will in every case remain Scandinavian, or rather Northern. At the same time I would recommend a closer study of the Diluvium in Texel and Wieringen, in order to ascertain, whether it contains erratics, the origin of which may be traced to other parts than of those which are found in the eastern parts of our country.

What I consider to be very "doubtful" indeed, is the right to trace the direction which the Northern glacial flow is supposed to have taken, solely from the examination of erratics, found at such a large distance from the rocks of their origin. In reference to this matter, I would strongly recommend "revision" and would especially suggest a wider field of investigation than the Hondsrug in Groningen.

Mathematics. — "HUYGENS' *sympathic clocks and related phenomena in connection with the principal and the compound oscillations presenting themselves when two pendulums are suspended to a mechanism with one degree of freedom.*" By Prof. D. J. KORTEWEG.

(Communicated in the meeting of October 28, 1905).

Introduction.

1. When in February 1665 CHRISTIAAN HUYGENS was obliged to keep his room for some days on account of a slight indisposition he remarked that two clocks made recently by him, and placed at a distance of one or two feet, had so exactly the same rate that every time when one pendulum moved farthest to the left the other deviated at that very moment farthest to the right [1]). Yet when the clocks were removed from each other one of them proved to gain daily five seconds upon the other.

At first HUYGENS ascribed this "sympathy" to the influence of the motion of the air called forth by their pendulums; but he soon discovered the real cause — the slight movability of the two chairs

[1]) „Ce qu'ayant fort admiré quelque temps"; he writes: „j'ay enfin trouvé „que cela arrivoit par une espèce de sympathie: en sorte que faisant battre les „pendules par des coups entremeslez; j'ay trouvé que dans une demieheure de „temps, elles se remettoient tousiours a la consonance, et la gardoient par apres „constamment, aussi longtemps que je les laissois aller. Je les ay ensuite eloignées „l'une de l'autre, en pendant l'une à un bout de la chambre et l'autre à quinze „pieds de là: et alors j'ay vu qu'en un jour il y avoit 5 secondes de difference „et que par consequent leur accord n'estoit venu auparavant, que de quelque „sympathie". *Journal des Sçavans du Lundy 16 Mars 1665. Oeuvres de* CHRISTIAAN HUYGENS, Tome V. p. 244.

over the backs of which rails had been placed with the clocks suspended to them [1]).

[1]) „J'ay ainsi trouvé que la cause de la sympathie . . . ne provient pas du „mouvement de l'air mais du petit branslement, du quel estant tout a fait insen- „sible je ne m'estois par apperceu alors. Vous scaurez donc que nos 2 horologes „chacune attachée a un baston de 3 pouces en quarré, et long de 4 pieds estoient „appuiées sur les 2 mesmes chaises, distantes de 3 pieds. Ce qu'estant, et les „chaises estant capables du moindre mouvement, je demonstre que necessairement „les pendules doivent arriver bientost à la consonance et ne s'en departir apres, „et que les coups doivent aller en se rencontrant et non pas paralleles, comme „l'experience desia l'avoit fait veoir. Estant venu a la dite consonance les chaises „ne se meuvent plus mais empeschent seulement les horologes de s'écarter par ce „qu'aussi tost qu'ils tachent a le faire ce petit mouvement les remet comme au- „paravant". Letter to MORAY of March 6th 1665. *Oeuvres,* T. V. p. 256.

Compare *Journal des Sçavans du Lundy 23 Mars 1665, Œuvres* T. V. p. 301, note (4), where HUYGENS withdraws his first explanation to replace it by the correct one and likewise his "*Horologium Oscillatorium*" where his experiments and his explanation are developed on one of the last pages of "*Pars prima*".

A somewhat more detailed account of those observations is moreover found in one of his manuscripts, from which we derive the diagrams found here and the explanation HUYGENS deemed he could give of the phenomenon:

Fig. I*a*

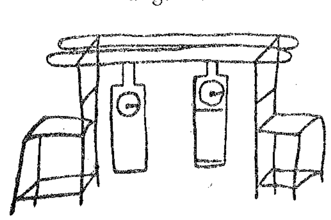
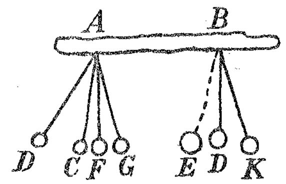


Fig. I*b*



„Utrique horologio pro fulcro erant sedes duae „quarum exiguus ac plane invisibilis motus pen- „dulorum agitatione exitatus sympathiae praedictae „causa fuit, coegitque illa ut adversis ictibus sem- „per consonarent. Unumquodque enim pendulum „tunc cum per cathetum transit maxima vi fulcra „secum trahit, unde si pendulum B sit in BD „catheto cum A tantum est in AC, moveatur „autem B sinistram versus et A dextram versus, „punctum suspensionis A sinistram versus im- „pellitur, unde acceleratur vibratio penduli A. Et „rursus B transiit ad BE quando A est in catheto „AF, unde tunc dextrorsum impellitur suspensio B, ideoque retardatur vibratio „penduli B. Rursus B pervenit ad cathetum BD quando A est in AG, unde „dextrorsum trahitur suspensio A, ideoque acceleratur vibratio penduli A. Rursus „B est in BK, quando A rediit ad cathetum AF, unde sinistrorsum trahitur sus- „pensio B, ac proinde retardatur vibratio penduli B. Atque ita cum retardetur „semper vibratio penduli B, acceleretur autem A, necesse est ut brevi adversis ictibus „consonent, hoc est ut simul ferantur A dextrorsum et B sinistrorsum, et contra. „Neque tunc ab ea consonantio recedere possunt quia continuo eadem de causa „eodum rediguntur. Et tunc quidem absque ullo fere motu manere fulcro mani- „festum est, sed si turbari vel minimum incipiat concordia, tunc minimo motu ful- „crorum restituitur, qui quidem motus sensibus percipi nequit, ideoque errori „causam dedisse mirandum non est".

We give this explanation for what it is. HUYGENS, who never published it, will probably himself, at all events later on, not have been entirely satisfied by it.

2. Although Huygens' observations were published in the *Journal des Sçavans* of 1665, and are moreover mentioned in his "Horologium oscillatorium", they seem to have been forgotten when in 1739 correlated phenomena were discovered by John Ellicott [1]). What he observed at first was this: of two clocks N°. 1 and N°. 2 placed in such a way that their backs rested against the same rail [2]), one, always N°. 2, took over the motion of the other, so that after a time N°. 1 stopped even if at first N°. 2 had been in rest and N°. 1 exclusively was set in motion. Later on he found that the mutual influence was greatly increased by connecting the backs of the clocks by a piece of wood [3]). He also made *both* clocks go on indefinitely by giving their pendulums the greatest possible motion, when alternately they took over a part of the motion from each other, according to a period becoming longer as the clocks being placed without connection with each other had a more equal rate [4]). At the same time he observed that both clocks when connected with each other in the way described above assumed a perfectly equal rate lying between those which they had each separately.

