Physics. — STOKES's *Theory of Aberration in the Supposition of a Variable Density of the Aether.* By Prof. H. A. LORENTZ.

In the theory of aberration that has been proposed by Prof. STOKES it must be assumed that the aether has an irrotational motion and that, all over the earth's surface, its velocity is equal to that of the planet itself, in its yearly motion. These two conditions are easily shown to contradict each other, if the aether is understood to have everywhere the same invariable density.

Prof. PLANCK of Berlin had the kindness to call my attention to the fact that both conditions might be satisfied at the same time, if the aether were compressible and subject to gravity, so that it could be condensed around the earth like a gas. It is true, a certain amount of sliding is not to be avoided, but the relative velocity of the aether with regard to the earth may be made as small as we like by supposing the condensation sufficiently large.

At my request Prof. PLANCK permitted me to communicate his treatment of the case; it is as follows.

Instead of considering the earth moving through the aether we shall suppose the planet to be at rest and the aether to flow along it; this comes to the same thing. Let this motion be steady and irrotational and let the velocity at infinite distance be c, constant in direction and magnitude. Let the aether obey BOYLE's law and be attracted by the earth according to the law of NEWTON.

We shall place the origin of coordinates in the centre of the planet and give to the axis of z the direction of the velocity c. Finally we shall call the distance to the centre r, the radius of the earth r_0, the velocity-potential φ, the pressure p, the density k, and the potential of gravity per unit mass V. We shall denote by μ the constant ratio $\frac{k}{p}$, and by g the value

$$\frac{\partial V}{\partial r}\,(r = r_0)$$

of the acceleration at the surface of the earth.

The motion will be determined by the equations

$$\frac{\partial}{\partial x}\left(k\,\frac{\partial \varphi}{dx}\right)+\frac{\partial}{\partial y}\left(k\,\frac{\partial \varphi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\,\frac{\partial \varphi}{\partial z}\right)=0 \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\int\frac{dp}{k}+V+\frac{1}{2}\left\{\left(\frac{\partial \varphi}{\partial x}\right)^2+\left(\frac{\partial \varphi}{\partial y}\right)^2+\left(\frac{\partial \varphi}{dz}\right)^2\right\}=const. \quad . \quad . \quad (2)$$

The problem becomes much simpler if, in the second equation, we suppose the variations of the square of the velocity to be much smaller than those of either of the first terms. We may then write

$$\int \frac{dp}{k} + V = const.,$$

or, since

$$V = -g\frac{r_0^2}{r},$$

$$\log k - \mu\, g\frac{r_0^2}{r} = const.$$

If k_0 be the density at the surface, and

$$\mu\, g\, r_0^2 = \alpha,$$

the last equation becomes

$$\log k - \log k_0 - \alpha\left(\frac{1}{r} - \frac{1}{r_0}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As we see, our simplification consists in this, that the distribution of the aether is independent of its motion, that is to say that it is condensed to the same degree as if it were at rest.

Substituting the value of k from (3) in (1), we find a differential equation for the determination of φ. It can be satisfied by

$$\varphi = z\left[a\left(\frac{\alpha}{2r} - 1\right) + b\left(\frac{\alpha}{2r} + 1\right)e^{-\frac{\alpha}{r}}\right], \quad . \quad . \quad . \quad (4)$$

the form of the solution being chosen with a view to the remaining conditions of the problem. These are

1°. for $r = \infty$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = 0, \quad \frac{\partial\varphi}{\partial z} = c,$$

2°. for $r = r_0$

$$\frac{\partial\varphi}{\partial r} = 0.$$

They give us the following relations between the constants of integration a and b:

$$-a+b=c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$a=\left(\frac{\alpha^2}{2\,r_0^2}+\frac{\alpha}{r_0}+1\right)e^{-\frac{\alpha}{r_0}}\,b. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The velocity with which the aether slides along the earth is found to be

$$v=\frac{\alpha^3}{4\,r_0^3}\,.\,e^{-\frac{\alpha}{r_0}}\,b\sin\vartheta\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where ϑ is the angle between the radius of the point considered and the direction of the velocity c. Now, Prof. PLANCK remarks that, by (6), if only $\frac{\alpha}{r_0}$ be large enough, a will be very small relatively to b, so that, as (5) shows, b is nearly equal to c. But then, the value of v given by (7) will be a very small fraction of c itself.

