumes, that the distal side shows an increase in growth when the tip has been illuminated unilaterally with 800 M.C.S.? In this case illumination of both sides, and also allsided illumination with 800 M.C.S. would have to cause an increase in growth of the whole coleoptile which is a flat contradiction of the facts.

3. With his last experiments VAN DILLEWIJN (9) found an increase in growth with 80000 M.C.S.

At the same time WENT found that with 100000 M.C.S. the production of growth-accelerating substances increases considerably in the tip.

When I illuminated the seedlings which had been treated in the above mentioned manner, unilaterally with 80000 M.C.S., 90 % showed a negative curvature after one hour (fig. 2).



Fig. 2. Curvature after illuminating with 80000 M.C.S. (100 sec. \times 800 M.C.) from the right hand side.

This time corresponds with that after which takes place an increase in growth after allsided illumination with 80000 M.C.S.

When we discuss shortly the acquired state of affairs then it is proved here that when a coleoptile is illuminated unilaterally with 800 M.C.S. and reaction of the distal side is excluded, the first positive curvature appears through a decrease in growth of the illuminated side.

This corresponds with the theory of PAAL but not with that of BOYSEN

JENSEN, while the results are wholly in accordance with the theory of BLAAUW.

Along three different lines, namely by the light-growth-responses, by the research about the production of tip-substances as well as by the above described experiments it has now been proved that the first positive curvature is the result of a decrease in growth of the illuminated side.

Besides, in the same way it has conclusively shown that with 80000 M.C.S. one half of the coleoptile independent of the other half may give an increase in growth.

Are there circumstances under which the distal side of coleoptile shows an increase in growth, while at the same time the proximal side grows in such a way, that a positive curvature results?

VAN DILLEWIJN has shown that these circumstances are indeed realized in the area of the second positive curvature.

From this it is therefore evident that the theory of BOYSEN JENSEN is valid for the second positive curvature.

So each of the theories of PAAL and BOYSEN JENSEN has been proved to be correct for a certain area, while both now fit in the theory of BLAAUW, which explains the phototropic curvatures by the difference in light-growth-response of the proximal and distal sides.

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Mathematics. — „Zur Entstehung meiner Arbeiten über Dimensions- und Kurventheorie.“ By KARL MENGER. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of May 29, 1926).

Im Folgenden werden (einem von mehreren Seiten geäußerten Wunsch zur Klärung von Prioritätsfragen entsprechend) einige Akten betreffend die Entstehung meiner dimensions- und kurventheoretischen Untersuchungen abgedruckt.

Bloss erwähnt werde ein im Juni 1921 Herrn Professor HAHN im Zusammenhang mit seinem damaligen punktmengentheoretischen Seminar übergebener Aufsatz, betitelt „Der Begriff der Kurve“¹⁾, in welchem im wesentlichen jene Gebilde eingeführt wurden, die ich später als reguläre Kurven bezeichnet habe (d.s. jene Kontinua, zu deren sämtlichen Punkten beliebig kleine Umgebungen mit endlichen Begrenzungen existieren), und in dem ferner End- und Knotenpunkte solcher Kurven definiert wurden.

Im Herbst 1921 hinterlegte ich bei der Wiener Akademie der Wissenschaften²⁾ ein kurzes Manuskript „Zur Theorie der Punktmengen“³⁾, welches hierunter nach einer notariell beglaubigten Abschrift abgedruckt wird:

Auf die Dimensionalität wird in der Theorie der Punktmengen im allgemeinen nur durch Zugrundelegung eines Raumes von bestimmter Dimensionszahl Rücksicht genommen. Es ist indes klar, dass, ganz unabhängig von dieser Zahl, von einer Dimensionalität gewisser Kontinua an sich gesprochen werden kann. Im folgenden Entwurf zu einer Abhandlung suche ich diesen Begriff zu präzisieren. Wir werden zu einer rekursiven Definition der Dimensionen gelangen und beginnen mit den eindimensionalen Kontinuen.

§ 1. Der Kurvenbegriff.

Der populäre Kurvenbegr. (nichtwissenschaftlichen Zwecken entsprungen) ist vage und nicht einheitlich. Wir suchen einen Begriff, der möglichst

¹⁾ Das noch vorhandene Manuskript trägt den Vermerk: „Eingegangen Juni 1921. H. HAHN“. — Sämtliche in dieser Note als noch vorhanden bezeichnete Manuskripte befinden sich gegenwärtig bei Herrn L. E. J. BROUWER.

²⁾ Vgl. Anzeiger d. Wiener Akad. d. Wissensch. 58, 1921, S. 224.

³⁾ Das noch vorhandene Manuskript wurde nach Eröffnung des Schreibens mit der Aufschrift versehen: „Dieses Blatt bildete den Inhalt des zur Wahrung der Priorität der Akademie der Wissenschaften übergebenen versiegelten Schreibens No. 778, 1921. — 9. April 1926. F. BECKE, Generalsekretär der Akademie d. Wissenschaften in Wien“.

viele Gebilde in sich fasst, die gemeinhin als Kurven bezeichnet zu werden pflegen und (von notwendigen Verallgemeinerungen abgesehen) möglichst wenige Gebilde, die gemeinhin nicht als Kurven gelten würden. Dass der Jordansche Kurvenbegriff diesen Forderungen nicht genügt, ist ohne weiteres klar, da er auch Flächen, ja n -dimensionale Räume umfasst (Peano-„Kurven“). Lennes andererseits beschränkt sich von vorneherein auf die Definition „einfacher“ Kurvenbögen.

Wir definieren: Ein Kontinuum \mathfrak{K} heisst Kurve, wenn in jeder Umgebung \mathfrak{U} jedes der Punkte P von \mathfrak{K} eine Umgebung \mathfrak{U}_1 enthalten ist, so dass der Durchschnitt \mathfrak{D} von \mathfrak{K} mit der Begrenzung von \mathfrak{U}_1 keinen zusammenhängenden Teil enthält.

§ 2. Sätze über Kurven.

1. \mathfrak{D} kann nicht für alle Punkte von \mathfrak{K} aus einem einzigen Punkt bestehen.

2. Wenn \mathfrak{D} für jeden P aus höchstens zwei Punkten bestehen soll, so gibt es höchstens zwei Punkte („End“-punkte von \mathfrak{K}) in denen \mathfrak{D} nur aus einem einzigen Punkt besteht.

3. Wenn \mathfrak{D} für alle P aus genau zwei Punkten bestehen soll, so ist \mathfrak{K} entweder eine geschlossene oder eine nicht kompakte offene Kurve.

4. Der Lennesbogen ist eine Kurve. — Zu einfachen Kurven geht man durch ein Studium der Zahl und Natur der „Knoten“- und „End“-punkte von \mathfrak{K} über (jener Punkte, für die \mathfrak{D} aus mehr bzw. weniger als zwei Punkten besteht).

5. In \mathfrak{K} gibt es keinen Teil, der umkehrbar eindeutiges und stetiges Abbild einer Kreisfläche wäre. Insbes. enthält also \mathfrak{K} im \mathfrak{R}_2 keinen inneren Punkt.

§ 3. Der Flächenbegriff.

Ein Kontinuum \mathfrak{F} heisst Fläche, wenn in jeder Umgebung \mathfrak{U} jedes Punktes P von \mathfrak{F} eine Umgebung \mathfrak{U}_1 enthalten ist, so dass der Durchschnitt \mathfrak{D} von \mathfrak{F} mit der Begrenzung von \mathfrak{U}_1 eindimensional ist, d.h. aus diskreten Kurven besteht.

(Dieses „diskret“ lässt sich präzisieren. — Dann Sätze über Flächen u.s.w.)

§ 4. n -dimensional heisst ein Kontinuum, wenn \mathfrak{D} $(n-1)$ -dimensional ist.

Die Definitionen von § 1, 3, 4 habe ich (angeregt durch Herrn Prof. Hahn's Seminar über den Kurvenbegriff) bereits im April 1921 gefunden, wie Herr Prof. Hahn und Herr Otto Schreier bestätigen können. Die Sätze, 1, 2, 3 von § 2 habe ich auf Anregung Herrn Prof. Hahn bewiesen. Er war es auch der im § 1 die Worte „keinen zusammenhängenden Teil“ an Stelle meiner ursprünglichen „höchstens abzählbar viele Punkte“ gesetzt hat.

Ende des Jahres 1921 übergab ich Herrn HAHN als Herausgeber der „Monatshefte für Mathematik und Physik“ den in der vorangehenden Note erwähnten Aufsatz zur Dimensions- und Kurventheorie. Ein unwesentliches Versehen in demselben veranlasste mich im Februar 1922 zur Einsendung einer Ergänzung folgenden Wortlautes ¹⁾:

Eine nicht leere Menge M des R_m heisst n -dimensional, wenn

1. *zu jedem ihrer Punkte P und jeder offenen Umgebung $U_1(P)$ eine offene Umgebung $U_2(P) < U_1(P)$ existiert, mit deren Begrenzung M eine höchstens $(n-1)$ -dimensionale Menge gemein hat; und wenn M*

2. *mindestens einen Punkt Q enthält, für den eine offene Umgebung U existiert derart, dass M mit der Begrenzung jeder offenen Umgebung $U_1(Q) < U$ eine Menge gemein hat, die einen $(n-1)$ -dimensionalen Teil enthält. (-1) -dimensional ist die leere Menge.*

Genau (ev. „homogen“) n -dimensional heisse eine Menge, die nur Punkte Q mit der Eigenschaft 2.) enthält. (Jede nulldimensionale Menge ist genau nulldimensional). Jeder Teil einer n -dimensionalen Menge ist höchstens n -dimensional. n -dimensionaler Kern einer n -dimensionalen Menge M heisse die Vereinigung aller genau n -dimensionalen Teile von M .

Diese Definition des allgemeinen Dimensionsbegriffes ist äquivalent und auch dem Wortlaute nach fast völlig übereinstimmend mit der in den Monatsheften für Mathematik und Physik Bd. 33 (S. 157 f.) enthaltenen Fassung.

Auf ein ausführliches im November 1922 Herrn HAHN eingesandtes Manuskript über Dimensions- und Kurventheorie werde ich gelegentlich näher eingehen.

¹⁾ Das noch vorhandene Manuskript trägt den Vermerk „Eingegangen unmittelbar nach Absendung (12. II. 1922); seither stets in meiner Verwahrung; an Herrn Dr. Menger ausgefolgt am 10. 4. 1926. H. HAHN“.

Mathematics. — "*Allgemeine Räume und Cartesische Räume*". Zweite Mitteilung: „*Ueber umfassendste n -dimensionale Mengen*". By KARL MENGER. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of June 26, 1926).

Wenn die n -dimensionale Menge N zu jedem n -dimensionalen kompakten metrischen Raum eine homöomorphe Menge als Teil enthält, dann wollen wir N eine *umfassendste n -dimensionale Menge* nennen. Wir weisen zunächst nach:

Es existieren umfassendste eindimensionale Mengen, darunter sogar beschränkte, stetig durchlaufbare Kurven des R_3 . Wir können eine solche Menge in folgender Weise konstruieren: Der Einheitswürfel des R_3 wird in 27 homothetische, einander kongruente Teilwürfel von der Seitenlänge $\frac{1}{3}$ zerlegt. Sodann tilgen wir (nebst ihren Oberflächen) den innersten dieser Würfel (d.h. jenen, dessen Oberfläche zu der des Einheitswürfels fremd ist) und die sechs Würfel, welche mit dem innersten eine Fläche gemein haben. Zu jedem der zwanzig nicht getilgten Würfel nehmen wir seine Begrenzung, soweit sie getilgt wurde, wieder hinzu und wiederholen in ihm denselben Vorgang der Zerlegung in 27 Teilwürfel und der Tilgung der sieben mittleren. Indem man so fortfährt, erhält man beim n -ten Schritt 20^n Würfel von einer Seitenlänge $\frac{1}{3^n}$, welche welche wir „Würfel des n -ten Schrittes von I “ nennen und deren Summe wir mit I_n bezeichnen. Die (offenbar eindimensionale stetig durchlaufbare) Menge $\coprod_{n=1}^{\infty} I_n$ ist unsere umfassendste eindimensionale Menge I .

Man kann I auch in folgender Weise darstellen: Die Summe der Kanten aller Würfel des n -ten Schrittes von I bezeichnen wir als G_n und nennen sie das n -te Gerüst von I . Der Raum I ist dann die abgeschlossene Hülle seines Gerüsts $G = \sum_{n=1}^{\infty} G_n$.

Der Nachweis der Einbettbarkeit jedes eindimensionalen kompakten metrischen Raumes A in den R_3 wurde in der ersten Mitteilung¹⁾ durch

¹⁾ Vgl. diese Proceedings, Bd. 29, 1926, S. 496. Es wurde daselbst das einfache, aber zahlreicher Anwendungen fähige Lemma bewiesen: Die Kerne homologer finiter Umgebungssysteme sind homöomorph. Die Umkehrung, dass nämlich in homöomorphen kompakten Räumen homologe finite Umgebungssysteme existieren, ist trivial. *Notwendig und hinreichend für die Homöomorphie zweier kompakter Räume ist also die Existenz homologer finiter Umgebungssysteme in ihnen.* Sind zwei Räume als Kerne von finiten Umgebungssystemen gegeben, so ist die Frage nach der Homöomorphie der beiden Räume zurückgeführt auf die Frage, ob aus den beiden definierenden Umgebungssystemen zwei homologe Umgebungssysteme hergeleitet werden können, und das ist eine Frage nach der Möglichkeit einer abzählbaren Folge von finiten Operationen.

die Angabe eines finiten Umgebungssystems in A und eines dazu homologen Systems von Polyedern des R_3 geführt. Bezeichnen wir alle Würfel irgend eines n -ten Schrittes von I kurz als *Würfel von I* und ein Polyeder, welches Summe endlich vieler Würfel von I ist, als *Intervallsystem von I* , so haben wir, um die Einbettbarkeit von A in die Menge I nachzuweisen, ein *finites Umgebungssystem in A* und ein *homologes System von Intervallsystemen von I* anzugeben.

Setzen wir nun voraus: Es sei A eine kompakte Menge und es seien B_1, B_2, \dots, B_m m paarweise fremde Teilmengen von A . Es sei ferner P ein Intervallsystem von I und es seien Q_1, Q_2, \dots, Q_m m paarweise fremde Quadratflächen auf der Oberfläche von P . Es sei endlich A Summe von n abgeschlossenen Umgebungen A_1, A_2, \dots, A_n , die zu je zweien keine inneren Punkte gemein haben und so dass jeder Punkt von A höchstens zwei Mengen A_i , jeder Punkt einer Menge B_k genau einer Menge A_i angehört. Auf Grund der Tatsache, dass für jedes n^{te} Gerüst G_n von I die Menge $G_n - G_{n-1}$ zusammenhängend ist, zeigt man durch elementargeometrische Ueberlegungen, dass unter den angegebenen Voraussetzungen folgendes möglich ist: Wählt man im Innern von P irgend n verschiedene Punkte p_1, p_2, \dots, p_n von P , dann kann man so kleine paarweise fremde Würfel von I W_1, W_2, \dots, W_n angeben, dass p_i im Innern von W_i liegt und dass sich aus dem Gerüst von I paarweise fremde Streckenzüge angeben lassen, welche die Oberfläche jedes W_i mit den Oberflächen aller übrigen W_k und mit allen Q_k verbinden. Daraus folgt aber sofort die Existenz von Intervallsystemen von I P_1, P_2, \dots, P_n , sodass alle $P_i \subset P$ sind und dass (für jedes i und k) P_i mit P_k , bzw. mit Q_k dann und nur dann Punkte u. zw. genau eine Quadratfläche gemein hat, wenn A_i mit A_k , bzw. mit B_k Punkte gemein hat. Damit sind in Hilfssatz 1 der ersten Mitteilung der Polyeder speziell durch Intervallsysteme von I ersetzt.

Hilfssatz 2 der ersten Mitteilung bedarf keiner Verschärfung. Die Verschärfung von Hilfssatz 3 besagt, dass man zu einem vorgelegten kompakten eindimensionalen Raum A eine bestimmt geartete Zerlegung in beliebig kleine Umgebungen und gleichzeitig in einem vorgelegten Intervallsystem P von I ein entsprechendes System von beliebig kleinen Intervallsystemen von I unter Erfüllung beiderseitiger Randbedingungen angeben kann. Um dies einzusehen, hat man ganz wie beim Beweis von Hilfssatz 3 zunächst *irgend* eine Zerlegung von A mit den gewünschten Eigenschaften in Teile von gewünschter Kleinheit anzugeben und kann zu dieser Zerlegung nach dem verschärften Hilfssatz 1 entsprechende Intervallsysteme von I in P angeben, welche auch die entsprechenden Randbedingungen erfüllen. Haben diese Intervallsysteme auch bereits die gewünschte *Kleinheit*, dann ist man am Ziel. Andernfalls ersetzt man jedes der zu grossen Intervallsysteme durch eine Kette von Intervallsystemen von I mit den Eigenschaften, die in Hilfssatz 2 vorausgesetzt werden, und bestimmt zu diesen Polyedern nach Hilfssatz 2

Teilumgebungen der Umgebung in A in entsprechender Lage. Damit ist die Verschärfung von Hilfssatz 3 bewiesen, auf Grund derer ein finites Umgebungssystem im vorgelegten eindimensionalen Raum und ein homologes System von Intervallsystemen von I konstruiert werden kann.

Aus diesen Überlegungen gehen zugleich die beiden folgenden Nebenresultate hervor: *Jeder eindimensionale kompakte metrische Raum ist homöomorph mit einer Menge des R_3 , zu deren sämtlichen Punkten beliebig kleine Intervalle existieren, mit deren Begrenzungen die Menge diskontinuierliche Durchschnitte hat.* Und: *Jeder kompakte eindimensionale Raum ist homöomorph mit einem Teil einer Fläche des R_3 .* Wir können nämlich, so wie bei der Konstruktion der Menge I , den Würfel in 27 Teile teilen, aber bloss den innersten dieser Würfel ohne seine Begrenzung tilgen und sodann dieses Verfahren in den 26 nicht getilgten Würfeln wiederholen und so ad infinitum fortfahren. Nennen wir II_n die Summe aller 26^n nicht getilgten Würfel von der Seitenlänge $\frac{1}{3^n}$, und bezeichnen wir mit II den Durchschnitt aller Würfelsummen II_n , dann ist die so entstehende Menge offenbar eine (im kleinen zusammenhängende) Fläche, welche die Menge I als Teil enthält¹⁾.

Von Interesse ist die Untersuchung dieser Verhältnisse in höheren Dimensionen. Der n -dimensionale Einheitswürfel kann in 3^n homothetische, einander kongruente Teilwürfel von der Seitenlänge $\frac{1}{3}$ zerlegt werden. Jeder dieser Würfel lässt sich charakterisieren durch ein n -Tupel gebildet aus den Zahlen 0, 1, 2, indem man jedem Würfel als k -te Koordinate seinen Abstand von der $(n-1)$ -dimensionalen Hyperebene durch den Ursprung, welche auf der k -ten Achse senkrecht steht, zuordnet. Es befinden sich unter den 3^n Würfeln $\binom{n}{n-k} \cdot 2^{n-k}$, unter deren Koordinaten genau k Einsen vorkommen. Wir tilgen nun samt ihren Oberflächen jene Würfel, unter deren Koordinaten mindestens $m+1$ Einsen vorkommen. Wir behalten $\mu(m) = \sum_{k=0}^m \binom{n}{n-k} \cdot 2^{n-k}$ Würfel zurück, nehmen zu jedem die Oberfläche, soweit sie getilgt wurde, wieder hinzu und wiederholen in jedem dieser Würfel dasselbe Verfahren der Unterteilung und Tilgung der $\sum_{k=m+1}^n \binom{n}{n-k} \cdot 2^{n-k}$ mittleren Würfel. Beim k -ten Schritt behalten wir $\mu(m)$ Würfel von einer Seitenlänge $\frac{1}{3^k}$, deren Summe wir mit ${}_k R_n^m$ bezeichnen.

¹⁾ Man kann die Mengen, welche umfassendste eindimensionale Mengen sind, in verschiedener Weise charakterisieren: dadurch, dass sie einen mit der Menge I homöomorphen Teil enthalten, — oder dadurch, dass sie einen Teil enthalten, in dessen sämtlichen offenen Teilen (vermöge ihrer Zusammenhangsverhältnisse) die Verkettungen endlich vieler Teilumgebungen durchgeführt werden können, die beim Beweis der Einbettbarkeit aller eindimensionalen Mengen in die Menge I verwendet wurden. — oder als abgeschlossene Hüllen eines hinreichend verzweigten Gerüsts u.s.w.

Die Menge $\prod_{k=1}^{\infty} R_n^m$, von der man leicht zeigt, dass sie stetig durchlaufbar und m -dimensional ist, nennen wir R_n^m .¹⁾

In Verallgemeinerung einer Schlussweise von SIERPINSKI²⁾ zeigt man unschwer, dass für jedes n die Menge R_n^{n-1} zu jeder nirgends dichten abgeschlossenen Teilmenge des R_n , also zu jeder höchstens $(n-1)$ -dimensionalen abgeschlossenen Menge des R_n , eine homöomorphe Menge als Teil enthält. Es gilt aber viel mehr: es zeigt sich nämlich, dass die Menge R_n^{n-1} ein topologisches Bild von jeder (auch nicht-abgeschlossenen) weniger als n -dimensionalen Menge des R_n als Teil enthält. Nach einem Satz der Dimensionstheorie ist jede weniger als n -dimensionale Menge des R_n enthalten in einer Menge, deren Komplement abzählbar und im R_n dicht ist. Einem bekannten Satz von FRÉCHET zufolge sind alle Mengen des letzteren Art untereinander homöomorph. Es genügt also zu zeigen, dass ein Teil der Menge R_n^{n-1} homöomorph ist mit einer Menge, deren Komplement abzählbar und im R_n dicht ist. Dies trifft aber, wie man nachweisen kann, für die G_s -Menge zu, welche entsteht, wenn man aus dem offenen Einheitswürfel von seinen 27 Teilwürfeln den innersten *nebst seiner Begrenzung* tilgt und dieses Verfahren in jedem der übrigen 26 Teilwürfel und so ad infinitum fortsetzt.

Ich halte es für sehr wahrscheinlich, dass allgemein die Menge R_n^m eine umfassendste m -dimensionale Menge hinsichtlich des R_n ist, d.h. zu jeder m -dimensionalen Menge des R_n eine homöomorphe Menge als Teil enthält. Der Beweis dieses Satzes dürfte keine prinzipiellen Schwierigkeiten bieten, allerdings für $m < n-1$ recht langwierig sein.

In einer folgenden Mitteilung wird bewiesen werden, dass jeder n -dimensionale kompakte (und vermutlich sogar jeder n -dimensionale separable) metrische Raum homöomorph ist mit einer Teilmenge des R_{2n+1} . Ein stetig durchlaufbares Kontinuum, welches ein topologisches Bild jedes kompakten endlichdimensionalen Raumes als Teil enthält, ist dann z.B.

die Menge $\sum_{n=1}^{\infty} I_n$, wobei I_n ($n=1, 2, \dots$) ein einen festen Punkt p enthaltendes

n -dimensionales Intervall mit der Seitenlänge $\frac{1}{n}$ ist.

1) R_1^0 ist die nirgends dichte perfekte Cantorsche Menge, R_2^1 die umfassendste ebene Kurve von SIERPINSKI (Comptes Rendus, 162, S. 629), R_3^1 die oben konstruierte umfassendste eindimensionale Menge.

2) a. a. O.

Mathematics. — "*Representation of a simplicial manifold on a locally simplicial manifold*". By WILFRID WILSON. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of September 25, 1926).

The object of the following investigation is to prove the extension of the theorem I of BROUWER quoted beneath. The proof, of which the leading idea was suggested to me by Prof. BROUWER, consists in repeated applications of the methods and results of his three papers

"Ueber Abbildung von Mannigfaltigkeiten",

"Beweis des n -dimensionalen Jordanschen Satzes", and

"Jordansche Mannigfaltigkeiten",

in Bd. 71 of the *Mathematische Annalen*; and which for brevity will be quoted as "A. v. M.", "J. S." and "J. M." respectively.

1. By an n -dimensional element ¹⁾ is understood the topological image of an n -dimensional simplex of the n -dimensional number space R_n .

By a simplicial n -dimensional manifold is understood one as defined by BROUWER ²⁾, that is, a connected set of points built up of n -dimensional elements any two of which have either no point in common, or only a p -dimensional face together with all its faces of lower dimensions in common ($0 \leq p \leq n-1$), while the incidence relations of elements having a common vertex are those of the simplexes of a simplex star in R_n . The manifold is said to be closed or open according as the number of elements is finite or infinite.

By a locally simplicial n -dimensional manifold is understood a connected topological space ³⁾ to whose defining system of neighbourhoods there exists a countable ⁴⁾ equivalent ⁵⁾ system of neighbourhoods each of which is an n -dimensional element. The manifold is said to be closed or open according as it is or is not a compact topological space. It follows from this definition that if E' and E'' be two elements of a closed locally simplicial manifold, a chain E_1, E_2, \dots, E_m of elements of the manifold exists such that $E_i E_{i+1} \neq 0$, $E_1 = E'$ and $E_m = E''$, ($i = 1, 2, \dots, m-1$).

2. The elements of a locally simplicial manifold may be assumed, by a suitable subdivision, to be so chosen that any two having common

¹⁾ BROUWER, "A. v. M." p. 97.

²⁾ BROUWER, "A. v. M." p. 97-98.

³⁾ HAUSDORFF, "Grundzüge der Mengenlehre", p. 213.

⁴⁾ HAUSDORFF's Grundzüge, Axiom (F), p. 263.

⁵⁾ HAUSDORFF's Grundzüge, Kap. VIII, §. 1. 260.

points are both contained in a third element and thus constitute a two-sided piece of the manifold.

The indicatrix of an element Δ of a locally simplicial manifold being assigned, the indicatrix of an element E of a simplicial division of Δ is determined by the rule used for simplicial manifolds. Let now E be an arbitrary element contained in Δ , P any point in E , Δ' the representative simplex⁶⁾ of Δ , E' and P' the images of E and P respectively in Δ' , and B' the boundary of E' . The order of P' with respect to the Jordan Manifold B' in Δ' is ± 1 ⁷⁾ and the positive indicatrix of E is defined as that for which this order is $+1$. Consider now two elements E' and E'' such that $E' E'' \neq 0$ and a third element $E \subset E' E''$ ⁸⁾. The indicatrix of E' being assigned, that of E is determined as above so that the order of a point $P \subset E$ with respect to the boundary of E in E' ⁹⁾ is $+1$ and that of E'' is assigned so that the order of P with respect to the boundary of E in E'' ⁹⁾ is also $+1$. Finally let E' and E'' be arbitrary elements of the manifold and $\Delta_1, \Delta_2, \dots, \Delta_m$ a chain of elements for which $\Delta_i \Delta_{i+1} \neq 0$, $\Delta_1 = E'$ and $\Delta_m = E''$. If then the indicatrix of E' be assigned, an indicatrix for E'' is determined by repeated applications of the above rule. When the indicatrix of E'' thus determined is independent of the chain $\Delta_1, \Delta_2, \dots, \Delta_m$ of elements used in connecting E' and E'' , the manifold is said to be two sided; otherwise it is called one sided.

It can now be proved that the following theorem of BROUWER¹⁰⁾ —

- I. If a closed, two-sided, simplicial, n -dimensional manifold μ be uniquely and continuously represented on a simplicial n -dimensional manifold μ' , there exists a finite whole number c invariant under continuous modification of the representation, with the property that the image of μ covers every region of μ' altogether c times.

If μ' be one sided or open c is always zero —

can be extended to the case where μ' is any locally simplicial n -dimensional manifold.

3. Let α be a unique continuous representation of the closed, two-sided, simplicial n -dimensional manifold μ on the locally simplicial n -dimensional manifold μ' , and first consider the case where μ' is closed and two-sided. Then the simplicial division ζ of μ , the fundamental simplexes, faces and vertices of μ , ordinary fundamental simplexes of μ ,

⁶⁾ BROUWER, "A. v. M.", p. 100.

⁷⁾ BROUWER, "J. M.", p. 323.

⁸⁾ Such an element E exists by HAUSDORFF's Axiom (B), Grundzüge p. 213.

⁹⁾ By the order of a point P with respect to a JORDAN manifold in any element is meant the order of the image of P with respect to the image of the JORDAN manifold in the representative simplex of the element. The indicatrix of an element is related to that of its boundary by the rule given in BROUWER, "A. v. M.", p. 107.

We are here using BROUWER's generalized indicatrix. "J. M.", §§ 4 and 5.

¹⁰⁾ "A. v. M.", Satz 1, p. 106.

simplicial and modified simplicial representations β and γ respectively, corresponding to ζ and approximating to a , the "inner simplex" of each element of μ' , the ordinary points of the inner simplexes of μ' and the numbers p and p' of positive and negative image simplexes respectively covering an ordinary point P of μ' , as defined and used by BROUWER lead to the results ¹¹⁾ —

- (1) The number $c = p - p'$ is constant in the ordinary points of an inner simplex of μ'
- (2) c has the same value for every simplicial representation approximating to a .

4. Let E' and E'' be elements of the manifold μ' such that $E' E'' \neq 0$ and let the corresponding inner simplexes J' and J'' be such that $J' J'' \neq 0$. Consider a sequence $\zeta_1, \zeta_2, \dots, \zeta_\lambda, \dots, \zeta_\nu, \dots$, ($\nu > \lambda$), of simplicial divisions of μ such that ζ_{i+1} is obtained by simplicial divisions of the elements of ζ_i , the maximum diameter of any element of ζ_i being $< \frac{1}{2^i}$.

Let β'_λ and β''_λ be the simplicial approximations ¹²⁾ to a in J' and J'' respectively corresponding to ζ_λ , γ'_λ and γ''_λ the corresponding modified simplicial approximations ¹³⁾, P any ordinary point ¹⁴⁾ of J' and J'' with respect to both γ'_λ and γ''_λ , and c'_λ and c''_λ the values of the number $p - p'$ at P for γ'_λ and γ''_λ respectively. It is required to prove that for sufficiently great λ , $c'_\lambda = c''_\lambda$.

Let E^i be any image simplex under γ'_λ contained in $E' E''$, E the corresponding ordinary simplex of the subdivision ζ_λ of μ ,

$E_1, E_2, \dots, E_{x_\nu}$ the simplexes of the subdivision ζ_ν of E ,

$E'_1, E'_2, \dots, E'_{x_\nu}$ their images in $E^i \subset E'$ under γ'_λ ,

$\Delta_1^\nu, \Delta_2^\nu, \dots, \Delta_{x_\nu}^\nu$ their images in E'' under γ''_λ ,

and $\Delta_1^\lambda, \Delta_2^\lambda, \dots, \Delta_{x_\nu}^\lambda$ those simplexes of E'' having the same vertices as $E'_1, E'_2, \dots, E'_{x_\nu}$ respectively. **.

The boundary B^i of $\Delta_1^\lambda + \Delta_2^\lambda + \dots + \Delta_{x_\nu}^\lambda$ consists of those $(n-1)$ -dimensional faces incident with only one of the Δ_i^λ , and constitutes a simplicial image in E'' of the boundary B of E^i . Let the number η be defined thus: if E^i does not contain P , $\eta = 0$; if $E^i \supset P$, $\eta = +1$ or -1 according as the indicatrix of E^i is positive or negative in E' . Then the indicatrix of E'' being determined from that of E' by the rule given in § 2, the order of P with respect to the Jordan manifold B in E'' is η .

¹¹⁾ BROUWER, "A. v. M." pages 101—104.

¹²⁾ BROUWER, "A. v. M.", p. 101—102.

¹³⁾ " " " " p. 102.

¹⁴⁾ " " " " p. 103.

***) That is, the Δ_i^λ are images in E'' of simplexes in the representative simplex of E' .

Therefore for sufficiently great ν any segment path in E'' ¹⁵⁾ from P to the boundary of E'' , no vertex and no segment of which is in B'' and no segment of which cuts an $(n-2)$ -dimensional face of B'' , crosses B'' p times positively and p' times negatively where $p-p'=\eta$. ¹⁶⁾ Since an indicatrix of an $(n-1)$ -dimensional face incident with two simplexes Δ_i' and Δ_j' is of different sign according to whether it is regarded as bounding Δ_i' or Δ_j' ¹⁷⁾, the above mentioned segment path also crosses the sum of the boundaries of the simplexes $\Delta_1', \Delta_2', \dots, \Delta_{\nu}'$, η times. If therefore the number of positive and negative of these simplexes containing P be q and q' respectively, we must have $q-q'=\eta$. Proceeding in this way with all the image simplexes such as E' it follows that $c_i' = r-r'$ where r and r' are the numbers of positive and negative simplexes respectively covering P from all sets of type $\Delta_1', \Delta_2', \dots, \Delta_{\nu}'$. If now the vertices of each simplex Δ_i' be moved continuously into the corresponding vertices of Δ_i'' , the number $r-r'$ remains constant if λ be sufficiently great, and is therefore equal to the corresponding number c_i'' for the image simplexes Δ_i'' under γ'' , that is, $c_i'' = r-r' = c_i'$. Therefore from (2) of § 3, $c_i = c_i''$, and since any two elements of μ' may be joined by a finite chain of elements as in § 1, it follows that —

(3) The number c has the same value in the ordinary points of all inner simplexes of μ' .

As with (1) and (2), the theorems (4), and (5) beneath, follow exactly as by BROUWER ¹⁸⁾: —

(4) When the image of μ under α is not everywhere dense in μ' , the degree c is zero.

(5) If μ' be one sided or open c is zero.

The above mentioned extension of I has thus been established. That this extension remains valid when μ is a closed, two-sided, n -dimensional pseudo-manifold, follows immediately from the definition of a pseudo-manifold ¹⁹⁾.

* See footnote 9).

¹⁵⁾ By a segment path in E'' is meant the image in E'' of a segment path in the representative simplex of E'' .

¹⁶⁾ BROUWER, "Beweis des n -dimensionalen JORDANSche Satzes", Math. Ann. **71**, p. 317 footnote and J. M. § 4. p. 323.

¹⁷⁾ BROUWER, "A. v. M.", p. 107.

¹⁸⁾ A. v. M.", p. 106.

¹⁹⁾ BROUWER, "Beweis der Invarianz des n -dimensionalen Gebiets", Math. Ann. Bd. **71**, p. 305—306.

ERRATUM.

In Vol. XXII of these Proceedings, on page 811 in the title of the paper: "*Ueber eindeutige, stetige Transformationen von Flächen in sich*", the words "Sechste Mitteilung" are to be replaced by "Siebente Mitteilung".

On page 814 of the same paper, the example given of a transformation of the first kind of degree n with a single invariant point, is irrelevant. Moreover, from footnote ²⁾ on page 865 of Vol. XXIX of these Proceedings, follows that by transformations of the first kind of positive degree, certainly at least two invariant points appear.

In Vol. XXIII of these Proceedings, on page 232 in the title of the paper: "*Ueber eindeutige, stetige Transformationen von Flächen in sich*", the words "Siebente Mitteilung" are to be replaced by "Achte Mitteilung".

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Physics. — “*Solid helium*”¹⁾. By W. H. KEESOM. (Communication N^o. 184*b* from the Physical Laboratory at Leyden).

(Communicated at the meeting of September 25, 1926).

§ 1. *Introduction.*

On the same day that KAMERLINGH ONNES liquefied helium for the first time, he investigated whether it would become solid on further cooling by evaporation under reduced pressure. He was at that time able to obtain a vapour pressure below 1 cm, probably 7 mm²⁾. The helium however remained liquid.

This attempt to solidify helium by reducing the pressure under which it evaporates, was repeated on several occasions. In 1909 KAMERLINGH ONNES was able to reduce the pressure to 2.2 mm (the temperature then was estimated at 2.5 to 2^o K., now put at 1.4^o K.)³⁾; in 1910 a pressure of even 0.2 mm was reached (temperature 1.15^o K.)⁴⁾. But the helium still remained a thin liquid.

The further development of the question, whether helium could be made to freeze on still further cooling, was then postponed in favour of more urgent questions, which could be dealt with the means available, while for the investigation of the solidification of helium new means were necessary.⁵⁾

A new attempt was made in 1919, but with little improvement. Then in 1921 KAMERLINGH ONNES⁶⁾ by making use of a battery of condensation pumps, reached a remarkably lower pressure, probably 0.013 mm, at any rate less than 1/50 mm, at which, after estimation, the temperature should be 0.82^o K.

Meanwhile helium still remained liquid, so that KAMERLINGH ONNES wondered whether helium would perhaps remain liquid even if it were cooled to the absolute zero. In order to obtain further data about this question KAMERLINGH ONNES still before his retirement procured a much more powerful mechanical pump installation, and further, a condensation pump of greater power was constructed with which we intend to reduce the temperature again.

1) Provisional communications were published in C. R. 183, 26 and 189, July 5 and 19, 1926, and Nature July 17, 1926.

2) These Proc. 11, 168, 1908; Comm. Leyden, N^o. 108.

3) These Proc. 12, 175, 1909; Comm. Leyden, N^o. 112.

4) Jubilee-Book VAN BEMMELEN 1910. See also Comm. Leyden N^o. 119, p. 12 and Comm. Leyden Suppl. N^o. 35, p. 29.

5) See KAMERLINGH ONNES, Comm. Leyden N^o. 159, § 4.

6) H. KAMERLINGH ONNES, Comm. Leyden N^o. 159.

Meanwhile the results of the provisional experiments of KAMERLINGH ONNES and VAN GULIK¹⁾ about the change of the melting point of hydrogen by pressure, had made me wonder, if it would not be possible at the temperatures already reached, to solidify helium by pressure. This idea became still more definite on considering the results of the measurements of SIZOO and KAMERLINGH ONNES²⁾ about the influence of allsided compression on the supraconductivity: so fig. 5 of Comm. N^o. 180*b* raised the question, whether the small difference between the lines for 193 and for 300 KG/cm² (according to the authors hardly more than experimental error), would not be explained thus: that the helium had become solid in the compression tube and that because of the adhesion to the wall an increase of the pressure put on had not caused a corresponding increase of the pressure upon the investigated thread³⁾.

§ 2. *Solidification of helium (June 25th 1926). Provisional determination of the melting curve.*

a. *Method.* The criterion for judging if helium had become solid, and

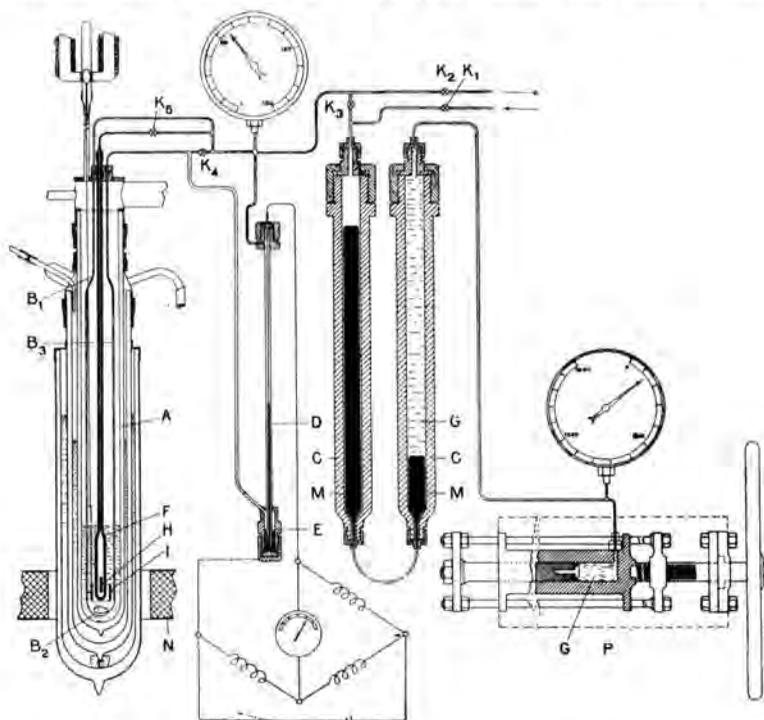


Fig. 1.

¹⁾ H. KAMERLINGH ONNES and W. VAN GULIK, These Proc. 29, p. 1184, 1926; Comm. Leyden N^o. 184*a*.

²⁾ These Proc. 28, 656, 1925; Comm. Leyden N^o. 180*b*.

³⁾ See also G. J. SIZOO, W. J. DE HAAS and H. KAMERLINGH ONNES, Comm. Leyden N^o. 180*c*, p. 32 and fig. 2.

the method for determining the melting curve of the solid helium were the same as those followed by KAMERLINGH ONNES and VAN GULIK (l.c.) in determining the melting curve of hydrogen.

In fig. 1, B_1 and B_3 are two tubes of new silver, which are connected at the bottom, in the cryostat vessel A , by a narrower brass tube B_2 . Here the helium is compressed. Therefore firstly the wheel of the hydraulic pump P , which is filled with glycerine G , is turned back. CC are two vessels, connected by a tube, which are half-filled with mercury M . By turning back the wheel, glycerine comes into the pump, the mercury rises in the right hand vessel C , and falls in the left hand one. So the left hand vessel gets filled with helium gas, which is supplied by the stop-cock K_1 (K_3 is shut). Then K_1 is shut, K_3 opened. K_4 is open, K_2 shut¹⁾. Now by turning the wheel of the hydraulic pump to the right the helium gas is compressed into the tubes B , in which it liquefies, and collects in the lower part of the tube. At a pressure of 400 KG/cm² all connections were still tight.

In order to determine if helium had solidified, the following apparatus was used, constructed by G. J. FLIM, chief of the technical staff of the cryogenic laboratory. D works as a differential manometer, and consists of a steel tube, which runs into the mercury-chamber E . When K_2 is opened slightly, K_4 and K_3 being shut, and the tube B is blocked by a piece of solid helium, so that a pressure difference appears between B_1 and B_3 , the pressure in D becomes less than the pressure in E , and the mercury in D rises. Along the axis of D a thin platinum wire was stretched, which formed one of the branches of a Wheatstone bridge. Now when mercury rises in D the resistance of this branch decreases, and the needle of the galvanometer becomes deflected.

b. The experiments. In the first experiment the pressure of the helium bath was lowered as much as possible by the means ordinarily at hand in the laboratory (Burckhardtpump, 360 m³/hour), namely to approximately half a millimeter mercury pressure. Within the capillary B the pressure was put at 250 KG/cm²²⁾. In the Wheatstone bridge the galvanometer was at equilibrium. When the stopcock K_2 was opened slightly, the galvanometer needle showed a deflection. This indicated that the capillary B was blocked.

The temperature of the bath was now raised till the pressure was 1 atm. The capillary remained blocked.

The pressure in the capillary was lowered. At 150 KG/cm² the capillary was still blocked. At 100KG/cm² blocking had disappeared. At

¹⁾ In the experiments of June 25th the side-tube with stopcock K_5 and the glass piezo-meter F were not present, in the experiments of July 1st K_5 was shut during the operation described above.

²⁾ Here are given the apparent readings of the metalmanometer. In table I the corrections are applied.

130 KG/cm² the capillary was blocked again, as also at 128. At 125 KG/cm² it was open, and at 126 KG/cm² also.

Inference: at 126 KG/cm² helium is liquid, at 128 KG/cm² it is solid.

As in this experiment there was no stirrer in the cryostat, the temperature of the bath under the liquid surface was uncertain ¹⁾.

The pressure of the helium bath was then reduced to about ²⁾ 400 mm. At 120 KG/cm² the capillary was blocked; so it was at 115.

| | | | |
|--|-----|----|-------------------|
| At 110 KG/cm ² capillary opens. | | | |
| .. | 112 | .. | .. blocked. |
| .. | 111 | .. | .. still blocked. |
| .. | 110 | .. | .. opens. |

This result clearly showed that we were on the track of the melting curve of helium.

The pressure of the helium bath was reduced to about 200 mm.

At 95, 92, 90, 88 KG/cm² capillary is blocked.

| | | | |
|----|----|----|-------------|
| .. | 85 | .. | .. opens. |
| .. | 9 | .. | .. blocked. |
| .. | 86 | .. | .. opens. |

Reduced to about 100 mm.

At 80, 74, 69, 67 KG/cm² capillary blocked.

| | | | |
|----|--------|----|-------------|
| .. | 63, 65 | .. | .. open. |
| .. | 70 | .. | .. blocked. |
| .. | 65 | .. | .. open. |

Reduced to about 50 mm.

At 55, 52 KG/cm² capillary blocked.

.. 50 .. blocking disappears gradually.

The bath was then brought to atmospheric pressure, but the observations were irregular; apparently the temperature of the bath was not uniform.

The bath was then reduced to about 400 mm. At 110 KG/cm² the blocking was gradually disappearing. On repetition the same was observed at 109 KG/cm². Thus the same result was obtained as in the first experiment at this temperature (see above). The phenomenon is reproducible.

¹⁾ Indeed it appears from the results, obtained in the repetition of the experiment on July 1st, that the temperature in the lower parts of the bath was below the boiling point of helium.

²⁾ Exact measurements were not made in this first series of experiments.

The provisional results, obtained on this day, are collected in the following table:

TABLE I.

| Pressure of the helium bath | Temperature ¹⁾ | Melting pressure | | | Difference with determination of July 1st |
|-----------------------------|---------------------------|-------------------|--|-----------------|---|
| | | Manometer reading | Corrected ²⁾ KG/cm ² | Atm. | |
| 400 mm | 3.61° K. | 110 | 109 ⁵ | 106 | — 2 |
| 200 | 3.12 | 86 | 83 | 81 | — 1 |
| 100 | 2.73 | 65 | 63 ⁵ | 61 ⁵ | — 1 ⁵ |
| 50 | 2.40 | 50 | 48 | 46 | — 1 |

§ 3. *Repetition on July 1st. Determination of the melting curve. Visual observations.*

a. Determination of the melting curve of helium. Notwithstanding the distinctness of the phenomena I wished to make quite sure and to repeat the experiment. This was done on July 1st.

This time a stirrer (*I* in fig. 1, a hollow soft-iron cylinder, see under *b*, provided with a couple of little paddle-boards in order to stir the liquid) was introduced into the bath, and further arrangements were made, in order to be able to take more exact measurements³⁾. Further it was intended to continue the melting curve to the lowest temperatures, which could be reached with the mechanical vacuum pumps.

Moreover, now that it had appeared that the pressures ought not to be so excessively high, I wished to make the experiment in a glass piezometer, so as to be able to see what happened.

The determination of the melting curve occurred in exactly the same way as the provisional determinations mentioned in § 2*a*. The phenomena took place regularly. The melting pressure, corresponding to a definite temperature, could be fixed to some few tenths of an atmosphere. In some cases the fusion-process could be followed on the pointer of the galvanometer.

For the sake of brevity only the results obtained are given here.

¹⁾ From the pressure of the bath calculated after the formula of Comm. Leyden N^o. 147*b*.

²⁾ The metal manometer used (reaching to 400 KG/cm²) was gauged to 100 KG/cm² with the aid of the closed hydrogen manometers M_{60} and M_{120} . Above this it was compared with the pressure balance of the VAN DER WAALS foundation at Amsterdam. I render therefore my cordial thanks to Dr. A. MICHELS, assistant of this foundation.

³⁾ The pressure of the bath, from which the temperature was derived, was read off by Miss J. L. SOLLEWIJN GELPKKE, phil. nat. cand.; for this aid I render to her my cordial thanks.

TABLE II.

| Melting curve of helium | | |
|-----------------------------|-------------|------------------|
| Pressure of the helium bath | Temperature | Melting pressure |
| 77.09 cm | 4.21 °K. | 140.5 atm. |
| 40.03 | 3.61 | 108.8 |
| 20.01 | 3.12 | 81.5 |
| 9.94 | 2.72 | 62.8 |
| 5.02 | 2.40 | 48.6 |
| 2.00 | 2.04 | 35.7 |
| 1.10 | 1.83 | 29.8 |
| 0.57 | 1.60 | 27.4 |
| 0.24 | 1.42 | 26.5 |
| 0.057 | 1.19 | 25.3 |

The temperatures from 4.21 down to and including 1.60° K. are derived from the pressures of the helium bath according to the formula of Comm. N^o. 147b¹⁾, the two lowest temperatures from the formula which is given by VERSCHAFFELT in Comm. Leyden Suppl. N^o. 49, p. 26²⁾.

The melting pressures were read off on a metal manometer (reading to 150 KG/cm²) which was gauged to 100 KG/cm² with the aid of a manometer, which was itself compared with the closed hydrogen manometers M_{60} and M_{120} ³⁾.

¹⁾ H. KAMERLINGH ONNES and S. WEBER. These Proc. 18, 493, 1915.

²⁾ In the provisional communication in the C.R. for these lowest temperatures, values are also communicated which were calculated from a former formula of VERSCHAFFELT (Thesis for the Doctorate TUIYN, Comm. Leyden N^o. 181). I prefer now the values only given here.

³⁾ When the experiments mentioned in § 4 were finished, it was found that the pointer of this manometer was not securely fastened on its axis, and that the zero reading was changed.

As it is not sure if this occurred before or after the experiment of July 1st, these pressure measurements must not be considered absolutely sure. Also for the same reason the corrections above 100 KG/cm² could not be determined afterwards. They were derived from comparison of the manometer with the pressure balance of the VAN DER WAALS foundation, done in Sept., by adopting a constant zero change. This zero change is derived from the difference between this gauging with the pressure balance and the gauging with M_{60} and M_{120} , mentioned in the text, for pressures below 100 KG/cm². I give here the results which for the moment I think to be the most probable ones, reserving the right to repeat this measurement.

[Such a repetition took place at Nov. 26th, 1926. The pressures of table II were confirmed within some tenths of an atmosphere. At the average the readings were 0.6 atm. higher at Nov. 26th. Added in the translation.]

These measurements are sufficiently in agreement with those of § 2a (table I) to form a confirmation of the observations of June 25th.

In fig. 2 the melting curve is represented.

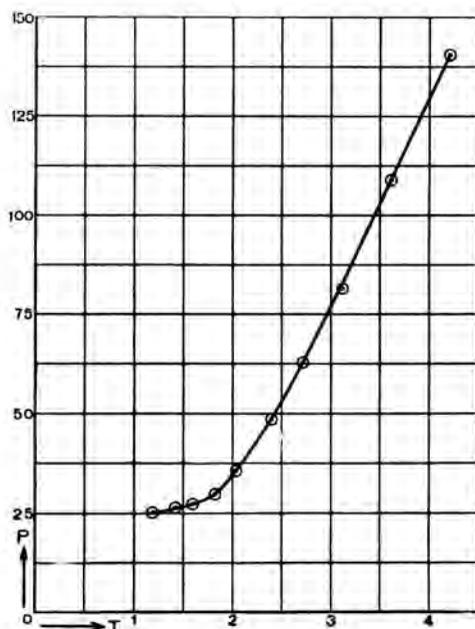


Fig. 2.

The melting curve shows an important peculiarity: it bends at the lowest temperatures so as to become more and more parallel to the T -axis. It shows no tendency at all to meet the vapour pressure curve in a triple-point ¹⁾. So unless the melting curve bends down again to the T -axis at still lower temperatures, it will never be possible to solidify helium, liquid under its own saturation pressure, by lowering the temperature alone. So the surmise expressed by KAMERLINGH ONNES (see § 1) that helium (under its own saturation pressure) remains liquid down to the absolute zero (perhaps gradually changing into the vitreous amorphous state), would be established.

In the supposition mentioned, that the melting curve does not bend down again at lower temperatures so that the melting curve and the vapour pressure curve do not meet, as long as we remain below the liquid-gas critical temperature, coëxistence between solid and gas is not possible. Then there is no sublimation curve; evaporation of the solid into the gaseous state will not be possible at those temperatures, the

¹⁾ As Prof. KRUYT pointed out to me, the possibility that this might occur with some substance was already foreseen by H. W. BAKHUIS ROOZEBOOM, *Die heterogenen Gleichgewichte vom Standpunkte der Phasenlehre*, Braunschweig 1901, p. 93. (Note added in the translation).

solid will always melt first¹⁾). This is yet another peculiarity, which distinguishes helium from all other substances as far as we can judge from observations already made.

The approaching to zero of dp/dT of the melting curve indicates according to the equation of CLAPEYRON, that the heat of fusion approaches in a higher power of T to zero than $T(\nu_{liq} - \nu_{sol})$. This is in agreement with the heat theorem of NERNST, when this is applied to the change liquid-solid helium, and perhaps forms, as EHRENFEST remarked, one of the most elementary confirmations of it.

b. Visual observations on the solidification of helium. For this purpose there was brought into the helium cryostat a glass piezometer F (inner diam. 7, outer diam. 13, length 90 mm), into which the helium was supplied by stopcock K_5 . Inside this piezometer was a small soft-iron rod H , which functioned as an electromagnetic stirrer (method of KUENEN). It was moved up and down with the aid of the cylinder I , which was magnetized by the current passing through the windings of the coil N . The cylinder I served also for stirring the helium bath, in order to obtain a uniform temperature (see under a).

After the measurements, mentioned under a, were finished helium was compressed into the glass piezometer.

The pressure of the helium bath was 13.3 mm, to which corresponds a temperature of 1.90° K. While the pressure within the piezometer was increased²⁾, the stirrer was constantly moved up and down. At a certain pressure the stirrer stuck; the helium had become solid. There was however nothing peculiar to be seen in the tube. Solid helium is perfectly transparent.

On decreasing the pressure the stirrer becomes free. Increase of pressure makes the stirrer stick again. These experiments were finished at 3^h.30.

The experiment was repeated at 5^h.30. The level of the helium bath was now somewhat higher (about $\frac{2}{3}$ of the piezometer was in the

¹⁾ When the melting curve continues to temperatures above the liquid-gas critical temperature, of course for those temperatures there will be a direct evaporation of the solid into the gaseous state.

²⁾ There now separated from the helium at the top of the tube a white flaky mass, which stayed for some time in this position, giving the impression of being of very loose structure. By the action of the stirrer H , small pieces occasionally got loose, and sank down in the liquid helium. They probably consisted of condensed oil vapours, which were present in the helium coming from the cycle in more than the usual (very small) quantity, because, during the preceding experiments the helium liquefactor was heated, and the oil vapours which had collected in it in the condensed state, were now scattered throughout the helium gas, particularly in that part of the cycle from which the gas to be compressed was sucked into the piezometer.

In the course of the experiment by the action of the stirrer this mass is spread over the inner wall as a nearly invisible deposit. On repetition of the experiment, in which the piezometer was filled, while the helium liquefactor was still cold, so that the oil vapours remained in the liquefactor, this phenomenon did not occur.

liquid ¹⁾). The pressure of the helium bath was now 200 mm, corresponding to a temperature of 3.12° K.

At 90 KG/cm² the stirrer is still loose, at 90.5 KG/cm² it sticks. On decreasing the pressure to 87 KG/cm² it becomes loose again. Evidently there is some delay in the phenomenon, as a result of the less rapid temperature adjustment owing to the thickness of the glass walls. Otherwise this observation is sufficiently in agreement with that of table II, for complete confirmation, that in these experiments we had observed helium solidifying.

Again there is nothing peculiar to be seen in the helium; no surface of demarcation between solid and liquid, nor between solid and gas or between liquid and gas ²⁾.

There was no indication of difference in refraction, nor change of volume. Evidently the densities and the refractive indices of the different phases under these pressures are nearly equal.

Helium solidifies to a homogeneous transparent mass. That it is a crystalline mass, seems to follow from the fact that the melting curve is sharply defined (comp. under a).

The following detail of the experiment is very instructive. At one moment the helium was liquid in the lowest part of the glass tube, where the stirrer was affixed; above it was a mass of solid helium. This appeared from the fact that the stirrer could be moved only to a definite height, where it collided against the solid block. We were able to hammer the solid helium. Very gradually the stirrer could be moved higher and higher; the block slowly melted down, probably in consequence of the temperature increasing slowly by the radiation of the lamp, which served for illumination.

Even now there was no limiting surface between solid and liquid to be seen.

§ 4. *Demonstration on July 7th.*

I had the pleasure on the occasion of the American week to demonstrate the solid helium to the American students who payed a visit to the laboratory and to some colleagues and others who were interested in it. The experiment of hammering against the block could now be repeated several times. When helium has solidified by increasing the pressure just a little above the melting pressure, and we then decrease the pressure a little, sufficiently to melt the helium slowly, the melting occurs firstly round the stirrer, the stirrer absorbing the radiation of the illumination lamp more than the transparent helium and so acquiring a slightly higher

¹⁾ In the preceding experiment $\frac{1}{4}$, about the length of the stirrer, 23 mm.

²⁾ As regards the last it should be noted, that immediately above the level of the helium in the bath there is a large temperature gradient, so there must be a rapid transition from either the liquid or the solid phase to the gaseous one.

temperature than the helium. So the stirrer gets loose before the helium melts "en masse".

The following observation is very well worth mentioning.

Towards the end of the demonstration, the helium being compressed in the tube more than ten times, there were in the compressed helium some very small dark particles, apparently oxide from the metal tube, which connects the glass experimenting tube with the other apparatus. Then on one occasion I noticed that, resulting from a blow of the stirrer, a mass of helium of about 15 to 20 mm in height was moving as one block over a distance of some millimeters. The small dark pieces made this visible.

In the experiments of this day slight indications of striae near the limit between liquid and solid were seen.

It is an agreeable duty to me to render my cordial thanks to G. J. FLIM, chief of the technical staff, and to L. and A. OUWERKERK, technicians to the cryogenic laboratory, for their intelligent aid.

Mathematics. — "*Projection of the lines in space into pairs of lines in a plane*". By Dr. HERMANC MULLEMEISTER. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of February 27, 1926).

Introduction. Since the ordinary space is fourdimensional in lines, and the plane is twodimensional in lines, it is possible to represent the lines in space by pairs of lines in a plane.

If a line l be projected from the centers C_1 and C_2 upon a plane a into the lines l_1 and l_2 , this pair of lines will be called the image of l .

Conversely; two arbitrary lines l_1, l_2 in a will in general determine one line l , which is the intersection of the planes C_1l_1 and C_2l_2 .

1. The lines in space (∞^4) are the intersections of the planes in the sheaf on C_1 (∞^2) with those in the sheaf on C_2 (∞^2). Each of these sheaves intersect a in ∞^2 lines.

A line l in space is represented by the two lines l_1 and l_2 in which the planes C_1l and C_2l meet plane a .

In case l lies in plane a , l_1 and l_2 will coincide with l , and conversely, a line in a , paired with itself ($l_1 \equiv l_2$), is the image of this line itself considered as ray in space.

Two lines l_1, l_2 in a will in general determine a unique line l : the intersection of C_1l_1 and C_2l_2 .

An exception will occur, if l_1 and l_2 coincide and this line passes through C_0 , the point where C_1C_2 meets a . In this case the planes C_1l_1 and C_2l_2 coincide, l and C_1C_2 (Call this line f) will intersect and l is indeterminate in the plane $fl_{1(2)}$.

Therefore every line l which meets f , has for its image two coincident lines, being the intersection of a and the plane fl .

Conversely, every line in a , passing through C_0 , may be considered as the image of *any* ray in $\lambda \equiv fl$. If we let λ revolve about f , its intersection with a will describe a flat pencil on C_0 .

The system of rays (∞^3) which meet f (an axial complex having f for its axis) is thus projected into the flat pencil on C_0 , every line of which is the image of a field of rays.

The line f has for its image *any* pair of lines through C_0 , i.e. any line l_1 through C_0 paired with any line l_2 through C_0 . Since it has ∞^2 images, the line f will be called a *fundamental line*.

Every line l through C_1 is *singular* (having ∞^1 images). There are ∞^1 planes through l and C_1 ; only one plane is determined by l and C_2 .

This plane (f) will meet a in $L_0 C_0$, where L_0 is the intersection of l and a .

The pencil of planes on l cuts a in a flat pencil on L_0 , and l has for its image: $C_0 L_0 \equiv l_2$ paired with *any* line l_1 through L_0 .

Similarly, every line l through C_2 is singular. One of these singular rays, $C_1 C_2$, is fundamental.

2. Image of a flat pencil (l) in a plane μ with vertex M .

Let P_1 be the intersection of MC_1 and a .

and P_2 the intersection of MC_2 and a .

Let l be any ray of the given pencil M .

The projecting planes $C_1 l$ and $C_2 l$ will meet a along the lines $P_1 L_0$ and $P_2 L_0$, where L_0 is the intersection of l and a , which will, of course, lie on the intersection m of the two planes a and μ .

The image of the pencil (l) will thus consist of two *perspective pencils* (l_1) and (l_2), whose corresponding rays meet on the intersection m of a and μ .

The centers P_1 and P_2 of the pencils (l_1) and (l_2) are collinear with C_0 ; the line $P_1 P_2$ is self-corresponding in the perspectivity and is the image ($l_1'' \equiv l_2''$) of l'' , this being the ray in the given pencil which meets f . (passing through C^* where f intersects the plane μ).

The projecting planes constitute two perspective axial pencils through $C_1 M$ and $C_2 M$, which cut a in the perspective flat pencils P_1 and P_2 .

Conversely, two perspective flat pencils in a , whose centers P_1 and P_2 are collinear with C_0 , will always be the image of a flat pencil, whose center M is found at the intersection of $P_1 C_1$ and $P_2 C_2$, and whose plane is the plane connecting this point M with the axis of perspectivity m .

If, in a special case, $\mu \parallel a$, m moves off to infinity and corresponding rays of the perspective flat pencils are parallel.

3. Two projective flat pencils (l_1) and (l_2) in a with vertices P_1 and P_2 , which are not perspective, are the image of a *quadratic ruled system*, whose axes r_1 and r_2 pass through C_1 and C_2 .

Indeed $P_1 C_1$ and $P_2 C_2$ are the axes of the projective axial pencils consisting of the planes which connect the pencil rays (l_1) with C_1 and (l_2) with C_2 . These axes do not intersect since $P_1 P_2$ is not coplanar with $C_1 C_2$.

The two projective axial pencils therefore generate a quadratic ruled system whose rulings rest upon the axes $P_1 C_1$ and $P_2 C_2$.

The projective flat pencils (l_1) and (l_2) will generate a conic which is the intersection of the ruled surface with a .

The plane through r_1 and f passes through C_2 , and therefore contains a ruling b of the system, whose image is the pair $B_0 P_1$ and $B_0 P_2$, corresponding rays in the pencils P_1 and P_2 , such that $P_1 P_2$ does not pass through C_0 .

4. *Image of a quadratic ruled system in general position.*

Let $(l)^2$ be a quadratic ruled system which does not pass through C_1 and C_2 .

This system will be projected from C_1 , by the tangent planes of a quadratic cone, and these will cut the plane α in a system of lines $(l_1)^2$ enveloping a conic.

In the same manner will the ruled system be projected from C_2 into a system of lines $(l_2)^2$ which envelop another conic.

The image of the ruled system therefore consists of two projective line conics in α .

In the ruled system there are two lines which rest upon f ; each one of these has for its image two coincident lines $l_{1,2}$ through C_0 .

This means that the two conics have two common tangents through C_0 , which are coincidences of the projective line conics.

5. *Image of a line-field $[l]$ in a plane β .*

The planes projecting a given line-field $[l]$ from C_1 intersect α in a line-field $[l_1]$ in α , and those projecting $[l]$ from C_2 produce another line-field $[l_2]$ in α , whose lines are in (1,1) correspondence with the lines of $[l_1]$.

The coplanar line-fields $[l_1]$ and $[l_2]$ are not only projective, but also perspective, and have the intersection α/β as axis of perspectivity and C_0 as center of perspectivity.

The pencil in C_0 consists of double lines $(m_1 \equiv m_2)$ and is the image of the pencil in β whose vertex is C^* , the intersection of C_1C_2 and β .

The intersection α/β also is a double line of the collineation; considered as a line l in β , it coincides with its image $(l_{1,2})$.

Let a second line-field be given in a plane γ , which cuts β in l' , then the image of this one again consists of two perspective line-fields in α which also have C_0 as center of perspectivity, and whose axis of perspectivity α/γ intersects α/β in L_0' , where l' passes through α .

6. *Image of a line-sheaf $[l]^*$ on a vertex O .*

Let P_1 and P_2 be the points where the lines OC_1 and OC_2 pass through α , then P_1 and P_2 will, of course, be collinear with C_0 .

The line-sheaf on O has for its image the ∞^2 pairs of lines formed by pairing any line of pencil P_1 with any line of pencil P_2 . Any such pair determines a ray of the line-sheaf on O .

A part of this line-sheaf is the flat pencil on O in the plane OC_1C_2 , whose image is the double line $P_1P_2(l_{1,2})$.

Let a second line-sheaf on S be pictured in the ∞^2 line pairs on Q_1 and Q_2 , the points where SC_1 and SC_2 pass through α .

The two line-sheaves on O and S have only one line in common, whose image is the pair P_1Q_1 and P_2Q_2 .

7. *Image of a bilinear congruence [1,1] with directrices r and s .*

A line l_1 in α determines a line l of the congruence which is found as the line joining the points in which the plane C_1l_1 cuts the axes r and s .

The plane joining l and C_2 then determines the corresponding l_2 .

If l_1 describes a pencil on P_1 , the plane C_1l_1 will describe an axial pencil on $C_1P_1 \equiv p$. The line l will then generate a ruled system on r, p and s .

The plane lC_2 will therefore envelop a quadric cone which has among its tangent planes C_2r, C_2s and C_2t' , where t' is the transversal through C_1 on r and s , resting at X and Y .

The line l_2 will then envelop a conic μ_2^2 in α which is tangent to r_2, s_2, t_2' , the intersections of the above named tangent planes with α .

To the pencil (l_1) on P_1 corresponds a conic μ_2^2 generated by its tangent lines (l_2) .

In the same way it can be shown that to a pencil (l_2) corresponds a conic μ_1^2 generated by its tangent lines (l_1) .

The point-field P_1 in α , each point of which may be considered as vertex of a pencil (l_1) , is thus transformed into a net of line conics μ_2^2 , all of whose members are tangent to r_2, s_2 and t_2' .

The line conic μ_2^2 is degenerated, if the pencil vertex P_1 is situated on one or two of the lines r_1, s_1 or t_1'' (where t'' is the transversal through C_2 on r and s , resting at U and V).

If P_1 is on r_1 , the corresponding line conic $[l_2]$ consists of a pencil P_2 (vertex on r_2) together with pencil Y_2 .

If P_1 is on s_1 , the corresponding line conic $[l_2]$ consists of a pencil P_2 (on s_2) together with pencil X_2 .

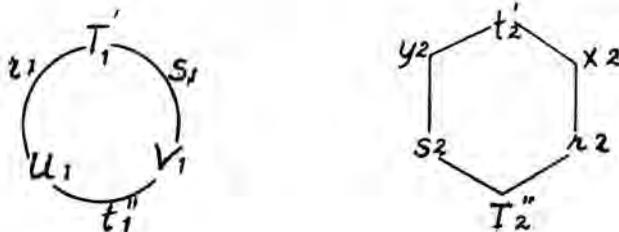
If P_1 is on t_1'' , the line conic $[l_2]$ consists of a pencil P_2 (on t_2') together with pencil T_2'' .

If P_1 is at $T_1'(r_1/s_1)$, the only corresponding l_2 is $X_2 Y_2 \equiv t_2'$.

If P_1 is at $V_1(s_1/t_1'')$, the only corresponding l_2 is $X_2 T_2'' \equiv r_2$.

If P_1 is at $U_1(r_1/t_1'')$, the only corresponding l_2 is $Y_2 T_2'' \equiv s_2$.

The following diagram shows the dual relation between the points and lines of triangle $T_1' V_1 U_1$ and the triangle formed by $t_2' r_2 s_2$.



indicating that:

Any l_1 along r_1 has corresponding l_2 through Y_2
 " " " s_1 " " " " X_2
 " " " t'_1 " " " " T'_2

and also:

Any l_1 through T'_1 has corresponding l_2 along t'_2
 " " " V_1 " " " " r_2
 " " " U_1 " " " " s_2

and vice versa.

The results which are here summarized follow from the consideration of lines and planes in space, and can be verified by a construction in the plane α , since every space construction for finding the l_2 corresponding to a given l_1 may be replaced by a plane construction in the plane α .

Astronomy. — *“Report on the expedition to Sumatra for observing the total solar eclipse of 1926 Jan. 14th.”* By J. VAN DER BILT, M. G. J. MINNAERT, W. J. H. MOLL and A. PANNEKOEK.

(Communicated at the meetings of October 31, 1925 and June 26, 1926)

I. *Organization and instrumental outfit.*

In the spring of 1924 the Royal Academy of Sciences appointed a committee which, after the nomination of some members from outside the Academy, consisted of Messrs J. VAN DER BILT, E. VAN EVERDINGEN, E. HERTZSPRUNG, W. H. JULIUS, M. G. J. MINNAERT, W. J. H. MOLL, J. J. A. MULLER, A. A. NIJLAND, A. PANNEKOEK, P. J. VAN RHIJN, W. DE SITTER and J. P. VAN DER STOK. This committee, on April 5th 1924, discussed the eclipse-problems thought most important and the general way along which they might be studied. At the following meeting, December 13th 1924, it was reported that the necessary funds for an expedition would be available; the members VAN DER BILT, MINNAERT, MOLL and PANNEKOEK were asked to form the observing party and, with the president of the committee, to act as an executive committee of the expedition. Unfortunately we could only for a short while enjoy the expert advice of our president Prof. Dr. W. H. JULIUS, who died on April 15th 1925; but we were able to draw up the general lines of research under his direction.

Thanks to Messrs NIJLAND and ORNSTEIN we were able to do all the preparatory work in the Astronomical Observatory and the Physical Laboratory of the University at Utrecht. The executive committee engaged Miss J. C. THODEN VAN VELZEN, a candidate for the Doctor's degree, as assistant, her duties to begin on May 1st 1925. Mr. H. C. BURGER, Lector in Physics at the Utrecht University has greatly obliged the Committee by giving his valuable advice in many difficulties.

The expedition was organized entirely through private gifts and facilities. The important gift of 30.000 guilders from a resident of Amsterdam, which made the expedition at all possible, has been already announced at the meeting of the Royal Academy on December 27th 1924. It was followed by a gift of 5000 guilders from the „Bataafsche Petroleum Maatschappij”, which, moreover, invited the observing party to its estate at Pladjoe near Palembang. Our assistant was enabled to join the expedition through funds from the „Ondernemersraad voor Nederl. Indië”, the „Nederlandsche Handel Maatschappij”, the „Koloniale Bank” and the „Deli Batavia Maatschappij”. The Directors of the Navigation Companies „Nederland”, „Rot-

terdamsche Lloyd" and „Kon. Paketvaart" allowed a free transport of the instruments and a reduction of 50 % to the personal passage.

Observing place.

The zone of total eclipse ran across Sumatra, Bangka and Borneo with a gradual decrease in the duration of totality. The central line on Sumatra ran South of Bengkoelen, over Tebing Tinggi and North of Palembang.

We were advised by employees of the „Bataafsche Petroleum Maatschappij" to build our camp near the coast or in the plains, at a sufficient distance from the mountains; observations concerning the gathering of clouds, which the Resident of Palembang, at our request, ordered to be made at different stations in January 1925, pointed in the same direction. In the neighbourhood of the mountains the chance of a clear sky proved to be much less than that along the coast, for which at the hour of total eclipse (2h. 30m. p.m.), the estimate was about 40 %. Though at Bengkoelen the chance for clear weather was a little more than at Palembang and the duration of total eclipse a little longer, we chose the neighbourhood of Palembang, on the ground that our instrumental outfit needed the vicinity of a well-equipped city; as such Bengkoelen stands decidedly behind Palembang. The hospitality offered by the „Bataafsche Petroleum Maatschappij" definitely settled our choice, since it would put the numerous technical facilities of Pladjoe at our disposal. So we decided to build an observing-camp on the central line near Talang Betoetoe, that seat of a Government official (Controleur), about 15 kms North West of Palembang. An additional advantage of this choice was that — most of the other parties having decided for Bengkoelen — it prevented the crowding of all the observers on a single spot of the central line.

Observing program.

In drawing up a programme of observations we thought it best to restrict ourselves to problems raised by the result of modern theoretical investigations.

Of these, the problem of the corona is doubtless the most important. Its spectrum consists of a continuous background with a number of bright lines. The first is probably formed by reflected sunlight and light from the corona itself; this is suggested by the fact that the Fraunhofer lines are weaker than those of the solar spectrum. A quantitative comparison of these two continuous spectra of the corona may add to our knowledge of its constitution.

As to the bright-line spectrum of the corona, not a single line of it is represented in the spectra of known elements. The best working hypothesis is to ascribe these lines to atoms with an abnormal number of electrons, i.e. to a multiple ionization of the atoms. The lines have not yet been reproduced in the laboratory; so their wavelengths are uncertain and suggestions as

to their origin, based upon their numerical relations could neither be proved nor disproved. Our knowledge, then, must be drawn from a study of the corona itself. For this it would be needful to get as many lines as possible and to measure their wavelengths with the highest precision. This will also enable us to judge of the reality of a number of doubtful lines, observed on a single occasion; and it will teach us — when duly repeated at future eclipses — whether the corona spectrum is variable or not. In the third place it is important to study the distribution of light in monochromatic images of the corona produced by different lines; this may teach us, which of them originate from the same kind of atoms and which from different kinds.

The spectrum of the chromosphere, too, must be observed from a modern point of view. The theory of ionization has emphasized the importance of accurate values for the intensities of the different lines of the same element. The great intensity of the enhanced lines suggests that in the outer atmosphere of the sun the atoms are more ionized than in the layers, which, by their absorption, yield the Fraunhofer spectrum. Thus also a variation of the ionization with increasing height may be expected. There exist, however, no reliable quantitative data about line-intensities. On the photographic plate a difference in intensity shows itself as a difference in the density of the silver deposit, which can be measured with a microphotometer. For the reduction of densities to intensities and for the interpretation of a change in intensity along each line, we must have some comparison line-spectra, weakened in a known ratio. Moreover, in order to compare the lines in different spectral regions, some standard-spectrum of known distribution of energy must be photographed on the same plate.

The usual flash-spectra obtained with a prismatic camera are not adequate to this investigation, because the light-source is an arc of the chromosphere of unequal width, and consequently the blackness of its image will depend not only on its intensity but also on the width of the arc. Therefore, in order to get accurate intensities, the spectrograms must be taken with a slit-spectrograph.

The continuous corona spectrum.

For the measurement of the intensities of the Fraunhofer lines, no great dispersion is wanted. For the sake of an increased brightness, a small dispersion together with a large angular aperture may be recommended. The two prism-spectrograph, built for the eclipse of 1901 seemed very appropriate for this purpose. The solar image is formed by an objective of 10 cm. with a focal length of 38 cm. The objectives of the collimator and the camera have a diameter of 36 mm. and a focal length of 13 cm.; the two compound prisms are 60 mm. wide and 38 mm. high. Upon examination, the prisms proved to give bad images, chiefly caused by large striae in the Canada balsam; these prisms were re-polished and freshly cemented by the firm of CARL ZEISS, and after this they gave very good images.

In their report on the observation of the eclipse of August 21th, 1914.

Messrs MIETHE, SEEGERT and WEIDERT have given a description of the spectrograph, with which they intended to photograph the continuous corona spectrum. In this instrument the focal ratio of the diameter of the camera-objective is only 1 : 2, and it, therefore, seemed to serve our purpose even better than our own spectrograph. The director of the GOERZ Works most kindly and disinterestedly lent us this instrument. We decided to compare the corona-spectrum which it would yield, with a number of comparison spectra, obtained by mixing sunlight and artificial white light in various known proportions.

The bright line-spectra of the corona and the chromosphere.

For the observation of these spectra we intended to use two spectrographs. One was the three prism-spectrograph, built for the eclipse of 1901 and described in the reports on that eclipse. The solar image is formed by an objective of 80 mm. with a focal length of 61 cm; the objectives of the collimator and the camera have a diameter of 55 mm and a focal length of 42 cm. The compound prisms are 60 mm wide and 55 mm high.

With this instrument, too, the prisms, at first, gave bad images. The ZEISS firm did not succeed in fully correcting these prisms, but they were so far improved as to yield very satisfactory images. We intended to place the slit along the East-West diameter of the solar image; thus, we got on both sides of the moon a radial section through the corona, nearly coinciding with the direction of the sun's equator. The plate would contain an iron-arc comparison spectrum, adjusted so as to get the iron lines only across the dark lunar disk and outside the corona.

The second instrument, with which we planned to photograph the spectrum of the corona and the chromosphere, was originally a large prismatic camera, built by Messrs TROUGHTON, COOKE and SIMMS for the eclipse of 1901. For our purpose we had it changed into an auto-collimating slit-spectrograph (Littrow-adjustment); its excellent TAYLOR-triplet (aperture 16 cm; focal length 260 cm.) now being at the same time collimator and camera objective. One of the two 45° prisms proved to give very bad images, probably the result of internal strains. We, therefore, ordered from COOKE a third prism of the same dimensions. At the same time, the rejected prism was sent to the SCHOTT Glass Works at Jena to be submitted to a renewed process of annealing and cooling (Präzisionskühlung) after which it was ground again at the ZEISS Works. The result was quite satisfactory; after its return the prism showed about the same qualities as the other ones and therefore all three were mounted in the new spectrograph.

We planned to observe the corona spectrum and the 1st and 2nd flash on plates, which would also contain iron-arc comparison-spectra for accurate wavelengths of the corona-lines and for reliable intensities for those of the chromosphere. At either contact a number of exposures would be made. For this purpose 12 plates were to be adjusted at the surface of a rotating cylinder. A solar image would be formed on the slit of this instrument by

the STEINHEIL photographic objective belonging to the Utrecht Observatory; this has a diameter of 26 cm and a focal length of 345 cm.

In order to get the successive layers of the chromosphere with maximum extension on the slit, the latter must be tangent to the solar image at the point of contact.

The relative positions of solar image and slit must remain exactly the same during the observations. A rotating coelostat is generally used for this purpose, but such an arrangement could not fulfil our requirements. The sun's image was not allowed to shift its position relative to the slit more than 1 or 2 seconds of arc, whereas the displacements, due to the imperfection of the usual coelostats and siderostats proved to be 5 to 10 seconds. Even an exceptionally well built coelostat, which was offered us by Prof. MIETHE at Charlottenburg, showed oscillations of the image, which were quite unallowable. In order to escape from these difficulties, we have chosen a fundamentally different arrangement. This consists in clamping the coelostat mirror and moving the objective in the opposite direction in which the image would travel in case of a stationary lens. If, then, the speed of the lens is made to equal that of the moving image, this will be kept in a fixed position. The screw, moving the lens, will have its errors, as well as that of the coelostat. But the effect of these errors on the steadiness of the image will be much less. In fact, compared with that, due to the coelostat-errors, it will be as many times smaller as the radius of the wormwheel is contained in the focal length of the lens. In our case this ratio is about 25. With a screw of the same quality as used in the coelostat, we, therefore, may expect the displacements of the sun's image to be much below one second of arc. The screw moves the lens by way of a small electro-motor, properly geared down, and regulated by a clock; a scheme, designed by GERRISH for driving the telescopes at the Harvard Observatory. In this way the sun's image will be kept in the same steady position for both the 1st flash and the corona; the coelostat-mirror will then be turned over an accurately determined angle for the second flash.

Monochromatic images of the corona.

In order to obtain the distribution of the radiation in the corona for various wavelengths, the use of a prismatic camera is indicated; this will yield a number of monochromatic images of the corona. The usual prismatic cameras having all been designed for the flash spectrum, we had to discuss the demands for work on the corona. The inner-corona sends us some faint monochromatic radiations and a rather strong continuous spectrum. The surface-brightness of the monochromatic corona images depends on the angular aperture of the camera; the contrast between these images and the continuous spectrum depends on the dispersion of the prisms. Hence, both the angular aperture of the camera and the dispersion had to be taken as large as possible. According to this principle we designed two prismatic cameras of a special type.

The first one has a camera with a ZEISS Astrotriplet, with an aperture of 6 cm and a focal length of 27 cm, making the inner diameter of the corona-rings 2.4 mm. In front of this objective are four flint prisms of 66° ; they give a dispersion of $5^\circ,5$ from H_β to H_γ . With such a long path inside the prisms, there is a danger that light-beams, far away from the central one, will partly be cut off; this was compensated by increasing the size of the prisms as they came farther from the camera. As a result of the strong astigmatism caused by such a series of prisms, the corona-ring is circular only for the central wave-length; for the shorter waves the rings are elongated in the direction of the spectrum; for the longer waves they are elongated perpendicular to the spectrum. Since the camera would be in the way of the incident light, we placed a small mirror in front of the first prism. This mirror sends the light reflected from a coelostat-mirror through the train of prisms. The instrument was constructed in the ZEISS Works at Jena.

In our second prismatic camera we used a liquid prism. As a consequence of the great dispersion of highly refractive liquids, a single liquid prism will give a dispersion equal to that of a number of glass prisms. Such prisms find no application in research work, because small differences of temperature cause striae in the liquid, which may distort the spectral images. By seeing to good isolation, we hope to prevent rapid changes of temperature. The same intention led us to the use of a direct-vision prism; the thick pieces of glass in front and behind the liquid, help to protect it from the heat. Since the instrument will only be exposed when the sun is eclipsed, the heating by the radiation itself will be negligible.

We, therefore, ordered from Messrs STEEG und REUTER at Homburg v. d. Höhe, a direct-vision "WERNICKE"-prism, to be filled with ethylcinnamate. We intend to use this in combination with a ZEISS camera of the same type as used in the last mentioned instrument.

We have thought it an interesting experiment to see whether, in eclipse-work, such a peculiar prismatic camera could be useful.

Solar radiation.

ABBOT derived the distribution of the energy over the sun's disc by direct measurements in a solar image. These do not include more than 95 % of the radius, reckoned from the centre, whereas just the sun's edge, from which comes all the radiation, emitted under angles of 72° to 90° , is of fundamental importance to our knowledge of the physical constitution of the photosphere.

JULIUS remarked that measurements in a solar image will not give correct results, since it is covered by light, diffused by the earth's atmosphere. He, therefore, proposed a method to deduce the distribution of intensity from measurements of the radiation during a solar eclipse. This method, which is entirely free from the disturbing effect, just mentioned, has been applied at the eclipses of 1905 (Burgos, Spain), 1912 (Maastricht, Holland)

and 1914 (Hernösand, Sweden) ; on the last occasion for a number of spectral regions.

The results of the 1914 eclipse, however, showed that the method is only reliable just before and after totality, i.e. when observing the radiation of the outermost regions. We originally planned to apply JULIUS' method, thus limited, at the Sumatra-eclipse. But since this eclipse did not seem to serve the purpose very well (difference between the solar and lunar diameter rather large ; a damp, tropical climate) we resolved to try measurements outside an eclipse, using refined instruments and choosing the most favourable observing site. As such we decided for the Gornergrat, Switzerland ; Messrs MOLL, BURGER and VAN DER BILT spent the month of August 1925 up there and got a series of preliminary measurements, the results of which have been published in the Bulletin of the Astronomical Institutes of the Netherlands (Vol. III N^o. 91). Measurements during a solar eclipse following JULIUS' suggestion remain desirable, even now, but only at an eclipse of short duration, observable in a good climate.

II. *The expedition to Sumatra.*

The journey.

The observing party left Marseilles for Singapore on the „Rotterdamsche Lloyd“ liner „Tabanan“, November 10th 1925. From Singapore the ss „van Swoll“ of the „Koninklijke Paketvaart Maatschappij“ conveyed the expedition to Palembang, where it arrived on Dec. 4th. A few hours earlier the ss „Wega“ of the Colonial Service Marine, put at our disposal by the Vice-admiral, Commander in chief of the naval forces at Batavia, had been anchored on the river. Its commanding officer, Mr. J. M. VOOREN and his staff, enlarged at our request with two additional officers, have regularly taken part in the mounting of the instruments, in the drills and the observations. The „Wega“ had also brought to Palembang Mr. N. P. MIEDEMA, chief-instrumentmaker in the Topografic-Service of the Army; we are much obliged to the chief of this Service for sending us this officer, whose zeal and capability have been of great value to us. On Dec. 31st Messrs P. VAN LEEUWEN BOOMKAMP and L. E. VAN LEEUWEN BOOMKAMP joined the party and at once took part in the work.

The members of the expeditions gladly accepted the invitations from the Directors of the „Bataafsche Petroleum Maatschappij“ to live in cottages of the Pladjoe-estate and from the Commander of the Navy to use the first class cabins of the „Wega“. Two days after the eclipse, the whole party left Palembang for Batavia on this ship. From there, Mr. PANNEKOEK went straight to the „Bosscha Observatory“ at Lembang in order to continue his observations of the southern Milky Way, which he had already started during the outward trip. The other members spent a few days in Batavia ; there they payed their respects to Their Excellencies the Governor General and the Commander in chief of the naval forces, and thanked them for the courtesies and assistance which all the various govern-

ment services had extended to the expedition. Mr. P. VAN LEEUWEN BOOMKAMP then, invited them for a short stay in Java (including a visit to the Bosscha Observatory) and in Deli (East-coast of Sumatra). From there they returned in Holland on Febr. 14th 1926 in the Nederland-liner „Koningin der Nederlanden“. Mr. PANNEKOEK left Java in the same ship on May 19th.

The eclipse-camp.

Prof. E. W. BROWN of Yale University had informed us that, judging from the 1925 eclipse-results, the central line on Sumatra would run about half a mile north of the almanac-line. We, therefore, choose for the installation of our instruments, a site about 18 kms from Palembang, by the road from Talang Betoetoe to Pangkalan Benteng. The coordinaties of this place were

Latitude = $2^{\circ} 52' 58''$ South
 Longitude = $6^{\text{h}} 58^{\text{m}} 40^{\text{s}.2$ East

After the brush had been cut and the ground cleared, the „Bataafsche“ started with the building of our camp according to our instructions. They had put their architect Mr. B. C. VAN DE WETERING in charge of the work ; he finished it promptly to our entire satisfaction. The relative position of the piers is shown in the diagram ; over these piers we had shelters, the upper part of which could be removed when the observations made this necessary.

To the North of the piers a large barrack was built, which contained a dark room, a workshop, a dining room, a packingroom, and a garage. Through the care of the Chief Engineer of the Civil Public Works, the camp was surrounded by a barbed wire railing. Outside this railing barracks were built for a non-commissioned officer and seven soldiers, who by order of the Military Officer in command, would have to guard our camp.

About 100 m. from the piers a gasmotor and dynamo were installed ; the latter supplied the current for two electric arc-lamps, the storage batteries, the illumination of the dark room and a number of dim lamps near the instruments. The storage batteries were lent to us by the „Borneo Sumatra Handel Maatschappij“ and charged by the „Nederlandsch Indische Gas Maatschappij“.

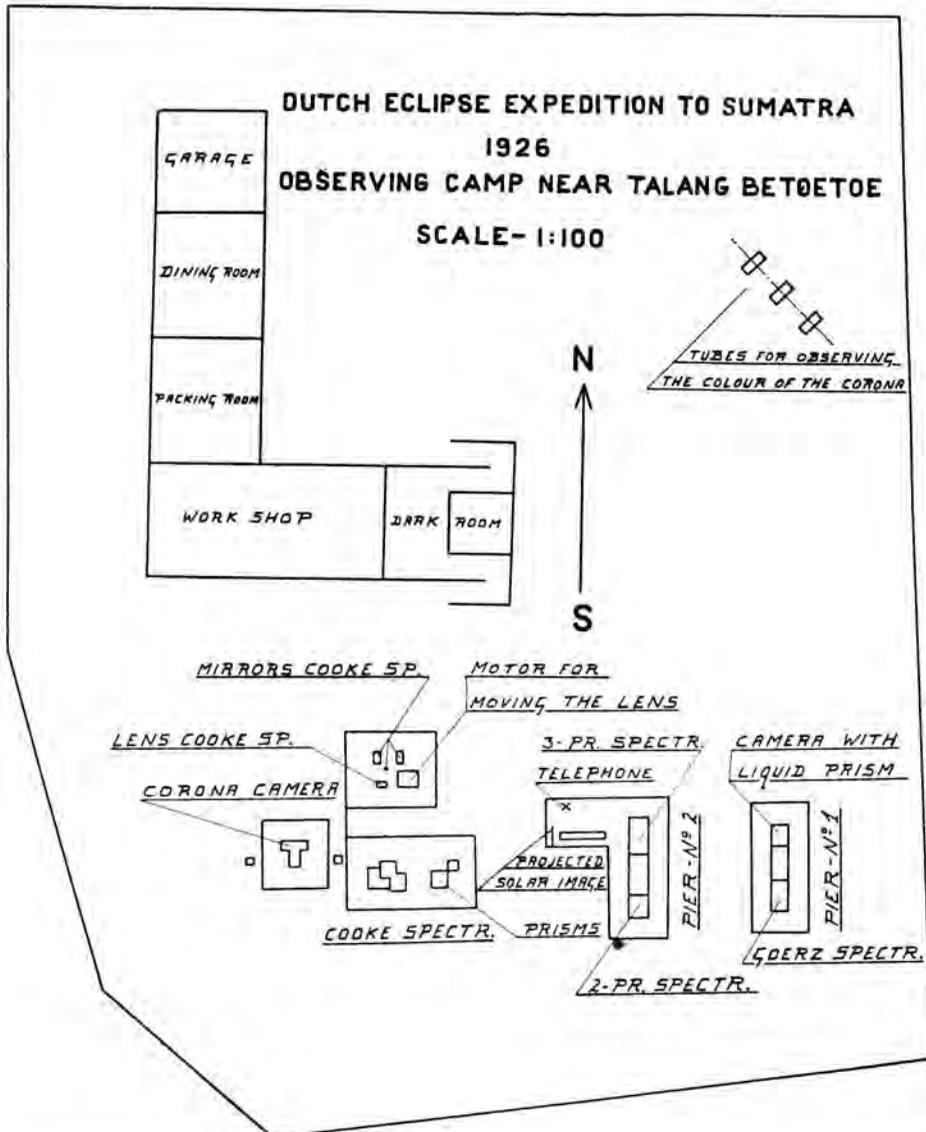
A motor-launch and a motor-car, both of the „Bataafsche“ secured a regular communication between Pladjoe, Palembang and the eclipse camp.

The instruments.

1. *Adjustment.* As indicated in the diagram, the prismatic camera with liquid prism and the Goerz-spectrograph were fed by the mirrors of the Eastern-most siderostat ; this was a double polar siderostat, clockdriven. The two- and three-prism spectrographs were fed by the mirrors of the second siderostat ; this too, was a double polar siderostat, driven by a motor which was regulated by a pendulum-clock.

The mirrors and the spectrographs were adjusted with respect to the pole of the heavens by using two universal instruments, borrowed from Prof. H. J. HEUVELINK of Delft and Prof. J. W. DIEPERINK of Wageningen. With a view to make these adjustments independent of weather conditions, we began with the measurement of the azimuth of a land-mark, a fork in a leafless tree, about 900 m Southward of the camp.

The siderostats and coelostats were adjusted by a method differing from that usually followed. In order to apply this method, both universal



instruments had been supplied with a Gaussian eye piece, through which auto-collimation became possible. Now, to adjust the axis of the siderostat

with respect to the pole, one of the mirrors was placed perpendicular to the axis ; then a universal instrument was placed in front of the mirror, and its telescope — by way of the known latitude of the camp and the azimuth of the landmark — carefully adjusted so as to be parallel to the earth's axis. We had then only to correct the inclination and the azimuth of the axis of the siderostat, till the observer at the telescope saw the cross-wires coincide with their reflected images.

The spectrographs were adjusted in the following way ; the universal instrument was placed on the pier of the spectrograph with its telescope properly adjusted so as to be parallel to the earth's axis, and the mirror adjusted so as to get a solar image on the cross-wires. Then, the driving mechanism of the siderostat was started, the universal instrument exchanged for the spectrograph and the latter adjusted so as to get the solar image on the slit.

With the coelostat used for the corona camera we proceeded as follows : the two universal instruments were placed on either side of the mirror, the telescopes pointing East and West and their optical axes coinciding. Then, by adjusting the mirror in its box and changing the azimuth of the axis of the coelostat, a position could be arrived at, in which the cross wires of both telescopes could be brought into coincidence with their reflected images. This meant that the plane of the mirror was parallel to the axis, and the axis was in the meridian plane. It remained to give the latter its proper inclination ; this was done with one universal instrument, whose telescope was in the meridian, and at right angles to the earth's axis. The inclination of the axis of the coelostat was then corrected, till the observer at the telescope saw the wires cover their reflected images.

