KONINKLIJKE AKADEMIE
VAN WETENSCHAPPEN
-:- TE AMSTERDAM -:-

PROCEEDINGS OF THE SECTION OF SCIENCES

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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday December 28, 1907.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 28 December 1907, Dl. XVI).

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Crystallography. — *"About the oblique extinction of rhombic crystals."* By J. Schmutzer. (Communicated by Prof. A. Wichmann).

(Communicated in the meeting of November 30, 1907).

As late as the year 1901 Alfred Harker in a brief communication [1]) pointed out the fact that with rhombic crystals the oblique extinction on planes making a small angle with the *c*-axis, is to be neglected only when the angle of the optic axes has no great value. That this was not superfluous is perhaps partly owing to the fact that, in the application of the theoretically deduced results concerning the extinction of crystal-sections mineralogists have confined themselves to monoclinic and triclinic minerals, preferably to the feldspars. [2]) It seems, therefore, that the circumstance that rhombic minerals as a rule show an oblique extinction and only exceptionally a straight one, is not laid sufficient stress on, though the fact is of course well-known. [3]) That is why, even in the younger petrographical literature, it is often alleged, in verification of the rhombic nature of a mineral, that all its sections show a straight extinction, whilst at partly straight, partly oblique extinction of the crystal-sections the monoclinic nature of the mineral is considered to have been proved.[4]) A separation of rhombic and monoclinic pyroxenes, olivine and diopside, zoisite and klinozoisite however on the ground of the character of the extinction is not to be insisted upon; only in case of small axis-angles this characteristic has some value as a criterion. What gave rise to the calculation of the angles of extinction for olivine was that considerable extinctions were found with respect to a particularly well developed pinacoidal cleavage of this mineral, whilst, to compare them with the results obtained from this, I have also made the same calculations for talc.

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1) Mineralogical Magazine, XIII, 1903, p. 66—68.

2) Michel Lévy, Ann. d. Mines, (7), XII, 1877, p. 392—471, Abstract Zeitschr. f. Kryst. III, 1879, 217—231; Minéraux des Roches, 1888. p. 9 seq.; Fouqué et Michel Lévy, Minéralogie Micrographique, Paris 1879; A. Harker, Min. Mag. X, 1893, p. 239—240; G. Césaro, Mém. cour. Acad. Roy. Belg. LIV, 1895; Daly, Proc. Americ. Acad. Arts a. Sc. XXXIV, 1899, p. 311—328; A. A. Ferro, Riv. di Min.; Padua XX, 1898; Atti Soc. Lig. di Sc. nat. Genova, IX, 1898, Abstract Zeitschr. f. Kryst. XXXII, 1900, 532; Vicente de Sousa Brandão, Communicações da direcção d serviç. geol. de Portug. IV, 1901, 13—126.

3) Cf. Fouqué et Michel Lévy, Minéralogie Micrographique, p. 55—57.

4) Cf. Lacroix, about Fouqué in Contributions à l'étude des gneiss à pyroxène et des roches à wernérite, Bull. Soc. franç. de Minéralogie XII, Paris 1889, p. 328.

Olivine and Talc.

If O be the intersection of the acute bisectrix with the globe of projection $\varrho = 1$, A and B the projections of the optic axes, ZO the axis of a zone, from which ZQb represents an arbitrary plane with its pole N, then, according to FRESNEL, the extinction on the plane ZQb, with respect to the zone-axis, is represented by the curve Zc, when the plane cN divides the angle BNA into two equal parts. Suppose we call $\frown OQ$, the inclination of the plane (N) with regard to the acute bisectrix, x, and the angle of extinction with respect to the zone-axis, $\frown Zc = y$, then, according to MICHEL LÉVY [1]) the value of y can be calculated from the equation:

$$cot\, 2y = cot\, (aZ + bZ).$$

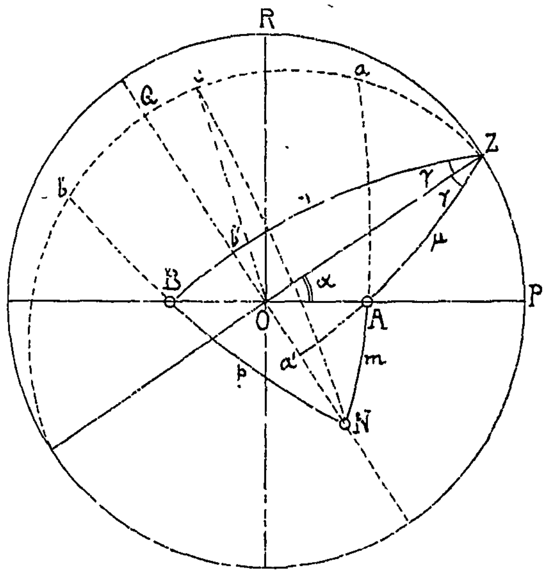

Fig. 1.

$$\frown aZ = \frac{\pi}{2} - \angle ANa'.$$

Now in the right-angled $\Delta ANa'$

$$tg\, ANa' = \frac{tg\, Aa'}{sin\, Na'} = \frac{cot\, \mu}{cos\, (x + \gamma)}$$

so that:

$$tg\, aZ = cot\, ANa' = tg\, \mu\, cos\, (x + \gamma).$$

In the same way we find:

[1]) Les Minéraux des Roches, p. 9.

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$$tg\ bZ = tg\ \nu \cos (x - \gamma).$$

Now

$$\cot 2y = \cot (aZ + bZ) = \frac{1 - tg\ aZ\ tg\ bZ}{tg\ aZ + tg\ bZ} = \frac{1 - tg\ \mu\ tg\ \nu \cos (x + \gamma) \cos (x - \gamma)}{tg\ \mu \cos (x + \gamma) + tg\ \nu \cos (x - \gamma)} \ . \ (1)$$

As is indicated in the figure, here the particular case is considered that the zone-axis lies in the plane forming right angles with the acute bisectrix, so that $\mu + \nu = \pi$. If we let the zone-axis successively make different angles α with OP, varying from 0 to π, and if, at the same time, we let this plane perform a revolution about this axis, then N passes through the whole surface of the globe and consequently the extinction with regard to OZ can be calculated as a function of α and x for each arbitrary section through the crystal. As $\mu + \nu = \pi$, the formula (1) can be simplified as follows:

$$\cot 2y = \frac{1 + tg^2 \mu \cos (x + \gamma) \cos (x - \gamma)}{tg\ \mu [\cos (x + \gamma) - \cos (x - \gamma)]}$$

from which we derive:

$$\cot 2y = \frac{-(\cos^2 \mu + \sin^2 \mu \cos^2 \gamma) + \sin^2 \mu \sin^2 x}{\sin 2\mu \sin \gamma \sin x} =$$

$$= -\frac{\cos^2 \mu + \sin^2 \mu \cos^2 \gamma}{\sin 2\mu \sin \gamma} \cdot \frac{1}{\sin x} + \frac{\sin^2 \mu}{\sin 2\mu \sin \gamma} \cdot \sin x \quad . \ . \ (2)$$

$$= \frac{A}{\sin x} + B \sin x \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (3)$$

Now in $\Delta\ ZOA$ $\cos \mu = \sin OA \cos \alpha = \sin V \cos \alpha$

and in $\Delta\ ZPA$ $\cos \angle AZP = \frac{tg\ PZ}{tg\ AZ}$

or

$$\cos \left(\frac{\pi}{2} - \gamma\right) = \sin \gamma = \frac{tg\ \alpha}{tg\ \mu}.$$

If in (2) we substitute the values in α and V for μ and γ, we get:

$$\cot 2y = -\frac{1 - \sin^2 \mu \sin^2 \gamma}{\sin 2\mu \sin \gamma} \cdot \frac{1}{\sin x} + \frac{\sin^2 \mu}{\sin 2\mu \sin \gamma} \cdot \sin x =$$

$$= -\frac{1 - tg^2 \alpha \cos^2 \mu}{2 \cos^2 \mu\ tg\ \alpha} \cdot \frac{1}{\sin x} + \frac{1 - \cos^2 \mu}{2 \cos^2 \mu\ tg\ \alpha} \cdot \sin x =$$

$$= -\frac{1 - \sin^2 V \sin^2 \alpha}{\sin 2\alpha \, . \, \sin^2 V} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cos^2 \alpha}{\sin 2\alpha \sin^2 V} \cdot \sin x \quad . \ . \ (4)$$

From this form we can deduce what follows. For $x = 0$, y

becomes $=\frac{\pi}{2}$ for all values of α, and the same thing takes place with $\alpha=0$ for all values of x. Consequently on all planes parallel to *the acute or to the obtuse bisectrix* the extinction with respect to these bisectrices is straight. If the direction of the mean index of refraction (OR) becomes zone-axis, with $\alpha=\frac{\pi}{2}$, a certain particularity shows itself. For this value of α (4) assumes the following form:

$$\cot 2y \underset{(2\alpha=\pi)}{=} \frac{1}{\sin 2\alpha}\left(-\frac{1-\sin^2 V}{\sin^2 V}\cdot\frac{1}{\sin x}+\frac{\sin x}{\sin^2 V}\right)=$$

$$\underset{(2\alpha=\pi)}{=} \frac{1}{\sin 2\alpha}\left(-\frac{\cos^2 V}{\sin^2 V.\sin x}+\frac{\sin x}{\sin^2 V}\right) \quad . \quad . \quad . \quad . \quad (5)$$

For $x=0$ becomes $y=\frac{\pi}{2}$.

For $x=\frac{\pi}{2}-V$ (5) changes into:

$$\cot 2y \underset{(2\alpha=\pi)}{=} \frac{1}{\sin 2\alpha}\left(-\frac{\cos V}{\sin^2 V}+\frac{\cos V}{\sin^2 V}\right)=\frac{0}{0}.$$

y becomes indefinite; the pole N of the plane at this moment coincides with an optical axis.

Finally for $x=\frac{\pi}{2}$ y becomes$=0°$. So the extinction is $\frac{\pi}{2}$ for a value of x between $0°$ and $\frac{\pi}{2}-V$, next becomes indefinite and remains $0°$ for $x=\frac{\pi}{2}-V$ to $\frac{\pi}{2}$, as the sign for $\cot 2y$ shows.

As to the values of y in general, the following may be observed. In (4), if $\frac{\pi}{2}>V>0$ is assumed, is always

$$1 \geqq 1-\sin^2 V \sin^2 \alpha > 0$$
$$1 \geqq 1-\sin^2 V \cos^2 \alpha > 0$$

For a given value of α $\cot 2y$ keeps the same sign, if x varies between 0 and π; it gets, however, negative values for x between 0 and $-\pi$. If we confine ourselves to a variation of x between the limits 0 and $\frac{\pi}{2}$, then the sign of $\cot 2y$ becomes negative for the values of α, lying between 0 and $\frac{\pi}{2}$; it becomes positive, however for $\frac{\pi}{2} < \alpha < \pi$, whilst the absolute values of y are equal for two poles, lying symmetrically with regard to the plane RO. The same thing holds good for the extinction on planes, lying symmetrically with regard to the plane OP, so that the isogyres drawn upon the globe will lie symmetrically with respect to the planes RO, OP and also RP. Just as the symmetry with regard to RO and OP is accompanied by a change of sign, so also for the plane RP.

The extinction with regard to the variable zone-axis OZ is easy to reduce to that with respect to the acute bisectrix, as the latter is yielded by $\angle ONc = \frown Qc = \frac{\pi}{2} - y = y'$.

$$y = \frac{\pi}{2} - y'$$

$$\cot 2y = \cot(\pi - 2y') = -\cot 2y'.$$

from which follows according to (3)

$$\cot 2y' = -\frac{A}{\sin x} - B \sin x \quad \ldots\ldots \quad (6)$$

in which:

$$A = -\frac{\cos^2 \mu + \sin^2 \mu \cos^2 \gamma}{\sin 2\mu \sin \gamma}$$

$$B = \frac{\sin^2 \mu}{\sin 2\mu \sin \gamma}.$$

For the determination of the greatest extinction with regard to the acute bisectrix with $x =$ constant and a variable angle α, we may set about as follows[1]). If we call $\angle ANO = \psi$, $\angle BNO = \psi'$, we find from the triangles ANO and BNO

$$tg\,\psi = \frac{\sin \angle AON}{\sin ON \cot V - \cos ON \cos \angle AON} =$$

$$= \frac{\sin V \cos \alpha}{\cos x \cos V - \sin x \sin V \sin \alpha}$$

and

$$tg\,\psi' = \frac{\sin V \cos \alpha}{\cos V \cos x + \sin V \sin x \sin \alpha}.$$

Now $2y' = \psi - \psi'$

$$tg\,2y' = tg(\psi - \psi') = \frac{-2 \sin^2 V \sin \alpha \cos \alpha \sin x}{\sin^2 V \cos^2 \alpha + \cos^2 V \cos^2 x - \sin^2 V \sin^2 \alpha \sin^2 x},$$

which gives:

1) A. HARKER, Min. Mag. XIII, 1903, p. 66—67.

$$\cot^2 \alpha (\sin^2 V + \cos^2 V \cos^2 x) - 2 \cot \alpha \sin^2 V \sin x \cot 2y' + (\cos^2 x - \sin^2 V) = 0$$

$$\cot \alpha = \frac{\sin^2 V \sin x \cot 2y'}{\sin^2 V + \cos^2 V \cos^2 x} \pm$$

$$\pm \sqrt{\left(\frac{\sin^2 V \sin x \cot 2y'}{\sin^2 V + \cos^2 V \cos^2 x}\right)^2 - \left(\frac{\cos^2 x - \sin^2 V}{\sin^2 V + \cos^2 V \cos^2 x}\right)}.$$

As long as the second term remains smaller than the first, the condition for which being $\cos x > \sin V$, or $x < \frac{\pi}{2} - V$, this equation will yield two positive roots, and accordingly two values of α between 0 and $\frac{\pi}{2}$ will satisfy it at a given value of $2y$ smaller than the maximum. The extinction will have reached the maximum, when the two roots are equal, so if

$$(\sin^2 V \sin x \cot 2y')^2 = (\sin^2 V + \cos^2 V \cos^2 x)(\cos^2 x - \sin^2 V)$$

or:

$$\sin 2y'_{max} = \frac{\sin V \, tg \, V}{\cos x \cot x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

whilst the corresponding value of α [1]) is found from

$$\cot \alpha_{max} = \frac{\sin^2 V \sin x \cot 2y'_{max}}{\sin^2 V + \cos^2 V \cos^2 x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The plane ON, in which lies the corresponding pole, then makes with the plane OP an angle $\left(\frac{\pi}{2} - \alpha\right)$.

If we take for olivine the value $2V = 87°$ [2]), this gives according to the above formulas the following figures:

TABLE I.

x	μ	γ	A	B
15°	48°19' 35"	13°48' 20"	—4.0852	2.3539
30	53 24 25	25 23 10	—2 1480	1.5708
45	60 52 23	33 51 47	—1.6103	1.6103
60	69 52 7	39 25 0	—1.5708	2.1482
75	79 44 14	42 30 39	—2.3548	4.0868

[1]) $\alpha_{(y'=max)}$ is denoted by α_{max}, wherever it could not give rise to ambiguity

[2]) Min. d. Roches, p. 248.

from which the following extinctions with respect to the acute bisectrix are calculated:

TABLE II.

x	Values of y' at $\alpha =$				
	$\alpha = 15°$	$\alpha = 30°$	$\alpha = 45°$	$\alpha = 60°$	$\alpha = 75°$
15°	1°53′ 6″	3°36′ 38″	4°53′ 14″	5° 8′ 26″	3°32′ 41″
30	4 4 28	7 56 59	11 14 42	12 54 21	10 16 52
45	6 49 59	13 42 48	20 38 46	27 27 29	33 6 56
60	10 14 10	20 52 53	32 32 7	46 20 3	64 40 45
75	13 32 29	27 22 46	41 47 36	57 4 59	73 14 21
90	15 (+18″)	30 (+11″)	45	60 (—4″)	75 (—1″)

The values for y' found by calculation with $x = 90°$, which accordingly represent the limit of extinction with regard to the acute bisectrix on the plane making right angles with the latter, give a measure for the exactitude of the values found. The errors successively amount to $+18''$, $+11''$, $0''$, $-4''$, $-1''$.

The greatest extinction for different values of x with the corresponding angle $\left(\frac{\pi}{2} - \alpha\right)$ are now to be calculated from the formulas (7) and (8); we come to the following result:

x	y'_{max}	$\frac{\pi}{2} - \alpha$
15°	5°13′ 12″	34°36′ 4″
30	12 54 29	29 25 6
45	33 44 35	10 40 21
$\frac{\pi}{2} - V =$ 46°30′	45°	0°

To calculate from

$$\cot \alpha_{max} = \frac{\sin^2 V \sin x \cot 2y'_{max}}{\sin^2 V + \cos^2 V \cos^2 x}$$

the value of α_{max} when $\sin x = 0$, we eliminate y'.

As

$$\sin 2y'_{max} = \frac{\sin V \, tg \, V}{\cos x \, \cot x}$$

we get

$$\cot \alpha_{max} = \frac{\dfrac{\sin^2 V \sin x \sqrt{1 - \left(\dfrac{\sin V \, tg \, V}{\cos x \, \cot x}\right)^2}}{\dfrac{\sin V \, tg \, V}{\cos x \, \cot x}}}{\sin^2 V + \cos^2 V \cos^2 x} =$$

$$= \frac{\sin^2 V \sin x \sqrt{\cos^2 x \cot^2 x - \sin^2 V \, tg^2 \, V}}{\sin V \, tg \, V (\sin^2 V + \cos^2 V \cos^2 x)} =$$

$$= \frac{\sin x \cos V \sqrt{\dfrac{\cos^4 x}{\sin^2 x} - \dfrac{\sin^4 V}{\cos^2 V}}}{\sin^2 V + \cos^2 V \cos^2 x} =$$

$$= \frac{\sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}}{\sin^2 V + \cos^2 V \cos^2 x} \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

When $x = 0$, is

$$\cot \alpha_{max} = \pm \cos V.$$

From which for olivine follows the value:

$$\left(\frac{\pi}{2} - \alpha\right) = bg \, tg \, (\pm) \cos V =$$
$$= bg \, tg \, (\pm) \cos 43°30'$$
$$= (\pm) \, 35°57'22''.$$

In the following figure these results are graphically represented. The black lines connect the poles of planes with equal positive, the lines in black and white those of planes with equal negative extinction. Herein the angles have been considered positive from the acute bisectrix in the direction of the hands of the clock; negative in the opposite direction.

The curves MM' and NN', going through the optic axes, connect the poles of the planes with the greatest (positive and negative) extinction and with the same inclination with regard to the acute bisectrix. The point in which the curves mentioned intersect an isogyre, has on that isogyre the greatest angulardistance from O. For the rest very little need be added to what is to be read from the figure. It shows clearly that an extinction with regard to the acute bisectrix, which deviates little from 0°, is confined to the immediate neighbourhood of the principal planes of symmetry.

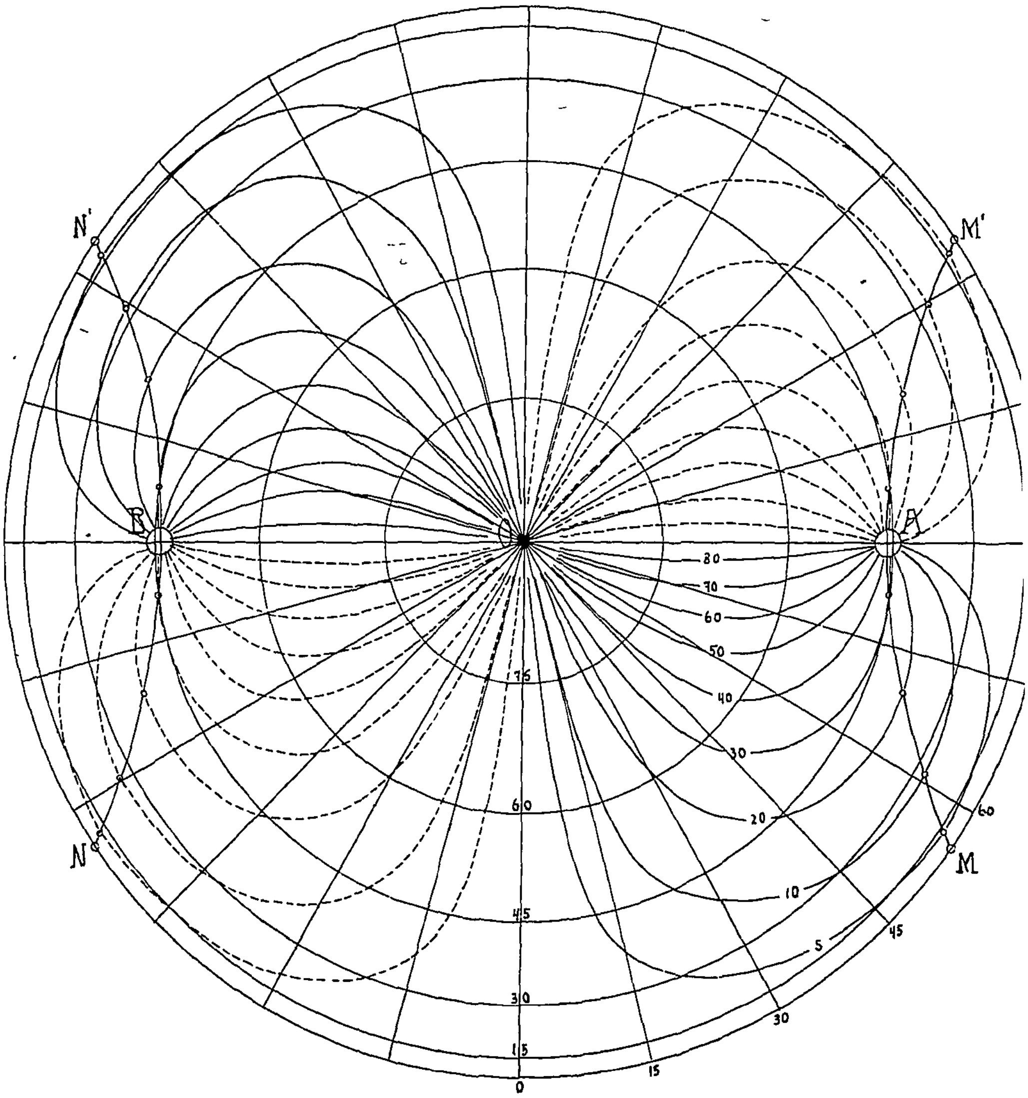


Fig. 2

For talc ($2V = 13°$) 1) the following values are calculated in the same way:

1) HINTZE, Handbuch der Mineralogie, II, 1897, p. 815, cf. BAUER, Pogg. Ann. 1869, CXXXVIII, 368.

	μ	i	A	B
15°	83°43′ 21″	1°42′ (± 1′)	— 156:05	154.32
30	84 22 26	3 15	— 89 846	89 268
35	84 40 45	3 44	— 82 815	82 45
40	85 1 25	4 11	— 78 895	78 718
45	85 24 31	4 36	— 77 628	77.628
50	85 49 38	4 59	— 78 717	78.891
55	86 16 38	5 20	— 82 31	82 672
60	86 45 19	5 38	— 89 268	89 846
65	87 15 5	5 53 30″	— 100.62	101 46
70	87 46 48	6 6	— 120.16	121 38
75	88 19 16	6 16 30	— 154 32	156.05
80	88 52 25	6 24	— 225.49	228.14
85	89 26 4	6 28 30	— 443.61	449 28

to which correspond the extinctions:

x	$\alpha = 15$	$\alpha = 30$	$\alpha = 35$	$\alpha = 40$	$\alpha = 45$	$\alpha = 50$	$\alpha = 55$
15	0° 3′ 4″	0° 4′ 54″	—	—	0° 6′ 9″	—	—
30	0 7 9	0 12 44	—	—	0 14 46	—	—
45	0 8 12	0 26 53	—	—	0 31 19	—	—
60	0 36 55	1 4 59	—	—	1 16 39	—	—
70	1 21 38	2 26 12	2°40′ 54″	2°51′ 34″	2 57 13	2°57′ 48″	2°52′ 14″
75	2 17 13	4 11 16	4 39 53	5 1 35	5 15 26	5 21 1	5 17 6
80	4 23 11	8 24 35	9 31 30	10 33 35	11 23 43	12 1 43	12 24 4
85	9 27 9	19 10 28	22 33 26	26 2 27	29 36 9	33 23 25	37 31 53
90	15°(+53″)	30°(—50″)	—	—	45°	—	—

x	$\gamma = 60$	$\gamma = 65$	$\gamma = 70$	$\gamma = 75$	$\gamma = 80$	$\gamma = 85$	$\gamma = 90$
15	0° 5′ 21″	—	—	0° 3′ 14″	—	—	0°
30	0 12 52	—	—	0 7 28	—	—	0
45	0 27 21	—	—	0 15 56	—	—	0
60	1 7 37	—	—	0 39 55	—	—	0
70	2 41 57	2°26′ 6″	2° 4′ 20″	1 7 40	1° 7′ 27″	0°34′ 26″	0
75	5 1 13	4 35 53	3 58 51	3 9 35	2 12 10	1 7 58	0
80	12 19 51	11 53 14	10 58 49	9 9 38	6 42 42	3 33 47	0
85	41 34 43	46 43 1	53 20 59	59 24 19	67 38 33	78 3 57	90
90	60°(+51″)	—	—	75°(— 8″)	—	—	90°

For the greatest extinction and corresponding angle we find:

x	$y'_{max.}$	$\frac{\pi}{2} - \gamma$
0°	0°	44°48′ 56″
15	0° 6′ 9″	44 48 31
30	0 14 47	44 41 24
45	0 31 21	44 27 2
60	1 16 50	43 38 28
75	5 21 33	39 38 9
$\frac{\pi}{2} - V =$ 83°30′	45°	0°

Figure 3 affords a general view of the results. Suppose the part of the globe-surface, falling outside the parallel-circle of 60° but within the isogyre of 1°, to be equal to the part falling within the same circle outside the isogyre, then it appears that at about $^7/_8$ of the sphere an extinction of less than 1° is observed, so practically a straight extinction. Now the sections, yielding greater extinctions, lie so much in the neighbourhood of the planes, making right angles with the optic axes, that they are for the greater part impracticable for the determination of the direction of extinction. A comparison of figures 2 and 3 shows the result that with rhombic

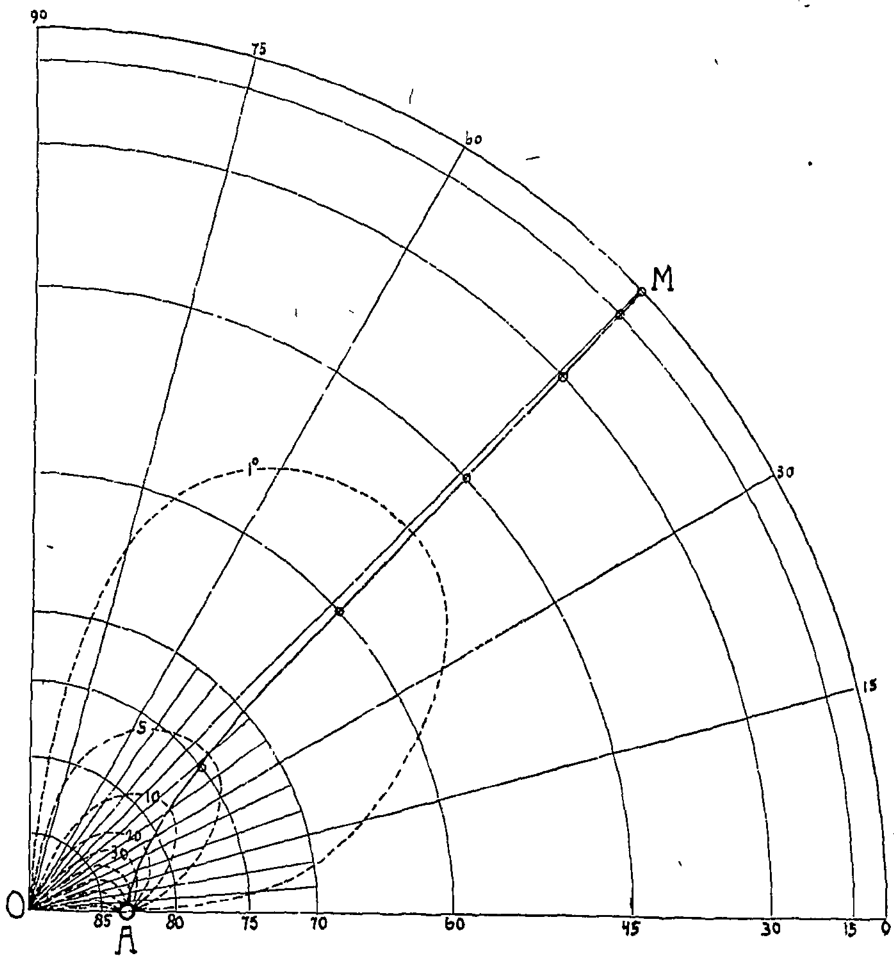


Fig. 3.

crystals with a great axis-angle the oblique, with those with a small axis-angle the straight extinction will predominate, when we have to do with arbitrary sections, as in a rock slice. However in the two cases we as rarely find an absolutely straight extinction. With hexagonal and tetragonal crystals, however, exclusively straight extinction with regard to the optic axis occurs, as for $V = 0$ the equation (4)

$$\cot 2y = \frac{1}{\sin^2 V}\left(-\frac{1 - \sin^2 V \sin^2 a}{\sin 2a} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cdot \cos^2 a}{\sin 2a} \sin x\right)$$

always becomes ∞.

In fig. 4 the maximum extinction as a function of x is represented for one globeoctant; MA_1 refers to talc, MA_2 to olivine, MA_3 to a mineral with an axis-angle $2V = 160°$. The values $O'A_1$, $O'A_2$

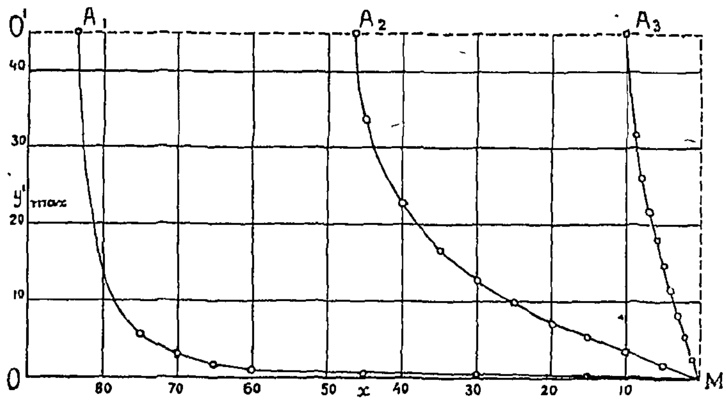


Fig. 4.

and $O'A_3$ give the size of V. The general èquation of the curves MA is:

$$\sin 2y'_{max} = \frac{\sin V \, tg\, V}{\cos x \cot x},$$

$$y'_{max} = \tfrac{1}{2}\, bg \sin\left(\frac{\sin V\, tg\, V}{\cos x \cot x}\right)$$

from which

$$\frac{dy'_{max}}{dx} = \frac{\sin V\, tg\, V \cdot}{2\sqrt{1-\left(\dfrac{\sin V\, tg\, V}{\cos x \cot x}\right)^2}} \cdot \left(\frac{1+2tg^2 x}{\cos x}\right) =$$

$$= \frac{\sin V\, tg\, V\,(1+2tg^2 x)\cot x}{2\sqrt{\cos^2 x \cot^2 x - \sin^2 V\, tg^2 V}} =$$

$$= \frac{\sin^2 V}{2} \cdot \frac{1+\sin^2 x}{\cos x \sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}} \quad . \; . \; . \; . \; . \; . \quad (10)$$

So for $x = 0$ the direction of the tangent is given by:

$$\frac{dy'_{max}}{dx} = \mathop{(-)}^{+} \frac{\sin^2 V}{2\cos V}.$$

For $x = \frac{\pi}{2} - V$ by

$$\frac{dy'_{max}}{dx} = \infty$$

as the term under the root-mark becomes $= 0$. So the tangents in A on the curves form right angles with the direction MO.

It further follows from the formula (10) that the rise of the curve for the same value of x grows smaller as the value of V diminishes, as is also shown by fig. 4. If V becomes $= 0$, as in the hexagonal and tetragonal system, then we also have

$$\frac{dy'_{max}}{dx} = 0\,,$$

so that the curve MA coincides with the abscissa-axis MO. Finally with regard to the form of the curve which represents the angle $\left(\frac{\pi}{2} - \alpha_{max}\right)$ as a function of x, it appears already from a comparison of figures 2 and 3, that this curve MA, the axis-angle becoming smaller, gradually approaches the straight line that divides into two equal parts the angle between OA and the normal to it in O.

Indeed (9) yields

$$\cot \alpha_{max} = \frac{\sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}}{\sin^2 V + \cos^2 V \cos^2 x}\,,$$

$$= tg\left(\frac{\pi}{2} - \alpha_{max}\right) = \frac{\sqrt{\cos^4 x - (\cos^4 x + \sin^2 x \sin^2 V) \sin^2 V}}{\cos^2 x + \sin^2 x \sin^2 V}\,.$$

If V becomes smaller, $tg\left(\frac{\pi}{2} - \alpha_{max}\right)$ increases, and with $V = 0$ reaches the greatest value $\overset{+}{(-)}$ 1, so that then α_{max} becomes $= 45°$. The curves $M'AM$ and NBN' then pass into two straight lines which intersect in O, thus forming right angles, whilst they have shifted 45° with regard to direction AB.

Of great practical importance is the solution of the problem, how

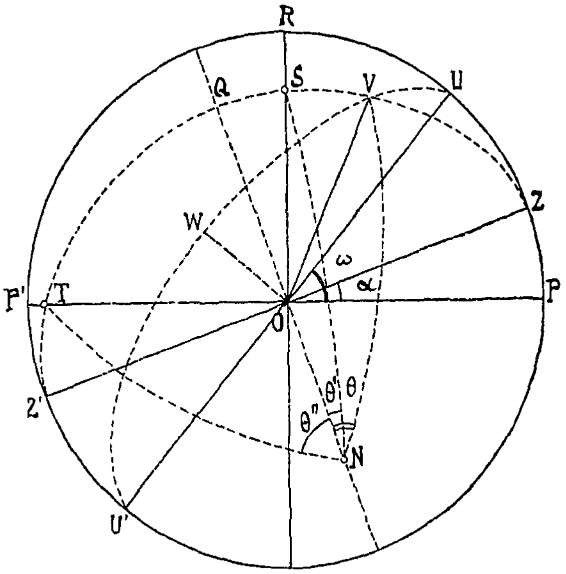


Fig 5.

great the extinction is with regard to the trace of a cleavage plane. For this, if the angle of extinction with regard to the acute bisectrix be known, is only necessary the value of the apparent angle between the trace mentioned and the same bisectrix. One value need only be subtracted from the other.

If ZZ' be the axis of a zone, in which ZQZ_1 represents an arbitrary plane, N the corresponding pole, and be the plane determined again by a and $OQ = x$; if UWU' be an arbitrary cleavage plane, determined by ω and $WO = y$, then VO is the line of intersection of both planes, $VQ = \angle QNV = \theta$, the apparent angle between the acute bisectrix (O) and VO.

Now $VQ = \frac{\pi}{2} - VZ$, and in ΔVUZ is

$$\angle Z = \frac{\pi}{2} - x$$

$$\angle U = \frac{\pi}{2} + y$$

$$UZ = \omega - a.$$

So

$$\cot VZ = tg\,\theta = \frac{\sin\left(\frac{\pi}{2} - x\right)\cot\left(\frac{\pi}{2} + y\right) + \cos\left(\frac{\pi}{2} - x\right)\cos(\omega - a)}{\sin(\omega - a)}$$

$$= \frac{-\cos x\, tg\, y + \sin x \cos(\omega - a)}{\sin(\omega - a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If we apply this formula to the cleavage planes h^1 (100) and g^1 (010) of olivine, then with

$$h^1\,(100) \;.\;.\;.\quad \omega = 0\,,\ y = 0$$

$$g^1\,(010) \;.\;.\;.\;.\quad \omega = \frac{\pi}{2}\,,\ y = 0$$

and (11) passes into:

$$tg\,\theta' = \sin x\, tg\, a$$

$$tg\,\theta'' = -\sin x \cot a$$

in which θ' and θ'' are successively the apparent angles between the traces of g' (010) and h' (100) on the plane (N).

Now if we think both x and a to vary between 0 and $\frac{\pi}{2}$, we find the following values for θ' and θ'':

$x=0$	$x=15$	$x=30$	$x=45$	$x=60$	$x=75$	$x=90$	φ at θ' (+)
indef.	90°	90°	90°	90°	90°	90°	90°
0°	86° 8′ 30″	88° 0′ 2″	88°35′ 50″	88°50′ 43″	88°57′ 53″	89	89
0	82 18 57	86 0 17	87 10 22	87 41 27	87 55 46	88	88
0	78 33 11	84 0 59	85 45 40	86 32 13	86 53 39	87	87
0	74 52 5	82 2 19	84 21 9	85 23 2	85 51 34	86	86
0	71 19 25	80 4 30	82 56 43	84 13 53	84 49 28	85	85
0	55 44 4	70 34 28	75 59 53	78 29 29	79 39 17	80	80
0	44 0 26	61 48 38	69 14 48	72 48 29	74 29 45	75	75
0	24 8 46	40 33 34	50 46 7	56 18 35	59 7 56	60	60
0	14 30 39	26 33 55	35 15 53	40 53 36	44 0 24	45	45
0	8 29 56	16 6 8	22 12 28	26 33 54	29 8 50	30	20
0	3 58 2	7 37 51	10 43 43	13 3 52	14 30 38	15	15
0	2 36 57	5 2 18	7 6 25	8 40 56	9 39 57	10	10
0	1 17 49	2 30 17	3 32 24	4 19 58	4 49 49	5	5
0	1 2 13	2 0 9	2 49 50	3 27 56	3 51 50	4	4
0	0 46 38	1 30 4	2 7 20	2 35 55	2 53 52	3	3
0	0 31 4	1 0 1	1 24 52	1 43 56	1 55 55	2	2
0	0 15 32	0 30 0	0 42 26	0 51 53	0 57 57	1	1
0	0°	0°	0°	0°	0°	0	0

s is also shown in the table

$$\theta' = bg\ tg\ sin\ x\ tg\ \alpha$$

mes:

$x = 0$, $\alpha < \frac{\pi}{2}$ equal to 0,

$x = 0$, $\alpha = \frac{\pi}{2}$ indefinite, and

$\alpha = 0$ also every time 0, whilst

$$-\theta'' = bg\ tg\ sin\ x\ cot\ \alpha$$

$x = 0$, $\alpha > 0$ equal to 0,

$x = 0$, $\alpha = 0$ indefinite, and lastly

$\alpha = \frac{\pi}{2}$ always becomes 0.

ı order to know the extinction of the plane (N) with respect to trace of g' (010) or h' (100), we combine this table with table II.

For $\frac{\pi}{2} \geqq \alpha \geqq 0$, $\frac{\pi}{2} \geqq x \geqq 0$ the extinctions successively become $(-\theta' + y')$ and $(-\theta'' + y')$; we find the following values:

$$\varphi' = y' - \theta^1.$$

α	$x=0$	$x=15$	$x=30$	$x=45$	$x=60$	$x=75$	$x=90$
0°	0°	0°	0°	0°	0°	0°	0°
15	0	— 2° 4′ 56″	— 3°33′ 23″	— 3°33′ 44″	— 2°49′ 42″	— 0°58′ 9″	0
30	0	— 4 53 18	— 8 9 9	— 8 29 40	— 5 41 1	— 1 46 4	0
45	0	— 9 37 25	—15 19 13	—14 37 7	— 8 21 29	— 2 12 48	0
60	0	—19 0 20	—27 59 13	—23 18 38	— 9 58 32	— 2 2 57	0
75	0	—40 27 45	—51 31 46	—36 7 52	— 8 7 44	— 1 15 24	0
90	indif.	—90°	—90°	—90°	0°	0°	0

When $\alpha = 90°$ and $x = \left(\frac{\pi}{2} - V\right) = 46°30'$ the extinction becomes indefinite; it is here that the transition from 90° to 0° takes place, In the same way we find for

$$\varphi'' = y' - \theta''.$$

α	$x=0$	$x=15$	$x=30$	$x=45$	$x=60$	$x=75$	$x=90$
0°	indefinite	90°	90°	90°	90°	90°	90°
15	0°	45°53′ 32″	65°53′ 6″	76° 4′ 47′	83° 2′ 39″	88° 2′ 14″	90
30	0	27 45 24	48 50 33	64 28 55	77 11 28	86 30 42	90
45	0	19 23 53	37 48 37	55 54 59	73 25 43	85 48 0	90
60	0	13 38 22	29 0 29	49 39 57	72 53 57	86 13 49	90
75	0	7 30 43	17 54 43	43 50 39	77 44 37	87 44 59	90
90	0	0°	0°	0°	90°	90°	90

Also here the extinction for $\alpha = 90°$ and $x = \left(\frac{\pi}{2} - V\right) = 46°30'$ becomes indefinite.

The shape of the φ-isogyres is represented in fig. 6. The black lines refer to φ'', those in black and white to φ', φ'' gives the value of the positive extinction with regard to the trace of h' (100), φ'

that of the negative extinction with reference to the trace of g' (010) on the plane (N).

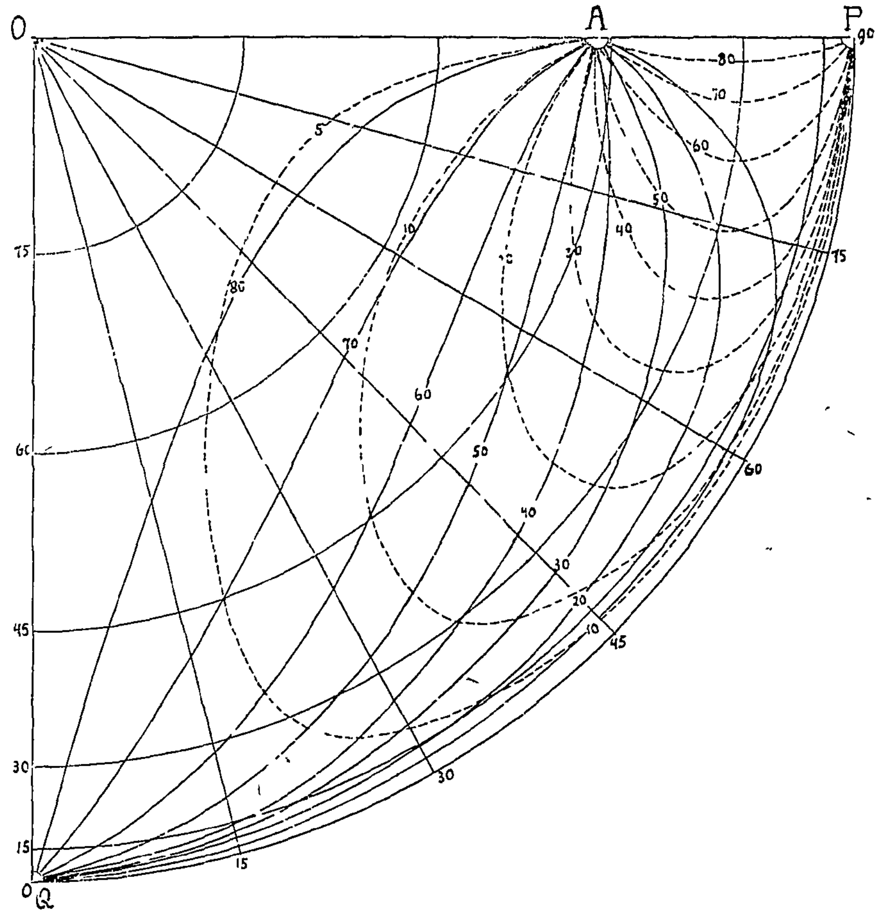


Fig. 6.

The figure is with regard to the axes OP and OQ quite symmetrical again, in the same way as fig. 2 shows for the isogyres with respect to the acute bisectrix, because the points here correspond to the value:

$$\varphi' = \tfrac{1}{2}\, bg \cot \left(\frac{1 - \sin^2 V \sin^2 \alpha}{\sin 2\alpha \,.\, \sin^2 V} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cos^2 \alpha}{\sin 2\alpha \sin^2 V} \,.\, \sin x \right) -$$

$$- \, bg \, tg \, (\sin x \, tg \, \alpha) = y' - \theta'.$$

$$\varphi'' = \tfrac{1}{2}\, bg \cot \left(\frac{1 - \sin^2 V \sin^2 \alpha}{\sin 2\alpha \,.\, \sin^2 V} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cos^2 \alpha}{\sin 2\alpha \sin^2 V} \,.\, \sin x \right) +$$

$$+ \, bg \, tg \, (\sin x \cot \alpha) = y' - \theta''.$$

26*

Now the signs of y', θ' and θ'' remain unchanged for $0 < x < \pi$ and for α varying between 0 and $\frac{\pi}{2}$, so that also here the sign of the extinction in adjoining globe octants will be alternately positive and negative. The points in which equivalent isogyres in the systems φ' and φ'' intersect, indicate the places of the poles of the sections, in which symmetrical extinction with respect to the cleavage planes h' (100) and g' (010) visible in the slice is observed. So there

$$\varphi' + \varphi'' = 2y' - (\theta' + \theta'') = 0.$$

$$2y' = \theta' + \theta''$$

The curve $OA = V$. If the axis-angle diminishes, A gradually approaches O; the isogyres φ' and φ'' approach a symmetrical direction with regard to the axes OP and OQ, so that the curve, connecting their points of intersection, draws nearer and nearer to the straight line, and at last, when V has become $= 0$, and A coincides with O, passes into the straight line which divides the angle POQ into two equal parts. For $V = 0$ y' becomes $= 0$, so:

$$\theta' + \theta'' = 0$$

$$\sin x \, tg \, \alpha - \sin x \cot \alpha = 0$$

$$tg \, \alpha = \cot \alpha$$

so that the geometrical place of the points of intersection of the isogyres φ' and φ'', i.e. that of the points of symmetrical extinction, is represented by the line

$$\alpha = 45^\circ.$$

Anatomy. — *"On ascending degeneration after partial section of the spinal cord."* By Dr. S. J. DE LANGE. (Communicated by Prof. C. WINKLER).

(Communicated in the meeting of November 30, 1907).

The following researches have been made with the purpose of investigating whether there exist any connections between the spinal cord and the ascending fasciculus longitudinalis dorsalis, and on the other hand to ascertain once more the course of the ascending anterolateral fascicle of GOWERS and its relation to the dorso-lateral fasciculi of the cerebellum.

From the extensive literature on this subject I will but rarely quote something, whenever the results obtained by others do not accord with my observations. The drawings are taken after four animals used in the experiments: a full-grown rabbit, in which a hemisection was made at the origin of the medulla oblongata (fig. 1) and three young cats, on which lesions were produced in different portions of the spinal cord (fig. 2, 3 and 4), indicated by C^7 C^4 and D^4.

The central nervous system of these animals was treated by the MARCHI-method, and afterwards cut into serial sections 25 μ thick.

In all these animals a plainly visible degeneration was found after the operation in the tractus spino-cerebellaris anterior or the fascicle of GOWERS. In the dorso-lateral fasciculus there is likewise found a certain degree of degeneration, but far less intense, excepting only in the first animal on which the hemisection was made near the origin of the medulla oblongata. For in this latter case there is found a very compact degeneration of the corpus restiforme, being caused however not only by spino-cerebellar fibres, but likewise and for a great part by bulbo-cerebellar fibres, that were injured when the lesion was produced [1]).

Figures 5—11 represent the degeneration as observed in my preparations. In the first sections we see the dorsal and the ventral spino-cerebellar system still undivided and taking a longitudinal course. Gradually the fibres of the dorso-lateral portion are deviating from the longitudinal direction, whilst the ventral portion still continues to follow it. In a slow spiral-line the dorsal fibres run towards the corpus restiforme, consequently by and by several important portions of the medulla oblongata are found to be encompassed between the two parts of the lateral fasciculi ad cerebellum. This fact becomes most plainly evident in fig. XV, taken from the first cat, on which an almost complete section had been made at C^7. There being no degeneration of the olivo-cerebellar tract in this case, the demarcation between corpus restiforme and antero-lateral fascicle is much more distinct.

I have represented from both these animals the radiation of the corpus restiforme into the cerebellum (dorsal portion of the lateral fasciculi to the cerebellum), and that of the fascicle of GOWERS (antero-lateral portion of the lateral fasciculi to the cerebellum): fig. 10 and 11, 17 and 18. The first radiation is produced through the pedunculus ad cerebellum inferior, that of the fascicle of GOWERS through the pedunculus ad cerebellum superior.

In order to reach this latter the fibres of the tractus spino-cerebel-

[1]) To understand this rightly, the exact place of lesion must be more accurately circumscribed. The lesion was made just underneath the most caudal root of the hypoglossus, and directed somewhat obliquely upward. Haemorrhage was still to be traced in those sections where the olivary body shows the largest profile.

laris anterior, that continued for a long while their longitudinal course, begin to assume an oblique dorsal direction in the region of the oliva superior and at the first appearance of fibres belonging to the corpus trapezoides. They cross the fibres of the corpus trapezoides in dorso-lateral direction, in consequence of which they present in the sections a peculiar distribution by layers (fig. 9). Until the place of exit of the trigeminus their situation remains nearly unaltered, the fibres lying only more closely together, till they take a sudden turn in dorsal direction, somewhat frontal from the trigeminus, thus joining the course of the lateral ribbon (laqueus lateralis) and staying for the greater part at its outside. Through the pedunculus ad cerebellum the greater part of the fibres now reach the cerebellum, where they take a retrograde direction, extending in the shape of a fan into the vermis inferior. Only a few fibres follow the course of the remaining portion of the lateral laqueus, and reach the corpus quadrigeminum posticum of the same side. Some authors assert that fibres may be traced likewise unto the corpus quadrigeminum anticum (THOMAS, WALLENBERG, BIANCHI, COLLIER and others), but VAN GEHUCHTEN, EDINGER and MOTT agree on this point with LOEWENTHAL, who has been the first to give an accurate description of the fascicle of GOWERS.

As I remarked before, the image of the lesions of the spinal cord presents differences only as regards the dorsal portion of the lateral fasciculi to the cerebellum, the degeneration found in this portion being far less intense than it was in the case, where the section was made near the origin of the medulla oblongata. This result, however, must be ascribed not only to the absence of degenerated bulbo-cerebellar fibres, but partly also to the fact that a great number of fibres from the dorsal fascicle find a provisory termination at the passage of the spinal cord into the myelencephalon, as is shown by the preparations. Whilst therefore in the ventral portion we find almost without exception only long fibres. in the dorsal part of the lateral fasciculi of the cerebellum long and short fibres are intermixed, and very probably the connection is composed for the greater part of two neurones.

In all my preparations, some degeneration is observed likewise in the fasciculus longitudinalis posterior. This receptacle of numerous fibres, ascending and descending ones, originating in very different portions of the nerve-trunk, shows degeneration over the whole of its length, and it seems as if from this fascicle all nuclei of motor nerve-fibres are provided with afferent fibres, the MARCHI-granulae being found even in the motor roots. In this paper I will not digress longer on the possible significance of this MARCHI-granulation, though perhaps it may be of very great importance. Other methods of investigation will be more efficient in bringing this problem nearer to its solution. In fig. 19 and 20 this granulation is represented, as it appeared in a cat after a central lesion of the 4th dorsal segment (fig. 3). Yet I think I am fully justified in concluding from my preparations to the existence of long ascending fibres, that have their origin probably somewhere near the central canal. In the spinal cord these fibres then continue their course in the anterior fascicle, part of them also in the anterolateral fascicle (fig. 12). The former ones turn more centralward at the origin of the medulla oblongata, and issue in the fasciculus longitudinalis dorsalis. The second ones take a bent near the most caudal root of the hypoglossus, and proceed along the course of that root centralward to the fasciculus longitudinalis (fig. 13, 14 and 25). These ascending fibres are still found even in the nucleus of the N. oculimotorius, as is shown in fig. 26.

Ramon y Cajal also assumes the existence of these fibres, v. Gehuchten however has found ascending degeneration only after a lesion of the nucleus of Deiter, but he does not mention whether he has made observations on sections obtained after central lesions of the spinal cord. In the laboratory of van Gehuchten similar experiments were made by Lubouschine, by the injection of a drop of water. But it was his purpose to destroy part of the posterior horn in order to trace the origin of the anterolateral fascicle. He found this fascicle to be degenerated, but there did not exist any degeneration of the fasciculus longitudinalis, because the central portion of the spinal cord remained uninjured.

After the hemisection near the origin of the medulla oblongata, there appeared also degeneration of the tractus bulbo-cerebellaris, which takes its course from the olivary body towards the periphery of the corpus restiforme as fibrae arcuatae internae and externae. The greater part of these fibres are crossed, still a few of them originate on the same side.

Besides these fibres there are found in the basal portion of the formatio reticularis a great number of degenerated fibres, ascending longitudinally and probably belonging to the secondary sensible tract (fig. 7—11). Part of these fibres transgress the median line and seek the median ribbon (laqueus medialis) of the non-injured side, the rest of them are united near the oliva superior and continue their course in the laqueus lateralis more centralward than the fibres of the fascicle of Gowers. They may be traced into the corpus quadrigeminum posterius.

The significance of the granulation in the corpus trapezoides, which was to be observed in all cases, did not become clear to me, still I will not pass in silence the fact of the constant appearance of this granulation.

As the summary of my results, I find that after onesided lesion of the spinal cord an ascending degeneration is observed in the following systems:

1. The homo-lateral posterior columns, where it may be traced as far as into the nuclei of GOLL and BURDACH.

2. The lateral fasciculi to the cerebellum,
    a. the dorsal portion almost without exception only on the operated side,
    b. the antero-lateral portion on both sides, but still principally on the operated side.

3. The fasciculus longitudinalis dorsalis on both sides.

4. The corpus trapezoides on both sides (?)

The descending degeneration is represented in figures 21, 22, 23 and 24. It includes:

1. The anterior columns, principally on the operated side, probably centrifugal fibres from the fasc. long. dors.

2. The pyramidal lateral fasciculus on the operated side.

3. The tractus rubro-spinalis, in the lateral columns (VAN GEHUCHTEN).

4. De tractus vestibulo-spinalis, frontal of the anterior horn (EDINGER).

5. Fibres in the posterior columns, being situated partly along the sulcus longitudinalis posterior, and partly along the entering posterior roots, to all probability presenting a homologon to the oval area and the comma of SCHULTZE.

Physics. — *"Motion of molecule-systems on which no external forces act."* By Dr. O. POSTMA. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the Meeting of November 30, 1907).

§ 1. Up till now two ways have been mainly followed to show that a gas mass left to itself, on which no external forces act, in consequence of the collisions of the molecules will finally pass into a state, in which the molecules are probably about uniformly distributed over the vessel and possess MAXWELL's distribution of velocities.

The first is the method of BOLTZMANN, who assuming that the density all through the vessel is already the same, and further starting from the assumption that there is no regular arrangement of the molecules as regards the velocity, demonstrates that a certain

Fig. I. HEMISECTIO. MEDULLA OBL.

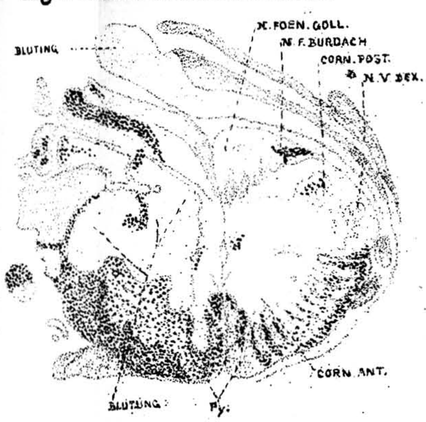


Fig. II. HEMISECTIO. IN C. 7

Fig. III. LAESIO CENTR. IN D. 4.

Fig. IV HEMISECTIO IN C. 4.

Fig. V

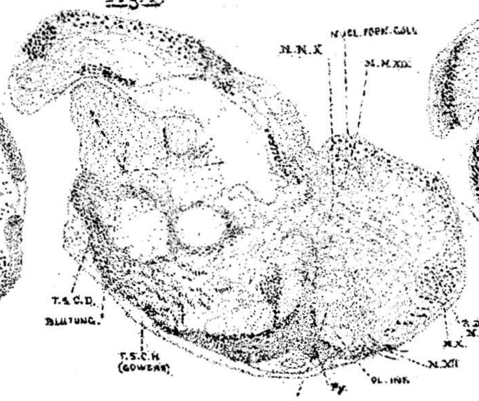


Fig. VI

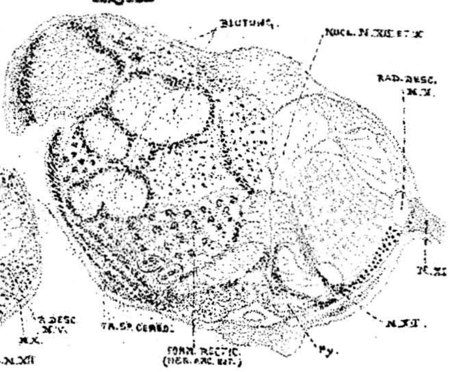


Fig. VII.

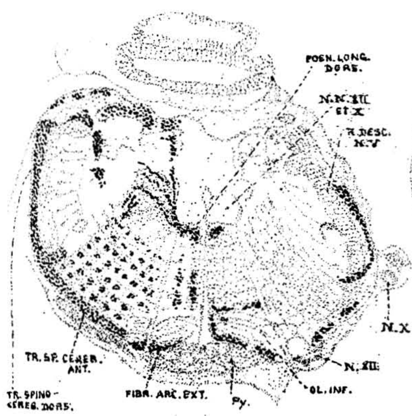


Fig. VIII.

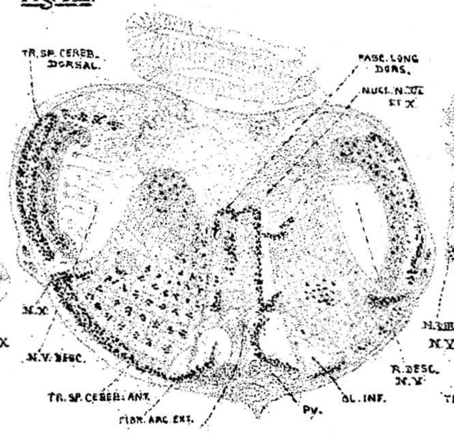


Fig. IX.

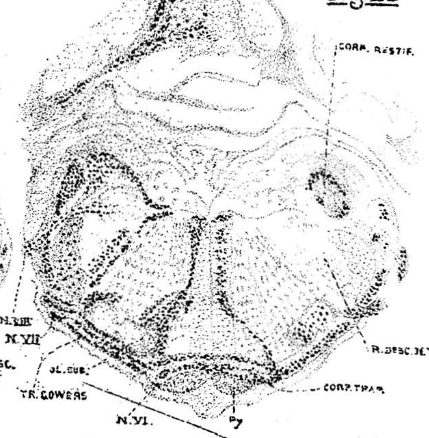


Fig. X

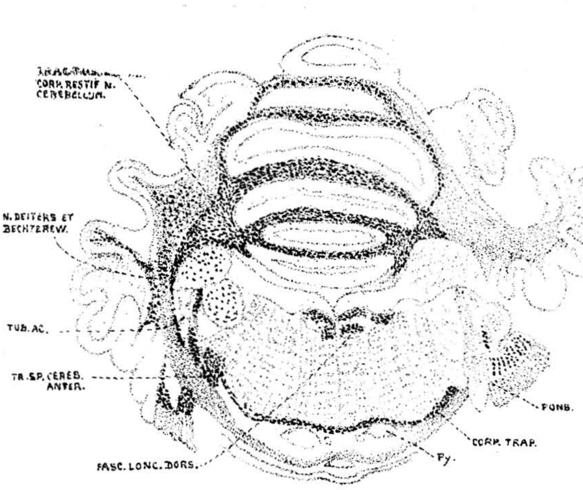


Fig. XI

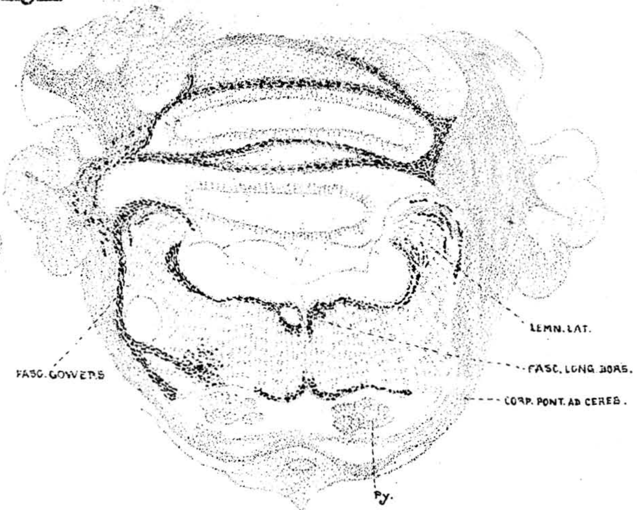


Fig. XII.

Fig. XIII.

Fig. XIV.

Fig. XV.

Fig. XVI. CORPUS TRAPEZOIDES.

Fig. XVII.

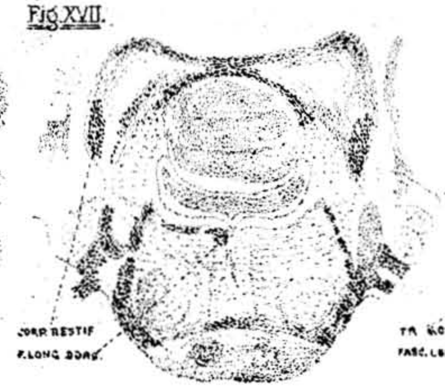


Fig. XVIII.

Fig. XIX.

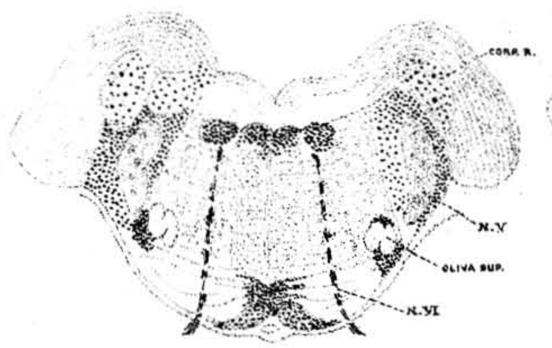


Fig. XX.

Fig. XXI.

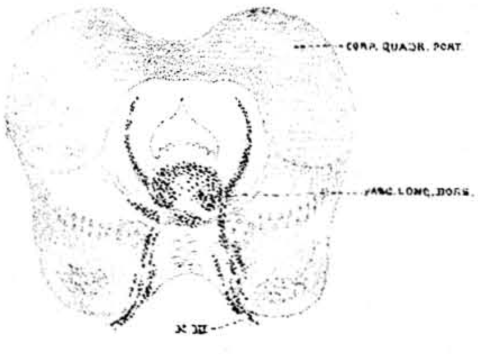
Fig. XXII.

Fig. XXIII.

Fig. XXIV.

Fig. XXV.

Fig. XXVI.



quantity $H = \int f.lf.d\omega$ constantly decreases by the collisions till the stationary state is reached, with which, as appears, MAXWELL's distribution of velocities exists.

The second is the method half followed by BOLTZMANN, and entirely by JEANS, by which it is demonstrated that on certain hypotheses the state with uniform density and that with Maxwell's distribution of velocities are the most probable. These hypotheses are, as regards the distribution of place, that every time there would be an equal chance to any place in the vessel for every molecule; with regard to the distribution of velocities that there would every time be an equal chance that the point of velocity of a molecule would get into any arbitrarily chosen volume-element, in which we should finally have to reckon with the fact that the total energy has a certain definite value.

I have tried to show[1]) that there is something contradictory in this, which might be avoided by assuming that the gasmass is arbitrarily chosen from a microcanonical ensemble, of which all the systems possess the energy which the gasmass must have. For in this all the combinations of place and all the combinations of velocities with the same energy are equally numerous, and so we have the same chance to hit upon them for the system chosen.

So another proof for the above mentioned result is furnished, when we show that an arbitrary ensemble of systems with the same energy, left to itself, passes into a microcanonical ensemble. GIBBS endeavours to demonstrate this in the XIIth Chapter of his "Statistical Mechanics"; the reasoning is made clearer by LORENTZ [2]), though the latter goes no further than calling the assumption that we should finally get a microcanonical distribution, very plausible.

However in a recent paper [3]) POINCARÉ called attention to a property in the light of which, in my opinion, the above reasoning is no longer tenable. There POINCARÉ shows, namely, that the quantity $S = \int P \log P dx_1 \dots dx_n$ (in which $x_1 \dots x_n$ represent the variables which determine every system of a certain ensemble, and $P = \frac{D}{N}$ the coefficient of probability, the integration being extended over the

[1]) These Proc. Febr. 21, 1906 and Jan. 24, 1907.

[2]) "Über den zweiten Hauptsatz der Thermodynamik"; Abhandlungen über Theoretische Physik, Leipzig 1906, p. 289.

[3]) "Réflexions sur la Théorie cinétique des gaz"; Journal de Physique, 1906, p. 369.

whole region occupied by the ensemble) is constant if the external circumstances are unchanged and the relation $\Sigma \frac{\partial X_i}{\partial x_i}$ exists (for which $X_i = \frac{dx_i}{dt}$). In this POINCARÉ takes as variables the coordinates and velocity-components of the molecules; so the quantity referred to above differs from the quantity $\bar{\eta}$ introduced by GIBBS only by a constant factor.

GIBBS shows that in a canonical ensemble $-\bar{\eta}$ has the properties of the entropy, POINCARÉ calls the quantity S itself entropy, also for an arbitrary ensemble. Hence this quantity will have to decrease, where $-\bar{\eta}$ increases.

The property in question may be derived as follows: P has the properties of a density, so:

$$\frac{\partial P}{\partial t} = -\Sigma \frac{\partial P X_i}{\partial x_i} = -\Sigma X_i \frac{\partial P}{\partial x_i} \left(\text{as } \Sigma \frac{\partial X_i}{\partial x_i} = 0\right).$$

Now, an arbitrary function of P will also behave as a density. Namely:

$$\frac{\partial f(P)}{\partial t} = f'(P)\frac{\partial P}{\partial t} = -f'(P)\Sigma X_i \frac{\partial P}{\partial x_i} = -f'(P)\Sigma X_i \frac{\partial P}{\partial x_i} - f(P)\Sigma \frac{\partial X_i}{\partial x_i} =$$

$$= -\Sigma X_i f'(P)\frac{\partial P}{\partial x_i} - \Sigma f(P)\frac{\partial X_i}{\partial x_i} = -\Sigma \frac{\partial [X_i f(P)]}{\partial x_i}.$$

So the $f(P)$ also satisfies an equation of the same form as the P itself, which equation represents the extension of the wellknown equation from hydrodynamics:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho u}{\partial x} + \frac{\partial \varrho v}{dy} + \frac{\partial \varrho w}{dz} = 0$$

to a space of n dimensions.

Now $S = \int P \log P d\tau$, (in which $d\tau = dx_1 \ldots dx_{n_1}) = \int f(P)\, d\tau$, is the integral of such an $f(P)$, integrated over the whole extension occupied by the ensemble. To ascertain the change of S with the motion of the ensemble, we must every time integrate over the variable space (though constant in size), over which the phases extend or where P and also $f(P)$ have values. So we can perfectly compare $\frac{dS}{dt}$ with the increase in time of a quantity of liquid, taken over all the places where it is. This increase, however, is equal to zero.

However, when $S = \int P \log P \, dx_1 \ldots dx_n$ is constant, the ensemble cannot move in the direction to the microcanonical distribution. For then P would become constant all over the phase-extension in course of time, and so the integral would get a minimum value.

Now, however, the question suggests itself: when the function S or $\bar{\eta}$ is constant, is there not another quantity characterising the ensemble, which by its variation in a certain direction indicates the motion of the ensemble in the direction of the uniform distribution of place and MAXWELL's distribution of velocities? For this motion is hardly open to doubt, and in a special case such a function has been found for one system by BOLTZMANN in the quantity H.

POINCARÉ supposes he has found such a function in his "entropie grossière", a quantity of the same form as S, but in which the elements of the area over which the summation is made, are not taken infinitely small, but so small that practically we cannot distinguish between systems lying within the same element. This quantity may, therefore, be represented by $\Sigma \Pi \log \Pi \, . \, \sigma$, in which σ represents the element and Π the mean density in it. In contradistinction with this entropy S is called the "entropie fine", and it may easily be shown that the "entropie grossière" is always smaller than the "entropie fine". It is less easy to see that the "entropie grossière" gradually decreases, nor does POINCARÉ prove this. For it is not easy to see that the quantity $S = \iint P \log P \, dl \, d\omega$, of which he tries to prove in some cases that it has decreased, represents an "entropie grossière", while the proof too rests on an assumption which is unjustified in my opinion. It is true that we shall demonstrate further on, that there are quantities of this form which decrease, but for them the name of "entropie grossière" is not very appropriate, as the elements of the extension over which the summation is made are just as well infinitely small, though of lower order of magnitude than the original elements.

§ 2. A very suitable introduction in the theory of gases is supplied by the *problem of the small planets*[1]) repeatedly treated by POINCARÉ. There the problem is discussed what in course of time the distribution along the ecliptic will become of a number of small planets, which at some time were placed in their orbit in such a way that chance has decided at least the distribution of the velocities. POINCARÉ shows

[1]) Cf. l. c. and also: "Calcul des Probabilités", Paris 1896 and "La Science et l'Hypothèse" Paris 1904.

that if the number is large, and the planets do not interfere with each other, in the long run the planets will most likely get about uniformly scattered over the ecliptic.

If we should wish to treat the problem in exactly the same way as GIBBS, we should have to consider an ensemble of systems each consisting of n planets. As, however, the planets do not interfere with each other, we may also take an ensemble of systems of only one planet, in which case the ensemble represents all possibilities which may occur in the placing of a planet. When now such an ensemble, satisfying certain simple conditions, gradually spreads uniformly over the ecliptic, there is for every planet, chosen at random from this ensemble, finally an equal chance to any place of the ecliptic, so that, if we have to choose a planet from such an ensemble n times, they will most probably be distributed about uniformly over the ecliptic, if n is large.

It is assumed that the orbits are circular, and lie in the plane of the ecliptic, so that every planet is determined by the variables l (length) and ω (angular velocity), in which ω is constant, and $l = l_0 + \omega t$, if also larger angles than 2π are admitted. The function S (POINCARÉ's entropie fine) is, accordingly, here $\iint P \log P \, dl d\omega$ integrated over all the phases.

As X_1 corresponding to l is equal to ω, and X_2 corresponding to ω is equal to 0, here $\sum \frac{\partial X_i}{\partial x_i} = \frac{\partial \omega}{\partial l} = 0$, so the function S remains constant.

Yet the ensemble approaches uniform distribution over the ecliptic, which, however, is an altogether different thing from the density P becoming constant. Then S would, of course, decrease (it must be observed that $\iint P dl d\omega$ and $\iint dl d\omega$ remain also constant). This approach to uniform distribution is perhaps most readily seen, when we consider only that part of the ensemble that had originally a length between l_0 and $l_0 + dl_0$, the angular velocities lying between ω_1 and ω_2. This part of the ensemble, being originally found in one point of the ecliptic, will get disintegrated by the different velocities, and gradually spread over the ecliptic, till finally the ecliptic is taken up a very large number of times. At a definite point of the ecliptic there are now parts, which were originally spread over a large number of elements of the extension, always at equal distances from each other, and it is easy to see, that if the function representing the original density, and its derivative are finite and continuous, the density along the ecliptic will finally be the same everywhere [1]). The adjoined fig. 1 represents the motion of the ensemble. Every point of the originally horizontal elementary area moves upwards in a vertical direction with constant velocity, so that the extension always occupies a slanting area with an inclination determined by $tg\,\alpha = t$.

The horizontal areas at distances 2π from each other indicate the parts of the extension, which are in the same point of the ecliptic. Originally these parts have been in parts of the original area at equal distances from each other. These distances become smaller and smaller, the surface elements becoming more numerous and at the same time smaller.

Instead of the constant quantity $S = \iint P \log P\, dl\, d\omega$ (l thought to be continuous) we get a variable, when we immediately take together the

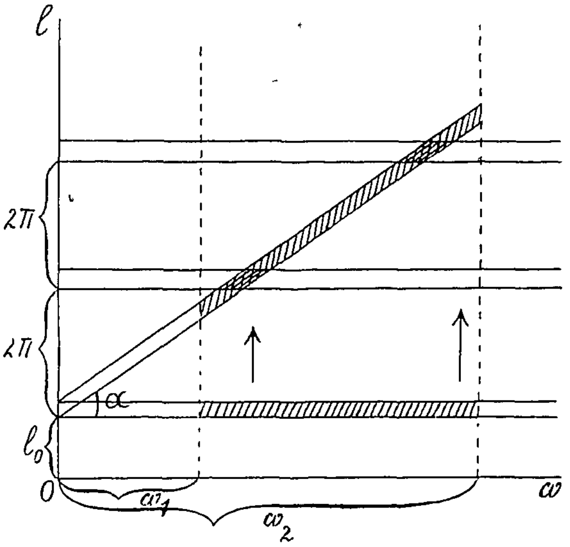


Fig. 1.

surface elements which come to the same thing with respect to the place in the ecliptic. So we get the quantity $S_p = \int_0^{2\pi} P' \log P'\, dl$, in which P' represents the quantity found per unit of length in the conjoint areas, of a width dl and at distances 2π apart, which give the same placing in the ecliptic. So in the case referred to, that originally a horizontal area whose width is dl_0 was occupied, $P'\, dl = \Sigma\, P_i\, dl_0\, dl \cot \alpha = \Sigma\, P_i \frac{dl_0\, dl}{t}$ so $P' = \frac{dl_0}{t} \Sigma\, P_i$, in which P_i re-

[1]) Cf. Poincaré l.c.

presents the density in each of the points of the original area which now come to the same thing.

Now this P'' becomes, as we saw, in course of time a constant and so S_p minimum. This quantity might be called *entropy of place*; "entropie grossière" seems less appropriate, because the elements of extension are infinitely small. This final approach to a minimum must not be taken as a continual decrease. It is easy to see that the original function of density might be chosen in such a way that there are also times, at least in the beginning, at which the densities P' rather diverge than draw nearer to each other. Then the quantity S_p would increase; so $\frac{dS_p}{dt}$ is not like BOLTZMANN's $\frac{dH}{dt}$ negative all through.

If, however, we compare times, in which first an angle 2π, then 4π etc. is occupied, we may say with a pretty high degree of certainty, that S_p has always diminished. If now instead of a horizontal area an arbitrarily chosen ensemble is considered, the above reasoning may be applied for every horizontal elementary area from it. So now too the ensemble is finally uniformly spread over the ecliptic, and the quantity $S_p = \int_0^{2\pi} P' \log P' \, dl$ becomes minimum. Now, however,

$P' \, dl = \Sigma \int P \, dl \, d\omega$ or $P' = \Sigma \int P \, d\omega$, every time integrated over all ω's which fall within the horizontal area determined by dl. This P' becomes finally constant. The motion of the ensemble in this more general case is expressed by fig. 2.

The above mentioned inaccuracy in POINCARÉ's reasoning is this: he considers the quantity $S = \iint P \log P \, dl d\omega$, in which he integrates with regard to l from 0 to 2π. So the P from this formula has evidently arisen by summation for the different values of l which come to the same thing, but not by integrating with respect to ω. Hence this P is the sum of the densities of the elements obtained by taking a definite ω, and then successively $l, l + 2\pi$ etc. In fig. 2 these elements are cross-hatched for one value of ω. However, POINCARÉ assumes further that finally for $t = \infty$ this P, or rather this ΣP no longer depends on l, but only on ω [1]). This seems to me incorrect. For then for every vertical elementary area in fig. 2 the sum of the densities in the elements, which are every

[1]) Réflexions sur la Théorie cinétique des gaz; p. 381, p. 385 etc.

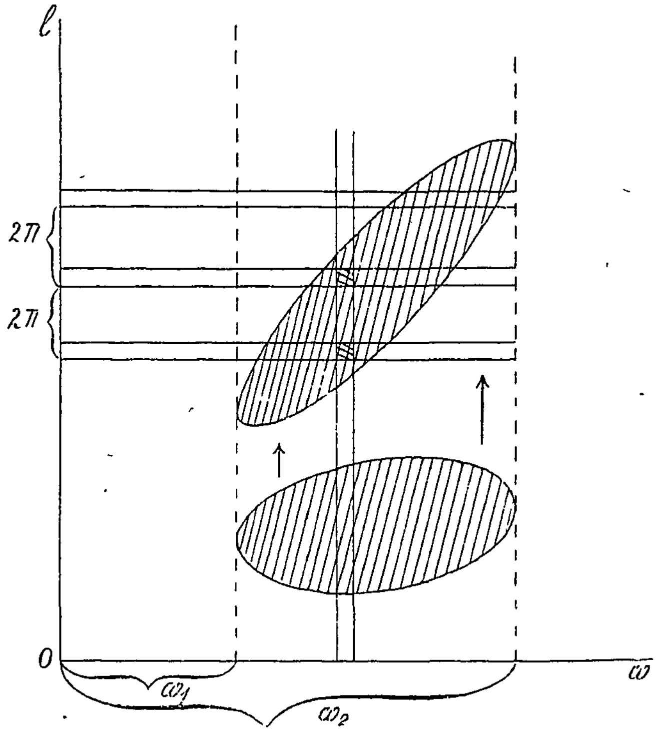


Fig. 2.

time at a distance 2π from each other, would finally have to become constant. This, however is only possible if the number of terms of this sum becomes at last infinitely large, which is by no means the case. Every vertical distance within the extension remains, namely, of the same length, so that the number of elements within each vertical area which are to be taken together, remains always finite. Only when the occurring ω's extend over a finite distance, the number of terms of the sum for $t = \infty$ can also become infinitely large.

A second partial entropy, that *with regard to the velocities* is obtained by taking these elements of the extension together which give the same velocity. So this is here $S_s = \int_{\omega_1}^{\omega_2} P'' \, log \, P'' \, d\omega$, in which $P'' = \int P \, dl$ is integrated along the vertical areas. This entropy, is indeed, also smaller than the "entropie fine", but the difference remains constant, and so also S constant.

§ 3. A transition case from that of the planets to that of the gas molecules is furnished by the case of a gas of one dimension. By this Poincaré understands a gas, all the molecules of which move parallel to each other to and fro between two parallel walls normal to the direction of motion of the molecules. We assume that perfectly elastic collision takes place, and that the dimensions of the molecules may be neglected. Now if we trace the whole way always in one direction, and introduce the distance to one of the walls as variable l, further consider the velocity ω as constantly positive, this case is identical with the preceding one, if we call the distance between the walls π, only with this difference, that now the distances $l, 2\pi - l, 2\pi + l$ etc. come to the same thing as regards the placing of the molecules. So twice as many areas must be taken together as before, and now we have only to integrate from 0 to π. Just as before we have now a quantity $S_p = \int_0^{\pi} P' \log P' dl$, which decreases because $P' = \Sigma \int P\, d\omega$ becomes at last a constant [1]).

Such molecules not interfering with each other in their motion, the case of a continuous ensemble of systems of one molecule each, and that of a real gas of n molecules is pretty much the same, just as for the planets. For a gas of three dimensions this is in general no longer the case in consequence of the collisions.

We meet with another transition case in an ensemble of systems consisting of n planets each. The "entropie fine" is now:

$$S = \int P \log P\, dl_1 \ldots dl_n\, d\omega_1 \ldots d\omega_n,$$

the S_p is given by

$$S_p = \int_0^{2\pi} \ldots \int_0^{2\pi} P' \log P'\, dl_1 \ldots dl_n,$$

in which $P' = \Sigma \int P\, d\omega_1 \ldots d\omega_n$, integrated with respect to $\omega_1 \ldots \omega_n$ and summed over all the combinations from l_1 to l_n which give the same arrangement of the planets. These combinations are obtained by combining the values l_1, $2\pi + l_1$ etc. with the values l_2, $2\pi + l_2$ etc. to l_n, $2\pi + l_n$ etc. in all ways possible. The number of these combinations, so the number of terms in the summation increases indefinitely during the motion, just as in the preceding cases; so P' becomes constant, and S_p approaches the minimum value. If

1) Here we might also have taken the coordinate x varying from 0 to π as variable instead of the continuous l. Then the ω shifts every time from $+$ to $-$ and vice versa, and we get the same terms $\int P d\omega$, but now at the same height x.

instead of the planets we imagine molecules moving in a one-dimensional motion, we get: $S_p = \int_0^{\pi} \cdot \cdot \int_0^{\pi} P' \log P' \, dl_1 \cdot \cdot \, dl_n$ in which l_1, $2\pi - l_1$, $2\pi + l_1$ etc. must be taken together in the calculation of P'.

We meet with still another transition case when we consider an ensemble of systems of one molecule each, moving freely in a vessel having the shape of a right-angled parallelopiped with the edges a, b and c. This motion may be thought as composed of three motions parallel to the edges, which we may each treat as in the first transition case. If we call the coordinates with respect to the sides x, y and z, the combinations of x, $2a—x$, $2a+x$ etc. with y, $2b—y$, $2b+y$ etc. and z, $2c—z$, $2c+z$ etc. come to the same thing with regard to place. So we now get the P' by integrating for the P with regard to the components of the velocity, and by then summing for all these combinations. The $S_p = \int_0^a \int_0^b \int_0^c P' \log P' \, dx \, dy \, dz$ decreases again till the minimum value is reached. A molecule chosen at random from the ensemble will have an equal chance to any place in the vessel. When the dimensions of the molecules are not neglected planes take the place of the walls of the vessel at a distance r parallel to them.

If we may disregard the collisions of the molecules inter se of a gas mass, we might always consider each of the n molecules as chosen arbitrarily from such an ensemble, and hence at last these n mole cules would probably be distributed over the vessel about uniformly.

§ 4. Finally we shall consider an ensemble of systems consisting of n molecules moving in a vessel having the shape of a right-angled parallelopiped. If we take the coordinates with regard to the side faces as variables, the components of the velocities may also be negative and the representing point of a system may occupy any place within the space

$$\int_r^{a-r} dx_1 \ldots dx_n \int_r^{b-r} dy_1 \ldots dy_n \int_r^{c-r} dz_1 \ldots dz_n \int_{-\infty}^{+\infty} d\xi_1 \ldots d\xi_n \int_{-\infty}^{+\infty} d\eta_1 \ldots d\eta_n \int_{-\infty}^{+\infty} d\zeta_1 \ldots d\zeta_n$$

in which we need only take into consideration that during the motion the kinetic energy or also the Σv^2 remains constant. So we shall now have to examine which parts of the phase extension will give the same arrangement of the molecules, and which the same molecular velocities, and to ascertain whether the P integrated and summed over these spaces becomes constant in course of time.

It stands to reason that here the problem of the distribution of velocities will be the simplest, because it is modified directly by the impact, and the distribution of place only indirectly.

In agreement with § 2 where we considered an area from the original extension lying between l_0 and $l_0 + dl_0$ in order to examine the placing, we shall take here a part from the extension determined by limits of velocity lying infinitely near each other, but covering a finite part of the $3n$-dimensional space of coordinates. In connection with the condition that $\Sigma v^2 = C$, we take from the $3n$-dimensional space of velocities an element of a spherical shell, whose radius is $\sqrt{\Sigma v^2}$. To this corresponds a prismatic or cylindrical part of the extension, the base of which is represented by the element in question. With regard to the distribution of velocities the points from these and similar prismatic or cylindrical tubes come to the same thing. The elements of the spherical shell represent the projection of the tubes on the space of velocities. Now it remains to investigate whether the quantity of substance, which originally is found above the element mentioned in a given tube, will not finally have spread uniformly over all the tubes, so that the same quantity will be found above every element of the same size. If so, $S_1 = \int P' \log P' d\tau$ will again become minimum, if $d\tau$ represents the size of such an element, and $P'd\tau$ the quantity which is projected in $d\tau$.

We may also call the points from the element of the spherical shell the points of velocity of the systems, and the vector, which joins the origin with such a point of velocity, represents all the velocities of the system both with regard to magnitude and to direction; the projections of the vector on the $3n$ axes of the space are the components of the molecular velocities.

The best way of setting forth the gradual uniform dispersion of these points over the mentioned hypersphere is perhaps by availing ourselves of BOREL's mode of representation, and by partially following his method. [1])

BOREL imagines that in the same $3n$-dimensional space in which the coordinates of the molecules are laid out (so that we get in this way a point representing the total arrangement for every system) also the components of velocity are projected starting from the representing point mentioned. The vector then obtained represents the velocity

[1]) "Sur les Principes de la Théorie cinétique des gaz" par EMILE BOREL. Annales de l'Ecole Normale Supérieure IIIe Série, 1906, N°. 1, p. 9.

of the representing point, and we may now examine what takes place with this vector during the motion. It is now obvious that the representing point moves in a space inclosed by surfaces and spaces as $x_1 = r$, $x_1 = a - r$ etc. with which it collides when one of the molecules of the system strikes against a wall of the vessel. When two molecules collide, the representing point strikes against a surface, the equation of which has the form:

$$(x_m - x_n)^2 + (y_m - y_n)^2 + (z_m - z_n)^2 = 4r^2.$$

Now BOREL shows that for this impact the same rules hold as for ordinary collisions, so that the lines along which the point moves before and after impact, lie in the same plane as the normal, the normal dividing the angle of the first two lines into two equal parts. The velocity, too, remains the same According to the above we must now imagine a finite space filled with such representing points, an infinitely small pencil of vectors of velocity starting from each point, mutually equal, and also equal for all the points. Now a number of these representing points strikes against one of the above mentioned surfaces, e.g. with equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 4r^2;$$

this implies then that for these systems the first and the second molecule collide. This surface is that of a cylindre of revolution of the $3n$—3^{th} degree, against the outside of which the points strike.[1]) The base of the cylindre is a sphere, the descriptive lines have here become descriptive spaces, namely plane spaces of $3n$—3 dimensions.

In the collision referred to, the extension from which the points come being large compared to the dimensions of the section of the cylindre (or at least of the same order) the infinitesimal pencil will extends in the directions of the perpendicular section of the cylindre, so here in 3 dimensions, to a pencil of finite width. If from this we take again an infinitesimal part, it comes from a definite point of each section of the sphere, and so from the points of a descriptive space of the cylindre.[2]) Part of this strikes again against another cylindre (which e.g. involves collision of the 1st and the 3rd molecule), and the infinitely narrow pencil extends again to finite width; etc.