If the quotient of the pressure and the density had the same value for aether as for air of 0°, and if the force of gravity acted with the same intensity on the aether as on ponderable matter, we should have

$$\frac{\alpha}{r_0}=800\,,\ \text{approximately.}$$

The sliding would then be absolutely imperceptible, but it should be noticed that this would be due to an enormous condensation, the ratio n between the densities for $r=r_0$ and $r=\infty$ being by (3)

$$e^{\frac{\alpha}{r_0}}.$$

In order that the aether may follow the earth in its motion in so far as is necessary for the explication of the phenomena, we need not require that the condensation should have such a high value. Of course, it would be less, if either $\frac{k}{p}$ or g were smaller than for air.

We can easily determine what degree of condensation must necessarily be admitted. Indeed, the constant of aberration may be reckoned to correspond to within $^1/_2$ pCt. to the value given by the elementary theory of the phenomenon; consequently, in the theory of STOKES, the velocity of sliding should be no more than about $^1/_2$ pCt. of the earth's velocity. Now, putting $\frac{\alpha}{r_0}=10$, I find for the maximum value of the velocity of sliding 0,011 c. If $\frac{\alpha}{r_0}=11$, this value would be 0,0055 c. Thus we are led to the conclusion that $\frac{\alpha}{r_0}$ cannot be much different from 11, so that the least admissible value of the condensation is nearly $n = e^{11}$.

Calculations which we shall omit here may serve to estimate the error that has been committed in simplifying the equation (2). It is found that far away from the earth the error may become rather large, but that nearer the surface, precisely on account of the smallness of the velocity in these parts, we need not trouble ourselves about it. Thus, what has been said about the condensation may be true, even though the state of motion in the rarefied aether, at great distances, depart widely from the equation (4).

Strictly speaking, the condensation must be still more considerable than the value we have found to be necessary. If the aether be attracted by the earth, it is natural to suppose that it is acted on likewise by the sun; thus, the earth will describe its orbit in a space in which the aether is already condensed. In this dense aether the earth must produce a new condensation.

Of course it is not necessary that the attraction follow precisely the law of inverse squares; any law which leads to a sufficient condensation will suffice for our purpose. To understand the connexion between the condensation and the velocity of sliding, we may consider a simple case. Let the aether have a constant small density k outside a certain sphere, concentric with the earth, and within this sphere a constant density $k' > k$.

If now the earth were at rest, and the aether flowed along it, a diametral plane of the sphere, perpendicular to the mean direction of flow, would be traversed by a quantity of aether, equal to that which enters the sphere on one side and leaves it on the other side. If this shall be the case, the velocities inside the sphere must be of the order $\frac{k}{k'} c$, if outside the surface they are of the order c.

If we wish to maintain the theory of Prof. Stokes by the supposition of a condensation in the neighbourhood of the earth, it will be necessary to add a second hypothesis, namely that the velocity of light be the same in the highly condensed and in the not condensed aether. This is the theory that may be opposed to that of Fresnel, according to which the aether has no motion at all. In comparing the two we should, I believe, pay attention to the following points.

1. The latter theory can only serve its purpose if we introduce the well known coefficient of Fresnel, concerning the propagation

of light in moving media. Now, this coefficient has been found to be true by direct measurements and may be calculated by means of well founded theoretical considerations. It might be deemed strange, if in these ways we arrived precisely at the value that is required by a wrong theory.

2. If we hope some time to account for the force of gravitation by means of actions going on in the aether, it is natural to suppose that the aether itself is not subject to this force.

On these and other grounds, I consider FRESNEL's theory as the more satisfactory of the two. Prof. PLANCK is of the same opinion. Nevertheless it will be of importance to consider the question from all sides, and it is for this reason that the following remarks may here be allowed.

1. If the large condensation that has been spoken of and the constancy of the velocity of propagation, whatever be the density, be taken for granted, one can indeed explain all observed phenomena. At least, I for one have been unable to find a contradiction. It is true, as has already been stated, that, far away from the earth, the equation (4) will no longer hold. In considering the motion in those distant regions, the square of the velocity in the equation (2) has to be taken into account, and the sun's attraction will have to be considered. But, after all, I find that there may always exist an irrotational motion, and this, in addition to a sufficient condensation near the earth, is all that is required.