The method of auto-collimation was again very useful on the day of the eclipse ; it allowed the coelostat mirrors, which could not be adjusted in hour-angle, to be clamped in the exact position on starting the driving mechanism.

All the mirrors have been resilvered a week before the eclipse. Mr. G. F. F. AVÉ LALLEMANT, in charge of the hospital at Pladjoe, had the courtesy to abandon his well lighted and dust-free operating room for this purpose.

2. *Focussing.* We devised the following method for obtaining a sharply focussed image on the slit of the spectrograph. A large screen with a central hole of about 1 cm was moved to and fro before the objective; if the slit is not exactly in the focal plane of the lens, the sun's image will be seen either to follow the motion of the screen, or to move in an opposite direction. Since the human eye is very sensitive to small displacements, this method affords a very sharp focussing. With regard to the spectral regions under observation, the use of a blue glass was necessary.

The two prismatic cameras were photographically focussed, using a collimator and an iron arc-lamp. About the camera with liquid prism we

want to say that the results came up to our expectations ; even without special precautions, the sharpness of the iron lines was very satisfactory. The ethylcinnamate in the brass tube, after some days, showed a slight precipitation and was therefore replaced by fresh liquid one day before the eclipse.

3. *Adjustment of the slit of the large spectrograph.* The mounting of the large auto-collimating Cooke-spectrograph required a number of piers (see diagram).

A mirror mounted as a coelostat sent a highly inclined beam of sunlight in the plane of the meridian to a second mirror, which reflected the light to the Southpoint of the horizon. Thus, the rotation of the sky caused the solar image to travel in a horizontal direction over the slit ; this motion, as has been stated above, could be compensated by a horizontal motion of the objective.

The mirrors were clamped in such a position, that two minutes before totality the solar image was 1 cm from the slit and was approaching it. On the instant it reached the slit, the carrier of the lens was coupled to the already working motor, causing the solar image to remain fixed. The observer at the viewing eyepiece of the spectrograph could visually observe the spectrum and move the slit parallel to itself with a micrometer-screw. By the length of the continuous spectrum he could judge what part of the slit covered the sun's image. Practical considerations induced us to adjust the slit parallel to the tangent, 1" inside the solar image. In this position the slit cut edge of the image $2^{\circ}.5$ from the point of contact in a point, where the chromosphere would be covered by the moon only by $0''.02$. This position would be kept while photographing the spectrum of the corona. In order to get the second flash, an assistant could turn the coelostat mirror through a previously determined angle. The moving lens compensated the rotation of the sky and the mirror could be adjusted for the second flash with a precision that left nothing to be desired.

The colour of the corona.

Some days before the eclipse, we decided to have colour observations of the corona made by amateur-observers. The contradictory results to which colour-estimates of the corona have led, are probably due to a lack of a suitable object for comparison. As such we choose a piece of white cardboard, illuminated by a Philips' daylight-lamp. The observer, looking in a cardboard tube could simultaneously view the corona and the illuminated screen and compare their colours. We found Messrs VAN DE WETERING, MOORMANN and WALLIS DE VRIES willing to make these observations.

Time-service.

The Navy Department in Holland, had granted us the loan of 3 chronometers. During our stay at the eclipse camp these were kept under control

by comparing them with one of the chronometers of the „Vega”, the correction for which was derived from daily wireless time signals. Since the calculated time of the first flash could not be trusted to a few seconds, we took two precautions so as to be exactly in time. Firstly we arranged to have a solar image of about 10 cm projected on a screen and to observe the moment at which the distance between the horns of the crescent was that computed for 12 seconds before totality. Secondly we had a telephonic connection made between the camp and a post on the central line, 10 kms distant. An observer at this point would see totality begin 12 seconds earlier than the observers in the camp. This responsible post was held by Mr. H. J. SCHMIDT, acting Resident of Palembang. His signal was received by Mr. J. VAN DER BILT, who also observed the projected solar-image, and with the assistance of Mr. ADAM, 4th officer of the Government's Navy, counted and called the seconds during the total phase of the eclipse.

Eclipse-day.

A week before the eclipse we decided upon the operations to be carried out on the eclipse-day; during the following days the observers went through regular rehearsals. On the eve of the 14th all the instruments stood carefully adjusted and tested; and we had the agreeable feeling that our rather complicated equipment, every detail of which had been cared for, would fully justify our expectations.

The arrangement for eclipse-day was the following :

a. *Corona camera with liquid prism* : Mr. E. K. VAN MELLE, 2nd officer Governm. Navy.

b. *Goerz-spectrograph* : Mr. A. J. LOUET FEISSER, 2nd officer G. N.

c. *Two-prism spectrograph* : Mr. J. M. VOOREN, Commanding officer G. N.

d. *Three-prism spectrograph* : Dr. A. PANNEKOEK, for the manipulation of the iron-arc lamp assisted by Mr. R. TROOST, 2nd officer G. N.

e. *Cooke-spectrograph* : Dr. M. G. J. MINNAERT, assisted by Mr. P. VAN LEEUWEN BOOMKAMP at the revolving plate-holder and the slit-shutter; by Mr. J. J. NEESSEN, acting 1st off. G. N. at the iron-arc lamp; by Mr. L. E. VAN LEEUWEN BOOMKAMP at the mechanism for moving the lens; and by Mr. B. M. D. FHIJNBEEN, non-commissioned officer of the Army, at the handle for turning the coelostat mirror.

f. *Corona-camera*. Miss J. C. THODEN VAN VELZEN.

Dr. W. J. H. MOLL superintended the correct execution of the program, ready to act in case of emergency. The instrumentmaker, Mr. MIEDEMA stood by him.

Before we left Holland, we had gathered from various informations, that we should encounter a rather regular change of weather in the course of a day, with little chance of a clear sky in the afternoon. But, to our surprise, the weather, during the six weeks of our stay at Palembang, was without any regularity. Entirely cloudy days were sometimes followed by rains,

sometimes by sunshine ; and this, apparently, regardless of the hour of the day. The last five days before the eclipse, conditions at eclipse-time were rather favorable.

On the morning of January 14th there were thin clouds in the sky. Shortly before noon, denser clouds came over from the North-West, soon covering the whole Western sky ; they brought us a shower about half an hour before 1st contact. It looked for a while as if there would be a change for the better ; but then a layer of cirro-cumuli began to cover the sun.

The mirrors and the projected solar image could still be adjusted, but towards 2nd contact the sheet became thicker and, though the beginning of totality could be seen with the naked eye, the flash spectrum was not visible in the viewing telescopes of the 3-prism spectrographs. The signal from the observer in the telephone-hut, where the sun was free from clouds, came exactly 12 seconds before totality ; the screen had since long ceased to show anything of the sun's image. Totality began two seconds later than was computed. Towards the time of third contact, the sheet of clouds became a little thinner. The observers duly went through all the operations, but were convinced of the fruitlessness of it ; development of the plates proved that they were right. The plates taken with the slit spectrographs revealed nothing. On those, taken with the slitless spectrographs, the prominences have left faint traces ; the plate of the corona camera showing the $H\beta$, $H\gamma$ and $H\delta$ images and that of the liquid prism camera showing in addition to these, the H and K lines. This fact is an indication of the great light-gathering power of the two instruments. Needless to say, that, in these circumstances, colour-estimates of the corona could not be made.

Final remarks. The expedition had no success ; but neither the effort nor the expense have been made for nothing. In the course of the preparative work new problems have been brought forward, new methods have been devised and new instruments have been constructed, which could be tested not only in the laboratory, but also under the peculiar conditions of an eclips-camp in a tropical climate.

This last experience is valuable in view of the eclipses of 1929 and 1934, which will be visible in the Dutch East Indies.

Our expedition, though missing its mark, has not been entirely fruitless. It led to the expedition to the Gornergrat and it offered to Mr. PANNEKOEK an opportunity to extend his researches on the structure of the Milky Way to the whole sky. We consider these two facts to constitute a positive gain to science. Moreover, thanks to the improvement of the older instruments and the construction of modern ones, the Royal Academy now possesses a valuable instrumental outfit, which is not solely useful for future eclipse-work, but, between eclipses, will be available for scientific work of different kind.

In concluding, we wish to express our appreciation and gratitude to

Mr. P. VAN LEEUWEN BOOMKAMP who, feeling sure that the Dutch Government would not be prepared to pay the heavy expenses of a solar eclipse expedition, generously provided the necessary funds, both for the Gornergrat and the Sumatra expeditions. He thus saved the honour of our national science at an international scientific event.

Furthermore, we are greatly indebted to the Directors of the „Bataafsche Petroleum Maatschappij“ and to all the other Companies mentioned in this and in our first report.

We also want to express our gratefulness to all the Government employees ; and to all the managers and employees of estates and factories, whom we met on our way ; they all did their best to be useful to our work and to make our ocean trips and our stay in India a real pleasure to us all.

Astronomy and Geophysics. — "*Ionisation Equilibrium in Stellar Atmospheres and in the Earth's Atmosphere*". By Prof. A. PANNEKOEK.

(Communicated at the meeting of September 25, 1926)

In 1920 the Bengalese physicist MEGH NAD SAHA derived from the equations of chemical reactions in thermodynamical equilibrium a formula for the ionization as a function of temperature and pressure:

$$\frac{x}{1-x} P_e = \frac{(2\pi m)^{3/2}}{h^3} (kT)^{3/2} e^{-\chi/kT}$$

where x is the fraction of the gas ionized, χ the ionization energy per atom, m the mass of an electron and P_e the electron pressure. This formula has already had an extensive application on the stellar atmospheres and it has provided a physical basis for the explanation of stellar spectra. There remained, however, an uncertainty. The stellar atmospheres are not gases in thermodynamical equilibrium, because a radial stream of energy goes outward causing the temperature to decrease to a boundary temperature. Thus the formula is not strictly applicable to these atmospheres; and where discrepancies are found between observational data and theory it cannot be decided whether this discrepancy is real or should be ascribed to this deficiency of theory.

This uncertainty can now be removed since formulas relating to ionization in non-isothermal gases have been derived, at first by SAHA and SUR, and then by J. WOLTJER ¹⁾, who made use of MILNE's researches. For the unknown mechanisms of ionization by incident radiation and of recombination of an electron with an ion MILNE introduces probability coefficients, which multiplied by the numbers of the particles give the numbers of the processes of ionisation and recombination per second. In a state of equilibrium these numbers must be equal; thus a relation between the numbers of the three kinds of particles is found. Applying this relation to the case of an isothermal gas, we must find the SAHA formula; this condition determines relations between the probability coefficients that have been deduced by MILNE ²⁾. Now we must assume that these coefficients express atomic properties not depending on statistical distribution quantities such as temperature and pressure. Then they must hold also in other cases, when there is no equilibrium and we may introduce them into the formulas representing these cases.

¹⁾ Physica 5, p. 406.

²⁾ Philosophical Magazine, (6) 47, p. 209, 1924.

If we put x_1 x_2 y the numbers of atoms, ions, and electrons per unit volume, $I(\nu)$ the intensity of the total radiation of frequency ν falling from all sides on the atoms, the number of ionizations by this radiation may be expressed by $x_1 \psi(\nu) I(\nu) d\nu$. This probability coefficient ψ has only values for $\nu > \nu_0$, if $h\nu_0$ denotes the energy of ionization; if the energy $h\nu$ is absorbed from the radiation, the part $h\nu_0$ is used for the ionization and the rest $h(\nu - \nu_0)$ is transformed into translation energy of the electron, amounting to $\frac{1}{2} m v^2 = h(\nu - \nu_0)$. Thus a continuous absorption band is formed, from ν_0 extending to $\nu = \infty$. The compensating process is the recombination of an electron with an ion by impact; the liberated ionization energy $h\nu_0$ with the kinetic energy of the electron is radiated as a quantum $h\nu = h\nu_0 + \frac{1}{2} m v^2$; thus a continuous emission band appears. The number of recombinations, proportional to the number of ions (assumed at rest) and the number of electrons, which by the partition law of MAXWELL have this velocity v , is given by

$$8\pi x_2 y \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-m v^2 / 2kT} \{ F(\nu) + I(\nu) G(\nu) \} v^3 d\nu,$$

where $F(\nu)$ and $G(\nu)$ denote the probabilities of spontaneous and of induced recombinations. If the total numbers of ionizations and recombinations, caused by radiation of all wavelengths, are equal, we have equilibrium. Then

$$x_1 \int_{\nu_0}^{\infty} \psi(\nu) I d\nu = x_2 y \int_{\nu_0}^{\infty} 8\pi^2 \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-h(\nu - \nu_0) / kT} \{ F(\nu) + I \cdot G(\nu) \} \frac{v^2 h}{m} d\nu.$$

In the case of isotropic radiation of temperature T , where I is given by PLANCK's formula, this equation must produce the SAHA equation

$$\left(\frac{x_2 y}{x_1} \right)_0 = \frac{(2\pi m)^{3/2}}{h^3} (kT)^{3/2} e^{-h\nu_0 / kT}.$$

In this case the integral signs may be dropped, because for each separate frequency there must be equilibrium between ionization and recombination. In this way relations between $F(\nu)$ and $G(\nu)$, and $\psi(\nu)$ are found, which introduced into the equilibrium formula give it the form

$$\frac{x_2 y}{x_1} = \left(\frac{x_2 y}{x_1} \right)_0 \frac{\int \psi(\nu) I d\nu}{\int \psi(\nu) e^{-h\nu / kT} \left(\frac{8\pi h\nu^3}{c^2} + I \right) d\nu}$$

where the integrations are taken from ν_0 to ∞ . This is the same formula as deduced by WOLTJER.

2. For the integration the function $\psi(\nu)$ is needed. It is not quite

unknown; from the experimental fact that the coefficient of absorption at the limit of the band, i.e. for $\nu \rightarrow \nu_0$ or $\nu = 0$ is neither zero nor infinite, MILNE deduces that $\text{Lim } F(\nu)/\nu^2$ for $\nu = 0$ must have a finite value. The simplified mechanism of electron capture used by EDDINGTON to compute the stellar absorption coefficient causes a strict proportionality of $F(\nu)$ with ν^2 . This corresponds to $\psi(\nu)$ being proportional to ν^{-3} . If this proportionality does not strictly hold, the differences become important only for large ν , where I is very small. Thus the result computed with this formula, if not rigidly, will certainly be practically right.

For a stellar atmosphere we compute I from the conditions of radiative equilibrium, for which the SCHWARZSCHILD approximation suffices. This means that I for the directions from the lower half of the sphere is taken equal to black radiation of the temperature T_1 , the effective temperature of the sun or the star, and for all directions from above it is taken 0. In equilibrium with this radiation the atmosphere has a temperature $T = T_1 \nu^{1/2}$. In WOLTJER's formula the second term I in the denominator, originating from $G(\nu)$, may be neglected for the same reason and in the same cases as the second denominator term in PLANCK's formula; it is only relevant for low frequencies, while here we have to integrate over high frequencies only. The integrals in the above formula now become:

$$\int \psi(\nu) I d\nu = \int C \nu^{-3} \frac{4\pi h\nu^3}{c^2} e^{-h\nu/kT_1} d\nu = \frac{4\pi CkT_1}{c^2} e^{-h\nu_0/kT_1}$$

$$\int \psi(\nu) e^{-h\nu/kT} \frac{8\pi h\nu^3}{c^2} d\nu = \int C \nu^{-3} \frac{8\pi h\nu^3}{c^2} e^{-h\nu/kT} d\nu = \frac{8\pi CkT}{c^2} e^{-h\nu_0/kT}$$

Thus the factor by which the formula of SAHA has to be multiplied, will be

$$\frac{1}{2} \frac{T_1}{T} e^{-h\nu_0(1/kT_1 - 1/kT)}$$

If we write χ for the ionization energy $h\nu_0$, the formula becomes

$$\frac{x}{1-x} P_e = \frac{1}{2} \frac{(2\pi m)^{3/2} k^{5/2}}{h^3} T_1 T^{3/2} e^{-\chi/kT_1}$$

Thus we find that the ionization, if computed after the Saha formula not with the atmospheric but with the effective temperature, needs only a constant correction factor $\frac{1}{2}(T/T_1)^{3/2}$, which in this case is $2^{-11/8}$.

3. In the same way also the ionisation of the upper layers of the atmosphere of the earth may be computed. From the propagation of wireless waves around the spherical earth it has been deduced that the highest layers of our atmosphere are conductive through the presence of a great number of free electrons or ions. The lower limit of this HEAVISIDE

1) The ionization by electron impacts for such small pressures, after R. H. FOWLER's formula (Philos. Magazine, (6) 47, p. 275), is wholly negligible compared with the photoelectric ionization.

layer is given usually at 90 kms and its electron density at 10^5 per ccm. at least; in the daytime the limit seems to be lower and the whole phenomenon seems to be less regular than by night. In order to explain this layer either ionization by impacts of swift moving particles is assumed, which ejected by the sun penetrate our atmosphere, or photoelectric ionization by the solar radiation. The latter effect can now be computed exactly.

The total radiation falling upon the particles is solar radiation proceeding from $1/184000$ of the sphere; this fraction will be called β . We assume a single gas with ionisation energy χ and temperature T ; the effective temperature of the radiation is T_1 . Then we have in the same way as in the former case

$$K = \frac{x^2}{1-x^2} p = \beta \cdot \frac{(2\pi m)^{3/2} k^{5/2}}{h^3} T_1 T^{3/2} e^{-\chi/kT_1},$$

where in the first part the total pressure has been introduced, because the numbers of electrons and ions are equal. The gases to be dealt with are nitrogen, oxygen, and hydrogen, for which the ionization potential for the molecules has to be taken. For nitrogen it is 16.9 after the measures of BRANDT; for hydrogen 16.1 ± 0.2 is given by FRANCK and JORDAN¹⁾ as an average of determinations by many observers; for oxygen, where 15.5 resulted from the older measures of MOHLER and FOOTE a recent accurate determination by LOCKROW²⁾ also gives 16.1. Assuming a temperature -55° ($T = 218^\circ \text{K}$) we find for $\log K$ values -18.65 for N , -17.98 for H and O . Thus in all practical cases x is very small, $1-x^2$ may be taken 1, and for K we may write $x^2 p$.

The energy used for ionizing the molecules is taken from the solar radiation having a frequency higher than $\nu_0 = eV/300h$; thus an absorption band is formed, beginning at $\lambda = 730 \text{ \AA}$ (for N) or $= 766 \text{ \AA}$ (for O and H). The absorption caused by the ionization of the higher layers weakens the radiation falling upon the lower layers: therefore the ionization of these lower layers is diminished also, and disappears below a certain limit, when all active wavelengths have been extinguished from the solar radiation. Putting a for the atomic absorption coefficient, N' for the number of atoms per ccm for one atmosphere and a temperature T , we find that a layer of the gas 1 cm thick absorbs the fraction $aN'p$ of the incident light. For the atomic absorption coefficient a a formula has been deduced by MILNE by making use of KRAMERS' computations; by omitting several factors of order zero it becomes

$$a(\nu) = \frac{16\pi^2 Z^2 e^6}{3\sqrt{3}} \frac{\chi}{ch (h\nu^3)}.$$

It has been derived for atoms; since factors of order zero are irrelevant here it may be used also for molecules. Computing its value for the limit

¹⁾ Anregung von Quantensprünge durch Stösse (Handwörterbuch der Physik 23, 749, 757).

²⁾ Astrophysical Journal 63, p. 205 (1926).

of the absorption band $\nu = \nu_0$ we find $\log a = -15.91$ for N, -15.71 for O, -17.56 for H.

The distribution of the ions over different layers of the atmosphere may now be computed. In a layer dh we have $N'p dh$ atoms; thus the absorption is $dI/I = aN'p dh = -aN'H dp$, if with H we denote the equivalent height of one atmosphere. Then

$$I/I_0 = e^{-xN'H p},$$

gives the weakening of the active solar radiation. This factor adds to the coefficient β in the ionization formula; the ionization is given by

$$x^2 p = K e^{-xN'H p}$$

and the number of electrons per ccm $n = xN'p$ becomes

$$n = N' \sqrt{pK} e^{-1/2 x N' H p}.$$

Introducing the numerical values $\log N'$ (for $T=218^\circ$) = 19.83; $\log H = 5.82$ for nitrogen, 5.76 for oxygen, 6.96 for hydrogen, and $\log K$ as given above, we have

$$n = [10.51] \sqrt{p} e^{-[9.44]p} \quad (\text{for nitrogen})$$

$$n = [10.84] \sqrt{p} e^{-[9.58]p} \quad (\text{for oxygen})$$

$$n = [10.84] \sqrt{p} e^{-[8.93]p} \quad (\text{for hydrogen}). \quad ^1)$$

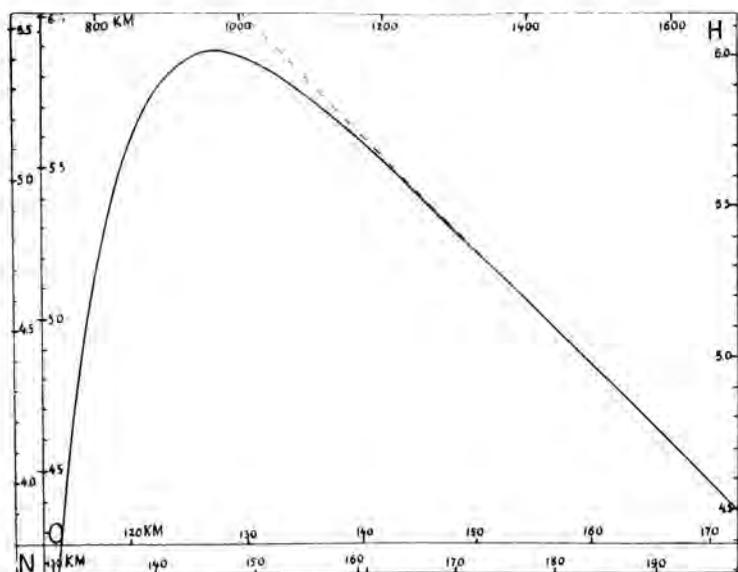
In the following table the value of $\log n$ has been computed for different values of the pressure p , and the corresponding heights in kilometres have been added.

Number of electrons per ccm.

| $\log p$ in atm. | Nitrogen | | Oxygen | | Hydrogen | |
|---------------------|----------|-----|----------|-----|----------|------|
| | $\log n$ | h | $\log n$ | h | $\log n$ | h |
| - 7 | | | | | - 30 | 500 |
| - 8 | - 5.5 | 120 | - 10 | 103 | + 3.14 | 710 |
| - 9 | + 4.81 | 135 | + 4.68 | 116 | + 5.97 | 920 |
| - 10 | + 5.39 | 150 | + 5.67 | 129 | + 5.80 | 1130 |
| - 11 | + 5.00 | 165 | + 5.32 | 142 | + 5.34 | 1340 |
| - 12 | + 4.51 | 180 | + 4.84 | 155 | + 4.84 | 1550 |
| - 13 | + 4.01 | 195 | + 4.34 | 168 | + 4.34 | 1760 |
| - 14 | + 3.51 | 210 | + 3.84 | 182 | | |

¹⁾ These computations are made for the limit of the band ν_0 . For greater values of ν the absorption is smaller, thus the intensity of the smaller wave lengths increases relative to this limit in the lower atmospheric layers. Computation shows, however, that this difference becomes perceptible only in depths where all action has disappeared.

The figure shows the variation of this number $\log n$ with the height, which for each gas is represented by the same curve, only referred to



different scales, which for each of them has been indicated ¹⁾. The dotted line represents the number of electrons if there had been no absorption. The figure shows that, descending from great heights the number of electrons at first increases regularly, then reaches a maximum between 10^5 and 10^6 , and falls rapidly to zero because of the increasing absorption. For nitrogen and oxygen the maximum occurs at 145 or 128 kms, the lower limit at 130 and 113 km.

Thus the number of electrons corresponds very well with what has been deduced from wireless experiments. The lower limit, however, found here, lies at a greater height than is usually adopted. Now in the newest literature greater heights are sometimes given; from experiments with short waves TUVE and BREIT deduced heights between 80 and 160 kms. From the experiments of TAYLOR and HULBURT and others HESS concludes: "Die Elektronenzahl pro Kubikzentimeter scheint nach diesen Untersuchungen in Höhen von 110 bis 250 km, ihren Maximalwert (10^5 pro ccm.) zu erreichen (bei Tage)" ²⁾. According to our computations electrons by photoelectric ionizations cannot occur below 100 kms. A moderate error in the adopted atomic coefficient of absorption has no appreciable influence on $\log n$ as a function of p ; the value a should be decreased 100 times to lower the limit 30 kms. A lower temperature than -55° C. for the upper atmospheric, however, which contracts the

¹⁾ The scale for O should be raised by 0.20; the values used in constructing the figure had been computed with an ionization potential of 15.5.

²⁾ V. T. HESS, Die elektrische Leitfähigkeit der Atmosphäre, p. 164.

gases more at the bottom of the atmosphere, will diminish the height corresponding to each p appreciably. If the temperature decreases regularly from -55° C. at 15 kms to -160° C. at 90 kms, the lower limit of the electron layer would be brought 30 kms lower. A considerable amount of electrons at a height of 90 kms or lower, in as much as it cannot be ascribed to ionization by impacts, thus points to a much lower temperature of these upper layers than is usually assumed.

For hydrogen the region of ionization begins only above 700 kilometers. To the HEAVISIDE layer ionization of hydrogen cannot contribute anything. Conversely the ionization of hydrogen at great heights extinguishes all solar wave lengths below 766 A., just the frequencies that are necessary for the ionization of nitrogen and oxygen. The existence of a hydrogen atmosphere above the nitrogen and oxygen atmosphere makes the existence of a HEAVISIDE layer by photoelectric ionization impossible. Therefore, unless this HEAVISIDE layer may be explained wholly by other causes, such as impacts by particles of high velocity, its existence shows, that hydrogen in the usually adopted quantity cannot occur in our atmosphere.

Postscript. From the refraction of sound waves in the higher layers of the atmosphere Prof. E. VAN EVERDINGEN in 1915¹⁾ concluded that the amount of hydrogen in our atmosphere usually adopted is too high; a partial pressure of 10^{-6} at the surface fits better to the observed outer radius of the silent region. This is still too much to allow ionization of nitrogen and oxygen. With a somewhat larger percentage of helium and absence of hydrogen it seems possible, however, to represent the data on the propagation of sound in the atmosphere.

¹⁾ These Proceedings Vol. 18, p. 933.

Zoology.—“Castration and implantation of gonads in *Xiphophorus Helleri* Heckel (Teleost).” (Preliminary note.) By G. J. VAN OORDT and C. J. J. VAN DER MAAS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of September 25, 1926).

The direct proof of the secondary sex characters of fishes, being influenced by the gonads, has not yet been furnished. However, we assume it on good grounds; we can deduce it a.o. from GERSCHLER's (1914) and BELLAMY's (1922) cross-breeding experiments with *Xiphophorus Helleri* and *Platypoecilus maculatus* [“criss-cross” inheritance of sec. sex characters]. Moreover we can prove it indirectly by BLACHER's investigations (1926), who described several male specimens of *Lebistes reticulatus* with atrophied gonads; in consequence of this fact these fishes lacked some male sec. sex characters (viz. male colouring).

However, the direct proof can only be furnished by castrating or by transplanting gonads in fishes of the other sex.

As far as we know only very few publications on such experiments, which involve many difficulties and a great mortality, have appeared. KOPEC (1918) has castrated many specimens of *Phoxinus laevis* Agass., but his results are not very convincing. In the first place because his animals only survived castration 3 weeks at most and secondly because some totally castrated fishes exhibited a nuptial colouring notwithstanding, while in other non-castrated control animals this sec. sex characters did not develop. CHAMPY's attempts (1924, p. 310) to castrate fishes by means of radium rays had no positive results.

So, contrary to Mammals, Birds and Amphibia, the hormonal relation of gonads and sec. sex characters has not yet been demonstrated satisfactorily in fishes. Consequently it is of importance to perform castrations and gonad-implantations in fishes with well-developed sec. sex characters. For this purpose we chose *Xiphophorus Helleri* Heckel, a species belonging to the viviparous Cyprinodontidae, which has been cultivated since 1923 in great numbers by one of us. In this species the ♂ and the ♀ are distinguished by several conspicuous sec. sex characters (cf. VAN OORDT, 1925) of which two male sec. sex characters viz. the strongly elongated caudal fin (the sword) and the anal fin, which has been transformed into an intromittent organ, the gonopod, must be mentioned in the first place.

Adult female specimens of *Xiphophorus* possess a relatively large ovary; attempts to castrate adult ♀♀ had to be given up, because it appeared to be impossible to remove the ovary — the ovaries are originally paired, but

soon fuse (ESSENBERG, 1923) — without making too large a wound. Up till now it proved also impossible to implant ovaries or parts of them into the narrow abdominal cavities of ♂♂. So for the present we decided to apply ourselves to :

- a. Castration of male *Xiphophorus* ;
- b. Implantation of testes into female *Xiphophorus*.

We intend soon to pass on to implantation of juvenile ovaries into male specimens of *Xiphophorus* and to injection of gonad-extracts.

a. *Castration of male specimens of Xiphophorus.*

Castration of male specimens of *Xiphophorus* involves many difficulties. In the first half of April 1926 14 specimens were castrated. For this purpose at one side of the body, just above the place where the testis is found — the testis is also originally a paired organ ; however, both testes fuse later on to a compact organ, in which the testes can be distinctly distinguished in cross-sections — an incision is made and the testis is carefully removed, without injuring the alimentary tract, which is situated close to the reproductive organs. We always examined microscopically, whether we had removed testis-tissue, because in the neighbourhood of the gonad, especially in large male specimens, much adipose tissue is found, which can only be distinguished macroscopically with great difficulty from testis-tissue. The wound is sewed up with common thread and with the aid of very small surgical needles. Generally precautions as to sterility were not taken ; we only washed the side of the body to be operated with a weak solution of KMnO_4 and the wound was covered with collodium after having been sewed up. Of these 14 specimens only one large adult ♂ survived. This fish, castrated on April 13 1926, was again quite normal after a week ; then it swam well and ate much. After having been sewed up, the wound closed totally and gradually became invisible.

Brought together with spec. N^o. 239 (see below), it behaved as a quite normal male from May to August 1926. The male sex characters remained at the same high grade of development : the fish was also psychically quite normal and tried to copulate with N^o. 239 again and again. From this it followed that incomplete castration or testis regeneration had taken place. This could only be ascertained by microscopical investigation. The animal was killed on August 20. On autopsy an extraordinarily large testis was found. Microscopical investigation revealed a very active spermatogenesis. The number of spermatozoa is very large, larger than normal ; all afferent ducts of the testis are full of spermatophores. Had not this specimen been totally castrated a difference between the old part of the testis and the regenerated part would have been visible. This is not the case, however ; only a very small part, situated close to the peritoneum shows a different structure. Evidently the regeneration of the testis has started from this part. Neither has a connection been established between the sperm ducts of the regenerated part of the testis and the ductus deferens.

Similar observations were made by one of us (VAN DER MAAS) in incompletely castrated cocks. In these animals the testes had also regenerated, appeared to be much larger than normal and the testis-canals were full of spermatozoa. In both cases these phenomena must be ascribed to the fact that the testis was unable to discharge the produced spermatozoa.

b. Testis-implantation in female specimens of Xiphophorus.

In the beginning of April 1926 18 adult female specimens of *Xiphophorus* were implanted with testes of adult ♂♂. The operation was performed by making an oblique longitudinal incision behind the ovary in one side of the body. Through this hole the testis or a part of it in case the testis was very large was cautiously introduced. For the rest we proceeded as described above.

The mortality of the animals operated was not extraordinarily great. On April 16, 6 specimens, swimming normally and eating well, were still living. In these specimens the wound was no longer visible either after some time. On May 13, 1926 one normal looking specimen was killed and investigated but no trace of the implanted testis could be detected. This was also the case in three other spec., killed on June 19. The last remaining spec. (one had died in the meantime) lived till the end of August 1926. This fish, N^o. 239, a very large specimen, showed a somewhat elongated caudal fin and an anal fin of which the ventral rays were a little longer than normal. These characteristics, which can be compared to male sec. sex characters in the first stage of development, have not necessarily developed under the influence of the implanted testis. As one of us has already communicated the caudal fin of adult female specimens is somewhat elongated (VAN OORDT 1925, p. 47). Moreover, these ventral finrays are a little yellower than the other rays of the anal fin. It appears, however, that in adult ♀♀, still capable of reproduction, the 3d, 4th and 5th ray of the anal fin (for terminology cf. ESSENBERG, 1923) elongates somewhat and grows thicker; consequently they make the impression of a young gonopod in the first stage of development. An accurate comparison of the anal fin of the implanted specimen, above mentioned, with those of normal adult control fishes did not reveal any difference between the anal fins of these fishes.

The testis-implantation has no great influence on the reproduction of the ♀♀. Two months after the operation one of these specimens gave birth to a large number of young. These young were not fully developed, but this is not necessarily a result of the implantation, because it happens also in non-operated ♀♀. N^o. 239, which had not given birth to young since the operation, died during the birth of its young or shortly after. A number of young were alive and quite normal; the others, however, were not capable of leaving the genital opening of the ♀, this probably causing the mother's death. Owing to this an investigation of the implanted testis could not take place.

Perhaps the fact that in these two cases the young were born late, can

be explained by an influence brought about by the implanted testis, for ESSENBERG communicates (1926) that the first sign of sex-reversal in *Xiphophorus* is that the adult females stop to produce young; in normal cases this takes place almost regularly, each month.

We have seen that operations necessary for testis implantations can be endured with success by adult females. *Xiphophorus* N^o. 239, which did not differ morphologically from control-specimens, survived operation for nearly 5 months.

From the above it follows that we succeeded in castrating male specimens of *Xiphophorus* and in implanting testes in female specimens of *Xiphophorus*. Some specimens survived operation, the wound healing totally. However, we have not yet been able to demonstrate a hormonal relation between the gonads and the sec. sex characters in *Xiphophorus*. The result obtained justifies us to proceed in the same way in order to demonstrate whether a relation exists between the sec. sex characters and the gonads and if so what the nature of this relation is.

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Physics. — "*Methods and apparatus used in the cryogenic laboratory. XIX. The methyl-chloride and ethylene circulations. The hydrogen liquefier and circulation. The helium liquefier and circulation.*"
By H. KAMERLINGH ONNES †. Communication N^o. 158 from the Physical Laboratory at Leyden¹⁾. (Communicated by Prof. W. H. KEESOM).

(Communicated at the meeting of September 25, 1926).

§ 1. *The methyl-chloride and ethylene circulations.*

Many years ago a description was given of the methyl-chloride circulation,²⁾ but this description no longer completely answers to the reality; the boiling flask in particular has undergone a change that renders it desirable to give a new illustration of this apparatus (fig. 1).

The ethylene circulation, although it has been in use for a great number of years, has never been described in these Communications. It is true that in the excellent article by Prof. MATHIAS in the "Revue générale des Sciences" of 1896 there is a short account and also a drawing of the apparatus at that time, but this account does not answer to the present conditions. A drawing of the ethylene boiling flask, by far the most important part of the circulation, is found in fig. 2. These two drawings speak so plainly for themselves that we believe a very brief description will be sufficient.

The methyl-chloride boiling flask (fig. 1).

The whole upper part of the boiling flask is the same as in the description given in the Communication quoted.

The lower part (beneath the narrowing) to which the drawing refers has been changed. The construction was made simpler and more compact. The liquid methyl-chloride was admitted along *A*; it boils in the chamber *B* (under a pressure of about 2 cm) and the vapour escapes through the upper part of the flask. *C* is a tube which opens into *B* and which serves for the emptying, cleaning, etc. of the apparatus. The compressed ethylene comes into the flask by a spiral in the methyl-chloride vapour-chamber and finally in *F* (in the liquid methyl-chloride); it leaves the flask as liquid at *G*.

The ethylene spiral has a side tube *I*; this is provided with a safety

¹⁾ As early as 1922, Prof. KAMERLINGH ONNES planned this publication, but much and various work during the last years of his life prevented him from carrying it out. When death overtook him in February of this year it was partly ready, but neither drawing nor manuscript were ready for press. The working up of his notes now published, is done by Prof. Dr. J. E. VERSCHAFFELT of Ghent and Dr. C. A. CROMMELIN of Leyden.

²⁾ These Proceedings 6, 668, 1903; Comm. Leyden N^o. 87.

apparatus, consisting of a copper plate *J* which breaks down when for any reason the pressure in the spiral tubes becomes too great and therefore dangerous.

The ethylene spiral has an inner diameter of 8 mm, an outer one of 10 mm.

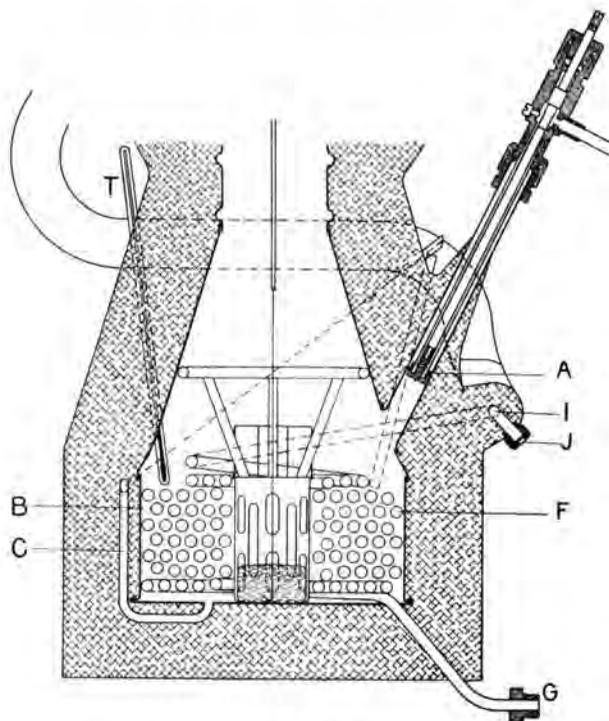


Fig. 1.

T is an alcohol thermometer, the reservoir of which is placed in a copper tube filled with alcohol, which tube is submerged into the boiling methyl-chloride.

The ethylene boiling flask (fig. 2).

The only description (with drawing) of this apparatus was given by MATHIAS in the article already quoted. It represents the old pattern. A new model was designed and constructed in 1909; since then it has been used without any alteration.

It shows a great resemblance to the methyl-chloride boiling flask. *A* is the entrance for the liquid C_2H_4 ; this boils (in a fairly shallow layer) in the lowest part of the chamber and the vapour rises along the spirals (wound round a central cylinder) to the escape *C*. The boiling takes place under greatly reduced pressure (± 2 cm). The compressed oxygen enters the boiling flask by *D*, and passes through a triple spiral (internal 3.5 mm, external 4.7 mm). The three parallel spirals unite into a single spiral *B* (internal 9 mm, external 11 mm), which is supported by a copper

frame and end at the bottom of the flask in a few coils *F*, which are completely immersed in the boiling ethylene. The liquefied oxygen leaves the flask through the outlet tube at the bottom.

M leads to a safety valve.

§ 2. *The hydrogen liquefier and circulation.*¹⁾

The hydrogen liquefier.

In Comm. N^o. 94f a liquefier was described (X, § 2) which was constructed in the spring of 1906 and used for many years after that. In this Communication it was already said that this liquefier was only a simplified form of a more efficient model that was already designed in 1906, but had not been constructed. A schematic drawing of this better model was given in the same Comm. (pl. I. fig. 1).

An improved liquefier according to the principles on which the schematic drawing was based, was constructed and taken into use in 1912. It is being used up to the present. This apparatus raised the capacity from 4 l to 13 l liquid hydrogen per hour; the capacity of the pumps had of course to be increased.

We here give a brief description of this new liquefier. We do not need to enter into such detail as is given in the Comm. referred to; the material used and the construction of the expansion tap, of the admission tap for the liquid air, of the syphon taps for the liquid hydrogen, of the safety arrangements, of the isolation, of the floats for air and hydrogen etc. are all exactly the same as in the old model so that it is not

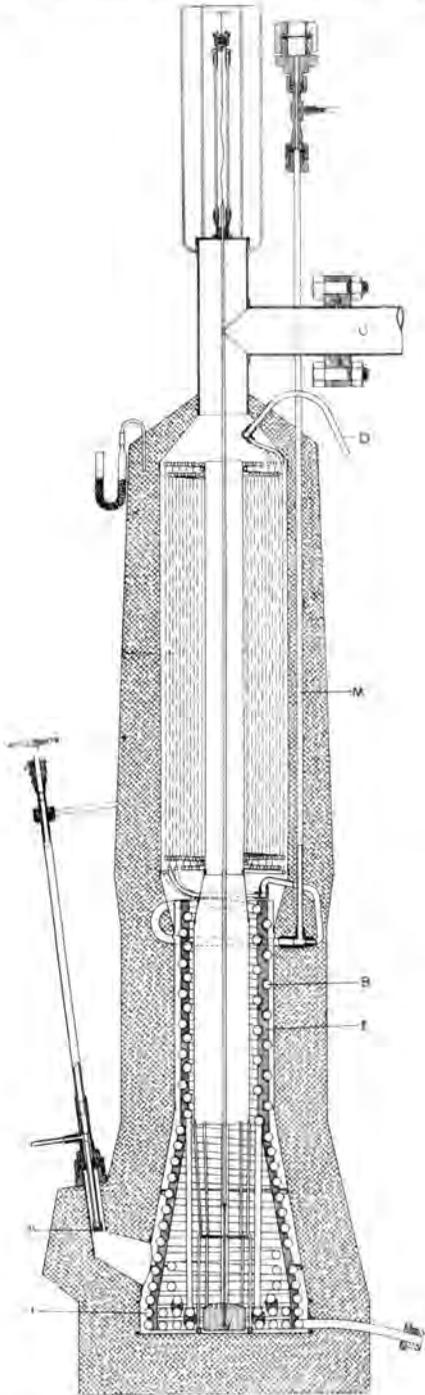


Fig. 2.

¹⁾ See also "Gedenkboek 1922" for a description and preliminary drawings of the hydrogen and helium circulations.

necessary to repeat the description. The construction of the whole and in particular the arrangement of the regenerator spirals is, however, different, as can be clearly seen in the drawing (fig. 3), which represents the new liquefier in section.

The hydrogen compressed to between 150 and 200 atm. enters at the top of the apparatus and divides at once into two tubes; one leads to a spiral *B* which, as can be clearly seen in the drawing, consists of three parallel spirally wound tubes (each 6 m long)¹⁾, the other to a quadruple spiral *B'* (also 6 m long). These two spirals join together again and the gas then passes through the windings *C* (quadruple spiral), *D* (triple)²⁾, *E* (single) and *F* (quadruple), till it reaches the tap *K*. *S* and *S'* are syphon taps by which the liquid hydrogen can be led off into vacuum vessels, to be taken to the rooms where the cryostats are built up.

Spiral *E* is completely immersed in liquid air, which is let in by the tap *L*. The drawing shows clearly, how the hydrogen is further successively cooled, when a stationary condition has been arrived at. The spirals *B'*, *C* and *F* are cooled in hydrogen vapour, *B*, *D* and *E* in air vapour and liquid air respectively.

The liquid air boils under a pressure of about 2 mm, being drawn in by a quick running horizontal vacuum pump (capacity 360 m³ per hour).

The safety valve *M* is a small glass instrument, made so that no mercury can escape when gas bubbles pass through the mercury.

The hydrogen circulation.

A description of the hydrogen circulation, as it could be put into action in the spring of 1906, was given in Comm. No. 94f (X, § 3).

In the course of time many improvements were made in the original apparatus; the capacity of the installation in particular was increased no less than threefold by the acquisition of new compressors and the construction of a new liquefier.

The gaseous hydrogen is now first compressed to 150—200 atm. by a series of 4 compressors. These compressors, delivered by the firm of BURCKHARDT of Bâle are slow horizontal pressure pumps (about 100 revolutions per minute), which bring the hydrogen to the pressure desired by five stages. The first pump, with a volume capacity of 40 m³ per hour, compresses 40 m³ gas³⁾ per hour from 1 to 2 atm.; the second (vol. cap. 20 m³ per hour) works with two stages, one from 2 to 6.5 and one from 6.5 to 25 atm.; the third and fourth pump (vol. cap. 0.9 m³ per hour) are connected in parallel and thus each deals with half of the gas introduced; they also compress in two stages, from 25 to 50 and finally from 50 to 250 atm.

¹⁾ These tubes, as well as those of the other spirals, are copper tubes of 2.4 mm inner and 3.8 mm outer diameter.

²⁾ The drawing is defective at the bottom of tube *C*, which there appears to be quintuple, and at the bottom of tube *D*, which seems to be there quadruple.

³⁾ It is intended to acquire another new pump, which will be able to work 80 m³, which would yield a total of 30 l liquid hydrogen per hour.

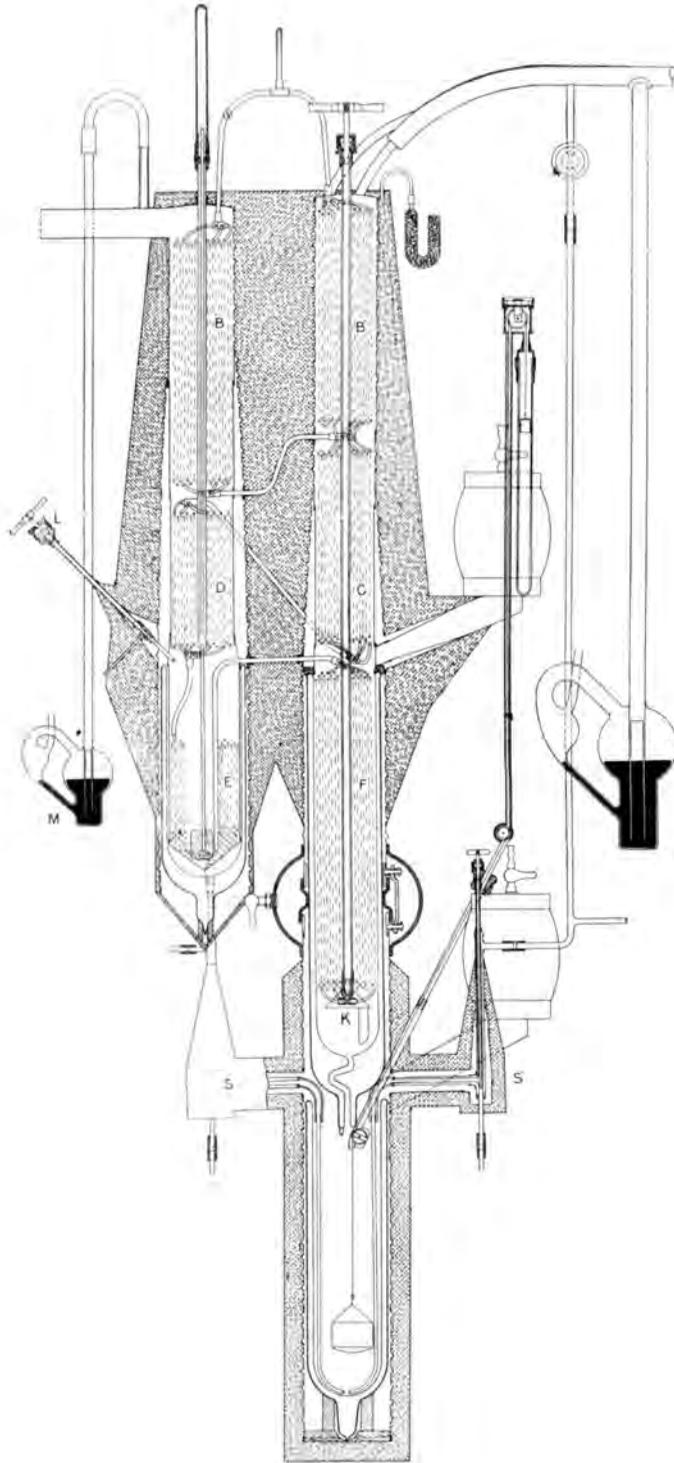


Fig. 3.

if necessary, but usually a final pressure of 150 to 200 atm. is used. Between the different stages the hydrogen is of course cooled by water.

§ 3. *The helium liquefier and circulation.*

The helium liquefier.

Up to December 1912 an old installation was used which has been described in Comm. N^o. 108. In the old installation the preliminary time, that is the time between the beginning of the expansion and the appearance of the first drops of liquid helium, lasted an hour and the installation, when it was once going, gave 0.28 l liquid helium per hour. In 1912, 2 small compressors (of which the one could also be used as a vacuum pump) with a capacity of 5 m³ per hour were purchased from the firm of BURCKHARDT in Bâle, which, when they were put into use considerably reduced the preliminary period, bringing it down to $\frac{1}{2}$ hour; the capacity of the liquefier was also increased, namely to $\frac{1}{2}$ l per hour. When moreover, in 1919 a new liquefier was brought into use, the circulation could be made with more than double the speed, 12 m³ per hour; the combination of the two compressors made this result possible; the preliminary time was reduced to 20 minutes and the capacity of the liquefier was raised to 1.7 l liquid helium per hour, while moreover the use of liquid hydrogen was reduced by half.

Fig. 4 shows the new liquefier in section.¹⁾

The gas compressed to about 30 atm. enters at *A* and is divided into two spirals *B* and *B'*, of which the first is double and the second triple; then the two streams of gas unite again and pass through the tube *L*, which is cooled at least at the beginning in liquid air and serves as a trap to catch oil or vapour, that is to say, easily condensed impurities. After this the gas returns to regenerating spirals and is divided between the tubes *M* and *M'* (the first double, the second triple); these spirals are placed immediately beneath *B* and *B'*.

The extremities of *M* and *M'* unite again and the gas now goes to a separator *O*, consisting of a *U*-formed arrangement of two metal tubes, which contain charcoal which has been heated to redness and are immersed in liquid air; this separator serves principally to remove the last traces of air and also traces of other very volatile substances. In stationary working of the liquefier, the gas is here previously cooled to the temperature of liquid air. Now the current of gas is again divided and the gas passes once more into two spirals, the double spiral *C* and the triple spiral *C'*. These spirals are placed in a peculiar way with regard to each other, namely so that *C'* occupies the central space of a cylindrical chamber and *C* the periphery; they are separated from each other by a metal wall. The object of this construction will presently be evident. The extremities of *C* and *C'* unite once more and the gas

¹⁾ The apparatus is represented in one plane; in reality the various parts are arranged in space round the central portion, that is round the actual liquefier.

enters the double spiral *D*, which is surrounded by vapour of liquid hydrogen, and further into the double spiral *E*, which is cooled in liquid

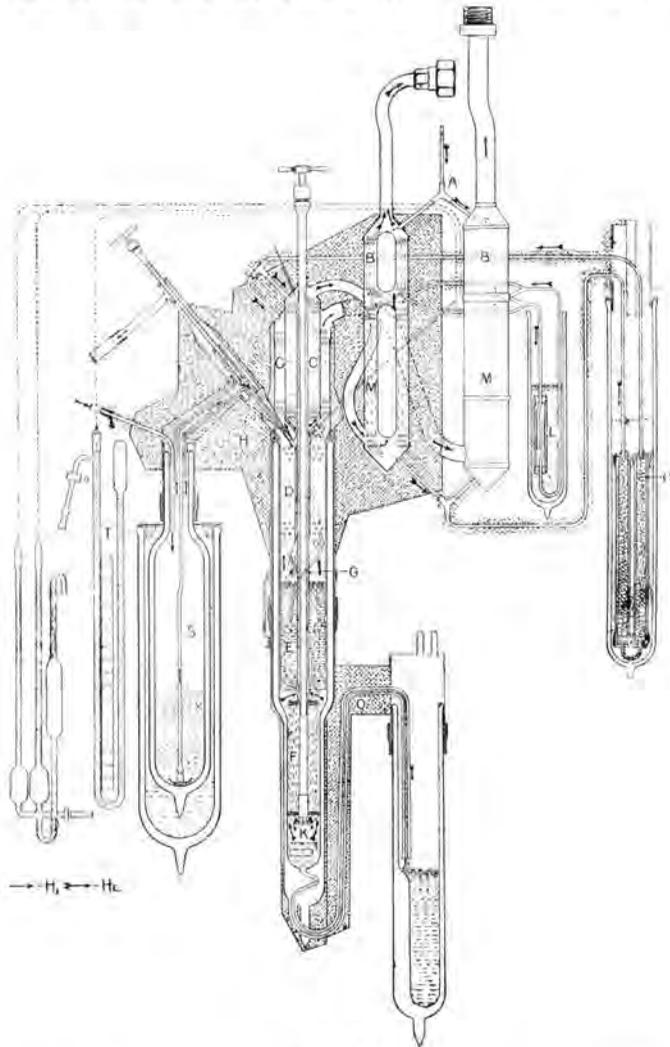


Fig. 4.

hydrogen. Here the gas is thus brought to the temperature of liquid hydrogen. The last spiral *F* is quadruple and terminates in the expansion tap *K*.

The liquid formed by expansion is collected in the space below *K* and from there it can be syphoned through the doubly bent vacuum tube *Q* into the cryostat.

The helium that is not condensed to liquid and the vapour rising from the liquid pass over the spiral *F*, which they cool; they then pass through a tube *G*, which embraces the wooden stem of the expansion tap rather closely and goes up along the axis of the hydrogen reservoir; after that this cold gas passes along the spirals *C'*, *M'* and *B'* to return finally to the compressor. The hydrogen vapour that comes out of the

hydrogen reservoir, in which spiral *E* is placed, is used to cool successively spirals *D*, *C*, *M*, and *B* and then returns to the hydrogen compressor.

S is a reservoir of liquid hydrogen, which is itself filled from vessels of liquid hydrogen and from which the liquid in which *E* is immersed can be syphoned out by the tap *H*. The stem of this tap is cooled by the hydrogen vapour from *S* which vapour is then caught in the hydrogen gas container.

The liquid coming from *S* flows into the reservoir *E* through a second ring-shaped tube round the stem of *K*.

A few more details. The lengths of the tubes *B*, *B'*, *M*, *M'*, *C*, *C'*, *D*, *E* and *F* are 2, 2, 4, 4, 2.7, 2.6, 2.5, 1.7 and 6 m respectively. The first six are again copper tubes of 2.4 mm inner and 3.8 mm outer diameter; *D* and *E* are 3.6 mm inner and 5.8 mm outer, *F* is 1.2 mm inner and 2.4 mm outer. The spirals *B*, *B'*, *M*, *M'*, have glass tubes as cores.

The height of the hydrogen surface in the reservoir round *E* is observed by means of a helium differential-thermometer one reservoir of which is placed at the lower end of *E* and the other at the lower end of *D* (the reservoirs are shown in the drawing). These reservoirs are connected by steel capillaries to the other parts of the thermometer, shown on the left of fig. 4.

This arrangement is simpler and easier than the thermo-needle formerly used.

There is also a thermometer reservoir with helium under the tap *K*. The capillary of this is connected with the thermometer *T*; this serves to follow the course of the regeneration process.

At the bottom of *S* there is a filter of metal gauze, which serves to hold back crystals of solid air and thus to prevent the hydrogen at *E* from being contaminated.

The vacuum glass in which the liquid helium is collected is only surrounded by a vessel containing liquid air; there is not room enough to place another vessel in between with liquid hydrogen; but it appeared that this was not necessary.

The helium circulation.

The most important particulars concerning the present circulation apparatus is found at the beginning of this §. It may be added that the large compressor has only a capacity of 5 m³ per hour, while the liquefier needs 15 m³ per hour. For this reason the small compressor is used as well as a forepump; it begins by compressing the gas to 2.5 atm.

For those who know the Leyden cryogenic laboratory it hardly needs saying (for those who do not know it intimately it may be said here) that the designs for the apparatus and circulations here described were made by the late Prof. KAMERLINGH ONNES in close co-operation with Mr. G. J. FLIM, technical chief of the cryogenic laboratory. The construction of the apparatus was done entirely by Mr. FLIM or under his supervision; some of the designs even are entirely his work.

Physics. — “*The melting-curve of hydrogen to 55 KG./cm².*” Communication N^o. 184a from the Physical Laboratory at Leyden. By H. KAMERLINGH ONNES † and W. VAN GULIK. (Communicated by Prof. W. H. KEESOM.)

(Communicated at the meeting of September 25, 1926).

§ 1. Knowledge of the course of the melting curve is of great importance, particularly in connection with the question of whether a continual change from the crystalline state of a substance into the amorphous state is possible¹⁾. The first question that arises then is, whether the melting curve of hydrogen continues beyond the critical temperature liquid-gas, as in the case of CO₂²⁾ and PH₄Cl³⁾.

In order to estimate the pressure necessary for raising the melting point so far, and to test the method, we have made some determinations of the melting curve of hydrogen, to ± 55 KG./cm².

§ 2. *Method.* We have used the method, which is usual for the determination of the viscosity of liquids and gases by streaming in a narrow tube.

In two vessels, connected by such a tube, an imposed pressure difference will not adjust itself when a part of the tube is blocked by a plug, fixed to the wall. This can be determined with the aid of a differential manometer, the limbs of which are connected with the two vessels. Also when the pressure is greatly increased the melting point can thus be determined.

It is also a great advantage, that by a proper choice of the dimensions, the viscosity can be derived from the rapidity with which the pressure difference adjusts itself.

Two compression vessels *A* and *B* (Fig. 1), placed in a water bath, were connected by means of copper leads to the German silver tubes *C* and *D* which protruded from the cryostat, and which at their other ends were connected to the ends of a steel capillary *E* about 35 cm in length (inner diam. 0.6 mm). Metal manometer *M* shows the pressure in the steel capillary, *K* is the differential manometer. A stirrer was introduced into the cryostat and also two resistance thermometers in overlapping shunts for determining the temperature of the capillary.

¹⁾ Comm. Suppl. 23, § 73. See also P. W. BRIDGMAN, Change of phase under pressure, Phys. Review (2) 3, 126, 153, 1914.

²⁾ P. W. BRIDGMAN, Phys. Review (2) 3, 160, 1914.

³⁾ G. TAMMANN, Kristallisieren und Schmelzen, Leipzig 1903 p. 284–294.

After evacuation of the compression vessels and tubes, which were heated where possible, and after washing with hydrogen, supplied from tube *H*, the apparatus was filled to $\pm 55 \text{ KG./cm}^2$. During this operation the cock *F* was open, in order to prevent the mercury of the differential manometer going over. The equilibrium pressure at a definite temperature, which temperature is kept constant as far as possible with the aid of the resistance thermometers, was now determined as follows. We assume the temperature to be such, that the capillary is already blocked

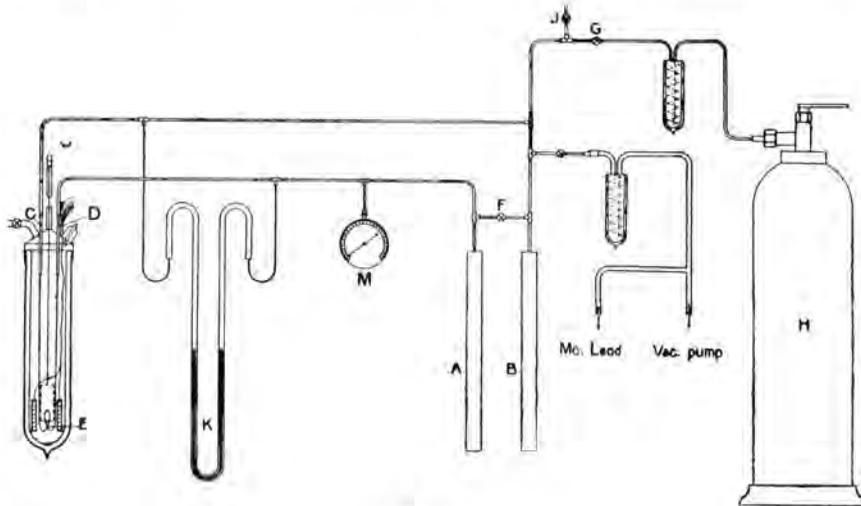


Fig. 1.

at the pressure mentioned. This is evident because, when the cock *F* is shut, and hydrogen is slowly admitted from *H* through cock *G* the differential manometer rises. Then, by opening cock *F* the difference in pressure is adjusted, and by blowing off through cock *J*, cock *F* open, the pressure is decreased until, on admitting hydrogen, cock *F* shut, the differential manometer does not rise, and so the hydrogen in the capillary is melted. The experiment is then carried out slowly in the opposite direction, till the rising of the differential manometer shows, that the hydrogen in the capillary has become solid. The pressure, which the metal manometer shows then, will be somewhat higher than the equilibrium pressure. When now, by opening and shutting *F* the pressure difference is removed and after that, hydrogen is blown off till the pressure difference, so formed, again disappears a pressure is obtained, which will be about as much lower. Thus it appeared, taking into account the temperature oscillations of the bath, that the observed pressures at which the capillary was just blocked, and at which flowing could first take place usually differed by less than 0.7 KG./cm^2 . The manometer corrections were determined by comparison with the open

manometer to 25 KG/cm², and with the manometer M_{60} for the higher pressures.

§ 3. *Results.* Six points were determined in the following order:

| $T_{HePt\ 32}$ | $T_{HePt\ 36}$ | T | p (KG./cm ²) | p (atm.) |
|----------------|----------------|----------|----------------------------|------------|
| 15.607 | 15.581 | 15.63 K. | 55.3 | 53.5 |
| 15.349 | 15.324 | 15.38 | 46.4 | 44.9 |
| 15.131 | 15.107 | 15.16 | 39.0 | 37.7 |
| 15.674 | 14.664 | 14.71 | 24.2 | 23.4 |
| 14.368 | 14.363 | 14.40 | 14.7 | 14.2 |
| 14.068 | 14.085 | 14.11 | 5.4 | 5.2 |

In Fig. 2 the melting curve is represented.

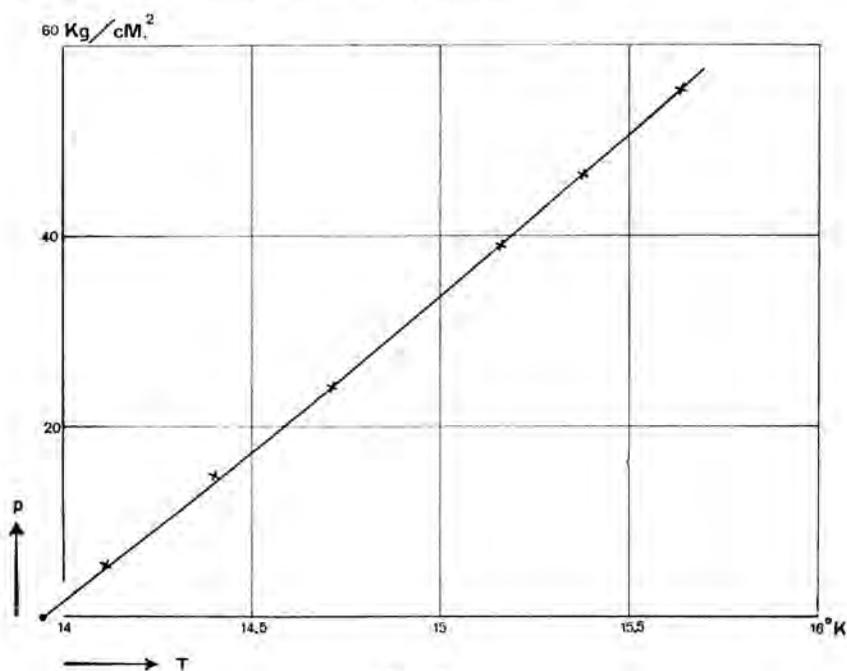


Fig. 2.

In columns 1, 2 and 4 are given means of temperature and pressure measurements (corrected) done just before and just after, or during the blocking of the capillary.

The triple point $p = 0.07$ KG/cm², $T = 13^{\circ}.95$ K. (Comm. N^o. 137d) forms with the points found here a sufficiently accurate continuous

curve. From the values obtained follows $\left(\frac{dp}{dT}\right)_{\text{coex.}} = 33$ ¹⁾. As the melting curve is almost straight, one obtains assuming that such an extrapolation is allowed, as a provisional estimation, for the melting pressure, corresponding with the critical temperature liquid-gas 600 KG/cm².

¹⁾ From the data of Comm. Nos. 137a and 153a for the specific volumina and the melting heat follows $\left(\frac{dp}{dT}\right)_{\text{coex.}} = 77.8$. The bad agreement is perhaps owing to the fact that the determination of the density of solid hydrogen has given too small a value, probably because the solid hydrogen in the dilatometer could not contract freely because of the adhesion to the wall and so was still in a state of tension.

Physics. — "*Preliminary measurements of the dielectric constants of liquid and solid nitrogen.*" By L. EBERT and W. H. KEESOM. (Communication N^o. 182*d* from the Physical Laboratory at Leyden).

(Communicated at the meeting of September 25, 1926).

§ 1. The dielectric constants of liquid and solid nitrogen have not previously been measured. In this communication some preliminary measurements will be published. The departure of one of us is the motive for communicating these measurements, which cannot yet be considered as definite.

§ 2. *Apparatus and corrections.* The apparatus used was the same as that used for the measurements on liquid oxygen by WERNER and KEESOM¹⁾. We always used condenser *B* (l.c. § 2). Since the existence region of liquid nitrogen is situated wholly within that of liquid oxygen, for measurements on liquid nitrogen the cryostat could always be filled with liquid oxygen. On one occasion we succeeded in obtaining solid nitrogen in the condenser, and in taking measurements on it.

The temperature exchange took place each time with satisfactory rapidity.

The accuracy in these measurements is not so great as in those with liquid oxygen (l.c. § 5), chiefly for the following two reasons.

In the first place we did not succeed in establishing that, after a series of measurements (this is after the apparatus had been cooled to -200°C . for 24 hours or longer) the capacity of the condenser empty returned to the same value, which it had at the beginning of the measurements. As an example, in table I, two measurements are given, which are done with particular care, in order to investigate whether the geometric capacity is reproducible. Probably this effect is due to a thermic deformation, adjusting itself very slowly. Now it is difficult to establish, which zero-value of the capacity must be taken into account for each moment of a series of measurements. This uncertainty of the zero-value (5 à 7 ‰) surpasses that of each capacity measurement by one order of magnitude.

In the second place it is not easy to see, how large the correction for the capacity of the leading wires must be chosen in each single case. The principal thing here is the condenser consisting of the wire *D* and the metal tube *H*. The question whether the tube *H* is totally

¹⁾ These Proceedings 29, 306, 1926; Comm. Leyden N^o. 178c.

TABLE I.

| Geometric capacity of the condenser before and after a series of measurements ¹⁾ . | | | |
|---|----------------------|-------------|---|
| Date | Time | Temperature | Capacity in arbitrary units ²⁾ |
| Jan. 18.26 | 11 ^h a.m. | — 183° C. | 32.76 ± 0.01 |
| .. 19.26 | 3 ^h p.m. | — 183 | 32.95 ± 0.01 |
| May 6.26 | 10 ^h a.m. | — 195 | 30.70 ± 0.008 |
| .. 7.26 | 10 ^h p.m. | — 195 | 30.97 ± 0.003 |

filled with the liquid to be investigated or not is then of great importance. In the first case the correction to ϵ is very small ($< 1\%$), in the second case it rises above 1% . The condenser was filled with nitrogen by condensation of the gas under 1 atm. overpressure, the pressure of the oxygen in the cryostat being sufficiently reduced. Filling of the cryostat with nitrogen was always continued for a long time after the liquid surface had disappeared behind the lower rim of the cap C, but for these measurements it was not possible to say to what height in H the liquid stood in each single case. Particular attention must be paid to the fact that the contraction of liquid nitrogen between boiling point and melting point amounts to 8% , so that the level of the surface in the upper, relatively narrow part of the apparatus may have fallen greatly during the course of a series of measurements. In what follows calculations were always made as if H was quite full. The correction for this not being so would *increase* the value of ϵ .

The temperature of the condenser was measured with the aid of the platinum thermometer T (l. c. fig. 1).

The nitrogen was taken from a cylinder, the contents of which appeared to contain according to an analysis about 3% oxygen. Before the condensation the nitrogen was led through a moisture trap which was cooled with liquid air.

That the nitrogen was indeed very pure follows also from the value, which was derived from the point of retardation in the cooling curve for the freezing point, namely $63.14^\circ \text{K.} \pm 0.1^\circ$, while EUCKEN ³⁾ obtained for it 63.19 , KEESOM and KAMERLINGH ONNES ⁴⁾ 63.06 , CATH ⁵⁾ 63.23 .

¹⁾ Each time at the same temperature, etc.

²⁾ For each number the probable error is indicated of the average derived from 5 to 10 readings.

³⁾ A. EUCKEN, Verh. d. D. physik. Ges. **18**, 4, 1916.

⁴⁾ These Proceedings **18**, 1247, 1916; Comm. Leyden N^o. 149a.

⁵⁾ These Proceedings **21**, 656, 1918; Comm. Leyden N^o. 152d.

§ 3. *The results.* During the preparations for the measurements several unexpected disturbances appeared, which took considerable time to overcome. Owing to this the number of measurements, which can be considered as trustworthy, is small. Table II gives the results of these measurements.

TABLE II.

| Dielectric constant of liquid nitrogen. | | | | | | | | |
|---|--------------------------|----------|-------|--------------|--------------|-------------------|---------------------|---------|
| Date | Temperature T (°K.) | Capacity | | D. C. | | Density ρ | Molec. polarisation | |
| | | C_0 | C | ϵ_1 | ϵ_2 | | P_1 | P_2 |
| Jan. 18.26 | 90.0 | 32.76 | — | — | — | — | — | — |
| | 66.7 | — | 47.95 | 1.464 | (1.453) | 0.8540 | 4.389 | (4.305) |
| | 63.9 | — | 48.21 | 1.472 | (1.461) | 0.8653 | 4.398 | (4.310) |
| .. 19.26 | 76.5 ₄ | — | 47.51 | (1.451) | 1.440 | 0.8108 | (4.446) | 4.351 |
| | 90.0 | 32.95 | — | — | — | — | — | — |
| May 6.26 | 78.0 | 30.70 | — | — | — | — | — | — |
| | 74.8 | — | 44.52 | 1.451 | 1.438 | 0.8185 | 4.471 | 4.358 |
| .. 7.26 | 78.0 | 30.97 | — | — | — | — | — | — |

Regarding the separate columns of this table, the following remarks may be made.

In the two series of measurements C_0 is different owing to a thorough cleaning of the interior of the condenser and to several small repairs, which took place meanwhile.

ρ has been interpolated from the measurements of MATHIAS, KAMERLINGH ONNES and CROMMELIN¹⁾; P indicates the expression $\frac{\epsilon-1}{\epsilon+2} \cdot \frac{M}{\rho}$.

The columns ϵ_1 , P_1 contain the values, calculated with the zero capacity *before*, ϵ_2 , P_2 those, calculated with the zero capacity *after* the experiment.

If one of the two numbers stands between (), then there is a large time-interval between the measurement and the zero capacity used, in this case probably the value not placed between () is preferable.

On Jan. 18, it was possible sufficiently to regulate the temperature of the cryostat for oxygen pressures of 7.6 and 5 mm respectively. So we could obtain some very preliminary values for solid nitrogen; see table III.

¹⁾ These Proceedings 17, 953, 1914; Comm. Leyden N^o. 145c.

TABLE III.

| Dielectric constant of solid nitrogen. | | | | |
|--|-------------|----------|--------------|--------------|
| Date | Temperature | Capacity | ϵ_1 | ϵ_2 |
| Jan. 18.26 | 61.9° K. | 48.12 | 1.469 | 1.458 |
| | 59.9 | 47.90 | 1.462 | 1.452 |

Here we do not know ϱ , so P cannot be calculated.

§ 4. *Discussion.* a. Liquid nitrogen.

The mean of the values P not placed between () from table II is 4.39. For comparison the following values can serve.

1. $\frac{n^2-1}{n^2+2} \cdot \frac{M}{\varrho}$ for liquid nitrogen at $T = 77.1^\circ$ K. From data of GEROLD ¹⁾ follows by extrapolation for $\lambda = \infty$: $n = 1.1983$; ϱ is 0.8082 (see above). Then

$$R_{liq} = \frac{n^2-1}{n^2+2} \cdot \frac{M}{\varrho} = 4.396.$$

2. The same expression for gaseous nitrogen at 77.97° K. and 752.3 mm. pressure. From the data of GEROLD (l.c. p. 88) follows $\varrho = 0.004556$ (recalculated after BOYLE-GAY-LUSSAC). For n_∞ 1.001073 was chosen. Then

$$R_{gas} = 4.395.$$

3. The same expression for gaseous oxygen under normal circumstances gives n_∞ being 1.000295:

$$R_{gas} = 4.404.$$

4. The expression $P_{gas} = \frac{\epsilon-1}{\epsilon+2} \cdot \frac{M}{\varrho}$ for gaseous nitrogen under normal circumstances is not quite certain as the values given by various investigators are somewhat different. Recent, very careful measurements of ZAHN ²⁾ gave $\epsilon = 1.000581$, the lowest value, obtained till now. From this follows:

$$P_{gas} = 4.34.$$

One can draw the conclusion, that the relation of MAXWELL as well as the equation of CLAUDIUS-MOSOTTI are approximately valid for

¹⁾ E. GEROLD, Ann. d. Phys. (4) 65, 93, 1921.

²⁾ CH. TH. ZAHN, Phys. Rev. 24, 400, 1924.

gaseous and liquid nitrogen¹⁾. Our measurements give no data regarding the question whether it is necessary to change the formula of CLAUSIUS-MOSOTTI into the more universal formula $\frac{\varepsilon-1}{\varepsilon+u} \cdot \frac{M}{\rho}$ (as WIENER²⁾ pleads), as this question is only to be answered by means of very accurate measurements. That the number ("form number") u should differ much from 2, seems however not probable because of the approximate validity of the equation of CLAUSIUS-MOSOTTI.

Further it should be remembered, that the validity of the rule of MAXWELL speaks against the existence of an appreciable dipole-moment and of an absorption, situated in the near infra-red.

b. Solid nitrogen.

In this case the meaning of the numbers is much less clear, because the density is not known. The values of ε are smaller than the values which would be valid for liquid nitrogen at the same temperatures. It is very possible, that this only appears to be so, and that those small values are only the result of the appearance of hollows in the solid dielectricum or of the crystals being more or less directed, by which an anisotropic dielectricum may be formed. These question could only be answered by a special investigation. If the values, obtained for solid nitrogen were valid for complete and isotropic filling, then the relation of CLAUSIUS-MOSOTTI could only hold for the change liquid \rightarrow solid, if the density of solid nitrogen were smaller than that of liquid nitrogen.

One of us (E.) wishes also to express his thanks to the International Education Board, for granting a stipendium, which made a sojourn in the Physical Laboratory at Leyden possible for him.

¹⁾ Be it remarked that the measurements of ZAHN for oxygen under normal circumstances give for $\frac{\varepsilon-1}{\varepsilon+2} \cdot \frac{M}{\rho}$ the value 0.1209 (see however the values of FRITS, Comm. Leyden N^o. 178c, p. 33, note 1). This value agrees practically with the value 0.1211, which WERNER and KEESOM have obtained for the greater part of the existence region of liquid oxygen.

²⁾ O. WIENER, Ber. d. Math. Phys. Kl. d. Kgl. Sächs. Ges. d. Wiss. 62, 256, 1910; Abh. d. Math. phys. Kl. etc. 32, N^o. 6, 509, 1912; E. GEROLD, l.c., p. 94.

Chemistry. — “*The Crystalstructure of Gallium*”. By Prof. F. M. JAEGER, P. TERPSTRA and H. G. K. WESTENBRINK.

(Communicated at the meeting of September 25, 1926).

§ 1. In this paper the results of the structure-determinations are published, which by means of the ROENTGEN-spectrographic method were obtained in studying the pure crystallized *gallium*-metal. The *gallium* used in these investigations was prepared in a spectroscopically pure condition from a residue, which was obtained in 1915 at the *Bartlesville Zinc-Company* in *Oklahoma* as an exsudation-product of the *zinc-lead*-plates ready for refining. This residu is left in the repeated distillation of the impure *zinc*: 12000 kilograms of raw material yield about 60 kilograms of the *lead*-containing residue, which do not contain more than a few grammes of *gallium*¹⁾. The final mixture contains, moreover, less than 8% *indium*, less than 1% *zinc* and only traces of *copper* and *cadmium*²⁾. It was solved in pure hydrochloric acid, then potassiumhydroxide added until almost neutral reaction, and the boiling solution treated with freshly precipitated copper-hydroxide, prepared from the sixfold weight of coppersulphate by means of the necessary quantity of *KOH*. When all hydroxide is solved and a white, gelatinous precipitate is settled, the liquor is filtered and repeatedly treated in the same way, until no precipitate any longer appears to be formed. The collected gelatinous residue is now thoroughly washed, then solved by hydrochloric acid and the copper still present in the solution eliminated by means of hydrogensulphide. After expelling the excess of hydrogensulphide, the solution is boiled with an excess of *KOH*, with the purpose to eliminate *indium* and traces of *iron* eventually present: both metals will then be precipitated, while the *gallium* remains in the solution and may be separated from the filtered liquor by means of carbonic acid³⁾. The precipitate thus obtained is thoroughly washed, converted on the waterbath by means of a slight excess of pure, strong sulphuric acid into *gallium-sulphate* (+ 16 H_2O), and then the calculated quantity of ammoniumsulphate is added. After evaporation of the solvent into a small volume, the solution is precipitated by means of its fivefold volume of 98%—

¹⁾ W. F. HILLEBRAND and J. A. SCHERRER, Journ. of Industr. and Engin. Chemistry, 8, 225, (1916).

²⁾ P. E. BROWNING and H. S. UHLER, Amer. Journ. of Science, 41, 351, (1916).

³⁾ All filtrates may be treated with potassiumferrocyanide and the precipitate eventually formed be decomposed by means of *KOH* or strong H_2SO_4 at higher temperature, to regain traces of *gallium* contained in it.

alcohol. The ammoniumgallium-alum thus prepared is solved in water, slightly acidified by some free sulphuric acid; this precipitation is repeated four or five times. The pure *gallium*-alum is solved in caustic potash, the ammonia driven out and the alkaline solution electrolyzed between platinum-electrodes. The metal thus obtained is once more converted into its hydroxide, solved in sulphuric acid, the solution, after addition of diluted sulphuric acid of 1,4 s.g., being brought to a volume of 350 ccm.; then it is electrolyzed with a current of 4 Amp. ¹⁾ After eventually separated *indium* being eliminated, the *gallium* obtained is anodically brought to solution in diluted (1 : 10) sulphuric acid (by a current of originally 1 Amp., later of 0,1 Amp. and 12 Volt) between a wire-shaped anode and a plate-shaped cathode. The solution is evaporated till a dry residue is formed; this is solved in water and, under continuous stirring by means of a stream of gas, now electrolyzed with 4 Amp. between a plate-shaped anode and a wire-shaped cathode. *Zinc* eventually present is thus totally eliminated; the solution is again during some hours electrolyzed with 0,6 or 0,8 Amp., the consecutive fractions of separated metal are successively eliminated and after having done this about seven times, all pure *gallium* is very slowly separated from the remaining solution by means of the electric current.

The already very pure metal thus obtained is now converted into its chloride: $GaCl_3$ after the method indicated by DENNIS and BRIDGMAN ²⁾ and this subjected to a fractional distillation at 230°—240° C. In this way the last traces of other metals were removed, even of *zinc* and *indium*, whose chlorides boil only at much higher temperatures ³⁾. The pure, distilled $GaCl_3$ was then again converted into the hydroxide, this solved in caustic potash and the solution again electrolyzed between platinum-electrodes. In the spectrum of the metal thus obtained neither the *indium*-lines, nor those of *zinc* or of any other of the accessory metals originally present in the raw material, could be observed any more now. ⁴⁾

§ 2. The pure *gallium* is a beautiful, silver-white metal, melting at 30°,2 C., and once molten, remains liquid far below that temperature, it being most susceptible to undercooling-phenomena. At about 0° C. it solidifies, depending on the particular conditions of the experiment, into aggregates of larger or smaller octahedral-shaped or flattened crystals.

¹⁾ LECOQ DE BOISBAUDRAN, Ann. de Chim. et Phys., **10**, 100, (1877); H. S. UHLER and P. E. BROWNING, Amer. Journ. of Science, **42**, 389, (1916).

²⁾ L. M. DENNIS and J. A. BRIDGMAN, Chem. News, **118**, 248, 256, 270, 281, 290, (1919).

³⁾ The boilingpoint of $ZnCl_2$ is 730° C., that of $GaCl_3$ is 215°—220° C. $InCl_3$ is volatile at 440°, but distills very slowly only at 600° C.; about the apparatus, cf. p. 257 of the paper mentioned.

⁴⁾ About the analytical determination of *Ga*, cf: L. E. PORTER and PH. E. BROWNING, Journ. Amer. Chem. Soc. **41**, 361, (1919); Chem. News **119**, 251, (1919).

It is very tough and cannot be powdered finely enough, so that the powder-method of HULL-DEBIJE originally seemed to promise very few results. However, it appeared possible to obtain easily somewhat larger and very good crystals of the metal, so that the crystal-structure could be investigated in details by BRAGG's spectrographic method, as well by means of rotation-spectrogramms (after SCHIEBOLD).

Crystals of *gallium* were for the first time obtained by LECOCQ DE BOISBAUDRAN ¹⁾, the discoverer of this element. He considered them as being monoclinic, but DESCLOIZEAUX got the conviction, that they possessed tetragonal symmetry. According to WYCKOFF ²⁾, a powder-spectrogram obtained by him, would indicate that the crystals are at least *not cubic*, as this appears to be the case with *aluminium* ³⁾, which belongs to the same group of elements and which has a face-centred cubic grating with 4 atoms in each fundamental cell. As far as concerns the other elements of this group of the periodic system, the element *indium* would, according to SACHS ⁴⁾, crystallize in regular octahedra; but HULL and DAVEY ⁵⁾ demonstrated that *indium* possesses a tetragonal grating with face-centred fundamental cells, whose dimensions were found to be $a_0 = 4,58$ A.U. and $c_0 = 4,86$ A.U., the axial ratio $a : c$ therefore being: 1 : 1,06; this proves that the crystals are pseudo-cubic. By means of a LAUE-spectrogram, LEVI ⁶⁾ recently was able to prove that *thallium* possesses hexagonal symmetry.

§ 3. *The Crystalform of Gallium.*

From the molten, strongly undercooled metal rather large, very lustrous crystals of the shape reproduced in Fig. 1a were obtained by inoculation of the molten mass and removing the rapidly grown individuals at the right moment. If, however, the crystals are allowed to grow *slowly* from the only slightly undercooled liquid, flat, often curved crystals, which are much poorer in limiting forms are deposited, as shown in Fig. 1b. Afterwards we will draw attention once more to this remarkable fact, as it will appear to be intimately connected with the typical crystalstructure of this metal.

Ditetragonal-bipyramidal.

$$a : c = 1 : 1,6753.$$

Forms observed: $o = \{111\}$, predominant, very lustrous and in general

¹⁾ LECOCQ DE BOISBAUDRAN, Compt. rend. Paris, **83**, 1044, (1876).

²⁾ R. W. G. WYCKOFF, *The Structure of Crystals*, (1924), 243. These data are until now not published.

³⁾ Cf. the corr. literature in P. P. EWALD, *Krystalle und RÖNTGENstrahlen*, (1923), 292. The edge of the cubic cell would be: 4,07 A.U.

⁴⁾ A. SACHS, *Zeitschr. Kryst.*, **38**, 495, (1904).

⁵⁾ A. W. HULL and W. P. DAVEY, *Phys. Rev.*, **17**, 571, (1921).

⁶⁾ G. R. LEVI, *Nuovo Cimento*, Ottobre, (1926); cf. also: *Zeits. f. Kryst.*, **61**, 561, (1925). Although the symmetry is right, the structure indicated by this author is apparently not so, a new investigation being, therefore, necessary.

well developed; $c = \{001\}$, well developed and yielding excellent reflections; $s = \{121\}$, small, but well reflecting, mostly with only part of its faces present; $\omega = \{115\}$, extremely narrow, but measurable; $r = \{101\}$, scarcely visible, just as $\varepsilon = \{313\}$ also; the indices could, however, be

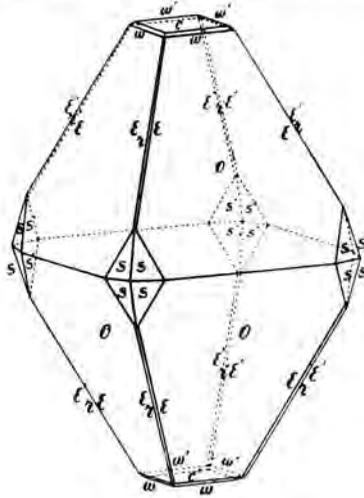


Fig. 1a.

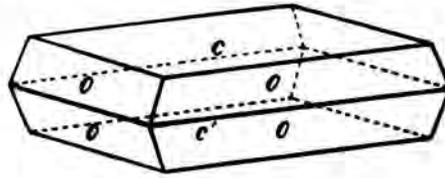


Fig. 1b.

determined by approximative measurements and ascertained by means of the general zone-relations. The habitus of the crystals is that of steep bipyramids, or short-prismatic according to an edge of the zone $[(001):(111)]$. The slowly developed crystals were thick tables parallel to $\{001\}$.

| <i>Angular values :</i> | <i>Observed :</i> | <i>Calculated :</i> |
|---|-------------------------------------|--------------------------|
| $c : o = (001) : (111) =$ | $^{\ast}67 \quad 7$ | — |
| $o : o = (111) : (\bar{1}\bar{1}1) =$ | $^{\ast}81 \quad 3$ | $81^{\circ} 18'$ |
| $o : o = (111) : (\bar{1}\bar{1}\bar{1}) =$ | $45 \quad 35$ | $45 \quad 46$ |
| $s : s = (121) : (\bar{1}\bar{2}1) =$ | $51 \quad 6$ | $51 \quad 12$ |
| $s : s = (121) : (1\bar{2}\bar{1}) =$ | $29 \quad 30$ | $29 \quad 53\frac{1}{2}$ |
| $s : s = (121) : (\bar{1}\bar{2}\bar{1}) =$ | — — | $119 \quad 34$ |
| $o : s = (111) : (121) =$ | $19 \quad 41$ | $19 \quad 8$ |
| $c : s = (001) : (121) =$ | $75 \quad 7$ | $75 \quad 3\frac{1}{4}$ |
| $o : r = (111) : (101) =$ | $\text{ca. } 40\frac{1}{2}^{\circ}$ | $40 \quad 39$ |
| $\varepsilon : o = (313) : (111) =$ | $\text{ca. } 24^{\circ}$ | $24 \quad 40\frac{2}{3}$ |
| $\varepsilon : r = (313) : (101) =$ | $\text{ca. } 16^{\circ}$ | $15 \quad 58\frac{1}{3}$ |
| $c : \omega = (001) : (115) =$ | $\text{ca. } 24\frac{1}{2}^{\circ}$ | $25 \quad 21$ |
| $o : \omega = (111) : (115) =$ | $\text{ca. } 42\frac{1}{2}^{\circ}$ | $41 \quad 46$ |

A stereographical projection of the forms observed is reproduced in Fig. 2.

The value of the angle $c : a$ is the mean value of five measurements and of four measurements of the angle $(111) : (11\bar{1})$. The smallest value

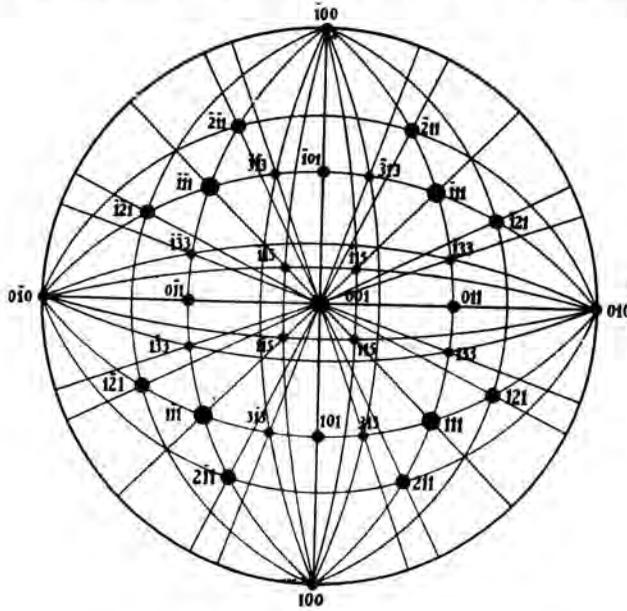


Fig. 2.

observed was $66^{\circ}45'$; from this last value, which differs only $22'$ with the fundamental angle used in the calculation of the axial ratio, this ratio would be found to be: $a : c = 1 : 1,646$.

§ 4. In the first place spectrograms after BRAGG's method were prepared by means of the K -radiation of a *copper*-anticathode on a face (001) of the basis and on a face of the bipyramid $\{111\}$. At the same time the spectrum of a piece of *rocksalt* on a face of the cube $\{100\}$

| TABLE I. | | | | | | | |
|-----------|-------------------------|-----------------------------|-----------------------------|-----------------------------|----------------------------|--------------------------------|-----------------|
| | $\frac{\theta}{2} :$ | | $\sin^2 \frac{\theta}{2} :$ | | Spacings d in A. U. : | | |
| | for $\lambda = Cu_x$ | for $\lambda = Cu_\beta$ | for $\lambda = Cu_x$ | for $\lambda = Cu_\beta$ | from : $\lambda = Cu_x$ | from : $\lambda = Cu_\beta$ | Mean values: |
| Basis | $11^{\circ}51'$ | $10^{\circ}39\frac{1}{2}'$ | 0.0422 | 0.0342 | 3.75 | 3.76 | 3.75^5 |
| Bipyramid | $15^{\circ}13'$ | $13^{\circ}41\frac{1}{2}'$ | 0.0689 | 0.0560 | 2.93 | 2.93 | 2.93 |

was photographed, with the purpose of having a spectrum for comparison, from which the true distances could be derived. The results of these measurements are given in Table I; with a distance of 60 mm between crystal and photographic plate, the accuracy of these data is about 0,02 A.U.

§ 5. Complete rotation-spectrograms were prepared with the purpose to determine 1. the identity-distances of the fundamental grating in different directions; 2. the symbols (indices and orders) of the diffracting planes in an accurate way. These spectrograms were obtained by using a modified theodolite-goniometer after GOLDSCHMIDT; this instrument is excellently adapted to this purpose, as a rotation in all desired directions may easily be effected by means of it. In most cases the crystal was moved round the axis of revolution to and fro to both sides over an angle of about 25° . In these experiments use was made of a flat photographic plate, which was suitably fixed at a distance of 40 or 50 mm. from the rotating crystal. The time of exposure was, on applying a copper-anticathode and a tension of about 50 K.V., ordinarily about 90 minutes.

To determine the identity-distance in the direction of the c -axis, a rotation-spectrogram was prepared, while the crystal was oscillating round this crystallographical axis; the distance from plate to crystal in this case was 40,5 mm. A distance of 16 mm was measured on the plate between the 1st positive and the 1st negative hyperbola; from this follows, in using the formula: $I = \frac{\lambda}{\sin \mu}$, — in which formula I is the

identity-distance required, $\lambda = Cu_\alpha = 1,540$ A. U. and $\mu = \text{arc tg } \frac{8}{40,5}$, — that the dimension I_c in the direction of the c -axis is: 8,00 A.U. In connection with the data of the spectrogram after BRAGG's method, mentioned above in Table I, the conclusion must be drawn that the diffraction-image observed there is really one of the 2nd order, the spacings between the planes parallel to (001) being actually halved. The accurate value for the identity-distance I_c in the direction of the c -axis is, therefore, evidently: $I_c = 2 \times 3,75^5$ A.U. = 7,51 A.U.

A preliminary determination of the identity-distance I_a in the direction of the a -axis was made by means of a rotation-spectrogram, the crystal being rotated round the a -axis. Here the distance of the plate from the crystal was: 38,5 mm; the distance of the first positive to the first negative hyperbola on the image was: 26,5 mm, from which data the required distance I_a is calculated to be: 4,7 A.U. The final accurate determination of I_a was executed as follows.