The representing points which have not taken part in these collisions strike again against another surface, and the pencil extends every time to one an infinite number of times wider, but every time in other

[1]) If the new coordinates $\xi, \xi', \eta, \eta', \zeta$ and ζ' determined by $\xi = \frac{1}{2}(x_1 - x_2)\sqrt{2}$, $\xi' = \frac{1}{2}(x_2 + x_2)\sqrt{2}$ etc. are introduced, the equation of the surface becomes $\xi^2 + \eta^2 + \zeta^2 = 2r^2$.

[2]) Properly speaking a narrow region in the direction of this descriptive space.

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spaces. However, when in a system the same molecule has had a great number of collisions, the extension has had a projection on the space determined by the axes of the coordinates of the 1st molecule for each of these collisions. So when every molecule has had a great number of collisions the above mentioned vector of velocity has passed round the sphere on which the points of velocity lie, a great many times in every direction. The points of velocity which originally covered an element of the spherical shell, will now occupy the whole spherical surface many times. As, however, the points of the sphere, where a point of velocity is after one, two etc. revolutions in a certain direction, come to the same thing with respect to the distribution of the velocities of the molecules, as in all the preceding cases we may again assume that finally the density is the same all over the spherical surface.

For the rest of the elements of the same spherical shell originally occupied the same reasoning holds. If there are also systems in the ensemble with another kinetic energy, the points disperse also here homogeneously in spherical layers; as, however, one kind cannot pass into another, the density may be different for the layers. It is the same as in the distribution of place when the gas masses are in different vessels. [1])

Now the problem of the placing of the molecules. For this purpose we consider a part of the phase-extension, originally determined with regard to place by limits lying indefinitely near each other, but occupying a finite part of the space of velocities. Now we have to demonstrate

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[1]) Also without BOREL's way of representation the above mentioned dispersion may be imagined to a certain extent. In each of a number of systems the molecules have the same velocities, but different places. Now it will entirely depend on the mutual situation e.g. of the molecules 1 and 2 in connection with their velocities, what the direction of the normal becomes in the collision, and so up to a certain extent. what the final velocities will be. In any case we get an infinitely large number from a single pair of velocities. Whereas we had before infinitely narrow limits between which the components of velocity had to lie, now we get a finite region. If we have chosen a definite one from the pairs of final velocities so also a definite velocity of the 1st molecule, this molecule may have all kinds of positions with regard to the 3rd molecule, against which it will presently strike, so also the normal of collision can have all kinds of directions, and so the limits thought infinitely narrow are again removed to a finite distance, etc If we now take as variables the angles of the general vector of velocity with the axes of velocity instead of the components of velocity, the moving apart of the limits will yield a larger amount of occupied angles, so that finally there is occupied an amount of a large number of times 2π. If we now take into account, that increase of such an angle by an amount 2π has no effect, we arrive at considerations and results of quite the same nature as in the problems treated above.

that finally this ensemble will be uniformly scattered over all the combinations of place. In this case there will again be a partial entropy S_p which has become minimum.

We may try to make use of BOREL's way of representation also here, and shall take as introductory case an ensemble of systems consisting of 2 molecules moving along the same line perpendicular to two parallel walls between these walls.

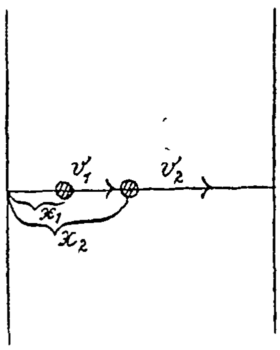

Fig. 3a

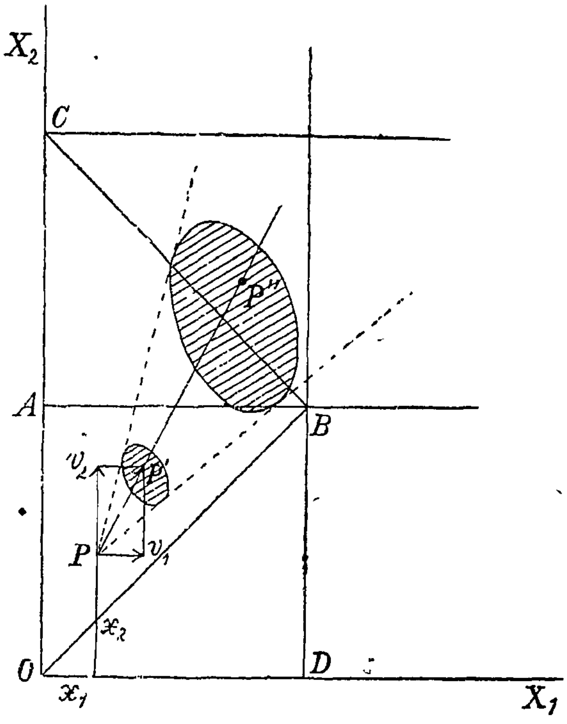

Fig. 3b

Fig. 3^a represents one system from the ensemble, fig. 3^b the motion of the whole ensemble, when the dimensions of the molecules are disregarded. Originally all the systems are in an element placed at $P(x_1, x_2)$, the points of velocity lying within an arbitrary figure. Here the representing points will be found after 1 sec., whereas they would occupy a continually extending figure if there was no collision. Collision of the molecules with the walls is here represented by collision of the representing points with the lines OA and AB, collision of the molecules inter se by collision with the line OB.

Now, however, we may also think the motion after the collision e.g. with AB continued past AB, if the triangle ABO which would then be passed through for the second time with the velocity after impact is thought to be turned over along AB. Now the point pursues its course uninterrupted. A following collision with OB now becomes collision with BC; now we may proceed again in the same way. The pencil then goes on without any disturbance, but we must take into account that the elements of surface, which in this way proceed from each other, represent the same placing of the two molecules. The continually extending figure of the representing points will finally contain a very large number of elements of surface of every kind coming to the same thing, or a very large number of turned over triangles, so that finally the points will be uniformly distributed over the sums of the elements of surface. So every *situation of the two molecules* represented by a point in ABO, occurs equally frequently. A point of BDO, however, is never reached: so every situation in which the 2[nd] molecule is on the right of the 1[st] is equally probable, but the 2[nd] cannot get to the left side of the 1[st].

Now we should have to extend this reasoning to the case of more than two dimensions. The reflection against the walls does not affect our reasoning. The striking of the molecules against each other, however, is now represented by striking against a cylindrical surface. Though this obstructs the way, it no longer shuts off a part of the space. The case may be compared with that of fig. 3_b if the line OB is replaced by a circle. I have not succeeded in solving the problem for this general case. However, it seems very plausible that the finite number of cylindres will not be able to prevent that the uniform distribution of the representing points over the sums of the elements of volume which come to the same thing, which distribution would finally come about as we saw in § 3 if there was no collision, will be established also now. So all the combinations of place of the molecules would then occur equally frequently.

§ 5. Finally it may still be shown that when all the combinations of place and all the combinations of velocity occur with equal frequency, it follows from this that for the great majority of the systems the molecules are distributed about uniformly over the vessel, and have MAXWELL's distribution of velocities. So far we have always distinguished between the individual molecules, now we shall have to take into consideration, that exchange of the molecules does not affect the *distribution* of place or velocity, so far as we can know it. So all the combinations which arise from each other by exchange of molecules, now come to the same thing. Hence if of s molecules s_1 are in the first element of volume, s_2 in the 2[nd] etc., there are

$\frac{s!}{s_1!\, s_2! \ldots s_n!}$ combinations yielding the same distribution of place. As BOLTZMANN has shown, the denominator may be represented by $ce^{\int f(xyz) \log f(xyz)\, dx\, dy\, dz}$ by approximation, in which $f(xyz)$ represents the function of distribution of the molecules over the vessel. The integral is minimum if $f(xyz) = C$, so the number of combinations is then maximum, or the uniform distribution is the most frequently occurring one. To show that the deviation from this distribution is not large as a rule, we may examine how many combinations yield a distribution, in which instead of $\frac{s}{n}$ molecules, $\frac{s}{n} + x_1$, $\frac{s}{n} + x_2$ etc. molecules occur in the elements. This number is:

$$\frac{s!}{\left(\frac{s}{n} + x_1\right)!\left(\frac{s}{n} + x_2\right)! \ldots \left(\frac{s}{n} + x_n\right)!}.$$

By putting $s! = s^{s+\frac{1}{2}} e^{-s} \sqrt{2\pi}$ etc., we get for this, taking into account that $x_1 + x_2 + \ldots x_n = 0$:

$$\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1}\left(1 + \frac{nx_1}{s}\right)^{\frac{s}{n}+x_1+\frac{1}{2}} \ldots \left(1 + \frac{nx_n}{s}\right)^{\frac{s}{n}+x_n+\frac{1}{2}}}.$$

Now by approximation

$$\log\left(1 + \frac{nx_1}{s}\right)^{\frac{s}{n}+x_1+\frac{1}{2}} = \left(\frac{s}{n} + x_1 + \tfrac{1}{2}\right)\left(\frac{nx_1}{s} - \frac{n^2 x_1^2}{2s^2} + \ldots\right) =$$

$$x_1 + \frac{nx_1}{2s} + \frac{nx_1^2}{2s} \ldots$$

So the $\log$ of the denominator (with the exception of the first factor) is by approximation:

$$\Sigma x + \frac{n\Sigma x}{2s} + \frac{n}{2s}\Sigma x^2 = \frac{n}{2s}\Sigma x^2.$$

The number of combinations becomes now:

$$\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1}} e^{-\frac{n}{2s}\Sigma x^2};$$

if we put $\Sigma x^2 = nu^2$, the chance, that the mean deviation is smaller than u, may be represented by:

$$c\,\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1}}\int_0^u e^{-\frac{\frac{u^2}{2s}}{n^2}}du=\frac{2}{\sqrt{\pi}}\int_0^{\frac{u}{\frac{1}{n}\sqrt{2s}}} e^{-t^2}\,dt.$$

Very soon, however, this value is very large, when u is only a few times $\frac{1}{n}\sqrt{2s}$ as yet; then u or the mean x, however, is still small compared to $\frac{s}{n}$, the mean number of molecules per volume element.

In a similar way the problem of the distribution of velocities may be treated. Here the denominator of the expression $\frac{s!}{s_1!\,.\,s_2!\,\ldots\,s_N!}$ may be reduced to $Ce^{\int f(\xi\eta\zeta)\log f(\xi\eta\zeta)d\xi d\eta d\zeta}$, in which $f(\xi\eta\xi)$ represents the function of distribution of the points of velocity. The integral is minimum, taking into consideration that Σv^2 is constant, when $f(\xi\eta\xi)=ae^{-b(\xi^2+\eta^2+\zeta^2)}$. Now it remains to investigate what is the chance to a given deviation from this distribution. We may define this deviation by the figures $x_1, x_1' \ldots x_2, x_2' \ldots$ etc. indicating the relative surplus of points of velocity in the elements of volume, respectively with the velocities v_1, v_2 etc. In the first element is then the quantity $s_1 = ae^{-bv_1^2}(1+x_1)$[1], in the second $s_2 = ae^{-bv_1^2}(1+x_1')$ etc., so that the number of combinations to be taken together:

$$\frac{s!}{[ae^{-bv_1^2}(1+x_1)]!\,[ae^{-bv_1^2}(1+x_1')]!\ldots[ae^{-bv_2^2}(1+x_2)]!\ldots}.$$

The first factor of the denominator is equal to (by approximation):

$$(ae^{-bv_1^2})^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times(1+x_1)^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times e^{-ae^{-bv_1^2}(1+x_1)}\times\sqrt{2\pi}=$$

$$=C_1\,(ae^{-bv_1^2})^{ae^{-bv_1^2}x_1}\times(1+x_1)^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times e^{-ae^{-bv_1^2}x_1}.$$

If we multiply by the other factors, the latter part vanishes, as $\Sigma ae^{-bv_1^2}x_1=0$. So we keep:

$$C\,(ae^{-bv_1^2})^{ae^{-bv_1^2}x_1}(1+x_1)^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times(ae^{-bv_1^2})^{ae^{-bv_1^2}x_1'}$$
$$(1+x_1')^{ae^{-bv_1^2}(1+x_1')+\frac{1}{2}}\times\ etc.$$

If we take the Nep. log. the former part vanishes, as also $\Sigma\,ae^{-bv_1^2}x_1\,.\,v_1^2=0$, so that log. denominator:

$$=\Sigma\,[ae^{-bv_1^2}(1+x_1)+\tfrac{1}{2}]\,log\,(1+x_1)\;+\;C.$$

1) This a is equal to the above one multipled by $d_\xi d_\eta d_\zeta$.

This is by approximation:

$$ae^{-bv_1^2}\left\{x_1 + \frac{x_1^2}{1.2} + \dots\right\} + ae^{-bv_1^2}\left\{x_1' + \frac{x_1'^2}{1.2} + \dots\right\} + etc. + C$$

or

$$ae^{-bv_1^2}\frac{\Sigma x_1^2}{1.2} + ae^{-bv_2^2}\frac{\Sigma x_2^2}{1.2} + \dots + C,$$

so that the whole form may be represented by

$$Ce^{-\left[ae^{-bv_1^2}\frac{\Sigma x_1^2}{1.2} + ae^{-bv_2^2}\frac{\Sigma x_2^2}{1.2} + \dots\right]}$$

or if we denote the most probable quantities per element by a_1, a_2 etc.:

$$Ce^{-\left[a_1\frac{\Sigma x_1^2}{1.2} + a_2\frac{\Sigma x_2^2}{1.2} + \dots\right]}$$

This exponent agrees perfectly with that obtained for the preceding problem. We may reduce the latter to the form $-\Sigma\left(\frac{ax}{\sqrt{2a}}\right)^2$, and the former to the form $-\Sigma\left(\frac{x}{\sqrt{\frac{2s}{n}}}\right)^2$; now they represent the negative sum of the squares of the absolute deviations divided by the root from twice the normal number. Just as in the preceding problem the chance to a combination of deviations for which the root from this latter sum does not amount to more than a few times unity is now very large. If we now take as measure for the deviation the mean relative deviation or $\sqrt{\frac{\Sigma x^2}{N}}$, we see that this value is very small compared to $\sqrt{\sum\frac{ax^2}{2}}$, so that this mean deviation will be very slight[1]).

To conclude we may still remark that in order to get in the end both uniform distribution of the molecules over the vessel and MAXWELL's distribution of velocities, originally both a finite part of the space of velocities and of the space of coordinates must have been occupied. Or there must be such an uncertainty as to the original situation and velocities of the molecules that we must consider possible a finite whole of combinations with regard to both. This finite whole of possible combinations constitutes the ensemble, which we follow in its course instead of the system unknown within certain limits.

[1]) It having been assumed in the calculation that a considerable number of points of velocity still occur in every element, we must not think of the whole of the space of velocities when estimating the number of elements N.

Crystallography. — "*On the permissible orders of the axes of symmetry in crystallography.*" By Prof. W. VOIGT at Göttingen. (Communicated by H. A. LORENTZ).

(Communicated in the meeting of November 30, 1907.)

In one of the articles of the second part of his collected papers, H. A. LORENTZ took up the question — which is equally important in both crystallography and crystalphysics — of the permissible order of an axis of symmetry of the first or the second kind. In this investigation he proceeds from the principle of the rational duplicate ratio, from which he first proves that it is consistent with itself, and therefore a suitable basis for crystallographical deductions.

The study of this interesting treatise led me to the thought, that for the purpose at hand another fundamental principle of crystallography — viz. that of the rational indices — might well form a more convenient starting point. The continuation of this thought led me to the following development, which, I believe, attains the end in view in a remarkably simple and short manner. I will prove for this useful fundamental principle, as LORENTZ did for the principle of the rational duplicate ratio, that it does not contradict itself and then derive from it the permissible orders of the axes of symmetry.

1. The principle of the rational indices, as is well known, is as follows.

If we select three arbitrary boundary surfaces of a crystal polyhedron and draw through any point O parallels to their lines of intersection to form a system of axes; if we choose further two other arbitrary positions through this system of axes, and then the intercepts of these planes upon the axes are

$$u = OA, \quad v = OB, \quad w = OC \quad \text{on the one hand,}$$
$$u' = OA', \quad v' = OB', \quad w' = OC' \quad \text{on the other,}$$

the principle of the rational indices maintains then, that,

$$\frac{u'}{u} : \frac{v'}{v} : \frac{w'}{w} = z_1 : z_2 : z_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

forms at all times a ratio of whole numbers.

In order that this principle should lead to no contradiction, it is necessary that if one proceeds from three *other* boundary surfaces of the polyhedron and uses *their* lines of intersection as the fundamental system of axes, then the polyhedral surfaces have *also on these axes* intercepts with the above mentioned relation, if the principle held with reference to the first system of axes.

The following consideration with reference to figure 1 proves that

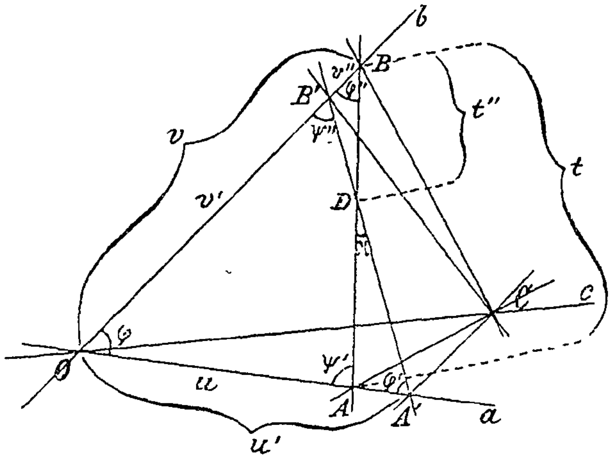

Fig. 1.

this is true. Oa, Ob, Oc form the first fundamental system of axes; ABC and $A'B'C$ represent the two planes for which the principle of the rational indices holds, i.e. their intercepts on the axes fulfil equation (1). For a simple proof it is essential that we let the two planes cut the Oc-axis in the same point, so that $w = w'$ and equation (1) assumes the form

$$\frac{u'}{u} : \frac{v'}{v} : 1 = z_1 : z_2 : z_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As second system of axes we take the lines BO, BA and BC, and as second pair of surfaces, which likewise cut the BC-axis in one point, the surfaces $A'B'C$ and AOC. If the principle is to lead to no contradiction, from (2) must follow

$$\frac{t''}{t} : \frac{v''}{v} : 1 = \zeta_1 : \zeta_2 : \zeta_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which the notation to the left is explained by the figure and ζ_1, ζ_2 and ζ_3 are likewise whole numbers.

If we understand by r and ϱ rational fractions we can expect that

$$\frac{u'}{u} : \frac{v'}{v} = r \text{ may follow from } \frac{t''}{t} : \frac{v''}{v} = \varrho,$$

while at the same time

$$\frac{v'}{v} = r' \text{ follows from } \frac{\overline{v''}}{v} = \varrho'.$$

The latter is apparent; for by the figure $v' + v'' = v$ and hence $r' + \varrho' = 1$, follows; consequently if r' is rational, then ϱ' is also rational.

For the former the proof follows by repeated application of the law of sines, which gives according to the figure

$$\frac{u}{\sin\varphi''}=\frac{v}{\sin\psi'}=\frac{t}{\sin\varphi},\quad \frac{u'}{\sin\psi''}=\frac{v'}{\sin\varphi'},\quad \frac{v''}{\sin\chi}=\frac{t''}{\sin\psi''},$$

while

$$\chi=\psi'-\varphi'=\psi''-\varphi'',\quad \pi=\varphi+\psi'+\varphi''=\varphi+\varphi'+\psi''.$$

From these we get

$$r=\frac{\sin(\varphi+\varphi')\sin(\varphi+\varphi'')}{\sin\varphi'\sin\varphi''},\quad \varrho=\frac{\sin(\varphi+\varphi')\sin(\varphi+\varphi'')}{\sin(\varphi+\varphi'+\varphi'')\sin\varphi}.$$

The relation between r and ϱ, is most easily obtained by determining φ' from the first formula and substituting this value in the second. We thus obtain

$$r/(r-1)=\varrho.$$

This shows that a rational r leads to a rational ϱ, which was to be proved.

The last part of the proof can still be simplified, according to a suggestion by LORENTZ, if we assume the MENELAUS' Theorem as known.

The desired proof is also given, when from

$$\frac{v'}{v}=r' \text{ and } \frac{u'}{u}=r''$$

$$\frac{v''}{v}=\varrho' \text{ and } \frac{t''}{t}=\varrho''.$$

follow.

The former of these we have considered above; relative to the latter, MENELAUS Theorem gives according to the figure

$$\frac{BD}{AD}=\frac{OA'}{AA'}\cdot\frac{BB'}{OB'},$$

i. e.

$$\frac{t''}{t-t''}=\frac{u'}{u'-u}:\frac{v-v'}{v'}.$$

Hence the rationality of u'/u and v'/v gives directly the rationality of t''/t.

2. The determination of the permissible order n of an axis of symmetry follows from any one of the fundamental principles of Crystallography, but only for the case when $n \geq 5$, because each of these principles places five similar crystallographic elements in relation. We usually so proceed, that the general property which the principle gives for the cases $n \geq 5$ is also demanded for the cases $n < 5$ We can however for the latter *limited* number of cases rely upon

experience, and apply the principle only for the former *unlimited* number of cases.

Since the principle of the rational indices permits surfaces of the crystal polyhedron to be translated parallel to themselves, therefore for its application axes of symmetry of the first and second kind have exactly the same value. A difference lies only in that for axes of the second kind n must necessarily be an even number.

We start with a construction upon a sphere of unit radius, through the center of which we lay all the directions that come into consideration. (Fig. 2). Let A be trace of the n-fold axis, $P_1, P_2 \ldots P_5$ the traces of the normals of 5 related surfaces (1), (2), ... (5) of the polyhedron, such that $\psi = 2\pi/n$. The P_h's are then designated as the *poles* of these surfaces.

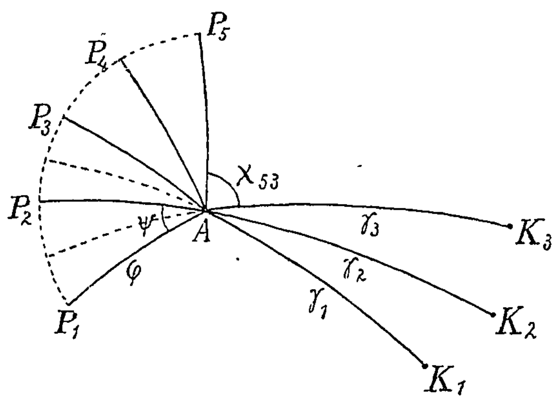

Fig. 2-

Let K_1, K_2, K_3, be the traces of the lines of intersection of the surfaces (2, 3), (3, 1), (1, 2), so that

$$K_1P_2 = K_1P_3 = \tfrac{1}{2}\pi, \quad K_2P_3 = K_2P_1 = \tfrac{1}{2}\pi, \quad K_3P_1 = K_3P_2 = \tfrac{1}{2}\pi.$$

Let K_1, K_2, K_3 form the system of axes, and (4) and (5) the pair of surfaces for the application of the principle of rational indices. It is now a question of determining the intercepts which the surfaces (4) und (5) make upon the edges K_i.

If we give the surfaces such positions that they are tangent to the sphere in their poles, then the sections σ_{hi} are identical with the reciprocals of the $\cos(K_i\ P_h)$ where $i = 1, 2, 3$, $h = 4, 5$. Consequently the values of these cosines are to be determined.

If we write γ_i, instead of AK_i, φ for AP_h and χ_{hi} for the $\angle\ P_h A K_i$ then from the $\Delta K_i A P_h$ in the figure we get

$$\cos(K\ P_h) = \cos\varphi \cos\gamma_i + \sin\varphi \sin\gamma_i \cos\chi^{hi}; \qquad (4)$$

further, from the ΔP_3AK_1, P_3AK_2, P_1AK_3 we get

$$tg\,\gamma_1 = tg\,\gamma_3 = \frac{ctg\,\varphi}{cos\,\frac{1}{2}\,\psi}\,,\; tg\,\gamma_2 = \frac{ctg\,\varphi}{cos\,\psi}\,, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

while direct from the figure we get

$$\left.\begin{array}{lll} \chi_{41} = \pi - \frac{3}{2}\psi, & \chi_{42} = \pi - 2\psi, & \chi_{43} = \pi - \frac{5}{2}\psi, \\ \chi_{51} = \pi - \frac{5}{2}\psi, & \chi_{52} = \pi - 3\psi, & \chi_{53} = \pi - \frac{7}{2}\psi. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we write the principle of rational indices

$$\frac{\delta_{41}}{\delta_{51}} : \frac{\delta_{42}}{\delta_{52}} : \frac{\delta_{43}}{\delta_{53}} = z_1 : z_2 : z_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and observe that in the quotients of each two δ_{hi}'s, since

$$1/\delta_{hi} = cos\,(K_iP_h) = cos\,\gamma_i cos\,\varphi\,(1 + tg\,\varphi\,tg\,\gamma_i\,cos\,\chi_{hi})$$

the factor standing before the brackets always cancel, then we can easily introduce the values (5) and (6) and obtain from (7)

$$\frac{cos\,\frac{1}{2}\,\psi - cos\,\frac{5}{2}\,\psi}{cos\,\frac{1}{2}\,\psi - cos\,\frac{3}{2}\,\psi} : \frac{cos\,\psi - cos\,3\psi}{cos\,\psi - cos\,2\psi} : \frac{cos\,\frac{1}{2}\,\psi - cos\,\frac{7}{2}\,\psi}{cos\,\frac{1}{2}\,\psi - cos\,\frac{5}{2}\,\psi} = z_1 : z_2 : z_3.$$

This gives directly

$$\frac{sin\,\frac{3}{2}\,\psi}{sin\,\frac{1}{2}\,\psi} : \frac{sin\,2\psi\,sin\,\psi}{sin\,\frac{3}{2}\,\psi\,sin\,\frac{1}{2}\,\psi} : \frac{sin\,2\psi}{sin\,\psi} = z_1 : z_2 : z_3 \quad . \quad . \quad . \quad (8)$$

If we now take the first and last members of this double proportion we have

$$\frac{sin\,\frac{3}{2}\,\psi}{sin\,\frac{1}{2}\,\psi} : \frac{sin\,2\psi}{sin\,\psi} = r$$

i. e. equal to a rational fraction, or also

$$\frac{1 + 2\,cos\,\psi}{2\,cos\,\psi} = r \text{ i. e. } cos\,\psi = \frac{1}{2(r-1)} = r',$$

where r' is also rational.

This requirement, when $\psi = 2\pi/n$ and $n \geqq 5$ is fulfilled only for $n = 6$.

If we introduce this value of ψ in equation (8) we get

$$2 : \frac{3}{2} : 1 = z_1 : z_2 : z_3 ;$$

which is entirely consistent with the double proportion.

Consequently $n = 6$ *is the only value* ≥ 5 *that is consistent with the principle of the rational indices.* If we extend the requirement that $\cos \psi$ must be rational to the case $n < 5$ then the values $n = 2$, 3, 4, are also permissible for axes of the first kind, and the values $n = 2$, 4 for axes of the second kind.

Göttingen, November 1907.

Physics. — "*Isotherms of diatomic gases and their binary mixtures. VI. Isotherms of hydrogen between* $-104°$ C. *and* $-217°$ C." (Continued). By Prof. H. KAMERLINGH ONNES and C. BRAAK. Communication N°. 100^a from the Physical Laboratory at Leiden.

(Communicated in the meeting of November 30, 1907).

§ 17. *Survey of the determinations. Remark on the apparatus.*

The measurements mentioned in this Communication comprise in the first place the supplementary determinations to which we already alluded in § 14 of Comm. N°. 99^a (Sept. 1907). These are three determinations at — 217° at a density of about 170 times the normal one.

The obvious thing to do further was to repeat the other determinations of series II with the same piezometer arranged for the determinations mentioned above, this piezometer being one of about the same dimensions as that of series II of Comm. N°. 97^a (March 1907). As a matter of fact a comparison of the values of pv_A obtained in this series with those yielded by the series III and IV teaches that the former lie somewhat, though only slightly, lower than the latter. This may be due to a systematical error as the filling in the later series was accomplished with more precautions (compare § 5 of Comm. N°. 97^a). In the series now given, just as in series IV, distilled hydrogen was used.

Both the steel tubes on the stem of the piezometer and those on the stem of the piezometer reservoir were soldered to the glass (cf. § 15 of Comm. N°. 99^a). This ensures a gas-proof connection with the steel capillary. With sealing wax it is difficult to make the connection gas-proof, because sometimes the nut begins to slide off when the flange is tightly screwed on.

§ 18. *Values of pv_A of series V.*

In table XX the results of the determinations have been represented in the same way as in table XII of Comm. N°. 97[a]. The temperatures at which the measurements were made were: —182°.74, — 195°.16, — 204°.62, — 212°.91 and — 215°.94. In table XX the reduction to the standard temperatures of table XII has been carried out. It was effected by interpolation by means of virial coefficients, which were derived in § 12 of Comm. N°. 97[a], which enabled us to abandon the elaborate method of § 8. The computation of the temperatures took place in the same way as in Comm. N°. 95[e] (Nov. 1906). They may be reduced to the normal hydrogen scale by means of table XVIII of Comm. N°. 97[b] (March 1907).

TABLE XX, H_2. Series V. Values of pv_A.

N°.	t_s	p	pv_A	d_A
1	— 182°.81	48 431	0 32746	147.90
2		55.499	0.32857	168.91
3		62 889	0.33028	190 41
4	— 195°.27	42.304	0.27362	154.61
5		47.782	0.27351	174.70
6		52.808	0 27360	193.01
7	— 204°.70	36.999	0.23165	159.72
8		41.258	0 23061	178 91
9		44.631	0.23001	194 04
10	— 212°.82	32.035	0.19414	165.01
11		34 611	0 19270	179 61
12		37.275	0 19149	194 66
13	— 217°.41	28.955	0 17318	167.63
14		31.191	0 17152	181.85
15		33.180	0.17005	195.12

§ 19. *Values of pv_A of series IV.*

If in the same way as in the preceding § the results of table XIX of Comm. N°. 99[a] are reduced to the standard temperatures the values of the adjoined table are obtained. For — 139°.88 this

TABLE XXI, H_2. Series IV. Values of pv_A.

N⁰.	t_s	p	pv_A	d_A
1	— 103°.57	28.447	0 63261	44.967
2		38.186	0.63702	59.944
3		48.724	0.64198	75.897
4		58.368	0.64694	90.222
5	— 139°.88	25.406	0.49459	51.368
6		33.774	0.49697	67.960
7		41.273	0.49967	82.600
8		48.558	0.50232	96.667
9	— 164° 14 [1])	22.818	0.40065	56.952
10		28.688	0.40164	71.427
11		34.387	0.40253	85.427
12		39 947	0.40376	98.936
13	— 182°.81	20.496	0.32704	62.670
14		24.818	0 32699	75.898
15		28.506	0.32672	87.248
16		32.568	0.32675	99.673
17	— 195°.27	18 527	0.27827	66.581
18		23.303	0.27724	84.055
19		27.837	0.27580	100.933
20	— 204°.70	16.749	0.24036	69.684
21		20.453	0.23876	85.658
22		24.015	0.23691	101.367
23	— 212°.82	15.416	0.20644	74.679
24		18.038	0 20430	88.296
25		20.643	0.20228	102.051
26	— 217°.41	14.635	0.18738	78.103
27		16.784	0.18491	90.766
28		18.853	0.18289	103.080

[1]) This temperature has been derived from the comparison of the resistance thermometer with the hydrogen thermometer of July 3, '07 (see table I, Comm. N⁰. 101a).

reduction has not been carried out, as it is better to take this ten perature as standard temperature instead of $-135^\circ.71$. Here th reduction would have to be made for a comparatively large difference c temperature, and would become inaccurate. It is, therefore, preferable t leave the values of table XIX for this temperature intact, and necessary apply the reduction to those of series I, which are muc less reliable. The temperature $-164^\circ.14$ has been adopted as ne standard temperature.

The determinations 5 and 9 as well as 14 and 18 of table XI have been united to a mean.

§ 20. *Comparison of the series I and II with the control-deter minations.*

For reasons mentioned in § 17 the points of series I and II hav been doubly determined in a mutually perfectly independent wa They can be easily compared with the control determinations c IV and V by reducing them to the same density and temperature b means of virial coefficients. If in this way for $-103^\circ.57$ Nos. 2, and 4 of series I (see table XII) are compared with 1, 2 and 3 c series IV (see table XXI), we find for the differences of pv_A fo IV—I respectively:

$$+0.00001 \quad , \quad +0.00007 \quad , \quad -0.00019$$

and for $-139^\circ.88$ for IV (5,6) — I (2,3):

$$-0.00085 \quad , \quad -0.00036.$$

Dealing in the same way with the series II and V (see table XII and XX), we find respectively for the temperatures $-182^\circ.81$ $-195^\circ.27$, $-204^\circ.70$, and $-212^\circ.82$,

V (1, 2) — II (2, 3) $= +0.00007, +0.00010$

V (4, 5, 6) — II (2, 3, 4) $= +0.00012, +0.00026, +0.00017$

V (7, 8, 9) — II (2, 3, 4) $= +0.00020, +0.00019, +0.00034$

V (10, 11, 12) — II (2, 3, 4) $= +0.00013, +0.00008, +0.00021$

The differences are to be ascribed chiefly to condensation o impurities, as they diminish with increase of the temperature. Thi was considered as sufficient ground to reject the results of the serie I and II for the further calculations as less reliable. This was als done for -104°, though the series I and IV harmonize very wel for this temperature. When we disregard the influence of th probable condensation the very regular course in the situation o the points is an indication about the accuracy of the measurement themselves also for the other isotherms.

So there remain the determinations of the series III, IV and V, which, reduced to the standard temperatures, occur in the tables XII, XX and XXI. With these data the further calculations have been carried out. Plate I gives a survey of the situation of the points in the diagram of isotherms; on this plate $\frac{pv_A}{T}$ has been given as function of the density. (T absolute temperature). By I and II the isotherms of 100°.20, and 0°, which will be treated in the following communication, are indicated, by the other Roman figures ascending with decrease of temperature, those to —217°.41.

§. 21. *Individual virial coefficients.*

In the same way as has been explained in § 12 of Comm. N°. 97[a] the first three virial coefficients of the development into series considered there were calculated for every isotherm, by means of the earlier and the new data. They have been put together in the subjoined table.

TABLE XXII. H_2 Individual virial coefficients.

t_s	A_A	$10^3 B_A$	$10^6 C_A$	$10^{12} D_A$	$10^{18} E_A$
—103°.57	0.62048	+0.24409	+0.5300	+0.9113	—0.648
—139°.88	0.48765	+0.11175	+0.4034	+0.6753	—0.378
—164°.14	0.39891	+0.00732	+0.4148	+0.4970	—0.208
—182°.81	0.33063	—0.07947	+0.3908	+0.3809	—0.088
—195°.27	0.28508	—0.12309	+0.3165	+0.2892	—0.016
—204°.70	0.25074	—0.17328	+0.3398	+0.2166	+0.031
—212°.82	0.22103	—0.22271	+0.3599	+0.1514	+0.066
—217°.41	0.20424	—0.24539	+0.3558	+0.1122	+0.082

In the same way as in Comm. N°. 97[a] the differences between the observed values of pv_A and those calculated by means of the found virial coefficients are put together in a table, which we subjoin.

The second column contains the differences for the points of the hydrogen thermometer (see table XII of Comm. N°. 97[a]), the following columns refer to the series IV, V and III in the order given here, the values being arranged according to the ascending densities for each series.

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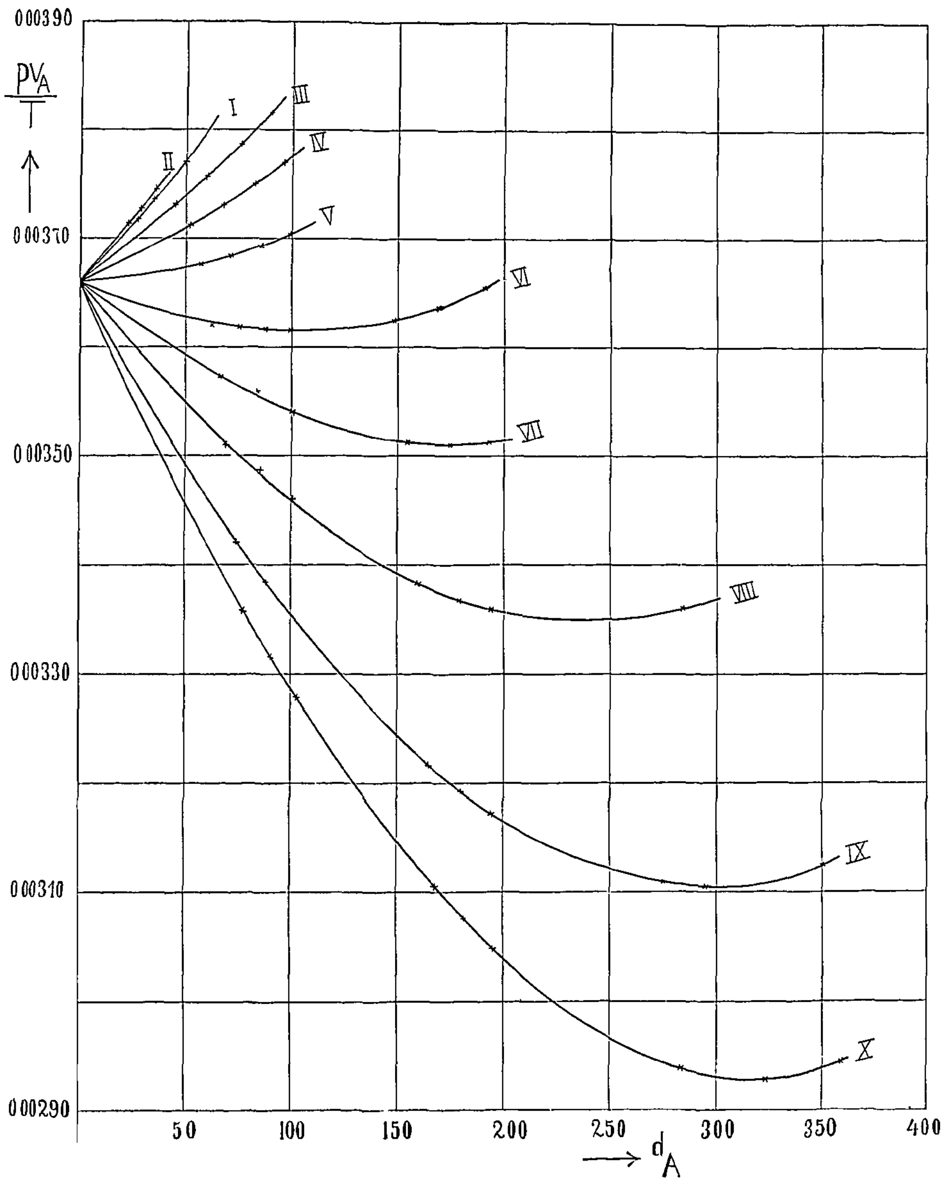
TABLE XXIII H_2. Deviations from the formula.										
t_s	$10^5(O_i - C_i)$									
—103° 57	— 1	+ 8	— 1	—11	+ 6					
—139°.88	— 2	+13	—15	0	+ 4					
—164° 14	0	— 3	+ 7	— 6	+ 2					
—182° 81	— 1	—15	+13	+ 3	+12	—15	— 9	+12		
—195° 27	— 4	— 2	+25	—11	—16	0	+ 8			
—204° 70	—18	+ 4	+36	+22	—22	—24	—21	+23		
—212° 82	—14	+ 3	+12	+22	— 5	—10	— 4	—22	+18	
—217° 41	-14	+13	0	+15	— 2	+ 1	— 3	—22	— 6	+18

It appears that on the whole series IV gives higher values than series V. The calculated curves may serve for the adjustment of the series mutually Undoubtedly their points will be more reliable than those of the separate determinations. In future we shall, therefore, start from the virial coefficients of XXII.

§ 22. *Minima of pv_A.*

With the now available data the minima of the pv_A curves were now again determined for the lowest five temperatures, and as before the coefficients P_0, P_1 and P_2 of a parabola calculated. The columns of table XXIV have the same meaning as those of table XV of Comm. N°. 97[a].

TABLE XXIV. H_2. Minima of pv_A.				
t_s	pv_A	d_A	p	$O—C$
— 182° 81	0 32663	99 70	32 57	—0 29
— 195° 27	0 27348	183 10	50 07	+1 36
— 204° 70	0 22945	238 27	54 67	—0 21
— 212° 82	0 18782	287 90	54 09	—0 84
— 217° 41	0 16342	321 51	42 54	+0 19
	0 30202	0	0	—0 23


0.00390
$\frac{PV_A}{T}$
0.00370
0.00350
0.00330
0.00310
0.00290
50
100
150
200
250
300
350
400
d_A
I
II
III
IV
V
VI
VII
VIII
IX
X

For the calculation of the coefficients of the parabola a sixth point was used which has been inserted at the bottom of the table, and was obtained by means of the isotherm of $-164°.14$. For this temperature the value of B_A is very slight, and by means of interpolation the BOYLE-point can be easily determined. For this is found, measured on the absolute scale:

$$\theta = -165°.72$$

to which corresponds a value of $pv_A = 0.39292$.

For the coefficients of the parabola we find:

$$P_0 = - \quad 14.8370$$
$$P_1 = + \quad 676.563$$
$$P_2 = - \quad 1624\ 31$$

The differences of the last column are slight, except for $-195°.27$. For this temperature C_A appears also to be too small (see table XXII).

Both deviations must be owing to the not quite accurate position of one or more of the points of the isotherms. From the diagram of Plate I · it is already to be seen that the middle point of series IV probably lies too high.

The parabola cuts the ordinate $p = 0$ in two points where pv_A is respectively $= 0.39330$ and 0.02323, with which agree the absolute temperatures:

$$T_1 = 6.3 \qquad T_2 = 107°.5.$$

For the top of the parabola $pv_A = 0.20826$, with which corresponds a pressure of 55.61 atmospheres. From this follows for the absolute temperature of the isotherm which passes through the top:

$$T = 64°.2.$$

Physics. — *"Isotherms of diatomic gases and their binary mixtures. VII. Isotherms of hydrogen between* 0° *C. and* 100° *C."* By Prof. H. KAMERLINGH ONNES and C. BRAAK. Communication N°. 100^b from the Physical Laboratory at Leiden.

(Communicated in the meeting of November 30, 1907).

§ 1. *Survey of the determinations.*

The reservoir of 5 cm.^{3} of the piezometer of series IV (Comm. N°. 99^a Sept. 1907) was replaced by one of 10 cm.^{3} With this apparatus two isotherms were determined, in ice and in vapour of boiling water. To obtain constant temperatures the same instruments were used as in Comm. N°. 60 (Sept. 1900). The water manometer (c.f. § 8 of Comm. N°. 27 (June 1896)) was read, but the difference of pressure amounting to no more than 0.5 mM., the corresponding correction for the temperature might be neglected. For the determination of the temperature of the steel capillary 3 thermometers were suspended along the capillary. For the determination of 100° a paper screen had been applied to turn off the rising current of heated air; the spiral with cold water, however, above the wool-packing of the boiling apparatus, had been omitted. The difference of temperature between the thermometers amounting to no more than 2°, this was permissible.

The corrected indication of the aneroid barometer amounted to 765.4 mM, from which for the temperature of the boiling-point follows 100°.20.

§ 2. *Values of* pv_A.

In the subjoined table the results of the determinations have been represented. The columns have the same meaning as in table XIX of Comm. N°. 99[a].

TABLE I. H_2. Values of pv_A.

N°.	t	p	pv_A	d_A
1	0°	27.333	1.01511	26.926
2		35.602	1.02002	34.903
3		43.413	1.02505	42.352
4		50.583	1.02964	49.127
5	100°.20	30 970	1.38619	22.342
6		39.796	1.39143	28.601
7		50.254	1.39788	35.951

§ 3. *Individual virial-coefficients.*

As has been done in § 12 of Comm. No. 97[a] we may avail ourselves of the data of table I to derive the first two virial coefficients for every isotherm. On account of the small densities which occur in these measurements, in formula (1) of § 12 of Comm. No. 97[a], D_A, E_A and F_A have been put $=0$, so that the formula reduces to:

$$pv_A = A_A + \frac{B_A}{v_A} + \frac{C_A}{v_A^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Only a small number of points having been given, and the densities being small, as was observed before, C_A cannot be determined with sufficient accuracy. We borrowed the values of this coefficient from Comm. No. 71 § 3, where $C_{A_0} = 0.0_6670$ and $C_{A_{100}} = 0.0_6606$.

In order to determine the course of the pv_A curves more accurately the value of pv_A was chosen for a density corresponding to that in the hydrogen thermometer of Comm. No. 60 with which (comp. Comm. No. 97*b* XV § 1) 0.0036629 was found for the pressure coefficient for hydrogen at 1090 mm. zero point pressure. By successive approximations this value of pv_A is to be derived by means of these determinations of isotherms. We find for 0°:

$$pv_{A\,0^\circ.1100\,mm.} = 1.000256$$

and for 100°.20 with the pressure coefficient 0.0036629 :

$$pv_{A\,100^\circ.2} = 1.367373.$$

For the density $d_A = 1.44$ may be put in both cases.

Now we obtain five values of pv_A and d_A for 0°, and four for 100°.20, from which by the method of least squares the coefficients A_A and B_A of equation (1) may be determined. These values are.

For 0°:

$$A_A = 0.99924$$
$$B_A = 0.5800 \times 10^{-3}.$$

For 100°.20:

$$A_A = 1.36626$$
$$B_A = 0.8632 \times 10^{-3}.$$

For 100°.00 we may calculate from this:

$$A_A = 1.36553$$
$$B_A = 0.8626 \times 10^{-3}.$$

The differences which remain between the values of pv_A of table I and those calculated according to formula (1) with the coefficients found here are respectively:

for 0°:

$$+0.00018, \quad -0.00023, \quad -0.00028, \quad +0.00004, \quad +0.00029$$

for 100°.20:

$$-0.00013, \quad +0.00034, \quad -0.00001, \quad -0.00019.$$

The first value always refers to the point calculated for the hydrogen thermometer. The differences are slight, and do not or only very slightly exceed $\frac{1}{4000}$ of pv_A.

Physics. — "*On the measurement of very low temperatures. XVII. Determinations for testing purposes with the hydrogen thermometer and the resistance thermometer.*" Communication N°. 101^a from the Physical laboratory at Leiden by Prof. H. KAMERLINGH ONNES, C. BRAAK and J. CLAY.

(Communicated in the meeting of November 30, 1907).

§ 1. *Introduction.*

In communication N°. 95^e (Nov. 1906) the results of a number of measurements are recorded which show the possibility of measuring temperatures down to $-217°$ with the hydrogen thermometer accurately to $\frac{1}{50}$ deg. The results obtained with several fillings showed that with our measurements to $-217°$ this accuracy has been reached indeed. It was our plan to make also the following measurements:

1st. more testing determinations between 0° and $-217°$ in order to establish still better the limit of the accuracy of the temperature measurements with the hydrogen thermometer and the accuracy of the definition [1]) once for all of special temperatures by definite resistances of a resistance thermometer;

2nd. the extension of the testing determinations to measurements in liquid hydrogen;

3rd. the determination of definite standard temperatures by means of the boiling points and melting points of hydrogen, oxygen and other substances that can be easily purified. [2])

4th (comp. Comm. N°. 95^a § 1, Sept. 1906) temperature measurements with the helium thermometer, *a.* for a direct or an indirect comparison with the hydrogen thermometer, *b.* in order to get a firm basis for the determination of the lowest temperatures, especially with a view to the reduction to the absolute scale.

The investigation mentioned sub 3 and 4 has advanced a good

1) In investigations the reading of temperatures with a resistance thermometer will as a rule for simplicity be preferred to reading them with the hydrogen thermometer.

2) When we possess the fixed points meant here, the hydrogen thermometer can for calibrations be replaced by boiling point apparatus filled with pure gas and placed in the same bath as the apparatus to be calibrated. This is a great simplification in cases where only these *few special* temperatures are required. Moreover in these fixed points we have the means for a comparison between gas thermometers filled with different gases (for instance H_2 and He) or between thermometers in different laboratories.

deal. We now intend to communicate some measurements relating to 1st and 2nd.

§ 2. *Survey of the determinations.*

With regard to the controls meant sub 2nd two independent determinations have been made with entirely the same apparatus for comparison. The measurements meant sub 1st did not entirely succeed owing to a small reparation which the resistance thermometer required. [1]) These measurements, however, have thereby acquired a signification in another respect, namely as a new calibration from — 104° to — 259° of the resistance thermometer used in Comm. N°. 95^e, they allow us to judge in how far after similar reparations, which in the long run will be inevitable, the same temperature coefficients will remain valid for the resistance thermometer.

The communicated measurements determine also with a greater accuracy a couple of temperatures (comp. however note 2 in § 3, 2°) which hitherto had not been determined with the desired reliability (comp. §§ 3 and 5) [2]).

The results have been combined in a table following below. The second and third columns contain the readings of the hydrogen thermometer and of the resistance thermometer. Those of the hydrogen thermometer are calculated in the way of N°. 95^e (designated by t) and therefore require the corrections mentioned in Comm. N°. 97^b (March 1907). They have not been applied here because this had not been done in any of the preceding communications and mutual comparison is thereby facilitated. The next column shows the resistances of column 3 recalculated with the factor 1.01806, which is the ratio between the resistances at 0° C. before and after the breaking of the wire. These values have been compared with formula A_I of § 6 of Comm. N°. 95^e. The fifth column contains the deviations from this formula. The sixth column contains the differences which were to be expected according to Comm N°. 95^e. The seventh colum contains the differences between the two resistance thermometers in Ω.

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1) When the resistance broke only $^1/_{60}$ of the wire was lost, yet on account of this one might allege that if the latter is not perfectly homogeneous variations in the coefficients of temperature are not entirely excluded. These are especially to be feared as a result of the new winding of the wire.

2) These measurements are used in table V, Comm. N°. 99^b (Sept. 1907) at — 252°.82 and — 255°.18 deviating from table I, Comm. N°. 95^e (Sept. 1906).

TABLE I. Comparison of the platinum thermometer $Pt_{I'}$ with the hydrogen thermometer.

Date 1907	Temperature according to the hydrogen thermometer	Resistance in Ω	Resistance recalculated	$O_{Pt'I'}-C_{AI}$	$O_{PtI}-C_{AI}$	$O_{Pt'}-C_{Pt}$
3 July	— 103°.671	79.131	80.551	— 0.000	— 0.023	+ 0.023
3 July	— 119°.730	70.190	71.450	+ 0.003		
3 July	— 164°.113	45.054	45.863	+ 0.047		
25 March	— 182°.352	34 492	35.111	— 0.008	— 0.029	+ 0.021
29 June	— 216°.610	14.936	15.204	+ 0.016	+ 0.028	— 0.012
19 March	— 252°.822	1.9208	1.9553	+ 2.4131	+ 2.432	— 0 019
1 July	— 252°.839	1.9243	1.9588	+ 2.4180	+ 2.432	— 0.014
19 March	— 255°.177	1.6852	1.7154	+ 2.0518		
1 July	— 258°.864	1.4522	1.4783	+ 0.4855	+ 0.199	+ 0.287

§ 3. *Results.*

In order to be able to make conclusions from table I we remark that the later determinations have been made with the same thermometer filled with distilled hydrogen [1]).

From the results of table I and some earlier determinations at the same temperatures made with the resistance thermometer before it was broken, we may derive.

1. A comparison between the indication of the resistance thermometer before and after the breaking of the wire, abstracting from the reading errors of the hydrogen and the resistance thermometer, by means of the determinations of March 25, June 29, and July 3. They give only small differences for the observations on the last two dates. Indeed if we compare the differences O—C_{AI} of table II of Comm. N°. 95

[1]) With the earlier determinations this was the case for only some among them. We may derive from former measurements (comp. Comm. 95e), that this will give no difference till — 217°, but for measurements in hydrogen this has not been proved experimentally. The method of filling by means of electrolytic hydrogen, provided it be done carefully, may be considered as perfectly satisfactory also for these temperatures. As this method, however, involves a more complicate system of auxiliary apparatus the other must be preferred with regard to the reliability.

with those of table I of this Communication, the differences for —103°, —183° and —217° [1]) respectively are:

$$+\,0.023 \quad +\,0.021 \text{ and } -\,0.012$$

corresponding to

$$0^{\circ}.040,\ 0^{\circ}.036 \text{ and } 0^{\circ}.022.$$

From this we derive that down to — 217°. the variations in the temperature coefficients owing to the new winding of the wire, though not imperceptible are extremely small.

2. A comparison between two resistance calibrations, for which the same hydrogen and resistance thermometers were used, in the neighbourhood of the boiling point of hydrogen by means of the determinations of March 19 and July 1. The difference is $0.0049\,\Omega = 0^{\circ}.046$ and exceeds $\frac{1}{50}$ deg. which has been derived as the limit of accuracy for measurements to — 217°. This must probably be ascribed to the fact that the measurements of the resistance are less accurate because they are made with the WHEATSTONE bridge and not with the differential galvanometer [2]).

3. A comparison between the indications of the thermometer filled with distilled hydrogen and the one used before and filled with electrolytical hydrogen, by means of the determination of July 1, abstracting from the errors of observation of the hydrogen

[1]) For —217° this difference just reaches the limit of accuracy derived in Comm N°. 95*e* and for the two other temperatures the difference only little exceeds this limit.

For — 183° another reason may be given for this difference. To a certain extent it must probably be ascribed to the circumstance that the earlier determinations (of June 30 and July 6 '06) like those at — 217° of June 30 '06 must be considered as less reliable. It appeared namely during an investigation started in Dec. 1906 that the steel capillary was no longer absolutely tight and this may also have been the case when the measurements under discussion were made. The latter becomes probable when we direct our attention to the great variations of the zero during these determinations, viz 0.33 mm., to which we alluded in § 11 of Comm. N°. 95*e* without being able to explain it then.

The fault may have arisen because at the end of May '06 the thermometer has partly been taken to pieces and the capillary was bent too much. The observations made before June '06 were not influenced by this fault.

[2]) The accuracy of the WHEATSTONE bridge is perfectly sufficient for higher temperatures below 0° C. (comp. Comm. N°. 99*b* § 2 for temperatures till —217°), but owing to the disadvantageous influence of the connecting resistances, it falls short for measurements in hydrogen where the variation of the resistance becomes so small. Therefore, simultaneously with the measurements of table I in hydrogen made with the WHEATSTONE bridge, we have calibrated another thermometer Pt_d with the differential galvanometer in order to fix the temperatures below — 217°

and the resistance thermometer and variations in the temperature coefficients of the resistance [1]).

The difference appears to be larger than we should expect after the experience made with the higher temperatures. It may be that in the measurement of May 5 '06, the first measurement made in liquid hydrogen, in the measurement of the resistance or the reading of the hydrogen thermometer a systematic error has crept in which escaped our attention. At any rate it will be necessary to repeat the calibration at these lowest temperatures.

The differences treated in this section, in so far as they go beyond the expected limit of accuracy, point partly to abnormal sources of error, partly to errors which in future may be prevented (as for instance by always measuring small resistances by means of the differential galvanometer) and it is probable that when we avail ourselves of the experience made we shall reach also for temperatures below $-217°$ an accuracy to 0°.02.

§ 4. In the same way as Comm. N°. 95^e § 7 the following observations, where two resistance thermometers were simultaneously immersed in the same bath, allow us to judge of the accuracy with which a temperature is fixed by a given resistance.

With Pt_I we have made an adjustment to a definite temperature at which the resistance of Pt_{III} was determined, then the temperature was changed a little and again read on Pt_I and then the resistance of Pt_{III} was determined and reduced to the first temperature.

	temp. on Pt_I	$-87°.54$,	resistance Pt_{III} 103.950 Ω [2])	
new	„ „ „ reduced	$-87°.54$,	„ „ 103.959	difference 0.009 Ω or 0°.014
	temp. on Pt_I	$-216°.65$	17.379	
new	„ „ „ reduced	$-216°.65$	17.385	difference 0.006 Ω or 0°.009

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[1]) Although it is not excluded that here the variation of the temperature coefficients is larger than to $-217°$, this can by no means explain the large deviation because the wire had previously been carefully annealed. Moreover it is difficult to assume that impurities of the gas in the thermometer would be the cause, for then we must accept that about 0.7 % of air has been present in the gas, which is rather impossible on account of the great carefulness observed when the thermometer is filled.

[2]) For small differences in the calibrations of Pt_{III} and Pt_V we refer to Comm. N°. 99^b where also the zero's are given. The differences result from a more accurate determination of the ratio between the arms of the Wheatstone bridge and of the resistance of the conducting wires. In observations which were made from 1905—1907 it appeared that the zeros had remained unchanged to less than $\frac{1}{20000}$ (comp. also Comm. N°. 99^b).

The probable error of an adjustment to the resistance thermometer appears to be equal to that of a reading on the hydrogen thermometer (comp. Comm. N°. 95^e § 7)[1]).

The following observations related to the defining of a temperature over a longer recording period.

Pt_{III} and Pt_V were calibrated immediately after each other with Pt_I, thence we derived the temperature reading according to Pt_I on Pt_{III} and Pt_V. Afterwards and adjusted to the same temperature we have compared the gold thermometer Au_0 with Pt_{III} and Pt_V immediately after each other. The readings were

temp. Pt_I on Pt_{III},	temp. Pt_I on Pt_V,	Au_0 resistance [2])
— 58°.56	— 58°.56	40.324
— 87°.43	[— 87°.50]	34.638
— 159°.07	— 159°.08	20.393
— 216°.27	— 216°.29	8.459

If we disregard the large deviation in brackets which indicate that it must be ascribed to an irregularity, it appears that the definition of a temperature by means of a single determination on a resistance thermometer has about the same probable error as a single determination with the hydrogen thermometer and is generally reliable to an amount which remains below 0°.02.

For the present the accuracy of the determination of a temperature which is kept constant during a long time by means of the hydrogen thermometer may be considered equal to the accuracy of the definition of a temperature by means of the resistance thermometer, where however the necessary readings, even when they are repeated require less time.

§ 5. According to § 3 the observations of June 30 and July 6, '06 must be rejected. They have been used for the calibration of the thermo-element and the resistance thermometer of Comms. N^{os}. 95^a, 95^c and 95^f (Nov. 1906) and hence the values — 217°.411 and — 182°.75 of the tables VIII of Comm. N°. 95^a, I, II of Comm. N°. 95^c and IX of Comm. N°. 94^f must be modified.

The determination of June 30, '06 at — 183° shows a perceptible difference with that of March 25, '07, the only one of the required

1) In Comm. N°. 95^e § 7 we for the time being took as a starting point that the error in the adjustment of the hydrogen thermometer was negligible.

2) These values deviate a little from the values given in table III of Comm. N°. 95^f owing to corrections which were afterwards determined.

reliability. In observation 11 of table VIII we must therefore replace — 182°.75 by — 182°.79 and an analogous modification is required in the tables I, II and IX.

Instead of the values which relate to — 182°.75 of table II we now get:

TABLE II. Values for — 183° instead of those of table II of Comm. N°. 97*c*.					
t	Number of observations	O.-C.$_{AI}$	O.-C.$_{AII}$	O.-C.$_{B}$	O.-C.$_{C}$
—182.79	2	—0.008	+0.048	+0.104	—0.014

The agreement is thereby greatly improved, as O.-C.$_{AI}$ is changed from — 0.029 to — 0.008.

In the tables VIII of Comm. N°. 95^a and IX of Comm. N°. 95^f the three temperatures — 183° from which a mean has been derived, are all diminished by 0°.04. O.-C. thereby becomes less by 0.0008 and the agreement improves in the same rate.

As to the determination of July 6, '06 the modification is very small.

In the tables VIII of Comm. N°. 95^a and IX of Comm. N°. 95^f no change occurs if we omit this last determination in the mean, in tables I and II it is not used.

With regard to this and to what has been said in note 1 of § 13 of Comm. N°. 95^a, no new calculation has been made.

As a supplement to columns 2 and 3 of table I towards a complete calibration of Pt_I' we can use Pt_I in table V of Comm. N°. 99^b with the zero 135.438 Ω (comp. note 1 of § 4), where we have to add the corrections of table XVIII of Comm. N°. 97^b.

Physics. — "*On the measurement of very low temperatures. XVIII. The determination of the absolute zero according to the hydrogen thermometer of constant volume and the reduction of the readings on the normal hydrogen thermometer to the absolute scale.*" Communication N°. 101[b] from the Physical Laboratory at Leiden by Prof. H. KAMERLINGH ONNES and C. BRAAK.

(Communicated in the meeting of November 30, 1907).

§ 1. *The determination of the absolute zero.*

D. BERTHELOT [1]) has used the observations of CHAPPUIS on the pressure coefficients between 0° C. and 100° C. and the slopes of the pv-lines at 0° C. and 100° C. to derive the mean relative pressure coefficient from 0° C. to 100° C. which the investigated gas would possess for densities in the state of AVOGADRO, (so we call for shortness the state in which the deviations from the law of BOYLE-GAY-LUSSAC-AVOGADRO may be neglected). In the same way we may use for this purpose the data of Comm. N°. 100[b] and Comm. N°. 60 (Sept. 1900).

If for the pressure coefficient of the hydrogen thermometer for a zero pressure of 1090 mm. and a density of 1.44 found in Comm. N°. 60 we derive the value 0.0036629, we may derive the pressure coefficient for the state of AVOGADRO (represented by α_{AV}) from the data of Comm. N°. 100[b] for B_A for 0° C. and for 100.°20 C. by means of the formula:

$$100 \times 0.0036629 = \frac{A_{A_0} \times 100\, \alpha_{AV} + (B_{A_{100}} - B_{A_0}) \dfrac{1090}{760}}{A_{A_0} + B_{A_0} \dfrac{1090}{760}} \quad . \quad (1)$$

where we must replace the value of A_{A_0} found in § 3 of Comm. N°. 100[b] by a more accurate value:

$$A_{A_0} = 1 - B_{A_0} - C_{A_0} = 0.999419$$

and where $B_{A_{100}}$ has been derived from B_{A_0} and $B_{A_{100.20}}$ by interpolation [2]). Hence follows for the desired pressure coefficient

$$\alpha_{AV} = 0.0036619.$$

The reverse gives the temperature of the freezing point measured on the absolute scale. Hence:

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[1]) Sur les thermomètres à gaz. Travaux et Mémoires du Bureau international des Poids et des Mesures, T. XIII.

[2]) In formula (1) the curvature of the pv-lines is left out of account, which is permissible.

$$T_{0^\circ C.} = 273^\circ.08\,K$$

where K (KELVIN) stands for degrees on the absolute scale of which 100 occur between the freezing point and the boiling point of water.

The data are not sufficiently accurate to allow us to determine the last decimal to less than unity [1] [2]. The value found here agrees very well with that which may be derived from the most reliable data of other observers [3].

The method followed here of deriving the pressure coefficient for infinitely small densities by means of determinations of isotherms at pressures between 25 and 50 atmospheres is preferable to using either the data of CHAPPUIS or those of AMAGAT. It is true that in the former case the coefficient C may be neglected without error arising, but the small difference of pressure has a bad influence on the determination of B. On the contrary with higher pressures, such as with AMAGAT's determinations, the coefficients C and the higher ones have too much influence to allow an accurate derivation of the value of B. In our determinations C is of so small account that an error in the estimation of C may be neglected for the determination of B [4].

While therefore the influence of errors in C_A may be neglected we find on the other hand that the pressures are so large that an error in the pressure coefficient passes diminished to about

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1) In discussing the isotherms we intend to refer to a small systematic difference between the isothermals of hydrogen at 20° C. according to Comm. N⁰. 70 (June 1901) and those at 0° C and 100° C. of Comm. N⁰. 100*b*. It rather points at $T_{0^\circ C.} = 273^\circ.07\,K$.

2) We intend to determine this value still more accurately with nitrogen and helium by means of determinations of isotherms at 0° C. and 100° C. and of pressure coefficients between 0° C. and 100° C. where we proceed according to Comm. N⁰. 60 (the determination of H_2 is also repeated), but as a higher degree of accuracy is wanted (designated by that now reached with the determinations of the isotherms) we now take a reservoir of 300 c.c.

3) Comp. for this the note of § 2. XIV Comm' N⁰. 97*b*.

4) If for instance in the adoption of C_A an error of 15% has been made, which with a view to the data of table XXII of Comm. N⁰. 100*a* probably includes the higher limit for the error for lower temperatures, this becomes only 0.0000001 which, considering that the greatest density which occurs in the determinations of Comm. N⁰. 100*b* amounts to about 50 times the normal one, would cause for 0° C. an error in B_A which remains below 0.000005. As such a systematic error would change the value of $C_{A.100^\circ}$ in nearly the same way, the error in this difference will be much smaller and, for instance, the error arising thence in the difference between $B_{A.0^\circ}$ and $B_{A.100^\circ}$ may be estimated at 0.000001, i. e. $^1/_5$ of the error in the absolute value of B_A. The error in the absolute zero arising from the two influences combined remains below 0°.01 C.

into the value of B_A. Hence in this way we have obtained data a rather large difference of pressure, from which B_A may be ·ived unambiguously.

§ 2. *Reduction of the readings on the normal hydrogen thermo-ter to the absolute scale.*

3y means of formula (3) of § 2 of Comm. N°. 97[b] and with the v values of B'_T (comp. formula (2) of § 2 Comm. N°. 97[b]) which y be derived from the data of table XXII, we have determined ɔw the corrections of the readings of the hydrogen thermometer of istant volume to the absolute scale. For this we have started from individual virial coefficients and not as in Comm. N°. 97[b] from a ieral temperature formula, because the course of the separate therms has now been ascertained sufficiently to render a similar vious equalization superfluous.

For B'_0 and B'_{100} we have adopted other values than in Comm. 97[b]. We have namely used the results obtained from direct erminations of isotherms at 0° C. and 100° C., of which the ults are laid down in Comm. N°. 100[b]. These values are:

$$B'_0 = 0.0005807 \qquad B'_{100} = 0.0006321.$$

the absolute zero we have adopted

$$t = -273°.08 \text{ C.}$$

TABLE XXV H_3. Corrections to the absolute scale.

t'_s	θ	$B'_T \cdot 10^3$	Δt_s	Δt	(O. C.)×10³
− 103°.56	− 103°.54	+ 0.394	+ 0.017	+ 0.016	0
− 139°.87	− 139°.84	+ 0.229	0.029	0.026	− 1
− 164°.13	− 164°.09	+ 0.018	0.043	0.039	+ 1
− 182°.80	− 182°.75	− 0.241	0.056	0.051	+ 3
− 195°.26	− 195°.20	− 0.432	0.059	0.054	− 3
− 204°.69	− 204°.62	− 0.692	0.069	0.063	− 2
− 212°.81	− 212°.73	− 1.009	0.079	0.072	+ 1
− 217°.40	− 217°.32	− 1.203	+ 0.083	+ 0.076	+ 1

ived in the previous § from the same determinations of isotherms. ɔ results have been combined in the preceding table in the manner table XVI of Comm. N°. 97[b].

The differences with the earlier values remain, notwithstanding we have used entirely different data, far within the limits of the accuracy mentioned in § 3 (loc. cit.). The coefficients of the formula given there become:

$$a = -0.007117$$
$$b = +0.005962$$
$$c = -0.000185$$
$$d = +0.001330$$

With this formula the temperatures of the second column have been determined. The differences between the data of the last column but one and the formula are given in the last column.

For -273° the new formula gives the same value as the one before of Comm. N$^\circ$. 97^b i.e. $\Delta t = +0^\circ.14$, for 0° C. and $+100^\circ$ C. it gives $\Delta t = 0^\circ$. For the temperatures between 0° C. and 100° C. the formula yields much larger negative values than those which BERTHELOT has derived with his equation of state (loc. cit. IX). For 20°, 50° and 80° are found: according to BERTHELOT:

$$\Delta t = -0^\circ.00046 \qquad -0^\circ.00053 \text{ and } -0^\circ.00033$$

according to our formula:

$$\Delta t = -0^\circ.0012 \qquad -0^\circ.0020 \text{ and } -0^\circ.0014$$

According to the general equation of state of hydrogen derived previously (comp. § 1 Comm. N$^\circ$. 97^b) these values would be respectively:

$$-0^\circ.0026 \quad . \quad -0^\circ.0047 \quad , \quad -0^\circ.0036.$$

Geology. — *"On the terms Schalie, Lei and Schist."* By Mr. J. SCHMUTZER. (Communicated by Prof. C. E. A. WICHMANN).

East-Indian mining engineers, by whom almost exclusively the existing descriptions of rocks in Dutch have been written, under the influence of the German literature relating to this subject, have used the terms *schiefer* and *lei* side by side, on the whole with hardly any difference. The cause of this is the want of a fixed geological terminology in our language; a want which of late has also been felt by the "Ned. Mijnbouwkundige Vereeniging", as appears from its attempt to create such a terminology under the supervision of Prof. G. A. F. MOLENGRAAFF. The purport of this communication is to aid in solving the problem, *what* terms are best fitted to denote in Dutch those rocks which in German bear the name of *Schiefer* in the amplest sense of the word.

The starting-point is formed by the consideration, that, for the sake of clearness, different notions should be rendered by different words, and that, in the choice of terms, — as long as there is a choice — everything that might give rise to ambiguity, should be avoided.

The distinct difference which in nature exists between exclusively sedimentary slate, metamorphosed in a relatively slight measure and the finer or coarser crystalline, more or less distinctly foliated metamorphosed rocks, partly of sedimentary, however also partly of eruptive origin, is greatly discounted by the application of the term *lei* also to the latter, although the term "crystalline" be added. This explains partly, how the word *schiefer* could introduce itself into the Dutch terminology, though in a meaning that is far from being a fixed one; the want was felt of another presentive word. A bad choice, however, was made with the word *schiefer*[1]), as the latter, by taking up a place by the side of the word *lei*, must necessarily assume a more limited meaning in our language, since in the language from which it was borrowed, it occurs in different widely-diverging combinations [2]). The consequence of this is an absurd state of things, which is not to be improved by reintroducing the disused *scheversteen*[3]), which has been tried in the shorter form of *schever*[4]). In order to prevent a possible confusion of ideas, which lies concealed in the analogy with the German cognate word, this word seems to be less fitted. The corresponding English *shiver*, which has maintained itself among miners in the meaning of "flake of stone, shale, slaty débris[5])", does not occur as a scientific term[6]) and could not as such be put side by side with *shale* and *slate*.

The solution of the problem is considerably simplified by the circumstance that the English *shale* is etymologically identical with the Dutch *schalie*, a word not used in the north of Holland. Whilst the English *shale* as a secondary form of *scale* and *shell* may be directly reduced to

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1) Already used in most Dutch dictionaries.

2) As is well known also the Danish-Norwegian *skifer*, the Swedish *skiffer* is used to denote the English terms *shale* as well as *slate* and *schist*.

3) PLANTIJN, cf. E. VERWIJS and J. VERDAM, Middelned. Woordb., IV, 336—337; VII, 224; J. u. W. GRIMM, Deutsch. Wörtb. IX, 1. sq. KILIAEN gives the word in the sense of 1⁰. *schalie*, 2⁰. *sicamb.*, i. e. the word used in Cleves for Latin *silex* (comm. of Prof. J. W. MULLER).

4) See also KLUGE, Etym. Wörtb. d. deutsch. Spr. 337; GRIMM, l. c., E. MÜLLER, Etym. Wörtb. d. engl. Spr. II, 378.

5) WRIGHT, Eng. Dial. Dict. V, 392.

6) Cf. ARCH. GEIKIE, Text Book of Geology, JAS. GEIKIE, Struct. a. Field Geology; TEALL, British Petrography.

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the old-English *scale, schale*[1]), this *schalie* is formed from the old-French *escaille* (Fr. *écaille*, Ital. *scaglia*), which, in its turn however, has been borrowed from Germanic[2]). Now, where these two words are closely connected both in origin and in meaning, it cannot be subject to serious objection to identify the Eng. *shale*: "applied to all argillaceous strata . . . which split up more or less perfectly in their line of bedding," [3]) with the Dutch term *schalie*.

Prof. J. W. MULLER had the kindness to communicate to me as follows[4]): As far as I can go into the question, the state of things with these two words is the following: Time out of mind *lei* in Holland, Utrecht and the (north and south) eastern provinces (cf. also Lorelei, Erpeler Lei on the Rhine, etc.), *schalie* on the other hand in the southwest: Flanders, Zeeland and South-Holland isles, have been the only word for both kinds of rocks, now more closely distinguished by geologists. Now you wish to confine the north-eastern word to the one, the south-western to the other kind. Of course this is something arbitrary, but in a scientific terminology, such unnatural and artificial distinctions are necessary, and there is nothing against it; the Flemish people will continue calling everything *schalie*, the Dutch *lei*, and there is no objection to this either". Prof. G. A. F. MOLENGRAAFF was so kind as to write to me that already at his lectures he availed himself of the term *schalie*, as an equivalent of the English *shale*[5]); a happy circumstance, which not only pleads for the weight of the grounds alleged, but will at the same time contribute much to make this term enter into general use[6]).

The English *slate*[7]) refers to argillaceous rocks, which by a metamorphosis, in which the pressure has predominated but a chemical

1) MÜLLER, II. 365.

2) Cf. Anglo-saxon *sceălu*, putamen, gluma; Gotic *skalja*, "Ziegel, eigentlich wohl Schindel, Schuppenartiges", KLUGE, 331; FRANCK, Etym. Woordb. d. ned. Taal, 827; MÜLLER, II, 365; KLUGE, 294, 351; VERWIJS and VERDAM, VII, 224; GRIMM, VIII, 2060—2064.

3) NICHOLSON cf. WRIGHT, V, 348. ARCH. GEIKIE's definition runs (op. cit. 2d edit 1885, 164), *shale*, (synon. Fr. *schiste*, G. *Schieferthon*), "clay that has assumed a thinly stratified or fissile structure," see further JAS. GEIKIE, 62. By the side of the French *argile schisteuse* we find the Ital. *scisto argilloso* (argilloscisto) and *argilla scagliosa*, lit. *kleischalie*.

4) Letter of 24 Dec. '07.

5) Dec. 16th '07: for argillaceous on other rocks "with indistinct, more or less shelly stratification."

6) The term *schalie* was used by the late Prof. J. L. C. SCHROEDER V. D. KOLK for the Eng. *crystalline schist*, but could not maintain itself in this meaning

7) Old-Eng. *slat, sclat, sklat*, old-Fr. *esclat*, Fr. *éclat*; COTGRAVE says: "*esclat*, a shiver, splinter, also a thin lath or shingle" cf. MÜLLER, II, 400—401.

change is not excluded, have got a thinly fissile structure, always parallel to the axis of the synclinal or anticlinal folds of system and consequently can make widely diverging angles with the original line of bedding) ("false bedding"). It is to be recommended, in accordance with the original [1]), now generally current, meaning of the word, to confine the term *lei* to these rocks, of which roof-slate in the best known representative. Therefore on the ground of the definition given, the above affix, apart from the well-known Rhenish devonian slates, ought to be given to the greater part of the so-called "Fleckschiefer", "Knotenthonschiefer", andalusite-, staurolite-, chiastolite slates, etc.

The English term *schist* [2]) is used for such heterogeneous rocks that the existence of this word by the side of *lei*, *slate* is fully justified. Not only does *schist* denote the more or less foliated peripherical facies of purely eruptive rocks, which often without any distinct boundary-line pass into adjoining strata of sedimentary rocks, — whether these eruptive rocks occur in smaller laccolithes and dykes or in extensive intrusions or effusions (as many "trapps"); — also sedimentary rocks can by contact change into *schists*, metamorphosing agencies in strongly disturbed regions can without any distinction on a large scale change sedimentary and eruptive rocks into crystalline rocks, more or less distinctly, sometimes excellently foliated, such as gneiss, eyegabbro, amphibolite, etc. Though the term *schist* in its various shades of meaning [3]) and especially the adjective derived from it [4]), has got a much wider meaning in Romance languages, yet it finds also there, by the simultaneous use of the equivalents of *lei* and *slate*, a more limited application than in German. In connection with this it is perhaps recommendable, more particularly after English

[1]) JAS. GEIKIE, op. cit. 76, 220, sq.; WRIGHT, 504; ARCH. GEIKIE, 125—126: "In England the term *slate* or *clay-slate* is given to argillaceous, not obviously crystalline rocks possessing this cleavage structure", (syn. *argillaceous schist*, Fr. *phyllite*, *phyllade*; *schiste ardoise*; G. *Thonschiefer*, *Thonglimmerschiefer*).

[2]) PLANTIJN, "schalie, leye oft scheversteen", une ardoise, ardosia, scandula, cf. VERWIJS and VERDAM, IV, 336-337; VII, 224, by the side of which stood the meaning "rots, leisteen", (cf. also "een leye der scandalisieringhe"); old-Saxon. *leia*, rots, cf. FRANCK, 558; KLUGE, 243. Mdl. Dutch had the word in the meaning of "Slate used for roofs", VERWIJS and VERDAM, IV, l. c.

[3]) From σχίζω, to split, to sever, to cleave, to divide; adj. verb. σχιστός; ARCH. GEIKIE, 124: "a rock possessing this crystalline arrangement into separate folia is in England termed a *schist*. As Prof. MOLENGRAAFF communicated to me, this term was already used by him.

[4]) Sp. also *pizarra cristalina*.

[5]) which coincides with Germ. *schiefrig*, Dan. Norw. *skifrig*, Swed. *skiffrig*.

usage, to denote also in Dutch the *krystalline Schiefer* by means of the term *schist* (pl. *schisten*) borrowed from the Greek, and to confine one's self in the application of the derived-adjective *schisteus* and the substantive *schistositeit* exclusively to this class of metamorphosed rocks.

While, in conclusion, I express my special gratitude to Prof. J. W. MULLER for his kindly furnished contributions and for some valuable hints, I cannot omit tendering my best thanks to Prof. A. WICHMANN and Prof G. A. F. MOLENGRAAFF for their lively interest and the support with which they obliged me.

Chemistry. — *"On the question as to the miscibility and the form-analogy in aromatic Nitro- and Nitroso-compounds".* By Dr. F. M. JAEGER. (Communicated by Prof. A. P. N. FRANCHIMONT).

1. The following communication contains a further contribution to the kowledge of the mutual behaviour of the aromatic nitro- and nitroso-derivatives, of which several particulars were given previously. The entire miscib ility in the solid condition, and the form-analogy bordering on isomorphism, were established in the case of *p-Nitro-* and *p-Nitroso-diethylaniline* [1]), while afterwards, in a more extended paper of a more general nature [2]), the mutual comparison of *p-Nitro-* and *p-Nitrosophenol* and of *o-Nitroso-* and *o-Dinitrobenzene* was discussed. It then appeared that no *general* rule could be laid down as to the connection of the two classes of compounds.

A new pair of similar comparable substances which are interesting from more than *one* point of view, namely *o-Nitro-* and *o-Nitroso-acetanilide* were now studied. The result was different from that obtained with *p-Nitro-* and *p-Nitrosodiethylaniline* although analogy is present in *one* of the axial relations, and a solid solution of the two components appears to be possible to some slight extent.

§ 2. **Ortho-Nitro-Aceto-Anilide.**

$C_6H_4(NO_2)_{(2)} . NH_{(1)} . CO CH_3$; meltingpoint: $87\frac{1}{2}°$ C.

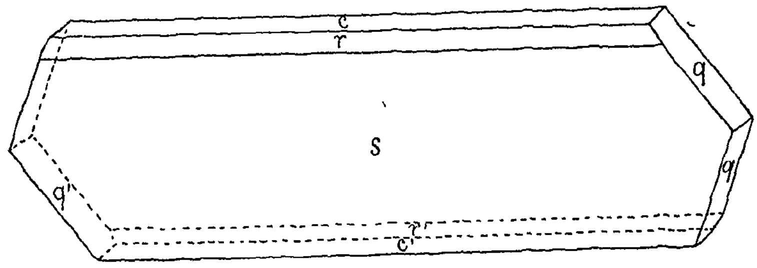

Fig. 1.

[1]) F. M. JAEGER, These Proc. VII, p. 660,

[2]) idem, Ueber Mischbarkeit von festen Phasen, Z. f. Kryst. **42**, 236—276 (1906).

On account of its great solubility in most solvents it is very difficult to obtain this compound in a properly crystallised form. The greatest success is met with by very slow evaporation of its solution in dilute alcohol when it crystallises in pale yellow, very thin laminae with extended hexagonal periphery; the crystals are very transparent and give sharp signal-images.

The symmetry is *monoclino-prismatic*. In the choice of the plane-symbols adopted here the axial relation is calculated on:

$$a : b : c = 0,8935 : 1 : 1,9198$$
$$\beta = 83°51'.$$

Forms observed: $s = \{101\}$, strongly predominant and yielding sharp reflexes; $r = \{101\}$ much narrower, but gives a good reflection; $c = \{001\}$, narrower than r; $q = \{011\}$, also narrow, but reflects well. The habit is flattened along $\{\bar{1}01\}$ with considerable elongation along the b-axis.

The following angular values were determined:

	Measured:	*Calculated*:
$c : q = (001) : (011) =$	* 62°21′	—
$q : s = (011) : (10\bar{1}) =$	* 80°57′	—
$c : r = (001) : (101) =$	* 60° 4′	—
$r : s = (101) : (10\bar{1}) =$	49°45′	49°45′
$s : c = (10\bar{1}) : (00\bar{1}) =$	70°11′	70°11′

A distinct plane of cleavage was not found.

In the orthodiagonal zone the direction of extinction is orientated everywhere perpendicular to the direction of the b-axis. On $\{\bar{1}01\}$ no perceptible dichroïsm is observable.

The optical properties of the substance in convergent polarised light are very remarkable.

For the red, yellow and most of the green rays of the spectrum the axial plane $= \{010\}$; extraordinarily strong, inclined dispersion: the axial angle for the red is much larger than that for the green. The character of the double refraction is positive.

On the other hand, the axial angle for the blue and violet rays is situated perpendicular on $\{010\}$ with a horizontal dispersion. The axial angle for all rays is but small.

The curious colourphenomena in white light exhibited by this substance, which thus possesses at the same time inclined and horizontal dispersion, lend themselves particularly well to the demonstration of anomalous dispersion in biaxial crystals.

The sp.gr. of the crystals is 1.419 at 15°; the equivalent volume is therefore 126.85.

Topic parameters: $\chi : \psi : \omega = 3,7578 : 4,2058 : 8,0744$.

§ 3. **Orthonitroso-Aceto-Anilide.**

$C_6H_4(NO)_{(2)} . NH\text{-}COCH_3$; m.p.: 108° C.

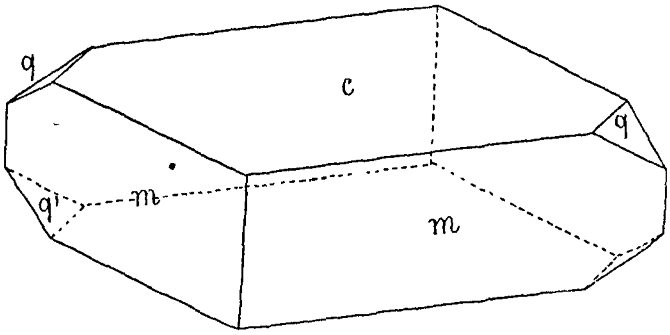

Fig. 2.

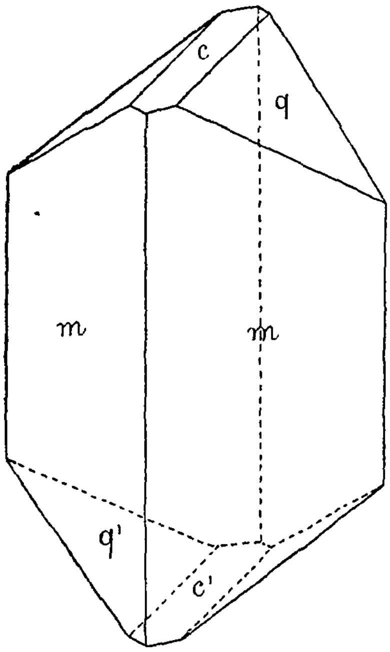

Fig. 3.

Through the kindness of Dr. F. Leuchs of *München*, who was the first to prepare this substance (Ber. 1907 40. 1083), I received a small quantity of the crystallised compound obtained by cooling the hot alcoholic solution. The crystals exhibited the habit of fig. 2; from a mixture of ether and benzene I obtained the thick prismatic crystals of fig. 3. The prism-planes, which often act as resting planes of the crystals in the mother liquor, were in consequence mostly curved and unsuitable for accurate measurement, whilst the forms c and q always gave ideal reflexes. The crystals have a brilliant emerald green colour and are quite transparent.

The symmetry is *monoclino-prismatic*; the axial relation was calculated on:

$$a : b : c = 0,8940 : 1 : 0,7295$$
$$\beta = 82° 6'$$

Forms observed: $c = \{001\}$, predominant in the crystals obtained from alcohol and always very shining; $m = \{110\}$, well developed but often with curved plane; $q = \{011\}$, giving ideal reflexes and mostly exhibiting rather large planes; $a = \{100\}$, exceedingly narrow and dim. The habit is flattened along c or long prismatic parallel the c-axis, with flattening along two parallel planes of m; perfectly cleavable towards $\{001\}$.

The following angles were measured:

	Measured:	*Calculated:*
$c : m = (001) : (110) =$	$^{*}84°\ 5^1/_4{}'$	—
$c : q = (001) : (011) =$	$^{*}35°\ 51'$	—
$m : m = (110) : (1\bar{1}0) =$	$^{*}83°\ 3'$	—
$m : q = (110) : (011) =$	$61°\ 55^1/_2{}'$	$61°\ 51^1/_2{}'$
$m : q = (110) : (0\bar{1}1) =$	$72°\ 18'$	$72°\ 15'$
$m : a = (110) : (100) =$	$41°\ 34'$	$41°\ 31^1/_2{}'$

On $\{001\}$ not observable but on $\{110\}$ very distinctly dichroïc; for vibrations parallel to the c-axis, grass-green; and yellowish-green for vibrations perpendicular thereon. The angle of extinction is very difficult to determine; it amounts to about 12° in regard to the c-axis on the planes of $\{110\}$.

The optical axial plane is $\{010\}$; on $\{001\}$ one axis is visible at a small angle with the normal to that plane; the inclined dispersion is extraordinarily strong: $\rho < v$.

The sp. gr. of the crystals is 1.351 at 15°, the equivalent volume is 121.39.