3. Since then different mechanisms where suchlike phenomena

Indeed, it is nothing but the friction which can finally cause that of the three possible principal oscillations only *one* remains. Every explanation in which friction does not play a part must thus from the outset be regarded as insufficient.

[1]) *Phil. Trans.* Vol. 51, p. 126—128: "An Account of the Influence which two "Pendulum Clocks were observed to have upon each other," p. 128—135: "Further Observations and Experiments concerning the two Clocks above mentioned."

[2]) "The two Clocks were in separate Cases, and . . . the Backs of them rested "against the same Rail."

[3]) "I put Wedges under the Bottoms of both the Cases, to prevent their bearing "against the Rail; and stuck a Piece of Wood between them, just tight enough "to support its own Weight."

[4]) "Finding them to act thus *mutually* and *alternately* upon each other, I set "them both a going a second time, and made the Pendulums describe as large "Arches as the Cases would permit. During this Experiment, as in the former, I "sometimes found the one, and at other times the contrary Pendulum to make the "largest Vibrations. But as they had so large a Quantity of Motion given them "at first, neither of them lost so much during the period it was acted upon by "the other as to have its Work stopped, but both continued going for several "Days without varying one Second from each other" . . . "Upon altering the Lengths "of the Pendulums, I found the Period in which their Motions increased and "decreased, by their mutual Action upon each other, was changed; and would be "prolonged as the Pendulums came nearer to an Equality, which from the Nature "of the Action it was reasonable to expect it would." Later on we shall see that there was probably an error in these observations. The continual transmissions of energy and the perfectly equal rate of the clocks exclude each other to my opinion.

of sympathy may appear have been investigated theoretically and experimentally; among others by Euler [1]) the case of two scales of a balance of which Daniel Bernoulli [2]) had observed that they in turns took over each other's oscillations; by Poisson [3]), by Savart [4]) and by Résal [5]) the case of two pendulums fastened with Poisson to the extremities of a horizontal elastic rod, or with Savart and Résal to the horizontal arms of a ⊤-shaped elastic spring; by W. Dumas [6]) the case of a pendulum, beating seconds, with movable horizontal cross rails, on which other pendulums were hung; by Lucien de la Rive [7]) and Everett [8]) the case of two pendulums joined by an elastic string; whilst finally Cellérier, Furtwängler and others developed the theory of the motion of two pendulums of about equal length of pendulum, placed on a common elastic stand, in order to determine experimentally, and to take into account in this way the influence exercised by the small motions of such a stand on the period of the oscillations [9]).

However, we see that the more recent investigations, with the exception of the work of W. Dumas, who does not purposely mention the phenomena of sympathy, relate to mechanisms where elasticity plays a part; whilst it seems probable that this was not the case or at least in only a slight degree in the experiments of Huygens and Ellicott.

[1]) *Novi commentarii Ac. Sc. Imp. Petropolitanae*, T. 19, 1774, p. 325—339. Routh, *Dynamics of a system of rigid bodies, Advanced part, Chapt.* II, Art. 94, giving the right solution, has justly pointed out an error in Euler's solution and likewise in the one signed D. G. S. appearing in *The Cambridge math. Journ.* of May 1840, Vol. 2, p. 120—128. Euler's treatment of the phenomenon of the transmission of energy is also defective, as he does not lay stress upon the necessity of the two almost equal periods, in this case of his quadratic equation admitting a root nearly equal to the length of the mathematical pendulum by which he replaces the scales.

[2]) *Nov. Comm.* l. c. preceding note, p. 281.

[3]) *Connaissance des tems pour l'an* 1833, Additions, p. 3—40. Theoretical. This memoir was indicated to me after the publication of the Dutch version of this paper.

[4]) *L'Institut*, 1e Section, 7e Année, 1839, p. 462—464. Experimental.

[5]) *Compt. Rend.* T. 76, 1873, p. 75—76; *Ann. Éc. Norm.* (2), II, p. 455—460. Theoretical.

[6]) "Ueber Schwingungen verbundener Pendel", *Festschrift zur dritten Säcularfeier des Berlinischen Gymnasiums zum grauen Kloster.* Berlin, Weidmann'sche Buchhandlung. 1874. The investigations themselves are according to this paper from the year 1867. Theoretical and experimental.

[7]) *Compt. Rend.* T. 118, 1894, p. 401—404; 522—525; *Journ. de phys.* (3), III, p. 537—565. Experimental and theoretical.

[8]) *Phil. Mag.* Vol. 46, 1898, p. 236—238. Theoretical.

[9]) See for this the *Encyclopädie der mathematischen Wissenschaften*, Leipzig, Teubner, Band IV, I_{II}, Heft 1, § 7, p. 20—22.

So it seemed worth while looking at the question from another side, and studying the behaviour of a very generally chosen mechanism [1]) with one degree of freedom, and with two compound pendulums attached to it; noting particularly the case that both pendulums have about equal periods of oscillation, whilst at the same time for the application of the phenomena of sympathy of clocks the influence of the motive works will have to be paid attention to.

Moreover it is worth noticing that the results obtained in this way will also be applicable to the case that the connection between the two pendulums is brought about by means of an elastic mechanism, every time when practically speaking only one of the infinite number of manners of motion is operating which such a mechanism can have. Such a manner of motion will have a definite time of oscillation for itself, which will play the same part in the results as if it belonged to a non-elastic mechanism with one degree of freedom.

Deduction of the equations of motion.

4. Let ζ represent for any point of the mechanism with one degree of freedom, to be named in future the "frame", the linear displacement out of the position of equilibrium common to frame and pendulums; let $\zeta^{(m)}$ be its maximum value for a definite oscillation to be regarded as equal on both sides for small oscillations; let ζ_1 and ζ_2 be its values for the suspension points O_1 and O_2 of the pendulums; let M be the mass of the frame; let m_1 and m_2 be that of the pendulums; a_1 an a_2 the radii of gyration of the pendulums about their suspension points; φ_1 and φ_2 their angles of deviation from the vertical position of equilibrium; x_1, y_1 and x_2, y_2 the horizontal and the vertical coordinates of O_1 and of O_2, h the vertical coordinate of the centre of gravity of the frame; taking all these vertical coordinates opposite to the direction of gravitation.

So we begin by introducing for the frame a suitable general coordinate u, for which we choose the quantity determined by the relation

$$Mu^2 = \int \zeta^2 \, dm \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the integration extends to all the moving parts of the frame; this quantity might therefore be called the mean displacement of the particles of the frame.