The quadratic form for $\sin^2 \frac{\theta}{2}$ in the tetragonal system can be written as:

$$\sin^2 \frac{\theta}{2} = \frac{\lambda^2}{4a^2} (h^2 + k^2) + \frac{\lambda^2}{4c^2} l^2,$$

where a and c represent the identity-distances I_a and I_c mentioned above, in the directions considered. Now $c = I_c = 7,51$ A.U.; therefore $\frac{\lambda^2}{4c^2}$ has the value: 0,0105. The value for $\sin^2 \frac{\theta}{2}$ for planes parallel to (111) can

be taken from Table I. If the diffraction-image after BRAGG's method is supposed to be of the 1st order, it follows from the quadratic form:

$$\sin^2 15^\circ 13' = \frac{(1,540)^2}{4a^2} \times (1+1) + 0,0105 \times 1;$$

or:

$$\frac{(1,540)^2}{4a^2} = 0,0292, \text{ from which is deduced: } a = I_a = 4,51 \text{ A. E.}$$

From the rotation-spectrogram the conclusion was already drawn, that this value for I_a needed not to be multiplied; therefore, it follows from $a = 4,51$ A.U. and $c = 7,51$ A.U., that the axial ratio $a:c$ = equal to: $1:1,667$ ($= \frac{2}{3}$ almost exactly). This value is in plain agreement with the results of the angular measurements of the *gallium*-crystals, by which $a:c$ was found to be: $1:1,6753$; for the fundamental angle $c:o$ in the latter case being: $67^\circ 07'$, it would be $67^\circ 02'$ for the axial ratio deduced in the above, — the difference being only $5'$.

The quadratic formula for $\lambda = 1,540$ A.U. takes, therefore, the form:

$$\sin^2 \frac{\theta}{2} = 0,0292 (h^2 + k^2) + 0,0105 l^2,$$

and thus for the wave-length Cu : $= 1,389$ A.U. in the same way:

$$\sin^2 \frac{\theta}{2} = 0,0238 (h^2 + k^2) + 0,0086 l^2.$$

§ 6. The dimensions of the elementary cell of the grating now being exactly known, the number N of atoms lying in each cell can be immediately computed from the formula:

$$N = \frac{V \times s}{A \times 1,64 \cdot 10^{-24}},$$

in which formula V is the volume of the cell in cm^3 , this being $(4,51)^2 \times (7,51) \cdot 10^{-24} \text{ cm}^3$; s = specific weight of crystallized *gallium*¹⁾, this number being: 5,904 and A = atomic weight of the element, i.e. 69,9. From these numbers N is calculated as: 7,85; it must be supposed, therefore, that in each fundamental cell *eight* atoms of *gallium* must find their places.

§ 7. The other rotation-spectrograms²⁾, in the preparation of which the crystal was rotated round a diagonal of the upper face of the parallelepipedon or respectively round a diagonal of one of the lateral faces of the elementary cell³⁾, enabled us to draw the conclusion, that the fundamental BRAVAIS' grating of the crystalstructure, must necessarily be a *simple tetragonal grating*⁴⁾, so that neither one of the faces of it is centred, nor the centre of the cell is occupied by an atom.

1) TH. W. RICHARDS and S. BOYER, Journ. Amer. Chem. Soc. 43, 274. (1921).

2) Cf. Tables V and VI.

3) These directions are resp. indicated by the zone-symbols: $[110]$ and $[\bar{1}01]$; i.e. round the basic. resp. the polar edge of the bipyramid $\{111\}$ as axes.

4) This will be confirmed by the detailed study of the symbols of the diffracting planes.

a. *Rotation-spectrogram with revolution round the zone-axis* [110].

The distance of crystal and photographic plate was: 41 mm.; the distance of the 1st hyperbola to the centre of the plate was: 10 mm., $tg \mu$ therefore being: $\frac{10}{41} = 0,2439$. From this follows: $I_{[110]} = 6,5$ A.U., while from the true value for $I_a (= 4,51$ A.U.), the diagonal of the basal face of the elementary cell is calculated to be: $4,51 \sqrt{2}$ A.U. = 6,38 A.U., which is in good agreement with the experimental result.

b. *Rotation-spectrogram with rotation round* $[\bar{1}01]$ (polar edge of {111}).

The distance of the crystal and the plate was here: 45 mm.; the distance of the 1st hyperbola to the centre of the plate was: 17 mm.; therefore: $tg \mu = \frac{17}{45} = 0,3777$ and $I_{[\bar{1}01]} = 8,2$ A.U. Calculation, on the other side, shows, that the diagonal of the lateral face of the elementary cell is: $\sqrt{(4,51)^2 + (7,51)^2}$ A.U. = 8,7 A.U. Here also there is a perfect agreement between calculated and observed value.

§ 8. With the aid of the quadratic formula found in the foregoing pages, it was now further possible to make a complete analysis as well of the different rotation-spectrograms, as of the powder-spectrogram finally obtained in the following way. As was said already, by the specific properties of the *gallium*-crystals, which are very tough and which probably possess, moreover, gliding-faces, — it was almost impossible to grind them down into a powder of sufficiently fine grains. A thin silk thread rubbed with this powder was covered with it, but if stretched along the axis of the camera, no interference-images after HULL-DEBIJE'S method could be obtained on the film. We succeeded, however, by grinding the *gallium* in a mortar under liquid air and smearing the silk thread,¹⁾ also under liquid air, with the finest *gallium*-dust in the mortar. *Gallium* gets very brittle at the temperature of liquid air and can be ground then easily into very fine powder; in the subsequent rising of the temperature, the original grating is preserved in the dust sticking to the silk thread, and now a real powder-spectrogram could be observed, only locally marred by small black patches, showing that some grains of the powder were not yet fine enough. However, the film could readily be made use of for the estimation of the relative intensities of the diffraction-lines.

The results of the analysis of the powder-spectrograms, as well of that of the different rotation-spectrograms already mentioned in § 5 are reviewed in the Tables II till VI. If necessary, besides the indices h , k and l with respect to the ordinary crystallographical axes, also the indices h' , k' and l' are mentioned, which are related to axes of which

¹⁾ It is remarkable, that the silk preserves its plasticity also at the temperature of the liquid air.

one coincides with the axis of rotation of the crystal in each case. The purpose of these last indices is to prove, that the indices h , k and l , as derived from the quadratic equation, are *not* contradictory to the sequence-numbers of the successive hyperbolae on the films, on which hyperbolae the corresponding interference-image is arranged. The necessary transformation-equations, by which h' , k' and l' are calculated from h , k and l , are also given in the tables.

§ 9.

| TABLE II. | | | | | | |
|--|------------------|----------------------|---------------------------|--|--|-------------------------|
| Analysis of the Powder-spectrogram of Gallium. | | | | | | |
| (Radius of the camera: 25 mm) | | | | | | |
| Distance $2l$ in m.m.: | Wave- length: | $\frac{\theta}{2}$: | $\sin \frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols ($h k l$): |
| 19.70 | Cu_{β_2} | 11°23' | 0.1970 | 0.0387 | 0.0324 | (002) |
| | | | | | | (101) * |
| 20.10 | Cu_{α_2} | 11°38' | 0.2016 | 0.0408 | 0.0420 | (002) |
| 23.48 | .. | 13°56' | 0.2351 | 0.0554 | 0.0562 | (111) |
| 26.24 | Cu_{β_1} | 15°11' | 0.2619 | 0.0686 | 0.0689 | (111) |
| 30.80 | Cu_{α_1} | 17°49' | 0.3060 | 0.0936 | 0.0952 | (200) |
| 34.48 | Cu_{β_3} | 19°50' | 0.3393 | 0.1151 | 0.1166 | (200) |
| 39.22 | Cu_{α_3} | 22°42' | 0.3859 | 0.1490 | 0.1460 | (210) * |
| | | | | | | (113) |
| 40.25 | .. | 23°18' | 0.3956 | 0.1565 | 0.1565 | (211) |
| | | | | | | (202) |
| 44.04 | .. | 25°30' | 0.4305 | 0.1853 | 0.1880 | (212) |
| 49.71 | .. | 28°46' | 0.4812 | 0.2316 | 0.2336 | (220) |
| 51.18 | Cu_{β_3} | 29°37' | 0.4942 | 0.2442 | 0.2466 | (311) |
| | | | | | | (302) |
| 55.09 | Cu_{α_4} | 31°53' | 0.5282 | 0.2792 | 0.2756 | (222) |
| 57.63 | .. | 33°21' | 0.5498 | 0.3023 | 0.3025 | (311) |
| | | | | | | (302) |

About ten weak lines, following the last mentioned one, were omitted here, as they have no significance for our purpose. The tension was here 55000 Volts; the time of exposure: 3 hours.

§ 10.

| TABLE III. | | | | | |
|--|--|----------------------|---------------------------------------|---|---------------------------|
| Rotation-spectrogram with revolution round the <i>c</i> -axis. | | | | | |
| (The distance from plate to crystal was: 41 mm) | | | | | |
| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols (<i>h k l</i>): |
| 0 | 62.5 | 28°22' | 0.2257 | 0.2336 | (220) |
| 1 | 24.0 | 15°10' | 0.0684 | 0.0689 | (111) |
| 1 | 42.4 | 22°59' | 0.1524 | 0.1565 | (211) |
| — 1 | 24.0 | 15°10' | 0.0684 | 0.0689 | (111) |
| 2 | 48.6 | 25°19' | 0.1810 | 0.1880 | (212) |
| — 2 | 50.1 | 25°21' | 0.1833 | 0.1880 | (212) |
| 3 | 40.3 | 22°15' | 0.1434 | 0.1529 | (113) |

§ 11.

| TABLE IV. | | | | | |
|--|--|----------------------|---------------------------------------|---|---------------------------|
| Rotation-spectrogram with revolution round the <i>a</i> -axis. | | | | | |
| (The distance from plate to crystal was: 40.5 mm) | | | | | |
| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols (<i>h k l</i>): |
| 0 | 17.8 | 11°52' | 0.0423 | 0.0420 | (002) |
| 0 | 43.8 | 23°37' | 0.1605 | 0.1680 | (004) |
| | | | | 0.1588 | (022) |
| 1 | 40.3 | 22°04' | 0.1452 | 0.1460 | (120) * |
| | | | | 0.1529 | (113) |
| 1 | 64.0 | 28°50' | 0.2326 | 0.2405 | (123) |
| | | | | 0.2264 | (114) |
| 2 | 50.0 | 25°30' | 0.1854 | 0.1880 | (122) |

§ 12.

| TABLE V. | | | | | | |
|---|--|----------------------|---------------------------------------|---|----------------------|---------------------------------------|
| Rotation-spectrogram with the revolution round the direction $[110]$. | | | | | | |
| (The distance from plate to crystal was: 41 mm) | | | | | | |
| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols ($h k l$): | ($h' k' l'$) (Reduced on the axis): |
| 0 | 45.5 | 24°0' | 0.1665 | 0.1680 | (004) | (004) |
| 0 | 18.6 | 12°12' | 0.0447 | 0.0420 | (002) | (002) |
| 1 | 53.5 | 26°16' | 0.1959 | 0.1972 | (104) | (1 $\bar{1}$ 4) |
| 1 | 25.5 | 15°56' | 0.0754 | 0.0712 | (102) | (1 $\bar{1}$ 2) |
| 2 | 42.5 | 23°1' | 0.1529 | 0.1529 | (113) | (203) |
| 3 | 51.0 | 25°36' | 0.1867 | 0.1880 | (212) | (3 $\bar{1}$ 2) |
| Transformation-formulae: $h' = h + k$; $k' = k - h$; $l' = l$. The index h' belongs to the axis of revolution. | | | | | | |

§ 13.

| TABLE VI. | | | | | | |
|---|--|----------------------|---------------------------------------|---|----------------------|---------------------------------------|
| Rotation-spectrogram with the revolution round the direction $[\bar{1}10]$. | | | | | | |
| (The distance from plate to crystal was: 45 mm) | | | | | | |
| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols ($h k l$): | ($h' k' l'$) (Reduced on the axis): |
| 0 | 26.6 | 15°17' | 0.0695 | 0.0689 | (111) | (110) |
| 0 | 39.5 | 20°38' | 0.1242 | 0.1168 | (020) | (020) |
| 1 | 56.5 | 25°44' | 0.1885 | 0.1880 | (122) | (121) |
| -1 | 47.6 | 23°19' | 0.1567 | 0.1565 | (211) | (21 $\bar{1}$) |
| 2 | 46.5 | 22°58' | 0.1522 | 0.1529 | (113) | (112) |
| -2 | 26.6 | 15°17' | 0.0695 | 0.0689 | (1 $\bar{1}$ 1) | (1 $\bar{1}$ 2) |
| -2 | 72.0 | 29°0' | 0.2350 | 0.2336 | (220) | (222) |
| -3 | 47.7 | 23°20' | 0.1569 | 0.1569 | (2 $\bar{1}$ 1) | (2 $\bar{1}$ 3) |
| -3 | 58.5 | 26°13' | 0.1952 | 0.1972 | ($\bar{1}$ 04) | ($\bar{1}$ 03) |
| 4 | 51.5 | 24°26' | 0.1711 | 0.1680 | (004) | (004) |
| -4 | 49.0 | 23°43' | 0.1618 | 0.1588 | (202) | (204) |
| Transformation-formulae: $h' = h$; $k' = k$; $l' = l - h$. The index l' belongs to the axis of revolution. | | | | | | |

§ 14. For the determination of the space-group the tables of ASTBURY and YARDLEY ¹⁾ were made use of. From the spectrograms described above, it is evident, that interference-images belonging to atomic planes with the symbols of the type: $(h k 0)$, in which $(h + k)$ is *odd*, and of the type: $(0 k l)$, in which l is *odd*, were *not* observed with certainty. Such uncertain symbols were indicated by an asterisk *) in the tables. All other combinations of indices were observed; but in the case of the symbols thus indicated with an asterisk, always another combination of indices appeared to be possible also.

As *gallium*, according to the crystallographical measurements, very probably possesses ditetragonal-bipyramidal symmetry, — as becomes evident, for instance, from the fact, that some crystals, as in Fig. 1a, manifest the full number of faces of each form in perfectly well-balanced development, — the four only space-groups possible here amongst the twenty groups belonging to this crystal-class, are, in WYCKOFF's notation ²⁾:

D_{4h}^1 : no halving of spacings between atomic planes what so ever;

D_{4h}^7 : halving of spacings between atomic planes of the type: $(h k 0)$, in which $(h + k)$ is *odd*;

D_{4h}^{10} : halving of spacings between atomic planes of the type: $(0 k l)$, in which l is *odd*;

D_{4h}^{16} : halving of spacings between atomic planes of the types: $(h k 0)$, in which $(h + k)$ is *odd*, and of the type: $(0 k l)$, in which l is *odd*.

To fix the choice between these, it is, therefore, of the highest importance now, to determine with full certainty the yet problematic indices of those interference-images, for which from the quadratic equation alone, hitherto no choice could be made between e.g. (101) (Cu_2) and (002) (Cu_3), or between (210) and (113) ; etc. The rotation-spectrograms, described in the tables III till VI, were principally prepared with the purpose of determining the identity-distances in the directions of revolution: but now it appeared necessary to prepare some others, with the intention to exclude with absolute certainty those indices, which were yet doubtful; the choice between the four space-groups mentioned can only be made, if the indices-triples really present, be determined beyond any doubt.

In the way followed up till now, it was, as already said, not possible to fix the choice between some of the symbols, as calculated from the quadratic equation, because the values of $\sin^2 \frac{\theta}{2}$ thus computed, did not differ sufficiently from each other. Therefore, in preparing the new rotation-spectrograms, which are reviewed in the tables VII till XI following here-after, — it was in each case necessary to determine accurately,

¹⁾ W. T. ASTBURY and K. YARDLEY, "Tabulated data for the Examination of the 230 Space-groups by homogeneous X-rays", London, (1924), Proceed. Royal Soc., A, 224, 221—257.

²⁾ R. W. G. WYCKOFF, "The Analytical Expression of the Results of the Theory of Space-groups", Washington, (1922), Carnegie-Publ. N^o. 318.

between what extreme angular limits in both directions the rotation round the axis of revolution really took place during the experiment. It was possible, by making a stereographical projection of the scattering planes with the aid of a WULFF's plot and rotating that projection through the same limiting angles as were used in the case of the rotating crystal, — to ascertain, whether these planes had really been in such positions, as possibly to give the diffraction-images of the symbols, which were supposed to have been observed in each special case. In this way it could be verified rigorously, whether atomic planes, to which simple indices were attributed, really had been in such favourable positions for giving interference-images or not, and whether some of them were absent, although those atomic planes might have been in the right positions for scattering the incident *X*-rays.

§ 15. In the spectrogram, to which table VII refers, the crystal was rotated round the direction $\bar{1}01$ in such a way, that in both directions an angle of 29° was covered, starting with a position, in which the primary pencil of *X*-rays was parallel to (111). By means of the stereographical projection it could be stated, that the atomic planes mentioned in table VII, had really been in the required „reflecting” positions. As

| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols (<i>h k l</i>): | Symbols (<i>h' k' l'</i>) (reduced): |
|---------------------------|--|----------------------|---------------------------------------|---|---------------------------|--|
| 0 | 32.1 | $15^\circ 15'$ | 0.0691 | 0.0689 | (111) | (110) |
| 0 | 45.5 | $19^\circ 56'$ | 0.1162 | 0.1168 | (020) | (020) |
| 1 | 57.5 | $23^\circ 16'$ | 0.1560 | 0.1565 | $\bar{2}11$ | $\bar{2}11$ |
| —1 | 33.0 | $15^\circ 36'$ | 0.0723 | 0.0712 | $\bar{1}0\bar{2}$ | $\bar{1}0\bar{1}$ |
| 2 | 45.5 | $19^\circ 56'$ | 0.1162 | 0.1168 | $\bar{2}00$ | $\bar{2}02$ |
| —2 | 25.0 | $12^\circ 19'$ | 0.0455 | 0.0420 | (00 $\bar{2}$) | (00 $\bar{2}$) |
| —2 | 55.5 | $22^\circ 46'$ | 0.1499 | 0.1529 | $\bar{1}1\bar{3}$ | $\bar{1}1\bar{2}$ |
| 3 | 58.0 | $23^\circ 23'$ | 0.1575 | 0.1565 | $\bar{2}11$ | $\bar{2}13$ |
| 4 | 56.0 | $22^\circ 53'$ | 0.1512 | 0.1529 | $\bar{1}1\bar{3}$ | $\bar{1}14$ |
| —4 | 61.0 | $24^\circ 7'$ | 0.1667 | 0.1680 | (00 $\bar{4}$) | (00 $\bar{4}$) |

Transformation-formulae:
 $h' = h ; k' = k ; l' = l - h.$
 The index *l'* refers to the axis of revolution.

the indices attributed to them appeared to be in full agreement with the subsequent hyperbolae on the film, as well with the quadratic equation, *their exactness is sufficiently assured in this case.*

§ 16. In the same way table VIII gives a review of the results obtained on rotating the crystal round the *c*-axis. If the position in which the primary pencil of *X*-rays is parallel to the plane (110), is called the position for which $\frac{\theta}{2} = 0^\circ$, — then the crystal appeared in this case to be rotated from $\frac{\theta}{2} = -11^\circ$ till $\frac{\theta}{2} = +35^\circ$. The plane (110) should have given a diffraction-image of the 1st order for $\frac{\theta}{2} = \text{circa } 14^\circ$, and a 2nd order image at about $\frac{\theta}{2} = +28^\circ$; but a 1st order image was *not* observed in this case.

| TABLE VIII. | | | | | |
|--|--|----------------------|--|--|------------------------------|
| Rotation-spectrogram with revolution round the <i>c</i> -axis. | | | | | |
| (Distance between crystal and plate: 42.5 mm) | | | | | |
| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols (<i>h k l</i>): |
| 0 | 67.0 | 28°48' | 0.2321 | 0.2336 | (220) |
| 0 | 35.0 | 19°44' | 0.1140 | 0.1168 | (200) |
| 1 | 25.0 | 15°14' | 0.0691 | 0.0689 | (111) |
| 1 | 44.0 | 23°0' | 0.1527 | 0.1565 | (211) |

§ 17. The table IX refers also to a rotation-spectrogram with revolution round the *c*-axis. But now the crystal was rotated between the

| TABLE IX. | | | | | |
|--|--|----------------------|--|--|------------------------------|
| Rotation-spectrogram with revolution round the <i>c</i> -axis. | | | | | |
| (Distance between crystal and plate: 42 mm) | | | | | |
| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$ (observed): | $\sin^2 \frac{\theta}{2}$ (calculated): | Symbols (<i>h k l</i>): |
| 0 | 26.7 | 19°47' | 0.1146 | 0.1166 | (200) |
| 0 | 66.3 | 28°50' | 0.2326 | 0.2336 | (220) |
| 1 | 43.9 | 23°16' | 0.1561 | 0.1565 | (211) |
| 3 | 43.5 | 23 0' | 0.1527 | 0.1529 | (113) |

limits: $\frac{\theta}{2} = -15^\circ$ and $\frac{\theta}{2} = +31^\circ$, if $\frac{\theta}{2} = 0$ is now the position, in which the plane (100) is parallel to the primary beam. The atomic planes (101), (100) and (210) will now all come into positions favourable for the occurrence of interference-images of the 1st order. Notwithstanding this, however, *no* such images were observed here.

§ 18. In table X the results are collected, which refer to a rotation-spectrogram with revolution round the *a*-axis. In the mean position the primary beam was parallel to the plane (001), and from this position the rotation took place in both directions through an angle of 23° . An interference-image of the plane (013) could now be expected, but on the film no trace of it could be discovered. The other planes of $\{013\}$ do not come into positions favourable for "reflection". Moreover, no interference-images of atomic planes can occur with the symbols: (102), (10 $\bar{2}$), (202), (20 $\bar{2}$) and neither of the planes of $\{210\}$; the indices given in table X, therefore, must be the only possible ones.

Another spectrogram of this kind was made, moreover, in which the only difference with the preceding case consisted in this, that the crystal, — having its mean position, when (001) was parallel to the primary pencil of *X*-rays, — was first turned 60° in the direction of this primary beam and then rotated from this new position over 23° to both sides of it. In this case it might namely have been possible for (210), to give a diffraction-image; but even now, this image was *not* present on the film. In this experiment the planes of $\{113\}$ could not give any confusion in the decision about the presence or absence of the image from (210), because the images belonging to this form would be situated on other hyperbolae; moreover, of all its planes, only (1 $\bar{1}$ 3) could arrive in a "reflecting" position, and its image was really observed on the film. Also an image from (110) could be expected in this case; but it was absent once more in the spectrogram obtained.

| Number of the hyperbolae: | Distance of Spots from the centre in mm: | $\frac{\phi}{2}$: | $\sin^2 \frac{\phi}{2}$ (observed): | $\sin^2 \frac{\phi}{2}$ (calculated): | Symbols (<i>h k l</i>): |
|---------------------------|--|--------------------|--|--|------------------------------|
| 0 | 20.0 | 11°59' | 0.0431 | 0.0420 | (002) |
| 1 | 27.0 | 15°29' | 0.0712 | 0.0689 | (111) |
| 1 | 46.0 | 22°49' | 0.1504 | 0.1529 | (113) |
| 2 | 48.5 | 23°36' | 0.1603 | 0.1565 | (211) |

| TABLE XI. | | | | | |
|--|---|--------------------|-------------------------------------|---------------------------------------|---------------------------|
| Rotation-spectrogram with revolution round the <i>a</i> -axis. | | | | | |
| (Distance between crystal and plate : 46 mm) | | | | | |
| Number of the hyperbolae: | Distance of Spots from the centre in mm : | $\frac{\phi}{2}$: | $\sin^2 \frac{\phi}{2}$ (observed): | $\sin^2 \frac{\phi}{2}$ (calculated): | Symbols (<i>h k l</i>): |
| 0 | 39.0 | 20°9' | 0.1187 | 0.1168 | (020) |
| 0 | 49.0 | 23°24' | 0.1577 | 0.1588 | (022) |
| 1 | 27.0 | 15°18' | 0.0696 | 0.0689 | (111) |
| -1 | 47.6 | 23°0' | 0.1527 | 0.1529 | ($\bar{1}$ 13) |
| 2 | 50.0 | 23°42' | 0.1616 | } 0.1588 0.1565 | (202) |
| | | | | | (21 $\bar{1}$) |

§ 19. After the analysis and study of the spectrograms reviewed in the tables VII till XI, the conclusion can be drawn with sufficient certainty, that really no other space-groups than D_{4h}^1 , D_{4h}^7 , D_{4h}^{10} and D_{4h}^{16} are possible in the case of the *gallium*-crystals, while all other space-groups can readily be excluded. On the other hand it becomes highly probable, that amongst those D_{4h}^{16} is the right one, because of the fact, that the interference-images corresponding with atomic planes having the symbols $\{h k 0\}$ and $\{0 k l\}$, in which respectively $(h + k)$ or l are *odd*, appear to have systematically an intensity equal to zero, at least in so far as this could be proved. However, it is advisable to investigate by means of calculating the relative intensities, whether perhaps one of the other space-groups might also be taken into account in this respect, so that the 8 atoms were arranged in such a way within each elementary cell, that a sufficient agreement between the calculated intensities and those observed in the powder-spectrogram were obtained. The powder-spectrogram is much better suited for this purpose than the rotation-spectrograms, for which an accurate calculation of the intensities is hardly possible.

The calculation of the intensities in the powder-spectrogram was effectuated by means of the well-known formula:

$$I = \frac{1 + \cos^2 \theta}{2 \cos \frac{\theta}{2} \cdot \sin^2 \frac{\theta}{2}} \cdot n \cdot |S|^2,$$

in which n is the number of planes of the complete crystalform $\{h k l\}$, and $|S|$ signifies the absolute value of the structural factor, in such a way, that:

$|S|^2 = |\sum e^{2\pi i(hm + kn + lp)}|^2 = |\sum \cos 2\pi(hm + kn + lp)|^2 + \sum |\sin 2\pi(hm + kn + lp)|^2$ is ¹⁾).

The estimated intensities of the lines of the powder-spectrogram are reviewed in the 2nd column of table XII. A practically unobservable small intensity was found to be present for lines, corresponding with (001), (100), (110), (101), (103) and (120); with exception of (110), all these symbols belong really to one of the types: $\{h k 0\}$ or $\{0 k l\}$, with respectively $(h + k)$, or $l = \text{odd number}$.

A review of all possible combinations of 8 equivalent points compatible with these space-groups, is given in WYCKOFF's tables. (Cf: *Analytical Expression*, etc., (1922), p. 89—102).

A. Space-group D_{4h}^1 .

In this case *five* combinations of 8 equivalent points are possible, which cases are discerned by the author as p , q , r , s and t (loco cit. p. 90).

The arrangement of the atoms according to the coordinates mentioned sub p is excluded, because of the fact that the spacing of the atomic planes parallel to (001) would not be halved, while experiment shows that this actually takes place. The same argumentation is valid for an arrangement of the atoms according to the coordinates mentioned sub q . Moreover, the combination r , s and t appear to be excluded, because if the parameter v in the direction of the c -axis be taken equal to $\frac{1}{4}$, the spacing, between the planes parallel to (001) is, indeed halved, but the identity-distance l_c appears to be shortened simultaneously to half its true value: then the cell would no longer contain 8, but only 4 atoms, what is in contradiction with experience. Therefore, the whole space-group D_{4h}^1 must be excluded.

B. Space-groups D_{4h}^7 .

Here *four* cases are possible of 8 equivalent places symmetrically arranged, which by the author are discerned as the cases g , h , i and j (loco cit., p. 93). The combinations g and h are excluded according the same arguments, which in the previous case led to the exclusion of the cases p and q . The combinations mentioned sub i and j are impossible, because the intensities of (110) and (001) must be both *zero*, which only occurs if $u = v = \frac{1}{4}$. But then the intensity of (101) in case i would be very great, while it is *zero* in reality; in case j the intensities of (113) and (212) become also *zero*, which is not in accordance with experience, as may be seen from the results formerly discussed. The space-group D_{4h}^7 is, therefore, also excluded.

C. Space-group D_{4h}^{10} .

Five cases, discerned in the tables as k , l , m , n and o , are possible

¹⁾ The question, several times already discussed in recent days, whether the intensity I may be supposed to be really proportional to $|S|^2$, or whether it be more in conformity with the actual situation, in supposing a proportionality to $|S|^n$, in which n lies between 1 and 2, — is now left out of consideration here.

here for arrangements of 8 equivalent points in a way compatible with the special symmetry of this space-group (loco cit., p. 95).

The case mentioned sub *k* must be excluded, because the mutual arrangement in this case would be such, that halving of spacings would occur, if *l* be odd; for in this case the sums ($hm + kn + lp$) differ pairwise just with an exact amount of $\frac{l}{2}$, so that in the structural factor *S* always sinus and cosinus of angles will appear combined, which differ from each other with the value π , with the result that these numbers will totally annihilate each other. For $l = \text{odd}$, all corresponding intensities, therefore, will get the value zero, which is contradictory to experience, because e.g. (111) and (113) are certainly observed with measurable intensity. The case *k* is thus excluded, and the same argumentation is valid for the combinations mentioned sub *l* and *m*; the cases being, therefore, also impossible here. Considering the cases *n* and *o*, it may be remarked, that the intensities of (100), (110) and (210) must be zero, which only takes place, if $u = v = \frac{1}{2}$. But then the intensity of (211) appears also to become zero, which is contradictory to the experimental results. From this follows, that both cases *n* and *o* are excluded, and therewith the whole space-group D_{4h}^{10} .

D. The space-group D_{4h}^{16} .

Within this group four cases of 8 equivalent positions must be discerned, which in the tables are indicated sub *f*, *g*, *h* and *i* (loco cit., p. 98). The combination mentioned sub *f* is excluded beforehand, because the coördinates corresponding to *h* and *k* are not variable here: thus, e.g. the intensity of (110) cannot get the value = 0, while experiment reveals this fact with certainty. The combinations of points mentioned sub *g* and *h* are equally excluded, because in these cases *p* must be taken equal to zero or to $\frac{1}{2}$. The sum of the sinus and that of the cosinus in the structural factor *S* must be equal to zero, therefore, not only for $l = 0$, but also if *l* be even, because in that case, for $p = \frac{1}{2}$, the corresponding angle is increased with a whole number, multiple of 2π . Now the intensity of (210) appears really to be zero, as might be expected in the space-group D_{4h}^{16} ; but according to what was said above, also the intensity of (212) should be zero in that case, while experiment has taught the contrary.

It therefore appears, that only the case mentioned sub *i* remains possible here. The parameters of the 8 equivalent points in this case are: $[u, (u + \frac{1}{2}), v]$; $[u, (\frac{1}{2} - u), (v + \frac{1}{2})]$; $[-u, (\frac{1}{2} - u), v]$; $[u, (u + \frac{1}{2}), (v + \frac{1}{2})]$; $[(u + \frac{1}{2}), u, -v]$; $[(u + \frac{1}{2}), -u, (\frac{1}{2} - v)]$; $[(\frac{1}{2} - u), -u, -v]$; en $[(\frac{1}{2} - u), u, (\frac{1}{2} - v)]$.

The symmetry of this atomic arrangement is characterized by the presence of tetragonal screw-axes: $[001]_{0,1/2}$ and $[001]_{1/2,0}$; diagonal rotation-axes: $[001]_{0,0}$ and $[001]_{1/2,1/2}$; eight symmetry-centra:

$[\frac{1}{4}, \frac{1}{4}, 0]$, $[\frac{3}{4}, \frac{1}{4}, \frac{1}{2}]$, $[\frac{3}{4}, \frac{3}{4}, 0]$, $[\frac{1}{4}, \frac{3}{4}, \frac{1}{2}]$, $[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}]$, $[\frac{3}{4}, \frac{1}{4}, 0]$, $[\frac{3}{4}, \frac{1}{4}, \frac{1}{2}]$ and $[\frac{1}{4}, \frac{3}{4}, 0]$;

gliding-mirror planes $(001)_0$ and $(001)_{1/2}$, with translations in (001) ; idem such: $(110)_0$, $(110)_{1/2}$, $(100)_0$, $(100)_{1/2}$, $(010)_0$, and $(010)_{1/2}$, with translations parallel to the c -axis; rotation-mirrorplanes: $(001)_{1/4}$ and $(001)_{3/4}$; and mirrorplanes: $(110)_{1/2}$ and $(\bar{1}\bar{1}0)_{1/2}$.

It could be shown, that indeed a sufficient agreement between the observed and calculated intensities could be obtained (cf. table XII), if for u the value: $\frac{8}{40}$ or $\frac{9}{40}$, and for v the value $\frac{6}{40}$ or $\frac{7}{40}$ be substituted.

A more accurate calculation of the parameters is impossible, while the observed, or rather: the estimated intensities are known only too inaccurately. It may be remarked, that in this case the smallest distance of two Ga -atoms appears to be: 2,56 A.U., while, according to W. L. BRAGG, the "atomic radius" of the Ga -atom should be 1,3 A.U., which is as perfect an agreement as one could desire.

Fig. 3, 4 and 5 may give an image of the structure of the elementary cell in perspective and in orthogonal projection on a face of (001) ; in Fig. 3 the four successive layers atoms A , B , C and D and the places of their centres are drawn and the different dimensions in the direction of the c -axis indicated. In each layer the atoms are arranged as indicated

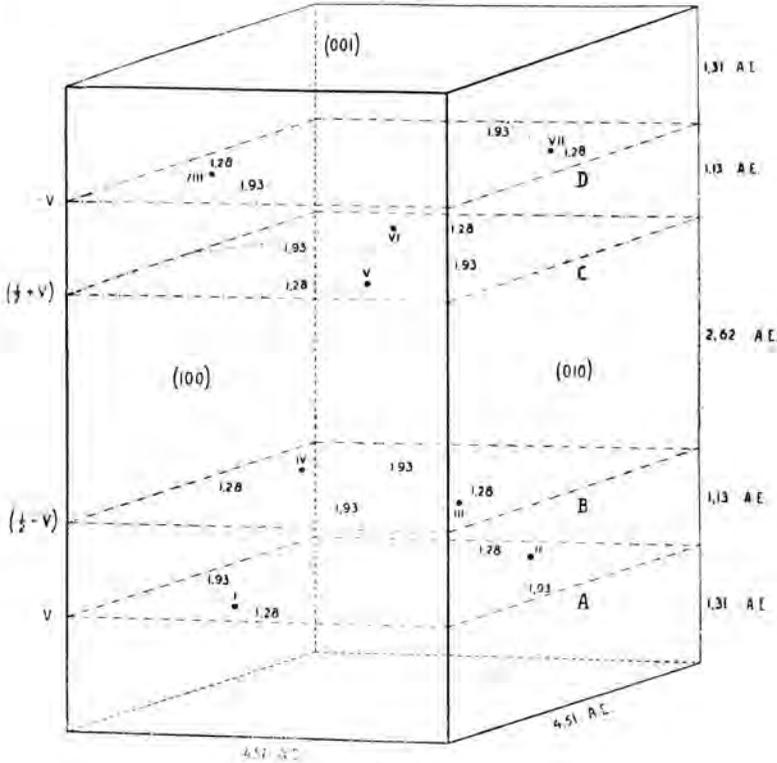


Fig. 3.

TABLE XII.

| Symbols (<i>h k l</i>): | Estimated intensities: | Calculated intensities ($\times 10000$): | | | Mean calculated intensities $\times \frac{1}{250}$: |
|------------------------------|---------------------------|--|--|--|---|
| | | ($\lambda = Cu_2$) | for: $u = \frac{8}{40}; v = \frac{6}{40}$: | for: $u = \frac{8}{40}; v = \frac{7}{40}$: | |
| (110) | 0 | 30 | 30 | 0 | 0.08 |
| (002) | 6 | 300 | 1100 | 1100 | 3 à 4 |
| (111); (102) | 8 | 2000 + 600 | 1100 + 200 | 1400 + 100 | 7 à 10 |
| (200) | 6 | 1250 | 1250 | 1700 | 5.5 à 6 |
| (113) | 7 | 1300 | 2200 | 2500 | 8 |
| (211) | 7 | 1100 | 1350 | 300 | 3 à 4 |
| (212) | 2 | 250 | 170 | 200 | 0.7 à 1 |
| (220) | 2 | 350 | 350 | 650 | 1.4 à 2.6 |
| (222) | 2 | 60 | 200 | 370 | 0.2 à 1.5 |
| (311); (302) | 4 | 80 + 700 | 170 + 500 | 370 + 160 | 2 à 2.5 |

Dividing the calculated values by 250 (column 6), numbers are obtained, which clearly show that the sequence of the relative intensities, as calculated, is principally the same as that of the estimated ones.

in Fig. 4, and 5; evidently the centres of each pair are distant from each other 2,56 A.U., while the distance of a next pair in the same straight line is 3,86 A.U.; etc. Although the arrangement of two immediately consecutive layers, such as *A* and *B* or *C* and *D* is essentially the same, each layer appears with respect to its neighbour turned round the *c*-axis over 90° and simultaneously shifted along a diagonal of (001) over a certain distance. As already mentioned, the smallest distance of the atomic centres is exactly that, which follows from W. L. BRAGG's values for the atomic radius of the *gallium*-atom: $2 \times 1,3 \text{ A.U.} = 2,6 \text{ A.U.}$ after BRAGG, while experimentally measured: 2,56 A.U. Four neighbouring atoms of one layer (cf. Fig. 5) are situated in the corners of a trapezium; just in its centre fits the *Ga*-atom of the immediately adjacent layer. If in the way indicated by BRAGG, round the atomic centres spheres be described with a radius $R = 1,3 \text{ A.U.}$, then the five spheres just mentioned appear to touch each other, so that in each double layer as *AB* or *CD*, all spheres are so closely packed, that the double layer could be thought to be built up completely by means of closest packed,

massive spheres¹⁾. It may be remarked, moreover, that the atomic planes of $\{110\}$ consist of three parallel ones, two of which have an identical atomic arrangement, the third being, however, at a different distance and having a somewhat different arrangement. If one considers the two

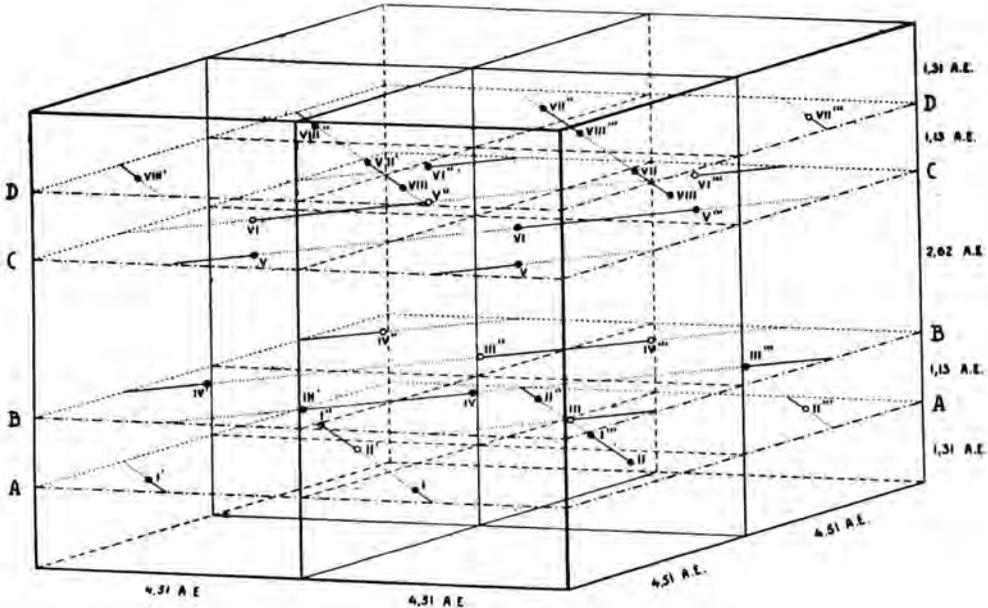


Fig. 4.

first, very closely approaching planes as a single one, the whole situation can also be described by saying, that in the directions (110) , $(\bar{1}\bar{1}0)$, etc. one *flat* plane of atoms interchanges with a plane, which appears to be slightly bent up and down in a regular way. It is this circumstance, which causes the intensities of $\{110\}$ to get *not exactly* the value zero; calculation teaches, that there remains an extremely small intensity of the corresponding diffraction-images.

The atomic planes parallel to the basis $\{001\}$ are, amongst all planes of the structure, those whose occupation by atoms is the densest.

¹⁾ Fig. 5 gives an orthogonal projection on (001) of two successive layers every time: Fig. 5a of A and B; Fig. 5b of C and D, which are identical with AB, but with respect to them turned round the c -axis over exactly 90° . The distance of two consecutive layers is: $\left(\frac{20}{40} - 2v\right) c_0 = \frac{6}{40} c_0 = 1.13$ A.U. The distance of an atomic centre in B to each of the nearest four centra in A is, in projection 2.3 A.U.; therefore, the true distance is: $\sqrt{(1.13)^2 + (2.3)^2} = 2.56$ A.U. From this follows, that the atomic spheres with a radius of 1.28 A.U. (according W. L. BRAGG) are situated in both layers in the way indicated in Fig. 5.

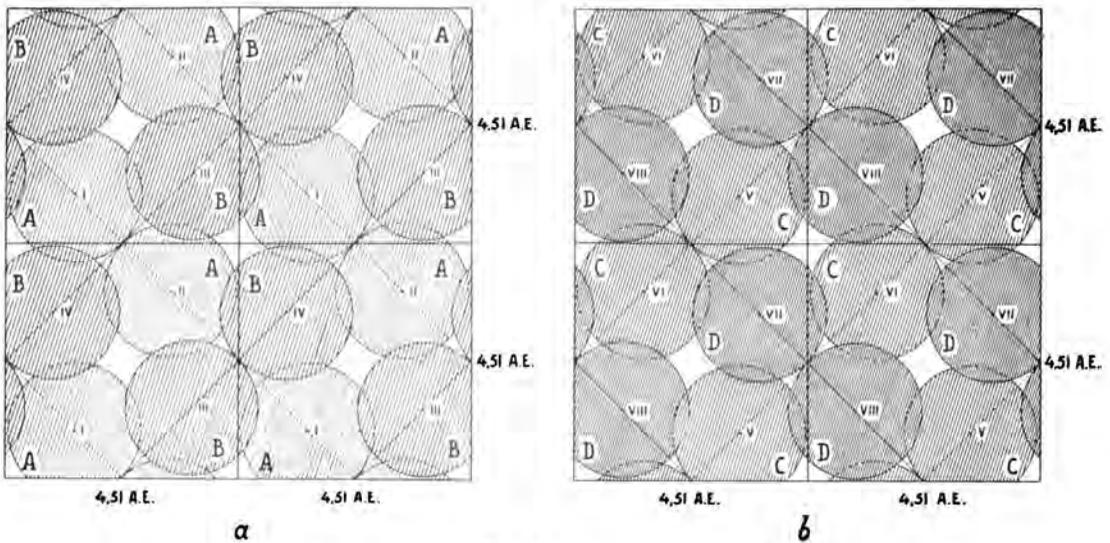


Fig. 5.

§ 20. It is remarkable, and highly in favour of the exactness of the here proposed structure of the metal, that from this structure several of the peculiarities of the solid *gallium* may be deduced.

As becomes clear from the Fig. 3 and 4, the double layer *AB* parallel to $\{001\}$, is in the cell separated from the double layer *CD*, — which has the same structure as *AB*, but differs from it only by the fact that it is turned with respect to it round the *c*-axis over 90° , — by a free space between the atomic spheres. As the layer *C* is turned with respect to *A* over 90° , *C* cannot approach as near to the layer *B*, as *A* does: the atomic spheres would touch each other only, if the distance from *C* to *B* gets value: 2,48 A.U. (cf. Fig. 6; the sphere *V* must have the position *V'* to touch the layer *B*). In reality this distance is, however, 2,62 A.U., so that a free space of 0,14 A.U. remains open between *AB* and *CD*. To bring both double layers in touch with each other, the upper double layer would have to be pressed down over 0,14 A.U., — the height of the elementary cell then getting the value: $7,51 - 0,14 - 0,07 - 0,07 = 7,23$ A.U.

Now, as is already known since its discovery, the specific volume of solid *gallium* is *greater* than that of the liquid metal, — just as the specific volume of *ice* is greater than that of water at the same temperature. The idea is easily suggested now, that this peculiar behaviour of *gallium* is intimately connected with the decrease in volume, which the crystalline metal suffers in melting, because of the fact, that at the meltingpoint, the structure being broken up and the atoms regaining their free mobility, the free space between the atomic layers, mentioned above, will disappear at the same time.

Indeed, the specific weight of the *liquid gallium* can, on this supposition,

be calculated from that of the *solid* metal, this being known as: 5,904. For, as the height of the cell is diminished from 7,51 A.U. to 7,23 A.U., this specific weight must change to: $\frac{7,51}{7,23} \times 5,904 = 6,124$; on the other hand, the specific weight of the *liquid gallium* actually observed is: 6,095. The difference between the calculated and observed numbers is only 0,5%, — this being an agreement, which may be called really perfect.

Recently EHRENFEST¹⁾ drew attention to the fact that some analogies in behaviour also occur in the case of such metals as *bismuthum*, *antimony*,

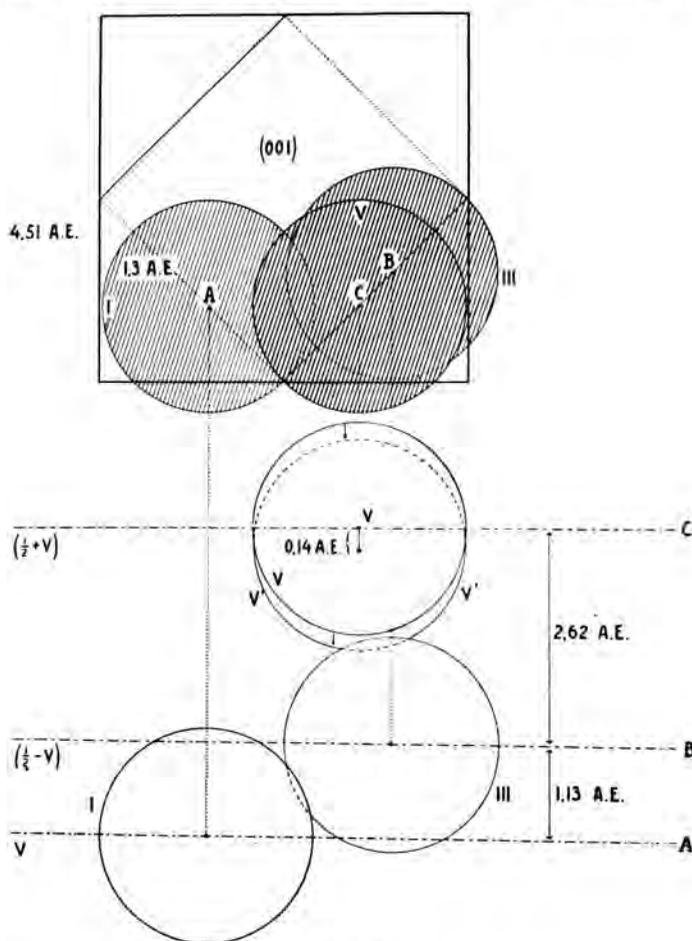


Fig. 6.

¹⁾ P. EHRENFEST, *Physica*, 5, (1925), p. 383.

From the fact that *bismuthum* shows an extraordinarily great diamagnetic susceptibility, the author concludes that the electronic orbits which cause this phenomenon, must surround more than one atom of the structure; because of this fact, the importance becomes clear of these layers of atoms approaching each other very nearly. In *gallium* the distance between two such atoms (2,56 A.U.) is even smaller than in *bismuthum*. All three metals (*antimony* included) have in common, that their electric conductivity increases, as they melt; all three have a specific volume in the solid state greater than in the liquid condition.

etc., whose crystalstructure is, like that of *gallium*, characterised by the occurrence of such double layers of atoms very nearly approaching each other. Perhaps the occurrence of these double layers might be related to the other fact, that, such metals as *bismuthum*, *antimony*, and *gallium* also, form oxides, which according to circumstances, may show a weak basic or a weak acid function ($Ga(OH)_3$ with *KOH* and *ammonia*; *antimonites*, *antimonates*, *bismuthites*; etc.): it is e.g. imaginable, that *two* modifications of the atoms, depending on their momentaneous electronic configuration, should really take part in building up these crystalstructures, of which modifications one kind had a more "metallic" character and the other a nature more tending to that of a "metalloid"; *together* they would then be able to form such double layers of atoms approaching very nearly to each other.

§ 21. But also in other respects the proposed structure enables us to give a satisfactory account of the special phenomena observed with the solid metal.

Attention has already been drawn to the fact, that the crystals of *gallium* show *two* different habits, according as they are formed *rapidly* or *slowly* from the more or less undercooled molten mass. (page 1196; Fig. 1a and 1b). This phenomenon must be interpreted in this way, that there exists evidently a great difference in crystallisation-velocity with these crystals in different directions perpendicular to the different sets of atomic planes. Evidently this velocity perpendicular to (001) is, under comparable circumstances, much smaller than in directions perpendicular to faces of $\{111\}$ and $\{121\}$. As a consequence, if the crystal is only allowed sufficient time to develop, — i.e. in *slow* crystallisation, — the relative development of the faces of $\{001\}$ will increase gradually in comparison with that of forms, like $\{111\}$, $\{121\}$, etc. According to BRAVAIS, the faces predominantly developed in the crystal are generally those of *greatest* atomic density. That this is really the case here also, may now be immediately seen from the structure derived for the *gallium*: for in (001) the eight metal-atoms appear to be distributed over *four* planes *A*, *B*, *C* and *D* of the structure, each layer containing *two* atoms (Fig. 5); therefore, the density $\Delta_{(100)}$ of these planes is: $\frac{2}{a_0^2} = \frac{2}{20,34} = 0,098$. On the other hand, for $\{111\}$ e.g., those 8 atoms are distributed over *six* planes, of which *four* contain only *one* atom, and *two* layers which have *two* atoms in a parallelogram. For $\{111\}$ the mean density of arrangement $\Delta_{(111)}$ is therefore: $\frac{4/3}{a_0 \sqrt{2c_0^2 + a_0^2}} = 0,026$ only. Thus it is evident¹⁾ that, independent of the special values of the parameters *u*

¹⁾ It may be remarked, that the density of the atomic arrangement of $\{111\}$ and $\{001\}$ can only be *the same*, if the 8 atoms be distributed for $\{111\}$ over only *two* atomic planes. This occurs, if to *u* the value $1/4$ is attributed. This parameter-value, however, is excluded here, because in that case the intensity of the diffraction-images on the planes parallel to (121), must be zero, the spacings then being halved for this form. Experience, however, proves, that the planes parallel to $\{121\}$ have a great intensity.

and *v.* the density of the atomic arrangement parallel to (001) is about four times greater than that in the planes parallel to $\{111\}$. It is evident for this reason, that the velocity of growth perpendicular to the atomic planes parallel to (001) is so much smaller than that in directions perpendicular to the planes of the form $\{111\}$; which involves, that the faces of this form must finally disappear, making gradually place for the more and more predominant form $\{001\}$. The specific density of the planes parallel to $\{121\}$ is, moreover, again smaller, so that this form *à fortiori* will be absent in the crystals of Fig. 1^b. The bipyramidal habitus of the crystals of Fig. 1^a depends undoubtedly on the special properties of the strongly undercooled, liquid medium, from which they are generated; in many cases the degree of undercooling has, without any doubt, a great influence on the particular shape of the crystals, which are deposited from such a medium.

Hitherto we have not been able to obtain sufficient information about the cleavability, the cohesion, the presence of gliding-planes, etc. of the *gallium*-crystals, in order to test also these properties by means of the here indicated crystalstructure; and the same is valid for some other physical properties, as e.g. thermic and electric conductivity; etc. Perhaps, however, it will appear possible to obtain before long, the data required to elucidate these phenomena also in the light of the crystalstructure deduced.

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Physical Chemistry of the University.*

Chemistry. — "*Some Remarks on the Crystalform of Boron-nitride and on the eventual Ambiguity in the Analysis of Powder-spectrograms.*" By Prof. F. M. JAEGER and H. G. K. WESTENBRINK.

(Communicated at the meeting of September 25, 1926)

§ 1. *Boron-nitride*: $(BN)_x$ commonly occurs in the form of a white, tallowy powder, which in literature is described as amorphous¹⁾. Even on magnifying 500 times, no crystalform of the particles present in the powder studied²⁾ is observed under the microscope. There is no doubt, however, about the fact, that the compound is really microcrystalline; this was, a short time ago, experimentally proved for the first time by TIEDE and TOMASCHEK³⁾. It is, however, impossible to provide better developed crystals by recrystallization from a solvent or molten mass, or by sublimation; therefore, on studying the substance by means of *X*-rays, it is *only* possible here to make use of powder-spectrograms after HULL-DEBIJE's method, without our having any opportunity to test the results obtained in any other way. In the following pages it is emphasized once more, how ambiguous results from the interpretation of such spectrograms may follow, if no other method in ROENTGEN-analysis can be made use of, or if neither goniometric measurements, nor sufficient data about the crystallographic symmetry are available for further control.

Another difficulty presented itself here in the fact, that the specific weight of the substance is not, or at least very inaccurately, known and its exact determination is much hampered by the peculiar nature of the compound. As the original preparation appeared to be not completely homogeneous under the microscope, but to contain small black grains, — it was necessary to separate these by fractional sedimentation with water. They consist principally of crystallized *boron*, which has a higher specific weight than the principal mass of the preparation. From the remaining mass about five fractions of a different degree of dispersion could be separated by fractional sedimentation; the last fraction gave with water a colloidal solution, from which the colloidal substance by addition of a trace of hydrochloric acid could be coagulated. After drying

1) L. MOESER and W. EIDMANN, Ber. d. d. chem. Ges., **35**, 536, 1902; A. STOCK and M. BLIX, Ber. d. d. Chem. Ges., **34**, 3039, (1901).

2) W. H. BALMAIN, Phil. Mag. (3), **21**, 170, (1842); **22**, 467, (1843), **23**, 71, (1843); **24**, 191, (1844); Journ. für prakt. Chemie, **27**, 422, (1842); **30**, 14, (1843).

F. WÖHLER, Ann. der Chem. **74**, 70, (1850); Pogg. Ann., **79**, 467, (1851).

3) E. TIEDE and H. TOMASCHEK, Zeits. f. Elektrochemie, **29**, 303, (1923); cf. also: E. TIEDE and F. BÜSCHER, Ber. d. d. chem. Ges., **53**, 2206, (1920).

at 100° C., also this last preparation was studied in the same way: it appeared, however, that all preparations gave *identical* ROENTGEN-spectrograms. The specific weight of these different fractions, however, differed considerably and appeared to decrease with an increasing degree of division. Perhaps this fact might be related to the particular circumstance, that the molecular weight of *boron-nitride* is, according to most investigations, certainly a *multiple* of the most simple formula: *BN*; the dissociating power of the water may be the cause of the occurrence of a mixture of several kinds of such polymeric molecules. The last three fractions obtained showed finally only a small difference of their specific weights, so that its final value with near approximation could be fixed at $d_{40}^{180} = 1.99$. But as will become clear, this value presents again a new difficulty for the exact interpretation of the true crystalstructure.

§ 2. A considerable number of diffraction-lines could be observed on the well developed spectrograms, which were obtained after HULL-DEBIJE's method in using the *K*-radiation of a *copper*-anticathode. The lines showed strongly different intensities on the film. The principal ones are reviewed in the following table; assuming a *cubic* symmetry as the most probable one in respect to the chemical composition, the indices were calculated from the suitable quadratic equation. Such a cubic arrangement may a priori be considered as probable here, taking

| TABLE I. | | | | | |
|----------------------|------------|--------------------------------------|-----------------------------|--|-------------------|
| Number of the lines: | Intensity: | Double distance to the centre in mm; | $\sin^2 \frac{\theta}{2}$: | Calculated values of $\sin^2 \frac{\theta}{2}$: | Indices: (MILLER) |
| 1 | 9 | 18.0 | 0.0321 | 0.0321 | (111) |
| 3 | 10 | 23.5 | 0.0542 | 0.0535 | (102) |
| 4 | 3 | 25.8 | 0.0651 | 0.0642 | (112) |
| 5 | 4 | 36.5 | 0.1275 | 0.1284 | (222) |
| 5 | 3 | 38.7 | 0.1425 | 0.1391 | (320) |
| 8 | 4 | 48.4 | 0.2166 | 0.2140 | (420) |
| 9 | 1 | 52.1 | 0.2477 | 0.2568 | (422) |
| 12 | 4 | 66.4 | 0.3794 | 0.3852 | (600) |

Quadratic equation: $\sin^2 \frac{\theta}{2} = 0.0107 (h^2 + k^2 + l^2)$ (α -radiation).
 $R = 25$ mm. For the lines N^o. 2 and 7, cf. Table II.
 Tension: 55000 Volt; time of exposure: 6 hours.

into consideration the relatively small difference in atomic number of the *boron-* and *nitrogen-*atoms¹⁾.

From the indices of the scattering atomic planes, which evidently occur in all possible combinations of odd and even numbers, it follows, that the underlying grating is a *simple cubic* one, the characteristic translation of it (edge of the cubic cell) a_0 being calculated, with respect to the wave-length used, as 7,44 A.U. But from the value for $d=1,99$, the number n of molecules *BN* present in such a cell, results as:

$$n = \frac{(7,44)^3 \times 1,99 \times 10^{-24}}{1,64 \times (11 + 14) \times 10^{-24}} = 20.$$

Therefore, within each cell 20 *boron-* and 20 *nitrogen-*atoms should necessarily find their places, — a number, which can hardly be considered to be in accordance with that of the equivalent positions, characteristic for the space-groups of cubic symmetry.

§ 3. However, as follows from Table II, the results of the measurements can be interpreted with the same degree of accuracy, also on the assumption, that here a *tetragonal* grating is present. (See Table II following page).

Supposing *this* interpretation to be the right one, *boron-nitride* appears to be tetragonal, with a grating built up from *simple tetragonal cells*. The edges of the cell are: $a_0=4,295$ A.U. and $c_0=5,176$ A.U.; so that the axial ratio has the value: $a:c=1:1,2052$. If again the value of d_0 at 18° C. be considered as: 1,99, the number n of molecules *BN*

per elementary cell is calculated as: $n = \frac{(4,295)^2 \times 5,176 \times 1,99 \times 10^{-24}}{1,64 \times 25 \times 10^{-24}}$

$= 4,6$; so that 4 or 5 *boron-*atoms and the same number of *nitrogen-*atoms must be present in each cell. The first number is certainly, but the second not in accordance with the number of equivalent positions compatible with the special symmetry of the tetragonal space-groups.

§ 4. As all further necessary data are lacking, it is impossible here to fix our choice and to decide, whether *boron-nitride* has *cubic* or *tetragonal* symmetry. An analogous case of ambiguity in the interpretation of powder-spectrograms occurred recently in the study of *thallium* by LEVI²⁾. This author attributed to the metal, with respect to his results obtained by the HULL-DEBIJE-method only, a hexagonal structure (trigonal closest packing), with the grating-constants: $a=3,47$ A.U.; $c=5,52$ A.U.; and $a:c=1:1,59$. TERPSTRA³⁾ drew, however, attention to the fact, that

1) The case of a trigonal structure, which could be supposed, appeared only to be compatible with the results of the measurements, if the axial ratio were $a:c=1:1,225$. Then, however, α must be 90°, and no difference is present then with a cubic grating, considered in the direction of a trigonal symmetry-axis.

2) G. R. LEVI, Nuovo Cimento, May, June, (1924); Zeits. f. Kryst. 61, 561, (2925).

3) P. TERPSTRA, Zeits. f. Kryst. 63, 318, (1926).

TABLE II.

| Number of the lines: | Intensities (estimated) | Double distance to the centre in mm : | $\theta/2$: | $\sin^2 \frac{\theta}{2}$: | $\sin^2 \frac{\theta}{2}$: | Calculated values of $\sin^2 \frac{\theta}{2}$: | Indices (MILLER). | Remarks: |
|----------------------|-------------------------|---------------------------------------|-----------------------|-----------------------------|-----------------------------|--|-------------------|----------------------------|
| 1 | 9 | 18.0 | 10°19' | 0.1791 | 0.0321 | 0.0321 | (100) | |
| 2 | 2 | 21.2 | 12°09' | 0.2105 | 0.0443 | 0.0442 | (101) | β -line. |
| 3 | 10 | 23.5 | 13°28' | 0.2329 | 0.0542 | 0.0542 | (101) | |
| 4 | 3 | 25.8 | 14°47' | 0.2552 | 0.0651 | 0.0642 | (110) | Only present on one film. |
| 5 | 4 | 36.5 | 20°55' | 0.3570 | 0.1275 | 0.1284 | (200) | |
| 6 | 3 | 38.7 | 22°10 $\frac{1}{2}$ ' | 0.3774 | 0.1425 | 0.1505 | (201) | |
| 7 | 2 | 43.8 | 25°6' | 0.4242 | 0.1799 | 0.1768 | (202) | β -line. |
| 8 | 4 | 48.4 | 27°44' | 0.4654 | 0.2166 | 0.2168 | (202) | |
| 9 | 1 | 52.1 | 29°51' | 0.4977 | 0.2477 | 0.2489 | (212) | |
| 10 | 1 | 63.9 | 36°37' | 0.5965 | 0.3558 | 0.3586 | (321) | β -line. |
| 11 | 3 | 65.8 | 37°42' | 0.6115 | 0.3740 | 0.3773 | (302) | Only observed on one film. |
| 12 | 4 | 66.4 | 38°3' | 0.6164 | 0.3799 | 0.3857 | (104) | |
| 13 | 5 | 71.8 | 41°8' | 0.6579 | 0.4327 | 0.4394 | (321) | |
| 14 | 1 | 76.4 | 43°46' | 0.6917 | 0.4785 | 0.4878 | (303) | Only present on one film. |
| 15 | 3 | 87.9 | 50°22' | 0.7701 | 0.5931 | 0.5999 | (331) | |

Quadratic form: $\sin^2 \frac{\theta}{2} = 0.0321 (h^2 + k^2) + 0.0221 l^2$ (α -radiation);
 and $\sin^2 \frac{\theta}{2} = 0.0262 (h^2 + k^2) + 0.0180 l^2$ (β -radiation).
R = 25 mm. Tension: 55 Kilovolt; time of exposure¹⁾: 6 hours.
 Line N^o. 2 is the β -image of N^o. 3; line 7 that of N^o. 8; N^o. 10 that of N^o. 13; all other lines are α -images.

these observations of $\sin^2 \frac{\theta}{2}$ could as well, if not better, be calculated from a quadratic equation for a tetragonal grating, whose parameters are: $a_0 = 5,2$ A.U. and $c_0 = 8,2$ A.U. Only after the Italian author¹⁾ had recently been able to obtain LAUE-spectrograms of cathodically deposited, very small *thallium*-crystals, it was possible for him to prove beyond any doubt, that the symmetry is certainly *hexagonal*. As the calculated and observed intensities for the structure of *thallium* attributed by him,

¹⁾ G. R. LEVI, „Nuovo Cimento“, October (1926).

do not agree, however, a reviewed investigation of this crystalstructure remains highly desirable.

These data and those given in this paper, may, therefore, be once more considered as a warning, not to draw conclusions about crystalstructures, where *only* the results of powder-spectrograms can be used. If it is required to give a final solution of problems concerning the crystalstructure of the compounds investigated, such data can only be considered as valuable, if used in connection with the results of other spectrographic methods and of crystallographic measurements.

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Chemistry. — "*The Space-Groups of the Rhombic and Monoclinic Heptahydrates of the Sulphates of the bivalent Metals.*" By H. G. K. WESTENBRINK. (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of September 25, 1926).

§ 1. The *heptahydrates* of the bivalent metals: *zinc, iron, magnesium*, etc., can form isomorphous or isodimorphous series of mixed crystals. For some of them the stable form has rhombic-bisphenoidal symmetry, for others the monoclinic-prismatic forms are the stabler ones.

The *rhombic modification* of $MgSO_4 \cdot 7H_2O$, of $MgCrO_4 \cdot 7H_2O$, of $ZnSO_4 \cdot 7H_2O$ and of $NiSO_4 \cdot 7H_2O$ were obtained by slow evaporation at roomtemperature of their solutions in water. Their habit is always strongly elongated in the direction of the *c*-axis. Far predominant is the prism $\{110\}$; of the three pinacoids only $\{010\}$ occurs. The experiments described in the following pages were chiefly made with the *magnesium-salt*; the properties of the other members of this series differ, however, only immaterially from those of the *magnesium-salt*, so that the conclusions drawn in that case, may safely be extended to the other salts.

For the determination of the identity-distances *I* in the three principal directions, three rotation-spectrograms were prepared, in which the crystal was successively rotated round the *c*-, the *a*- and the *b*-axis. In these experiments a flat photographic plate was made use of. The results obtained are reviewed in table I.

| Axis of revolution: | Measured distances between the hyperbolae 0 and 1: | Identity-distances in the direction of the axes: |
|------------------------|--|--|
| <i>c</i> ¹⁾ | 9.5 mm | c_0 ¹⁾ = 6.7 A.U. |
| <i>b</i> | 5 mm | b_0 = 13.0 A.U. |
| <i>a</i> | 5 mm | a_0 = 13.0 A.U. |

The data for a_0 , b_0 and c_0 in this table give only the *order of magnitude* of the required distances, not their exact values. To get more accurate data for the dimensions of the elementary cell, BRAGG-spectrograms for the best developed faces of $\{110\}$ were studied. The exact

¹⁾ The axial ratio is expressed by: $a : b : c$; the dimensions of the elementary cell in the corresponding directions are indicated by a_0 , b_0 and c_0 .

distance from plate to crystal was determined in this case by simultaneously photographing the spectrum on $\{100\}$ of a piece of *rock-salt*. The spacings $d_{(110)}$ could in this way be determined with an accuracy of circa 0,02 A.U., when the distance between plate and crystal happened to be 60 mm. The value of $\frac{\theta}{2}$ for the most intense line of $MgSO_4 \cdot 7 H_2O$ was found to be: $10^\circ 30'$ if *copper-K α* -radiation of 1,540 A.U. was used here. The quadratic equation in the rhombic system having the form:

$$\sin^2 \frac{\theta}{2} = \frac{\lambda^2}{4a_0^2} h^2 + \frac{\lambda^2}{4b_0^2} k^2 + \frac{\lambda^2}{4c_0^2} l^2,$$

this equation becomes in the case that the said interference-image is supposed to be one of the 2nd order ($h = k = 2$):

$$\sin^2 10^\circ 30' = \frac{(1,540)^2}{a_0^2} + \frac{(1,540)^2}{b_0^2};$$

for the planes parallel to $\{110\}$.

From the very accurately known axial ratio: $a : b : c = 0,9901 : 1 : 0,5709$, it may be, moreover, deduced, that $a_0 = 0,9901 b_0$, and therefore: $b_0 = 12,01$ A.U.; so that $a_0 = 11,89$ A.U. and $c_0 = 6,86$ A.U. respectively.

It is, of course, *a priori* not quite sure, that $a_0 : b_0 : c_0$ is equal to $a : b : c$; but certainly $a_0 : b_0 : c_0$ may be supposed to be equal to: $ma : nb : pc$, in which ratio m , n , and p are integer numbers. But from the rotation-spectrograms it could here be deduced already, that $m = n = p = 1$ in this case.

If V be the volume of the elementary cell ($= 12,01 \times 11,89 \times 6,86$ A.U.³); s the specific weight: 1,677, and M the molecular weight of the *magnesium-salt*: 246,49, — then the number N of molecules per elementary cell is calculated to be:

$$N = \frac{V \times s}{1,64 \times M \times 10^{-24}} = 4,06,$$

i.e. four molecules of the hydrated salt.

§ 2. For the identification of the interference-images the rotation-spectrogram obtained by rotating the crystal round the c -axis, appeared to be the most suitable one. As a_0 is practically equal to b_0 , instead of the rigorous quadratic formula:

$$\sin^2 \frac{\theta}{2} = 0,00419 h^2 + 0,00411 k^2 + 0,01260 l^2 \text{ (for } K_{(\alpha)}\text{-radiation)}$$

the simplified equation:

$$\sin^2 \frac{\theta}{2} = 0,00415 (h^2 + k^2) + 0,0126 l^2$$

could be used in the analysis of the spectrograms.

In table II the calculated and observed values for $\sin^2 \frac{\theta}{2}$ are put

TABLE II.

| (Distance of crystal to plate: 39 mm) | | | |
|---------------------------------------|---|-----------------------------|--|
| MILLERian Symbol : | $\text{Sin}^2 \frac{\theta}{2}$ (calculated): | Distance to the centre : | $\text{Sin}^2 \frac{\theta}{2}$ (observed) : |
| (200) | 0.0166 | 10. — m.m. | 0.0157 |
| (210) | 227 | 13. — " | 256 |
| (220) | 332 | 14.5 " | 314 |
| (300) | 373 | — | — |
| (310) | 415 | 16.5 " | 395 |
| (320) | 539 | 19.5 " | 528 |
| (330) | 587 | — | — |
| (400) | 664 | 22.5 " | 670 |
| (410) | 705 | 23.7 " | 728 |
| (420) | 830 | — | — |
| (430) | 0.1037 | 30.2 " | 0.1050 |
| (440) | 1328 | — | — |
| (500) | 1037 | — | — |
| (510) | 1079 | — | — |
| (520) | 1203 | 32.5 " | 1160 |
| (530) | 1411 | 37.5 " | 1400 |
| (600) | 1494 | — | — |
| (101) | 0.0168 | — | — |
| (111) | 210 | 11.5 " | 0.0204 |
| (201) | 294 | 14.5 " | 314 |
| (211) | 336 | — | — |
| (221) | 462 | — | — |
| (301) | 504 | 19. — " | 505 |
| (311) | 546 | 20. — " | 552 |
| (321) | 672 | 22.5 " | 670 |
| (331) | 882 | — | — |
| (401) | 798 | 25. — " | 791 |
| (411) | 840 | 26. — " | 840 |
| (421) | 966 | 28. — " | 938 |
| (431) | 0.1176 | 32. — " | 0.1301 |
| (441) | 1470 | — | — |
| (501) | 1176 | 32. — " | 1130 |
| (511) | 1218 | — | — |
| (521) | 1344 | 36.5 " | 1350 |
| (531) | 1554 | — | — |

Principal spectrum

1st Accessory spectrum

TABLE II (Continued).

| (Distance of crystal to plate 39 mm) | | | |
|--------------------------------------|---|-------------------------|---|
| MILLERian Symbol: | $\text{Sin}^2 \frac{\theta}{2}$ (calculated): | Distance to the centre: | $\text{Sin}^2 \frac{\theta}{2}$ (observed): |
| (102) | 0.0546 | — | — |
| (112) | 588 | 20.5 m.M. | 0.0574 |
| (202) | 672 | 23. — .. | 693 |
| (212) | 714 | — | — |
| (222) | 840 | 26. — .. | 840 |
| (302) | 882 | — | — |
| (312) | 924 | — | — |
| (322) | 0.1050 | 30. — .. | 0.1037 |
| (332) | 1260 | — | — |
| (402) | 1176 | — | — |
| (412) | 1218 | — | — |
| (422) | 1344 | 36. — .. | 1326 |
| (432) | 1554 | 41. — .. | 1554 |
| (442) | 1848 | — | — |
| (502) | 1554 | 41. — .. | 1554 |
| (512) | 1596 | — | — |
| (522) | 1722 | 44.5 .. | 1704 |
| (532) | 1932 | — | — |
| (103) | 1176 | 32. — .. | 1135 |
| (113) | 1218 | — | — |
| (203) | 1302 | — | — |
| (213) | 1344 | 37. — .. | 1373 |
| (223) | 1470 | 39. — .. | 1465 |
| (303) | 1512 | 40. — .. | 1510 |
| (313) | 1554 | — | — |
| (323) | 1680 | 43.5 .. | 1663 |
| (333) | 1890 | 49. — .. | 1888 |
| (403) | 1806 | — | — |
| (413) | 1848 | 48. — .. | 1846 |
| (423) | 1974 | — | — |
| (433) | 2184 | — | — |
| (443) | 2478 | — | — |
| (503) | 2184 | — | — |
| (513) | 2226 | 58. — .. | 2209 |
| (523) | 2352 | — | — |

2nd Accessory spectrum

3rd Accessory spectrum

together, with the corresponding MILLERian symbols eventually multiplied by the order n . Of these indices the numbers h and k can here, if desired, be interchanged. Only the diffraction-lines for K_α -radiation are reviewed in the table; to show with what accuracy the indices can be attributed to the observed lines, also some values of $\sin^2 \frac{\theta}{2}$ of lines not observed in the spectrograms are included with the other ones.

From table II it becomes clear that no certainty is obtained, whether the observed lines correspond to (201) or (211) and to (202) or (212). A slight difference in the measured distance between plate and crystal has here a relative great influence on the values of $\sin^2 \frac{\theta}{2}$. But leaving these uncertain values aside for the moment, it becomes in every case evident, that a number of lines occur, for which $(h+k)$ is *odd*; so e.g.: (210), (410), (430), (520), (301), (321), (411), (431) or (501), (521), (322), (432), (522), (213), (303), (323) and (413); moreover, such for which $(h+l)$, as well as $(k+l)$ are *odd*¹⁾: (310), (530), (401), (421), and (223); in the third place such lines, for which $(h+k+l)$ is an *odd* number, e.g.: (210), (410), (430), (520), (111), (311), (401), (421) and (223).

From this it becomes sufficiently certain, that the underlying type of grating can be no other than that built up by *simple rhombic cells*.

This conclusion is contradictory to an hypothesis made by JACKSON²⁾ concerning his results of the study of paramagnetism of these sulphates at low temperatures. According to this hypothesis, the type of grating present here should be that with body-centred cells. But if this were true, the intensities of all diffraction-images for which $(h+k+l)$ is *odd*, should appear to be systematically equal to zero. As becomes clear from the above, this is not the case, there being, therefore, no reason to sustain the hypothesis mentioned.

§ 3. There are *four* rhombic-bisphenoidal space-groups, which, in WYCKOFF's notation, are discerned as V_1, V_2, V_3 and V_4 and which have, therefore, the symmetry-elements of structures built up by intercalated BRAVAIS' gratings with simple rhombic cells. Of these space-groups V_1 has only axes of rotation; V_2 rotatory axes parallel to the directions [100] and [010] and screw-axes parallel to the direction [001]; V_3 possesses rotatory axes parallel to [100] and screw-axes parallel to [001] and [010]; finally V_4 has only screw-axes. Now from the spectrograms after BRAGG's method, obtained for (100), (010) and (001), it followed already that the *odd* orders of the diffraction-images on these planes have all the intensity zero. Therefore, *all* spacings of the planes parallel to the

¹⁾ As a_0 and b_0 are approximately identical, h and k , and also $(h+l)$ and $(k+l)$, cannot be discerned from each other.

²⁾ L. C. JACKSON and H. KAMERLINGH ONNES, Phil. Trans. Royal Soc., A, 224, 1, (1924).

three pinacoids are evidently halved by the other planes. The only space-group possible here can, as a consequence, be no other than V_4 .

Table III gives a review of the measured distances $d_{(110)}$, from which values the dimensions a_0 , b_0 and c_0 and the specific weight are calculated, with the use of the accurately known axial ratio ¹⁾.

| Salts : | $d_{(110)}$: | Axial ratio $a : b : c :$ | a_0 | b_0 | c_0 | Specific weight (calculated): | Specific weight (literature) : |
|------------------------|---------------|------------------------------|-------|-------|-------|----------------------------------|-----------------------------------|
| $MgSO_4 \cdot 7 H_2O$ | 8.45 | 0.9901 : 1 : 0.5709 | 11.89 | 12.01 | 6.86 | 1.717 | 1.677 |
| $MgCrO_4 \cdot 7 H_2O$ | 8.45 | 0.9901 : 1 : 0.5735 | 11.89 | 12.01 | 6.89 | 1.827 | 1.695 |
| $NiSO_4 \cdot 7 H_2O$ | 8.46 | 0.9804 : 1 : 0.5631 | 11.86 | 12.08 | 6.81 | 1.882 | 1.953 |
| $ZnSO_4 \cdot 7 H_2O$ | 8.46 | 0.9815 : 1 : 0.5656 | 11.85 | 12.09 | 6.83 | 1.933 | 1.974 |

§ 4. *The monoclinic modification.* The stabler form, e.g. of $FeSO_4 \cdot 7 H_2O$, of $CoSO_4 \cdot 7 H_2O$, etc., which were the only salts suitable for the following experiments, is monoclinic-prismatic. The flat crystals of $FeSO_4 \cdot 7 H_2O$ ordinarily grow out on a face of $\{001\}$, this form, therefore, always being predominant. The *cobalto-salt* crystallizes mostly with predominant development of $\{101\}$. Most experiments were made with the *iron-salt*. The results of the analysis of three rotation-spectrograms, in which the crystal was successively rotated round the *a*-, the *b*- and the *c*-axis, are reviewed in table IV.

| Axis of revolution : | Distance of the first hyperbolae to centre : | Identity-distances : |
|----------------------|--|----------------------|
| <i>a</i> -axis | 4 mm | $a_0 = 15.50$ A.U. |
| <i>b</i> -axis | 4.75 mm | $b_0 = 13.07$ A.U. |
| <i>c</i> -axis | 3 mm | $c_0 = 20.60$ A.U. |

For the accurate measurements of the grating-constants a spectrogram after BRAGG's method was made on (001), this plane being the best

¹⁾ The axial ratios and specific weights are taken from: P. GROTH, "Chemische Kristallographie", (1908), Volume II.

suit for that purpose. On this spectrogram was found one line belonging to the $FeSO_4 \cdot 7H_2O$ besides the lines of *rocksalt*, which were simultaneously photographed. Supposing that this line is respectively of the 1st, 2nd, 3rd and 4th order, the results obtained in all these cases are reviewed in table V.

| TABLE V. | | |
|--------------------|---|---|
| Order number n : | $d_{(001)} = \frac{n \cdot \lambda}{2 \sin \frac{\theta}{2}}$: | $c_0^1) = \frac{d_{(001)}}{\sin 104^\circ 15' / 2}$: |
| 1 | 4.85 A.U. | 5.00 ⁵ A.E. |
| 2 | 9.70 A.U. | 10.01 A.E. |
| 3 | 14.55 A.U. | 15.01 ⁵ A.E. |
| 4 | 19.40 A.U. | 20.02 A.E. |

From this table it becomes evident, that the identity-distance in the direction of the c -axis becomes really equal to the value found by means of the rotation-spectrogram, if calculated on assuming it to be of the *fourth* order. To eliminate each doubt about this, attempts were made, by much longer exposure, to photograph the lines of lower order, their places on the plate now being exactly known. But no traces of one of these lines was ever observed. As the axial ratio of the monoclinic $FeSO_4 \cdot 7HO_2$ is: $a : b : c = 1,1828 : 1 : 1,5427$, the distances c_0 , a_0 and b_0 were calculated to be: $c_0 = 20,02$ A.U.; $a_0 = 15,34$ A.U. and $b_0 = 12,98$ A.U.

The number N of molecules per unit-cell was then calculated by:

$$N = \frac{V \times s}{M \times 1,64 \cdot 10^{-24}} = 16$$
; in this formula $V = 15,34 \times 20,02 \times 12,98 \times \sin 104^\circ 15' / 2$; $s = 1,899$ and $M = 278,01$.

The quadratic equation in the monoclinic system can be written:

$$\sin^2 \frac{\theta}{2} = \frac{\lambda^2}{4 a_0^2 \sin^2 \beta} h^2 + \frac{\lambda^2}{4 c_0^2 \sin^2 \beta} l^2 - \frac{\lambda^2 \cos \beta}{2 a_0 c_0 \sin^2 \beta} hl + \frac{\lambda^2}{4 b_0^2} k^2.$$

Using K_α -copper-radiation, the coefficients for $FeSO_4 \cdot 7H_2O$ can be easily calculated and the formula becomes:

$$\sin^2 \frac{\theta}{2} = 0,00268 h^2 + 0,00158 l^2 + 0,00101 hl + 0,00352 k^2.$$

The coefficients in it are, however, so small, that by its aid the indices cannot be attributed to the diffraction-lines with sufficient accuracy. But

¹⁾ The obtuse angle β between c - and a -axis is in the case of the *iron-salt*: $104^\circ 15' / 2$.

yet notwithstanding this inconvenience, the space-group could be determined in this special case in the following way.

§ 5. The simple monoclinic grating I'_m is the fundament of 4 of the 6 spacegroups of monoclinic-prismatic symmetry: C_{2h}^1 , C_{2h}^2 , C_{2h}^4 and C_{2h}^5 . In each elementary cell of these four space-groups never more than 4 equivalent points are present. The two remaining space-groups: C_{2h}^3 and C_{2h}^6 are built up by means of the BRAVAIS' grating I'_m , the centre of the planes (001) of its elementary cells being also occupied by a point. In these unit-cells no more than 8 equivalent points can be present. As the choice of the angle β in the monoclinic system is arbitrary within certain limits, this grating can also be considered as being built up by means of cells of the type I_m''' , which is characterized by the fact, that *all* faces of it are centred. In such an elementary cell 16 equivalent positions can be present indeed. The monoclinic $FeSO_4 \cdot HO_2$, having 16 molecules per cell, must, therefore, belong to one of the space-groups C_{2h}^1 or C_{2h}^6 . Now it was found, that the 1st, 2nd and 3rd order images on (001) have the intensity zero: the space-group C_{2h}^6 , therefore, appears to be the only possible one, because only there the spacings between the planes (001) of the grating I_m''' are really *quartered* by other atomic planes, as a consequence of the presence of gliding-mirrorplanes. The edges of the elementary cell I_m''' coincide with the ordinary crystallographic axes: although the edges of the smallest elementary cell I'_m , therefore, do *not* coincide with those axes, it appears, however, unnecessary to make another choice of the coördinate-axes here. If the crystals were placed in such a position as to make the edges of the cells I'_m coincide with the crystallographic axes, it would appear, by using LEWIS' transformation-formule¹⁾, that e.g. the form $\{110\}$ would get the symbol: $\{\bar{2}21\}$. As the faces of $\{110\}$ are predominant, the more complicated indices for them are not feasible; so that the new choice of axes can hardly be considered as an advantageous alteration.

In the case of $CoSO_4 \cdot 7H_2O$, whose axial ratio is:

$$a : b : c = 1,1815 : 1 : 1,5325, \text{ with } \beta = 104^\circ 40'.$$

the spacing $d_{(001)}$ was found to be: 19,39 A.U. From this it follows, that:

$$c_0 = 20,04 \text{ A.U.}; a_0 = 15,45 \text{ A.U. and } b_0 = 13,08 \text{ A.U.}$$

The specific weight of the salt is, from these data, calculated at: 1,889; by GOSSNER and PETERSON values were found between 1,948 and 1,964, a result which may be considered to be in sufficient agreement with the calculated value, concerning the rather appreciable inaccuracies of the measurements of these authors.

¹⁾ Cf. F. M. JAEGER, „*Inleiding tot de Studie der Kristalkunde*“, Groningen, (1924), p. 66.

§ 6. *Comparison of the monoclinic and rhombic gratings with each other.* Both types of structures may be best compared in considering together the simple rhombic cell on the one hand, and the monoclinic cell of type I'_m on the other hand. The data required are taken from the observations with the rhombic *magnesium-* and the monoclinic *iron-* salts.

In table VI these numbers are reviewed:

| TABLE VI. | | |
|-----------|------------------|---------------------|
| | Rhombic grating: | Monoclinic grating: |
| a_0 | 11.89 A.U. | 15.34 A.U. |
| b_0 | 12.01 A.U. | 12.98 A.U. |
| c_0 | 6.86 A.U. | 20.02 A.U. |

In Fig. 1 the projection is given of the elementary cells of the BRAVAIS' gratings I''_m and I'''_m on the plane (010). $AB = 15.34$ A.U.; $BC = 20.02$ A.U.; $\angle ABC = 104^\circ 15\frac{1}{2}'$. From this follows: $DB = 22.02$ A.U. and $BE = 11.01$ A.U. The length of the perpendicular AG , which is drawn from A to BD , is calculated to be: 13.32 A.U.; therefore, $\frac{1}{2}AG = 6.66$ A.U.

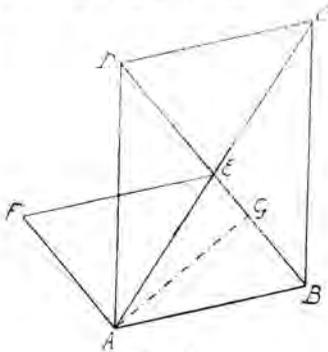


Fig. 1.

Now the dimensions BE and $\frac{1}{2}AG$ appear to be almost identical with a_0 and c_0 of the rhombic cells, while also b_0 has in both cases values which differ only slightly. If, therefore, BE be called: c and $\frac{1}{2}AG$ in the same way: a , — then the corresponding distances in both cases may be compared

as is done in Table VII:

| TABLE VII. | |
|--------------------|---------------------|
| Rhombic grating: | Monoclinic grating: |
| $a_0 = 11.89$ A.U. | $c_1 = 11.01$ A.U. |
| $b_0 = 12.01$ A.U. | $b_0 = 12.98$ A.U. |
| $c_0 = 6.86$ A.U. | $a_1 = 6.66$ A.U. |

Suspicion arises, that both structures may be transformed into each

other by relatively small translations and rotations only. If, therefore, a yet closer comparison be required, it appears feasible to put both gratings in such positions with respect to each other, as is drawn in Fig. 2.

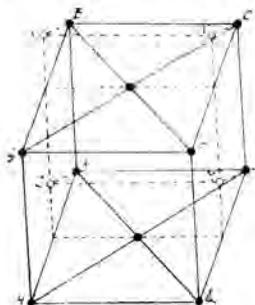


Fig. 2.

In this figure $ABCDEFGH$ is the monoclinic cell (I'_m); the lines KL and NM divide the faces $ABGH$ and $DCFE$ into two equal parts; $\angle HAB = 119^\circ 42'$; and $\angle LAB = 90^\circ$. The parallelepipedon $ABCDMNKL$ now is almost the same as the rhombic cell, the original a -axis of it having the position AB , and the original c -axis the position AL .

Finally in table VIII a review is given of the values of the so-called "molecular-volumes", i.e. of the volumes of the elementary cells, divided by the number of molecules present in each of them:

| TABLE VIII. | | |
|-------------|---------------|---|
| Salts: | Modification: | Molecular volume in A.U. ³ : |
| Mg-sulfate | Rhombic | 244.9 |
| Mg-chromate | .. | 245.9 |
| Ni-sulfate | .. | 243.9 |
| Zn-sulfate | .. | 244.7 |
| Fe-sulfate | Monoclinic | 241.4 |
| Co-sulfate | .. | 244.8 |

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Palæontology. — "*Manis Palaejavanica, the Giant Pangolin of the Kendeng Fauna.*" By Prof. EUG. DUBOIS.

(Communicated at the meeting of October 30, 1926).

Near Kedung Brubus, 40 km E.S.E. of Trinil, in the same tract of the Kendeng formation from which a piece of a very peculiar human mandible had been obtained, which, on closer consideration, after the discovery of a lower premolar at Trinil, I have ascribed to *Pithecanthropus erectus*, I found, almost two years later, a number of bones of a gigantic *Manis* species.¹⁾ In all I found 28 bones or large pieces of bones, some of them coherent, still in their natural mutual position or only very little dislocated. They all were spread over an area of a few square meters, and as regards form, bear a striking resemblance to the homonymous bones of the present *Manidae*. They are certainly parts of one skeleton of a *Manis* species and agree most closely with *Manis javanica* Desm., with this difference that the dimensions are truly gigantic. The long bones of the hand and the foot are, indeed, more thickly built, comparatively short for their length, which, it seems, must be ascribed to the claws being necessarily relatively stronger in a so much heavier species. But yet from what is available (among which also pieces of vertebrae and of the skull) the total length of the animal may be estimated to have been at least $2\frac{1}{2}$ meters, i.e. three times the length of a middle-sized Javanese pangolin, and more than once and a half that of a fairly sized adult existing Giant Pangolin of West Africa. The specimen found was, besides, not even entirely full-grown, as appears from the fact, that the epiphyses of the ulna, the radius, the femur, the tibia, and the calcaneus have not yet entirely united with the body of the bone.

I have called the species *Manis palaejavanica*²⁾, denoting by this name the closer relationship with the Javanese species of to-day, from which it is, however, very well to be distinguished by characters of minor importance.

The available parts of the skeleton³⁾ are:

1. A considerable piece of the left *temporal bone* with the *processus zygomaticus*, in connection with a large part of the *frontal bone* and a

1) Verslag van het Mijnwezen over het 2e Kwartaal 1892. Batavia. Landsdrukkerij. 1892.

2) EUG. DUBOIS, Das geologische Alter der Kendeng- oder Trinil-Fauna. Tijdschrift van het Kon. Ned. Aardrijkskundig Genootschap, 2nd Series, Vol. 25. (1908), p. 1267.

3) Belonging to the Collection-DUBOIS of the Leyden University.

small piece of the *parietal bone* with its antero-inferior angle. (Plate, middle row, third figure from the left. Above this a skull of *M. javanica*).

The breadth of the *processus zygomaticus*, immediately before the place of the lost *tympanicum*, is 20.5 mm; the corresponding breadth of the skeleton of *Manis gigantea* Ill. in the Rijks Museum van Natuurlijke Historie at Leyden (Cat. N^o. 3) is 14.3 mm, and of that of a more than middle-sized *Manis javanica* Desm., in my possession, 10 mm.¹⁾ The length of the *sutura fronto-temporalis* is 25 mm, in the *Manis gigantea* it is 20 mm, and in the more than middle-sized *Manis javanica* 14.5 mm. In this it should be considered that from species to species the size of the neurocranium does increase relatively less than the size of the body. The plaster cast of the available part of the endocranial surface shows sharp separation of the *lobus olfactorius*, very pronounced folding of the *pallium*, especially in the *lobus hippocampi*, which is large also here, and in the parts lying before this, to the *lobus olfactorius* — undoubtedly as a consequence of the large size of the body —, further a very remarkable wide and deep depression, which implies a considerable deficiency of brain mass, round the place where the *fossa Sylvii* of the *Manidae* was described. The bone wall of the cranial vault is thickened there to 9.3 mm, on account of local smaller *growth of the brain* (for the outer surface of the skull is smooth, and there is no mechanic cause for this growth of the bone), whereas behind it, at the thinnest place of the fragment, only 5 mm can be measured. An endocranial plaster cast of *M. javanica* shows a similar condition, only the folding and also the depression of the cerebral surface are not so considerable — no doubt on account of the smaller body size; the depression, especially, is comparatively less wide. The thickness of the bone wall reaches there a maximum of 3.6 mm, and behind it, at the thinnest place, it is scarcely 1 mm.

It seems that from the local shortage of brain mass described for *M. palaejavanica*, it is justifiable to conclude that also this pliocene giant species was covered with scales.

2. A piece of the right half of the *occipital bone*, round the asterion in connection with a piece of the *parietal bone* at its postero-inferior angle, and a piece of the *temporal bone* at its postero-superior angle. The inner side shows the dividing ridge between the *pallium* and the *cerebellum*.

3. The right half of the ventral arch of the *atlas*. The distance between the outer edges of the articular surfaces is 36 mm, and must have been at least 37 mm in the perfectly intact bone. The corresponding distance in *M. gigantea* is 25 mm, in a more than middle-sized *M. javanica* 15 mm.

¹⁾ In the dimensions of the bones I compare the fossil species with *Manis javanica*, because morphologically it resembles this most closely among all existing species, and with *M. gigantea*, because this living species comes nearest to it in the size of the body.

4. The left processus articularis superior and some adjacent bone substance of one of the basal *caudal vertebrae*, and the adherent extremity of the processus articularis inferior of the preceding caudal vertebrae. The dimensions are almost $2\frac{1}{2}$ times the corresponding ones in the more than middle-sized *Manis javanica*.

5. The upper end of the right *ulna* (Pl. upper row, fourth figure). The olecranon to the distal edge of the *cavitas sigmoidea major* and *minor*; the *processus coronoides* is broken off. In this articular cavity is a large part of the *trochlea* and the *capitulum* of the humerus, somewhat dislocated to the inside. The epiphysis of the olecranon was not yet united with the body of the bone. The length from the beak to the top of the olecranon is 65 mm. The corresponding measure is 33 mm in *M. gigantea*, 22 mm in the large *M. javanica*, 20.5 mm in a middle-sized one. Through its length, which is considerable compared with its thickness, the olecranon resembles that of *M. javanica* very closely; it is only still greater than proportional to the size of the body estimated by the vertebrae. This process is relatively much shorter and broader in *M. gigantea*.