Topic parameters: $\chi : \psi : \omega = 5.1206 : 5.7277 : 4.1784$.

4. A small addition of the nitroso-compound to the nitro-compound causes a perceptible depression of the melting point of the latter substance. As, therefore, no certainty is obtained as to the formation of mixed crystals of these closely related derivatives — for a melting curve with an absolute minimum might be also present — a few preliminary quantitative experiments were carried out, which showed that we have here indeed an ordinary binary meltingpointline with eutecticum. It was further shown by more detailed microscopic tests that from mixed fusions of the two compounds is always deposited a *mixture* of the *yellow* nitroaceto-anilide crystals and the *green* nitrosoaceto-anilide crystals with a totally different aspect. At the side of the nitroso-compound a perceptible quantity of the nitro-derivative is carried down by the deposited crystals as solid solution; at the side of the nitro-derivative a formation of solid solutions could not be ascertained in this way. In any case, if there should be a slight mixing, it is limited on the side of the nitroso-compound to a few percent of the nitro-compound; the hiatus is therefore enormously extended.

Finally be it observed here that both substances are very volatile; the vapour of the nitroso-compound is *yellowish green.*

A few experiments were tried to sublime mixtures of the two compounds. The deposit on the concave coveringglass then consists of a network of dentritic, strongly pleochroïc (colourless — yellowish-green) little crystals, between which are found the square-shaped crystals of the nitroso-compound, besides the yellow individuals of the nitro-compound, united in clusters.

The first mentioned crystals contain chiefly the nitroso-compound and probably to a very small amount also the nitro-compound, so this may be a new case of the formation of solid solutions by sublimation. Probably, this pair of compounds lends itself to the measuring of the vapour tensions of these solid solutions.

Zaandam, December 1907.

Physics. — *"Observation of the magnetic resolution of spectral lines by means of the method of* FABRY *and* PEROT." By Prof. P. ZEEMAN.

The interference method of the parallel semi-silvered plates, worked out with so much ingenuity by FABRY and PEROT[1]) excels above all other spectroscopic modes of procedure by the accuracy with which its theoretical foundations may be practically realized.

The principal task of the experimenter in applying this method has become to effect the perfect parallelism of the reflecting silvered plates.

In order to test by an independent method some recent results obtained in an investigation of the magnetic resolution of spectral lines[2]) the method of FABRY and PEROT seemed most appropriate. Especially it appeared possible to extend at the same time the investigation to the behaviour of the lines in weak fields. The present paper is preliminary to a discussion of numerical results. I think it beforehand very improbable that errors of ruling of the ROWLAND grating will turn out to be the reason of the asymmetrical resolution of some lines, which I have described.

The method of FABRY and PEROT is applied in the present paper for the first time to an investigation of the magnetic separation of spectral lines. In some places in the literature of the subject the opinion is expressed that the method of interference fringes of silvered layers cannot be used for the subject under review. The

1) FABRY et PEROT, Ann. de Chimie et de Physique 1899—1904.

2) ZEEMAN, These Proceedings, November 1907.

main objection is derived from the great loss of intensity in the apparatus of FABRY and PEROT. The present paper proves that this objection is not insuperable.

2. Of the two forms in which the method of the parallel plates is employed, the simplest, also requiring the least costly apparatus, has been used for the actual measurement of wave-lengths by FABRY and PEROT[1]), lord RAYLEIGH [2]) and EVERSHEIM [3]). This form of instrument called *étalon*, I also have preferred. The distance of the silvered surfaces is here constant. The glass plates are held up to rounded distance-pieces (made of steel), by adjustable springs, which permit to regulate the pressure. By variation of the pressure the steel and the glass can be deformed in an extremely small degree and the accurate parallelism of the glass plates effected; the parallelism being secured already very approximately beforehand by the accuracy or finish of the distance-pieces.

3. The theory of the comparison of wave-lengths by means of this apparatus is simple, and has been given by FABRY and PEROT. We will apply it to the magnetic resolution of spectral lines and especially to the most simple case, the division into a triplet.

Let λ_0 be the wave-length of the original line (afterwards therefore the middle line of the triplet). To this corresponds a system of rings; let P_0 be the ordinal number of the first from the centre. The ordinal number at the centre p_0 is then the integer number P_0, augmented with a fraction ε_0, hence $p_0 = P_0 + \varepsilon_0$. Ordinarily $0 < \varepsilon_0 < 1$.

The diameter of a ring increases with ε. Let e be the thickness of the plate of air, the order of interference at the centre is $p_0 = \frac{2e}{\lambda_0}$. At an angle i with the normal to the plate the order of interference is $p_0 \cos i$.

If x_0 be the angular diameter of the ring P_0, then we have, observing in the focal plane of a lens, $p_0 \cos \frac{x}{2} = P_0$. Developing the cosinus

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1) FABRY et PEROT, Ann. de Chim. et de Phys. T. 25, Janvier 1902. C.R. 27 Mars 1904. FABRY et BUISSON, C. R. 16 Juillet 1906. BARNES Astrophysical Journal. Vol. 19. p. 190. 1904.

2) Lord RAYLEIGH, Phil. Mag. Vol. 11, p. 685, 1906.

3) EVERSHEIM, Zeitschr. f. wissenschaftl. Photographie, Band 5 p. 152, 1907.

$$p_0 = P_0\left(1+\frac{x_0^2}{8}\right)$$

or

$$\varepsilon_0 = P_0\,\frac{x_0^2}{8} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Let λ_r be the wavelength of the outer component of the triplet towards the red then, if P_r, ε_r and x_r have a significance corresponding to that of P_0, ε_0 and x_0,

$$\varepsilon_r = P_r\,\frac{x_r^2}{8}.$$

We have however $\lambda_0\,(P_0+\varepsilon_0) = \lambda_r\,(P_r+\varepsilon_r)$, whence

$$\lambda_r = \lambda_0\,\frac{P_0}{P_r}\left(1+\frac{x_0^2}{8}-\frac{x_r^2}{8}\right) \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

In like manner, λ_v, P_v, x_v determining the component of the triplet towards the violet, we have

$$\lambda_v = \lambda_0\,\frac{P_0}{P_v}\left(1+\frac{x_0^2}{8}-\frac{x_v^2}{8}\right) \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

In the case of radiation in a magnetic field this expression may often still be simplified. In many cases we may choose

$$P_0 = P_v = P_r \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

Looking at the system of rings corresponding to λ_0, while the magnetic force is slowly but gradually increased one sees at the same time rings which proceed from the system λ_0 and are moving outwards and others which are moving inwards. The rings corresponding to λ_r are contracting, those corresponding to λ_v are expanding.

It depends upon the value of p of the étalon and upon the intensity of the magnetic field how great the expansion and contraction of the rings, in comparison with the distance of the rings λ_0, will be.

The value of p and the maximum magnetic force will determine whether in the centre new rings will appear or respectively will disappear.

In the case one does not select for measurement the smallest rings but if the rings λ_r and λ_v, *which originate from the same ring* λ_0, are suitable to be measured, ε can become larger than unity.

When we select the rings thus specified the equality (4) applies and then we may determine λ_r and λ_v from the angular diameters of the rings and the value of λ_0, regarded as known; the result is then independent of the accurate value of the thickness of the plate of air.

Of course the position of the new rings between the rings λ_0 will, with a given value of the magnetic force, be determined by the thickness of the plate of air and what might be called "the sensibility" of the system of rings to magnetic forces will increase with the thickness of the plate of air. A limit of this sensibility is (often too soon) attained by the effective width of the spectral lines under consideration.

In some cases it will be desirable to select for measurement rings different from the three specified ones. There are no difficulties about the significance of P; it always means the ordinal number of the measured ring.

However if P_0 differs from P_r or P_v, their values must be known for the calculation according to (2) and (3).

4. Besides the simplification resulting from equation (4) there is still another one to be considered in the investigation of the radiation in a magnetic field.

I mean that the quantity $e = p\frac{\lambda}{2}$, the optical thickness of the plate of air may be treated as an absolute constant.

Ordinarily this thickness depends upon λ. In consequence of the change of phase by reflection upon the silver, which varies with wavelength, the comparison of different coloured systems of rings finally requires a knowledge of the optical thickness for each separate colour.

It is clear that in the application to the subject now under review only systems of rings corresponding to rays differing extremely little in wavelength are considered, hence the variation of thickness with wavelength needs not to be taken into account.

5. Figures 1 and 2 may give an idea of the aspect of the magnetic resolution of the spectral lines observed by means of the method of Fabry and Perot. They are about sixfold enlargements of negatives taken with an étalon with an interval of nearly 5 m.m. between the plates. The source of light in the magnetic field was a small vacuum tube charged with mercury. The order of interference at the centre for the mercury line 5791 is at 16° about 17265,7.

The system of rings was formed in the focal plane of a small achromatic lens of 18 m.m. aperture and of 12 c.m. focus. Its focal plane coincides exactly with the plane of the slit of a small spectroscope. When the slit is opened widely each spectral line is seen as a rectangle with bright rings or parts of rings as the case may be. The part of the spectrum in the figures refers to the two yellow and the green mercury lines. In fig. 1 the two rectangles corresponding to the two yellow mercury lines are superposed. The green mercury line is largely overexposed. I have reproduced it also in order to give an idea of the dispersion used. The intensity of the magnetic field in figures 1 and 2 was about 5000 Gauss.

It is a very beautiful sight to watch the moving system of rings, while the magnetic force is slowly increased. The rings λ_r and λ_v are first seen approaching, then coïnciding, separating, coïnciding for a value of the field of about 15000 Gauss with the next ring λ_0, passing over this ring, etc.

For measurements it is necessary to reduce the width of the slit, as in Fig. 2. Owing to rise of temperature the rings have somewhat expanded.

6. For measurements, which I hope to communicate in a future paper, I have used not only the method of diameters resumed above (§ 3) but also *the method of the coïncidences* [1]) for the distinct values of the magnetic force, which bring to coincidence λ_r and λ_v, or λ_r and λ_v with λ_0.

Concerning the difficulties attending the use of the method of coïncidences FABRY and PEROT [1]) remark:

"Même avec ce perfectionnement, la méthode présentait des inconvénients assez graves :

1. La necessité d'éclairer simultanément l'appareil par les deux sources entrâine des pertes de lumière assez importantes ;

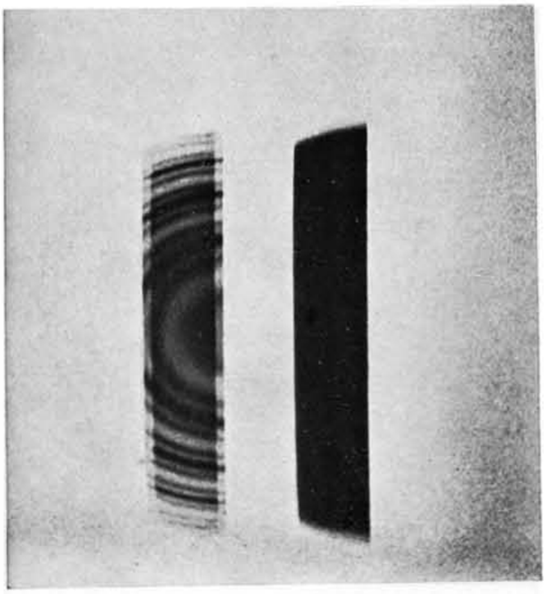
2. Les coïncidences ne sont bien observables que lorsque les deux systèmes d'anneaux ont des éclats comparables, et cette condition n'est pas toujours facile à réaliser ;

3. La recherche de la coïncidence entrâine toujours des tâtonnements et l'on n'est jamais sur (lorsque la période est courte) d'en rencontrer une qui soit exacte."

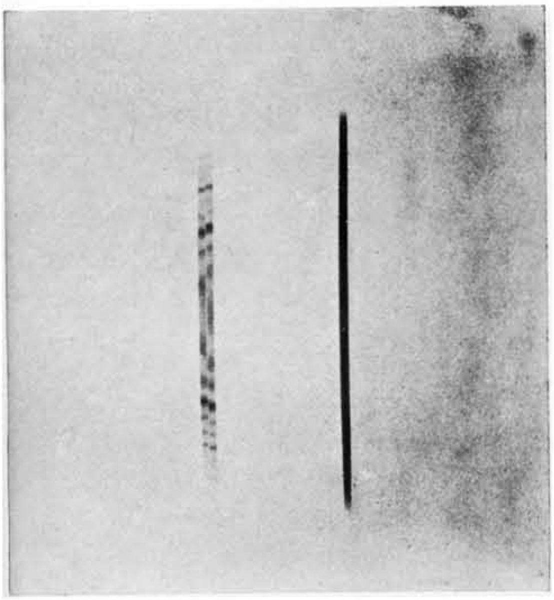
The drawbacks to the method, mentioned sub 1. and 2. are eliminated in the application to radiation in a magnetic field, now under consideration. By variation of the current in the electromagnet the coincidence can be attained with the desired degree of accuracy and hence also the third objection is obviated.

[1]) FABRY et PEROT, Ann. de Chim. et de Phys. p. 12, T. **25**, Janvier 1902

P. ZEEMAN. "Observation of the magnetic resolution of the spectral lines by means of the method of **Fabry** and **Perot**."



1. The yellow mercury lines in magnetic field. Very wide slit. Green mercury line overexposed.



2. The same lines. Narrow slit for measurement of the yellow mercury lines.

7. Some further details concerning the apparatus may finally be given. The mounting and the plates of the 5 m.m. étalon are by JOBIN. The inner surfaces of the plates are accurately flat. The outer surfaces need only ordinary flatness, they are inclined at an angle of 1′ to the inner ones. The plates of the étalon are vertical, and the whole apparatus is capable of the necessary adjustments in azimut, while also a horizontal sliding motion parallel to the plates of the étalon was provided for.

An image of the vacuum tube was focussed upon the étalon by means of an achromatic lens of 12 cm. focus, the enlargement being four times. All optical pieces were mounted upon double *T*-pieces and therefore rigidly connected.

The figures clearly indicate that for the investigation of the magnetic separation of the yellow mercury lines, it would be of no value to use an étalon of greater optical thickness of the plate of air. On the contrary the effective width of the yellow mercury lines when under magnetic influence is rather large, so that the limits of the method in this case are being rapidly approached.

Physics. — "*Isotherms of monatomic gases and their binary mixtures. I. Isotherms of helium between* $+100°$ *C. and* $-217°$ *C.*" Communication N°. 102^a from the Physical Laboratory at Leiden. By Prof. H. KAMERLINGH ONNES.

§ 1. On account of the important rôle, which VAN DER WAALS' theory plays in many chapters of thermodynamics, experimental data concerning the equation of state of a substance are of the greater value as the interaction of the molecules of this substance conforms the better to the hypotheses from which VAN DER WAALS started. The knowledge of the equation of state of the monatomic gases, whose molecules we must consider as the simplest for the present, is of the greatest importance from this point of view.

In Comm. N°. 69 (April 1901) on the isotherms of diatomic gases and their binary mixtures it was already observed that the investigation of the net of isotherms of argon and of helium promised still more important results than the completion of the net of isotherms of the gases formerly called permanent, particularly of hydrogen, at low temperatures, on which subject my attention had been chiefly fixed since the establishment of the cryogen laboratory (cf. Comm. N°. 14, Dec. '94). But the difficulty of obtaining argon and helium in so pure a state and in such quantities as are required for the determination of isotherms, retarded the determination of the equation of state of helium and argon for a long time after Comm. N°. 69.

The investigations on the isotherms of hydrogen are in progress, and yielded already results laid down in communications N°. 78, 97^a, 99^a and 100^a, which, I hope, will soon be followed by others. In the meantime, however, also the difficulty of obtaining pure helium has been quite, that of obtaining pure argon nearly overcome. The successful preparation of pure helium was chiefly due to the hydrogen circulation (Comm. N°. 94^f) yielding the required liquid hydrogen. So the first measurements from the series which will refer to the monatomic gases and their binary mixtures, can already be communicated.

They concern the isotherms of helium, which have now taken the place occupied by the isotherms of hydrogen before the hydrogen was liquefied. Among others the isotherms must lead to the calculation of the critical quantities for helium. From the now communicated determinations of the compressibility along different isotherms at densities which are comparatively small and differ only slightly, the critical temperature can already be calculated by approximation.

§ 2. *Survey of the determinations.*

This investigation comprises some six determinations of isotherms. The temperatures at which they were made, were kept constant and determined in the same way as in the determinations of isotherms for hydrogen published in preceding Communications N°. 97^a (April 1907), N°. 99^a (Sept. 1907), N°. 100^a (Jan. 1908). The readings of the hydrogen thermometer were reduced to the absolute scale by means of formula (4) of Comm. N°. 97^b (Sept. 1907) with the new coefficients of § 2 of Comm. N°. 101^b. The six temperatures thus reduced to which the isotherms refer, are:

$$+100°.35,\ 20°.00,\ 0°,\ -103°.57,\ -182°.75 \text{ and } -216°.56.$$

Besides the measurements at the two standard temperatures 0° C. and 100° C.[1]) and those at low temperatures a determination was made at 20° C. to obtain data for the calculation of the quantity of gas in the stem of the piezometer and in the other parts, which remain at the ordinary temperature during the measurements.

For all these isotherms the densities, at which the pressure was observed, lie about between the same limits which were set by the

[1]) The results at 0° C. and 100° C. are incompatible with those of RAMSAY and TRAVERS, which, indeed, show strange deviations.

dimensions of the piezometer and by the manometer. The utmost limits of the density are 25 and 54 times the normal one. The piezometer and further auxiliary apparatus were perfectly the same as have served for the determinations of C. BRAAK and me (see Comm. N°. 100^b Jan. 1908) with hydrogen at 0° C. and 100° C. The satisfactory results obtained then, enhance at the same time the reliability of the measurements considered now.

§ 3. *Results for* pv_A.

The subjoined table contains the results of the determinations. The first column gives the number of observation, the second the temperature measured above 0° C. on the absolute scale, the third the pressure in atmospheres, the two following ones the product pv_A, and the density d_A, in which the volume of the gas v_A is expressed in the normal volume (that at 0° C. and 1 atmosphere) and the density d_A in the normal density (that at 0° C. and 1 atmosphere). (Compare the corresponding tables of the above mentioned determinations of isotherms of hydrogen).

The calculation of these results was made as follows:

First the points of the isotherm of 20° C. were calculated, (cf. also § 8 of Comm. N°. 79 (April 1902)) and the coefficients A_A and B_A of the curve

$$pv_A = A_A + \frac{B_A}{v_A} + \frac{C_A}{v_A^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

were determined by the 3 points by the aid of the method of least squares. For C_A a definite value was assumed, the densities being too small for this coefficient to be determined with sufficient certainty. If we write VAN DER WAALS' equation with the second correction for the size of the molecules in the form:

$$pv = RT + \frac{RT\,b - a}{v} + \frac{5}{8}\,\frac{RT\,b^2}{v^2},$$

where v is the volume of the gas under the pressure p at the absolute temperature T, expressed in the theoretical normal volume (see Comm. N°. 71 § 3) and if we put the value of A_A at 0°, $A_{A_0} = 1$, which approximation is allowed for our purpose, we find (cf. Comm. N°. 71 § 3) for the value C_{A_T} of C_A at T

$$C_{A_T} = {}^5/_8\, RT\, b^2,$$

where $R = 0.0036619$. The value of b was first estimated at 0.0005 (cf. the note to § 6 of Comm. N°. 96^c Jan. 1907), afterwards at 0,000432, see § 4. With the coefficients A_A and B_A obtained for 20°

TABLE I. He. Values of pv_A.

N°.	ϑ	p	pv_A	d_A
1	+ 100°.35	42 574	1 38725	30.089
2		54 459	1 39314	39.091
3		66 590	1 39929	47.589
4	+ 20°.00	27.539	1.08664	25.343
5		36.303	1.09028	33.297
6		53.708	1.09918	48 862
7	0°	26.634	1.01392	26.268
8		38.565	1.01851	37 864
9		50 240	1 02521	49 004
10	— 103° 57	20.580	0 63135	32 597
11		24.100	0 63296	38.075
12		29.185	0.63597	45.891
13		33 383	0 63845	52 288
14	— 182° 75	13 751	0.33787	40 699
15		16 019	0.33898	47.257
16		18.189	0 34025	53.457
17	— 216° 56	9 564	0.21132	45 259
18		10.502	0 21171	49.606
19		11.448	0 21219	53.951

in this way, the reductions to 0° for the gas which is outside the reservoir at a temperature of 20° and for a small part at the temperature of the room, were carried out in first approximation. With the three points which were thus found on the isotherm of 0°, the virial coefficients A_A and B_A were then calculated also for this temperature.

From this follows A_{A_0}, the value of pv_A for $d=0$ by means of the formula:

$$A_{A_0} = 1 - B_{A_0} - C_{A_0}.$$

With the pressure coefficient from 0° C. for the state of AVOGADRO, 0.0036619 (cf. § 1 of Comm. N°. 101^b) follows for 20°:

$$A_{A_{20°}} = A_{A_0}(1 + 0.0036619 \times 20),$$

so that on the isotherm of 20° a fourth point is acquired, which renders the slope of the pv_A line a great deal more certain (cf. the conclusion of § 1 of Comm. N°. 101[b]). Then the calculation of A_A and B_A was repeated, and by the aid of these corrected coefficients the isotherm of 0° C. was again calculated, and this calculation by approximation was continued till it caused no longer any change. In this way we found for 20° C. (for b=0.000432):

$$pv_{A_{20}} = 1.07273 + 0.0005337\, d_{20} + 0.000000125\, d_{20}^2 \quad . \quad . \quad (3)$$

With this formula the corrections have been calculated for the determinations of isotherms. For the rest the latter were treated as in the preceding communications.

§ 4. *Individual virial coefficients.*

We may avail ourselves of the data of table I in order to derive the coefficients A_A and B_A by the aid of the method of least squares. C_A was assumed according to formula (2) of the preceding §. For every isotherm $pv_{A,d=0}$ was calculated, and this value was added to the others as if it concerned a new observed point. This calculation was effected by the aid of the value $A_{A_0} = 0.99949$, which may be derived from the value for the coefficients B_A and C_A for 0° finally obtained in the calculation by approximation from the conclusion of the preceding §. Table II contains the virial coefficients and at the same time the differences between the given pv_A's and the calculated ones. These differences are arranged according to the ascending densities. So the first column of differences refers to $pv_{A,d=0}$, the others to the data of table I in the above succession.

The calculation of the B_A's is still uncertain, because for C_A estimated values have been assumed. Determinations of pv_A at greater densities, which will render an independent determination of C_A possible, are in preparation.

That the estimations of C_A are not too inaccurate, may be made probable as follows. For 100° follows from table II $B_{A\,100°} = 0.000673$. On the suppositions on which VAN DER WAALS' equation rests, the value of b may be derived from the value for two temperatures of $B = RTb - a$ with $B = B_A\,(A_{A_0})$ and then $b = 0.000432$ is found, which does not differ much from the value 0.0005, which was first assumed by way of estimation on other grounds. Though the calculation followed here is very uncertain, yet the found value was preferred to the first estimated one, and for this reason the calculations which were first made with 0,0005, have been repeated with this new estimation. The differences of the results lie within the limit of errors of observation.

TABLE II. He. Individual virial coefficients.
Deviations of the pv_A from the calculated ones.

θ	A_A	$10^3 B_A$	$10^6 C_A$	10^5 (O—C)				
+100°.35	1.26667	+0.673	+0.16	+10	—21	—10	+22	
+ 20°.00	1.07273	+0.534	+0.13	— 3	+31	— 36	+ 8	
0°	0.99970	+0.512	+0.12	—20	+80	—75	+15	
—103°.57	0.62026	+0.337	+0.07	+11	— 7	— 32	0	+27
—182°.75	0.33066	+0.176	+0.04	1	— 1	— 8	+ 8	
—216°.56	0.20693	+0.096	+0.02	0	+ 1	— 3	+ 3	

§ 5. *Determination of the critical temperature of helium.*

From the data of table II we may arrive at a first estimation concerning the critical temperature of helium, which will be found from determinations of isotherms within the now accessible region of temperatures.

Extrapolation proves that the BOYLE-point will lie in the neighbourhood of — 250° C. For hydrogen — 166° was found for this. (cf. Comm. N°. 100^a). If we assume 30° K. for the critical temperature of hydrogen, then follows from this for helium

$$T_{kHe} = 6° K.$$

If this value of T_k is adopted, the region of temperature — 217° to — 183° for helium corresponds with that of 0° to + 200° for hydrogen. By applying the law of the corresponding states to the slopes of the pv_A-lines for the two substances in the neighbourhood of these equivalent limits of temperature, we arrive at a slightly lower value of the critical temperature viz.

$$T_{kHe} = 5°.3 K.$$

This value too I think I may still consider as a highest limit for the critical temperature of He, as it seems probable to me, that He with respect to H_2 will deviate from the law of the corresponding states in this sense that the critical temperature will be found lower than would follow from the application of this law to corresponding states for values of the reduced temperatures many times larger than 1.

Now there can only be question of a first estimation based on determinations of isotherms. The determination of the isotherms of — 253° and — 259°, which is in progress, will, I hope, soon lead to a more reliable estimation.

In conclusion I gladly express my thanks to Mr. C. BRAAK for his assistance in this investigation.

(January 23, 1908).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday January 25, 1908.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 25 Januari 1908, Dl. XVI).

CONTENTS.

Zoology. — "*Physiological regeneration of neurofibrillar endnets (tactile discs) in the organ of Eimer in the mole.* By Dr. J. BOEKE and Dr. G. J. DE GROOT. (Communicated by Prof. G. C. J. VOSMAER).

(Communicated in the Meeting of November 30, 1907).

In recent years several authors (RANVIER, VON LENHOSSEK inter alia) have called attention to the fact, that there where intraepithelial nerves in mucous membranes or in the epidermis are found penetrating even between the superficial layers of epithelium cells covering the sensory surface (so for example in the peribulbar nerve-endings between the taste-buds in the papillae of the tongue, etc.), we have to draw the conclusion, that at the same time as the superficial cells degenerate and are cast off, the sensory nerves with their knob-like end-swellings or end-loops of the neurofibrillae must undergo a perpetual change and growth. But then these are always the fine ramifications and endings of the nerves, which branch between the deeper layers of epithelium cells. Real neurofibrillar endnets like those which are formed round the base of the tactile cells of MERKEL, are always found in the deeper layers of the epidermis, where they lie protected by the other epithelium cells. These tactile cells nowhere degenerate so quickly as it is the case with the superficial cells of the upper layer of the epithelium, and need not be replaced by other cells coming from the deeper strata. There is no need of a quick regeneration of the neurofibrillar endnets (and the tactile cells).

But suppose we had a tissue, where in the uppermost strata of a stratified epithelium, in which the superficial cells quickly degenerate and are cast off, we find real tactile cells with distinct neurofibrillar endnets, which therefore must degenerate at the same time as the surrounding cells, how would the process of regeneration of the neurofibrillae take place there?

In the course of investigations carried on in the histological laboratory at Leiden we found a favourable object to study this question in the sensory organs in the snout of the earth-mole (Talpa europaea).

Here we find an extremely sensitive tissue (the organ of EIMER) the elements of which are only protected by a very thin horny layer, and which by reason of its lying at the tip of the snout must, on account of the well-known habits of the animal, continuously form new horny cells for the protecting horny layer above, because otherwise the functional cells would very soon come to lie at the surface and be liable to be injured.

The structure of the peculiar organ first described by EIMER (1870) and the innervation of it, have been studied in the course of this year (1907) by two authors [1]) by means of the recent improved methods of staining the neurofibrillae. Both give about the same description but arrive at different conclusions.

As is well known, the organ of EIMER consists of thickenings of the epidermis, formed by columns of epithelial cells in the shape of an hourglass, which form small round prominences on the surface of the snout, and which, because the columns of cells are longer than the thickness of the epidermis at the place where they are found, project with their base into the corium, and form here a bulging out of the epithelium, generally described as "buffershaped". Each of the columns is made up of several strata of more or less flattened epithelial cells, which at the base of the column do not reach from one side to the other, but are wedgeshaped and overlapping each other with the thinned-out ends. Nearer the surface the cells gradually become flattened and larger, until only two cells lying at the same niveau, fill out the entire cross-section of the sensory column (fig. 1, 5). There the column ends as it reaches the horny layer. The cells of the column are, according to BOTEZAT, true spiny cells like the other cells of the epidermis (fig. 3).

In the axis of the column a thick nerve fibre, the axial fibre, runs through the whole length of it, penetrating into the epithelium at its base. Sometimes there are two or three axial fibres. Around the column of cells a set of 18 or 19 thin, unbranched nerve fibres, closely set, somewhat zigzag, run upwards between the outer ends of the cells of the column and the adjoining epidermis-cells, until they reach the horny layer. These are called rand-fibres to distinguish them from the axial fibre. At the base of the column between the epidermis-cells a small number of tactile cells of MERKEL are found, and underneath the epidermis in the corium one or two small Pacinian corpuscles.

EIMER already described small varicosities or knoblike swellings of the nerve-fibres in the upper part of the columns. The nerve-fibres run more or less zigzag between the cells. EIMER himself and after him HUSS (1898) thought that these knoblike varicosities were lying intracellular, the nerve-fibres running between the cells. The varicosities are therefore attached laterally to the nerve-fibres.

[1]) EUGEN BOTEZAT. Anat. Anzeiger, 30 Bd. 1907.

M. BIELSCHOWSKY. Anat. Anzeiger, 31 Bd. 1907.

31*

In this year (1907) BIELSCHOWSKY [1]) has investigated the nerves of the organ of EIMER by means of his method of staining the neurofibrillae, and although he does not give much that is new, as he says himself in his paper, his study is interesting because with that by BOTEZAT it is the only one, in which the new staining methods for the nervous system are used for this organ. We may quote here what he says about the course and the peculiarities of the nerve-fibres, because this makes clear his opinion better than a long description. The course of both the axial fibre and the randfibres he assumes to be entirely intercellular: "irgend ein näherer Konnex der Fasern zu den Epithelzellen findet nicht statt; ihr Verlauf ist ein rein intercellulärer. Im Bereiche der äusseren Schicht weisen sie in scheinbar regelmässigen Abständen die bekannten punktförmigen Varikositäten auf...... Die Varikositäten sind offenbar nur auf Zerfallsvorgänge zurückzuführen. Dafür spricht der Umstand, dass sie immer erst in der Verhornungszône des Epithels deutlich hervortreten. Aehnliche Beobachtungen kann man auch am Schweinerüssel and anderen rüsselformigen Säugerschnauzen machen". (l.c. p. 189).

In his last paper, published some months before the paper by BIELSCHOWSKY appeared (April 1907), BOTEZAT [2]) who in his paper of 1903 pronounced the same opinion as EIMER and HUSS, viz. that the knoblike thickenings of the nerve fibres penetrate into the cells of the column of EIMER, adopts the view that they are epicellular, after a study of the nerves coloured with methylene blue and after the method of RAMON Y CAJAL. "Der Beweis hierfür lässt sich am besten dadurch erbringen, dass man die Terminalknöpfchen fast genau zwischen den Zellen des Organs liegen sieht." BOTEZAT states that the varicosities possess a netlike structure. Because they are excessively small, the extreme sensibility of the snout must be due to the very large number of the terminal knobs ("tactile discs") and not to their great perceptibility. A column of EIMER consists of about 15 layers of cells, and in each layer about 20 of these tactile knobs are to be found. The total number therefore is in each organ of EIMER 300, and for the entire snout more than 100000 [3]). According

[1]) M. BIELSCHOWSKY, Ueber sensibele Nervenendigungen in der Haut zweier Insectivoren (Talpa europaea und Centetes ecaudatus). Anat. Anzeiger. Bd. 31, p. 187—194, 1907.

[2]) EUGEN BOTEZAT. Ueber die epidermoidalen Tastapparate in der Schnauze des Maulwurfs etc. Archiv für Mikroskopische Anatomie. Bd. 61. p. 730—764. 1903.

[3]) EUGEN BOTEZAT. Die fibrilläre Struktur von Nervenendapparaten in Hautgebilden. Anat. Anzeiger. Bd. 30. p. 321—344. 1907.

to Bielschowsky the total number is $\pm 150{,}000$, together with more than 5000 end-bulbs and numerous cells of Merkel.

Although he does not ascribe to the varicosities a high degree of perceptibility, Botezat assumes all of them (both of the axial fibre and of the rand-fibres) to be tactile discs, in accordance with most authors. A difference in structure between the different tactile discs or knobs he mentions without paying much attention to it.

Now the facts seem to us to point to a different conclusion.

The opinion of Bielschowsky, that the varicosities of the nerve-fibres are due to "Zerfallsvorgänge", seems to us to be erroneous. In the first place these varicosities do not appear first in the horny zone. On the contrary, as soon as the cells are transformed into horny cells, the fibres and their varicosities degenerate, and the first varicosities appear seven to eight layers of cells lower down. In the second place the varicosities are much too regular and are distributed with a far too great regularity to be the mark of degeneration, and are always present in nearly the same number. In the third place their structure does not point at all to "Zerfallsvorgänge."

But in his description Botezat too does not seem to have hit the point. He does not give an explanation of the difference in structure of the varicosities and of their mode of attachment to the nerve-fibres, and of the fact that they are only to be found in the peripheral part of the nerve-fibres and not in the basal half.

When we treat a small piece of the snout of the mole, after fixation in formaline, according to the method of Bielschowsky—Pollack, and study a correctly differentiated preparation in thin (6 μ) longitudinal sections (that is a longitudinal section of the nerve-fibres and of the column of cells, the section being made at right angles to the surface of the epidermis of the snout), the following details will be seen: the structure and form of the varicosities ("Terminalknöpfchen, Seitenknöpfchen") are not the same in the course of the nerve-fibres. When we follow a rand-fibre from the base of a column of Eimer to the top, the first swellings appear at a distance of 10 to 12 cell-layers from the top (fig. 1, 5). The swellings are here only loosely built small nets, lying in the course of the nerve-fibres, nothing but a local slackening of the bundle of neurofibrillae in the fibres, the fibrillae probably forming a few anastomoses. From this point upwards we see these networks appearing with great regularity in the course of the nerve-fibres where the fibre passes another cell of the column, and each time the reticular structure becomes finer and more distinct (fig. 1, 3).

In the upper four to five rows of cells a change in the form and arrangement of the networks becomes visible. The small swellings of the nerve-fibre no more lie in the course of the nerve-fibres, but more and more pass to the side of it (fig. 1, 3) and at last they lie entirely beside the nerve-fibre, being connected with it by means of a very small and short stalk (fig. 1, 2, 3). The swellings of the rand-fibres always pass to that side of the fibre lying close against the cells of the column of EIMER, and so project centripetally (fig. 1, 6). So when we look at a rand-fibre from the outside of the column, as in fig. 5, we see nothing or only very little of this change of place of the varicosities, and only when we play up and down with the micrometer-screw of the microscope, we are able to make out that the peripheral rows of varicosities lie in reality underneath the fibres.

So in the first place we see a very regularly occurring change of place of the varicosities, as the fibre approaches the surface of the epithelium. When we only take the place of the fibre we are examining in the section into account, this change is always found to take place with perfect regularity.

In the second place the following change may be seen: the nerve-fibres of the organ of EIMER (both the rand-fibres and the axial fibres) run between the cells of the epithelium. The first swellings or varicosities, the small loose nets lying in the course of the fibres, of course also appear between the cells. But as soon as these varicosities get larger and change their places, so that they come to lie besides the fibres, they push their way *into* the substance of the cells of the sensory column and not between these cells. They become *intracellular*. In the preparations stained after the method of BIELSCHOWSKY the cells and their margins and nuclei are so clear and distinct, that when we only take care to examine thin sections (5—6 μ), this fact may be stated with perfect clearness. Fig. 1, 2 and 3 give a good idea of it; when we examine longitudinal sections of the rand-fibres, the section passing through the axis of the sensory column, we see as it were the varicosities or knobs push their way into the protoplasm of the cells. In cross-sections now and then we come across places, one of which is figured in fig. 4. The tactile knob growing into the flat epithelial cell, pushes its way into the protoplasm apparently with some force so that the flattened nucleus is curved in by it. Similar drawings are given by HUSS.

Another question is, whether these varicosities or tactile knobs lie in the protoplasm of the cell, become an integrating part of it. The facts seem to point to the contrary. On observing the fibres and their tactile knobs closely under the highest power, we get the impression that even there where the knobs lie intracellularly, the neurofibrillae are still surrounded by a very thin layer of perifibrillar substance, taking a different stain from the protoplasm of the cell itself. But of course this layer of perifibrillar substance must be continuous with the surrounding protoplasm. The neurofibrillar network remains entirely independent, but a trophic connection of the perifibrillar substance and the protoplasm surely must be present. This seems to us to be beyond doubt, and we may venture to suggest, that only now the varicosities reach their full development, are real tactile discs; as long as they lie between the cells, the varicosities are only parts of the nerve-fibres where the neurofibrillae are getting looser and growing out, but only when they pass to the side of the fibres and grow into the cells, they become real tactile neurofibrillar end-nets. The rows of varicosities are merely stages of development of the tactile discs.

The end-knobs or terminal discs in the upper row of cells of the sensory column, which are already on the point of passing into the horny layer, are for the greater part already lying loose in the cells, the nerve-fibres themselves and the connecting stalks atrophying. So in fig. 5 the four knobs, represented by black spots in the upper row of cells, are entirely separated from the nerve-fibres below, and the same fact is to be seen in the fig. 1 and 3, where a part of the nerve-fibre (the stalk of the end-knob) was still stained. The argument, that this independence of the terminal knobs is due to the connecting stalks not being cut in the section examined, is annihilated by a close study of many sections. Thus we can state with perfect accuracy, that the connecting fibre really does not exist any more (at least, is not stained as the functional fibres are).

The axial fibre shows the same peculiarities as the rand-fibres, but the tactile nets are larger and more rounded; the axial fibre too runs between the cells until its end; even there where, in the upper part of the column, the entire cross-section is composed of two cells, the line between these cells runs just through the middle of the transverse plane (cf. Huss) and leaves a small room just in the axis of the column, occupied by the axial fibre (fig. 6). The tactile nets grow out from the fibre now at one side and then at the other, and grow into the cells of the sensory column just as it was described for the rand-fibres.

So we find the same peculiarities of structure in all the nerve-fibres and their tactile neurofibrillar networks. The same cause seems to us to underlie all these differentiations, which we may describe as a getting more and more differentiated and independent of the tactile discs (or varicosities) as we draw nearer to the surface of the sensory column.

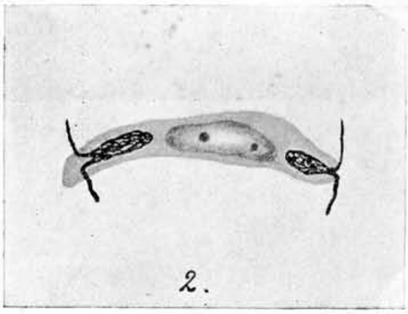
When we see now, that the horny layer above the columns of the organ of EIMER is always thinner than in the adjacent parts of the epidermis (so for example in fig. 1, st. c.), as it was the case in all the preparations examined, and when we bear in mind, that these prominences on the surface of the snout of the mole are continually exposed to all sorts of mechanical insults, the question, put at the beginning of this paper, may be answered in the following manner:

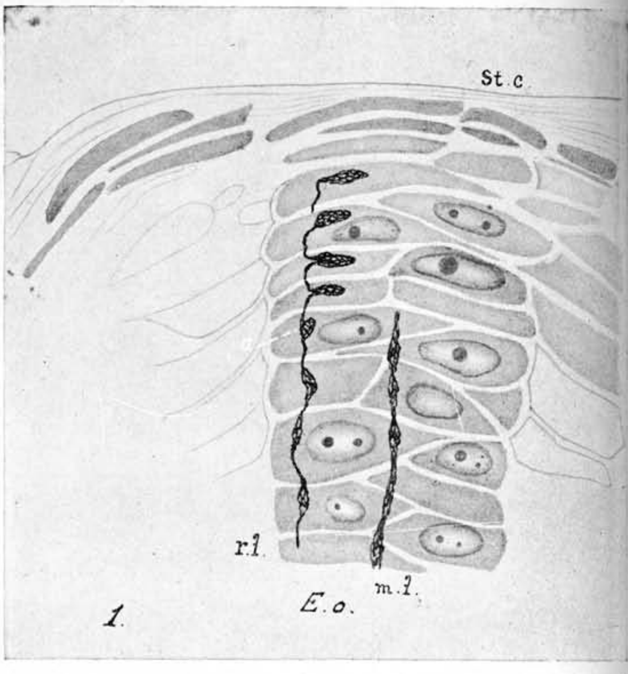
The horny layer above the cells of the column of EIMER being very thin and composed of a few layers of cells, and the horny scales being lost very soon by desquamation, there must be a continual moving upwards of the cells of the deeper layers of the column of EIMER, to take the place of the thrown off cells. With these cells the nerve-fibres must grow upwards at the same rate. About in the middle of their course these nerve-fibres begin to form tactile discs. These corpuscles first appear as places in the course of the fibres where the neurofibrillar structure is looser; these first varicosities little by little pass out of the course of the fibre and grow into the cells of the column of EIMER, and so become real tactile discs. These tactile discs lying at the side of the nerve-fibres and remaining attached to them by a short stalk, are a direct argument for the growing upwards of the nerve-fibres together with the cells of the column. Otherwise the cells would take with them the tactile corpuscles and sever them from the fibres they belong to or draw out the stalks in an oblique direction. Of this no trace is to be found anywhere. It is very probable, that only when the varicosities grow out to small tactile discs and come to lie intracellularly, they acquire a heightened perceptive faculty. As they are continually travelling upwards to the surface, new varicosities are formed underneath in the course of the same nerve-fibre. As soon as the cells undergo the transformation into horny scales, the tactile discs and their connecting stalks and the nerve-fibres atrophy, the former remaining visible longer than the latter. Only the upper rows of tactile discs, of the form of the networks of fig. 2 and fig. 3, seem to be fully developed.

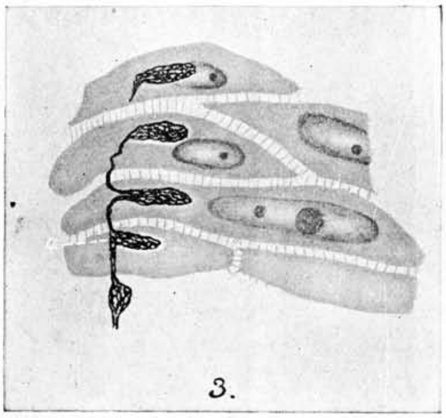
They are continually replaced by others, coming from below. The nerve-fibres of the column of EIMER chiefly grow at the base of the column and atrophy at its top.

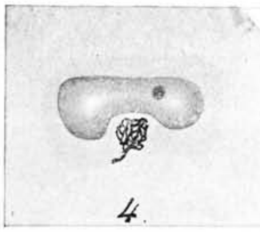
Perhaps these views may be extended to other intraepithelial

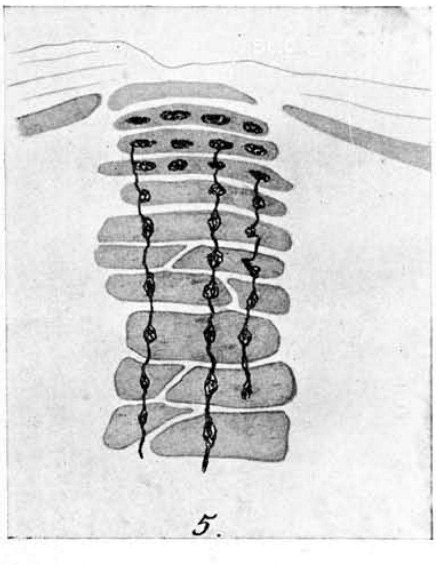
J. BOEKE and G. J. DE GROOT, "Physiological regeneration of neurofibrillar endnets (tactile discs) in the organ of Eimer in the mole."

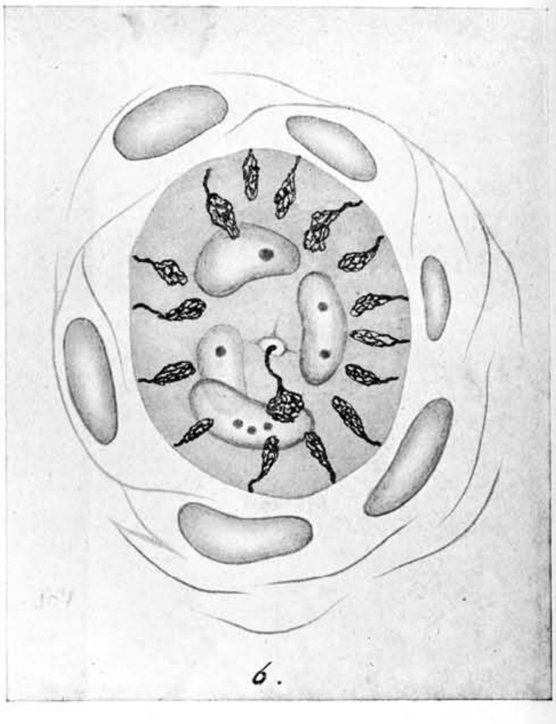












J. Boeke a. n. del.

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nerve-endings. It will be difficult to find an object of study as favourable as the organ of EIMER.

Leiden, Anatomical Cabinet.

DESCRIPTION OF THE FIGURES ON THE PLATE

All the figures are drawn from life from preparations made after the method of BIELSCHOWSKY-POLLACK, with a camera lucida of ABBE. Fig. 1 and 5 are enlarged 1200 times, the others 1600 times. Apochromate-oil-immersion. Sections 5 and 6 μ.

Fig 1 Longitudinal section of the upper part of a column of EIMER of the earth-mole. A rand fibre *(rf)* and a part of an axial fibre *(mf)* are seen. The horny layer *(stc)* above the column of EIMER is distinctly thinner than at both sides of it.

Fig. 2. Longitudinal section of a flat cell of the upper part of a column of EIMER, with two tactile discs, growing into the same cell. The netlike structure and the curious drawing in of the connecting fibre, is clearly shown.

Fig. 3. Longitudinal section of the upper part of a column of EIMER, to show the developing of the tactile discs, and the final atrophying of the nerve-fibre.

Fig. 4. From a cross-section through the upper part of a column of EIMER. A nucleus curved in by a tactile disc

Fig. 5. Longitudinal section through the peripheral part of a column of EIMER. Three rand-fibres are shown. The tactile discs lie behind the nerve-fibres. The intracellular position of the tactile discs is clearly to be seen. The upper cell, in which lie four tactile discs, is being transformed into a horny cell. The nerve-fibres degenerate.

Fig. 6. Cross-section through the upper cells of a column of EIMER. In the section of 6 μ four cells were to be seen, lying two and two in the same niveau. The tactile discs of the rand-fibres all grow centripetally into the cells, the axial fibre runs between the cells.

Astronomy. — *"β Lyrae as a double star."* By J. STEIN, S. J. at Rome. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

1. As far as I know, Professor E. C. PICKERING was the first who, led by his spectroscopic investigations, suggested that β Lyrae might be a close double, the components of which describe circular orbits in a light-period [1]).

This surmise was confirmed by BELOPOLSKY [2]) in 1892. He measured the displacement of the luminous F-line on some fourteen spectographs. They were found to show a minimum (in absolute value) at the time of the minima and a maximum at the time of the maxima of

[1]) Spectrum of β Lyrae. By Prof EDWARD C. PICKERING. A. N. 3051 (1891).

[2]) Les changements dans le spectre de β Lyrae. A. BÉLOPOLSKY. Memorie della Società degli Spettroscopisti Italiani. Vol. XXII, 1893.

the star's light, in such a way that they correspond to an approach before the principal minimum and to a recession after that time. From these observations he derived a circular orbit for that component which eclipses the other at the time of this minimum. The investigation of the Potsdam spectographs equally led Prof. VOGEL [1]) to the conclusion that the displacement of the lines can hardly be explained otherwise than as a consequence of the motion of different bodies having unequal spectra. He does not succeed however in determining the position of the lines with sufficient accuracy. He thinks that the photometric data would lead to the assumption of two bodies of unequal luminosity moving either in a fairly circular orbit or in an ellipse having its major axis in the visual line. On the other hand the spectroscopic investigations would lead to the assumption of two bodies, one showing a spectrum with luminous, the other a spectrum with absorption-lines, which would describe very excentric orbits the major axes of which would make a considerable angle with the visual line. It would be impossible, in his opinion, to satisfy the two phenomena at the same time. In 1896 Dr. MYERS [2]) subjected ARGELANDER's lightcurve ("vera" pro 1850) to an elaborate theoretical investigation. His result is that the whole curve of the lightvariation is represented satisfactorily by assuming two elongated revolution ellipsoids the major axes of which are in each other's prolongation, circulating around each other in nearly circular orbits.

The next year BELOPOLSKY found the duplicity confirmed [3]). This time it was the displacements of the dark *Mg*-line ($\lambda = 448.2$ $\mu\mu$), which enabled him to derive a slightly excentric orbit for the second component viz. of that component which is eclipsed during the principal minimum. Father W. SIDGREAVES, in his latest spectrographic investigation of β Lyrae [4]) arrives at the same result as Prof. VOGEL: rather considerably excentric orbit, the major axis of which makes a great angle with the visual line.

In conformity with what had already been suggested before by

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1) Ueber das Spectrum von β Lyrae. Von H. C. VOGEL. Sitzungsberichte der K. Preussischen Ak. der Wiss zu Berlin. 8 Februar 1894

2) Untersuchungen über den Lichtwechsel des Sternes β Lyrae. Inauguraldissertation... von G. W MYERS, München 1896. — The system of β Lyrae. id. The Astroph. Journ. Vol. VII N⁰. 1.

3) Recherches nouvelles du spectre de β Lyrae, par A. BÉLOPOLSKY. Memorie della Società degli Spettrosc. It. vol. XXVI, 1897. — New Investigations of the Spectrum of β Lyrae, id Astroph. J. Vol. VI N⁰. 4.

4) A spectrographic Study of β Lyrae. By Rev. WALTER SIDGREAVES S. J. Monthly Notices of R. A. S., Jan. 1904.

Dr. MYERS, Prof. CH. ANDRÉ [1]), basing himself on different numerical data, thinks himself justified in assuming, that the excentricity of the orbit has increased since the time of ARGELANDER, and also that the major axis has been displaced. On this supposition ANDRÉ tries to found an explanation of the terms of a higher order in the formula of ARGELANDER as corrected by Dr. PANNEKOEK [2]). Finally Dr. L. TERKAN has brought forward some short considerations in A. N. n° 4067 [3]). Afterwards a more elaborate investigation has appeared in the Memoirs of the Hungarian Academy of Sciences [4]).

We think that this enumeration covers the principal literature about what has been put forward in *explanation* of the light-variation.

2. The original plan of the author of the present paper was a treatment by the method of MYERS of the light curve derived by Dr. PANNEKOEK, in order to ascertain whether any important change of the elements of the orbits since the time of ARGELANDER, might be established.

The first part of MYERS' thesis in which, as a first approximation, a circular orbit is derived, is generally fairly correct. But the second part in which this orbit is changed to a slightly excentric one, by the aid of differential formulae, appeared to call urgently for a fresh treatment. Erroneous normal equations have been derived from incorrect differential formulae. The former have been wrongly solved and finally the close adjustment of the theoretical curve to that of ARGELANDER, chiefly in the vicinity of the principal minimum, seems to have been obtained by a happy coincidence of numerical errors. It is of no use to enter into further particulars on the subject. As an instance we give in the 2nd column of the following table the light-intensities (I_B), as derived by MYERS from the observed grades (Stufen) of ARGELANDER during the period of from 30 hours before to 30 hours after the principal minimum. In the next column are contained the light-intensities (I_{C_1}) given by MYERS as resulting from the definitive elements of his orbit [5]), the 4th col. shows these same quantities freed from numerical errors. In the three last columns the

[1]) Traité d'Astronomie Stellaire par CH. ANDRÉ, 2me p. NN. 460—1.

[2]) Untersuchungen über den Lichtwechsel von β Lyrae. Dr. A. PANNEKOEK. Verhandelingen der Kon. Ak. van Wetensch. te Amsterdam, Vol. 5, N°. 7. id. A. N. N°. 3456.

[3]) Beitrag zur Berechnung der Bahnelemente van β Lyrae. Dr L. TERKÁN.

[4]) β Lyrae pályaelemeinek kiszámítása spektroskopikai és photometriai adatokból. TERKAN Lajostól. — Mathematikai és Természettudományi Ertesitö, XXIV kötet 3 füzetéböl Budapest 1906.

[5]) Inaugural-dissertation, p. 48; A. J. l.c. p. 16.

same quantities have been given reduced to light-grades (σ). I do not find mentioned what is the value of a light-grade of ARGELANDER according to MYERS. From the light-intensities in the two minima I find 0.130 magnitudes, a value to which I have adhered. The intensity of the maximum has been taken for unit. The light-grades of A which, from 3.35 in the principal minimum, rise to the value 12.35 at a maximum, have been reduced to the interval of 3.00 to 12.00 for the sake of convenience.

t	I_B	I_{C_1}	I_{C_2}	σ_B	σ_{C_1}	σ_{C_2}
-30^h	0 7296	0 7525	0.7586	9.27	9.61	9.67
-24	.5836	.6019	.6674	7.40	7.73	8.60
-18	.4336	4993	.5627	4.83	6.15	7.16
-12	.3661	.4275	.4506	3.55	4.85	5 29
-6	.3484	.3487	.3500	3.10	3.13	3.16
0	.3433	.3433	.3433	3.00	3.00	3 00
$+6$	.3490	3488	3477	3.15	3.12	3.11
$+12$	.3988	.4275	.4462	4.30	4.85	5.21
$+18$	.5306	5591	.5586	6 67	7.11	7.10
$+24$	.6572	.6624	6635	8.46	8 54	8.55
$+30$	.7644	.7528	.7553	9.70	9.61	9.64

In what follows we have tried, first of all, to give correct formulae for the derivation of a slightly excentric orbit from the variation of the light. These have then been used for the curve of ARGELANDER and for that of Dr. PANNEKOEK. Afterwards the spectroscopic data of BELOPOLSKY have also been freshly reduced, because there is some uncertainty about the resulting orbit [1]). This is perhaps to be attributed to the method of LEHMAN—FILHÉS [2]). This method is excellent for a satisfactory determination of the excentricity if it is large; but it is less suitable for a very small excentricity. For the drawing of the graphical velocity-curve remains always slightly arbitrary and this fact exerts too strong an influence in the case that e is small.

3. We thus start from the following hypothesis:

1) In the "Recherches nouvelles" (Memorie etc.) BELOPOLSKY gives $e = 0.04$; in his "New investigations" (A. J.) $e = 0.07$, as the result of the same observations.

2) A. N. n. 3242.

Two similar, elongated revolution-ellipsoids move about their common centre of gravity in elliptic orbits. We assume that the major axes of the ellipsoids are continually in each other's prolongation while their centres move about their common centre of gravity in obedience to the laws of KEPLER. Required the intensity of the light as it appears to our eye, if we assume that the ellipsoids may be exchanged for their uniformly illuminated projections on the sphere.

As unit of length we take the semi major axis of the larger ellipsoid (E_1); as unit of brightness the maximum of β-Lyrae.

Further let be:

$\varkappa$ the semi major axis of the smaller body:

q the proportion of the major axis to the diameter of the equator;

f the proportion of the major axis of the ellipse, which is the projection of one of the ellipsoids on the sphere, to the major axis of that same ellipsoid;

a the semi major axis of the relative orbit of the smaller body (E_2), e the excentricity, v the true anomaly, r the radius vector in the true relative orbit.

β the angle formed by the radius vector in the true orbit with the projection of the visual line on the plane of the orbit (on the further side of the sphere); this angle increases with the motion in the orbit;

ω the longitude of the periastron, counted in the same way as β;

i the angle between the plane of the orbit and a plane tangent to the sphere;

ϱ the projection of r on the sphere;

M the common part of two circles the radii of which are resp. $= 1$ and $= \varkappa$, having their centres at a distance of $\varrho' = \dfrac{\varrho}{f}$;

λ the proportion of the brightness (per unit of surface) of the larger of the elliptic projections to the smaller one;

J the apparent total light-intensity at the time t, (as seen from the earth).

As long as E_1 and E_2 do not cover each other, we have:

$$J = f.$$

When E_2 is covered by E_1

$$J = f\left(1 - \frac{M}{\pi(\lambda + \varkappa^2)}\right).$$

When E_1 is covered by E_2

$$J = f\left(1 - \frac{\lambda M}{\pi(\lambda + \varkappa^2)}\right).$$

Let $2\varphi'$ and 2φ be the angles formed by the common chord of the circles, which define M, as seen from their respective centres, then

$$M = \tfrac{1}{2}\{(2\varphi' - \sin 2\varphi') + \varkappa^2 (2\varphi - \sin 2\varphi)\}$$

$$\sin\varphi' = \varkappa \sin\varphi\,; \qquad \cos\varphi = \frac{\varrho'^2 + \varkappa^2 - 1}{2\varkappa\,\varrho'}.$$

φ' is always $< \frac{\pi}{2}$; φ may become $= \pi$, in the case that the smaller disc is seen projected wholly within the larger one.

Furthermore:

$$\varrho^2 = r^2(1 - \cos^2\beta \sin^2 i)\,; \qquad \beta = \omega + v.$$

These formulae agree with those of Dr. MYERS.

Computation of f.

The equation of the cylindre, enveloping the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ the axis of which makes the angles φ, χ, ψ with the X-, Y- and Z-axis, is:

$$\left(\frac{\cos^2\varphi}{a^2} + \frac{\cos^2\chi}{b^2} + \frac{\cos^2\psi}{c^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) =$$

$$= \left(\frac{x\cos\varphi}{a^2} + \frac{y\cos\chi}{b^2} + \frac{z\cos\psi}{c^2}\right)^2.$$

The surface of an orthogonal section of this cylindre is:

$$\Omega = \pi\sqrt{a^2 b^2 \cos^2\psi + b^2 c^2 \cos^2\varphi + c^2 a^2 \cos^2\chi}.$$

or, putting

$$a = b = \frac{1}{q},\ c = 1$$

$$\Omega = \frac{\pi}{q}\sqrt{1 - \varepsilon^2\cos^2\psi}\,, \qquad \varepsilon^2 = \frac{q^2 - 1}{q^2}.$$

The semi minor axis of the section is $\frac{1}{q}$, therefore the semi major axis

$$f = \sqrt{1 - \varepsilon^2\cos^2\psi}\,; \qquad \cos^2\psi = \cos^2\beta\sin^2 i\,;$$

because ψ is the angle between the major axis of the ellipsoid and the visual line.

In the computation of f Dr. MYERS, instead of taking the instantaneous projection of the ellipsoid on the sphere, takes the intersection of the ellipsoid with a plane through the centre, at right angles to the visual line. In his opinion this is allowable "wenn die Abplattung nicht ungeheuer gross ist". He therefore puts

$$f = \frac{1}{\sqrt{1 + (q^2 - 1)\cos^2\psi}}.$$

As the rigorous formula is at least equally simple there is no reason for the substitution. If we put the expression of MYERS $= f'$, we have

$$\frac{f}{f'} = \sqrt{1 + \left(\frac{q^2-1}{2q}\right)^2 \sin^2 2\psi}.$$

Greatest values, for $\psi = \frac{\pi}{4}$

q	1.2	1.3	1.4	1.5
$\frac{f}{f'}$	1.02	1.03	1·06	1.08

There thus is introduced a systematic error, which, already for small elongations, cannot be neglected.

4. As soon as, with the aid of provisional elements, a light-curve has been calculated, we try to vary these elements in such a way that the differences between observation and computation are diminished. We have to investigate, therefore, in what way the light-intensity varies with the elements.

We have already: $J = F(f, M, \lambda, \varkappa^2)$. We will now, first of all, express df and dM in function of $d(\varkappa^2)$, $d\beta$, $d\varrho$.

In the first place we have to consider that λ is fully determined by $\varkappa^2$, in the case, which as we shall presently see must be admitted, that during the minima E_2 is projected wholly on E_1. For, if $f_m = \sqrt{1 - \varepsilon^2 \sin^2 i}$ (= value of f in both the minima), then, on the same supposition:

$$f_m \cdot \frac{\lambda}{\lambda + \varkappa^2} = \text{const.}_1 \text{ (= intensity at the principal minimum)}$$

$$f_m \left(1 - \frac{\varkappa^2 \lambda}{\lambda + \varkappa^2}\right) = \text{const.}_2 \text{ (= „ „ „ secondary „)}$$

From these, after division:

$$d\lambda = \frac{\lambda(1-\lambda)}{\varkappa^2}\, d(\varkappa^2) \text{ and } \frac{df_m}{f_m} = \frac{\lambda}{\lambda + \varkappa^2}\, d\varkappa^2 = -\frac{d(\varepsilon^2 \sin^2 i)}{2 f_m^2}.$$

With the aid of the latter formula we get without difficulty

$$df = \frac{\varepsilon^2 \sin^2 i}{2f} \sin 2\beta d\beta + \frac{\cos^2\beta}{f}(1 - \varepsilon^2 \sin^2 i)\frac{\lambda}{\lambda + \varkappa^2}\, d(\varkappa^2);$$

which is independent of the variation of i.

The computation of dM directly from the formulae is rather lengthy[1]; by considering the geometrical meaning of M, dM is found at once. Evidently M is purely a function of $\varkappa$ and ϱ'. If $\varkappa$ increases by the amount $\Delta\varkappa$, the increment of M is a strip $2\varkappa\varphi\Delta\varkappa$; if ϱ' increases by $\Delta\varrho'$, the increment of M is *negative* and equal to a strip (crescent) $2\,sin\,\varphi'\,.\,\Delta\varrho' = 2\varkappa\,sin\,\varphi\,.\,\Delta\varrho'$. Therefore

$$dM = \varphi d(\varkappa^2) - 2\,sin\,\varphi'\,.\,d\varrho'$$
$$= \varphi d(\varkappa^2) - \frac{2\,sin\,\varphi'}{f}\,d\varrho + \frac{2\varrho\,sin\,\varphi'}{f^2}\,.\,df$$

If in this expression we substitute the value, given above, of df, we get dM expressed as a function of $d(\varkappa^2)$, $d\beta$, $d\varrho$.

5. *Calculation of* $d\beta$ *and* $d\varrho$ *in function of the variations of the elements of the orbit and of the epoch.*

If $sin\,\varphi = e$, then (*vide* BAUSCHINGER, die Bahnbestimmung der Himmelskörper, n°. 197):

$$dv = \left(\frac{a}{r}\right)^2 cos\,\varphi\,\{(t-T)\,d\mu - \mu dT\} + \frac{a}{r}\,cos\,\varphi\,sin\,v\left(1 + \frac{r}{p}\right)d\varphi.$$

$$\frac{d\varrho}{\varrho} = \frac{da}{a} - sin\,(P-☊)\,cos\,(P-☊)\,sin\,i\,tg\,i\,.\,d\omega - sin^2\,(P-☊)\,tg\,i\,.\,di$$
$$+ \left(\frac{a}{r}\right)^2\{e\,sin\,E - sin\,(P-☊)\,cos\,(P-☊)\,sin\,i\,tg\,i\,cos\,\varphi\,\{(t-T)\,d\mu - \mu dT\}$$
$$- \left(\frac{a}{r}\right)^2\left\{(cos\,E-e)\,cos\,\varphi + sin\,(P-☊)cos(P-☊)\,sin\,i\,tg\,i\,sin\,E\left(\frac{r}{a} + cos^2\varphi\right)\right\}d\varphi.$$

According to the definition of ω adopted above, we have to put:

$$\varrho\,sin\,(P-☊) = r\,cos\,i\,cos\,(\omega + v) = r\,cos\,i\,cos\,\beta$$
$$\varrho\,cos\,(P-☊) = -\,r\,sin\,(\omega + v) = -\,r\,sin\,\beta$$

We now pass to the following particular case:

a. the original orbit is circular,

b. if $i = 90° - \iota'$, then ι' is so small that 3rd and higher powers may be neglected; the same is true for $sin\,\varphi$.

Furthermore let $d\mu = 0$; $d\omega = 0$; $2e\,cos\,\omega = x$, $2e\,sin\,\omega = y$; $n = \frac{2\pi}{U}$ (U = period = 12.91 days), t_1 = time counted from "superior conjunction"; $dM_0 = -\,\mu dT$.

Then with sufficient approximation

[1]) See: Untersuchungen über den Lichtwechsel des Sternes β Persei, von J. HARTING. (München 1889) p. 41.

$$d\beta = dM_0 + x \sin nt_1 - y \cos nt_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\frac{d\varrho}{\varrho} = \frac{da}{a} + \frac{a^2}{2\varrho^2} \cos^2 nt_1\, i'^2 + \frac{a^2}{2\varrho^2} \sin 2nt_1 \,.\, dM_0 -$$

$$- \tfrac{1}{2} x \left(\cos nt_1 - \frac{a^2}{\varrho^2} \sin 2nt_1 \sin nt_1 \right) - \tfrac{1}{2} y \left(\sin nt_1 + \frac{a^2}{\varrho^2} \sin 2\, nt_1 \sin nt_1 \right) \quad . \quad (b)$$

As these differential expressions have led several astronomers [1]) into error, we will derive them in still another way.

From:

$$\beta = v + \omega$$

we get:

$$d\beta = dv + d\omega.$$

In the circular orbit $v = M$; in the elliptic orbit this becomes:

$$v = M + 2e \sin M + \ldots + dM_0.$$

If we substitute $M = nt_1 - \omega$, and put $d\omega = 0$, we get, neglecting higher powers of e:

$$d\beta = dM_0 + x \sin nt_1 - y \cos nt_1$$

If in:

$$\varrho^2 = r^2 \sin^2 \beta + r^2 \cos^2 i \cos^2 \beta,$$

$$i = 90^\circ - i',$$

then, neglecting higher powers of i':

$$\frac{d\varrho}{\varrho} = \frac{dr}{r} + \frac{r^2}{2\varrho^2} \sin 2\beta\, d\beta + \frac{r^2}{2\varrho^2} \cos^2 \beta .\, i'^2 \quad . \quad . \quad . \quad . \quad (c)$$

In the elliptic orbit we have:

$$r = \frac{a(1-e^2)}{1 + e \cos v} = a_0 + da - ae \cos(\beta - \omega) + \ldots =$$

$$= a_0 + da - \frac{1}{2} a\, x \cos nt_1 - \frac{1}{2} ay \sin nt_1 \ldots$$

Therefore:

$$dr = da - \frac{1}{2} ax \cos nt_1 - \frac{1}{2} ay \sin nt_1$$

and, substituting this in (c), we get the expression already given of $\dfrac{d\varrho}{\varrho}$.

---

1) Dr. Myers puts $d\beta = 0$ for $t_1 = 0$ and at the same time $dM_0 = 0$; this is incompatible with (a). Prof. Hartwig, in his paper: "Der veränderliche Stern vom Algoltypus Z Herculis" (Bamberg 1900) p. 39, puts $\sin (P\text{-}\Omega) \cos (P\text{-}\Omega) \sin i \, tg\, i$ $= 0$ for $i = 90^\circ$, whereas, according to our formulas, it becomes $- \dfrac{r^2}{2\varrho^2} \sin 2\beta$. (See also A.N. 3644).

Dr. Pannekoek quotes another instance in his Thesis on Algol (p. 22—3).

6. We thus have consecutively expressed dM and df in function of $d(\varkappa^2)$, $d\beta$ and $d\varrho$; and afterwards $d\beta$ and $d\varrho$ in function of da, i'^2, dM_0, x and y. If now we differentiate the expression

$$J_1 = f\left(1 - \frac{M}{\pi(\lambda + \varkappa^2)}\right),$$

valid in the vicinity of the first minimum, we find, by consecutive substitution, the following expression for dJ_1:

$$\pi(\lambda + \varkappa^2)\, dJ_1 = K_1\, d(\varkappa^2) + A_1\, da + I_1\, i'^2 + X_1 x + Y_1 y + \Delta_1 (dM_0 - y),$$

in which:

$$K_1 = \frac{\pi\lambda}{q^2 f^2} J_1 \cos^2 nt_1 + \pi\left(1 + \frac{\lambda - \lambda^2}{\varkappa^2}\right)(f - J_1) - f\varphi - \frac{2\lambda}{q(\lambda + \varkappa^2)} \cdot \frac{\varrho}{f^2} \cdot \cos^2 nt_1 \sin\varphi' ;$$

$$A_1 = \frac{2\varrho \sin\varphi'}{r} \quad ; \quad I_1 = \frac{r^2 \cos^2 nt_1}{\varrho} \sin\varphi' ;$$

$$X_1 = \Delta_1 \sin nt_1 - \varrho \sin\varphi' \cos nt_1 \quad ; \quad Y_1 = \Delta_1 (1 - \cos nt_1) - \varrho \sin\varphi' \sin nt_1 \quad ;$$

$$\Delta_1 = \frac{r^2 \sin 2nt_1 \sin\varphi'}{\varrho} + \frac{\varepsilon^2}{2f^2} \sin 2nt_1 \{\pi(\lambda + \varkappa^2) J_1 - 2\varrho \sin\varphi'\} .$$

If we treat in the same way the expression:

$$J_2 = f\left(1 - \frac{\lambda M}{\pi(\lambda + \varkappa^2)}\right),$$

valid in the vicinity of the second minimum, we find, putting $t_2 = t_1 - \frac{U}{2}$:

$$\pi \frac{\lambda + \varkappa^2}{\lambda} dJ_2 = K_2\, d(\varkappa^2) + A_2\, da + I_2 i'^2 + X_2 x + Y_2 y + \Delta_2 (dM_0 + y)$$

in which:

$$K_2 = \frac{\pi}{q^2 f^2} J_2 \cos^2 nt_2 + \pi(f - J_2) - f\varphi - \frac{2\lambda}{q(\lambda + \varkappa^2)} \cdot \frac{\varrho}{f^2} \cdot \cos^2 nt_2 \sin\varphi' ;$$

$$A_2 = \frac{2\varrho \sin\varphi'}{r} \quad ; \quad I_2 = \frac{r^2 \cos^2 nt_2}{\varrho} \sin\varphi' ;$$

$$X_2 = -\Delta_2 \sin nt_2 + \varrho \sin\varphi' \cos nt_2 ; \quad Y_2 = -\Delta_2 (1 - \cos nt_2) + \varrho \sin\varphi' \sin nt_2$$

$$\Delta_2 = \frac{r^2 \sin 2nt_2 \sin\varphi'}{\varrho} + \frac{\varepsilon^2}{2f^2} \sin 2nt_2 \left(\pi \frac{\lambda + \varkappa^2}{\lambda} J_2 - 2\varrho \sin\varphi'\right).$$

7. If the observations do not give the light-intensity, but the brightness expressed in magnitudes or in grades, then we have still to express the variation of the number indicating the magnitude or the grade, in the variation of the light-intensity.

Let J_0 represent the intensity at the maximum. G_0 the corresponding magnitude, J and G the same quantities at the time t, then, by the formula of POGSON:

$$G - G_0 = 2.512\ (\log J_0 - \log J).$$

Consequently:

$$dG = -2.512\ m. \frac{dJ}{J},\ (m = \textit{modulus of Brigg's log.})$$

$$dG = -1.092\ \frac{dJ}{J}.$$

Now, if $-\frac{1}{v}$ is the equivalent in magnitudes of a grade, then, σ_0 and σ being the number of grades:

$$\sigma_0 - \sigma = v(G - G_0) = 2.512v\ (\log J_0 - \log J)$$

Therefore:

$$d\sigma = 1.092v \frac{dJ}{J}$$

Putting the value of ARGELANDER's grade for the light-curve of β Lyrae at 0.130 magnitudes, then:

$$d\sigma = 8.413\ \frac{dJ}{J}.$$

8. In the hypothesis which we adopted, the main phases (min.$_1$, max.$_1$, min.$_2$, max.$_2$) take place for the values $\beta = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ of β_1. Let v_1, v_2, v_3, v_4 represent the true anomalies for these values; M_1, M_2, M_3, M_4 the corresponding mean anomalies. If, as is the case with β Lyrae, the intervals are nearly equal, e must be small and we may put approximately:

$$v_1 = M_1 - y;\ v_2 = M_2 + x;\ v_3 = M_3 + y;\ v_4 = M_4 - x.$$

$$(x = 2e \cos \omega;\ y = 2e \sin \omega)$$

or:

$$v_2 - v_1 = \frac{\pi}{2} = (M_2 - M_1) + x + y$$

$$v_3 - v_2 = \frac{\pi}{2} = M_3 - M_2) - x + y$$

$$v_4 - v_3 = \frac{\pi}{2} = (M_4 - M_3) - x - y$$

If the differences $M_2 - M_1 \ldots$ [1]) are known with equal and

[1]) The time-equation for the reduction to the common centre of gravity, computed from the spectroscopic orbit, is found to reach a value of somewhat over $\pm$ 100 seconds and may consequently be neglected.

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sufficient precision, we find from these formulae the most probable values of x and y as follows:

$$\left.\begin{aligned} 4x &= -\pi + (M_4 - M_3) + 2(M_3 - M_2) - (M_2 - M_1) \\ 4y &= \quad \pi + (M_4 - M_3) - 2(M_3 - M_2) - (M_2 - M_1) \end{aligned}\right\} \quad . \quad . \quad I$$

If we combine only similar phases, we get

$$\left.\begin{aligned} 2x &= -\pi + (M_4 - M_2) \\ 2y &= \quad \pi - (M_3 - M_1) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad II$$

The two solutions are identical, if

$$(M_2 - M_1) + (M_4 - M_3) = \pi.$$

In A. N. n°. 3456 Dr. PANNEKOEK summarises the intervals, counted from the principal minimum, for different observers between 1842 to 1895.

Dividing this period in two, he finds on an average: ($U = 12^d.91$)

	max$_1$—min$_1$	min$_2$—min$_1$	max$_2$—min$_1$
1842—1870	$3^d.12$	$6^d.40$	$9^d.54$
1870—1895	$3^d.32$	$6^d.48$	$9^d.73$.

From these values we find, for the first period:

according to form. (I):

$$\left\{\begin{aligned} e \sin \omega &= -0.0052; \quad e = 0.009 \\ e \cos \omega &= +0.0076; \quad \omega = 326° \end{aligned}\right.$$

according to form. (II):

$$\left\{\begin{aligned} e \sin \omega &= +0.0067; \quad e = 0.008 \\ e \cos \omega &= -0.0043; \quad \omega = 123°. \end{aligned}\right.$$

Similarly for the second period:

$$\left\{\begin{aligned} e \sin \omega &= +0.0040; \quad e = 0.013 \\ e \cos \omega &= -0.0125; \quad \omega = 162° \end{aligned}\right.$$

$$\left\{\begin{aligned} e \sin \omega &= -0.0030; \quad e = 0.006 \\ e \cos \omega &= -0.0055; \quad \omega = 209° \end{aligned}\right.$$

The only conclusion to be derived from these results is that e was very minute in both periods, and hardly exceeding 0.01.

9. A single glance at the numbers communicated by Dr. PANNEKOEK shows that a trial to derive something more definite from the results of the *separate* observers would be quite hopeless. In particular we may allege the considerable difference between the results obtained by LINDEMANN and PANNEKOEK, in their reduction of the observations of PLASSMANN. It thus seems to be out of place, from these observations alone, to draw the conclusion that the excentricity has increased.

Dr. L. TERKAN has proposed the following method of deriving the inclination of the orbit.[1])

[1]) A. N. nr 4067.

The minimum or maximum of light takes place when ϱ takes "extreme" values, consequently when

$$\frac{d\varrho}{d v} = \cos v \sin v \sin^2 i + e \sin v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

In formula (2) $\sin v = 0$ for the principal minimum, $\cos v = -\frac{e}{\sin^2 i}$ for the "secondary maximum". This is in the assumption that the time of the principal minimum coincides with the time of periastron. Therefore if, at the moment of the maximum, we know e and v, then we know also i. TERKAN adopts the value 0.07, derived for e by BELOPOLSKY from his spectroscopic observations[1]). He determines the mean anomaly at the maximum from the interval found by PLASSMANN[2]):

$$\text{II min.} - \text{II max.} = 3{,}05 \text{ days},$$

and then expands this anomaly in a series[3]) by the aid of $\cos v = -\frac{e}{\sin^2 i}$. This series has an argument α, which contains $\sin^2 i$. He thus finds

$$i = 51°.3.$$

Afterwards, in his Hungarian paper[4]), he takes $e = 0.06$. From his own observations he derives: I min. — I max. $= 3{,}48$ days and then finds, using the usual equations of KEPLER, by the aid of $\cos v = -\frac{e}{\sin^2 i}$:

$$i = 30°.$$

Even if we disregard the very doubtful value of the numerical data, the hypothesis seems unfounded that the maximum of the light occurs at the moment that ϱ is a maximum. If, moreover, we assume with Dr. TERKAN, both the celestial bodies to be spherical, then the light must be constant as long as the two spheres do not cover each other as seen by the observer. This is not confirmed by observation. Besides there can be no question of a clearly defined epoch of maximum in such a case. The way in which Dr. TERKAN meets this objection by saying: "that our eye or the telescope is unable to separate the system and that the rays of light which in space come

[1]) See p. 462. 1st footnote.

[2]) A. N. nr 3242.

[3]) In this series e has been erroneously substituted for $\sin\varphi \cos\varphi$ and $\frac{\sqrt{4-e^2}-e}{\sqrt{4-e^2}+e}$ (instead of $\sqrt{\frac{1-e}{1+e}}$) for $tg\,(45°-\frac{1}{2}\varphi)$.

[4]) β Lyrae palyaelemeinek etc. p. 412.

from the same distance, but from a larger field, are united to a larger disc" [1]), seems little satisfactory.

10. *Determination of the elements of the orbit etc. by means of the light-curve of* ARGELANDER [2]).

As a first approximation we put $i = 90°$, $e = 0$.

An approximate value of q is furnished by the general course of the curve in the vicinity of the maxima. As long as it is symmetrical in regard to the ordinate of the maximum, we may assume that the eclipse has not yet begun, so that

$$J = \sqrt{(1-\varepsilon^2 \sin^2 nt_M)}\,;$$

t_M being the time counted from the maximum.

From the light-curve we take the decrease of $(\sigma_0-\sigma)$ in grades, for equal intervals of time before and after the two maxima.

t_M	$(\sigma_0-\sigma)_I$	$(\sigma_0-\sigma)_{II}$
-30^h	0 76	0.50
−24	0 47	0.31
−18	0.25	0.18
−12	0 10	0.07
− 6	0 02	0 02
+ 6	0 03	0.03
+12	0.09	0 11
+18	0.21	0.24
+24	0.38	0 43
+30	0.59	0.67

t_M	$(\sigma_0-\sigma)_{mean}$
$\pm 6^h$	0.025
±12	0.093
±18	0 220

t_M	$(O-C)_I$	$(O-C)_{II}$
-18^h	−0.03	+0.04
−12	0 00	+0.03
− 6	0.00	0.00
+ 6	−0.01	−0.01
+12	+0.01	−0.01
+18	+0.01	−0.01

An increasing dissymmetry begins to show itself for both the maxima at about 24 hours distance from these epochs.

With the mean values $(\sigma_0-\sigma)_{mean}$ the light-intensities were now computed by the formula:

$$\sigma_0-\sigma = -\frac{2.512}{0.13}\log J$$

and the relation

$$\varepsilon^2 \sin^2 nt_M = 1-J^2$$

[1]) β Lyrae palyaelemeinek etc. p. 417.

[2]) De Stella β Lyrae variabili commentatio altera. Scripsit FREDERICUS ARGELANDER Bonnae a. 1859. — Curva "vera" pro 1850.

furnished the data:

$$0.015\,\varepsilon^2 = 0.006$$
$$0.058\,\varepsilon^2 = 0.022$$
$$0.127\,\varepsilon^2 = 0.051$$

leading to the most probable values $\varepsilon^2 = 0.397$; $q = 1.288$.

The deviations Obs.-Comp. have been given in the two last columns.

Having found q, we get $\varkappa$ and λ from the light-intensities at the two minima. The values of these being 0.3433 and 0.6365, we obtain, for $i = 90°$, the two relations

$$\frac{1}{q} \cdot \frac{\lambda}{\lambda + \varkappa^2} = 0.3433\,; \quad \frac{1}{q}\left(1 - \frac{\varkappa^2\lambda}{\lambda + \varkappa^2}\right) = 0.6365,$$

whence:

$$\varkappa = 0.6387\,; \quad \lambda = 0.3233.$$

Finally, at the moment at which the eclipse begins:

$$\frac{\varrho}{f} = \varrho' = 1 + \varkappa.$$

The consideration of the asymmetry, shows that this must be the case shortly after 18^h ($nt_M = \pm\, 20°55'$). We therefore put:

$$\frac{\varrho}{f} = \frac{a \cos 21°}{\sqrt{1 - \varepsilon^2 \sin^2 21°}} = 1.6387 = 1 + \varkappa$$

from which:

$$a = 1.710.$$

We thus have, as a first approximation, the following elements: $\varkappa = 0.6387$; $\lambda = 0.3233$; $q = 1.288$; $a = 1.710$; $e = 0$; $i = 90°$ and, as for the "epoch", we assume, that the central eclipse of E_2 by E_1, coincides with the principal minimum of Argelander's curve.

11. In the following table the 2nd column, headed O_1, shows the light-grades of Argelander's curve for equal intervals before and after the principal minimum; the 7th column, headed O_2, similarly shows the same element before and after the half period $= 6^d.455$ (*not* therefore before and after the secondary minimum, which Argelander places at $6^d.375$ from the principal minimum). The columns C_{s_1} and C_{s_2} contain the light-grades, computed by the aid of the elements given just now.

12. As will be remarked, the deviations $O - C_s$ are in the main negative before, positive after the two minima. We conclude that, by shifting the theoretical light-curve in a negative direction with regard to the time, we may obtain improved agreement. The excen-

t	O_1	C_{S_1}	C_{M_1}	O_1-C_{S_1}	O_1-C_{M_1}	O_2	C_{S_2}	C_{M_2}	O_2-C_{S_2}	O_2-C_{M_2}
−72	11.95	11.98	11.98	−0.03	−0.03	11.87	11.98	11.98	−0.11	−0.11
−66	11.84	11.91	11.93	−0.07	−0.09	11.79	11.91	11.93	−0.12	−0.14
−60	11.69	11.80	11.84	−0.11	−0.15	11.66	11.80	11.84	−0.14	−0.16
−54	11.48	11.58	11.72	−0.10	−0.24	11.48	11.62	11.72	−0.14	−0.24
−48	11.20	11.34	11.45	−0.14	−0.25	11.25	11.37	11.52	−0.12	−0.27
−42	10.82	10.75	11.04	+0.07	−0.22	10.99	11.06	11.26	−0.07	−0.27
−36	10.29	10.07	10.44	+0.22	−0.15	10.68	10.68	10.92	0.00	−0.24
−30	9.27	9.14	9.64	+0.13	−0.37	10.30	10.22	10.50	+0.08	−0.20
−24	7.40	7.91	8.55	−0.51	−1.15	9.78	9.70	10.00	+0.08	−0.22
−18	4.83	6.28	7.13	−1.45	−2.30	9.04	9.13	9.44	−0.09	−0.40
−12	3.55	4.22	5.26	−0.67	−1.71	8.42	8.56	8.82	−0.14	−0.40
− 6	3.10	3.05	3.15	+0.05	−0.05	8.21	8.23	8.25	−0.02	−0.04
0	3.00	3.00	3.00	0.00	0.00	8.20	8.19	8.19	+0.01	0.00
+ 6	3.15	3.05	3.15	+0.10	0.00	8.38	8.23	8.25	+0.15	+0.13
+12	4.30	4.22	5.26	+0.08	−0.96	8.91	8.56	8.82	+0.35	+0.09
+18	6.67	6.28	7.13	+0.39	−0.46	9.64	9.13	9.44	+0.51	+0.20
+24	8.46	7.91	8.55	+0.55	−0.09	10.25	9.70	10.00	+0.55	+0.25
+30	9.70	9.14	9.64	+0.56	+0.06	10.75	10.22	10.50	+0.53	+0.25
+36	10.50	10.07	10.44	+0.43	+0.06	11.14	10.68	10.92	+0.46	+0.22
+42	10.97	10.75	11.04	+0.20	−0.07	11.43	11.06	11.26	+0.37	+0.17
+48	11.31	11.34	11.45	−0.03	−0.14	11.64	11.37	11.52	+0.27	+0.12
+54	11.57	11.58	11.72	−0.01	−0.15	11.79	11.62	11.72	+0.17	+0.07
+60	11.75	11.80	11.84	−0.05	−0.09	11.91	11.80	11.84	+0.11	+0.07
+66	11.88	11.91	11.93	−0.03	−0.05	11.98	11.91	11.93	+0.07	+0.05
+72	11.91	11.98	11.98	−0.07	−0.07	12.01	11.98	11.98	+0.03	+0.03

tricity, besides displacing the maxima and minima, also causes a slight dissymmetry in regard to the minima. In order to separate the influence of the excentricity on the asymmetry from that of the epoch, we may divide the equations of condition into two groups. For the coefficients K, A, I and X are even functions; Y and Δ uneven functions of nt_1, resp. nt_2. If, therefore, we take the sum and the difference of two equations of condition, corresponding to the times $+t_1$ and $-t_1$, the former of the resulting equations will only contain the quantities $d(\varkappa^2)$, δa, ι'^2 and x, the latter only y and dM.

In this way the following equations have been derived for successive intervals of six hours counted from the 1[th], resp. 2[nd] minimum.