1) We assume with respect to this mechanism no other restriction than that the motions of each of its material parts just as those of the two pendulums take place in mutually parallel vertical planes, i.o.w. we restrict ourselves to a problem in two dimensions.

For small oscillations of the frame we can put: $u = nu^{(m)}$, $\zeta = n\zeta^{(m)}$, where n is a function of time, but the same for all the points of the frame.

So we have for such vibrations:

$$M\dot{u}^2 = M(\dot{n}u^{(m)})^2 = \int (\dot{n}\zeta^{(m)})^2 \, dm = \int \dot{\zeta}^2 \, dm \, ;$$

so that $\frac{1}{2}M\dot{u}^2$ proves to represent the vis viva of the frame.

For the vis viva of the first pendulum we find, if k_1 denotes the distance between its suspension point O_1 and its centre of gravity, and if φ_1 is reckoned (like φ_2) in such a way that a positive value of φ_1 increases the horizontal coordinate of the centre of gravity:

$$\tfrac{1}{2}\left[m_1 \dot{\zeta}_1^2 + 2 m_1 k_1 \dot{x}_1 \dot{\varphi}_1 + m_1 a_1^2 \dot{\varphi}_1^2\right] =$$

$$= \tfrac{1}{2} m_1 \left[\left(\frac{d\zeta_1}{du}\right)^2 \dot{u}^2 + 2 \, k_1 \, \dot{\varphi}_1 \frac{dx_1}{du} \dot{u} + a_1^2 \dot{\varphi}_1^2\right];$$

therefore for the entire vis viva of the whole system:

$$T = \tfrac{1}{2}\left[M + m_1\left(\frac{d\zeta_1}{du}\right)^2 + m_2\left(\frac{d\zeta_2}{du}\right)^2\right]\dot{u}^2 + \tfrac{1}{2} m_1 a_1^2 \dot{\varphi}_1^2 + \tfrac{1}{2} m_2 a_2^2 \dot{\varphi}_2^2 +$$

$$+ m_1 k_1 \frac{dx_1}{du} \dot{u} \dot{\varphi}_1 + m_2 k_2 \frac{dx_2}{du} \dot{u} \dot{\varphi}_2 \, ; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and further for the potential energy [1])

$$V = \tfrac{1}{2} g \left[M \frac{d^2 h}{du^2} + m_1 \frac{d^2 y_1}{du^2} + m_2 \frac{d^2 y_2}{du^2}\right] u^2 + \tfrac{1}{2} m_1 g k_1 \varphi_1^2 + \tfrac{1}{2} m_2 g k_2 \varphi_2^2 \, . \quad (3)$$

5. To simplify further we introduce the new variable u' determined by:

$$M' u'^2 = \left[M + m_1\left(\frac{d\zeta_1}{du}\right)^2 + m_2\left(\frac{d\zeta_2}{du}\right)^2\right] u^2 = Mu^2 + m_1 \zeta_1^2 + m_2 \zeta_2^2 \, ; \quad (4)$$

where

$$M' = M + m_1 + m_2 \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

represents the entire mass of the whole system; this variable u' is proportional to u, because for small vibrations $\frac{d\zeta_1}{du}$ and $\frac{d\zeta_2}{du}$, as indeed all such derivatives appearing in the formulae, may be regarded as constant.

1) Indeed that potential energy amounts to $Mgh + m_1 g y_1 + m_2 g y_2 - m_1 g k_1 \cos\varphi_1 - m_2 g k_2 \cos\varphi_2 +$ a constant. By developing according to u, taking note that on account of the equilibrium $M\frac{dh}{du} + m_1 \frac{dy_1}{du} + m_2 \frac{dy_2}{du}$ is equal to 0 and by proper choice of constant, we can easily deduce (3) from it.

Out of this proportionality follows easily :

$$M'\dot{u}'^2 = \left[M + m_1\left(\frac{d\zeta_1}{du}\right)^2 + m_2\left(\frac{d\zeta_2}{du}\right)^2\right]\dot{u}^2 = M\dot{u}^2 + m_1\dot{\zeta}_1{}^2 + m_2\dot{\zeta}_2{}^2 \quad . \quad (6)$$

which proves that $\frac{1}{2} M'\dot{u}'^2$ represents the vis viva of what we shall call the *reduced system*, which system consists of the frame and of the masses of the pendulums each transferred to the corresponding suspension point O_1 or O_2.

If now likewise we introduce the vertical coordinate h' of the centre of gravity of the reduced system, so that $M'h' = Mh + m_1 y_1 + m_2 y_2$, the first term of (3) transforms itself into $\frac{1}{2} g M' \frac{d^2h'}{du^2} u^2$, for which, however, on account of the mutual proportionality of u and u' we may write : $\frac{1}{2} g M' \frac{d^2h'}{du'^2} u'^2$.

So for the reduced system it holds that $T' = \frac{1}{2} M' \dot{u}'^2$ and $V' = \frac{1}{2} g M' \frac{d^2h'}{du'^2} u'^2$; if now we write for this system the equations of motion, and if we then introduce the length l' of the simple pendulum which is synchrone to this system [1]) we shall easily find:

$$\frac{d^2h'}{du'^2} = (l')^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus we finally may write for (2) and (3):

$$T = \tfrac{1}{2} M'\dot{u}'^2 + \tfrac{1}{2} m_1 a_1{}^2 \dot{\varphi}_1{}^2 + \tfrac{1}{2} m_2 a_2{}^2 \dot{\varphi}_2{}^2 + m_1 k_1 \frac{dx_1}{du'} \dot{u}' \dot{\varphi}_1 + m_2 k_2 \frac{dx_2}{du'} \dot{u}' \dot{\varphi}_2 ; \quad . \quad (8)$$

$$V = \tfrac{1}{2} g M' (l')^{-1} u'^2 + \tfrac{1}{2} m_1 g k_1 \varphi_1{}^2 + \tfrac{1}{2} m_2 g k_2 \varphi_2{}^2 \quad . \quad . \quad . \quad (9)$$

Application of the equations of LAGRANGE and substitution of the expressions :

$$u' = u'^{(m)} \sin \sqrt{\frac{g}{\lambda}}\, t ; \quad \varphi_1 = \varkappa_1 \sin \sqrt{\frac{g}{\lambda}}\, t ; \quad \varphi_2 = \varkappa_2 \sin \sqrt{\frac{g}{\lambda}}\, t \quad . \quad . \quad (10)$$

leads further easily to the equations

$$M'(l' - \lambda) u'^{(m)} + m_1 k_1 l' \frac{dx_1}{du'} \varkappa_1 + m_2 k_2 l' \frac{dx_2}{du'} \varkappa_2 = 0 ; \quad . \quad . \quad (11)$$

$$\frac{dx_1}{du'} u'^{(m)} + \left(\frac{a_1{}^2}{k_1} - \lambda\right) \varkappa_1 = 0 ; \quad . \quad . \quad (12)$$

$$\frac{dx_2}{du'} u'^{(m)} + \left(\frac{a_2{}^2}{k_2} - \lambda\right) \varkappa_2 = 0. \quad . \quad . \quad (13)$$

[1]) Should the reduced system be in indifferent equilibrium as was probably the case in ELLICOTT's experiments l' is infinite; if it were in unstable equilibrium this would correspond to a negative value of l'. We shall again refer to these cases in the notes. In the text we shall always consider l' positive, hence the reduced system stable.

where $\varkappa_1$ and $\varkappa_2$ denote the maximum deviations of the pendulums and λ the length of the pendulum synchrone to one of the principal vibrations.