2. If we apply to the moving aether the equations which HERTZ has proposed for moving dielectrics [1]) the propagation of light will obey very simple laws. Suppose the earth to be at rest, and the aether to flow, and let the axes of coordinates be fixed in space. Then, if $\mathfrak{d}$ be the dielectric displacement, $\mathfrak{H}$ the magnetic force, $\mathfrak{v}$ the velocity of the aether and V that of light, and if the electromagnetic properties of the aether be supposed to be wholly independent of its density, the equation may be put in the form

$$Div\ \mathfrak{d} = 0\,,$$

$$\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z} = 4\pi \left[\frac{\partial \mathfrak{d}_x}{\partial t} + \frac{\partial}{\partial y}(\mathfrak{v}_y \mathfrak{d}_x - \mathfrak{v}_x \mathfrak{d}_y) - \frac{\partial}{\partial z}(\mathfrak{v}_x \mathfrak{d}_z - \mathfrak{v}_z \mathfrak{d}_x)\right], \text{ etc.}$$

$$Div\ \mathfrak{H} = 0,$$

$$4\pi V^2 \left(\frac{\partial \mathfrak{d}_z}{\partial y} - \frac{\partial \mathfrak{d}_y}{\partial z}\right) = -\frac{\partial \mathfrak{H}_x}{\partial t} - \frac{\partial}{\partial y}(\mathfrak{v}_y \mathfrak{H}_x - \mathfrak{v}_x \mathfrak{H}_y) + \frac{\partial}{\partial z}(\mathfrak{v}_x \mathfrak{H}_z - \mathfrak{v}_z \mathfrak{H}_x), \text{ etc.}$$

[1]) Wied. Ann. Bd. 41, p. 369.

We shall apply these equations to a steady motion with velocity-potential φ, without supposing that $Div\, \mathfrak{v}$ vanishes. We shall however neglect quantities of the order $\mathfrak{v}^2$.

Now if, instead of t, we introduce as a new independent variable

$$t' = t + \frac{\varphi}{V^2} ,$$

and instead of $\mathfrak{d}$ and $\mathfrak{H}$ the vectors $\mathfrak{F}'$ and $\mathfrak{H}'$, defined by

$$\mathfrak{F}'_x = 4\pi V^2 \mathfrak{d}_x + (\mathfrak{v}_z \mathfrak{H}_y - \mathfrak{v}_y \mathfrak{H}_z), \text{ etc.},$$

and

$$\mathfrak{H}'_x = \mathfrak{H}_x - 4\pi (\mathfrak{v}_z \mathfrak{d}_y - \mathfrak{v}_y \mathfrak{d}_z), \text{ etc.},$$

the equations become

$$Div\, \mathfrak{F}' = 0 ,$$

$$\frac{\partial \mathfrak{H}'_z}{\partial y} - \frac{\partial \mathfrak{H}'_y}{\partial z} = \frac{1}{V^2} \frac{\partial \mathfrak{F}'_x}{\partial t'} , \text{ etc.}$$

$$Div\, \mathfrak{H}' = 0 ,$$

$$\frac{\partial \mathfrak{F}'_z}{\partial y} - \frac{\partial \mathfrak{F}'_y}{\partial z} = - \frac{\partial \mathfrak{H}'_x}{\partial t'} , \text{ etc.}$$

These formulae have the same form as those that would hold for an aether without motion, and this is sufficient to obtain in a moment the well known theorems concerning the rotation of the wave-fronts and the rectilinearity of the rays of light. At the same time we see that at the boundary of the different layers of the aether, which slide one over the other, there is never a reflection of light.

It is curious that in the two rival theories somewhat the same mathematical artifices may be used.

3. There seems to be nothing against the assumption that, while the aether may be condensed by gravitation, molecular forces are incapable of producing this effect. In this way it might be explained that small masses, e. g. the flowing water in FIZEAU's experiments, cannot drag the aether along with it. In these cases the coefficient of FRESNEL would remain of use.

4. A decision between the two theories would be soon obtained, if the phenomena of the daily aberration were sufficiently known. Unfortunately, this is by no means the case; even, as Prof. VAN DE SANDE BAKHUYZEN assures me, one has never purposely examined what the existing observations teach us concerning this aberration.

Mathematics. — "*On reducible hyperelliptic Integrals.*" By Prof. J. C. KLUYVER.

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