They do not rest however on the above elements but on those

PRINCIPAL MINIMUM.

t_1	$d(\varkappa^2)$	da	$(10\iota'^2)$	x	=	y	(dM_0-y)	=
$\pm 72^h$	+ 0.03	+ 0.00	+ 0.00	+ 0.70	— 0.05	+ 0.62	+ 0.70	— 0.02
66	+ .13	+ .00	+ .00	+ 1 44	— .07	+ 1.12	+ 1.45	+ .02
60	+ .30	+ .00	+ .00	+ 2.02	— .12	+ 1.41	+ 2.15	+ .03
54	+ .04	+ 60	+ .02	+ 2.59	— .19	+ 1.55	+ 3.20	+ 04
48	— .40	+ 1 46	+ .06	+ 3.16	— 19	+ 0.94	+ 4.76	+ .05
42	— .50	+ 1.92	+ .14	+ 3.55	— .14	+ .76	+ 6.37	+ 07
36	— .40	+ 9 34	+ .28	+ 3 80	— .04	+ 60	+ 8.18	+ .10
30	— .06	+ 2.58	+ .50	+ 3.89	— .15	+ .44	+10.32	+ .21
24	+ .60	+ 2.70	+ .91	+ 3.78	— 62	+ .31	+12.92	+ 53
18	+ 1.75	+ 2.60	+ 1 70	+ 3.43	— 1.38	+ .18	+16.08	+ .92
12	+ 3.42	+ 2.20	+ 3.38	+ 2.69	— 1.33	+ .07	+19 57	+ .37
6	+ 2.97	+ 0 79	+ 4 98	+ 1 02	— 0.02	+ .01	+14.11	+ .02

SECONDARY MINIMUM.

t_2	$d(\varkappa^2)$	da	$(10\iota'^2)$	x	=	y	(dM_0+y)	=
$\pm 72^h$	+ 0.03	+ 0.00	+ 0.00	— 0 05	— 0.04	— 0.04	+ 0.05	+ .02
66	+ .12	+ .00	+ .00	— .10	— .04	— .07	+ .10	+ .09
60	+ .30	+ .00	+ .00	— .14	— .05	— .09	+ .15	+ .12
54	+ .33	+ .24	+ .00	— .26	— .08	— 01	+ .40	+ .15
48	+ .30	+ .58	+ .02	— .44	— 07	+ .07	+ .90	+ .19
42	+ 38	+ 78	+ .05	— .59	— .05	+ .06	+ 1.42	+ .22
36	+ .52	+ .89	+ .10	— .71	— .01	+ .05	+ 2 01	+ .23
30	+ .70	+ .94	+ .18	— .79	+ .02	+ .04	+ 2.65	+ .22
24	+ .90	+ .91	+ 30	— 80	+ .01	+ .03	+ 3.35	+ 23
18	+ 1.11	+ .80	+ .52	— .73	— .10	— .01	+ 4.02	+ .30
12	+ 1.22	+ .58	+ .89	— .55	— .15	— .00	+ 4.48	+ .24
6	+ 0.62	+ .17	+ 1.09	— .17	+ .04	+ .00	+ 2.71	+ .08

which have been derived by repeated approximations from ARGELANDER's curve by Dr. MYERS. In his opinion these are the best possible circular elements:

$$a = 1\ 8955\ ;\ \varkappa = 0.7580\ ;\ q = 1.1993\ ;\ \lambda = 0.4023\ ;\ i = 0$$

By their aid I computed the light-grades C_{M_1} and C_{M_1} of the preceding table. As will be remarked, the deviations $O - C_{M_1}$ are rather considerable in the vicinity of the principal minimum.

In deriving the following normal equations, the equations of condition for $t_1 = \pm 6^h$ and $t_2 = \pm 6^h$ have been neglected.

Normal-equations:

$$\begin{array}{llllll}
18.54\, d(\varkappa^2) & + 15.17\, da & + 16.98\, (10\iota'^2) & + \ 15.31\, x & = - & 7.670 \\
15.17 \ \ ,, & + 41.68 \ ,, & + 18.14 \ \ ,, & + \ 53.17 \ ,, & = - & 9.673 \\
16.98 \ \ ,, & + 18.14 \ ,, & + 20.29 \ \ ,, & + \ 20.75 \ ,, & = - & 7.720 \\
15.31 \ \ ,, & + 53.17 \ ,, & + 20.75 \ \ ,, & + 101.96 \ ,, & = - & 13.227
\end{array}$$

$$255.69\, y + \ 160.37\, (dM_0 - y) = 10.242$$

$$160.37\, y + 1125.52\, (dM_0 - y) = 37.759$$

Solution of the first four equations:

$$x = -0.026\ ;\quad 10\iota'^2 = -0.044\ ;\quad da = -0.0741\ ;\quad d(\varkappa^2) = -0.2915.$$

As ι' becomes imaginary, we put $\iota' = 0$ in the equations of condition, and then find:

$$x = -0.026\ ;\ da = -0.0799\ ;\ d(\varkappa^2) = -0.3268.$$

$$y = +0{,}021\ ;\ dM_0 - y = +0.031,$$

which lead to the improved elements:

$$a = 1.8156\ ;\ \varkappa = 0.4978\ ;\ \lambda = 0.2249\ ;\ q = 1\ 3859\ ;\ e = 0.017;\ \omega = 141^\circ.3.$$

The correction for $\varkappa^2$ is particularly large, more than half its original value (0.5746). As in such a case dJ cannot any longer be considered to be proportinal to $d(\varkappa^2)$, we should have to compute a new light-curve by the aid of the new elements; we should then have to calculate the differential coefficients in order — if necessary — to find a new approximation.

In the following table the columns C_1 and C_2 show the light-grades calculated by means of the improved elements.

In fig. I has been given a graphical representation of these numbers. The agreement in the vicinity of the principal minimum is considerably improved. It is true that there remains a deviation exceeding a light-grade, at 18 hours before the minimum. It might perhaps be further diminished by a repetition of the whole process. If, however, we take into account the uncertainty mostly existing when we draw the curve for the vicinity of the minimum, then it seems hardly worth while to repeat the elaborate calculation. At all

t	C_1	O_1	O_1-C_1	O_2	C_2	O_2-C_2
—72h	11.95	11.95	0.00	11.87	11.97	—0.10
—66	11.84	11 84	.00	11 79	11.88	— .09
—60	11.69	11.68	+ .01	11.66	11.73	— .07
—54	11.48	11.47	+ .01	11.48	11.52	— .04
—48	11.20	11.22	— .02	11.25	11.26	— .01
—42	10.82	10.93	— .11	10.99	10.96	+ .03
—36	10.29	10.42	— .13	10.68	10.55	+ .13
—30	9.27	9.47	— 20	10.30	10 05	+ .25
—24	7.40	8.06	— .66	9.78	9 47	+ .31
—18	4 83	6 01	—1.18	9.04	8.84	+ .20
—12	3.55	3.46	+ .09	8.42	8.35	+ .07
— 6	3.10	3.05	+ .05	8.21	8.22	— .01
0	3.00	3.00	.00	8.20	8 21	— .01
+ 6	3.15	3 06	+ .09	8 38	8.30	+ .05
+12	4 30	3.88	+ .42	8.91	8.73	+ .18
+18	6.67	6.39	+ .28	9.64	9.35	+ .29
+24	8.46	8.32	+ .14	10.25	9.94	+ .31
+30	9.70	9.66	+ .04	10.75	10 45	+ .30
+36	10.50	10.54	— .04	11.14	10.87	+ .27
+42	10.97	11.01	— .04	11.43	11.18	+ .25
+48	11.31	11 30	+ .01	11.64	11.44	+ .20
+54	11.57	11.55	+ 02	11.79	11 66	+ .13
+60	11.75	11.75	.00	11.91	11.83	+ .08
+66	11.88	11.89	— .01	11.98	11.94	+ .04
+72	11.91	11.98	— .07	12.01	12 00	+ .01

events the small coefficients of y in the equations of condition, show clearly that it is impossible to explain any appreciable asymmetry by an excentricity of a few hundredths. The improved agreement near the principal minimum is obtained in the main by shifting the theoretical principal minimum by $\frac{dM_0 - y}{2\pi} \times U = 0^d.063$ in the direction of the negative time-axis, while at the same time the secondary minimum occurs 0.069 days before the minimum of ARGELANDER's curve.

13. A second set of elements was derived by making the plausible assumption, that the first minimum in ARGELANDER's curve occurs 0.08 days earlier. We thus get rid of the greater part of the dissymmetry. The second theoretical minimum is assumed to coincide with the observed minimum. The interval between the two minima thus becomes just half the period ($6^d.375 + 0^d.08 = 6^d.455$). As a consequence the orbit must be either circular or elliptic with

t	O_1	C_1	C'_1	$O_1 - C_1$	$O_1 - C'_1$	O_2	C_2	C'_2	$O_2 - C_2$	$O_2 - C'_2$
— 72^h	11.97	11.97	11.97	0.00	0.00	11.89	11.97	11.97	—.08	— 08
— 66	11.88	11.89	11 83	—.01	+.05	11 83	11.89	11 95	—.06	—.12
— 60	11.74	11.75	11.66	—.01	+.08	11.71	11.75	11.84	— 04	—.13
— 54	11.54	11.56	11.44	— 02	+ 10	11.54	11 56	11 67	— 02	— 13
— 48	11.29	11.31	11.18	—.02	+ 11	11.33	11 31	11.44	+ 02	—.11
— 42	10 95	10.98	10 86	— 03	+.09	11 08	11.01	11.15	+.07	—.07
— 36	10.48	10.35	10.16	+.13	+ 32	10.78	10.62	10.71	+.16	+.07
— 30	9.65	9 42	9 20	+.23	+ 45	10.42	10.26	10 40	+.16	+.02
— 24	8 06	8.09	7.85	—.03	+.21	9 95	9 63	9.75	+ 32	+.20
— 18	5.48	6.29	6 05	—.81	—.57	9.28	9.06	9.15	+.22	+ 13
— 12	3 80	4.07	3 90	—.27	— 10	8.56	8.52	8.57	+.04	—.01
— 6	3.18	3.05	3.05	+.13	+.13	8.25	8.25	8.26	.00	—.01
0	3.00	3.00	3.00	00	.00	8.19	8.20	8.20	—.01	— 01
+ 6	3.07	3.05	3 05	+.02	+ 02	8.30	8.25	8 26	+.05	+.04
+ 12	3.76	4 07	3 90	—.31	— 14	8 71	8 52	8.57	+ 19	+.14
+ 18	6 01	6.29	6.05	— 28	—.04	9.40	9.06	9.15	+ 34	+.25
+ 24	7.98	8 09	7.85	— 11	+ 13	10.08	9.63	9.75	+ 45	+.33
+ 30	9 37	9.42	9.20	—.05	+.17	10.61	10.26	10.40	+ 35	+ 21
+ 36	10 31	10.35	10 16	— 04	+.15	11 04	10 62	10 71	+.42	+ 33
+ 42	10.84	10.98	10 86	—.14	— 02	11 35	11.01	11.15	+.34	+.20
+ 48	11.22	11.31	11.18	—.09	+.04	11.58	11.31	11 44	+ 27	+.14
+ 54	11.50	11.56	11 44	—.06	+.06	11 75	11 56	11 67	+.19	+ 08
+ 60	11 71	11.75	11.66	—.04	+.05	11 87	11 75	11.84	+.12	+.03
+ 66	11.84	11.89	11.83	—.05	+.01	11 96	11 89	11 95	+.07	+.01
+ 72	11.91	11.97	11.97	—.06	—.06	12 00	11.97	11 97	+.03	+.03

the major axis at right angles to the line of the nodes. In this way we find:

$a=1.7209$; $\varkappa=0.5015$; $\lambda=0.2276$; $q=1.3944$; $i'=7°,25$; $e=0.04$; $\omega=180°$.

In the next tables the columns C_1 and C_2 show the light-grades computed by means of the first five elements, neglecting the excentricity. The columns C_1 and C_3 contain the same quantities taking into account the excentricity.

The mean deviation of the values in the columns $O_1—C_1'$ and $O_2—C_2'$ is $\pm$ 0.17 light-grades, whereas ARGELANDER assigns the value $\pm$ 0.16 to the mean error of the ordinates of his light-curve (prob. error 0.1095). It would be quite illusory therefore to endeavour to obtain an improved agreement. Against the elliptic orbit there is however the grave objection that it gives the first maximum 0.18 days after —, the second maximum 0.10 days before the corresponding maxima of ARGELANDER's light-curve. In the circular orbit the first maximum lies only 0.02 days, the second 0.06 days later, whereas the agreement is still very satisfactory.

14. Finally we communicate a set of circular elements obtained by a repeated approximation from the light-curve of Dr. PANNEKOEK:

$$a = 1.5378\,;\ \varkappa = 0.5378\,;\ \lambda = 0.2900\,;\ q = 1\ 4609.$$

In deriving them we assumed that a cannot fall short of $1 + \varkappa$. We further assumed the theoretical principal minimum to coincide with the observed minimum.

In the following table t is the number of hours before and after the *theoretical* principal and secondary minimum; O_1 and O_2 are the light-grades at the same moments, as read off from Dr. P.'s light-curve; C_1 and C_2 the light-grades of the theoretical curve.

The results have been graphically represented in Fig. II. The remaining deviations are mainly positive before the first minimum; after that they are negative. At the secondary minimum the signs are reversed. The deviations might be rather considerably diminished if, with a small excentricity ($e \sin \omega = 0.016$), we place the principal minimum in Dr. PANNEKOEK's light-curve 0.063 days later, the secondary minimum 0.069 earlier. In this way, however, the interval in time min. I — min. II is diminished more considerably than seems admissible.

For the rest it need not be said, that in the present case, where two gaseous bodies seem to be in contact, the Keplerian equations of motion must give only a rough approximation, while the action of the tides must contribute its part to mask the influence of the

t	O_1	C_1	$O_1—C_1$	O_2	C_2	$O_2—C_2$
—72h	11.96	11.97	—0 01	11.96	11 97	—0.01
—66	11.88	11.87	+ .01	11 87	11.88	— .01
—60	11.72	11 67	+ .05	11.62	11.71	— 09
—54	11.53	11.37	+ 16	11.33	11.46	— .13
—48	11.26	10.95	+ 31	10.89	11.15	— 26
—42	10.87	10.37	+ .50	10 42	10.75	— .33
—36	10.23	9.61	+ .62	9 92	10 27	— .35
—30	9.06	8.60	+ .46	9 38	9 72	— 34
—24	7 17	7.27	— 10	8 86	9.10	— 24
—18	5.15	5.53	— .38	8.34	8.43	— .09
—12	3.80	3.53	+ .27	7 92	7.82	+ .10
— 6	3 20	3 09	+ .11	7 61	7.57	+ .04
0	3.00	3 00	.00	7.50	7.50	.00
+ 6	3.16	3.09	+ .07	7.75	7.57	+ .18
+12	3.77	3 53	+ .24	8.11	7.82	+ .29
+18	5.07	5.53	— .46	8.68	8.43	+ .25
+24	6.75	7.27	— 52	9.26	9.10	+ .16
+30	8 36	8.60	— .24	9.80	9.72	+ .08
+36	9 40	9.61	— .21	10 30	10.27	+ .03
+42	10.07	10.37	— .30	10.75	10.75	.00
+48	10.62	10.95	— .33	11.08	11 15	— .07
+54	11.13	11.37	— .24	11.42	11.46	— .04
+60	11.52	11 67	— .15	11.61	11.71	— .10
+66	11.78	11.87	— .09	11.87	11.88	— .01
+72	11.93	11.97	— .04	11.94	11.97	— .03

excentricity on the course of the light-curve. We conclude that, from the light-curves we can only infer that the orbit is nearly circular.

At all events there is no reason to assume an increase or decrease of the, certainly very small, excentricity. A comparison of the elements a and q might lead us to conjecture that the distance of the two celestial bodies has diminished since the time of ARGELANDER. The increase in q is in agreement with such a supposition, but the continual lengthening of the period seems to clash with it.

15. *Computation of the orbit from the spectrographs of* Belopolsky.

In the computation of the orbit from the velocities in the line of sight as derived by B. from the displacement of the bright *F*-line in the spectrographs of 1892, the method of Wilsing [1]) has been adopted. For very small excentricities it is to be preferred to that of Lehman-Filhés.

The first column contains the mean time of observation at Pulkowa; the 2nd gives the phase in the light-period of 12.91 days. We have assumed, in accordance with the formula of Argelander, as corrected by Pannekoek, that the principal minimum occurred on 1892 Sept 25.781 M. T. Greenwich (= 25d.865 M. T. Pulkowa).

The 3rd column contains the velocities, expressed in geographical miles, reduced to the sun. They have been taken, with slight modifications, from the Memorie della Soc. d. Spettr. It., vol. XXII. For Belopolsky has applied a constant correction — 2.1 G.M. for the velocity of the earth, whereas in reality this velocity varies between — 1.6 and — 2.3 G.M.

T	Phase	Veloc. in G.M.	V_r	O-C
	d			
Sept. 23 3	10.34	—11.2	—11.25	+0 05
24.4	11.44	—11 6	—10.09	—1.51
25.4	12.44	— 4.4	— 2.58	+1.82
27.3	1.44	+ 4.8	+ 5.28	—0 48
30.3	4.44	+10.7	+10.50	+0.20
Oct. 2.3	6.44	+ 1 7	+ 2.09	—0.39
3.3	7.44	— 3.6	— 2 71	—0.89
7.3	11 44	— 9.5	—10.09	+0.59
11.3	2.53	+10.1	+10.29	—0.19
19.3	10 53	—12.4	—11.29	—1.11
20.3	11.53	—10.3	— 9.84	—0.46
26.3	4.62	+10.6	+ 9 98	+0.62
Nov. 25.2	8.70	— 6.7	— 7.85	+1.15
26.2	9.70	—10.2	—10 62	+0 42

[1]) Dr. J. Wilsing. Ueber die Bestimmung von Bahnelementen enger Doppelsterne aus spectroskopischen Messungen der Geschwindigkeitscomponenten. A. N. no. 3198.

With the notations of WILSING these observations lead to the following normal equations:

$$\begin{array}{rrrrrcr}
+14\,g_0 & -3.17\;a_1 & +1.56\;b_1 & -3.41\,a_2 & -1.28\,b_2 & = & -42.00\\
-3.17\;g_0 & +7.64\;a_1 & -1.68\;b_1 & +1.81\,a_2 & +1.02\,b_2 & = & +90.78\\
+1.56\;g_0 & -1.68\;a_1 & +6.36\;b_1 & -2.71\,a_2 & -0.27\,b_2 & = & -37.81\\
-3.41\;g_0 & +1.81\;a_1 & -2.17\;b_1 & +8.52\,a_2 & +0.32\,b_2 & = & +30.74\\
-1.28\;g_0 & +1.02\;a_1 & -0.27\;b_1 & +0.32\,a_2 & +5.48\,b_2 & = & +8.57
\end{array}$$

Solution:

$g_0 = -0.097$ G.M. $=$ constant velocity towards the sun.

$a_1 = -an\,sin\,i\,sin\,(\omega' + M_0) = +11.196$; $b_1 = an\,sin\,i\,cos\,(\omega' + M_0) = -2.953$

$a_2 = -ean\,sin\,i\,sin\,(\omega' + 2M_0) = +0.498$; $b_2 = ean\,sin\,i\,cos(\omega' + 2M_0) = -0.708$

ω' is the longitude of the periastron, counted from ☊; M_0 the mean anomaly at the beginning of the light-period, consequently:

$$an\,sin\,i = 11.579\,;\ \omega' = 115°20'\,;\ M_0 = 139°54'\,;\ e = 0.075.$$

As $\omega' + M_0 = 180° + 75°12'$, the elements belong to the body which, during the principal minimum, eclipses the other. Conjunction takes place, when $\omega' + v = 270°$, i.e. $0^d.39$ days after the time of the principal minimum, as computed from the empirical formula.

In the following table the 4th column shows the computed velocities, the 6th the outstanding deviations.

16. *Spectrographs of* 1897.

The velocities (in G. M), derived by *B.* from the displacements of the dark *Mg*-line $\lambda = 448.16\,\mu\mu$, have been taken unchanged from the Memorie della Soc. degli Spettr. It. vol XXVI. The empirical formula leads to the epoch 1897 June 22 $16^h.24$ M. T. PULKOWA for the principal minimum.

In a first approximation I determined a circular orbit and found:

$$g_0 = -2.094\,;\ an\,sin\,i = 24.210\,;\ \omega' + M_0 = 89°.30'.1.$$

Afterwards corrections were derived by the aid of the formula:

$$d\frac{dz}{dt} = dg_0 + KndT\,sin\,(\omega' + v) + dK\,.\,cos\,(\omega' + v) + \\ + Ke\,cos\,\omega'\,cos\,2\,(\omega' + v) + Ke\,sin\,\omega'\,sin\,2\,(\omega' + v).$$

in which $K = an\,sin\,i$, $T =$ time of periastron-passage.

This formula is obtained from the general differential-formula [1]), by putting $d\mu = 0$, $d\omega = 0$ and by further neglecting 2nd and higher powers of e. We thus find the following normal equations:

[1]) Vide: BAUSCHINGER, Die Bahnbestimmung der Himmelskörper, N⁰. 199.

$$
\begin{array}{llllll}
+\ 26dg_0 & -\ 1.35K\mu dt & -\ 2.01dK & -\ 2.22Ke\cos\omega' & -\ 2.03Ke\sin\omega' & =-0\ 05 \\
-1.35\ ,, & +13.18\ ,, & -\ 1.02\ ,, & +\ 1.31\ ,, & +\ 0.16\ ,, & =+6.17 \\
-2.01\ ,, & -\ 1.02\ ,, & +12.82\ ,, & -\ 0.28\ ,, & +\ 0.26\ ,, & =\ 0\ 00 \\
-2.22\ ,, & +\ 1.31\ ,, & -\ 0.28\ ,, & +13.88\ ,, & -\ 0.31\ ,, & =+4.75 \\
-2.03\ ,, & +\ 0.16\ ,, & +\ 0.26\ ,, & -\ 0.31\ ,, & +12.12\ ,, & =-5.98
\end{array}
$$

	T	Phase	Veloc. in G.M.	V_r	$O-C$
	h	d h			
June 20	11.5	10 17.0	+18.27	+19.29	—1.02
22	12.0	12 17.6	— 2.60	+ 0.28	—2.88
23	12.4	0 20.2	—10.62	—10 98	+0.36
24	12.1	1 19.9	—20.40*)	—19.92	—0.48
28	11.6	5 19.4	—11.14	—10.77	—0.37
30	11.1	7 18.9	+14.16	+12.41	+1.75
July 2	11.9	9 19.7	+21.38	+22.42	—1.04
8	12.3	2 22.3	—24.97*)	—25.56	+0.59
9	11.4	3 21.4	—25.68	—25.32	—0 36
10	11.1	4 21.1	—21.27	—19.87	—1.40
11	11.0	5 21.0	— 8.83	—10.00	+1.17
12	11.5	6 21.5	+ 3.24	+ 2.34	+0.90
13	11.4	7 21.4	+13.15	+13.43	—0.28
15	11.4	9 21.4	+24 15	+22.34	+1.81
17	11 2	11 21.2	+10.34	+ 9.35	+0.99
21	11.2	2 23.4	—27.52	—25.65	—1.87
22	11.2	3 23.4	—23.48	—25.07	+1.59
24	10.3	5 22.3	— 9.28	— 9.37	+0.09
25	10.2	6 22.2	+ 0.53	+ 2.70	—2.17
26	10.0	7 22.0	+12.77	+13.68	—0.91
27	10.2	8 22.0	+21.03	+20.86	+0.17
30	10.1	11 22.1	+10.11	+ 9.16	+0.95
31	10.2	0 0.6	— 1.03	— 2.01	+0.98
Aug. 2	9.7	2 0.1	—20.36	—21.64	+1.28

*) Mean of two observations.

Solution:

$$dg_0 = +0.0124 \quad ; \quad K\mu dT = +0.450 \quad ; \quad dK = +0.054 ;$$
$$Ke \cos \omega' = +0.292 \quad ; \quad Ke \sin \omega' = -0.489.$$

from which we get the elements:

$$g_0 = -2.082 \; GM. \quad ; \quad an \sin i = 24.264 \quad ; \quad e = 0.0235 ;$$
$$\omega' = 300°51' \quad ; \quad \omega' + M_0 = 88°26' ;$$

whereas the conjunction coincides perfectly with the principal minimum, *the difference in time amounting to less than* 0.01 *days*. Evidently this is the principal minimum. This is in accordance with the fact that the difference of the longitudes found for the two periastra deviates but slightly from 180° (300°51′ — 115°20′). This may be partly due to a fortunate coincidence.

Meanwhile the excentricity of the 2nd orbit is more than three times *smaller* than that of the 1st, while the velocity in the direction towards the sun found for the whole system is 2 Geogr. miles greater in the 2nd case.

If the latter difference is real, this acceleration would have caused a *shortening* of the period between 1892 and 1897. As, however the measures of 1892, according to the judgment of Prof. H. C. VOGEL "nicht als ganz einwurfsfrei angesehen werden können" [1]), we suspend our judgment to the time that Prof. BELOPOLSKY will again have taken up his beautiful investigations on the spectrum of β Lyrae, particularly about the F- line. Already in 1897 he communicated his intention to do so.

If we put $i = 90°$, the semiaxes major are:

$$a_1 = 2056000 \; G.M. \quad ; \quad a_2 = 4307000 \; G.M.$$

From KEPLER's third law we derive therefrom, roughly

$$m_1 = 17.1 \text{ sun's masses} \quad ; \quad m_2 = 8.1 \text{ sun's masses.}$$

17. In our opinion the preceding considerations justify the conclusion that the data about β Lyrae do not furnish a sufficient basis for a decision about any change in the elements, in particular in the excentricity. For the rest, owing to our ignorance on the circumstances in such a close system, the adopted explanation of the light variation can only claim to give a rough approximation — a rude imitation of a very complicated mechanism.

1) Ueber das Spectrum von β Lyrae. Sitzungsb. Ak. Berlin. 1894 VI.

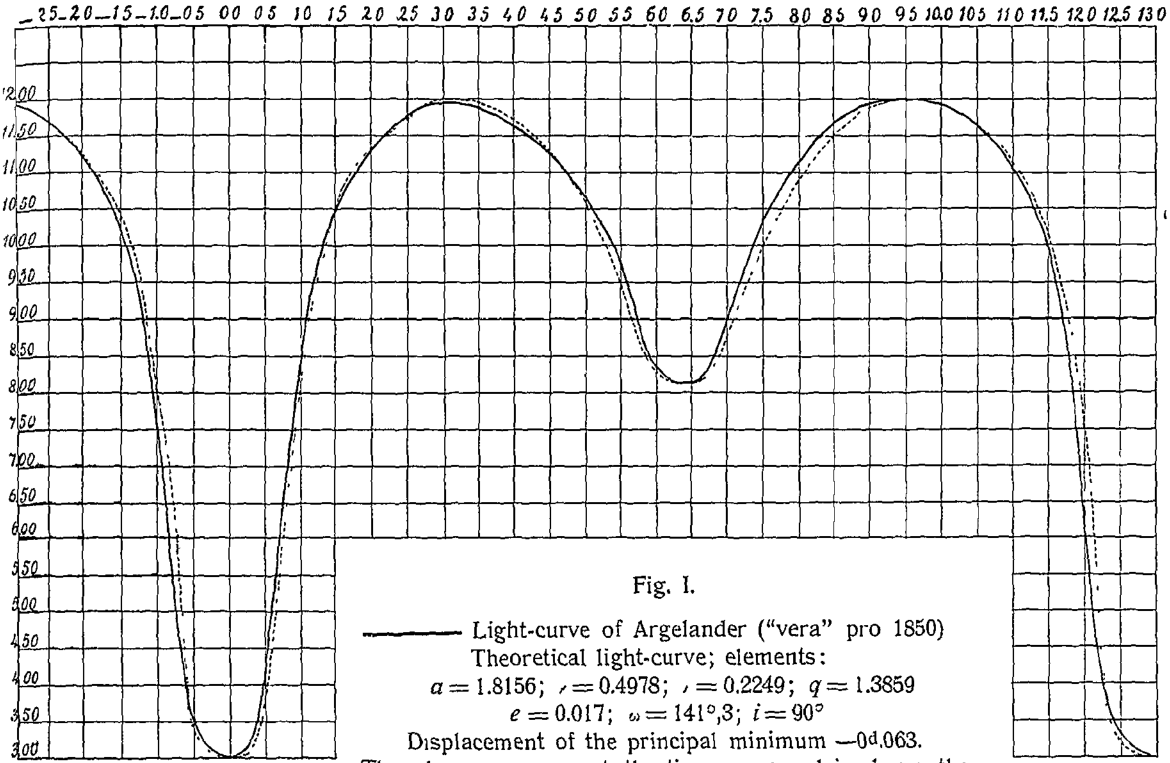

Fig. I.

——— Light-curve of Argelander ("vera" pro 1850)
Theoretical light-curve; elements:
$a = 1.8156$; [illegible] $= 0.4978$; [illegible] $= 0.2249$; $q = 1.3859$
$e = 0.017$; $\omega = 141^\circ{,}3$; $i = 90^\circ$
Displacement of the principal minimum $-0^d.063$.
The abcissae represent the time expressed in days; the ordinates the light-intensity in grades.

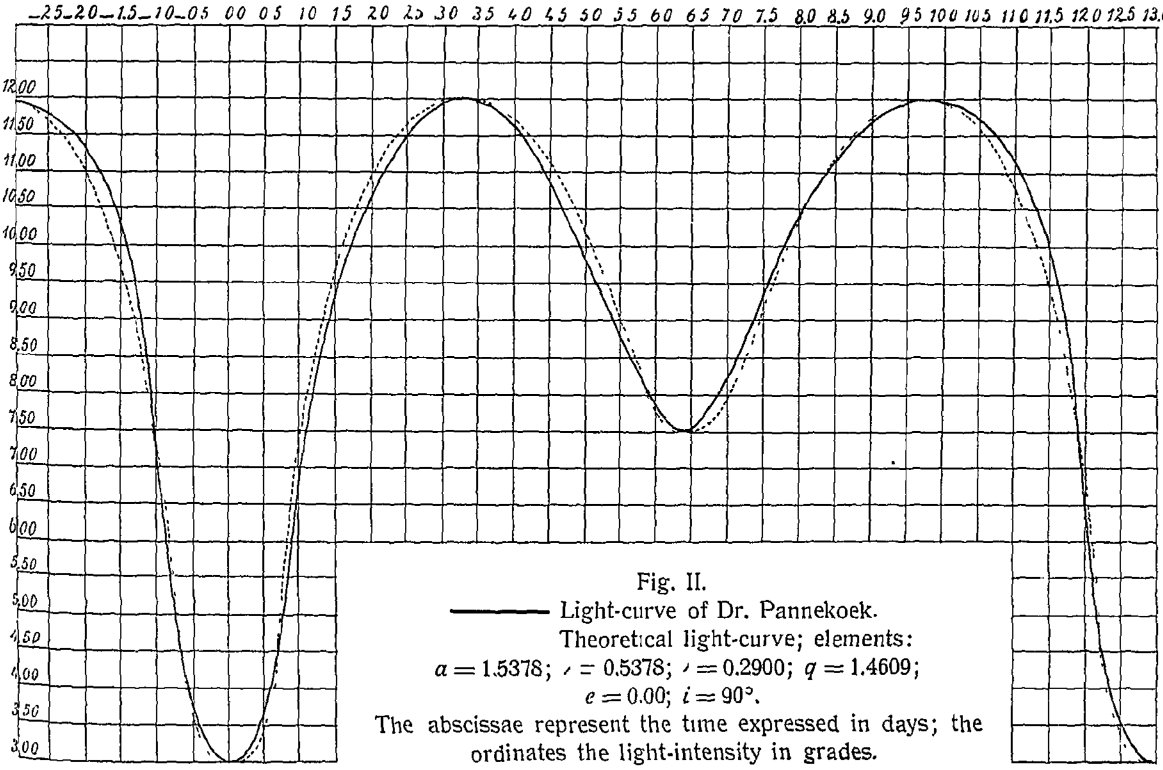

Fig. II.

——— Light-curve of Dr. Pannekoek.
Theoretical light-curve; elements:
$a = 1.5378$; [illegible] $= 0.5378$; [illegible] $= 0.2900$; $q = 1.4609$;
$e = 0.00$; $i = 90^\circ$.
The abscissae represent the time expressed in days; the ordinates the light-intensity in grades.

Mathematics. — "*The section of the measure-polytope M_n of space Sp_n with a central space Sp_{n-1} perpendicular to a diagonal.*" By Prof. P. H. SCHOUTE.

(Communicated in the meeting of December 28, 1907).

We determine the indicated section in three different ways:

1. by means of the projection of M_n on the diagonal,
2. with the aid of the projection of M_n on a plane through two opposite edges intersecting the diagonal,
3. by regarding regular simplexes.

I. *The projection of M_n on a diagonal.*

1. We can easily prove both analytically and synthetically the following theorem:

"The vertices of the measure-polytope M_n project themselves on a "diagonal in $n+1$ points, namely in the ends of the diagonal and "in the $n-1$ points, which divide the latter into n equal parts, in "these $n+1$ points are projected successively

$$1, n, \tfrac{1}{2} n (n-1), \ldots . \tfrac{1}{2} n (n-1), n, 1$$

"points, where these numbers are the coefficients of the terms "of $(a+b)^n$".

From this general theorem ensue the results for $n = 4, 5, 6, 7, 8$ given in the diagrams added here (see the expanding plate). An explanation of the sketch belonging to $n = 4$ will sufficiently explain the others.

The horizontal lines of this figure always represent the same diagonal on which the projection takes place; on these ten lines are successively indicated the projections of vertices, of edges, of faces and of bounding bodies In order to find space for the figures indicating the numbers, the thick projection-lines have been broken off, where such was necessary.

If we designate the five points on the diagonal by a, b, c, d, e, — see the bottom line of the ten horizontal ones — then in these places — see the topmost of the ten lines — 1, 4, 6, 4, 1 vertices are projected there — bear in mind $(1+1)^4$.

On the four equal segments ab, bc, cd, de are projected successively 4, 12, 12, 4 edges — think of $4(1+1)^3$.

In like manner the three equal segments ac, bd, ce are successively the projections of 6, 12, 6 faces — think of $6(1+1)^2$.

Finally on the two equal segments ad, be are projected successively 4, 4 bounding bodies — think of $4(1+1)$.

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It is easy to deduce from this the results given in the other diagrams for $n = 5, 6, 7, 8$, if we keep in mind, that the coefficients by which $(1+1)^4$, $(1+1)^3$, $(1+1)^2$, $(1+1)$ are multiplied are $1, 4, 6, 4$ and so by addition of unity at the end pass into a repetition of $(1+1)^4$.

2. More generally holds the following theorem, comprising the preceding:

"The vertices of each bounding M_p of M_n $(p \leqq n)$ are projected on "the diagonal of M_n in $p+1$ successive points of division of that "diagonal; here again the projections are distributed according to "the coefficients $1, p, \frac{1}{2} p (p-1), \ldots$ of $(a+b)^p$ over these $p+1$ "successive points."

The vertices of a bounding square are projected in three of the $n+1$ points, which naturally demands the division $1, 2, 1$. The vertices of a bounding cube are projected in four of the $n+1$ points, which of necessity must lead to the division $1, 3, 3, 1$ as by the preceding the division $2, 2, 2, 2$ is excepted.

From this ensues then directly the following theorem:

"The section of a space Sp_{n-1} perpendicular to the diagonal of M_n "forming the axis of projection, with the space Sp_p bearing a bounding "M_p of M_n is an Sp_{p-1} in Sp_p perpendicular to the diagonal of "M_p connecting the two vertices of M_p projecting themselves in the "ends of the projection of M_p." [1])

But there is more. If $p'(M_p)$ represents the section of a measure-polytope M_p with a space Sp_{p-1} of its space Sp_p perpendicular to one of its diagonals in a point of which the distance to the centre of the diagonal in the diagonal as unity amounts to $\frac{1}{2} - p'$, from which is evident that $p' \leqq \frac{1}{2}$, the two theorems hold:

"For even n a bounding measure-polytope M_p of M_n is intersected "by the central space Sp_{n-1} perpendicular to the diagonal of M_n

[1]) The indicated diagonal d_p of M_p is the projection of the axis of projection d on the space Sp_p of M_p; so we can obtain the projections of the vertices of M_p on d by projecting these vertices first in Sp_p on d_p and projecting afterwards on d the points found on d_p by the preceding means.

As d_p and d in the edge of M_n as unity are represented by $\sqrt{p}$ and $\sqrt{n}$ and d_p is projected on d as $\frac{p}{n}$ of d, the cosine of the angle between d and Sp_p is equal to $\frac{1}{n}\sqrt{np}$.

"according to an $\frac{a}{p}(M_p)$, where a according to circumstances can "assume for even p one of the $\frac{p}{2}$ values $1, 2, \ldots \frac{p}{2}$, for odd p one "of the $\frac{p-1}{2}$ values $1, 2, \ldots \frac{p-1}{2}$."

"For odd n the measure-polytope M_p is intersected under the same "circumstances according to a $\frac{2a-1}{p}(M_p)$ where a can assume for "even p one of the $\frac{p}{2}$ values $1, 2, \ldots \frac{p}{2}$, for odd p one of the $\frac{p+1}{2}$ "values $1, 2, \ldots \frac{p+1}{2}$."

We shall now, instead of losing ourselves in further generalities, give the full results of the diagrams for the cases $n = 4, 5, 6, 7, 8$ to make clear the above. In order to be able to indicate easily ratios of measure we shall suppose the edge of M_n to be unity of length.

3. Case $n = 4$. The space — see first diagram — perpendicular in the centre c of diagonal ae to this diagonal contains the six vertices of M_4 projecting themselves in c and cuts — see lines 3 and 4 — no edge; so the section has six vertices. This same space cuts twelve faces — see line 7 — according to $\frac{1}{2}(M_2)$ and eight bounding bodies — see lines 9 and 10 — according to $\frac{1}{3}(M_3)$; so the section has twelve edges with a length $\sqrt{2}$ and eight equilateral triangles as faces. So the section is a (6, 12, 8) and, indeed, the regular octahedron with edges $\sqrt{2}$.

Case $n = 5$. We find — see second diagram — thirty vertices generated by intersection of edges, sixty edges, forty faces and ten bounding bodies, so a (30, 60, 40, 10). The vertices are of the same kind, the edges have as $\frac{1}{4}(M_2)$ the length $\frac{1}{2}\sqrt{2}$. The forty faces consist of twenty $\frac{1}{2}(M_3)$ and two times ten $\frac{1}{6}(M_3)$, i. e. of twenty hexagons and twenty triangles, both regular[1]) with sides $\frac{1}{2}\sqrt{2}$.

[1]) Where the regularity is obvious — as e. g. with the triangles by the equal length of all edges, etc. — the additional "equilateral" or "regular" will in future be left out.

Each of the ten bounding bodies is as $\frac{3}{8}(M_4)$ — compare in the first diagram the section with a space perpendicular to ae in the point in the middle between c and d — a (12, 18, 8) bounded by four $\frac{1}{2}(M_3)$ and four $\frac{1}{6}(M_3)$, i. e. by four of the hexagons and four of the triangles, and therefore a tetrahedron truncated regularly at the vertices, i. e. the first of the equiangular semi-regular (Archimedian) bodies.

Case $n = 6$. Out of the third of the diagrams we read that the section is a (20, 90, 120, 60, 12). All the edges have a length $\sqrt{2}$, all the faces are triangles. The bounding bodies are for one half (30) as $\frac{1}{2}(M_4)$ octahedra, for the other half (15 + 15) as $\frac{1}{4}(M_4)$ tetrahedra. The twelve bounding polytopes are as $\frac{2}{5}(M_5)$ — compare now again the second diagram — polytopes (10, 30, 30, 10) bounded by five of the octahedra and five of the tetrahedra, which can be regarded as regular five-cells, regularly truncated at the vertices as far as half of the edges, so as to lose all the original edges by this truncation.

Case $n = 7$. We arrive at a (140, 420, 490, 280, 84, 14). The length of the edges is $\frac{1}{2}\sqrt{2}$. The 490 faces consist of 210 hexagons and 280 triangles, the 280 bounding bodies of 210 truncated tetrahedra and 70 tetrahedra, the 84 four-dimensional bounding polytopes of 42 polytopes $\frac{1}{2}(M_5) = (30, 60, 40, 10)$ found already under $n = 5$ and 42 polytopes $\frac{3}{10}(M_5) = (20, 40, 30, 10)$ bounded by five truncated tetrahedra and five tetrahedra — regular five-cells truncated at the vertices as far as a third of the edges. The 14 five-dimensional bounding polytopes are as $\frac{5}{12}(M_6)$ polytopes (60, 150, 140, 60, 12) bounded by six (30, 60, 40, 10) and six (20, 40, 30, 10).

Case $n = 8$. Here a (70, 560, 1120, 980, 448, 112, 16) is the result. The length of the edges is $\sqrt{2}$, all faces are triangles. The

980 bounding bodies consist of 420 octahedra and 560 tetrahedra the 448 four-dimensional bounding polytopes of 336 polytopes $\frac{2}{5}(M_5)$ and 112 polytopes $\frac{1}{5}(M_5)$, i. e. of 336 five-cells truncated as far as half of the edges, found under $n = 6$, and 112 five-cells. The 112 five-dimensional bounding polytopes are as far as one half is concerned $\frac{1}{2}(M_6) = (20, 90, 120, 60, 12)$ already found above, as far as the other half is concerned $\frac{1}{3}(M_6) = (15, 60, 80, 45, 12)$ bounded by six five-cells truncated as far as half the length of the edges and six five-cells. Finally the sixteen six-dimensional bounding polytopes are as $\frac{3}{7}(M_7)$ polytopes (35, 210, 350, 245, 54, 84) bounded by seven (20, 90, 120, 60, 12) and seven (15, 60, 80,45, 12) [1]).

From this all we easily deduce the following general laws:

"The vertices of the section are vertices of M_n for even n, for odd n they are centres of edges of M_n; they are always of the same kind [2])."

"The common length of the edges is $\sqrt{2}$ for even n and $\frac{1}{2}\sqrt{2}$ for odd n; they are always of the same kind[3])."

"The faces are triangles for even n, hexagons and (smaller) triangles [4]) for odd n."

"The bounding bodies are octahedra and tetrahedra for even n, truncated tetrahedra and (smaller) tetrahedra for odd n".

"The four-dimensional bounding polyhedra are five-cells truncated as far as halfway the edges and five-cells for even n, five-cells

1) If we had set to work, when enumerating the results, in that sense inversely that with each new value of n of the bounding polytopes with the greatest number of dimensions we had descended to the vertices, we should have furnished a geometrical variation of the well known nursery-book: "the house that Jack built". However with two differences. When descending from every one round higher of the ladder we pass *every other time* again the same stadia and the ladder is a Jacob's ladder with an infinite number of rounds.

2) That is, in each vertex as many edges meet in the same way, etc.

3) The cases $n =$ odd seem to be an exception to this, as there are for the truncated tetrahedra two kinds of edges, namely: sections of two hexagonal faces and sections of an hexagonal and a triangular face. However, this is only apparently. For, for each edge we find that in the section itself always again the number of faces passing through it of each of the two sorts is steadfast, thus for $n = 5$ two hexagonal faces and one triangular one.

4) We do not mention here, that for $n = 3$ only an hexagon appears. Neither that of the bounding bodies the tetrahedra do not appear for $n = 4$, etc.

truncated as far as a third of the edges and (smaller) five-cells for odd n."

Etc., etc. [1]).

The above results are for the greater part given in the general theorems mentioned above.

II. The projection of M_n on a plane through two opposite edges cutting the diagonal.

4. For each value of n the indicated projection — see fig. 1 for $n = 8$ and $n = 9$ — is a rectangle $PQQ'P'$ with the sides 1 and

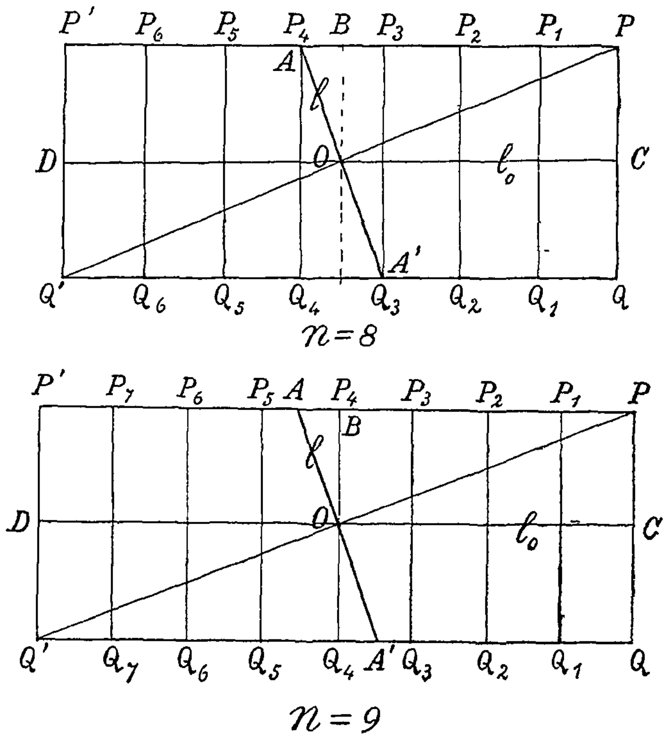


Fig 1.

$\sqrt{n-1}$, which is divided by n—2 lines P_1Q_1, P_2Q_2, . . . $P_{n-2}\,Q_{n-2}$ parallel to the shorter sides PQ, $P'Q'$ into n—1 equal rectangles.[2])

[1]) We break off here because not until the third division do we indicate that everything making its appearance in the section can be regarded as simplex or truncated simplex.

The symbol which indicates the numbers of vertices, edges, faces, etc. for arbitrary n is purposely omitted as its form is rather complicated.

[2]) To compare the treatise "On the sections of a block of eightcells, etc." (Verhandelingen der K. A. v. W., vol IX, N . 7).

The diagonal on which the intersecting space Sp_{n-1} is at right angles is one of the diagonals of the rectangle, e.g, PQ'. If the normal erected in the centre O of PQ' on this line, representing the projection of the intersecting space Sp_{n-1}, cuts the side PP' in A, this point A always lies at a distance $\frac{1}{2\sqrt{n-1}}$ from the centre B of PP'. For in the right-angled triangle AOP we find that B is the foot of the normal let down out of O on AB and from this ensues $AB.BP = OB^2$ and therefore $AB = \frac{1}{4} : \frac{1}{2}\sqrt{n-1}$. So A coincides for even n with the point of division $P_{\frac{n}{2}}$ and this point lies for odd n in the middle between $P_{\frac{n-1}{2}}$ and $P_{\frac{n+1}{2}}$. From this it is again evident that the vertices of the section are vertices of M_n for even n and centres of edges of M_n for odd n.

In the paper quoted above which restricts itself to the case $n=4$ we find in a note how we can regard the section under observation as a "rhombotope" truncated at both sides; the course of thoughts is as follows. Let us imagine in the direction of the edges $PQ, P'Q'$ on either side an infinite number of measure-polytopes M_n piled on each other and let us then remove the measure-polytopes M_{n-1}, projecting themselves on PP', QQ' and lines parallel to these, with which the successive polytopes M_n bound each other; then a prism is formed with M_{n-1} as right section. If this prism is intersected by a space Sp_{n-1} which projects itself along the perpendicular l_0 let down out of O on PQ, the section is thus an M_{n-1}. What variation does this section M_{n-1} of the prism undergo when we substitute for the intersecting space projecting itself along l_0 an other one which projects itself along a line l_φ through O, enclosing with l_0 an angle φ? As is easy to see from the figure this variation consists of a regular enlargement of the perpendiculars let down out of the boundary of M_{n-1} on the space Sp_{n-2}, projecting itself in O, which enlargement means a multiplication of those perpendiculars by *sec* φ and can be regarded as a *stretching* in the direction of the diagonal CD. As for $n=4$, where M_{n-1} becomes a cube, such a stretching makes a rhombohedron of a cube, out of M_{n-1} is formed in general what we call a rhombotope.

Just as the rhombohedron regarded as a whole passes into itself when it is revolved 120° about the axis, or — in other words — just as the axis of the rhombohedron has a period three, the axis of the rhombotope under consideration has a period $n-1$. Let us now imagine this rhombotope, for the special case that the projection of the intersecting space Sp_{n-1} — so also the projection of the rhombotope itself — falls along OA and let us truncate it by the two spaces Sp_{n-2} standing normal to the plane of projection in the ends A, A' of the segment AA' of that projection lying inside the rectangle and cutting the axis of the rhombotope therefore at right angles; we then find the required section, to be indicated according to the number of its dimensions by D_{n-1}. We directly determine the length of the axis of the untruncated rhombotope and of D_{n-1}, but before this we shall deduce some general theorems easy to find.

5. The edges of M_n project themselves on the assumed plane *either* along one of the n lines $PQ, P_1Q_1, P_2Q_2, \ldots P_{n-2}Q_{n-2}, P'Q'$, *or* as parts of PP' or QQ'. Because the vertices of D_{n-1} must be vertices of M_n or points of intersection with edges of M_n, these points project themselves — compare fig. 1 for $n=8$ and for $n=9$ — for even n exclusively in the ends A, A', for odd n exclusively in those ends and in the centre O.

From this ensues for $n=2n'$ the general theorem:

"The section $D_{2n'-1}$ of $M_{2n'}$ is a $2n'-1$-dimensional prismoid with respect to each pair of opposite bounding spaces $Sp_{2n'-2}$ and so in $2n'$ ways".

Here follow two theorems holding for arbitrary n:

"Each line through the centre O normal to two opposite bounding spaces Sp_{n-2} is axis of D_{n-1} with the period $n-1$."

"Each space Sp_{n-2} through O parallel to a bounding space Sp_{n-2} divides D_{n-1} into two congruent $n-1$-dimensional prismoids."

In the demonstration of these three theorems the entire equivalence of a pair of opposite bounding spaces Sp_{n-2} with any other pair has the chief part; moreover the third causes us to inquire how the space Sp_{n-2} through the centre parallel to a bounding space intersects D_{n-1}. We prove as follows that this section is a D_{n-2}.

If the projection l of the intersecting space Sp_{n-1} revolves round O, the $Sp^{(0)}_{n-2}$ normal to the plane of projection in O remains in its place and Sp_{n-1} thus describes a pencil with this $Sp^{(0)}_{n-2}$ as axial space. Therefore then the varying section keeps going through the section of $Sp^{(0)}_{n-2}$ with M_n. We can easily know the nature of this section of $n-2$ dimensions by regarding the case in which l coincides with l_0. Then our D_{n-1} is an M_{n-1} and this measure-polytope projecting itself along l_0 is intersected according to a D_{n-2} by the space $Sp^{(0)}_{n-2}$, which is in O normal to the plane of projection and which therefore bisects the diagonal CD of this M_{n-1}. This D_{n-2} is the section of D_{n-1} with the space $Sp_{n-2}^{(0)}$, through O parallel to the spaces $Sp_{n-2}^{(0)}$, which are in A_1 and A' normal to the axis and which truncate the rhombotope. So we find:

"Each space $Sp_{n-2}^{(0)}$ through the centre O parallel to a bounding space Sp_{n-2} intersects D_{n-1} according to a D_{n-2} of which O is again the centre."

From this follows again more generally·

"Each space $Sp_p^{(0)}$ $(0 < p < n-1)$ through the centre O parallel to a bounding space Sp_p intersects D_n according to a D_{p-1}, of which O is again the centre".

Thus we find ascending from below:

"Each chord of D_{n-1} through O parallel to an edge has a length $\sqrt{2}$, each plane through O parallel to a face intersects D_{n-1} according to a regular hexagon with sides $\frac{1}{2}\sqrt{2}$, each space through O parallel to a bounding body intersects D_{n-1} according to a regular octahedron with edges $\sqrt{2}$, etc."

6. We retrace our steps and determine of the above mentioned rhombotope the length of the axis before and after the truncation. Out of the similitude of the triangles AOB and POC follows in connection with the length $\frac{1}{2}\sqrt{n-1}$, $\frac{1}{2}\sqrt{n}$, $\frac{1}{2}$ of OC, OP, OB for OA the value $\frac{1}{2(n-1)}\sqrt{n(n-1)}$ and so for half of the unmutilated axis which is $n-1$ times as large $\frac{1}{2}\sqrt{n(n-1)}$. If we represent by $Rh_p\,[q, r]$ a rhombotope with p dimensions of which q is the length of the axis, r are the parts of the axis removed by the truncation, the section D_{n-1} has to be represented by the symbol $Rh_{n-1}\left[\sqrt{n(n-1)}, \frac{n-2}{2(n-1)}\right]$

So the theorem holds:

"We obtain the section D_{n-1}, if we allow the measure-polytope M_{n-1} to pass in the indicated way by stretching in the direction of a diagonal as far as $\sqrt{n}$ times the original length into a rhombotope with a length of axis $\sqrt{n(n-1)}$ and if we truncate this rhombotope by two spaces Sp_{n-2}. normal to the axis to a

$$Rh_{n-1}\left[\sqrt{n(n-1)}, \frac{n-2}{2(n-1)}\right]$$ [1])

III. Explanation in details of the connection of D_{n-1} with regular and regularly truncated simplexes.

7. We consider in the space Sp_n a rectangular system of coordinates with an arbitrary point O as origin and $OX_1, OX_2, \ldots OX_n$ as axes, and we now call the $2^{n\text{th}}$ part of that space which is the locus of the point with only positive coordinates the "n-edge $O(X_1 X_2 \ldots X_n)$".

If A, A' are two opposite vertices of a measure-polytope M_n of Sp_n and if $AA_1, AA_2, \ldots AA_n$ are the edges passing through A and $A'A'_1, A'A'_2, \ldots A'A'_n$ the edges parallel to these but directed oppositely, then M_n can be regarded as the part of the space Sp_n common to the two n-edges $A(A_1 A_2 \ldots A_n)$ and $A'(A'_1 A'_2 \ldots A'_n)$.

If we intersect this figure of the two oppositely orientated n-edges and the measure-polytope M_n common to both by an arbitrary space Sp_{n-1}, the two n-edges are intersected along two oppositely orientated simplexes and the section of M_n with that space Sp_{n-1} appears as the part of that space that is enclosed at the same time by both simplexes situated in that space. If the selected space is normal to the diagonal AA', connecting the vertices of the n-edges, the simplexes are regular and they have the point of intersection P of the intersecting space Sp_{n-1} with AA' as common centre of gravity. So the general theorem holds:

"The section of M_n with a space Sp_{n-1} normal to a diagonal can always be regarded as a part of that space Sp_{n-1} enclosed by two definite concentric, oppositely orientated, regular simplexes of that space".

If we wish to make use of this theorem we must determine in a more detailed way the length of the edges of those oppositely orientated regular simplexes with common centre of gravity.

8. If we think the intersecting space Sp_{n-1} to be normal to the

[1]) This theorem shows distinctly why the sections of an octahedron parallel to two faces must be identical to those of a cube by planes normal to a diagonal in points of the middle third part of that line. The same in other words: If we truncate a cube with the unity of edge at two opposite vertices by planes normal to the connecting line in the points dividing this diagonal into three equal parts and if we compress an octahedron with edges $\sqrt{2}$ in the direction of the normal on two parallel faces as far as halt the thickness, then we cause the same solid to be generated in two different ways.

diagonal AA' in the first point of division A_1, at a distance $\frac{1}{n}\sqrt{n}$ from A, the section is a simplex with edge $\sqrt{2}$. So the two simplexes, generated when an arbitrary point P of AA' is substituted for point A_1, have edges of a length of $AP\sqrt{2n}$ and $A'P\sqrt{2n}$, wherefore we indicate them, also with reference to the number of vertices, by $S_n(AP\sqrt{2n})$ and $S'_n(A'P\sqrt{2n})$. So the theorem holds:

"If we shove an M_n, of which the diagonal AA' is normal to a given space Sp_{n-1}, in the direction of that diagonal through that space Sp_{n-1}, so that the spaces Sp_{n-1} of the bounding polytopes M_{n-1} move parallel to themselves, the section of Sp_{n-1} with the moving polytope M_n is at every moment the part of that space Sp_{n-1} that is enclosed within two concentric, yet oppositely orientated, regular simplexes $S_n(p\sqrt{2n})$ and $S'_n(p'\sqrt{2n})$ where p and p' are connected in such a way that the sum $p+p'$ is equal to $\sqrt{n}$. During that movement of M_n the common centre of gravity of the two simplexes remains in its place and the spaces Sp_{n-2} of the bounding simplexes S_{n-1} and S'_{n-1} move parallel to themselves; whilst simplex S_n expands itself from this common centre of gravity to a simplex $S_n(n\sqrt{2})$, simplex S'_n inversely contracts from a simplex $S'_n(n\sqrt{2})$ to this point".

At the moment when this process has got halfway and the two simplexes are of the same size we find:

"The section D_{n-1} is the part of the intersecting space Sp_{n-1} enclosed by two definite equal concentric yet oppositely orientated regular simplexes $S_n\left(\frac{1}{2}n\sqrt{2}\right)$ and $S'_n\left(\frac{1}{2}n\sqrt{2}\right)$."

Thus for $n=3$ the regular hexagon with sides $\frac{1}{2}\sqrt{2}$ is the figure enclosed by two triangles with sides $\frac{3}{2}\sqrt{2}$ — think of the well-known trademark —, thus for $n=4$ the regular octahedron with edges $\sqrt{2}$ is the figure enclosed by two tetrahedra with edges $2\sqrt{2}$ — think of the two tetrahedra described in a cube and the octahedron common to both. So in general the problem in the space of n dimensions is reduced to another problem in space of $n-1$ dimensions and moreover the connection of the result with regular simplexes is explained.

If we think the simplex S_n to be white and the simplex S'_n to be black, the n bounding spaces Sp_{n-2} of D_{n-1} originating from S_n will be white, those originating from S'_n will be black. From this ensues that it must be possible to colour the $2n$ bounding spaces

Sp_{n-2} of D_{n-1} in such a way in turns white and black, that two opposite bounding spaces Sp_{n-2} have a different colour. The octahedron is really the only one of the regular bodies that allows this operation. [1])

9. If the simplex S_n expands from a point to an $S_n(n\sqrt{2})$ and at the same time S'_n contracts from an $S'_n(n\sqrt{2})$ to a point, then S_n lies at the beginning of the process within S'_n and at the end inversely S'_n lies within S_n. Gradually first the vertices, then the edges, then the faces, etc. of S_n have passed outward. We shall now investigate when that takes place.

From the diagrams of the expanding plate given in the first part it is evident, that the section of M_n with a space Sp_{n-1} changes its nature when the point of intersection P of that space Sp_{n-1} with the diagonal AA' passes one of the $n-1$ points of division $A_1, A_2, \ldots$ As the nature of the section of course also changes when bounding elements of S'_n lying inside S_n pass outward, the latter must take place at those moments when those points of division of the diagonal AA' of the moving M_n pass through the fixed space Sp_{n-1}. This theorem then really holds:

"In the translation of M_n in the direction of AA' through the space Sp_{n-1} in succession the vertices, the edges, the faces, bounding bodies, etc. of S_n come entirely outside S'_n at those moments that the point of intersection P of the diagonal AA' with the space Sp_{n-1} coincides successively with the points of division A_1, A_2, A_3, A_4, etc."

We regard — in order to prove this theorem — the arbitrary stadium of the simplexes $S_n(AP\sqrt{2n})$ and $S'_n(A'P\sqrt{2n})$, divide the n vertices of S_n in an arbitrary way into two groups β and γ of p and $n-p$ points, and indicate by β' and γ' the groups of the p and $n-p$ corresponding vertices of S'_n, by B, C, B', C' (fig. 2) the centres of gravity of the point-groups $\beta, \gamma, \beta', \gamma'$ — i.e. the

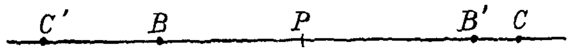


Fig. 2.

[1]) In contradiction to this seems that for $n=5$ through each edge *three* faces pass and thus *three* bounding bodies (12, 18, 8) lie around it. This contradiction however is only apparent; it is annulled by the remark that two bounding bodies (12, 18, 8) having a face in common agree or differ in colour according to the face being triangular or hexagonal. Of the three faces one is triangular, two are hexagonal; the bounding bodies to which the two hexagonal faces belong, differ in colour from the two others, these agreeing in colour.

centres of the bounding simplexes $S_p, S_{n-p}, S'_p, S'_{n-p}$ with these points as vertices. Then the five points B, C, B', C', P lie in such a way upon the same right line, that B and C' lie on one side of P and B' and C on the other side, and we have

$$\left.\begin{array}{r} p \;\; BP = (n-p)\, PC \\ (n-p)\, C'P = \quad p \,.\, PB' \end{array}\right\}, \; \frac{AP}{PA'} = \frac{BP}{PB'} = \frac{CP}{PC'}.$$

We can now assert that the bounding simplex S_p of the vertices β of S_n lies entirely or partly inside S'_n when B is between C' and P, whilst S_p lies entirely outside S'_n when C' lies between B and P. In other words. as AP increases, the bounding simplex S_p of S_n comes entirely outside S_n when B coincides with C' and the spaces Sp_{p-1} and Sp_{n-p-1} of S_p and S'_{n-p}, crossing each other in general entirely perpendicularly, become incident because they get the point $B = C'$, then common centre of gravity, as point of intersection.

Under the condition $BP = C'P$ follows from the equations

$$\frac{BP}{PC} = \frac{n-p}{p}, \; \frac{PC}{C'P} = \frac{AP}{PA'}$$

the relation

$$(n-p)\, AP = p \,.\, PA',$$

which shows that P must coincide with the p^{th} dividing point A_p of AA'.

10. If P coincides with A_p the spaces Sp_{p-1} and Sp_{n-p-1} of S_p and S'_{n-p} have, as we saw above, the common centre of S_p and S'_{n-p} in common. As this point of intersection of S_p and S'_{n-p} becomes vertex of the section, — if we call this again $\frac{p}{n}(M_n)$ in connection with preceding investigations — the theorem holds:

"The centres of the $\binom{n}{p}$ bounding simplexes S_p of a regular simplex $S_n(p\sqrt{2})$ form the vertices of a polytope congruent to $\frac{p}{n}(M_n)$ for $p = 1, 2, \ldots, n-1$."

For even $n = 2n'$ we have specially:

"The centres of the $\binom{2n'}{n'}$ bounding simplexes $S_{n'}$ of a regular simplex $S_{2n'}\,(n'\sqrt{2})$ form the vertices of a $D_{2n'-1}$."

11. If P lies between A_p and A_{p+1} the vertices of the section of the two simplexes S_n and S'_n are furnished by the points of intersection of each bounding simplex S_{p+1} of S_n with the $p+1$

bounding simplexes S'_{n-p} of S'_n which have the property of counting among their $n-p$ vertices only one vertex corresponding to a vertex of this S_{p+1}; in each bounding simplex S_{p+1} these $p+1$ points of intersection form the vertices of a new regular simplex $\overline{S}_{p+1}$ which is concentric to the assumed one but oppositely orientated. We determine the length of the edges of this new simplex, for the definite case that P lies just in the middle between A_p and A_{p+1}, with the aid of reflections in quite close connection with the preceding.

If B, C, B', C' (fig. 3) are successively the centres of gravity of

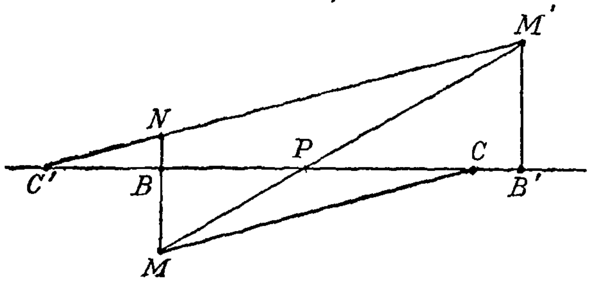


Fig. 3.

the bounding simplex S_{p+1}, of the bounding simplex S_{n-p-1} of the remaining vertices of S_n and of the bounding simplexes S'_{p+1}, and S'_{n-p-1} of the groups of vertices of S'_n corresponding with the vertices of S_{p+1} and S'_{n-p-} these points lie on a same right line through P again, viz.: B and C' on one side and C and B' on the other side of P. If furthermore M and M' are corresponding vertices of S_{p+1} and S'_{p+1} these points lie in parallel normals erected in B and B' on BB' and the line connecting M and M' passes through P. The point of intersection N of BM and $C'M'$ is the vertex of $\overline{S}_{p+1}$ corresponding to the vertex M of S_{p+1}. From CM and $C'M'$ being parallel follows

$$\frac{BN}{MB}=\frac{C'B}{BC}=\frac{C'P-BP}{BP+PC},$$

whilst the relations

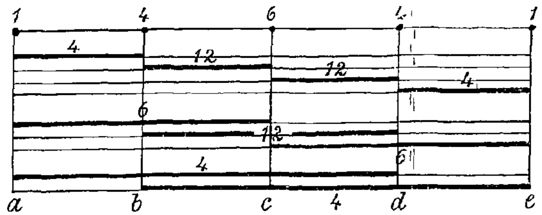
$$\frac{AP}{PA'}=\frac{BP}{PB'}=\frac{CP}{PC'}=\frac{2p+1}{2n-2p-1}$$

and

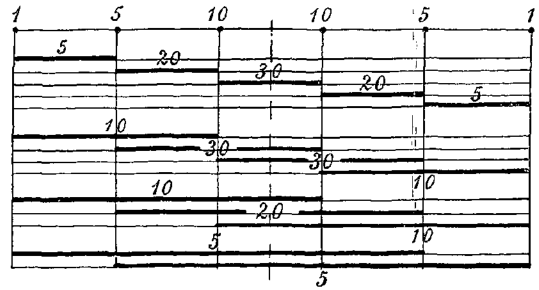
$$\frac{BP}{PC}=\frac{B'P}{PC'}=\frac{n-p-1}{p+1}$$

enable us to express $C'P$ and BP in PC. Substitution gives the result

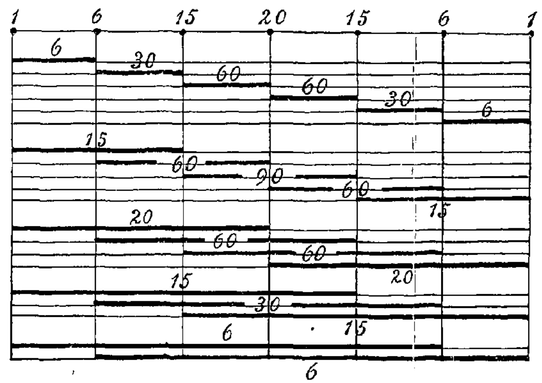
P. H. SCHOUTE. "The section of the measure-polytope M_n of space Sp_n with a central space Sp_{n-1} perpendicular to a diagonal."



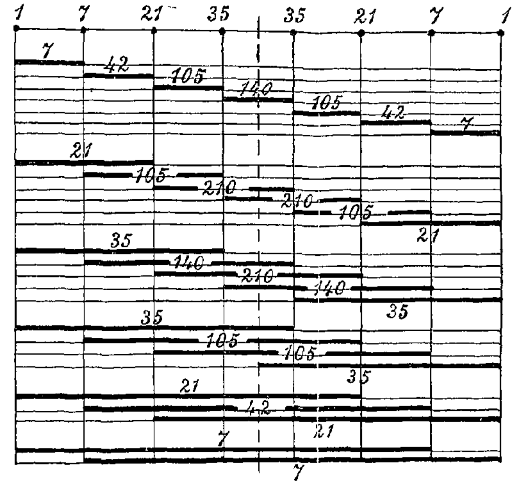
$n=4$



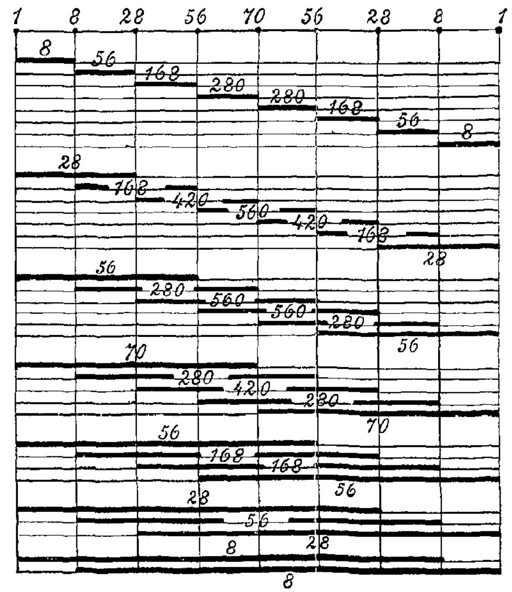
$n=5$



$n=6$



$n=7$



$n=8$

$$\frac{BN}{MB}=\frac{1}{2p+1}.$$

So the theorem holds:

"If we describe in the spaces Sp_p bearing the bounding simplexes $S_{p+1}\left(\frac{2p+1}{2}\sqrt{2}\right)$ of a regular simplex $S_n\left(\frac{2p+1}{2}\sqrt{2}\right)$ simplexes $S_{p+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(p+1)\binom{n}{p+1}$ vertices of a $\frac{2p+1}{2n}(M_n)$."

For odd $n=2n'+1$ we have in particular:

"If we describe in the spaces $Sp_{n'}$, bearing the bounding simplexes $S_{n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ of a regular simplex $S_{2n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ simplexes $S_{n'+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(n'+1)\binom{2n'+1}{n'+1}$ vertices of a $D_{2n'}$."

In connection with the results found above the length $\frac{1}{2}\sqrt{2}$ appearing here for the edges of the new simplexes contains a confirmation.

Mathematics. — *"On five pairs of four-dimensional cells derived from one and the same source."* By Mrs. A. Boole Stott and Prof. P. H. Schoute.

(Communicated in the meeting of December 28, 1907).

Introduction.

As this paper must be regarded as a short completion of the handbook of the "Mehrdimensionale Geometrie" included in the Sammlung Schubert we keep the notation used there.

We regard in succession each of the six regular cells C_5, C_8, C_{16}, C_{24}, C_{120}, C_{600} of the space Sp_4 and derive from these two new four-dimensional cells. The first, which has the centres K_0 of the edges of the regular cell as vertices is formed by a regular truncation at the vertices as far as the centres of the edges; the second is the reciprocal polar of the first with respect to the spherical space of the points K_0.

Because the regular C_{16} leads us to find the regular C_{24}, the number of pairs of new cells is not six but five.

I. *General observations.*

1. If we understand for the regular cells by e, k, f, r successively the number of the vertices, edges, faces, bounding bodies, by p, q the number of bounding bodies through an edge, through a point, by e', k', f' the number of vertices, edges, faces of the bounding bodies, then besides the relations

$$e+f=k+r \quad , \quad e'+f'=k'+2$$

of EULER the equations hold

$$qe=re' \quad , \quad pk=rk' \quad , \quad 2f=rf' \; ,$$

out of which number of five we can easily deduce the relation

$$(q-2)\,e=(p-2)\,k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

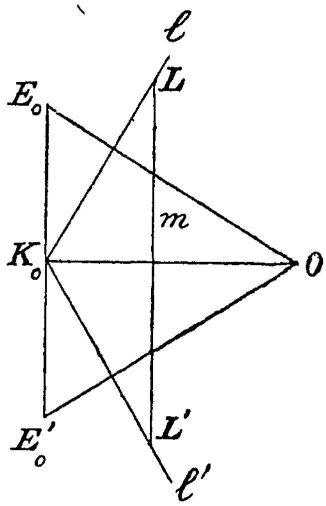
The following table furnishes these quantities for the six regular cells of Sp_4.

e	k	f	r	p	q	e'	k'	f'
5	10	10	5	3	4	4	6	4
8	24	32	16	4	8	4	6	4
120	720	1200	600	5	20	4	6	4
16	32	24	8	3	4	8	12	6
24	96	96	24	3	6	6	12	8
600	1200	720	120	3	4	20	30	12

2. We shall now endeavour to express the characteristic numbers E, K, F, R of the first of the two new cells — and what is also possible for these P, Q — in the characteristic numbers e, k, f, r, p, q of the regular cell.

"The number of vertices of the new cell is k, i. e. $E=k$."

If we project the regular cell (see the diagram) on the plane through one of the edges E_0E_0' and the centre O, the two new bounding spaces passing through the centre K_0 project themselves according to the perpendiculars l, l' let down out of K_0 on the axes OE_0, OE_0'. The section of the regular cell with a plane normal to the plane of projection in a point lying close to E_0E_0' being an equilateral triangle, a square or a regular pentagon, with the assumed point always as centre, according to p having the value 3, 4 or 5, the section of the system of the $p+2$ bounding spaces of the new cell passing through K_0 with a space normal to OK_0 — e.g. with a space which according to the normal m to OK_0 is perpendicular to the plane of projection -- is a right p-lateral prism, of which the segment LL' of m enclosed between l, l' is the axis and the perpendicular endplanes project themselves in L and L'. From this ensues:



"Through a vertex pass $p+2$ bounding spaces, i.e. $Q = p + 2$".

"Through an edge pass three bounding spaces, i.e. $P = 3$".

"Through a vertex pass $2p$ edges, so pk is the number of edges, i.e. $K = pk$."

"The system of the bounding spaces consists of two groups, namely of e regular polyhedra with q faces, and r semi-regular polyhedra (e', k', f') with equivalent vertices truncated at the vertices as far as the centres of the edges, i.e. $R = e + r$".

"As the polyhedra of the second group have $e' + f' = k' + 2$ faces and a face is common to two bounding spaces, the number of faces is half the sum of qe and $r(k' + 2)$ or qe and $pk + 2r$, i.e. $2F = qe + pk + 2r$".

Thus the result is:

"The first of the two cells, (E, K, F, R, P, Q), deduced out of the regular cell (e, k, f, r, p, q) has the characteristic numbers

$$E = k, \quad K = pk, \quad F = \tfrac{1}{2}(qe + pk) + r, \quad R = e + r,$$
$$P = 3, \quad Q = p + 2."$$

Here the law of EULER $E + F = K + R$ may serve as verification. In reality the difference of the two members of this equation

$$\begin{aligned} E + F - (K + R) &= k + \tfrac{1}{2}(qe + pk) + r - (pk + e + r) \\ &= k - e + \tfrac{1}{2}(qe - pk) \\ &= \tfrac{1}{2}\{(q-2)e - (p-2)k\} \end{aligned}$$

is equal to zero in consequence of the relation (1).

3. The second of the new cells deduced out of the regular cell is enclosed by the polar-spaces of the centres K_0 of the edges with respect to the spherical space through those points, i.e. by the

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tangential spaces to that spherical space in those points, i. e. by the spaces in the points K_0 normal to the axes OK_0[1]). By polar inversion of the result found above we arrive with respect to this second new cell at the following results:

"The number of bounding spaces of the new cell is k, i. e. $R' = k$."

"The bounding bodies have $p + 2$ vertices and are double pyramids with a regular polygon with p-sides as base lying in a plane bisecting the connecting line of the vertices at right angles."

"The faces are isosceles triangles."

"In a bounding space lie $2p$ faces, so pk is the number of faces, i. e. $F' = pk$."

"The system of the vertices consists of two groups, namely of e regular vertices and r semi-regular vertices, i. e. $E' = e + r$".

"The number of edges K' is $\frac{1}{2}(qe + pk) + r$."

So the result is:

"The second of the two cells, (E', K', F', R'), deduced out of the regular cell (e, k, f, r, p, q) is bounded by double pyramids with a regular polygon with p-sides as base and has the characteristic numbers

$$E' = e + r, \quad K' = \tfrac{1}{2}(qe + pk) + r, \quad F' = pk, \quad R' = k."$$

It might appear as if it were possible to deduce more pairs of new cells out of the regular cells by doing for the ends F_0 of the axes OF_0 the same as has been done above for the points K_0. This is, however, not the case, because for each regular cell the centres F_0 of the faces form the centres K_0 of the edges of another regular cell which is for the cells C_5, C_{24} dualistically related to themselves a cell of the same kind, for the cells related in pairs to one another (C_8, C_{16}) (C_{120}, C_{600}) a cell dualistically related. And as is immediately evident, the pointgroups E_0 and R_0 can neither lead to new results.

We conclude these general observations with the remark that the two cells deduced from the regular cell (e, k, f, r) show much regularity; of the former the vertices and the edges, of the latter the faces and the bounding bodies are mutually equivalent groups of elements, whilst the faces and the bounding bodies of the former and the vertices and edges of the latter form groups of elements consisting of two subgroups. Do these new cells furnish the maximum amount of regularity for polytopes not entirely regular? We do not intend to go into further details here, as the Mathematical Society at Amsterdam is proposing a prizequestion about what is to be understood by "semi-regular polytopes".

1) The handbook quoted above contains in Vol II pages 256—261 some communications about the corresponding polytopes in the space Sp_n.

The following table shows the results which are obtained by substituting the values of e, k, f, r, p, q for the five different cases. For the sake of completeness those quantities are also included which indicate how many vertices are situated in face and bounding space. We must here notice that, the first cell having two kinds of faces and bounding bodies, we are obliged to take four new quantities, namely the numbers of vertices S_1 and T_1 in face and bounding body of one, the numbers of vertices S_2 and T_2 in face and bounding body of the second kind. Here S_2 and T_2 will relate to the truncating bodies with faces of the same kind and S_1 and T_1 refer to the truncated bodies, where we must then consider as far as S_1 is concerned those faces which the truncated bodies keep in common. We must likewise for the second cell, with two kinds of vertices and edges, introduce the four new quantities P_1', P_2', Q_1', Q_2',

As is evident $T_1 = Q_1' = k'$, whilst $T_2 = Q_2'$ is the number of vertices of the regular polyhedron with q faces.

	e	E, R'	K, F'	F, K'	R, E'	P, S'	Q, T'	S_1, P_1'	S_2, P_2'	T_1, Q_1'	T_2, Q_2'
C_5	5	**10**	**30**	**30**	**10**	3	5	3	3	6	4
C_8	16	**32**	**96**	**88**	**24**	3	5	4	3	12	4
C_{24}	24	**96**	**288**	**240**	**48**	3	5	4	4	12	8
C_{600}	120	**720**	**3600**	**3600**	**720**	3	7	3	3	6	12
C_{120}	600	**1200**	**3600**	**3120**	**720**	3	5	5	3	30	4

In a second part we shall submit each of these five pairs of new cells to a separate investigation.

Mathematics. — *"The analogon of the Cf. of* KUMMER *in seven-dimensional space".* By Dr. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of December 28, 1907).

§ 1. In a preceding communication a method was indicated to generate the *Cff.* in spaces of (2^p-1) dimensions, which can be regarded as analoga of the *Cf.* of KUMMER [1]).

[1]) A quotation in HUDSON's *Kummer's Quartic Surface* (p. 187) drew my attention to the fact, that these *Cff.* have already been obtained by an altogether different method by W. WIRTINGER (*Göttinger Nachrichten*, 1889, page 474; *Monatshefte fur Mathematik und Physik* I, page 113; *Mathem. Annalen* 40, page 74). In these papers the varieties are also investigated, for which the elements of such *Cff.* are singular in the same way as those of the *Cf.* (16_6) for KUMMER's quartic surface.

In the following the K^{VII} is under closer investigation especially with a view to the *Cff.* obtainable out of it by omission of certain elements. To this end it is necessary to construct an other diagram than that of eight simplexes, which can only clearly show the *Cff.* (56_{26}); (48_{22}); (40_{19}); (32_{16}); (24_{13}) and (16_{10}), formed by omission of the elements of 1, 2, 3, 4, 5, 6 simplexes, all (except the first) in different types, likewise of *Cff.* (24_9) and (32_{12}), constructed exclusively of fillings (8_3).

§ 2. If we isolate in the *Cf.* of KUMMER a point and a plane not incident to it, the remaining fifteen elements of each kind are divided into a sextuple incident to one of the isolated elements and a nonuple. Each one of the two sextuples forms with the 15 elements of the other kind a free *Cf.* $(6_5, 15_2)$ which means nothing else but that each of the fifteen right lines connecting the *Cf.*-points in one plane bears another *Cf*-plane and reciprocally.

The two nonuples of elements, however, of both kinds together form one *Cf.* (9_4), the structure of which is identical to that of *Cf.* (9_4) III of the classification of MARTINETTI [2]).

This arrangement can be done in $16 \times 10 = 160$ ways.

We can likewise isolate out of K^{VII} in $64 \times 36 = 2304$ ways a point and an Sp_6 not incident to it, by which the sixty-three other elements of each kind are divided into a group of twenty-eight incident to the isolated element of the other kind and a remaining group of thirty-five. The two groups of twenty-eight form together a scheme (28_{12}); each group of twenty-eight with that of thirty-five of another kind a scheme $(28_{15}, 35_{12})$; addition of (28_{12}) and $(28_{15}, 35_{12})$ furnishes a scheme $(28_{27}, 63_{12})$; the two groups of thirty-five form together a scheme (35_{16}).

This arrangement made for the two elements $A1$ is shown in the plate, where the same notations are assigned to the elements as in the diagram of simplexes of which it is a transformation.

We have but to explain how the regular composition indicated by the thicker lines is obtained.

§ 3. Let us first take into consideration the scheme $(28_{27}, 63_{12})$ of points (columns) and Sp_6 (rows). Every number of twelve points on a row lying in two different *Cf.*-Sp_6 lies in an Sp_5, so we can take the *Cf.* to consist of twenty-eight points and sixty-three Sp_5 in Sp_6; each of the sixty-four *Cf.*-Sp_6 of the K^{VII} contains such a *Cf.* (and reciprocally).

[2]) Atti della R. Accademia Peloritana XV.

We find for the complete notation of such a *Cf.* in a way analogous to that of § 6 of the former paper:

	Sp_0	Sp_1	Sp_2	Sp_3	Sp_4	Sp_5
	28	**378**	**2016**	**5040**	**1008**	**63**
incid. to:						
Sp_0	—	2	3	4	6	12
Sp_1	27	—	3	6	15	66
Sp_2	216	16	—	4	20	160
Sp_3	720	80	10	—	15	240
Sp_4	216	40	10	3	—	32
Sp_5	27	11	5	3	2	—

By projection and intersection we find from this in Sp_2 a *Cf.* $(378_{16}, 2016_3)$; a $(2016_{10}, 5040_4)$ and a $(5040_3, 1008_{15})$ of points and right lines, in Sp_3 a $(378_{80}, 5040_6)$ and a $(2016_{10}, 1008_{20})$ of points and planes.

Although the number of *Cf*-points is $28 = 4 \times 7$ one cannot succeed in forming four simplexes S_6 out of the *Cf.*-elements; after isolation of such a simplex (which is possible in several ways) we can form out of the remaining elements (also in several ways) at most a scheme S_5, and then an S_4, S_3, S_2 and S_1 after which of the (28_{12}) an element of each kind is left, mutually not-incident; which we join to an "S_0". In the figure $S_3 + S_2 + S_1 + S_0$ are taken together to a scheme (10_6) which we indicate by T.

The arrangement of the thirty-five remaining elements follows now by our regularly putting down the combinations 3 by 3 of the seven points chosen for S_6; it is evident that the entire diagram contains along the chief diagonal only schemes S or T, whilst outside a couple of new fillings appear amongst which we notice a (10_4), complementary to T.

It is by addition of these partial schemes that we can obtain a great number of *Cff.* included in the total figure; we restrict ourselves to the *forced Cff.* which are those of which each element shows more incidences than are sufficient to determine it and of which for this reason the existence is remarkable from a geometric point of view.

Of the *Cf.* $(28_{27}, 63_{12})$ in Sp_6 the twenty-eight points form evidently with twenty-eight Sp_5 a dual *Cf.* (28_{12}), the same points with the thirty-five other Sp_5 a *Cf.* $(28_{15}, 35_{12})$.

By omission out of (28_{12}) of a S_6 remains a (21_{10}); by omitting S_6 and S_5 a (15_8) remains the scheme of which is *anallagmatic*: each couple of its *Cf.*-Sp_5 has four *Cf.*-points in common. The same number of 15 points forms with 15 other Sp_5 (namely $H1$ as far as $B1$ included out of the number of thirty-five) a *Cf.* (15_7) of which the scheme is complementary to the anallagmatic (15_8).

Out of the *Cf.* (35_{16}) is formed by omission of S_4 a *Cf.* (30_{14}), by omission of T a *Cf.* (25_{12}), of S_4 and T a *Cf.* (20_{10}); of two different T a *Cf.* (15_8), identical to the already mentioned one, its points lie in an Sp_6, the (35_{16}) has in each of the twenty-eight other *Cf.*-R_6 such a (15_8).

If we add to the *Cf.* (30_{14}) a system T out of (28_{12}) a *Cf.* (40_{18}) is formed.

The *Cf.* (35_{16}) is also obtainable out of the diagram of simplexes of the K^{VII}, the simplex A then falls away entirely, of each of the seven other ones three elements of each kind disappear. The diagram (35_{16}) consists of seven systems S_4 in the chief diagonal, mutually connected by fillings (5_2), which all degenerate into (3_2) and (2_2).

By omitting 1, 2, 3, 4 from this S_4 we obtain *Cff*: (30_{14}), (25_{12}), (20_{10}) and (15_8). The (30_{14}) is identical to the already mentioned one, the (15_8) however is of another type. not anallagmatic, neither do its points lie in *one* Sp_6.

§ 5. In each of the Sp_5 formed by intersection of two *Cf.*-Sp_6 of K^{VII} lie 12 *Cf.*-points, of which thirty-two sextuples are also common to a third *Cf.*-Sp_6; such a sextuple lies thus in an Sp_4, the twelve points form with the thirty-two Sp_4 a *Cf.*$(12_{16}, 32_6)$.

We can give to the diagram of such a *Cf.* the following form· (see table p. 507).