6. The right *radius*, without the proximal epiphysis. (Pl. upper row, third figure 1). This had got detached, being still separately ossified. Nor are the distal epiphysis and the diaphysis united together. The length is 103 mm, in *M. gigantea* 69 mm, in a middle-sized *M. javanica* 53 mm. The smallest (dorso-volar) breadth of the distal epiphysis is 29 mm, and the greatest (at right angles to it) 35 mm, in *M. gigantea* resp. 19 mm and 25 mm, in a middle-sized *M. javanica* resp. 11 mm and 13 mm, and in a small one (but also full-grown) 9 and 10 mm. The smallest diameter of the collum is 13 mm, in *M. gigantea* 6.7 mm, and in a large *M. javanica* 4.5 mm.

7. The proximal extremity of the left *radius* (Pl., upper row, second figure). The epiphysis is slightly dislocated. Its smallest breadth (dorso-volar) is 19.3 mm, its greatest breadth (at right angles to the latter) 32.2 mm. In *M. gigantea* these measures are resp. 13.6 mm and 20 mm, in a middle-sized *M. javanica* 9 mm and 13 mm, in a small adult *M. javanica* 6.5 mm and 11 mm.

The total length of the radius (with the two epiphyses) may be estimated at about 115 mm. In the *M. gigantea* (also with epiphyses not yet joined by bone) it is 73 mm, in a middle-sized *M. javanica* 57 mm, and in the small adult 45.5 mm.

In relation to its length the bone is, compared with that of *M. javanica*, particularly thick and strong.

8. The *metacarpal bone* of the left *middle finger* (Pl., bottom row, fifth figure). Against the proximal articular surface a piece of the *os capitatum*. Length 37 mm, breadth in the middle 16.5 mm, in a large *M. javanica* resp. 15.5 and 5.5 mm. Accordingly the bone in the fossil species is thicker in relation to its length than that of the living species.

9 and 10. The *metacarpal bone* of the right *middle finger*, connected with the somewhat defect *first phalanx* (Pl., bottom row, second figure).

11 and 12. The connected *first and second phalanx* of the left *middle finger*. (Pl., bottom row, fourth figure). Length of the first phalanx in the middle of the side surfaces, on the medial side 15.5 mm, on the lateral side 16 mm. Thickness in the middle 22 mm. In the large *M. javanica* these measures are 8.5 mm, 8.7 mm, and 6 mm. Hence the fossil bone is relatively to its length much thicker. Measured in the same way the length of the second phalanx is on the medial side 22 mm, on the lateral-side 22.5 mm, the breadth at the base 21 mm. In the large *M. javanica* the corresponding measures are 9 mm, 9 mm, and 6.3 mm. Also this fossil bone is, therefore, thicker relatively to its length than that of *M. javanica*.

The second phalanx of a middle finger in the Indian Museum at Calcutta, ascribed originally (1880) by LYDEKKER to a *Manis sindiensis*, and considered later (1886) to belong to *Macrotherium*, and then again (1891) assigned to *Chalicotherium*, is, as regards length, measured as above, and as regards dorso-volar dimension of the trochlea, in good agreement with *M. palaejavanica*, but the trochlea is broader on the volar side, and the base differs greatly in form; the latter lacks the long dorsal point, and the very broad volar rim ending knobby on every side in the pliocene and the recent species of Java.

13. The third or *nail phalanx* of the left *middle finger* (Pl., bottom row, third figure). Of this cleft phalanx the ulnar point has broken off and has got lost, probably over a third part of the total length of the bone. Likewise the radial point is absent, which is broken off still about $1\frac{1}{2}$ cm nearer the base. The length of what remains is 53 mm, and that of the whole bone may be estimated at 80 mm by comparison with the homonymous bone of other species of *Manis*, especially *M. javanica*, which the fossil bone resembles most closely. The thickness, between the dorsal and the volar edge of the base, is 27 mm, the breadth at the base 17.5 mm. In *M. gigantea* the last three measures are: 53 mm, 18 mm, and 11.5 mm. In *M. javanica* I find mostly 26 mm (as maximum 27 mm), 8.5 mm, and 6.2 mm in more than middle-sized specimens. In the plistocene *M. Lydekkeri* of India (cave deposits in the Presidency of Madras)¹⁾, only known through the homonymous bone, the length is 57 mm and the width 12.5 mm. The cleaving begins dorsally in *M. palaejavanica* at 17 mm distance from the transversely truncated base, i.e. about $\frac{1}{5}$ of the total length of the phalanx, in *M. gigantea* at 17 mm from the rounded base point, or about $\frac{1}{3}$ of the total length, in *M.*

¹⁾ Tijdschr. Kon. Ned. Aardrijksk. Genootschap, loc. cit. p. 1268. The characters mentioned here, and besides the pointed form of the volar process of the base, sharply distinguish the Indian plistocene species from the *M. gigantea*. It is certainly also different from *M. palaejavanica*; I was able to establish this in the Indian Museum at Calcutta, where this fossil is preserved.

Lydekkeri at 10 mm distance from the truncated base or little more than $\frac{1}{6}$ of the total length, and in *M. javanica* at 10.5 mm or $\frac{2}{5}$ of the total length. Apart from this more proximal cleaving in *M. palaejavanica*, there exists the closest similarity in the shape of this bone with *M. javanica*. Both species are distinguished by the possession of small bars of bone, bounding the volar entrances to the vascular furrows on both sides. On the fossil bone they are broken off, but surfaces of fracture clearly betray their former presence. Besides, only the existing *M. javanica* has the sharp, antler-like ramified vascular furrows on the side surfaces of the nail phalanges in common with *M. palaejavanica*.

14. The *metacarpal bone* of the right *fourth finger* (Pl., bottom row, last figure). Length 33.5 mm. Breadth in the middle 9.5 mm. In *M. javanica* length 14 mm, breadth in the middle 3 mm; the bone is, accordingly, much slenderer. In *M. gigantea*, on the other hand, the metacarpal IV is still relatively thicker than in *M. palaejavanica*, for the length is 17.5 mm and the breadth in the middle 6.5 mm.

15 and 16. The *phalanx prima* and *phalanx secunda* of the right fourth finger connected in their natural mutual position (Pl., bottom row, sixth figure). The length of the first bone, in the middle of the radial side, is 15 mm, in the middle of the ulnar side 14 mm, the breadth at the base 17 mm. The length of the second bone is 15 mm on the radial side and 19 mm on the ulnar side, the breadth at the base 15 mm. Accordingly the capitulum of this phalanx points very obliquely to the middle finger, as it does in *M. javanica*, which proves, that also *M. palaejavanica* supported itself, in its walk, on the outside edge of the forefeet (hands), turning its fingers inwards.

In *M. javanica* both phalanges are much slenderer. The corresponding lengths are 6.3 mm, 6 mm and 6 mm, 7.4 mm, the corresponding breadths 5 mm and 4.7 mm. In *M. gigantea* the first phalanx is as short and thick as in the fossil species of Java, but the second somewhat less.

17, 18, 19, 20. The *metacarpal bone* and the *three phalanges* of the left *fourth finger* (Pl., bottom row, first figure). Somewhat dislocated from their natural mutual position; on the ulnar side of the capitulum of the metacarpal bone, a *sesamoid bone*. The nail phalanx, broken off at 45 mm length, was probably as a whole 65 mm long. As in *M. javanica* and *M. gigantea*, it is somewhat obliquely flattened sidelong. The base measures from the back side to the palm side 23.5 mm, transverse 12.5 mm. The radial half is slightly defect, probably as a consequence of a morbid process during life. For the rest this phalanx has many characters in common with the nail phalanx of the middle finger.

21. The medial half of the upper end of the right *femur* with almost the entire caput and the trochanter minor; the lateral half is broken off. (Pl., middle row, fifth figure). The epiphysis of the caput and that of the trochanter minor were not yet united by bone with the body of the femur. Length of the fragment 86 mm; diameter of the caput,

measured from the front backward, 40 mm; in a middle-sized *M. javanica* 15.5 mm, in *M. gigantea* 28 mm. Thickness of the diaphysis, measured in the same way, 27 mm, in the other two species 9 mm and 18 mm. Distance from the top of the trochanter minor to the top of the caput 52 mm resp. 24 mm and 40 mm. In comparison with those two living species the thickness of the caput and of the diaphysis and also the length of the collum is about proportional to the body length estimated.

22. The upper half of the left *tibia*. (Pl., upper row, first figure). The epiphysis is not yet united together with the shaft. Length of the fragment 122 mm. The length of the whole tibia was probably about 245 mm. The antero-posterior dimension of the epiphyses from the tuberositas to the middle of the concavity on the back side, is about 44 mm; the transverse dimension 64 mm, and the circumference of the diaphysis, measured at the fragment as low as possible, about at the thinnest place of the tibia, is 73 mm. These three dimensions are in a middle-sized *M. javanica* 15 mm, 24 mm, and 25 mm; in the *M. gigantea* 26 mm, 43 mm, and 40 mm. The tibia of the last is apparently not entirely full-grown.

23. The left *calcaneus*. (Pl., middle row, first figure and Fig. 1 Ca. In this figure, accurate outlines of a photograph, four tarsal bones are placed in their natural position). The total length is 72 mm, that of the part behind the articular surface for the astragalus, i.e. the length of the collum and the tuberositas (the epiphysis of which is not united by bone), is 39 mm or 55% of the total length, the width of the collum at its narrowest is 15 mm, the height idem 25 mm. In a middle-sized *M. javanica* these measures are resp. 28 mm, 15 mm, 5 mm, and 9,5 mm; the posterior part occupies 53,6% of the total length.

In *M. gigantea* the measures are resp. 44 mm, 19 mm, 14 mm, and 19 mm; the posterior part occupies 43.2% of the total length. The form of the calcaneus of *M. palaejavanica* presents a striking resemblance to that of *M. javanica*; especially on the lower side it is to be seen that, as regards body size, both are equally long, narrow, and high. On the other hand the calcaneus of *M. gigantea* is short, and at the bottom in its front half, not narrow and angular, but broad and flatter. The processus (in man "spina") peronealis is relatively smaller than in *M. javanica*.

A very important difference between *M. palaejavanica* and *M. javanica* on one side, and *M. gigantea* on the other side, consists in this that the calcaneus of the two first-mentioned species does not articulate only with the astragalus but in a surface forming a lateral continuation of the articulating surface for the astragalus, the

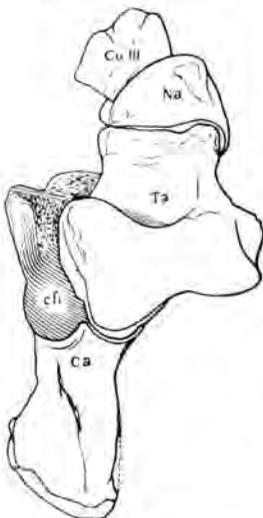


Fig. 1.
2/3 nat. size.

width of which is $\frac{2}{3}$ of that of the last-mentioned (fig. 1, the hatched part *cfi*), also with the lower border of the maleolus of the fibula, and chiefly with the back part of that lower border. This articular surface slopes down to the outside, that for the astragalus, with which it, however, forms a whole, to the inside. It is absent in the African species *M. gigantea* and *M. temmincki* Smuts, and also in the Asiatic *M. pentadactyla* L. and *M. aurita* Hodgs, but it is found in the African species *M. tetradactyla* L. and *M. tricuspis* Rafin. Through this particular articulation with the fibula the foot obtains the guidance and the firm support which it requires in its outwardly-turned (somewhat supine) position, especially when its claws, which in these particular species are distinguished by their large size, are active. All this — it also appears from other parts of the skeleton of the foot — applied undoubtedly also to the *M. palaejavanica*.

24. The left *talus* or astragalus (Pl., middle row, fourth figure and Fig. 1 *Ta*). The length measured normal to the tangent on the back-side of the trochlea, is 44 mm, the greatest breadth is 42 mm. In a middle-sized *M. javanica* the length, measured in the same way, is 14.5 mm, in the *M. gigantea* 26 mm. The collum is relatively much narrower than in *M. javanica*, and does not point so obliquely inward. As in all *Manidae*, the caput is for the greater part concave, only near the inner edge convex to about the middle.

25. The left *navicular bone* (Pl., middle row, second figure. With the dislocated ectocuneiform bone, and Fig. 1, *Na*). Breadth, total 30 mm; in a middle-sized *M. javanica* 11 mm, and in the *M. gigantea* 19 mm. Length of the free dorsal surface 13 mm; in *M. javanica* 4.5 mm, in *M. gigantea* 8 mm.

26. The left *ectocuneiform bone* (Pl., middle row, second figure, and Fig. 1, *Cu III*). Breadth 22 mm. Length of the free dorsal surface 11 mm; in *M. javanica* 4.5 mm, in *M. gigantea* 6.5 mm. The anterior articular surface is, as in *M. javanica*, somewhat saddle-shaped, but on account of the strong outward turned, position of the articulating-surface with the navicular bone, turned also much more outward. The metatarsal of the middle toe and



Fig. 2.
 $\frac{2}{3}$ nat. size.

the toe itself was, however, not pointing much more outwards, as its base must have been very oblique to the axis of the bone, for the extant fourth metatarsal possesses such a very oblique basal plane.

27. The *metatarsal* of the right *second toe* (Fig. 2, *Mt II dext*. The outlines of these figures, too, are accurate photographic ones). Though this bone is slightly weathered at the surface, yet it is to be recognized with certainty as such, inter alia by the characteristic excavation of the base at the outside. The length measured in the direction of the axis,

dorsally to the large plantar process of the base, is 38 mm, (the total length, with that plantar process, is 40.5 mm), the breadth of the diaphysis at its narrowest is 10.5 mm. In a middle-sized *M. javanica* the length is 16 mm and the breadth 3.2 mm.

28. The *metatarsal bone* of the left *fourth toe* (Pl. above the end of the top row, and Fig. 2, *Mt IV sin.*) The length, measured dorsally to the large plantar process of the base (here at the same time the greatest length), is 49 mm, the breadth of the diaphysis at its narrowest 10 mm; in a middle-sized *M. javanica* the length is 20,7 mm and the breadth 3,2 mm. The slightly convex basal articular surface is much *more obliquely turned outward* than in *M. javanica*, through which the base possesses a very sharp edge as outer rim, and the dorsal free surface presents a much acuter angle than ever occurs in *M. javanica*.

The *much more considerable length of the fourth* than of the second metatarsal bone the extinct giant pangolin of Java has in common only with *M. javanica* among all the existing *Manis* species. Except the existing Javanese pangolin all the recent species have these two metatarsalia of about the same length. This similarity in structure of the foot of the pliocene and of the living *Manis* of Java renders it already probable that also the pliocene *Manis* had large claws on its hind-feet as well as on its fore-feet, which implies similarity in the function of scratching open termite-hills or ant-dwellings. The outer border of the foot is turned somewhat downward, so that the animal throws the unearthed termites and ants under it, and not behind it. This resemblance in construction and function of the foot is corroborated by a peculiarity in the construction of the foot, which *distinguishes* *M. palaejavanica* from *M. javanica*. In the extinct giant pangolin the metatarsal of the fourth toe and also the metatarsal of the middle toe and the navicular bone were evidently at the same time particularly adapted to offer resistance to strong pressures exerted by the anterior part of the outer border of the foot — it is to be seen in the articular surfaces — in contrast with the homonymous bones of *M. javanica*. It may be inferred from this that by turning both the hind claws and the fore-claws under its feet, *M. palaejavanica* walked permanently on the outside also of its hind-feet, not only "sometimes", as is stated of *M. javanica*¹⁾. But this permanent mode of walking is, at the same time, a sure indication of the possession of large claws also on the hind-feet.

It may, therefore, be assumed that in both species the two pairs of extremities were less specialized than in the other *Manidae*; moreover the recent species of Java lives chiefly on the ground, but it does also climb. For climbing the extinct giant pangolin will undoubtedly have been too heavy, certainly even more so than the existing African *M. gigantea*.

¹⁾ By W. T. BLANFORD, according to Tickell, in "The Fauna of British India. Mammalia", p. 600. London 1891.

Although constituting a true (natural) genus, the existing Manidae are divided into two tribes, further they are differently specialized.

Of the four existing African species, which as appears from the common possession of certain distinguishing characters (as the form of the xiphisternum and the arrangement of the scales on the tail) certainly belong to one tribe, whereas the three Asiatic species classified according to these peculiarities, form another tribe, two species are adapted exclusively to a life on the ground, through specialisation of their fore- and hind limbs. They find their food on the ground: termites and ants, chiefly termites, and the hills of these latter mostly reaching considerable heights in Africa, the closely allied *M. temmincki* Smuts and *Manis gigantea* Ill. have assumed to a certain extent bipede locomotion. The first-mentioned species, which inhabits the steppes and savannas of South Africa, north of the Vaal-river, and East-Africa to 17° N.L., even walks almost exclusively on its hind-legs, the heavy, broad tail being used not as a support, but to keep its balance. As in the plantigrade mammals their hind feet are provided with toes and claws differing comparatively little in size inter se, whereas on the fore-feet the three middle ones, especially the middle toe and claw, have grown to gigantic dimensions. Standing on their hind feet more or less erect, they use there fore-feet like pickaxes to scratch, or rather hew, open very hard termite-hills, often some meters high. Also *M. temmincki* is of considerable size, and can therefore reach high with its fore-claws.

The two other African species: *M. tricuspis* Rafin. and *M. tetradactyla* L. (*M. longicaudata* Briss), which, with *M. gigantea*, inhabit West Africa between the Gambia and the Kunene, are small, possess a long prehensile tail, which in its functions exceeds the tail of *M. javanica* no less than the admirably perfect prehensile tail of *Ateles* that of *Mycetes*, and live chiefly in the trees, where they, being excellent climbers, seek termites (which build there round nests, the size of a mans head, on the branches out of their own excrements) and ants. They have long middle toes, both on the hind- and on the fore-feet, but since they do not, or only rarely, use their hind feet to scratch open termite- or ant-nests on the ground, their fourth metatarsal is not longer, or longer only in an insignificant degree than the second. Their calcaneus, however, does resemble that of *M. javanica* somewhat.

Of the three existing Asiatic species, *M. javanica*, which inhabits besides the Great Sunda Islands, also the continental area east from the Bay of Bengal and south from the habitat of *M. aurita*, has not to do with very high termite hills; those in Java, at least, are hardly higher than a meter. This species chiefly lives on the ground, and does not climb very nimbly. The fore-feet and the hind-feet, both provided with large claws, are used for scratching open the low termite-hills and ant-dwellings, which moreover are not so very hard. *M. pentadactyla* L. of India and Ceylon, and *M. aurita* Hodgs of Nepal, Assam and South China, Hainan and

Formosa, have, on the other hand, again to deal with high termite hills; as far as I have been able to ascertain they climb seldomer than *M. javanica*, live more exclusively on the ground. The three middle claws of their hind feet have scarcely half the length of those of the fore-feet¹⁾. It is known at least of *M. pentadactyla* that it habitually stands on its hind-feet. They have their extremities already more specialized than *M. javanica* and *M. palaejavanica*. Accordingly these two species are really to be considered as the least specialized of all the Manidae known. This finds a striking confirmation in the fact that the cephalisation of *M. javanica* is only as high (the volume of the brain calculated for the same body weight, half as great) as that of *M. gigantea* and *M. tetradactyla*. Wanting data on the body weight, I had no opportunity to examine the cephalisation of *M. temmincki* and *M. tricuspis* properly, nor of *M. pentadactyla* and *M. aurita*; apparently the two first-mentioned may be considered equal to the two other African species of which I was able to calculate the cephalisation of specimens in the Rijks-Museum van Natuurlijke Historie at Leyden. Among the Asiatic species, *M. pentadactyla* and *M. aurita* on the other hand, do not rise above *M. javanica*, judging from the few available data.

The species arose by adaptation to the different particular conditions of life, imposed on the animal by the world in which it had to live, especially with regard to the regional character of the termite nests, which contain their principal food, and the circumstances of life in the original home of the Asiatic tribe of the genus must actually have differed little from those which exist to-day in the habitat of *M. javanica*.

Probably these were more favourable to that tribe in the Pliocene time than they are at present, and this time coincides with its flourishing period in that region, as may be deduced from the occurrence of the giant form. As all the *Manis* species live in open and comparatively dry regions, it may be assured, that the then climate of what is now Java, was still somewhat drier. This corroborates what was already to be inferred from the continental character of the Kendeng-fauna as a whole, viz. that Java at the time of its existence constituted a part of the Asiatic continent, and formed, with the other Great Sunda Islands an appendage to the present continental area east from the Bay of Bengal, of which geographical condition a climate less humid than at present was a necessary consequence.

¹⁾ The significance of the relative length of the fore- and hind claws for the systematization of the Manidae had already been realized by F. A. JENTINK („Revision of the Manidae in the Leyden Museum“, Notes from the Leyden Museum. Vol. IV, p. 193—209, 1882) and P. MATSCHIE („Die natürliche Verwandtschaft und die Verbreitung der Manis-Arten“. Sitzungs-Berichte der Gesellschaft naturforschender Freunde zu Berlin, Jahrgang 1894, p. 1—11).

EUG. DUBOIS: "MANIS PALAEJAVANICA, THE GIGANTIC PANGOLIN OF THE KENDENG-FAUNA".



Manis palaejavanica Dubois and *Manis javanica* Desm. (Somewhat more than $\frac{2}{5}$ nat. size).

EXPLANATION OF THE PLATE.

Photographic figures of 24 (of the 28 collected) bones of *Manis palaejavanica* and the homonymous bones of *Manis javanica*, mostly near each other; the radii of the former species, however, lie in the upper row, of the latter in the lower row. Partly the bones were found connected in their natural mutual position, some slightly, the ectocuneiform bone considerably, dislocated. The corresponding bones of *Manis javanica* are connected with each other in the same way as in the fossil bones found.

Chemistry.—“*On Lyotropy.*” By H. R. KRUYT and CONMAR ROBINSON.

(Communicated at the meeting of April 24, 1926) ¹⁾.

The behaviour of electrolytes in solution is only partially explained by the osmotic theory. There are several properties of solutions of electrolytes which cannot be connected with the number of particles in solution. These properties are found for instance where one investigates the influence of electrolytes on the solubility of non-electrolytes, on reaction velocities, on the salting out of proteins, on the swelling of gels, on the surface tension of water and on the maximum density of water. In all these cases the effect of equimolecular solution is not the same and the influence which the salt exerts is a function of both the cation and the anion. The ions can be arranged in a series which gives the order in which they exert their influence.

A phenomenon not yet mentioned that shows this series is the electric mobility of the ions, which for the alkali metals is just in the reverse order to what we might expect from the size of the atoms. For a long time this has been accounted for by the property of the ions of binding water molecules. The sequence of the lyotropic series will therefore be that of the hydration of the ions. While not denying the truth of this explanation, it is certainly not complete; it can, however, suffice to explain most of the above mentioned phenomena. But to explain, for example, the influence of electrolytes on the surface tension of water one is inclined to turn to the influence of these substances or the molecular aggregation of the water.

Seeing that lyotropy plays so important a part in colloid chemistry, we have set out to investigate those phenomena which can be most easily controled. We first of all wished to show that LOEB's ²⁾ explanation that the lyotropic series is simply a consequence of alterations in the hydrogen ion concentration is wrong. For this purpose we investigated the influence of the addition of chlorides on the limited miscibility of phenol and water.

As in the meantime DUCKET and PATTERSON ³⁾ published a similar research, we shall not here quote our results except to say that they showed that KCl and HCl have exactly the same influence, which in itself is a complete refutation of LOEB's theory when extended to cover substances others than proteins.

Subsequently we investigated the solubility influence. We found that

¹⁾ Published in this Academy's "Verslag der gewone vergaderingen", Vol. 35, p. 812, (meeting of September 25, 1926).

²⁾ J. LOEB, *Proteins and the Theory of colloidal Behavior*, New-York 1922.

³⁾ J. DUCKET and W. H. PATTERSON, *J. physic. chem.* **29**, 295 (1925).

systematic data were only available for one substance, namely phenylthiourea¹⁾. In recent literature there is a very interesting investigation by LINDESTRÖM-LANG²⁾ which prompted us to extend this research in order to establish for certainty certain conclusions which one is inclined to come to when considering his experiments. From his results combined with ours, one is able to draw the following noteworthy conclusions. The influence of salts on the solubility of quinone is only dependent on their anions. With what cation the anion is combined is almost immaterial. With hydroquinone the facts are just the reverse. Here one has a cation lyotropy while the anion plays a very subordinate role. In tables 1 and 2 are given the results of our experiments; those marked with a * are taken from the work of LINDESTRÖM-LANG (interpolated from 1.5m). In the third column in these tables the solubility in water is given as equal to 100; as the solubility in water in our experiments was not exactly the same as in the experiments of LINDESTRÖM-LANG, we only brought these relative numbers into our tables. In fig. 1 and 2 the results are shown graphically, from which one may see directly the typical anion lyotropy for quinone and the cation lyotropy of hydroquinone.

From LINDESTRÖM-LANG's research it seems that succinic acid and boric acid belong to the hydroquinone type. We have tried to find other substances which belong to the quinone type. Firstly we chose substances which should have a more or less basic character, since the acids mentioned above belong to the type exhibiting cation lyotropy. In tables 3 and 4 are given the figures for meta and para nitroaniline respectively, in table 5 those for para phenylenediamine. In fig. 3 are given the results for salt additions of 0.2 m.

The analysis of the saturated solutions in the case of the nitroanilines was carried out by a specially developed method (potentiometric titration with $TiCl_3$) which has already been published elsewhere³⁾. The solubility of the diamines was determined by precipitation of the diamine with picric acid, filtering off the picrate and back titrating with baryta using methyl red as indicator.

From the figure one sees directly that in all these three cases we have to do with anion lyotropy, exactly as in the case with quinone; there is a great influence of anions and a very limited influence of cations⁴⁾.

Finally in table 6 are given the results with nitrophenol (see also fig. 3). This gives the impression that here we are dealing with an intermediate case, as the influence of both anions and cations are quite marked.

¹⁾ V. ROTHMUND, *Z. physik. Chem.* **33**, 401 (1900); W. BILTZ, *ibidem* **43**, 41 (1903).

²⁾ K. LINDESTRÖM-LANG, *C. r. Lab. Carlsberg* **15**, Nr. 4 (1924).

³⁾ I. M. KOLTHOFF et C. ROBINSON, *Rec. Trav. Chim.* **46**, 169 (1926).

⁴⁾ With phenylenediamine we have a reversal of the series K—Na—Li. We cannot say with certainty how this should be interpreted. There are other cases where Li is found out of its place (cf. V. ROTHMUND, *Löslichkeit und Löslichkeitsbeeinflussung* p. 153, Leipzig 1906).

TABLE 1.

| Quinone | | Temperature: 23.75° |
|-----------------------------------|-------------------------------|---|
| Added electrolyte | Solubility in grams per litre | Solubility as percentage of that in water |
| Water | 13.97 | 100.0 |
| 1.5 m KSCN | 23.77 | 170.1 |
| .. KI | 20.89 | 149.6 |
| .. KNO ₃ | 18.46 | 132.1 |
| .. KBr | 15.15 | 108.5 |
| .. KCl | 12.63 | 90.4 |
| .. K ₂ SO ₄ | 8.90 | 63.7 |
| .. *LiCl | — | 77.3 |
| .. *NaCl | — | 80.3 |
| .. *RbCl | — | 93.8 (18°) |
| .. CsCl | — | 97.6 (18°) |

TABLE 2.

| Hydroquinone | | Temperature: 23.75° |
|--|-------------------------------|---|
| Added electrolyte | Solubility in grams per litre | Solubility as percentage of that in water |
| Water | 67.64 | 100.0 |
| 1.5 m KI | 50.62 | 74.8 |
| .. KNO ₃ | 54.09 | 80.0 |
| .. KBr | 47.91 | 70.8 |
| .. *KCl | — | 67.0 |
| 1/2 m K ₂ SO ₄ ¹⁾ | 52.72 | (1/2 m) = 78.0 of $\left(\frac{1.5 \text{ m}}{2}\right) = 68$ |
| 1.5 m *LiCl | — | 56.6 |
| .. *NaCl | — | 59.1 |
| .. *CsCl | — | 95.0 |

¹⁾ On account of incomplete solution of the salt, only one determination with 1/2 m. K₂SO₄ was made. This gave a solubility for hydroquinone of 52.72 grams per litre or 78 0/0. This gives by linear extrapolation a value of 68 0/0 for $\frac{1.5}{2}$ m. K₂SO₄.

TABLE 3.

| <i>m</i> -Nitroaniline | | Temperature 25° |
|--|-------------------------------|---|
| Added electrolyte | Solubility in grams per litre | Solubility as percentage of that in water |
| Water | 0.8929 | 100.0 |
| 0.2 m KI | 0.9527 | 106.7 |
| .. KBr | 0.8857 | 99.2 |
| .. KCl | 0.8619 | 96.5 |
| $\frac{0.2 \text{ m}}{2}$ K ₂ SO ₄ | 0.8349 | 93.5 |
| 0.2 m LiCl | 0.8449 | 94.6 |
| .. NaCl | 0.8514 | 95.3 |

TABLE 4.

| <i>p</i> -Nitroaniline | | Temperature 25° |
|--|-------------------------------|---|
| Added electrolyte | Solubility in grams per litre | Solubility as percentage of that in water |
| Water | 0.5735 | 100.0 |
| 0.2 m KI | 0.6053 | 105.5 |
| .. KBr | 0.5834 | 101.7 |
| .. KCl | 0.5606 | 97.7 |
| $\frac{0.2 \text{ m}}{2}$ K ₂ SO ₄ | 0.5430 | 94.7 |
| 0.2 m LiCl | 0.5430 | 94.7 |
| .. NaCl | 0.5480 | 95.5 |

TABLE 5.

| <i>p</i> -Phenylenediamine | | Temperature 25° |
|--|-------------------------------|---|
| Added electrolyte | Solubility in grams per litre | Solubility as percentage of that in water |
| Water | 47.23 | 100.0 |
| 0.2 m KI | 51.68 | 109.4 |
| .. KBr | 48.68 | 103.0 |
| .. KCl | 46.85 | 99.2 |
| $\frac{0.2 \text{ m}}{2}$ K ₂ SO ₄ | 43.21 | 91.5 |
| 0.2 m LiCl | 48.32 | 102.3 |
| .. NaCl | 47.23 | 100.0 |

TABLE 6.

| p.-Nitrophenol, | | Temperature 25° |
|---|-------------------------------|---|
| Added electrolyte | Solubility in grams per litre | Solubility as percentage of that in water |
| Water | 11.82 | 100.0 |
| 0.2 m KI | 12.45 | 105.3 |
| .. KBr | 11.91 | 100.8 |
| .. KCl | 11.55 | 97.7 |
| $\frac{0.2 \text{ m}}{2} \text{K}_2\text{SO}_4$ | 11.50 | 97.3 |
| 0.2 m LiCl | 10.90 | 92.2 |
| .. NaCl | 11.05 | 93.5 |

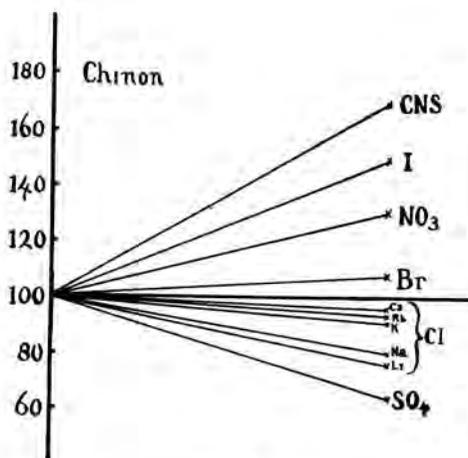


Fig. 1.

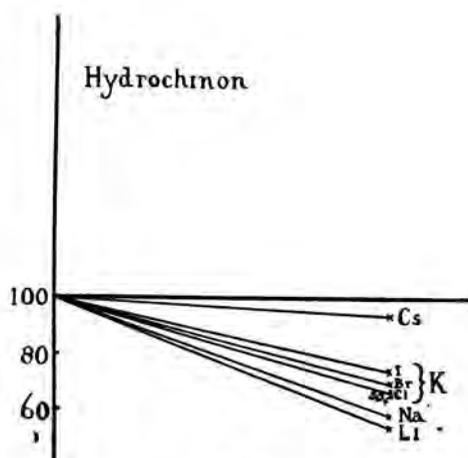


Fig. 2.

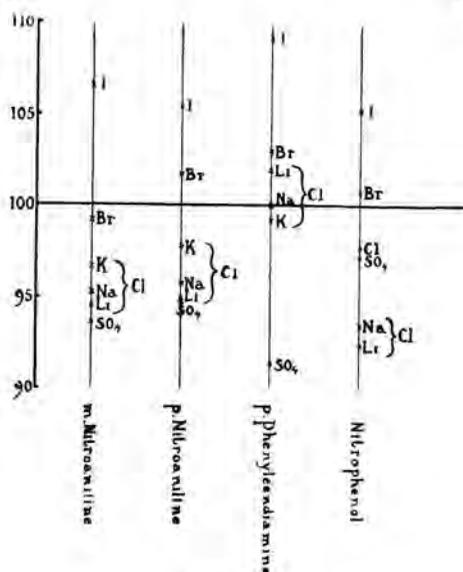
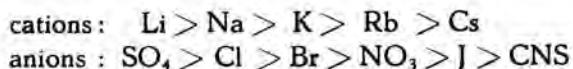


Fig. 3.

The investigations described above are of interest in the understanding of lyotropy. Thus while one usually bases theories of lyotropy on the "neutral salt influence" and the „salting out effect", here we have a marked specific influence of both cation and anion and also the frequent occurrence of an increase of solubility by the added salt.

DEBIJE ¹⁾ has recently connected the lowering of the solubility of organic substances by addition of electrolytes with the fact that these organic substances lower the dielectric constant of water. Increase of solubility is not so easily accounted for by this theory and it does not lead us to expect a specific influence of the ions. It seems to us that an explanation may more easily be found in the orientation of the dipoles of the water in the immediate neighbourhood of the ions, that is to say the orientation of the water of hydration, although at present one can only make surmises.

We have already stated that various physico-chemical phenomena show that hydration decrease in the following order :



This in general will be the sequence for the salting out effect, Li and SO_4 having a strong, and Cs and CNS a weak salting out effect. Hence we always find this series.

The solution of a substance implies an attractive force between the molecules of the substance and those of the solvent. As we may consider this attraction electrical in origin, the water molecules will not only be attracted, but in consequence of their dipolar character they will be orientated. Two kinds of orientation are of course possible: either the positive end or the negative end of the water molecule may be turned towards the solute molecules. Assuming a polar character of the dissolved organic substance, one of these two orientations must preponderate. At the same time however there will be an attraction for one of the ions of the dissolved salt. But also there will be orientated water molecules round these ions.

Now if for example the molecule is anionphilic (we assume that quinone and the amines investigated belong to this class) then the orientation of the dipoles round the anions will favour the orientation round the organic molecules and hence increase their solubility. Cations on the other hand will bring about a decrease in the solubility but their influence will be small since their concentration will be less in the neighbourhood of the organic molecules.

Thus, as well as the ordinary salting out effect there is an increase of solubility brought about (in this case) by the anions. The CNS lowers the solubility very little on account of its low hydration and the increase in solubility will therefore preponderate. The sulphate ion has a strong

¹⁾ P. DEBIJE, *Physik. Z.* **26**, 22 (1925).

lowering influence and its lowering influence will therefore preponderate. The influence of the cations is quite subordinate since they are kept further from the molecule. What small influence they do exert will be in the order of their salting out effect. In fact here we have the behaviour of quinone and the nitroanilines investigated. For cationphilic substances (hydroquinone and the acids) an exactly corresponding argument holds.

Seeing the present hypothetical character of these considerations we are at present further developing the theory. We hope later to return to the subject when we will publish our investigations on colloidal systems.

VAN 'T HOFF *Laboratory.*

Utrecht, April 1926.

Mathematics. — "*Ueber stetige Kurven.*" By JULIA RÓZANSKA. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of May 29, 1926).

Das Ziel dieser Arbeit ist, ausgehend von einigen Eigenschaften einer stetigen Kurve „im kleinen“, d.h. in der Umgebung irgend eines ihrer Punkte, auf den Verlauf der Kurve „im grossen“ zu schliessen.

Anknüpfend an die Arbeiten von PAUL URYSOHN über die Cantorsche Kurven, deren im Folgenden verwendeten Resultate sich auch bei K. MENGER befinden, ¹⁾ gebe ich die Definition vom Integralindex für die Punkte einer beliebigen Cantorsche Kurve. ²⁾ Die Vergleichung des URYSOHN-MENGER'schen Verzweigungsindex mit dem Integralindex gestattet mir eine neue lokal-integrale Charakteristik der sogenannten „Baumkurven“, sowie die notwendigen und hinreichenden Bedingungen dafür zu geben, dass eine stetige Kurve eine und nur eine einfache geschlossene Linie enthält.

Herrn PAUL ALEXANDROFF, der mich auf die Unabhängigkeit dieser Resultate von der Topologie der Ebene aufmerksam gemacht und viele wertvolle Ratschläge bezüglich der Redaktion dieser Arbeit gegeben hat, spreche ich an dieser Stelle meinen Dank aus.

Definition des Integralindex.

Es sei C eine Cantorsche Kurve, x ein Punkt von C .

1. x ist vom Integralindex m (wobei m eine natürliche Zahl, \aleph_0 oder c ist) falls m die kleinste so beschaffene Kardinalzahl ist, dass es für jedes $\varepsilon > 0$ eine ε -Aussonderung $C = A + B + D$ ³⁾ gibt, von der Art, dass die abgeschlossene Menge D aus m Komponenten besteht.

¹⁾ URYSOHN, Comptes Rendus t. 175 (1922), p. 440, 481: „*Mémoire sur les multiplicités Cantoriennes*“, II. Teil (erscheint demnächst in den Verhandlungen der Kgl. Akademie der Wissenschaften zu Amsterdam; diese Arbeit wird in folgendem einfach als URYSOHN, II zitiert); K. MENGER, „*Grundzüge einer Theorie der Kurven*“, Proceed. Ak. Amsterdam, XXVIII, SS. 67-71 und Math. Annalen, 95, SS. 277-306.

²⁾ Ein Hinweis in dieser Richtung findet sich bei MENGER, Math. Ann. 95, S. 281, Fussnote ⁸⁾.

³⁾ Die Zerlegung $C = A + B + D$ in zueinander fremde Mengen heisst nach URYSOHN eine ε -Aussonderung von x auf C wenn

$$x \in A \subset A + B \subset S(x, \varepsilon) \dots \dots \dots (1)$$

$$A \cdot \overline{D} + \overline{A} \cdot D = O \dots \dots \dots (2)$$

Die Menge B wird im Folgenden stets eine abgeschlossene nulldimensionale (also in C nirgends dichte) Menge sein. Vgl. wegen der Terminologie URYSOHN, Mémoire mult. cant., I, Introduction (§§ 15-21, Fund. Math., VII, SS. 49-64).

2. x ist vom Integralindex ω , wenn für jedes $\varepsilon > 0$ eine ε -Aussonderung $C = A + B + D$ vorhanden ist von der Art, dass \bar{D} aus endlich vielen Komponenten besteht, wobei die Anzahl dieser Komponenten mit $\frac{1}{\varepsilon}$ notwendig gegen ∞ strebt.

Es wird (wie in der Verzweigungstheorie) die Verabredung getroffen

$$n < \omega < \aleph_0 < c.$$

Bemerkung 1. Die Aussonderungsmengen B , die zum gegebenen ε die Minimaleigenschaft in bezug auf die Komponentenanzahl der Menge \bar{D} besitzen, brauchen nicht notwendig eine, der Indexdefinition entsprechende Mächtigkeit zu haben. Doch werden wir später sehen, dass wir stets im Falle der stetigen Kurven die betreffenden Aussonderungsmengen gebrauchen können.

Es gilt aber ohne weiteres der:

Satz I. Für jeden Punkt x der Kurve C ist $\text{Ind}_x C \leq \text{ind}_x C$.

Um diesen Satz zu beweisen, genügt es zu zeigen, dass jede Komponente K der Menge \bar{D} (wo $C = A + B + D$ eine beliebige ε -Aussonderung des Punktes x ist) mit B wenigstens einen Punkt gemeinsam hat. Falls nun letztere Behauptung falsch wäre, würden die beiden Mengen $B \cdot \bar{D}$ und K zueinander fremd sein; da aber K eine Komponente von \bar{D} ist, so folgt daraus die Existenz einer Zerlegung

$$(1) \quad \bar{D} = F_1 + F_2$$

in zwei zueinander fremde, den Bedingungen $F_1 \supset B \cdot \bar{D}$, $F_2 \supset K$ genügende abgeschlossene Mengen. Die Zerlegung (1) hat aber die widerspruchsvolle Zerlegung $C = (A + B + F_1) + F_2$ des Kontinuums C in zwei zueinander fremde abgeschlossene Mengen $A + B + F_1$ und F_2 zur Folge. Der Satz I ist hiermit bewiesen.

Bemerkung 2. Wir können stets annehmen, dass jeder Punkt y von B gleichzeitig Häufungspunkt der beiden Mengen A und D ist. In der Tat ist jeder Punkt y von B ein Häufungspunkt mindestens einer der beiden Mengen A und D ; denn andernfalls wäre C kein Kontinuum¹⁾. Alle Punkte y , die nur einer der Mengen \bar{A} bzw. \bar{D} angehören, fügen wir zu den entsprechenden Mengen A bzw. D hinzu, ohne dass $B^* = B - \{y\}$ ihre Eigenschaften als Aussonderungsmenge verliert. Ausserdem ist B^* wieder abgeschlossen; denn jeder Häufungspunkt von B^* gehört gleichzeitig zu den abgeschlossenen Mengen B , \bar{A} und \bar{D} .

Nur solche Aussonderungsmengen werden wir von hier an betrachten. Also haben wir immer:

$$\bar{D} = D + B \quad \bar{A} = A + B^2).$$

¹⁾ Die Menge B war ja als nulldimensionale abgeschlossene Menge vorausgesetzt.

²⁾ Diese Bedingung hat wie leicht ersichtlich keinen Einfluss auf die Bestimmung von $\text{ind}_x C$ und $\text{Ind}_x C$.

Wir wenden uns jetzt zu den stetigen, d. h. im kleinen zusammenhängenden¹⁾ Kurven C .

Satz II. *Für jeden Punkt x einer stetigen Kurve C ist $\text{Ind}_x C \leq \omega$.*

Der Beweis des Satzes stützt sich auf den folgenden

Hilfssatz: *Es sei $C = A + B + D$ eine ε -Aussonderung des Punktes x der stetigen Kurve C . Dann enthält D nur eine endliche Anzahl von Komponenten, deren Durchmesser eine vorgegebene positive Grösse α übertrifft.*

Beweis. Nehmen wir im Gegenteil an, es existiere eine Zahl $\alpha > 0$, so dass \bar{D} unendlich viele Komponenten K mit $\delta(K) > \alpha$ enthält.

Infolge eines bekannten Satzes²⁾ ist es möglich eine (im Sinne von HAUSDORFF) konvergente Folge aus $\{K\}$ zu wählen. Es sei $K_1, K_2, \dots, K_n, \dots$ eine solche Folge und $\text{lt}(K_i) = K^*$ ihr topologischer Limes. K^* ist ein Kontinuum. Dann ist:

1°. $\delta(K^*) \geq \alpha$; 2°. $K^* \subset \bar{D}$; 3°. $K_i \cdot K^* = 0$ (mit Ausnahme vielleicht einer einzigen Komponente, die dann K^* enthält und die wir in Folgendem vernachlässigen können).

Da B eine nulldimensionale abgeschlossene Menge ist, ist die abgeschlossene Menge $B \cdot K^*$ auf dem Kontinuum K^* nirgends dicht. Es gibt also einen Punkt $y \in K^*$ mit $\varrho(y, B) \geq \eta > 0$. Es sei $\delta > 0$ beliebig gegeben. Infolge der Definition des topologischen Limes ist es möglich einen Punkt $z \in K_r$ (bei genügend grossem r) zu finden von der Art, dass $\varrho(y, z) < \delta$ ist.

Es sei nun Q irgendein, die beiden Punkte y und z verbindendes Teilkontinuum von C . Wir wollen zeigen, dass $Q \cdot B \neq 0$ ist. In der Tat, falls $Q \cdot B = 0$ wäre, so würde Q (da es $y + z$ enthält) in $C - A = \bar{D}$ enthalten sein. Da aber $z \in K_r$ ist, so wäre auch $Q \subset K_r$, und also $K_r \cdot K^* \supset Q \cdot K^* \supset z \neq 0$, was der Bedingung 2°, widerspricht. Aus der soeben bewiesenen Relation $Q \cdot B \neq 0$ folgt, dass $\delta(Q) \geq \eta$ ist; da aber δ und Q beliebig gewählt waren, so heisst dies, dass

$$\varrho_c(y, z) \geq \varrho(y, B) \geq \eta \text{ ist.}$$

Also ist der Punkt y kein Stetigkeitspunkt der Kurve. Der Hilfssatz ist bewiesen.

Aus diesem Hilfssatz folgt unsre Annahme, dass die Menge der Komponenten von \bar{D} bei jeder ε -Aussonderung der stetigen Kurve C höchstens abzählbar ist.

Es sei jetzt ε_1 eine beliebige positive Zahl und $C = A_1 + B_1 + D_1$

¹⁾ Der Begriff des "Zusammenhanges im kleinen" (Stetigkeit) wurde zuerst von HAHN eingeführt, unabhängig auch von MASURKIEWICZ behandelt und lautet folgendermassen: die Kurve C heisst "zusammenhängend im kleinen" (stetig) im Punkte x , wenn es für jedes ε eine Zahl δ gibt von der Art, dass für zwei Punkte x und y , deren Abstand $\rho(x, y) < \delta$ ist, der relative Abstand $\rho_c(x, y) < \varepsilon$ ist, wobei $\rho_c(x, y)$ die untere Grenze der Durchmesser aller Teilkontinua von C , die x und y gleichzeitig enthalten, bedeutet.

²⁾ URYSOHN, II, Ch. II. VIETORIS, Stetige Mengen (Monatsh. f. Math. u. Phys., 1921).

irgendeine ε_1 -Aussonderung des Punktes x . Wir bezeichnen dann durch δ die positive Zahl $\varrho(x, B_1 + D_1)$ (es ist stets $0 < \delta < \varepsilon_1$).

Wir setzen nun $\varepsilon_2 = \frac{\delta}{2}$ und betrachten irgend eine ε_0 -Aussonderung des Punktes x

$$C = A_2 + B_2 + D_2, \quad x \in A_2 \subset A_2 + B_2 \subset S(x, \varepsilon_2) = S\left(x, \frac{\delta}{2}\right).$$

Es seien jetzt $\{K\}$ bzw. $\{K^*\}$ die Komponenten von \bar{D}_1 bzw. \bar{D}_2 . Die Menge $\bar{D}_1 = D_1 + B_1$ (und folglich jede Komponente K) ist dann zu $S(x, \delta)$ fremd. Jede Komponente K^* hat notwendig Punkte mit $S\left(x, \frac{\delta}{2}\right)$ gemeinsam, nämlich die Punkte von B_2 . $K^* \neq \emptyset$.

Es ist ausserdem $A_2 \subset A_1$ und folglich $\bar{D}_2 \supset \bar{D}_1$. Jede Komponente K ist in einer Komponente K^* enthalten. Es gibt aber nur eine endliche Anzahl von Komponenten K^* , die mindestens ein K enthalten, weil Komponenten K^* , da sie gleichzeitig Punkte ausserhalb $S(x, \delta)$ und innerhalb $S\left(x, \frac{\delta}{2}\right)$ enthalten müssen, sämtlich einen Durchmesser grösser als $\frac{\delta}{2}$ haben. Also kann ihre Anzahl dem Hilfssatz gemäss unmöglich unendlich sein.

Es seien nun $K_1^*, K_2^* \dots K_m^*$ diese Komponenten. Es ist

$$D_2 = \sum K^* = \sum_1^m K_i^* + \sum_{K^* \subset A_1} K^* \quad 1)$$

und

$$B_2 = B_2 \cdot \bar{D}_2 = B_2 \cdot \sum_1^m K_i^* + B_2 \cdot \sum_{K^* \subset A_1} K^* = B_1^* + B_2 \cdot \sum_{K^* \subset A_1} K^*.$$

Die Identität $C = A_1^* + B_1^* + D_1^*$, wobei

$$B_1^* = B_2 \cdot \sum_1^m K_i^*$$

$$A_1^* = A_2 + \sum_{K^* \subset A_1} K^*$$

$$D_1^* = \sum_1^m K_i^* - (B_1^* + A_1^*)$$

gesetzt ist²⁾, gibt uns eine ε_1 -Aussonderung des Punktes x .

1) Letztere Summe bezieht sich auf alle diejenigen K^* die zu \bar{D}_1 fremd, und also in A enthalten sind.

2) Man bemerkt sofort, dass $D_1^* = \sum_1^m K_i^* - B_1^*$ ist.

In der Tat haben wir:

$$\begin{aligned}
 1) \quad A_1^* \supset A_2 \supset x; \quad & 2) \quad A_1^* + B_1^* \subset A_2 + B_2 + \sum_{K^* \subset A_1} K^* \subset S(x, \varepsilon_1); \\
 3) \quad A_1^* \cdot \overline{D_1^*} + \overline{A_1^*} \cdot D_1^* \subset & (A_2 + \sum_{K^* \subset A_1} K^*) \cdot \sum_1^m K_i^* + \overline{A_2} \cdot (\sum_1^m K_i^* - B_1^*) + \\
 & + \sum_{K^* \subset A_1} \overline{K^*} \cdot (\sum_1^m K_i^* - B_1^*).
 \end{aligned}$$

Aus unsern Definitionen folgt leicht, dass der erste und zweite Summand von (3) leer ist. Es bleibt also zu beweisen, dass $\sum_{K^* \subset A_1} \overline{K^*} \cdot (\sum_1^m K_i^* - B_1^*) = 0$ ist. Der zweite Faktor enthält weder Punkte von B_2 , noch von $\sum_{K^* \subset A_1} K^*$; die gemeinsamen Punkte der beiden Faktoren könnten also nur diejenigen Häufungspunkte von $\sum_{K^* \subset A_1} K^*$ sein, die weder zu $\sum_{K^* \subset A_1} K^*$, noch zu B_2 gehören. Solche existieren aber überhaupt nicht. In der Tat streben die Durchmesser der in A enthaltenen K^* dem Hilfssatz gemäss gegen Null, d.h. dass jede konvergente unendliche Folge von Komponenten $K^* \subset A_1$ gegen einen Punkt konvergiert. Da aber jede Komponente notwendig einen Punkt auf der abgeschlossenen Menge B_2 besitzt, gehört dieser Limespunkt auch zu B_2 .

Wir sehen jetzt, dass die Zerlegung $C = A_1^* + B_1^* + D_1^*$ wirklich eine ε_1 -Aussonderung des Punktes x ist. Die Menge $\overline{D_1^*}$ besteht dabei aus einer endlichen Anzahl von Komponenten. Da ε_1 aber eine beliebige positive Zahl war, so heisst das, dass $\text{Ind}_x C \leq \omega$ ist, w. z. b. w.¹⁾

Wir wollen nun den Integralindex mit dem Verzweigungsindex etwas näher vergleichen.

Satz III. *Bei der Bestimmung des Integralindex der stetigen Kurve C können wir uns auf die Aussonderungsmengen B von minimaler d.h. der, dem Indexe des betreffenden Punktes entsprechenden, Mächtigkeit beschränken.²⁾*

Beweis. Es seien $\varepsilon > 0$ und eine ε -Aussonderung $C = A + B + D$ des Punktes x gegeben, wobei die Mächtigkeit der Komponentenmenge von $\overline{D} = \sum_1^m K_i$ den kleinstmöglichen Wert annimmt. Wir wollen zeigen, dass es stets eine ε -aussondernde abgeschlossene Menge B_1 von minimaler Mächtigkeit gibt, die in bezug auf die Komponentenanzahl von $\overline{D_1}$ die Eigenschaften von B besitzt. In der Tat gibt es nach der Definition des Verzweigungsindex eine den Punkt x ε -aussondernde, in A enthaltene

¹⁾ Aus diesem Satz folgt insbesondere, dass falls für wenigstens einen Punkt $x \in C$ $\text{ind}_x C \geq \aleph_0$ ist, C unendlich viele geschlossene Jordankurven enthält (da in diesem Falle wenigstens ein $K_j^* \subset \overline{D_1^*}$ mit A_1^* unendlich viele Punkte gemeinsam hat).

²⁾ Falls dabei $\text{ind}_x C = \omega$ ist, so betrachten wir jede endliche Menge B als eine die erwähnten Minimaleigenschaft besitzende.

Menge \bar{B} von der, dem Werte $\text{ind}_x C$ entsprechenden Mächtigkeit. Wir betrachten nun die entsprechende ε -Aussonderung $C = \bar{A} + \bar{B} + \bar{D}$. Wir haben erstens die Inklusionen $\bar{A} \subset A$, $\bar{D} \supset D$; ferner ist jede Komponente K (von \bar{D}) in einer Komponente \bar{K} (von \bar{D}) enthalten. Wir bezeichnen jetzt diejenigen Komponenten \bar{K} , die mindestens ein K enthalten, mit K^* und setzen Q_* gleich der Vereinigungsmenge sämtlicher K^* ; wir bezeichnen weiter durch K^\odot die von K^* verschiedenen Komponenten der Menge \bar{D} ; ihre Vereinigungsmenge soll Q_\odot heissen.

Dann ist:

$$\bar{D} = Q_* + Q_\odot \quad ; \quad \bar{B} = \bar{B} \cdot Q_* + \bar{B} \cdot Q_\odot.$$

Die Zerlegung

$$C = A_1 + B_1 + D_1$$

mit

$$A_1 = A + Q_\odot,$$

$$B_1 = \bar{B} \cdot Q_*,$$

$$D_1 = Q_* - B_1.$$

ist eine ε -Aussonderung des Punktes x , was man in derselben Weise, wie beim Beweise des Satzes II, zeigt.

Diese ε -Aussonderung genügt aber den beiden Voraussetzungen in bezug auf die Komponentenzahl der Mengen \bar{D}_1 und B_1 , w.z.b.w.

Es sei hierzu noch bemerkt, dass man stets verlangen kann, die Menge A_1 sei zusammenhängend: im entgegengesetzten Falle hätte man nur \bar{A}_1 durch $\bar{A} = \text{Komp}_x A_1$ zu ersetzen¹⁾ und der Reihe nach $\bar{B} = \bar{A} - \bar{A}$, $\bar{D} = C - (\bar{A} + \bar{B})$, und schliesslich

$$A_2 = \bar{A} + (B - \bar{D})$$

$$B_2 = \bar{B} \cdot \bar{A} \cdot \bar{D}$$

$$D_2 = \bar{D} + (\bar{B} - \bar{A})$$

zu definieren²⁾. Man kann dies wie folgt zusammenfassen:

Satz IV. Für jeden Punkt x einer stetigen Kurve C und für jedes $\varepsilon > 0$, gibt es ε -Aussonderungen, welche gleichzeitig Minimaleigenschaften in Bezug auf die Komponentenzahl der Mengen \bar{A} , B und \bar{D} besitzen³⁾.

¹⁾ Es ist zu bemerken, dass letztere Menge auf Grund eines bekannten Satzes von HAHN in bezug auf C offen ist.

²⁾ Man beweist in der Tat leicht, dass erstens $B_2 \subset B_1$ ist, zweitens jede Komponente von D_2 mit B_2 , also mit B_1 gemeinsame Punkte hat, und folglich eine Komponente von \bar{D}_1 enthält. Daraus folgen aber beide Minimaleigenschaften für $C = A_2 + B_2 + D_2$.

³⁾ Nur 0-dimensionale Mengen B werden dabei zur Konkurrenz zugelassen (vgl. auch die Fussnote ¹⁾, Seite 1252).

Diese ε -Aussonderungen werden wir als „normal“ bezeichnen.

Wir sind jetzt im Stande, die beiden Indices zu vergleichen, und führen dazu die neue Zahl

$$\Delta_x C = \text{ind}_x C - \text{Ind}_x C$$

ein.

Wenn $\text{ind}_x C$ endlich ist, ist die Bedeutung von $\Delta_x C$ klar.

Falls $\text{ind}_x C = \omega$ ist, setzen wir $\Delta_x C$ definitionsgemäss gleich der kleinsten Zahl $a \leq \infty$ von der Beschaffenheit, dass es für jedes $\varepsilon > 0$ eine normale ε -Aussonderung $C = A + B + D$ gibt, für die $\beta - \delta \leq a$ ist (wobei β bzw. δ die Anzahl der Komponenten von B bzw. D bedeutet).

Falls endlich $\text{ind}_x C \geq \aleph_0$ ausfällt, so ist das einzig-vernünftige $\Delta_x C$ (dem Satze II entsprechend) gleich $\text{ind}_x C$ zu setzen.

Satz V. Wenn C und C^* zwei stetige Kurven sind und $C \supset C^* \supset x$, so ist $\Delta_x C \geq \Delta_x C^*$.

Beweis. Offenbar ist $\text{ind}_x C \geq \text{ind}_x C^*$. Die Ungleichung $\Delta_x C \geq \Delta_x C^*$ folgt alsdann für $\text{ind}_x C \geq \aleph_0$ aus der Definition selbst. Bleibt übrig den Satz für den Fall $\text{ind}_x C \leq \omega$ zu beweisen.*

Es sei $C = A + B + D$ eine normale, dem Werte von $\Delta_x C$ entsprechende ε -Aussonderung des Punktes x . Die Zerlegung $C^* = A^{**} + B^{**} + D^{**}$, wobei $A^{**} = A \cdot C^*$, $B^{**} = B \cdot C^*$, $D^{**} = D \cdot C^*$ gesetzt ist, ist dann wieder eine ε -Aussonderung von x in bezug auf C^* . Es ist ausserdem $A^{**} \subset A$, $B^{**} \subset B$; $D^{**} \subset D$.

Falls B^{**} Punkte ausserhalb $\overline{A^{**} \cdot D^{**}}$ enthält, so fügen wir diese Punkte zu D^{**} bzw. A^{**} hinzu. Die so erhaltenen Mengen bezeichnen wir wieder mit A^{**} , B^{**} , D^{**} . Es seien β, δ bzw. β^{**}, δ^{**} die diesen Aussonderungen entsprechenden Anzahlen von Komponenten (wobei nicht zu vergessen ist, dass die Menge B (also auch B^{**}) aus endlichvielen Punkten besteht).

Wir haben immer $\beta \geq \beta^{**}$. Wenn dabei $\delta \leq \delta^{**}$, so ist $\beta - \delta \geq \beta^{**} - \delta^{**}$. Wir werden aber gleich sehen, dass letztere Ungleichung auch für den Fall $\delta > \delta^{**}$ gilt. Es seien in der Tat $K_1^{**}, K_2^{**}, \dots, K_{\delta^{**}}^{**}$ die Komponenten von $\overline{D^{**}}$, und $K_1, K_2, \dots, K_{\beta^{**}}$ die, durch die Inklusionen $K_i \supset K_i^{**}$ ($i = 1, 2, \dots, \delta^{**}$) eindeutig bestimmten Komponenten von \overline{D} . (Zwei mit verschiedenen Indizes versehene K_i können dabei selbstverständlich identisch sein). Unserer Voraussetzung gemäss gibt es jedenfalls $\delta - \delta^{**}$ von allen $K_1, \dots, K_{\beta^{**}}$ verschiedene Komponenten $K_{\beta^{**}+1}, \dots, K_{\beta}$ der Menge \overline{D} . Jede dieser Komponenten hat mit B wenigstens einen Punkt gemein. Daraus folgt, dass

$$\beta - \beta^{**} \geq \delta - \delta^{**} > 0,$$

und also

$$\beta - \delta = [\beta^{**} + (\beta - \beta^{**})] - [\delta^{**} + (\delta - \delta^{**})] \geq \beta^{**} - \delta^{**}$$

ist.

Die Aussonderung $C^* = A^{**} + B^{**} + D^{**}$ ist im allgemeinen keine normale. Man kann sie aber normalisieren, so wie es beim Beweise der Sätze III und IV gezeigt wurde. Es sei $C^* = A^* + B^* + D^*$ die so ent-

standene normale ε -Aussonderung und β^* , δ^* ihr entsprechende Anzahlen von Komponenten. Dabei nimmt, für genügend kleine ε , die Zahl $\beta^{**} - \delta^{**}$ während des ganzen Prozesses der Normalisierung niemals zu,¹⁾ so dass $\beta^{**} - \delta^{**} \geq \beta^* - \delta^*$ ist.

Also haben wir $\Delta_x C \leq \beta - \delta \geq \beta^{**} - \delta^{**} \geq \beta^* - \delta^*$, wobei $\beta^* - \delta^*$ einer normalen ε -Aussonderung auf C^* entspricht. Die Zahl $\Delta_x C$ genügt der für $\Delta_x C^*$ charakteristischen Ungleichung. Da $\Delta_x C^*$ die kleinste Zahl von dieser Beschaffenheit ist, so ist $\Delta_x C \geq \Delta_x C^*$, w.z.b.w.

Aus dem Satze V folgt

Satz VI. *Damit eine stetige Kurve C eine Baumkurve²⁾ sei, ist notwendig und hinreichend, dass für sämtliche Punkte $x \in C$ die Gleichung $\Delta_x C = 0$ gilt.*

Die Bedingung ist hinreichend. In der Tat, falls C eine einfache geschlossene Linie C^* enthält, so gilt für jeden Punkt $x \in C^*$ die Gleichung

$$\Delta_x C^* = 1;$$

also ist nach dem Satze V $\Delta_x C \geq 1$.

Die Bedingung ist notwendig. Wir setzen voraus, dass C eine Baumkurve ist, für die in irgend einem Punkte $x \in C$, $\Delta_x C \geq 1$ ausfällt.

Es sei nun $C = A + B + D$ eine normale ε -Aussonderung (wobei ε klein genug ist damit $\beta - \delta > 0$ sei). Mindestens eine Komponente K von \bar{D} enthält zwei Punkte von B , z.B. b_1 und b_2 . Da K ein im kleinen zusammenhängendes Kontinuum ist¹⁾, kann man b_1 mit b_2 durch einen in K enthaltenen einfachen Bogen S_1 verbinden. Da aber $A + B$ auch ein im kleinen zusammenhängendes Kontinuum ist, so kann man auch einen in $A + B$ enthaltenen, b_1 mit b_2 verbindenden einfachen Bogen S_2 finden. $Q = S_1 + S_2$ ergibt dann eine in C enthaltene einfache geschlossene Linie, was unsrer Voraussetzung widerspricht.

Wir wollen jetzt folgenden Satz beweisen:

Satz VII. *Damit eine stetige Kurve C eine und nur eine einfache geschlossene Linie Q enthält, ist notwendig und hinreichend, dass die beiden folgenden Bedingungen gleichzeitig erfüllt sind:*

1^o. Für jeden Punkt $x \in C$ ist $\Delta_x C < 1$.

2^o. Die Menge M aller Punkte $x \in C$, in denen $\Delta_x C > 0$ ist, ist zusammenhängend.

Bevor wir die Notwendigkeit unserer beiden Bedingungen beweisen, bemerken wir, dass falls C nur eine einfache geschlossene Linie enthält, sicher für alle Punkte $x \in C$ die Ungleichung $ind_x C \leq \omega$ gilt³⁾. Beim

¹⁾ Für den Fall $ind_x C^* = \omega$ besteht der erwähnte Prozess aus einem einzigen Schritte, man setzt nämlich $A^* = Komp_x A^{**}$ (vgl. auch Fussnote 1) auf Seite 1255).

²⁾ Man versteht unter einer *Baumkurve* eine, keine einfache geschlossene Linie (= geschlossene JORDANKurve) enthaltende stetige Kurve. Diese Kurvenart wurde zuerst von MAZURKIEWICZ und sodann von MENGER (l.c.), von dem auch die Bezeichnung "Baumkurve" herrührt, untersucht. (Vgl. Fund. Math. 2, S. 119).

³⁾ Siehe Fussnote 1) auf S. 1255.

Beweise der Notwendigkeit unserer Bedingungen dürfen wir also annehmen, dass C eine endlich verzweigte¹⁾ Kurve ist.

Die Bedingung 1⁰ ist notwendig. Es sei in der Tat C eine endlich verzweigte stetige Kurve, die eine einzige einfache geschlossene Kurve Q enthält und die in wenigstens einem Punkte x der Bedingung $\Delta_x C \geq 2$ genügt. Wir wollen zeigen, dass dies unmöglich ist.

Zwei Fälle sind a priori möglich.

1⁰. Der Punkt x ist nicht in Q enthalten.

2⁰. Der Punkt x gehört zu Q .

Wir untersuchen zuerst den Fall 1⁰. Es sei $U(x)$ eine der Bedingung $\bar{U}(x), Q=0$ genügende *zusammenhängende* Umgebung des Punktes x (rel. C). Dann ist $C_0 = \bar{U}(x)$ eine Baumkurve und es gibt (infolge des Satzes VI) eine normale ε -Aussonderung ($\varepsilon < \rho(x, C - U(x))$) des Punktes x (in Bezug auf C_0)

$$C_0 = A_0 + B_0 + D_0,$$

wobei die Komponentenzahl δ_0 von \bar{D}_0 gleich der (endlichen) Anzahl β_0 von Punkten der Menge B_0 ist. Wir setzen nun $A = A_0, B = B_0, D = C - (A + B)$ und überzeugen uns ohne Schwierigkeit davon, dass

$$C = A + B + D$$

eine ε -Aussonderung der Punktes x in bezug auf C ist.

Ich behaupte nun, dass die Komponentenzahl δ von \bar{D} gleich δ_0 ist.

In der Tat, da jede Komponente von \bar{D} mit $B = B_0$, also mit \bar{D}_0 gemeinsame Punkte hat, so ist sicher $\delta \leq \delta_0$. Falls aber $\delta < \delta_0$ wäre, so würden gewiss zwei Komponenten K_1 und K_2 von \bar{D}_0 zu einer Komponente K von \bar{D} gehören; da aber (wegen $\beta_0 = \delta_0$) K_1 und K_2 in verschiedenen Punkten b_1 und b_2 von B münden, so hat K mit B wenigstens diese beiden Punkte gemein. Indem wir nun b_1 und b_2 zuerst in K und dann in C_0 mittels zweier einfacher Bogen verbinden²⁾, erhalten wir sofort eine einfache geschlossene Linie Q_1 , die mit C_0 gemeinsame Punkte hat und also sicher von Q verschieden ist. Im Falle 1⁰ würde also C im Widerspruch zu unserer Voraussetzung mindestens zwei verschiedene einfache geschlossene Linien Q und Q_1 enthalten.

Wir wenden uns jetzt zum Falle 2⁰: $x \in Q$ und betrachten eine bestimmte ε -Aussonderung $C = A + B + D$ des Punktes x , wobei

$$B = \sum_1^{\beta} b_i, \quad \bar{D} = \sum_1^{\delta} K_i, \quad \beta - \delta \geq 2$$

ist.

Ohne die Allgemeinheit einzuschränken dürfen wir in unsrem Falle

¹⁾ Eine Kurve C heisst *endlich-verzweigt* (URYSOHN) falls für jeden ihrer Punkte $\text{ind}_x C \leq \omega$ ist. Bei MENGER (l.c.) wird eine derartige Kurve „regulär“ genannt.

²⁾ Dies ist stets möglich, da jedes Teilkontinuum einer endlich verzweigten Kurve selbst endlich verzweigt, also im kleinen zusammenhängend ist (Siehe URYSOHN, II, ch. III; MENGER, Math. Ann. 95, S. 300.

annehmen, dass die Menge $B \cdot Q$ nur aus zwei Punkten, z.B. aus b_1 und b_2 besteht. Dann folgt aus $\beta - \delta \geq 2$ leicht, dass

I. entweder die Komponente von $Q - A$ in \bar{D} , d.h. die grösste zusammenhängende, $Q - A$ enthaltende Teilmenge K von \bar{D} , mindestens einen, von beiden Punkten b_1 und b_2 verschiedenen Punkt $b_i \in B$ enthält

II. oder eine von K verschiedene Komponente K^* der Menge \bar{D} zwei (von den Punkten b_1, b_2 nicht notwendig verschiedene) Punkte $b_p + b_q \in B$ enthält.

In beiden Fällen gibt es, wie leicht beweisbar, eine von Q verschiedene einfache geschlossene Linie $Q^* \subset C$, was der Annahme widerspricht.

Die Bedingung 2^0 ist notwendig. In der Tat, falls C nur eine geschlossene JORDANKurve Q enthält, so folgt aus VI leicht, dass für $x \in C - Q$ stets $\Delta_x C = 0$ ist. Da andererseits für jeden Punkt $x \in Q$ $\Delta_x C \geq \Delta_x Q = 1$ ist, so ist M mit Q identisch, also zusammenhängend.

Die Bedingungen 1^0 und 2^0 sind hinreichend. Aus 1^0 folgt wieder $ind_x C \leq \omega$. Nehmen wir an, C enthalte zwei oder mehrere einfache geschlossene Linien, $C \supset Q_1 + Q_2 + \dots$. Wir beweisen zuerst, dass wenigstens zwei dieser Linien gemeinsame Punkte haben. Im Falle, wo C nur endlichviele einfache geschlossene Kurven enthält, folgt letztere Behauptung unmittelbar aus unsren Voraussetzungen (weil dann, wie leicht ersichtlich, M mit der Vereinigungsmenge aller in C enthaltenen geschlossenen Kurven identisch ist).

Es seien nun in C unendlich (und dann notwendig nur abzählbar) viele geschlossene JORDANKurven

$$Q_1, Q_2, \dots, Q_n, \dots$$

enthalten, die paarweise zueinander fremd sind. Offenbar ist (auf Grund von Satz V) $\Sigma Q_n \subset M$. Ich behaupte, dass auch umgekehrt $M \subset \Sigma Q_n$ ist, m. a. W., dass für jeden Punkt $x \in C - \Sigma Q_n$

$$\Delta_x C = 0$$

ist. Es sei in der Tat C^* diejenige *Baumkurve*, die aus C dadurch entsteht, dass man jede Kurve Q_n durch einen einzigen Punkt x_n^* ersetzt und alle sonstigen Limesbeziehungen in C unverändert in C^* aufnimmt (so dass C^* ein eindeutiges stetiges Bild von C , und $C^* - \sum_{n=1}^{\infty} x_n^*$ ein eineindeutiges Bild von $C - \Sigma Q_n$ ist).

Dann ist für jeden Punkt $x \in C - \sum_{n=1}^{\infty} Q_n$

$$\Delta_x C = \Delta_{x^*} C^* = 0 \quad (x^* \text{ ist der Bildpunkt von } x \text{ in } C^*),$$

womit unsre Behauptung bewiesen ist.

Es können also in C unmöglich lauter zueinander fremde geschlossene JORDANKurven enthalten sein, so dass es in C mindestens zwei gemeinsame Punkte besitzende einfache geschlossene Linien Q_1 und Q_2 gibt.

Die Summe $Q_1 + Q_2$ bildet ein Kontinuum (stetige Kurve) $C_0 \subset C$. Nach dem Satze V bleibt nur noch übrig folgenden Hilfssatz zu beweisen:

Hilfssatz. Wenn der Durchschnitt $Q_1 \cdot Q_2$ von zwei verschiedenen einfachen geschlossenen Linien nicht leer ist, gibt es in $Q_1 + Q_2 = C_0$ Punkte, für die $\Delta_x C_0 \geq 2$ ist.

Beweis. Da Q_1 und Q_2 verschieden sind, gibt es z.B. auf Q_1 einen offenen Bogen, der von Punkten von Q_2 frei ist. Wir bezeichnen jetzt mit S einen grössten diese Eigenschaft besitzenden Bogen. Dann ist für die Endpunkte a_1 und a_2 (die auch zusammenfallen können): $\Delta_{a_1} C_0 \geq 2$, $\Delta_{a_2} C_0 \geq 2$. Es genügt offenbar die Ungleichung $\Delta_{a_1} C_0 \geq 2$ zu beweisen. Ist a_1 ein isolierter Punkt von $Q_1 \cdot Q_2$, so können wir ihn für genügend kleine ε durch vier (und nicht weniger) Punkte (zwei auf Q_1 und zwei auf Q_2) aussondern. Die zu A komplementäre Menge \overline{D} enthält aber höchstens zwei Komponenten $Q_1 - A$ und $Q_2 - A$.

Ist dagegen a_1 kein isolierter Punkt von $Q_1 \cdot Q_2$, so braucht man um ihn für ein hinreichend kleines ε auszusondern drei Punkte (zwei auf Q_2 , von denen einer auf $Q_1 \cdot Q_2$ zu wählen ist, und einen auf $S \subset Q_1$). Die zu A komplementäre Menge D kann aber dabei stets zusammenhängend vorausgesetzt werden. In beiden Fällen haben wir $\Delta_{a_1} C^* \geq 2$. Der Hilfssatz und der Satz VIII sind damit vollständig bewiesen.

Ist C eine ebene stetige Kurve, so gibt uns die Anwendung des Jordanschen Kurvensatzes die den Sätzen VI und VII analogen Sätze VI' und VII':

Satz VI'. Damit eine ebene stetige Kurve C die Ebene nicht zerlegt, ist notwendig und hinreichend, dass für sämtliche Punkte $x \in C$ $\Delta_x C = 0$ ist.

Satz VII'. Damit eine ebene stetige Kurve C die Ebene in zwei Gebiete zerlegt ist notwendig und hinreichend, dass die beiden folgenden Bedingungen gleichzeitig erfüllt seien:

1°. Für jeden Punkt $x \in C$ ist $\Delta_x C \geq 1$.

2°. Die Menge $M = \{x, \Delta_x C > 0\}$ ist zusammenhängend.

Diese beiden Sätze können auch direkt (aber ziemlich umständlich) durch systematische Anwendung der Sätze von JANISZEWSKI und STRASZEWICZ bewiesen werden.

Moskau, 15. V. 1926.

Biochemistry. — "*On the spreading of fatty acids, fats and proteins.*"
By Prof. E. GORTER and F. GREDEL. (Communicated by Prof.
P. EHRENFEST.)

(Communicated at the meeting of October 30, 1926).

I.

The spreading of fatty acids.

The results of our experiments on the spreading of different substances on a watersurface, are not completely comparable to those obtained by other workers, because we have examined substances which are insoluble in ether, petroleumether or other like substances.

The greater part of the experimenters have dissolved the substances of which they studied the spreading on a watersurface in a solvent, that spreads itself and evaporates very quickly from the watersurface. We however have examined the area of the spreading of substances dissolved in water; all be it that these solutions were mostly colloidal ones.

In order to enable us to put our results on a firm basis, we have made a first series of experiments with substances which are soluble in petroleumether and of which also colloidal watery solutions can be prepared.

We have chosen for this purpose different fatty acids and have studied separately:

1^o. the influence of the acidity of water on which the spreading takes place;

2^o. the influence of the temperature of water on which the spreading was performed.

As fatty acids we have chosen palmitic acid, oleic acid and lauric acid and we have compared the spreading of these substance of their solutions in petroleumether with the spreading of the watery soap solutions which have been prepared by dissolving the corresponding fatty acid in the minimum amount of caustic soda solution. Both solutions were of the same strength $1/100$ mol.

As a result of these experiments one important fact is to be noted, that it is possible to obtain a spreading which is considered by everybody as a monomolecular one, by making use of a colloidal soap solution. It is however necessary to observe certain precautions. One of the technical precautions that are of great importance is that only a very small quantity measured in a capillary pipette (we made use of a pipette of 0.005 c.c. divided in 10 parts) must be used, and this pipette must be held at the surface of the water in the tray and blown out in a horizontal position. A second condition which is necessary to observe is that the water in the tray is strongly acid. This acid not only serves to set free the fatty acid, but

also to promote the spreading so as to make it maximal. The third condition is that the suitable temperature is chosen, which is different for different acids. Fatty acids with a short chain must be examined at a lower temperature than fatty acids with a longer chain. For these latter substances it is necessary to take account of the fact, described by ADAM that fatty acids exist in two forms: condensed and expanded.

From the correspondence between the numbers for the maximal spreading, obtained in the two series of experiments it is allowed to conclude that also the results with colloidal watery solutions are reliable.

Let us examine first of all the influence of the acidity of the water at 15° on the spreading of a certain fatty acid like *palmitic acid*, and observe if there is a difference between the spreading of a measured quantity of the petroleum ether solution and the colloidal watery palmitic-soap solution.

On fairly acid solutions of a PH 2 ($1/100$ n. HCl acid) the spreading is maximal: the area per molecule is 20 sq. A. U. If we take solutions of hydrochloric acid that are of a lower acidity we find that in both cases the area is smaller. Whereas with petroleum ether solutions these differences are of minor importance, the area occupied, when spreading from a watery soap-solution, are much smaller, so that a palmitic soap spread on distilled water occupies a surface that is only a small part ($1/3$ or $1/4$) from the maximal monomolecular-area. Moreover the results are not far as constant as when the maximal spreading is studied. This imperfect spreading that can be obtained under different conditions seems to be due to the existence of colloidal aggregates of molecules which orientate their polar groups to an outer surface.

Now the explanation of these too small numbers is a very dangerous question, for there are several other reasons for finding a too small area. We mention the solubility of the soap in the water of the tray and insufficient spreading due to insolubility, so that the polar group is so to speak insoluble. We have met with both reasons for imperfect spreading in our experiments. It is possible to recognize the first reason of smaller numbers than the mono-molecular layer would have given, by studying the influence of time and of compression. If the substance is too soluble to give a beautiful spreading the area tends to diminish in size by waiting. Besides compression of the films by laying more weights in the pan of the balance produces a considerable decrease of the area occupied, and this change is not wholly reversible. Often the pressure-area curve is not straight as with well-spreading substance but convex to the axis, so that it has more or less the form of a $p \cdot A = k$ -line. We leave out of consideration whether this solution is a molecular or colloidal one, although the latter form of solution seems to have the highest probability.

The second reason why smaller numbers are found than a mono-molecular film would have given, can sometimes be recognized by a careful inspection of the surface and by seeing some small solid particles floating on the surface. In this case compression has the effect to break the film at a

lower force than usually, whereas the pressure-area-curve tends to become concave towards the axis, thus presenting the reverse form from the foregoing case.

Soap solutions or fatty acids set free from these soap-solutions form surface films presenting a rather high compressibility, and often show an increase in size as a function of time, and are influenced by the same factors such as acidity temperature and pressure as ordinary mono-molecular films do. But not always the effect of these factors is the same in both cases.

It seems possible to describe all observed phenomena using a scheme of the following form.

There exists a gradual transition between substances of a certain series, say the fatty acids of the saturated series ($C_n H_{2n} O_2$) also with regard to their spreading. Whereas the lower members are soluble the next ones form colloidal solutions, then we meet with fatty acids showing what we should like to call a colloidal spreading, followed by the group of well-spreading members, and ultimately we meet with substances, showing an imperfect spreading. We can thus give this series :

| | | | | |
|-----------------------|-----------------------|------------------------|-----------------------------|-------------------------|
| 1 | 2 | 3 | 4 | 5 |
| molecular solution | colloidal solution | colloidal spreading | mono-molecular spreading | imperfect spreading. |

A fatty acid having a high number of C-atoms in the chain is placed to the right side whereas a lower member is placed more to the left hand side. If several chains are linked together, a displacement to the right, results. (If the polarity of a substance is lessened e.g. through esterification, the effect is a displacement to the right.) If now we study the influence of temperature we find that a lowering of temperature has the effect of a displacement to the right side of the above series. So a substance which shows a colloidal spreading at say 15° , can present a normal mono-molecular spreading at zero, and another fatty acid or fat, which shows a colloidal spreading at 15° , is dissolved in a colloidal form by heating. Also the influence of a change in the hydrogenion concentration of the water may be described in the same simple form, and we are justified to say that an increase of the acidity has the effect of a displacement to the right side of the series. A fatty acid or soap showing a colloidal spreading at a PH 4 will spread beautifully when blown out on water of PH 1. In other terms acidity promotes the spreading of a fatty acid.

How are all these influences to be explained? The tendency to spread is the consequence of two antagonistical factors: the solubility of the polar group in the water acts in orders to dissolve the molecule in the water, but its influence is counter-balanced by the action of the bounds between the C-atoms in the chain, that tends to keep the molecule out of the water.

This view makes it comprehensible that a substance with a lower number of C-atoms is more easily soluble than a substance with a higher number

of C.-atoms. It explains that a fall of temperature has the same effect as an increase of the length of the chain, because the solubility diminishes through a fall of temperature. It makes plausible that an increase of the acidity of the water promotes the spreading because the polar groups are attracted with a smaller force, diminishing the solubility of these groups.

In all the foregoing résumé no mention had been made of the change of a substance from the condensed to the expanded condition. If we study this phenomenon, we first of all must mention the remarkable fact already observed by ADAM, that the point of half-expansion is not the same as the melting point, but is in all members of the series much lower. We have come to the conclusion that nevertheless this transition between expanded and condensed condition is so to speak a bi-dimensional melting. We must try to explain why this bi-dimensional melting takes place at a lower temperature than the tri-dimensional. It seems that this difference is principally due to the fixation of the polar groups of the fatty acids to the water. This fixation at the water has the effect of diminishing the strength, with which the molecules are kept together and so it must be admitted that the COOH-groups contribute to this latter property. This seems not improbable, if we compare the melting points of a series of fatty acids to the same series of methylated fatty acids or to the paraffine-series. Always the melting points are highest in the series of substances having the COOH-groups in the molecule.

This hypothesis is strengthened by the very remarkable observation, that the acidity of the water in the tray has a very distinct influence on this bidimensional melting point: On acid water ($1/10$ n. HCl.) the melting point is much lower than on distilled water. The acid produces this effect by loosening the bounds between the molecules, as it has a greater influence on the polar groups than water has.

We give the figures for palmitic and acid (fig. 1, 2 and 3).

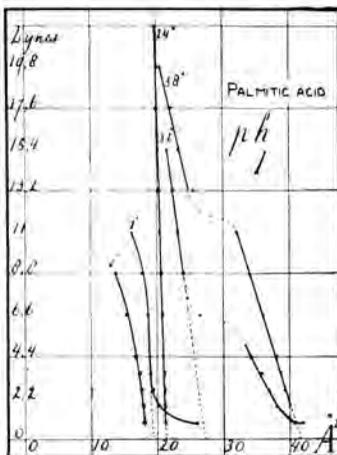


Fig. 1

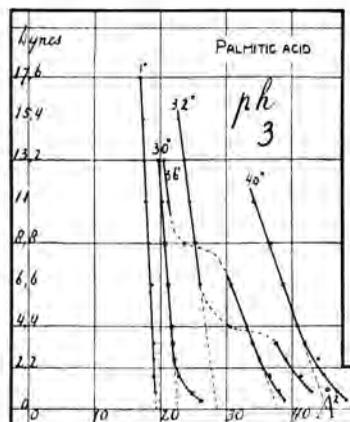


Fig. 2.

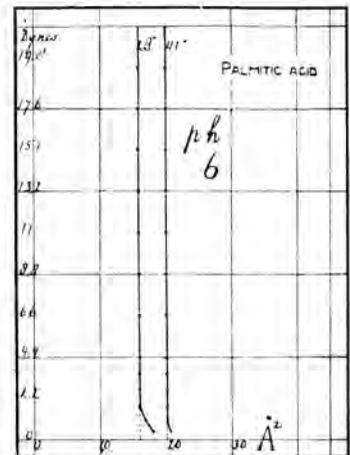


Fig. 3.

In concordance with this hypothesis is the fact that the compressibility of the "expanded" substance, spread on water, is much higher than that of the same substance in its condensed condition. Besides it is striking that lower members of the series of fatty acids, whose spreading is promoted by lowering the temperature or by using strongly acid water, always show a behavior of expanded substances and thus occupy a larger area and present a greater compressibility (fig. 4, 5 and 6).

We give as an example the spreading of lauric acid on strongly acid water.

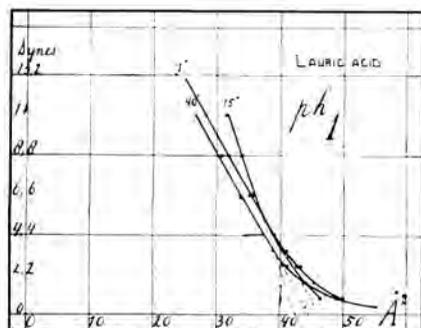


Fig. 4.

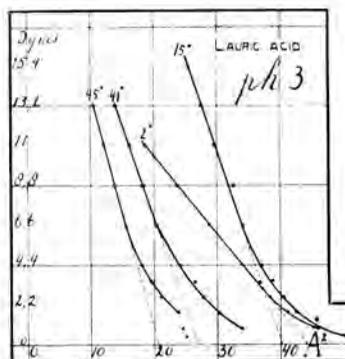


Fig. 5.

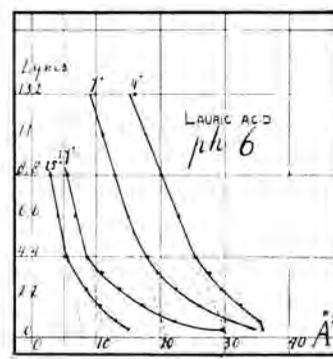


Fig. 6.

The following figures show how the mono-molecular spreading of soaps can only be obtained on strongly acid water (fig. 7—12).

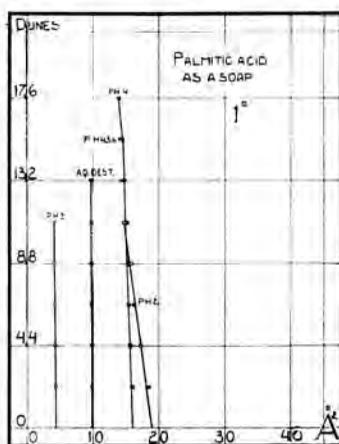


Fig. 7.

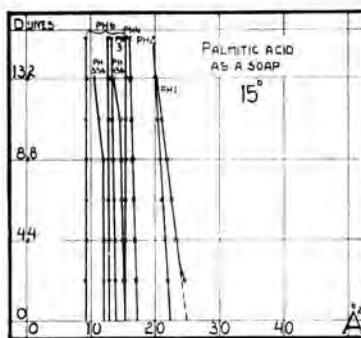


Fig. 8.

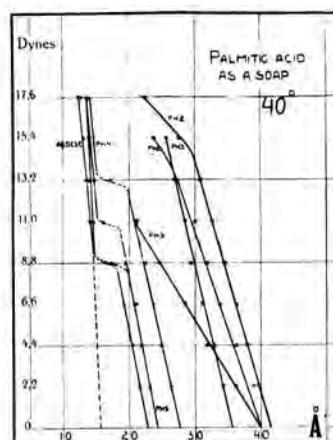


Fig. 9.

A very clear impression is given by plotting the PH of the water in the tray against the area occupied by the soaps (fig. 13 and 14).

We now come to the very important question as to the meaning of the small numbers encountered for the soap-spreading. We are inclined to ascribe these small areas to an association of the soap-molecules in some

form that persists on the surface of the water in the tray. By using strongly acid water, this association is broken, and on ordinary distilled

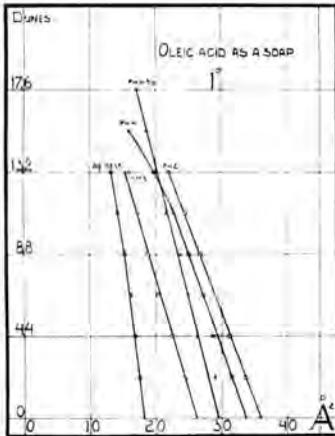


Fig. 10.

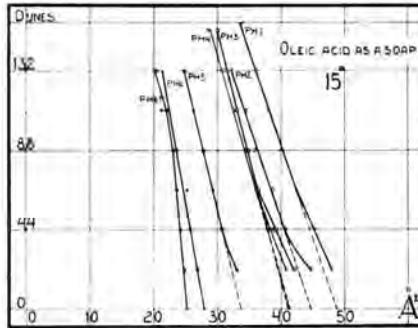


Fig. 11.

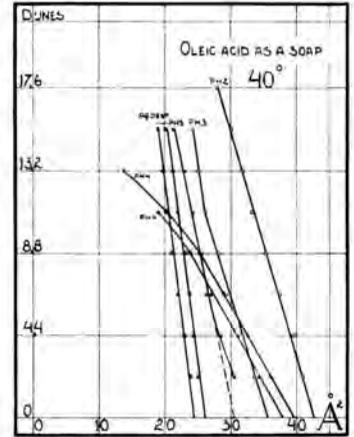


Fig. 12.

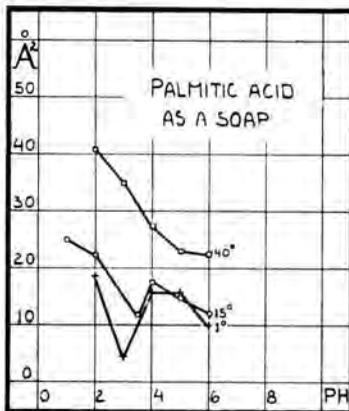


Fig. 13.

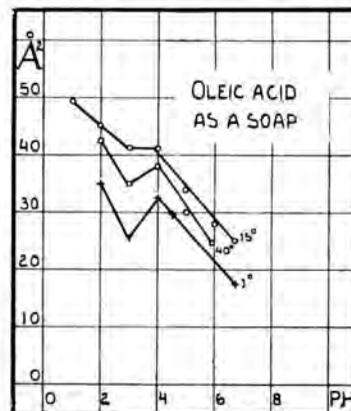


Fig. 14.

water or on a hydrochloric acid solution of a PH 3, the molecules remain associated.

In this line of thought it is possible to admit that the associated molecules turn the polar groups towards an outer surface and that the spreading of these aggregates is due to these superficially placed groups. In good agreement with this view is the fact that the smaller spreading often is $1/3$ of the mono-molecular one. This number is to be expected if the soap-molecules are placed with the carbon chains in touch of each other and the (broader) COONa-groups in the outside: so forming globes, whose outer surface is entirely constituted of COONa polar groups, thus rendering the particle soluble.

Another fact deserves mention in this respect that fatty acids dissolved

in alcohol and spread out on distilled water, often give values, that are far too small and again show a real mono-molecular spreading if blown out on strongly acid water. This is what might be expected from the boiling point determinations of such kind of substances, which are abnormal and are generally explained by admitting an association of molecules.

II.

On the spreading of fats.

Before describing the experiments with proteins, we wish to add a few words on the spreading of fats.

If we examine different representatives of the triglycerides of the saturated series, we find the same influence of the length of the chain and of the temperature on the area occupied at a water-surface whereas, however, the influence of a change in acidity of the water is of far less importance owing to the COOH groups being replaced by CO—OC groups.

In comparing the length of the chain, where the mono-molecular spreading begins, we find that already 6 C-atoms suffice to render the

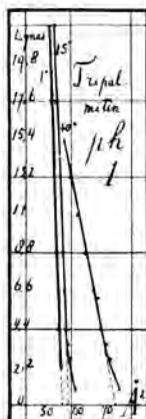


Fig. 15.

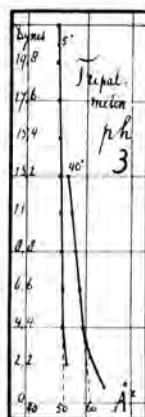


Fig. 16.

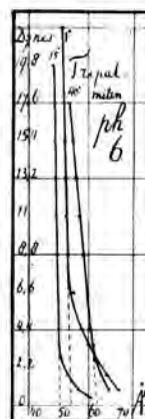


Fig. 17.

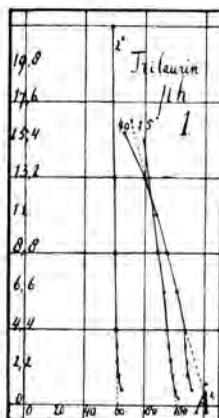


Fig. 18.

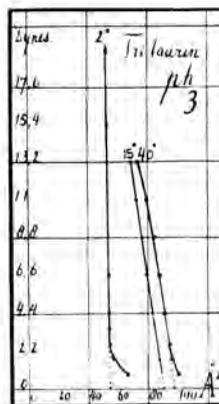


Fig. 19.

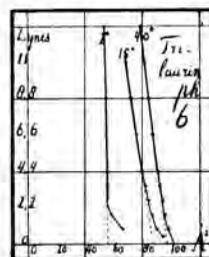


Fig. 20.

molecule sufficiently stable to enable a perfect spreading. This leads to the conclusion that the tendency to spreading is highly ameliorated by the linkage of three chains. We also detect the same difference in area and compressibility between the lower and the higher members of the series.

