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Mathematics. — "The theorem of JOACHIMSTHAL for the normal curves", by Prof. P. H. SCHOUTE.

The circle through the feet of the three normals, which we can let fall from any point of the plane of a parabola on this curve, passes through the vertex of the curve. In other words:

"The circles of JOACHIMSTHAL presenting themselves for a parabola form a net with one basepoint, the vertex of the parabola".

And the relation between the point \( P \) through which the three normals pass, and the centre \( M \) of the corresponding circle of JOACHIMSTHAL can be expressed as follows:

"If \( P \) describes the point-field of which the plane of the parabola is the bearer, \( M \) generates in the same plane a point-field affinely related to this."

We shall now investigate how far these theorems can be extended to the normal curve \( N_n^2 \) of the space \( S_n \) with \( n \) dimensions, and we shall commence this investigation with the simple case \( n = 3 \) of the skew parabola.

1. The spheres of JOACHIMSTHAL for the skew parabola. If the skew parabola is represented by the equations

\[
x = t, \quad y = v^2, \quad z = v^3,
\]

then

\[
3 t^5 + 2 t^3 - 3 z v^3 + (1 - 2 y) t - x = 0.
\]  
(1)

is the equation of the normal plane in point \( t \). This equation in \( t \) being of degree five, through any point \( P \) five normal planes pass; the feet of these normal planes we shall call "conormal points" of the curve. These conormal points form on the curve an involution of degree five with three dimensions, for, if three points of such a quintuple are taken arbitrarily, the point \( P \) in space through which the five normal planes must pass, and in this way likewise the supplementary pair of feet, is unequivocally determined. So there must exist two relations between the parameter values \( t \) of five conormal points. If in general \( \sum m \) represents the sum of the products \( l \) by \( l \) of \( k \) quantities \( m \), we deduce immediately from (1)

\[
\sum_{5,1} t = 0, \quad 3 \sum_{5,2} t = 2.
\]  
(2)

On the other hand the six points of intersection of the given curve with the sphere

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are determined by the equation

\[(t^2-p^2) - (t^2-q^2) + (t^2-r^2) - s^2 = 0,\]

or

\[t^6 + t^4 - 2rt = (1-2q) t^2 - 2pt + p^2 + q^2 + r^2 - s^2 = 0. \quad (3)\]

These "conspherical points" form on the skew parabola an involution of degree six with four dimensions; for, if four of the six points of intersection be chosen, the sphere is determined and together with it the supplementary pair of points of intersection. From (3) follows immediately that six conspherical points are connected by the two relations

\[\sum t = 0, \quad \sum t = 1. \quad \ldots \ldots \quad (4)\]

between their parameter values.

We now prove the following theorems:

The spheres determined by the quadruples of points of the skew parabola, conormal with a given point \( t_1 \), intersect this curve still in two fixed points, determined by the equation

\[3t^2 - 3t_1 t + 1 = 0;\]

so they form a net of which these two points are the basepoints. The point \( t_1 \) describing the curve, these basepoints generate on it a quadratic involution of one dimension, of which the two points \( \pm \frac{1}{3} \sqrt{3} \) are the double points.

If \( r_1, r_2, r_3, r_4 \) are four points conormal with \( t_1 \), then according to (2) we have the relations

\[\sum r + t_1 = 0, \quad \sum r + t_1 \sum r = \frac{2}{3}. \quad \ldots \ldots \quad (5)\]

If \( r_1, r_2, r_3, r_4 \) are four points conspherical with \( t_2, t_3 \), then according to (4) we have

\[\sum r + t_2 + t_3 = 0, \quad \sum r + (t_2 + t_3) \sum r + t_2 t_3 = 1. \quad (6)\]

So from (5) and (6) follows immediately:

\[t_2 + t_3 = t_1, \quad t_2 t_3 = \frac{1}{3}, \quad \ldots \ldots \quad (7)\]

with which is proved what was asserted. For the spheres belonging
in the indicated way to the point \( t_1 \), which we name the spheres of \( \text{JOACHIMSTHAL} \) of this point, pass through two fixed points \( t_2, t_3 \) and their number is twofold infinite, it being possible to assume arbitrarily besides \( t_1 \) still two of the other four points \( \tau \).