6. In order to put these equations still more simply, we first introduce the lengths of pendulum $l_1 = \frac{a_1{}^2}{k_1}$ and $l_2 = \frac{a_2{}^2}{k_2}$ of the two suspended pendulums, secondly the maximum deviations in horizontal direction of their suspension points:

$$\xi_1{}^{(m)} = \frac{dx_1}{du'_1} u'^{(m)} \text{ and } \xi_2{}^{(m)} = \frac{dx_2}{du'_2} u'^{(m)} .$$

It is then easy to find the following system of equations equivalent to the equations (11), (12) and (13), namely:

$$F(\lambda) \equiv (l' - \lambda)(l_1 - \lambda)(l_2 - \lambda) - c_1{}^2 l' l_1 (l_2 - \lambda) - c_2{}^2 l' l_2 (l_1 - \lambda) = 0 ; \quad (14)$$

$$\varkappa_1 = \frac{\xi_1{}^{(m)}}{\lambda - l_1} \quad ; \quad \varkappa_2 = \frac{\xi_2{}^{(m)}}{\lambda - l_2} \quad ; \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where :

$$c_1{}^2 = \frac{m_1}{M'} \cdot \frac{k_1}{l_1} \cdot \frac{(\xi_1{}^{(m)})^2}{(u'^{(m)})^2} \quad ; \quad c_2{}^2 = \frac{m_2}{M'} \cdot \frac{k_2}{l_2} \cdot \frac{(\xi_2{}^{(m)})^2}{(u'^{(m)})^2} \quad . \quad . \quad (16)$$

We must notice here that c_1 and c_2 are numerical coefficients, the first of which depends only on the first pendulum and its manner of suspension, the second on the second pendulum.

Taking note of the signification of u' and ξ_1, and observing that for instance $\xi_1{}^{(m)} : u'^{(m)} = \xi_1 : u'$ on account of the supposed smallness of the vibrations, we can write for the above after some reducing:

$$c_1{}^2 = \frac{m_1 \xi_1{}^2}{m_1 \zeta_1{}^2 + m_2 \zeta_2{}^2 + \int \zeta^2\, dm} \cdot \frac{k_1}{l_1} , \quad c_2{}^2 = \frac{m_2 \xi_2{}^2}{m_1 \zeta_1{}^2 + m_2 \zeta_2{}^2 + \int \zeta^2\, dm} \cdot \frac{k_2}{l_2} \quad (17)$$

holding at any moment of the oscillation, where ξ denotes the horizontal, ζ the linear deviation out of the position of equilibrium of an arbitrary point of the frame, and where the indices relate to the suspension points O_1 and O_2, whilst the integrations must be extended over the whole frame.

If we finally remark that the relation between every ξ and every ζ is the same as that of the fluxions, we can give the signification of $c_1{}^2$ and $c_2{}^2$ also in the following words:

$c_1{}^2$ *is equal to the proportion, remaining constant during the motion, between on one side the vis viva of the horizontal motion of the suspension point O_1 in which the mass of the first pendulum is con-*

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centrated and on the other side the entire vis viva of the reduced system multiplied by the distance between suspension point and centre of gravity of the first pendulum and divided by its length of pendulum; and in the same way c_2^2.

Discussion of the general case.

7. Passing to the discussion of equation (14) we notice that in the supposition $l_1 > l_2$ we have: $F(+\infty)$ neg.; $F(l_1)$ pos.; $F(l_2)$ neg.; $F(0) = l' l_1 l_2 (1 - c_1^2 - c_2^2)$, and therefore with reference to (17) where $k_1 : l_1$ and $k_2 : l_2 < 1$, $F(0)$ always positive.

So there are three principal oscillations. The slowest, which we shall call the *slow principal one* has a synchrone length of pendulum greater than the greatest length of pendulum of both suspended pendulums; of the *intermediate principal one* the length of pendulum lies between that of these two pendulums; of the *rapid principal one* it is shorter than the shorter of the two [1]). Further we can note that when $l' > l_1 > l_2$ the length of pendulum of the slow principal one is greater than l' and that for $l_1 > l_2 > l'$ the rapid principal one has a smaller length of pendulum than l'.

The following graphic representation gives these results [2]) for the case $l' > l_1 > l_2$, practically the most important.

[1]) This is the case for l' positive and this proves that when the reduced system is stable, this must also be the case for the original system with the two suspended pendulums. If l' is infinite, thus the reduced system at first approximation in indifferent equilibrium, then the slow principal oscillation has vanished or rather has passed into an at first approximation uniform motion of the entire system, which would soon be extinguished by the friction. The two other principal ones remain and their lengths of pendulum are found out of the quadratic equation:

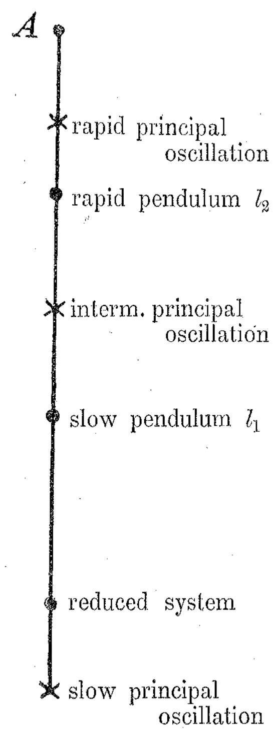
$$(l_1-\lambda)(l_2-\lambda) - c_1^2 l_1 (l_2-\lambda) - c_2^2 l_2 (l_1-\lambda) = 0.$$

For l' negative $F(0)$ becomes negative too, but $F(-\infty)$ positive, so then always one of the principal lengths of pendulum is negative. From this ensues that when the reduced system is unstable, this is also the case for the original one.