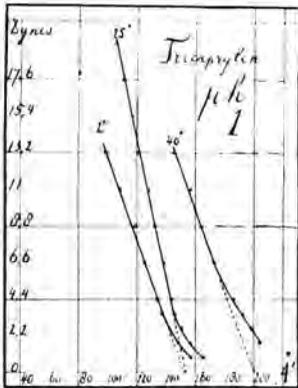


Fig. 21.

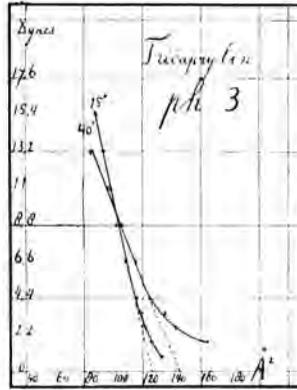


Fig. 22.

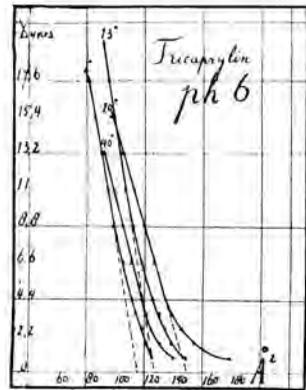


Fig. 23.

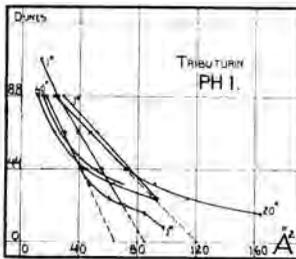


Fig. 24.

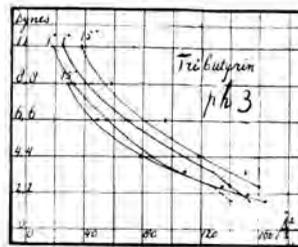


Fig. 25.

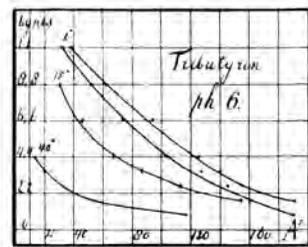


Fig. 26.

Tripalmitin (C 16) spreads in a condensed form even at a temperature of 40°, trilaurin (C 12) has a condensed form only at 1°, but is in an intermediary condition at 15°. Tricaprylin (C 8) already at 15° C. occupies an area of 120 sq. A.U. (60 sq. A.U. being the value for the spreading of a fat in a condensed state). Finally tributyrin only spreads at a low temperature, 1°, whereas at a higher temperature it spreads colloiddally or not at all. (Fig. 15—26.) Neither on the transition of the condensed to the expanded state, nor on the spreading itself, has the acidity of the water any important influence. Where it exists the influence is of the same order as with fatty acids: acidification of the water in the tray promotes the expansion and increases the area occupied on the watersurface.

III.

On the spreading of proteins.

The observations on the spreading of proteins are based on the results obtained in studying soaps and fatty acids. It has been shown that it is

possible to obtain a mono-molecular spreading even when using colloidal solutions, if only care is taken to use strongly acid water and to choose a suitable temperature.

The results were proven to be highly dependent on the length of the fatty acid chain: at a length of 14 or 16 C-atoms mono-molecular spreading easily occurs at any temperature. This temperature only exerts its influence on the mode of spreading, namely whether in a condensed or in an expanded form. With the lower members of the series (C 12) mono-molecular spreading is only seen at a higher acidity or at a lower temperature. Now in comparing fatty acids with fats having the same number of C-atoms, we found that spreading is far more easily arrived at as with fats, as already short-chained members give a perfect spreading. Moreover with the lower fatty acids (under C 10) and fats (under C 6) the numbers indicating the area occupied are too small and by increasing the temperature the area diminishes still more. This fact already indicates a solution.

Now we still believe that at a certain temperature and for a certain length of the chain it is possible to admit a colloidal mode of spreading, an expression used by us to indicate that associated molecules having their polar groups orientated to an outer surface can prefer to stay at the surface instead of disappearing in the water of the tray under the surface. They can remain at the surface even when conglomerated in globes in which case values for the area occupied must be found of about $\frac{1}{3}$ that of the mono-molecular spreading, as has actually been found.

Let us now study the results of the protein-spreading in the light of the foregoing experiments. It appears first of all, that most proteins can be spread on a hydrochloric acid solution of a PH 1 (or 2) in a layer whose thickness is 7 to 8 A.U. In this calculation the assumed value for the molecular weight drops out. We admit that this maximal spreading is a real mono-molecular one. One often succeeds in obtaining this maximal spreading on water of a lower acidity, if the temperature of the water in the tray is higher.

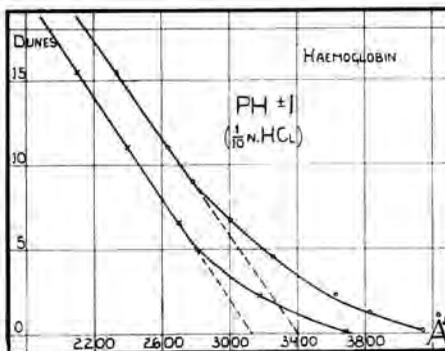


Fig. 27.

Haemoglobin (Mol. w. 16000)
Thickness at 3000 sq. A.U. 7 A. U.

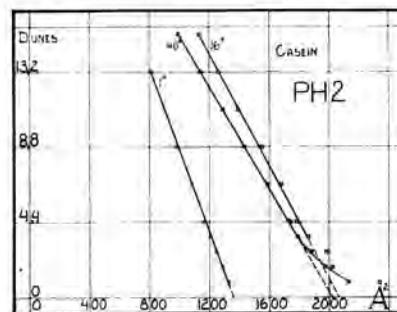


Fig. 28.

Casein (Mol. w. 12600)
Thickness at 2000 sq. A.U. 8 A.U.

We give some examples of this mono-molecular spreading. (Fig. 27, 28 and 29.)

Now it is impossible to obtain this maximal spreading in such thin films with every protein. We tend to admit that in analogy to what has been

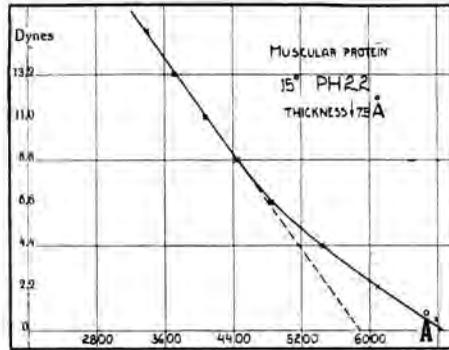


Fig. 29.
Serumprotein (Mol. w. 34000)
Thickness at 5800 sq. A.U. 7.6 A.U.

observed while studying fats, this different behavior of some proteins is due to a too small number of connected amino-acids or to a too great prevalence of short-chained amino-acids in the molecule. We give as examples *gelatin* and *gliadin*. (Fig. 30 and 31.)

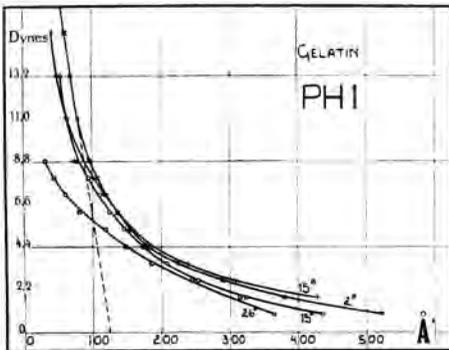


Fig. 30.
Gelatin (Mol. w. 10800)
Thickness at 450 sq.A.U.; 30 A.U.

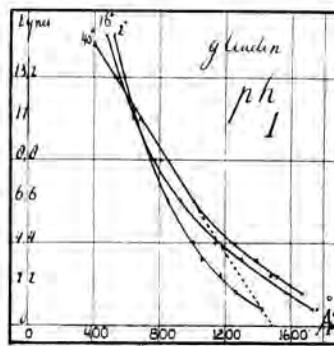


Fig. 31.
Gliadin (Mol. w. 12600)
Thickness at 1500 sq.A.U.; 10.5 A.U.

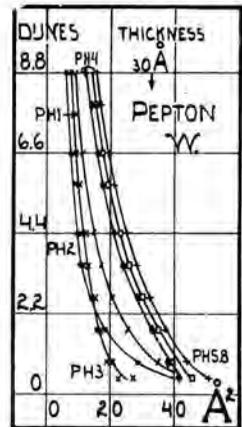


Fig. 32.
Peptone (Mol. w. 860)

Also *peptone* shows this same imperfect spreading. (Fig. 32.)

It is apparent that the spreading of gelatin is comparable to the spreading of tributyrin. They resemble us not only in so far that the size of the spreading is much too small, but also with regard to the form of the pressure-area curve, which is identical with both substances whereas heating has the same paradoxal influence to diminish the area by promoting the solution. In

good agreement with this explanation is the fact that gelatin has a considerable quantity (25%) of glycocoll, thus possessing too many short-chained (2 C-atoms) amino-acids in its molecule.

Perhaps the circumstance that it is necessary to heat the gelatin-solution before putting it in the surface, has an influence on the final result. In peptone we meet with an example of a substance where the imperfect spreading is the consequence of both factors: a too small number of connected amino-acids must have the greatest importance here. Gliadin seems in a certain sense an intermediary form.

Before leaving the subject of the mono-molecular spreading we wish to emphasize that this spreading is the consequence of the attraction towards the water of all polar groups principally the CO-NH groups. If we calculate what is the area on the average occupied by each amino-acid we obtain a value of 18 sq. A.U. in the case of such proteins as hemoglobin, or casein, which is exactly the expected value if all the CO-NH groups are orientated towards the water. Each carbon chain in a fat occupies an area of about the same value ± 21 sq. A.U.

We must add a few words on the remarkable fact, that according to the classical view the protein molecules are very large. For hemoglobin 16000 is too small a value, and from ADAIR's experiments 64000 seems the most probable value, so that each molecule would contain 4 Fe-atoms. For casein 34000 is the generally admitted value, but even here some experimental work seems to indicate, that a higher value is more probable. But even when we take the smallest number one is forced to consider the protein molecules — probably very plastic — as flat platelets or needles, when spread out on a water surface, whose thickness depends on the average length of the constituting amino-acids.

If now we examine the spreading of proteins at a different acidity or a lower temperature we encounter in the beginning considerable irregularity as to the results. Generally speaking we may say, that by diminishing the acidity of the water or by lowering temperature the area tends to diminish and hence the calculated thickness of the film is increased. Before getting any reproducible results it is however indispensable to take into

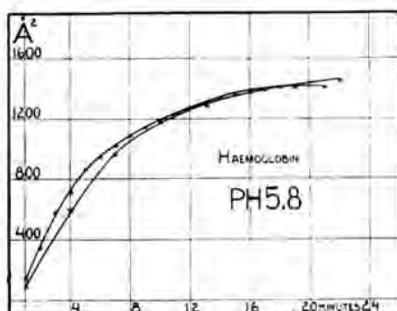


Fig. 33.

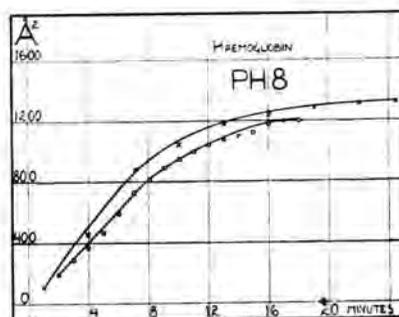


Fig. 34.

account the sometimes considerable influence of the time on the size of the area occupied.

This influence appeared most clearly in studying hemoglobin. (Fig. 33 and 34.)

But also in measurements of casein it is always necessary to reckon with the possibility of a slow reaching of the end-point of the spreading although it takes seldom more than 10 minutes, before the final definite spreading is obtained.

As a rule the mono-molecular maximal spreading either due to a higher temperature or to a higher acidity of the water in the tray, is almost immediate. This influence of time already indicates an association of molecules in the original solution.

We have very carefully studied the influence of acidity and temperature on the spreading of very pure casein (prepared in the laboratory of Prof. KRUIJT). We give the following figures expressing the very remarkable influence of change in acidity on the spreading of casein at different temperatures.

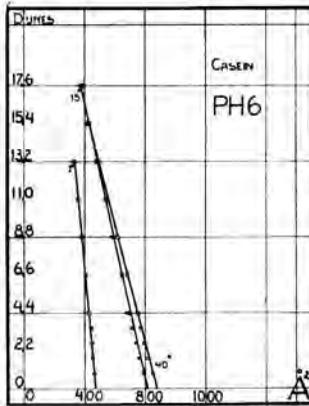


Fig. 35.

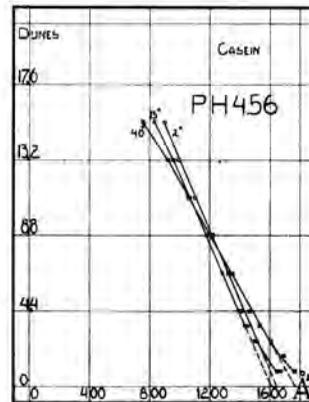


Fig. 36.

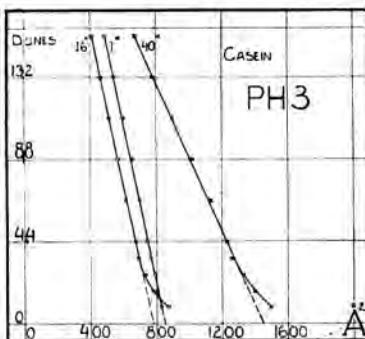


Fig. 37.

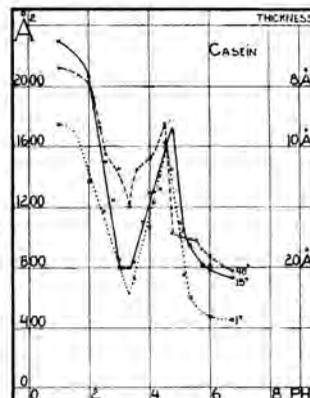


Fig. 38.

To give an idea of the compressibility of the protein films and the slope of the pressure-area curves under influence of temperature and acidity we give these experiments. The compressibility is of the same order of magnitude as of the „expanded” films of fatty acids (18—24 dynes). This fact does not imply, that proteins are considered as „expanded”, because it is difficult to decide this point.

With regard to the figure indicating the influence of PH variations on the size of the protein-molecule, we wish to remark, that both casein and hemoglobin show two minima on both sides of the iso-electric point. Both proteins were dissolved with caustic soda and are therefore exactly comparable to the soap-solutions described above. We will show that the same explanations hold good. In the casein-sol several molecules are associated but in such a manner that they turn their polar groups to the outside. If we bring a very small amount of this casein-sol on freshly boiled distilled water this association remains unaltered. But if we spread the same casein-sol on more acid water, the casein is gradually set free, till at the iso-electric point the casein spreads in a mono-molecular layer. By further adding acid to the water, we obtain a change into strongly associated casein hydrochloride-molecules, who spread only in a mono-molecular layer by considerable increasing the acidity of the water.

In certain experiments almost fortunately, with hemoglobine or with casein, muscle protein or serum protein we find a much smaller area per molecule, probable due to a higher degree of association. Only at a definite PH of the water and at a low temperature this small spreading is somewhat stabile. In most of the experiments in the beginning this small area is seen, but increases gradually when time is allowed to interfere. In those cases one is reminded the experiment of *Cary and Rideal*, who found the same mode of very slow spreading by making use of a crystal of stearic acid. We believe that this minimal spreading, where each molecule occupies an area of 100 sq. A.U. must be ascribed in the case of hemoglobin to an orientation of the molecules in such a manner that the prosthetic group is turned to the surface, and that the CO-NH groups of the different molecules lay side on side perhaps separated by a layer of water molecules. For the spreading of haematin itself gives a value of about 72 sq. A.U.

To increase the analogy between fats and proteins we add a few results obtained with lower members of the series of saturated fats. By X ray experiments we know that the transverse dimension of a fatty acid chain must be 4 A.U. Now some values obtained with tricaproin and tricapyrin are exactly those which would be expected by admitting a horizontal position of these fats on the water.

Palæontology. "*Figures of the Femur of Pithecanthropus Erectus*".
(Plates I to IV). By Prof. EUG. DUBOIS.

(Communicated at the meeting of June 26, 1926).

The photographic figures 1 tot 6 and the röntgenograms, figures 8 and 9, of the femur of *Pithecanthropus erectus*, with the photographic figure 7 of a human femur for comparison, (Pl. I to IV) herewith published, serve to illustrate "the Principal Characters of the Femur of *Pithecanthropus Erectus*". (Proceedings Vol. 29, p. 630—743), and conclude the illustration of all the fossil remains of *Pithecanthropus* hitherto discovered.

In the same way as the before published figures of the calvarium etc. ¹⁾ these photos natural size were taken with the aid of the stereorthoscope (Proceedings Vol. 27, p. 459), thus exactly according to certain planes, and with a lens of 3 m focal distance, consequently with the object placed at 6 m from the lens, in which way figures were obtained approaching geometrical projections.

The two röntgenograms, one of the upper end, the other of the lower end of the femur were taken by the kind help of the late distinguished röntgenologist, Professor J. K. A. WERTHEIM SALOMONSON.

EXPLANATION OF THE PLATES.

PLATE I.

Fig. 1. — (Left) Femur of *Pithecanthropus erectus*, front view. Plane of figure the tangent plane of the most prominent back parts of the bone.

Of points not mentioned in "The Principal characters of the Femur", the figure shows, in the upper two third part of the shaft, a certain roughness caused by adherent small pyritoid concretions, moreover, as consequences of the excavating work, some superficial loss of small flakes and a somewhat larger one at the inner side of the lower end (about 9 cm from below). Of the anterior extremity of the condylus medialis a small fragment of bone got lost, exposing the spongiosa; this part appears smooth in the figure, as a consequence of the preparation for making the plaster cast. Conspicuous is on the epicondylus lateralis a flat eminence for insertion of the lower part of the intermuscular septum behind the vastus lateralis and crureus.

Fig. 2. — (Left) femur of *Pithecanthropus erectus*, inner view. At right-angles to Fig. 1. Particulars mostly mentioned in "The Principal Characters of the Femur". The rounded

¹⁾ "Figures of the Calvarium and Endocranial Cast, a Fragment of the Mandible and three Teeth of *Pithecanthropus Erectus*", Proceedings Vol. 27, p. 459—464. (Plates I—XI).

form of the shaft (complete absence of an *angulus medialis*) is especially noteworthy. The lower arterial foramen (in upward direction) at 19 cm above the edge of the *condylus medialis*. This edge is incomplete anteriorly through the loss of the mentioned fragment.

Fig. 3. — (Left) femur of *Pithecanthropus erectus*, outer view. At right-angles to Fig. 1.

The peculiar arching of the shaft from above downwards in both figures, 2 and 3, clearly recognizable. Also the flattening of the posterior surface above the middle of the shaft. In this aspect also the *trochanter major* appears with a form different from the human *trochanter*. The strong eminence for the insertion of the lower end of the lateral inter-muscular septum is a prominent feature in the distal extremity of the bone. The apparent roughness of the shaft in its upper two thirds is still more conspicuous than in Fig. 1.

PLATE II.

Fig. 4. — (Left) femur of *Pithecanthropus erectus*, from behind. Plane of figure the tangent plane of the most prominent back parts of the bone.

The great exostosis is seen in its full extension, clearly also the well marked *linea aspera* and the strong external supracondylar line, the total absence of a real internal supracondylar line. At the inner side of the former, clearly separated, the median buttress of the *planum popliteum*.

Fig. 5. — (Left) femur of *Pithecanthropus erectus*. View from above. The condyli below tangent to the plane of figure.

Shows the „angle of torsion” and the transversal inclination.

Fig. 6. — (Left) femur of *Pithecanthropus erectus*. View from below. The condyli from above tangent to the plane of figure.

Shows the shape of the articular surface in that aspect, the prominence of the lateral lip of the patellar surface, and the transversal inclination.

Fig. 7. — Right femur of a man, with similar exostosis. The exostosis was found by SCHWALBE in the anatomical theatre of Strassburg¹⁾. It seems that the man, a comedian who died at 50 years of age, never knew that his leg was abnormal. The dissection proved entirely sound surroundings of the exostosis and normal muscles. It appears that it does not in the least affect the normal forms of the other parts of the bone. The exostosis is certainly of the same character as that of the *Pithecanthropus* femur, also as regards place and extension. Only in the *Pithecanthropus* femur its upper part is much heavier and in the Strassburg-femur the lower portion more extended. Certainly SCHWALBE was right in remarking that the upper part of the Strassburg-exostosis belongs to the extensive connective tissue between the *vastus medialis* and the adductor muscles.

PLATE III.

Fig. 8. — Röntgenogram of the upper end of the (left) femur of *Pithecanthropus erectus*, from behind.

Although darkened in a large degree by the filling of many of the bone cavities with calcite and pyrite, the general character of the trajectoria is clearly recognizable. This is human. The static trajectorium of the erect attitude of WALKHOFF, ascending in oblique direction from the compact wall at the internal angle between the shaft and the neck and penetrating the head of the femur to arrive at the articular surface, is unmistakably present. Also the great arching trajectorium, the „Zug”trajectorium of WOLFF, which, starting from the *trochanter* side and arching upwards and inwards to the inferior part of the articular surface of the head, crosses the former trajectorium at right-angles. WALKHOFF attributes it to the working of the pelvis-fixing muscles. Moreover the meshes are narrow and the lamellae longitudinal as in Man; no large, rounded meshes as in the Anthropoid Apes.

¹⁾ See note in „the Principal Characters of the Femur of *Pithecanthropus Erectus*”, p. 734.



Fig. 2

Fig. 1

Fig. 3

EUG. DUBOIS: "FIGURES OF THE FEMUR OF PITHECANTHROPUS ERECTUS"



Fig. 4



Fig. 7



Fig. 5

Fig. 6

EUG. DUBOIS: "FIGURES OF THE FEMUR OF PITHECANTHROPUS ERECTUS"

Heliotypie v. Leer & Co., Amsterdam

EUG. DUBOIS: "FIGURES OF THE FEMUR OF PITHECANTHROPUS ERECTUS"



Fig. 8

EUG. DUBOIS: "FIGURES OF THE FEMUR OF PITHECANTHROPUS ERECTUS"



Fig. 9

PLATE IV.

Fig. 9. — Röntgenogram of the lower end of the (left) femur of *Pithecanthropus erectus*, from before.

Shows the narrow-meshedness, the vertical rectilinear lamellae ascending from the medial and the lateral condylus, and the lying down time-glassform inferior and transversal superior trajectory of Man.

Certainly the femur of *Pithecanthropus erectus*, if not entirely constructed for the same locomotion as that of Man, was adapted for an erect gait and probably only a deviating locomotion for which the femur of Man is also fit, although not without artificial aid to the rest of his frame.



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Physics. — P. EHRENFEST and G. E. UHLENBECK: "On the connection of different methods of solution of the wave equation in multi-dimensional spaces".

(Communicated at the meeting of April 24, 1926).

Introduction.

The general equation for wave motion in space under the influence of an external force $k(t, x y z)$:

$$\frac{1}{c^2} \frac{\partial^2 s}{\partial t^2} - \Delta s = k(t, xyz)$$

has, when for $t = -\infty$ everything is at rest $\left(s_{,r=-\infty} = \left(\frac{\partial s}{\partial t} \right)_{,r=-\infty} = 0 \right)$, the well known solution, the retarded potential:

$$s = \frac{1}{4\pi} \int \int \int_{-\infty}^{+\infty} \frac{k(t - \frac{r}{c}, \xi \eta \zeta)}{r} d\xi d\eta d\zeta.$$

The peculiarities of this solution, especially its close connection to the three dimensional space, become apparent when we consider the analogous problem for the wave equation in multidimensional spaces:

$$\frac{\partial^2 s}{\partial t^2} - \sum_{h=1}^n \frac{\partial^2 s}{\partial x_h^2} = k(t, x_1 \dots x_n) \dots \dots \dots (1)$$

Various methods are available, but they all give the solution in *very different* analytical forms, whose identity unlike the case with three dimensions is not at once clear. In the following we will endeavour to show the connection and the identity of these solutions. ¹⁾ The difference between spaces with an even and an odd number of dimensions will be especially brought to the front. ²⁾

§ 1. *Method of HERGLOTZ* ³⁾.

Putting $t = i\omega$, in (1), then the latter is transformed to the equation of POISSON in $n + 1$ dimensions. Now the use of the known solution for this, suggests in our case the trial of:

$$s(t, x_1 \dots x_n) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{1}{2}n-\frac{1}{2}}} \int \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_n \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{k(t-\vartheta, \xi_1 \dots \xi_n)}{(r^2-\vartheta^2)^{\frac{n-1}{2}}} d\vartheta \dots (2)$$

as a solution. By transforming the integration path of the last integral, we can write it without complex variables. Here enters the difference between spaces with even and odd numbers of dimensions.

a. When n is odd (put $n = 2p + 3$), we transform the path of integration to a small circle round one of the poles $\vartheta = +r$ or $\vartheta = -r$ of the integrand, and obtain with the help of the theorem of residues the solution represented by a "retarded" or an "advanced" potential respectively. In the first case for example:

$$s(t, x_1 \dots x_{2p+3}) = \frac{\Gamma(p+1)}{2\pi^{p+1}} \int_{-\infty}^{+\infty} \dots \int d\xi_1 \dots d\xi_{2p+3} H_{2p+3} \dots \quad (3)$$

is obtained, where:

$$H_{2p+3} = \frac{1}{p!} \sum_{l=0}^p \frac{(2p-l)!}{l!(p-l)!} \cdot \frac{1}{2^{2p-l+1}} \frac{k^{(l)}(t-r, \xi_1 \dots \xi_n)}{r^{2p-l+1}} \dots \quad (4)$$

b. When n is even (put $n = 2q$), we can no longer close the integration path, as $\vartheta = \pm r$ are now also branching points, and we can only transform it to a loop round $\vartheta = +r$ or $\vartheta = -r$. The real expressions can now be written with the help of the so called "partie finie" of an infinite integral, as defined by HADAMARD.⁴⁾ We obtain:

$$s(t, x_1 \dots x_{2q}) = \frac{\Gamma\left(q - \frac{1}{2}\right)}{2\pi^{q-\frac{1}{2}}} \int_{-\infty}^{+\infty} \dots \int d\xi_1 \dots d\xi_{2q} H_{2q} \dots \quad (5)$$

where

$$H_{2q} = \frac{(-1)^q}{\pi} \left[\int_r^\infty d\vartheta \frac{k(t-\vartheta, \xi_1 \dots \xi_{2q})}{(\vartheta^2 - r^2)^{q-\frac{1}{2}}} \dots \right] \quad (6)$$

the bracket indicating the "partie finie". This becomes then in ordinary symbols:

$$H_{2q} = \frac{(-1)^q}{\pi} \int_r^\infty \frac{f(\vartheta) - \left[f(r) + (\vartheta-r)f'(r) + \dots + \frac{(\vartheta-r)^{q-2}}{(q-2)!} f^{(q-2)}(r) \right]}{(\vartheta-r)^{q-\frac{1}{2}}} \quad (7)$$

where:

$$f(\vartheta) = \frac{k(t-\vartheta)}{(\vartheta+r)^{q-\frac{1}{2}}}$$

Remarks:

1. If the solution for a certain n is known, then the solution for $n - 1$ can always be found by the consideration of the cylindrical problem (the so called "methode de la descente" of HADAMARD). Formula (6) can be derived in this way from the solution (3).

2. The application of this method to (2) gives the solution in one dimension lower again in exactly the same form.

3. Still (2) and the transformations of the integration path indicated under a and b must only be considered as a heuristic method of arriving at the real expressions (3) and (5), which as can be shown are really

the solutions of our problem. It is difficult to give the exact meaning of (2) and the justification of the transformations.

§ 2. "Polarisationmethod".⁵⁾

Physically the solution of our problem can be represented as the superposition of all the „spherical waves“ excited in the various points of the phase space by the „force“ $k(t, x_1 \dots x_n)$. It can be easily shown⁵⁾ that, in analogy to $\frac{\Phi(t-r)}{r}$ for three dimensions, a solution of the equation for spherical waves in $(2p+3)$ dimensions is given by :

$$u_{2p+3} = \left(\frac{1}{r} \frac{\delta}{\delta r}\right)^p \frac{\Phi(t-r)}{r} \dots \dots \dots (8)$$

which at once suggests as a solution :

$$s(t, x_1 \dots x_{2p+3}) = A_{2p+3} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_{2p+3} \left(\frac{1}{r} \frac{\delta}{\delta r}\right)^p \frac{k(t-r, \xi_1 \dots \xi_{2p+3})}{r} \quad (9)$$

Here A_{2p+3} is a constant. The solution is essentially limited to an odd number of dimensions.

The identity of (9) and (3) can be shown as follows:

Let:
$$P_{2p+3} = \left(\frac{1}{r} \frac{\delta}{\delta r}\right)^p \frac{k(t-r)}{r} \dots \dots \dots (10)$$

From the definition of H_n :

$$H_3 = \frac{1}{2\pi i} \int d\vartheta \frac{k(t-\vartheta)}{r^2 - \vartheta^2} = \frac{1}{2} \frac{k(t-r)}{r} = \frac{1}{2} P_3$$

and therefore

$$\left. \begin{aligned} P_{2p+3} &= \left(\frac{1}{r} \frac{\delta}{\delta r}\right)^p \frac{k(t-r)}{r} = \left(\frac{1}{r} \frac{\delta}{\delta r}\right)^p (2H_3) = \\ &= \frac{1}{\pi i} \int d\vartheta \frac{k(t-\vartheta)}{(r^2 - \vartheta^2)^{p+1}} 2^p (-1)^p p! = (-1)^p 2^{p+1} p! H_{2p+3} \end{aligned} \right\} \quad (11)$$

Equation (11) can also be derived by direct differentiation, H_{2p+3} then being given by (4). From (3) and (11) then follows the constant in (9):

$$A_{2p+3} = \frac{(-1)^p}{2^{p+2} \pi^{p+1}} \dots \dots \dots (12)$$

§ 3. Method of RIEMANN-HADAMARD⁶⁾.

This method starts from the identity of GREEN, and is therefore in many ways analogous to the usual method of solving POISSON's equation. Hence in our case it is actually not very different from the method of HERGLOTZ, except that now everything is kept real from the beginning.

HADAMARD, however, has applied it to much more general hyperbolic equations and boundary conditions. Specialising his results to our problem then:

a) for an even number of dimensions ($n = 2q$)⁷⁾:

$$s(t, x_1 \dots x_{2q}) = \frac{(-1)^{q+1} \Gamma(q-\frac{1}{2})}{2\pi^q + \frac{1}{2}} \sqrt{\int_T \dots \int_T \frac{1}{\Gamma^{q-\frac{1}{2}}} k(t, \xi_1 \dots \xi_{2q}) d\xi_1 \dots d\xi_{2q} dt} \quad (13)$$

where:

$$\Gamma \equiv (t-\tau)^2 - \sum_1^{2q} (x_h - \xi_h)^2 = (t-\tau)^2 - r^2$$

and T is the cone $\Gamma = 0$. Putting $t-\tau = \vartheta$, we get (5) and (6), i. e. the solution of HERGLOTZ for an even number of dimensions.

b) for an odd number of dimensions ($n = 2p + 3$)⁸⁾:

$$s(t, x_1 \dots x_{2p+3}) = \frac{(-1)^{p+1}}{2\pi^{p+1}} \lim_{\gamma \rightarrow 0} \left[\frac{d^p}{d\gamma^p} \int \dots \int_{\sigma} k(t, \xi_1 \dots \xi_{2p+3}) d\sigma \right] \quad (14)$$

where σ represents the surface of the hyperboloid:

$$\Gamma \equiv (t-\tau)^2 - r^2 = \gamma$$

and the element $d\sigma$ thereof is defined by:

$$d\sigma d\Gamma = d\xi_1 \dots d\xi_{2p+3} dt.$$

Hence:

$$\begin{aligned} d\sigma &= - \frac{d\xi_1 \dots d\xi_{2p+3}}{2(t-\tau)} = - \frac{d\xi_1 \dots d\xi_{2p+3}}{2\sqrt{r^2 + \gamma}} \\ \tau &= t - \sqrt{r^2 + \gamma} \\ s(t, x_1 \dots x_{2p+3}) &= \frac{(-1)^p}{4\pi^{p+1}} \lim_{\gamma \rightarrow 0} \left[\frac{d^p}{d\gamma^p} \int \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_{2p+3} \frac{k(t - \sqrt{r^2 + \gamma})}{\sqrt{r^2 + \gamma}} \right] = \\ &= \frac{(-1)^p}{2^{p+2} \pi^{p+1}} \lim_{\gamma \rightarrow 0} \left(\frac{\delta}{r\delta r} \right)^p \int \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_{2p+3} \frac{k(t - \sqrt{r^2 + \gamma})}{\sqrt{r^2 + \gamma}} \\ &= \frac{(-1)^p}{2^{p+2} \pi^{p+1}} \int \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_{2p+3} \left(\frac{\delta}{r\delta r} \right)^p \frac{k(t-r)}{r}. \end{aligned}$$

We thus get the polarisation solution (9).

§ 4. Method of FOURIER-POISSON⁹⁾.

From the identity of FOURIER, we have:

$$k(t, x_1 \dots x_n) = \frac{1}{(2\pi)^n} \int \dots \int_{-\infty}^{+\infty} dv_1 \dots dv_n \int \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_n e^{i\sum v_h(x_h - \xi_h)} k(t, \xi_1 \dots \xi_n). \quad (15)$$

Similarly we now put for the solution :

$$s(t, x_1 \dots x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dv_1 \dots dv_n \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_n e^{i \sum v_h (x_h - \xi_h)} H(t, \xi_1 \dots \xi_n, v_1 \dots v_n) \quad (16)$$

where $H(t, \dots \xi_1 \dots \xi_n, v_1 \dots v_n)$ is an *unknown* function. Substituting (15) and (16) in (1), we get for H the equation :

$$\frac{\delta^2 H}{\delta t^2} + v^2 H = k(t, \xi_1 \dots \xi_n) \dots \dots \dots (17)$$

where $v^2 = \sum_1^n v_h^2$. With the limiting values $H_{t=-\infty} = \left(\frac{\partial H}{\partial t}\right)_{t=-\infty} = 0$, we get:

$$H = \int_{-\infty}^t dt \frac{\sin v(t-\tau)}{v} k(\tau, \xi_1 \dots \xi_n).$$

as the solution of (17).

This in (16) gives the required solution. If we introduce polar coordinates into the v -space, then we can easily perform all the integrations except those over the radiusvector v and one angle γ [defined by $\cos \gamma = \sigma = \frac{1}{vt} \sum v_h (x_h - \xi_h)$]. Also putting $t - \tau = \vartheta$, then :

$$s(t, x_1 \dots x_n) = \frac{n-1}{2^{n-1} \pi^{\frac{1}{2}n + \frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right)} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d\xi_1 \dots d\xi_n F_n \quad (18)$$

where :

$$F_n = \int_0^\infty dv v^{n-2} \int_0^\infty d\vartheta \sin v\vartheta k(t-\vartheta, \xi_1 \dots \xi_n) \int_0^1 d\sigma (1-\sigma^2)^{\frac{n-3}{2}} \cos v r \sigma \quad (19)$$

For an *odd* number of dimensions the identity of (18) with (3) and (4) can be shown by direct calculation of (19). For optional values of n the demonstration of the identity with (2) can also be given (perhaps not mathematically irreproachable *) as follows: The comparison of (2) with (18) and (19) shows, that we must demonstrate :

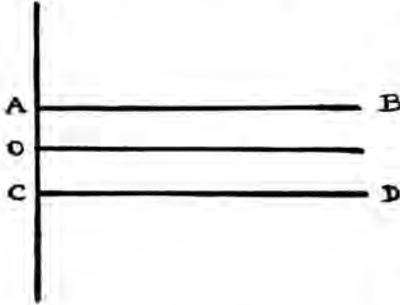
$$\left. \begin{aligned} & \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{k(t-\vartheta)}{(t^2-\vartheta^2)^{\frac{n-1}{2}}} d\vartheta = \\ & = \frac{1}{2^{n-3} \pi \Gamma^2\left(\frac{n-1}{2}\right)} \int_0^\infty dv v^{n-2} \int_0^\infty d\vartheta \sin v\vartheta k(t-\vartheta) \int_0^1 d\sigma (1-\sigma^2)^{\frac{n-3}{2}} \cos v r \sigma \end{aligned} \right\} \quad (20)$$

*) In the transformations of the integration paths we have the same incertainties as in § 1. (See remark 3).

We start from the identity ($\alpha > 0$):

$$\int_0^\infty dv v^{n-2} e^{-\alpha v} \int_0^1 d\sigma (1-\sigma^2)^{\frac{n-3}{2}} \cos v r \sigma = \frac{2^{n-3} \Gamma^2\left(\frac{n-1}{2}\right)}{(r^2 + \alpha^2)^{\frac{n-1}{2}}} \quad (21)$$

Now putting in (20) $\sin v \vartheta = \frac{1}{2i} (e^{i v \vartheta} - e^{-i v \vartheta})$, we can divide the right



hand side into two integrals. Instead of the interval $(0, \infty)$ we now take in the first integral for ϑ the path OAB , and in second the path OCD . Changing in each integral the integration over ϑ with that over v , then the resulting integrals are convergent and can be calculated by (21). Finally, by combining both integrals into one over the loop $DCAB$, (20) is demonstrated.

LITERATURE AND FURTHER REMARKS.

1. The analogue for the equation of heat conduction has recently been shown by M. C. GRAY (Proc. Edinb. Roy. Soc. XLV, 230, 1925).
2. The principle of HUYGENS is, as is well known, no longer valid in spaces of an even number of dimensions. See for two dimensions RAYLEIGH, The Theory of Sound (Sec. edition 1896) Vol. II, Ch. XIV, § 275, and for further differences: P. EHRENFEST: In what way does it become manifest in the fundamental laws of physics, that space has three dimensions? Proc. Acad. Amst. 20, 200, 1917.
3. HERGLOTZ: Gött. Nachr. p. 549, 1904.
4. HADAMARD: Acta Math. 31, 339, 1908; Lectures on Cauchy Problem (London, Humphrey Milford, 1923) p. 133.
5. Comp. J. COULON: Sur l'intégration des équations aux dérivées partielles par la méthode des caractéristiques (Paris, Hermann, 1902) p. 56.
6. RIEMANN: Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Ges. Werke p. 157.
KIRCHHOFF: Zur Theorie der Lichtstrahlen. Ges. Abh., Nachtrag No. 3, p. 22.
VOLTERRA: Atti Lincei 1892; Acta Math. 18, 214, 1894.
TEDONE: Annali di Matematica, Serie 3, T. 1 (1898) p. 1.
HADAMARD: Ann. Sc. de l'Ec. Norm. Sup. 21, 535, 1904; 22, 101, 1905.
Journ. de Phys. 1906 (survey); Acta Math. 31, 333, 1908.
- HADAMARD has given a survey of all these researches in his book: Lectures on Cauchy Problem in linear hyperbolic partial differential equations (London, Humphrey Milford, 1923). This is cited below.
7. Follows from form. (39), p. 166 l.c.
8. Follows from form. (28), p. 232 l.c., which is got by the „méthode de la descente“ from form. (39) p. 166.
9. For the papers of POISSON, see: H. BURKHARDT, Entwicklung nach oscillierenden Funktionen (Teubner, 1908) p. 606. Comp. further: RAYLEIGH, Theory of Sound (Sec. Edit. 1896), Ch. XIV. RIEMANN-HATTENDORF, Partielle Differentialgleichungen § 106.

Chemistry. — "*The Scattering Power of Lithium and Oxygen, Determined from the Diffraction-Intensities of Powdered Lithiumoxide*".

By J. M. BIJVOET, A. CLAASSEN and A. KARSSSEN. (Communicated by Prof. W. H. KEESOM).

(Communicated at the meeting of September 25, 1926).

1. Li_2O crystallizes in the Fluor-spar type. It is very fit for the determination of the scattering power F of the light oxygen particle as the structure contains no parameter crystallographically undetermined and the effect of the light atoms is not dominated here by heavy ones¹⁾. The fact that the scattering power of Lithium is found at the same time, has two more advantages.

First it enables us to test the intensity formula²⁾. When determining the scattering power a theoretical formula is used, which contains the reflecting power and some other factors changing according to the glancing-angle. In deducing the scattering power from the experimental intensities every error in the factors taken into account enters into the scattering power thus calculated³⁾. If an ionic-lattice were assumed, we can expect the scattering power of lithium to fall off only slightly with increasing glancing-angle, the distances of the electrons from the nucleus being small. In interpreting the intensities we found this dependence to agree with the theoretical estimation; this is considered a proof of the intensity formula used. There are but few substances adapted for the testing of the powder-formula. In the case of light atoms the dependence of the scattering power, in that of heavy atoms the absorption in the rod causes great uncertainty⁴⁾.

Secondly the simultaneous determination of F_{Li} has the advantage of giving a measure for the absolute value of the scattering power of oxygen

¹⁾ J. M. BIJVOET and A. KARSSSEN. Rec. Trav. Chim. **43**, 680 (1924).

²⁾ (Note added September 1926). Lately this test was performed by comparing the powder intensities of NaCl with the well known measurements on the BRAGG-method. The powder intensities were measured ionometrically by R. J. HAVIGHURST (Proc. Nat. Acad. Sc. **12**, 375 1926), photographic measurements being made by L. HARRIS, S. J. BATES and D. A. MAC INNES (Phys. Rev. **28**, 235 1926). The powder-formula was found to hold good. Also c.f. A. H. COMPTON and N. L. FREEWAN, Nature **110**, 38 (1922).

³⁾ J. M. BIJVOET, Werken van het Genootschap ter bev. van nat. genees- en heilkunde, XI, 512 (1926).

⁴⁾ We are investigating whether in the case of a heavy powder the intensities of a rotating rod agree with the calculated intensities, assuming that the substance be homogeneously distributed over the surface. This is essential for the question to which degree quantitative measurements may be used in ordinary structure determinations.

(i. e. the ratio of the amplitude diffracted by the particle to that diffracted by a free electron). In general extrapolation of scattering powers as far as $\vartheta=0$ is very uncertain because of their strong angle-dependence; for the *Li*-ion this extrapolation was possible, the dependence being found to be much smaller in this case. Putting $F_{Li_{\vartheta=0}}^+ = 2$, F_O can be expressed in these units ¹⁾).

2. The intensity formula reads

$$I = \int I_s d\vartheta = \frac{1 + \cos^2 \vartheta}{\sin^2 \frac{\vartheta}{2} \cos^2 \frac{\vartheta}{2}} \nu S^2 \quad 2)$$

Here I_s is the intensity for diffraction-angle ϑ ; the integration extends over the small domain in which reflection takes place. ν is the plane-number-factor, S the structure factor.

$\int I_s d\vartheta$ was measured photographically.

In general a factor must be added, which gives the influence of absorption in the rod. In our case — thin celluloid cylinders filled with loose powder of Li_2O , diameter 0.4—0.8 mm — calculation shows this factor to be constant within a few percents ³⁾. This, together with the accuracy of our measurements, permits to neglect the absorption. For the present the influence of the heat-motion is taken up in the F 's.

The sample being very finely powdered the extinction may be neglected ⁴⁾. This important advantage over the one-crystal method causes the examination of the applicability of the powder-method to quantitative measurements.

3. *Photographic intensity-measurements.*

The powder films were made with the apparatus described *altero loco* ⁵⁾, $Cu_{K\alpha}$ radiation was used, sometimes the β -radiation was filtered out by a *Ni*-filter.

The blackening of the photographic plate by homogeneous *X* rays was found proportional to the incident radiation if the blackening is less than 0.6 ⁶⁾. The blackening was measured with the fotometer of MOLL.

¹⁾ We are investigating whether the powder method may give the absolute values of the scattering power from photograms of the substance in known mixture with a substance measured absolutely (e. g. NaCl).

²⁾ P. DEBYE and P. SCHERRER. *Phys. Zs.* **19**, 474 (1918).

J. M. BIJVOET. *Rec. Trav. Chim.* **42**, 886 and 898 (1924).

³⁾ A. CLAASSEN, Dissertation, Amsterdam 1926.

⁴⁾ C. G. DARWIN. *Phil. Mag.* **43**, 800 (1922).

W. L. BRAGG, C. G. DARWIN and R. JAMES. *Phil. Mag.* (7) **1**, 897 (1926).

⁵⁾ J. M. BIJVOET and A. KARSEN, l.c.

⁶⁾ P. P. KOCH. *Ann. de Phys.* **45**, 392 (1924).

R. GLOCKER and W. TRAUB. *Phys. Zs.* **22**, 345 (1921).

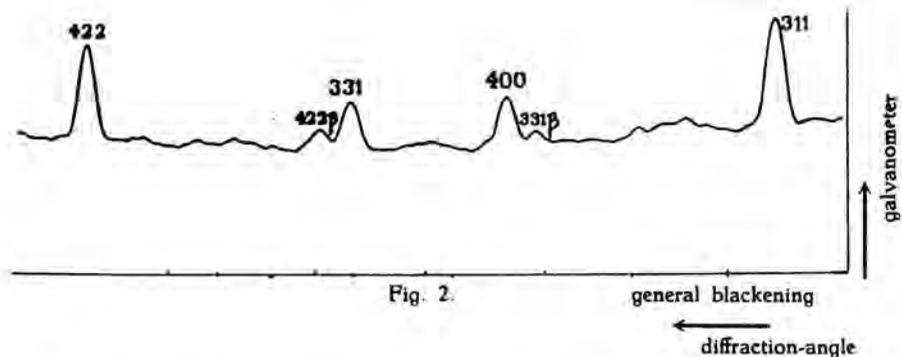
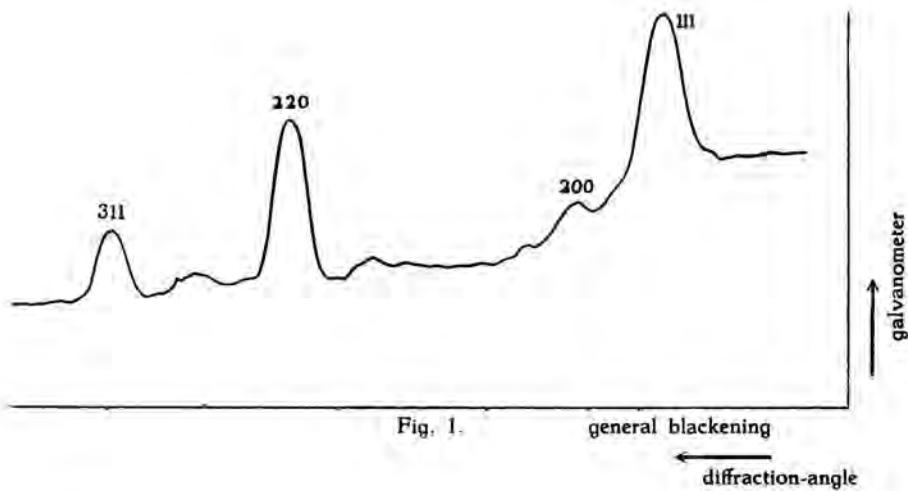
A. BROUWERS. *Z. f. Phys.* **14**, 374 (1923).

R. BLUNCK. *Ann. d. Phys.* **77**, 477 (1925).

The registered deviations of the galvanometer were transposed in terms of the blackening S , the quantity $\int S d\theta$ found by measuring the concerning area of the S -curve. In this way I of the intensity equation was found, S being proportional to I_g .

4. Accuracy of the measurements.

The fainter lines were measured on long exposed films, the lines 111 and 220 on shorter exposed ones. The photograms reproduced — of half size, transmission X -ray film to photogram 1:10 — are examples of two of these film-parts; on a few films all the lines were well measurable.



The average intensities of column 5 of table I were deduced from the photograms of the right and left halves of some ten films. The extreme values in column 5 show the reproducibility of the measurements. This is better than we expected. For 111 the discrepancies are large; this might have been caused by the great general blackening round this line or perhaps

by its coincidence with the strongest diffraction line of $LiOH$. Li_2O being hygroscopic the hydroxide might have been formed in some small quantities in preparing the rod. The other diffraction lines of $LiOH$ being absent, only very small quantities of $LiOH$ must have been formed; no other $LiOH$ -lines coincide with Li_2O lines but 111.

TABLE I.

| 1 | 2 | 3 | 4 | 5 | | 6 |
|---------------|----------|------------------------------|-----|----------|----------------|-----------|
| $h_1 h_2 h_3$ | S | $\sin^2 \frac{\vartheta}{2}$ | A | I | | S |
| | | | | average | extreme values | |
| 111 | O | 0.290 | 168 | 165 | 145—185 | 9.9 |
| 200 | O — 2 Li | 0.334 | 91 | ± 13 | | ± 3.8 |
| 220 | O + 2 Li | 0.472 | 80 | 100 | | 11.2 |
| 311 | O | 0.555 | 108 | 34 | 33—35 | 5.6 |
| 400 | O + 2 Li | 0.669 | 18 | 14 | 13—15 | 8.8 |
| 331 | O | 0.729 | 66 | 14 | 12—15 | 4.6 |
| 422 | O + 2 Li | 0.819 | 70 | 30 | 29—31 | 6.6 |
| { 333 511 | O | 0.870 | 109 | 16 | 16—17 | 3.8 |

Of the reflections with structure factor $2Li-O$ the line 200 was the only one which could be measured on a few films.

For most reflections the accuracy in S may be estimated at a few percents, as a great number of films was measured and the relative error in S amounts to half that in I ($S = \sqrt{I}$).

§ 5. Calculation of the scattering powers.

In table I column 1 contains the indices of the reflecting planes, column 2 the structure factor, column 3 the values of $\sin^2 \frac{\vartheta}{2}$, column 4 the values of $\nu \frac{1 + \cos^2 \vartheta}{\sin^2 \frac{\vartheta}{2} \cos^2 \frac{\vartheta}{2}} = A$, column 5 the intensities I measured photographically, column 6 values proportional to $\sqrt{I/A}$ or S . In fig. 3 the values of S are plotted against $\sin^2 \frac{\vartheta}{2}_{Cu}$.

Then from the curves F_O and $2F_{Li} + F_O$ that for F_{Li} is deduced. The extrapolated part is dotted in the fig. ¹⁾

In fig. 4 the curves for F_O and F_{Li} are reproduced once more, and here the value 2 is assigned to the ordinate of the intersection of latter curve with the axis.

¹⁾ This extrapolation assumes ions (see 6).

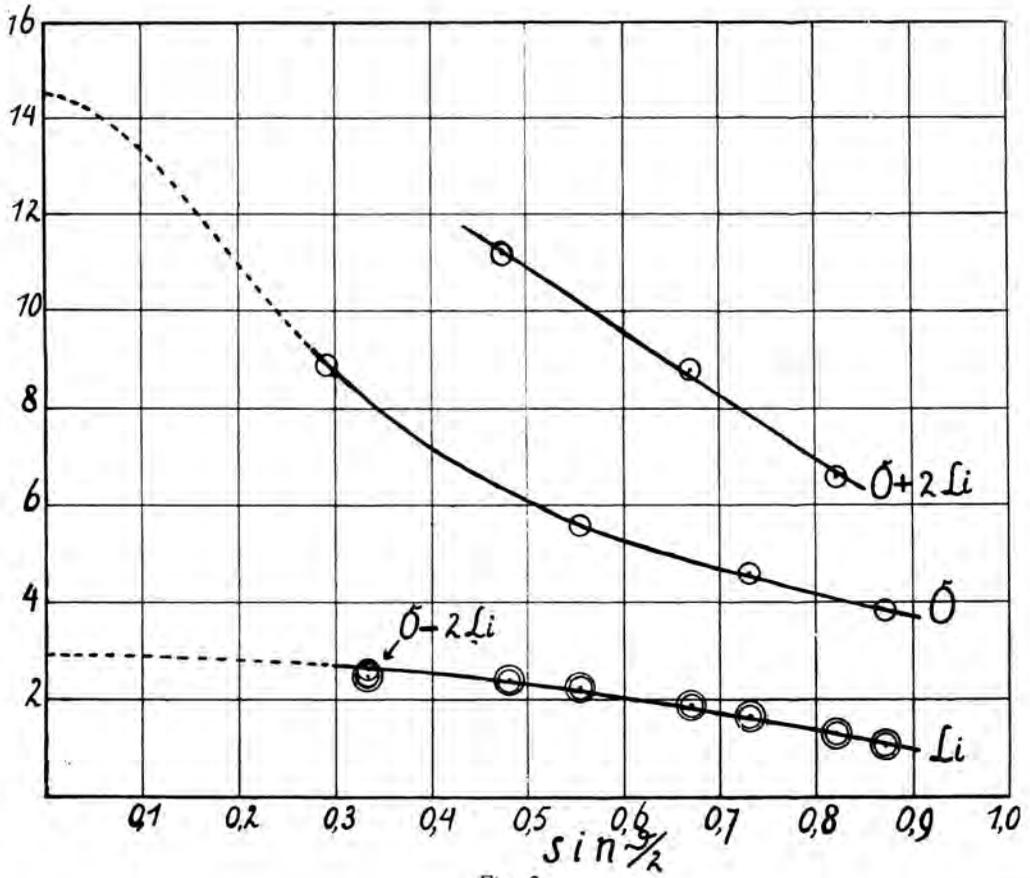


Fig. 3.

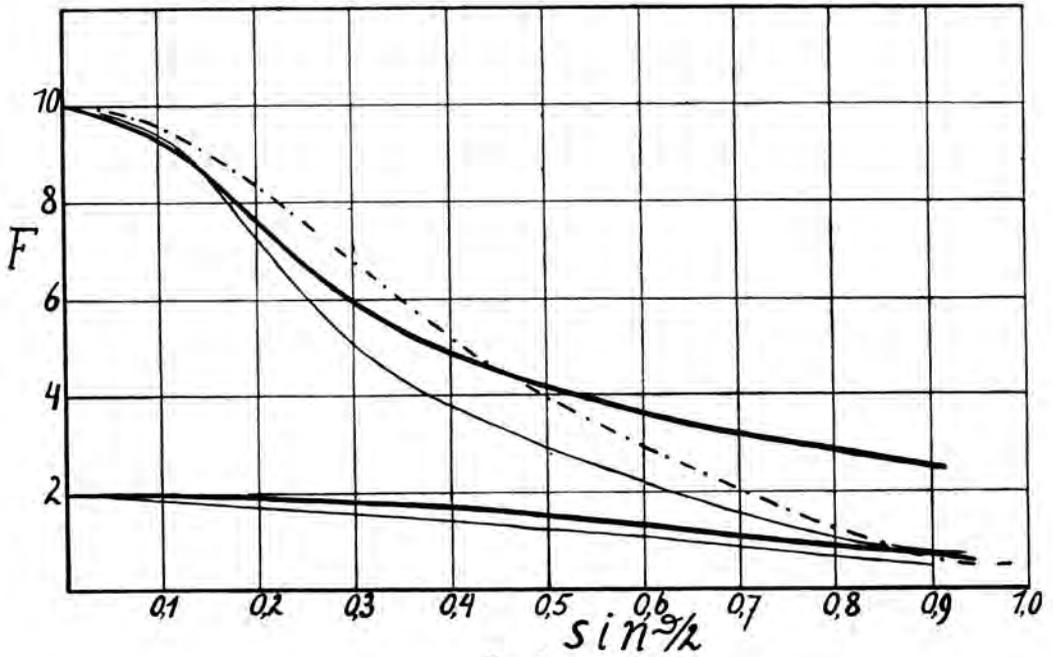


Fig. 4.

Table II gives some values of the scattering power of lithium and oxygen taken from fig. 4.

TABLE II.

| $\sin \vartheta/2$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|--------------------|----|------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| F_{Li} | 2 | 1,9 ⁵ | 1.9 | 1.8 | 1.7 | 1.5 | 1.3 | 1.1 | 0.9 | 0.7 |
| F_O | 10 | 9.3 | 7.6 | 6.0 | 4.9 | 4.2 | 3.7 | 3.2 | 2.9 | 2.6 |

6. Discussions of the scattering power of lithium and oxygen.

As stated above the slight angle-dependence we found in the case of F_{Li} confirms the intensity formula used.

The non-extrapolated part of the F curve of oxygen shows the following features:

1. The decline is so great as to make the assumption of constant scattering power in crystal analysis quite impermissible for light atoms ¹⁾.

2. The decline is much smaller than the one calculated for this domain by HARTREE, notwithstanding our F -values contain the temperature factor too. HARTREE's theoretical values, converted into Cu -radiation, give 3,3:0,9 for the ratio $F_{O_{311}}:F_{O_{333}}$; we found 3,9:2,7. On our basis of calculation the ratio $\frac{I_{311}}{I_{333}}$ should have been found some six times greater

to get HARTREE's decline. Even without quantitative measurements this must undoubtedly be rejected.

Extrapolation of the F -curve is possible without great inaccuracy in the case of lithium if ions be assumed. Then a decline as given in the figure is the only one possible. This corresponds with the calculated dependence if the distance of both electrons to the nucleus be about one half the radius of the one-quantum-orbit of hydrogen. This result seems to be plausible; more accurate calculation is not possible without surpassing the accuracy of the measurements, which also contain the effect of the heat motion. It appears fully possible that the F_O curve will end for $\vartheta=0$ at $F=10$.

In Fig. 4 the — — — line represents HARTREE's calculated values. As stated above, for $\sin \vartheta/2 > 0,4$ our decline is much smaller than the one calculated.

Should an atomic lattice be assumed, than the X -ray data do not seem to be able to reject this assumption with certainty. ²⁾

¹⁾ N. H. KOLKMEYER, J. M. BIJVOET and A. KARSEN. Z. f. Phys. **20**, 82 (1923).

²⁾ In our case one can easily ascertain that this is true. In the well-known example of LiF (P. DEBIJE and P. SCHERRER. Phys. Zs. **19**, 474 (1918)) too the deduction of ions being present is by no means conclusive.

7. *Comparison with other results.*

F -values for lithium can be derived from the photometrical data of DEBIJE and SCHERRER on LiF . The data of their Table III l.c. give a decline about equal to ours; their values are represented in Fig. 4 by a thin line.

On oxygen one of us made absolute measurements with Fe_3O_4 (magnetite) in the laboratory of Prof. W. L. BRAGG ¹⁾. A decline was found still greater than that calculated by HARTREE. In Fig. 4 this decline is given by the thin line. The difference between both experimental curves seems to indicate that the scattering power of a particle is strongly dependent on its environment. For the oxygen ion with its relatively great number of outer electrons this is not astonishing.

8. We intend to continue our measurements with different wavelengths, at low temperature, on some analogous compounds and mixtures of known composition.

9. *Summary.* The scattering power F of lithium and oxygen is determined by photographic intensity measurements on Li_2O powder with $CuK\alpha$ rays.

F_{Li} was found to agree with the results of DEBIJE and SCHERRER on LiF . In the observed region F_o appeared to be much less dependent on the diffraction-angle as found by CLAASSEN in Fe_3O_4 ; the calculated values of HARTREE for a free oxygen-ion lie between both sets.

We wish to express our hearty thanks to our amanuensis Mr. A. KREUGER for his aid in the experimental part of this investigation.

We are much indebted to Prof. SMITS for his kind interest.

Lab. of General and Inorganic Chemistry
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Amsterdam, July 1926.

¹⁾ A. CLAASSEN. Proc. Phys. Soc. London August 1926.

Chemistry. — “Equilibria in systems, in which phases, separated by a semipermeable membrane”. XVIII. By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of November 27, 1926).

Influence of the pressure on osmotic systems.

We take a system E_P (of one or more phases) under the pressure P . We bring this system, while its temperature and total composition remain constant, under the pressure $P + dP$. Then a new system E'_{P+dP} arises, in which the phases can differ a little in composition with those of the first system; this difference in composition is defined by the value of dP .

In order to compare the *O.W.A.* of the two systems:

$$E_P \qquad E'_{P+dP}$$

we represent the total thermodynamical potential of the first system by Z and that of the second system by Z' . We then have:

$$Z' = Z + VdP. \dots \dots \dots (1)$$

in which V represents the total volume of system E under the pressure P . The *O.W.A.* of the first system is defined by:

$$\varphi = \frac{\partial Z}{\partial w} \dots \dots \dots (2)$$

if viz. $\frac{\partial Z}{\partial w} \cdot \delta w$ represents the change of the thermodynamical potential, which the first system receives, if this takes in δw quantities of water. The *O.W.A.* of the second system now is:

$$\varphi' = \frac{\partial Z'}{\partial w} = \frac{\partial Z}{\partial w} + \frac{\partial V}{\partial w} \cdot dP \dots \dots \dots (3)$$

We now put:

$$\frac{\partial V}{\partial w} = \Delta V_E$$

Hence is apparent that $\Delta V_E \cdot \delta w$ represents the change, which gets the total volume V of system E , if this takes in δw quantities of water. If we put further $\varphi' = \varphi + d\varphi$ then (3) passes into:

$$d\varphi = \Delta V_E \cdot dP \dots \dots \dots (4)$$

by which the change of the *O.W.A.* is defined. We can also deduce (4) at once from (2) by differentiating this with respect to P .

As φ and the *O.W.A.* of a system change in opposite direction, we shall substitute in the following φ by $-\xi$; then the ξ and the *O.W.A.* of a system change in the same direction; then they become viz. at the same time larger or smaller. Consequently we may write instead of (2) and (4)

$$\xi = -\varphi = -\frac{\partial Z}{\partial w} \quad (5^a) \qquad d\xi = -d\varphi = -\Delta V_E \cdot dP \quad (5^b)$$

It follows from (5^b):
the *O.W.A.* of a system E becomes smaller on increase of pressure, if ΔV_E is positive, and larger if ΔV_E is negative.

If we indicate by an arrow the direction, in which the water diffuses, then we can express this in the following way:

$$\Delta V_E > 0 \qquad E_P \leftarrow E_{P+\Delta P} \quad \dots \quad (6^a)$$

$$\Delta V_E < 0 \qquad E_P \rightarrow E_{P+\Delta P} \quad \dots \quad (6^b)$$

in which E_P represents the system under the pressure P and $E_{P+\Delta P}$ this same system under the pressure $P + \Delta P$, and in which ΔP is positive.

If both the systems E_1 and E_2 have the same *O.W.A.* under the pressure P , then we have the osmotic equilibrium:

$$(E_1)_P \mid (E_2)_P \quad \dots \quad (7^a)$$

If we raise the pressure to $P + dP$ then both systems will have no more the same *O.W.A.*; that of the left system will increase with:

$$d\xi_1 = -\Delta V_{E_1} \cdot dP$$

and that of the right system with:

$$d\xi_2 = -\Delta V_{E_2} \cdot dP$$

If we take $\Delta V_{E_1} > \Delta V_{E_2}$ then, on increase of pressure, the *O.W.A.* of the right system becomes greater than that of the left system; consequently water will diffuse from left to right. We can represent this by:

$$\Delta V_{E_1} > \Delta V_{E_2} \qquad (E_1)_{P+dP} \rightarrow (E_2)_{P+dP} \quad \dots \quad (7^b)$$

in which ΔP is positive. Consequently we may say:

if we increase the pressure of an osmotic equilibrium, then the water diffuses in such direction that contraction of volume occurs.

We can deduce also in another way the influence of a change in pressure on the *O.W.A.* of a system. In previous communications (f.i. in Comm. VI) we have seen a.o. that the *O.W.A.* of a system E :

in which one or more liquids occur, is equal to the *O.W.A.* of each of those liquids:

in which by taking in — or losing a little water a liquid arises, is equal to the *O.W.A.* of this liquid ;

in which by taking in — or losing water a phases-reaction occurs, is equal to the *O.W.A.* of a liquid with which that system can be in equilibrium.

The same is true also for systems with a vapour or for those which can be in equilibrium with a vapour.

If we represent the composition of the liquid or vapour, above-mentioned, by :

$$xX + yY + zZ + \dots + (1-x-y-z\dots) W. \dots (8^a)$$

or f.i. by :

$$wW + xX + yY + \dots + (1-w-x-y\dots) Q \dots (8^b)$$

in which *X, Y* etc. (except *W*) may be components or composants, then the *O.W.A.* of the system *E* is, therefore, defined by :

$$\xi = -\varphi = -\zeta + x \frac{\partial \zeta}{\partial x} + y \frac{\partial \zeta}{\partial y} + \dots \dots \dots (9^a)$$

$$\xi = -\varphi = -\zeta - (1-w) \frac{\partial \zeta}{\partial w} + x \frac{\partial \zeta}{\partial x} + y \frac{\partial \zeta}{\partial y} \dots \dots (9^b)$$

If we bring the system *E* from the pressure *P* to *P + dP* then the *O.W.A.* of this system changes with :

$$d\xi = \frac{\partial \xi}{\partial P} \cdot dP + \frac{\partial \xi}{\partial x} \cdot dx + \frac{\partial \xi}{\partial y} \cdot dy + \dots \dots \dots (10)$$

It now depends on the number of freedoms of the system *E* whether *dx dy* etc. are completely defined by the change in pressure *dP*; if this is not the case, then we can put still different conditions, which must be satisfied by the changes of the system.

In order to apply those general considerations to some simple cases, we take the systems :

$$E = G (11^a) \qquad E = L \dots \dots \dots (11^b)$$

which consist each of one phase only; the change of the *O.W.A.* at a change of pressure is defined by (5^b). For system (11^a) $\Delta V_E \cdot \delta w$ now represents the increase of volume, which gets one quantity of vapour, if this takes in δw quantities of water; for system (11^b) it is the increase of volume if one quantity of liquid takes in δw quantities of water. This increase of volume is positive for vapours and also in general for liquids, excepted in the special case that the contraction occurring with the mixture would be still greater than the volume of the taken δw quantities of water. In the following, unless the opposite is precisely said, we shall take this increase of volume positive. We then find :

the O.W.A. of a vapour and that of a liquid becomes smaller on increase of pressure and larger on decrease of pressure.

In the osmotic systems:

$$G_P \leftarrow G_{P+\Delta P} \quad L_P \leftarrow L_{P+\Delta P} \quad \dots \quad (12^b)$$

the water diffuses, therefore, in the direction of the arrows [compare also system (6^a)].

In order to express the change in volume ΔV_E , mentioned above, we take a vapour G or liquid L of the composition:

$$x \text{ Mol } X + y \text{ Mol } Y + (1-x-y) \text{ Mol } W \quad \dots \quad (13)$$

with the volume V . If we mix this with δw quantities of water, then arise $1 + \delta w$ quantities of a new vapour or liquid, which differs infinitely little (dx and dy) in composition from the original. Consequently we have:

$$\Delta V_E \cdot \delta w = (1 + \delta w) \left(V + x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} \right) - V \quad \dots \quad (14)$$

For dx and dy we find:

$$dx = \frac{x}{1 + \delta w} - x = \frac{-x \cdot \delta w}{1 + \delta w} \quad dy = \frac{y}{1 + \delta w} - y = \frac{-y \cdot \delta w}{1 + \delta w} \quad (15)$$

If we substitute those values in (14), then follows:

$$\Delta V_E = V - x \frac{\partial V}{\partial x} - y \frac{\partial V}{\partial y} \quad \dots \quad (16)$$

the second part of which in the case of a vapour, which follows the gas-laws, passes into the volume V of that vapour.

We are able to deduce this also in the following way, The O.W.A. of a vapour or liquid (13) is defined by:

$$\xi = -\eta = -\zeta + x \frac{\partial \zeta}{\partial x} + y \frac{\partial \zeta}{\partial y} \quad \dots \quad (17)$$

Hence follows, in connection with the value of ΔV_E from (16):

$$d\xi = -\Delta V_E dP + (rx + sy) dx + (sx + ty) dy \quad \dots \quad (18)$$

If we keep constant the composition of this vapour or liquid, so that dx and dy are zero, then (18) passes into (5^b).

As we can give still arbitrary values to dx and dy in (18), we choose them in such a way, that the new system E' arises from E by taking in or losing a little water. We then have to put:

$$dx : x = dy : y = d\lambda \quad \dots \quad (19)$$

so that $d\lambda$ is positive if the liquid or vapour loses water. With the aid of (19) now (18) passes into:

$$d\xi = -V_E dP + (rx^2 + 2sxy + ty^2) d\lambda \quad \dots \quad (20)$$

We define the system E' in such a way that its *O.W.A.* under the pressure $P + dP$ is equal to that of system E under the pressure P . As then $d\xi$ must be zero, follows from (20)

$$d\lambda = \frac{V_E}{rx^2 + 2sxy + ty^2} dP \dots \dots \dots (21)$$

in which the denominator, as long as we consider stable states, is positive. As, therefore, $d\lambda$ and dP have the same sign, it follows :

if we increase the pressure of a liquid or vapour with a definite amount dP , then this must lose a definite quantity of water, in order to keep the same *O.W.A.*

We take a liquid c of fig. 1 under the pressure P ; all liquids, which have under this same pressure the same *O.W.A.* as this liquid L_c are situated on the isotonic curve acb going through point c . If we take the same liquid L_c under a pressure $P + dP$, then all liquids which have the same *O.W.A.* under this pressure $P + dP$ as L_c , will be situated also on an isotonic curve a_1cb_1 going through point c , which curve, as follows from the previous considerations, does not coincide with acb .

If we take viz. an arbitrary liquid q of curve acb , then this is isotonic with L_c under the pressure P ; consequently we have the osmotic equilibrium :

$$(L_c)_P \mid (L_q)_P \dots \dots \dots (22)$$

If we represent the increase of volume of the liquids L_c and L_q when

taking in δw quantities of water, by $\Delta V_c \cdot \delta w$ and $\Delta V_q \cdot \delta w$, then, if we bring the pressure of (22) to $P + dP$, the *O.W.A.* of the left system increases with $-\Delta V_c \cdot dP$ and that of the right system with $-\Delta V_q \cdot dP$. As ΔV_c and ΔV_q are different in general, L_c and L_q are no more isotonic, therefore, under the pressure $P + dP$. The isotonic curve of the pressure $P + dP$ which goes through the point c must be therefore, an other than that of the pressure P .

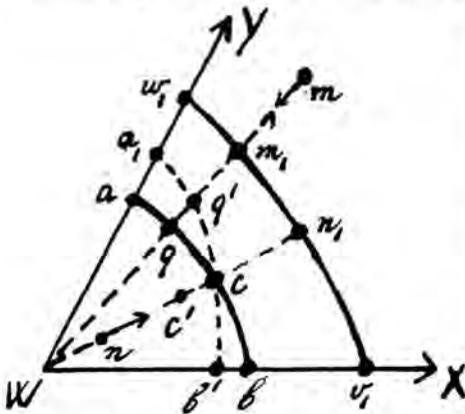


Fig. 1.

We are able to show this also in the following way. For the osmotic equilibrium (22) is true :

$$\left(\zeta - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} \right)_c = \left(\zeta - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} \right)_q \dots \dots \dots (23)$$

For an osmotic equilibrium :

$$(L_c)_{P+dP} \mid (L_q)_{P+dP} \dots \dots \dots (24)$$

in which L'_q represents a liquid, which differs infinitely little (dx and dy) from L_q then follows from (23):

$$\Delta V_c \cdot dP = \Delta V_q \cdot dP - (rx + sy)_q dx - (sx + ty)_q dy \quad \dots \quad (25)$$

If we choose L'_q in such a way that it is represented by a point q' (fig. 1) on the line Wq then dx and dy satisfy (19); (25) now passes into:

$$(rx^2 + 2sxy + ty^2)_q d\lambda = (\Delta V_q - \Delta V_c) \cdot dP \quad \dots \quad (26)$$

$(\Delta V_q - \Delta V_c) \delta w$ is the change in volume if δw quantities of water diffuse from L_c towards L_q . As in general this change is not zero, $d\lambda$, therefore, has a value, different from zero. If $d\lambda$ is positive, then point q' is situated as is drawn in fig. 1.

If L_q proceeds along the curve acb , then the value of $\Delta V_q - \Delta V_c$ changes; if q coincides with c , then this value is zero of course. If we take this value positive, if q is situated between a and c and negative if q is situated between c and b , then we can represent the isotonic curve of the pressure $P + dP$ ($dP > 0$) by $a_1 c b_1$. For small values of dP those curves acb and $a_1 c b_1$ have the same direction with approximation in the point c .

Above we have seen that both the liquids of (22) are no more in equilibrium with one another under a pressure $P + \Delta P$. If we keep constant the total composition, then, as we have deduced already above in general for system (7^a) in (7^b), the water will diffuse under the pressure $P + \Delta P$ in such direction that the total volume decreases. If we take $\Delta V_q > \Delta V_c$ then a little water must diffuse, therefore, from L_q towards L_c ; we shall represent this, just as in (7^b) by:

$$\Delta V_q > \Delta V_c \quad (L_c)_{P+\Delta P} \leftarrow (L_q)_{P+\Delta P} \quad \dots \quad (27)$$

Under the pressure $P + \Delta P$ a new osmotic equilibrium:

$$(L'_c)_{P+\Delta P} \mid (L'_q)_{P+\Delta P} \quad \dots \quad (28)$$

is formed, therefore.

If we represent the liquids of this latter osmotic equilibrium in fig. 1 by c' and q' , then the line $q'c'$ must go through the point (situated on the line cq) which represents the complex of the liquids L_q and L_c . The position of the points q' and c' depends, therefore, not only on the value of dP , but also on the ratio of the quantities of the two liquids. We can deduce this also as follows.

For the osmotic equilibrium (22) equation (23) is valid; for an osmotic equilibrium (28), of which pressure and composition of the liquids differ a little from (22) then is valid:

$$\Delta V_c \cdot dP - (rx + sy)_c \cdot dx_c - (sx + ty)_c \cdot dy_c = \left. \begin{aligned} & \dots \dots \dots \end{aligned} \right\} \dots \quad (29)$$

$$= \Delta V_q \cdot dP - (rx + sy)_q \cdot dx_q - (sx + ty)_q \cdot dy_q \left. \begin{aligned} & \dots \dots \dots \end{aligned} \right\} \dots$$

In our case we have to define in such a way the changes of the concentrations: dx_c etc., that L'_c arises from L_c by taking in δw quan-

tities of water and L'_q from L_q by giving δw quantities of water (δw positive or negative). If there are n quantities of L_c and m quantities of L_q then we find:

$$\left. \begin{aligned} dx_c &= -\frac{x_c}{n} \cdot \delta w & dy_c &= -\frac{y_c}{n} \cdot \delta w \\ dx_q &= +\frac{x_q}{m} \cdot \delta w & dy_q &= +\frac{y_q}{m} \cdot \delta w \end{aligned} \right\} \dots \dots \dots (30)$$

If we substitute those values in (29) and if we put:

$$(x^2r + 2xys + y^2t)_c = K_c \qquad (x^2r + 2xys + y^2t)_q = K_q$$

then follows: $\left(\frac{K_c}{n} + \frac{K_q}{m}\right) \delta w = (\Delta V_q - \Delta V_c) dP \dots \dots \dots (31)$

so that δw has the same sign as dP . In accordance with (27) we find, therefore, that for positive values of dP the water diffuses from L_q towards L_c , (31) defines, however, also δw and in accordance with (30), therefore, also dx_c etc., as function of the quantities n and m of both the liquids.

If we take infinitely large the quantity of L_c in (22) then its composition rests unchanged when taking in δw quantities of water; equilibrium (28) then is the same as (24). As n becomes infinitely large, $K_c : n$ in (31) becomes zero, therefore, and as $\delta w : m$ in (31) is equal to $d\lambda$ in (26), then (31) passes into (26).

In the osmotic equilibrium

$$(W)_P \mid (L_1)_{P_1} \text{ (fig. 1) } \dots \dots \dots (32)$$

we find at the left side of the membrane pure water under the pressure P and at the right side of the membrane a liquid L_1 under the pressure P_1 . In this special case, therefore, $\pi = P_1 - P$ is the osmotic pressure of the liquid L . The osmotic equilibrium (32) is defined by:

$$(\zeta_w)_P = \left(\zeta_1 - x_1 \frac{\partial \zeta_1}{\partial x_1} - y_1 \frac{\partial \zeta_1}{\partial y_1} \right)_{P_1} \dots \dots \dots (33)$$

If we consider stable states only: then we can show in a similar way as f.i. in communication I that (33) can be satisfied only if $P_1 > P$. We now give a definite value to both pressures; then it follows from (33) that the liquids L_1 of (32) are represented by a curve, f. i. curve $w_1 v_1$ in fig. 1.

As under this pressure P_1 all liquids of the region $Ww_1 v_1$ have a smaller *O.W.A.* than the liquids of curve $w_1 v_1$ under this same pressure P_1 and as those of the region $w_1 v_1 XY$ have a greater *O.W.A.* than the liquids of curve $v_1 w_1$, it follows, therefore:

- under a pressure $P_1 > P$ all liquids
- of curve $w_1 v_1$ have the same,
- of region $Ww_1 v_1$ have a smaller,
- and of region $w_1 v_1 XY$ have a greater,
- O.W.A.* than that of pure water under the pressure P .

If we take, therefore, the osmotic systems:

$$(W)_P \rightarrow (L_m)_{P_1} \quad (W)_P \leftarrow (L_n)_{P_1} \quad (\text{fig. 1}) \dots \dots (34)$$

then in the first system the water diffuses from the pure water towards the liquid; in the second system, however, water diffuses from the liquid towards the pure water; the liquids m and n move in fig. 1, therefore, in the direction of the arrows, till they reach curve $w_1 v_1$ (if in the first of those systems a sufficient quantity of pure water is present). Then the systems (34) pass into the osmotic equilibria:

$$(W)_P \mid (L_m)_{P_1} \quad (W)_P \mid (L_n)_{P_1} \quad (\text{fig. 1}) \dots \dots (35)$$

We now take in fig. 2, in which $w_1 v_1$ represents the corresponding curve of fig. 1, a liquid L of curve ab . If ab is an isotonic curve of the pressure P , then all liquids of this curve ab have the same *O.W.A.* under the pressure P . If we replace in (32) the pure water by this liquid L , then an osmotic equilibrium:

$$(L)_P \mid (L_1)_{P_1} \dots \dots \dots (36)$$

arises, in which of course the liquids L_1 now must have an other composition than in (32). As the *O.W.A.* of the liquids of curve ab under the pressure P is greater than that of the pure water under this same pressure, curve $a_1 b_1$ which represents the liquids L_1 of (36) must be situated in fig. 2 further from the point W than curve $w_1 v_1$.

Every arbitrary liquid L of curve ab has, therefore, under the pressure

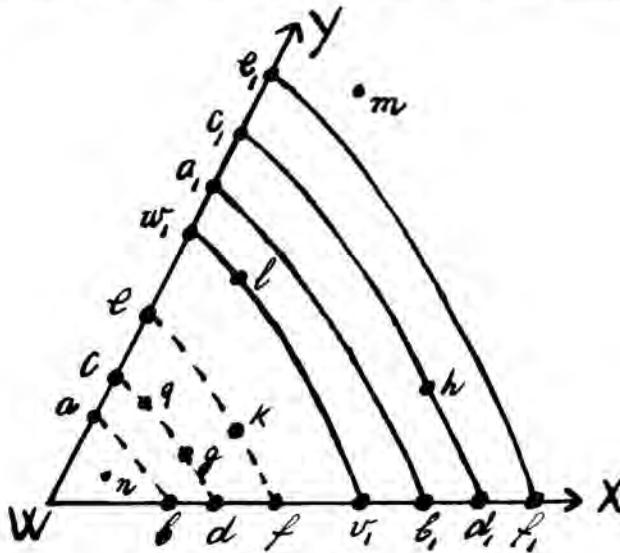


Fig. 2.

P the same *O.W.A.* as every arbitrary liquid of curve $a_1 b_1$ under the pressure P_1 ; we may call those curves ab and $a_1 b_1$ conjugated isotonic curves of the pressures P and P_1 . The same is true for the curves cd

and $c_1 d_1$, for ef and $e_1 f_1$ etc. Consequently we find in fig. 2 a.o. the osmotic equilibria:

$$\begin{array}{ccc} (W)_P \mid (L_i)_{P_1} & (L_a)_P \mid (L_{a_1})_{P_1} & (L_a)_P \mid (L_{b_1})_{P_1} \\ (L_g)_P \mid (L_h)_{P_1} & (L_q)_P \mid (L_{d_1})_{P_1} & (L_k)_P \mid (L_{f_1})_{P_1} \end{array}$$

If we consider two conjugated isotonic curves of the pressures P and P_1 f.i. the curves cd and $c_1 d_1$ then follows from this:

- under a pressure $P_1 > P$ all liquids
- of the isotonic curve $c_1 d_1$ have the same
- of the region $W_{c_1 d_1}$ have a smaller
- and of the region $c_1 d_1 XY$ have a greater
- O.W.A. than the liquids of curve cd under the pressure P .

Hence follows that in the osmotic system:

$$(L_g)_P \xrightarrow{\quad} (L_m)_{P_1} \quad (\text{fig. 2}) \dots \dots \dots (37)$$

the water must diffuse from left to right. It now depends on the ratio of the quantities of both liquids on which conjugated isotonic curves the equilibrium will be formed. If this is the case f.i. on the curves ef and $e_1 f_1$ then is formed the osmotic equilibrium:

$$(L'_g)_P \mid (L'_m)_{P_1} \quad (\text{fig. 2}) \dots \dots \dots (38)$$

in which the left liquid in fig. 2 is represented by the point of intersection of the line W_g with curve ef and the right liquid by the point of intersection of W_m with $e_1 f_1$.

This conversion of (37) into (38) is possible, of course, only then, when the complex of the liquids L_g and L_m is situated within the region $eff_1 e_1$; if this is not the case, then the system (37) also can not be formed on the curves ef and $e_1 f_1$.

We now take the osmotic systems:

$$\left. \begin{array}{cc} (L_g)_P \mid (L_g)_P & (L_g)_P \mid (L_q)_P \\ (L_g)_P \xrightarrow{\quad} (L_i)_P & (L_g)_P \xleftarrow{\quad} (L_n)_P \end{array} \right\} (\text{fig. 2}) \dots \dots (39)$$

in which in all at the left side of the membrane the liquid L_g is present; the pressure is equal on both sides of the membrane, viz. P . It appears from fig. 2 that both the first systems then form osmotic equilibria, in the third system the water diffuses towards the right and in the last system towards the left.

If, however, we bring at the right side of the membrane the pressure from P to P_1 then all liquids, at the right side of the membrane are situated within the region $W_{c_1 d_1}$ of fig. 2; consequently they have all a smaller O.W.A. than the liquid L_g under the pressure P . Instead of (39) we then get the systems:

$$\left. \begin{array}{cc} (L_g)_P \xleftarrow{\quad} (L_g)_{P_1} & (L_g)_P \xleftarrow{\quad} (L_q)_{P_1} \\ (L_g)_P \xleftarrow{\quad} (L_i)_{P_1} & (L_g)_P \xleftarrow{\quad} (L_n)_{P_1} \end{array} \right\} (\text{fig. 2}) \dots \dots (40)$$

in which the water diffuses towards the left. If the ratio of the quantities of both liquids in each of those systems is chosen in such a way, that the equilibrium is formed on the conjugated isotonic curves ab and a_1b_1 , then the left liquid is represented by the point of intersection of the line Wg with curve ab ; the right liquids are represented by the points of intersection of the lines Wg, Wq, Wl and Wn with curve a_1b_1 .

We now take the osmotic equilibrium:

$$(G)_P \mid (G_1)_P \dots \dots \dots (41)$$

in which on both sides of the membrane a vapour under the pressure P . If we bring the pressure on both sides of the membrane to $P + \Delta P$, then [comp (7^a) and (7^b)] the water must diffuse in such direction, that the total volume decreases. If, however, those vapours, follow the gas-laws, then the total volume does not change at diffusion of water, consequently (41) passes, without diffusion occurs, into the osmotic equilibrium,

$$(G)_{P+\Delta P} \mid (G_1)_{P+\Delta P} \dots \dots \dots (42)$$

This appears still also as follows. In the previous communication we have seen that two vapours are in osmotic equilibrium, when the partial vapour-pressure of the water-vapour is equal in both. If this is the case under the pressure P then this is also the case under the pressure $P + \Delta P$; if (41) is an osmotic equilibrium, then (42) it is also, therefore.

At last we still consider the osmotic equilibrium:

$$G_P \mid L_P \dots \dots \dots (43)$$

We now imagine that fig. 1 (XVII) is valid for this pressure P ; then the vapour G is represented by a point of the gas-branch and the liquid L by a point of the liquid-branch of an isotonic curve; if, therefore, G is represented by a point f.i. of curve hc then L is situated anywhere on curve c_1l_1 .

We now bring the pressure of (43) to $P + \Delta P$ and we take ΔP positive; we then get the osmotic system:

$$G_{P+\Delta P} \rightarrow L_{P+\Delta P} \dots \dots \dots (44)$$

in which both the phases have no more the same *O.W.A.* That of the left system decreases viz. with $\Delta V_G \cdot \Delta P$ and that of the right system with $\Delta V_L \cdot \Delta P$ and as in general we may assume that $\Delta V_G > \Delta V_L$, the right system has a greater *O.W.A.* than the left, therefore. Consequently the water diffuses in (44) in the direction of the arrow. This follows also, when applying the rule, that the water on increase of pressure diffuses in such direction that the total volume decreases.

(To be continued).

Physics. — "On the Maximum and Minimum Density and the Heat of Evaporation of Helium". (First Part). By Dr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

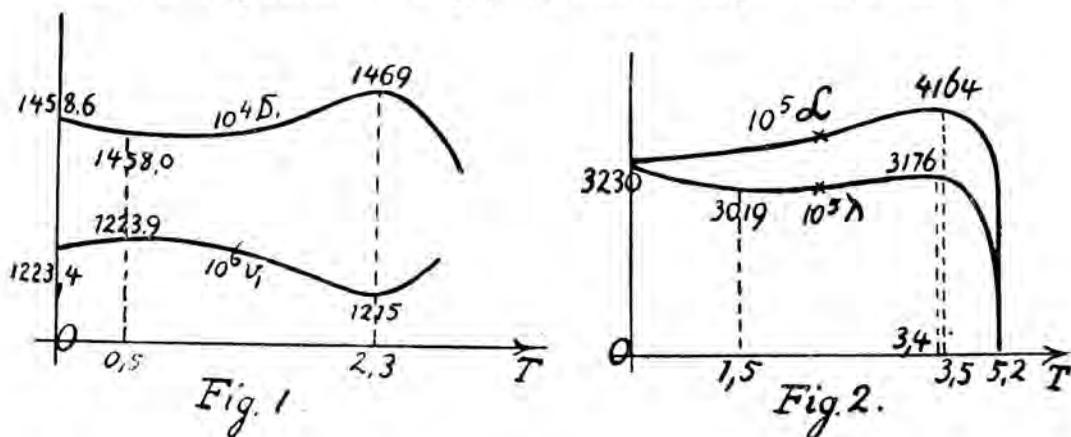
(Communicated at the meeting of September 25, 1926).

A. Theoretical Part.

1. Introduction.

It has become known through the investigations of KAMERLINGH ONNES (Comm. Leiden 119 (1911)), and later through those of KAMERLINGH ONNES and BOKS (Comm. 170^b (1924)), that the liquid density of Helium shows a *maximum* at $2^{\circ},3$ abs. The course sketched in this latter paper leads us, however, to expect that, at still lower temperature, another *minimum* will appear; accordingly after this D_1 will again increase up to the absolute zero (See Fig. 1). The following simple theoretical considerations really confirm this supposition. We see from the values, calculated in § 3, that the minimum (at $0^{\circ},52$) is exceedingly flat, and that the liquid density between $T=0$ and somewhat more than 1° abs. varies only very little — quite conformable to the diagram given p. 23 of Comm. 170^b.

Presently it will also appear theoretically, that the internal molecular heat of evaporation $\lambda = L - p(v_2 - v_1)$ does not only — like L itself —



present a *maximum* at $3^{\circ},4$ (that of L lies at $3^{\circ},5$), but that at lower temperature λ also passes through a *minimum*, and this at $1^{\circ},47$. It can easily be shown that L does not pass through a minimum, but has a *point of inflection* in the neighbourhood of $2^{\circ},3$. (Cf. Fig. 2; the values given there are expressed in "normal" units; multiplication by 542,63 yields L and λ in gr. cal.). The experiments of KAMERLINGH ONNES and DANA

recently published (These Proc. 29, 1051—1060 (1926)) only give the maximum of L at $3^{\circ},5$, for which we theoretically calculate 22,6 gr. cal.; experimentally also 22,6 is found, but at about 3° . The experiments are, however, not accurate enough to determine this temperature with certainty; compare on this point the Second Part of this Article.

The course of both, that of Fig. 1 and Fig. 2 — and to this is added the low temperature coefficient of the *constant* of EÖTVÖS (cf. VAN URK, KEESOM and KAMERLINGH ONNES, *ibid.* 28, 958 (1925)) and the too low values of the *specific heat* (DANA and KAMERLINGH ONNES, *ibid.* 29, 1061—1068 (1926)), but this we will treat in a later paper — is only accounted for by the fact, that in Helium below T_k the quantity a does not diminish with increasing temperature, as in all other substances, but *increases*. As secondary factor we have then the in itself *small value* of a in Helium. The quantity b *always* decreases with increasing temperature (this decrease is not to be confused with the increase of b with the *volume*, which may, however, be neglected in Helium at the lower temperatures, since then v_1 remains *practically* unchanged); and this, like the decrease of a , in consequence of exponential temperature functions occurring by the side of a and b (BOLTZMANN'S factors of distribution; c.f. among others "Zustandsgl.", p. 55-64). In all other substances the two decreases will counteract each other in the equation of state, and about neutralize each other, so that no special complications will appear, but in Helium the increase of a will act in the same direction as the decrease of b , and as we shall see, there the action of the two influences will give rise to the above-mentioned maxima and minima of D_1 and λ , in consequence of alternately stronger and weaker coöperation.

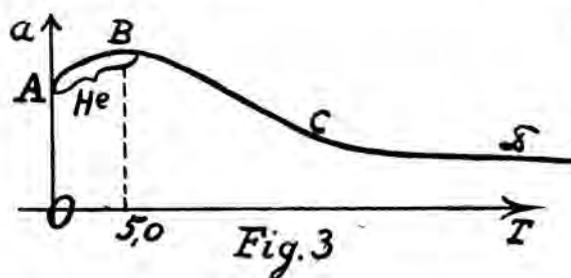
It need scarcely be stated, that in the fact of the relatively *too small* liquid density at low temperatures lies also the key to the phenomenon that Helium cannot become solid at the ordinary vapour pressures. Not until the *external* pressure is raised, through which the too small internal pressure a/v^2 (in consequence of the decrease of a at fall of temperature) is compensated, there can be question of solidification. Which has, indeed, proved to be true through KEESOM'S recent experiments.¹⁾

That the phenomenon of the maximum density is a consequence of the decrease of a with falling temperature, I showed already in 1920 (Recueil Tr. Chim., 34, 382), though at the time I still shared KAMERLINGH ONNES' (erroneous) view, that the liquid density could again diminish to a very small value at very low temperature, in other words might give rise to a second vapour phase. This would be the consequence of KAMERLINGH ONNES' assumption (Comm. 119 (1911); cf. particularly § 6, p. 15—19), that a would be $= cT^2$ (*loc. cit.* p. 17), in other

¹⁾ Cf. on this subject my article in the N. Rott. Ct. of July 21st (Evening paper A. p. 1) and (without the numerous confusing printer's errors that occur in the N. R. Ct.) in the Journal de Genève of July 26th 1926. Cf. also Chem. Weekbl. of July 31st (under "Personalia").

words would become $=0$ at $T=0$. The later, above-mentioned investigations of KAMERLINGH ONNES and BOKS have contradicted these, in fact pretty improbable, assumption. We shall see in § 2 that $10^6 a$, which quantity has the value of 67,4 at T_k , approaches to 39,5 at $T=0$, hence by no means $=0$. The assumption $a = cT^2$ would not be able to explain the minimum at $0^{\circ},5$, nor that of λ at $1^{\circ},5$. No more is the so-called „energy-degeneration” at lower temperatures, on which e.g. VERSCHAFFELT (see the cited paper by DANA and KAMERLINGH ONNES) refers, able to do so. I, therefore, disputed this view already in the paper of 1920 mentioned before (p. 382 under: “L’effet des “quanta” n’y est pour rien, etc.”). Later on I worked out the theory on a better basis (L’Hydrogène et les gaz nobles, Leiden, Sijthoff, 1921, p. 39–41), and this theory is the same as will be developed in what follows (with only slight modification as regards db/dt).