2. The affinely-related point-fields \((P)\) and \((M)\). If the point \( P \), through which the five normal planes pass, describes the normal plane of point \( t_1 \), and \( t_1 \), as was assumed above, is always one of the five conormal points, the centre \( M \) of the sphere of \( \text{JOACHIMSTHAL} \) passing through the four other points moves in the plane that bisects orthogonally the distance of the points \( t_2, t_3 \) belonging to \( t_1 \). Indicating the first plane by \( \pi \) and the second plane by \( \mu \), we have the theorem:

"The point-fields \((P)\) and \((M)\) in the planes \( \pi \) and \( \mu \) corresponding "with each other are affinely related."

From (3) ensues

\[
2 x_m = \sum_{6,3}^{} t, \quad 1 - 2 y_m = \sum_{6,4}^{} t, \quad 2 z_m = \sum_{6,3}^{} t,
\]

where the sums refer to six conspherical points.

If \( t_2, t_3 \) appear among these and if we call the others again \( \tau_1, \tau_2, \tau_3, \tau_4 \), we find

\[
\begin{align*}
2 x_m &= t_2 t_3 \sum_{4,3}^{} \tau + (t_2 + t_3) \sum_{4,4}^{} \tau \\
1 - 2 y_m &= t_2 t_3 \sum_{4,2}^{} \tau + (t_2 + t_3) \sum_{4,3}^{} \tau + \sum_{4,4}^{} \tau \quad \ldots \ldots \ldots (8) \\
2 z_m &= t_2 t_3 \sum_{4,1}^{} \tau + (t_2 + t_3) \sum_{4,2}^{} \tau + \sum_{4,3}^{} \tau 
\end{align*}
\]

Moreover, if \( t_1 \) is conormal with \( \tau_1, \tau_2, \tau_3, \tau_4 \), we obtain according to (1)

\[
\begin{align*}
t_1 + \sum_{4,1}^{} \tau &= 0 \\
t_1 \sum_{4,1}^{} \tau + \sum_{4,2}^{} \tau &= \frac{2}{3} \\
t_1 \sum_{4,2}^{} \tau + \sum_{4,3}^{} \tau &= \tau_p \\
t_1 \sum_{4,3}^{} \tau + \sum_{4,4}^{} \tau &= \frac{1}{3} (1 - 2 \gamma_p) \\
t_1 \sum_{4,4}^{} \tau &= \frac{1}{3} \tau_p
\end{align*}
\]
So in connection with (7) we find by elimination of the quantities \( \sum \alpha \) the relations

\[
\begin{align*}
18 \alpha_m &= 3(x_p + x_m) - 3 t_1^2 - 2 t_1 \\
18 \alpha_p &= 6 y_p + 3 t_1^2 + 4 \\
6 \alpha_p &= 3 x_p - t_1
\end{align*}
\]

proving what was asserted.

However the two point-fields \( (P) \) and \( (M) \) are not in perspective. For on the line of intersection of the planes \( \pi \) and \( \mu \) not a single point corresponds to itself. For the conditions \( x_p = x_m, y_p = y_m, z_p = z_m \) involve

\[
x = -\frac{t_1(3 t_1^2 + 1)}{15}, \quad y = \frac{3 t_1^2 + 4}{12}, \quad z = \frac{t_1}{3}
\]

and this point is not situated in the normal plane of \( t_1 \). So the connecting lines \( PM \) of the corresponding points \( P \) and \( M \) of \( \pi \) and \( \mu \) form a system of rays \((3,1)\).

3. Relation between the spacial systems \( (P) \) and \( (M) \). To a point \( P \) taken arbitrarily five points \( M \) correspond. For, if \( P \) is given, the five conormal points, the normal planes of which intersect in \( P \), are given and any point of this quintuple may be regarded as the above given point \( t_1 \).