[2]) Of course these results are in perfect harmony with and partly reducible from the well-known theorem according to which when removing one or more degrees of freedom by the introduction of new connections the new periods must lie between the former ones. To show this we can 1. fix the frame, 2. bring about two connections in such a way that the pendulums are compelled to make a translation in a vertical direction when the frame is moved. In the latter case it is easy to see that the time of oscillation of the reduced system must appear.

For the rest these same results are found back in the main, extended in a way easy to understand for more than two suspended pendulums, in the work of W. Dumas, quoted in note 6, page 439 which I did not get until I had finished my investigations. By him also the length of pendulum of the reduced system is introduced. However, he has not taken so general as we have done the mechanism of one degree of freedom, on which the pendulums were suspended.

Fig. 2.



8. With respect to the manner of oscillating of the two suspended pendulums we shall call it the *antiparallel* mode when the simultaneous greatest deviations are on different sides as was the case in the observations of Huygens, in the reverse case we shall call it the *parallel* mode.

It is easy to see then from (15) that the following three possible combinations will always appear, namely: for one of the three principal oscillations the mode of oscillating of the pendulums is the antiparallel one, for the two other ones the parallel one, but in such a way that for a definite greatest deviation of the pendulums in a given sense the frame takes for each of these two other principal oscillations an opposite extreme position [1]).

If thus for instance $\xi_1^{(m)}$ and $\xi_2^{(m)}$ have equal signs as was certainly the case in the mechanism used by Huygens (see fig. I*a*) and also in that of Ellicott, the antiparallel mode of oscillation observed by Huygens belongs to the intermediate principal one.

9. For the application to the behaviour of two clocks connected in the manner described we first consider l_1 and l_2 as very different from each other, and that neither c_1 nor c_2 is small. In that case it is evident from the values of $F(l_1)$ and $F(l_2)$ differing greatly from naught that neither of the principal lengths of pendulum nearly corresponds to l_1 or l_2; however from (15) then ensues that the oscillations of the frame are of the same order as those of the pendulums at every possible mode of oscillating.

Now it is of course not at all impossible that the principal oscillations or certain combinations of them once set moving, might remain sustained by the action of one or of both motive works under favourable circumstances with sufficiently powerful works and when means have

[1]) Dumas has: „dass, wenn .... die Aufhängepunkte der Nebenpendel tiefer als „die Drehungsaxe des Hauptpendels liegen, alle Nebenpendel von kürzerer als der „zu erzielenden [principalen] Schwingungsdauer in gleichen Sinne mit dem Haupt„pendel Schwingen müssen, alle anderen im entgegengesetzen Sinne". This too follows immediately from the formulae (15) which, indeed, correspond essentially to those of Dumas.

been taken to decrease sufficiently the frictions in the frame. However in such a case the behaviour of the two clocks would differ greatly from what was observed concerning the phenomena of sympathy; and in the more probable supposition that the motive works will prove to be unable to sustain a considerable motion of the frame, which motion would absorb a great part of the energy, each of the principal oscillations as well as each combination of them will after a certain time have to come to a stop.

So we shall leave this general case, and pass to the discussion of three special cases, which are more important for the consideration of the phenomena of sympathy, namely A the case that l_1 and l_2 differ rather much, but where c_1 and c_2 are small numbers, B the case, that l_1 and l_2 differ but little, but c_1 and c_2 are not small, C the case where l_1 and l_2 differ but little and c_1 and c_2 are both very small. In all these discussions we shall suppose $l' > l_1 > l_2$ and l' differing considerably from l_1 and l_2. The treatment of other special cases, e.g. c_1 small but c_2 not, will not furnish any more difficulties if such a mechanism were to present itself [1]).

A. *Discussion of the case that l_1 and l_2 differ rather much but where c_1 and c_2 are small* [2]).

In this case $F(l')$, $F(l_1)$ and $F(l_2)$ are all very small, from which is evident that each of the three roots of equation (14) is closely corresponding to one of these three quantities, so that the graphic representation of Fig. 2 looks as is indicated in Fig. 3.

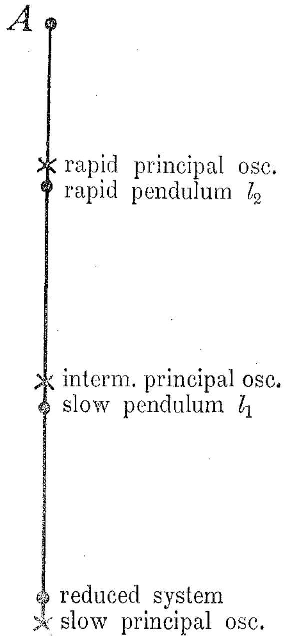
From this then ensues according to (15) that for the rapid principal oscillation the oscillations of the rapid pendulum are much wider than of the slow one [3]), and that for the intermediate principal oscillation

[1]) Also the case $l' = \infty$ differs in nothing, as far as the results are concerned, from the cases treated here but by the vanishing of the slow principal oscillation.

[2]) The smallness of each of these coefficients may according to (16) be due to three different causes, namely 1. to the smallness of $k_1 : l_1$ which will not easily appear in clocks, 2. to the fact that the masses of the pendulums are small with respect to that of the frame, 3. to the fact that the pendulums are suspended to points of the frame whose horizontal motion is a slight one compared to that of other points of that frame. It is remarkable that this difference of cause has hardly any influence on the considerations following here, and therefore on the phenomera which will present themselves.

[3]) Then still when in (15) $\xi_2^{(m)}$ might prove to be very small compared to $\xi_1^{(m)}$; for as a first approximation for $l_2 - \lambda$ we find: $c_2^2 \, l' l_2 : (l' - l_2)$, and therefore $\varkappa_2 = -M' (l' - l_2) (u'^{(m)})^2 : m_2 k_2 l' \xi_2^{(m)}$. So the motion of the frame determined by $u'^{(m)}$ is slight compared to that of the rapid pendulum and consequently $\varkappa_1$ is small compared to $\varkappa_2$.

Fig. 3.



the opposite is the case. For the slow principal oscillation the oscillations of both pendulums are either of the same order as those of the frame or smaller still; the latter is the case when the third cause mentioned in note 2 of page 446 is at work.

Suppose now π', π_1 and π_2 to be small oscillations belonging respectively to each of the three types of the principal oscillations, namely the slow one, the intermediate one and the rapid one, each having the same small quantity of total energy $\varepsilon = T + V$; then every compound oscillation can be represented by $\omega = K'\pi' + K_1\pi_1 + K_2\pi_2$ and its total energy will be equal to $(K'^2 + K_1^2 + K_2^2)\,\varepsilon$.