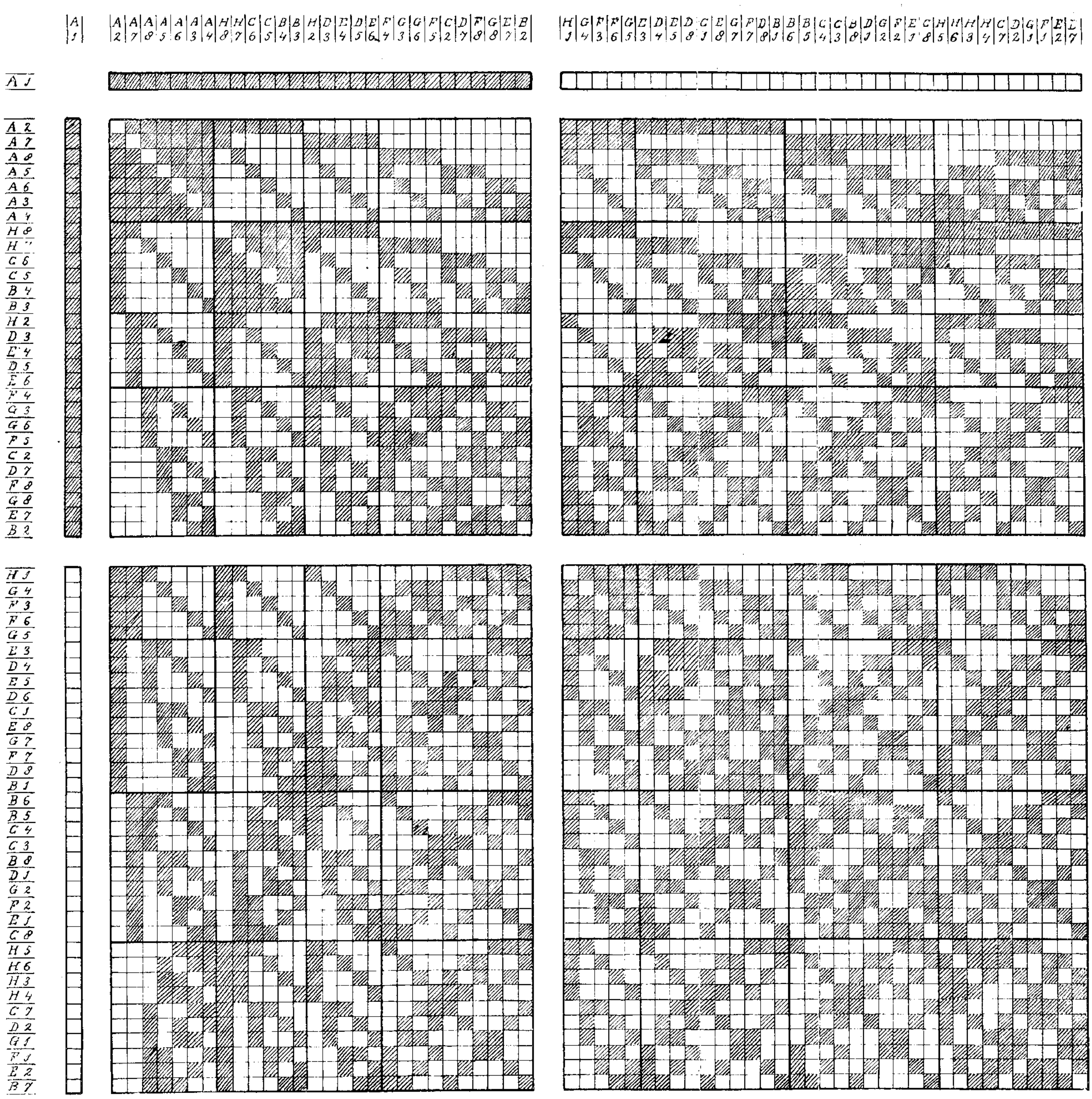
If e. g. we take the *Cf*-Sp_5, formed by the intersection of the *Cf*-Sp_6. $A\,1$ and $A\,2$, the twelve points become respectively:

$$\begin{array}{ll} A3 = P1 & B4 = Q1 \\ A4 = P2 & B3 = Q2 \\ A5 = P3 & C6 = Q3 \\ A6 = P4 & C5 = Q4 \\ A7 = P5 & H8 = Q5 \\ A8 = P6 & H7 = Q6 \end{array}$$

The entire *Cf.* consists evidently of two simplexes $S_5 : P$ and Q in Möbius-position forming together the part (12_6) whilst moreover every triplet of vertices of one simplex with the three non-conjugate ones of the other lie in *one* Sp_4, i.o.w., each face of one intersects the non-conjugate one of the other.

P *Q*

	2	3	4	5	6	1					
1		3	4	5	6		2				
1	2		4	5	6			3			
1	2	3		5	6				4		
1	2	3	4		6					5	
1	2	3	4	5							6
1							2	3	4	5	6
	2					1		3	4	5	6
		3				1	2		4	5	6
			4			1	2	3		5	6
				5		1	2	3	4		6
					6	1	2	3	4	5	
1	2	3							4	5	6
1	2		4					3		5	6
1	2			5				3	4		6
1	2				6			3	4	5	
1		3	4				2			5	6
1		3		5			2		4		6
1		3			6		2		4	5	
1			4	5			2	3			6
1			4		6		2	3		5	
1				5	6		2	3	4		
	2	3	4			1				5	6
	2	3		5		1			4		6
	2	3			6	1			4	5	
	2		4	5		1		3			6
	2		4		6	1		3		5	
	2			5	6	1		3	4		
		3	4	5		1	2				6
		3	4		6	1	2			5	
		3		5	6	1	2		4		
			4	5	6	1	2	3			



This connection is for the first time possible in Sp_5, the analogon in Sp_3 would be: two tetrahedra in MÖBIUS-position, where each edge of one intersects the non-conjugate one of the other; of this *Cf.* $(8_7, 14_4)$, although it is possible to design the diagram, the execution is evidently impossible.

We find for the complete notation of *Cf.* $(12_{16}, 32_6)$:

	Sp_0	Sp_1	Sp_2	Sp_3	Sp_4
	12	**60**	**160**	**240**	**32**
incid. to					
Sp_0	—	2	3	4	6
Sp_1	10	—	3	6	15
Sp_2	40	8	—	4	20
Sp_3	80	24	6	—	15
Sp_4	16	8	4	2	—

By projecting and intersecting are formed out of these e.g. in Sp_2 a *Cf.* $(60_8, 160_3)$ and a *Cf.* $(160_6, 240_4)$ of points and right lines; in Sp_3 a *Cf.* $(60_{24}, 240_6)$ and a *Cf.* $(160_4, 32_{20})$ of points and planes.

§ 6. The points of Sp_5 can be conjugated one to one to the linear complexes of the usual three-dimensional space, the Sp_4 become linear systems of ∞^4 of these complexes, the *Cf.* $(12_{16}, 32_6)$ can be represented in our space.

It is however possible to take the twelve complexes simultaneously special and to regard them as right lines, the thirty-two Sp_4 then become linear complexes which each contain a sextuple of the right lines; the *Cf.* $(12_{16}, 32_6)$ is then realized in right lines and linear complexes of our space.

We can easily give line-coordinates for such a number of twelve right lines by omitting from the point-coordinates of the twelve points a couple, e. g. X_7 and X_8, and by letting the six remaining ones satisfy the fundamental relation

$$X_1 X_6 + X_2 X_5 + X_3 X_4 = 0$$

So we obtain e. g. the right lines

$$P1 = (c, -d, -a, \ 0, -g, \ h) \qquad Q1 = (d, \ c, \ 0, -a, -h, -g)$$
$$P2 = (d, \ c, \ 0, -a, \ h, \ g) \qquad Q2 = (c, -d, -a, \ 0, \ g, -h)$$
$$P3 = (0, -f, \ g, -h, -a, \ 0) \qquad Q3 = (f, \ 0, \ h, \ g, \ 0, -a)$$
$$P4 = (f, \ 0, -h, -g, \ 0, -a) \qquad Q4 = (0, -f, -g, \ h, -a, \ 0)$$
$$P5 = (g, -h, \ 0, \ f, \ c, -d) \qquad Q4 = (h, \ g, -f, \ 0, \ d, \ c)$$
$$P6 = (h, \ g, \ f, \ 0, -d, -c) \qquad Q6 = (g, -h, \ 0, -f, -c, \ d);$$

if besides is satisfied

$$ch + dg = af - gh = 0.$$

The peculiarity appearing with this example taken for simplicity's sake, that the right lines show mutually some incidences, is lost by submitting the coordinates in Sp_5 first to a linear transformation.

In the same way, indeed, we can formulate for all *Cff.* indicated in spaces of a lower number of dimensions an analytical definition by deducing the coordinates of their elements from those of the elements of K^{VII}.

Chemistry: — "*On the constitution of* VAN GEUNS's *oxymethyldinitrobenzonitrile*". By Dr. J. J. BLANKSMA. (Communicated by Prof. A. F. HOLLEMAN).

By the action of potassiumcyanide on meta-dinitrobenzene in methylalcoholic or ethylalcoholic solution, LOBRY DE BRUYN [1]) obtained in 1882 the oxymethyl- or oxyethylnitrobenzonitrile $C_6H_3(OCH_3)CN\,NO_2$ 1. 2. 3.

The investigation of these substances was continued afterwards by VAN GEUNS [2]) who succeeded in saponifying these nitriles to acidamines and in preparing the corresponding acids thereof. At the same time VAN GEUNS showed that in both substances a further nitro-group can be introduced by nitration with nitric and sulphuric acids thus yielding the compounds $C_6H_2(OCH_3)CN(NO_2)_2$ m.p. 113° and $C_6H_2(OC_2H_5)CN(NO_2)_2$ m.p. 63°, These two compounds contain a movable nitro-group which may be readily replaced by OH, OCH_3, NH_2, $NHCH_3$, NHC_6H_5 etc.

As, however, the place where the nitro-group had been introduced had remained unknown, the constitution of these derivatives was consequently also unknown.

When VAN GEUNS, owing to his departure for India was obliged

1) Recueil 2, 205.
2) Dissertation Amsterdam 1903.

to discontinue this research I tried at the request of the late Prof. LOBRY DE BRUYN to determine this constitution. After a few trials which led to no result the method was followed which had proved successful in the determination of the constitution of 2.3.4 trinitrophenetol[1]). The constitution of that substance was shown to be:

$$C_6H_2(OC_2H_5)(NO_2)_3\ 2.3.4 \rightarrow C_6H_2(OC_2H_5)NH_2(NO_2)_2\ 1.3.2.4 \rightarrow C_6H_3(OC_2H_5)(NO_2)_2\ 2.4.$$

Oxymethyldinitrobenzonitrile was now treated in an analogous manner; by the action of alcoholic ammonia one NO_2-group was replaced by NH_2 and this was then in turn removed by diazotation and boiling with alcohol. In this manner was obtained an oxymethylnitrobenzonitrile (m.p. 126°) $C_6H_2(OCH_3)\ CN(NO_2)_2 \rightarrow C_6H_2(OCH_3)$ $CN.NH_2NO_2 \rightarrow C_6H_3(OCH_3)CN.NO_2$.

This shows that the NO_2-group at 3 is replaced by NH_2 as otherwise the original oxymethylnitrobenzonitrile $C_6H_3(OCH_3)CN.NO_2$ 1.2.3 m.p. 171° would have been reobtained. Now it remained only to determine the constitution of this substance. On treatment with nitric and sulphuric acids an oxymethyldinitrobenzonitrile was obtained which melts at 71° and which possesses the following constitution: $C_6H_2(OCH_3)CN(NO_2)_2$ 1.2.4.6 [2]).

The constitution of this substance was determined in the following manner. If this compound is treated in alcoholic solution with ammonia or methylamine the OCH_3 group is readily substituted by NH_2 or $NHCH_3$ and dinitrocyano-aminobenzene m.p. 219° or dinitrocyanomethylaminobenzene m.p. 161° is formed which substances were prepared previously from the corresponding oxyethyl compound [3]).

The oxymethylnitrobenzonitrile m.p. 126° was then heated at 150° with hydrochloric acid for 5 hours. On opening the tube a gas escaped which burnt with a green-bordered flame (CH_3Cl) whilst in the tube there were present crystals which after recrystallisation from water melted at 228° and proved to be 5-nitrosalicylic acid (C_6H_3 COOH, OH, NO_2, 1. 2. 5.) In the motherliquor the presence of NH_3 was detected, formed by saponification of the cyano-group. For the purpose of identifying the substance obtained a little of the preparation was mixed with an equal quantity of 5-nitrosalicylic acid (m.p. 228° prepared by nitration of salicylic acid [4]). The melting point

1) Recueil **27**, 49.

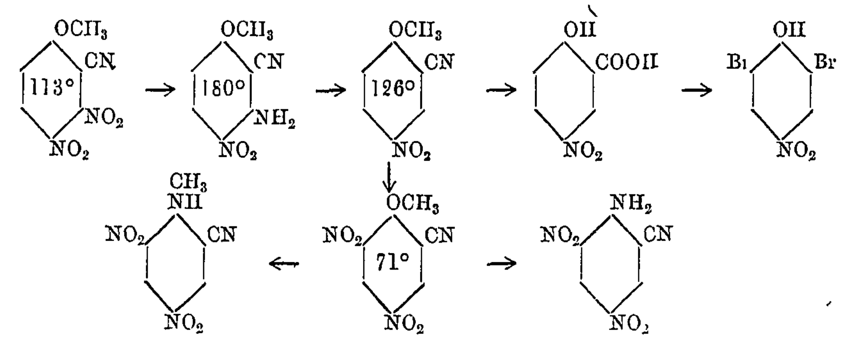
2) This shows that in oxymethylnitrobenzonitrile m.p. 126° the nitro-group is placed on 4 or 6.

3) BLANKSMA. Rec. **20**, 413. **21**, 274.

4) HÜBNER. Ann. **195**, 31.

was not altered thereby. Both preparations could also be converted readily into 2-6-dibromo-4-nitrophenol m.p. 141° by treatment with bromine water [1]).

The following reactions were applied:

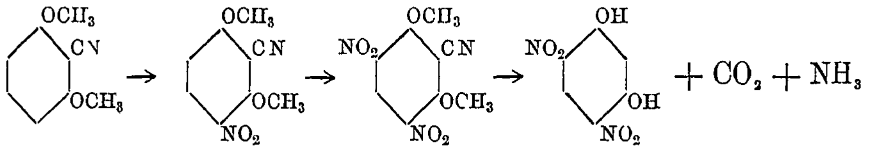


This proved that the constitution of the oxymethyldinitrobenzonitrile prepared by VAN GEUNS is $C_6H_2(OCH_3), CN, (NO_2)_2$. 1, 2, 3, 4.

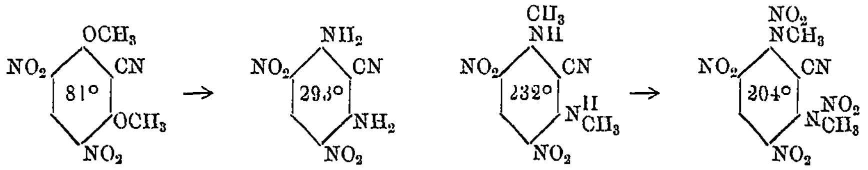
At the same time it was shown that the movable NO_2-group in this substance is placed at 3; consequently we now know the constitution of the compounds obtained from it by substitution of the NO_2-group by OH, OCH_3 etc.

Finally, the constitution was determined of the dinitrodimethoxybenzonitrile obtained by the nitration of $C_6H_3(OCH_3)_2$ CN 1, 3, 2, or of the nitrodimethoxybenzonitrile $C_6H_2(OCH_3)_2$ CN. NO_2 1, 3, 2, 4. [2]).

This compound was converted into 4-6-dinitro-resorcine m.p. 215° by being heated for 5 hours at 150°–160° with hydrochloric acid (30 °/₀ HCl); from this follows that its constitution is $C_6H(OCH_3)_2CN(NO_2)_2$ 1. 3. 2. 4. 6.



4.6 dinitro-2-cyano 1.3 dimethoxybenzene on treatment with alcoholic ammonia or methylamine readily yields compounds which perfectly resemble the compounds which have been obtained in a similar manner from 2.4.6 trinitroresorcinoldimethylether.



1) LELLMANN and GROTHMANN. Ber. 17, 2731.

2) Dissertation VAN GEUNS. p. 69.

Geophysics. — *"Registration of earth-currents at Batavia for the investigation of the connection between earth-current and force of earth-magnetism."* By Dr. W. VAN BEMMELEN at Batavia. (Communicated by Mr. J. P. VAN DER STOK).

(Communicated in the meeting of December 28, 1907.)

Notwithstanding the great progress in our knowledge of the phenomena of earth-magnetism, the desired improvement has not yet been noticed in the explanation of these phenomena.

That the different variations to which the magnetic needle is liable are the consequence of the changes of electric currents has become highly probable and the place, too, where in that case the currents are to be found is no longer entirely unknown to us.

So SCHUSTER[1]) has proved that the daily variation is in general caused by extra-terrestrial currents, whilst I myself have indicated[2]) that this is likewise the case for that part of the magnetic disturbances which shows a regular daily variation.

Suchlike electric currents have, however, not been shown experimentally and their indication in those unapproachable regions is for the present not to be expected.

Only *one* part of the earth is accessible to us, viz. the outer crust and numerous are the investigations on the electric currents circuiting in that crust.

However, all these investigations have but poorly advanced our knowledge about the connection between those currents and the magnetic variations.

The reason is not only to be found in the great experimental obstacles and the lack of cooperation in the various investigations, but especially in the complicated relations of the system of currents in those zones of the earth where those investigations have been made, viz. between 40° and 70° latitude.

My supposition that in the equatorial zone, just as for other geophysical phenomena, simpler relations must exist, has been proved to be right by the investigations which I wish to communicate here.

The annotations of the earth-current executed by me these last three years at the observatory of Batavia can be divided into two series.

During the first period, March—November 1905, I registered the

1) Phil. Trans. Vol. 180, p. 667.

2) Natuurk. Tijdschrift voor N. I. Dl. 63, p. 227.

earth-current between Cheribon and Batavia, with the aid of the intercommunal telephone-line. The variations were written down photographically, the velocity of the registering strip amounted to the usual 15 mm. an hour.

The important results obtained by this method incited me to go on and in the period now come to a close I could at night make use, by the kind cooperation of the officials of the Telegraph Service, of different telegraph-lines (to Anjer, Buitenzorg, Soekaboemi, Billiton, Poerwakarta, Cheribon, Samarang, Soerabaja and Makassar); greater velocity of registration was applied too (60 and 240 mm. an hour).

Besides continuing the different registrations of the earth-current during longer or shorter time to obtain statistic results, I also made experiments. When the registration pointed to a new connection between earth-current and magnetic variation, other registrations of the earth-current were organised to get a closer investigation of that connection. When questions on the influence of wire or groundplate cropped up, it was tried to answer them by experimentation.

The instrumental arrangement was contrived in such a way, that on a strip 20 cm. wide beside the variations of two earth-current circuits those of the corresponding magnetic components were noted down. Corresponding means here: the component normal to the direction of the earth-current circuit.

The sensibility was chosen in such a way, that the corresponding variations of earth-current and magnetic force did not differ too much in magnitude. To this end great sensibility of the magnetic variation instruments (up to 0.1 γ an mm.) was necessary; as however only the relative position during one night was considered it was easy to arrange those variation instruments quite simply.

I intend to give a more extensive communication about this at some other time.

The daily variation.

The obstacles met with in investigating the daily variation were very great. For the most important part of that variation takes place about noon, but during the day time the electric field of Batavia is disturbed by the electric tram and moreover I was allowed the use of the lines only at night.

If still I succeeded in coming to useful results, this is due to the kindness of the Superintendent of the Intercommunal Telephone Company, Mr. S. W. Baints, who allowed hourly readings of the amount of the earth-current to be done at the office of the Company at Batavia. The hours were 8^h45^m A. M., 9^h45^m A. M. etc. until

4^h45^m P. M.; an ordinary galvanometer with pointer was used.

I have chosen from these readings those falling on magnetically very calm days and evincing moreover not to have suffered from disturbances on the line or from other irregularities. For those same days I have used the observatory-notation during the hours of the night (6 P. M.—5 A. M.).

Two unknown quantities remained, viz. the ratio of the values of the scale division and the difference of the central position.

The former I determined one evening during a magnetic storm at the office of the Company by alternately reading the galvanometer and allowing the Observatory to register. The reduction to a same central position I got to a plausible result by using the Sunday notations. For, on those days I could use the line already after 12 at noon and from a score of magnetically fairly calm Sundays I deduced the difference between the hours $4^3/_4$ and 6 p.m.

Graphically I then interpolated the 24 values of the hours of the day.

For the employed magnetically calm days finally was calculated the daily variation of the magnetic component normal to the direction Cheribon-Batavia from the Buitenzorg magnetograms.

Daily variation of the earth-current Cheribon-Batavia and of the magnetic horizontal component normal to that direction.

Earth-current in Volt per Kilometer $\times 10^{-5}$ (Direction Ch.-Bat. = +)		Magnetic Component in 10^{-5} C. G. S. (NE = +)	Earth-current in Volt per Kilometer $\times 10^{-5}$ (Direction Ch.-Bat. = +)		Magnetic Component in 10^{-5} C. G. S. (NE = +)
1 a. m.	— 38	—11 1	1 p. m.	+ 33	22.2
2	— 36	— 9.6	2	— 6	13.9
3	— 40	— 9.3	3	— 33	3.3
4	— 33	— 7.9	4	— 40	— 4.5
5	— 32	- 6.5	5	— 32	— 9.4
6	— 22	— 3.0	6	— 23	—10.2
7	11	3 4	7	— 51	—11 0
8	81	8.8	8	— 60	—13.1
9	153	15.9	9	— 51	—13.6
10	154	24.2	10	— 47	—14.6
11	117	29.5	11	— 46	—13.7
12	84	28.5	12	— 39	—12.9

Out of the curves, indicating according to the above numbers the daily vibration of earth-current and magnetic component, is evident:

that this vibration for them corresponds;

that the direction of the earth-current is such that it can be regarded as causing the variations of the magnetic component;

further: *that the magnetic component is retarded with respect to the earth-current* and finally;

that the ratio of the amplitudes of corresponding vibrations decreases with the duration of that vibration, so that those of the earth-current are relatively larger with a shorter duration.

The chief maximum in the afternoon is reached by the earth-current about an hour and a half earlier, the chief maximum at night about two hours earlier.

The secondary vibration in the evening-hours is for the earth-current much stronger.

It is an indication to apply here the harmonic analysis and to employ for it the formula

$$A = A_n \sin n\,(t + c_n).$$

The results of the harmonic analysis confirm in full what the mere observation taught us.

Especially the increase of the earth-current as the duration of the corresponding variation of the earth-magnetism becomes shorter is very distinctly expressed.

This dependence can be pretty accurately expressed by the following formula.

Let the amplitude of the magnetic component be M; the duration expressed in days T, and the amplitude of the earth-current A, then

$$A = 0.8 \sqrt[4]{\frac{1}{TM}}.$$

The values in the above column "calc" (on the next page) have been computed with the aid of this formula.

Yet not much value must be attached to that correspondence, as the higher terms of the harmonic analysis are very untrustworthy on account of the inaccuracy of the hourvalues employed.

The difference of phase increases regularly as far as the 5^{th} term, and then drops again to the value it had for the 3^{rd} term; but the phase differences found for the higher terms deserve little confidence.

I have been successful in obtaining a confirmation of a part of these results with the aid of the cable Batavia—Billiton. The four months March—June 1906 gave for the nightly course proper results.

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	AMPLITUDE						PHASE.		
	Earth-current Volt. p. K.M. $\times 10^{-5}$	Magn. Comp. C.G.S. $\times 10^{-5}$	$\frac{\text{Earth-current}}{\text{Magn. Comp.}}$ Observ.	$\frac{\text{Earth-current}}{\text{Magn. Comp.}}$ Calc.	Δ C—O		Earth-current	Magn. Comp.	Earth-current — Magn. Comp.
A_1	75	18.6	4.0	3.9	—0.1	C_1	299°	286°	13°
A_2	44	7.6	5.7	5.9	0.2	C_2	73	53	20
A_3	28	3.0	9.4	8.0	—1.4	C_3	122	98	24
A_4	9	0.9	10 0	10.8	0.8	C_4	53	15	38
A_5	7	0.5	13.1	13.6	0 5	C_5	103	56	47
A_6	6	0.4	15.7	15.9	0.2	C_6	57	34	23

The variation of the corresponding magnetic component changed its nature pretty much (as was to be expected) during these months.

The earth-current really followed this variation whilst the maxima and the minima kept preceding those of the magnetic component.

Out of the average for the 4 months this is obvious.

	7	8	9	10	11	Midnight	1	2	3	4	5
Earth-current Volt. p. KM. $\times 10^{-5}$	0	—27	—25	—29	—17	12	20	28	26	0	—14

Magnet. comp. 10^{-5} C. G. S.	0	−2.1	−3.7	−5.3	**−5.5**	−4.9	−4.1	**−3.3**	−3.4	−4.5	−5.4

For the earth-current the maximum comes one hour, the minimum about half an hour earlier. Let us suppose this minimum to belong to the preceding vibration of $3^h\,30^m$ duration, then the difference in phase is 26°.

The ratio of the amplitude is 26°.0, whilst I deduced roughly out of the Sunday notation for the great vibration 16.0 when the cable was at my disposal from 0^h till 4^h p.m.

So we meet here too with decrease of the ratio between earth-current and corresponding magnetic component together with increase of the duration of the vibration, but by the side of it a much stronger earth-current than for the line Cheribon—Batavia.

Annual inequality in the daily vibration.

At Batavia the amplitude of the daily vibration of the magnetic force is liable to a single- and a double-yearly inequality, where the maxima are attained in March and September, the minima in January and in June. The two maxima and the two minima are of the same magnitude.

From the continued measurements at the office of the Intercommunal Telephone Company I could deduce that the variations of the earth-current show the same annual inequality.

This series of measurements shows two breaks.

First in January '06 the lines were permanently disturbed and secondly in August '06 errors seem to have slipped into the observations, on account of which repeatedly improbably large values were read. After my having pointed this out, the readings in December next were again serviceable.

Out of each month I have taken those days which were in the first place magnetically calm and for which in the second place as much as possible complete and useful readings of earth-currents were at hand.

Of the mean hourvalues for each month was then taken the difference of the smallest and the greatest value.

The maximum generally fell in with the $8^3/_4$ a.m. or $9^3/_4$ a.m. observation, the minimum with that of $3^3/_4$ p.m. or $4^3/_4$ p.m.

These differences expressed in Volt pro kilometer $\times 10^{-6}$ follow here.

	J.	F.	M.	A.	M.	J.	J.	A.	S.	O.	N.	D.
1905		**266**	194	208	190	129	**127**	**127**	170	**173**	167	131
1906		171	**177**	127	125	**109**	135	233 (?)				122
1907	**81**	**118**	113	90	88.							

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Notwithstanding the imperfection in these measurements, the double annual period and its correspondence with that of the magnetic component is so distinctly expressed that doubt is not possible.

Variations of short duration.

The second period of registration, November 1905—October 1907, was chiefly devoted to the study of the connection of the vibrations of short duration in earth-current and magnetic component.

The usual velocity of registration was here 1 mm. a minute, which with sharp photographic lines allows the measurement of variations with the duration of half a vibration of 0.2 to 0.3 minute, but in numerous nights the velocity was enlarged to 4 mm. a minute, when it was possible to measure accurately differences of time of 0.1 minute.

By the continued registration of the earth-current along different lines, each one accompanied by that of the corresponding magnetic component, an extensive material of curves was collected, from which in general the following could be gathered.

For the earth-current along about east-west lines [1]) to each vibration answers a similar one of the magnetic component. For that of the nearly north-south lines that correspondence seems also to exist in part, but it is greatly disturbed by the circumstance that the earth-current keeps following more or less the vibrations of the east-west line.

So also near the equator we find complicated phenomena, but only in part, for as far as the east-west current is concerned we meet with such a striking correspondence that it is possible to deduce simple laws; the two same laws, which were found for the daily variation:

1. *the vibration in the earth-current precedes that of the magnetic component with a certain difference in phase;*

2. *the ratio of amplitude of earth-current and magnetic component decreases when the duration of the vibration increases;*

1) The east-west lines were:

	Direction	Distance
Bat. — Anjer	W 6° N	106 Km.
„ — Poerwakarta	E 40 S	78 „
„ — Cheribon	E 18 S	200 „
„ — Semarang	E 12 S	406 „
„ — Soerabaja	E 10 S	665 „
„ — Makassar	E 5 N	1486 „

The north-south lines were:

	Direction	Distance
Bat. — Billiton	N 13° E	392 K.m.
„ — Buitenzorg	S 5 E	46 „
„ — Soekaboemi	S 9 E	84 „

appear distinctly from this material of registering curves with its thousands and thousands of shorter and slower vibrations.

To deduce the real amplitudes and phases of those variations we should have to execute for each separate case an immensely extensive harmonic analysis and, this being quite impossible, corresponding variations had to be chosen and measured discriminately. Therefore all the measurements have been done by me personally.

The precedence of the earth-current.

This precedence was rule; it was quite exceptional if it was not met with. When choosing cases for measurement I always avoided those where by a superposed oscillation of greater length and amplitude the time of the turning point was made to appear much earlier or later.

The difference of phase proved to vary from case to case but to be already constant in the mean of a small number.

For the lines Batavia—Soerabaja and Batavia—Poerwakarta the difference in phase was determined with respect to that of Batavia—Cheribon and not with respect to the magnetic component.

For the lines Batavia—Buitenzorg and Batavia—Billiton the deduction was accompanied by great difficulties, as perfectly corresponding cases between earth-current and the magnetic component seldom made their appearance on account of the interference of the east-west component.

For Batavia—Soekaboemi I have therefore desisted from making a calculation and the difference in phase for the two first lines must be mentioned with reserve.

Difference in phase.

Batavia—Poerwakarta	22°	Batavia—Buitenzorg	23° (?)
„ —Anjer	14	„ —Billiton	28 (?)
„ —Cheribon	22		
„ —Semarang	36		
„ —Soerabaja	32		

The difference in phase found here for Cheribon and Billiton shows a striking resemblance to that found for the daily variation.

Cheribon.

Variations of short duration.	Average of 6 terms of the daily variation.
22°	27°

Billiton

Variations of short duration.	Nightly variation.
28°	26° (?)

Ratio of amplitude.

The accurate indication of the moment of maximum or minimum of a vibration is sooner impossible than the measurement of an amplitude on account of the interference of smaller superposed variations.

I have therefore been able to select a much greater number — 346 — of cases for measurement; the results are as follows:

Batavia—Cheribon.

$$\frac{\text{Amplitude Earth-current in Volt p. Kilom.}}{\text{Amplitude magnetic component in C.G.S.}}$$

Duration of half vibration.	Magnetic Component in 10^{-5} C. G. S.							
	0 23	0 37—0 40	0 50—0 79	0.90—1.50	1 7—1.8	2 8—3 2	5 5—7 6	15.1—18.6
0.3 m.	23 3	22.1	22 4					
0 5	24 3	23 8	21 4	22 8				
0 8		23.6	22 3	19 4				
1.2		22 4	21.1		22 5			
3 7			19.2	21.1		19.1		
7 6			18 3		16.9	17 8	15.6	
15 2				14 8		14.3	14 0	
29 0						15.9		
36 3							12.1	
39.6								10.8
120 —		15 7						
144.—			13 1					
180.—				10.0				
240 —						9.4		
360.—							5 7	
720 —								4.0

According to this table the increase of the amplitude when the duration of the vibration diminishes seems to reach a maximum value at 0.5 min. and moreover the ratio of amplitudes seems to be dependent on the amplitude itself and in such a way that with equal duration it increases with the diminishing of the amplitude.

A complete confirmation of these results was found in 312 cases for the Anjer-line.

Batavia-Anjer.

$$\frac{\text{Amplitude Earth-current in Volt p. K.m.}}{\text{Amplitude Magnetic Component in C. G. S.}}$$

Duration of half vibration.	Magnetic Component in 10^{-5} C. G S.				
	0 19—3.0	0.53—0.68	1.36—1.83	2 30—3.89	5.20—5.48
0.4 min.	96	95	97		
0.6	97	104	95		
0.9	86	90	89		
1.8	92	84			
2.3			64		
5.3		86	74		
6.3				68	
7.7	83				
9.2			68		
9.7					60
10.8		68			
12.3			73		
13.6		58			
19.9				75	
22.0					63

Here too with a short duration a maximum is found, viz. with 0.6 min.

Beside this correspondence we find the unexpected result, *that the Batavio—Anjer current is about four times stronger than the Batavia —Cheribon current.*

That ratio is constant for different duration of vibration, as was proved by the arrangement according to groups of mean equal duration.

Duration of half a vibration	$\dfrac{\text{Batavia-Anjer}}{\text{Batavia-Cheribon}}$
0.33 min.	4.28
0.54	3.94
0.66	4.36
0.76	4.21
0.96	4.21
1.67	3.79
5.6	4.25
7.6	3.84
15.1	5.21
Mean	4.23

When we wish to compare the values for the ratio of amplitudes of magnetic force and earth-current of vibrations of shorter duration with those of the six terms of the daily variation, it is best to inscribe all values in one diagram with abscis $\sqrt[4]{M}$ and ordinate $\sqrt[4]{1/T}$. When now the formula $A = 0.8\sqrt[4]{\dfrac{1}{TM}}$ found above for the 6 terms of the daily variation will still hold, then the values which lie on the same radius-vector through the origin must be the same.

It is evident that this is only the case for the middle part of the diagram, namely for the radius-vector where $0.8\sqrt[4]{\dfrac{1}{TM}} = 150$. The ratio $\sqrt[4]{1/T}$ and $\sqrt[4]{M}$ is therefore $= 2$.

If the amplitude of the magnetic component is relatively larger, then the radii with equal values are bent gradually to the axis of abscissae, and if $\dfrac{1}{T}$ is relatively larger, then they are bent to the axis of ordinates.

For a great $\dfrac{1}{T}$; i. e. for a duration of vibration of about one minute, they turn again to the axis of ordinates and a maximum seems to be formed.

It will be possible to force these curves in a formula, but we must not expect that that formula will give the real formula, as it must be very complicated on account of the nature of the phenomenon.

What according to me is clear from the diagram is that the ratio of amplitudes for the vibrations of shorter duration will gradually pass into those of longer duration (the six terms of the daily variation) and so that between these two phenomena there is also a gradual transition.

The results for the earth-current Buitenzorg-Batavia and Billiton-Batavia are again uncertain on account of the lack of agreement with the magnetic component. I found:

Buitenzorg-Batavia.

Duration of half a vibration	Amplitude Magn. Comp.	Amplit. earth-current in V. p Km. / Amplit. magn. Comp. in C. G. S.	Number of cases
1.1 min	0.38 J.	75	20
3.7	0 96	79	20
7.6	1.20	71	28
			68

Billiton—Batavia.

0.7	0.39	63	14
1.3	0.29	74	14
3.5	0.49	66	14
7.0	0.50	58	23
22.4	1.44	41	5
			70

If indeed these figures are trustworthy then the ratio of amplitudes of the earth-current with respect to the magnetic component decreases here too when the duration increases.

The increase at the outset with very small duration is also found in the above figures, even in much greater ratio than for the Anjer- and Cheribon-current.

The numbers for Buitenzorg are a little smaller and for Billiton smaller than for Anjer, but still larger than for the Cheribon-current.

The line to Poerwakarta I had but for two nights at my disposal.

I have then allowed the Poerwakarta-current to be registered at the same time as the Cheribon-current and I have found a complete correspondence between them for vibrations of a duration from 0.8 to 15.5 minutes.

Not before the last months of the registering-period have I extended the investigations to the lines to Semarang and Soerabaja and to my surprise I found a pretty great deviation from the circumstances which appear on the lines Cheribon and Anjer.

Semarang—Anjer.

Duration of half a vibration	Amplitude Anjer-current 10^{-6} V. p Km.	$\frac{\text{Amplitude Anjer}}{\text{Amplitude Semarang}}$	Number of cases
0.33 min.	38	1.70	20
0.69	74	2.06	20
1.04	61	2.16	20
1.71	57	2 32	20
7.43	160	3.96	20
20.81	290	4.61	16
			116

Soerabaja—Anjer.

0.37	43	2.17	20
0 93	92	2.87	20
2.65	110	3.27	20
11.85	326	4.71	20
34.28	739	7.16	7
			87

So the influence of the duration on the amplitude of the earth-current is here much greater for the Semarang- and Soerabaja-current than for the Anjer-current.

It is remarkable that here too the increase of the influence with the duration takes place about according to $\sqrt[4]{1/T}$.

With respect to the first value for $t = 0.33$ min., respect. 0.37 minutes duration we get:

	$\sqrt[4]{\frac{t_n}{t_0}}$	$\sqrt[4]{\frac{A_x}{A_0}}$		$\sqrt[4]{\frac{t_n}{t_0}}$	$\sqrt{\frac{A_x}{A_0}}$
Semarang	1.20	1.21	Soerabaja	1.26	1.32
	1.33	1.27		1.64	1 51
	1 51	1.36		2.38	2 17
	2.18	2.33		3.10	3.30
	2.82	2.71			

Direct comparison of the Batavia-Semarang current to the magnetic component furnished: (June 18—21, '07):

Duration of half a vibration	Amplit. of the Magnet. Comp. in 10×10^{-5} C.G.S.	$\dfrac{\text{Amplit. earth-current in Volt p. K.M.}}{\text{Amplit. earth-current in C.G.S.}}$	Number of cases
0.6 min.	1.3	43	14
1.0	0.9	35	15
1 4	0.9	31	13
5.0	0.7	18	10
11.5	2.2	13	9

The last two values fit in very well with the scheme of the values found for the Cheribon-current, but the first three show a much quicker increase when the duration of the vibration is shorter.

This peculiar increase immediately strikes one when regarding the registered lines. In order to investigate whether that increase of the influence of the duration was connected with the increase of the distance of the two stations between which the earth-current was measured, I asked for and obtained direct connection with Makassar. The loss by defective isolation on the line however was so great, that the real distance had not obtained any lengthening of importance.

On the trustworthiness of the results.

A certain doubt has always been left when observing earth-currents whether the results arrived at do give an idea of the real existing earth-current.

According to SCHUSTER (Terr. Magn. III, p. 130) the intensity of the current is really to be determined by switching on in the circuit a cell of known E. M. F. I have therefore always used this simple means and connected to it often, by introducing a resistance of 100 or 1000 Ohm, a measurement, though a rough one, of the total resistance of the whole circuit.

The results generally showed a mutual correspondence, only for longer lines a distinct loss by defective isolation was often discernible.

For the earth-current this loss by defective isolation is of not much consequence; for, if two points lying at a distance L from each other with potentials P and $P + L_p$ are connected by a wire the potential will vary along that wire proportional to the distance of P to $P + L_p$ and will be in a point between the two, say at $1/a$ of the distance, $P + \frac{L_p}{a}$. But there the potential of the earth will also be $P + \frac{L_p}{a}$ if that earth potential likewise varies proportionally to the distance (which we shall suppose to be true at first computation). There will thus be no difference of potential between line and point of contact with the ground, neither loss of current.[1])

However there is loss of current, when I switch on a cell, thus when I generate a drop of potential along the wire, that does not at all run parallel to the earth potential.

This explains that the image of the earth-current rose and fell so regularly with the magnetic component, whilst so often a great loss by defective isolation took place on the line, so that the determination of the values of the scale division by means of inserting a cell gave abnormal values.

When, however, an investigation must be made of the regular or non-regular increase of the earth-potential with the distance, then this loss by defective isolation is disturbing. That is why the registering with the continuous connection Batavia—Makassar shed no light upon the subject.

Influence of the material of the line.

I could experiment accurately on the possibility of the influence of the material of the conductive wire on amplitude and phase of the earth-current by registering simultaneously the currents between Cheribon and Batavia, resp. through the copper telephone wire and the iron telegraph one.

---

1) If we suppose the earth-current to form a closed circuit passing round the earth and our wire to have contact with it in three points viz at the two end stations and the point of contact, there is a distribution of current according to WHEATSTONE and the contact is the bridge of WHEATSTONE.

The resistance of the former circuit was generally about 10 times smaller than that of the other.

At Batavia the two circuits were on the same groundplate, at Cheribon the two groundplates hung in the same well. Moreover the wires ran for the greater part on the same telegraphpoles.

An all but perfect correspondence was now found, so that all influence of the material (especially with respect to magnetic induction) may be regarded as non-existing.

Influence of the current of polarisation.

I was more anxious about a disturbing influence of the polarisation of the ground-plates to which repeatedly from various sides attention has been drawn. For, the polarisation might be able to explain the difference in phase and the change in the ratio of amplitudes.

Let us suppose that the earth-current together with the magnetic component increases, then the resisting current of polarisation also grows. If now the increase of the earth-current and of the magnetic component passes into a decrease, the current of polarisation will for the moment keep increasing and consequently the observed current (i.e. earth-current minus current of polarisation) will sooner reach its turning-point than the magnetic component. If the vibration increases in duration the current of polarisation will also increase first faster, then slower, and therefore the observed current will always lose with respect to the magnetic component and the ratio of amplitudes — as was really found — will decrease first faster, then slower.

Though the influence to be expected of the polarisation had therefore to agree with what was found, yet we could not believe that it could be the cause of those phenomena, as for the observations made at the office of the Telephone Company the connection with the earth was made every hour only for a few moments and so there could be no question about a continual increase of the polarisation.

To find out the influence of the polarisation I have taken the following experiment. To begin with I measured the current of polarisation directly. To that end I buried a second ground plate a few meters from the old one and made a new connection: old earthplate—galvanometer—new ground-plate.

The old ground-plate I polarised strongly by switching on a cell into the line (Cheribon—Batavia). After breaking the connection with Cheribon I immediately closed the new one, in consequence of which the depolarisation-current passed through the second galvanometer.

I actually found the polarisation with its characteristic qualities, but its intensity was hardly more than a few percents of the chief current and thus really too small to serve as cause of difference in phase and change in ratio of amplitudes.

After this investigation I have placed a set of non-polarising ground-plates (amalgamated zinc plates immersed in a solution of $Zn\,SO_4$ in porous pots)[1]) on the garden of the Observatory, and repaired to Cheribon to place a corresponding set there. The repetition of the experiment described above showed really the non-appearance of polarisation.

After this I connected one of the two telephonewires between Batavia and Cheribon with the old polarising ground-plates, the other with the new non-polarising ones, and allowed the two earth-currents to register simultaneously on the same strip with the same sensibility and a velocity of registration of 24 cm. an hour together with the magnetic component. The experiment could hardly have been taken more accurately.

As I expected the result for the difference in phase was a very slight influence in the sense mentioned above; for the ratio of amplitude I found for *one* night also a very small influence in the expected direction, but during two other nights a somewhat greater difference in opposite sense. I think I must attribute those last influences to the unavoidable inaccuracy of the determination of the values of the scale division (by switching on a cell of known E. M. F.).

At any rate I had proved sufficiently that the current of polarisation was not the cause of the found phenomena, so I can take those phenomena to be real.

Connection between earth-current and magnetic force.

If we wish to investigate more closely the connection between the variation of earth-current and magnetic component it is necessary to regard the variations of the latter quite by themselves.

The general rule holds at Batavia that the two horizontal components change simultaneously, i. e. that generally between the turning-points of X and Y only a small difference in time exists and that on the other hand Z generally has a difference of phase of 90° with X and Y.

AD. SCHMIDT (Met. Zeitschrift 1899) has pointed out, that the

[1]) C. A. BRANDER, Inaugural Dissertation, Helsingfors, 1888.

variations of the magnetic component might be explained by the passage of electric current vortices.

Following this explanation we should have to conclude for Batavia to the passing of vortices whose centre remains far from Batavia, so that only the outside slightly bent pieces of current pass by.

So as a first approximation we may assume that the extra-terrestrial current is almost rectilinear at Batavia and when varying in intensity has but slight oscillations in direction.

The average direction must be $WSW — ENE$ for almost without exception an increase of the N-component is accompanied by a weakening of the E-component and so $\Delta X > \Delta Y$.

That current we really find back in the diagram of the equipotential lines of the daily variation according to SCHUSTER-VON BEZOLD, [1]) which equipotential lines follow at first approximation the current-lines.

Also for the explanation of the phenomenon found by me of the *earthmagnetic after-disturbance*, it was a matter of fact to take a current encircling the earth and this current too had to have a suchlike direction as was mentioned above, but the angle with the equator was at Batavia much smaller than is found now.

Each varying extra-terrestrial current will induct an intra-terrestrial one and the magnetic variation observed at the surface is the sum of the influence of the two. LAMB (see the paper quoted above of SCHUSTER on the daily variation) proves that the ratio of the potential of the primary and the secondary field is complex and that therefore difference in phase exists. The horizontal component caused by the extra-terrestrial current is in advance compared to the one generated by the inducted currents; so the resulting component will be in retardation compared to the extra terrestrial current.

By SCHUSTER however no difference of phase is found for the vertical force and LAMB has pointed to the fact that this can be the consequence of increasing electric conductivity of the earthstrata towards the depth. The results of the new seismological observations point to an iron nucleus of the earth and therefore to a very great increase.

We may therefore probably assume that the difference in phase is very little, at any rate that difference in phase is slight for variations of short duration.

The magnetic force observed consisted of a primary and of a secondary part, which have the same sign as far as the horizontal component is concerned,

[1]) Sitz. Ber. der Berliner Akademie für 1895.

The ratio c' between the secondary and primary part increases with the frequency of the vibrations of the current.

Let us call the magnetic force X, then

$$X = \text{primary} + \text{secondary}$$
$$X = \text{primary}\ (1 + c'),$$

so the primary part of $X_0 = \frac{X}{1 + c'}$.

The extra-terrestrial current which we can put $S = s \sin 2\pi \frac{t}{T}$ will induct in the upper earthstrata a current S':

$$S' = \varrho f(ST) \frac{2\pi}{T} \cos 2\pi \frac{t}{T}.$$

The induction will depend on the distance and the latter possibly on intensity and duration of the vibration of the current, moreover on the conductivity ϱ of the upper earthstrata.

The primary magnetic force will at first approximation (the distance being about the same) depend in the same manner on the extra-terrestrial current. So:

$$X_0 = f(sT) \sin 2\pi \frac{t}{T}$$

and

$$X = (1 + c') f(sT) \sin 2\pi \frac{t}{T}.$$

The existence of a vertical conducting current having been proved we must also take for granted that part of the extra-terrestrial current is closed by the earth and that a current is generated equally directed as the current of induction.

Already the properties of the conductivity of the atmosphere point to a dependence of magnitude and duration of the extra-terrestrial current, also on the conductivity of the upper earthstrata.

So we put for the current

$$\varrho \psi (sT) \sin 2\pi \frac{t}{T}.$$

And for the total current we find:

$$A = \varrho \left[\frac{2\pi}{T} f(sT) \cos 2\pi \frac{t}{T} + \psi (sT) \sin 2\pi \frac{t}{T} \right]$$

or

$$A = \varrho \left[\psi^2 (sT) + \frac{4\pi^2}{T^2} f^2 (sT) \right]^{1/2} \sin \frac{2\pi}{T} \left(k + \frac{T}{2\pi} Bg\ tg \frac{f(sT)}{\psi(sT)} \frac{2\pi}{T} \right),$$

whilst we found above

$$X = (1 + c') f(sT) \sin 2\pi \frac{t}{T}.$$

For the difference in phase we found a constant part of T or

$$\frac{T}{2\pi} Bg\, tg \frac{f(sT)}{\psi(sT)} \frac{2\pi}{T} = \frac{T}{K}.$$

For angles of $\pm 23°$, found for Cheribon for a shorter duration of the vibration, we may write here approximately

$$\frac{f(s\,T)}{\psi(s\,T)} = \frac{T}{K}.$$

So the ratio of amplitudes becomes

$$\frac{A}{X} = \frac{\varrho}{1 + c'} \frac{\psi(s\,T)}{f(s\,T)} \left[1 + \frac{4\,\pi^2}{K^2}\right]^{1/2},$$

$$\frac{A}{X} = \frac{\varrho}{1 + c'} \frac{1}{T} \frac{1}{\sqrt{K^2 + 4\,\pi^2}} = C \frac{\varrho}{1 + c'} \frac{1}{T}.$$

For $1 + c'$ we find according to Lamb-Schuster for that part of the potential which is to be expanded *in terms of a spherical function* of order 2 (for that part which is to be expanded in terms of spherical functions of a higher order, the increase is quicker).

$\sigma = \frac{constant}{T}$	$1 + c'$
10	1.172
20	1.278
30	1.337
40	1.374
50	1.399
100	1.466
900	1.605
6400	1.643

So for higher frequency (for duration of half a vibration $=$ 1 minute σ is 7200) $\frac{1}{1 + c'}$ is about constant.

So we get $\frac{A}{X}$ proportional to $\frac{1}{T}$.

The observations, however, give for Cheribon and Anjer proportionality with $\sqrt[8]{\frac{1}{T}}$ (for still smaller T, even inversion, and for Semarang and Soerabaja proportionality to $\sqrt[4]{\frac{1}{T}}$.

AD. SCHMIDT (Met. Zeitsch. 1902 p. 94) brings the supposition forward that the current can be generated in the wire by induction only, thereby supposing the wire to be closed by the earth.

Then putting the case very simply we arrive by application of the rule of AMPÈRE at :

$$A_{E-W} = -\gamma \frac{dX}{dt} + \sigma \frac{d^2 Z}{d\varphi\, dt}.$$

This gives the difference with respect to the above that the variation Z makes its appearance.

The Z, however, changes but little in equatorial regions, so it cannot make the theory correspond to the observations.

The slow increase of the earth-current when the frequency increases does *not* point to induction, but rather to direct connection with the primary current.

The quantity $\frac{1}{1+c'}$ has indeed, compared to $\frac{1}{T}$, rather a slow course.

That difference in phase is according to SCHUSTER-LAMB rather decreasing for quicker vibration whilst for the earth-current it proves to be constant.

But whence the difference in phase?

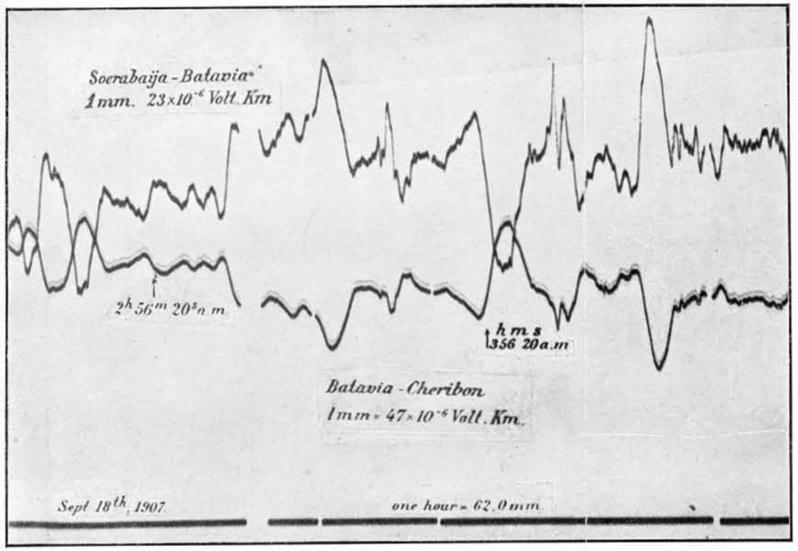
The differences in the intensity and the difference in phase of the earth-current for the lines to Anjer, Cheribon, Billiton and Buitenzorg can be explained also by the difference in conductivity of the ground between those places and Batavia.

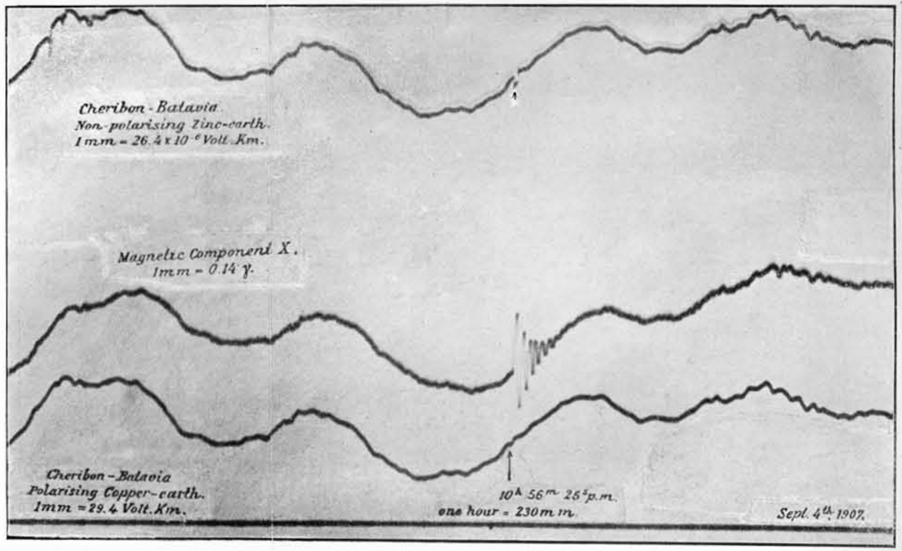
For instance between Anjer and Batavia lies the volcano Karang and therefore the conductivity is probably greater than between Batavia and Cheribon, and the fact that the earth-current is four times as strong can be attributed to it.

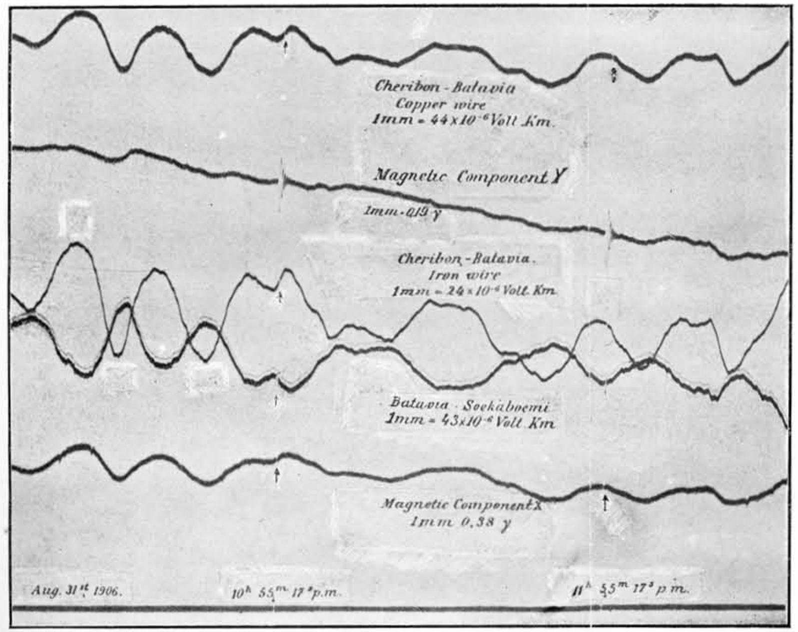
The great intensity of the earth current for Buitenzorg—Batavia may be partly attributed to the same reason and moreover to the difference in height (280 M), that between Batavia and Billiton to the well conducting seawater.

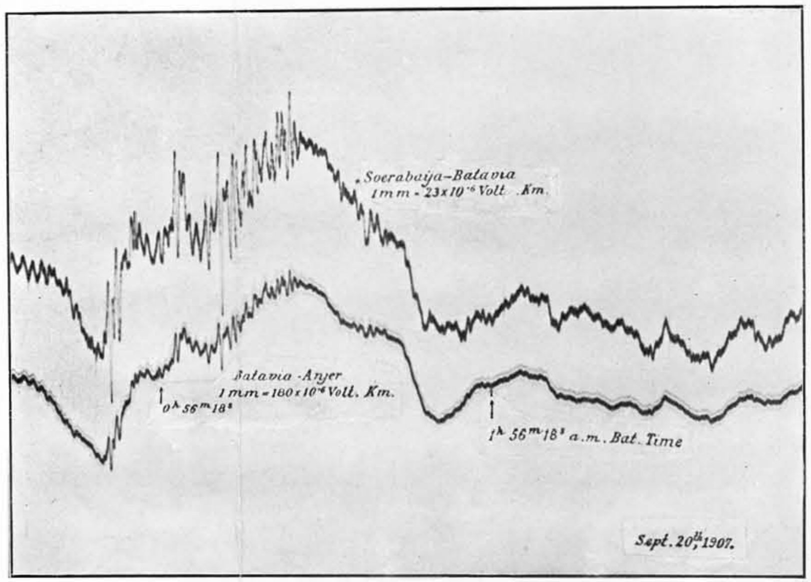
For the lines Semarang—Batavia and Soerabaja—Batavia we find however for the ratio of amplitude a distinct difference for vibrations of short duration. Each attempt at explanation of the connection between earth-current and magnetic variation will be in vain as long as this has not been confirmed and expounded.

To explain it out of the loss by isolation is impossible, as the difference would have to appear less for the lines Anjer-Batavia (106 K. M.) and Cheribon-Batavia (200 K. M.) which is not the case.


Soerabaija - Batavia
1mm. 23×10⁻⁶ Volt. Km
2h 56m 20s a.m.
h m s
356 20a.m
Batavia - Cheribon
1mm = 47×10⁻⁶ Volt. Km.
Sept 18th, 1907.
one hour = 62.0mm


Cheribon - Batavia
Non-polarising Zinc-earth.
1mm = 26.4×10⁻⁶ Volt. Km.
Magnetic Component X.
1mm = 0.14 γ.
Cheribon - Batavia
Polarising Copper-earth.
1mm = 29.4 Volt. Km.
10h 56m 25s p.m.
one hour = 230mm.
Sept. 4th 1907.


Cheribon - Batavia
Copper wire
1mm = 44×10⁻⁶ Volt. Km.
Magnetic Component Y
1mm = 0.19 γ
Cheribon - Batavia.
Iron wire
1mm = 24×10⁻⁶ Volt. Km.
Batavia - Soekaboemi
1mm = 43×10⁻⁶ Volt. Km
Magnetic Component X
1mm 0,38 γ
Aug. 31st 1906.
10h 55m 17s p.m.
11h 55m 17s p.m.


Soerabaija - Batavia
1mm = 23×10⁻⁶ Volt. Km.
Batavia - Anjer
1mm = 180×10⁻⁶ Volt. Km.
0h 56m 18s
1h 56m 18s a.m. Bat. Time
Sept. 20th 1907.

Neither can it be explained by mutual induction of the two lines, passing partly along the same telegraphpoles, as that influence would just work inversely.

There is a circumstance which causes the lines to Anjer and Cheribon to differ greatly from those to Semarang and Soerabaja; that is the greatest depth below the surface of the earth which the chord reaches between those places and Batavia.

It is for	Batavia—Anjer	1 K. M.
	Batavia—Cheribon	3 „
	Batavia—Semarang	14 „
	Batavia—Soerabaja	37 „

When thus the variations of short duration cause a current chiefly at a greater depth, where the conductivity is very different from than at the surface, a distinct difference might appear. The opposite however is more to be expected.

To conform that difference it will be necessary to register at Semarang the current between Cheribon-Semarang and Soerabaja-Semarang.

If we find for that the same as for Batavia-Cheribon and Batavia-Anjer, then indeed we must attribute the greater increase of amplitude with short duration for the lines Batavia-Semarang and Batavia-Soerabaja to the greater distance.

ERRATUM.

In the Proceedings of the Meeting of March 30, 1907:

p. 770 l. 3 from the bottom: for 46.419 read 46.491.
p. 779 l. 10 from the top: for VII H_2 1 read VII H_2 2.

(**February 20, 1908**).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday February 29, 1908.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 29 Februari 1908, Dl. XVI).

CONTENTS.

Mathematics. — "*Fourdimensional nets and their sections by spaces*". (First part). By Prof. P. H. SCHOUTE.

(Communicated in the meeting of January 25, 1908).

Out of the table

$C_5 \ldots . 75° 31' 21''$, $C_{16} \ldots . 120°$, $C_{120} \ldots . 144°$

$C_8 \ldots . 90°$, $C_{24} \ldots . 120°$, $C_{600} \ldots . 164° 28' 39''$

of the angles formed by two bounding bodies meeting in a face of the regular cells of space Sp_4 it is immediately evident that only for the cells C_8, C_{16}, C_{24} can there be any question about each respectively filling that space. It is well known, that this is really the case. In the handbook included in the Sammlung SCHUBERT "*Mehrdimensionale Geometrie*" (vol. II, page 241) is indicated how the two nets of the cells C_{16} and C_{24} can be deduced by transformation from the net of cells C_8, the existence of which is clear in itself. We repeat this here in a somewhat different form to add new considerations to it.

1. The points with the coordinates $(\pm 1, \pm 1, \pm 1, \pm 1)$ are the vertices of an eightcell $C_8^{(2)}$ with double the unit of length as length of edge, the origin of the coordinates as centre and the directions of the axes as directions of the edges. These vertices can be easily arranged in two groups of eight points, one group of which contains the points with a positive product of coordinates, the other group the points with a negative one. Each of these groups has the property that no two of the eight points are united by an edge of $C_8^{(2)}$; therefore we call them groups of non-adjacent vertices. Let us join for each of these groups the two points lying in the same face of $C_8^{(2)}$ by a diagonal, then the systems of edges of two cells $C_{16}^{(2\sqrt{2})}$ are generated; as each of the bounding cubes of $C_8^{(2)}$ is circumscribed about one of the 16 bounding tetrahedra of each of the two $C_{16}^{(2\sqrt{2})}$, we call these last inscribed in $C_8^{(2)}$, where one may be called positive, the other negative.

Let us now suppose the net of the C_8 to be composed of alternate white and black eightcells, so that two C_8 with a common bounding body differ in colour — from which it follows, that two C_8 in contact of edges do this too, whilst on the other hand two C_8 in face or in vertex contact bear the same colour —, and let us assume that in each white C_8 is inscribed a positive C_{16} and in each black C_8 a negative one; then it is clear that both groups of C_{16} do not yet fill the whole space Sp_4. For to make of a C_8 the inscribed C_{16} we must truncate from this measure polytope at each of the eight vanishing vertices a fivecell rectangular at this point, of which the four edges passing through this point have a length 2. Because a vertex which vanishes for one of the sixteen cells C_8, to which it belongs, does this for all, there will remain round this point sixteen alternate white and black rectangular five-cells and these will form together a $C_{16}^{(2\sqrt{2})}$ of which this point is the centre. Thus a space-filling for Sp_4 is formed by three equally numerous groups of cells $C_{16}^{(2\sqrt{2})}$ with the property that all cells C_{16} of the same group can be made to cover one another by translation.

To show how striking the regularity of the net of the C_{16} is we must suppose three cells $C_{16}^{(2\sqrt{2})}$, of which no two belong to the same group, to be removed parallel to themselves to a common centre, the origin of coordinates. We then see immediately that the vertices of the three $C_{16}^{(2\sqrt{2})}$ together form the vertices of a $C_{24}^{(2)}$. For the two inscribed cells $C_{16}^{(2\sqrt{2})}$ together again furnish the vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$ of the original eightcell $C_8^{(2)}$ and the coordinates of the vertices of the third cell $C_{16}^{(2\sqrt{2})}$ are

$$(\pm 2, 0, 0, 0),\ (0, \pm 2, 0, 0),\ (0, 0, \pm 2, 0),\ (0, 0, 0, \pm 2),$$

from which is evident what was assumed (compare *"Mehrdimensionale Geometrie"*, vol II, p. 205).

We shall presently use this observation to trace the connection between the four groups of axes of the three systems of cells C_{16} with the groups of axes of C_8.

2. To transform the net of the cells C_8 into a net of cells C_{24} we must again suppose the cells of the former alternately coloured white and black in order to break up each of the black cells into eight congruent pyramids with the centre of the eightcell as common vertex and the eight bounding cubes as bases. By adding to each white eightcell the eight black pyramids having a bounding cube in common with it, the net of the cells $C_{24}^{(2)}$ is generated; in reality to the sixteen vertices of the eightcell supposed to be white with the origin of coordinates as centre, viz. to the points $(\pm 1, \pm 1, \pm 1, \pm 1)$ the eight vertices mentioned above

$$(\pm 2, 0, 0, 0),\ (0, \pm 2, 0, 0),\ (0, 0, \pm 2, 0),\ (0, 0, 0, \pm 2)$$

are added.

The transformation of the net of the $C_8^{(2)}$ into that of C_{24} can also take place in the following simple way. Divide each of the cells $C_8^{(2)}$

37*

into 16 equal and similarly placed cells $C_8^{(1)}$ by means of four spaces through the centre O parallel to the pairs of bounding spaces. Then divide each of the sixteen parts $C_8^{(1)}$ (fig. 1) by the space in the midpoint of the diagonal concurring in the centre O of $C_8^{(2)}$ normal to this line into two equal halves; here the section as is known is a regular octahedron $A_{12}\,A_{13}\ldots A_{34}$. We now direct our attention first to the half cells $C_8^{(1)}$ surrounding the point O; they form a $C_{24}^{(\sqrt{2})}$. Of the 24 bounding octahedra sixteen are furnished by the sections $A_{12}\,A_{13}\ldots A_{34}$, whilst the eight remaining ones are obtained by joining

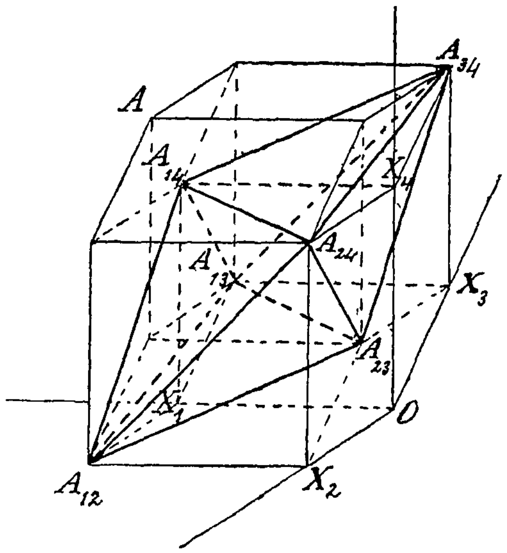


Fig. 1.

in each of the eight ends of the chords along the four axes OX_1, OX_2, OX_3, OX_4 through O, e.g. in X_1, the eight rectangular tetrahedra $X_1\,(A_{12}\,A_{13}\,A_{14})$, where it is clear that in X_1 eight of those tetraheda really meet, because we can reverse the direction of each of the segments X_1A_{12}, X_1A_{13}, X_1A_{14}. Furthermore we observe that around an arbitrary vertex A of the original cell also 16 half cells $C_8^{(1)}$ are lying and that these form in exactly the same way a $C_{24}^{(\sqrt{2})}$. By this the net of the $C_8^{(2)}$ has been transformed into a net of cells $C_{24}^{(\sqrt{2})}$, where the centres and the vertices of the cells $C_8^{(2)}$ form the centres of the cells $C_{24}^{(\sqrt{2})}$ placed in the same way.

If we add to the considered sixteenth part $C_8^{(1)}$ (fig. 1) the three parts generated by reversing the sign of one of the two axes OX_1 and OX_2 or of both, it is immediately evident that A_{34} is the centre of a face of the original cell $C_8^{(2)}$. From this is evident to the eye the truth of the wellknown theorem, that the centres of the faces of a $C_8^{(2)}$ — and therefore also the centres of the edges of each of the two inscribed cells $C_{16}^{(2\sqrt{2})}$ — are the vertices of a $C_{24}^{(\sqrt{2})}$.

3. Before examining more closely the nets of the cells C_8, C_{16}, C_{24} — or, as we shall express ourselves, the nets (C_8), (C_{16}), (C_{24}) — in their mutual connection we put to ourselves the question whether it is possible to fill Sp_4 entirely with *different* regular cells. Here the table given above points to two possibilities. We can either complete the sum of the angles 75° 31′ 21′ and 164° 28′ 39″ with 120° to 360° or by combination of one of the two cells C_{16}, C_{24} with twice the other arrive at 360°. The latter is however already excluded by the fact that C_{16} and C_{24} differ in bounding bodies, which obstacle does not occur when one tries to arrange the three cells C_5, C_{16}, C_{600} with the same length of edges around a face. Yet, though this is possible, neither in this way does one arrive at the object in view. If the indicated space-filling had taken place then two bounding tetrahedra of C_5, having always a face in common, would have to differ from each other in this, that one would at the same time have to belong to a C_{16} and the other to a C_{600} and this is impossible. For one cannot colour the bounding tetrahedra of a C_5 alternately white and black for the mere reason, that the number five of those tetrahedra is odd. So there is no space-filling of Sp_4 where *different* regular cells appear.

4. We shall now consider more closely the systems of points formed by the centres of the regular cells of the nets (C_8), (C_{16}), (C_{24}) which we shall indicate by the symbols (P_8), (P_{16}), (P_{24}).

Of the systems of points (P_8), (P_{16}), (P_{24}),, which we might call fourdimensional "assemblages of Bravais", (P_8) is the simplest. If the axes of coordinates are assumed through the centre of a definite cell $C_8^{(2)}$ parallel to the edges of this cell, then (P_8) is the system of the points $(2a_1, 2a_2, 2a_3, 2a_4)$ with only even integer coordinates which we indicate by means of abbreviated symbols by the equation $(P_8) = (2a_i)$.

Of the two other systems of points, (P_{24}) can be most simply expressed in (P_8). Out of the second mode of transformation of the cells $C_8^{(2)}$ into the cells $C_{24}^{(\sqrt{2})}$ it was clear to us that (P_{24}) is found by joining the system (P_8) to the system of the vertices of the cells $C_8^{(2)}$. Now this system of the vertices can be deduced out of (P_8) by a translation indicated in direction and magnitude by the line-segment connecting the centre of the eightcell, which served to determine the system of coordinates, with one of the vertices; thus this system of vertices is indicated in the same symbols by $(2a_i + 1)$ and we find $(P_{24}) = (2a_i) + (2a_i + 1)$, i. e. (P_{24}) is the system of the points with integer coordinates which are either all even or all odd.

Finally (P_{16}) is derived from (P_{24}) by adding to (P_8) not the whole system of the vertices of the cells $C_8^{(2)}$, but only that half which is not occupied by the vertices of the inscribed $C_{16}^{(2\sqrt{2})}$. We express this by means of the equation $P_{16} = (2a_i) + \frac{1}{2}(2a_i + 1)$. Here we have to understand by $\frac{1}{2}(2a_i + 1)$ that system of points of which the coordinates are only odd integer numbers under the condition that half the sum is either always even or always odd. If in the cell $C_8^{(2)}$ which furnished us above with the system of coordinates a positive $C_{16}^{(2\sqrt{2})}$ is inscribed, which for the future we shall always suppose, then the point (1, 1, 1, 1) is occupied by a vertex of the inscribed $C_{16}^{(2\sqrt{2})}$ and so for the non-occupied vertices $\frac{1}{2}(2a_i+1)$ half the sum of the four quantities a_i is odd.

If we make the connection between the systems of points (P_8), (P_{16}), (P_{24}) in the indicated way, then the number of points of (P_{24}) is twice, and the number of points (P_{16}) is one and a half times as large as that of (P_8) and so the fourdimensional volumes of $C_8^{(2)}$, $C_{16}^{(2\sqrt{2})}$, $C_{24}^{(\sqrt{2})}$ have to be in the same ratio as the numbers $1, \frac{2}{3}, \frac{1}{2}$. This can be easily verified. To make a $C_{16}^{(2\sqrt{2})}$ of $C_8^{(2)}$ we have truncated at eight vertices a rectangular fivecell, which is $\frac{1}{24}$ of $C_8^{(2)}$; so $\frac{2}{3}$ of $C_8^{(2)}$ remains. And to make of $C_8^{(2)}$ the cell $C_{24}^{(\sqrt{2})}$ contained in the former we have halved each of the sixteen parts $C_8^{(1)}$.

5. By the "transformation-view" of each of the nets (C_8), (C_{16}) and (C_{24}) with respect to a space Sp_3 of the bearing space Sp_4 as screen we understand the intersection varying every moment, of this non-moving space with the fourdimensional net moving along in the direction normal to this space. If for this movement we interchange the relative and the absolute, we can also take this transformation-view to be generated by the intersection of the non-moving fourdimensional net with a space Sp_3, moving along in a perpendicular direction and remaining parallel to itself; there we can again assume that this view is observed by one who shares the movement of the space

Sp_3. The chief aim of this communication is to indicate how we can connect the transformation-views of the nets (C_{16}), (C_{24}) with that of the net (C_8), which is by far the simplest. Because the three views furnish at every moment a filling of the intersecting space, this investigation can lead to new threedimensional space-fillings, even though they be not entirely regular.

To be able to design a transformation-view of the net (C_{16}) we must know for each of the component cells C_{16} the place *of* the centre and the position *about* the centre; as the coordinates of the centres of the cells are given above, we have only to occupy ourselves further with the position about the centre. We designate that position by means of the four diagonals of each C_{16} and we then notice that these four lines for each of the two kinds of inscribed cells C_{16} are also diagonals — groups of non-adjacent diagonals — of the circumscribed cells C_8, whilst for the cells C_{16} of the third group they are parallel to the axes of coordinates.

If we suppose the centre of a cell $C_{16}^{(2\sqrt{2})}$ of the third group to be at the same time the centre of a cell $C_8^{(4)}$, the edges of which are parallel to the axes of coordinates, the $C_{16}^{(2\sqrt{2})}$ is inscribed in this new eightcell in such a sense, that the vertices of $C_{16}^{(2\sqrt{2})}$ are the centres of the eight bounding cubes of $C_8^{(4)}$. For an obvious reason we call this $C_{16}^{(\sqrt{2})}$ *polarly* inscribed in $C_8^{(4)}$ — and now to distinguish, we call the cells of the two other groups *bodily* inscribed in the cells $C_8^{(2)}$. For, as was observed above, in each of the eight bounding cubes of $C_8^{(2)}$ a bounding tetrahedron of $C_{16}^{(2\sqrt{2})}$ is inscribed, whilst each of the remaining eight bounding tetrahedra of $C_{16}^{(2\sqrt{2})}$ has with respect to each of the four pairs of opposite bounding cubes of $C_8^{(2)}$ three vertices of one and one vertex of the other cube as vertices.

In this way each of the cells $C_{16}^{(2\sqrt{2})}$ of the net (C_{16}) is packed up in a C_8 as small as possible, of which the edges are parallel to the axes of coordinates; here the fourdimensional *cases* of the "erect" cells C_{16} of the third group are cells $C_8^{(4)}$, those of the "inclining" cells C_{16} of the first and the second group are cells $C_8^{(2)}$. Whilst the cases $C_8^{(2)}$ of the inclining cells C_{16} fill the space Sp_4, the cases $C_8^{(4)}$ of the erect cells C_{16} do so eight times, because $C_{16}^{(2\sqrt{2})}$ is the $\frac{1}{24}$th part of $C_8^{(4)}$, — as is immediately evident when one divides the erect $C_{16}^{(2\sqrt{2})}$ and its case $C_8^{(4)}$ by spaces through the common centre parallel to the pairs of bounding spaces of $C_8^{(4)}$ into sixteen equal parts and when one compares the rectangular fivecell of $C_{16}^{(2\sqrt{2})}$ to the $C_8^{(2)}$ of $C_8^{(4)}$ —, and the erect C_{16} together fill a third of Sp_4.

In the second mode of transformation of the cells $C_8^{(2)}$ of the net (C_8) into the cells $C_{24}^{(\sqrt{2})}$ of a net (C_{24}) the vertices of the $C_{24}^{(\sqrt{2})}$ concentric to $C_8^{(2)}$ are the centres of the faces of these $C_8^{(2)}$, from which it follows that the six centres of the faces of each of the eight bounding cubes of $C_8^{(2)}$ are vertices of a bounding octahedron of $C_{24}^{(\sqrt{2})}$ and so this cell may again be called inscribed — and *bodily* inscribed too — in $C_8^{(2)}$. Also the remaining bounding octahedra can be directly indicated with respect to these circumscribed $C_8^{(2)}$; through each of the sixteen vertices of $C_8^{(2)}$ pass six faces of this cell, of which the centres form the vertices of a bounding octahedron of $C_{24}^{(\sqrt{2})}$. [1])

From the preceding it follows, that the fourdimensional cases, inclosing the cells $C_{24}^{(\sqrt{2})}$ and having edges parallel to the axes of coordinates, consist of two nets (C_8) of cells $C_8^{(2)}$, which by exchange of centres and vertices pass into each other.

6. We conclude this first part by indicating the connection existing between the systems of axes of the five different cells with the origin of coordinates as common centre, which can be obtained by parallel translation of one of the cells $C_8^{(2)}$, one of each of the three groups of cells $C_{16}^{(2\sqrt{2})}$ and one of the cells $C_{24}^{(\sqrt{2})}$. We indicate these cells for brevity by C_8, C_{16}, C'_{16}, C''_{16}, C_{24} where C_{16} represents the polarly inscribed sixteencell and C'_{16} and C''_{16} successively the positive and the negative bodily inscribed one. Further here too — according to the notation of the handbook mentioned above — E, K, F, R will denote a vertex, midpoint of edge, centre of face, centre of bounding body and therefore OE, OK, OF, OR will have to denote the axes converging in these points. Thus OE_8 is an axis OE of C_8, OK_{16} an axis OK of C_{16}, OF'_{16} an axis OF of C'_{16}, etc.

The numbers of axes OE, OK, OF, OR of each of the three different cells are always the halves of the numbers of the elements E, K, F, R; they are contained in the following table.

Here C_{16} of course represents the three cells C_{16}, C'_{16}, C''_{16}.

We now indicate the connection of the systems of axes of the

[1]) By doubling the radii vectores of the six centres of the faces from the chosen vertex of these $C_8^{(2)}$ we find the central section normal to the diagonal of this point.

	OE	OK	OF	OR
C_8	8	16	12	4
C_{16}	4	12	16	8
C_{24}	12	48	48	12

five cells C_8, C_{16}, C'_{16}, C''_{16}, C_{24} by giving the coordinates of the points E, K, F, R belonging to these concentric cells with respect to two systems of axes of coordinates with the common centre of the cells as origin, the systems (OX_i) of the four axes OR_8 and the system (OY_i) of the four axes OE'_{16} (fig. 2) between which the relations

$$\left.\begin{aligned} 2y_1 &= x_1 + x_2 + x_3 + x_4 \\ 2y_2 &= x_1 - x_2 - x_3 + x_4 \\ 2y_3 &= -x_1 + x_2 - x_3 + x_4 \\ 2y_4 &= -x_1 - x_2 + x_3 + x_4 \end{aligned}\right\}$$

exist. [1])

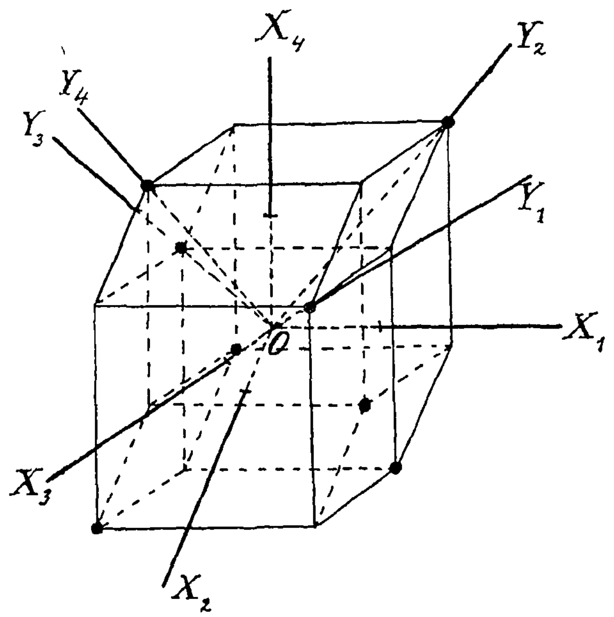


Fig. 2.

[1]) We selected this transformation T, because it causes the octuples of vertices of C_{16} and C'_{16} to pass into each other and those of C''_{16} into itself. It satisfies the condition $T^4 = -1$, so that first T^8 gives unity. We find that T^2 is a rectangular double-rotation round O by which (x_1, x_4) passes into $(-x_4, x_1)$ and (x_2, x_3) into $(-x_3, x_2)$.

We shall now give in both systems of coordinates the coordinates of the vertices of the five concentric cells and we divide in doing so — see the following table — the sixteen vertices of $C_8^{(2)}$ into the eight vertices of C'_{16} and the eight vertices of C''_{16}; to that end it is necessary for distinction to indicate whether the product of the coordinates is positive on negative.

Cells	Number of vertices	Coordinates (OXi)	Product	Coordinates (OYi)	Product
C_8 and C'_{16}	8	$(\pm 1, \pm 1, \pm 1, \pm 1)$	$+$	$(\pm 2, 0, 0, 0)$	
C_8 and C''_{16}	8	$(\pm 1, \pm 1, \pm 1, \pm 1)$	$-$	$(\pm 1, \pm 1, \pm 1, \pm 1)$	$-$
C_{16}	8	$(\pm 2, 0, 0, 0)$		$(\pm 1, \pm 1, \pm 1, \pm 1)$	$+$
C_{24}	24	$(\pm 1, \pm 1, 0, 0)$		$(\pm 1, \pm 1, 0, 0)$	

With the aid of this it is easy to find both quadruples of coordinates of the systems of the points K, F, R of the five cells. They are given in the following table, which after all the preceding is clear in itself.

Cells					Number of axes	Coordinates (OXi)	Product	Coordinates (OYi)	Product
C_8	C_{16}	C'_{16}	C''_{16}	C_{24}					
E	$2R$	E	$\frac{4}{3}R$	$2R$	4	$(\pm 1, \pm 1, \pm 1, \pm 1)$	$+$	$(2, 0, 0, 0)$	
E	$2R$	$\frac{4}{3}R$	E	$2R$	4	$(\pm 1, \pm 1, \pm 1, \pm 1)$	$-$	$(\pm 1, \pm 1, \pm 1, \pm 1)$	$-$
K	$\frac{3}{2}F$	—	—	$\frac{3}{2}F$	16	$(\pm 1, \pm 1, \pm 1, 0)$		$\left(\pm \frac{3}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\right)$	$-$
F	K	K	K	E	12	$(\pm 1, \pm 1, 0, 0)$		$(\pm 1, \pm 1, 0, 0)$	
R	$\frac{1}{2}E$	R	R	R	4	$(\pm 1, 0, 0, 0)$		$\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\right)$	$+$
—	—	F	—	F	16	$\left(\pm 1, \pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3}\right)$	$-$	$\left(\pm \frac{2}{3}, \pm \frac{2}{3}, \pm \frac{2}{3}, 0\right)$	
—	—	—	F	F	16	$\left(\pm 1, \pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3}\right)$	$+$	$\left(\pm 1, \pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3}\right)$	$+$
—	—	—	—	K	48	$\left(\pm 1, \pm \frac{1}{2}, \pm \frac{1}{2}, 0\right)$		$\left(\pm 1, \pm \frac{1}{2}, \pm \frac{1}{2}, 0\right)$	

Of course the axes, of which the number is given each time, agree in nature with the points connected by them with O. So the four axes given in the first row are axes OE for C_8 and C'_{16}, axes OR for C_{16}, C'''_{16} and C_{24}; moreover the coefficients $2, \frac{4}{3}, 2$ of $2R, \frac{4}{3}R, 2R$ indicate that the quadruples of coordinates appearing in this row relate to the point which is obtained by multiplying the observed axis OR of C_{16}, C'''_{16}, C_{24} as far as the length from O goes by $2, \frac{4}{3}, 2$.

With the preceding we have pointed out the position of each axis of one of the cells of the three nets (C_8), (C_{16}), (C_{24}) with reference to each of the two systems of coordinates and so we have furnished in connection with the preceding the material by which it is possible to deduce easily all the spacial sections of these three regular nets connected in a simple way with these axes. To give an example here already we observe that a space normal to one of the twelve axes OF_8 is normal to an axis OK for all the cells of the net (C_{16}); if it now proves possible to determine such a space in such a way that it is equally distant from the centres of all the cells C_{16} which are intersected, then in the intersecting space a more or less regular space-filling is generated by a selfsame body in three different positions.

In a future part we hope to commence with the determination of the remarkable spacial sections of the nets (C_8), (C_{16}), (C_{24}).

Mathematics. — "*Contribution to the knowledge of the surfaces with constant mean curvature*". By Dr. Z. P. BOUMAN. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of January 25, 1908).

§ 1. As is known the great difficulty connected with the study of the surfaces with constant mean curvature is the integration of the differential equation

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = - \sinh\theta \,.\, \cosh\theta .$$

The course followed here leads to two simultaneous partial differential equations of order one and of degree two.

In Gauss' symbols the value of the mean curvature H of a surface is indicated by

$$H = \frac{2FD' - ED'' - GD}{EG - F^2} .$$

As independent coordinates on the surface we choose those which are invariable along the lines with length zero and we represent them by ξ and η. So we find.

$$H = -2\frac{D'}{F}, \quad \text{whilst } E = G = 0 .$$

Let us multiply both members of the first equation by X (cosine of the angle of the normal with the X-axis); we then find:

$$FHX = -2D'X .$$

But

$$D'X = \frac{d^2x}{\partial\xi\partial\eta} \text{ }^{1)}$$

and moreover [2]:

$$FX = \frac{1}{i}\begin{vmatrix} \frac{\partial y}{\partial\xi} & \frac{dz}{\partial\xi} \\ \frac{\partial y}{\partial\eta} & \frac{\partial z}{\partial\eta} \end{vmatrix} = \frac{1}{i}\begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix},$$

where x, y, z represent the Cartesian coordinates of the surface with respect to a rectangular system of axes.

So we find

$$\frac{H}{i}\begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix} = -2\frac{\partial^2 x}{\partial\xi\partial\eta},$$

or:

and likewise:

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial\xi\partial\eta} &= -\frac{H}{2i}\begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix} \\ \frac{\partial^2 y}{\partial\xi\partial\eta} &= -\frac{H}{2i}\begin{pmatrix} z & x \\ \xi & \eta \end{pmatrix} \\ \frac{\partial^2 z}{\partial\xi\partial\eta} &= -\frac{H}{2i}\begin{pmatrix} x & y \\ \xi & \eta \end{pmatrix} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (I)$$

Moreover x, y and z must satisfy

$$E = G = 0,$$

therefore

$$\left.\begin{aligned} \Sigma\left(\frac{\partial x}{\partial\xi}\right)^2 &= 0 \\ \Sigma\left(\frac{\partial x}{\partial\eta}\right)^2 &= 0 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (II)$$

1) Bianchi, Vorlesungen über Differential-Geometrie, translation into German by Max Lukat, page 89.

2) l. c. page 86.

The equations (I) and (II) give back for $H=0$ the problem of the minimal surfaces.

For $-\frac{H}{2i}$ we shall introduce for brevity the symbol Q.

§ 2. To satisfy beforehand (II) we put

$$\left.\begin{array}{ll}\frac{\partial x}{\partial \xi}+i\frac{\partial y}{\partial \xi}=u\frac{\partial z}{\partial \xi}\,, & \frac{\partial x}{\partial \eta}+i\frac{\partial y}{\partial \eta}=v\frac{\partial z}{\partial \eta}\\ \frac{\partial x}{\partial \xi}-i\frac{\partial y}{\partial \xi}=-\frac{1}{u}\frac{\partial z}{\partial \xi}\,, & \frac{\partial x}{\partial \eta}-i\frac{\partial y}{\partial \eta}=-\frac{1}{v}\frac{\partial z}{\partial \eta}\end{array}\right\}, \quad . \; . \; (III)$$

where u and v are functions to be determined of ξ and η.

When we substitute the equations (III) into (I) we find the equations which u and v must satisfy, whilst moreover $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$, derived from (III) must obey the conditions of integrability.

The latter furnish

$$\frac{\partial\, u\frac{\partial z}{\partial \xi}}{\partial \eta}=\frac{\partial\, v\frac{\partial z}{\partial \eta}}{\partial \xi},$$

and

$$\frac{\partial\, \frac{1}{u}\frac{\partial z}{\partial \xi}}{\partial \eta}=\frac{\partial\, \frac{1}{v}\frac{\partial z}{\partial \eta}}{\partial \xi},$$

which is clear.