To investigate how many points \( P \) correspond to any point \( M \), we deduce the equation of the plane \( \mu \) belonging to the normal plane \( \pi \) of the point \( t_1 \). The plane \( \mu \) bisecting the distance of the points \( t_1, t_2, t_3 \) orthogonally, it is represented by the equation

\[
(x-t_2)^2 + (y-t_2^2) + (z-t_2^2) = (x-t_3)^2 + (y-t_3^2) + (z-t_3^2),
\]

which reduces itself in connection with (7) to

\[
3 t_1^2 - t_1 - 6 x t_1 + 2 (1 - 3 y) t_1 + 2 (z - 3 x) = 0 . \tag{11}
\]

As \( t_1 \) presents itself to degree five in this equation, any given point \( M \) is centre of five spheres of JOACHIMSTHAL and so a quintuple of points \( P \) corresponds to this point \( M \). So we find:

"The relation of the spacial systems \( (P) \) and \( (M) \) is a correspondence \((5,5)\)."
Although it would not be difficult to trace by means of the equations (10) the complex of the connecting lines \( PM \), we shall avoid this for brevity’s sake.

4. Cyclographic representation of the spheres of JOACHIMSTHAL.

If we wish to extend Fiedler’s cyclographic representation of circles lying in a plane to spheres in space, we must suppose that the three-dimensional space containing the spheres forms part of a space \( S_4 \) with four dimensions. We represent in \( S_4 \) a sphere lying in \( S_3 \) and having \( M \) as centre and \( q \) as radius by the two points \( M_1, M_2 \) of the normal in \( M \) on \( S_3 \) at a distance \( MM_1 = MM_2 = q \) from \( M \).

We shall now first investigate what is the representation of the net of the spheres of JOACHIMSTHAL passing through the points \( t_2, t_3 \) of the skew parabola. If the ordinate in the direction perpendicular on \( S_3 \) is indicated by \( w \), the equations

\[
\begin{align*}
(x - t_2)^2 + (y - t_3)^2 + (z - t_2)^2 &= w^2 \\
(x - t_3)^2 + (y - t_2)^2 + (z - t_3)^2 &= w^2
\end{align*}
\]

will indicate the two quadratic hypercones forming successively the representation of all the spheres through \( t_2 \) and all the spheres through \( t_3 \). By subtracting these equations from each other we find that the section of the two hypercones lies in a three-dimensional space perpendicular to \( S_3 \) along the plane (11). So the locus of pairs of points \( M_1, M_2 \) corresponding to this net of spheres of JOACHIMSTHAL is, as the section of a hypercone of revolution with a three-dimensional space parallel to the axis of the hypercone, a hyperboloid of revolution with two sheets, and the orthogonal one.

Passing on to the investigation of the curved space containing the pairs of images of all the spheres of JOACHIMSTHAL, we have to deal with a simple infinite number of orthogonal hyperboloids of revolution with two sheets. To find the degree of that curved space we have but to observe that the point common to all the normals on \( S_3 \) does not belong to the locus and that the number of points of that curved space situated on a definite one of those normals is double the number of planes (11) passing through the point \( (M) \) where that normal meets \( S_3 \). This number being five, the curved space must be of order ten. It is not difficult to deduce the equation of the locus; we have but to eliminate \( t_2 \) and \( t_3 \) between the two equations (12) and \( t_2 t_3 = \frac{1}{3} \). To this end we give the equations (12) the form
\begin{align*}
T_2 &= t_2^2 + t_2^4 - 2 x t_2^2 + (1 - 2 y) t_2^2 - 2 z t_2 + v^3 = 0 \\
T_3 &= t_3^2 + t_3^4 - 2 x t_3^2 + (1 - 2 y) t_3^2 - 2 z t_3 + v^3 = 0
\end{align*}

where \(v^3\) stands for \(a^2 + y^2 + z^2 - w^2\). Reducing \(t_2^3 T_2 - t_3^3 T_3 = 0\) and \(t_2^2 T_2 - t_3^2 T_3 = 0\) by means of the relation \(3 t_2 t_3 = 1\), we find in \(t_2\) as variable

\begin{align*}
3 (1 - 27 v^3) t_2^3 + 54 x t_1 + 27 v^3 + 18 y - 7 &= 0 \\
6 (3 - 27 v^3) t_2 + 6 (3 z - x) &= 0
\end{align*}

and so after elimination of \(t_1\) we obtain the equation

\begin{align*}
&3 (1 - 27 v^3), \quad 54 x, \quad 27 v^3 + 18 y - 7, \quad 0, \quad 0 \\
&0, \quad 3 (1 - 27 v^3), \quad 54 x, \quad 27 v^3 + 18 y - 7, \quad 0 \\
&0, \quad 0, \quad 3 (1 - 27 v^3), \quad 54 x, \quad 27 v^3 + 18 y - 7, \quad 0 \\
&3, \quad 0, \quad 1 - 27 v^3, \quad 6 (3 z - x), \quad 0 \\
&0, \quad 3, \quad 0, \quad 1 - 27 v^3, \quad 6 (3 z - x)
\end{align*}

which is really of degree ten. For by developing we find

\[315 v^{10} + \ldots \ldots + 267 = 0.\]