Let us then start from an arbitrary compound oscillation for which K', K_1 and K_2 have moderate and mutually comparable values; it is then clear that the motion of one clock, namely the one with the rapid pendulum will be dependent almost exclusively on the rapid principal oscillation, that of the other clock on the intermediate one. It is true, that slight periodical deviations in the amplitudes will present themselves, which are due to the two other principal oscillations, but these can have no influence of any importance on the periods according to which the motive works regulate their action; so that therefore one of the motive works will be able to contribute to the sustenance of the motion $K_1\pi_1$, the other to the motion $K_2\pi_2$, but neither of them to the sustenance of the motion $K'\pi'$. So this will vanish first.

What takes place furthermore will depend on the power of the motive works, and on the frictions presenting themselves during the motion of the frame. If those powers are great enough to conquer the frictions when the pendulums deviate sufficiently to keep the motive works in movement, a motion $K_1\,\pi_1 + K_2\,\pi_2$ will remain, where the values of K_1 and K_2, thus also of their proportion, will finally depend exclusively on the power of those motive works and on the frictions. A theorem the proof of which we shall put off to § 14, to be able to give it at once for all cases, shows that in general such a motion can be sustained rather easily; it is the theorem that for principal oscillations whose λ differs but slightly from l_1 or l_2 whatever may be the cause, the kinetic energy of the motion of the frame

will be small compared to that of the corresponding pendulum. For such a motion $K_1 \pi_1 + K_2 \pi_2$ remaining in the end, the two clocks will each have their own rate [1]) whilst however slight periodic variations in their amplitudes are noticed, caused by the cooperation of the two remaining principal oscillations whose periods differ considerably if l_1 and l_2 are sufficiently unequal.

11. Let us now however suppose that l_1 and l_2, differing at first considerably, are made to correspond more and more, for instance by displacement of the pendulum weights. The chief consequence will have to be that, according to equation (15), the amplitudes of both pendulums will become more and more comparable to each other, for $K_1\pi_1$ as well as for $K_2\pi_2$, in consequence of which to obtain their motion for the compound oscillation $K_1\pi_1 + K_2\pi_2$ we shall finally have to compose for each of them two oscillations with comparable amplitudes, and whose periods of oscillation differ but slightly. As is known this leads for both pendulums alternately, to periods of relatively greater and smaller activity, i. o. w. to the phenomenon of transference of energy of motion from one pendulum to another and back again; the period in which this alternation of activity takes place will be the longer according as l_1 and l_2 differ less [2]).

Now however a suchlike behaviour of the two pendulums according as it gets more and more upon the foreground when l_1 and l_2 approach each other, becomes less and less compatible with the regular action of the two clockworks. For, during the period of smaller activity of one of the pendulums the motive work corresponding to it will finally, when the remaining activity has become much smaller than the normal, come to a stop. Then one of the two will take place: *either* the principal oscillation which is sustained particularly by this work is powerful enough to keep on till the period of greater activity has been entered upon, and this will be deferred the longer according as l_1 and l_2 differ less, *or* it is not so. In the first case the clock can keep going with alternate periods in which it ticks and in which it does not tick, which phenomenon may of course present

1) Both rates however a little more rapid than for independent position.

2) These phenomena remind us of what Ellicott observed later on (see note (4) p. 438). However the correspondence is not complete, as in the case treated here both clocks retain their different rate, whilst Ellicott mentions emphatically that the two clocks did not differ a second for many days. We shall therefore have to again refer to these observations at case C.

itself in both clocks [1]). In the second case the clockwork stops entirely; the corresponding principal oscillation vanishes, and the pendulum performs only passively the slight motion which is its due in that principal oscillation, which can now be sustained indefinitely by the other motive work.

This is the phenomenon remarked by Ellicott in his first experiment when the clock n° 2 regularly made n° 1 stop.

We have now gradually reached case C where c_1 and c_2 are small and where l_1 and l_2 differ but slightly; this case demands, however, separate treatment, for which reason we shall discuss it later on.

B. *Discussion of the case that l_1 and l_2 differ but very little, but where c_1 and c_2 are not small* [2]).

Before passing to the case C we shall treat the simpler case now mentioned which will lead us to phenomena corresponding to those found by Huygens.

To this end we put $l_1 = l_2 + \Delta$, and substitute this in the cubic equation (14). Then by writing for one of the roots of that equation $l_2 + \sigma$ and by treating Δ and σ as small quantities we shall easily find for the length of pendulum of the intermediate principal oscillation the value

$$l_2 + \frac{c_2^2}{c_1^2 + c_2^2} \cdot \Delta, \qquad \qquad (18)$$

from which is evident that this length of pendulum divides the distance between l_1 and l_2 in ratio of $c_1^2 : c_2^2$.

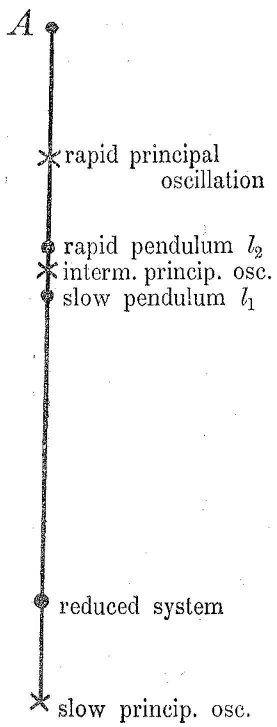
The two other roots satisfy approximately the quadratic equation:

$$(l' - \lambda)(l_2 - \lambda) - (c_1^2 + c_2^2)\, l' \, l_2 = 0 \qquad \qquad (19)$$

1) This was really observed by Ellicott (l. c. p. 132 and 133) for both clocks, however only temporarily, for at last the work of the first clock came entirely to a stop. Compare for the rest the experiment of Daniel Bernoulli with the two scales mentioned in § 3.

2) If l_1 is perfectly equal to $l_2 = l$, then of course (14) has a root $\lambda = l$ for whose principal oscillation according to (15) the frame remains in rest. The remaining roots are found by means of the quadratic equation $(l' - \lambda)(l - \lambda) - (c_1^2 + c_2^2)\, l' l = 0$. One of them will nearly correspond to l if c_1 and c_2 are both small fractions. All this in accordance with Routh's solution (l.c. note (1) page 439) which refers exclusively to this case and also to that of Euler (barring what is remarked in that note).

Fig. 4.



They correspond to the slow and the rapid principal oscillation differing considerably in general in length of pendulum from l' and l_2[1]) and therefore by reason of (15) giving rise to oscillations of the frame which are of the same order of magnitude as those of the pendulums.

So unless special measures are taken with respect to the decrease of the friction of the frame, these oscillations will have to stop, the more so as they are not sustained by the action of the motive works.