Writing out we find

$$\left.\begin{array}{ll}(a). & \frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}+u\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta}=\frac{\partial v}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}+v\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta}\\ (b). & \frac{1}{u^2}\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}-\frac{1}{u}\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta}=\frac{1}{v^2}\frac{\partial v}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}-\frac{1}{v}\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta}\end{array}\right\} \quad . \; . \; (IV)$$

If we now also substitute the values of $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$ into the equations (I) whilst we put $Q=-\frac{H}{2i}$ we find:

$$\left(\frac{\partial u}{\partial \eta}+\frac{1}{u^2}\frac{\partial u}{\partial \eta}\right)\frac{\partial z}{\partial \xi}+\left(u-\frac{1}{u}\right)\frac{\partial^2 z}{\partial \xi \partial \eta}=\frac{Q}{i}\left(u+\frac{1}{u}-v-\frac{1}{v}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta},$$

$$\frac{1}{i}\left(\frac{\partial u}{\partial \eta}-\frac{1}{u^2}\frac{\partial u}{\partial \eta}\right)\frac{\partial z}{\partial \xi}+\frac{1}{i}\left(u+\frac{1}{u}\right)\frac{\partial^2 z}{\partial \xi \partial \eta}=Q\left(v-\frac{1}{v}-u+\frac{1}{u}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta},$$

$$\frac{\partial^2 z}{\partial \xi \partial \eta}=\frac{Q}{2i}\left(\frac{u}{v}-\frac{v}{u}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}.$$

From these three last equations we derive directly with the aid of (IV):

$$\left.\begin{aligned}
&(a). && \frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}+u\,\frac{\partial^2 z}{\partial\xi\partial\eta}=Q\,i\,(v-u)\,\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}\\
&(b). && \frac{1}{u^2}\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}-\frac{1}{u}\frac{\partial^2 z}{\partial\xi\partial\eta}=Q\,i\left(\frac{1}{v}-\frac{1}{u}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}\\
&\text{whilst} && \frac{\partial^2 z}{\partial\xi\partial\eta}=\frac{Q}{2i}\left(\frac{u}{v}-\frac{v}{u}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}
\end{aligned}\right\}\quad\ldots\quad(V)$$

We can easily show that one of the equations (V) is dependent on the two others, as is clear.

If we divide both members of (V,a) by u^2 and if we add (V,b), we find:

$$\frac{\partial z}{\partial \eta}=\frac{2v}{Qi(v-u)^2}\cdot\frac{\partial u}{\partial \eta}.$$

From (IV,a) follows:

$$\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}-\frac{\partial v}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}=(v-u)\frac{\partial^2 z}{\partial\xi\partial\eta}=(v-u)\left(\frac{u^2-v^2}{uv}\right)\frac{Q}{2i}\cdot\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}.$$

By substituting here $\dfrac{\partial z}{\partial \eta}$ we find:

$$\frac{\partial z}{\partial \xi}=-\frac{2u}{Qi(v-u)^2}\cdot\frac{\partial v}{\partial \xi}.$$

We can now write down out of (III) the following set of equations:

$$\left.\begin{aligned}
\frac{\partial x}{\partial \xi}&=\frac{1}{2}\left(u-\frac{1}{u}\right)\cdot\frac{\partial z}{\partial \xi}=\frac{-(u^2-1)}{Qi(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\\
\frac{\partial x}{\partial \eta}&=\frac{1}{2}\left(v-\frac{1}{v}\right)\cdot\frac{\partial z}{\partial \eta}=\frac{v^2-1}{Qi(v-u)^2}\cdot\frac{\partial u}{\partial \eta}\\
\frac{\partial y}{\partial \xi}&=\frac{1}{2i}\left(u+\frac{1}{u}\right)\cdot\frac{\partial z}{\partial \xi}=\frac{u^2+1}{Q(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\\
\frac{\partial y}{\partial \eta}&=\frac{1}{2i}\left(v+\frac{1}{v}\right)\cdot\frac{\partial z}{\partial \eta}=\frac{-(v^2+1)}{Q(v-u)^2}\cdot\frac{\partial u}{\partial \eta}\\
\frac{\partial z}{\partial \xi}&= \qquad\qquad\qquad\quad =\frac{-2u}{Qi(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\\
\frac{\partial z}{\partial \eta}&= \qquad\qquad\qquad\quad =\frac{2v}{Qi(v-u)^2}\,\frac{\partial u}{\partial \eta}
\end{aligned}\right\}\quad\ldots\quad(VI)$$

So, as soon as u and v are known, the problem will be solved.

§ 3. In order now to write down the equations which u and v must satisfy, we can make use of (IV) and (VI), or we can use the conditions of integrability.

(IV,a) gives:

$$\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}=\frac{-2u}{iQ(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\frac{\partial u}{\partial \eta}+\frac{2v}{iQ}\left(\frac{2}{(v-u)^2}\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{v-u}\frac{\partial^2 u}{\partial \xi\partial \eta}\right).$$

(IV,b) gives:

$$\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}=\frac{-2u}{iQ(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\frac{\partial u}{\partial \eta}+\frac{2u}{iQ}\left(\frac{2}{(v-u)^2}\cdot\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{v-u}\cdot\frac{\partial^2 u}{\partial \xi\partial \eta}\right).$$

Out of (VI) we find:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \eta}\right)=-\frac{2}{iQ}\frac{v+u}{(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}+\frac{2v}{iQ(v-u)}\left(\frac{2}{(v-u)^2}\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{v-u}\frac{\partial^2 u}{\partial \xi\partial \eta}\right),$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \xi}\right)=-\frac{2}{iQ}\frac{v+u}{(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}+\frac{2u}{iQ(v-u)}\left(\frac{2}{(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}-\frac{1}{v-u}\frac{\partial^2 v}{\partial \xi\partial \eta}\right),$$

and

$$\frac{\partial^2 z}{\partial \xi\partial \eta}=\frac{Q}{2i}\left(\frac{u}{v}-\frac{v}{u}\right)\cdot\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}\text{ gives }\frac{\partial^2 z}{\partial \xi\partial \eta}=-\frac{2}{iQ}\frac{v+u}{(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}.$$

The equations given above show that all the conditions of the problem can be satisfied in the only way by putting:

$$\frac{2}{(v-u)^2}\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{(v-u)}\cdot\frac{\partial^2 u}{\partial \xi\partial \eta}=0\text{ and }-\frac{2}{(v-u)^2}\frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}+\frac{1}{(v-u)}\frac{\partial^2 v}{\partial \xi\partial \eta}=0,$$

which equations we write in the form:

$$\left.\begin{aligned}2\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+(v-u)\frac{\partial^2 u}{\partial \xi\partial \eta}&=0\\ 2\frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}-(v-u)\frac{\partial^2 v}{\partial \xi\partial \eta}&=0\end{aligned}\right\}\quad\ldots\ldots\quad(VII)$$

So the problem is entirely reduced to the integration of these two sumultaneous differential equations which are of order two and non-linear.

It is easy to deduce from (VII), that the conditions

$$\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\right)=\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\right)\text{ and }\frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\right)=\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\right),$$

are satisfied.

We find namely always:

$$\frac{\partial^2 x}{\partial \xi\partial \eta}=-\frac{2(uv-1)}{Qi(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}\quad,\quad\frac{\partial^2 y}{\partial \xi\partial \eta}=\frac{2(uv+1)}{Q(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi},$$

whilst

$$\frac{\partial^2 z}{\partial\xi\partial\eta} = -\frac{2(v+u)}{iQ(v-u)^3}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi}.$$

After substitution we get:

$$D' = -\frac{H}{2}F \text{ and } X^2 + Y^2 + Z^2 = 1,$$

so that really all the conditions of the problem prove to be satisfied by the equations (VII). Thus only the solution of (VII) is left to be found.

§ 4. We already know, that for the coordinates ξ and η

$$D' = -\frac{H}{2}\cdot F$$

must be satisfied.

But moreover follows from the equations of CODAZZI[1]:

$$\frac{\partial D}{\partial\eta} = 0 \quad\text{and}\quad \frac{\partial D''}{\partial\xi} = 0$$

So

$$D = f_1(\xi) \text{ and } D'' = f_2(\eta), \quad . \quad . \quad . \quad . \quad (VIII)$$

where f_1 and f_2 are respectively functions of ξ and η only.

The case that either D or D'' is equal to zero offers no difficulties, but nothing remarkable either.

The case that D and D'' are both equal to zero, leads, as is immediately clear, to the sphere as the simplest form of a surface with constant mean curvature. We can namely write down the condition for umbilical points, which is as follows with the omission of infinitesimals of higher order: [2])

$$\frac{E}{D} = \frac{F}{D'} = \frac{G}{D''}.$$

When for each point of the surface $E = G = 0$ then each point is an umbilical point, as soon as always $D = D'' = 0$, and these surfaces are (in as far as it concerns the real solution) spheres only.

§ 5. We shall now take the matter a little more generally.

Let us regard the total curvature of a surface as a simultaneous differential-invariant of both groundforms, we then find[3]):

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1) BIANCHI, l. c. p. 91. In using the coordinates ξ and η the CHRISTOFFEL symbols are all zero, except $\left\{\begin{matrix}1 & 1\\ & 1\end{matrix}\right\}$ and $\left\{\begin{matrix}2 & 2\\ & 2\end{matrix}\right\}$. By making use of $D = -\frac{H}{2}F$, we prove what was said in the text.

2) See e g. V. and K. KOMMERELL, Allgemeine Theorie der Raumkurven und Flächen, II, p. 21.

3) BIANCHI, l.c. p. 68.

$$\text{Total curvature} = \frac{DD''-D'^2}{EG-F^2} = \frac{H^2}{4} - \frac{f_1(\xi)f_2(\eta)}{F^2} =$$

$$= \frac{1}{2iF}\frac{\partial}{\partial\eta}\left(\frac{2}{iF}\frac{\partial F}{\partial\xi}\right) = -\frac{1}{F}\frac{\partial^2(lF)}{\partial\xi\partial\eta} \quad (IX)$$

(We notice moreover that, as is directly to be seen,

$$\frac{2}{F} = \frac{1}{r_1} - \frac{1}{r_2},$$

where r_1 and r_2 are the principal radii of curvature).

Let us now deduce from (VI) the value of F, we then find:

$$F = -\frac{2}{Q^2(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi},$$

or:

$$F = \frac{8}{H^2(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi}.$$

We substitute this value of F into (IX) by means of the following calculations. Out of (VII) follows

$$\frac{1}{F}\frac{\partial F}{\partial\eta} = \frac{\dfrac{\partial^2 u}{\partial\eta^2}}{\dfrac{\partial u}{\partial\eta}} + \frac{2}{v-u}\cdot\frac{\partial u}{\partial\eta},$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{F}\frac{\partial F}{\partial\eta}\right) = \frac{\partial}{\partial\xi}\frac{\dfrac{\partial^2 u}{\partial\eta^2}}{\dfrac{\partial u}{\partial\eta}} - 2\,\frac{\dfrac{\partial u}{\partial\xi}\cdot\dfrac{\partial u}{\partial\eta} + \dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}}{(v-u)^2}.$$

This must be equal to

$$-\frac{H^2}{4}F + \frac{f_1(\xi)\cdot f_2(\eta)}{F} = -\frac{2}{(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi} + \frac{H^2(v-u)^2 f_1(\xi)f_2(\eta)}{8\dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}},$$

and so we find:

$$\frac{H^2\cdot f_1(\xi)\cdot f_2(\eta)\cdot(v-u)^2}{8\dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}} = \frac{\partial}{\partial\xi}\frac{\dfrac{\partial^2 u}{\partial\eta^2}}{\dfrac{\partial u}{\partial\eta}} + \frac{\dfrac{\partial^2 u}{\partial\xi\,.\,\partial\eta}}{v-u}.$$

The second member can be once more reduced by means of (VII), and we find:

$$\frac{H^2 f_1(\xi) f_2(\eta)(v-u)^2}{8\dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}} = \frac{2}{(v-u)^2}\cdot\frac{\partial v}{\partial\eta}\cdot\frac{\partial u}{\partial\xi}.$$

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So

$$H^2 f_1(\xi) f_2(\eta) = \frac{16}{(v-u)^4} \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} \quad . \; . \; . \quad (X)$$

§ 6. Let us now return to equation (VII). We see immediately that a solution, which does not cause $F = \frac{8}{H^2 (v-u)^2} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial u}{\partial \eta}$ to vanish, is given by

$$u = \varphi(\eta) \quad , \quad v = \psi(\xi),$$

where φ and ψ are respectively functions of η and ξ only.

It is clear that equation (IX) is satisfied, when $f_1(\xi) = f_2(\eta) = 0$, so when $D = D'' = 0$ (§ 4).

It is worth noticing, that when $u = \varphi(\eta)$ and $v = \psi(\xi)$ are substituted into the equation for F, this form becomes a solution of

$$-\frac{H^2}{4} F = \frac{\partial}{\partial \xi}\left(\frac{1}{F} \cdot \frac{\partial F}{\partial \eta}\right) \quad .$$

and so this tallies perfectly, because we have here the differential equation of LIOUVILLE. Indeed, the problem of the surfaces with constant mean curvature always leads to an extended equation of LIOUVILLE, as (IX) does, in whatever way we treat it.

That we really find a sphere here must follow from (VI). These equations give for $u = \varphi(\eta)$ and $v = \psi(\xi)$,

$$z = \frac{1}{Qi} \frac{v+u}{v-u},$$

$$x = \frac{1}{Qi} \frac{uv-1}{v-u},$$

$$y = -\frac{1}{Q} \frac{uv+1}{v-u},$$

the wellknown formulæ for the sphere in minimal coordinates.

We find ·

$$x^2 + y^2 + z^2 = -\frac{1}{Q^2} = \frac{4}{H^2},$$

i.e. a sphere with radius $\frac{2}{H}$, as is necessary.

Now that we have regarded the special case $f_1(\xi) = f_2(\eta) = 0$, we can put both functions equal to 1 by introducing new functions

$$f_1(\xi) = \xi_1 \text{ and } f_2(\eta) = \eta_1,$$

which we shall again indicate by ξ and η. This is of high importance, if eventually the solution of equation (VII) were to be found.

§ 7. We can now put the question whether the equations (VII) can be solved by putting u equal to $f(v)$, where for the present f is arbitrary.

From (VII) can be deduced

$$\frac{\dfrac{\partial^2 v}{\partial\xi\partial\eta}}{\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}+\frac{\dfrac{\partial^2 u}{\partial\xi\partial\eta}}{\dfrac{\partial u}{\partial\xi}\cdot\dfrac{\partial u}{\partial\eta}}=0.$$

For $u=f(v)$ this leads to

$$\frac{\partial u}{\partial\xi}=f'(v)\cdot\frac{\partial v}{\partial\xi},$$

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=f''(v)\cdot\frac{\partial v}{\partial\xi}\cdot\frac{\partial v}{\partial\eta}+f'(v)\cdot\frac{\partial^2 v}{\partial\xi\partial\eta}.$$

So:

$$\frac{\dfrac{\partial^2 v}{\partial\xi\partial\eta}}{\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}+\frac{f''(v)\cdot\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}+f'(v)\cdot\dfrac{\partial^2 v}{\partial\xi\partial\eta}}{f'^2(v)\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}=0,$$

or

$$\{f'^2(v)+f'(v)\}\frac{\partial^2 v}{\partial\xi\partial\eta}+f''(v)\cdot\frac{\partial v}{\partial\xi}\cdot\frac{\partial v}{\partial\eta}=0.$$

Then, according to (VII), $\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}=\dfrac{v-u}{2}\cdot\dfrac{\partial^2 v}{\partial\xi\partial\eta}$.

So:

$$f'^2(v)+f'(v)+\frac{v-f(v)}{2}f''(v)=0.$$

One integral of this is sufficient to recognize the nature of the surfaces found. We find that satisfies

$$f(v)=-v\ ^{1)}.$$

1) Prof. W.KAPTEYN was so kind as to draw my attention to the following general solution of the differential equation.

Put

$$f(v)=y,$$

then

$$\frac{v-y}{2}\frac{d^2y}{dv^2}+\left(\frac{dy}{dv}\right)^2+\frac{dy}{dv}=0.$$

Now put

$$y=v+w,$$

so

$$\frac{dy}{dv}=1+\frac{dw}{dv}\ ,\quad \frac{d^2y}{dv^2}=\frac{d^2w}{dv^2},$$

so that

The equations (VII) become

$$\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}-u\frac{\partial^2 u}{\partial \xi \partial \eta}=0 \text{ and } \frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}-v\frac{\partial^2 v}{\partial \xi \partial \eta}=0\,,$$

which are satisfied by a function and its opposite. From this we deduce:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial\,(lu)}{\partial \eta}\right)=0\,.$$

Therefore e. g.

$$u=e^{\psi(\eta)+\varphi(\xi)}\;,\;v=-\,e^{\psi(\eta)+\varphi(\xi)}\,.$$

By quadratures we find out of (VI),

$$2Qiz=-\,\psi(\eta)+\varphi(\xi),$$
$$4Qix=\quad e^{\psi(\eta)+\varphi(\xi)}+e^{-\psi(\eta)-\varphi(\xi)},$$
$$4Qy=-\,e^{\psi(\eta)+\varphi(\xi)}+e^{-\psi(\eta)-\varphi(\xi)}.$$

The surface is a cylinder of revolution. Its section with the plane XOY is a circle, as we find

$$y^2+x^2=-\frac{1}{4Q^2}=\frac{1}{H^2}\,.$$

The radius of the circle is therefore $\frac{1}{H}$, as it has to be.

We can furthermore easily show that our solution agrees with the differential equations (IX), when we put

$$f_1(\xi)=f_2(\eta)=1.$$

We find namely that the second member becomes zero, so that

$$\frac{1}{F}=\frac{H}{2}=\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right).$$

As moreover $\frac{1}{F}=\frac{1}{2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$, as we saw before, r_2 is therefore $=\infty$.

§ 8. We can now investigate what in the equations (VII) the significance would be of a solution $u=\chi\,(\xi)$, if it were possible.

$$-\frac{w}{2}\frac{d^2w}{dv^2}+\left(\frac{dw}{dv}\right)^2+3\left(\frac{dw}{dv}\right)+2=0.$$

Let $\frac{dw}{dv}=p$, so $\frac{d^2w}{dv^2}=p\,\frac{dp}{dw}$,

then:

$$-\frac{w}{2}\,p\,\frac{dp}{dw}+(p+1)\,(p+2)=0,$$

from which ensues, $\frac{(p+2)^2}{p+1}=kw^2$ ($k=$ const.)

For $k=0$ this solution gives the one used in the text.

The equation

$$\frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + (v-u)\ 2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

is satisfied by $u = \chi(\xi)$.

So there remains to be integrated

$$2\frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} - (v-u)\frac{\partial^2 v}{\partial \xi \partial \eta} = 0\,,$$

when $u = \chi(\xi)$.

We find:

$$\frac{\partial v}{\partial \xi} = \frac{1}{2}(v - \chi(\xi))^2 \cdot f(\xi),$$

with $f(\xi)$ as arbitrary function of ξ.

The solution $u = \chi(\xi)$ furnishes (see (VI)) the value zero for $\frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}$ and $\frac{\partial z}{\partial \eta}$; whilst for $\frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}$ and $\frac{\partial z}{\partial \xi}$ the wellknown formulæ are found back for the minimal curves.

Entirely the same (with exchange of u and v, ξ and η) is found by putting $v = \chi_1(\eta)$.

This solution therefore shows what relations there are between the minimal surfaces and those under consideration. For the former we have but to join the two solutions found to get the complete solution with two arbitrary functions. So we see that the minimal surfaces are translation surfaces, generated by moving a minimal curve out of a set along the various points of a curve out of the second set; i. o. w. we have found back the integration of the minimal surfaces and in the usual form too.

Because of H tending to zero there is in this case no fear of F becoming 0.

§ 9. Now that the special cases of sphere (plane), cylinder and minimal surfaces are excluded, the integration of the equations (VII) would remain. I have not been able to attain more than the lowering of the order of the two differential equations, which is perhaps a step onward to a complete solution or to solutions for definite series of surfaces.

To this end we put:

$$\frac{\partial v}{d\xi}\,\frac{1}{(v-u)^2} = \frac{w_1}{2}\quad,\quad \frac{\partial u}{\partial \eta}\frac{1}{(v-u)^2} = -\frac{w_2}{2},$$

where w_1 and w_2 are functions of ξ and η.

From these we derive, by differentiation with respect to ξ and η respectively

$$\frac{\partial^2 v}{\partial\xi\partial\eta}=(v-u)\cdot w_1\cdot\left(\frac{\partial v}{\partial\eta}-\frac{\partial u}{\partial\eta}\right)+\frac{1}{2}(v-u)^2\cdot\frac{\partial w_1}{\partial\eta},$$

and

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=-(v-u)\cdot w_2\cdot\left(\frac{\partial v}{\partial\xi}-\frac{\partial u}{\partial\xi}\right)-\frac{1}{2}(v-u)^2\cdot\frac{\partial w_2}{\partial\xi}.$$

By means of the two non-differentiated equations and by equation (VII), we deduce from our last equation:

$$w_1\cdot\frac{\partial v}{\partial\eta}=w_1\left(\frac{\partial v}{\partial\eta}-\frac{\partial u}{d\eta}\right)+\frac{1}{2}(v-u)\cdot\frac{\partial w_1}{\partial\eta}$$

$$\text{and } -w_2\cdot\frac{\partial u}{\partial\xi}=w_2\left(\frac{\partial v}{\partial\xi}-\frac{\partial u}{\partial\xi}\right)+\frac{1}{2}(v-u)\cdot\frac{\partial w_2}{\partial\xi},$$

or:

$$w_1\frac{\partial u}{\partial\eta}=\frac{1}{2}(v-u)\cdot\frac{dw_1}{\partial\eta}\text{ and } w_2\frac{\partial v}{\partial\xi}=-\frac{1}{2}(v-u)\cdot\frac{\partial w_2}{\partial\xi},$$

from which ensues:

$$-w_1 w_2(v-u)=\frac{\partial w_1}{\partial\eta}\text{ and } -w_1 w_2(v-u)=\frac{\partial w_2}{\partial\xi}.$$

So we may put:

$$w_1=\frac{\partial f}{\partial\xi}\text{ and } w_2=\frac{\partial f}{\partial\eta},$$

where f is a function of ξ and η which has however to satisfy a new differential equation.

So we have:

$$\frac{\partial v}{\partial\xi}\cdot\frac{1}{(v-u)^2}=\frac{1}{2}\frac{\partial f}{\partial\xi}\text{ and }\frac{\partial u}{\partial\eta}\cdot\frac{1}{(v-u)^2}=-\frac{1}{2}\frac{\partial f}{\partial\eta},$$

whilst moreover:

$$v-u=-\frac{\dfrac{\partial^2 f}{\partial\xi\partial\eta}}{\dfrac{\partial f}{\partial\xi}\cdot\dfrac{\partial f}{\partial\eta}}.$$

Out of (VII) follows:

$$v-u=2\frac{\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}{\dfrac{\partial^2 v}{\partial\xi\partial\eta}}\text{ and } v-u=-2\frac{\dfrac{\partial u}{\partial\xi}\cdot\dfrac{\partial u}{\partial\eta}}{\dfrac{\partial^2 u}{\partial\xi\partial\eta}}.$$

By substitution of $v-u$, $\frac{\partial v}{\partial \xi}$ and $\frac{\partial v}{\partial \eta}$ we thus find:

$$1=\frac{(v-u)\frac{\partial f}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}}{\frac{\partial^2 v}{\partial\xi\partial\eta}}, \qquad 1=\frac{(v-u)\frac{\partial f}{\partial \eta}\cdot\frac{\partial u}{\partial \xi}}{\frac{\partial^2 u}{\partial\xi\partial\eta}}$$

$$1=-\frac{\frac{\partial^2 f}{\partial\xi\partial\eta}\cdot\frac{\partial v}{\partial \eta}}{\frac{\partial^2 v}{\partial\xi\partial\eta}\cdot\frac{\partial f}{\partial \eta}}, \qquad 1=-\frac{\frac{\partial^2 f}{\partial\xi\partial\eta}\cdot\frac{\partial u}{\partial \xi}}{\frac{\partial^2 u}{\partial\xi\partial\eta}\cdot\frac{\partial f}{d\xi}}.$$

$$\frac{\frac{\partial^2 v}{\partial\xi\partial\eta}}{\frac{\partial v}{\partial \eta}}=-\frac{\frac{\partial^2 f}{\partial\xi\partial\eta}}{\frac{\partial f}{\partial \eta}}, \qquad \frac{\frac{\partial^2 u}{\partial\xi\partial\eta}}{\frac{\partial u}{\partial \xi}}=-\frac{\frac{\partial^2 f}{\partial\xi\partial\eta}}{\frac{\partial f}{\partial \xi}}.$$

After integration we find:

$$\frac{\partial v}{\partial \eta}\cdot\frac{\partial f}{\partial \eta}=F_2(\eta) \text{ and } \frac{\partial u}{\partial \xi}\cdot\frac{\partial f}{\partial \xi}=F_1(\xi).$$

Joining these equations to the values of $\frac{\partial v}{\partial \xi}$ and $\frac{\partial u}{\partial \eta}$, we find:

$$\frac{\partial v}{\partial \eta}\cdot\frac{\partial u}{\partial \eta}=-\frac{1}{2}(v-u)^2 F_2(\eta) \text{ and } \frac{\partial u}{\partial \xi}\cdot\frac{\partial v}{\partial \xi}=\frac{1}{2}(v-u)^2 F_1(\xi).$$

These equations must be regarded as the intermediate integrals; they contain the arbitrary functions F_2 and F_1, and it is easy to prove that by differentiation they lead back to the two equations (VII) of order two.

It goes almost without saying that F_2 and F_1 appearing here are closely connected to f_1 and f_2 appearing in (VIII).

From the equations just found follows:

$$\frac{\partial v}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial u}{\partial \xi}=-\frac{1}{4}(v-u)^4 \cdot F_2(\eta)\cdot F_1(\xi),$$

or

$$F_2(\eta)\cdot F_1(\xi)=-\frac{4\frac{\partial v}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial u}{\partial \xi}}{(v-u)^4}.$$

If we compare this to (X), then:

$$-4F_2(\eta)\cdot F_1(\xi)=H^2 f_1(\xi) f_2(\eta).$$

The first integrals found satisfy therefore all the conditions entirely. We have transformed our original coordinates in such a way that

$f_1(\xi)$ and $f_2(\eta)$ both became 1 and so now we can take in accordance with it:

$$F_2(\eta) = \frac{H}{2i} \text{ and } F_1(\xi) = \frac{H}{2i},$$

so that the first integrals become:

$$\frac{\partial v}{\partial \eta} \cdot \frac{\partial u}{\partial \eta} = -\frac{H}{4i}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = \frac{H}{4i}(v-u)^2,$$

or

$$\frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \eta} = \frac{Q}{2}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = -\frac{Q}{2}(v-u)^2 \quad . \quad . \quad . \quad (A)$$

By replacing moreover $v-u$ by s_1 and $v+u$ by s_2 the final equations become:

$$\left(\frac{\partial s_1}{\partial \eta}\right)^2 = \left(\frac{\partial s_2}{\partial \eta}\right)^2 - 2\,Q\,s_1^2 \text{ and } \left(\frac{\partial s_1}{\partial \xi}\right)^2 = \left(\frac{d s_2}{\partial \xi}\right)^2 + 2\,Q\,s_1^2 \quad . \quad (B)$$

These are still to be solved.

Mathematics. — *"On the multiplication of trigonometrical series."* By Prof. W. Kapteyn,

1. If $f(x)$ and $\varphi(x)$ are two functions which are finite and continuous in the interval from $x=0$ to $x=\pi$, we have

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots.$$
$$f(x) = b_1 \sin x + b_2 \sin 2x + \ldots.$$
$$\varphi(x) = \tfrac{1}{2}a'_0 + a'_1 \cos x + a'_2 \cos 2x \ldots.$$
$$\varphi(x) = b'_1 \sin x + b'_2 \sin 2x + \ldots.$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi f(\omega)\cos n\omega\, d\omega \qquad b_n = \frac{2}{\pi}\int_0^\pi f(\omega)\sin n\omega\, d\omega$$

$$a'_n = \frac{2}{\pi}\int_0^\pi \varphi(\omega)\cos n\omega\, d\omega \qquad b'_n = \frac{2}{\pi}\int_0^\pi \varphi(\omega)\sin n\omega\, d\omega.$$

In the same way the product $f(x) \cdot \varphi(x)$ may be developed, this product being finite and continuous in the same interval; therefore

$$f(x) \cdot \varphi(x) = \tfrac{1}{2}A_0 + A_1 \cos x + A_2 \cos 2x + \ldots.$$
$$f(x) \cdot \varphi(x) = B_1 \sin x + B_2 \sin 2x + \ldots.$$

where

$$A_n = \frac{2}{\pi}\int_0^\pi f(\omega)\,\varphi(\omega)\cos n\omega\, d\omega, \qquad B_n = \frac{2}{\pi}\int_0^\pi f(\omega)\,\varphi(\omega)\sin n\omega\, d\omega.$$

We shall now investigate the relations which exist between the integrals A_n, B_n and the coefficients a_n, b_n, a'_n, b'_n.

Substituting in A_n for $\varphi(\omega)$ the series of cosines, we obtain

$$A_n = \frac{2}{\pi}\int_0^\pi f(\omega)\cos n\omega\,[\tfrac{1}{2}a'_0 + a'_1\cos\omega + a'_2\cos 2\omega + \ldots]\,d\omega$$

$$= \tfrac{1}{2}a'_0 a_n + \sum_1^\infty \frac{a'_m}{2}\cdot\frac{2}{\pi}\int_0^\pi f(\omega)\,[\cos(m+n)\,\omega + \cos(m-n)\,\omega]\,d\omega$$

$$= \tfrac{1}{2}a'_0 a_n + \sum_1^\infty \frac{a'_m}{2}(a_{m+n} + a_{m-n}).$$

This equation may be written in another form; for, because $a'_{-p} = a'_p$,

$$\sum_1^\infty \frac{a'_m}{2}a_{m-n} = \sum_1^\infty \frac{a'_m}{2}a_{n-m} + \tfrac{1}{2}\sum_{n+1}^\infty a'_m a_{m-n}$$

or, putting $m+n$ instead of m in the summation from $n+1$ to ∞

$$\sum_1^\infty \frac{a'_m}{2}a_{m-n} = \tfrac{1}{2}\sum_1^n a'_m a_{n-m} + \tfrac{1}{2}\sum_1^\infty a_m a'_{m+n}.$$

Hence

$$A_n = \tfrac{1}{2}\sum_0^n a'_m a_{n-m} + \tfrac{1}{2}\sum_1^\infty (a'_m a_{m+n} + a_m a'_{m+n}) \quad . \quad . \quad . \quad (I)$$

If now we substitute in A_n for $\varphi(\omega)$ the series of sines, we have

$$A_n = \frac{2}{\pi}\int_0^\pi f(\omega)\cos n\,\omega\,[b'_1\sin\omega + b'_2\sin 2\omega + \ldots]\,d\omega$$

$$= \sum_1^\infty \frac{b'_m}{2}\cdot\frac{2}{\pi}\int_0 f(\omega)\,[\sin(m+n)\,\omega + \sin(m-n)\,\omega]\,d\omega$$

$$= \sum_1^\infty \frac{b'_m}{2}(b_{m+n} + b_{m-n})$$

or, as $b'_{-p} = -bp$

$$A_n = -\tfrac{1}{2}\sum_1^n b'_m b_{n-m} + \tfrac{1}{2}\sum_1^\infty (b'_m b_{m+n} + b_m b'_{m+n}). \quad . \quad . \quad (II)$$

In the same way we find

$$B_n = \frac{2}{\pi}\int_0^\pi f(\omega)\sin n\omega\,[\tfrac{1}{2}a'_0 + a'_1\cos\omega + a'_2\cos 2\omega + \ldots]\,d\omega$$

$$= \tfrac{1}{2}a'_0 b_n + \sum^\infty \frac{a'_m}{2}(b_{m+n} - b_{m-n})$$

or, after a slight reduction

$$B_n = \tfrac{1}{2}\sum_0^n a'_m b_{n-m} + \tfrac{1}{2}\sum_1^\infty (a'_m b_{m+n} - b_m a'_{m+n}) \quad . \quad . \quad . \quad . \quad . \quad (III)$$

and

$$B_n = \frac{2}{\pi}\int_0^\pi f(\omega)\, sin\, n\omega\, [b'_1\, sin\, \omega + b'_2\, sin\, 2\omega + \ldots]\, d\omega$$

$$= \sum_1^\infty \frac{b'_m}{2}(a_{m-n} - a_{m+n})$$

or

$$B_n = \tfrac{1}{2}\sum_1^n b'_m a_{n-m} + \tfrac{1}{2}\sum_1^\infty (a_m b'_{m+n} - b'_m a_{m+n}) \quad . \quad . \quad . \quad . \quad . \quad (IV)$$

2. If we suppose

$$f^2(x) = \tfrac{1}{2}\mathfrak{A}_0 + \mathfrak{A}_1\, cos\, x + \mathfrak{A}_2\, cos\, 2x + \ldots$$
$$= \mathfrak{C}_1\, sin\, x + \mathfrak{C}_2\, sin\, 2x + \ldots$$

the four preceding equations give immediately, by putting $\varphi(x) = f(x)$

$$\mathfrak{A}_n = \tfrac{1}{2}\sum_0^n a_m a_{n-m} + \sum_1^\infty a_m a_{m+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\mathfrak{A}_n = -\tfrac{1}{2}\sum_1^n b_m b_{n-m} + \sum_1^\infty b_m b_{m+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\mathfrak{C}_n = \tfrac{1}{2}\sum_0^n a_m b_{n-m} + \tfrac{1}{2}\sum_1^\infty (a_m b_{m+n} - b_m a_{m+n}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

3. From the four equations of Art. 1, the beautiful theorem of Parseval may be easily deduced. For, supposing that for all the values on the circumference of the circle $mod\, z = 1$, we have

$$\tfrac{1}{2}a_0 + a_1 z + a_2 z^2 + \ldots = \varphi(z)$$
$$\tfrac{1}{2}a'_0 + \frac{a'_1}{z} + \frac{a'_2}{z^2} + \ \ldots = \psi(z),$$

it is evident, if we assume in succession $z = e^{i\omega}$ and $z = e^{-i\omega}$, that

$$F_1(\omega) + i\,F_2(\omega) = \varphi(e^{i\omega}) \quad G_1(\omega) - i\,G_2(\omega) = \psi(e^{i\omega})$$
$$F_1(\omega) - i\,F_2(\omega) = \varphi(e^{-i\omega}) \quad G_1(\omega) + i\,G_2(\omega) = \psi(e^{-i\omega}).$$

Multiplying these equations and adding the results we obtain

$$2\,[F_1(\omega)\,G_1(\omega) + F_2(\omega)\,G_2(\omega)] = \varphi(e^{i\omega})\,\psi(e^{i\omega}) + \varphi(e^{-i\omega})\,\psi(e^{-i\omega})$$

where

$$F_1(\omega) = F_1 = \tfrac{1}{2}a_0 + a_1\, cos\, \omega + a_2\, cos\, 2\omega + \ldots$$
$$G_1(\omega) = G_1 = \tfrac{1}{2}a'_0 + a'_1\, cos\, \omega + a'_2\, cos\, 2\omega + \ldots$$
$$F_2(\omega) = F_2 = a_1\, sin\, \omega + a_2\, sin\, 2\omega + \ldots$$
$$G_2(\omega) = G_2 = a'_1\, sin\, \omega + a'_2\, sin\, 2\omega + \ldots$$

If now we put $n=0$ in the equations (I) and (II) we find

$$\frac{2}{\pi}\int_0^\pi F_1\, G_1\, d\omega = \tfrac{1}{2}\, a_0\, a'_0 + a_1\, a'_1 + a_2\, a'_2 + \ldots.$$

$$\frac{2}{\pi}\int_0^\pi F_2\, G_2\, d\omega = \qquad a_1\, a'_1 + a_2\, a'_2 + \ldots$$

thus

$$\frac{1}{2\pi}\int_0^\pi \{\varphi\,(e^{i\omega})\,\psi\,(e^{i\omega}) + \varphi\,(e^{-i\omega})\,\psi\,(e^{-i\omega})\}\,d\omega = \frac{1}{4}\,a_0 a'_0 + a_1 a'_1 + a_2 a'_2 + \ldots$$

which is the theorem in question.

4. From the preceding formulae we may also deduce the values of several interesting series. For, if the series for $f\,(x)$ and $\varphi\,(x)$ are given, and the integrals

$$\int_0^\pi f\,(\omega)\,\varphi\,(\omega)\cos n\omega\, d\omega \ \text{ and} \int_0^\pi f\,(\omega)\,\varphi\,(\omega)\sin n\omega\, d\omega$$

are to be found, the values of the series in the second members of the given equations may be determined. To show this, we shall make the following application of the formulae (1), (2) and (3).

Suppose $f\,(x) = x$, then

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} - \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \ldots.\right)$$

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots.\right)$$

and

$$\mathfrak{A}_n = \frac{2}{\pi}\int_0^\pi \omega^2 \cos n\omega\, d\omega = 4\,\frac{\cos n\pi}{n^2}$$

$$\mathfrak{C}_n = \frac{2}{\pi}\int_0^\pi \omega^2 \sin n\omega\, d\omega = -\,\frac{2\pi\cos n\pi}{n} - \frac{4\,(1-\cos n\pi)}{\pi n^3}.$$

Now the formula (1) gives, because

$$a_2 = a_4 = a_6 = \ldots = 0$$

$$\mathfrak{A}_0 = \tfrac{1}{2}\, a_0^{\,2} + a_1^{\,2} + a_3^{\,2} + a_5^{\,2} + \ldots$$

$$\mathfrak{A}_2 = \tfrac{1}{2}\, a_1^{\,2} + a_1 a_3 + a_3 a_5 + a_5 a_7 + \ldots$$

$$\mathfrak{A}_4 = a_1 a_3 + a_1 a_5 + a_3 a_7 + a_5 a_9 + \ldots$$

$$\mathfrak{A}_6 = a_1 a_5 + \frac{a_3^{\,2}}{2} + a_1 a_7 + a_3 a_9 + a_5 a_{11} + \ldots$$

. . . .

therefore

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots \qquad =\frac{\pi^4}{96}$$

$$\frac{1}{1^2.3^2}+\frac{1}{3^2.7^2}+\frac{1}{5^2.9^2}+\dots=\frac{\pi^2}{16}-\frac{1}{2}$$

$$\frac{1}{1^2.5^2}+\frac{1}{3^2.7^2}+\frac{1}{5^2.9^2}+\dots=\frac{\pi^2}{64}-\frac{1}{9}$$

$$\frac{1}{1^2.7^2}+\frac{1}{3^2.9^2}+\frac{1}{5^2.11^2}+\dots=\frac{\pi^2}{144}-\frac{137}{4050}$$

. . . .

According to formula (2) we have

$$\mathfrak{A}_0=\sum_1^\infty b_m^2$$

$$\mathfrak{A}_1=\sum_1^\infty b_m b_{m+1}$$

$$\mathfrak{A}_2=-\tfrac{1}{2}\,b_1^2+\sum_1^\infty b_m b_{m+2}$$

$$\mathfrak{A}_3=-\,b_1 b_2+\sum_1^\infty b_m b_{m+3}$$

$$\mathfrak{A}_4=-\,\tfrac{1}{2}\,(2b_1 b_3+b_2^2)+\sum_1^\infty b_m b_{m+4}$$

$$\mathfrak{A}_5=-\,(b_1 b_4+b_2 b_3)+\sum_1^\infty b_m b_{m+5}$$

$$\mathfrak{A}_6=-\,\tfrac{1}{2}\,(2b_1 b_5+2b_2 b_4+b_3^2)+\sum_1^\infty b_m b_{m+6}$$

. . . .

From which may be deduced

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6}$$

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots=1$$

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\dots=\frac{3}{4}$$

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\dots=\frac{11}{18}$$

$$\frac{1}{1.5}+\frac{1}{2.6}+\frac{1}{3.7}+\dots=\frac{25}{48}$$

$$\frac{1}{1.6}+\frac{1}{2.7}+\frac{1}{3.8}+\dots=\frac{137}{300}$$

$$\frac{1}{1.7}+\frac{1}{2.8}+\frac{1}{3.9}+\dots=\frac{49}{120}$$

. . . .

In the same way the formula (3) gives

$$\mathfrak{C}_1 = \tfrac{1}{2}\, a_0 b_1 + \tfrac{1}{2}\,(a_1 b_2 - b_2 a_3 + a_3 b_4 - b_4\, a_5 + ..)$$

$$\mathfrak{C}_2 = \tfrac{1}{2}\,(a_0 b_2 + a_1 b_1) + \tfrac{1}{2}\,(a_1 b_3 - b_1 a_3 + a_3 b_5 - b_3 a_5 + a_5 b_7 - b_5 a_7 + .\ )$$

$$\mathfrak{C}_3 = \tfrac{1}{2}\,(a_0 b_3 + a_1 b_2) + \tfrac{1}{2}\,(a_1 b_4 - b_2 a_5 + a_3 b_6 - b_4 a_7 + a_5 b_8 - b_6 a_9 + .\ )$$

$$\mathfrak{C}_4 = \tfrac{1}{2}\,(a_0 b_4 + a_1 b_3 + a_3 b_1) + \tfrac{1}{2}\,(a_1 b_5 - b_1 a_5 + a_3 b_7 - b_3 a_7 + a_5 b_9 - b_5 a_9 + ..)$$

$$\mathfrak{C}_5 = \tfrac{1}{2}\,(a_0 b_5 + a_1 b_4 + a_3 b_2) + \tfrac{1}{2}\,(a_1 b_6 - b_2 a_7 + a_3 b_8 - b_4 a_9 + a_5 b_{10} - b_6 a_{11} + ..)$$

$$\mathfrak{C}_6 = \tfrac{1}{2}\,(a_0 b_6 + a_1 b_5 + a_3 b_3 + a_5 b_1) + \\ + \tfrac{1}{2}\,(a_1 b_7 - b_1 a_7 + a_3 b_9 - b_3 a_9 + a_5 b_{11} - b_5 a_{11} + ..)$$

. . . .

from which the following relations may be obtained

$$\frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} + .. \qquad = \frac{\pi^2}{16} - \frac{1}{2}$$

$$\frac{1}{4.1^2} - \frac{1}{2.5^2} + \frac{1}{6.3^2} - \frac{1}{4.7^2} + .. = \frac{\pi^2}{12} - \frac{31}{54}$$

$$\frac{1}{5.1^2} - \frac{1}{1.5^2} + \frac{1}{7.3^2} - \frac{1}{3.7^2} + .. = \frac{\pi^2}{16} - \frac{4}{9}$$

$$\frac{1}{6.1^2} - \frac{1}{2.7^2} + \frac{1}{8.3^2} - \frac{1}{4.9^2} + .. = \frac{7\pi^2}{60} - \frac{347}{900}$$

$$\frac{1}{7.1^2} - \frac{1}{1.7^2} + \frac{1}{9.3^2} - \frac{1}{3.9^2} + .. = \frac{\pi^2}{24} - \frac{187}{675}$$

. . . .

Chemistry. — *"On a crystallised d. fructose tetracetate"*, by Dr. D. H. Brauns. (Communicated by Prof. A. P. N. Franchimont).

Very few crystallised derivatives of *d.* fructose have as yet been obtained. A pentacetate was described by Erwigs and Koenigs as a gummy substance. A number of researches have shown, however, that the high temperature at which the reactions generally took place causes a conversion or decomposition of the fructose. As no satisfactory results were obtained with acetic anhydride and acetyl chloride acetyl bromide was employed which reacts at a comparatively low temperature. The greatest possible precautions were taken to exclude moisture and to let the reaction take place at a low temperature. The details will be published in full later on.

Refrigerated d. fructose in fine powder was mixed with a little more than 5 mols. of acetyl bromide at -15° and after starting the reaction by touching one spot with a tube having the ordinary temperature, I waited until most of the hydrogen bromide had been evolved and the reaction was consequently nearly over. The excess of acetyl bromide was then distilled in a high vacuum and the *product*, consisting of a tenaceous, yellow mass, treated with iced water, then dissolved in alcohol and placed in a desiccator containing caustic potash and kept at a low temperature. A crystallised mass was obtained which after being submitted to pressure was recrystallised at a low temperature when beautiful crystals, free from bromine, were deposited.

These crystals are colourless, odourless, taste bitter and melt at 131°—132°. In a high vacuum they may be sublimed even at 95° more rapidly at 105°, the sublimate has the same melting point.

The ultimate analyses gave a mean result of C 48.26°/₀, H 5.86°/₀.

The molecular weight determination by the lowering of the freezing point of benzene gave a mean of 355.

The acetyl determination was carried out by saponification with $n/_{10}$ sodium hydroxide at a low temperature. Blank experiments made under similar conditions showed that fructose is not altered or converted into acids. The saponification was nearly complete after two hours and quite so in 18 hours; after 28 hours no sensible decomposition of the fructose had set in and about the same figures were obtained as those in 18 hours. The average amount of acetic acid found was 69.42°/₀.

It is, therefore, a fructose tetracetate $C_{14}H_{20}O_{10}$ for which theory requires C. 48.25°/₀ H. 5.86°/₀, molecular weight 348 acetic acid 68.96°/₀.

This compound is but little soluble in water, ether, benzene and ligroin, readily so in alcohol and chloroform.

The chloroform solution was used to determine the rotatory power. It polarises to the left and the specific rotation of d. fructose tetracetate at 20° was found $[\alpha]^{20}_{D} - 91°.38$.

Dr. F. M. JAEGER was kind enough to investigate the crystals and reported as follows:

d. Fructose tetracetate (BRAUNS).

$C_{14}H_{20}O_{10}$; Melting point 132° C.

Sp. Gr. of the crystals at 15° = 1.388; Mol. Vol. = 250.72.

From ethyl alcohol + ether, it is obtained on slow evaporation, in beautiful, colourless, shining little crystals which may be readily measurèd and which possess a pure geometrical structure.

The compound is *hemimorphous*; its symmetry is that of the *monoclino-sphenoidic* class. It, therefore, does not possess a single symmetry-plane or a symmetry centre; but only one single unipolar, twin axis. All the crystals which I investigated represented the same variety of the two possible enantiomorphous forms.

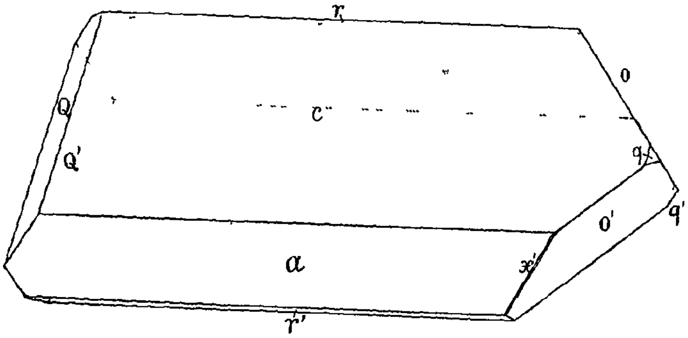

Fig. 1. d. Fructose tetracetate (BRAUNS).

The symmetry assigned here to the crystals is not only proved by their habit, but also proved beyond all doubt by the investigation of the etched figures obtained by means of 95°/₀ alcohol; these were very distinct particularly on {100} and {001}.

Parameters: $a:b:c = 1,3463:1:1,5733$

$\beta = 52^\circ.12'$

Forms observed: $c = \{001\}$, broad and very shining; $a = \{100\}$, somewhat narrower; $o = \{\bar{1}11\}$, large and yielding sharp reflexes; $q = \{011\}$, small but reflecting well; $Q = \{0\bar{1}1\}$, large and shining; $r = \{\bar{1}02\}$, very narrow and dull; $x = \{\bar{9}11\}$ exceedingly narrow and measurable only with difficulty. Once or twice one plane of $\{\bar{1}\bar{1}1\}$ was observed, rudimentary and striped parallel to the plane {001}.

Angular values:	*Measured*:	*Calculated:*
$c:a = (001):(100) =$	* 52°.12′	—
$o:o = (\bar{1}11):(11\bar{1}) =$	* 75 .41	—
$c:o = (001):(\bar{1}11) =$	* 79 .37	—
$a:q = (100):(011) =$	67 .21½	67°.24½′
$q:o = (011):(\bar{1}11) =$	43 .10½	43 .17¾
$x:o = (91\bar{1}):(11\bar{1}) =$	60 .44 (about)	60 .53½
$x:a = (91\bar{1}):(100) =$	8 .36 (about)	8 .27¼
$a:o = (100):(11\bar{1}) =$	69 .29½	69 .20¾
$q:q = (011):(01\bar{1}) =$	77 .39	77 .37½
$c:q = (001):(011) =$	51 .10½	51 .11¼
$c:r = (001):(\bar{1}02) =$	35 .44	35 .43½
$r:a = (\bar{1}02):(\bar{1}00) =$	92 .4	92 .4½

Readily cleavable parallel to *a* and *c*.

The optical axial plane is {010}. Very faint, inclined dispersion: $\varrho > \nu$; double refraction negative. On c one optical axis emerges at a small angle with the normal.

Topic axial relation : $\chi : \psi : \omega = 7.1503 : 5.3109 : 8.3556$.

Physics. — *"New observations concerning asymmetrical triplets"*. By Prof. P. Zeeman.

Asymmetry investigated by means of Fabry *and* Perot's *method.*

1. In the second part of the paper "Magnetic resolution of spectral lines and magnetic force" I [1]) investigated, by means of a method, which I called that of the non-uniform field, the asymmetry predicted from theory by Voigt [2]) in the case the original line is resolved into a triplet.

A glance at Plate II of my paper immediately shows that observation seems to confirm strikingly Voigt's theoretical result that the component of the triplet towards the red is at a somewhat smaller distance from the middle line than the one towards the violet.

In order to exclude however all doubt as to the reality of this experimental result I thought it desirable to continue my work in a direction independent of Rowland's method.

I have shown [3]) that the resolution of spectral lines by magnetic forces can be investigated by means of the semi-silvered parallel plates of Fabry and Perot.

Using the special form of instrument in which the distance of the silvered surfaces is constant, the étalon, we may yet choose between two ways of comparison of the wavelengths of the centre line and of the components, originating by the action of the magnetic field.

Firstly we may measure, the intensity of the field being arbitrarily chosen, the diameters of the interference rings. By combining only measurements of rings originating from the same ring the calculation becomes very simple; for as shown in my last paper even a knowledge of the ordinal number of the rings then is unnecessary.

2. We may use however also the method of coincidences, regulating

[1]) Zeeman. These Proceedings 30 November 1907.

[2]) Voigt. Ann. d. Phys. 1. p. 376. 1900, see also the last paper by Voigt. Physik. Zeitschrift 9. p. 122. 1908.

[3]) Zeeman. These Proceedings 28 December 1907.

the magnetic force in such a manner that a ring which expands by increasing magnetic intensity coincides with a contracting ring.

The rings corresponding to components towards the red then coincide with rings due to components towards the violet side of the spectrum. The intensity of the coïnciding rings is then only slightly inferior to that of the unmodified one, a circumstance favourable to the accuracy of the measurements.

Let λ_0 be the wavelength of the middle component of the triplet, λ_r that of the component towards the red, λ_v that of the component towards the violet then we may perform the calculation, ignoring the value of the ordinal numbers of the rings, by the following procedure.

Let P_0, P_r, P_v be the ordinal numbers of rings with angular diameters x_0, x_r, x_v then we have in general:

$$\lambda_r = \lambda_0 \frac{P_0}{P_r}\left(1 + \frac{x_0^2}{8} - \frac{x_r^2}{8}\right)$$

$$\lambda_v = \lambda_0 \frac{P_0}{P_v}\left(1 + \frac{x_0^2}{8} - \frac{x_v^2}{8}\right).$$

If the magnetic force is increasing a contracting ring corresponds to λ_r, an expanding one to λ_v. As I remarked on a former occasion we can put in the case of radiation in a magnetic field $P_0 = P_r$ or $P_0 = P_v$ if only rings λ_r and λ_v are considered, which originate from the same ring λ_0.

Hence in applying the method of coïncidences the simplest procedure is to consider the ring formed by superposition of two other rings, once as a ring λ_v derived from a smaller ring λ_0, and again as a ring λ_r derived from a larger one λ_0.

By measuring three rings viz. the one due to the coïncidence of the rings λ_r and λ_v (diameter $x_c = x_r = x_v$), further the larger ring with diameter x_0 and finally the smaller one with diameter x'_0, the result may be found by the simple formulae:

$$\lambda_r = \lambda_0 \left(1 + \frac{x_0^2}{8} - \frac{x_c^2}{8}\right)$$

and

$$\lambda_v = \lambda_0 \left(1 + \frac{x_0'^2}{8} - \frac{x_c^2}{8}\right).$$

3. Using an étalon, with an interval of nearly 5 mm. between the plates, I have made by means of the method of coïncidences some measurements of the magnetic resolution of the yellow mercury lines 5791 and 5770.

The system of rings was formed in the focal plane of a small achromatic lens of 18 m.m. aperture and of 12 cm. focus. This focal plane coincided exactly with the plane of the slit of a one-prism spectroscope. The width of the slit was so far reduced that the rings of the two yellow mercury lines could be observed separately. Reproductions of negatives (somewhat enlarged) are given on the Plate, the first with field off; the second showing the first coincidence (superposition of rings λ_r and λ_v); the third gives the second coincidence, the rings λ_r and λ_v being now in coincidence with λ_0. The plate refers to coincidences for 5770; negatives showing the coincidences for 5791 however scarcely present any difference with those now given.

By measurements on half a dozen of negatives concerning the first coincidence, the result was obtained that a separation equal to 0.166 Ångström units for line 5770, corresponds to a separation of line 5791 towards the red of 0.160 A. U., towards the violet of 0.177 A. U.

Now a separation of 0.166 A. U. corresponds, according to the data given in § 6 of my paper cited in § 1 above sub [1]), to a strength of field of 9130 Gauss.

Considering as the object of the investigation the determination of the numerical value of the asymmetry we infer from the given data that it is equal to 0.017 A. U. A discussion of the systematic errors of observations to be feared, shows that the values 0.015 A. U. and 0.019 A. U. are yet possible, that however the values 0.011 A. U. and 0.023 A. U. are very improbable.

Some measurements made by means of the method of diameters tend to show that the accuracy of results obtained by that method is somewhat superior to that now found.

The accuracy obtained is however in excellent accordance which what might be expected from data given by Fabry and Perot [1]) if applied to our case

By our experiments with the method of silvered plates two points are clearly shown viz. first that the positive results concerning asymmetrical resolution in strong fields obtained on a former occasion by Rowland's method have a real significance, secondly that also in lower fields the asymmetry remains and has an amount such as to be expected, if strength of field and asymmetry are nearly proportional.

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[1]) Fabry et Perot. Ann. de Chim. et de Phys. Janvier 1902.

Determination of the total charge of the electrons.

4. Taking for granted the existence and also the nature of the asymmetrical resolution as being in accordance with VOIGT's theory, it certainly is extremely interesting to interprete the result in the language of electronic theory.

LORENTZ [1]) has deduced VOIGT's equations from the theory of electrons or more accurately expressed he gives a system of equations which come to the same thing as those of VOIGT.

Let H be the intensity of the magnetic field, λ the wavelength, $d\lambda_1$ and $d\lambda_2$ the differences of wavelengths between the middle component and those towards violet and red, V the velocity of light in the aether, and $\frac{e}{m}$ the well-known ratio of charge and mass of the electron, then according to LORENTZ:

$$\frac{e}{m} = \frac{4\pi V}{H\lambda^2} \sqrt{d\lambda_1 \cdot d\lambda_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For $d\lambda_1 = d\lambda_2$ this formula changes into the equation, which first enabled us to determine $\frac{e}{m}$. This ratio is found in electromagnetic units.

If N denote the number of molecules per unit volume, one electron vibrating in each molecule, we have also according to LORENTZ

$$Ne = \frac{H}{2\lambda V} \frac{d\lambda_1 - d\lambda_2}{\sqrt{d\lambda_1 \cdot d\lambda_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These formulae were already communicated by GEHRCKE and VON BAEYER [2]).

My own observations concerning asymmetry (§ 5 of my paper cited [3]) and § 3 above) seem at first sight to be in contradiction with this formula. One of my results being that the asymmetry varies with strength of field, according to (2) Ne must vary also, because H and $\sqrt{d\lambda_1 \cdot d\lambda_2}$ change in nearly the same ratio. Now an increase of Ne, or of the number of radiating particles per unit volume, must manifest itself in the radiating power of the vacuum tube. An inspection of Plate II (paper cited sub [3])) shows that in my experiments the intensity of the light of the components really has been a maximum in the strongest part of the field. We must therefore conclude that the circumstances of the luminous mercury vapour in the Geissler tube were slightly different in the various parts of the non-uniform magnetic field.

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[1]) LORENTZ. Rapports présentés au congrès international de physique 1900.

[2]) GEHRCKE u. v. BAEYER. Verhandl. deutsch physik. Gesellsch. 7. p. 401. 1906.

[3]) ZEEMAN. These Proceedings 30 November 1907.

39*

It therefore seems probable to accept with Prof. VOIGT [1]) that the change of value of the assymmetry is due to differences in the circumstances of the radiating vapour.

5. The following table embodies the result of the calculations according to (1) and (2) of my observations concerning line 5791.

MERCURY LINE 5791.

$\frac{e}{m}$	Ne	Mean resolution 5770	H
1.92×10^7	8.10×10^{-4}	0.532 A. E.	29220
1.92	6.24	0.440 „	24140
1.90	5.97	0.399 „	21910
1.87	5.03	0.328 „	18020
1.87	4.33	0.270 „	14800
(2.07	4.58	0.166 „	9130)

The last line in this table refers to the observations recorded in § 1 of this paper.

Dividing the numbers of the second column by those of the first one we infer that the vibrating mass (probably wholly electromagnetic) only amounts from $4 \cdot 10^{-11}$ to 2.10^{-11} gram per cm^3.

Accepting J. J. THOMSON's value of e viz. $1{,}1.10^{-20}$ electromagnetic units, we may find the number N. The number of electrons per unit volume causing the radiation of the mercury line 5791 in a vacuum tube, appears then in the circumstances of our experiments and according to the magnetic force to lay between 8.10^{16} and 4.10^{16} per cm^3.

In these experiments the temperature of the vacuum tube may be taken as somewhere between 100° and 120°. According to HERTZ the pressures of mercury vapour corresponding to these temperatures are 0.29 resp. 0.78 m.m. From these facts in connection with other well known data we may conclude that the number of electrons participating in the emission of line 5791 is of the same order of magnitude as the number of atoms present.

There seems to be no obstacle in accepting this result and the hypothesis that all atoms participate simultaneously in the emission of light might even seem the most natural. It is however of some

[1]) VOIGT. Physik. Zeitschr. 9. S. 120. 1908.

interest to compare with this result some consequences issuing from work done in the Amsterdam laboratory by HALLO on the magnetic rotation of the plane of polarisation in sodium vapour [1]), and by GEEST on magnetic double refraction in the same substance [2]), and from one of the results of JEAN BECQUEREL [3]) in his remarkable experiments concerning the behaviour of tysonite and other crystals at low temperature and in a magnetic field.

These physicists come to the conclusion that in the substances they have experimented on, only a small part of the atoms are participating simultaneously in the emission or absorption phenomena.

Of course there is not the least improbability in accepting that in a Geissler tube the circumstances are quite different, and to admit that in a vacuum tube the number of atoms vibrating at a given instant is very large.

Asymmetries of Wolframium and Molybdenum lines.
Observations of Mr. JACK.

6. Not only the lines of mercury and iron, which I investigated, but also those of other substances give in the magnetic field asymmetrical triplets. Some examples of very pronounced asymmetries, have been met with by Mr. JACK in the physical laboratory at Göttingen, and I am indebted to the kindness of Prof. VOIGT in being able to communicate these here. In the annexed table the wavelengths are given in ÅNGSTROM units, the separations however in m.m. as measured on the plates. For a knowledge of the relative asymmetry this is sufficient.

With some lines the asymmetry is reversed, the component towards the red being at a larger distance. According to the remarks of Mr. JACK it is not excluded however that in these cases the structure of the lines is not quite simple.

The intensities given can only have a relative value according to the results of my paper in these Proceedings of October 1907.

Observation parallel to the lines of force.

7. In a direction parallel to the magnetic force the two components of the doublet must be placed, according to the elementary theory, symmetrically relatively to the unmodified line. It seemed rather superfluous to test this point. However at the very outset

1) HALLO. Thesis, Amsterdam 1902. Arch. Néerl. (2) T. **10** p. 148. 1905.

2) GEEST. Thesis, Amsterdam 1904 Arch. Néerl. (2). T. 10, p. 291, 1905.

3) See especially JEAN BECQUEREL. Influence des variations de Température sur la dispersion. Le Radium. 1907.

Substance	Wave-length (λ)	Separation in mm. [— tow. violet / + „ red]	Intensity	Substance	Wave-length (λ)	Separation in mm [— tow violet / + „ red]	Intensity
Wolframium	2488.89	— .1474 0 + .1155	4 3 4	Wolframium	2856.20	— .1559 0 + .1375	3 1 3
	2522.14	— .1458 0 + 1172	3 2 3		3049.80	— .2892 0 + .2519	6 2 6
	2555.23	— .1524 0 + .1140	3 2 3		3311.53	— .1590 0 + .1814	8 10 8
	2580.63	— .1281 0 + .1012	3 2 3		3361.25	— .1239 0 + .1394	3 4 3
	2606.50	— .1487 0 + .1553	3 2 3		*3373.88	— .0780 0 + .0923	6 6 4
	2633.24	— .1353 0 + .1010	3 1 3	Shade on violet side of violet component, but no other line to give component.	*3413.09	— .0844 0 + .1080	6 6 6
	2697.81	— .1695 0 + .1498	5 3 5		*3429.79	— .0687 0 + .0837	6 6 3
	2774.12	— .1769 0 + .1332	4 2 4		3448.96	— 0770 0 + .0879	3 3 1
	2774.60	— .1530 0 + .1364	4 2 4		4022.27	— .2324 0 + .2032	3 8 3
	2792.85	— .1720 } 889 — .0831 } 0 + .0828 } 759 + .1586 }	2 2 2 2 2		4298.55	— 5339 — .1235 0 + .1242 + .4561	1 2 2 2 1
				Molybdene	2672.93	— .2224 0 + .1674	10 5 10

of my experiments in this direction I made an observation which seemed irreconcilable with a symmetrical position of the components of the doublet.

Looking at the doublets of the lines 5791 and 5770, which were very brilliant, I observed a narrow and extremely weak line between the components of the two lines. This weak line seemed with 5770 precisely midway between the components, with 5791, however it seemed to be displaced somewhat towards the red.

These weak lines evidently are due to reflection of light, radiating nearly at right angles to the direction of the magnetic force, from the inner surface of the capillary of the Geissler tube. LOHMANN [1]) investigating the neon lines, observed a similar, but in his case entirely symmetrical perturbation. I found the weak line to be linearly polarized, as was to be expected.

The whole image, apart then from the ratio of intensities and the character of the polarization, strikingly resembles the type of effect observed at right angles to the magnetic force. No good photographs showing the extremely weak line at the same time with the two components of the doublet were obtained.

I therefore tried to bring into the field of view the unmodified line at the same time with the doublet. It is well known that the use of a spectrum of comparison in measurements where a high degree of precision is wanted, is not without serious objections. KAYSER [2]) therefore recommends as the most suitable method to produce the lines necessary for comparison in the source itself. In our case this is naturally out of question.

The sidelong displacement, which the luminous line in the vacuum tube undergoes by the action of the field, makes it already impossible, even if the position of the vacuum tube remains unchanged, accurately to compare a negative taken with the field off, with one taken when the field is on.

The best manner of procedure in the given circumstances therefore seemed to reflect into the spectroscope by means of a semi-silvered mirror the light of a separate vacuum tube placed sideways and to analyse this light simultaneously with that of the tube between the poles. However also this comparison succeeded only incompletely in view of the extreme accuracy wanted. In some comparisons the line of the unmodified source seemed to be in a symmetrical position for line 5770 as well as for 5791. I hesitate however to attach even a very moderate value to this result. The experiments however forcibly suggested the question :

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[1]) LOHMANN, Beiträge zur Kenntniss des ZEEMAN-Phänomens. Dissertation. Halle a. d S. S. 62. 1907. Zeitschr. f. Wissensch. Photographie. Band 6. Heft 1 u. 2. 1908.

[2]) KAYSER, Handbuch der Spectroscopie. Band I. p. 732.

Has the middle line of a triplet the same wavelength as the unmodified line?

9. The change of wavelength here contemplated undoubtedly must be extremely small, for no one of the physicists occupied with the radiation phenomena in a magnetic field has, to my knowledge, come across phenomena which decide the question put above this paragraph.

Some observations made with an echelon spectroscope have given me evidence, that different spectral lines and among these the mercury lines undergo in very strong fields displacements of the order of 6 or 10 thousandth parts of an Angström unit, in most cases towards the violet. The matter seems of sufficient interest to be treated in a separate paper, which I hope to give rather soon.

Physics. — *"Change of wavelength of the middle line of triplets."* (First Part). By Prof. P. ZEEMAN.

1. In dealing with radiation in a magnetic field it has been tacitly assumed by all experimentalists I know of, that the middle line of triplets or of other symmetrical separation figures occupies the same position in the spectrum as the unmodified line. During a rather detailed investigation of the asymmetrical separation shown by some lines (see the paper immediately preceding) experiments on the light emitted in the direction of the magnetic force showed that symmetry was not always present where it was expected.

The interest attaching to the encountered anomaly suggested the question whether the original line is displaced during magnetisation. The following paper gives sufficient evidence to assert that such is the case. The asymmetrical position of the very weak line observed between the components of the doublet of line 5791 (see § 8 of the paper immediately preceding) is not explained however by this displacement. The contrary is the case. The theoretical interest of the subject is probably intimately connected with the existence of couplings between vibrations parallel and perpendicular to the field [1]).

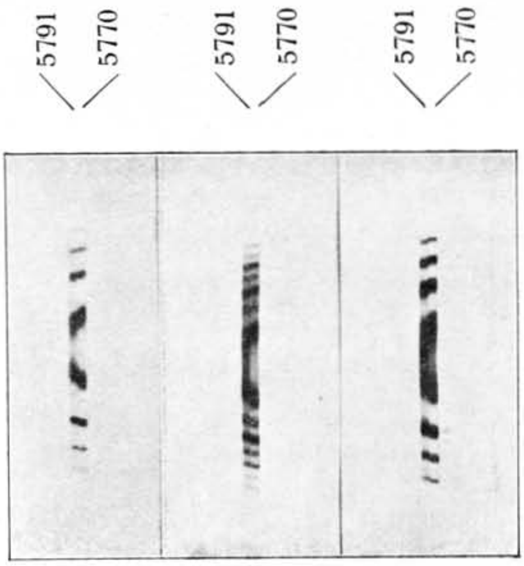
2. For the further discussion I will recapitulate here very briefly the formulae [2]) giving for MICHELSON's echelon grating the angular

[1]) Cf. however VOIGT. Annalen d. Phys Bd. **24**, p. 195, 1907.

[2]) MICHELSON. Journal de Physique, (3), Vol. 8, p. 305, 1899.

Fürst B. GALITZIN. Zur Theorie des Stufenspectroscops. Bull. de l'Acad. Imp. des Sciences St. Pétersbourg 1905 (5) T. 23. N⁰. 1 et 2.

P. ZEEMAN. New observations concerning asymmetrical triplets.



field off. 1st coïncidence 2d coïncidence.

dispersion and the distance of adjacent orders of the same line.

Let λ denote the wavelength of the light considered, μ being the index of refraction, θ the angle of diffraction, t the thickness of the glassplates, s the width of the steps of the echelon.

Then in the case of normal transmission:

$$\frac{d\theta}{d\lambda} = \frac{t}{s.\lambda}\left((\mu-1) - \lambda \, . \frac{d\mu}{d\lambda}\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

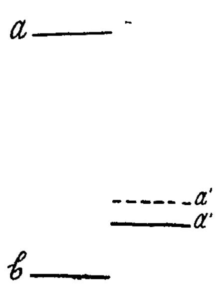
Δ being the distance of adjacent orders, we have further·

$$\Delta = \frac{\lambda^2}{t\left((\mu-1) - \lambda \, . \frac{d\mu}{d\lambda}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

3. The most simple hypothesis that could be made was, that of the two lines under consideration only 5791, which exhibits asymmetrical separation and not line 5770, would show a displacement of the middle line.

In order to test this hypothesis I used the echelon spectroscope in such a manner that different lines could come *simultaneously* under comparison. The ordinary manner of using the echelon only permits the examination of one line at the same time or at least only of lines which differ by a small fraction of an ÅNGSTRÖM unit.

We may however place the steps of the echelon in a horizontal position, the slit of the echelon collimator being also horizontal, thus rotating these parts through 90° from the position commonly used. The slit of the auxiliary spectroscope may remain vertical. This arrangement, which is principally that of NEWTON's crossed prisms or that of GEHRCKE's "Interferenzpunkte", has the advantage of showing simultaneously the behaviour of different lines. To every spectral line correspond small horizontal lines, the length of which is determined by the width of the slit of the auxiliary spectroscope. It depends upon the position of the echelon whether two or one of the orders of a line will be visible. Fig. 1 represents that part of the field of view, which relates to the yellow mercury lines. The lines a and b represent the successive orders of line 5770, a' the only visible order of line 5791, supposing that the small vacuum tube, charged with mercury vapour, is out of the field.



During the establishment of the magnetic field the well known components are seen moving upwards and downwards. Moreover any change in. wavelength of line 5791, that of the other line remaining constant, must manifest itself in a relative displacement, determined by equations (1) and (2).

Taking two negatives with the smallest possible interval of time any change of position of line a' can be made out by measurement. A small displacement of line a' to a position a'', was noticed.

4. The annexed table gives in detail the results of measurements on negatives, taken according to the method of § 3 on different days and under somewhat different circumstances.

The echelon used, was described on a former occasion [1]; it has 30 plates 7,8 m.m. thick, the depth of the steps being 1 m.m.

The distance of line a to line b is measured in m.m. and indicated as distance $a—b$ and so on. H denotes the strength of field in Gauss.

Plate Nr.	Field on — Distance			H in Gauss	Field off — Distance		
	$a—b$	$a—a'$	$b-a'$		$a—b$	$a—a'$	$b—a'$
135	1.215	0 896	0.319	7830	1.219	0.898	0.321
139	1.200	0.891	0 309	10920	1.208	0.898	0.310
140	1.214	0.882	0.332	8580	1.221	0.905	0.316
141	1 147	0.861	0.286	7700	1.150	0.867	0.283
142	1.140	0 849	0.291	7180	1.147	0.862	0.285
144	1.140	0.855	0.285	15120	1.145	0.872	0.273
146	1 136	0.819	0.307	20340	1 143	0.861	0.282
150	1.093	0.746	0.347	23470	1.116	0.818	0.298

It appears from this table that the position of line a' relatively to a and b is changed by magnetization and that the displacement increases with the field intensity.

It is not less clear however that the observed displacement is due not only to a change of wavelength of line 5791, but to a superposition of change of frequency of the two lines considered. Indeed the distance $a—b$, i. e. the distance of the adjacent orders of line 5771, is always smaller in the first section of the table than in the second one. Hence we must conclude to a change of wavelength of

[1] ZEEMAN. These Proceedings 30 November 1901.

line 5771. The amount can easily be given in A. U. The change of $a-b$ amounts to 0,023 m.m. in the strongest field of 23470 Gauss. *Half* this amount determines the change of wavelength. It becomes 0,007 A. U., the distance of two orders of the echelon = 1.116 m.m. corresponding to 0,689 A. U.

5. The simplicity of the results obtained by means of the method of § 4 is considerably diminished by the fact, that line 5770 undergoes a change of wavelength as well as line 5791. The sensibility of the method for discovering *relative* changes of wavelength is very clearly seen by a comparison of the two columns under $a-a'$.

In order however to be sure of a simple interpretation of results and also on account of gain in the intensity of the light I returned, the reality of a change of wave-length being now rather evident, to the arrangement as it is most commonly used. The slit of the auxiliary spectroscope is then parallel to that of the echelon.

The results obtained for the yellow mercury lines are given in the table.

Plate Nr	λ	Distance of orders, field off in m.m.	Distance of orders, field on in m.m.	H	δ_r in A. U.
160*a*	5770	1.081	1.067	20100	0.004
160*b*	5791	1.061	1.044	20100	0.006
160*c*	5791	1.058	1.050	8900	0.003
161*a*	5770	1 110	1.089	23800	0.007
161*b*	5770	1 110	1.106	9000	0.001
164	5770	0.855	0.834	24400	0.009
165*a*	5791	0 856	0.847	13750	0.004
165*b*	5770	0.859	0.843	16450	0.007

The observations recorded in the last three columns have been taken with other orders of the echelon. δ_r gives in A. U. the change of wavelength by magnetization. The largest change observed is one of 0.009 A. U. recorded on plate 164 for a field of 24400 Gauss.

The evidence from these experiments tends to confirm those obtained in § 4.

The separate numbers show some discrepancies which needs a discussion, which will be given later on. Before proceeding further, I think it appropriate however to call attention to the fact that the change of wavelength of the middle line of a triplet seems not to be confined to the light emitted by a Geissler tube.

During the writing of this paper my attention was arrested by a passage in a Thesis of W. HARTMANN [1]:

"Es mag schon an dieser Stelle erwähnt werden, dass der Abstand der Ordnungen beim Einschalten des Magnetfeldes sich mehrfach änderte, und zwar im allgemeinen mit wachsender Feldstärke kleiner wurde.

Dieser Änderung würde rein ausserlich betrachtet eine Verkurzung der Wellenlänge entsprechen, doch konnte eine wirkliche Gesetzmässigkeit nicht constatirt werden."

The observations of HARTMANN were made by means of an echelon spectroscope, the source of light being the self-induction spark in vacuum after MICHELSON's arrangement. HARTMANN's negatives concerning copper, iron, gold and chrominum were made with fields ranging from 8000 to 12000 Gauss. Perhaps the author would have expressed his opinion with less reserve, if he had operated with stronger fields, in which case the phenomenon is more definite. In the light however of our own observations there seems to be sufficient evidence to conclude, that also the middle lines of the triplets of other metals undergo the kind of change existing in the case of mercury.

Physics. — *"The influence of temperature and magnetisation on selective absorption spectra"*. By Prof. H. E. J. G. DU BOIS and G. J. ELIAS. (Communication from the Bosscha-Laboratory.)

§ 1. As soon as the unequalled paramagnetic properties of the compounds of so-called rare earth-metals had been demonstrated [2]), attention was drawn to the fact that most likely also the magneto-optic phenomena would show important peculiarities; this was done in the following words: "La polarisation rotatoire magnétique "a le signe positif ou négatif pour les composés des différents "métaux de cette série, comme d'ailleurs pour ceux de la série "du fer. Je n'ai pas pu constater jusqu'ici un effet particulier de "l'aimantation sur le spectre d'absorption très caractéristique d'une

[1]) WALTHER HARTMANN. Das ZEEMAN-Phaenomen im sichtbaren Spectrum von Kupfer, Eisen, Gold und Chrom. Dissertation, Halle a. d. S. 1907. p. 10.

[2]) H. DU BOIS & O. LIEBKNECHT, Ann. d. Physik (4) 1 p. 196, 1900; St. MEIJER, Ann. d. Physik (4) 1 p. 664, 1900.

"solution d'erbium fortement paramagnétique; d'ailleurs M. ZEEMAN "lui-même l'avait déjà cherché en vain pour le spectre d'émission "de l'erbine chauffée. Des expériences sont en préparation pour déter-"miner la rotation dans les raies d'absorption mêmes et aux alentours "immédiats". [1])

After results had been published by SCHMAUSS, BATES and WOOD, the agreement of which left much to be desired, the experiments in question were taken up again in this laboratory in 1906, when one of us actually obtained a very peculiar dispersion curve of the magnetic rotation within and near a narrow region of absorption. [2]). At the same time such determinations proved to be subject to many difficulties, which could only be surmounted by means of specially adapted apparatus; moreover, simultaneous measurements of other optical properties of the absorbing substances are desirable for completeness' sake; this more extensive investigation is now being continued with such improved apparatus.

In the first negative experiments referred to a direct influence of magnetisation in the form of a displacement of the dimly defined absorption bands of an aqueous erbium-solution, in other words a ZEEMAN-effect in the usual sense, could not be observed, the grating used, however, being the same as that used in the present experiments. Naturally the observation of the last-mentioned effect is much simpler and more easily feasible than an adequate and trustworthy measurement of the rotation. However, the relation between those two modes of looking at one and the same phenomenon is so close that either remains undetermined without a rather complete knowledge of the other.

Mr. JEAN BECQUEREL Jr. [3]) resumed such an investigation on the narrower and more sharply defined absorption bands of some exceedingly rare and small crystal fragments, which we had not at our disposal: xenotime, tysonite, parisite and others, the spectra of which had been formerly determined by HENRI BECQUEREL Sr. [4]). Besides, the influence, already more or less known, of the temperature on such spectra was more fully investigated. The important results obtained may be assumed to be known.

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[1]) H. DU BOIS, Rapp Congr de Phys, **2** p. 499, Paris 1900; Ann. d. Phys. (4) **7** p. 944, 1902.

[2]) G. J. ELIAS, Physik. Zeitschr. **7** p. 931, 1906 (chloride of erbium).

[3]) J. BECQUEREL, Compt. Rend. **142** pp. 775, 874, 1144, 1906. **143** pp 769, 890, 962, 1133, 1906. **144** pp. 132, 420, 592, 682, 1032, 1336, 1907. **145** pp. 413, 795, 916, 1150, 1412. Also Physik. Zeitschr. **8** pp. 632, 929, 1907.

[4]) H. BECQUEREL, Ann. Chim. & phys. (6) **14** p. 170, 1888.

§ 2. In a former attempt to classify all elements according to their magnetic properties and those of their compounds, the following remarks were made:

"Les éléments nouvellement reconnus: hélium, argon [1]), néon, "krypton, xénon, n'ont pas encore été déterminés; il n'est guère "probable qu'ils soient autres que diamagnétiques. On peut classer "63 autres éléments, dont 37 diamagnétiques, 22 paramagnétiques, "4 ferromagnétiques à la température ambiante; tandis qu'en 7 cas "(Be, Mg, Sc, Nb, La, Ta, Th) la classification nous paraît encore "plus ou moins douteuse. Dans le système naturel à masses atomi-"ques croissantes, on peut distinguer 7 séries d'éléments paramag-"nétiques consécutifs, qui les comprennent tous, le signe de l'élément "ouvrant chaque série étant seul encore incertain; les séries d'ordre "pair sont moins prononcées au point de vue paramagnétique que "celles à numéro impair" [2]).

These last-mentioned uneven series are:

1) *O*.

3) *Sc* (?), *Ti*, *V*, *Cr*, *Mn*, *Fe*, *Co*, *Ni*, *Cu* [3])

5) *La* (?), *Ce*, *Pr*, *Nd*, *Sa*, *Eu*, *Gd*, *Tb*, *Dp*, *Er*, *Yb*.

7) *Ra* (?), *Th* (?), *U*.

Now the anorganic compounds [4]) which chiefly absorb light selectively, evidently belong to these paramagnetic series; this connection is so remarkable that it can hardly be an accidental one.

From this profuse supply of material only a few samples could be chosen; we have thought that we ought to extend the investigation in the first place to matter which was comparatively easily to be had in larger pieces; among others to some coloured gems, which are to be considered as dilute solid solutions, to certain micro-crystalline salts, but also to amorphous solidified molten matter and to glassy solid solutions in an amorphous substratum, e. g. borax or glass. The crystalline structure gives rise to complications which render the phenomena very intricate, though they are most interesting in themselves. All this yields a rather extensive material of observation, possibly of importance in connection with molecular theories on solid and liquid substances.

With the available cryomagnetic arrangement we could expose matter in liquid air to a strong magnetic field; in many cases we

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1) Recently confirmed by P. Tänzler, Ann. d. Physik. (4) **24** p. 931, 1907.

2) Rapp. Congr d. Phys. **2** p. 487, Paris 1900.

3) O Liebknecht & A. P. Wills, Ann. d. Physik (4) **1** p. 186, 1900.

4) Kayser, Handb. d. Spectroscopie **3**; Leipzig 1905.

could thus study the *simultaneous* influence of the two factors, temperature and magnetisation.

§ 3. For the observation or measurement of the absorption spectra we used, besides a hand-spectroscope, according to circumstances:

1. A RAPS[1]) spectrometer with a prism of heavy flint glass; dispersion C—F about 7.5°.

2. An autocollimator made according to our directions by the firm of C. ZEISS, a description of which will soon appear; dispersion C—F, about 25°.

3. The concave grating presented by ROWLAND to Berlin university, and kindly placed at our disposal by Prof. RUBENS; the radius is about 4 m., the number of lines 5684 per cm. The arrangement was as usual in a right-angled triangle with movable constant hypotenuse; a unit of the scale in the spectrum of the first order (half mm). corresponded to 0.23 $\mu\mu$.

The calibration of these instruments in wave-lengths was made by means of lines of hydrogen, helium, potassium, and those of a mercury arc lamp between the limits 434 and 770 $\mu\mu$.

The sources of light were according to the required strength of illumination 1. a "Nernst lamp", 2. a "Lilliput" arc lamp (2 Amp.), 3. an arc lamp with horizontally directed and slowly rotating positive-carbon (25 Amp.), 4. sunlight.

The polar pieces of the large ring-electromagnet had slitted or rectangular openings, to which attention was paid in the dioptric determination of the path of the rays along the field's axis.

The cryomagnetic arrangement was that used before[2]); the level of the liquid air was kept at the lower edge of the openings; the sample could be enclosed in a thick copper frame in order to let it have a temperature as uniform as possible.

Outside the field a small vacuum vessel was used with unsilvered strips for observation; the variations of temperature had to take place as gradually as possible to prevent the samples from bursting and cracking.

The use of higher temperatures up to about 200° does not give rise to any difficulty with this apparatus which may also be arranged pyromagnetically; we hope to revert to this question later on.

Results.

§ 4. We reserve the detailed description of the measured absorption

1) A. RAPS, Zeitschr. f. Instr. Kunde, **7** p. 269, 1887.

2) H. DU BOIS & A. P. WILLS, Verh. D. Phys. Ges. **1** p. 169, 1899. — F. C. BLAKE, ibid. **9** p. 295, 1907.

spectra for a future occasion, and here confine ourselves to the principal characteristics. The wave-lengths have been expressed in $\mu\mu$ with an uncertainty of no more than 0,3 $\mu\mu$.

First series. The well-known rather narrow absorption bands of oxygen (among others 476,7—477,6) only played a secondary and inconvenient part in this preliminary investigation; for when the samples under observation are immersed in liquid air their absorption spectra become impure, which should be duly taken into account. We have not yet succeeded in observing the absorption spectrum of the strongly paramagnetic liquid oxygen in a field of sufficient intensity. O_3, NO_2 and NO_3 are also of importance [1]).

§ 5. Third series. Here chromium is of special importance. It derives its name from its coloured compounds, which mostly show dichroism and the well-known transmutation of colour with change of temperature. We examined:

Chromium alum $[Cr\,K\,(SO_4)_2]$; diluted green aqueous solution. At 18° light band 662,7—672,5; fainter band 688,1—726,4. A plate of alum 2 m.m. thick exhibited pretty narrow bands in liquid air, some of which were slightly affected by magnetism.

Chromium-potassium oxalate $[Cr_2\,K_6\,(C_2\,O_4)_6 + 6\,H_2\,O]$; strongly dichroitic (red-blue) small monoclinic crystals, which were cemented on a covering glass and ground to a thickness of about $^1/_2$ m.m.

Plane of polarisation || long sides: at 18° a bright band 698,1—703,7; at — 193° it lay 696,4—701,4.

Plane of polarisation ⊥ long sides: at 18° bright band 697,5—703,5; at — 193° it lay 696,4—701,4.

An aqueous solution exhibited at 18° a broad band 693,2—702,3, the maximum of which lay at 695,4—699,3; moreover a very faint band 708,4—711,0.

A solution in glycerin had this broad band at 18° from 694,9-699,4. At a temperature higher than that of liquid air (roughly estimated at — 130°): faint band 659,3—664,9 (possibly not simple), faint band 669,0—671,2, stronger band 674,7—676,8; halfshade limit at 681,8. Very strong band 694,8—698,1, shade to 700,3; beginning of region of absorption 706,0 [2]).

"*Chromium borax*" obtained by melting together 5—15 °/₀ chromium fluorite with anhydrous borax, in the way of the borax-pearls used in analytical chemistry; ground, polished and varnished in order to

[1]) E. Warburg & G. Leithäuser. Ann. d. Physik (4) **23**, p. 209, 1907.

[2]) E. Wiedemann, Wied. Ann. **5** p. 515, 1878. W. Lapraik, Journ. f. prakt. Chemie (2) **47** p. 307, 1893. G. B. Rizzo, Nuov. Cim. (3) **35** p. 132, 1894.

prevent decomposition on exposure to air: smaragdine amorphous plates about 3 m.m. thick.

At 18 light band 673,7—681,4; vague indistinct band 695,3—736,9. At — 193° light band 672,6—680,8; dimly defined band 692,3—747,6.

Natural emerald. $[Be_3 Al_2 (Si O_3)_6]$; hexagonal, coloured by a few permilles Cr_2O_3; sensibly dichroitic (grassgreen-seagreen). Worthless light green specimen not quite transparent, provided at the laboratory with parallel facets, thickness 6 mm.

In the *ordinary* spectrum at 18° rather strong band 679,0—680,7; somewhat stronger band 682,4—685,0; at — 193° strong band 678,2—679,5, and still stronger band 681,8—683,4.

In the extraordinary spectrum the bands were much paler but at the same places, their relative widths being interchanged.

§ 6. *Ruby.* $[Al_2 O_3]$; rhombohedral, solid solution of a little Cr_2O_3; dichroitic (purplered-brickred). By the kindness of Mr. M. A. WOLFF—DE BEER, manager of the Amsterdam diamond factory, several natural and several artificial rubies were placed at our disposal. The last-mentioned rough material is imported from Paris in the form of cones [1]); by cleaving them we got pieces of about uniform colour and crystallographic orientation, as is to be observed by means of the dichroscope. With carborundum a square plate (7 × 7 × 3 mm.) was ground and polished at the laboratory, a side of which contained the direction of the axis; most of the experiments were made with it. There is no reason to suppose that natural ruby would show other properties than the artificial material.

This stone absorbs green and yellow light. In the above investigations of J. BECQUEREL we only found it briefly stated that "a group lying between 657 and 676 disappears in liquid air, that the band at 697 becomes thinner and the band at 705 broader and intenser than at ordinary temperature;" no mention is made of any magnetic influence [2]).