Of this curved space the sphere, according to which the hypercone \(v^2 = x^2 + y^2 + z^2 - w^2 = 0\) intersects the space at infinity, is a fivefold surface, etc.

In passing we remark that the plane \((11)\) envelops the developable of which the rational skew curve of degree five represented by the equations

\begin{align*}
6 x &= t (18 t^4 + 9 t^2 - 1) \\
2 (1 - 34) &= 3 t^3 (15 t^2 - 1) \\
2 z &= t (10 t^2 - 1)
\end{align*}

is the cuspidal edge; as follows from the common factor \(30 t^2 - 1\) of the derivatives of \(x, y, z\) according to \(t\) this curve has two real cusps, etc.

5. The normal curve \(N^a\) of \(S^a\). If we represent the curve by the equations
xi = ei, (i = 1, 2, ..., n),

the following theorems are proved in an entirely similar way:

"The hyperspheres \( H_{n-1} \) with \( n-1 \) dimensions determined by 
"groups of \( n+1 \) points of the curve \( N^n \) conormal with \( n-2 \) given 
"points \( t_1, t_2, \ldots, t_{n-2} \) of that curve, intersect the curve still in 
"\( n-1 \) fixed points \( s_1, s_2, \ldots, s_{n-1} \) and form therefore a net, of 
"which the hypersphere \( H_{n-3} \) determined by those \( n-1 \) points \( s \) is 
"the base. And if the system of the \( n-2 \) given points describes 
"the curve \( N^n \), the groups of \( n-1 \) points \( s \) determine on \( N^n \) an invo-
"lution of degree \( n-1 \) with \( n-2 \) dimensions."

"If the point \( P \) describes the plane \( \pi \) common to the normal 
"spaces of \( n-2 \) given points \( t_1, t_2, \ldots, t_{n-2} \), the centre \( M \) of the 
"corresponding hypersphere of JOACHIMSTHAL describes the plane \( \mu \), 
"which is the locus of the points at equal distance from the 
"\( n-1 \) points \( s \) depending in the indicated way on the \( n-2 \) given 
"points \( t \); then \( P \) and \( M \) describe in the planes \( \pi \) and \( \mu \) affinely 
"related point-fields (\( P \)) and (\( M \))."

"Between the spacial systems (\( P \)) and (\( M \)) with \( n \) dimensions 
"exists a correspondence (\( 2n-1, 2n-1 \))."

"The cyclographic representation of all the hyperspheres of 
"JOACHIMSTHAL demands the given space \( S_n \) to be supposed to be 
"part of a \( S_{n+1} \); it leads to a curved space of order 2 (\( 2n-1 \)) with 
"\( n \) dimensions in this \( S_{n+1} \) as locus of the pairs of images, etc."

We believe we can suffice with these general indications; we 
only wish still to observe that the coefficients of the equation deter-
mining the \( n-1 \) fixed points are connected in a simple way with the 
symmetrical functions \( \Sigma_t \) of the \( n-2 \) points \( t \) taken arbitrarily.

Mathematics. — "Approximation formulae concerning the prime 
numbers not exceeding a given limit". By Prof. J. C. KLUYVER.

RIEMANN's method for determining the total number of the prime 
numbers \( p \) less than a given number \( c \) is equally serviceable, when 
it is required to evaluate other arithmetic expressions involving these 
prime numbers, for example the sum \( \Sigma p^{-s} \) of their \((-s)\)th powers, 
or the least common multiple \( M(c) \) of all integers less than \( c \). The 
different results, thus deduced, constantly contain a set of terms 
depending on the complex zeros \( \mu \) of the RIEMANN \( \xi \)-function. The 
most important of these terms is always one and the same disconti-