So the only oscillation which will be able to continue for some time is the intermediate principal one whose length of pendulum is lying between l_1 and l_2; entirely in accordance with the observations of HUYGENS[2]) and also with those of ELLICOTT described in note (4) p. 438 when for the latter we overlook for a moment the observed periodic transference of energy.

C. *Discussion of the case that l_1 and l_2 differ but very little and that at the same time c_1 and c_2 are small numbers.*

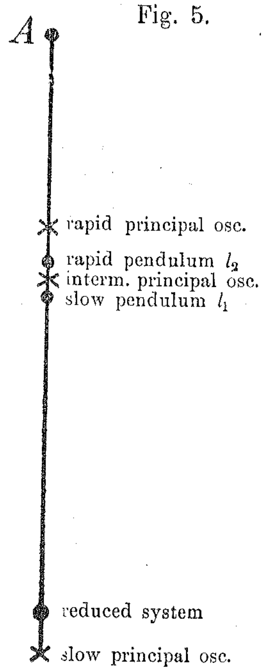
13. The remarkable thing in this case is that now the remaining quadratic equation (19) is also satisfied by a root differing but little from l_2. So there are now *two* roots of the original cubic equation situated in the vicinity of l_2, one found just now and expressed by (18) and the other which is likewise easily found by approximation and represented by the expression

$$l_2 - \frac{(c_1^2 + c_2^2)\, l' l_2}{l' - l_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This root is, at first approximation, independent of $\Delta = l_1 - l_2$; so when the lengths of the pendulums approach each other sufficiently, it is, though small, yet many times larger than Δ. These

1) See the graphic representation of Fig. 4.

2) See however note (3) p. 452; from which is evident that the case which really presented itself in HUYGENS' experiments is probably not the one discussed here, but the more complicated case C.



conditions are represented by Fig. 5, where we have moreover to notice that the third root belonging to the slow principal oscillation differs but little from l'.

We can now show that for the rapid principal oscillation as well as for the intermediate one, *although not in the same measure*, the oscillations of the frame remain small compared with those of the pendulums.

Generally this is already directly evident from the equations (15); this is however not the case when the pendulums are suspended to points of the frame whose horizontal motion is an exceptionally slight one[1]). In that case we refer to the general theorem to be proved in the following paragraph, and from which what was assumed ensues immediately.

Let us note before continuing that now for the rapid as well as for the intermediate principal oscillation the two pendulums possess amplitudes which are mutually of the same order of magnitude.

14. The indicated theorem can be formulated as follows: *when the length of pendulum of a principal oscillation approaches closely to l_1 of l_2 then the vis viva of the reduced system, thus a fortiori of the frame alone, is continually small with respect to that of the pendulum corresponding to l_1 or l_2.*

To prove this we compare in formula (8) the three terms: $\frac{1}{2} M' \dot{u}'^2$; $m_1 k_1 \frac{dx_1}{du'} \dot{u}' \dot{\varphi}_1$ and $\frac{1}{2} m_1 a_1^2 \dot{\varphi}_1^2$. For the proportion of the second to the third can be written $2 \frac{dx_1}{du'} \dot{u}' : l_1 \dot{\varphi}_1$, or on account of equation (10), $2 \frac{dx_1}{du'} u'^{(m)} : l_1 \varkappa_1 = 2\, \xi_1^{(m)} : l_1 \varkappa_1 = 2 (\lambda - l_1) : l_1$. The second is therefore, when λ approaches l_1 closely, small with respect to the third, which can thus be regarded in such a case to represent at first approximation the vis viva of the first pendulum.

1) That is to say, when the third cause mentioned in note (2) p. 446 has given rise to the smallness of c_1 and c_2.

For the proportion of the vis viva of the reduced system to that of the pendulum referred to we can write [1]:

$$M' \dot{u}'^2 : m_1 a_1^2 \dot{\varphi}_1^2 = M' (u'^{(m)})^2 : m_1 a_1^2 \varkappa_1^2 =$$
$$= M' (u'^{(m)})^2 (l_1 - \lambda)^2 : m_1 a_1^2 (\xi_1^{(m)})^2 = (l_1 - \lambda)^2 : c_1^2 l_1^2. \quad . \quad (21)$$

If now c_1 is not small, as in case B, then we have in this manner already proved what was put. In case A we substitute $\lambda = l_1 - \delta$ in the cubic equation (14) after which we find easily at first approximation, c_2 being likewise small [2]), $\delta = l_1 - \lambda = c_1^2 l' l_1 : (l' - l_1)$, by which what was put is likewise proved.

In case C finally, which occupies our attention at present, ensues from (20) for the rapid principal oscillation $l_2 - \lambda = (c_1^2 + c_2^2) l' l_2 : (l' - l_2)$; from which is evident after substitution of l_2 and c_2 for l_1 and c_1 in (21) the correctness of the theorem also for this principal oscillation, hence *a fortiori* for the intermediate one; unless c_1 be small but yet much larger than c_2, which restriction does not exist for the intermediate principal oscillation.

15. From these results must be inferred that in the case C under consideration the rapid principal oscillation as well as the intermediate one when once set in motion will each be able to maintain themselves under the influence of the motive works, when the conditions of friction in the frame are not too unfavourable. However, the intermediate principal oscillation will have, if the difference in rate between the two clocks was originally very slight, a considerable advantage on the rapid one, the motion of the frame being much slighter still in the former case than in the latter. And this will probably be the reason that in the experiments of Huygens as well as in the later ones of Ellicott evidently the intermediate principal oscillation exclusively [3]) or at least chiefly [4]) presented itself.

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[1]) According to (10), (15) and (16) taking at the same time note of the signification of l_1, a_1 and k_1.

[2]) For c_1 small and c_2 not, the proof runs in the same way, although the expression for δ becomes a little less simple.

[3]) With Huygens. In his experiments the masses of the pendulums were certainly slight with respect to those of the frame, so that without doubt c_1 and c_2 were small and the case C was present.

[4]) With Ellicott, where at least at first according to the observed transferences of energy also the rapid principal oscillation must have been present. Although Ellicott used according to his statement very heavy pendulums, we have probably also the case C with him. If we do not assume this then it is more difficult still to make the perfectly equal rate of his clocks tally with the observed transferences of energy. The presence of two principal oscillations evident from these would have been continued indefinitely in case B, so the clocks would have retained an unequal rate.

Savart on the contrary has effected with the aid of his T-shaped spring at whose ends almost equal pendulums were attached both principal oscillations [1]).

But besides these two principal oscillations which deviate in their periods of oscillation, and moreover by the circumstance that the pendulums will move in a parallel mode for one and in an antiparallel mode for another, there is still a third manner of motion which must be able to continue indefinitely.