Through my researches on the equation of state I have, of late, entirely relinquished the idea that in the case of *liquids* there could be question of “energy-degeneration”, even at the lowest temperatures. ¹⁾ Not only that all the phenomena of ordinary substances, and even the so greatly diverging ones of Helium can be explained qualitatively and quantitatively by the dependence of a and b on the temperature, and the dependence of b on the volume; this being *not* the case on assumption of energy-degeneration (the coefficient of expansion, inter alia, would simply rapidly approach to 0 without maximum nor minimum volume; the minimum for λ remained unexplained, etc.) — but in principle I protest against applying PLANCK’s and DEBYE’s considerations, which are especially valid for “oscillators” moving round fixed positions of equilibrium, to liquids (and even to gases!), where the molecules can move freely with regard to each other. Especially in Germany, where, in spite of all that has been written about it — even by VAN DER WAALS himself already —



a and b are still almost universally considered as constant, physicists have recourse to the assumption of energy degeneration for liquids, in imitation of NERNST c. s. (vapour-pressure equations with $+1,75 \log T$, even at the highest temperatures, etc.).

With regard to this decrease of a in Helium at fall of temperature it may be pointed out, that this is a *quite general* phenomenon. There is (see Fig. 3) between B and D , i. e. almost over the total course of a ,

¹⁾ PALACIOS MARTINEZ and KAMERLINGH ONNES (Comm. 164 (1923)) showed already for H_2 and He that — in contradiction to the opinion of some authors — there is not yet question of any quantum effect in the gaseous state at very low temperatures ($20^{\circ},5$ abs.).

a decrease with increasing temperature (see above: BOLTZMANN'S distribution factor); only at *very* low temperatures there will be found decrease after a maximum on further fall of the temperature (Cf. on this point also my considerations in These Proc. 21, 19 (1918)). As a is practically independent of v , the results found there for large volumes may also be applied to small volumes.

What is now the case? For almost all substances the region *between melting point and critical temperature* lies in the descending part, e. g. in the neighbourhood of C ; only for hydrogen the part left of B is encroached on a little. Helium is, however, the only substance where the region between 0° and $5^\circ,2$ lies *almost entirely* between A and B . (The maximum lies at exactly 5° , so that T_k lies slightly on the right of B). Also for ordinary substances deviating phenomena would make their appearance at very low temperatures, but this is prevented by the appearance of the *solid* state long before the time.

2. The maximum of density at $2^\circ,3$. Formulae for a and b .

As $\left(\frac{dv}{dt}\right)_p = \left(\frac{dp}{dt}\right)_v : -\left(\frac{dp}{dv}\right)_t$, and the coefficient of compressibility $-\frac{1}{v}\left(\frac{dv}{dp}\right)_t$ always remains finite, hence never becomes 0 or ∞ , $\left(\frac{dv}{dt}\right)_p$ will necessarily become 0 (maxima and minima of v) at the same time with $\left(\frac{dp}{dt}\right)_v$ or $T\left(\frac{dp}{dt}\right)_v$. From $p = \frac{RT}{v-b} - \frac{a}{v^2}$ follows:

$$T\left(\frac{dp}{dt}\right)_v = \frac{RT}{v-b} - \frac{Ta'}{v^2} + \frac{RT^2 b'}{(v-b)^2} = \frac{a-Ta'}{v^2} + \frac{RT^2 b'}{(v-b)^2} + p,$$

when for shortness a' is written for $\left(\frac{da}{dt}\right)_v$ and b' for $\left(\frac{db}{dt}\right)_v$.

If for the present — for the calculation of different quantities — we confine ourselves to the minimum of v at $2^\circ,3$, we shall have there:

$$a - Ta' = -\left(\frac{v}{v-b}\right)^2 RT^2 b' - pv^2 \quad (2^\circ,3).$$

From the equation of state follows, however:

$$\frac{v}{v-b} = \frac{a/v + pv}{RT} = \frac{a/v}{RT} \left(1 + \frac{p}{a/v^2}\right),$$

so that we may also write:

$$a - Ta' = -\left(\frac{a^+}{v}\right)^2 \frac{b'}{R} - pv^2,$$

where, accordingly, a^+ means $a(1 + p : a/v^2)$. We shall see that for Helium at $2^\circ,3$ the different correction quantities may not be neglected.

Putting now

$$a = a_0 + \alpha T - \gamma T^2 \quad ; \quad b = b_0 - \beta T + \delta T^2,$$

we have the following equation at 2°3, with $a-Ta' = a_0 + \gamma T^2$:

$$a_0 + \gamma T^2 = \frac{\left(\frac{a^+}{v}\right)^2 \beta - 2 \frac{\delta T}{R} - p v^2}{\dots} \quad (1)$$

This is already a first relation between a_0, α, γ and $-b' = \beta - 2 \delta T$, when v is experimentally known.

A second relation is found from (for a is practically independent of v , and the above expression for a is, therefore, certainly valid up to T_k)

$$a_k = a_0 + \alpha T_k - \gamma T_k^2.$$

i.e. with $T_k = 5^{\circ}19$ and $10^6 a_k = 67,42$ (to be calculated from T_k and p_k)

$$a_0 + 5,19 \alpha - 26,94 \gamma = 67,42 \cdot 10^{-6} \quad (2)$$

A third relation follows from the theoretical value of the internal molecular heat of evaporation λ . For this, as the difference of two energies, the following equation is evidently valid — when, as we suppose, no energy degeneration occurs in liquids:

$$\lambda = \int_{v_1}^{v_2} \left(T \left(\frac{dp}{dt} \right)_v - p \right) dv.$$

which, accordingly, yields with the above-found value of $T \left(\frac{dp}{dt} \right)_v$:

$$\lambda = \int_{v_1}^{v_2} \left(\frac{a-Ta'}{v^2} + \frac{RT^2 b'}{(v-b)^2} \right) dv = \frac{a-Ta'}{v_1} + \frac{RT^2 b'}{v_1-b} - \text{corr. } v_2,$$

when we may consider b and b' as independent of v in Helium, at least on the liquid branch. When we again substitute $a^+/v : RT$ for $v:(v-b)$ (see above), we get (substituting v for v_1):

$$\lambda v = (a-Ta') + \frac{a^+}{v} T b' - \text{corr. } v_2. \quad (a)$$

The value of λ in this equation can be calculated from the experimental values of p, v_1 and v_2 . (Cf the Second Part of this paper).

In this equation $b' = -(\beta - 2 \delta T)$ can be eliminated at 2°3 by means of (1). We then obtain:

$$\lambda v = (a_0 + \gamma T^2) - \frac{a_0 + \gamma T^2 + p v^2}{a^+/v} RT - \text{corr. } v_2,$$

i. e.

$$\lambda v \left(1 + \frac{v_1}{v_2} \right) = (a_0 + \gamma T^2) \left(1 - \frac{RT}{a^+/v} \right) - p v^2 \frac{RT}{a^+/v} \quad (2^{\circ}3). \quad (3)$$

1) For ordinary substances, where the terms with a' and b' about cancel each other, this expression becomes simply $\lambda = \frac{a}{v_1} - \frac{a}{v_2}$, which relation was first derived by BAKKER in his Thesis for the Doctorate (1885).

in which with regard to the very slight correction for v_2 it has been taken into account, that both in (a) and in (3) the principal term is $(a_0 + \gamma T^2)$, so that in approximation λ will be $= (a_0 + \gamma T^2) : v_1 - \text{ibid} : v_2$,

which leads to $\lambda v_1 = (a_0 + \gamma T^2) \left(1 - \frac{v_1}{v_2}\right)$ or $\lambda v_1 \left(1 + \frac{v_1}{v_2}\right) = a_0 + \gamma T^2$,

i. e. — with the addition of the other terms — to (3).

A *fourth* relation follows from (a), when $T=0$ is put. Then follows

$$\lambda_0 v_0 = a_0. \quad \dots \quad (4)$$

in which λ_0 and v_0 are known from experimental data. For we have from the vapour-pressure equation $\log^{10} p = -A/T + \text{etc.}$, in which $2.3026 A = 3.8308 \times 2.3026 = v_0/R$ (see § 1 of the Second Part), with $R = 1 : 273.09 = 0.0036618$:

$$\lambda_0 = 3.8308 \times 0.0036618 \times 2.3026 = \underline{0.03230} \text{ (in norm. units).}$$

To reduce this to gr. cal. 15° , it must be multiplied by $273.09 \times 1.9866 = 542.63$, R being $= 1 : 273.09$ in normal units, and $= 1.9866$ in gr. cal. Then $\lambda_0 = L_0 = \underline{17.53}$ gr. cal. is obtained, and not $4 \times 3.5 = 14$ gr. cal., as VERSCHAFFELT calculated on the ground of in my opinion unfounded suppositions.

Now from $\underline{D_0 = 0.1458^6}$ (this value will be found in § 3) follows in normal units:

$$v_0 = \frac{4}{22416 D_0} = \frac{0.000178444}{D_0},$$

so that $10^6 v_0$ becomes $= \underline{1223.4}$, and further according to (4):

$$a_0 = 0.03230 \times 1223.4 \cdot 10^{-6} = \underline{39,515 \cdot 10^{-6}},$$

against $a_k = \underline{67.42 \cdot 10^{-6}}$; in consequence of which $a_0 : a_k$ becomes $= 0.5861 = 0.6$. For ordinary substances this relation is about 1.6.

With this value of a_0 we can now determine a and γ from (2) and (3). According to table IX in the Second Part λ is at $2^\circ,3 = 0,03060$, $v_1 = 1215 \cdot 10^{-6}$, hence $\lambda v_1 = 37,18 \cdot 10^{-6}$. And $v_1 : v_2$ being $= 1 : 93$ at $2^\circ,3$, $\lambda v_1 (1 + v_1/v_2)$ becomes $= 37,58 \cdot 10^{-6}$.

At $2^\circ,3$ the correction term $p v_1^2 (RT : a^+ / v_1)$ is $= 0,06959 (1215)^2 \cdot 10^{-12} \times (0,008422 \times 1215 \cdot 10^{-6} : 59,41 \cdot 10^{-6})$, in which p is expressed in atm. (Table V), while for a at $2^\circ,3$ presently $59,31 \cdot 10^{-6}$ will be found, to which then $0,10 \cdot 10^{-6}$ is added as correction on a , viz. (see before (1)) $p v_1^2 = 0,06959 \times (1215)^2 \cdot 10^{-12} = 0,06959 \times 1,476 \cdot 10^{-6} = 0,1027 \cdot 10^{-6}$.

We then find for the correction term in question:

$$0,103 \cdot 10^{-6} \times (10,233 \cdot 10^{-6} : 59,41 \cdot 10^{-6}) = 0,103 \cdot 10^{-6} \times 0,172 = \\ = 0,0177 \cdot 10^{-6} = 0,02 \cdot 10^{-6}.$$

Hence (3) becomes:

$$37,58 \cdot 10^{-6} + 0,02 \cdot 10^{-6} = \\ = (39,515 \cdot 10^{-6} + 5,29 \gamma) \left(1 - \frac{0,008422 \times 1215 \cdot 10^{-6}}{(39,515 \cdot 10^{-6} + 2,3a - 5,29 \gamma)^+} \right).$$

Now according to (2) $5,19 \cdot 10^6 a - 26,94 \cdot 10^6 \gamma = 67,42 - 39,515 = 27,905$, hence $10^6 a - 5,19 \cdot 10^6 \gamma = 5,3767$, and therefore $2,3 \cdot 10^6 a = 12,366 + 11,94 \cdot 10^6 \gamma$. Substituting this in the above equation, the result is

$$37,60 = (39,515 + 5,29 \cdot 10^6 \gamma) \left(1 - \frac{10,233}{(51,881 + 6,65 \cdot 10^6 \gamma)^+} \right).$$

in which the correction factor at a^+ amounts to $= 0,103 : 59,31 = 0,00173 = 1 : 600$ of the value, according to the above.

In consequence of this we get:

$$39,515 + 5,29 \cdot 10^6 \gamma = \frac{37,60}{1 - \frac{10,233}{51,971 + 6,661 \cdot 10^6 \gamma}}$$

i.e.

$$(39,515 + 5,29 \cdot 10^6 \gamma) (41,738 + 6,661 \cdot 10^6 \gamma) = 37,60 (51,971 + 6,661 \cdot 10^6 \gamma),$$

leading to

$$35,237 (10^6 \gamma)^2 + 233,54 \cdot 10^6 \gamma - 304,8 = 0,$$

or $(10^6 \gamma)^2 + 6,622 \cdot 10^6 \gamma - 8,650 = 0$, yielding $10^6 \gamma = 1,1177$; and further (see above) $10^6 a = 5,3767 + 5,19 \times 1,1177 = 11,178$. We then finally find for a :

$$\underline{10^6 a = 39,515 + 11,178 T - 1,1177 T^2} \text{ (valid to a little further than } T_k \text{). (5)}$$

in which accidentally the coefficient of T^2 is exactly $= 0,1$ of that of T . Hence for a the formula gives a maximum at exactly 5° , i.e. $= 67,46 \cdot 10^{-6}$. After this a will decrease, approaching a limiting value at $T = \infty$ (see Fig. 3), which is, of course, not rendered by (5). Now the following values for a are calculated from (5).

$$\begin{array}{cccccc} T = & 0^\circ & 0^\circ,5 & 1^\circ,4 & 2^\circ,3 & 5^\circ,0 & 5^\circ,2 \text{ abs.} \\ 10^6 a = & 39,515 & 44,82\bar{5} & 52,973 & 59,311 & 67,46 & 67,42 \end{array}$$

The value 59,31 at $2^\circ,3$ has already been used above for the calculation of the small correction term.

From (1) the value of $-b' = \beta - 2 \delta T$ at $2^\circ,3$ can now be calculated. We find:

$$-b' = \frac{(a_0 + \gamma T^2 + \rho v^2) R}{(a^+ / v)^2} = \frac{(39,515 + 5,913 + 0,103) 10^{-6} \times 0,0036618}{(59,414 : 1215)^2}.$$

For at $2^\circ,3$ $10^6 a$ is $= 59,311$, which becomes $= 59,414$ with the correction 0,103 (see above). Hence we get:

$$-10^6 b' = 0,16673 \times (20,450)^2 = \text{ibid.} \times 418,20 = \underline{69,73} (2^\circ,3).$$

Thus from the fact of the maximum density at $2^{\circ},3$, and making use of the values of λ_0 and $\lambda_{2,3}$ to be calculated from the experimental material, we have determined the whole course of a , and have succeeded at the same time in finding out something about the quantity b , viz. $-b' = \beta - 2\delta T$.

Assuming again $10^6 v_0 = 1223,4$, the coefficients β and δ in $b = b_0 - \beta T + \delta T^2$ can easily be determined, now that a is known, from two observations of the liquid density, e.g. at $1^{\circ},4$ and at $2^{\circ},3$. Of course we might also have determined $b_0 = v_0$, β and δ from three such observations, but — now that the value of v_0 has once been established (see § 3) — we prefer the method of calculation indicated, the more so because we then also possess a means to test the value of $-10^6 b'$ at $2^{\circ},3$ found just now, i. e. about 70.

From the equation of state follows:

$$\begin{array}{l} T=1^{\circ},4 \left\{ 0,003822 + \frac{52,973 \cdot 10^{-6}}{(1222,5)^2 \cdot 10^{-12}} (1222,5 - 10^6 b_{1,4}) 10^{-6} = 0,005127 \right\} \\ T=2^{\circ},3 \left\{ 0,06959 + \frac{59,311 \cdot 10^{-6}}{(1215)^2 \cdot 10^{-12}} (1215 - 10^6 b_{2,3}) 10^{-6} = 0,008422 \right\} \end{array}$$

in which the values of a were determined just now, those of p and v (calculated from D) having been experimentally given (Table V and VII). This yields therefore:

$$\begin{array}{l} (0,004 + 35,445) (1222,5 - 10^6 b_{1,4}) = 5127 \\ (0,070 + 40,178) (1215 - 10^6 b_{2,3}) = 8422 \end{array}$$

from which $1222,5 - 10^6 b_{1,4} = 144,63$, $1215 - 10^6 b_{2,3} = 209,25$, hence

$$10^6 b_{1,4} = 1077,87 \quad ; \quad 10^6 b_{2,3} = 1005,75.$$

From this, and with $10^6 b_0 = 1223,4$, follows immediately:

$$\underline{10^6 b = 1223,4 - 118,45 T + 10,356 T^2} \quad (\text{valid to } \pm 2^{\circ},3) \quad ; \quad (6)$$

From this follows, therefore, $10^6 b'_{2,3} = 118,45 - 20,71 \times 2,3 = 70,82$, against 69,73 above, or round 71 against 70. Hence the maximum of density lies possibly not *exactly* at $2^{\circ},3$; it is also possible that the slight dependence of b on the volume at $2^{\circ},3$, which has been neglected in the beginning, is already perceptible¹⁾.

3. The minimum of density at $0^{\circ},52$. Formula for D_1 .

From the equation for $T \left(\frac{dp}{dt} \right)_v$ in the beginning of § 2, after substi-

¹⁾ In this connection we remark that from the above formulae, which of course remains valid to T_k , when only the volume is kept = v_0 , it would follow that $10^6 (b_{v_0})_k = 888$, $10^6 b_k$ being = $\frac{1051}{888}$ (from T_k and p_k). It would then follow from this, according to a formula derived by me (Zeitschr. f. anorg. u. allg. Chem. **149**, p. 349 (1925)), that $(b_{\infty})_k : b_k$ is = $\frac{1}{2} \left(\frac{1051}{888} + 1 \right) = 1,09$, hence $10^6 (b_{\infty})_k = 1146$. From this it is possible, at least at T_k , to judge to some extent about the variability of b with the volume. Between $v = v_0$ and $v = \infty$ the value of b then varies from 888 to 1146, i. e. in the ratio of 1 : 1,3.

tution of the value for $\left(\frac{v}{v-b}\right)^2$, follows:

$$T\left(\frac{dp}{dt}\right)_v v^2 = a_0 + \gamma T^2 + pv^2 - \left(\frac{a^+}{v}\right)^2 \frac{\beta - 2 \delta T}{R}$$

At $T=0$ this becomes:

$$a_0 - \left(\frac{a_0}{v_0}\right)^2 \frac{\beta}{R} = a_0 \left(1 - \frac{a_0}{v_0^2} \frac{\beta}{R}\right)$$

Now

$$1 - \frac{a_0}{v_0^2} \frac{\beta}{R} = 1 - \frac{39,515 \cdot 10^{-6}}{(1223,4)^2 \cdot 10^{-12}} \times 118,45 \cdot 10^{-6} \times 273,09,$$

i.e. with $(1,2234)^2 = 1,4967$:

$$1 - 26,401 \times 118,45 \cdot 10^{-6} \times 273,09 = 1 - 0,854 = 0,15.$$

And $\left(\frac{dv}{dt}\right)_v$ always having the same sign as $\left(\frac{dp}{dt}\right)_v$ (see § 2), hence also as $T\left(\frac{dp}{dt}\right)_v v^2$, with Helium $\left(\frac{dv}{dt}\right)_p$ will be **positive** at the absolute zero. And since there is a minimum at $2^{\circ}.3$, there will necessarily be also a maximum volume at lower temperature, i.e. a **minimum density** (See Fig. 1), and this in the neighbourhood of $0^{\circ}.52$, as we shall presently see.

In general it follows from the above relation, that $\left(\frac{dv}{dt}\right)_p$ will be *positive* at $T=0$, when

$$\frac{\beta}{b_0} < \frac{R}{a_0/b_0} \dots \dots \dots (7)$$

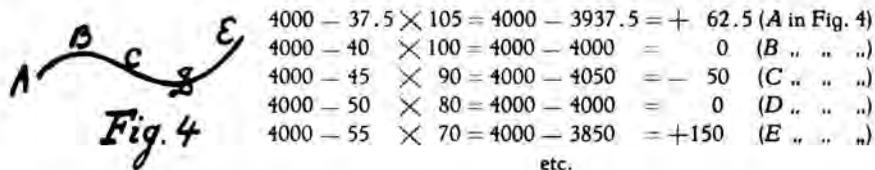
This is fulfilled in Helium, owing to the small value of a (low critical temperature). But how is this with ordinary substances with comparatively high critical temperature? For such substances $a_0 = 1,6 a_k$, $b_0 = 0,5 b_k$, hence $a_0 : b_0 = 3,2 (a_k : b_k)$. With $\lambda = \frac{27}{28}$ it follows from $RT_k = \frac{8}{27} \lambda a_k/b_k$, that $a_k : b_k$ is $= \frac{27}{2} RT_k$, so that for ordinary substances $a_0 : b_0$ will be about $= 11 RT_k$. For a positive initial course $\beta : b_0$ must, therefore, be $< 1 : 11 T_k$, which would yield $\beta : b_0 < 1 : 6000$ with e.g. $T_k = 550^{\circ}$ abs.

Now it is, of course, *possible* that β is so small, and then v will immediately increase, without maximum or minimum, however. For if this were the case, it would follow that $(a_0 + \gamma T^2) - \dots$ could become $= 0$, which with such small values of β is evidently impossible.

But generally β will, no doubt, be greater. And then v begins with a *decrease*, because then $\left(\frac{dv}{dt}\right)_p$ is negative. But because v must of course, increase later on, there must necessarily be a minimum volume (maximum density) in this (most frequently occurring) case. Accordingly, the

phenomenon of a maximum density would not remain restricted to Helium, but would have to be found for any substance (provided β be not too small) at low temperature — without the minimum, however, at still lower temperature. But unfortunately an "ordinary" substance becomes solid long before the temperature, at which this phenomenon could appear, is reached. And above the melting-point, a just as b decreases with the temperature, so that then the phenomenon cannot possibly be realized (see below).

The possibility of a maximum and a minimum in Helium can very simply be represented in a diagram in the following way. If the terms γT^2 and pv^2 , whose values are small compared with the other terms, are disregarded, the sign of $\left(\frac{dv}{dt}\right)_v$ is chiefly determined by that of $a_0 - q \times r$, in which for Helium $q = (a/v)^2$ is an *increasing* quantity owing to $a = a_0 + \alpha T \dots$ (v varies then exceedingly little), and $r = (\beta - 2 \delta T) : R$ is a *decreasing* quantity. This may e. g. give rise to what follows (the figures are entirely arbitrary).



Here the quantity b itself must, of course, also decrease (β positive). For else we should get (see above) $a_0 + q \times r$ (instead of $a_0 - q \times r$), which of course, could never become = 0. Nor can the phenomenon, of course, be found with a diminishing at the same time with b (e. g. for ordinary substances above the melting-point, (see above), since $\left(\frac{dv}{dt}\right)_p$ being once positive, becomes then greater and greater positive (q and r both decrease).

Let us now examine *where* the minimum of density in Helium lies. For this purpose T must be solved from

$$(a_0 + \gamma T^2) - \left(\frac{a}{v}\right)^2 \frac{\beta - 2 \delta T}{R},$$

where a is $a = a_0 + \alpha T - \gamma T^2$, the correction term pv^2 being certainly negligible at $0^\circ,5$ ($p = 0,073$ atm.!). After some trials the following value is soon found:

$$\underline{T_{\text{min.}} = 0^\circ,521.}$$

For then the above expression becomes:

$$(39,515 + 0,304) 10^{-6} - \left(\frac{45,039 \cdot 10^{-6}}{1223,9 \cdot 10^{-6}}\right)^2 (118,45 - 10,79) \cdot 10^{-6} \times 273,09,$$

or after division by 10^{-6} :

$$39,819 - (36,801)^2 \times 107,66 \cdot 10^{-6} \times 273,09 = 39,819 - 1354,3 \times \text{etc.},$$

hence

$$39,819 - 39,820, \text{ i.e. } 39,82 - 39,82 = 0.$$

If, now, at low temperatures the liquid density is represented by the quadratic formula $D = D_0 - \varepsilon T + \eta T^2$, already $\varepsilon : 2\eta$ is $= 0,521$ on account of the minimum at $0^\circ,521$. If, further, we assume the experimental values $D_{1^\circ,2} = 0,1459$ and $D_{1^\circ,9} = 0,1462$, we find easily from

$$D = D_0 - 1,042 \eta T + \eta T^2:$$

$$10^4 \eta = 2,0825, \quad 10^4 D_0 = 1458,605, \quad \text{and further } 10^4 \varepsilon = 2,1700,$$

so that at low temperatures the liquid densities may be represented by

$$\underline{10^4 D_1 = 1458,6 - 2,170 T + 2,082 T^2} \text{ (valid up to } \pm 2^\circ). \quad (8)$$

It is the above value of D_0 , that we have assumed in our calculations in § 2. If we had assumed a slightly different value, the value found now would *not* have been in perfect harmony with it. Through some repeated new calculations the right value, which leads to a value in (8) identical with it, is finally found. This formula gives the following values of D_1 and $v_1 = 178,444 \cdot 10^{-6} : D_1$ (see § 2).

TABLE A.

| T | $10^4 D_1$ | $10^6 v_1$ | T | $10^4 D_1$ | $10^6 v_1$ |
|-----|----------------|----------------|-----|------------|------------|
| 0 | 1458.61 | 1223.39 | 1.2 | 1459.00 | 1223.1 |
| 0.5 | 1458.04 | 1223.86 | 1.3 | 1459.30 | 1222.8 |
| 0.6 | 1458.05 | 1223.85 | 1.4 | 1459.65 | 1222.5 |
| 0.7 | 1458.11 | 1223.80 | 1.5 | 1460.04 | 1222.2 |
| 0.8 | 1458.20 | 1223.7 | 1.6 | 1460.46 | 1221.8 |
| 0.9 | 1458.34 | 1223.6 | 1.7 | 1460.93 | 1221.4 |
| 1.0 | 1458.52 | 1223.5 | 1.8 | 1461.45 | 1221.0 |
| 1.1 | 1458.74 | 1223.3 | 1.9 | 1462.00 | 1220.5 |

At 2° the formula gives 1462,6; experimentally round 1463 has been found. (At $1^\circ,6$ the value according to the formula is 1460,5, round 1460 having been found experimentally). Hence the minimum of density at $0^\circ,52$ is $= 0,1458$; we know that D_1 at $2^\circ,3$ reaches the maximum 0,1469. The values of D_1 between 0° and 1° , therefore, do not differ much, they are always in the neighbourhood of 0,1458; only after this they begin to rise somewhat. The course is then *quite conformable* to the graphical representation given in Comm. 170^b.

4. The minimum of λ at $1^{\circ},5$.

Also the molecular internal latent heat λ , which passes through a maximum at $3^{\circ},4$, will again pass through a *minimum* at about $1^{\circ},5$, as does not only follow from the values of λ calculated from $\frac{dp}{dt}$ and $v_2 - v_1$ (see Table IX), but can also be demonstrated theoretically.

We saw above in § 2, that

$$\lambda = \frac{a_0 + \gamma T^2}{v} - \frac{a}{v^2} T (\beta - 2 \delta T),$$

when the corrections are omitted, which disappear at very low temperatures, and of course at $T=0$. Then it follows from this, when we suppose v along the saturation line to be constant, which is about fulfilled in the neighbourhood of $T=0$, that

$$\frac{d\lambda}{dT} = \frac{2\gamma T}{v} - \frac{a + Ta'}{v^2} (\beta - 2\delta T) + \frac{aT}{v^2} 2\delta.$$

Hence at $T=0$, $a + Ta' = a_0 + 2aT - 3\gamma T^2$ then being $= a_0$, we have:

$$\left(\frac{d\lambda}{dT}\right)_0 = -\frac{\beta a_0}{v_0^2}, \dots \dots \dots (9)$$

which is, accordingly, **always negative**. (See Fig. 2).

For *ordinary* substances, where the variability of v cannot be neglected, there is, besides, still a term $-\frac{a_0}{v_0^2} \left(\frac{dv}{dt}\right)_0$. This is then even the principal term, as β is then small in comparison with $\left(\frac{dv}{dt}\right)_0$. It now follows from the equation of state, that when a and b are constant (ideal substances), this term is $= -R$, and *greater negative*, when the substances are not ideal. λ , therefore, decreases continually, from $T=0$ to T_k , without minimum or maximum, as in Helium (see further).

As regards $L = \lambda + RT$ (at low temperatures $p(v_2 - v_1) = RT$), this will be $\left(\frac{dL}{dt}\right)_0 = -R + R = 0$ for ordinary substances, when they are ideal substances, and *negative* when they are not ideal substances.

Now from (9) follows for *Helium*:

$$\left(\frac{d\lambda}{dT}\right)_0 = -\frac{118,45 \cdot 10^{-6} \times 39,515 \cdot 10^{-6}}{(1223,4)^2 \cdot 10^{-12}} = \underline{\underline{-0,003127}},$$

theoretically. Experimentally the following value is found (see Table IX) between 0° and $0^{\circ},5$:

$$(0,03091 - 0,03230) : 0,5 = -0,0028.$$

This is the chord between the said temperatures. Hence for the tangent a somewhat greater (negative) value must be found (see Fig. 2), so that

-0,0031 may be right. If to this $R=0,0036618$ is added, then the following value holds for Helium:

$$\left(\frac{dL}{dT}\right)_0 = +0,000535,$$

i.e. **positive**. Also this value proves to be in good agreement. For according to Table IX (0,03274 - 0,03230): 0,5 = 0,00088 is found for it between 0° and $0^\circ,5$, and (0,03399 - 0,03274): 0,5 = 0,00250 between $0^\circ,5$ and 1° (which points to a *point of inflection* between $T=0$ and the maximum at $3^\circ,5$; see Fig. 2). The value of $10^5 \left(\frac{dL}{dT}\right)_0$ will, therefore, be pretty considerable smaller than 88, so that again 53,5 may be correct.

L increases, therefore, at first, and as this quantity will necessarily rapidly approach to 0 in the neighbourhood of T_k , in consequence of the factor $\frac{1}{v_1} - \frac{1}{v_2}$, there must also necessarily be a **maximum** at L , which is, indeed, found at $3^\circ,5$ (cf. Table IX).

Now it follows from the experimental data, that not only L presents a *maximum*, but also λ , and this at $3^\circ,4$ (Table IX). And because λ at first decreases, there must necessarily, besides this maximum, be also present a **minimum** for λ at low temperature. This, too, is confirmed experimentally. According to Table IX this minimum lies between $1^\circ,4$ and $1^\circ,5$, but nearer to $1^\circ,5$. *Theoretically* $1^\circ,47$ is found for it, as appears from what follows.

At this temperature

$$(2,2354 \cdot 10^{-6} \times 1,47 \times 1222,3 \cdot 10^{-6}) - (a_0 + 2\alpha T - 3\gamma T^2) \times \\ \times (118,45 - 20,71 \times 1,47) 10^{-6} + (a_0 + \alpha T - \gamma T^2) \times 1,47 \times 20,71 \cdot 10^{-6}$$

is found for

$$v^2 \frac{d\lambda}{dT} = 2\gamma T v - (a_0 + 2\alpha T - 3\gamma T^2) (\beta - 2\delta T) + (a_0 + \alpha T - \gamma T^2) T \cdot 2\delta.$$

Now $10^6 (a_0 + \alpha T - \gamma T^2) = 39,515 + 11,178 \times 1,47 - 1,1177 \times (1,47)^2 = 39,515 + 16,432 - 2,415 = 53,532$. And $10^6 (a_0 + 2\alpha T - 3\gamma T^2) = 39,515 + 32,864 - 7,245 = 65,134$. Hence, multiplying by 10^{12} , the above becomes, with $118,45 - 30,444 = 88,006$:

$$10^{12} v^2 \frac{d\lambda}{dT} = 4016,5 - 5732,2 + 1629,9 = -85,8.$$

To this should be added 101,2 of a correction term with $\frac{dv}{dt}$, viz.

$$\left[-\frac{a_0 + \gamma T^2}{v^2} + \frac{2\alpha T(\beta - 2\delta T)}{v^3} \right] \frac{dv}{dT} \times 10^{12} v^2,$$

i.e.

$$\left[(a_0 + \gamma T^2) - \frac{2\alpha T(\beta - 2\delta T)}{v} \right] \times \left(-\frac{dv}{dT} \right) \times 10^{12}.$$

Now $-\frac{dv}{dt}$ is at $1^{\circ},47$ (see Table A in § 3, or formula (8) for D_1) $= 3,307 \cdot 10^{-6}$. With $10^6(a_0 + \gamma T^2) = 39,515 + 2,415 = 41,93$ this correction term, therefore, becomes:

$$\left[41,93 - \frac{2 \times 53,532 \times 1,47 \times 88,006}{1222,3} \right] \times 3,307,$$

or

$$(41,93 - 11,33) \times 3,307 = 30,60 \times 3,307 = 101,2.$$

Hence we get at $1^{\circ},47$:

$$10^{12} v^2 \frac{d\lambda}{dT} = -85,8 + 101,2 = +14,4.$$

At $1^{\circ},48$ we would have found in the same way:

$$(4043,7 - 5729,5 + 1643,2) + (-100) = +57.$$

And at $1^{\circ},4$ we get $(3826 - 5747 + 1536) + (-90) = -295$, from which it is therefore seen, that at slightly below $1^{\circ},47$ $\frac{d\lambda}{dT}$ passes through 0. At $1^{\circ},46$ this quantity is already about -30 ; all this in perfect harmony with Fig. 2 in connection with the values of λ , calculated from the experimental data (Table IX).

We once more point out how accurate the results are (also quantitatively) to which the formulae for a and b , determined from observations at $2^{\circ},3$, lead; also in the neighbourhood of $T=0$ and at very low temperatures (minimum D_1 and λ resp. at $0^{\circ},5$ and $1^{\circ},5$). We may, therefore, safely put aside the value $L_0 = \lambda_0 = 14$ gr. cal. of VERSCHAFFELT (based on NERNST's quite erroneous theory), and accept the value 17,5, following necessarily from the different data; which value, indeed, is in much better harmony with the observations of DANA and KAMERLINGH ONNES, whereas the value 14 would be in conflict with them. (See § 4 of the Second Part).

Physics. — „On the Maximum and Minimum Density and the Heat of Evaporation of Helium.” (Second Part). By Dr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of October 30, 1926).

B. Experimental Part.

1. The equation of the vapour pressure.

The experimental material is here, indeed, not very large, but still sufficient to derive a fairly reliable vapour pressure equation from it. In the following table the vapour pressures measured are given, and by the side of them the values of p calculated from the two formulae:

$$\left. \begin{aligned} \log^{10} p^{mm} &= -\frac{3,8308}{T} + 2,9044 + 0,2107 T \quad (0^\circ \text{ to } 4^\circ, 2) \\ \log^{10} p^{mm} &= -\frac{4,9874}{T} + 3,5083 + 0,1325 T \quad (3^\circ, 5-5^\circ, 2) \end{aligned} \right\} \dots (10)$$

The observations marked with an asterisk (bold type) are considered as the most accurate by the authors¹⁾. For this reason they have been used for the calculation of the constants in both formulae. When the

TABLE I.

| T | $p_{exp.}^{mm}$ | $p_{form.}$ | T | $p_{exp.}^{mm}$ | $p_{form.}$ |
|---------------|-----------------|-------------|---------------|-----------------|-------------|
| 1.35 | (3) | 2.25 | *3.516 | 359.5 | 359.5 |
| *1.475 | 4.15 | 4.15 | 3.89 | 565 | 552 |
| 2.24 | 51 | 46.4 | *4.205 | 757.5 | 757.5 |
| 3.17 | (197) | 231 | 4.21 | 760 } 767 } | 761 |
| *3.516 | 359.5 | 359.5 | 4.90 | (1329) | 1380 |
| 3.89 | 565 | 549 | 5.04 | 1520 | 1536 |
| *4.205 | 757.5 | 757.5 | 5.09 | 1569 | 1596 |
| 4.21 | 760 } 767 } | 761 | 5.16 | 1668 | 1681 |
| | | | *5.19 | 1718 | 1718 |

¹⁾ K. ONNES, Comm. 119^a (1911) and 124^b (1911); K. ONNES and S. WEBER, Comm. 147^b (1915).

experimental values are represented grafically ($T \log p$ against T), it is at once perceived that the pressures at $1^{\circ},35$, $3^{\circ},17$ and $4^{\circ},90$ (placed between brackets by us) are *not* in conformity with the others (one too high, the two others too low), and can, therefore, be left out of consideration in the calculation of the formulae. If this is overlooked, it leads of course to great disappointment, and the method of least squares will be of no use; which method is mathematical nonsense with a comparatively small number of observations. In contrast with the said values, those at $2^{\circ},24$, $3^{\circ},89$, $5^{\circ},04$ and $5^{\circ},16$ are in pretty good harmony with the others.

The coefficient 4,9874 in our second formula, of course, lacks the physical meaning of the corresponding coefficient 3,8303 in the first formula, viz. $= \lambda_0 : R$, because the second formula is only valid to $3^{\circ},5$ downwards.

The onlogically built formula of Comm. 147*b*, where the constant 7,98 instead of 3,83 is much too high, suffers from the same evil. Terms with $1/T^2$ and $1/T^3$ may occur neither in a theoretical, nor in an empirical formula. At low temperatures $\log p$ must approach to $-A/T$, whereas according to the formula mentioned $\log p$ would approach to $-A'/T^3$, which is an absurdity. Already in the neighbourhood of $2^{\circ},4$ it gives much too low values for p ¹⁾.

If we wish to construct a theoretical formula, which is valid throughout the *whole* region, the variability of a with T , of b with T and v , and of v_1 with T must be taken into account over a great range; in any case an equation arises of the form $\log p = -\frac{A}{T} + B \log T + C + DT + ET^2 \dots$. But this problem is particularly difficult for Helium with its deviating behaviour at lower temperatures.

In the following table the values of p have been calculated according to both formulae from 0,1 to 0,1 degree. At the same times the values of $F = \frac{T dp}{p dt} = T \frac{d \log p}{dt}$ are then obtained. For from our equation

$$\log^{10} p = -\frac{A}{T} + C + DT \text{ follows immediately } F = \left(\frac{A}{T} + DT \right) \times 2,3026,$$

of which the terms $\frac{A}{T}$ and DT are, therefore, already known from the calculation of $\log^{10} p$. The accurate knowledge of F is necessary for the calculation of

$$L = T \frac{dp}{dt} (v_2 - v_1) = F \times p (v_2 - v_1) = F \times W \left\{ \begin{array}{l} . . . (11) \\ \lambda = L - p (v_2 - v_1) = L - W = (F - 1) W \end{array} \right.$$

and

which quantities we shall calculate in § 4.

¹⁾ Cf. also Note 2) added to Table IV.

TABLE II.
a. Values of ρ and F between 0° and $4^\circ.2$.

| T | $\frac{3.8308}{T}$ | $0.2107 T$ | $\log^{10} \rho^{mm}$ | ρ^{mm} | $F: 2.3026$ | F |
|-----|--------------------|------------|-----------------------|----------------------|-------------|----------|
| 0 | ∞ | 0 | $-\infty$ | 0 | ∞ | ∞ |
| 0.5 | 7.6616 | 0.1054 | - 4.6518 | 0.0 ₄ 223 | 7.7670 | 17.88 |
| 0.6 | 6.3847 | 0.1264 | - 3.3539 | 0.0 ₃ 443 | 6.5111 | 14.99 |
| 0.7 | 5.4726 | 0.1475 | - 2.4207 | 0.0 ₂ 380 | 5.6201 | 12.94 |
| 0.8 | 4.7885 | 0.1686 | - 1.7155 | 0.0193 | 4.9571 | 11.41 |
| 0.9 | 4.2564 | 0.1896 | - 1.1624 | 0.0688 | 4.4460 | 10.24 |
| 1.0 | 3.8308 | 0.2107 | - 0.7157 | 0.1924 | 4.0415 | 9.306 |
| 1.1 | 3.4825 | 0.2318 | - 0.3463 | 0.4505 | 3.7143 | 8.553 |
| 1.2 | 3.1923 | 0.2528 | - 0.0351 | 0.9224 | 3.4451 | 7.933 |
| 1.3 | 2.9468 | 0.2739 | + 0.2315 | 1.704 | 3.2207 | 7.416 |
| 1.4 | 2.7363 | 0.2950 | 0.4631 | 2.90 $\bar{5}$ | 3.0313 | 6.980 |
| 1.5 | 2.5539 | 0.3161 | 0.6666 | 4.641 | 2.8700 | 6.608 |
| 1.6 | 2.3943 | 0.3371 | 0.8472 | 7.034 | 2.7314 | 6.289 |
| 1.7 | 2.2534 | 0.3582 | 1.0092 | 10.21 | 2.6116 | 6.013 |
| 1.8 | 2.1282 | 0.3793 | 1.1555 | 14.31 | 2.5075 | 5.774 |
| 1.9 | 2.0162 | 0.4003 | 1.2885 | 19.43 | 2.4165 | 5.564 |
| 2.0 | 1.9154 | 0.4214 | 1.4104 | 25.73 | 2.3368 | 5.381 |
| 2.1 | 1.8242 | 0.4425 | 1.5227 | 33.32 | 2.2667 | 5.219 |
| 2.2 | 1.7413 | 0.4635 | 1.6266 | 42.33 | 2.2048 | 5.077 |
| 2.3 | 1.6656 | 0.4846 | 1.7234 | 52.89 | 2.1502 | 4.951 |
| 2.4 | 1.5962 | 0.5057 | 1.8139 | 65.1 $\bar{5}$ | 2.1019 | 4.840 |
| 2.5 | 1.5323 | 0.5268 | 1.8989 | 79.23 | 2.0591 | 4.741 |
| 2.6 | 1.4734 | 0.5478 | 1.9788 | 95.24 | 2.0212 | 4.654 |
| 2.7 | 1.4188 | 0.5689 | 2.0545 | 113.4 | 1.9877 | 4.577 |
| 2.8 | 1.3681 | 0.5900 | 2.1263 | 133.8 | 1.9581 | 4.509 |
| 2.9 | 1.3210 | 0.6110 | 2.1944 | 156.5 | 1.9320 | 4.449 |
| 3.0 | 1.2769 | 0.6321 | 2.2596 | 181.8 | 1.9090 | 4.396 |
| 3.1 | 1.2357 | 0.6532 | 2.3219 | 209.8 | 1.8889 | 4.349 |
| 3.2 | 1.1971 | 0.6742 | 2.3815 | 240.7 | 1.8713 | 4.309 |
| 3.3 | 1.1608 | 0.6953 | 2.4389 | 274.7 | 1.8561 | 4.274 |
| 3.4 | 1.1267 | 0.7164 | 2.4941 | 312.0 | 1.8431 | 4.244 |
| 3.5 | 1.0945 | 0.7375 | 2.5474 | 352.7 | 1.8320 | 4.218 |
| 3.6 | 1.0641 | 0.7585 | 2.5988 | 397.0 | 1.8226 | 4.197 |
| 3.7 | 1.0354 | 0.7796 | 2.6486 | 445.2 | 1.8150 | 4.179 |
| 3.8 | 1.0081 | 0.8007 | 2.6970 | 497.7 | 1.8088 | 4.165 |
| 3.9 | 0.9823 | 0.8217 | 2.7438 | 554.4 | 1.8040 | 4.154 |
| 4.0 | 0.9577 | 0.8428 | 2.7895 | 615.9 | 1.8005 | 4.146 |
| 4.1 | 0.9343 | 0.8639 | 2.8340 | 682.3 | 1.7982 | 4.141 |
| 4.2 | 0.9121 | 0.8849 | 2.8772 | 753.7 | 1.7970 | 4.138 |

b. Values of p and F between $3^{\circ}.5$ and $5^{\circ}.2$.

| T | $\frac{4.9874}{T}$ | $0.1325 T$ | $\log^{10} p^{mm}$ | p^{mm} | $F: 2.3026$ | F |
|-----|--------------------|------------|--------------------|----------|-------------|-------|
| 3.5 | 1.4250 | 0.4638 | 2.5471 | 352.5 | 1.8888 | 4.349 |
| 3.6 | 1.3854 | 0.4770 | 2.5999 | 398.0 | 1.8624 | 4.288 |
| 3.7 | 1.3479 | 0.4903 | 2.6507 | 447.4 | 1.8382 | 4.233 |
| 3.8 | 1.3125 | 0.5035 | 2.6993 | 500.4 | 1.8160 | 4.182 |
| 3.9 | 1.2788 | 0.5168 | 2.7463 | 557.6 | 1.7956 | 4.135 |
| 4.0 | 1.2469 | 0.5300 | 2.7914 | 618.6 | 1.7769 | 4.091 |
| 4.1 | 1.2164 | 0.5433 | 2.8352 | 684.2 | 1.7597 | 4.052 |
| 4.2 | 1.1875 | 0.5565 | 2.8773 | 753.9 | 1.7440 | 4.016 |
| 4.3 | 1.1599 | 0.5698 | 2.9182 | 828.3 | 1.7297 | 3.983 |
| 4.4 | 1.1335 | 0.5830 | 2.9578 | 907.4 | 1.7165 | 3.952 |
| 4.5 | 1.1083 | 0.5963 | 2.9963 | 991.5 | 1.7046 | 3.925 |
| 4.6 | 1.0824 | 0.6095 | 3.0336 | 1080 | 1.6937 | 3.900 |
| 4.7 | 1.0611 | 0.6228 | 3.0700 | 1175 | 1.6839 | 3.877 |
| 4.8 | 1.0390 | 0.6360 | 3.1053 | 1274 | 1.6750 | 3.857 |
| 4.9 | 1.0178 | 0.6493 | 3.1398 | 1380 | 1.6671 | 3.839 |
| 5.0 | 0.9975 | 0.6625 | 3.1733 | 1490 | 1.6600 | 3.822 |
| 5.1 | 0.9779 | 0.6758 | 3.2062 | 1608 | 1.6537 | 3.808 |
| 5.2 | 0.9591 | 0.6890 | 2.2382 | 1731 | 1.6481 | 3.795 |

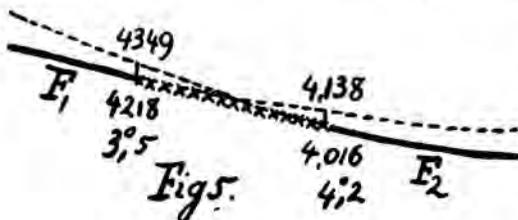
Hence we find for the value of F at the critical temperature $5^{\circ}.19$:

$$F_k = 1,6487 \times 2,3026 = 3,7963 = \underline{3,80}.$$

The almost identical values of p between $3^{\circ}.5$ and $4^{\circ}.2$, obtained with the two formulae, may now simply be averaged. The values of

F should be treated in another way. Care should be taken, that the two parts F_1 and F_2 in the added graphical representation (see Fig. 5) pass continuously into each other (denoted by the line $\times\times\times\times$). For this

purpose the differences Δ should only be gradually decreased from 35 (between $3^{\circ}.2$ and $3^{\circ}.3$) to 27 (between $4^{\circ}.4$ and $4^{\circ}.5$). Cf. Table c.



c. Adjusted values between 3°.5 and 4°.2.

| <i>T</i> | <i>p</i> ₁ | <i>p</i> ₂ | <i>p</i> | <i>F</i> ₁ | <i>F</i> ₂ | <i>F</i> |
|----------|-----------------------|-----------------------|----------|-----------------------|-----------------------|---------------------|
| 3.2 | 240.7 | — | 240.7 | 4.309 ^Δ | — | 4.309 ^Δ |
| 3.3 | 274.7 | — | 242.7 | 4.274 ³⁵ | — | 4.274 ³⁵ |
| 3.4 | 312.0 | — | 312.0 | 4.244 ³⁰ | — | 4.242 ³² |
| 3.5 | 352.7 | 352.5 | 352.6 | 4.218 ²⁶ | 4.349 ^Δ | 4.212 ³⁰ |
| 3.6 | 397.0 | 398.0 | 397.5 | 4.197 ²¹ | 4.288 ⁶¹ | 4.183 ²⁹ |
| 3.7 | 445.2 | 447.4 | 446.3 | 4.179 ¹⁸ | 4.233 ⁵⁵ | 4.154 ²⁹ |
| 3.8 | 497.7 | 500.4 | 499.1 | 4.165 ¹⁴ | 4.182 ⁵¹ | 4.125 ²⁹ |
| 3.9 | 554.4 | 557.6 | 556.0 | 4.154 ¹¹ | 4.135 ⁴⁷ | 4.096 ²⁹ |
| 4.0 | 615.9 | 618.6 | 617.2 | 4.146 ⁸ | 4.091 ⁴⁴ | 4.067 ²⁹ |
| 4.1 | 682.3 | 684.2 | 683.3 | 4.141 ⁵ | 4.052 ³⁹ | 4.038 ²⁹ |
| 4.2 | 753.7 | 753.9 | 753.8 | 4.138 ³ | 4.016 ³⁶ | 4.009 ²⁹ |
| 4.3 | — | 828.3 | 828.3 | — | 3.983 ³³ | 3.980 ²⁹ |
| 4.4 | — | 907.4 | 907.4 | — | 3.952 ³¹ | 3.952 ²⁸ |
| 4.5 | — | 991.5 | 991.5 | — | 3.925 ²⁷ | 3.925 ²⁷ |

2. The second Virial-Coefficient *B* and the vapour volume *v*₂.

For the calculation of the values of *v*₂ (required for $W = p(v_2 - v_1)$), the values of *B* in

$$pv_2 = RT + \frac{B}{v_2} \dots \dots \dots (12)$$

should be rather accurately known. Only at very low temperatures (to about 0°.7) $pv_2 = RT$ may be put; the deviations soon become pretty great for Helium, much greater than for "ordinary" substances. The values of *B*, calculated in the following table, have been derived from the formula

$$10^6 B = (-44.9 + 2,032 T) e^{\frac{1.155}{T}}, \dots \dots \dots (13)$$

the coefficients of which has been calculated from the pretty certain values 512 at 0°C.; 0 at 22°.1 abs. (*Boyle point*); and -47.7 at 4°.23 abs.; marked in the subjoined table with an asterisk and printed in bold type. This theoretical formula was derived¹⁾ by me at the time in the form

$$B = (RT \varphi_1 b_k - \varphi_2 a_k) e^{\frac{\alpha}{T} - \frac{\alpha}{T_k}} = (-\beta_1 + \beta_2 T) e^{\frac{\alpha}{T}},$$

the efficiency of which has been tested by me by a number of substances

¹⁾ J. de Ch. ph. 17 (1919), p. 266-324. Cf. also Zustandsgl. p. 21 et seq.

(He, H₂, Ne, Ar, O₂, N₂, CO₂, CH₃Cl). The available material of facts for *Helium* is combined in Table III.

TABLE III. 1)

| <i>T</i> | $10^6 B_{exp.}$ | $10^6 B_{formula}$ |
|------------------|---|--------------------|
| 673 (400° C.) | + 1126 (from 457 H.O.) | + 1325 |
| 573 (300° ..) | + 986 (.. 470 ..) | + 1125 |
| 473 (200° ..) | + 854 (.. 493 ..) | + 919 |
| 373 (100° C.) | + 701 (from 513 H.O.); K.O. 493 at 100°.35 | + 716* |
| 323 (50° ..) | + 619 (.. 523 ..) | + 614* |
| 293 (20° ..) | + 534 (.. 448 K.O.) | + 553* |
| *273.09 (0° ..) | + 512 (K.O.; 529 H.O.) | + 512 |
| 169.52 | + 337 (from 543 K.O.) | + 302 |
| 90.34 | + 176 (.. 532 ..) | + 141 |
| 69.86 | + 100.3 (v. A.; K.O.) | + 98.7* |
| 56.53 | (+ 96.0) (from 464 K.O.) | + 71.4 |
| 55.68 | (+ 97.0) (.. 476 P.; K.O.) | + 69.6 |
| *22.1 (BOYLE-p.) | 0 (v. A.; K.O. grafically from †) | 0 |
| 20.55 | - 4.2 | - 3.4 (20°.5)* |
| 53 | - 9.0 } † (v.A.; K.O.) | |
| 51 | - 9.1 | |
| 48 | - 7.4 (from - 99 Pal.; K.O.) | |
| 20.37 | (+ 3.0) (.. + 40 K.O.*) | - 3.7 |
| 18.22 | (- 24.4) } † (v.A.; K.O.) | - 8.4 |
| 16.55 | (- 24.5) } | - 12.1 |
| 14.27 | - 11.4 (from - 211 K.O.*) | - 17.3 |
| 4.71 | - 45.9 | - 45.2 |
| 59 | - 46.9 } (from D ₂ M.; Cr.; K.O.; Sw.) | |
| 29 | - 47.0 (from - 2990 K.O.**) | |
| *4.23 | - 47.7 (.. D ₂ M. - Sw.) | - 47.7 |

1) The addition: "from 457 H. O." means that 457 must be multiplied by $1 + 0,0036618 t$. For these values refer to $p v_2 = RT \left(1 + \frac{B'}{v} \right)$, so that our $B = RTB' = (273 + t) : 273 \cdot B' = (1 + 0,0036618 t) B'$. ($R = 1 : 273,09$ in norm. units).

H.O. means HOLBORN en OTTO 1924; K.O. = K. ONNES, Comm. 102a; K.O.* =

In particularly good agreement are the values (marked with an asterisk) at 100° , 50° and 20° C., at $69^{\circ}.9$ abs., $20^{\circ}.5$ abs. (-4.2 against calculated -3.4) and the three values at about $4^{\circ}.5$ abs. The values at 56° and at $20^{\circ}.4$, $18^{\circ}.2$ and $16^{\circ}.6$, placed between parentheses, on the other hand, fall outside the schema, also of a graphical representation, and may, therefore, be ignored. We may, therefore, safely use the formula (13) for the calculation of the B -values at some temperatures below the critical temperature, occurring in Comm. 172*b*. These values, and those below $2^{\circ}.3$ are necessary for the determination of D_2 for

TABLE IV.

| T | $10^6 RT$ | $p_{form}^{atm.}$ | $10^6 B_{form.}$ | $10^6 v_2 calc.$ | $10^4 D_2 calc.$ | $10^4 D_2$ Comm. 172 <i>b</i> | $10^6 pv_2$ <i>calc.</i> | Δ BOYLE in % |
|------|-----------|-------------------|---------------------------|----------------------|---------------------|----------------------------------|-----------------------------|------------------------|
| 5.19 | 19005 | 2.2605 | (-42.9) ¹⁾ | 2572 ³⁾ | 693.8 ³⁾ | 693.0 ³⁾ | 5815 | 69.4 |
| 4.71 | 17247 | 1.5592 | -45.2 | 6800 | 262.4 | 269.9 | 1060.10 | 38.6 |
| 4.59 | 16808 | 1.4092 | -45.7 | 7737 | 230.6 | 238.9 ⁴⁾ | 1090.10 | 35.2 |
| 4.23 | 15489 | 1.0197 | -47.7 | 1090.10 | 163.7 | 163.7 | 1111.10 | 28.2 |
| 4.22 | 15453 | 1.0111 | -47.8 | 1098.10 | 162.5 | 161.8 | 1112.10 ⁵⁾ | 28.0 |
| 3.90 | 14281 | 0.7316 | -49.7 | 1499.10 | 119.0 | 117.6 | 1097.10 | 23.2 |
| 3.30 | 12084 | 0.3614 | -54.2 | 2810.10 | 63.50 | 64.35 | 1016.10 | 16.0 |
| 2.56 | 9374 | 0.1169 | -62.3 | 7288.10 | 24.48 | (20.79) | 8520 | 9.11 |
| 2.37 | 8678 | 0.08088 | -65.2 ²⁾ | 9916.10 | 17.99 | (13.68) ²⁾ | 8020 | 7.58 |
| 2.30 | 8422 | 0.06959 | -66.5 | 1125.10 ² | 15.86 | (11.59) | 7830 | 7.03 |
| 2.29 | 8386 | 0.06812 | -66.7 | 1146.10 ² | 15.57 | — | 7804 | 6.94 |
| 2.21 | 8093 | 0.05699 | -68.1 | 1330.10 ² | 13.42 | — | 7580 | 6.34 |
| 2.10 | 7690 | 0.04384 | -70.4 | 1657.10 ² | 10.77 | — | 7264 | 5.54 |
| 1.93 | 7067 | 0.02788 | -74.5 | 2425.10 ² | 7.358 | — | 6761 | 4.33 |
| 1.92 | 7031 | 0.02709 | -74.8 | 2484.10 ² | 7.184 | — | 6729 | 4.30 |
| 1.59 | 5822 | 0.008897 | -86.2 | 6392.10 ² | 2.792 | — | 5687 | 2.32 |
| 1.28 | 4687 | 0.001997 | -104 | 2325.10 ³ | 0.7675 | — | 4643 | 0.94 |
| 1.20 | 4394 | 0.001214 | -111 | 3595.10 ³ | 0.4964 | — | 4363 | 0.70 |

ibid. Comm. 102*c*; K.O.** = ibid. Comm. 119*a*; P.; K.O. = PENNING and K.O., Comm. 165*c*; Pal.; K.O. = PALACIOS MARTINEZ and K.O., Comm. 164; v. A.; K.O. = VAN AGT and K.O., Comm. 176*b*, p. 28 (1925); M.; Cr.; K.O.; Sw. = MATHIAS, CROMMELIN, K.O. and SWALLOW, Comm. 172*b*. Compare also KEESOM and K.O., Arch. Néerl. IX, N^o. 1 (1925).

Cf. the notes 1, 2, 3, 4 and 5 of Table IV on the following page.

the calculation of the diameter. For this reason the values of v_2 , calculated from (12), resp. $D_2=4:22416 v_2=178,444 \cdot 10^{-6}:v_2$ (see § 2, 1st Part, after (4)), are inserted in the foregoing table by the side of those of B ; and besides those of pv_2 , to judge about the deviations from the law of BOYLE.

After these preliminary calculations (as said before, necessary for the calculation of the diameter, see § 3), we have again inserted all the quantities of Table IV in Table V, but now over the whole region and from 0.1 to 0.1 degree. The values of v_2 (and D_2) above $4^{\circ}.7$ cannot be calculated accurately by means of (12), because then the vapour volumes become too small. We have, therefore, calculated the values of D_2 (and of D_1) at these temperatures from formulae for $\frac{1}{2}(D_1+D_2)$ and $\frac{1}{2}(D_1-D_2)$ (see § 3), and then v_2 (and v_1) from D_2 (and D_1). The values of v_2 and D_2 from $4^{\circ}.3$ to $4^{\circ}.7$ (inclusive), placed between brackets, have been calculated in the same way; it is seen that they differ but little from the values calculated from (12), which may be considered to be very accurate; since the values of B on which they are based (see Table III) have been calculated directly from the *observed* values of D_2 according to Comm. 172b, and therefore refer to the saturation line.

The deviations from the law of BOYLE are comparatively great, and negligible only up to $0^{\circ}.7$. Above this pv_2 may no longer be put equal to RT . The same applies to v_2-v_1 (see § 3), for which v_2 may be put only to $0^{\circ}.9$. For benzene the deviation from the law of BOYLE at 70° ($m=0.61$) is 2% (Zustandsgl. p. 156); this already amounting to almost 15% for Helium at the same reduced temperature ($T=3^{\circ}.2$)! This is

1) This value is no longer valid on the saturation line, as the critical volume can no longer be considered as large. Terms should then still be added with $C: v_2^2$ etc. From $B=(RT_k-p_k v_k)v_k$ is calculated the correct value -34.0 .

2) The B -values, used by M.; Cr.: K.O. and Sw. for the calculation of their D_2 -values, viz. resp. -93.40 , -98.05 and -99.87 (according to a kind communication from Dr. CROMMELIN), are much too great negative in my opinion, at least on the saturation line. But chiefly the p -values, used by the said authors, viz. resp. 71.10 , 46.37 and 38.46 mm. (instead of 88.84 , 61.47 and 52.89 mm., calculated according to (10), cf. also Table IIa), are much too low. For the experimental value at $2^{\circ}.24$ is already 51 mm. (calculated 46.6 , see Table I); the pressure at the *higher* temperature $2^{\circ}.30$, therefore, *cannot possibly* be $=38.46$! The reason is, that the authors have used the formula of Comm. 147b (cf. § 1), which is quite inadmissible at these temperatures. It is chiefly these erroneous p -values, that have caused the great errors in the D_2 -values calculated by them, through which e.g. 11.59 was found for $10^4 D_2$ at $2^{\circ}.30$ instead of 15.86 , which value is almost 50% higher.

3) Calculated from the diameter, see further.

4) These values have been directly observed: they are in pretty good agreement with the calculated values. The following values have been calculated: the first three are still correct, but the last three are, in my opinion, entirely wrong. Comp. Note 2).

5) Here a maximum for pv_2 at $m=T: T_k=4.22:5.19=0.81$ is duly found. For "ordinary" substances this maximum lies at the same place: for benzene e.g. at $m=0.80$ (Cf. Zustandsgl. p. 255, Fig. 8). It appears from Table IX, that the maximum of $W=p(v_2-v_1)$ is situated at $m=0.73$ ($T=3^{\circ}.8$), against 0.77 for benzene.

owing to the fact, that with equal m the deviations are about proportional to the *reduced pressures* $\varepsilon = p : p_k$. And since for Benzene $\varepsilon = 548 : 36486 = 0.015$ at $m = 0.61$, and for the Helium $= 231.4 : 1718 = 0.135$,

TABLE V.

| T | m | $10^6 RT$ | $p^{atm.}$ | $10^6 B$ | $10^6 v_2$ | $10^4 D_2$ | $10^6 p v_2$ | Δ BOYLE (0/0) ($RT - p v_2$) : : $RT \times 100$ |
|-----|---------|-----------|------------|----------|-------------------|------------|--------------|---|
| 0 | 0 | 0 | 0 | ∞ | ∞ | 0 | 0 | 0 |
| 0.5 | 0.09634 | 1831 | 0.072934 | -442 | $6241 \cdot 10^7$ | 0.0429 | 1831 | 0.0339 |
| 0.6 | 0.1156 | 2197 | 0.065825 | -299 | $3772 \cdot 10^6$ | 0.0347 | 2197 | 0.0236 |
| 0.7 | 0.1349 | 2563 | 0.054995 | -226 | $5131 \cdot 10^5$ | 0.0235 | 2563 | 0.017 |
| 0.8 | 0.1541 | 2929 | 0.042533 | -183 | $1156 \cdot 10^5$ | 0.015 | 2927 | 0.055 |
| 0.9 | 0.1734 | 3296 | 0.049053 | -155 | $3636 \cdot 10^4$ | 0.049 | 3292 | 0.13 |
| 1.0 | 0.1927 | 3662 | 0.032532 | -136 | $1443 \cdot 10^4$ | 0.12 | 3653 | 0.26 |
| 1.1 | 0.2119 | 4028 | 0.035928 | -122 | $6764 \cdot 10^3$ | 0.26 | 4010 | 0.45 |
| 1.2 | 0.2312 | 4394 | 0.021214 | -111 | $3595 \cdot 10^3$ | 0.50 | 4363 | 0.70 |
| 1.3 | 0.2505 | 4760 | 0.02242 | -103 | $2101 \cdot 10^3$ | 0.85 | 4711 | 1.03 |
| 1.4 | 0.2698 | 5127 | 0.023822 | -96.0 | $1322 \cdot 10^3$ | 1.35 | 5054 | 1.42 |
| 1.5 | 0.2890 | 5493 | 0.026107 | -90.4 | $8827 \cdot 10^2$ | 2.02 | 5391 | 1.86 |
| 1.6 | 0.3083 | 5859 | 0.029255 | -85.7 | $6180 \cdot 10^2$ | 2.89 | 5721 | 2.37 |
| 1.7 | 0.3276 | 6225 | 0.01343 | -81.8 | $4500 \cdot 10^2$ | 3.97 | 6043 | 2.92 |
| 1.8 | 0.3468 | 6591 | 0.01883 | -78.4 | $3377 \cdot 10^2$ | 5.28 | 6359 | 3.52 |
| 1.9 | 0.3661 | 6957 | 0.02557 | -75.4 | $2608 \cdot 10^2$ | 6.84 | 6668 | 4.16 |
| 2.0 | 0.3854 | 7324 | 0.03386 | -72.8 | $2059 \cdot 10^2$ | 8.67 | 6970 | 4.83 |
| 2.1 | 0.4046 | 7690 | 0.04384 | -70.4 | $1657 \cdot 10^2$ | 10.77 | 7264 | 5.54 |
| 2.2 | 0.4239 | 8056 | 0.05570 | -68.3 | $1356 \cdot 10^2$ | 13.16 | 7552 | 6.26 |
| 2.3 | 0.4432 | 8422 | 0.06959 | -66.5 | $1125 \cdot 10^2$ | 15.86 | 7830 | 7.03 |
| 2.4 | 0.4624 | 8788 | 0.08572 | -64.8 | $9452 \cdot 10$ | 18.88 | 8102 | 7.81 |
| 2.5 | 0.4817 | 9155 | 0.1043 | -63.2 | $8022 \cdot 10$ | 22.24 | 8367 | 8.61 |
| 2.6 | 0.5010 | 9521 | 0.1253 | -61.8 | $6882 \cdot 10$ | 25.93 | 8623 | 9.43 |
| 2.7 | 0.5202 | 9887 | 0.1492 | -60.5 | $5944 \cdot 10$ | 30.02 | 8869 | 10.3 |
| 2.8 | 0.5395 | 10253 | 0.1761 | -59.3 | $5172 \cdot 10$ | 34.50 | 9108 | 11.2 |
| 2.9 | 0.5588 | 10619 | 0.2059 | -58.1 | $4535 \cdot 10$ | 39.35 | 9338 | 12.1 |

which is 9 times greater, the $7^{1/2}$ times greater deviation for Helium is accounted for.

TABLE V (Continued).

| T | m | $10^6 RT$ | $p^{atm.}$ | $16^6 B$ | $10^6 v_2$ | $10^4 D_2$ | $10^6 pv_2$ | Δ BOYLE (%) ($RT - pv_2$): : $RT \times 100$ |
|------|--------|-----------|------------|----------|-------------------|---------------|-------------|---|
| 3.0 | 0.5780 | 10985 | 0.2392 | -57.0 | 3997.10 | 44.64 | 9561 | 13.0 |
| 3.1 | 0.5973 | 11352 | 0.2761 | -56.0 | 3545.10 | 50.34 | 9772 | 13.9 |
| 3.2 | 0.6166 | 11718 | 0.3167 | -55.1 | 3147.10 | 56.70 | 9967 | 14.9 |
| 3.3 | 0.6358 | 12084 | 0.3614 | -54.2 | 2810.10 | 63.50 | 1016.10 | 16.0 |
| 3.4 | 0.6551 | 12450 | 0.4105 | -53.4 | 2516.10 | 70.92 | 1033.10 | 17.0 |
| 3.5 | 0.6744 | 12816 | 0.4639 | -52.6 | 2261.10 | 78.92 | 1049.10 | 18.2 |
| 3.6 | 0.6936 | 13182 | 0.5230 | -51.8 | 2034.10 | 87.73 | 1064.10 | 19.3 |
| 3.7 | 0.7129 | 13549 | 0.5872 | -51.1 | 1833.10 | 97.35 | 1076.10 | 20.6 |
| 3.8 | 0.7322 | 13914 | 0.6567 | -50.4 | 1655.10 | 107.8 | 1087.10 | 21.9 |
| 3.9 | 0.7515 | 14281 | 0.7316 | -49.7 | 1499.10 | 119.0 | 1097.10 | 23.2 |
| 4.0 | 0.7707 | 14647 | 0.8121 | -49.1 | 1359.10 | 131.3 | 1104.10 | 24.7 |
| 4.1 | 0.7900 | 15013 | 0.8991 | -48.5 | 1232.10 | 144.8 | 1108.10 | 26.2 |
| 4.2 | 0.8093 | 15380 | 0.9918 | -47.9 | 1119.10 | 159.5 | 1111.10 | 27.8 |
| 4.3 | 0.8285 | 15746 | 1.090 | -47.3 | 1019.10 (1021.10) | 175.1 (174.8) | 1112.10 | 29.4 |
| 4.4 | 0.8478 | 16112 | 1.194 | -46.8 | 9271 (9298) | 192.5 (191.9) | 1109.10 | 31.2 |
| 4.5 | 0.8671 | 16478 | 1.305 | -46.2 | 8425 (8461) | 211.8 (210.9) | 1102.10 | 33.1 |
| 4.6 | 0.8863 | 16844 | 1.421 | -45.7 | 7649 (7658) | 233.3 (233.0) | 1088.10 | 35.4 |
| 4.7 | 0.9056 | 17210 | 1.546 | -45.2 | 6887 (6878) | 259.1 (259.4) | 1064.10 | 38.2 |
| 4.8 | 0.9249 | 17577 | 1.676 | (-44.7) | 6108 | 292.1 | 1024.10 | 41.8 |
| 4.9 | 0.9441 | 17943 | 1.816 | (-44.2) | 5340 | 334.1 | 9697 | 46.0 |
| 5.0 | 0.9634 | 18308 | 1.961 | (-43.8) | 4564 | 391.0 | 8950 | 51.1 |
| 5.1 | 0.9827 | 18675 | 2.116 | (-43.3) | 3744 | 476.6 | 7922 | 57.6 |
| 5.19 | 1 | 19005 | 2.261 | (-42.9) | 2572 | 693.8 | 5815 | 69.4 |

From $pv_2 = RT + \frac{B}{v_2}$ follows $\Delta = (RT - pv_2) : RT = -\frac{B}{v_2} : RT$. And B being $= RTb_g - a_g$ for large volumes, Δ becomes $= (a_g - RTb_g) : RTv_2$, in which a_g and b_g are the values of a and b at the temperature T

and at $v = \infty$. As a is practically independent of the volume, a may simply be written instead of a_g . Now at lower temperatures $\leq a_k$ may be substituted for $a - RTb_g = \theta a_k - RTb_g$ (in which for ordinary substances θ is always > 1 , but for Helium < 1); hence we get ($m = T; T_k$):

$$\Delta \leq \frac{a_k}{RTv_2} \leq \frac{a_k \cdot p}{RT \cdot pv_2} \leq \frac{a_k \cdot p}{(RT)^2} \leq \frac{a_k \cdot p_k}{m^2 (RT_k)^2}$$

With $p_k = \frac{1}{27} \frac{a_k}{b_k^2}$, $RT_k = \frac{8}{27} \frac{a_k}{b_k}$ this becomes:

$$\Delta \leq \frac{27}{64} \frac{\varepsilon}{m^2} \dots \dots \dots (14)$$

From this follows, therefore, that with equal m , Δ is "about" proportional to ε . With his usual lack of thoroughness NERNST writes simply ε for $[RT - p(v_2 - v_1)] : RT$, which does not differ much from Δ up to near T_k . This is only correct in approximation at $m^2 = 27/64$, $m = 0,63$, but of course not with arbitrary values of m . How great the differences, accordingly, are between NERNST and reality, especially at higher reduced temperatures, appears from the table on p. 257 of my "Zustands-gleichung".

Let us take as example Helium at $m = 0,5$ ($T = 2^{\circ},6$). Here ε is $= 0,1253 : 2,261 = 0,0554$, so that Δ is somewhat $< \frac{27}{64} \times 0,2216$, or $< 0,109$. We really found $9,43 \%$, i.e. $0,094$.

The explanation, why ε for Helium is e.g. 9 times greater than for benzene at the same value of m (see above), follows from the approximated vapour pressure formula $-\log \varepsilon = f \left(\frac{1}{m} - 1 \right)$ or $\varepsilon = e^{-f \left(\frac{1}{m} - 1 \right)}$. Now $f = 3,16$ for Helium at $m = 0,61$, and $f = 6,59$ for benzene¹⁾, so that ε becomes resp. $= e^{-2,02}$ and $e^{-4,20}$, i.e. for Helium $e^{2,18} = 8,8 = 9$ times greater than for benzene.

The same ratios are found for $v_2 : v_1$ (see Table IX). Thus e.g. $v_2 : v_1$ is only $= 23$ for Helium ($3^{\circ},26$) at $m = 0,63$, this ratio being 300 at the boiling-point (80° C.) of benzene (m likewise $= 0,63$).

3. The values of $\frac{1}{2}(D_1 + D_2)$ and $\frac{1}{2}(D_1 - D_2)$ in the neighbourhood of the critical temperature.

For the calculations of the so-called "Diameter" we only use those data, that are sufficiently far from the maximum liquid density at $2^{\circ},3$; we take the following values of D_1 and D_2 (see also Table IV).

¹⁾ This depends again on the family of the substances, as $f = 8\gamma$, and γ is $= 0,9$ or 1 for ordinary substances and $< 0,5$ for Helium.

TABLE VI.

| T | D_1 (C.172 ^b) | $D_{2\text{ca.c.}}$ (Table IV) | $\frac{1}{2}(D_1 + D_2)$ | Ibid formula |
|-------|-----------------------------|--------------------------------|--------------------------|----------------------|
| *3.30 | 0.1395 | 0.0063 ⁵ | 0.0729 ²⁵ | 0.0729 ²⁵ |
| *3.90 | 0.1311 | 0.0119 ⁰ | 0.0715 ⁰ | 0.0715 ⁰ |
| 4.22 | 0.1255 | 0.0162 ⁵ | 0.0708 ⁷⁵ | 0.0708 ⁵⁴ |
| *4.23 | 0.1253 | 0.0163 ⁷ | 0.0708 ³⁵ | 0.0708 ³⁵ |
| 4.59 | 0.1165 | 0.0230 ⁶ | 0.0697 ⁸ | 0.0702 ⁰⁶ |
| 4.71 | 0.1139 | 0.0262 ⁴ | 0.0700 ⁷ | 0.0700 ¹⁸ |

From the values at 3^o,30, 3^o,90 and 4^o,23, marked by us with an asterisk, we calculate:

$$10^4 \cdot \frac{1}{2}(D_1 + D_2) = 857,41 - 51,60 T + 3,868 T^2,$$

which, accordingly, gives a course *convex* towards the T -axis (*concave* at lower temperatures, where D_1 quite predominates with its maximum). We may also write:

$$\underline{10^4 \cdot \frac{1}{2}(D_1 + D_2) = 693,79 + 11,45 (T_k - T) + 3,868 (T_k - T)^2,} \quad (15)$$

or also with $T = mT_k$:

$$10^4 \cdot \frac{1}{2}(D_1 + D_2) = 693,79 + 59,426 (1 - m) + 104,19 (1 - m)^2,$$

or with $D = d \times D_k$, in which d is, therefore, the reduced density:

$$\underline{\frac{1}{2}(d_1 + d_2) = 1 + 0,085654 (1 - m) + 0,15017 (1 - m)^2.}$$

From this follows:

$$\gamma = 0,085654 + 0,30034 (1 - m),$$

which, at T_k ($m=1$) gives $\gamma=0,086$, but at $T=3^o,90$ ($m=0,75$) already $0,86 + 0,075 = 0,161$, i.e. almost *double* the value!

We find for D_k :

$$\underline{10^4 D_k = 693,8.} \quad (\text{Comm. 172b: } 693,0).$$

If the values, found for $\frac{1}{2}(D_1 + D_2)$ according to the formula, are assumed to be correct, and also the calculated values of D_2 , the value $1254^6 = 1255$ (exp. 1255), follows for $10^4 D_1$ at 4^o,22, the value $1173^{52} = 1174$ (exp. 1165) at 4^o,59, and the value $1137^{96} = 1138$ (exp. 1139) at 4^o,71. Accordingly it follows that there is no objection to calculating D_2 to 4^o,7 from the values of B and p calculated by us, which we, accordingly, have done in Table V. The corresponding values of D_1 between 3^o,3 and 4^o,7 can then be calculated from (15).

Past $4^{\circ},71$ up to $5^{\circ},19$ recourse must be had to a theoretical formula for $\frac{1}{2}(D_1 - D_2)$, valid in the neighbourhood of T_k , viz. (cf. also Zustandsgl. p. 332 and 346)

$$\frac{\frac{1}{2}(D_1 - D_2)}{\sqrt{T_k - T}} = a' - \beta'(T_k - T) + \delta'(T_k - T)^2 \dots$$

For the calculation of the coefficients a' , β' and δ' we use the following data.

TABLE VIa.

| T | D_1 | D_2 | $D_1 - D_2$ | $\sqrt{T_k - T}$ | $\frac{1}{2}(D_1 - D_2) : \sqrt{T_k - T}$ |
|------|---------|---------|-------------|------------------|---|
| 4.23 | 0.1253 | 0.01637 | 0.10893 | 0.9798 | 555.9 |
| 4.59 | 0.11735 | 0.02306 | 0.09429 | 0.7746 | 608.6 |
| 4.71 | 0.1138 | 0.02624 | 0.08756 | 0.6928 | 631.9 |

And from this follows:

$$\frac{1}{2}(D_1 - D_2) : \sqrt{T_k - T} = 753.77 - 301.67(T_k - T) + 99.54(T_k - T)^2, \quad (16)$$

for which may also be written, with $D = d \times D_k$:

$$\begin{aligned} \frac{1}{2}(d_1 - d_2) : \sqrt{T_k - T} &= 1.0865 - 0.43481(T_k - T) + 0.14347(T_k - T)^2 \\ &= 1.0865 - 2.2567(1 - m) + 3.8645(1 - m)^2, \end{aligned}$$

hence

$$\frac{1}{2}(d_1 - d_2) : \sqrt{1 - m} = 2.475 - 5.141(1 - m) + 8.804(1 - m)^2.$$

The constant term a is therefore = 2,475. This is 3,6 to 4 for ordinary substances ($\gamma = 0,9$ to 1), about 3 for argon ($\gamma = 0,75$) and 2 for "ideal" substances (a and b constant, $\gamma = 0,5$). Hence a is always $\sim 4\gamma$.

Now for Helium $s = R T_k : p_k v_k = 0,019005 : 0,005815 = \underline{3,268}$ (see Table V), and s being = $8\gamma : (1 + \gamma)$, the value 0,69 would follow for the "theoretic" value of γ for Helium, so that 4γ becomes = 2,76, which is, therefore, here slightly more than 2,475. For the quantity $r = v_k : b_k$ we find for Helium $0,002572 : 0,001051 = \underline{2,447}$. For rs we find for Helium the general value 8, as $3,268 \times 2,447 = \underline{7,997} = 8,00$. (Compare also Zustandsgl. p. 141). [For "ideal" substances s is = $\frac{8}{3}$ and r is = 3; for ordinary substances s is = 3,6 to 4, r is = 2,22 to 2].

From (15) and (16) the following values are now calculated between $4^{\circ},3$ and $5^{\circ},2$; those of D_2 and v_2 have already been recorded in Table V.

As regards D_1 , we have already the values between $4^{\circ},3$ and $5^{\circ},19$ in the following Table. Further we have the experimental material of Comm. 170 (K. ONNES and BOKS) and 172b (M., CR., K.O. and Sw.) between $1^{\circ},2$ and $4^{\circ},2$ (Table VIII), and also the values calculated by us in Table A of the first Part of this paper.

TABLE VII.

| T | $10^4 \cdot \frac{1}{2}(D_1+D_2)$ | $10^4 \cdot \frac{1}{2}(D_1-D_2)$ | $10^4 D_1$ | $10^4 D_2$ | $10^6 v_1$ | $10^6 v_2$ |
|------|-----------------------------------|-----------------------------------|------------|------------|------------|------------|
| 4.3 | 707.0 | 532.2 | 1239 | 174.8 | 1440 | 10206 |
| 4.4 | 705.2 | 513.3 | 1219 | 191.9 | 1464 | 9298 |
| 4.5 | 703.5 | 492.6 | 1196 | 210.9 | 1492 | 8461 |
| 4.6 | 701.9 | 468.9 | 1171 | 233.0 | 1524 | 7658 |
| 4.7 | 700.3 | 440.9 | 1141 | 259.4 | 1564 | 6878 |
| 4.8 | 698.8 | 406.7 | 1106 | 292.1 | 1614 | 6108 |
| 4.9 | 697.4 | 363.3 | 1061 | 334.1 | 1682 | 5340 |
| 5.0 | 696.1 | 305.1 | 1001 | 391.0 | 1782 | 4564 |
| 5.1 | 694.9 | 218.2 ⁺ | 913.1 | 476.6 | 1954 | 3744 |
| 5.19 | 693.8 | 0 | 693.8 | 693.8 | 2572 | 2572 |

The values of D_1 according to Comm. 170 and 172*b* are the following. Those of Comm. 170 have all been increased by 7 units of the last decimal, in accordance with the last correction in Comm. 172*b* on account of an improved determination of the normal volume.

TABLE VIII.

| T | $10^4 D_1$ | T | $10^4 D_1$ | T | $10^4 D_1$ |
|------|------------|------|-------------------|------|------------|
| 1.20 | 1459 | 2.10 | 1464 ⁵ | 2.56 | 1457 |
| 1.28 | 1459 | 2.21 | 1466 | 3.30 | 1395 |
| 1.59 | 1460 | 2.30 | 1496 (max) | 3.90 | 1311 |
| 1.92 | 1462 | 2.37 | 1466 | 4.22 | 1255 |
| 1.93 | 1462 | | | | |

From this, interpolations can easily be made from 1^o,2 to 3^o,3. But between 3^o,3 and 4^o,2 some D_1 -values must still be calculated. This is done by subtracting the values of D_2 from the values of $D_1 + D_2$ calculated according (15), through which the following table is obtained.

TABLE VIII^a.

| T | $10^4 (D_1+D_2)$ | $10^4 D_2$ | $10^4 D_1$ | T | $10^4 (D_1+D_2)$ | $10^4 D_2$ | $10^4 D_1$ |
|-----|------------------|------------|------------|-----|------------------|------------|------------|
| 3.3 | 1458.5 | 63.5 | 1395 | 3.8 | 1434.4 | 107.8 | 1327 |
| 3.4 | 1453.4 | 70.9 | 1382 | 3.9 | 1430.0 | 119.0 | 1311 |
| 3.5 | 1448.4 | 78.9 | 1369 | 4.0 | 1425.8 | 131.3 | 1294 |
| 3.6 | 1443.6 | 87.7 | 1356 | 4.1 | 1421.7 | 144.8 | 1277 |
| 3.7 | 1438.9 | 97.4 | 1342 | 4.2 | 1417.8 | 159.5 | 1258 |

4 The values of $W = p(v_2 - v_1)$, L and λ .

Now the whole material is complete for the determination of W (the values of p in atm. should be taken from Table V), $L = F \times W$, and

TABLE IX.

| T | $10^4 D_2$ | $10^4 D_1$ | $10^6 v_2$ | $10^6 v_1$ | $v_2 : v_1$ | $10^6 W = 10^6 \cdot p(v_2 - v_1)$ | F | $10^5 L$ (norm. units) | L' (Gr. cal.) | $10^5 \lambda = L - W$ (norm. units) |
|-----|---------------------|-------------------------|----------------------|-------------------------|--------------------|------------------------------------|----------|----------------------------|--------------------|---|
| 0 | 0 | 1458 ⁶ | ∞ | 1223 ⁴ | ∞ | 0 | ∞ | 3230 | 17.53 | 3230 |
| 0.5 | 0.0 ₄ 29 | 1458 ⁰ (min) | 6241.10 ⁷ | 1223 ⁹ (max) | 5.10 ⁷ | 1831 | 17.88 | 3274 | 17.77 | 3091 |
| 0.6 | 0.0 ₃ 47 | 1458 ¹ | 3772.10 ⁶ | 1223 ⁹ | 3.10 ⁶ | 2197 | 14.99 | 3293 | 17.87 | 3073 |
| 0.7 | 0.0 ₂ 35 | 1458 ¹ | 5131.10 ⁵ | 1223 ⁸ | 4.10 ⁵ | 2563 | 12.94 | 3317 | 18.00 | 3061 |
| 0.8 | 0.015 | 1458 ² | 1156.10 ⁵ | 1223 ⁷ | 1.10 ⁵ | 2927 | 11.41 | 3340 | 18.12 | 3047 |
| 0.9 | 0.049 | 1458 ³ | 3636.10 ⁴ | 1223 ⁶ | 3.10 ⁴ | 3292 | 10.24 | 3371 | 18.29 | 3042 |
| 1.0 | 0.12 | 1458 ⁵ | 1443.10 ⁴ | 1223 ⁵ | 12.10 ³ | 3652 | 9.306 | 3399 | 18.44 | 3034 |
| 1.1 | 0.26 | 1458 ⁷ | 6764.10 ⁴ | 1223 ³ | 55.10 ² | 4009 | 8.553 | 3429 | 18.61 | 3028 |
| 1.2 | 0.50 | 1459 ⁰ | 3595.10 ³ | 1223 ¹ | 29.10 ² | 4362 | 7.933 | 3459 | 18.77 | 3023 |
| 1.3 | 0.85 | 1459 ³ | 2101.10 ³ | 1222 ⁸ | 172.10 | 4708 | 7.416 | 3491 | 18.95 | 3021 |
| 1.4 | 1.3 | 1459 ⁶ | 1322.10 ³ | 1222 ⁵ | 108.10 | 5049 | 6.980 | 3524 | 19.12 | 3019 |
| 1.5 | 2.0 | 1460 ⁰ | 8827.10 ² | 1222 ² | 722 | 5384 | 6.608 | 3558 | 19.31 | 3019 (min) |
| 1.6 | 2.9 | 1460 ⁵ | 6180.10 ² | 1221 ⁸ | 506 | 5710 | 6.289 | 3591 | 19.49 | 3020 |
| 1.7 | 4.0 | 1460 ⁹ | 4500.10 ² | 1221 ⁴ | 369 | 6027 | 6.013 | 3624 | 19.67 | 3021 |
| 1.8 | 5.3 | 1461 ⁴ | 3377.10 ² | 1221 ⁰ | 277 | 6336 | 5.774 | 3658 | 19.85 | 3022 |
| 1.9 | 6.8 | 1462 ⁰ | 2608.10 ² | 1220 ⁵ | 214 | 6637 | 5.564 | 3693 | 20.04 | 3029 |
| 2.0 | 8.7 | 1563 ⁰ | 2059.10 ² | 1219 ⁷ | 169 | 6929 | 5.381 | 3728 | 20.23 | 3035 |
| 2.1 | 11 | 1464 | 1657.12 ² | 1219 | 136 | 7211 | 5.219 | 3763 | 20.42 | 3042 |
| 2.2 | 13 | 1466 | 1357.10 ² | 1217 | 111 | 7484 | 5.077 | 3800 | 20.60 | 3052 (point of inflection) |
| 2.3 | 16 | 1469 (max) | 1125.10 ² | 1215 (min) | 93 | 7744 | 4.951 | 3834 (point of inflection) | 20.80 | 3060 |
| 2.4 | 19 | 1465 | 9452.10 | 1218 | 78 | 7998 | 4.840 | 3871 | 21.01 | 3071 |
| 2.5 | 22 | 1460 | 8022.10 | 1222 | 66 | 8239 | 4.741 | 3906 | 21.20 | 3082 |
| 2.6 | 26 | 1454 | 6882.10 | 1227 | 56 | 8469 | 4.654 | 3941 | 21.39 | 3094 |
| 2.7 | 30 | 1448 | 5944.10 | 1232 | 48 | 8685 | 4.577 | 3975 | 21.57 | 3107 |
| 2.8 | 34 ⁵ | 1442 | 5172.10 | 1237 | 42 | 8890 | 4.509 | 4009 | 21.75 | 3120 |
| 2.9 | 39 | 1435 | 4535.10 | 1244 | 36 | 9081 | 4.449 | 4040 | 21.92 | 3132 |

TABLE IX (Continued).

| T | $10^4 D_2$ | $10^4 D_1$ | $10^6 v_2$ | $10^6 v_1$ | $v_2 : v_1$ | $10^6 W =$ $10^6 \cdot p (v_2 - v_1)$ | F | $10^5 L$ (norm. units) | L' (Gr. cal.) | $10^5 i = L - W$ (norm. units) |
|------|------------------|------------|------------|------------|-------------|--|-------|---------------------------|-----------------------|-----------------------------------|
| 3.0 | 45 | 1427 | 3997.10 | 1250 | 32 | 9262 | 4.396 | 4072 | 22.09 | 3146 |
| 3.1 | 50 | 1418 | 3545.10 | 1258 | 28 | 9440 | 4.349 | 4105 | 22.28 | 3161 |
| 3.2 | 57 | 1407 | 3147.10 | 1268 | 25 | 9565 | 4.309 | 4122 | 22.36 | 3165 |
| 3.3 | 63 ⁵ | 1395 | 2810.10 | 1279 | 22 | 9693 | 4.274 | 4143 | 22.48 | 3174 |
| 3.4 | 71 | 1382 | 2516.10 | 1291 | 19 | 9798 | 4.242 | 4156 | 22.55 | 3176 (max) |
| 3.5 | 79 | 1369 | 2261.10 | 1303 | 17 | 9884 | 4.212 | 4163 (max) | 22.59 | 3175 |
| 3.6 | 88 | 1356 | 2034.10 | 1316 | 15 | 9950 | 4.183 | 4162 | 22.58 | 3167 |
| 3.7 | 97 | 1342 | 1833.10 | 1320 | 14 | 9982 | 4.154 | 4147 | 22.50 | 3149 |
| 3.8 | 108 | 1327 | 1655.10 | 1345 | 12 | 9985 (max) | 4.125 | 4119 | 22.35 | 3120 |
| 3.9 | 119 | 1311 | 1499.10 | 1361 | 11 | 9971 | 4.096 | 4084 | 22.16 | 3087 |
| 4.0 | 131 | 1294 | 1359.10 | 1379 | 10 | 9917 | 4.067 | 4033 | 21.89 | 3041 |
| 4.1 | 145 | 1277 | 1232.10 | 1397 | 8.8 | 9821 | 4.038 | 3966 | 21.52 | 2984 |
| 4.2 | 159 ⁵ | 1258 | 1119.10 | 1418 | 7.9 | 9692 | 4.009 | 3886 | 21.08 | 2917 |
| 4.3 | 175 | 1239 | 1020.10 | 1440 | 7.1 | 9559 | 3.980 | 3804 | 20.64 | 2848 |
| 4.4 | 192 | 1219 | 9284 | 1464 | 6.4 | 9337 | 3.952 | 3697 | 20.06 | 2762 |
| 4.5 | 211 | 1196 | 8443 | 1492 | 5.7 | 9071 | 3.925 | 3570 | 19.37 | 2661 |
| 4.6 | 233 | 1171 | 7654 | 1524 | 5.0 | 8711 | 3.900 | 3399 | 18.45 | 2527 |
| 4.7 | 259 | 1141 | 6882 | 1564 | 4.4 | 8222 | 3.877 | 3185 | 17.28 | 2363 |
| 4.8 | 292 | 1106 | 6108 | 1614 | 3.8 | 7532 | 3.857 | 2905 | 15.76 | 2152 |
| 4.9 | 334 | 1061 | 5340 | 1682 | 3.2 | 6643 | 3.839 | 2550 | 13.84 | 1886 |
| 5.0 | 391 | 1001 | 4564 | 1782 | 2.6 | 5456 | 3.822 | 2085 | 11.32 | 1539 |
| 5.1 | 477 | 913 | 3744 | 1954 | 1.9 | 3788 | 3.808 | 1442 | 7.83 | 1063 |
| 5.19 | 694 | 694 | 2572 | 2572 | 1 | 0 | 3.795 | 0 | 0 | 0 |

$\lambda = L - W$ (see (11) in § 1). The values of D_2 and v_2 , already given in Table V, have been repeated in the foregoing Table IX, as also the values of F from Table II. To reduce L expressed in normal units to L' in gr. cal., we must multiply by 542,63 (see § 2 of the first Part of this paper).

When we represent the values, found for L in this table — which, as appears from what precedes, have been calculated with the greatest care

and accuracy from the available data concerning p, B, D_2 and D_1 — by a graph, we obtain a continuous curve, which has a *maximum* at $3^{\circ},5$, and a *point of inflection* at $2^{\circ},3$. The presence of this point had already been stated in § 4 of the first Part. The maximum had necessarily to appear, because the initial course at $T=0$ is *ascending*. (cf. there formula (9)).

On the contrary the initial course of λ is *descending* (loc. cit.), and leads necessarily to a *minimum* (at about $1^{\circ},5$), because λ in conformity with L has a *maximum* at almost the same place ($3^{\circ},4$). Of course there is again a *point of inflection* between minimum and maximum, and this about at the same temperature as that of the point of inflection of L (c.f. Fig. 2).

DANA and KAMERLINGH ONNES (loc. cit., cf. § 1 of the first Part) also made a calculation of the values of L between 3° and 5° , but in consequence of the less accurate values of dp/dt and v_2-v_1 , used by them, they only obtained a rough approximation (see the following table). They find, indeed, also the maximum at $3^{\circ},5$, but their values at 3° , $3^{\circ},5$ and 4° are slightly too high. That at 5° is almost identical with ours, the value at $4^{\circ},5$, viz. 16,0, being quite incongruous. In their graphic representation they have, accordingly, substituted 19 (i.e. 4.75 gr. cal. per gr. He) for this value (i. e. 4 gr. kal. per gr. He), through which they have obtained a continuous curve (loc. cit. p. 1056).