Further ruby was more closely investigated by MIETHE, who found the two principal absorption bands at 694 and 696, with a breadth of about 0.4 $\mu\mu$, besides 6 bands of less importance. Moreover he described the remarkable fluorescence spectrum; the latter we have no further examined, as Prof. MIETHE intended to proceed with his experiments on this subject [3]).

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1) M. DUBOIN, Compt. Rend. **134** p. 840, 1902; A. VERNEUIL, ibid. **135** p. 791, 1902.

2) J. BECQUEREL. Physik. Zeitschr. **8** p. 932, 1907 (Sept.).

3) A. MIETHE, Verh. D. physik. Ges. **9** p. 715, 1907 (Nov.)

§ 7. With the spectrometer we found in the *ordinary* spectrum of ruby: At 18° strong band 692,4—692,6; very strong band 693.9—694,2; at —193° strong line 691,7; stronger line 693,2. So with decrease of temperature a displacement of 0.7 $\mu\mu$ takes place towards violet, the mutual distance of the double line retaining the value 1.5 $\mu\mu$, however, as it is also found between the centre-lines of the two bands. In the extraordinary spectrum, at 18° band 692,5, fainter band 694,1; at —193° faint line 691,7, fainter line 693,2. In both spectra about 8 more bands of less importance, which cannot all be discussed here.

In the extraordinary grating spectrum the bands have faded away so much that an exact observation is impossible; hence only the *ordinary* spectrum was observed here. Though the bands are not sharply defined, they were estimated at about the same breadth by the two observers, viz. at 18°: band 692.5 at 0.23, band 694,1 at 0.33 $\mu\mu$. In a longitudinal magnetic field, estimated at about 30 kilogauss the breadths amounted to 0,37 resp. 0.49 $\mu\mu$ in a situation slightly shifted towards violet; the increase in breadth was therefore 0,14 and 0.16 $\mu\mu$ respectively.

With unpolarised light the phenomena were pretty much the same as the photo shows; it was obtained on a "LUMIÈRE B-plate" sensitized for red light with alizarin blue and nigrosin with an exposition of 20 minutes: 1. indicates the position of the ruby bands in the solar spectrum; 2. those in the arc-lamp spectrum outside the field; 3. the same in the field. The apparent broadening is evidently caused by a doublet of vaguely outlined bands, as sufficiently appears from the connection with what follows.

If the ruby was cooled down to about —193°, band 692,5 appeared to contract to a breadth of about 0,06, band 694,1 to 0,08 $\mu\mu$. In fact they have then become *absorption lines*, which though thicker than FRAUNHOFER lines, are yet no broader than e.g. those of a dense sodium vapour; by this gradual transition any distinction between absorption bands and lines vanishes.

In the field an ordinary ZEEMAN-effect was now observed as follows: line 692,5 resolves into two components at a mean distance of 0,25 $\mu\mu$, the space between somewhat less bright than the spectral background; the components of line 694,1 had a distance of 0,35 $\mu\mu$ with a somewhat darker space between, possibly due to the intermediate lines of a quadruplet; for both a slight displacement towards violet was to be perceived. So the separation at —193° seems to be greater than the above broadening at 18°. These numerical determinations cannot yet be considered as definite or accurate.

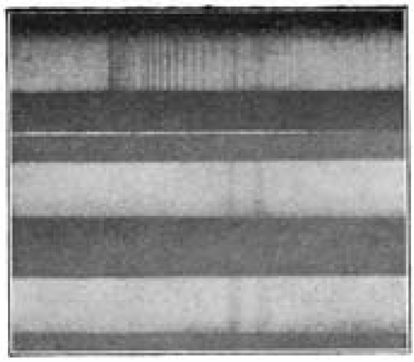
H. E. J. G. DU BOIS and G. J. ELIAS. The influence of temperature and magnetisation on selective absorption spectra.

← violet red

1

2

3



1. Ruby bands in the solar spectrum.
2. Ruby bands outside the field at 18°.
3. Ruby bands in the field at 18°.

The phenomenon was very clearly visible, and the amount of the separation is evidently pretty large. Chiefly in consequence of rime only a short observation was possible, which frustrated photographing with long exposition. Our $^1/_4$ plate proving unfit, we could not yet determine the circular polarisation; nor did our arrangements allow the observation of a transversal effect for equatorial direction of rays; we hope to be able to remove these difficulties in course of time.

§ 8. Of the other metals of the third series there exist among others also the following compounds with characteristic absorption spectra [1]: Sapphire, $CoCl_3$, $KMnO_4$, $FeCl_3$, the last mentioned with strongly negative rotation, and some compound rhodanides. Of these we examined:

Cobalt-ammonium rhodanide $[(NH_4)_2Co(CNS)_4]$ in rather diluted alcoholic solution: at 18° dimly outlined band 594—663; limit of absorption at 696. At —193° the blue solution became solid, but remained transparent, though of a paler colour; the bands become narrower as the temperature falls; at last vague bands are seen 580—586, 597—605, and 618—620; shade up to 645.

§ 9. Fifth series. This now includes, arranged according to increasing atomic weights from 140 to 175: Cerium, *Praseodymium, Neodymium, Samarium,* [*Europium*]. Gadolinium, [*Terbium, Dysprosium*], *Erbium,* Ytterbium [2]; the three placed between brackets were not yet obtainable in 1899; the compounds of the others then proved strongly paramagnetic, with a maximum for erbium. The visible absorption spectra of the metal compounds printed in italics exhibit the well-known highly selective properties; possibly an accurate investigation would yield a similar result also for the "white" compounds of Ce, Gd and Yb, perhaps only in the ultra-violet or the infra-red spectrum. We have confined ourselves for the present to compounds of neodymium and erbium, which had been used for the magnetic measurements in 1899. We wish to express our indebtedness to Prof. ROSENHEIM and Dr. R. J. MEIJER for their kind assistance in this special chemical department.

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1) Cf. J. M. HIEBENDAAL, thesis for the doctorate, Utrecht 1873.

2) Cf. R. J. MEIJER, Handb. der anorg. Chemie 3 p. 129—338, Leipzig 1906. Scandium, Yttrium and Lanthanum have lower atomic weights and diamagnetic compounds. Holmium and Thulium have not yet been sufficiently chemically determined. Further the following investigations have lately been published on the absorption spectra: W. RECH Zeitschr wiss. Photogr 3 p. 411, 1906; — HELEN SCHAEFFER, Physik. Zeitschr 7 p. 822, 1906; B. SCHAEFFERS, Dissert. Bonn 1907.

40*

Neodymium nitrate [$Nd(NO_3)_3 . 6H_2O$]. The water of crystallisation is expelled during moderate heating, and on cooling a fine rosy-red amorphous transparent mass is retained, which remains unchanged for some time. In course of time a kind of crystalline light foam deposits on the surface and the walls of the test-tube attended by absorption of water, which foam could be collected and be pressed into thin films by the aid of zapon-varnish.

The amorphous nitrate exhibited numerous bands, among which a rather well defined one at about 625 (with companion 626). At —193° it had a breadth of 0,3, in a field of about 25 kilogauss one of 0,4 $\mu\mu$. By chance we noticed that the crystalline nitrate foam at —193° exhibits much narrower bands than an amorphous layer of the same thickness; among others in the neighbourhood of *D*: 577.0, 578.7, 579.7, 582.0, 583.0; the three last of these have a breadth of about 0,15 $\mu\mu$; in the field a very slight, hardly measurable broadening seems to take place. The 5 sharply defined bands in the green between 500 and 525 behave in an analagous manner.

Neodymium magnesium nitrate [$2Nd(NO_3)_3 . 3Mg(NO_3)_2 . 24H_2O$]. Yields analogous products when treated as before.

It has a very complex absorption spectrum with a great number of narrow bands, which are very pronounced particularly at the temperature of liquid air. Particularly worth mentioning is the following, observed with a thickness of about 10 m.m. of the amorphous mass:

At 18° a region of absorption extending from 499.2—537.7, consisting of 499.2—513.9 very strong band, very badly defined towards violet; 516.9—528.2 very strong band, then shade; and 534.8 narrow faint line; 535.6—537.1 band with maxima 535.6—536.4 and 536.7—537.1.

At —193° this region extends from 498.1—526.6 and consists: of 498.1—512.6 very strong band, then bands at 513.7—514.1, 514.7 —515.3, 516.1—516.7, then shade, 517.6—518.5 band, then again shade, and 519.2—526.0 very strong band.

Another remarkable region extends at 18° from 616.9— 629.3, which consists of 616.9—618.9 faint band; 621.0—622.9 ditto; 624.8—626.9 pretty strong band; 628.3—629.3 very faint band. At —193° we see what follows: 616.7—617.6 faint band; 618.9 thin, faint line; 621.1—623.0 faint band; 624.7—625.4 band with strong fine centre; 625.9—626.4 ditto, 627.1 thin faint line; 628.6 —629.5 faint band.

In thinner layers these regions diverge even more; e. g. a rather narrow absorption region branches off from the region 560.1—587.5

on the red side, which with decrease of temperature first becomes more distinct, but then vanishes at still lower temperature. Also for this double nitrate the crystalline foam exhibited more sharply defined bands at -193°; the 5 bands in the green from 500—525 particularly the first and the fourth were broadened in the field.

Neodymium borax, obtained by melting together about 5—10 % neodymium oxide with anhydrous borax; pink amorphous mass.

With decrease of temperature the spectrum is also subjected to important modifications, but displays in general wider bands than the preceding one. Analogous in this respect are:

Neodymium glass, prepared for us by the firm of SCHOTT & Co. at Jena (V.S. 5255 and 5256), with 15 % and 20 % cerite; it contains so much neodymium that it has a pink colour, and exhibits strong selective absorption.

§ 10. *Erbium nitrate* [$Er(NO_3)_3 . 6H_2O$]. By evaporation of the solution this, too, may be obtained as an amorphous transparent mass, which has an absorption spectrum rich in narrow bands, which, however, has not yet been further examined.

Erbium magnesium nitrate [$2\,Er(NO_3)_3 . 3Mg(NO_3)_2 . 24\,H_2O$].

Treated just as the neodymium salt; yellow transparent mass; readily absorbs water of crystallisation and becomes crystalline and opaque.

Like the preceding salt it shows a very complex spectrum, of which the following groups, measured for a sample of about 10 m.m. thickness, are particularly noteworthy:

At 18°: 514.5—527.3 group of bands, consisting of three intense bands, at 516.9—517.2; 517.7—519.5; 520.5—521.7. At —193° this becomes as follows: 513.7—521.8 group of bands, consisting of: 513.8—514.0 bands; 514.8—514.9 intense band; 515.1—515.7 band; 516.3—517.2 intense band; 517.7—519.3 intense band; 520.1 faint band; 521.0—521.8 intense band.

At —193° a number of double lines in the red are of importance, the principal of which lie at 641.4 and 642.6; 643.7 and 645.6; 647.7 and 649.4. At 18° these lines have entirely vanished.

The whole visible spectrum of this salt contains about 40 bands and lines, partly very faint, which vanish for a thinner layer.

Erbium borax obtained by melting together 15—20 % erbium oxide with anhydrous borax; yellow, amorphous, transparent.

Here decrease of temperature has not so important an influence.

The most remarkable group of bands, extending at 18° from

516.7—525.6 contains three bands at 516.9—517.2; 518.0—519.1; 520.6—522.2.

At —193° this group extends from 516.2—523.9 and then exhibits three bands too, at 516.6—517.0; 517.8—519.0 (these two pretty sharply defined); 520.4—522.4 (less sharply defined, perhaps double).

Erbium glass was prepared for us by the firm SCHOTT & Co. at Jena (V.S. 5257, about the same as V.S. 3524).

It shows some narrow, not very sharply defined bands.

The well-known group of bands in the green extends at 18° from 516,8—523.0, and consists of: 517.0—517.3 band; 517.8 line; 518.4—518.9 band; 519.1 indistinct line; 520.0—520.2 band; 521.1—521.8 band.

At —193° it extends from 516.5—522.6 and shows what follows; 516.5—517.2 band; 517.9 line; 518.2—518.6 band; 519.0 line; 519.8—520.0 band; 521.0—521.2 band; 521.8—522.3 band.

Further the following lines are found at —193°:

648.9; 651.6; 655.5; 657.6, all faint. At 18° faint bands are seen at 650.0—654.0 and 656.4—661.1.

§ 11. Seventh series. We have until now investigated:

Uranyle nitrate [$UO_2(NO_3)_2 + 6H_2O$], monoclinic crystals, not sensibly dichroitic; ground plate 2 m.m. thick. In the blue the well-known bands, of which we mention: at 18° two faint undefined bands 467.5—471.6 and 484.9—488.0; at —193° strong band 467.9—469.7 (possibly not simple), shade to 470.3; strong sharply defined band 484.5—484.9, then shade; idem 485.3—485.7, shade to 486.3

Berlin, Febr. 27, 1908.

Physics. — "*On the measurement of very low temperatures.* XIX. *Derivation of the pressure coefficient of helium for the international heliumthermometer and the reduction of the readings on the heliumthermometer to the absolute scale*". Communication N°. 102[b] from the Physical Laboratory at Leiden. By Prof. H. KAMERLINGH ONNES.

§ 1. *Pressure coefficients of helium.* As the absolute zero is known with sufficient accuracy — from the Leiden observations on hydrogen may be derived $T_{0^\circ C.} = 273^\circ.10\ K.$ [1]) a value which, because it agrees with other determinations, is probably not far from the true one — we may by means of the virial coefficients B_A for helium at 0° C. and 100° C., determined in the preceding communication (102[a]), derive the pressure coefficients of helium at different densities for this range of temperature. For the pressure coefficient of the *international helium thermometer* [2]) i. e. the mean relative pressure coefficient from 0° C. to 100° C. for helium with the density belonging to the zero pressure of 1000 m.m. $\left[{}^{0^\circ\ C. - 100^\circ\ C.}\ \alpha_v\right]_i$ or for shortness ${}_i\alpha_v$, the formula

$$100 \,.\, {}_i\alpha_v = \frac{A_{A_0} \times 0.36617 + (B_{A,100^\circ C} - B_{A0^\circ C.})\,\frac{100}{76}}{A_{A_0} + B_{A,0^\circ C.}\,\frac{100}{76}} \quad . \quad . \quad . \quad (1)$$

yields

$$ {}^{He}_{\;i}\alpha_v = 0.0036613. $$

If one considers that according to table II of Comm. N°. 102[a] the isotherm of 0° gives rather large values for Obs.—Comp., then it seems that the isotherm of 20° C., where the Obs.—Comp. are only small, are more reliable for the derivation given above.

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[1]) In Comm. N°. 101[b] the value 273°.08 is found, but as will be explained in Comm. N°. 102[f], an erratum to Comm. N°. 97[b] XV, the pressure coefficient 0.0036627 for hydrogen at 1090 mM. must be restored instead of 0 0036629 which was derived in the above mentioned communication and used for a certain time. It is to be noted that the difference introduced by this recalculation is not greater than the other observational errors. The small differences between some numbers of this communication with the Dutch text are the consequence of this correction.

[2]) The scale of the hydrogen thermometer of constant volume at 1000 m m. zero pressure is generally called the scale of the normal hydrogen thermometer (this was also done in Comm. N°. 97[b]). As 0° C. and 760 m.m. are accepted as the normal state for gases, it seems to me preferable to call the scale just mentioned the scale of the *international* hydrogen thermometer. In the same way we must speak of the international helium thermometer.

Therefore I have calculated $\left[{}^{0^\circ C.-100^\circ C.}\,\alpha_v\right]_{1000\text{ mm.}}$ by means of the data for 20° C. and 100°.35 C. With neglection of the deviations from the absolute scale for the hydrogen thermometer at 20° C., $B_{A,0^\circ C.}$ was determined by means of rectilinear extrapolation. This gave

$$B_{A,\,0^\circ C.} = 0.0_8 499,$$

whence

$$A_{A_0} = 0.99950.$$

With these new data we derive from formula (1) of this section

$$^{He}{}_{1}\alpha_v = 0.0036616.$$

From the data for B_A of table II of the preceding Comm. and $T_{0^\circ C.} = 273^\circ.10\ K$ we may determine in the way mentioned in § 2 of Comm. N°. 97[b] the corrections of the readings of the helium thermometer of constant volume with a given zero pressure to the absolute scale. They have been calculated for a zero pressure of 1000 m.m. and are combined in table I where the remaining columns have the same signification as the corresponding ones of table XVI of Comm. N°. 97[b].

TABLE I. Correction of the international helium thermometer to the absolute scale.

θ	$10^3 . B'_T$	$(\Delta t)_a$	$(\Delta t)_b$
100°.00	+ 0.492		
0° (a)	+ 0.513		
0° (b)	+ 0 500		
— 103°.57	+ 0 544	+ 0°0034	— 0° 006
— 182 .75	+ 0 532	+ 0 0158	+ 0 .002
— 216 56	+ 0.463	+ 0.0252	+ 0 010

The corrections indicated with (a) are derived by means of the values of $B_{A.0^\circ C.}$ from the direct determination, for (b) we have used the value which is recalculated with B_{A20° (comp. the preceding Comm.). It is probable that on account of what has been said in the preceding section those of column (b) are the most reliable.

§ 3. *Determinations of other observers.*

For a comparison with the results of the two preceding sections we can only use the determinations of TRAVERS, SENTER and JACQUEROD.[1]) They have found:

1^{st}. for the pressure coefficient of the helium thermometer at 700 m.m. zero pressure $\left[\overset{0^\circ C-100^\circ C}{\alpha_v}\right]_{700} = 0.00366255$ which agrees with 0.0036628 for $\left[\overset{0^\circ C-100^\circ C}{\alpha_v}\right]_i$ and

2^{nd}. for the difference between the indications of the helium thermometer t_{He} and the hydrogen thermometer t_{H_2} (each of about 1000 m.m. zero pressure) at the boiling point of oxygen $(t_{H_2} - t_{He})_{-180^\circ C} = 0^\circ.10$, and at that of hydrogen $(t_{H_2} - t_{He})_{-252^\circ C} = 0^\circ.20$, which differences are so considerable that CALLENDAR[2]) concludes thence that the corrections of the helium thermometer to the absolute scale are negative.

The two results which strongly deviate from mine may be readily explained if one adopts that the determination of the coefficient of pressure variation of helium by TRAVERS, SENTER and JACQUEROD has not yielded the true value. For if the differences in indication found by them between their helium- and their hydrogen thermometer are reduced by means of the corrections of each of these thermometers to the absolute scale which are given in Comm. 100^a and in Table I of this Comm., to the difference in readings on the absolute scale, which are found at the same temperature by means of the hydrogen thermometer which gives θ_{H_2} and by means of the helium thermometer which gives θ_{He}, there remains at -182° a difference

$$(\theta_{H_2} - \theta_{He})_{-182^\circ} = 0^\circ.10 - 0^\circ.049 - 0^\circ.002 = 0^\circ.05$$

while by extrapolation of the corrections found to -217° for -252° one would find

$$(\theta_{H_2} - \theta_{He})_{-252^\circ} = 0^\circ.20 - 0^\circ.12 - 0^\circ.02 = 0^\circ.10.$$

When calculating the temperatures t_{iH_2} and t_{iHe} the investigators mentioned have taken the pressure coefficient of the helium thermometer $\left(\overset{He}{\underset{i}{\alpha_v}}\right)_{Travers}$ to be equal to that of the hydrogen thermometer at the same zero pressure (for 1000 m.m. therefore 0.0036626 according to our value of Comm. N°. 60). If the corrections applied by me are right that pressure coefficient must therefore, at -182° in order that $\theta_{H_2} - \theta_{He} = 0$ be diminished by 0.0000010 so that

$$\overset{He}{\underset{i}{\alpha_v}} = 0.0036616$$

1) Phil Trans. Ser. A. Vol. 200 p. 105—180. KUENEN and RANDALL (Proc. Roy Soc. Vol. 59) have made a determination, which, being only intended to show whether the helium behaved normally, is not made to the high degree of accuracy which is required for a comparison with isothermal determinations.

2) Phil. Mag. [6] 5. 1903.

and in order $\theta_{H_2} - \theta_{He}$ at — 252° by 0,0000013 to that $\alpha_v^{He} = 0,0036614$.

The first value which has been derived without extrapolation and which is therefore the most reliable, appears to agree perfectly with the one derived by me from the isothermals in § 1.

With regard to the method of derivation followed here we may remark that it allows of a fairly large accuracy. Though the certainty of the determinations of temperature on which it is based may be doubted to the absolute value, yet the only difference which comes into account here is known with sufficient certainty. The calculation mentioned above therefore not only gives an explanation of the too large differences found by TRAVERS, SENTER and JACQUEROD, but is also a welcome control for the coefficient of pressure variation of helium found in section 1.

Physics. — "*The absorption spectra of the compounds of the rare earths at the temperatures obtainable with liquid hydrogen, and their change by the magnetic field*", by JEAN BECQUEREL and H. KAMERLINGH ONNES. Communication N°. 103 from the Physical Laboratory at Leiden.

§ 1. *Introduction.* The investigations of one of us (J. B.) [1]) proved that the absorption spectra of the compounds of the rare earths, cooled down to the temperature of liquid air, may serve to acquire new data for the nature, the number, and the motion of the electrons which play a part in the formation of these spectra. So it seemed to us of great importance to continue these investigations at the temperatures obtainable with liquid hydrogen, which are so many times lower and seem particularly adapted [2]) to reveal the forces which the ponderable substance exerts on the electrons. For this purpose the apparatus used at Paris for the observation of the spectra were conveyed to the cryogenic laboratory at Leiden, so that we were enabled to obtain some three hundred of spectrograms which represent the observed phenomena. The study of these photographs will take a long time; we shall therefore confine ourselves on this occasion to the communication of some facts which immediately draw the attention.

[1]) JEAN BECQUEREL, Radium IV. 9, p. 328 and IV, 11, p. 385 (1907).

[2]) H. KAMERLINGH ONNES, The importance of accurate measurements at very low temperatures. Comm. of the phys. lab. of Leiden Suppl. no. 9, p. 25 sqq. (1904).

§ 2. *Apparatus.* In the first place a few words about the arrangement of the experiments. It was the same for the experiments without and with the magnetic field. The crystals, fixed with wax on a small piece of platinum foil a, (a_2, a_3 fig. 3^a, Pl. I), which was carried by a rod a_4, were immersed in liquid hydrogen in a double-walled tube (b fig. 2, fig. 3^a), which is the continuation of a non-silvered vacuum glass b_2, which contained liquid hydrogen and which is surrounded by another double-walled (c_{10}, c_{20}) tube c, also the continuation of a non-silvered vacuum glass with liquid air, on which it rests on pieces of cork b_3. A clearance of $^1/_2$ mm. between the two glasses (fig 3^a) proved sufficient to allow the liquid air to circulate along the hydrogen tube. This protects the hydrogen so effectively from access of heat that the evaporation is insignificant, even when the two tubes are placed between the hot coils of the magnet and the crystal is exposed to strongly concentrated electric light.

The walls of the narrow part of the tubes are very thin, and because the radiation of heat is independent of the distance of the walls they have been brought to an exceedingly small distance from each other (0.5 mm.), but without being anywhere in contact. Owing to the skill of Mr. KESSELRING, glassblower of the laboratory, who succeeded in doing this, we had at our disposal a tube of 4 mm. inner diameter filled with liquid hydrogen, protected by a tube of liquid air, the outer diameter of which is no more than 8 mm., which allows us to bring the poles of the magnet so near together that very strong fields are obtained even with hollow poles. [1].

The hydrogen tube must be closed hermetically. For this purpose it is fastened in a cap, d, which may be adjusted by means of a levelling board, f, with screws and sliding groove. The tube is brought from below into the cap, where it rests against a wooden cylinder, within d_0 (fig. 2), and it is fastened with a thin rubber ring e_1, which lies round d_0 doubled over and is turned down when the tube is put in. To ensure tightness a rubber solution is put between ring and glass, and the rubber is pressed tight against the glass and the cap with copper wire. The cap is provided with: 1. the tube d_{30}, to which at d_{31} a head with packing cap d_{32} is screwed, in which the rod a_6 can turn (by means of a_7), and move up and down (by means of the nut d_{33}). 2. a tube d_1 to siphon over liquid hydrogen as

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1) Instead of the usual poles of the WEISS magnet we have used auxiliary pieces, p_{30} (see figs. 2 and 3), which prolong the cone to a section of 6 mm. diameter, with conic perforations, which have a diameter of 3 mm. on the side of the crystal.

indicated in Comm. N°. 94 from the supply bottle into the apparatus, which tube is closed in other cases with a rubber tube with cock. 3. an outflow tube d_2 (fig. 2), which leads along cock k (fig. 1 and fig. 4) to the gasholder with pure hydrogen, to a safety tube l (fig, 1), along k_2 to an airpump, and past k_4 to the vacuum bottle r, from which the liquid hydrogen is siphoned over (the operation is elucidated by the diagrammatic fig. 4, which does not call for a further description).

We first have convinced ourselves that when the air has been exhausted from the hydrogen tube surrounded by c, this tube exactly occupies its place between the poles, without being strained by the supports q and i, when these have a suitable position, we then fill it along k_1 with hydrogen from the gasholder, exhausting it repeatedly, then we pour liquid air through a funnel with filter into b_1, which is covered with some cotton wool. The apparatus is then filled with liquid hydrogen through d_1. In order to pass to the melting point of hydrogen, k_2 is opened till crystals appear on the surface of the liquid hydrogen, through which the gas bubbles which rise from the heated crystal, are seen to make their way. If the apparatus has been filled in the way described before, observations with the crystals may be made uninterruptedly for several hours. The precautions taken to prevent mixing of hydrogen and air are indispensable. Air entering the apparatus, would sink down, and be sucked up in front of the crystal as soon as the magnetic field is applied, and intercept the light.

For every filling of the apparatus $^1/_4$ liter of liquid hydrogen from the supply is generally used, and it was sufficient to do this twice a day to be able to observe all the day in case of ordinary as well as of low pressure; twice a week a quantity of 5 liters was prepared for these experiments, which was just sufficient to fill the apparatus also the second day after the preparation. As it was impossible to entirely prevent the hydrogen which evaporated at lowered pressure from being contaminated with air, it was not admitted again into the cycle. The hydrogen cycle proved its reliability by never failing us a single time in all these weeks.

I. Phenomena which depend solely on the temperature.

§ 3 *Simplification of the spectra.* On cooling to the temperature of liquid air ($T = 85°$) one of us had found[1]) that almost all bands become narrower and divide, some new ones also appearing. In

[1]) Jean Becquerel, l. c.

general their intensity increases. The bands which decrease in intensity or which vanish altogether, are exceedingly few in number. The measurements on anomalous dispersion in the neighbourhood of some bands of tysonite had proved that this increase of intensity is not only the consequence of the bands becoming narrower, but also of a modification which, according to the theory of electrons on the supposition of quasi-elastic forces, indicates the increase of the dielectric constant in every band, and implies that the number of electrons which determine such a band, has increased.

Passing to the temperature of liquid hydrogen ($T = 20)^{\circ}$, we saw some bands continue to increase in intensity, but also others which showed an increasing absorption with fall of temperature down to that of liquid air, decrease both in intensity and in breadth. There are even bands having appeared in liquid air, which become almost invisible in liquid hydrogen. An example of such a change with the temperature is furnished by the bands 523.5 and 479.1 of tysonite.

The measurements of the anomalous dispersion in the neighbourhood of these bands had shown that the electrons belonging to these bands are about twice or three times as numerous at the temperature of liquid air as at the ordinary temperature. In liquid hydrogen the number has already become very small, and at the temperature of solid hydrogen (14°) hardly any electrons of this kind take part in the motion. Fig. 1, Pl. II, which represents the compensator fringes [1]) in the neighbourhood of band 523,5 of tysonite at different temperatures and with different thickness, allows us to measure the disturbance in the fringe with regard to height and breadth. Figs. 2 and 3, which we treat in § 8 and 9, and which represent the magneto-optic phenomena, may elucidate this.

§ 4. *Maximum of intensity of every band for a definite temperature.* It follows from the foregoing that several bands pass through a maximum of intensity with decrease of the temperature. In general the place of this maximum is different for different bands. When in the experiment with tysonite described in § 3 we wait till the last traces of hydrogen evaporate from the crystal, immediately after when the temperature of the crystal rises, the band 523,5 is seen to greatly increase in intensity. Without doubt the maximum for this band lies at a temperature not far above the boiling point of hydrogen. All the crystals of xenotime, tysonite, parisite, apatite, monazite, didymium sulphate, praseodymium sulphate, neodymium sulphate, exhibit

[1]) JEAN BECQUEREL, Radium IV no. 9 p. 328.

similar phenomena. The green line 523,5 of neodymium which is exceedingly fine and sharp at $T = 20^\circ$, has almost vanished at $T = 14^\circ$.

We have further examined the influence of the fall from $T = 91^\circ$ to $T = 58^\circ$ by immersing the crystals in liquid oxygen boiling at the airpump. The change in this region is only slight. This confirms the conclusion drawn from what was observed in heating from $T = 20^\circ$ upwards that the maximum must lie near this latter temperature and at all events far below $T = 58^\circ$.

Naturally the question obtrudes itself whether those few bands, whose intensity diminishes between the ordinary temperature and that of liquid air, do not also pass through a maximum either between $T = 290^\circ$ and $T = 95^\circ$, or at a temperature above $T = 290^\circ$. It will be difficult to decide the question, because in consequence of the broadening and overlapping of the bands the change of each of these bands in itself escapes observation.

§ 5. *Change in width.* In the previous experiments[1]) it had been found generally valid for all bands measured down to the temperature of liquid air, that the width of the bands was proportional to the square root of the absolute temperature. This is the law which for the case of a gas may be deduced from the formulae formerly developed by LORENTZ[2])

When we pass to the temperature of liquid hydrogen this law appears to be no longer valid for some bands, whereas for others the order of magnitude of the change seems to remain the same. In the figures 1, Pl. II obtained by the method of the compensator fringes, it is very clearly to be seen, that 523.5 of tysonite is not half as broad at $T = 20^\circ$ as at $T = 85^\circ$, as the law of the $\sqrt{T}$ would require. And it was this very band which had served to show experimentally, that this law held down to $T = 85^\circ$ with a high degree of approximation.

The question whether there is a *minimum of width*, could not be solved yet. At first sight some bands do not seem to contract any further between $T = 20^\circ$ and $T = 14^\circ$, two of xenotime seem even to get wider.

With regard to the totality of the phenomena of change of width in liquid and solid hydrogen we may further observe that in these even more than in liquid air[3]) the spectra manifest a pronounced

1) JEAN BECQUEREL, Radium IV no. 9 p. 328.

2) H. A. LORENTZ. Kon. Akad. v. W. VI p 506 and p. 555 (1898).

3) That tysonite and xenotime have this tendency has been observed by JEAN BECQUEREL, Radium l. c.

tendency to assume the character of gas spectra when the temperature decreases. Some absorption lines of praseodymium and neodymium sulphate, cleared of broad bands that covered them, are even finer than the D-lines.

§ 6. *The approach to a limit of the double refraction of crystals in the non absorbed parts of the spectrum.* If we watch the bands, by the aid of which the double refraction is investigated, with change of temperature, we observe the following. If the crystal is heated above the ordinary temperature, they are greatly displaced. When the temperature is lowered to that of liquid air they move in the opposite direction. For a crystal of tysonite we have also examined them with further cooling with liquid hydrogen. In spite of the great difference of temperature the displacement is then hardly perceptible. This may point to the fact that the difference of the expansion of the crystals in the different directions approaches a limit at very low temperatures.

§ 7. *Connection of the change of the absorption bands occurring at very low temperatures with the electronic theory.* Already in § 3 we pointed out the connection of the change of the bands with that of the number of the electrons which are concerned with a certain band according to the electronic theory coupled with the assumption of quasi-elastic forces. The experimental problems raised by § 3 and § 4 may be defined as follows in the language of this theory [1]: to determine as functions of T on one side the number and on the other side the damping coefficient (proportional to the width of the band) of the electrons which belong to a certain band. We might make use of the position of the maxima to find mutually related bands, in the first place in the different spectra of one crystal. An investigation into the connection between what we already know about these functions and what the change of the electrical resistance of the metals leads us to expect about the action of forces exercised by the ponderable substance on the electrons naturally suggests itself.[2]) At very low temperatures we shall no longer be justified in considering the electrons as a perfect gas, but we shall rather have to compare them to a vapour which precipitates on parts of the atoms (dynamides (LENARD)), and solidifies at still lower temperature [3]). When we approach these centres the paths of the electrons are subjected to changes which modify the free

[1]) Cf JEAN BECQUEREL. Radium l. c

[2]) H. KAMERLINGH ONNES. Loc. cit.

[3]) A metal would become transparent at very low temperature.

ength of path in the same way as VAN DER WAALS' quantity b is subjected to a change by the forces exerted by the molecules on each other. [1])

The three states of aggregation which we used just now as an illustration of the behaviour of the electrons, might perhaps be considered as referring to the stability of different paths of the electrons, and the quasi-elastic force might be connected with the conditions for the electrons moving in these paths.

If we further note that it is the ratio of the absolute temperatures on which the degree of change of the spectra depends (compare the transition from $T=20$ to $T=14$ with that from $T=290$ to $T=95$), we may accept for the present as a heuristic image the idea that we may speak of corresponding states according to different units of temperature caused by mechanic similarity of the motion of the electrons round the centres.

II. Phenomena depending on the temperature and on the strength of the magnetic field.

§ 8. *Constancy of the change of the frequency of vibrations under the influence of the magnetic field at all temperatures.*

According to the experiments made by one of us previously (J. B.), when a uniaxial crystal is placed with its axis in the direction of the lines of force and of the ray of light, some absorption bands are resolved into two components, which belong to the absorption of two circularly polarized rays of opposite sense. The difference of frequency of vibration of the two components had then proved to be independent of the temperature. It follows now in a still more convincing way from the comparison of the divergence of the two bands at the temperature of liquid hydrogen with the divergence at the temperature of liquid air, that within the limits of errors of observation, the difference of frequency of vibration is entirely independent of the temperature. According to the theory of LORENTZ this constancy of the divergence of the bands, which is observed both for those which behave in the sense of the ZEEMAN-effect as for those which behave in opposite sense, must be considered as proceeding from the invariability of the relation e/m. Accordingly the observations in liquid hydrogen seem to furnish a strong support to the argument in favour of the existence of positive electrons derived from the constancy of this quotient. [2])

[1]) Calculated by REINGANUM according to the theory of BOLTZMANN.

[2]) *Le Radium* tom V. p. 17 1908.

§ 9. *Partial polarisation of the components of some bands.* In a foregoing communication (CR. 19 Aout 1907) one of us (J. B.) has demonstrated, that the band 624,97 of tysonite becomes double in each of the two spectra of left-handed and right-handed circularly polarized light, which are obtained by means of a plate of a quarter wavelength and a rhombohedron. Therefore in both components of the magnetic doublet of the band the polarisation is not perfectly circular. The band behaves as if it were owing both to positive and to negative electrons with the same period of vibration, and the same ratio e/m, in which the number of positive electrons is to be put as the largest, because the strongest component belongs to it.

At the temperature of liquid hydrogen the same phenomenon is observed with some bands which become at the same time fine and bright (fig. 2 Pl. I band 522. 1). In general the same thing is found on reexamining the spectra at the temperature of liquid air and at the ordinary temperature, though it is more difficult to see. Some time ago DUFOUR again found the same phenomenon in emission bands of fluorcalcium put into the flame.

§ 10. *Asymmetry of the right- and left-handed components.* The experiments at the temperature of liquid air had proved [1]) that when the rays of light run parallel to the lines of force the right- and left-handed components very often differ in strength. No regularity had been found in these differences, the asymmetry was now in one, then in the other sense.

If we pass to the temperature of liquid, or better still, to that of solid hydrogen, the asymmetries, which sometimes change their sign, become exceedingly great; one component increases in intensity at the expense of the other, even to such a degree, that some components vanish almost entirely on the side of the greater wave lengths. An example is furnished by fig. 3, Pl. III referring to 654,2 and 643,4 of xenotime, one component of which is very intense, the other very faint. Apatite shows the same thing.

In solid hydrogen almost all the components which diverge towards the small wave lengths, become very sensibly intenser than those of opposite sign.

§ 11. *Variation of the magnetic rotation of the plane of polarisation in the neighbourhood of the absorption bands.*

a. Simple bands. The experiments of MACALUSO [2]), H. BECQUEREL [3]),

[1]) JEAN BECQUEREL *Le Radium* V. No. 1. p. 9. 1908.

[2]) CR. CXXVII p. 548, 1898.

[3]) CR. CXXV p. 679. 1897 CXXVII p. 899. 1891.

ZEEMAN [1]) have proved that in the neighbourhood of the bands which exhibit the ZEEMAN-phenomenon, the rotation of the plane of polarisation on both sides of the band is positive, and in the inside of the magnetic doublet negative. The experiments made with uniaxial crystals [2]) with the axis placed parallel to the lines of force and to the beam of light either at the ordinary temperature, or at the temperature of liquid air, have proved that the regular change of the magnetic rotatory power with the wavelength of the light is subjected to a disturbance of the same kind on both sides of the band, and to an opposite disturbance at the middle of the band. This disturbance is positive outside the band for the bands belonging to negative electrons, and negative for the bands of positive electrons.

At the temperatures of liquid and solid hydrogen the same phenomena are observed, at least when the asymmetry of the left- and right-handed components is not too large. In the neighbourhood of some bands whose components are very unequal, opposite disturbances are observed on both sides of the band — as is easily explained by means of the usual figures of the anomalous dispersion. These phenomena are clearly visible on the figures. 4 Pl. III and 5 Pl. IV.

These figures have been obtained by a method which was already used in former experiments [3]). Against the slit of the spectroscope a BABINET compensator was fixed between two crossed Nicols in such a way that the fringes were perpendicular to the slit. Before the compensator a plate of a quarter of a wave-length is placed in such a way that the two opposite circularly polarized vibrations are changed into two rays rectilinearly polarized parallel and normal to the principal direction of the compensator. The deviations of the fringe in the spectrum in the neighbourhood of the bands are proportional to the difference of phase of the circularly polarized rays in the crystal plate.

In the figures we find for band 522.15 fig. 4 the symmetrical case, for band 523.7 fig. 4, and 642.3 fig. 5 the dissymmetrical case with disturbance in the same direction, for band 537 fig. 4, and 654.2 fig. 5 the opposite disturbance on both sides of the band.

b. Compound bands. The phenomena of absorption at lower temperatures have shown that several bands may be resolved into two or more. These components behave differently with respect to the magnetic field, because some belong to positive, others to negative

[1]) Arch Neerl. VII p. 465. 1902.

[2]) JEAN BECQUEREL, Radium IV No. 2 p. 49. 1907, V No. 1 p. 5. 1908.

[3]) JEAN BECQUEREL, C.R. May 21 1906.

electrons. Therefore we meet with disturbances in the magnetic rotation which are different for the different bands, and whose effects are superposed. Thus two bands placed side by side, one of positive and the other of negative electrons, may give rise to disturbances in opposite direction in the dispersion of magnetic rotation. It is perhaps to this that we must look for the explanation of what is observed in band 577 of tysonite, which is clearly double in liquid hydrogen.

In general we may say that with regard to the theory of the magnetic rotation for absorption bands, the conclusions drawn from experiments at the ordinary temperature do not lead to a definite result. For at the ordinary temperature it is uncertain whether we have really to deal with a simple band. On the other hand at the low temperatures, at which the bands become narrow, and their change in the magnetic field may be closely followed, it is easy to find the true explanation of the different types of disturbances in the magnetic dispersion of rotation for the bands in the different cases.

§ 12. *Magnetic rotatory power of the paramagnetic crystals.* One of us (J. B.)[1]) had previously shown that the negative magnetic rotatory power of the crystals of tysonite and parisite increases considerably with decrease of temperature. The rotatory power is about inversely proportional to the absolute temperature. If this is brought into connection with the law of CURIE that the paramagnetic susceptibility is inversely proportional to T, it appears that the negative rotation of these crystals is probably a consequence of the increase of the paramagnetic polarisation of the crystal.

If these crystals are placed in liquid hydrogen we find that the increase continues in the same way with decrease of temperature, and the rotatory power rises to exceedingly high values. The exact numbers will be given later, but in round numbers the rotation of the plane of polarisation of the blue light amounts to 150° for a plate of tysonite of 1 mm. in a field of 10000 Gauss at the boiling point of hydrogen. Xenotime, which gives a very slight rotation at the ordinary temperature, shows a considerable rotatory power in liquid hydrogen.

§ 13. *Connection between the phenomena of the asymmetry of the left- and right-handed polarized components by the magnetic field at very low temperatures, and the electronic theory.*

In connection with § 4 the phenomena taken together give rise

[1]) JEAN BECQUEREL, Radium. Tom. V, N°. 1, p. 5, 1908.

to the supposition that for the paths of the electrons there exist conditions (fields) of stability, which are determined by the temperature. The action of the magnetic force and the change in the rate of vibration would then bring about that some electrons enter these fields of stability or leave them, both changes occurring either in the direction of greater union with or further separation from the centres which determine the paths, and the increase of this action at low temperature would be in connection with the small velocity. The influence on the stability of the paths, which is here considered, would be the same as manifests itself in the change by temperature of the number of electrons (see § 7) which satisfy the conditions of the motions which may be ascribed to quasi-elastic forces.

In this connection the question suggests itself if the greater stability of vibrations in a certain direction will not give rise to paramagnetic properties.

§ 14. *Variability of the mass of the electrons with the direction of the movement.* The theory of the magneto optic phenomena in crystals (VOIGT [1]), JEAN BECQUEREL [2])) leads to the following results.

The magnetic field gives rise to certain connections between the motions of the electrons in the different principal directions of the crystal. Let us consider the simple case which is repeatedly met with, viz. that the corresponding bands in the different spectra occupy the same place. In that case according to the theory the magnetic doublets will have to be symmetrical, and when the bands are sufficiently narrow to allow us to neglect the breadth, the deviations will be proportional to the square root of the product of the two magnetic constants which belong to the corresponding bands of the two spectra. If the beam of light and one of the principal directions 1, 2, 3 of the crystal are made to coincide with the direction of the magnetic field, those two of the three spectra of the crystal are observed which correspond with the vibrations normal to the lines of force.

Observation shows that both for the uniaxial crystals of xenotime and tysonite and for the biaxial crystals of didymium sulphate, neodymium sulphate, and praseodymium sulphate (which last exhibits some lines in liquid hydrogen as sharp as vapour lines) the doublets of the common band have the same divergence. A phenomenon of great importance is observed, when the spectra of vibrations normal to the lines of force are combined in different ways. If the directions 1, 2, 3 successively are placed in the direction of the

1) Nachr Kön. Ges. d. Wiss. Gottingen Juli 1906.
2) C.R. 19 Nov. 3. 10. 24 Dec. 1906. Radium IV n°. 3 Mars 1907.

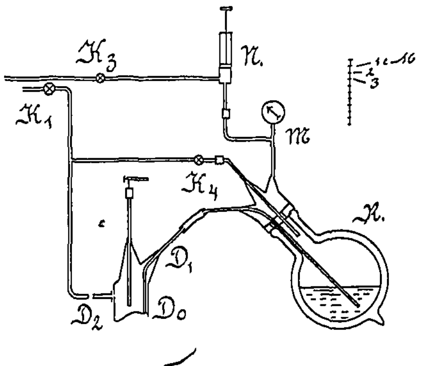


Fig. 4.

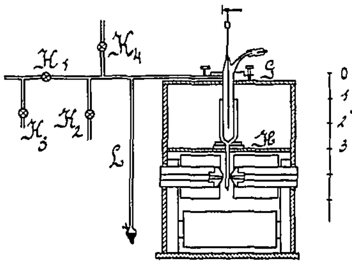


Fig. 1

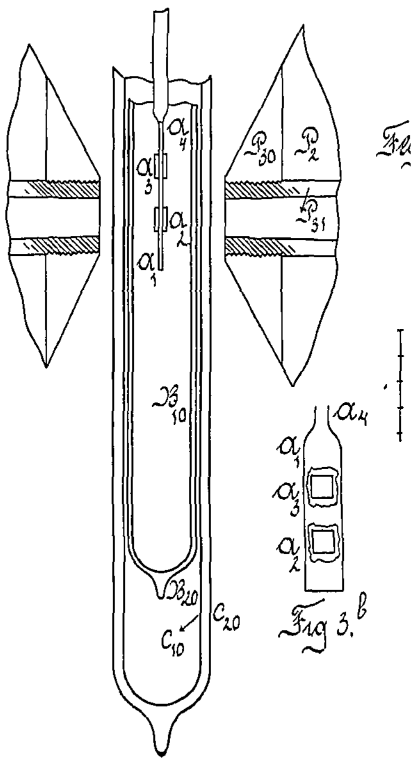


Fig 3.

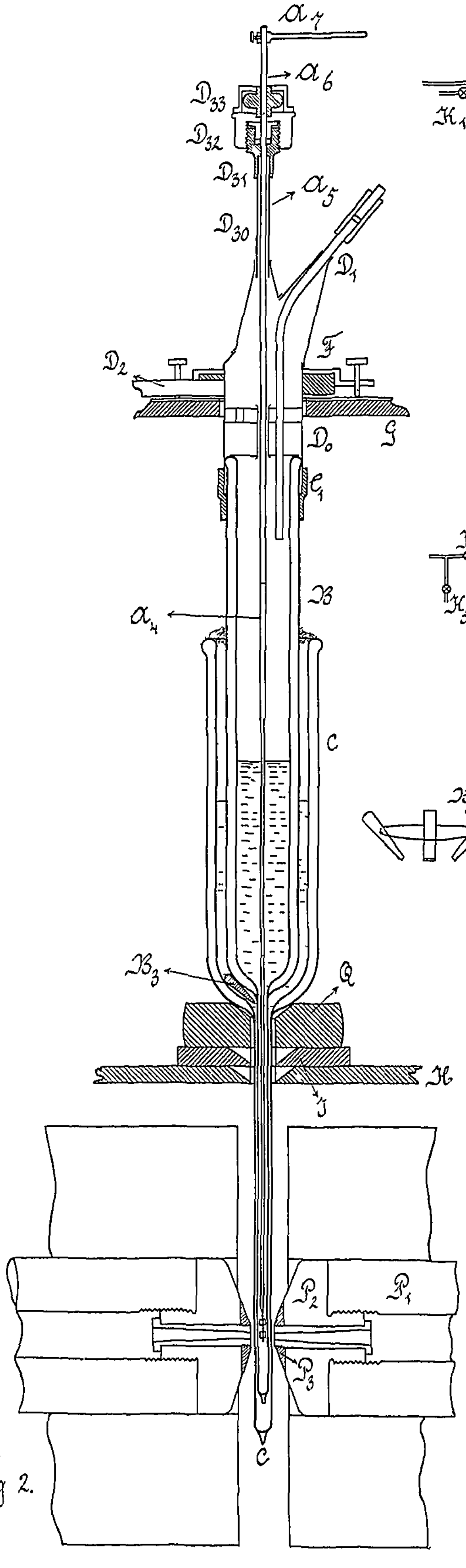


Fig 2.

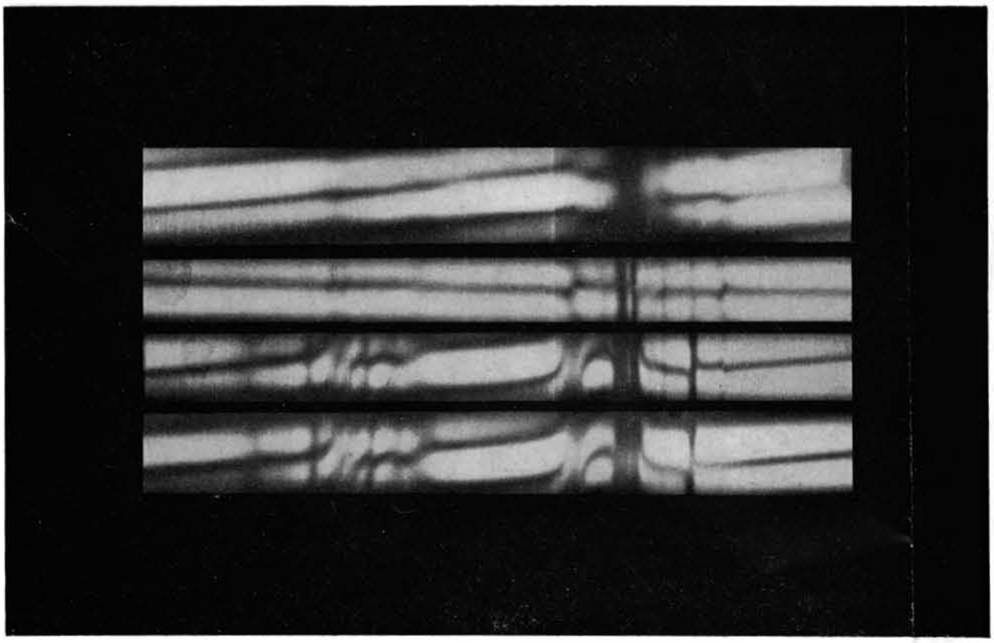
0 1 2 3 4 c M

JEAN BECQUEREL and H. KAMERLINGH ONNES. On the absorptionspectra of the compounds of the rare earths at the temperatures obtainable by liquid hydrogen and their change by the magnetic field.

Pl. II.

517.6 523.5

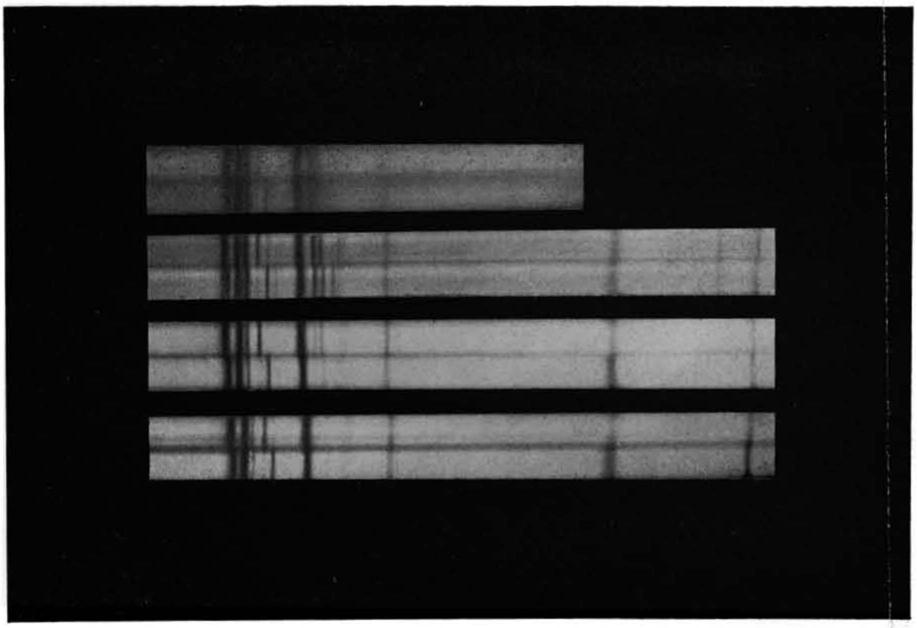
1 | 2 | 3 | 4



$T = 293°$ abs.
$T = 85°$ „
$T = 20°$ „
$T = 14°$ „

Fig. 1. *Anomalous birefrigency,* tysonite, group in the green 2d spectrum (ROWLAND grating), thickness op plates 1.71 mM. in 1, 2, 3, 4 and 0.41mM. in 2 (in 2 the ordinary and extraordinary ray are interchanged).

520.6 523.7 537



$T = 293°$ abs.
$T = 85°$ „
$T = 20°$ „
$T = 14°$ „

522.1

Fig. 2. *Left- and righthanded vibrations* in a field of 18000 Gauss nearly. Xenotime, group in the green, 2d spectrum (ROWLAND grating).

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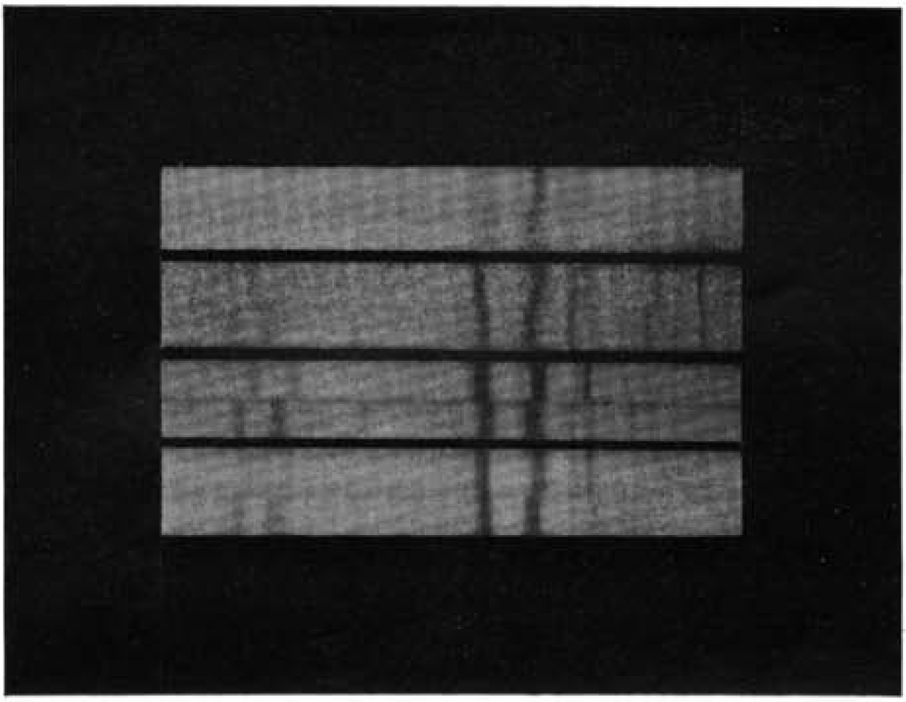
Pl. III.

642.27 643.45 650.56 654.25 658.10



$T = 290°$ abs

$T = 85°$ „

$T = 20°$ „

$T = 14°$ „

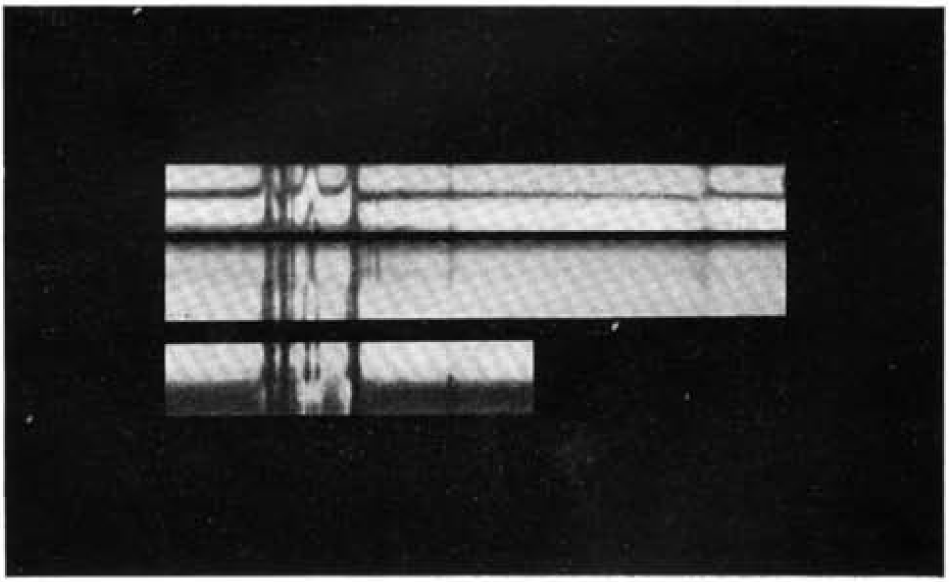
Fig. 3. *Left and righthanded vibration* in 3 a field of 18000 Gauss nearly, Xenotime, group in the red, 2d spectrum (ROWLAND grating); panchromatic plates of WRATTEN and WAINWRIGHT.

520.6 522.15 537

1

2

3



$T = 20°$abs

$T = 20°$abs

$T = 14°$abs

5237

Fig. 4. *Xenotime*, group in the green, 2d spectrum (ROWLAND grating)
1. magnetic circulair birefringency, plate thick 0.80 mM., field 15000 Gauss.
2. images by rhombohedron before slit, the incident light polarized to give equal intensities to the regions in the transparent part in the middle of the group. Field 15000 Gauss.
3. images given by rhombohedron before slit, incident light polarized under 45° with the horizon; field 18000 Gauss.

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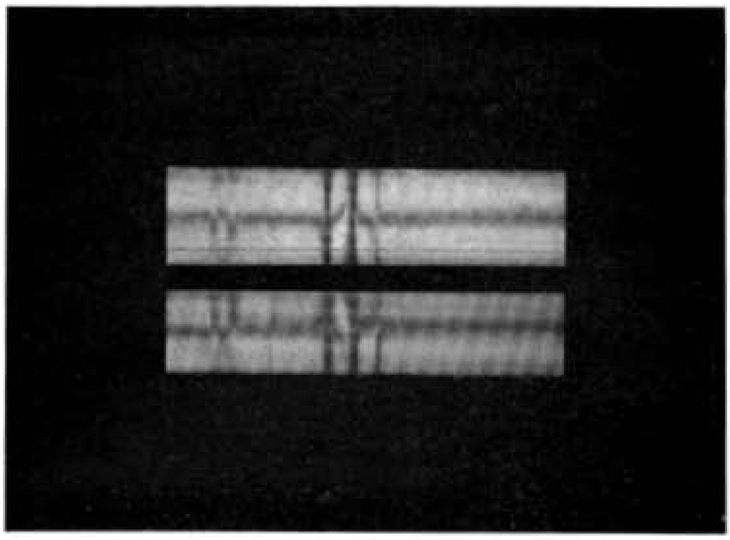
Pl. IV.

642.3 654.3

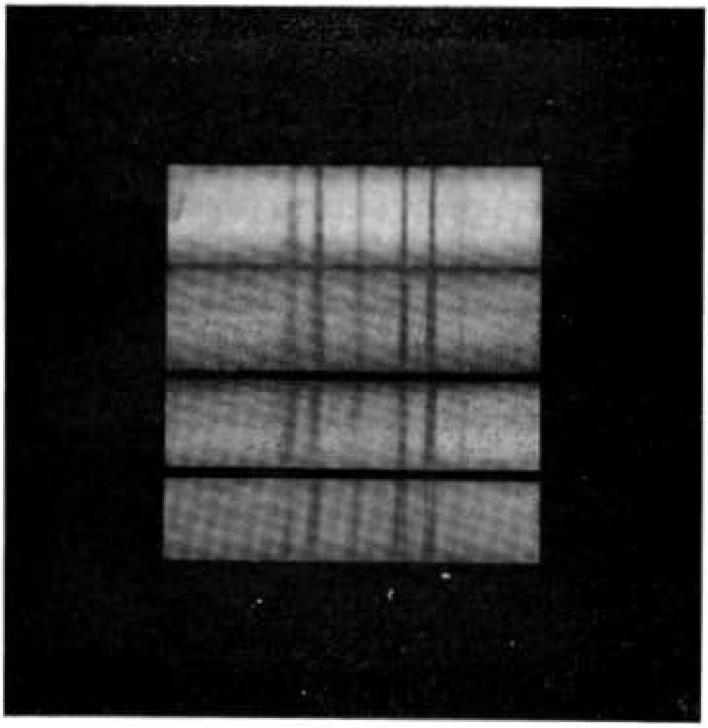


$T = 20°$ abs.

$T = 14°$ „

Fig. 5. *Magnetic rotation of the plane of polarization.* Xenotime 2d spectrum (ROWLAND grating); thickness 0.80 mM., field 18000 Gauss; (quarter of wavelength plate turned 90° in the one in respect to the other).

1
2
3
4
5
6
7
8



$T = 14°$ abs.

$T = 14°$ „

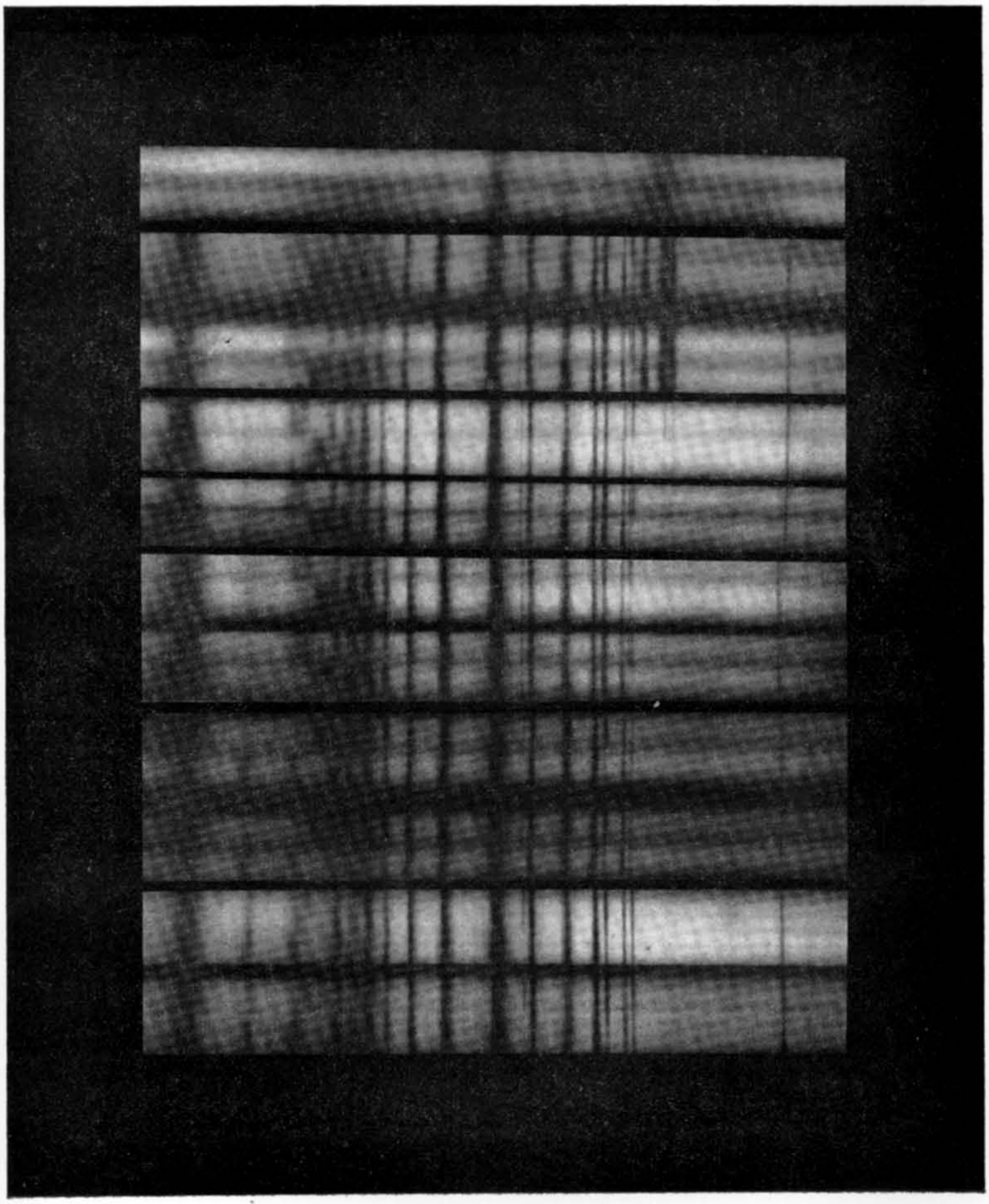
$T = 14°$ „

$T = 14°$ „

Fig. 6. *Sulphate of Neodyme* group in the orange, 2d spectrum ((ROWLAND grating) spectra of the vibrations:

1 in the principal direction *a*, field = 0.
2 „ „ „ „ *b*, field = 0.
3 „ „ „ „ *a*, *a* and *b* normal to the field (18000 Gauss).
4 „ „ „ „ *b*, *a* and *b* „ „ „ „ „
5 „ „ „ „ *a*, *a* and *c* „ „ „ „ „
6 „ „ „ „ *c*, *a* and *c* „ „ „ „ „
7 „ „ „ „ *b*, *b* and *c* „ „ „ „ „
8 „ „ „ „ *c*, *b* and *c* „ „ „ „ „

Pl. V.



1 2	$T = 293°$ abs.
3 4 5 6	$T = 85°$ „
7 8	$T = 20°$ „
9 10	$T = 20°$ „
11 12	$T = 14°$ „
13 14	$T = 14°$ „
15 16	$T = 14°$ „
17 18	$T = 14°$ „
19 20	$T = 14°$ „
21 22	$T = 14°$ „

Fig. 7. *Sulphate of neodyme;* spectra of vibrations in the directions α, β, γ group in the blue, 2d spectrum (ROWLAND grating)

1, 3, 7, 11, 19	α without field
2, 4, 8, 12, 15	β „ „
16, 20	γ „ „
5 and 6, 9 and 10, 13 and 14	α and β normal to field (18000 Gauss)
17 and 18,	β and γ „ „ „ „
21 and 22	α and γ „ „ „ „

field, we get the combinations 2.3, 1.3, 1.2 for the vibrations normal to the field. Experiment shows that the divergences of the pairs of doublets in these three cases are very different. Thus for a band of spectrum 1, the vibration being normal to the field, the magnetic doublet is different according as the direction normal to the field has the principal direction 2 or 3. The phenomenon is clearly seen in the figure which represents the group of bands in the orange for neodymium sulphate at — 259°. Fig. 7 Pl. V gives a survey of the phenomena of the changes with the temperature and the magnetic field in the blue of neodymium sulphate. According to theory it follows from this that each of the three different directions has a different magnetic constant, and that therefore the vibrating system presents three different masses for the three kinds of vibrations.

As the corresponding bands in the two spectra occupy the same or only slightly different places, it follows that in first approximation the constant of the quasi-elastic force in each of the three directions must be proportional to the mass in that direction.

Physics. — *"On the equation of state of a substance in the neighbourhood of the critical point liquid-gas. I. The disturbance function in the neighbourhood of the critical state."* By Prof. KAMERLINGH ONNES and Dr. W. H. KEESOM. Communication N°. 104^a from the Physical Laboratory at Leiden.

§ 1. The great compressibility of a substance in the neighbourhood of the critical point liquid-gas and the properties connected with this, (such as the small variation of the thermodynamical potential at isothermal compression etc.) — which are derived from VAN DER WAALS' original equation of state and better still from his latest considerations about the compressibility of a molecule [1]) — render it necessary that in deriving conclusions from observations in the neighbourhood of that condition we must take into account various circumstances, otherwise unnecessary for the experimental investigation of the equation of state of a homogeneous substance consisting of one component, which investigation includes that of the quantities of saturation etc.

It is well-known that owing to the great compressibility the thermodynamic equilibrium is difficult to attain, in fact it has often

[1]) Comp. VAN DER WAALS, Proceedings June '03,

happened that phenomena [1]) at the critical point have been described as abnormal and as being at variance with the views of ANDREWS-VAN DER WAALS, in cases where the thermodynamic equilibrium had not yet been attained either because small differences in composition had remained owing to the slow diffusion of very small quantities of admixture (KUENEN Comms. N°. 8, Oct. '93, N°. 11, May and June '94), or because differences of temperature resulting from variations of volume in different portions of the substance during the passage from one condition of temperature and pressure to the next had not yet been equalized (KAMERLINGH ONNES, Comm. N°. 68, March and April '01 and KAMERLINGH ONNES and FABIUS, N°. 98, May '07).

When the thermodynamical equilibrium is obtained either by keeping the substance in the neighbourhood of the critical point during a long time at a constant temperature or by repeated reversals of the sealed tube containing the substance (GOUY), or by stirring it electromagnetically (KUENEN) we must pay regard to the gravitation which on account of the great compressibility of the substance in that condition becomes of great influence [2]) and also to small quantities of admixture which may occur and of which the nature and the quantity are known [3]).

The consideration of these influences and those of capillarity and absorption phenomena near the walls of the vessel [4]), things which in other cases are hardly to be considered, is indispensable at the critical point liquid-gas for the determination of the *experimental equation of state of a substance,* i. e. the relation between p, v and T for a substance consisting of one component in thermodynamic equilibrium subject to no other external forces than the pressure on the walls of the vessel.

§ 2. In this communication we intend to bring into connection some peculiarities in the experimental equation of state in the neighbourbood of the critical state with the great compressibility

[1]) For a survey of these phenomena comp. GRAETZ, WINKELMANN's Handbuch, III, 2te Aufl. p. 837.

[2]) GOUY. C R. 115 (1892) p. 720 and 116, p. 1289. J. P. KUENEN, Comm. N°. 17 May '95.

[3]) Cf. Comm. N°. 75, Nov. '01, N°. 79, March '02, N°. 88 Nov. '03 (KEESOM), N°. 81, June and Sept. '02, Suppl. N°. 6, Febr. and May '03, N . 18, Dec '04, N°. 12, Jan. '07 (VERSCHAFFELT). On the influence of gravity a small quantity of admixture being present, cf. KUENEN, Comm. N°. 17, May '95 and KEESOM, Comm. N°. 88 VI, Nov. '03.

[4]) Comp. VAN DER WAALS, l.c. p. 106 and 107.

in this area. Therefor we compare the experimental equation of state of a substance near the critical point liquid-gas with an equation of state which we shall call *the special undisturbed equation of state for that substance* and which is derived by adjusting interpolation formulae to observations in areas where no disturbances occur such as in the neighbourhood of the critical point.

For we presume to be able to derive from the results of data at our disposal that the experimental equation of state differs from the special undisturbed one by the presence of terms which for the accuracy reached in the observations meant only deserve notice in the neighbourhood of the critical point, and which are intimately connected with the great compressibility in this area. We shall call the compound of these terms *the disturbance function in the equation of state in the neighbourhood of the critical point.*

In order to be able to derive from the special undisturbed equation of state and the disturbance function at the critical state the conditions of coexistence, vapour pressures, liquid and vapour densities, we must have investigated whether in that condition MAXWELL's criterium for a substance consisting of one component may be applied unmodified or not.

For the present we must include in this disturbance function the disturbances caused by admixtures which chemically may have an existence of their own, but which it was not possible to remove and which always occur in definite quantities, as long as the nature and the quantity of these admixtures are unknown. The investigation of substances with small quantities of admixture [1]) may help us towards a better judgment of the question whether this disturbance function may be entirely ascribed to admixtures which may exist separately. As long as this has not yet been decided it will be indispensable to pay regard also to admixtures which can have no existence of their own but which may always occur as electrically charged particles, or as portions of the substance of greater density which may give rise to differences of density distributed as nebulous drops and which in this area might be kept up by capillary force. It will also be necessary to take into account differences of density depending on the statistic equilibrium.

In order to arrive at some knowledge of such a disturbance

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1) Comp. p. 604 note 3. For the influence of small quantities of admixture of substances of small volatility the following investigations are important: M. CENTNERSZWER, ZS. physik. Chemie 46 (1903) OSTWALD Jubelb. p. 427, 61 (1907) p. 356; M. CENTNERSZWER and A. PAKALNEET, ibid. 55 (1906) p. 303, M. CENTNERSZWER and A. KALNIN, ibid. 60 (1907) p. 441.

function, observations of greater accuracy are required over an area which comprises the critical state and also approaches it sufficiently. These observations must be accurate to within $^1/_{5000}$, as is usual in the Leiden laboratory in the investigations of bi- and monatomic substances and their binary mixtures, while the nature and the quantity of the separable admixtures ought to be known to $^1/_{10000}$ of the whole mass [1]).

§ 3. Our conclusion about the existence of a disturbance function in the equation of state in the neighbourhood of the critical point liquid-gas is based on the following data which may be arranged into three groups.

a. In Comm. N°. 74 (Arch. Néerl. (2) 6 (1901) livre jub. BOSSCHA p. 874) has been pointed out that AMAGAT's observations of the isothermals of carbon dioxide near the critical point show systematic deviations from the values derived from the special undisturbed equation of state. This equation of state was derived from the empiric equation of state introduced in Comm. N°. 71, June '01, by choosing the virial coefficients so (Comm. N°. 74 § 4) that the agreement with the observations over the whole area of observations is as good as possible while the agreement with the general reduced equation of state at a reduced temperature lying far outside the area of observation was retained.

We get a similar series of observations if we compare the observations of carbon dioxide in the neighbourhood of the critical point — described in Comm. N°. 88 (Jan. '04) — with the special undisturbed equation of state, while using the reduced virial coefficients V s. 1 (Comm. N°. 74, p. 884) and the critical temperature and pressure found in Comm. N°. 88.

It really appeared in Suppl. N°. 14, Jan. '07 (KAMERLINGH ONNES and Miss JOLLES) that the critical quantities, derived according to $\partial p/\partial v = 0$, $\partial^2 p/\partial v^2 = 0$ from the special undisturbed equation of state V s. 1, show great deviations from those derived experimentally.

A similar difference was found by AMAGAT (Journ. de phys. (3) 8 (1899) p. 353) when he derived the densities of saturated liquid and vapour from the equation of state (containing 10 constants) formed by him for carbon dioxide. The curve which represents the densities calculated thus as a function of the temperature, at lower tempera-

1) For such an investigation carbon dioxide would be fittest owing to the comparably small difficulties in preparing it perfectly pure and keeping its temperature sufficiently constant, and also because much is already known about its equation of state over a large area.

tures coincides almost with the curve given by the observations: according as we approach the critical temperature the calculated curve shows a displacement towards the small densities with regard to the observed curve. That this displacement is much larger than follows from the calculation of Suppl. N°. 14 mentioned above must be ascribed at least partly to the circumstance that AMAGAT probably did not derive the liquid and vapour densities from his equation of state by means of the criterium of MAXWELL, but for shortness' sake calculated by means of his equation of state the densities for which p has the value for the saturation pressure furnished by experiment.

b. In Comm. N°. 75 Nov. '01 attention was directed to the difference between the $C_5 = \left[\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_v\right]_k$ derived from AMAGAT's net of isothermals and the $C_6' = \left[\frac{T}{p}\frac{dp_{coex.}}{dT}\right]_k$ resulting from his determinations of the saturated vapour pressure, which values must be equal for the undisturbed equation of state [1]). One of the reasons to undertake the observations about carbon dioxide of Comm. N°. 88, Jan. '04, was the wish to obtain more certainty about these peculiarities in the behaviour of the substance in the neighbourhood of the critical point (comp. l.c. p. 566). The same difference viz. $C_5 = 7.12$, $C_6 = 6.71$ followed from these determinations. BRINKMAN (Thesis for the doctorate, Amsterdam 1904, p. 43) confirmed this difference not only for carbon dioxide, but he also found it for methyl chloride, while MILLS (Journ. phys. Chem. 8 (1904) p. 594, 635; comp. also 9 (1905) p. 402) for ethyl oxide (RAMSAY and YOUNG), isopentane and normal pentane (YOUNG) finds differences of 10 percent between C_6 derived by means of the formula of BIOT for the saturated vapour pressures and C_5 which with regard paid to the regular variation of b with temperature [2]), follows from the data collected by RAMSAY and YOUNG (ethyl oxyde), YOUNG (isopentane) and ROSE-INNES and YOUNG (normal pentane) in order to judge of the equation of isochors $p = bT - a$.

c. In Comm. N°. 88, Jan. '04, p. 575 table XXII the saturated vapour pressures of carbon dioxide between 25°.55 C. and the critical temperature (30°.98 C.) are compared with the formula $log\,\frac{p}{p_k} = f\,\frac{(T - T_k)\,T_k}{T^2}$ which was obtained by keeping in the develop-

[1]) M. PLANCK, Wied. Ann. 15 (1882) p. 457; comp. also Comm. N°. 75 § 3. The quantities C_5 and C_6 are both obtained by an extrapolation, C_5 at $v = v_k$ of a higher temperature to T_k, C_6 along the vapour pressure curve of lower T to T_k.

[2]) S. YOUNG, Proc. Phys. Soc. London 1894/95, p. 602; comp. also Comm. Phys. Lab. Leiden No. 88 p. 54 note 1, KEESOM Thesis p. 86.

ment of $log\ p$ in ascending powers of T^{-1} the second power[1]). While for the other temperatures in the table mentioned the deviations did not exceed 0.01 atm., a deviation of Obs.—Comp. = 0.05 atm. was found for 30°.82 C.[2]). Although it was then held probable that this deviation was to be ascribed to an accidental error of observation, we have afterwards found that a deviation in the same sense and of about the same size also occurs in the results of other observers about saturated vapour pressures of carbon dioxide near the critical point.

A comparison of the results of BRINKMAN's observations (Thesis Amsterdam 1904 pp. 41 and 42) of saturated vapour pressures of methyl chloride and carbon dioxide with the pressures derived by him according to a formula of the same form as the one mentioned above, yields the following differences:

for methyl chloride ($t_k = 143.^\circ 12$):

$$\text{at } t = 137.^\circ 54,\quad 138.^\circ 92,\quad 140.^\circ 26,\quad 141.^\circ 66,\quad 142.^\circ 02$$
$$O—C = +0.02,\quad -0.01,\quad -0.02,\quad +0.03,\quad +0.08;$$

for carbon dioxide ($t_k = 31.^\circ 12$):

$$\text{at } t = 24.^\circ 24,\quad 26.^\circ 08,\quad 28.^\circ 46,\quad 29.^\circ 86,\quad 30.^\circ 40$$
$$O—C = +0.02,\quad -0.02,\quad +0.03,\quad +0.08,\quad +0.07.$$

In both substances investigated one finds below the critical temperature an obvious deviation resembling that found in Comm. N°. 88.

The observations of AMAGAT, Journ. de phys. (2) 1 (1892) p. 288, of the saturated vapour pressure of carbon dioxide fail to give any definite indication about the question treated here because AMAGAT has rounded off the pressures to 0.1 atm. In connection with the preceding statements, however, it deserves attention that TSURUTA, Journ. de phys. (3) 2 (1893) p. 272, when comparing these data with the formula $p = 34.3 + 0.8739t + 0.01135t^2$, also there found an obvious difference $O—C$ at 31.°25 which exceeds by 0.06 atm. that at 31.°35 (crit. temp. according to AMAGAT).

From the data mentioned here one might draw the conclusion that for carbon dioxide and methyl chloride the curve of the saturated vapour pressures, continued to near the critical point, with extrapolation to this point would lead us to expect a p_k somewhat larger than the critical pressure found experimentally. From the very careful

[1]) In Physik. ZS. 8 (1907) p. 944, BOSE went still farther and kept the third power in this development which had been given by RANKINE, Misc. Scientif. Papers pp. 1 and 410,

[2]) As it appears from the columns Obs. and Comp. all the numbers in the column $O.—C.$ have wrong signs.

observations of YOUNG of isopentane Proc. Phys. Soc. London 1894/95, p. 613, however, a deviation as found above for carbon dioxide cannot be derived.

It may be that some connection exists between the above mentioned disturbance in the saturation pressure in the immediate neighbourhood of the critical point of carbon dioxide and a disturbance in the observations of Comm. N°. 88 of the densities of saturated liquid and vapour of carbon dioxide. Plate I represents these densities d_{liq} and d_{vap}, expressed in the theoretical normal density. $\frac{1}{2}(d_{liq}+d_{vap})$ is also represented. The straight line is the line which was drawn for the determination of the critical volume after the method of the rectilinear diameter of CAILLETET and MATHIAS in Comm. N°. 88 (comp. Comm. N°. 88 p. 574). The middle of the chord belonging to 30.°8 lies clearly below this line. If for the determination of the rectilinear diameter only the three points at lower temperature are used, the difference is much larger. If this deviation cannot be ascribed to an error of observation, it would follow hence that the diameter of CAILLETET and MATHIAS for carbon dioxide shows a curvature in the immediate neighbourhood of the critical point [1]). K_{98} indicates the critical density which in Comm. N°. 98 (KAMERLINGH ONNES and FABIUS) was derived from determinations less than 0.°002 deg. below the critical temperature. If we might assume that the carbon dioxide of Comm. N°. 98 and that of N°. 88 possessed the same degree of purity, an assumption to which the agreement between the critical temperatures entitles, and also that the difference in the methods of density determination has not given rise to a systematic difference, then the situation of the point K_{98} would confirm the curvature of the diameter in the neighbourhood of the critical point.

A similar disturbance as we remarked above for the saturation volumes of carbon dioxide in the immediate neigbourhood of the critical point, cannot be derived either from YOUNG's observations of isopentane (comp. Proc. Phys. Soc. London 1894/95 p. 636) or from those of normal pentane (Trans. Chem. Soc. 71 (1897) p. 455),

[1]) This curvature is in another sense than the curvature found by KUENEN and ROBSON, (Phil. Mag. (6) 3 p. (1902) p 624) at lower temperatures in the diameter for carbon dioxide and which agrees with the general rule given by YOUNG (Phil. Mag. (5) 50 (1900) p. 291) about this curvature at lower temperatures in connection with the value of $\frac{RT_k}{p_k v}$ and the slope of the diameter as compared with the temperature axis.

which are continued down to 0°.05 below the critical temperature [1]). It would be very desirable to investigate more closely in how far the disturbances mentioned sub *c* are connected with a disturbance in the equation of state, or must be ascribed to special circumstances of those experiments themselves (such as the difficulty to determine the moment at which begin condensation occurs).

§ 4. The disturbances mentioned in § 3 apparently point to the fact that the substance in the neighbourhood of the critical point occupies a smaller volume than would be expected according to the special undisturbed equation of state. In Comm. N°. 88 p. 555 the possibility is mentioned that these disturbances are connected with differences of density which occur in the substance near the critical state, as it is indicated by the mist (the blue opalescence) in the neighbourhood of that state. The question was left aside whether those differences of density are to be interpreted either as condensations round condensation centres with an existence of their own (dust according to KONOWALOW [2]), electrically charged particles [3]) a third phase separated in small drops and for the greater part consisting of an admixture), or simply as spontaneously formed differences of density, either as accidental aggregations caused by molecular motion and governed by the statistic equilibrium (SMOLUCHOWSKI [4]), or because small drops still have a positive surface tension at temperatures at which larger drops are no longer stable (DONNAN [5]).

Whatever may be the cause of the blue mist, in all cases we may expect a close relation between the compressibility and the occurrence of it. In order to form a better judgment about this matter it was considered to be desirable to start an investigation of the conditions of existence of this mist in a substance consisting of one component in the neighbourhood of the critical point liquid-gas. For an optical research of these conditions of existence we refer to the next communication.

[1]) Nor can a similar disturbance be derived with certainty from BRINKMAN's observations of carbon dioxide and methyl chloride, which observations, however, are not continued so near to the critical point as those of comm. N°. 88.

[2]) D. KONOWALOW. Ann. d. Phys. (4) 10 (1903) p. 360.

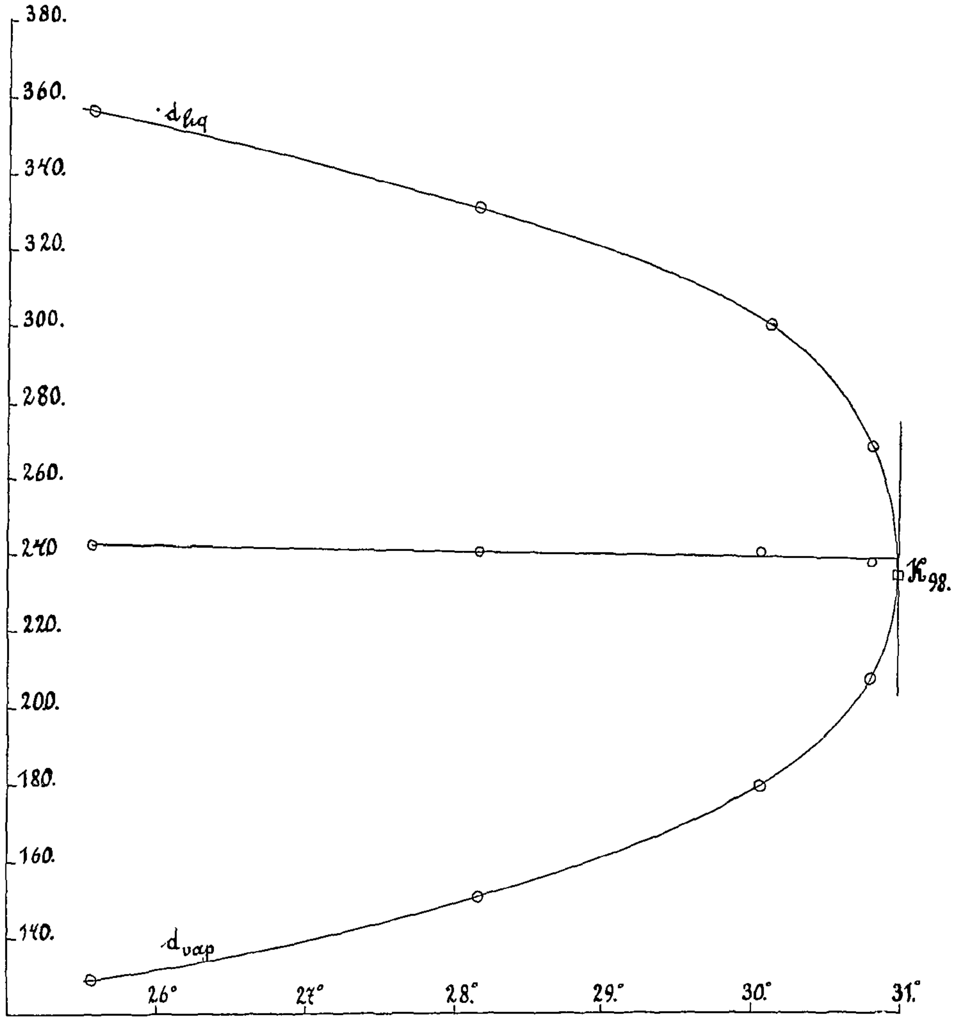
[3]) Owing to the highly penetrating radiation from the radio-active portions of the crust of the earth (EVE, Phil. Mag. (6) 12 (1906) p. 189) in the atmosphere (STRONG, Physik. ZS. 9 (1908) p. 170), or in the surroundings of the building where the experiments are made, these particles would always be present to almost the same amount. In the mean time it follows from the experiment of FRIEDLÄNDER ZS physik. Chem. 38 (1901) p. 385, on the stability of the mist in an electric field, that the particles which cause the opalescence are not charged.

[4]) M v. SMOLACHOWSKI, Ann. d. Phys. (4) 25 (1908) p. 205.

[5]) F. G. DONNAN. Chem. News 90 (1904) p. 139.

Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM. On the equation of state of a substance in the neighbourhood of the critical point liquid-gas. I. The disturbance function in the neighbourhood of the critical state.

Plate I.