16. To prove this let us again start from an arbitrary compound oscillation $\omega = K'\pi' + K_1\pi_1 + K_2\pi_2$; then unless the friction in the frame be extremely slight the oscillation $K'\pi'$ will soon disappear. When however in the remaining motion K_2 is much smaller than K_1, it is clear that as the intermediate principal oscillation is then the chief one for the motion of the two pendulums, the motive works of *both* clocks will regulate themselves according to it, so that they will not be able to contribute to the sustenance of the principal oscillation $K_2\pi_2$ which will thus likewise have to die away, so that finally only a pure oscillation $K_1\pi_1$ will be left, for which both clocks will follow the rate of the intermediate principal oscillation.

If on the contrary after the disappearance of the slow principal oscillation K_1 is much smaller than K_2, it will have to be the intermediate principal oscillation, which dies away, whilst the rate of the clocks will finally regulate itself entirely according to the rapid one.

But in the intermediate case, when the proportion of K_1 to K_2 lies within certain limits, also a manner of motion will be able to appear under favourable circumstances where both principal oscillations are sustained for indefinite time, whilst each of them will govern the behaviour of one of the two clocks; for from the equations (15) it is easy to deduce that in general the proportion between the amplitudes $\varkappa_1$ and $\varkappa_2$ is different for both principal oscillations [2]). Then the values of K_1 and K_2 and so also their proportion will in the long run be entirely governed by the power of the motive works,

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[1]) *l.c.* note (4) page 439. Savart had however $l' < l_1 = l_2$; therefore with him it is the *slow* principal oscillation which plays the part given here in the supposition $l' > l_1 > l_2$ to the rapid one.

[2]) By substitution of the value (18) for λ we find for the intermediate principal oscillation $\varkappa_1 : \varkappa_2 = c_1^{-2} \xi_1^{(m)} : c_2^{-2} \xi_2^{(m)}$; whilst the substitution of (20) furnishes for the rapid principal oscillation

$$\varkappa_1 : \varkappa_2 = \left[\Delta + \frac{(c_1^2 + c_2^2) l' l_2}{l' - l_2} \right]^{-1} \xi_1^{(m)} : \left[\frac{(c_1^2 + c_2^2) l' l_2}{l' - l_2} \right] \xi_2^{(m)};$$

so for very small values of Δ we have for this one $\varkappa_1 : \varkappa_2 = \xi_1^{(m)} : \xi_2^{(m)}$.

connected with the frictions presenting themselves, i. e. these values will be independent of the initial condition. At the same time the two clocks will show a different rate [1]), of which clocks one therefore will have to sustain the rapid principal oscillation, the other the intermediate one. Periodic transference of energy will then take place.

Probably it will not be easy to realize this condition, characterizing itself particularly by the fact, that one of the clocks goes considerably faster than would be the case when placed independently [2]). The initial conditions will then have to be chosen in such a manner that from the very beginning one oscillation will predominate for one clock, the other for the other clock. And this will become all the more difficult as c_1 and c_2 become more and more equal, therefore according as the two clocks become more and more alike and are suspended in a more symmetric way. For, so much smaller will, according to what was mentioned in note (2) p. 453 be the difference in proportion of the amplitudes $\varkappa_1$ and $\varkappa_2$ at each of the oscillations. [3])

17. Finally we wish to point out how we must represent to ourselves the transition of case A into case C. In case A in which the rate of the clocks differs greatly, the manner of motion which is most difficult to realize in case C, namely the one, where the clocks have each their own rates, is the normal one. Yet the two other manners of motion also are possible, i. e. those where exclusively *one* of the principal oscillations appears; however in these cases, the pendulum of the least active of the two clocks will still perform a slight oscillation though not sufficient to set its motive work in motion.

If now starting from case A we reach case C, i. e. if the rate of the clocks is taken more and more equal, the state of motion with mutually different rate of the clocks becomes continually more difficult to realize, finally perhaps impossible; whilst for the two other possible manners of motion the pendulum of the second clock too keeps performing greater and greater deviations till these deviations are finally sufficient to set its motive work also in motion, so that both clocks go quite alike, either with the rate belonging to the rapid principal oscillation or, what is more easily realized, with that or the intermediate one.

1) So this differs again from what ELLICOTT observed in his last experiments, so that these cannot be regarded as the realisation of this case, though they have the transferences of energy in common with it. However, between the fact of those transferences and the assurance that both clocks have entirely the same rate exists a contradiction, as we have already seen, which is not to be solved. Indeed, those transferences can be explained by interference only, so they require the cooperation of two oscillations of different periods; but these oscillations must both be sustained if the state is really to continue indefinitely, and then each of them by one of the motive works where the oscillation referred to will predominate the other one. See also the last note.

To me it seems most probable that with ELLICOTT the transferences of energy existed only at first indicating the *temporary* presence of the rapid principal oscillation. ELLICOTT'S wording is not emphatically against this conviction.

2) The difference from case A is of course only quantitative. In both cases the clocks go faster than when placed independently, but in case C the acceleration of the quickest clock becomes much greater than that of the less rapid one (see § 13). A gradual transition presents itself then, and the case of ELLICOTT was probably situated on that transition-line.

3) The idea that perhaps each of the motive works might be able to take over one principal oscillation and the other in turns had to be set aside after a closer investigation. If we compose in the well-known graphical way two oscillations of unequal amplitudes and of periods of oscillation differing but little, it is evident that the motive work will go alternately somewhat quicker and somewhat slower than will correspond to the period of oscillation of the greatest amplitude, but this can never go so far that the rate of the smaller amplitude is taken over, not even for a short time.

Chemistry. — *"The different branches of the three-phase lines for solid, liquid, vapour in binary systems in which a compound occurs."* By Prof. H. W. BAKHUIS ROOZEBOOM.

(Communicated in the Meeting of October 28, 1905)

A chemical compound, formed from two components, need not to be regarded as a third component, when this compound is somewhat dissociated, at least when it passes into the liquid or gaseous state. Instead of the triple point we then get a series of triple points, the three-phase line, indicating the co-related values of temperature and pressure at which the compound can exist in presence of liquid and vapour of varying compositions [1]) This was advanced for the first time in 1885 by VAN DER WAALS. The equation for that line was deduced by him [2]) and shortly afterwards [3]) applied by me in a few instances where it was always admitted that the vapour tension of the liquid mixtures gradually diminished from the side of the most volatile (A) towards that of the least volatile component (B).

In the first considerations as to the course of the three-phase line

1) There exist several other three-phase lines which are not considered here.
2) Verslag Kon. Akad. 28 Febr. 1885.
3) Rec. Tr. Chim. **5**, 334 (1886)