The values found experimentally by DANA and K.O. are, indeed, in harmony, as far as order of magnitude is concerned (see Table X), with our calculated values; but the place of the maximum can hardly be ascertained from them, since their values are about constant between 2° and $3^{\circ},5$.

TABLE X.

| | $T=0$ | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.19 |
|----------------------|--------|------|------|------|------|------|------|------|------|---------|------|------|
| $L_{theor.}$ v. $L.$ | = 17.5 | 17.8 | 18.4 | 19.3 | 20.2 | 21.2 | 22.1 | 22.6 | 21.9 | 19.4 | 11.3 | 0 |
| Ibid. D.; K.O. | — | — | — | — | — | — | 23.2 | 24 | 22.8 | (16.0?) | 11.2 | 0 |
| $L_{exp.}$ | „ | — | — | — | 21.8 | 22.3 | 22.4 | 22.6 | 22.2 | 20.2 | — | — |

More accurate measurements will be required to establish the course experimentally with more certainty. From their measurements the authors also conclude (see the footnote on p. 1057), that there happens "*something particular*" in the neighbourhood of $2^{\circ},3$ (which happens to be exactly the place of the maximum liquid-density). But the degree of accuracy of their experiments does not allow them here either to elucidate this point further. But after our considerations and calculations we now know, that really something particular happens here, i. e. the occurrence of a *point of inflection*, below which the course of L is *convex* with regard to the temperature-axis, and above which it becomes *concave*, proceeding to the maximum at $3^{\circ},5$.

As we have seen, the appearance of this maximum can be explained in a *perfectly natural* way, of which the clue lies in formula (9), viz. (augmented by R , to transform $d\lambda/dt$ to dL/dt)

$$\frac{dL}{dt} = -\frac{\beta a_0}{v_0^2} + R = -0,003127 + 0,003662 = +0,000535,$$

through which, owing to the abnormally low value of a_0 for Helium, dL/dt proves to be *positive*, instead of negative, as in all other substances. Consequently the appearance of this maximum does not give in my opinion — in opposition to DANA and K. O.'s opinion — the slightest support to the calculations of VERSCHAFFELT cited in their paper, which are in connection with a possible “degeneration of energy” of the liquid at very low temperatures. The degeneration of energy is, in my opinion, not only *improbable* (it does not even explain the maximum and minimum in the liquid density), but its assumption is also *unnecessary*, because the ordinary theory, taking the deviating behaviour of the quantity a into consideration (rise instead of fall on increasing temperature), accounts for all the particularities, even in detail.

Also the abnormally low specific heat, but this will be treated on a subsequent occasion.

Tavel sur Clarens, Suisse, Summer 1926.

Physiology. — "*Veratrine poisoning of degenerated mammalian muscle.*"

By ARIE QUERIDO. (Communicated by Prof. G. VAN RIJNBEEK.)

(Communicated at the meeting of October 30, 1926).

An investigation into the action of veratrine on degenerated muscle will give the opportunity to throw some light on two till now unsolved difficulties in the problem of the action of veratrine on skeletal muscle. The structure of muscle changes in a definite and well-known way during degeneration; if we find that veratrine has on degenerated muscle an effect different from the normal, we may be able to demonstrate a causal relation between the changed structure and the change in function. Furthermore, severance of the motor innervation is the only method possible to obtain a muscle free from nervous elements; in this way we may study the action of veratrine on striated muscle without poisoning simultaneously any nerve fibres.

As far as I know, the previous work on this subject consists of two publications only, both by FONTÈS¹). He cut the sciatic nerve in frogs and poisoned after a varying interval their muscles. The contraction, obtained from the M. Gastrocnemius had the typical shape we also find in the intact muscle after veratrine poisoning.

FONTÈS made his investigation in winter; the longest interval between the section of the nerve and the poisoning of the muscle was three months. This is certainly not long enough for a frog's muscle to degenerate. I cut in some frogs the sciatic in November, and was not able to find any indication of degeneration in the muscles on the operated side in April²).

Therefore it is probable that FONTÈS worked on non-degenerated muscles; the conclusions drawn by him from his experiments are not justified; I repeated the procedure, but with warm-blooded animals.

In aether narcosis under aseptic precautions I cut the N. Ischiadicus in a number of albino-mice on the left side.

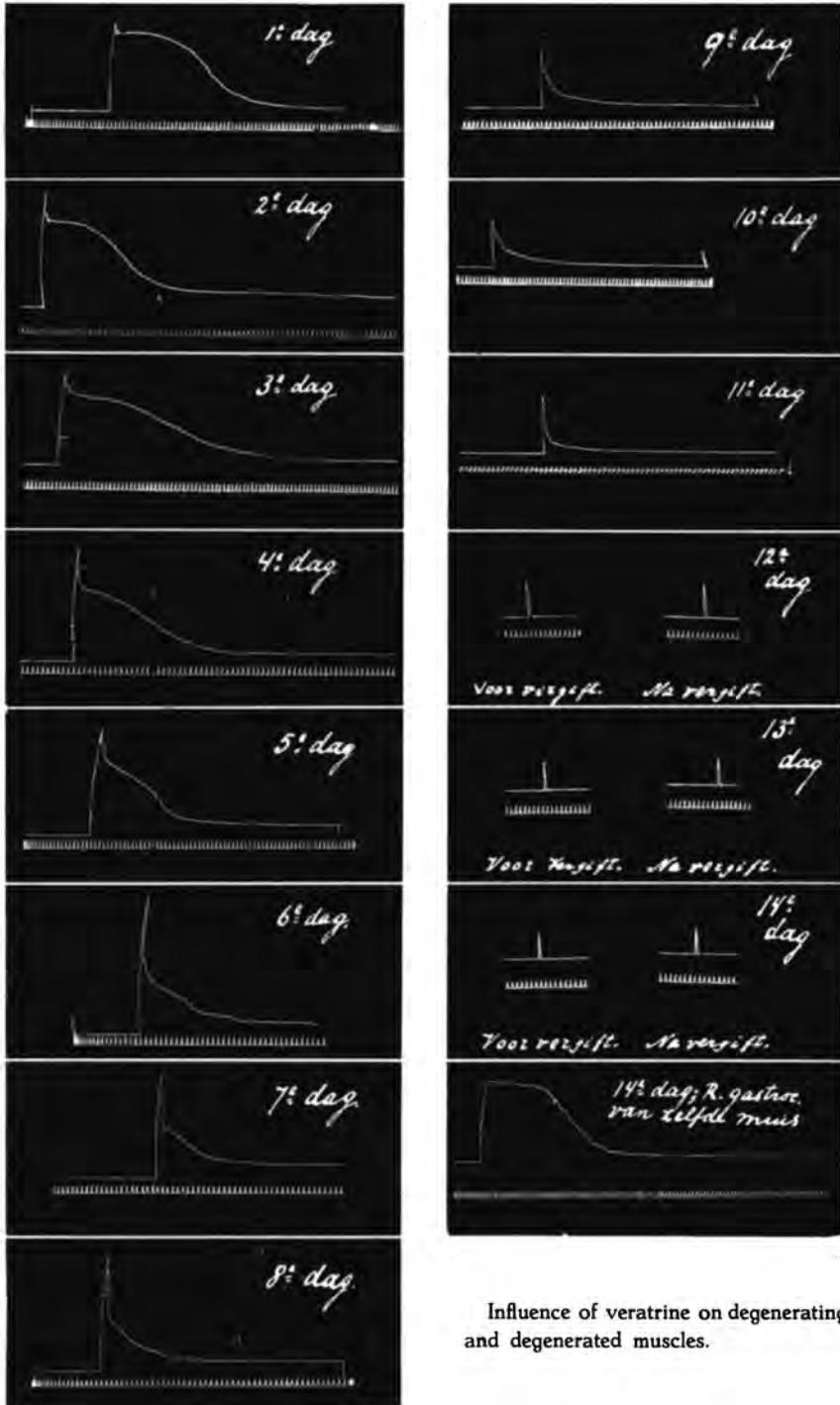
The muscles of the hind legs of those animals were tested with intervals of one day, the first 24 hours after operation.

The animals were anaesthetised with urethane (1 cc. of a 10 % solution, injected intra-peritonally per 100 Gr. mouse). I freed the cut end of the sciatic from surrounding tissue and cut the sciatic on the other side; both tendons of Achilles were cut loose from their insertion and connected with a writing lever, providing a load of 3 Gr. The sciatic nerves were placed

¹) Comptes-rendus Soc. Biol. 1921, LXVI, 84; Journ. de Physiol. et de Pathol. gén. 1925, XXIII, 292.

²) Cf. BREMER and GERARD, Comptes-rendus Soc. belge de Biol. 1926, XCIV, 1035.

on a pair of stimulating electrodes; in both the Gastrocnemius muscles a pair of silver needle-electrodes were thrust. Three double-throw double-pole switches allowed a rapid change from faradic to galvanic stimulation,



Influence of veratrine on degenerating and degenerated muscles.

from direct to indirect stimulation, and made it possible to stimulate right or left without disturbing the preparation. Faradic stimuli were provided by a DUBOIS—REYMOND induction coil, the strength of stimulus being varied by changing the resistance of the primary circuit; in the table the strength of stimulus is expressed by the size of the primary current which is direct proportional to this value. In wiring the set-up care had been taken that one electrode acts as cathode on the break, both of induction shocks and of constant current; this electrode was always distal to its fellow.

When the narcotised mouse had been prepared as described above, I first compared the threshold for indirect stimuli right and left; after this the threshold for direct stimuli right and left was compared, and thereupon the behaviour of both muscles in regard to constant current.

In this way a complete insight was obtained into the stage of degeneration of the left Gastrocnemius of the tested mouse. Now I injected into both muscles 0.1 cc. veratrine solution 1 : 20,000, and compared after 3 minutes the contraction right and left on direct stimulation with induction shocks.

No clear differences are found in the first three days following the operation; after this time the differences are unmistakable. The second part of the contraction of the muscle on the operated side takes less time and is lower; small oscillations are superposed on the second part of the contraction curve. Day after day the points of difference between the contractions right and left become more distinct; the characteristic shape of the veratrine contraction disappears quite gradually in the curve from the muscle on the operated side. Twelve days after the operation we find after poisoning a contraction that is in all aspects similar to the contraction of the same muscle before poisoning. No indication of an influence of the drug can be found in the shape of the shortening curve.

We may follow the process of degeneration until the muscle has lost its direct irritability — eight weeks after section of the nerve — and during this period the effect of veratrine on the degenerated muscle does not change, and remains nil.

We observe that the first muscle that does not show any typical effect of the poisoning is the same to show a complete electrical reaction of degeneration (Vide Table). From the twelfth day following the section of the nerve the chief points of electrical degeneration are regularly present, and from this day on no difference can be found between the contraction before and after poisoning on the operated side.

Microscopic examination. BIELSCHOWSKY-preparations, made by Dr. H. C. VOORHOEVE from the tested muscles, showed, that 14 days after the section of the nerve no motor end-organs were to be found in the muscles. But in those muscles, and a fortiori in muscles that were examined at a shorter interval after operation, no difference could be detected in the structure (aspect of striation, number and distribution of nuclei, amount of sarcoplasm) compared with normal muscle.

TABLE.

Development and progress of degeneration on the operated side.

| Day | Strength of current milli-Amps, used for stimulation of nerve and muscle. | | | | | | |
|-----|---|--------|------------|-------------|------------|--------------|-------------|
| | Faradic | | Galvanic | | | | |
| | Nerve | Muscle | Cath. make | Anod. break | Anod. make | Cath. tetan. | Cath. break |
| 1 | 0.5 | 0.8 | 2.0 | 3.2 | 5.5 | 10.4 | 22 |
| 2 | 0.8 | 1.1 | 1.8 | 3.0 | 4.8 | 12.2 | 22 |
| 3 | 0.8 | 1.2 | 2.2 | 3.5 | 5.0 | 12.0 | 20 |
| 4 | 1.3 | 1.5 | 2.5 | 4.0 | 4.2 | 12.0 | 21 |
| 5 | 1.3 | 1.8 | 2.5 | 3.8 | 4.5 | 10.8 | 18 |
| 6 | 2.0 | 1.3 | 2.7 | 4.1 | 4.5 | 10.6 | 18 |
| 7 | 2.2 | 2.8 | 3.1 | 5.0 | 4.0 | 12.4 | 19 |
| 8 | 3.5 | 3.5 | 3.5 | 5.2 | 4.0 | 11.3 | 22.5 |
| 9 | 3.4 | 3.7 | 3.8 | 5.4 | 3.8 | 12.0 | 22 |
| 10 | 4.6 | 4.0 | 4.0 | 5.5 | 3.8 | 10.5 | 22 |
| 11 | 5.7 | 4.0 | 4.5 | 6.0 | 4.1 | 12.2 | 24 |
| 12 | — | 4.2 | 6.0 | 7.4 | 5.0 | 11.6 | 22.5 |
| 13 | — | 4.2 | 5.1 | 9.0 | 4.8 | 8.2 | 18 |
| 14 | — | 4.5 | 5.4 | 6.8 | 6.0 | 9.4 | 21 |
| 15 | — | 6.0 | 5.4 | 7.2 | 4.8 | 10.2 | 15 |
| 16 | — | 5.5 | 7.0 | 8.5 | 6.2 | 8.5 | 18 |
| 17 | — | 6.2 | 7.0 | 8.5 | 5.7 | 7.9 | 17 |

Sympathetic. When the sciatic nerve is sectioned, of course the sympathetic innervation of the Gastrocnemius muscle is interrupted at the same time.

Therefore I performed some experiments in which the spinal innervation was left intact, and only the sympathetic nerve paths were interrupted.

In a number of rats the sympathetic chain was extirpated on the left side from the solar plexus downward.

The first of those rats was examined two weeks after the operation, the others with intervals of a fortnight after the first; the last rat twelve weeks after the operation.

Some of those animals showed a difference in temperature and in aspect of both hindlegs; one rat had a marked edema of the left hindleg and foot.

The contractions obtained from the Mm. Gastrocnemii did not show any difference before nor after poisoning between the muscles from the operated and from the intact side.

Conclusions. Muscles that are caused to degenerate by cutting their motor nerve show gradually a diminishing influence of veratrine poisoning.

From muscles that are completely degenerated as shown by electrical test, a veratrine contraction cannot be elicited; after poisoning they yield on induction shock a single muscle twitch, just as before poisoning.

This is also true for muscle that do not show any structural change when examined microscopically.

Muscles, lacking sympathetic innervation, yield veratrine contractions that are identical to those of intact muscles after poisoning.

Physiology. — "*Veratrine contraction of the frog's heart.*" By ARIE QUERIDO. (Communicated by Prof. G. VAN RIJNBERK.)

(Communicated at the meeting of October 30, 1926).

The function of the heart of a cold- or warm-blooded animal will be changed profoundly when the animal has been poisoned by veratrine.

Those changes have a twofold character: the rhythm of the pulsations is affected — it becomes irregular, more frequent at first, to slow down afterwards, and the coordination between the subdivisions of the heart is impaired — as the poisoning progresses, a partial heartblock becomes more and more apparent.

The activity of the heart muscle as indicated by the shape of each contraction is influenced only quantitatively, in accordance to the changes of the rhythm in which the pulsations follow each other; the character of each contraction however is similar to that of the non-poisoned organ, contrary to the conduct of the skeletal muscle after poisoning. When we bear in mind that a poisoned skeletal muscle which is stimulated at a sufficient rapid rate will yield contractions that are identical to normal twitches, and that the heart muscle is under analogous conditions as a frequently stimulated skeletal muscle, it is clear that we cannot expect a functioning heart to yield veratrine contractions.

In order to obtain this result it will be necessary to suspend for a period the normal activity of the heart.

The faradisation of the Nervus Vagus seems to be the simplest way to reach this. BUSQUET and PACHON¹⁾ have demonstrated that veratrine poisoning of an animal results in loss of influence of the Vagus on the activity of the heart.

This is true only, if veratrine be applied on the heart in a concentration higher than 1:100.000. If more diluted solutions are used the heart will still stop on vagal stimulation, at least in the first minutes following the application of the drug. I have studied the nature of the reaction on an extra stimulus, when the heart has been poisoned previously by a slight dose of veratrine and has been stopped by faradisation of the N. Vagus.

Method. The brain of a large-sized frog is destroyed by a needle introduced in the median line behind the acoustic membranes. The animal is pinned on the back, and the heart is exposed by removing a part of the sternum and opening the pericard. I connect the apex of the ventricle with

¹⁾ Comptes-rendus Soc. Biol. 1906, LXI, 89.

Ibid. 1907, LXII, 943.

a light writing lever, and at one side (which is irrelevant) I expose the N. Vagus by means of a partial resection of the lower jaw; the nerve is cut, and its peripheral end is placed on a pair of stimulating electrodes, connected with the secondary of an induction coil. After ascertaining that faradisation of the Vagus causes the heart to stop, I inject into the peritoneal cavity 1 cc. of a veratrine solution 1 : 10.000. While the movements of the heart are being registered on the smoked drum, faradic stimuli are applied on the N. Vagus; the heart stops; after a few seconds, I touch the ventricle lightly with a glass rod which is drawn out into a fine point. Faradisation of the Vagus is continued until the contraction set up by the contact has come to end.

Fig. 1 shows the result of such an experiment; the last contraction of



Fig. 1.

1. Suspension curve from ventricle. 2. Signal for mechanic stimulation.
3. Signal for vagal stimulation. 4. Time, 1 sec.

the heart before it responds to vagal stimulation and stops, shows already a peculiar lengthening; on mechanical stimulation a contraction is elicited that is undoubtedly of the veratrine type, viz. reaches a greater height than the other contractions, that is considerably prolonged, and that shows two consecutive shortenings of the muscle, the second one being the slower. When faradisation of the Vagus ceases, the heart resumes its former activity.

In those experiments we meet the difficulty that the heart sometimes „escapes” the influence of the Vagus, and contracts once or twice during the vagal stimulation; this is seen repeatedly in non-poisoned preparations, but after veratrine poisoning this tendency to escape becomes much stronger. Sometimes it appears to be impossible to force the heart to rest long enough to obtain a veratrine contraction on mechanic stimulation. The contractions may have a somewhat longer duration and may be slightly larger than their predecessors, but the differences are not clear enough. Therefore I used also another method to obtain veratrine contractions from the heart after a period of artificial rest.

A frog's heart is removed from the body and is pinned on a cork by the vessels that arise from it.

The apex of the ventricle is connected with a lever. Now I perform

the so-called first ligature of STANNIUS, viz. with a silk thread the Sinus venosus is separated from the rest of the heart. This causes the heart to stop. After waiting a couple of minutes — not too long, since the heart

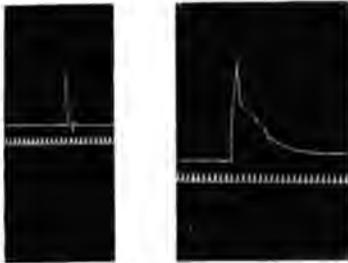


Fig. 2.

will resume its activity after a certain interval, and the duration of this interval varies widely in the experiments — a few drops of a veratrine solution 1 : 10,000 are trickled on the heart; 2—3 minutes afterwards the heart is touched by the glass rod.

Fig. 2 shows the outcome of such an experiment; two contractions are represented, yielded by the heart after the STANNIUS ligature on mechanic

stimulation, the first before, the second after veratrine poisoning. It is clear that the second curve is identical with the shortening curve of a skeletal muscle after veratrine poisoning.

The first series of experiments was repeated 18 times, the second series 26 times; the results were similar in all single cases.

When a contraction was elicited after stopping the heart and applying veratrine, this contraction always showed the shape that is characteristic for the shortening curve of veratrine-poisoned skeletal muscle.

In the cases in which stoppage of the heart was obtained by stimulating the Vagus, I have not investigated the shape of a contraction on mechanic stimulation before veratrine was applied; but in the second series of experiments this control was taken regularly, and the difference in shape of contraction curve before and after poisoning could be demonstrated in each experiment.

Conclusion. When the rhythmic activity of the frog's heart is suppressed, either by stimulating the N. Vagus or by the first ligature of STANNIUS, and the heart is poisoned by veratrine, the ventricle will give a contraction on mechanic stimulation that is identical to the contraction of a skeletal muscle after veratrine poisoning.

Botany. — "*Rapid Flowering of Darwin-tulips.*" I. By Prof. A. H. BLAAUW. (Laboratory for Plantphysiol. research. Communic. N^o. 21, Wageningen.)

(Communicated at the meeting of September 25, 1926).

§ 1. *Introductory.*

Our experiments on "the results of the temperature-treatment in summer for the Darwin-Tulips" have only been made on the red variety *Pride of Haarlem*. From those experiments follow various indications for culture, also for early flowering. On exposure to 44 different combinations of temperature in summer from lifting to planting (1922), it was already a striking fact for the botanist to discover that an exposure of 11 weeks to so low a temperature as 9°, gives the earliest bloom the next spring (1923). (See N^o. 17, Tab. 2 and fig. 2.)

Meanwhile exposures to other combinations occurred, described in N^o. 19; the question was a.o. whether more rapid flowering might be obtained in another way than by a permanent exposure to 9° in summer. Not that this could be of economical interest for field-cultures, but it might give an indication with respect to earliest bloom. Moreover on permanent exposure to 9° those flowers, though early, were rather frequently irregular or crooked, on account of transitional forms between the upper foliage-leaf and the first tepal. If we compare the number of temperature-combinations tried in 1924, among which 21 which had not yet been applied in 1922, we see in table 14, last column, that the tulips exposed to 9° for 11 weeks in summer, and those exposed to 20° for 3 weeks + 9° for 8 w. flower earliest and simultaneously in spring; that just as before, those of 11 w. 9° flowered very irregularly, but those exposed first to 20° for 3 w. and then to 9° were quite uniform.

So that was the most suitable summer-treatment for Darwin-tulips, that were to be forced. And this is reasonable. For in 20° (and 17°) the flower-formation progresses most rapidly and the flowers show less irregularities than in 9°.

Let us now further consider this summer-treatment: 3 w. 20° and next 9°. After those 3 weeks 20° the flower had averagely reached stage III in 1924, i.e. the outward tepals are all visible as individual organs; the bulbs in 9° are still merely leaf-forming. (See N^o. 19, Tab. 22.) If we had exposed to 20° for 5 w. and then to 9°, the flower-formation was quite finished on transition from 20° to 9°. (See also Tab. 22.) But already by

that time the successive number of flowering in the field has become 4 instead of 1. Accordingly it is not at all necessary, nay undesirable, to wait for a celerrimal flowering (most rapid flowering) till all floral parts have been formed in 20°. Besides it is worth notice (see tab. 15 in N^o 19) that the joint number of floral parts, if we transfer after 3 w. or 5 w. from 20° C. to 9°, amounts resp. to 16.10 and 16.55, i.e. shows no difference (0.45) worth mentioning, whereas the flower is in stage III after 3 w. 20° in 1924, quite finished after 5 w. On regarding the mean errors of Tab. 16 and the fact (s. Tab. 15), that the chief difference between 3 w. and 5 w. 20° is still found in the tepals already for the greater part formed (0.35 in the tepals and 0.45 for the joint number), whereas more floral parts might be expected "earlier in 9°", the noticeable fact is observed, that when the flower is well-started in 20°, transition to 9° may take place. It is striking that 3 w. 20°, besides a rapid finishing of the foliage-leaves has such results, that the number of floral parts is brought to 16—16.5 instead of ± 21 , when flower-formation is started and continued both in 9°. Even this increased number often produces transitional forms of the organs, frequently attended with the flower growing crookedly on the stalk.

So starting in 20° and an early transition to 9°, has been the base for the experiments on early-flowering, described in this first part. (For the sake of application this was already communicated in Laboratory-communication N^o. 19, p. 7; = p. 1073 of these Proceedings, Vol. 34, 1925).

We shall see in this paper, that for successful experimenting we are indeed compelled to mind the stage, in which the flower is, before the temperature is changed, and that it does not suffice to fix a certain date, as in the successive years this varies according to the climate to which the plant has been exposed in the field in the previous months.

What can be given in this first part on Early flowering of Darwin tulips is by no means complete. Though the results were an uncommonly early flowering for the Pride of Haarlem — yet I did not intend publishing until further modified experiments were made this year. But this division in at least two parts seemed better to me after all: 1^o. the result as far as it could be ascertained hitherto, is sooner known to everybody who wants to apply or to experiment in this sphere; 2^o. a description of the experiments of 2 or 3 years at a stretch would grow too long-winded for the readers; 3^o. if for shortness' sake at the end of the experiments we give only the results of the exposures which ultimately lead to earliest flowering, we lose the whole connection with the preliminary experiments which indicated the direction how we could gradually attain that celerrimal flowering. Besides the proof-material is lacking which treatments are wholly unfavorable, which fairly favorable, e.g. in producing a somewhat later flowering. Besides various results of no value for practical purposes, may be of botanic-scientific importance. In such results economical interest is concentrated to those treatments having a celerrimal or optimal effect, but scientifically it is

of at least as much importance to know the extreme treatments, the limits, either high or low temperatures, at which the flower-formation becomes impossible or shows abnormalities.

§ 2. *The beginning of the experiments and the influence of one and the same temperature.*

We received the lifted bulbs (from VAN TUBERGEN of Haarlem) already early in 1925, but the internal condition round the growing-point accorded fairly well with what we found in the bulbs of Pride of Haarlem a few weeks later in previous years. I.e. the temperature-exposures were started in 1922 on July 20, number of new foliage-leaflets in formation 2 or 3 — in 1924 on July 15 number of foliage-leaflets in formation 1 or 2, i.e. about 1 less — in 1925 already 2 foliage-leaflets had been formed on July 3. *From this it appears that the more or less advanced interior state of the bulb must be ascertained and we cannot rely on the date.* In 1925 namely the state of the growing-point on July 3 was intermediate between July 15, 1924 and July 20, 1922, whilst in those years the main bulbs were sent to us as soon as possible after lifting and had not been kept in a high temperature at the grower's.

The bulbs were kept in 20° from July 3—July 31, i.e. 4 weeks, then put in 9° for a long time. Besides on July 31 10 bulbs were fixed in alcohol 96 % and examined *later*.

It appeared *later on*, that in that transition from a short time 20° to a long time 9° the length of the outer young foliage-leaf in the bulb (which is to appear from the bulb later with the other leaflets as "nose" (tip)), was as a rule 3.0 to 3.5 mms. long. For the rest the flower was already in stage VI to VII. This appeared *later on*, and on comparison with other years (see above) we could safely (according to the field-cultures) have put the bulbs in 9° a little earlier, i.e. when stage III has been attained (outer whorl of tepals split off as 3 *individual* organs). This will be taken into account in the new series of experiments of this year. Doing so is also of practical importance; but if this sure method is thought to take up too much time or to be too difficult, the transition from 20° to 9° may fairly safely take place 3 or 4 weeks after lifting.

These bulbs were planted for early flowering on Sept. 7, 1925, in groups of 8 in boxes measuring 25 × 40 cms. and 21 cms. deep. (This year we are going to use groups of 6 in much smaller boxes). On Sept. 7, so on planting, the length of the outer leaflet, i.e. of the "nose", varied from 17—20 mms. in 9 bulbs. It seems to me, as far as we know at present, certainly not advisable to plant later. But fixing the date of *transferring* the bulbs from 20° to 9° (see above) requires much greater precision. For by that time they are in the midst of the rapidly progressing flower-forming period; — if the planting occurs at the beginning of September a week earlier or later *may* have its

influence, but the organs of foliage-leaves and flowers are simply more or less long ; by planting somewhat earlier or later the chance of producing an injurious effect is by no means so great and the temperature remains the same (9°). Yet it is important for studies on earliest flowering to consider if that flowering may be accelerated by *planting* more or fewer weeks after the *transmission* from 20° to 9°.

The Darwin tulips thus treated were all planted on Sept. 7, 1925, viz. groups of 8 in 16 boxes. Besides the division in the actual experiments was started by that time. For the theoretical study of most rapid stretching the 1st was placed in 17°, the 2nd in 12°, the 3rd in 9° and the 4th in 5° C. — moreover 6 boxes were placed in 9° and 6 in 5° C. These 'stores' in 9° and 5°, served for experiments to be further described, in which after a shorter or longer period, i.e. with a shorter or longer "nose" length (= the outer leaf growing from the bulb), the boxes were transferred to higher temperatures (see § 3). We shall first describe how the stalks with leaves mainly behave, if they remain from the planting on Sept. 7 in 5°—9°—12° à 13° and 17°. From this it may be concluded whether *change of temperature is necessary* to attain the earliest flowering and cannot be substituted by continuous application of one certain average temperature.

The first shooting of the roots.

After about 4 weeks i.e. at the beginning of Oct. we noticed that the bulbs in 5°, 9° and 12° lifted up the soil, i.e. it appeared that the shooting roots did not go down well, but lifted up the bulb and some cms. of soil on top of it. This had to be redressed directly ; all the bulbs were carefully replanted in looser sandy soil and the rest of the empty hard *outer brown scale* was removed. This namely greatly impedes the downward growth of the roots. For early flowering it is advisable as, I think, some growers also do — to carefully remove that outer scale *on planting*. Besides we could now observe the effect of the temperature on the shooting of the roots after 4 weeks. In 17° not a single root had shot ; in 12° all the bulbs had roots the longest of which were already 4—7 cms. ; in 9° the shooting of the roots is most advanced, the longest is already 6—10 cms. ; in 5° the longest roots are 4—5 cms.

Accordingly 9° already shows a "celerrimum" in the shooting of the roots (most rapid ; we might call it an optimum, but who knows whether this is really most favorable, whereas we do know it is most rapid, so a *celerrimum*).

Besides 12° and 5° are also well-advanced, 12° slightly more so than 5°, while 17° is lagging behind. We shall revert to this subject, so remarkable with a view to the revision of the current botanical conceptions of optima, in connection with the other organs.

Only when the foliage-leaves ("the noses") had grown 9 cms. or more from the bulb, they were put in the light in constant hothouses. *In the bulb itself* we find by that time 4 cms. of the stalk with leaves. As throughout 5°

remained far behind other temperatures, they were not continued in 5° in the light, because this experiment would have been too expensive, but they were transferred to 13° (light).

The results of continuous 5°—9°—13°—17° from Sept. 7 follows in the table subjoined.

Length from the tip of the bulb to the apex of the leaf in mms.

Length from the tip of the bulb to the apex of the leaf in mms.

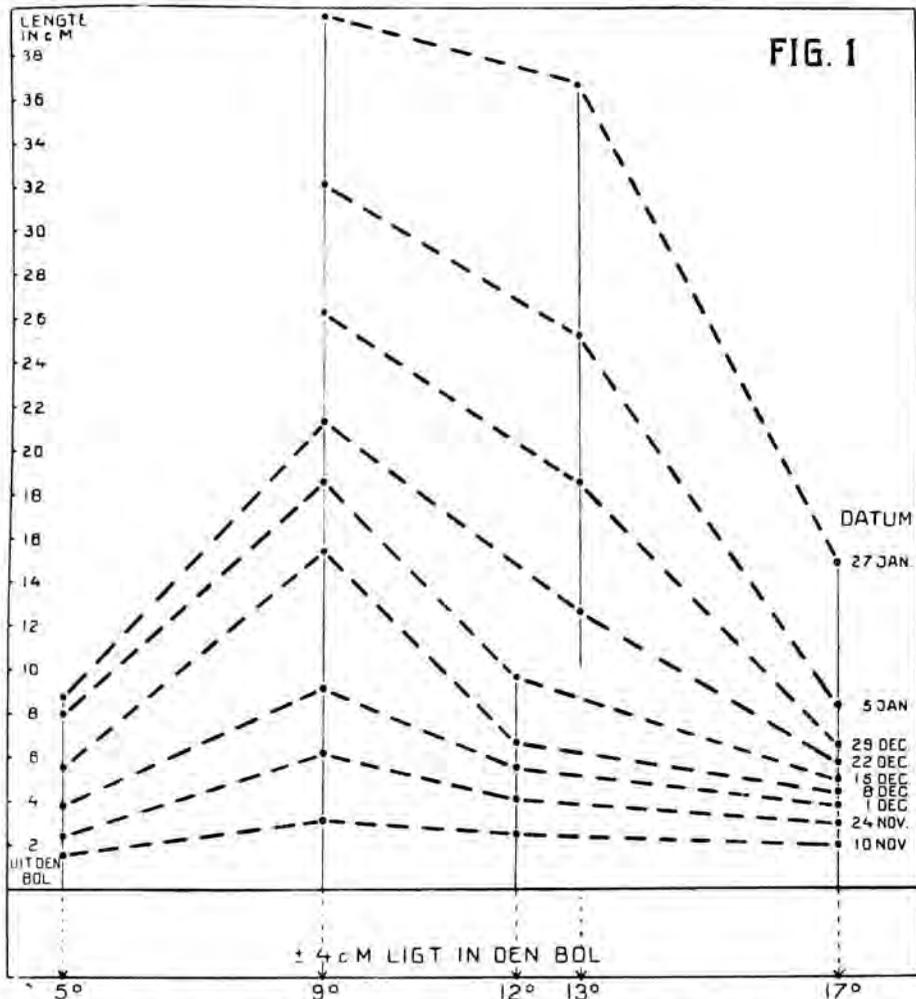
| | 5° | 9° | 12°/13° | 17° |
|-------------|-------------|-----------|-------------|-------------------|
| Nov. 10 '25 | 16.7 mms. | 30.3 mms. | 25.5 mms. | 20.9 mms. |
| Nov. 24 | 26.5 | 60.5 | 41.1 | 30.8 |
| Dec. 1 | 38.6 | 91.6 | 54 | 38.4 |
| Dec. 8 | 53.7 | 157.0 L | 68.6 | 44.5 |
| Dec. 15 | 81.1 | 188 | 100.3 L | 50.6 |
| Dec. 22 | 90.5 L. 13° | 216 | 130 | 59 |
| Dec. 29 | ↓ 152 | 266 | 189 | 66.5 |
| Jan. 5 '26 | 276 | 327 | 257 | 85 — L Jan. 12 |
| Jan. 27 | 411 | 403 (310) | 373 (444) | 149 |
| Febr. 9 | 428 | 435 (340) | 386 (531) * | 190 |
| March 10 | | 444 (358) | | 210 (450) * |

The data of Tab. I up to and including Jan. 27 are also graphically represented in Fig. 1. The temperatures are plotted on the abscis. Those of 12° C. have got to 13° C. on transmission to light for technical reasons. The temperatures give the degrees of heat of the soil in the boxes, as long as the largest growing part is still found there. When leaves and stalk have grown some way and the boxes are moved into the light, the temperature among the plants above ground is constantly kept at 9°, 13°, 17°, because by that time (at least with the tulip) the most important part of the stretching takes place in the air. On the day that the boxes are put in light, an *L* is placed in the table behind the linear measure. On the ordinate (Fig. 1)

the linear measure is placed to the left: to the extreme right the date of observation. In the table the figure between brackets from Jan. 27 denotes the distance from the tip of the bulb to the lower side of the flower, i.e. the length of the stalk to which is added ± 40 mms. inside the bulb. The period of full bloom has been indicated in the table with an *. In small print the length of the plants outside the bulb has been given, after transmission from 5° to 13° .

Length in cM. = Length in cms.

Datum = Date.



Uit den bol = out of the bulb.

cm. light in den bol = cms. are in the bulb.

For explanation the text should be consulted.

From table and figure the following conclusions may be drawn. (The figures always represent an average of 8 bulbs.)

10. The stretching of the main-axis with leaves, e.g. from Nov. 10 to Dec. 22 yields exactly the same picture as we have observed in the shooting

of the roots: in 9° most rapidly, in 12° distinctly less rapid than in 9°, in 5° less rapid than in 12°, in 17° slowest.

2°. Those in 12° and later (in light) in 13°, were the first to flower of these groups (Febr. 9); those in 5°, were continued in light in 13° (small printed figures in the table) and flowered some days later than those constantly kept in 12°—13° C. Those of continuously 17° flowered as late as March 10. But in the temperature, which causes "most rapid" progress for months together, i.e. 9°, the flowers remain in bud, green, even up to ± March 15; while both light and moisture of soil and air are perfectly identical to those in 13°.

3°. Accordingly none of these temperatures, *continuously* applied, is conducive to *early* flowering. And 9° is the only suitable temperature for months at a stretch. When the Darwin tulip has reached stages III to V (the progress is very quick then) in 20°, i.e. about 3 or 4 weeks after lifting, the bulbs remain from Aug. 1 to e.g. some days in Nov. in 9° to promote celerrimal stretching (and have meanwhile also been planted into boxes ± Sept. 1).

4°. Starting with 20° followed by one temperature, e.g. 9° for months together will not do for a celerrimum, but we must find out at what time of growth we had better transfer from a low temperature to another. This is also known to the growers, but we must try to approximate and fix this point of time in the growth as exactly as possible.

This will be discussed in the next §.

But it should be borne in mind that this is a first part and a new series of experiments follows made in 1926.

§ 3. *By what change of temperature may the earliest flowering be attained?*

Simultaneously with the experiments described above in constant 5°—9°—13°—17° some (6) boxes (planted on Sept.) were left in 9° and 6 transferred from 9° to 5°.

From these two boxes were taken from 5° and two from 9° and one of each placed into 17° and into 20°, when the length of the "nose", i.e. the distance from the tip of the bulb to the apex of the leaf (in which measure extension of stalk and leaf are mingled) amounted to *averagely* 3 cms. Next again 4 boxes were transferred, one from 5° and one from 9°, respectively to 17° and 20° when this measure amounted to ± 6 cms. On reaching a length of 9 cms. (or longer) all were put in the light and besides those from 20° somewhat cooler, viz. in 17°. Those still kept in the dark in 5° and 9°, protruding ± 9 cms. from the bulb, were directly moved into 17° light and by way of control one box from 5° and 9° to 23°. They remained there for a short time till the flower-bud became visible, when they were also transferred to 17°.

In order to get a survey of this set of experiments, this is repeated in the following schematic representation :

After lifting all 20° for 4 weeks, next all 9°, on Sept. 7 planted into boxes (each containing 8 bulbs). One box was constantly kept in 5°, one in 12°—13°, one in 17°, as has been described in the preceding §. Moreover on Sept. 7 6 boxes were planted in 5° and 6 in 9°.

These were treated as follows :

Mode of treatment. Length of the plant out of the bulb.

| | Upwards of 3 cms. | Upwards of 6 cms. | > 9 cms. light. | Flower-bud visible. |
|----|-------------------|-------------------|-----------------|---------------------|
| 9° | → 17° | | | → |
| 9° | → 20° | | → 17° | → |
| 9° | | → 17° | | → |
| 9° | | → 20° | → 17° | → |
| 9° | | | → 17° | → |
| 9° | | | → 23° | → 17° |
| 5° | → 17° | | | → |
| 5° | → 20° | | → 17° | → |
| 5° | | → 17° | | → |
| 5° | | → 20° | → 17° | → |
| 5° | | | → 17° | → |
| 5° | | | → 23° | → 17° |

The results of these experiments are summarised below from figures 2—6, giving a graphical representation of them, also from § 2, from Nov. 10.

For comparison with figs. 3—6 fig. 2 repeats the results of constant 5° (in light 13°, then dotted line), constant 9°, 13° and 17°. The figures are found in Tab. I, so they are not repeated here, the graphical representation is connected with figs. 3—6, consequently differing from Fig. 1. Here the data of observation are placed along the basal line at proportional distances of time and the length of the part grown from the bulb erected vertically upon it, indicated by a dot. The temperature has been given at the line of growth, and on transferring to another temperature, the new temperature has been recorded behind that date, behind that dot. A transfer to light is indicated by a cross-stroke with an *L* above it. The date on which the group concerned was in full flowers, has been indicated by a rosette with a central dot.

A. H. BLAAUW: "RAPID FLOWERING OF DARWIN-TULIPS". I.

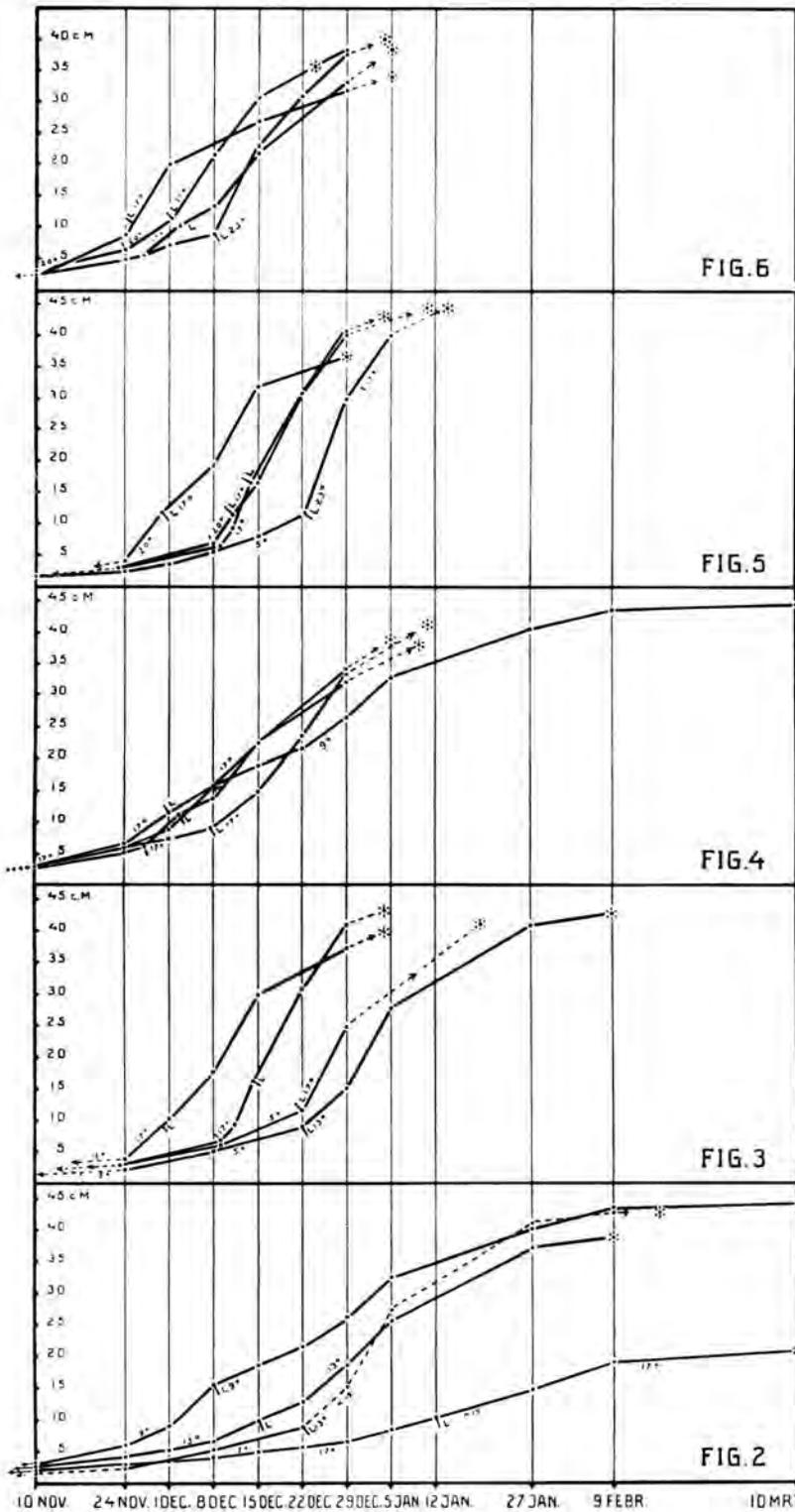


Fig. 3 gives four lines of growth, viz. it first repeats a curve of fig 2, concerning some 8 tulips stored in 5° and transferred to light in 13° on Dec. 22 (length from the bulb 90.5). Besides three curves, when from 5° (since Sept. 7) the bulbs are transferred to 17° C. at a "nose"-length of upwards of 3 cms. — upwards of 6 cms. — and upwards of 9 cms.

1^o. On conveying from 5° to 17° we directly notice a considerable acceleration of growth. An increasing acceleration of growth and earlier flowering compared to 5° and afterwards 13° .

2^o. The strongest acceleration of growth we find on transferring from 5° , upwards of 6 cms. to 17° . Then this group equals in growth and bloom the one put in 17° already at 3 a 4 cms. The full flowering occurs on the same days (Jan. 4). The third group transferred to 17° does not flower until Jan. 19.

3^o. Whereas we saw in Tab. I and Fig. 1, that on Dec. 8, 15 and 22 a constant exposure to 17° was unfavorable compared to 5° , we see the striking fact, that when the bulbs have *first* been exposed to a *lower* temperature (here 5°) and *next* 17° , a much more vigorous growth and earlier bloom is attained than when the low temperature is still continued.

Accordingly about that time the celerrimum of growth and [flowering is shifted from a lower to a much higher temperature.

We shall not add any separate tables of figures from which all lines of growth have been composed. Every one can read and measure these lengths and data from the figures.

Fig. 4 repeats "constant 9° ", even on March 10 still closed as green flower-buds. The difference with 17° is not so striking now as in fig. 2, also because "constant 9° " were first rather vigorous growers, but yet it is evident that earlier or later transmission to 17° after some time causes more rapid growth than 9° , while in previous months 9° most of all others excelled 17° far in rate of growth (see fig. 1). Here too the celerrimal temperature of 9° is moved upwards at a certain point of time, i.e. with a certain length of the plant. All three groups transferred from 9° to 17° attain full bloom from Jan. 5—12; the earliest (Januari 5) those which were moved to 17° at upwards of 6 cms.

Now let us compare fig. 5 with fig. 3. In fig. 5 we do not transfer to 17° but to 20° from 5° , at an average length of upwards of 3 cms. and 6 cms, besides 8 tulips with a length of nearly 12 cms. were transferred from 5° to 23° C. The curve of fig. 3, where we transferred from 5° to 17° at a length of upwards of 6 cms., was repeated, as giving the quickest result in those experiments of fig. 3.

What may be concluded from this? 1^o. That the first transmission from 5° to 20° (and a short time after in light to 17°) gave the most rapid flowering (Dec. 29); 2^o. that when at a "nose"-length upwards of 6 cms. the bulbs were put in 20° for some days, the progress of growth is fairly identical

to the growth of the bulbs directly put in 17°; — the flowering of the group 15° — 6 cms. 17° was a week earlier (Jan. 4) than of 5° — at 6 cms. 20° (for 3 days, next light 17°). We must not attach too much value to this difference, because 5° — 6 cms. — 17° was not put in light until a length of upwards of 16 cms. had been reached and 5° — 6 cms. — 20° already at a length of 11.5 cms. We are now investigating whether light or dark in these days plays an important part; this will be discussed in the 2nd part on early Flowering. 3^o, transferring from 5° to 23° when a length of 12 cms. was attained gave us a hint. For though this group was kept a long time (up to Dec. 22) in 5°, before that length had been attained, 23° (in light) gives a considerable acceleration of growth; only when the bud shows from the leaves, they are transferred to 17° and are in full flowers on Jan. 14. The stretching however is so forcedly rapid, that the insertion-spot of the upper leaves along the stalk often grows together with the stalk and causes the upper leaves or outer tepals to be torn. This method is not recommended, though it proves, how accelerating an influence a high temperature can have in *that later period*. It might be expressed in this way, that here for a short period of time the celerrimum (most rapid) is found in a higher temperature than the optimum (most favorable).

Finally Fig. 6 gives the most important result. On transferring after that initially celerrimal temperature of 9° to 20° at ± 3 cms. or upwards of 6 cms., it appears that "transferring to 20° at 6 cms." gives *the earliest flowering* of all experiments hitherto made, viz. on Dec. 24. Indeed those transferred to 20° already at 3 cms. first exceed a good deal in growth, but they are overtaken by those moved to 20° a fortnight later (± 6.8 cms.), as to growth in length and early flowering. And though they remain in 20° *but one week* before being removed to 17° in light, they do flower 10—12 days earlier, than the group directly put in 17° at 6 cms.

A photo of the first red Darwin-tulips flowering at Christmas, is shown in fig. 7 (Plate).

In various ways the experiments are continued on this base. That base consists in this, that the Tulips, 3 or 4 weeks after lifting, preferably controlled, viz. till at least the outer tepals have been formed, are kept in 20° next in 9°, after 4½ to 6 weeks planted in boxes in 9°, and only then, when the plant has grown from the bulb ± 6 cms., are transferred to a higher temperature (preferably 20°).

The above is however repeated and varied to discover whether e.g. that temperature should be more or less raised or lowered; whether at 6 cms. an exposure to dark up to ± 10 cms. (as we did now) is really better than directly to light at 6 cms. etc. These subjects will be treated in the 2nd part on Early Flowering of Darwin-tulips and later likewise in a paper on Early Flowering of Early Tulips.



Fig. 7. Pride of Haarlem, flowering on Christmas in Holland; — $3\frac{1}{2}$ week in 20° — afterwards 9° until 6 cm. above the bulb (Nov. 24th) — then 20° at first dark and in light 17° (see the text).



Fig. 8. Flowering in the first week of January, with example of a withering Flowerstalk (the right one).

§ 4. Conclusion first part.

So it is possible to get the Darwin-tulips *Pride of Haarlem* flowering at Christmas without abnormally early lifting or particular technical means. In the above it has been sufficiently described (summarily), how from 65 temperature-combinations between lifting and planting one most rapid and favorable mode was chosen (9° was most rapid, but not best) and how this summer-treatment served as a base for treating in 16 various modes after the planting in boxes to the flowering. That of those 16 there was one exceptional group flowering at Christmas and some other groups flowering 5—10 days later.

In the second part this celerrimal flowering will be repeated and different slight variations will be made to investigate whether the results are corroborated and may possibly be somewhat accelerated or improved.

Now we have still to point out a reverse in our experiments. Two "mistakes" appeared in the various groups, so that only half the number of the plants got to full bloom. Firstly in part of them the extension lags behind the others at a certain point of time and the flower-bud with the top-part of the stalk gets a withered strawlike appearance. This is not due to forcing, it occurred in the lower temperatures as well as in the higher. Nor can it be possibly owing to a dry atmosphere, for it occurred in a degree at least equally strong in very moist air in small glass boxes (e.g. thermostats of 13°). Secondly we also observed in our experiments the phenomena of the tipping flowerstalks. Just as the withered flower-buds, it occurred in low temperatures as frequently as with the tulips transferred to a high temperature, also in very moist rooms.

Fig. 8. (Plate) represents a group on Jan.13 (transferred from 9° at 9 cms. length to 17°). It shows the Tulip to the extreme right in full bloom completely broken. The different hue of the rest of the Tulips is but seeming, as two are in the shade, others in sunlight. We also notice a short stalk with "straw-like" flower-bud between the leaves. The tipping occurred in our experiments, when the flowers were still in bud, but as well when the flower was in full bloom or even when it was beginning to fade. The spot where the stalk snapped lay as a rule 8—12 cms. below the flower. For further description of this phenomenon VAN SLOGTEREN's papers and further investigation should be consulted; and further STEVENS a. PLIMKETT "Tulip Blossom Blight" Bull. N^o. 265, Agr. Exper. Station, Univ. of Illinois.

Certainly it is not due to forcing in higher temperature, and therefore we mention, that it did not occur in the box flowering at Christmas. This did yield however four flower-buds that remained strawy. But as already stated this likewise occurred in constant 13° , etc. For the rest we shall not go into these phenomena any further; we shall only record in the second part whether this year in somewhat altered treatments the two phenomena re-appear in an equally strong degree.

Besides we wish to mention here that a box, come to bloom with some broken flowers and a few withered buds was emptied. It appeared that the soil was sufficient and moderately moist and the root-system looked equally healthy in all of them.

Besides on the description of the experiments and the way in which we attained this flowering so early for Darwin-tulips, I mainly emphasized those parts essential for practical purposes. Meanwhile in the data of this and of previous papers there are found many things interesting from a scientific-physiological viewpoint and requiring profounder investigation. All this may be found in the preceding; but I wish to emphasize some cardinal points, because they make us ask about the inner causes of these phenomena.

1^o. We know now that the formation of the last leaflets and of the floral parts progresses most rapidly in 20° C. (also fairly rapidly in 17°), in the first 3 or 4 weeks after lifting. This concerns the organs formation; that celerrimum is comparatively low, considering the "optimal" temperatures found for other processes, as respiration and assimilation. But in our researches we have to deal with *the plant as a whole*.

2^o. Even more astonishing it is to the botanist used to those high "optima" of simpler physiologic-chemical processes, that the plant, studied *as a whole*, shows us — when the first floral parts have been formed — a very low celerrimum of 9°. That which according to the standing phrase is called the optimum, is shifted rapidly downwards, when the first floral parts have been formed and is found at ± 9° C. And this for months at a stretch. A higher temperature, as 13°, gives a slower development, 17° still slower. In that 9° C. the flower is finished, the stalk, the young foliage-leaflets, the floral parts increase in size most rapidly in that temperature. What is the cause of this, that the celerrimum falls so strongly and continues so for months together? Because from the process of *organ-formation* we proceed to the process of *organ-enlargement*? But even then a celerrimum ("optimum") of 9° is particularly low.

3^o. After ± 3 months, when the length of the plant (the part still inside the bulb, plus length of the leaves growing from the bulb) is 8—10 cms. — the optimum of growth is shifted to a higher temperature and we are obliged to transfer to ± 20° for a celerrimal stretching and flowering. This fact is almost more surprising. Whereas for so many months, for what we might call the "little stretching" the celerrimum is found in ± 9°, at a certain length, which requires rather great precision, the celerrimum is moved rapidly upwards to ± 20°. Where it will be exactly found, we shall further investigate in the second part. But to what is it due, that the "little stretching" with a celerrimum of 9° for months together passes into the "great stretching" progressing most rapidly at ± 20°? If in Aug. the bulbs had been left and planted in 20° or 17°, everything would have progressed very slowly and the flowering would have occurred in March. If after 20° they had remained in 9°, the result had even been worse (see

above "constant 9°"), It is exactly 9° for months together followed by $\pm 20^\circ$, which yields flowers as early as the end of Dec.

Such instances of exposure to cold then finally to heat with a view to early flowering are familiar to practice. To determine these temperatures and periods more exactly is the first task of these researches. But here botanical science is put before questions with which it has been too rarely occupied and which teach us, that *from a botanical point of view we have first of all to mind the behaviour of the plant as a whole*, before entering upon the deeper and more detailed processes of finding an explanation for that behavior. With this botany and application will both be served.

August 1926.

Geology. — "*On Fossil Cetacea and Pinnipedia in the Netherlands.*". By
A. B. VAN DEINSE. (Communicated by Prof. MAX WEBER.)

(Communicated at the meeting of October 30, 1926).

In "*De Bodem van Nederland*" (1855—1860) by W. C. H. STARING it says on p. 216, that as early as 1837, so well-nigh a century ago, molars and vertebrae of *Squalodon grateloupi*, Herman von Meyer were found in the environs of Eibergen. On the following pages 217—219 STARING enlarges upon findings of fossil Cetacea in our country, and on p. 209 we find some additional data concerning them. On p. 219 he concludes by saying: "We have dwelt for some length on this *Squalodon* in order to emphasize the great significance of finding and preserving even a few bones or fragments of bones for our knowledge of primeval animals. A comparative study of some bones enabled us so to speak to reconstruct to all appearance the whole animal. It is desirable, therefore that similar finds should by all means be made accessible to science and not be left dispersed in possession of private persons, or be kept isolated in sundry remote places, where no books or collections are available for an indispensable comparative examination."

Thus STARING wrote already 70 years ago. What has been done in this field since that time? The very opposite of what he suggested. Numberless persons and corporations in Twente, Achterhoek, and Westphalia have been making collections during many decades without any guiding principle, or without any cognizance of the work of other observers in the same domain. The exact localities have been vaguely indicated, or are altogether unknown. Rare fragments have been carelessly shifted to other places or have got lost; only a few of the hundreds of unearthed fragments have been saved from oblivion in our State-Universities, in Museum Teyler, or in collections of Educational Institutions. Hardly any have been properly determined.

True, some collectors have endeavoured, with partial success, to save some beautiful fragments: amongst them R. E. DE HAAN, 1882, the learned Director of the H. B. School at Winterswijk, who in his "*Beknopt Leerboek der Delfstof- en Aardkunde*" also treats this subject on pp. 45 and 212, and who corresponded with Prof. HUBRECHT on teeth of *Squalodon* (now *Scaldicetus grandis*, du Bus). A letter from DE HAAN to HUBRECHT is still preserved together with a tooth of *Scaldicetus* at Utrecht in the Zoological Museum and Laboratory.

Furthermore WICHMAN, LORIÉ, P. TESCH, DUBOIS, MAX WEBER, and a few others have collected material, but have not or only partly denominated it. Since after STARING, next to nothing has appeared in our country on

this subject, STARING's "Bodem" was still the standard work relied on, so that even now the majority of the fossil remains of Cetacea in our country are given the name of Squalodon, or have no name at all. I have seen numerous fragments with the invariable inscription "Squalodon", while nobody knew what could really be classed as such and what not.

What adds to the chaos is that the name "Zeuglodon" is occasionally met with (De Haan Beknopt Leerboek, vide supra p. 212 and STARING's "Bodem", p. 216). In the Geol. Mineral Collection of the Utrecht University a fragment of a skull of a fossil cetacean was called "Zeuglodon verdense" derived from Zwiilbroek near Groenlo, while on the skull a label was stuck with the name "Squalodon Grateloupi (?) v. Meyer", miocene 1887. In August 1925 I found in the Museum for Natural History of Brussels that this skull-fragment, much injured alas, originates probably from Eurhinodelphis Cocheteuxi du Bus, Upper Miocene. So far as we know now, Zeuglodon occurs in Europe only in England, to wit Zeugl. Wanklyni Seeley (1). Some relation was also often suspected between the Pinnipedia and the fossil Cetacea (STARING pp. 217, 218). This looks the more ominous, as genuine specimens of this group really occur in our country as fossils in the same localities where fossil cetacea are found. We shall discuss this later on.

Of the copious material of fossil Cetacea from our country only three fragments have been properly described and pictured. Plate III in STARING's "De Bodem" shows us a caninus of Squalodon Grateloupi H. v. Meyer, now called Scaldicetus grandis du Bus in the new nomenclature of Abel. On the same plate we see a molar, according to STARING also from Squalodon Grateloupi H. v. Meyer, now after Abel (5) from Squalodon Antwerpiensis P. J. van Beneden. Both pictures are excellent. The two fragments are now in the Geol. Museum of the Leyden-University. Not until 1917 did M. WEBER (2) give a description and a picture of the third, and up to this day the last specimen of the native fossil Cetacea, viz. the skull of Choneziphius planirostris G. Cuv. found in the West-Schelde and preserved in the above-named Leyden-Museum.

In VAN BAREN's new edition of "De Bodem van Nederland" (3) we read on p. 386 something about the Miocene of the Netherlands and about the fossil Cetacea found there. Already in 1778 J. A. DE LUC found among other things dorsal vertebrae of whales, which were recognized by P. CAMPER. They were found below Delden near Twikkel; these findings have also been mentioned by STARING (4). I suppose some of these vertebrae are still present in Teyler's Museum. Others, which found their way in the Museum of the Groningen-University with Camper were destroyed in a fire in August 1906. VAN BAREN also makes mention (p. 386) of the genera Squalodon and Scaldicetus; he does not mention other genera.

For some years we know already from WEBER, TESCH and others that not all that has been referred to the genus Squalodon can be classed as

such; the said investigators have warned me before against this. In August 1924 I started my examination from Enschedé with unexpected succes. Through Miss P. S. L. v. DEINSE and Mr J. J. v. DEINSE I made the acquaintance of Mr M. J. v. SAMBEEK, Lonneker near Enschedé, who on my very first visit kindly lent me a humerus of a fossil cetacean, while we found on the same day a mass of material at Hengelo at the houses of Mr SCHIEDAM, VAN HOUTEN, LANGERHUIZEN and SCHOENMAKER. Part of a trunk, two humeri, an enormous cavitas glenoidea of a scapula, many vertebrae were shown me and kindly given me as a loan.

At Enschedé two very large and fine collections of Messrs VINKENBORG and REGELINK came under my notice through the kind assistance of Mr VAN SAMBEEK, who also granted me free access to the collection of the "Enschedesche Museumvereniging". Moreover Mr J. J. VAN DEINSE stood me in good stead as President of the Archaeological Society "Twente" where also much material is present. Afterwards I still got acquainted with the collections of the TEN BOKKEL HUININK family of Nede, Mr JANSEN and Mr STEGEMAN both of Winterswijk, while also Mr. TEN HOUTEN of Winterswijk and Messrs LANDMAN and VAN BEMMEL both of Winterswijk rendered me great service. Finally I received material from Messrs J. G. C. DE GROOT of Eibergen, from J. W. TE WINKEL of Miste near Winterswijk and from Mr J. WAGNER of Ellewick near Vreden in Westphalia. Truly, material, data and assistance enough and to spare!

In October 1924 I was examining part of this material in the Museum for Nat. Hist. at Brussels with Prof. L. DOLLO Sc D., who assisted me wherever he could. Also the skilled preparator H. MENSCHAERT, who spent a lifetime at the Museum, was of great help to me. I feel greatly indebted to all those for their most useful assistance in my work.

At Brussels I found that many fossil species of Cetacea occur in our country, besides the three familiar to us. With much more material, about 200 numbers, weighing \pm 150 k.g. I was at Brussels again in 1925. In July 1926 I was able to study in Brussels another lot of bones together 40 fragments.

A small percentage of the fossil remains, not 10 %, could not possibly be found on account of a considerable amount of wear and the total absence of typical processes, articular surfaces etc. The species that were established, have been included in the subjoined list, every fragment coincided well with the corresponding pieces at Brussels. Besides the material lent to me by many collectors I myself have collected fossil remains in Aug. 1924 and 1925 in the Needschen Berg north of Nede, Gelderland, and in the famous quarry of Wiegerink, near Zwolle, not far from Groenlo, municipality of Eibergen. The latter locality is sometimes called "de Koerboom", anyhow by the Enschedé collectors. This may create confusion with "the Koerboom" meant by Staring and mentioned by him also on pp. 218 and 219. We have to do here with two different

places, which in common parlance got the same name. "De Koerboom" of STARING was a brick-kiln (p. 218) that ceased to exist long, long ago, and was situated to the north of the road from Groenlo to the German border, about halfway. The loam-pits are fallen into disuse there. On STARING'S geological map it is made mention of and in two places the Upper-Miocene is indicated as "tertiary loam of Eibergen". Those grounds are part of the municipality of Eibergen. Now the quarry of Wiegerink, i.e. the new "Koerboom" lies to the south of the said road and has large loam-pits, partly in disuse, partly still in exploitation. By this new "Koerboom" is meant the stone quarry "de Hoop" (STARING p. 218), now the property of the firm FERNIER, OVERKAMP and WIEGERINK, formerly WIEGERINK BROTHERS. Here also the loam is Upper-Miocene. On STARING'S Geological Map the Wiegerink quarry is located due north of the quarter of Zwolle. The loam of both places is of the same formation and there is sure to be some similarity in the underground of all this grey loam. Nowadays fossil remains of Cetacea and Pinnipedia occur only in the Wiegerink-quarry.

It is extremely regrettable that all the material handled by me, belongs to so many persons. After all had been studied it had to be distributed again and was broken up for ever. I was allowed to keep only a few fragments. Of the material that had to be returned I had good photo's taken, or I had plaster-casts made if the fragments were rare or very beautiful. I made 14 casts myself after DR. W. F. J. MILATZ of Rotterdam had taught me that art. About 60 fragments have been preserved in 180 photos taken from various sides. Future material will be treated as the above, for which I have received many promises of collaboration. In a later paper I hope to revert to this subject in detail. It goes without saying that in spite of my assiduous searching in Twente and Achterhoek, many pieces must have escaped my notice, but this will now occur less frequently with the present systematic collaboration.

The fossil Cetacea from Achterhoek are of the same antiquity as the great abundance of remains, found between 1860 and 1863 during the building of the fortifications round Antwerp (6), (7), the greater part of which is now at Brussels. In the interjacent grounds near Reek, in the neighbourhood of Grave on the Meuse, boring 41 of the "Rijks Opsporing van Delfstoffen", a crown of a canine of *Scaldicetus grandis du Bus* has been discovered which is now preserved at Harlem in the collection of the "Rijks Geologische Dienst". In November 1924 Dr. P. TESCH was so kind as to draw my attention to this find. This crown agrees entirely with the one, pictured by Staring in "De Bodem" on plate III (vide supra). From the following list of recognized fossil Cetacea and Pinnipedia in our country it appears that, truth to tell, we do not possess an ossuarium such as Belgium possesses in the environs of Antwerp, but that we have quite a good number of species and very much valuable material.

As said above, up to now only three genera were determined in Holland, every one with 1 species (*Squalodon Antwerpiensis*, *Scaldicetus grandis* and *Choneziphius planirostris*) (2, 3).

PROF. RUTTEN still obliged me by sending me for inspection a very good canine of *Scaldicetus grandis* du Bus, "found between Groenlo and Eibergen" by Dr. J. Lorié, the property of the Geol. Mineral Institute of Utrecht. Some fragments are still at Leyden. Beyond the above-named fragments of *Squalodon* and *Scaldicetus*, the molar and the canine pictured and treated by STARING nothing (so far as we are aware) exists in our country of these two genera. Neither is anything known of vertebrae up to now. It is remarkable, though, that for 70 years such a great amount of material from our country has been referred to "*Squalodon*", and that we know only now, that what STARING indicated as such, are still the only fragments to which the term can be applied.

It is just the same with *Choneziphius*. Besides the skull, treated by WEBER nothing is known of the other parts of this animal in our country. The three known genera and species just mentioned, are not included in the list. The odontoceti and mystacoceti have been recorded separately, while the list closes with a few fossil Pinnipedia, which are by far the least known in our country.

† ODONTOCETI.

| Name: | Locality | Remains found |
|---|--------------------------------------|---|
| 1. <i>Acrodelphis macrospondylus</i> Abel, | Wiegerink quarry. | 8 caudal vertebrae, 4 lumbal vertebrae, 1 dorsal vertebra, 2 humeri. |
| 2. <i>Acrodelphis scheynensis</i> Du Bus. | " " | 1 caudal vertebra, 10 lumbal vertebrae, 2 dorsal vertebrae, 2 epistropheus, 1 humerus, 1 right radius, 1 caudal vertebra, |
| | Dieters (?) | |
| 3. <i>Acrodelphis denticulatus</i> Probst? or <i>scheynensis</i> Du Bus? | Meddeho near Winterswijk. | 2 caudal vertebrae. |
| 4. <i>Acrodelphis</i> sp.? | Wiegerink quarry. | 1 epistropheus, 1 perioticum. |
| 5. <i>Cyrtodelphis sulcatus</i> Gerv.? | " " | 4 bullae, 4 periotica. |
| 6. <i>Cyrtodelphis</i> sp.? | " " | 1 bulla, 1 perioticum. |
| 7. <i>Eurhinodelphis Cochetuxi</i> Du Bus. | " " and Meddeho near Winterswijk. | 1 lumbal vertebra, 1 humerus, 1 skull-fragment, 1 proc. zygomaticus of the right temporal-bone. |

† ODONTOCETI (*Continuation*).

| Name : | Locality: | Remains found: |
|--|--|--|
| 8. <i>Eurhinodelphis cristatus</i> Du Bus | Wiegerink quarry Meddeho near Winterswijk and Miste. | 5 caudal vertebrae, 2 lumbal vertebrae, fragment of the right maxillaire, intermaxillaire right and left; fragment of the right temporalis bone. |
| 9. <i>Eurhinodelphis longirostris</i> Du Bus. | Quarry of Wiegerink and Miste near Winterswijk. | 2 lumbal vertebrae. 2 caudal vertebrae. |
| 10. <i>Eurhinodelphis</i> sp.? | Miste, Hooge Lutte near Oldenzaal, Wiegerink quarry | 2 caudal vertebrae, 1 dorsal vertebra. |
| 11. <i>Thalassocetus</i> sp.? | Wiegerink quarry. | 3 teeth. |
| 12. Unknown genera of Odontoceti? | " " | 2 teeth. |

13. Ziphioids, not completely determinable, of which several caudal vertebrae and 1 dorsal vertebra were found in the Needsche Berg, Nede, Gelderland, at Meddeho near Winterswijk and in the quarry of WIEGERINK. A caudal vertebra, perfectly intact, from Nede, Upper Miocene, reminds us forcibly of a corresponding one of *Ziphius cavirostris*, which is recent. Among these Ziphioids no doubt some species occur. Furthermore some fragments of † Odontoceti could not be determined in detail. When adding *Scaldicetus* and *Squalodon* (vide supra) we have 13 species and about 7 genera.

† MYSTACOCETI.

| Name : | Locality : | Remains found: |
|---|-------------------------|---|
| 1. <i>Amphicetus</i> sp? van Beneden. | Wiegerink quarry, Nede. | 2 lumbal vertebrae, 1 caudal vertebra. |
| 2. <i>Amphicetus verus</i> ? van Beneden. | " " " | 2 caudal vertebrae, 1 lumbal vertebra. |
| 3. <i>Balaenoptera</i> sp., van Beneden. | " " " | piece of lower jaw and pieces of skull-bone. |
| 4. <i>Balaenoptera borealina</i> , v. Beneden. Bal. bor.? of musc.? | " " " | 3 lumbal vertebrae, 1 dorsal v., 1st dorsal v. or the 7 th cervical vertebra, 2 caudal vertebrae, 1 symphysis of the left lower jaw, 1 dorsal part of a costa. |
| 5. <i>Balaenoptera muscu- loides</i> , van Beneden. | " " " | 1 symphysis of the right lower jaw, 1 head humerus, 2 caudal vertebrae, 1 finger phalanx. |

† MYSTACOGETI (Continuation).

| Name : | Locality : | Remains found : |
|--|--|---|
| 6. <i>Balaenoptera rostratella</i> van Beneden. | Wiegerink quarry, Nede. | 5 pieces of costa, 1 dorsal vertebra, 1 caudal vertebra. |
| 7. <i>Balaenula balaenopsis</i> , van Beneden. | " " " | 6 lumbal vertebrae. |
| 8. <i>Burtinopsis minutus</i> van Beneden. | " " " | 3 dorsal vertebrae, 1 caudal vertebra. |
| 9. <i>Burtinopsis similis</i> en Burt. Sp? van Beneden | " " " | 2 caudal vertebrae. |
| 10. <i>Heterocetus affinis</i> , v. Bened. en <i>Heterocetus affinis</i> ? | Wiegerink quarry and Winterswijk (?), Meddeho. | 9 dorsal vertebrae, 2 lumbal vertebrae, 1 caudal vertebra, 1 humerus, 1 petrosal bone, 2 bullae, 1 piece of the lower jaw right, costa? |

The humerus of N^o. 10 is a remarkably fine large specimen with a well preserved head. It is present in the Museum of Teyler's Genootschap, Harlem. It is the only one in our country. Prof. DUBOIS kindly informed me that the locality is probably the environs of Winterswijk. A plaster cast and three photo's were made. Brussels has similar specimens to this one.

| | | |
|--|--|---|
| 11. <i>Heterocetus brevifrons</i> ? van B. en <i>Heterocetus</i> sp? | Wiegerink quarry. | fragment of petrosal bone. Cervical vertebra. |
| 12. <i>Idiocetus laxatus</i> , v. B. | " " and locality (?) | 4 lumbal vertebrae. |
| 13. <i>Isocetus</i> sp. v. B. | Wiegerink quarry. | 1st 2 nd and 3 rd dorsal vertebra, 1 lumbal vert. |
| 14. <i>Mesocetus latifrons</i> ? van B. | Nede, Needsche Berg. | 1st caudal vertebra, 1 dorsal vertebra. |
| 15. <i>Mesocetus longirostris</i> van Ben. | Wiegerink quarry very probable, Nede. Miste near Winterswijk. | 1 atlas, 2 epistropheus, cervical vertebra no. 3 and 4 or 5, 1 lumbal vertebra, 2 caudal vertebrae. Right ulna, proximal part; head of right humerus, piece of lower jaw, left; 3 rd , 4 th , 12 th dorsal vertebra, 4 th , 8 th lumbal vertebra, 1st caudal vertebra, fragm. left temporal bone, 3 fragm. costa; piece of the right lower jaw near the symphysis, 1 finger phalanx, (eventually of <i>Balaenoptera</i>). |

† MYSTACOCETI (*Continuation*).

| Name : | Locality : | Remains found : |
|---|----------------------------|---|
| 16. Mesocetus sp. van B. | Probably Wiegerink quarry. | 1 lumbal vertebra |
| 17. Plesiocetus brialmonti? van Ben. | Wiegerink quarry. | 1 " " |
| 18. Plesiocetus dubius, vanB. | " " | 1 " " |
| 19. Plesiocetus hupschii? van Beneden. | " " | cervical vertebra 3 or 4, 1 cervical vertebra, the 1 st dorsal vertebra. |
| 20. Plesiocetus sp. van B. | Probably Wiegerink quarry. | 1 caudal vertebra, 1 lumbal vertebra. |

As appears from the above of the †Mystacoceti 9 genera and about 18 species were found. When adding to these numbers the †Odontoceti we get 16 genera and ± 30 species †Cetacea in our country. These figures must be considered provisional. The †Mystacoceti invariably bear the name of the author P. J. VAN BENEDEN and it can hardly be questioned that not all the names of species and genera can reasonably be maintained. Most likely some of them have to be combined, so that the number will diminish. In the Brussels Museum — where I made my determinations — they stick to the names of VAN BENEDEN. I am trying to ascertain from the literature which of those names should be replaced by others.

Of the following Pinnipedia remains have been found in our country.

† PINNIPEDIA.

| Name : | Locality : | Remains found : |
|---------------------------------------|---------------------------|---|
| 1. Monotherium affinis Van Beneden. | Wiegerink quarry. | Calcaneus. |
| 2. Monotherium aberratum Van Beneden. | " " | 1 Metacarpus or metatarsus. |
| 3. Alachterium Cretsii Du Bus. | " " | Phalanx and left metatarsus of the 5 th toe. |
| 4. Phoca vitulinoides Van Ben. | " " | A fragm. of humerus and a piece of the radius. |
| 5. Palaeophoca? Nysti Van Ben. | " " | Tooth. |
| 6. Phocanella minor Van Ben | " " | Fragm. of femur. |
| 7. Gryphoca similis? Van Ben. | Meddeho near Winterswijk. | Calcaneus. |

Compared with the remains of † Cetacea, the remains of † Pinnipedia are far less significant. It is striking that I have not met with any remains of † Pinnipedia from the Needsche Berg.

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Mathematics. — “Weights of the most probable values of the Unknowns in the case of Direct Conditioned Observations”. By Prof. M. J. VAN UVEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of October 30, 1926).

In adjusting direct conditioned observations, the reliableness of the “solutions”, i. e. of the values obtained by adjustment, is indicated by their weights.

In the following paper we propose deriving for these weights an expression which gives them in a closed form as a function of the data, and which moreover holds good under an arbitrary number of conditions.

The quantities observed (m in number) will be designated by x, y, z, \dots ; the observed values may be ξ, η, ζ, \dots , resp., their weights $g_\xi, g_\eta, g_\zeta, \dots$.

For simplification's sake we will assume that these m quantities x, y, z, \dots must satisfy a priori μ linear conditions:

$$a_j x + \beta_j y + \gamma_j z + \dots = \alpha_j \quad j=1, 2, \dots, \mu, \quad \mu < m \quad \dots \quad (1)$$

We form the differences

$$\alpha_j - (a_j \xi + \beta_j \eta + \gamma_j \zeta + \dots) = \alpha_j - [a_j \xi] = V_j, \quad j=1, 2, \dots, \mu \quad (2)$$

where $[]$ denotes a summation over the variables x, y, z, \dots : Let X, Y, Z, \dots be m provisionally assumed comparative values for x, y, z, \dots which, by agreement, satisfy the μ conditions, so that

$$[a_j X] = \alpha_j \quad j=1, 2, \dots, \mu,$$

Then, by forming the differences

$$\xi - X = v_x, \quad \eta - Y = v_y, \quad \zeta - Z = v_z, \dots \quad \dots \quad (3)$$

we have, according to (2):

$$[a_j v_x] = -V_j \quad j=1, 2, \dots, \mu. \quad \dots \quad (4)$$

We next consider those comparative values X, Y, Z, \dots as the most probable values for x, y, z, \dots , for which $[g_\xi v_x v_x]$ takes the least value allowed by the conditions.

Then the equation $[g_\xi v_x dv_x] = 0$ must be dependent on the μ equations $[a_j dv_x] = 0$ ($j=1, \dots, \mu$), whence

$$[g_\xi v_x dv_x] \equiv \sum_{j=1}^{\mu} \lambda_j [a_j dv_x];$$

here the λ_j are multipliers to be determined; the summation sign Σ is used here (and will be used also in future) to designate a summation over the μ conditions. Thus the sum Σ runs always from 1 to μ .

Now we have, as a consequence of the above identity:

$$g_{\xi} v_x = \sum \lambda_j a_j, \quad g_{\eta} v_y = \sum \lambda_j \beta_j, \quad g_{\zeta} v_z = \sum \lambda_j \gamma_j, \dots m \text{ equations} \quad (5)$$

and the μ equations already found:

$$[a_j v_x] = -V_j \quad j = 1, \dots, \mu \dots \dots \dots (4)$$

hence altogether $m + \mu$ equations to determine the $m + \mu$ unknowns $v_x, v_y, v_z, \dots, \lambda_1, \lambda_2, \dots, \lambda_{\mu}$.

The usual solution runs as follows:

From (4) and (5) we derive the μ equations:

$$\sum_{j=1}^{\mu} \lambda_j \left[\frac{a_j a_k}{g_{\xi}} \right] = -V_k, \quad k = 1, \dots, \mu$$

or, putting for the sake of abbreviation:

$$\left[\frac{a_j a_k}{g_{\xi}} \right] = s_{jk}, \quad (s_{jk} = s_{kj}) \dots \dots \dots (6)$$

$$\sum_{j=1}^{\mu} s_{kj} \lambda_j = -V_k, \quad k = 1, \dots, \mu. \dots \dots \dots (7)$$

These μ equations determine the μ multipliers λ_j .

Denoting the solution of these equations by $\bar{\lambda}_j$, and representing the values of v_x, v_y, v_z, \dots , corresponding with them, according to (5), by u_x, u_y, u_z, \dots , we have:

$$u_x = \frac{1}{g_{\xi}} \sum a_j \bar{\lambda}_j, \quad u_y = \frac{1}{g_{\eta}} \sum \beta_j \bar{\lambda}_j, \quad u_z = \frac{1}{g_{\zeta}} \sum \gamma_j \bar{\lambda}_j, \dots \dots (8)$$

the most probable values of x, y, z, \dots being

$$\bar{x} = \xi - u_x, \quad \bar{y} = \eta - u_y, \quad \bar{z} = \zeta - u_z, \dots \dots \dots (9)$$

and $[g_{\xi} u_x u_x]$ being the least value which $[g_{\xi} v_x v_x]$ may assume, so far as the conditions allow.

The mean (true) error σ of the unity of weight follows from

$$\sigma^2 = \frac{[g_{\xi} u_x u_x]}{\mu} \dots \dots \dots (10)$$

The sum $[g_{\xi} u_x u_x]$ may also be expressed directly into the data:

$$\begin{aligned} [g_{\xi} u_x u_x] &= \sum_k \sum_j \left[g_{\xi} \cdot \frac{a_k \bar{\lambda}_k}{g_{\xi}} \cdot \frac{a_j \bar{\lambda}_j}{g_{\xi}} \right] = \sum_k \sum_j \left[\frac{a_k a_j}{g_{\xi}} \bar{\lambda}_k \bar{\lambda}_j \right] = \\ &= \sum_k \sum_j \left[\frac{a_k a_j}{g_{\xi}} \right] \bar{\lambda}_k \bar{\lambda}_j = \sum_k \sum_j s_{kj} \bar{\lambda}_k \bar{\lambda}_j \end{aligned}$$

(see (8) and (6)), or, since $\bar{\lambda}_j$, being the solution of (7), satisfies

$$\begin{aligned} \sum_j s_{kj} \bar{\lambda}_j &= -V_k, \\ [g_{\xi} u_x u_x] &= -\sum_k V_k \bar{\lambda}_k. \end{aligned}$$

Representing the determinant $|s_{jk}| = \begin{vmatrix} s_{11} & s_{12} & \dots & s_{1\mu} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1\mu} & s_{2\mu} & \dots & s_{\mu\mu} \end{vmatrix}$ by S , and the minor (algebraic complement) of the element s_{jk} by S_{jk} (whence $S = \sum_j s_{jk} S_{jk}$, $0 = \sum_j s_{jk} S_{jl}$ ($l \neq k$)), we have

$$\bar{\lambda}_k = \frac{-\sum_j S_{jk} V_j}{S},$$

hence

$$[g_\xi u_x u_x] = \frac{\sum_k \sum_j S_{jk} V_j V_k}{S} = -\frac{1}{S} \begin{vmatrix} s_{11} & \dots & s_{1\mu} & V_1 \\ \vdots & \vdots & \vdots & \vdots \\ s_{1\mu} & \dots & s_{\mu\mu} & V_\mu \\ V_1 & \dots & V_\mu & 0 \end{vmatrix},$$

or, putting for the sake of abbreviation:

$$\begin{vmatrix} s_{11} & \dots & s_{1\mu} & q_1 \\ \vdots & \vdots & \vdots & \vdots \\ s_{1\mu} & \dots & s_{\mu\mu} & q_\mu \\ p_1 & \dots & p_\mu & r \end{vmatrix} = \begin{vmatrix} s_{jk} & q_j \\ p_k & r \end{vmatrix} \dots \dots \dots (11)$$

$$[g_\xi u_x u_x] = -\frac{1}{S} \begin{vmatrix} s_{jk} & V_j \\ V_k & 0 \end{vmatrix} \dots \dots \dots (12)$$

The uncertainty of the "solutions" \bar{x} , \bar{y} , \bar{z} , ... (see (9)) depends, in the last instance, on the uncertainty of the observed values ξ , η , ζ , ...: these latter have the weights g_ξ , g_η , g_ζ , ... thus the mean errors:

$$\sigma_\xi = \frac{\sigma}{\sqrt{g_\xi}}, \quad \sigma_\eta = \frac{\sigma}{\sqrt{g_\eta}}, \quad \sigma_\zeta = \frac{\sigma}{\sqrt{g_\zeta}}, \dots$$

If we imagine that, in eventually repeating the series of observations, the observed values ξ , η , ζ , ... undergo the variations $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, ... respectively, then the mean square of $\Delta\xi$ (represented by $\overline{\Delta\xi^2}$) is nothing else than the square of the mean error of ξ , etc., in other words:

$$\overline{\Delta\xi^2} = \frac{\sigma^2}{g_\xi}, \quad \overline{\Delta\eta^2} = \frac{\sigma^2}{g_\eta}, \quad \overline{\Delta\zeta^2} = \frac{\sigma^2}{g_\zeta}, \dots \dots \dots (13)$$

the mutual independency of the observed values ξ , η , ζ , ... having as a consequence that the mean products, two by two, of the variations $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, ... are zero, so that

$$\overline{\Delta\xi \cdot \Delta\eta} = 0, \quad \overline{\Delta\xi \cdot \Delta\zeta} = 0, \dots, \overline{\Delta\eta \cdot \Delta\zeta} = 0 \text{ etc.} \dots (14)$$

In consequence of the variations $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, ... the V_j undergo (according to (2)) the variations

$$\Delta V_j = -[a_j \Delta\xi] = -(a_j \Delta\xi + \beta_j \Delta\eta + \gamma_j \Delta\zeta + \dots).$$

The variations $\Delta\lambda_k$ of $\bar{\lambda}_k$, deriving from these ΔV_j , satisfy (see (7))

$$\sum s_{kj} \Delta\lambda_j = -\Delta V_k \quad k = 1, \dots, \mu \dots \dots (15)$$

Further, from

$$u_x = \frac{1}{g_\xi} \sum_j a_j \bar{\lambda}_j = \xi - \bar{x} \dots \dots \dots (8), (9)$$

follows

$$\sum \frac{a_j}{g_\xi} \Delta \lambda_j = \Delta \xi - \Delta \bar{x} \dots \dots \dots (16)$$

We now multiply the μ equations (15) and the equation (16) by $\Delta \xi$. Then, taking the mean value, we obtain (by (13) and (14))

$$\begin{aligned} \sum_j s_{kj} \overline{\Delta \lambda_j \cdot \Delta \xi} &= -\overline{\Delta V_k \cdot \Delta \xi} = +(\alpha_k \overline{\Delta \xi^2} + \beta_k \overline{\Delta \xi \cdot \Delta \eta} + \gamma_k \overline{\Delta \xi \cdot \Delta \zeta} + \dots) \\ &= \frac{\alpha_k}{g_\xi} \sigma^2 \quad k = 1, \dots, \mu \end{aligned}$$

$$\sum \frac{a_j}{g_\xi} \overline{\Delta \lambda_j \cdot \Delta \xi} = \overline{\Delta \xi^2} - \overline{\Delta \bar{x} \cdot \Delta \xi} = \frac{\sigma^2}{g_\xi} - \overline{\Delta \bar{x} \cdot \Delta \xi}.$$

By eliminating the μ variables $\overline{\Delta \lambda_j \cdot \Delta \xi}$ from these $\mu + 1$ equations and using the notation (11), we arrive at

$$\begin{vmatrix} s_{kj}, & \frac{\alpha_k}{g_\xi} \sigma^2 \\ \frac{a_j}{g_\xi}, & \frac{\sigma^2}{g_\xi} - \overline{\Delta \bar{x} \cdot \Delta \xi} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} s_{kj}, & \frac{\alpha_k}{g_\xi} \sigma^2 \\ \frac{a_j}{g_\xi}, & \frac{\sigma^2}{g_\xi} \end{vmatrix} - \begin{vmatrix} s_{kj}, & 0 \\ \frac{a_j}{g_\xi}, & \overline{\Delta \bar{x} \cdot \Delta \xi} \end{vmatrix} = 0,$$

or

$$\frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj}, & \frac{\alpha_k}{g_\xi} \\ a_j, & 1 \end{vmatrix} - S \cdot \overline{\Delta \bar{x} \cdot \Delta \xi} = 0,$$

whence

$$\overline{\Delta \bar{x} \cdot \Delta \xi} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj}, & \frac{\alpha_k}{g_\xi} \\ a_j, & 1 \end{vmatrix} \dots \dots \dots (17)$$

In the same manner we obtain, by multiplying the μ equations (15) and the equation (16) by $\Delta \eta$, and then taking the mean,

$$\begin{aligned} \sum_j s_{kj} \overline{\Delta \lambda_j \cdot \Delta \eta} &= -\overline{\Delta V_k \cdot \Delta \eta} = +(\alpha_k \overline{\Delta \eta \cdot \Delta \xi} + \beta_k \overline{\Delta \eta^2} + \gamma_k \overline{\Delta \eta \cdot \Delta \zeta} + \dots) \\ &= \frac{\beta_k}{g_\eta} \sigma^2 \quad k = 1, \dots, \mu \end{aligned}$$

$$\sum_j \frac{a_j}{g_\xi} \overline{\Delta \lambda_j \cdot \Delta \eta} = \overline{\Delta \xi \cdot \Delta \eta} - \overline{\Delta \bar{x} \cdot \Delta \eta} = -\overline{\Delta \bar{x} \cdot \Delta \eta},$$

From these $\mu + 1$ equations we derive by eliminating the μ variables $\overline{\Delta \lambda_j \cdot \Delta \eta}$:

$$\begin{vmatrix} s_{kj}, & \frac{\beta_k}{g_i} \sigma^2 \\ \frac{a_j}{g_\xi}, & 0 - \overline{\Delta x \cdot \Delta \eta} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} s_{kj}, & \frac{\beta_k}{g_i} \sigma^2 \\ \frac{a_j}{g_\xi}, & 0 \end{vmatrix} - \begin{vmatrix} s_{kj}, & 0 \\ \frac{a_j}{g_\xi}, & \overline{\Delta x \cdot \Delta \eta} \end{vmatrix} = 0,$$

or

$$\frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj}, & \frac{\beta_k}{g_i} \\ a_j, & 0 \end{vmatrix} - S \cdot \overline{\Delta x \cdot \Delta \eta} = 0,$$

whence

$$\overline{\Delta x \cdot \Delta \eta} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \begin{vmatrix} s_{kj}, & \frac{\beta_k}{g_i} \\ a_j, & 0 \end{vmatrix} \dots \dots \dots (18)$$

Likewise we find

$$\overline{\Delta x \cdot \Delta \zeta} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \begin{vmatrix} s_{kj}, & \frac{\gamma_k}{g_\zeta} \\ a_j, & 0 \end{vmatrix} \dots \dots \dots (18 \text{ bis})$$

etc.

We next multiply (15) by $\overline{\Delta x}$ and take the mean; so we obtain (replacing the index k by i):

$$\sum_j s_{ij} \overline{\Delta \lambda_j \cdot \Delta x} = -\overline{\Delta V_i \cdot \Delta x} = + (a_i \overline{\Delta x \cdot \Delta \xi} + \beta_i \overline{\Delta x \cdot \Delta \eta} + \gamma_i \overline{\Delta x \cdot \Delta \zeta} + \dots), i = 1, \dots, \mu$$

or, on account (17), (18), (18 bis) etc.

$$\begin{aligned} \sum_j s_{ij} \overline{\Delta \lambda_j \cdot \Delta x} &= \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \left\{ a_i \begin{vmatrix} s_{kj}, & \frac{a_k}{g_\xi} \\ a_j, & 1 \end{vmatrix} + \beta_i \begin{vmatrix} s_{kj}, & \frac{\beta_k}{g_i} \\ a_j, & 0 \end{vmatrix} + \gamma_i \begin{vmatrix} s_{kj}, & \frac{\gamma_k}{g_\zeta} \\ a_j, & 0 \end{vmatrix} + \dots \right\} \\ &= \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj}, & \left[\frac{a_k a_i}{g_\xi} \right] \\ a_j, & a_i \end{vmatrix} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj}, & s_{ki} \\ a_j, & a_i \end{vmatrix}, i = 1, \dots, \mu \end{aligned}$$

or

$$\sum_j s_{ij} \overline{\Delta \lambda_j \cdot \Delta \bar{x}} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{11}, \dots, s_{1i}, \dots, s_{1\mu}, s_{1i} \\ \vdots \\ s_{\mu 1}, \dots, s_{\mu i}, \dots, s_{\mu\mu}, s_{\mu i} \\ a_1, \dots, a_i, \dots, a_\mu, a_i \end{vmatrix} \equiv 0 \quad i = 1, \dots, \mu$$

The determinant $|s_{ij}| = S$ of the μ homogeneous equations

$$\sum_j s_{ij} \overline{\Delta \lambda_j \cdot \Delta \bar{x}} = 0 \quad i = 1, \dots, \mu$$

which we obtain in this way, being different from zero, we have separately

$$\overline{\Delta \lambda_j \cdot \Delta \bar{x}} = 0. \quad j = 1, \dots, \mu \quad \dots \dots \dots (19)$$

Multiplying at last the equation (16) by $\overline{\Delta \bar{x}}$ and then taking the mean, we find

$$\overline{\Delta u_x \cdot \Delta \bar{x}} = \sum \frac{a_j}{g_\xi} \overline{\Delta \lambda_j \cdot \Delta \bar{x}} = \overline{\Delta \xi \cdot \Delta \bar{x}} - \overline{\Delta \bar{x}^2}$$

thus, by (19),

$$\overline{\Delta u_x \cdot \Delta \bar{x}} = \overline{\Delta \xi \cdot \Delta \bar{x}} - \overline{\Delta \bar{x}^2} = 0 \quad \dots \dots \dots (20)$$

or, on account of (17),

$$\overline{\Delta \bar{x}^2} = \overline{\Delta \xi \cdot \Delta \bar{x}} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj} & \frac{a_k}{g_\xi} \\ a_j & 1 \end{vmatrix} \dots \dots \dots (21)$$

Now, $\overline{\Delta \bar{x}^2}$ being nothing but the square of the mean error $\sigma_{\bar{x}}$ of \bar{x} , we have

$$\sigma_{\bar{x}}^2 = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj} & \frac{a_k}{g_\xi} \\ a_j & 1 \end{vmatrix} \dots \dots \dots (22)$$

We shall moreover put this latter determinant into another form. In

$$\begin{vmatrix} s_{kj} & \frac{a_k}{g_\xi} \\ a_j & 1 \end{vmatrix} = \begin{vmatrix} s_{11}, \dots, s_{1j}, \dots, s_{1\mu}, a_1 : g_\xi \\ \vdots \\ s_{k1}, \dots, s_{kj}, \dots, s_{k\mu}, a_k : g_\xi \\ \vdots \\ s_{\mu 1}, \dots, s_{\mu j}, \dots, s_{\mu\mu}, a_\mu : g_\xi \\ a_1, \dots, a_j, \dots, a_\mu, 1 \end{vmatrix}$$

we take from the 1st column a_i times the last column

$$\begin{matrix} \vdots \\ \dots & j^{th} & \dots & a_j & \dots & \dots & \dots & \dots \\ \vdots \\ \dots & \mu^{th} & \dots & a_\mu & \dots & \dots & \dots & \dots \end{matrix}$$

Then we retain for the element of the j^{th} column and the k^{th} row:

$$s_{kj} - \frac{a_j a_k}{g_\xi} = \left[\frac{a_k a_j}{g_\xi} \right] - \frac{a_k a_j}{g_\xi} = \frac{\beta_k \beta_j}{g_\eta} + \frac{\gamma_k \gamma_j}{g_\zeta} + \dots = a_{kj} \quad (23)$$

the j^{th} element of the last row becoming zero ($j=1, \dots, \mu$). Representing the determinant $|a_{kj}|$ by A , we obtain

$$\begin{vmatrix} s_{kj} & \frac{a_k}{g_\xi} \\ a_j & 1 \end{vmatrix} = \begin{vmatrix} a_{kj} & \frac{a_k}{g_\xi} \\ 0 & 1 \end{vmatrix} = |a_{kj}| = A.$$

So we find

$$a_x^{-2} = \frac{A}{S} \cdot \frac{\sigma^2}{g_\xi}$$

and for the weight g_x of x :

$$g_x = \frac{S}{A} \cdot g_\xi.$$

Putting in a similar manner:

$$s_{kj} - \frac{\beta_j \beta_k}{g_\eta} = \left[\frac{a_k a_j}{g_\xi} \right] - \frac{\beta_k \beta_j}{g_\eta} = \frac{a_k a_j}{g_\xi} + \frac{\gamma_k \gamma_j}{g_\zeta} + \dots = b_{kj} \quad (23 \text{ bis})$$

$$s_{kj} - \frac{\gamma_j \gamma_k}{g_\zeta} = \left[\frac{a_k a_j}{g_\xi} \right] - \frac{\gamma_k \gamma_j}{g_\zeta} = \dots = c_{kj} \quad (23 \text{ ter})$$

etc.,

and further:

$$|b_{kj}| = B \quad , \quad |c_{kj}| = C, \text{ etc.,}$$

we arrive at last at:

$$a_x^{-2} = \frac{A}{S} \cdot \frac{\sigma^2}{g_\xi} \quad , \quad a_y^{-2} = \frac{B}{S} \cdot \frac{\sigma^2}{g_\eta} \quad , \quad a_z^{-2} = \frac{C}{S} \cdot \frac{\sigma^2}{g_\zeta} \quad , \text{ etc.,} \quad (24)$$

or

$$g_x = \frac{S}{A} \cdot g_\xi \quad , \quad g_y = \frac{S}{B} \cdot g_\eta \quad , \quad g_z = \frac{S}{C} \cdot g_\zeta \quad , \text{ etc.,} \quad (25)$$

Thus we have expressed the weights of the solutions in a closed form into the data $g_\xi, g_\eta, g_\zeta, \dots; a_j, \beta_j, \gamma_j, \dots (j=1, \dots, \mu)$

If there is given, in particular, but one condition

$$ax + \beta y + \gamma z + \dots = \varkappa$$

so that $\mu=1$, then we have

$$S = \left[\frac{a a}{g_\xi} \right], \quad A = \left[\frac{a a}{g_\xi} \right] - \frac{a a}{g_\xi} = \frac{\beta \beta}{g_\eta} + \frac{\gamma \gamma}{g_\zeta} + \dots,$$

$$B = \left[\frac{a a}{g_\xi} \right] - \frac{\beta \beta}{g_\eta} = \frac{a a}{g_\xi} + \frac{\gamma \gamma}{g_\zeta} + \dots, \quad C = \dots, \text{ etc.,}$$

thus

$$\sigma_x^2 = \frac{\sigma^2}{g_\xi} \cdot \frac{\beta\beta + \gamma\gamma + \dots}{g_\alpha + g_\beta + g_\gamma + \dots}, \sigma_y^2 = \frac{\sigma^2}{g_\alpha} \cdot \frac{\alpha\alpha + \gamma\gamma + \dots}{g_\xi + g_\beta + g_\gamma + \dots}, \sigma_z^2 = \dots, \text{etc.} \quad (24\text{bis})$$

The determinants S, A, B, C, \dots may yet be put into another form.

Of the m variables x, y, z, \dots , represented by $g_\xi, g_\alpha, g_\beta, \dots$ and by $\alpha_j, \beta_j, \gamma_j, \dots$, we consider the combinations μ by μ . Be x, z, \dots, t ($g_\xi, g_\beta, \dots, g_\tau; \alpha_j, \gamma_j, \dots, \theta$) such a combination.

Then we may form determinants such as

$$\Delta(x, z, \dots, t) = \begin{vmatrix} \alpha_1, \gamma_1, \dots, \theta_1 \\ \alpha_2, \gamma_2, \dots, \theta_2 \\ \vdots \\ \alpha_\mu, \gamma_\mu, \dots, \theta_\mu \end{vmatrix}$$

According to the theory of determinants, we may write S as follows:

$$S = |s_{kj}| = \left| \left[\frac{a_k a_j}{g_\xi} \right] \right| = \sum \frac{\Delta^2(x, z, \dots, t)}{g_\xi \cdot g_\beta \cdot \dots \cdot g_\tau} \dots \dots \dots (27)$$

where the sum Σ runs over all the $C_\mu(m) = \frac{m!}{\mu!(m-\mu)!}$ combinations μ by μ of the m elements x, y, z, \dots .

In the same manner we may also reduce the determinant A .

Here all the m elements x, y, z, \dots appear, except the element x . With these $m-1$ elements y, z, \dots we can build determinants of the form:

$$\Delta_x(y, z, \dots, t) = \begin{vmatrix} \beta_1, \gamma_1, \dots, \theta_1 \\ \vdots \\ \beta_\mu, \gamma_\mu, \dots, \theta_\mu \end{vmatrix} \dots \dots \dots (28)$$

wherin the elements α_j are always missing.

So we have

$$A = |a_{kj}| = \left| \left[\frac{a_k a_j}{g_\xi} \right] - \frac{a_k a_j}{g_\xi} \right| = \sum_x \frac{\Delta_x^2(y, z, \dots, t)}{g_\alpha \cdot g_\beta \cdot \dots \cdot g_\tau} \dots \dots (29)$$

where the sum Σ_x extends over all the $C_\mu(m-1) = \frac{(m-1)!}{\mu!(m-\mu-1)!}$ combinations μ by μ of the $m-1$ elements y, z, \dots (x absent).

Similarly:

$$B = |b_{kj}| = \sum_y \frac{\Delta_y^2(x, z, \dots, t)}{g_\xi \cdot g_\beta \cdot \dots \cdot g_\tau} \dots \dots \dots (29 \text{ bis})$$

$$C = |c_{kj}| = \sum_z \frac{\Delta_z^2(x, y, \dots, t)}{g_\xi \cdot g_\alpha \cdot \dots \cdot g_\tau} \dots \dots \dots (29 \text{ ter})$$

etc.

So the determinants S, A, B, C, \dots are written as sums of positive terms. The terms of the sum A also appear in the sum S , which contains besides those terms depending upon the element $x(g_{\xi}, a_j)$. Thus the value A is certainly less than that of S . Likewise B, C, \dots are altogether less than S .

The fractions $\frac{A}{S}, \frac{B}{S}, \frac{C}{S}, \dots$ are therefore altogether less than unity, so that the mean errors $\sigma_x, \sigma_y, \sigma_z, \dots$ of the solutions are less than the mean errors of the respective observations, viz:

$$\sigma_{\xi} = \frac{\sigma}{\sqrt{g_{\xi}}}, \quad \sigma_{\eta} = \frac{\sigma}{\sqrt{g_{\eta}}}, \quad \sigma_{\zeta} = \frac{\sigma}{\sqrt{g_{\zeta}}}, \dots$$

thus:

$$\sigma_x < \sigma_{\xi}, \quad \sigma_y < \sigma_{\eta}, \quad \sigma_z < \sigma_{\zeta}, \quad \dots \quad (30)$$

If all the terms of the sum S (27) are nearly equal, then the sums A and S are almost proportional to the numbers of their terms, thus as $C_{\nu}(m-1)$ to $C_{\nu}(m)$, that is: as $m-\mu$ to m . In this case the mean errors of the solutions are about $\sqrt{\frac{m-\mu}{m}}$ times as large as the mean errors of the corresponding observations. The larger μ is in comparison to m , the more the solution surpasses the observation in accuracy.

Chemistry. — "On Double-Salts in the Series of isomorphous Mixed Crystals of Magnesium- and Zinc-sulphate, and on the Occurrence of Chemical Compounds in the Solid State." By H. G. K. WESTENBRINK. (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of October 30, 1926).

§ 1. According to HOLLMANN's investigations¹⁾, two double-salts: $2 MgSO_4, ZnSO_4, 21 H_2O$ and $MgSO_4, ZnSO_4, 14 H_2O$, would be inclosed in the continuous series of mixed crystals, which are formed in all proportions on crystallisation of mixed solutions of *magnesium* and *zinc-sulphate* in water. To this view he was led by the results of his measurements of the vapour-tensions of these mixed crystals, executed after the so-called "indirect" method, which consists in the determination of the concentration of mixtures of water and sulphuric acid, if these mixtures were in equilibrium with the hydrates of definite composition, when put together in an exsiccator.

In the curve, representing the vapour-tension as a function of the composition of the mixed crystals, HOLLMANN observed two discontinuities, which, according to him, were situated at the stoichiometric proportions: $Mg : Zn = 1 : 1$ and $2 : 1$. As in general, from earlier experiments it was seen, that the vapour-tension varies continuously with the composition of the solid phase, he drew the conclusion that the discontinuities observed must be considered to be the points of intersection of three continuous vapour tension-curves, which refer successively to a series of mixed crystals between $MgSO_4, 7 H_2O$ and the first mentioned double-salt, to such between both double-salts indicated above, and to such between the second double-salt and pure $ZnSO_4, 7 H_2O$. Later-on he tried to corroborate these conclusions by fixing the particular temperatures of decomposition of the mixed crystals into *hexahydrates* and water, applying the dilatometric method²⁾. If these temperatures be graphically represented in function of the chemical composition, they form again a curve with two discontinuities, situated at the same stoichiometric proportions as mentioned above. HOLLMANN³⁾, following the method of VAN RIJN VAN ALKEMADE⁴⁾ and BAKHUIS ROOZEBOOM⁵⁾,

1) R. HOLLMANN, Zeits. f. phys. Chemie, **37**, 193, (1901).

2) A. C. VAN RIJN VAN ALKEMADE, Zeits. f. phys. Chemie, **11**, 289, (1893).

3) H. W. BAKHUIS ROOZEBOOM, Zeitschr. phys. f. Chemie, **30**, 385, (1899).

4) R. HOLLMANN, Zeits. f. phys. Chemie, **40**, 577, (1902).

5) R. HOLLMANN, Zeits. f. phys. Chemie, **40**, 561, (1902).

deduced three types of temperature-concentration-curves for the case of the decomposition of a mixed crystal with definite content of crystal-water into a mixed crystal with a smaller number of water-molecules and a saturated solution. The three types deduced by him are: 1st. The curve rises continually from 100% *A* to 100% *B*; 2nd. the curve shows a minimum; 3rd. the curve shows a maximum. The two discontinuities in the curve relating to the mixed crystals $Mg(Zn)SO_4 \cdot 7H_2O$ are, according to HOLLMANN, again the points of intersection of three continuous curves. Indeed, such a shape of the curve, as really observed, might be expected beforehand, if double-salts were present in the series. However, several objections can be made with respect to his way of experimenting. One could put the question, for instance, whether the same hydrate is always formed from the heptahydrate? Whether this lower hydrate always and in all proportions forms mixed crystals? Whether the equilibria set in rapidly enough, to allow the appliance of the dilatometric method? Etc. Moreover, the author has drawn his curves rather arbitrarily through the 19 points observed. It can be doubted, whether the tendency to find his compound at the proportions: $MgSO_4 : ZnSO_4 = 1 : 1$ and $2 : 1$, has not had a real influence on his interpretation of the facts? It is, moreover, very difficult to answer the question, whether the minima observed are to be considered as real discontinuities in his curve; etc.

Later-on BARCHET¹⁾ made a series of solubility-measurements of the same salts when forming mixed crystals. In his results he could not see a confirmation of the occurrence of HOLLMANN's two compounds. The last author²⁾ again emphasised, that BARCHET's interpretation of the observed facts was to be considered as erroneous, and that, on the contrary, from his results the existence of the compounds might be deduced. Here HOLLMANN again makes use of VAN RIJN VAN ALKEMADE's method, who deduced the solubility-curve for two salts and for two salts and two double-salts, with or without water of crystallisation, making use of GIBBS' graphical representation of the ζ -function. However, VAN RIJN VAN ALKEMADE did not draw the curve which must be present if mixed crystals form the solid phase; also the simultaneous occurrence of mixed crystals and double salts is left completely out of consideration by him. For this last mentioned case, HOLLMANN deduces a curve, which agrees with that deduced from BARCHET's observations. Theoretically, the solubility-curve is found by drawing the tangents to the ζ -curve of the mixed crystals and the ζ -surface of the saturated solutions: the points of touching at the last surface form together the desired solubility-curve. But the accurate shape of both ζ -functions being not known, it is impossible to get convinced of the fact, that a curve, agreeing in shape with that determined experimentally, should not as well be valid

¹⁾ E. BARCHET, N. Jahrb. f. Miner., etc. Beil. Bnd. 18, 377, (1904).

²⁾ R. HOLLMANN, Centr. Bl. f. Miner., (1904), 513.

for the case that only mixed crystals, and no compounds whatsoever, represent the solid phase of the system.

It seems evident from the foregoing, that little hope is left to settle these questions by a repetition of HOLLMANN's and BARCHET's experiments. Quite independently from this, the problem of the possible occurrence of these compounds may, however, also be treated by the aid of the results of the structure-theory of the solid state. Then it will be necessary to ascertain in the first place, how under these circumstances must be understood the expression solid "compound", as contrasted by the conception of "mixed crystal".

§ 2. In the first place is it, therefore, necessary to remember, what the ROENTGENographic investigations have taught us about the structures of mixed crystals in general.

Mixed crystals may, *a priori*, be conceived as being built up in three fundamental ways:

1st. They represent heterogeneous conglomerates of very small, crystalline parts of the components.

2nd. They are homogeneous, or quasi-homogeneous, crystals, the constituent kinds of atoms or atom-groups substituting each other in the regular structure in a completely *hasardous* way.

3rd. They are homogeneous crystals, in which the substituting atoms, replace each other in a perfectly regular way.

The way of structure as mentioned sub 1^o, which is, already from the point of view of the phase-rule, highly improbable, is not confirmed by any result of the ROENTGEN-analysis.

It is true, that the regular distribute of the atoms in mixed crystals, as mentioned sub 3rd, is defended more in particulars by TAMMANN¹⁾. This view was expressed by him in trying to explain the occurrence of apparent limits of concentration in such series of mixed crystals, at which limits their resistance against chemical attack appeared suddenly much greater than before or afterwards. Such cases were found in the case of mixed crystals between metals, or such salts as *NaCl* and *AgCl*. As a result of many observations made in his laboratory, it seemed to be evident, that, if in the case of these cubic mixed crystals their concentration, expressed in molecular fractions of the whole, the composition corresponded to $\frac{1}{8}$ mol. of the nobler metal or to a multiple $\frac{n}{8}$ of it, — suddenly an increase of resistance in chemical respect occurred; in the case of syncrystallizing metals, simultaneously a sudden change in potential with respect to a suitable electrolyte could be observed. Doubt has rightly arisen, whether the limits indicated here, are really situated at $\frac{n}{8}$ molecules so exactly as TAMMANN believes; but the occurrence of definite limits of maximum chemical resistance seems to be sure, as also other

¹⁾ G. TAMMANN, *Zeits. f. anorg. Chem.*, **107**, 1, (1919).

experimenters, quite independently of him, have found analogous facts. It is worth remarking, that MASING ¹⁾ and BORELIUS ²⁾ succeeded in deducing theoretically the possible occurrence of such limits in a series of mixed crystals, if it was assumed beforehand, that the way of arrangement and substitution of the "vicarious" atoms or atoms-groups in such mixed crystals is completely haphazardous. But the limits calculated by them are *not* situated at exactly $n/8$ molecules.

§ 3. TAMMANN ³⁾ assumes a real difference between solid compounds and mixed crystals only at higher temperatures, where appreciable "diffusion" can take place: from a mixed crystal both species of atoms will then diffuse into an outer medium *independently* from each other, while from a crystallized compound they will diffuse together and only in a fixed proportion.

However, with the view, mentioned above, concerning the regular distribution of the substituting atoms in a mixed crystal, TAMMANN comes in conflict with the general structure-theory of solid matter, as may be seen from the following considerations.

A "regular" distribution, as meant by TAMMANN, can have no other significance in this connection, than that also a mixed crystal is considered to be an interpenetration of "homogeneous discontinua", in which each point is surrounded by the other points in an identical way; "identical" here being understood in the sense of "congruent" or "equal to its mirror-image". Therefore, the atoms must be arranged in this case in such a way, as is described by means of one of the 230 possible space-groups. But then there must always be present *identical* atoms in *crystallographically-equivalent* positions. In a mixed crystal of a quite arbitrary composition, this is evidently impossible; for to each point of the elementary cell, determined by a definite space-group, correspond a number of $(n-1)$ equivalent points, if n be the number indicating the special degree of equivalence of the point first considered. If one of these n atoms be replaced by another, then the other $(n-1)$ places must be substituted simultaneously by the new atom, if the symmetry-character of the structure will remain preserved. In most of the real "compounds", n equal atoms are situated in such positions, as are characterized by a same degree of equivalence n ; sometimes in points, for which the sum of the degrees of equivalence is equal to n . It is, therefore, often observed, that the ratio of the numbers of atoms or atomgroups of a compound, appears to be equal to the ratio of the degrees of equivalence of the positions in the elementary cell, in which places these atoms are situated. Thus e.g. compounds with *three* (or a multiple of three) equal atoms or atomgroups arranged round a central atom, often appear to be *trigonal* or

1) G. MASING, Zeits. f. anorg. Chem., **118**, 293, (1922).

2) G. BORELIUS, Ann. d. Phys., **74**, 216, (1924).

3) G. TAMMANN, Zeits. f. Electrochemie, **28**, 36, (1922).

hexagonal; such with *four* equal atoms or atom-groups often appear to be tetragonal¹⁾ or cubic; etc.

Because the numbers of the different atoms in a chemical compound are always in proportion of simple integers, the structure of the crystals is possible according to a simple scheme. But in mixed crystals, in which *all* mixing proportions of the vicarious atoms or atom-groups are possible, yet for each concentration a structure is imaginable in which the symmetry-relations of one of the space-groups are valid; but, on the other hand, it will often appear necessary in such cases, to distribute *identical* atoms over numerous places which are characterized by different values of their degree of equivalence; moreover, that distribution will be different for each special concentration of the mixed crystal. Also it will be necessary, almost in each case, to assume elementary cells of abnormal size, — the last also necessarily varying with each concentration. This kind of structure of the mixed crystals is so improbable, that this view of it may be quietly left out of consideration here.

No difficulties of this kind arise, however, if an irregular, i. e. *statistical* distribution of the substituting atoms be assumed in a grating, whose fundamental dimensions may be derived in an additive way²⁾ from the grating-dimensions of the components of the mixed crystal. VON LAUE³⁾ has shown, starting from a set of premisses, that a mixed crystal built up in this way, will give the same, well defined diffraction-images, as the pure components do. In accordance with the additive character of the fundamental dimensions, the interference-lines will be shifted somewhat with respect to the lines met with in the case of the pure components. Although the original symmetry-elements are, properly speaking, no longer present, because of the unequal nature of the constituting atoms, their absence is in no way manifested in the spectrograms obtained, because they manifest themselves in a statistical way in the crystal-symmetry, and consequently also in the external form.

§ 4. However, for a rigorous and exhaustive understanding of what has been said here about the constitution of mixed crystals, it must be remarked in this connection, that the view of the mixed crystal, just as the crystalline pure components, being characterized by a rigorously definite periodicity with a mean value between that of the components, must be understood with the necessary reserve. Recently FRIEDEL⁴⁾ has drawn attention to the fact, that the view, according to which in the mixed crystal a grating with "mean" values for its parameters is be

¹⁾ Cf. the numerous instances of this in: P. GROTH, *Elemente der physikalischen und chemischen Krystallographie*, p. 273 and following, (1920).

²⁾ This (apparent) additive character was hitherto stated in all cases, in which mixed crystals were studied by means of ROENTGENrays.

³⁾ M. VON LAUE, *Ann. d. Physik*, [4], **56**, 497, (1918).

⁴⁾ G. FRIEDEL, *Compt. rend. Paris*, **182**, 741, (1926).

present, may be regarded neither as a logical deduction from the results of the experiment, nor as agreeing with the real state of things. As soon as the well known and generally assumed fact be taken into consideration, that the place of each atom in the structure depends wholly or almost exclusively on the influence of the immediately surrounding atoms, at distances which do not exceed a simple multiple of the parameter-lengths in all directions; and that the far distant particles have, therefore, no appreciable action upon it, — the conclusion mentioned above ceases to be a necessary postulate from the experimental results. For in the case of an irregular, statistical substitution, it may happen in a domain or volume-element with the dimensions of only a few parameter-distances in each direction, that occasionally, for instance, *only* atoms *A* or *only* atoms *B* will be present in succession of each other; or that, as in most cases, *A* and *B* will occur in very different, quite accidental mutual succession and in different proportions. In a domain, which consists only of atoms *A*, the parameters can, as a consequence of what was said above, only be those, which are valid in the case of pure *A*; in a domain, in which only the atoms *B* occur, these parameters can only have the values valid in the case of pure *B*; and in all intermediate mixtures of *A* and *B*, a finite number of parameter-values will occur, which lie between those of *A* and *B*, and depending as well on the proportions of *A* and *B*, as on their accidental way of distribution within the volume-element considered. Therefore, the apparently definite chemical constitution of the mixed crystal can only be a statistical "mean" value of all local compositions, which vary from one domain to another. The "homogeneity" of the mixed crystal is thus only a *quasi*-homogeneity, its periodicity a *quasi*-periodicity. Properly speaking, *no* grating whatsoever in the strict sense of the word is present in it any more, showing constant parameters of intermediate, „mean" values. This could only occur, if *all* particles, also those far distant in the structure in its whole extension, — or at least within a sphere whose radius should be very large in comparison with a few parameter-distances, — manifested their influence in fixing the place of the atom originally considered here; — a supposition, which is certainly not true. It can only be proved, that there is a chance considerably greater than all others, that a distribution of the atoms within a microscopical part of the mixed crystal will actually occur, such, that its action on the pencil of ROENTGEN-rays and the incident wave-trains will cause a way of interference, *as if* a grating with a "mean" value of the parameters, determined by the statistical constitution, were really present. The chance for that distribution of atoms is incomparably much greater than that for all other kinds of distribution, deviating from the one mentioned above. It is that circumstance, which causes the behaviour of the mixed crystal with respect to the ROENTGEN-radiation to appear to be that of an apparently real homogeneous discontinuum, with an apparent "grating" of "mean" value, analogous to

that of the pure components. However, this phenomenon is in fact a mere illusion: the mixed crystal, as has been said already, does not possess any real grating at all in the strict sense of the structural theory. It is more comparable with a kind of mosaic, built up by parts of unequal magnitude and different chemical constitution; it is, as said above, *quasi-homogeneous*, *quasi-periodical*, and possesses only a *quasi-grating* as structural fundament.

§ 5. Only in the case of some mixed crystals with very simple mixing proportion, an atomic distribution as TAMMANN supposes, would be possible, although it is by no means a necessary one. Because of the un-equal nature of the constituent components, the symmetry will, in the case of an analogous structure like that of the components, necessarily be of a *lower* degree than that of those components. In the ROENTGEN-spectograms a greater number of diffraction-lines would, therefore, be observed than in the corresponding spectograms of the pure components. Such crystals would appear, therefore, built up after a totally different scheme, in comparison with the other members of the series. Such crystals would be no longer mixed crystals, but they would just possess the true characteristics of real chemical compounds; *the characteristic feature* for these, — contrary to what is the case for mixed crystals, — *is, that crystallographically-equivalent places of the space-group must be always occupied by identical atoms or atom-groups*. It was, indeed, experimentally shown, that in those cases, in which *new* diffraction-lines were observed in the ROENTGEN-spectograms, the occurrence of chemical compounds had already been proved before by means of thermic or electric methods. BAIN¹⁾, for instance, found such new lines in the series of mixed crystals of *gold* and *copper*, in the stoichiometric proportion: 1 : 3, — a fact, which he erroneously considered to be a corroboration of TAMMANN's theory of mixed crystals. BAIN's results were also stated by JOHANSSON and LINDE²⁾, who found new lines in the proportion: 1 : 1 also. But the last mentioned investigators remark rightly, that just at the same concentrations, thermic analysis as well as electrical conductivity-measurements by KURNAKOW, ZEMCZUSNY and ZASEDATALEV³⁾ have proved the existence of two chemical compounds between the two metals. Even after a long annealing of other mixtures of this series, if having simple stoichiometric proportions, JOHANSSON and LINDE could not prove the occurrence of new diffraction-lines in the ROENTGEN-spectograms.

§ 6. The question has often been discussed, whether compounds and mixed crystals, if formed by the same components, must necessarily exclude each other, or not? Also in some cases it appeared to be very

1) E. C. BAIN, Chem. and Metall. Engin., 26, 655, (1922); 28, 65, (1923).

2) C. H. JOHANSSON and J. C. LINDE, Ann. d. Physik, [4], 78, 439, (1925).

3) N. KURNAKOW and collaborators, Journ. Inst. of Metals, 155, 305, (1916).

difficult to decide, whether a chemical compound or a mixed crystal were present in a special case.

The not numerous cases, in which these questions could be studied, are:

1⁰. The system: *calcite* (CaCO_3) — *dolomite* ($\text{CaMg}(\text{CO}_3)_2$) — *magnesite* (*giobertite*: MgCO_3). The external habit of these minerals is completely analogous¹⁾. By means of the method of specific weight-determination, RETGERS²⁾ was not able to decide, whether *dolomite* is a mixed crystal of the series or a compound. However, the complete proof, that *dolomite* is a chemical compound has been given by accurate crystallographical investigation, in connection with the ROENTGEN-analysis. *Calcite* and *magnesite* (*giobertite*) crystallise both in the *ditrighonal-scalenohedral* class. By means of the method of corrosion-figures it had already become probable, that the symmetry of *dolomite* is not higher than that of the *rhombohedral* class. Indeed, HAGA and JAEGER³⁾ by means of LAUE-radiograms, obtained by a pencil of ROENTGENrays incident perpendicularly to a crystal plate cut parallel to $\{0001\} = \{111\}$, could prove, that the three vertical symmetry-planes present in *calcite*, are *absent* in *dolomite*. The structure of *calcite* now, is such, that, if the planes of gliding symmetry in it be thought eliminated, by the presence of which the one half of the Ca-atoms can be brought to coincidence with the other half, the crystallographically-equivalent places, originally occupied by Ca-atoms, will be simultaneously differentiated in two kinds of positions, which are no longer equivalent to each other. Thus at the same time the opportunity is created, to substitute both kinds of places by Ca- and Mg-atoms alternately; and this is exactly what happens in the "compound": $\text{CaMg}(\text{CO}_3)_2$. Indeed, ROENTGEN-analysis showed, that the crystalstructure of *dolomite* is in full agreement with this view⁴⁾.

2⁰. The second known case of this kind, refers to the system: *sodium-sulphate* (Na_2SO_4) — *glaserite* ($3\text{K}_2\text{SO}_4, \text{Na}_2\text{SO}_4$) — *potassiumsulphate*: (K_2SO_4). By determination of the specific weights, RETGERS⁵⁾ could corroborate the view, that *glaserite* is a "compound". Also in this case, only a slight deviation from the stoichiometric composition occurs, which is a consequence of a limited formation of mixed crystals.

3⁰. More extensive series of mixed crystals were observed in the system: $\{\text{Al}(\text{C}_2\text{O}_4)_3\} \text{Na}_3, 4\frac{1}{2} \text{H}_2\text{O} - \{\text{Al}(\text{C}_2\text{O}_4)_3\} (\text{NH}_4)_3, 3\text{H}_2\text{O}$ in which system at the same time a double salt: $\{\text{Al}(\text{C}_2\text{O}_4)_3\}_2 \text{Na}_3(\text{NH}_4)_3, 7 \text{H}_2\text{O}$ occurs.⁶⁾

1) The composition of these minerals sometimes deviates a little from the stoichiometric proportion, as a consequence of a mixing of both components within narrow limits.

2) J. W. RETGERS, Zeits. f. phys. Chem., 6, 191, (1890).

3) H. HAGA and F. M. JAEGER, Proceed. Kon. Acad. v. Wetensch. Amsterdam, 18, 548, (1916); cf. Plate II.

4) W. H. and W. L. BRAGG, *X-Rays and Crystal-Structure*, 3d Edit., (1918), 112.

5) J. W. RETGERS, Zeits. f. phys. Chemie, 6, 191, (1890).

6) G. WYROUBOFF, Bull. Soc. fr. Minér., 23, 65, (1900); W. STORTENBEKER, Rec. 32, 226, (1913).

4°. In the systems: $SrCl_2-SrF_2$; $BaCl_2-BaF_2$; and $CaCl_2-CaF_2$ ¹⁾, besides extensive mixing series, also the occurrence of compounds between the components in the proportion: 1 : 1 was proved by means of thermic analysis.

5°. Finally by the same method, the occurrence as well of continuous mixing series from 0% till 100%, as of intermediate compounds was established in the following systems:

a. $Br-I$; compound: BrI . ²⁾

b. $Mg-Cd$; compound: $MgCd$. ³⁾

c. $Au-Cu$; compounds: $AuCu_3$ and $AuCu$. ⁴⁾

As has already been said, the last mentioned system has been also investigated after the ROENTGENographic method by JOHANSSON and LINDE. The new diffraction-lines on the ROENTGENspectrograms were not only observed in such proportions of both components, as correspond with the composition of the compounds mentioned above; but they appeared on the films also, be it with slighter intensities, if the composition differed from these of the compounds themselves, until at last they disappeared completely. This behaviour is such, as might be expected according the theory of the statistical distribution of the isomorphously substituting atoms: the absence or presence of certain symmetry-elements manifests itself *gradually*, on approaching the composition of the compounds or on withdrawing from it. The transition from the one space-group to the other occurs evidently gradually.

§ 7. If now the system: $MgSO_4 \cdot 7H_2O-ZnSO_4 \cdot 7H_2O$ be considered more in detail in the light of what was said in the foregoing, we can remark that the occurrence of a continuous mixing series from 0% till 100% does not exclude the possibility of the eventual occurrence of "compounds" between the components within this series. It was proved in a convincing way, that in the case considered, the crystal-class remains always the same throughout the whole series. Not only was it impossible to state any deviation in the crystal form ⁵⁾ of the mixed crystals of different composition, but, — what is of high interest, — DUFET ⁶⁾ was able to prove that a complete *proportionality* exists between the composition of the mixed crystals and their middle refraction-coefficient β ; the same appeared to be true with respect to their specific weights.

As the pure sulphates crystallize in the class of the lowest symmetry of the rhombic system (bisphenoidal), and as it may safely be assumed

1) W. PLATO, Zeits. f. phys. Chem., 58, 350, (1907).

2) P. C. E. MEERUM TERWOGT, Zeits. f. anorg. Chemie, 47, 203, (1905).

3) G. GRUBE, Zeits. f. anorg. Chem., 49, 72, (1905); G. URASOW, *ibid.* 73, 31, (1912); G. BRUNI and C. SANDONINI, *ibid.* 78, 277, (1912).

4) N. KURNAKOW and collaborators, Journ. Instit. of Metals, 155, 305, (1916).

5) C. VIOLA, Zeits. f. Kryst., Festband P. GROTH, 58, 583, (1923).

6) H. DUFET, Bull. Soc. fr. Minér., 12, 22 (1889).

that compounds which are formed by partial substitution of one metal by the atoms of another, will surely have no higher symmetry than the pure components, — it appears evident that an eventually occurring compound, within the series considered, of *another* symmetry than that of the components, would only crystallize either in the monoclinic or in the triclinic system. A more or less appreciable alteration of the optical behaviour would undoubtedly be than observed at the same time.

In the rhombic system a compound of the type: $Mg_2 Zn (SO_4)_3$, $21 H_2O$ is already highly improbable, because of the fact, that in the space-groups belonging to this system, only 1-, 2-, 4-, 8-, 16-, and 32-fold equivalent places can occur. A number of groups (SO_4), which is equal to 3 or to a multiple of 3, can find its place in an elementary cell of rhombic symmetry therefore only in positions, which are *not* equivalent to each other. In this particular case it can be proved, however, that a compound: $Mg_2 Zn (SO_4)_3$, $21 H_2O$ is completely impossible here.

For it has been proved already ¹⁾, that the space-group of the pure crystallized components: $MgSO_4$, $7 H_2O$ and $ZnSO_4$, $7 H_2O$ can be no other than V_4 , and that the elementary cell contains 4 molecules.

Of mixed crystals of different concentrations, rotation-spectrograms were made with successive rotation of the crystals round each of the crystallographic axes. From these it was seen, that the identity-distances of the mixed crystals did not differ appreciably from those in the same direction in the case of the pure components. Moreover, DUFET showed, that the specific weights varied practically proportionally to the concentration. In each case the elementary cell contains 4 molecules: $Mg (Zn) SO_4$, $7 H_2O$. A molecule: $Mg_2 Zn (SO_4)_3$, $21 H_2O$ is, therefore, impossible in this series, because the specific weight of it should necessarily be much smaller than that found experimentally, if the cell only contained a single molecule of this composition, and much greater than the observed value, if the cell contained two molecules of it.

§ 8. By the study of these rotation-spectrograms and of such obtained after BRAGG's method, in the case of mixed crystals, whose compositions deviated only slightly from the proportions: $MgSO_4 : ZnSO_4 = 1 : 1$ and $2 : 1$, — it was undeniably demonstrated, that the space-group in these cases always remains quite the same, i.e.: V_4 .

From the dimensions of the grating and the specific weights observed, it becomes evident, that 2 molecules of a compound: $MgZn(SO_4)_2$, $14 H_2O$ would have to be placed into the elementary cell. But this cannot be done in any way so, that the Mg - and Zn -atoms shall come pairwise in equivalent positions, because the space-group V_4 has only *fourfold* equivalent positions. From this the conclusion must, therefore, be drawn, that also this so-called "compound", indicated by HOLLMANN as such, can, by no means be a true compound. In this connection it is of interest to

¹⁾ H. G. K. WESTENBRINK, these Proceed. 29, (1926), p. 1223.

remember, that BECKENKAMP ¹⁾ in his criticism of HOLLMANN's work, drew the conclusion, that the elementary cells of the heptahydrates of the rhombic sulphates of this kind must contain of 6 molecules or a multiple of 6, making the supposition, that the compounds indicated by this author really existed. In the case of the presence of 6 molecules $MgSO_4, 7 H_2O$, then respectively 3×2 or 2×3 of these molecules must be thought to be substituted by 3 molecules of $MgZn(SO_4)_2, 14 H_2O$ and 2 molecules of $Mg_2Zn(SO_4)_3, 21 H_2O$. Reversely, in this paper, it has been proved, that the elementary cell contains 4 molecules $MgSO_4, 7 H_2O$, and from this the conclusion has been drawn, that some errors must have been introduced into HOLLMANN's experiments or into his reasonings, the occurrence of these two compounds being, from the structure point of view, impossible.

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Physical Chemistry of the University.*

¹⁾ J. BECKENKAMP, Zeits. f. Kryst., **37**, 523, (1903).

Histology. — “Concerning an Optic Phenomenon in Tendons and on double refraction Gratings.” By M. MINNAERT Sc.D. and Prof. G. C. HERINGA M.D. (Communicated by Prof. A. PANNEKOEK.)

(Communicated at the meeting of September 25, 1926).

When looking through a longitudinal section of a tendon at a source of light, placed at some distance, we shall observe a row of diffraction images resembling those viewed through a grating of which the grooves are *perpendicular* to the direction of the fibres of the tendon. Between crossed nicols the central image and the diffraction spectra of even number are extinguished; between parallel nicols the uneven numbered spectra disappear. Under the microscope the fibres exhibit a regular undulation, whose period (40μ) corresponds with the one calculated from the diffraction angle. We have described the phenomenon in an earlier number of these Proceedings¹). We now purpose to look more closely at the relation between the internal structure of the tissue and the optic diffraction phenomenon.

We start from the conception that the fibres constituting the tendon are to be considered as quasi-uniaxial crystals, their axis being directed like the fibres²). The microscopic picture goes to show that these crystal filaments are either undulating like *sine lines* in a plane, or winding in *helices*. A similar structure possesses all the properties of an optic grating, of which, however, the diffraction is caused by double refraction, for which reason we shall call it *double refraction grating*. We shall now try to ascertain what diffraction images must originate with sine lines and with helices.

1st hypothesis.

The fibres are disposed like sine lines in a plane.

Let the x -axis be in the longitudinal direction of the tendon and let the course of the fibres be expressed in the equation: $y = a \sin x$.

The angle which the fibres make with the x -axis then is: $\vartheta = \arctg \frac{dy}{dx}$
so that

$$\sin \vartheta = \frac{a \cos x}{\sqrt{1 + a^2 \cos^2 x}} \quad , \quad \cos \vartheta = \frac{1}{\sqrt{1 + a^2 \cos^2 x}} .$$

¹) Proc. Acad. Amst. 1926, **35**, 763.

²) For our observations it does not matter whether this birefringency is really caused by micellar double refraction or by bacillar double refraction (“Stäbchen doppelbrechung”), or by a combination of the two.

This preparation is now illuminated by a parallel beam of polarized light, of which the vector may have the amplitude 1, and may travel

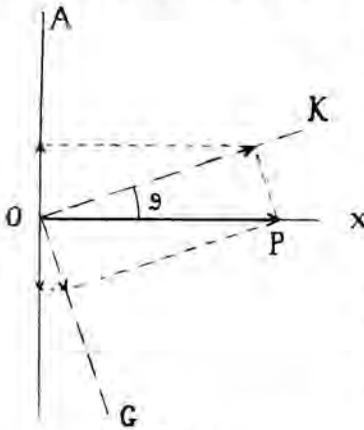


Fig. 1.

along the x -axis (Fig. 1). When the rays pass through the crystal, the vibration is decomposed into OK and OG normal to OK . With crossed nicols both waves are recombined by the analyzer along OA . Their amplitude is now $\sin \theta \cos \theta$. In case the preparation be very thin they would destroy each other by interference. With a thicker preparation a difference of phase originates, inducing a wave $c \sin \theta \cos \theta$; the interference phenomena are strongest when c reaches its maximum value 2; after this they will decrease again.

Likewise the transmitted vibration is with parallel nicols: $1 - c \cos^2 \theta$, c again reaching no higher than 2.

In these two cases, then, the transmitted amplitude is respectively

$$\frac{ac \cos x}{1 + a^2 \cos^2 x} \quad \text{en} \quad 1 - \frac{c}{1 + a^2 \cos^2 x}$$

ergo a periodic function of the place along the tendon.

It appears, then, that our *double refraction grating* practically works as a sort of *absorption grating* of which the transparency fluctuates periodically. It is remarkable, though, that in the first case the average transmitted amplitude is zero; the consequence of this is that the central image is lacking, a phenomenon that can never occur with true absorption gratings, whatever may be the shape of their transmission curve.

Now, the general grating theory tells us that the transmitted amplitude which is a function of x must be analyzed into a series after FOURIER; let N be the coefficient of a definite term, then $\frac{N^2}{4}$ is the relative intensity of the light of the corresponding diffraction image relative to the incident beam.

In this way we find the following relative intensities, assuming a to be $=0.5$ (consequently rather flat sine lines), and $c=2$, which holds approximately for our thickest preparations.

| | 1 st diffraction image | 2 nd diffraction image | 3 th diffraction image |
|-----------------|-----------------------------------|-----------------------------------|-----------------------------------|
| Crossed nicols | 0,178 | 0 | 0,0005 |
| Parallel nicols | 0 | 0,0100 | 0 |

We see thus that even this simple theory gives a very satisfactory explanation of the observed phenomena.

When rotating the analyzer the uneven-numbered diffraction maxima and the even-numbered maxima with the central image are extinguished alternately. When the analyzer is removed, the different spectra are seen simultaneously. The light intensities found seem to be of the right order of magnitude.

Finally it is also easy to see that the phenomena must be quite the same when we work without an analyzer and let the incident light vibrate not in the longitudinal direction of the tendon, but in any given direction; and our observation confirms this conclusion, so that we have also found an explanation for the image in natural light.

But the phenomena are somewhat less defined than is suggested by the theory. The central image and the spectra are never extinguished completely; the changes of intensity of the second diffraction image do not seem to be so considerable as could be expected. These differences will be readily understood if we reflect that the shape of the fibres is not that of a pure sine line.

2nd hypothesis.

The fibres are wound in helices.

When we are examining the preparation, the angle between our line of sight and the optic axis varies with the distance x along the tendon. We know that in a crystal the vibrations of the ordinary and of the extraordinary ray are parallel and perpendicular to the principal plane, and therefore they must be in each point of our preparation parallel and perpendicular to the projection of the optic axis (the direction of the fibres).

Now let us conceive a uniaxial crystal in the centre of a unit sphere with its optic axis directed along OA (fig. 2). We rotate the crystal round a mechanical axis OC , which makes with OA the angle ψ , while the observer keeps looking at the crystal from OB . A still simpler method is to conceive of the crystal as being in rest, but to let OW travel round the great circle normal to OC .

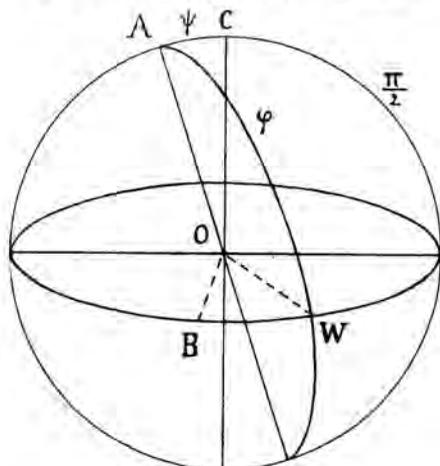


Fig. 2.

of observation and the optic axis the angle $AOW = \varphi$ is given by:

$$\cos \varphi = \cos \left(\psi + \frac{\pi}{2} \right) \sin BOW = -\sin \psi \cdot \sin x.$$

When counting all the azimuth angles from OB , where the axis OA is seen at its greatest elongations, the arc BOW is x .

Therefore, between the direction

The velocities of propagation of the ordinary and the extraordinary wave in the crystal are:

$$V_o = a$$

$$V_e = \sqrt{a^2 \cos^2 \psi + c^2 \sin^2 \psi} = c + \frac{a^2 - c^2}{2c} \sin^2 \psi \cdot \sin^2 x$$

when the double refraction is not strong. The velocities a and c are then respectively the largest and the smallest that can occur in the crystal. During the passage of a thickness D of the crystal there originates between the ordinary and the extraordinary ray a difference of phase

$$\frac{2\pi D}{\lambda} (n_e - n_o) (1 - \sin^2 \psi \cdot \sin^2 x),$$

in which λ represents the wave-length in air.

Now in order to examine the action of such a preparation as a grating, we again make the direction of the vibration of the incident light parallel to the X -axis (OC in fig. 2), and we suppose the nicols to be crossed.

We have now to compound:

$$\frac{a \cos x}{1 + a^2 \cos^2 x} \sin 2\pi \frac{t}{T};$$

and

$$- \frac{a \cos x}{1 + a^2 \cos^2 x} \sin \left[2\pi \frac{t}{T} + \frac{2\pi D}{\lambda} (n_e - n_o) (1 - \sin^2 \psi \cdot \sin^2 x) \right].$$

For further calculations we again assume that the phase-difference for $x=0$ has reached the value π . Then the resultant vibration is:

$$\frac{2a \cos x}{1 + a^2 \cos^2 x} \cos \left(\frac{\pi}{2} \sin^2 \psi \cdot \sin^2 x \right) \cdot \sin \left(2\pi \frac{t}{T} - \frac{\pi}{2} \sin^2 \psi \cdot \sin^2 x \right).$$

Likewise we find with *parallel nicols*:

$$\left(1 - \frac{2a^2 \cos^2 x}{1 + a^2 \cos^2 x} + \frac{\pi^2}{2} \sin^4 \psi \cdot a^2 \cdot \frac{\cos^2 x \cdot \sin^4 x}{1 - a^4 \cos^4 x} \right) \cdot \sin \left(2\pi \frac{t}{T} + \pi \sin^2 \psi \cdot a^2 \frac{\sin^2 x \cos^2 x}{1 - a^2 \cos^2 x} \right).$$

So *this* double refraction grating also acts as an absorption grating. If we compare the amplitude with the one for the birefringent sine lines (p. 1386) we shall see that there is an additional small term. But besides this there is also a periodic change in the phase. This double refraction grating acts as an absorption grating and at the same time as a *retardation grating*. The calculation of the FOURIER-coefficients gives us, as before, the intensity of the light of the different diffraction images. The result of the somewhat tedious calculations is

| | 1 st diffraction image | 2 ^d diffraction image |
|------------------------|-----------------------------------|----------------------------------|
| <i>Crossed nicols</i> | 0.174 | 0 |
| <i>Parallel nicols</i> | 0 | 0.0094 |

A comparison with the corresponding table on p. 2 shows that the diffraction images must be always the same in the 1st and in the 2^d hypothesis. The differences are so small as to elude measurement with our equipment. So the optic phenomenon can be explained on the hypothesis of the sine lines as well as on that of the helices; the latter hypothesis is supported only by what has been brought forward in our previous paper (p. 767).

*(From the Physical Laboratory and the Laboratory for
Embryology and Histology of the State University at Utrecht).*

Utrecht, September 1926.

Physiology. — „*On the Isolation of the anti-beri-beri Vitamin.*” By B. C. P. JANSEN and W. F. DONATH. (Communicated by Prof. C. EYKMAN.)

(Communicated at the meeting of September 25, 1926).

The first attempts to isolate the substance that, according to EYKMAN's ¹⁾ fundamental investigations, possess a prophylactic property against beriberi, were made by G. GRYNIS ²⁾, but they were ineffectual. EYKMAN ³⁾ found still a number of properties of this substance, among which the very remarkable one that it is easily soluble in water and in 80° alcohol, and readily dialysable. This is of vital importance, as it goes to show that this substance has a comparatively small molecule, and on that account cannot be classed under the proteins, nucleoproteids, and the like. In view of its considerable physiological activity it probably belongs to the group that BARGER has included under the general name of the "simpler natural bases" and was termed by GUGGENHEIM "die biogenen Amine".

In 1911 C. FUNK ⁴⁾ published his first study on this substance which he designated by the name of "vitamin". In this publication he described a body, 20—40 mgrms. of which could cure a pigeon that had developed polyneuritis after a diet of polished rice. It appeared later on that this was not the "vitamin" sought for. Furthermore the symptoms that occurred on a diet of purified proteins + fats + carbohydrates + salts warranted the assumption of several of such "vitamins", which were differentiated by the designations A and B, afterwards also C, D and E, etc. vitamins. It is still a moot point, which may now be expected to be soon set at rest, whether the B-vitamin, described by the American and English observers, which was noted for its influence on the growth of young rats, is identical with the anti-beriberi- or anti-neuritic-vitamin, that protects man against beriberi and birds from polyneuritis ⁵⁾. This identity was discredited by authors as EYKMAN ⁶⁾ and MENDEL ⁷⁾. After FUNK a good many observers have tried

1) C. EYKMAN, *Geneesk. Tijdschr. voor Ned.-Indië*, **36**, p. 214 (1896).

2) G. GRIJNS, *Geneesk. Tijdschr. voor Ned.-Indië*, **41**, p. 1 (1901).

3) C. EYKMAN, *Arch. f. Hygiene*, **58**, p. 150 (1906).

4) C. FUNK, *Journ. physiology*, **43**, p. 395 (1911).

5) See H. H. MITCHELL, *Journ. biol. Chem.*, **40**, p. 399 (1919) and A. D. EMMETT and G. O. LUROS, *Ibid.*, **43**, p. 265 (1920).

6) C. EYKMAN, C. J. VAN HOOGENHUIJZE and T. J. G. DERKS, *Journ. biol. Chem.*, **50**, p. 311 (1921).

7) CROLL and L. B. MENDEL, *Americ. Journ. Physiol.*, **74**, p. 675 (1925).

B. C. P. JANSEN and W. F. DONATH: "ISOLATION OF THE ANTI-BERIBERIVITAMIN".



Fig. 1.
Ricebird (*munia maja*) with polyneuritis; about $1\frac{1}{2}\times$ natural size.



Fig. 2.
Ricebird (*munia maja*) with polyneuritis; about natural size.

to isolate the B-vitamin, and especially the anti-beriberi-vitamin, without any positive results, so far as we know. A review of their work can be found in the modern books on vitamins, see e.g. "The Vitamins" by H. C. SHERMAN, and L. L. SMITH, New York 1922 pp. 18—48. Last year two articles appeared in the "Chemisch Weekblad" by American authors¹⁾, which contained a short survey of the work on the isolation of the antineuritic, resp. B-vitamin. This prompted us to publish in the "Chemisch Weekblad" a paper on the state at that time of our attempts to isolate the antineuritic vitamin²⁾. At the same time we could set our mind more deliberately to the investigations that had occupied us for more than nine years already in the intervals of other work. To our great satisfaction we can say that we have succeeded in isolating the anti-beriberi-vitamin in the same laboratory where EYKMAN and GRYNs have worked.

For our earlier work we refer to the above-named article in the "Chemisch Weekblad" and to the various Annual Reports of the "Geneeskundig Laboratorium" of Weltevreden. In this paper we intend to describe the method that led to our ultimate success in isolating the anti-beriberi-vitamin.

The material we started with were the fine rice-polishings (dedek) that can be procured in India at a very low price. An extract of it was fractionated in different ways. In order to determine the vitamin-content of the various fractions, we worked with small ricebirds called bondols (munia maja). Groups of ten of these birds, caged together, were fed with polished rice, to which was added a known quantity of the fraction under examination. The polished rice had previously been washed in running water for 2×24 hours, while the deficiency of mineral salts and A-vitamin was compensated by addition of 2 % of a salt-mixture, about equal to that used by OSBORNE and MENDEL, and of $\frac{1}{4}$ % codliver oil. These ricebirds are very fit reagents to testify the presence of the antineuritic vitamin, and to determine its content³⁾. With a rare exception (no more than 1 or 2 % of the total number) all the ricebirds fed on washed, polished rice develop polyneuritis in from 9 to 13 days. (See Plate Fig. 1 and 2.) If 5 % of a definite sort of dedek is added to the rice, polyneuritis reveals itself only after from 15 to 23 days.

We now determined every time the quantity of the different that had to be added to the washed, polished rice, so that the 10 rice-birds fed with this mixture contracted polyneuritis within from 15 to 23 days. As we found that a ricebird ingests on an average 2 grms of rice a day, we could compute the quantity of a definite fraction per bird and per day that had to be added to the rice, to guard the animals from polyneuritis for 15—23 days.

¹⁾ A. SEIDELL, Chem. Weekblad 22, p. 353 (1925) and P. A. LEVENE and B. VAN DER HOEVEN, Chem. Weekblad 22, p. 575 (1925).

²⁾ B. C. P. JANSEN and W. F. DONATH, Chem. Weekblad 23, p. 201 (1926).

³⁾ Vide B. C. P. JANSEN, Mededeelingen Geneeskundig Laboratorium te Weltevreden 1920, p. 40.

For some kinds of dedek a quantity of 100 mgrms per ricebird and per day sufficed; however, with most commercial sorts this quantity amounted to 140 mgrs. Finely ground rice-polishings (dedek) were now extracted with tap-water. (The Batavian tapwater contains only about 130 mgrs of dissolved salts per Litre), to which $3\frac{1}{2}$ cc of sulphuric acid had been added per Litre, to obtain the precise acidity (p_H ultimately = ± 4.5), and 200 cc of spiritus fortior to neutralize the action of micro-organisms. It seemed to us that formol was a less favourable disinfectant. The extraction was performed in a row of 4 wooden casks of ± 40 Litres each in accordance with the principle of the counter-current. The bottom and the lid of each cask was provided with an opening resp. for the affluence and the effluence of the liquid. A little above the bottom a second loose bottom had been applied for the greater part consisting of copper-gauze. On this gauze rests a layer of mineral grit, and on top of this the dedek (16 kg per cask). Every day one of the casks was refreshed, and the other three were moved up, so that the fresh dedek came in contact with the liquid that has already extracted the dedek in the other three casks, and the most extracted dedek was extracted with the fresh water-alcohol-sulphuric acid mixture. In this way almost 100 kg of dedek could be extracted every week. This extract contained ± 20 kg of solid matter and approximately all the vitamin contained in the dedek. SEIDEL showed in 1916 that the vitamin is absorbed selectively by Lloyd's reagent, a kind of fuller's earth. After a prolonged enquiry we discovered in the "acid clay" put on the market by the BÜNING concern of Cheribon, a variety of fuller's earth that possessed the same absorbing properties with respect to the vitamin as Lloyd's reagent.

The extract from 100 kg of dedek was now diluted to 300 Litres (the p_H was if necessary raised to 4.5) and mixed with 300 kg of very finely ground acid clay. After some hours' stirring of this mixture by means of an electromotor, the acid clay was allowed to subside and the supernatant fluid was pipetted off. Finally the activated acid clay was sucked off in a filter of a surface of 45×60 cm acid, washed repeatedly with small amounts of water acidulated to $p_H = 4-5$, the residue was removed from the filter and treated with so much baryta that a paper, soaked with tropaeolin 0, is turned reddish brown, when immersed in the fluid; the p_H then amounts to ± 12 or 13. This baryta-extract, which contains the vitamin, was again sucked off on the filter and then washed, first with baryta, subsequently with distilled water; to the filtrate sulphuric acid was added to an amount that made the acid red on congopaper. That treatment with baryta should pass off as rapidly as possible, because the vitamin is being deteriorated in alkaine solution. However, the solution, acidulated with sulphuric acid, may safely be left to stand for one or two days, after which the barium-sulphate will have subsided. Most of the supernatant fluid was then decanted off, and the remaining fluid was filtered off on a hardened paper filter. This filtrate still contains ± 100 grms of solid matter, and 80 % of the original vitamin present in the dedek. Now the acid clay-extract

was again fractionated after KOSSEL and KUTSCHER¹⁾ with silver sulphate and baryta; their method was, however, refined by determining the p_H of the liquid. To this end the acid-clay-extract obtained from 100 kg of dedek was evaporated in a large porcelain dish placed in a heated water-bath. A very strong current of air was blown over the liquid by means of an exhauster, so that at a temperature of from 30° to 40° C. about 1½ to 2 Litres of water was evaporated per hour. The liquid was evaporated to a volume of ± 4 Litres. This was transmitted to a thick walled glass beaker of Pyrexglass of ± 10 Litres, and was subsequently treated with silversulphate or with silvernitrate. In our first experiments we used silversulphate partly in solution, partly as a solid. However, as it took some days of incessant stirring with an electromotor to dissolve a sufficient quantity of silversulphate, we used silvernitrate later on. Most often a quantity of ± 50 grms of silvernitrate was required. As known, the sufficiency of the addition will be attained when a sample of the liquid gives a brown precipitate directly on the addition of an excess of baryta. The addition of the silvernitrate to the acidulated acid-clay-extract yields a thick precipitate, not containing any vitamin. By violent stirring with an electromotor baryta was now allowed to flow in slowly from a burette or pipette. Then the p_H of the liquid (at the beginning only 2 or 3) rises slowly. The p_H was mostly determined with the aid of the capillator procured by the British Drug House, which gave us great satisfaction.

We are now able to supply so much baryta that the p_H rises to ± 4.5 . In the resulting precipitate there is no appreciable quantity of vitamin. Now the liquid was filtered off and an amount of baryta was allowed to flow into it until the p_H became ± 6.5 . In this second silverfraction there is rather more than 50 % of the vitamin contained in the acid-clay-extract. After filtering this deposit baryta is again added to the filtrate, until the $p_H = \pm 8.0$. In this 3rd silverfraction there is still a considerable amount of vitamin, but it is mixed with a relatively much larger quantity of inactive substances than in the 2nd silverfraction. Therefore, this 3rd silverfraction was not worked up directly, but the 3rd fraction of different preparations together, was again fractionated with silvernitrate and baryta after analysis with hydrochloric acid. In the filtrate of the 3rd silverfraction there is still a large amount of solid matter but practically no vitamin.

The 2nd silverfraction was decomposed with an excess of hydrochlorid acid, and filtered off from the consequent silverchlorid. This filtrate still included ± 20 grms of solid matter. The quantity required daily for a rice-bird, amounts to ± 0.080 mgrm = 80 γ . In a testing sample of the filtrate the excess of hydrochloric acid was determined by titration with alkali and an amount of sulphuric acid was then still added to raise the total acid-concentration, computed on sulphuric acid, to 5 %. Now a solution of phosphotungstic acid in 5 % sulphuric acid was added until a new addition

¹⁾ A. KOSSEL and P. KUTSCHER, *Zeitschr. f. physiol. Chem.* 31, p. 165 (1900).

no longer yielded a precipitate. The phosphotungstic acid was first purified after WINTERSTEIN¹⁾, by shaking up the aqueous solution with sulphuric acid and ether, the phosphotungstic acid passing into the latter fluid. This treatment with phosphotungstic acid may also precede the fractionation with silver and baryta, with the same favourable result. Then, however, much more phosphotungstic acid is required, and our store of this acid was limited, whereas silver nitrate could be procured in sufficient quantity. When the fractionation with silver + baryta is carried out before the treatment with phosphotungstic acid, no more than ± 40 grms of the latter is wanted per 100 kg of dedek. The phosphotungstic acid-precipitate was left to subside during 34 hours; it was subsequently filtered off on a suction-filter, and washed with 5 % sulphuric acid. Then the residue was sucked off completely and after this, for further purification, it was dissolved in acetone + water. This acetone-solution was, if need be, filtered and then poured into a large excess of 5 % sulphuric acid. Most of the phosphotungstic acid will then precipitate. A portion of it, however, remains in solution, but this no longer contains any vitamin worth mentioning.

After filtration this new precipitate of the phosphotungstic acid was decomposed with baryta. Once we tried to decompose it with ether and hydrochlorid acid after WINTERSTEIN, but we were not more successful, while the procedure was much more time-consuming.

When we decomposed with baryta, the precipitate was first dissolved again in acetone + water, concentrated baryta being added until, also after prolonged stirring, phenolphthalein-paper turned deep red. Rapid filtration of the barium-phosphotungstic acid precipitate followed, and a quantity of sulphuric acid was added to the filtrate, enough to throw down all the excess of baryta. A fortuitous surplus of the added sulphuric acid was removed with barium chlorid. Lastly the liquid was acidulated with hydrochloric acid. The solution still contains ± 5 grms of dry residue, while per ricebird and per day $\pm 30 \gamma$ is required to guard animals from polyneuritis for from 15 to 23 days.

It appears then that with this treatment with phosphotungstic acid no considerable purification is attained, but neither has a large loss of vitamin been caused, while substances have been removed, that are not eliminated in the succeeding procedure. This hydrochlorid solution was now evaporated to dryness, first on the waterbath, subsequently in vacuo over quick lime.

The sticky residue was now taken up in absolute alcohol, by which it was almost completely dissolved. A small quantity of insoluble material was filtered off and to the filtrate was added an alcoholic solution of platinic-chlorid. This yields a thick deposit of an orange tint. Virtually this deposit contains all the vitamin and only about $\frac{1}{4}$ of the solid matter present in the original alcoholic solution.

¹⁾ E. WINTERSTEIN, *Chemiker Zeitung*, 1898, p. 539.

After 24 hours the platinum precipitate was filtered off, washed with absolute alcohol, taken up in water and decomposed with hydrogen-sulphid. Also after an excess of hydrogen sulphid is passed through the liquid one has to wait at the very least another 24 hours, before all the platinum-sulphid is thrown down. The filtrate of platinum-sulphid was evaporated first on the waterbath, then in vacuo over lime, a considerable portion of the dry residue resulting from it, is crystalline. Of the original quantity of solid matter in 100 kg. of dedek only about 1.4 gramme is left. The vitamin-content of the dedek used, was such as to necessitate the addition of $\pm 7\%$ dedek to the washed polished rice to protect the rice-birds against polyneuritis for from 15 to 30 days. It follows then that for 2 grms of rice i.e. the quantity ingested daily by a ricebird, 140 mgrms is required. So in 100 kg of dedek there is enough for about 700.000 ricebirds per day.

Of the platinum-precipitate analyzed with hydrogen sulphid $\pm 8\gamma$ is required for a rice-bird. So in the 1.4 gr. there is enough for ± 175.000 ricebirds. It will be seen then that after all the above operations as much as $\pm \frac{1}{4}$ is left of the vitamin contained in the dedek.

Although it took us long before we got so far, the knowledge obtained facilitates further experimentation. The substance thus isolated, is comparatively pure, as we shall see lower down that of the pure vitamin $\pm 3\gamma$ per ricebird is wanted daily to guard these animals from polyneuritis for from 15 to 23 days. As about 8γ of the decomposed platinum-precipitate is required per bird daily, this precipitate appears to consist for $\pm \frac{1}{4}$ of the vitamin we are trying to isolate. However, further purification takes up a good deal of time and great loss results from it, as besides vitamin at least two more substances are to be found in this decomposed platinum-precipitate. This is evidenced by what follows: When we evaporate the solution that is obtained by treating the platinum-precipitate with hydrogen-sulphid after filtering off the PtS_2 , subsequently evaporate it to dryness over quick lime, and finally dissolve the residue in absolute alcohol, part of it remains undissolved. The animal experiment shows that it is not the vitamin we are in quest of.

Treatment of the alcoholic solution with acetone produces a milky cloudiness from which after one or two days a partly viscous, partly crystalline deposit is set free against the bottom and the wall of the cask. Now this deposit is much richer in vitamin than the solution which, however, contains still a great part of the solid matter. So in the platinum precipitate we find anyhow: 1^o, a substance insoluble in absolute alcohol; 2^o, a substance that dissolves in absolute alcohol but is insoluble in acetone (this appeared to be the vitamin) and 3^o, a substance soluble in alcohol as well as in acetone. When we dissolve the precipitate with acetone in alcohol, part of it again remains undissolved: this then, is the substance insoluble in alcohol, which at the first treatment with alcohol remained in solution, in consequence of the presence of a large quantity of substances soluble in alcohol. We are now in a position to crystallize fractionally by dissolving

as much of the decomposed platinum-salt as passes into solution in absolute alcohol. To this a little acetone is added, some of the precipitate adhering to the wall of the flask is decanted off and once more acetone is added to the liquid. Another precipitate appears and again some of it is decanted off, etc. Each of the precipitates is dissolved in absolute alcohol, which has been filtered off from the undissolved portion, and precipitated every time with renewed quantities of acetone, etc. By repeating this process several times we get at length the pure vitamin in the form of hydrochloric acid salt. However, after addition of the acetone it generally takes 24 hours, sometimes many days, before the initial milky cloudiness has completely subsided, so that the whole procedure takes months and months. When the vitamin is approximately pure, the operation progresses better, because during the treatment of the alcoholic solution with acetone a crystalline precipitate ensues directly, which does not cling to the wall.

We have endeavoured to correct this lengthy and sparingly productive process, but to little purpose so far. Picrolonic-acid gives a yellow-coloured precipitate, consisting especially of picrolonates of the vitamin, and of the substance insoluble in alcohol, the latter being most difficult of solution in water. Through recrystallization from diluted alcohol here also a separation can be attained, but not in a much easier way than through the alcohol-acetone fractionation, described above. With this picrolonic-acid fractionation the fractions can be recognized by their melting-point. The optimum-temperature of the picrolonate of the constituent insoluble in alcohol is 340° C.; that of the picrolonate of the vitamin is 165° (uncorrected).

The several picrolonic-acid-fractions (mostly no more than some tens of milligrammes) were filtered off to advantage in Gooch's mugs with porous bottom of sintered glass, from which the precipitate can be scraped off almost quantitatively with a platinum spatula without causing any pollutions through paper fibres.

By treatment with ether or ethylacetate and hydrochloric acid it is easy to separate the hydrochlorid from the picrolonate. Via the picrolonic-acid as well as with fractionation with alcohol and acetone we have ultimately obtained from 300 kg of dedek \pm 100 mgrm of a crystalline substance, which also after recrystallization had a melting point of 250° C., as determined with the Anschütz thermometer: the so-called corrected melting-point. Already at rather more than 200° C. the substance begins to turn brown, but then melts rather sharply at 250° C. A portion of this substance was transferred by means of gold-chlorid to a beautifully crystallizing double-salt.

Now for the evidence that these crystals are, indeed, the antiberiberi vitamin hydrochlorid.

Of some hundreds of groups of 10 ricebirds each, fed on a mixture of washed, polished rice and inactive fractions, there was not one group of which all the birds or even most of them kept healthy for more than 12 days, while

only very few individuals could hold out longer than 14 days. We now conducted the following experiments with the vitamin-hydrochlorid, besides a number of preliminary tests (all of which gave the same result as the experiments proper): Every time ten ricebirds were fed during ± 3 weeks with washed polished rice, to which different quantities of the vitamin-hydrochlorid had been added.

A. with vitamin purified through crystallization from absolute alcohol and acetone:

Cage I: 1 part vitamin-hydrochlorid to 1 million parts washed, polished rice; since the average daily diet of a ricebird is two grms of rice, this comes to 2 γ per bird and per day. One of the ricebirds contracts polyneuritis after 18 days, two after 23 days, the others are still in good health after 23 days.

Cage II: 1½ parts vitamin-hydrochlorid to 1 million parts washed, polished rice, i.e. 3 γ per bird and per day. After 3 weeks all the animals are healthy.

B. with vitamin-hydrochlorid purified via picrolonic acid:

Cage III: 1 part vitamin-hydrochlorid to 1 million parts washed polished rice; i.e. 2 γ per bird and per day: three ricebirds polyneuritis after resp. 13, 18 and 20 days. The others still healthy after 3 weeks.

Cage IV: 1½ parts vitamin-hydrochlorid to one million parts washed, polished rice; i.e. 3 γ per ricebird and per day. After 3 weeks all are healthy.

C. with vitamin-hydrochlorid obtained through decomposition of the double salt with hydrogen-sulphid:

Cage V: 1 part vitamin-hydrochlorid to 1 million parts washed polished rice; i.e. 2 γ per ricebird and per day. This experiment has been in progress for 14 days now, but all the animals still keep in good health.

Cage VI: 2 parts vitamin-hydrochlorid to 1 million parts washed, polished rice; i.e. 4 γ per ricebird and per day. After 3 weeks all the animals are quite well.

The foregoing justifies us in assuming 2 γ per ricebird and per day to be the critical limit; of the 30 ricebirds, fed with it, only one develops polyneuritis inside of 15 days; the majority keep in good health even for more than 3 weeks. Of the ricebirds that received 3 γ or 4 γ hydrochlorid daily not one developed polyneuritis.

We have repeated these experiments with pigeons:

A. with hydrochlorid purified via the picrolonate.

Pigeon 1 and 2: 1 part vitamin-hydrochlorid to 400.000 parts washed, polished rice. For a pigeon that ingests ± 12 grms a day, this comes to 30 γ a day. After 5 weeks the foodstuff was finished; the pigeons looked quite well, but their bodyweight was reduced resp. from 267 to 222 and from 270 to 235 grms.

Pigeon 3 and 4: 1 part vitamin-hydrochlorid to 200.000 parts washed, polished rice; i.e. 60 γ per pigeon per day. After 6 weeks the food was

finished; the pigeons still looked quite well; the weight of the one pigeon had remained quite the same, that of the other had been slightly lowered, viz. from 270 to 213 gms.

In these pigeon-experiments the ratio of vitamin to rice was taken somewhat liberally, as we do not know beforehand whether ricebirds and pigeons require a relatively equal quantity of vitamin per day. However, the results being so favourable, we afterwards took the same ratio for both birds in the preparation purified by absolute alcohol and acetone.

Pigeon 5 and 6: 1 part vitamin-hydrochlorid purified with alcohol and acetone, to 1 million parts washed, polished rice, i.e. $\pm 12 \gamma$ per pigeon and per day. The test is proceeding for 4 weeks only, but the animals look well, only the weight is lessened a little, viz. from 271 to 245 grms, and from 292 to 235 grms.

Pigeon 7 and 8: 2 parts vitamin-hydrochlorid purified with alcohol and acetone to 1 million parts washed polished rice, i.e. $\pm 24 \gamma$ per pigeon and per day. This test is also in progress only 4 weeks, but the animals look thriving while their weight is hardly lessened, viz. from 322 to 314 grms and from 302 to 291 grms. Since $\pm 1 \%$ proteins is lost in the twice 24 hours' washing of the polished rice, so that the protein-content of the washed rice amounted to only 5.5 %, it is not impossible that a deficiency in protein was responsible for the decrease in the weight of some animals. For this reason we added in the last two experiments $3\frac{1}{2} \%$ thoroughly minced meat, extracted repeatedly with boiling water, which meat, as appeared from advisedly arranged experiments, contained no appreciable quantity of antineuritic vitamin.

Now if we consider that the pigeons fed only on washed, polished rice develop polyneuritis most often within 2 or 3 weeks, mostly with a marked loss in bodyweight if not forcibly fed, the great influence of the slight addition of the vitamin-hydrochlorid becomes very evident. To make assurance double sure, we also administered to two pigeons the same washed polished rice, to which, as with the last pigeons, had been added 2 % salts + $2\frac{1}{2} \%$ extracted minced meat, but no vitamin chlorid. The one developed polyneuritis after 24 days, the other after 25 days, while their weight decreased resp. from 312 to 207 gms and from 289 also to 207 gms.

The pigeons to which an admixture of vitamin-hydrochlorid was administered, also behaved differently from those that had washed polished-rice alone. The latter practically cease eating spontaneously already after a few days, while it looks as if they are rummaging for the few grains to which a trace of silverlayer still adheres, and throw away the other grains, thus making a mess of their diet; the former, including the pigeons on only 1 part vitamin-hydrochlorid to 1 million parts washed polished-rice, finish their allowance with great relish.

From these experiments, therefore, we may safely conclude that the hydrochlorid, detected by us, with a melting point at 250°C. , is in a high degree instrumental in warding off polyneuritis. But the question may

be asked whether this salt is indeed the vitamin-hydrochlorid, or whether perhaps the salt is contaminated with a small quantity of a still more active component that may be the vitamin that we endeavour to isolate. We believe that the latter supposition is altogether erroneous, if we consider that the salt has been purified through recrystallization, while the mother-lye appeared to be much less active. If the supposition were right, we should have to assume that the crystals had absorbed by selection the active component from the liquid, and had detained it after recrystallization, and transference to goldsalt and back again to hydrochlorid. To be sure this idea must really be precluded.

So a proportion of 1 or 2 parts vitamin to 1,000,000 parts rice would accordingly create in a man, who consumes \pm 500 gms of rice a day, a want of $\frac{1}{2}$ or 1 mgrm of vitamin. This is of the same order of magnitude as what we know of the consumption of other substances with great physiological activity, such as thyroxin, adrenalin, and the like.

Until it is quite purified the vitamin-hydrochlorid crystallizes in bundles of needles. The pure salt consists of rosettes of bundled rodlets. The hydrochlorid in a pure state is not hygroscopic; but it is readily soluble in very little water; it is also soluble in ethyl-, and methyl-alcohol, the solution in ethyl-alcohol gives a precipitate with an admixture of amyl-alcohol, acetone, ether, chloroform, benzol, petroleum-ether or ethyl-acetate.

A 2 % -solution of the hydrochlorid in water produces the following reactions: Sublimate yields a slight precipitate that becomes stronger by adding sodium-acetate; from a solution of mercuric-sulphate in dilute sulphuric acid a thick precipitate ensues; with iodine potassium-iodid a fine black precipitate originates; with picric acid a cloudiness arises; styphnic-acid yields a precipitate; Dragendorff's reagent, a solution of bismuth-iodid in potassium-iodid solution, gives a thick, reddish precipitate; with zinc-chlorid, cadmium-chlorid, lead-acetate, copper-acetate, potassium-chromate, potassium-salicylate, and with perchloric acid, no precipitate comes forward. As is evident from the preparation, phosphotungstic acid, picrolonic acid and gold chlorid give a precipitate, the last two consisting of needle-shaped crystals. The aqueous solution does not give a precipitate with platinum-chlorid, the solution in absolute alcohol with an alcoholic solution of platinum-chlorid, does. By treating with sulphanilic acid and nitric-acid, and subsequently with soda, we observe an intensely red colour (diazo-reaction of Pauly).

In the hydrochlorid no other elements appeared to occur than C, N, H, O and Cl. After transference of the hydrochlorid to the nitrate we could demonstrate that no halogens occur in the vitamin itself.

Of the hydrochlorid and of the gold-double salt we made some elementary-analyses by means of PREGL—MÜLLER—WILLENBERG's micro-apparatus with quartz-tube and ground-in absorption tubes. We publish these results only provisionally. Within a few months we hope to be able to repeat these analyses with larger quantities of material.

We found :

Of the gold-salt :

13.775 mgrms give 2.864 mgrms H₂O, 7.447 mgrms CO₂ and 5.852 mgrms Au;
found: 2.31 % H, 14.74 % C, and 42.48 % Au.

Duplicate: 14.876 mgrms give 3.505 mgrms H₂O, 8.072 mgrms CO₂ and
6.278 mgrms Au;

found: 2.07 % H, 14.80 % C, and 42.20 % Au.

10.589 mgrms gives with micro-Kjeldahl 0.461 cc³ $\frac{n}{10}$ NH₃;

found: 6.1 % N.

8.991 mgrms give 0.50 cc (measured over water) N of 760 mm and
32° C.; found 6.08 % N 7.293 mgrms give 8.901 mgrms AgCl.

found: 30.21 % Cl.

Computed for C₆H₁₀ON₂.HCl.AuCl₃:

2.36 % H, 15.45 % C, 42.48 % Au, 6.02 % N and 30.54 % Cl.

Of the hydrochlorid :

7.198 mgrms give 4.515 mgrms H₂O; carbon was lost;

found: 6.97 % H.

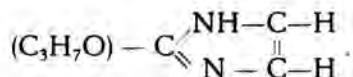
3.289 mgrms give 2.894 mgrms AgCl:

found: 21.77 % Cl.

Computed for C₆H₁₀ON₂.HCl:

6.77 % H, 21.84 % Cl.

We see, then, that analyses correspond (not very beautifully though) to the formula C₆H₁₀ON₂.HCl for the hydrochlorid. Now, if we consider that fractionation with silver and baryta causes the body to precipitate in the histidingroup, whereas it gives a very intense red-coloration with the diazo-reaction of PAULY¹⁾, the suggestion is obvious that possibly there may be an imidazol-nucleus in the body, that also occurs in other substances essential to life, such as histidin, histamin and carnosin. The formula may then be imagined to be :



But this has to be substantiated by later inquiry.

In conclusion we would take this opportunity of tendering our thanks to our analysts, especially to Raden SOEDARSONO, for their valuable assistance in the experimental work.

Weltevreden (Batavia), Medical Laboratory.

August 1926.

¹⁾ H. PAULY. Zeitschr. f. Physiol. Chemie. **42**, p. 508 (1904); **44**, p. 159 (1905); **44**, p. 284 (1915).

Mathematics. — "A property of 2-dimensional Elements" by M. H. A. NEWMAN. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of October 30, 1926).

In a paper recently published in these Proceedings¹⁾ it was stated without proof that if $n > 2$ not every n -element can be transformed into an n -simplex by moves of type 2 alone, i.e. in certain n -elements there is no unit (n -simplex) whose removal leaves an n -element. It is the purpose of this note to prove this assertion by the production of an example (§ 2), and also to shew (§ 1) that if E_2 and S_2 are respectively a 2-element and a 2-simplex, then $E_2 \xrightarrow{2} S_2$.

A proof of the 2-dimensional JORDAN's Theorem, (in the combinatory sense), is included.

§ 1.

LEMMA. If E_2 is a 2-element and $a\beta$ an internal edge²⁾ whose extremities belong to $\overline{E_2}$ and divide it into the parts E_1' and E_1'' , then E_2 is the sum of two 2-elements with no common units, whose boundaries are $a\beta + E_1'$ and $a\beta + E_1''$.

THEOREM 1. If the 2-element E_2 is neither a simplex nor a complete star it has two detachable³⁾ units with no common internal vertex.

THEOREM 2. If $O_{12}(E_2) = q$,⁴⁾ and S_2 is a detachable unit of E_2 , then $O_{12}(E_2 - S_2) = q - 1$.

It is convenient to prove the lemma and the two theorems together. Let them be assumed true of elements of order less than q , (1, 2); and suppose that $O_{12}(E_2) = q$ in all three cases.

Proof of the Lemma. From the inductive hypothesis about Theorem 2 it follows that $E_2 \xrightarrow{2} S_2$. For if E_2 were derived from E_2^* , of order $q - 1$, by the removal of a unit, $O_{12}(E_2)$ would be $q - 2$, not q . Suppose then that T_2 is such that $O_{12}(E_2 - T_2) = q - 1$.

If T_2 contains $a\beta$, (whose extremities lie in $\overline{E_2}$), the contact of T_2 with the rest of E_2 can only be regular if its remaining vertex is free; and then the required 2-elements are T_2 and $E_2 - T_2$.

If T_2 does not contain $a\beta$, the part of $\overline{T_2}$ in $\overline{E_2} - \overline{T_2}$ belongs to one

¹⁾ M. H. A. NEWMAN, Proc. Roy. Ac. Amsterdam, 29 (1926) pp. 611 and 627, two parts here called FI and FII. The definitions of regular contact, moves of types 1, 2 and 3, etc., are given in FI.

²⁾ An edge is a 1-component.

³⁾ A unit, S_n , of Γ_n , is detachable if it has regular contact with $\Gamma_n - S_n$.

⁴⁾ " $O_{rs}(E) = q$ ", or " E is of order q , (r, s)", means that q is the smallest number of moves of types r and s by which E can be transformed into a simplex.

of the parts E_1' or E_1'' . For if it had an edge in each they would necessarily be the pair of edges of $\overline{E_2 - T_2}$ at α , or the pair at β ; and since the contact of T_2 is regular this is contrary to the supposition that α and β are in the boundary of E_2 . Suppose then that the part V, \overline{U} , of $\overline{T_2}$ belongs to E_1' . $\alpha\beta$ is interior to $E_2 - T_2$ and has its extremities on the boundary, and therefore divides it into two elements, E_2'' and E_2''' , with boundaries $E_1' + \alpha\beta$ and $(E_1' - V, \overline{U} + U, \overline{V}) + \alpha\beta$ respectively. Hence E_2 is divided by $\alpha\beta$ into the elements E_2'' and $E_2''' + T_2$, with boundaries $E_1' + \alpha\beta$ and $E_1' + \alpha\beta$.

Proof of Theorem 1. Suppose, as before, that T_2 is a unit of E_2 such that $O_{12}(E_2 - T_2) = q - 1$, (and therefore $O_2(E_2 - T_2) = q - 1$). Let T_2 be $\alpha\beta\gamma$.

If $\alpha\beta$ is internal and γ free, $E_2 - \alpha\beta\gamma$, being of order $q - 1$, contains two detachable units, of which at least one does not contain $\alpha\beta$. This simplex and $\alpha\beta\gamma$ are detachable units of E_2 ; and $\alpha\beta\gamma$ has no internal vertex.

If γ is internal and $\alpha\beta$ free, let $\xi\eta$ be a boundary edge of E_2 such that the unit $\xi\eta\zeta$ containing it does not contain γ . (There is such an edge, since E_2 is not a complete star). If ζ is interior to E_2 , $\xi\eta\zeta$ is a second detachable unit, not containing γ . If ζ is in $\overline{E_2}$, $\xi\zeta$ divides E_2 into two 2-elements, both of order less than q (since $E_2 \xrightarrow{2} S_2$), of which one, say E_2' , does not contain γ . E_2' has two detachable units, of which one at least does not contain $\xi\zeta$, and is therefore detachable from E_2 . This is a second detachable unit, not containing the internal vertex, γ , of $\alpha\beta\gamma$.

Proof of Theorem 2. If E_2 is a complete star the theorem is obviously true. We suppose then that it is not a complete star.

Let T_2 be as before, and let $\rho\sigma\tau$ be the given detachable unit S_2 .

If $\rho\sigma\tau$ has a free vertex, τ , the connected array $E_2 - T_2$, which contains $\rho\sigma\tau$, must have $\rho\sigma$ as an internal edge, save in the trivial case where $E_2 - T_2$ is $\rho\sigma\tau$. Hence $E_2 - T_2 \xrightarrow{2} E_2 - T_2 - \rho\sigma\tau$, which is therefore of order $q - 2$; and $(E_2 - T_2 - \rho\sigma\tau) + T_2$, which is $E_2 - \rho\sigma\tau$, is of order $q - 1$.

If τ is internal and $\rho\sigma$ free, and T_2 does not contain τ , a similar argument shows that $O_2(E_2 - \rho\sigma\tau) = q - 1$. If τ belongs to T_2 , let U_2 be a detachable unit of E_2 not containing τ . U_2 is then detachable from $E_2 - T_2$; $(E_2 - T_2) - U_2$ is of order $q - 2$, and therefore $O_2(E_2 - U_2) = q - 1$. The unit $\rho\sigma\tau$ is detachable from $E_2 - U_2$, and therefore

$$O_2(E_2 - U_2 - \rho\sigma\tau) = q - 2, \text{ whence } O_2(E_2 - \rho\sigma\tau) = q - 1.$$

JORDAN'S Theorem in two dimensions, of which the lemma is a special case, is now easily proved in its unrestricted form.

Theorem 3a. *If e_1 is a 1-element consisting of internal edges of E_2 but having its extremities, α and β , in $\overline{E_2}$, then E_2 is the sum of two 2-elements with no common unit, whose boundaries are $e_1 + E_1'$ and $e_1 + E_1''$, where E_1' and E_1'' are the parts into which $\overline{E_2}$ is divided by α and β .*

Suppose the theorem true of 2-elements of order less than q , and let $O_2(E_2) = q$. Let T_2 be such that $O_2(E_2 - T_2) = q - 1$. Clearly the part, (say $U \cdot \bar{V}$), of T_2 in \bar{E}_2 belongs either to E_1' or to E_1'' — say to E_1' .

If T_2 contains no internal vertex of e_1 , $E_2 - T_2$ is divided by e_1 into two 2-elements, with boundaries $E_1' - U \cdot \bar{V} + V \cdot \bar{U} + e_1$ and $E_1'' + e_1$; and on adding T_2 to the first, the division of E_2 by e_1 is established.

T_2 cannot contain two edges of E_1 save in the trivial case where $V \cdot \bar{U}$ is e_1 . If it contains the edge $a\xi$ and no other, $e_1 - a\xi$ is interior to $E_2 - T_2$, and the proof is completed as before.

If T_2 contains an interior vertex, ξ , but no edge, of e_1 , let e_1^1 and e_1^2 be the parts, (containing a and β respectively), into which ξ divides e_1 , and ε_1^1 and ε_1^2 the corresponding parts into which it divides $E_1' - U \cdot \bar{V} + V \cdot \bar{U}$.

The 1-element e_1^1 is interior to $E_2 - T_2$, save for its extremities, and so divides it into, say, E_2^1 and E_2^3 with boundaries $e_1^1 + \varepsilon_1^1$ and $e_1^1 + \varepsilon_1^2 + E_1''$. e_1^2 is interior to E_2^3 , save for its extremities, and divides it into, say, E_2^2 and E_2'' , with boundaries $e_1^2 + \varepsilon_1^2$ and $e_1^2 + e_1^1 + E_1'$. T_2 has regular contact with E_2^1 , and the 2-elements E_2^2 and $E_2^1 + T_2$ have a single boundary edge in common. Their sum, E_2' , is a 2-element, (F II Theorem 8a), whose boundary is $E_1' + e_1$. The two elements E_2' and E_2'' are the parts into which E_2 is divided by e_1 .

Theorem 3b. *If Σ_1 is a circle contained in a 2-sphere, Σ_2 , Σ_2 is the sum of two 2-elements, without common units, of both of which Σ_1 is the boundary.*

Let a unit, S_2 , of Σ_2 containing an edge, $a\beta$, of Σ_1 be replaced by $\xi \cdot \bar{S}_2$, where ξ is a new vertex, (move of type 3). Theorem 3b is easily proved by applying Theorem 3a to $\Sigma_2 - \xi a\beta$, then replacing $\xi a\beta$, and finally reversing the move of type 3.

§ 2.

Consider the 3-array, I_3 , with the vertices

$$a, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \delta', \varepsilon', \zeta', \eta';$$

and the units

$$\begin{array}{lll} a \beta \varepsilon' \eta' & \beta \gamma \zeta' \eta' & a \beta \varepsilon \eta \\ a \beta \zeta' \eta' & a \beta \zeta \zeta' & a \beta \zeta \eta \\ a \delta' \varepsilon' \eta' & \beta \gamma \zeta \zeta' & a \delta \varepsilon \eta \\ a \delta' \zeta' \eta' & \beta \gamma \zeta \eta & a \delta \zeta \eta^1) \end{array}$$

(The significance of the dots will be explained later).

¹⁾ See fig. 1.

Let Δ_3 be the solid ring formed by placing Γ_3 and three arrays congruent to it end to end in cyclical order, the "ends" being $\delta\zeta\eta + \delta\epsilon\eta$

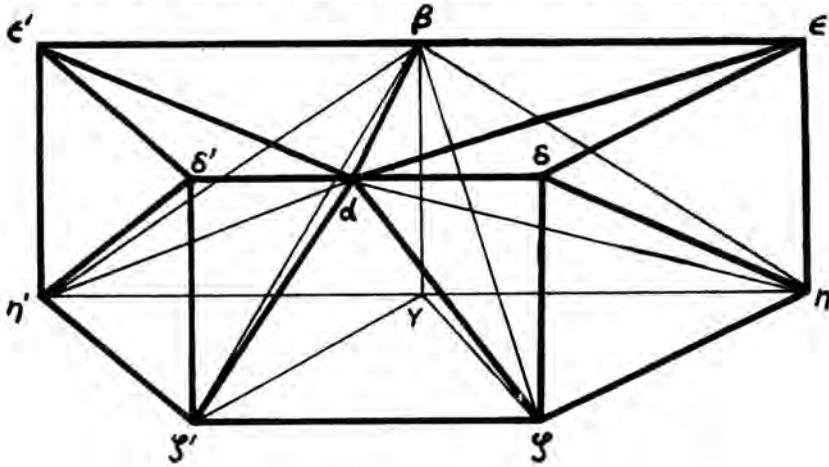


Fig.1.- Γ_3 .

and $\delta'\zeta'\eta' + \delta'\epsilon'\eta'$ in Γ_3 , and their correlates in the other arrays. Fig. 2 will make the arrangement clear.

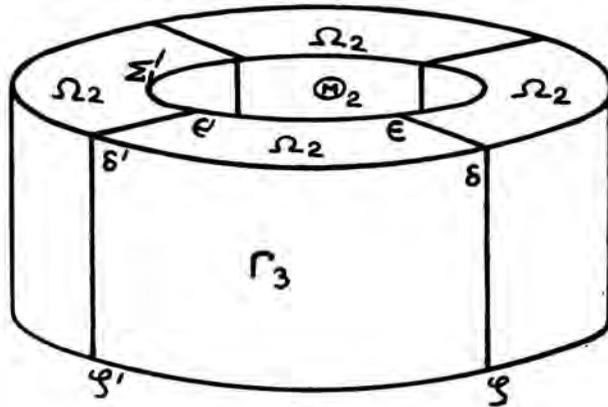


Fig 2 - Δ_3

Part of the boundary of Δ_3 is the "cylinder" Θ_2 , formed by adding together the four "rectangles" of the type $\beta\epsilon\eta + \beta\gamma\eta + \beta\epsilon'\eta' + \beta\gamma\eta'$; and the boundary of Θ_2 is made up of two circles, viz. Σ_1 , (the sum of the four 1-elements of the type $\gamma\eta + \gamma\eta'$), and Σ_1' , (the sum of the 1-elements of the type $\beta\epsilon + \beta\epsilon'$).

Let σ and τ be two new vertices, and consider the array, E_3 , defined to be $\Delta_3 + \sigma(\Theta_2 + \tau\Sigma_1)$.

a. E_3 is a 3-element.

Let Ω_2 be the sum of the four 2-elements of type $a\beta\epsilon + a\delta\epsilon + a\beta\epsilon' + a\delta'\epsilon'$, and let ω be a new vertex. Then

$$E_3 + \omega (\Omega_2 + \sigma\Sigma'_1) \xrightarrow{2} \omega (\Omega_2 + \sigma\Sigma'_1).$$

For the units of $\sigma\tau\Sigma_1$ can be removed according to the order of their edges in Σ_1 ; then the units of $\sigma\Theta_2$ in the order $\sigma\beta\gamma\eta$, $\sigma\beta\epsilon\eta$, $\sigma\beta\gamma\eta'$, $\sigma\beta\epsilon'\eta'$, etc.; and then the units of Γ_3 in the following order:

- | | |
|---|---|
| 1. $\beta\gamma\zeta\eta$ ($\gamma\eta$) | 7. $a\delta\epsilon\eta$ ($a\eta$) |
| 2. $\beta\gamma\zeta\zeta'$ ($\gamma\zeta$) | 8. $\beta\gamma\zeta'\eta'$ (γ) |
| 3. $a\beta\zeta\zeta'$ ($\zeta\zeta'$) | 9. $a\beta\zeta'\eta'$ ($\beta\zeta'$) |
| 4. $a\beta\zeta\eta$ ($\beta\zeta$) | 10. $a\delta\zeta'\eta'$ ($a\zeta'$) |
| 5. $a\delta\zeta\eta$ ($a\zeta$) | 11. $a\beta\epsilon'\eta'$ ($\beta\eta'$) |
| 6. $a\beta\epsilon\eta$ ($\beta\eta$) | 12. $a\delta'\epsilon'\eta'$ ($a\eta'$) |

It will be verified that just before any one of these twelve moves the component which appears after the name of the unit to be removed is free, while the opposite component is interior to what remains of $E_3 + \omega (\Omega_2 + \sigma\Sigma'_1)$.

The reduction is completed by dismantling in the same way the other three blocks making up A_3 .

Now $\Omega_2 + \sigma\Sigma'_1$ is clearly a 2-element; the method of reducing it to a 2-simplex is obvious. It follows that $\omega (\Omega_2 + \sigma\Sigma'_1)$ and therefore also $E_3 + \omega (\Omega_2 + \sigma\Sigma'_1)$ are 3-elements. But $\omega (\Omega_2 + \sigma\Sigma'_1)$ has the 2-element $\Omega_2 + \sigma\Sigma'_1$ in common with the remainder, E_3 , of $E_3 + \omega (\Omega_2 + \sigma\Sigma'_1)$. Hence, (*F II* Theorem 8b), E_3 is a 3-element.

b. E_3 contains no detachable unit.

E_3 has clearly no internal vertex, and it will be verified that no unit has more than one face in the boundary ¹⁾.

¹⁾ In the original specification of Γ_3 every face belonging to E_3 is indicated by a dot below the opposite vertex of the unit containing it. All other faces of Γ_3 belong to two units of Γ_3 , or to $\sigma\Theta_2$, or to an array congruent to Γ_3 ,

ERRATA.

- page 1095, line 1 from top, for "Rott", read: "Rotti".
" 1098. " 15 " " " " "by rotating of", read: "by rotating".
" 1098. " 15 from bottom, for "which diverge in the placoliths".
read: "which in the placoliths diverge".
" 1104. " 4 from top, for "very circumstances", read: "very
special circumstances".
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