Physics. — *"On the equation of state of a substance in the neighbourhood of the critical point liquid-gas.* II. *Spectrophotometrical investigation of the opalescence of a substance in the neighbourhood of the critical state"*, by Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM. Communication N°. 104*b* from the Physical Laboratory at Leiden.

§ 1. *Introduction.* The spectrophotometrical investigation [1]) of the opalescence will have to give an answer to the question how the intensity of the light of a certain wavelength scattered in a certain direction with respect to the incident light and included in a certain angle of vision, in connection with the polarisation state depends on the temperature and the density of the single substance in the neighbourhood of the critical point liquid-gas (cf. Comm. N°. 104*a* § 4). A first quantitative contribution to this investigation is given in this communication [2]).

We have confined ourselves in this first investigation to the determination of.

1. for different temperatures the ratio in which the rays of light of different wave lengths at the same temperature are scattered in a certain direction ;

2. the way in which the intensity of the light of definite wavelength scattered in a certain direction and included in a certain angle of vision changes with the temperature.

On the supposition *a*, that the light emitted by the blue mist is owing to the scattering of the incident light in consequence of part of the substance condensing to particles of the same size (e.g. spheres) round centres which are uniformly spread through the space, the results of the investigation mentioned under 1 will enable us to form an opinion on the size of these particles [3]), that under 2 on the way in which the total quantity of substance which has condensed, varies with the temperature.

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[1]) This investigation was carried out before the interesting article of SMOLUCHOWSKI Ann. d Phys. (4) 25 (Febr. 7, 1908) p. 205, appeared. We still had an opportunity, however, to compare it with some parts of the text (c.f. also the preceding Communication). [Added in the English translation].

[2]) The colorimetric determinations of FRIEDLANDER, ZS. physik. Chem. 38 (1901) p. 385 constitute such a contribution for a mixture of two liquids in the neighbourhood of the critical point of separation.

[3]) Already FRIEDLANDER loc. cit. p. 438 called attention to the importance of such an investigation for the knowledge of the internal structure of the critically turbid media

On the more general supposition b, that the opalescence is the consequence of differences of density e.g. governed by the statistic equilibrium, which extend over parts of the volume of irregular shape and size, a distance can be pointed out which is connected with the average size of these parts of the volume and so with the substance being more or less coarse-grained in that state; and which determines the optical phenomenon in a similar way as the size of the particles on supposition a. The investigation mentioned under 1 will then enable us to judge about this distance. When in future we speak of the size of the light-scattering particles, we shall refer to this distance. In this case the investigation mentioned under 2 will teach us something about the mean deviation of the density in these parts of the volume. This too will be implied in future in "the quantity of condensed substance".

The measurements made by us can, however, be only considered as preliminary ones. As, however, we have to put off the continuation of these measurements for some time, we think that we must not postpone the communication of these provisional results any longer.

§ 2. *The arrangement of the experiments* is represented in Pl. II fig 1. After having passed through a layer of water the light emitted by the luminous body of the NERNST-lamp *Ner* (70 HK) is concentrated by the lenses L_1 and L_2 (to an image of $\pm$ 1 c.M. height) in the tube *Et* filled with ethylene [1]).

The light scattered upwards in the direction of the axis of the tube [2]) by the cloud is concentrated by means of the system of

1) This was obtained by distilling over so much from the ethylene circulation of the cryogenic laboratory into the glass tube with cock fused to it, which had been cooled in liquid air, and rinsed with ethylene, that after the gas phase left above the solid ethylene had been drawn off, and the tube was heated to the melting point, $^1/_3$ of it was filled. Then with the cock closed the tube was removed from the ethylene circulation, and fused together at a previously narrowed place while still partially placed in liquid air. When the temperature rose to that of the room, it appeared, when at 0° the rime cleared away, that in the gas space a thin white deposit was visible on the wall, which evaporated some degrees below the critical temperature of the ethylene. This deposit points to the presence of an admixture which is slightly less volatile than ethylene (cf. VILLARD, Ann. de chim. et de phys. (7) 10 (1897) p. 389). That it was not visible in the liquid space, probably points to a small difference in refrangibility with liquid ethylene.

When we stirred, and then slowly cooled the tube to below the critical temperature, the meniscus appeared in the top of the tube.

2) The top of the tube is surrounded by a black cylindre in order to prevent rays of light received by this part from being reflected upwards, and being thrown into the spectroscope.

lenses L_3, L_4 and L_5, L_6 and the totally-reflecting prism Pr to an image of the beam of light crossing the ethylene tube on the slit of the spectroscope Sp (a direct-vision spectroscope of HILGER-CHRISTIE giving a spectrum of great intensity [1]), in which an eye-piece slit has been made in order to confine a certain portion of the spectrum; by means of the screw Scr_2 different portions of the spectrum may be brought into the field). A beam of the light emitted by the NERNST-lamp is thrown on the slit of the spectroscope by means of plane mirrors through the polarizing prisms Nic_1, Nic_2, Nic_3 (after having been first made parallel by lens L_7) and then through lens L_8 and a totally-reflecting prism. The prisms Nic_1 and Nic_3 are rigid, which ensures that the light thrown into the spectroscope by the reflections on the mirrors with different positions of Nic_2 is reduced in the same proportion; the prism Nic_2 can turn round, and is provided with a graduated circle, which could be read up to 3′. The plane of polarisation of Nic_3 is horizontal so that the condition of polarisation in the two beams thrown into the spectroscope agrees in the main [2]). The plane of polarisation of Nic_1 has been put parallel to that of Nic_3. After the tube of ethylene had been brought to the required temperature, and the temperature of the room had been regulated in such a way that the temperature of the tube of ethylene (read to 0°.01) could be kept sufficiently constant (up to some hundredths of a degree) by the addition, when necessary, of some cold or hot water into the vacuum glass, the prism Nic_2 was adjusted by rotation so as to obtain equal intensity of the considered portions

[1]) See ZEEMAN Comm. N°. 5, June '92, more detailed Arch. Néerl. 27 (1893) p. 259 and Pl. V. The "halfprism" was used in our experiments with a view to the intensity in the magnifying position (CHRISTIE, Proc. Roy. Soc. 26 (1877) p. 8). Moreover the dispersion is greater in this position, whereas the loss of purity in the spectrum is of no importance here.

[2]) Not too near the critical state the light emitted by the blue mist in a direction normal to the incident light, is polarized in the plane of incidence (RAMSAY, ZS. physik. Chem. 14 (1894) p. 486). It is to be expected that on approach to the critical state the light emitted in the direction mentioned becomes more and more partially polarized (cf. TYNDALL, Phil. Trans. 160 (1870) p. 348). It would be interesting to examine if then TYNDALL's residual blue (l. c.) could be observed (on the connection of this with the difference in refractivity of the scattering particles and the surrounding medium see RAYLEIGH, Phil. Mag. (4) 41 (1871) p. 454). Measurements on the condition of polarisation might also lead to an opinion on the size of the particles, see BOCK, Wied. Ann. 68 (1899) p. 674 (spectrophotometrical investigation of the light scattered by a jet of steam, measurement of the condition of polarisation, and determination of the size of the particles by means of diffraction rings) and PERNTER, Denkschr. Kais. Ak. d. Wiss. Wien 73 (1901) p. 301.

of the two spectra. With a view to this adjustment care had been taken that the two spectra were as close above each other as possible [1]) and had about the same height. The adjustment and reading were made in the four different positions of Nic_2 which gave equality of intensity.

§ 3. *Observations*. Only observations above the critical temperature have been communicated here; in order to get unambiguous data for the dependence of the intensity of the opalescence on the temperature and the density below the critical temperature, a stirring-apparatus, or an arrangement to keep the temperature constant till the thermo-dynamic equilibrium should have been reached, would have been required. The observations were made after the tube of ethylene had been kept at higher temperature for 15 hours or longer, and had then been slowly cooled down to the temperature of observation. The measurements have been made for two wavelengths, corresponding to D and F in the solar spectrum [2]). In order to give an idea of the degree of accuracy of the adjustments, we have communicated the data of an observation at a mean intensity of the scattered light in table I.

TABLE I.

Series VI, No. 3, 13 Nov 1907						
Wave length	Temperature	Adjustments of Nic_2				?
D	11° 69	63 48′	36°36′	154°36′	125° 9′	14°10′
	11 66	64 24	36 15	153 30	126 18	13 50
	11. 70	63 18	36 54	154 9	125	13 53
	11. 69					—
		temp. mean 11° 68				mean: 13°58′
F	11 66	155°45′	124°48′	65°18′	34°33′	15°15′.5
	11. 68	156 15	124 54	66 9	33 54	15 54
	11. 67	157 18	122 57	67 50	34 6	16 56
	11 61					—
		temp. mean 11°.66				mean: 16°5′

[1]) The use of a HÜFNER's prism would render more accurate adjustments possible.

[2]) When the experiments are repeated under circumstances which admit of a more accurate spectrophotometric adjustment, an extension of the measurements to more wavelengths will be desirable.

The last column contains the angle of the plane of polarisation of Nic_2 derived from the other columns, for the adjustment at equal intensity, with this plane of polarisation when Nic_2 crosses Nic_1 and Nic_1. In general the adjustment for the wavelength F was less accurate than for D on account of the slighter intensity in the spectrum for the former wavelength. The greater deviation which the latter angle φ shows for the wavelength F in table I from the preceding ones, may be explained from the difference in temperature.

The results thus obtained have been joined in table II.

TABLE II.

Wavelength D		Wavelength F	
Temperature	φ	Temperature	φ
Series V, 12 Nov. 1907			
13°.53	8°27′	13°.59	10°11′
12 .55	9 45.5		
Series VI, 13 Nov. 1907			
12°.54	10°36′	12°.54	12°39′
11 .86⁵	12 37	11 .83	14 58.5
11 .68	13 58	11 .66	16 5
11 .42	17 52	11 43	18 24
11 .24	22 18		

The observations of series VI ceased after the adjustments for the wavelength D at 11.°24, because after this the temperature fell below the critical temperature, which was determined at 11.°18 [1]) (cf. § 3 beginning).

The difference between the angles φ for Series V 12.°55 and Series VI 12.°54, wavelength D, is owing to this that between these observations a slight modification in the position of the lenses L_7, L_8 has taken place. The observations mentioned here may serve

[1]) Comparison of this value of the critical temperature with that of other investigators indicates that the critical temperature of the admixture (cf. § 2, p. 612, note 1) does not lie much higher than that of ethylene.

to bring connection between the series V and VI. The results of other series of observations are not communicated here, because for them all the precautions mentioned had not yet been taken.

From the data of table II the course of the intensity of the scattered light with the temperature (§ 2 2nd) will be derived in the first place. Let us call $H_{D,t}$ the intensity in the spectrum of the light scattered by the cloud at the temperature t of the wave-length D for a certain arrangement of the apparatus, which is further thought to be unmodified, $H_{D comp}$ the intensity in the comparison spectrum when Nic_2 is parallel with Nic_1 and Nic_3, then $i_{D,t} = H_{D,t}/H_{D,11.^{\circ}68} = sin^4\, \varphi_{D,t}/sin^4\, \varphi_{D,11.^{\circ}68}$. An investigation of the absolute intensity of the light scattered by the mist compared with that of the incident light (cf. § 6*b*) will have to reveal how to derive a quantity from $i_{D,t}$ which determines the intensity of the scattered light, independent of the particular circumstances of the arrangement. For an examination of the way in which the intensity of the scattered light depends on the temperature, the quantity $i_{D,t}$ is very suitable.

Table III contains the results obtained on this from table II:

TABLE III.

t	$i_{D,t}$	t	$i_{D,t}$
13°.53	0.190	11.°68	1
12. 54	0.337	11. 42	2.61
11. 86⁵	0.671	11. 24	6.11

These results have been represented in Pl. II fig. 2, where also a curve has been traced through the points of observation (see further p. 620).

The ratio $r_{F:D,t} = \dfrac{H_{F,t}/H_{F\,comp.}}{H_{D,t}/H_{D\,comp.}} = \dfrac{sin^4\varphi_{F,t}}{sin^4\varphi_{D,t}}$ yields data for the inquiry mentioned in § 2 1st. into the ratio in which the light of different wavelengths is scattered. Table IV contains the results.

TABLE IV.

t	$r_{F:D,t}$	t	$r_{F:D,t}$
13°.59	2.00	11°.68	1.66
12 .54	2.01	11 .43	1.18
11 .56⁶	1.85		

To this purpose the angles φ for D and F have been reduced to the same temperature by interpolation.

Above 12°.54 the ratio of the intensities of D and F seems to be constant. The fact already observed by several earlier observers that on approach of the critical temperature the mist changes from blue to almost white, is clearly set forth in the table. Measurements on this change of colour, however, have been communicated here for the first time.

§ 4. *On the size of the light-scattering particles* [1]). To be able to derive from $r_{F:D}$ the ratio of the intensities F and D of the light scattered in a certain direction by the mist compared with the ratio of the intensities F and D of the incident light on the mist, we must bear in mind: 1st that the two beams of light which are compared with each other in the spectroscope are subjected to different reflections and absorptions outside the spectroscope, which might bring about a change in the ratio of the intensities D and F, 2nd that the optical apparatus for observation of the scattered light not being perfectly achromatic might cause a similar change in the ratios of intensities, 3rd that if the condition of polarisation of the two beams is not exactly the same on their arrival in the spectroscope, the reflections in the spectroscope may also give rise to such a change [2]).

The influences mentioned under 1 and 2 may be determined and eliminated by measurements of the scattered light when the substance in the neighbourhood of the critical state has been replaced by a suspension for which the ratios of intensities of the scattered light are known [3]). Then it will have to appear in how far the deviation of the values 2,00 found in table IV at the higher temperatures from that which according to RAYLEIGH (Phil. Mag. (4) 41 (1871) p. 107) would be found if the scattering were brought about by non-conducting particles the dimensions of which are small with

1) Cf. § 1 p. 612.

2) Cf. CHRISTIE loc. cit.

3) Suspensions for which the intensity of the transmitted light is: according to RAYLEIGH $I = I_0\, e^{-k\lambda^{-4}x}$: mastic, Ag Cl, Cu_2S in water, emulsion of lemon-essence in water: ABNEY and FESTING, Proc. Roy. Soc. 40 (1886) p. 378, LAMPA, Wien. Sitz. ber. [2a] 100 (1891) p. 730, HURION, C.R. 112 (1891) p. 1431, COMPAN, C.R. 128 (1899) p. 1226; according to CLAUSIUS $I = I_0\, e^{-k\lambda^{-2}x}$: Ba SO_4 in a mixture of glycerin and water, etc.: COMPAN loc. cit. To ensure that in this experiment the light is subjected to the same reflections as in the experiments with the mist we should have to take a suspension in ethylene of the critical density.

respect to the wavelength: $\lambda^4{}_D/\lambda^4{}_F = 2.129$, is to be explained in this way [1]).

About the influence of what was mentioned under 3 we have made a separate measurement. See for this § 5.

After the corrections indicated in this § have been applied, the data of table IV may serve to give an idea of the size of the particles by the aid of developments such as are given by LORENZ [2]). From the change of $r_{F:D}$ in table IV on approach to the critical temperature may already be deduced that the light-scattering particles must no longer be considered as small with regard to the wavelength at and below 11°.86[5] (i.e. 0°.5 above T_k).

§ 5. *On the quantity of substance which is condensed in the light-scattering particles at different temperatures* [3]). To get to know the intensity of the scattered light at different temperatures, only a correction has to be applied to table III on account of the circumstance mentioned p. 617 under 3. Therefor the condition of polarisation of the scattered light at different temperatures must first be known (cf. p. 613 note 2). An upper limit for this correction may already be given as follows.

In the measurement mentioned in § 4 it appeared that light polarized normal to the slit was weakened to a greater degree in the spectroscope than light polarized parallel to the slit, in such a way that the ratio of the intensities in the spectrum is [4]):

$$H_{D\perp}/H_{D//} = 0.82, \quad H_{F\perp}/H_{F//} = 0.70.$$

If we now suppose that at 13°.53 all the light of wavelength D scattered in a direction normal to the incident light is polarized in the plane of incidence, and that at 11°.24 this light would be totally unpolarized, it follows from this measurement, that at 11°.24 the weakening of the D-light in the spectroscope would be 1.10 times the weakening of the D-light at 13°.53.

To be able to derive from the intensity of the scattered light at different temperatures how the quantity of condensed substance depends on the temperature, we should have to get a somewhat complete insight into the way in which the light is scattered by such

1) Also the fact that the light scattered by the mist must pass through a layer of a certain thickness (± 2 cM.) in the direction of propagation, may cause a deviation in the same direction.

2) L. LORENZ. Vidensk. Selsk. Skr. Copenhagen 6 (1890). Oeuvres Scientifiques 1 p. 405.

3) Cf. § 1 p. 612.

4) Cf. with this the calculations of CHRISTIE Proc. Roy. Soc. 26 (1877) p. 24.

particles, and hence be acquainted with the structure of the particles (cf. § 1), in which also the origin (cf. Comm. N°. 104^a, § 4) would come in for discussion. However, it is to be expected that when the particles are small compared with the wavelength of the light, the intensity of the scattered light will increase proportional to the square of the quantity of condensed substance, whereas when the particles are no longer so small, the increase will take place more slowly.

To whatever cause we may attribute the occurrence of the differences in density, the great compressibility of the substance in the neighbourhood of the critical state will have a preponderating influence on it. Thus e.g. the mean deviation in density governed by the statistic equilibrium (SMOLUCHOWSKI)[1]) will be proportional to $\sqrt{\partial p/\partial \varrho}$ ($\varrho =$ density). If we assume that the substance condenses round centres of attraction which exert forces on the surrounding particles of the substance which per unit of mass are only dependent on the distance, the quantity which is condensed round every centrum of attraction is proportional to [2]) $\partial p/\partial \varrho$.

In order to examine what information the data in table III give on a connection between the intensity of the scattered light and the compressibility, we notice that in the neighbourhood of the critical point $\partial p/\partial \varrho = q_{11}(T - T_k)$, if the average density of the substance differs so little from ϱ_k that the following term $3q_{30}(\varrho - \varrho_k)^2$ may be neglected (so $T - T_k$ not too small).

TABLE V.

t	i_{DtO}	i_{DtC}	$O-C$ in % of O
13.°53	0.190	0 213	— 12
12. 54	0.337	0.368	— 9
11. 86^5	0.671	0.730	— 9
11. 68	1	1	
11. 42	2.61	2.08	+ 20
11. 24	6.11	8.33	— 36

[1]) M. v. SMOLUCHOWSKI, Ann. d. Phys. (4) 25 1908 p. 205.

[2]) In this it is supposed that the condensation is so insignificant that ϱ in a condensed part remains sufficiently near ϱ_k.

In table V the data of table III have been compared with the formula: $i_{D,t} = \frac{0\ 5}{T - T_k}$ ($t_k = 11°.18$, see p. 615).

The – – — — — curve in Pl. II fig. 2 represents i_{DtC}.

The differences $O - C$ are of two kinds:

1. The deviation at 11°.24: this was to be expected in the immediate neighbourhood of the critical temperature, as the formula for T_k would give an infinite intensity; here the influence makes itself felt of following terms in the development of $\partial p / \partial \varrho$, or of the intensity of the scattered light as function of the quantity of substance (see p. 619);

2. also at temperatures further from the critical temperature there is a systematic deviation: the observed curve of intensity ascends here more rapidly than the calculated one. This might among others be in connection with the observation of TRAVERS and USHER [1]), who found that the maximum of the intensity of the opalescence should not lie at T_k, but for SO_2 0°.05 above T_k.

Leaving these deviations out of account we may conclude that on the main the observations conform to the mentioned equation. The deviations from a formula $i_{D,t} = \frac{0.25}{(T - T_k)^2}$ would have been much larger. The correction mentioned in the beginning of this § will not affect this conclusion.

On the supposition that at least when the dimensions of the volume elements in which appreciable condensations or rarefactions are found, are small with respect to the wavelength, the intensity of the scattered light is proportional to the square of the quantity of substance which has condensed round every centrum, or to the square of the mean deviation in density governed by the statistic equilibrium, it follows that our observations rather support the hypothesis of the condensations and the rarefactions caused by the molecular movement and governed by the statistic equilibrium, than the hypothesis of centres of attraction whose number remains constant with varying temperature.

If it appears from further investigations that the absolute value of the intensity of the light scattered by the mist is in harmony with what is to be expected according to the distribution law of BOLTZMANN (cf. SMOLUCHOWSKI, loc. cit.) a connection may be formed between the observations of the intensity of the light scattered by the mist and the disturbance function in the equation of state in

1) M. W. TRAVERS and F. L. USHER. Proc. Roy. Soc. A. 78 (1906), p. 247.

the neighbourhood of the critical point through considerations on the increase of the virial of attraction in consequence of the differences of density [1]).

§ 6. *Remarks on further experiments on the mist in the neighbourhood of the critical state.*

a. When through measurements as treated in § 3 the way in which the intensity of the light scattered by the mist depends on temperature and density, will have been sufficiently brought to light, the determination of this intensity at different heights in a CAGNIARD-LATOUR tube may be substituted for the method of the floating bulbs for the determination of the density at different heights in the tube (See Comm. N°. 98, Sept. '07). If the establishing of the thermodynamic equilibrium is effected by keeping the temperature for a long time sufficiently constant, the determination of the intensity of the scattered light as function of the height in the tube would supply a method for the accurate determination of the experimental equation of state in the immediate neighbourhood of the critical state (cf. Comm. N°. 98 § 1 p. 218).

b. Besides the before-mentioned measurements on the condition of polarisation (§ 2) and the measurements for the sake of the corrections mentioned in § 4, measurements on the ratio between the intensity of the scattered light and that of the incident light would also be desirable. (Cf. § 3 p. 616 and § 5 p. 620). For this purpose measurements might serve in which the ethylene is replaced by a silver mirror forming an angle of 45° with the axis of the tube [2]).

[1]) Cf. M. v. SMOLUCHOWSKI, BOLTZMANN Festschrift 1904, p. 626

[2]) We have in the meantime made a preliminary measurement of the absolute intensity of the scattered light by comparing it with the light reflected from a silver mirror (reflection constants for light polarized perpendicular and parallel to the plane of incidence calculated according to QUINCKE, Pogg. Ann. 128 (1866) p. 541 from determinations of the principal angle of incidence and the principal azimuth by JAMIN, Ann. chim. phys. (3) 22 (1848) p. 311). For this measurement the comparison spectrum had to be intensified by replacing the systems of lenses L_7 and L_8 by stronger combinations. From the angles $\varphi_{Ag} = 31°33'$ and $\varphi_{Et} = 5°4.'5$ we derive that at $t = 11°.93$ the intensity of the light of wavelength D scattered by 1 cM.^{3} of ethylene perpendicular to the direction of incidence per unit angle of vision is $s_D = 0{,}0007$, if the intensity of the incident (unpolarized) light $= 1$.

If we calculate according to RAYLEIGH, Phil. Mag. (5) 12 (1881) p. 86—88, LORENZ, Oeuvres Scientif. I p. 496, $s = \frac{2\pi^2}{N\lambda^4}\frac{(\Delta\mu)^2}{\mu_0^{\,2}}$ (N = number of light-scattering *c.* If would be of interest to investigate whether for a single substance in the neighbourhood of the critical point an increase of viscosity is found as has been noted by OSTWALD [1]) for a liquid mixture in the neighbourhood of the critical point of separation from measurements of STEBUTT, and has been further determined by FRIEDLÄNDER (see p. 611). Perhaps the increase of the viscosity and the size of the light-scattering particles (§ 1) might be brought into relation, and so also the colour of the scattered light.

d. We could not ascertain an influence of RÖNTGEN-rays on the blue mist in ethylene. An investigation might be made as to whether the α-rays or the emanation of radium exert an influence on the mist.

e. FÜCHTBAUER [2]) investigated a mixture of iso-butyric acid and water in the neighbourhood of the critical point of separation ultramicroscopically; he did not succeed in dissolving the cone of light. Nor could we ascertain [3]) the presence of separate light-scattering particles in the mist for a mixture of amylene-aniline with the objective Homog. Imm. $\frac{1}{12}$, eye-piece 4, condenser AA (ZEISS) and as source of light an electric arc lamp (30 Ampère) or solar light (10 Dec. '07), We consider a repetition of this experiment with more intense solar light and with more precautions taken to keep the temperature of the mixture that is ultramicroscopically examined,

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particles per cM³, $\Delta\mu$ deviation from the average refractive index μ_0), and if we express $\Delta\mu$ in terms of the deviation in density according to LORENTZ-LORENZ, and if according to SMOLUCHOWSKI we write $\overline{\delta^2} = -\frac{RT_0}{\nu v_0^2 \left(\frac{\partial p}{\partial v}\right)_0}$ (ν = number of molecules in the light-scattering particle), in which $\left(\frac{\partial p}{\partial v}\right)_0$ for $v_0 = v_k$ can be developed as $\frac{p_0}{v_0 T_0} \mathfrak{p}_{11} (T - T_k)$ (Suppl. N⁰. 6, May '03), we find at $T - T_k = 0.75$ for ethylene $s_D = 0.00075$.

Although our measurement is but preliminary, it leads us to conclude that, at least as far as the order of magnitude is concerned, the intensity of the light scattered by the blue mist in a single substance in the neighbourhood of the critical state agrees with the hypothesis of SMOLUCHOWSKI, that light is due to differences in density caused by molecular motion and governed by statistical equilibrium. [Note added in the English translation].

1) W. OSTWALD. Lehrbuch der allgemeinen Chemie II 2 (2te Aufl. p. 684).

2) CHR. FÜCHTBAUER, Zeitschr. physik. Chem. 48 (1904) p. 552.

3) We express our hearty thanks to Prof. M. DE HAAS of Delft for his kindness to lend us his ultramicroscopic apparatus.

Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM. On the equation of state of a substance in the neigbourhood of the critical point liquid-gas. II. Spectrophotometrical investigation of the opalescence of a substance in the neigbourhood of the critical state.

Plate II.

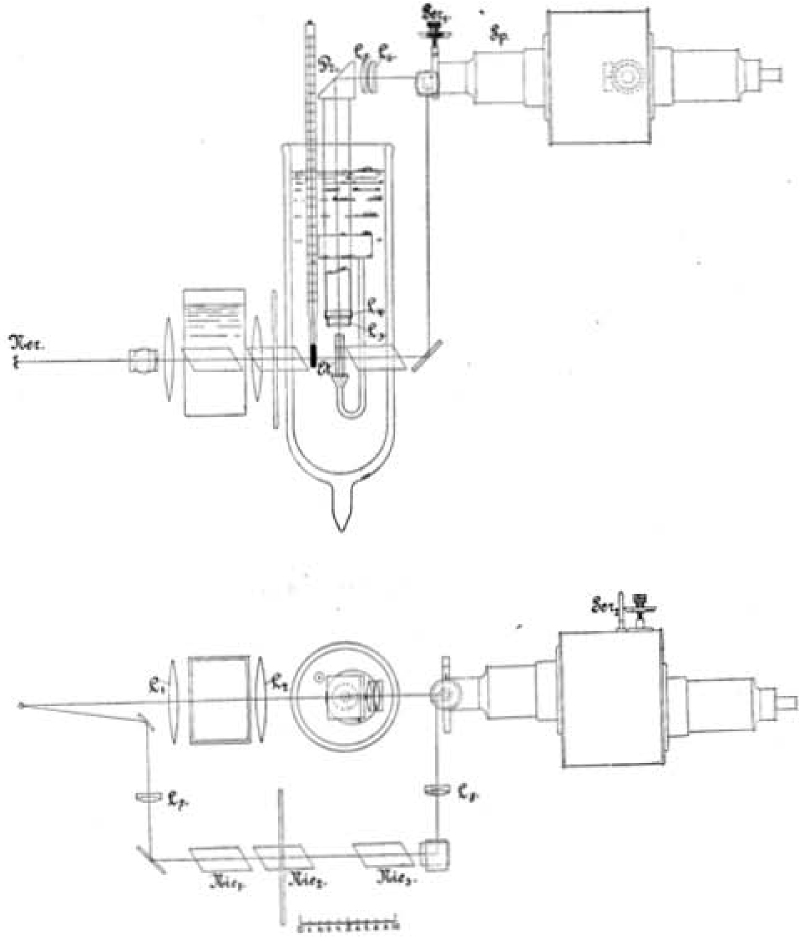

Fig. 1.

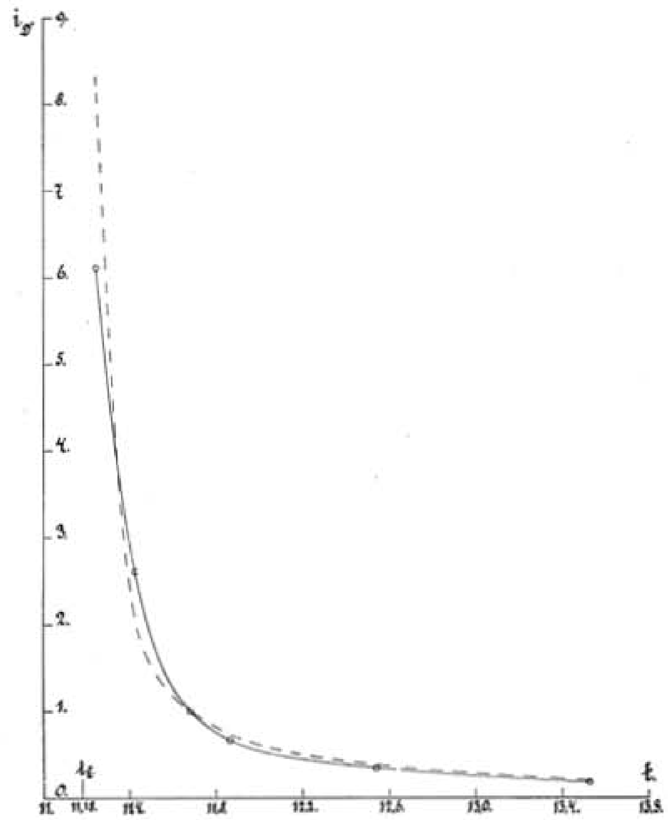

Fig. 2.

at a constant temperature near the critical temperature of separation, desirable, and also such an investigation for a single substance in the neighbourhood of the critical point gas-liquid [1]).

This investigation in connection with what follows (see § 4) from measurements as mentioned in § 3 on the size of the light-scattering particles might give us an idea of the velocity of motion of the light-scattering particles or of the mean time of existence of definite aggregations governed by the statistic equilibrium.

Chemistry. — *"On the form-analogy of Halogene-derivatives of Hydro-carbones with open chains"*. By Dr. F. M. JAEGER. (Communicated by Prof. A. P. N. FRANCHIMONT).

(This paper will not be published in these Proceedings).

ERRATA.

In the Proceedings of the meeting of March 30, 1907:

Plate II belonging to the Communication of Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM: for $\tau = 1.18$ read $\tau = 1.08$; for $\tau = 1.05$ read $\tau = 1.035$.

p. 798 l. 4 from the bottom: for 0.966 read 0.996.

In the Proceedings of the meeting of September 28, 1907:

p. 211 l. 12 from the bottom: for 0.16822 read 0.25234.

In the Proceedings of the meeting of December 28, 1907:

p. 414 l. 7 from the bottom: for 28.955 read 29.030.

1) The possibility is namely not excluded that then the light-scattering particles have larger dimensions and a greater mutual distance than at the critical point of separation of two liquids. To form an opinion on this point a spectrophotometric investigation for a liquid mixture, in the same way as we have made for a single substance (§ 3) would be useful.

(March 27, 1908).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday March 28, 1908.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 28 Maart 1908, Dl. XVI).

CONTENTS.

Astronomy. — *"On the mean star-density at different distances from the solar system."* By J. C. KAPTEYN.

(Communicated in the meeting of February 29, 1908).

In the meeting of April 20, 1901, I derived not only the so-called *luminosity-curve* but also the law according to which the star-density, i. e. the number of stars per unit of volume, diminishes with increasing distance from the solar system [1]). I assumed, and the assumption will again be made in the present paper, that there is no absorption of light in space. I then pointed out that the luminosity-curve is not very appreciably modified if we change, within admissable limits the data from which it was derived. On the other hand it was expressly stated that the determination of the change of density was only quite *provisional.* Its discussion was deferred to a subsequent communication, in which, along with the data then used, other elements might be taken into account. (l.c. p. 731).

These other elements are mainly: the total numbers of stars of different magnitude and their mean parallaxes. As to the first, the numbers: a short time ago I treated all the materials accessible to me (see Publications of the Astr. Lab. at Groningen N°. 18) and I think that I obtained very reliable results for the stars brighter than 11.5, fairly reliable ones down to the 15th magnitude. Now, though the mean parallaxes are still wanting, we are already able, by the numbers alone, to arrive at a considerable improvement in the distribution of the densities, at least for the larger distances. Such a derivation will be given in what follows.

As formerly a separate treatment of the regions of different galactic latitude was not yet attempted, because I think that it will be desirable that in carrying out such a separate treatment we investigate at the same time whether it be admissable or not to assume the same luminosity curve (mixture-law) for the different parts of the system.

Already I have collected fairly extensive materials for this purpose, but still some time will have to elapse before the investigation can be carried out with advantage.

In the main my purpose, in making the following determination, was simply to get first notions about the star-density at still greater distances than could be reached in our former investigation. The determination embraces also the smaller distances but it remains to be seen whether for these the correction found is or is not an improvement.

In the *Astron. Journ.* N°. 566 I derived analytical expressions,

[1]) See also: Publications of the Astr. Lab. at Groningen No. 11.

which fairly represent the numbers found in the communication just mentioned.

Let

$\delta = 2\,512 \ldots = \dfrac{\textit{Apparent brightness of a star of mag } m}{\text{,, \quad ,, \quad ,, ,, \quad ,, \quad ,, ,, } m+1}$ $(log.\ \delta = 0.4)$;

$\varrho =$ distance from the solar system ($\varrho = 1$ for parallax $= 0''1$);

$N_m =$ number of stars in the whole of the sky between the apparent magnitudes $m - \frac{1}{2}$ and $m + \frac{1}{2}$.

$\Delta(\varrho) =$ star-density $=$ number of stars per unit of volume at distance ϱ (unit of volume $=$ cube, each side of which $=$ unit of distance).

$h_m =$ apparent brightness of a star of the magnitude m ($h_{5.5} = 1$).

$L =$ Luminosity $=$ total quantity of light emitted ($L = 1$ for Sun).

$\varphi(L)\,dL =$ probability that the luminosity of a star, chosen at random, is contained between L and $L + dL$.

$\psi(L) = \int_{\frac{L}{\sqrt{\delta}}}^{L\sqrt{\delta}} \varphi(z)\,dz =$ probability that the luminosity is contained between the limits $L \pm \frac{1}{2}$ mag.

Now, if we assume that $\varphi(L)$ is not dependent on ϱ, we shall have

$$N_m = 4\pi \int_0^\infty \varrho^4 \Delta(\varrho)\,d\varrho \int_{\frac{h_m}{\sqrt{\delta}}}^{h_m\sqrt{\delta}} \varphi(h\varrho^2)\,d\varrho = 4\pi \int_0^\infty \varrho^2 \Delta(\varrho)\,\psi(h_m\varrho^2)\,d\varrho \quad (1)$$

The expressions derived in *Astron. Journ.* N°. 566 are:

$$\psi(L) = \frac{\alpha^2 \,.\, mod.}{\sqrt{\pi}\,L}\, e^{-\alpha^2 [log\, L - T]^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\Delta(\varrho)}{\Delta(0)} = e^{-\beta\varrho} + \beta\varrho\, e^{-\gamma\varrho^2} \quad . \quad . \quad . \quad . \quad (3)$$

in which

$$\left.\begin{aligned} T &= 1.400 \\ \alpha^2 &= 0.385 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\Delta(0) = 111.0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\left.\begin{aligned} \beta &= 0.0220 \\ \gamma &= 0.0052 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In a subsequent part of the same paper, the *numbers* of the stars as given by PICKERING led to a new value of $\Delta(0)$ *viz.*

$$\Delta(0) = 136.9 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the difference of this value and the value (5) is wholly explained by the constant difference of the photometric scale of Potsdam, which wa

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used for the determination (5) and that of Harvard which served for the derivation of (7).

In what follows the magnitudes have also be reduced to the Harvard scale. I have adopted the luminosity-curve (2), in which the constants have the values (4), without any change. For the density curve (3), however a new determination was obtained by the aid of the total numbers of the stars of different apparent magnitude. In other words: by the aid of formula (1) I derived Δ as a function of ϱ from the *given* values N_m ($m = 2$ to 15) and the *given* form of ψ.

The introduction of the analytical functions (2) and (3) has the advantage of greatly facilitating the computations. Of course we have not to forget, however, that they can be relied on only just to the same extent as that for which we possess observational data. For the luminosity-curve, with the exception only of the stars belonging to the classes of the very greatest apparent brightness, the unlimited use of the formula will not easily give rise to appreciable errors, because extrapolation is only necessary for a very small fraction of the total. On the contrary, the density-curve (3) (which, as we already remarked, is not very accurately determined) furnishes values, which, for ϱ exceeding 60, are to be considered as wholly obtained by *extrapolation*. It will appear from what follows that up to $\varrho = 60$ the values derived from the new materials do not differ from those formerly obtained more than seems in accordance with their uncertainty. That on the other hand, the values for $\varrho > 60$, which we may extrapolate by means of formula (3), are *far* too small; to such an extent that for these greater distances the formula is evidently quite unsatisfactory.

To begin with, I ascertained how the formula (3), in which the constants have the values (6) and (7), represents the N_m of publication 18. A table of the integrals entering in the formula (1) has been given in *Astronomical Journal* N°. 566 for values of m between 0 and 11. (table III) [1]).

[1]) In the calculation of the values of T_1 and T_3 a mistake has been discovered:

			T_1					T_3		
For $m =$ 3 0,	instead	of	9.12	read	9.13					
4.0	„	„	12.69	„	12.71					
8.0	„	„	17.04	„	17.10					
6.0	„	„	21.86	„	21.99					
7.0	„	„	26.63	„	26.88					
8.0	„	„	30.54	„	30.96					
9.0	„	„	32.80	„	33.42	Instead	of	1.71	read	1.72
10.0	„	„	32.84	„	33.64	„	„	1.37	„	1.38
11.0	„	„	30.54	„	31.51	„	„	1.04	„	1.05

For $m = 14$ the values were now expressly computed. The result is as follows:

Total number of stars.

m	Obs. (Publ. 18)	Comp. I. (by Form. (3))	Comp. II.
4.5 tot 5.5	1 848	1 897	1 788
6.5 „ 7.5	17 940	18 420	18 650
8.5 „ 9.5	159 200	140 200	169 500
10.5 „ 11.5	1 275 000	808 200	1 335 000
13.5 „ 14.5	23 680 000	6 500 000	20 800 000

. . (8)

The deviation increases strongly with diminishing brightness and is excessive for magnitude 14. We conclude at once, that for the greater distances the formula (3) furnishes a value of the star-density which is much too small. Calculation shows that some approximate agreement is already obtained if we take the stars between $\varrho = 140$ and $\varrho = \infty$ to be 21 times more numerous.

As it thus appears that formula (3) is useless for considerable values of ϱ, I began by retaining that formula exclusively for the values of ϱ below 70 whereas for the values exceeding 70 I assumed that the density diminishes regularly (linearly) from 0.214 to *zero*.

It was easily ascertained that, if we choose the decrease of the density in such a way that it vanishes for $\varrho = 557$, we get considerably nearer to the truth, especially if we take:

$$\Delta_0 = 125.$$

The values obtained in this way were put down in the above table under the head Comp. II.

As a further approximation I also derived corrections for the star-density at distances below 70. It appeared that the results become more satisfactory if the linear decrease of the densities is assumed to begin for distances somewhat smaller than 70.

Having obtained this result I have no further continued these approximations, but I have given up the formula (3) altogether and have tried to determine the luminosity-curve directly in the assumption that, for the intervals between $\varrho = 0$ and $\varrho = 10$; $\varrho = 10$ and $\varrho = 30$; $\varrho = 30$ and 50; $\varrho = 50$ and $\varrho = g$ the density changes linearly in such a way that it vanishes for $\varrho = g$.

In this way the problem is reduced to the derivation of the 5 unknown quantities:

$$\Delta(0)\ ;\ \Delta(10)\ ;\ \Delta(30)\ ;\ \Delta(50)\ ;\ g.$$

For reasons given in the paper quoted above we have to assume that $\frac{\partial \Delta}{\partial \varrho}$ vanishes for $\varrho = 0$. As a consequence $\Delta(10)$ will certainly be little different from $\Delta(0)$. Therefore, in order to reduce the number of unknown quantities as much as possible, I took, in agreement with what was formerly found:

$$\Delta(10) = 0.97\, \Delta(0) \quad \dots \dots \dots \quad (9)$$

The number of unknown quantities is thus lowered to 4.

Putting

$$\frac{\Delta(\varrho)}{\Delta(0)} = D_\rho \quad \dots \dots \dots \quad (10)$$

we have

$$D_\rho = A\varrho + B \quad \dots \dots \dots \quad (11)$$

in which, for the several intervals, A and B have the following values:

$$\left.\begin{array}{l|c|c}
 & A & B \\
\varrho = 0 \text{ to } 10 & \frac{1}{10} D_{10} - \frac{1}{10} & 1. - \\
10 \;,,\; 30 & \frac{1}{20} D_{30} - \frac{1}{20} D_{10} & \frac{3}{2} D_{10} - \frac{1}{2} D_{30} \\
30 \;,,\; 50 & \frac{1}{20} D_{50} - \frac{1}{20} D_{30} & \frac{5}{2} D_{30} - \frac{3}{2} D_{50} \\
50 \;,,\; g & -\frac{1}{g-50} D_{50} & \frac{g}{g-50} D_{50}
\end{array}\right\} \quad \dots \quad (12)$$

The practical advantage of the present form is that the expression (1) for N_m can now be reduced to the well known integral

$$\Theta(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-x^2} dx \quad \dots \dots \dots \quad (11)$$

Numerical integration is thus avoided and we obtain relations which are linear in respect to the unknown quantities D_{30}, D_{50} and $\frac{1}{\Delta(0)}$.

If we denote by $(N_m)_0^\rho$ the number of stars between apparent magnitude $m - \frac{1}{2}$ and $m + \frac{1}{2}$ existing between the distance 0 and ϱ, we get, substituting (2) (10) (11) in (1):

$$(N_m)_0^\rho = [G\,A + H\,B]\, \Delta(0) \quad \dots \dots \quad (12)$$

in which $\mu = mod.$ of the Nep-Log.

$$G = \frac{2\pi a}{h_m} e^{-\frac{0.4\,m - 2.20 + T}{\mu} + \frac{1}{4a^2\mu^2}} \Theta\left[a(2.20 - T - 0.4m + 2 log\varrho) - \frac{1}{2a\mu}\right] \quad (13)$$

$$H = \frac{2\pi a}{h_m} e^{-\frac{0.4\,m - 2.20 + T}{2\mu} + \frac{1}{16a^2\mu^2}} \Theta\left[a(2.20 - T - 0.4m + 2 log\varrho) - \frac{1}{4a\mu}\right] \quad (14)$$

and

$$log\ h_m = 2.20 - 0.4\,m \quad . \ . \ . \ . \ . \ . \ . \quad (15)$$

As soon as the $(N_m)_0^{\varrho}$ have become known we find the $(N_m)_{\varrho_1}^{\varrho_2}$ by simple subtraction.

I have carried through the solution for the values 400, 600, 800 and 1000 for g. It appeared that only when we come to the last value we get satisfactory results.

It seems superfluous to give all my calculations in full. I will only communicate some of the values obtained with the constant

$$g = 1000 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (16)$$

which was finally adopted. The value of the G and H were found to be as follows. (See table I p. 632).

Now, if for the stars of magnitude 2, 3, 4, 5, we take the numbers found by PICKERING for the whole of the sky, *viz.* resp. 58, 172, 577, 1848 [1]) and for the remaining magnitudes, the numbers which we derive from table 2 of the Groningen Publication N°. 18, by simply multiplying with 41 253 (the number of square degrees for the whole of the sky), we find equations of condition for the derivation of the unknown quantities $\Delta(0)$, D_{80}, D_{60}, such as this:

$$58 \frac{1}{\Delta(0)} = 0.2962 + 0\ 0411\, D_{80} + 0.0244\, D_{60}$$

etc. They get a more convenient form if we put $\frac{140}{\Delta} = Z$ and if we then divide all the equations by the coefficient of Z. In this way the equations of condition become as follows:

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1) In Publ. 18. p. 8 I found, by countings made on the materials of PICKERING:

$$N_{1.495}^{2.495} = 58;\ N_{2.495}^{3.495} = 171;\ N_{3.495}^{4.495} = 574;\ N_{4.495}^{5.495} = 1837.$$

With the aid of the computed values communicated in the same publication it is easy to pass from these to the numbers $N_{1.5}^{2.5}$ etc. The results thus found are those of the text.

TABLE I.

	G.				H.			
	$\rho = 10$	30	50	1000	$\rho = 10$	30	50	1000
$m = 2$	0.9341	2.365	3.105	4.834	0.2463	0 3302	0.3496	0 3670
3	3.406	10.75	15.39	30.29	0.7842	1 203	1.324	1.461
4	11.22	44.74	70.58	188.1	2.314	4.179	4.847	5.813
5	33.29	169.5	296.8	1 151.	6.275	13.66	16.94	23.12
7	210.9	1 795.	3 937.	39 430.	34.79	116.5	171.0	361.9
9	850.9	12 380.	34 570.	1 107 000.	128.0	696.0	1 254.	5 425.
11	2 163.	54 580.	195 700.	23 250 000.	304.6	2 778.	6 288.	72 510.
13	3 436.	151 700.	703 600.	340 600 000.	460.9	7 203.	20 780.	784 150.
15	3 387.	264 000.	1 589 000.	3 329 000 000.	438.6	11 900.	44 120.	6 275 000.

$$\left.\begin{array}{llll}
(m=2) & 0.099\,D_{30} & +\,0.059\,D_{60} & -Z=0.715 \\
(m=3) & 0.186 & +\,0\ 146 & -Z=0.836 \\
(m=4) & 0.272 & +\,0.287 & -Z=0\ 817 \\
(m=5) & 0.375 & +\,0.534 & -Z=0.781 \\
(m=7) & 0.527 & +\,1.474 & -Z=0.595 \\
(m=9) & 0.508 & +\,3\ 108 & -Z=0.345 \\
(m=11) & 0.341 & +\,5.187 & -Z=0.149 \\
(m=13) & 0.159 & +\,6.926 & -Z=0.047 \\
(m=15) & 0.050 & +\,7.148 & -Z=0.010
\end{array}\right\} \quad . \quad . \quad (17)$$

In solving these equations I have neglected those corresponding to the magnitudes 2 and 3. The reason is that for these the influence begins to be sensible of stars of so great a luminosity that *extrapolation* beyond the directly determined part of the luminosity-curve becomes necessary. These stars might therefore rather be used for a correction of this curve at its brighter extremity.

The remaining equations have been condensed into three by combining those for $m = 4$ and 5, those for 7, 9, 11 and those for 13 and 15. The solutions of these three equations is:

$$\left.\begin{array}{l}
Z = 1.002 \quad \text{therefore } \Delta(0) = 139.7 \\
D_{30} = 0.460 \\
D_{60} = 0.1315
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (18)$$

whereas we already assumed

$$D_{10} = 0\ 970 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If with these values we compute the numbers N_m and if further we interpolate those for $m = 6$, 8, 10, 12, 14 we get the following comparison between theory and observation:

TABLE II. TOTAL NUMBER OF STARS.

m	Obs.	Comp.	$O-C$ in fraction of whole obs. numb.
2	58	44 5	+ 0.233
3	172	161.5	+ 0.061
4	577	564.3	+ 0.022
5	1 848	1 889	— 0.022
6	5 816	6 025	— 0.036
7	17 940	18 450	— 0 028
8	54 040	54 580	— 0 010
9	159 200	157 200	+ 0.013
10	457 900	448 000	+ 0.022
11	1 275 000	1 256 000	+ 0 015
12	3 453 000	3 490 000	— 0.011
13	9 157 000	9 419 000	— 0.028^5
14	23 680 000	24 100 000	— 0.018
15	60 225 000	58 500 000	+ 0 029

If, in accordance with what has been said, we except the very brightest magnitudes, the deviations are doubtlessly below the uncertainties in the determination of the observed numbers of the stars. The somewhat irregular course of the numbers is probably due to the discontinuities in the density-curve as definitively adopted.

The following table may serve to get at least some insight in the distribution of the stars of a determined apparent magnitude over the different distances.

TABLE III. NUMBER OF STARS $(N_m)_{\rho_1}^{\rho_2}$

ρ	$m=3$	5	7	9	11	13	15
0 to 10	108.	863	4770	17 500	42 000	63 000	60 000
10 » 30	45.5	779	8340	56 100	236 000	625 000	1 030 000
30 » 50	5.5	145	2350	3 400	144 000	543 000	1 250 000
50 » 1000	2 5	103	3000	59 900	835 00	8 188 000	56 150 000

or expressed in fractions of the totals

TABLE IV.

ρ	3	5	7	9	11	13	15
0 to 10	0.669	0.457	0.259	0 112	0.033	0.007	0.001
10 » 30	.282	.412	452	.357	.188	066	.018
30 » 50	.034	.077	.127	.150	.115	.058	.021
50 » 1000	015	.054	162	382	.665	.870	.960

Summing up we find: that the total numbers of stars of different magnitude (Harvard scale) as derived from observation in *Publications of the Groningen Laboratory No.* 18, are well represented by adopting the luminosity-curve (2), with the values (4) of the constants and the following values of the star-density:

TABLE V. STAR-DENSITY.

ρ	Δ	ρ	Δ
0	1.000$\Delta(0)$	100	0.125$\Delta(0)$
10	0 970	200	0.111
20	0 715	300	0.097
30	0 460	400	0.083
40	0.296	500	0.069
50	0.131^{5}	600	0 055
60	0.130	700	0.042
70	0 129	800	0.028
80	0.127	900	0 014
90	0.126	1000	0 000

in which $\Delta(0) = 139.7$.

It would be interesting to investigate what are the changes that can be made in these values without their ceasing to represent the observed numbers satisfactorily. I have deferred this investigation for the present because it will be desirable in such a discussion to include also the data furnished by the parallaxes.

Mathematics. — *"On fourdimensional nets and their sections by spaces."* (Second part). By Prof. P. H. Schoute.

(Communicated in the meeting of February 29, 1908).

The net (C_8).

1. The problem to determine the section of the net (C_8) with a given space can be naturally divided into two parts. The first part occupies itself with the question, how a series of spaces parallel to the given one intersects an eightcell; in the second is indicated, how the section of each of the eightcells intersected by the given space can be deduced from that section which determines this space in the eightcell assumed in the first part. Of course the four series of parallel spaces normal to an axis of the eightcell come here to the fore and then in the first part of the problem are investigated in the first place the so-called "transition forms" where the intersecting space contains one or more vertices of the eightcell, whilst between each pair of transition forms adjacent to each other a single intermediary form is introduced, namely that one by the space which bisects the distance between the two spaces bearing those transition forms. Generally this is sufficient for our end; moreover it is not difficult to interpolate where necessary other intermediary forms.

In the preceding communication of the same title we have packed up each of the cells C_{16} of the net (C_{16}) and each of the cells C_{24} of the net (C_{24}) in the smallest possible eightcell with edges parallel to the axes of coordinates, with the intention to connect the spacial sections of the nets (C_{16}) and (C_{24}) with those of the net (C_8) by cutting with each C_{16} and each C_{24} also the case C_8 enclosing these cells. With a view to this application we add to the above indicated four series of parallel intersecting spaces two others, viz. those normal to one of the two lines connecting the origin of coordinates with the point $(3, 1, 1, 1)$ and the point $(2, 1, 1, 0)$; indeed, these lines are — see the last table of the preceding communication — axes of one or more of the cells C_{16} and C_{24} enclosed in a cell C_8. Also for these two new series we restrict ourselves to the forms of transition and the intermediate forms lying in the middle between two adjacent forms of transition.

In order to simplify the survey of the sections appearing in the six series of parallel spaces we give the results to which the first part — the determination of the section with one C_8 — leads in two different ways. In the first place we project all vertices, edges, faces, bounding bodies of the cell C_8 on the axis normal to each of the six series of spaces to deduce the sections from this tabularly; in the second place we indicate the sections themselves in parallel perspective in the eightcell. To each of those two closely allied modes of transacting an extending plate is given.

To promote the uniformity we indicate the axes OE, OK, OF, OR by their ends $(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 0)$, $(1, 0, 0, 0)$. Then we have to deal successively with the six series

$(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 0)$, $(1, 0, 0, 0)$, $(3, 1, 1, 1)$, $(2, 1, 1, 0)$

and we have now to investigate for each of those six cases the two parts into which the problem was above divided.

2. *Case* $(1, 1, 1, 1)$. — This case was, as far as the first part of the problem is concerned, completely solved in a foregoing study (*Proceedings*, Jan. 1908, page 485). Hence the first part of the first plate with the superscription $(1, 1, 1, 1)$ OE_8 is an extension of the first diagram $n = 4$ of the plate given then. In order to be able to indicate together with the projections of all bounding elements the projections of the vertices of these elements, which considerably promotes the insight into the spacial figure, the numbers of edges, faces, bounding bodies are denoted here outside the scheme on the righthand side. Moreover the sections of the eightcell with the spaces of transition and the intermediate spaces perpendicular to the diagonal of projection are mentioned tabularly; here use has been made of a method formerly (*Verhandelingen*, volume IX, n°. 4) developed in all details which acquaints us not only with the characteristic numbers (e, k, f) of each section, but also with the nature of the faces. Thus the central section is a $(6, 12, 8)$, because it contains 6 vertices and does not cut an edge, intersects 12 faces and contains no edges, intersects 8 bounding cubes and contains no faces; this section is a regular octahedron in connection with which each cube of the two quadruples of bounding bodies is cut according to an equilateral triangle of the same size. In this way the adjacent intermediary section is a $(12, 18, 8)$, because 12 edges, 18 faces and 8 bounding cubes are intersected, viz. a tetrahedron regularly truncated at the vertices, i. e. the first of the semi-regular Archimedian polyhedra (*Proceedings*, page 488) because four of the bounding cubes are intersected according to regular hexagons, the four remaining ones according to equilateral triangles. Here the number of edges is found back as half of the total number of sides of the faces, thus 12 as half the product of eight and three, 18 as half the sum of four times six and four times three. Moreover, when indicating the polygons lying in the faces, we have underlined the figure of each group of regular polygons.

The second plate indicates the obtained sections in parellel perspective. The first diagram on the top leftside, represents an eightcell which indicates besides the diameters normal to the different series of parallel intersecting spaces a few other lines appearing in the solution; for our case (1, 1, 1, 1) to which the four following diagrams refer the axis EE' is this diameter. To characterize this case the mark $(1, 1, 1, 1)$ is noted down to the right at the bottom in the rectangle; moreover the fractions $\frac{4}{8}, \frac{3}{8}, \frac{2}{8}, \frac{1}{8}$ placed to the left at the top of each diagram indicate the part of the axis EE' lying with E on the same side of the intersecting space. It is easy to follow in these diagrams the changes in form which each face of the regular octahedron forming the central section undergoes, when the point of intersection of the intersecting space with the axis OE moves from O to E. Thus the face lying in the upper cube of the eightcell, which is at the same time the visible upper plane of the octahedron regarded by itself, transforms itself first into a regular hexagon, then into an equilateral triangle of opposite orientation, etc.; if the eightcell is a $C_8^{(2)}$, then the sides of the triangles of the first and third diagrams are $2\sqrt{2}$, those of the hexagons and the triangles of the second and fourth diagrams are $\sqrt{2}$, whilst the series closes with the transition form consisting of the single vertex E to which the fraction $\frac{0}{8}$ answers.

We now arrive at the question how the remaining eightcells that are likewise cut by the intersecting space are intersected in each of the considered cases. To this end we suppose the above intersected eightcell to be the central one of the net and so we assume the centre of this cell to be the origin of the system of coordinates with respect to which we have determined in the first communication the coordinates of the centres of the remaining cells in the symbolic form $(2a_i)$. The equation of the central space perpendicular to the axis OE_8 towards the point $(1, 1, 1, 1)$ is $x_1 + x_2 + x_3 + x_4 = 0$; the length of the normal let down out of the centre $(2a_i)$ on to this space is therefore Σa_i. So the eightcell with the centre $(2a_i)$ is cut by the central space $\Sigma x_i = 0$, when $-2 \leqq \Sigma a_i \leqq 2$, and here the five cases occur where Σa_i has one of the values $-2, -1, 0, 1, 2$. In other words: if with the central cell the central section $\frac{4}{8}$ makes its appearance, then with the remaining cells the sections $\frac{0}{8}, \frac{2}{8}, \frac{4}{8}, \frac{6}{8}, \frac{8}{8}$ occur and no others. The sections $\frac{0}{8}$ and $\frac{8}{8}$ being points and therefore not under consideration, we find as section of the net (C_8) a threedimensional space-filling consisting of two groundforms, octahedron and tetrahedron, where the tetrahedron occurs in two positions of opposite orientation. From a close consideration of this result follows now that the fractional symbols of the intersected cells furnish *in general* differences of multiples of quarters with that of the central cell and are thus represented by $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ when the symbol of the central cell is $\frac{3}{8}$ or $\frac{1}{8}$. We find then again a threedimensional space-filling consisting of two groundforms each of which appearing in two oppositely orientated positions, the first semi-regular Archimedian body and the tetrahedron. As we arrive again at eightcell and tetrahedron when starting from the section $\frac{2}{8}$ of the central cell, the above-mentioned two cases are for this series the only ones where the threedimensional space-filling consists of two groundforms. In every other case — as e. g. the one answering to the fractions $\frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{13}{16}$ — we find four different groundforms and never more; we recommend the designing of the just mentioned quadruplet of sections as a good practice.

If we exchange the infinite system of cells $C_8^{(2)}$ by a finite block of k^4 cells $C_8^{(2)}$ forming together a $C_8^{(2k)}$, if we divide a diagonal of this block into eight equal parts and if we suppose the block to be intersected by a space standing in one of the points of division perpendicular to the diagonal, we then find according to circumstances either a finite system of octahedra $O^{(2\sqrt{2})}$ and tetrahedra $T^{(2\sqrt{2})}$ with edges $2\sqrt{2}$, or a finite system of Archimedian bodies $A^{(\sqrt{2})}$ and tetrahedra $T^{(\sqrt{2})}$ with edges $\sqrt{2}$, enclosed in an octahedron, a tetrahedron or an Archimedian body of greater size, viz., in the section of the block $C_8^{(2k)}$ with the intersecting space. In connection with the notes joined to the pages 15, 16 and 24 of the study "On the sections of a block of eightcells, etc." (*Verhandelingen*, volume IX, n°. 7) we here indicate how large in each of those cases the number of the component parts $O^{(2\sqrt{2})}$, $A^{(\sqrt{2})}$, $T^{(2\sqrt{2})}$, $T^{(\sqrt{2})}$ is. We restrict ourselves here to mentioning the results and we only remind the readers that the deduction of these are based on the actual connection

$\frac{1}{4}\, C_8^{(2k)} = T^{(2k\sqrt{2})}$	$T_p^{(2\sqrt{2})} \ldots\ (k+2)_3$ $T_n^{(2\sqrt{2})} \ldots\ (k)_3$	$O^{(2\sqrt{2})} \ldots\ (k+1)_3$
	Sum $\frac{1}{2}k(k^2+1)$	
$\frac{3}{8}\, C_8^{(4k)} = A^{(2k\sqrt{2})}$	$T_p^{(2\sqrt{2})} \ldots \frac{1}{6}k(23k^2+6k-2)$ $T_n^{(2\sqrt{2})} \ldots \frac{1}{6}k(23k^2-6k-2)$	$O^{(2\sqrt{2})} \ldots \frac{1}{6}k(23k^2-1)$
	Sum $\frac{1}{6}k(69k^2-5)$	
$\frac{1}{2}\, C_8^{(2k)} = O^{(2k\sqrt{2})}$	$T_p^{(2\sqrt{2})} \ldots\ 4(k+1)_3$ $T_n^{(2\sqrt{2})} \ldots\ 4(k+1)_3$	$O^{(2\sqrt{2})} \ldots \frac{1}{3}k(2k^2+1)$
	Sum $k(2k^2-1)$	
$\frac{1}{8}\, C_8^{(4k+2)} = T^{(2k+1)\sqrt{2}}$	$T_p^{(\sqrt{2})} \ldots (k+3)_3$ $T_n^{(\sqrt{2})} \ldots (k)_3$	$A_p^{(\sqrt{2})} \ldots (k+2)_3$ $A_n^{(\sqrt{2})} \ldots (k+1)_3$
	Sum $\frac{1}{3}(2k^3+3k^2+7k+3)$	
$\frac{3}{8}\, C_8^{(4k+2)} = A^{(2k+1)\sqrt{2}}$	$T_p^{(\sqrt{2})} \ldots \frac{1}{6}k(k+1)(23k+34)$ $T_n^{(\sqrt{2})} \ldots \frac{1}{6}k(23k^2+12k-11)$	$A_p^{(\sqrt{2})} \ldots \frac{1}{6}(k+1)(23k^2+19k+6)$ $A_n^{(\sqrt{2})} \ldots \frac{1}{6}k(23^2+27k+10)$
	Sum $\frac{1}{3}(46k^3+69k^2+29k+3)$	

between the coefficients of the different powers of x in the development of $(1+x+x^2+\ldots+x^{k-1})^4$ and the numbers of cells $C_8^{(2)}$ of the block $C_8^{(2k)}$ which agree with each other in projection on a diagonal. In the following table of results we have separated from one another the three cases leading to sections $\frac{1}{4}C_8^{(2)}=T^{(2\sqrt{2})}$, $\frac{1}{2}C_8^{(2)}=O^{(2\sqrt{2})}$ and the two cases leading to sections $\frac{1}{8}C_8^{(2)}=T^{(\sqrt{2})}$, $\frac{3}{8}C_8^{2}=A^{(\sqrt{2})}$. Moreover, the two positions of opposite orientation appearing for T and A are distinguished from each other as T_p, T_n and A_p, A_n, and then those parts $T^{(\sqrt{2})}$ and $A^{(\sqrt{2})}$ get the same foot-index which answer not only as regards volume but also as regards position of juncture to the relation

$$A^{(\sqrt{2})}+4\,T^{(\sqrt{2})}=T^{(3\sqrt{2})},$$

whilst this index is a p (positive) for $T^{(2\sqrt{2})}$ when this tetrahedron agrees in position to $T^{(2k\sqrt{2})}$ and $A^{(2k\sqrt{2})}$, and can be taken arbitrarily in the third case $O^{(2k+2)}$, where the two amounts are indeed equal.

In this table the symbols $(k+2)_3$, etc. represent binomial coefficients. The coming to the fore of the numerical factor 23 is connected with the relation holding only for the volume

$$A^{(\sqrt{2})}=23\,T^{(\sqrt{2})},$$

which ensues immediately from the one given above. It forms part of

$$\frac{O^{(2\sqrt{2})}}{32}=\frac{A^{(\sqrt{2})}}{23}=\frac{T^{(2\sqrt{2})}}{8}=\frac{T^{(\sqrt{2})}}{1},$$

of which we have availed ourselves when arranging the preceding table, either as an aid in the calculation or as control.

The cases (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0). — These three cases are so much simpler than the preceding one, that we can treat them collectively, now that the application of the results appearing here to the nets (C_{16}) and (C_{24}) make a short treatment necessary. The projection of the bounding elements on the corresponding axes OK, OF, OR are immediately found; in order to take into account the duality, appearing on one hand between OE and OR and on the other hand between OK and OF, the projections on OR are placed on the first plate next to those on OE, whilst the projections on OK and OF find a place there side by side. A single glance given to these diagrams already arouses the conviction that the sections in the direction of DE over OK and OF to OR must keep on becoming simpler. That this is really the case — and for what reason — is clearly evident from the second plate, giving the sections for the cases OK and OF. As is shown in the three diagrams with the fractional symbols $\frac{3}{6}, \frac{2}{6}, \frac{1}{6}$ belonging to OK here one of the dimensions of the section, viz. the dimension in the direction of the edge with K as centre, is of constant length, by which the sections become prisms with a height 2, namely an hexagonal prism $H^{(\sqrt{2})}$, a triangular prism $P^{(2\sqrt{2})}$ and a triangular prism $P^{(\sqrt{2})}$; with these symbols H and P the indices $\sqrt{2}$ and $2\sqrt{2}$ indicate the length of the sides of the bases. As a matter of fact we can now assert that with these prisms of which the endplanes are the determining variable elements, the problem of the intersection has lost a dimension; for, in order to determine the prism we have only to ask how the ground-cube is intersected by a plane perpendicular to a diagonal of this bounding body of the eightcell, i. o. w. the problem has become threedimensional. In the same way we find in case OF rectangular prisms of which two dimensions remain constant, which has been indicated for the section of transition $\frac{2}{4}$ and the intermediary section $\frac{1}{4}$, whilst the section in case OR is an invariable cube, which is of course not designed.

It is almost superfluous to stop for the two space-fillings of case OK, that by $H^{(\sqrt{2})}$ and $P^{(\sqrt{2})}$ together and that by $P^{(2\sqrt{2})}$ alone, as they appear indeed as well-known plane-fillings. We suffice by giving the following relations:

$$\left.\begin{array}{llll} P^{(2k+1)\sqrt{2}} = & (k+2)_2\, P_p^{(\sqrt{2})} + & (k+1)_2\, H^{(\sqrt{2})} + & (k)_2\, P_n^{(\sqrt{2})} \\ H^{(2k+1)\sqrt{2}} = & 6(k+1)_2\, P_p^{(\sqrt{2})} + & (3k^2+3k+1)\, H^{(\sqrt{2})} + & 6(k+1)_2\, P_n^{(\sqrt{2})} \\ P^{(k\sqrt{2})} = & (k+1)_2\, P_p^{(\sqrt{2})} & + & (k)_2\, P_n^{(\sqrt{2})} \\ H^{(k\sqrt{2})} = & 3k^2\, P_p^{(\sqrt{2})} & + & 3k^2\, P_n^{(\sqrt{2})} \end{array}\right\}$$

4. *Case* (3, 1, 1, 1). — If the vertex A of the eightcell $C_8^{(2)}$ — see first diagram of second plate — is point $(1, 1, 1, 1)$ then the point $\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is obtained by dividing the inner diagonal AB of the cube lying in the space $x_1 = 1$ into three equal parts and then to take the first point of division C[1]). The line OC is for this case

[1]) By mistake in the diagram for AC has been taken $\frac{1}{3} AR$ instead of $\frac{1}{3} AB$.

the axis upon which we must project to determine the projec of the bounding elements. Now it is clear that the projection of cube with AB as a diagonal is obtained by projecting first bounding body on the projection AB of the axis OC on its sp $x_1 = 1$ which furnishes with regard to the vertices the stratifica 1, 3, 3, 1 and by determining then the projection on OC of tl new points lying on AB. Now, angle BOC is a right one, for of the coordinates $(1, -1, -1, -1)$ and $\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ of B C follows immediately $OB^2 + OC^2 = BC^2$. So B projects itself OC in O and so this of course is also the case with the ve $(-1, 1, 1, 1)$ of the eightcell lying opposite B. So we find — the first plate under head (3, 1, 1, 1) C_8 — the stratification the 16 vertices by causing the group of points 1, 3, 3, 1 upon the axis of projection at equal intervals to be followed a second group of the same structure in such a way that the 1 of this second group coincides with the last 1 of the first gro It is from this that this projection has its type, as is indicatec the foot. One really finds without any difficulty all that is giver the scheme by representing to oneself the two bounding cubes i cated in the typical image — here lying in the spaces $x_1 = \pm 1$ and to suppose that their corresponding vertices, edges, faces united by edges, faces and bounding bodies.

If again we do not take the isolated point A into considerat then we have to deal here with six different forms of the section, th intermediary forms and three forms of transition ; these are given v the addition of the corresponding fractional symbols $\frac{6}{12}, \frac{5}{12}, \ldots,$ on the second plate. We shall indicate somewhat in details h these diagrams are deduced by drawing, independently of the res of the first plate, and to this end we immediately notice that the sp through A perpendicular to OC is represented by $3x_1 + x_2 + x_3 + x_4 =$ and that this space after a slight parallel displacement to O trunca from the edges of the eightcell passing through A segments wh are in the ratio to each other of $1 : 3 : 3 : 3$. If now the edge A drawn horizontally is parallel to OX_1, we begin to set off, in or to obtain the first intermediary form, on the other edges throu A — see the last of the six diagrams — segments AP_2, AP_3, A to the length of half the edge, i. e. of the unit, on the edge A a segment AP_1 with a length of a third of the unit, which cau the tetrahedron $P_1P_2P_3P_4$ corresponding to the symbol $\frac{1}{12}$ to be ge

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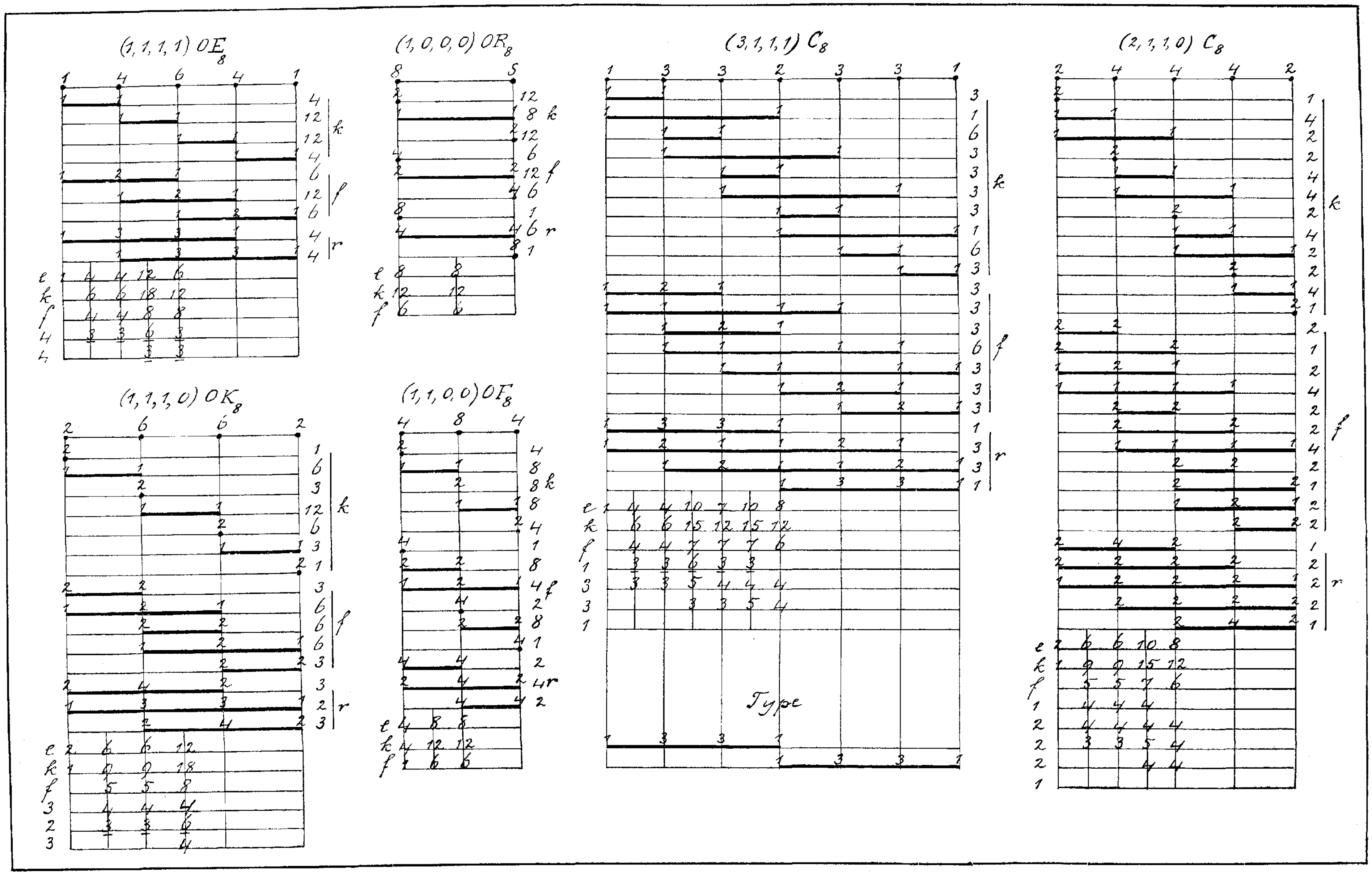
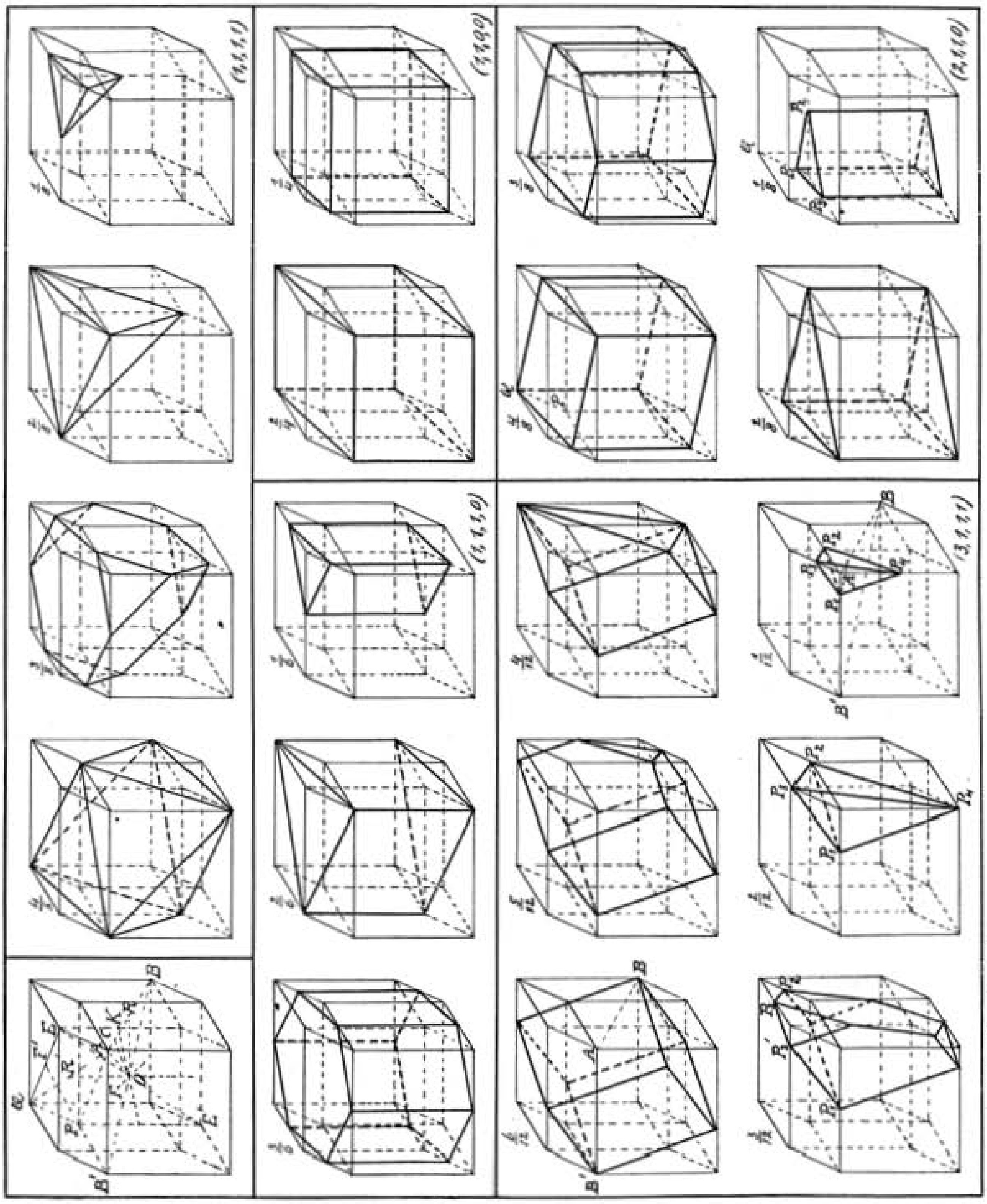
rated. The space $x_1 = 1$ contains of this tetrahedron the equilateral triangle $P_2P_3P_4$ with the side $\sqrt{2}$; the other faces $P_1P_3P_4$, $P_1P_2P_4$, $P_1P_2P_3$ lying in the spaces $x_2 = 1, x_3 = 1, x_4 = 1$ are isosceles triangles with basis $\sqrt{2}$ and sides $\frac{1}{3}\sqrt{10}$. So this section is not a regular *tetrahedron* but a regular *triangular pyramid*, of which the perpendicular let down out of the vertex P_1 on to the groundplane $P_2P_3P_4$ is an axis with the period three; because the foot of this perpendicular lies on the diagonal AB of the right cube at a distance from A forming a sixth part of AB and as AP_1 is likewise a sixth of AB' this axis is parallel to the diagonal BB' of the eightcell. It is now easy to deduce the changes of the section following from the displacement of the intersecting space by investigating either the parallel displacement of the edges of the section over the faces of the eightcell or the parallel displacement of the faces of the section through the bounding cubes of the eightcell. If the intersecting space has removed itself as far as double the distance from A, then — as is evident from both considerations — the tetrahedron of intersection has simply been multiplied by two from A. Passing on from this section $\frac{2}{12}$ it seems preferable to watch more closely the edges. If the edges P_3P_2 and P_3P_1 of the section $\frac{2}{12}$ have arrived in the positions P_3P_2 and $P_3'P_1$ of the section $\frac{3}{12}$ when the intersecting space has come at the threefold distance from the starting point A, it is sufficiently evident that the connection of the points P_3P_3' must furnish a new edge. So we see gradually how the entire rhombohedron forming the section $\frac{6}{12}$ develops itself. We yet point to the fact that the section in each position of the intersecting space during its parallel motion has an axis with period three, parallel to the diagonal BB' and at last passing into this line. Indeed, the diagonal AB of the bounding cube lying in space $x_1 = 1$ being an axis of revolution with the period three for that cube, so the plane through AB and AB' is a "plane of revolution" with the period three for the eightcell. As now the moving intersecting space is and remains normal to the line OC lying in this plane — see the first of the 20 diagrams — the line of intersection of this plane with the intersecting space, which line is of course normal to OC, must be an axis with the period three for the section. As was found

already above the line OB is really normal to OC and so the obt
axis is parallel to OB. Because the plane through AB and B' con
the perpendiculars OC and OR out of O on to the interse
space and the space $x_1 = 1$ of the righthand cube, each line of i
therefore also OB must be normal to the plane determined by
intersecting space in the space $x_1 = 1$; so if we move the interse
space in an opposite sense and return from $\frac{6}{12}$ by $\frac{5}{12}$, etc. to $\frac{1}{1}$
rhombohedron forming the central section, and then moving in
direction of the edge $B'A$ through the eightcell, is truncated no
to the axis by the plane determined in the space of that right
In fact, in the above mentioned paper (*Verhandelingen*, vol. IX,
has been found that the section is always a rhombohedron or a trun
rhombohedron when the intersecting space is normal to a
through two opposite edges, which is here the case, as the
through AB and B' contains the edge AB' and the opposite

We now indicate the body corresponding to the fractional sy
$\frac{n}{12}$ by D_n, where n can take one of the values $1, 2, \ldots, 11, 12$
D_n and D_{12-n} represent the two oppositely orientated positions
selfsame body, with a view to then investigating which of those
make their appearance when the net (C_8) is cut by the central s
$3x_1 + x_2 + x_3 + x_4 = 0$. From the distances of the points with
coordinates $(2a_i)$, forming the system of centres of the net, fo
immediately that the parts $D_2, D_4, D_6, D_8, D_{10}$ appear together
that thus the corresponding threedimensional space-filling co
of three — and if we notice the orientation even of five — diff
groundforms. Now, as we know, the form D_6 alone already is
to fill the space and so this is also the case with the forms D
D_8 and the forms D_4 and D_{10} together. What is more, from
condition that in the obtained space-filling with the three or
different groundforms the face of one of those forms must con
itself in faces of the surrounding forms, follows immediately
beside each D_2 must lie a completing D_8, beside each D_4 a compl
D_{10} and that recomposition of those parts completing each othe
D_6 must lead to a net of rhombohedra D_6. We really cause th
of rhombohedra to be generated in a simpler way if, before cu
the net (C_8) by the assumed space, we suppose the series o
spaces $x_1 = 2a_1 + 1$ to have disappeared, a thing to which th
of the plane of projection through the two edges, here AB' an
opposite one, has led us involuntarily in the paper quoted last
this the net (C_8) transforms itself into a threefold infinite net


(1, 1, 1, 1) OE8
(1, 0, 0, 0) OR8
(3, 1, 1, 1) C8
(2, 1, 1, 0) C8
(1, 1, 1, 0) OK8
(1, 1, 0, 0) OF8
Type



infinite series of rectangular prisms which have a cube with the edge two as basis, and the section of this net of prisms is exactly the net of rhombohedra. That the sections which, when the intersecting space has an arbitrary position, are quite irregular parallelopipeda, here become rhombohedra is the result of the fact that the intersecting space forms with each of the three spaces $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ equal angles, angles with a cosine of the value $\frac{1}{6}\sqrt{3}$. Out of the diagram with the symbol $\frac{6}{12}$ it is furthermore evident that the ends BB' of the axis of this rhombohedron lie in two consecutive spaces $x_1 = 2a_1 + 1$ and that the distance of the parallel spaces of intersection of the intersecting space with these spaces, which spaces cut the net of rhombohedra in the intersecting space into pieces, must amount to 4. This tallies; for the angle between the spaces $3x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 = 0$ has $\frac{1}{2}\sqrt{3}$ as cosine and therefore $\frac{1}{2}$ as sine, so that the distance of the planes must be $2 : \frac{1}{2}$.

From the preceding follows now likewise that the section with the space $3x_1 + x_2 + x_3 + x_4 = 1$ furnishes a space-filling consisting of the parts $D_1, D_3, D_5, D_7, D_9, D_{11}$; of course also this space-filling consisting of three groundforms each of which appearing in two opposite positions can be obtained by cutting up a net of rhombohedra. It is also clear that by taking an intermediary position of the space of intersection we are led to six quite different groundforms, which can be indicated by $D_{\frac{1}{2}}, D_{2\frac{1}{2}} \ldots D_{10\frac{1}{2}}$, or in opposite orientation by $D_{1\frac{1}{2}}, D_{3\frac{1}{2}} \ldots D_{11\frac{1}{2}}$.

By cutting a block of k^4 cells C_8 instead of a fourfold infinite net (C_8) we can also deduce how one of the forms $D_n^{(k)}$ of k-times greater linear size can be built up out of the above mentioned segments D_n. We avoid this not to become too longwinded.

5. *Case* (2, 1, 1, 0). — When treating the case (1, 1, 1, 0) we have seen that the appearance of nought in the symbol causes prisms to be found with the constant height 2, by which the fourdimensional problem is reduced to a threedimensional one. Thus we are placed before the consideration of the section (2, 1, 1) of the net of cubes which in various respects for the threedimensional space forms the analogon of that of the section (3, 1, 1, 1) in Sp_4.

If we suppose that the space, in which the section (2, 1, 1) is to be taken, contains the upper cube of the eightcell and the vertex P — see the first of the 20 diagrams — is taken as origin of a rectangular system of coordinates with the edges passing through this point as axes, the edge PQ as axis corresponding to the figure 2 of $(2, 1, 1)$, then the centre F' of the upper plane of that cube is the point $(2, 1, 1)$ and PF' is therefore the axis normal to the series of intersecting planes [1]). Now it follows from the rectangle $APQE$ with the sides $AE = 2$, $AP = 2\sqrt{2}$, that AQ is normal to PF' and that the points A and Q project themselves on PF' in the same point. Thus we find the projection of the eight vertices of the cube under consideration on PF' by placing the projections (1, 2, 1) of the faces with PA and QE as diagonals so side by side that the last 1 of the first coincides with the first 1 of the last, by which the stratification 1, 2, 2, 2, 1 is arrived at, which, with a view to upper and lower cube, passes by doubling into 2, 4, 4, 4, 2. From this ensue then the results given on the first plate. If we now — returning to the second plate — set off on the three edges of the cube passing through P, in the assumed supposition that PQ agrees with the 2 of $(2, 1, 1)$, from P segments $\frac{1}{2}$, 1, 1 then — see the last diagram — the triangle $P_1P_2P_3$ appears forming the upper plane of the triangular prism corresponding to the fraction $\frac{1}{8}$ and out of this the sections $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$ are developed in the same way as was pointed out above. Of triangle $P_1P_2P_3$ the line connecting P_1 with the middle of P_2P_3 is an axis with the period two, or to express it more simply a line of symmetry, and this line is parallel to the diagonal AQ of the first diagram. In each position of the intersecting plane the section has the line of intersection of this plane with the plane $APQE$ as line of symmetry; in connection with this the lozenge, unmutilated for the case $\frac{4}{8}$, which when following the reverse way to the case $\frac{1}{8}$ moves parallel to itself through the cube in such a way that the vertex Q describes the edge QP, is cut by the groundplane of the cube according to a perpendicular on the line of symmetry. If we imagine in the chosen space of the upper cube of the eïghtcell the threefold net of cubes and if we remove before passing to the intersection by the series of parallel planes the partitions parallel to the endplanes, we obtain in the intersecting plane a net of lozenges which are cut by the removed partitions into segments of the found form, etc.

In the ensuing parts we shall pass on to the intersection of the nets (C_{16}) and (C_{24}).

[1]) It is really inaccurate to speak of an upper plane of the upper cube; of course the plane is meant, which appears in the diagram as upper plane to the eye.

Mathematics. — *"Some constructions deduced from the motion of a plane system."* By Prof. J. CARDINAAL.

(Communicated in the meeting of February 26, 1908.)

1. We precede two wellknown principles of motion.

a. Let the motion be generated by rolling a curve C_u (body centrode) with which the system S is connected over a curve C_u' (space centrode). If $A, B, C, \ldots\ldots$ are moving points of S and $\alpha, \beta, \gamma \ldots\ldots$ the centres of curvature of the orbits which they describe at a certain moment there exists between the system S and the system $\alpha, \beta, \gamma \ldots\ldots (\Sigma)$ a quadratic correspondence and such that if $\alpha, \beta, \gamma \ldots$ were moving points $A, B, C \ldots$ would be the centres of curvature of their orbits. The conics of S corresponding to the right lines of Σ touch the tangent of body centrode and space centrode in the pole and osculate each other. The inflectional circle belongs to it. The reverse theorem is easy to deduce.

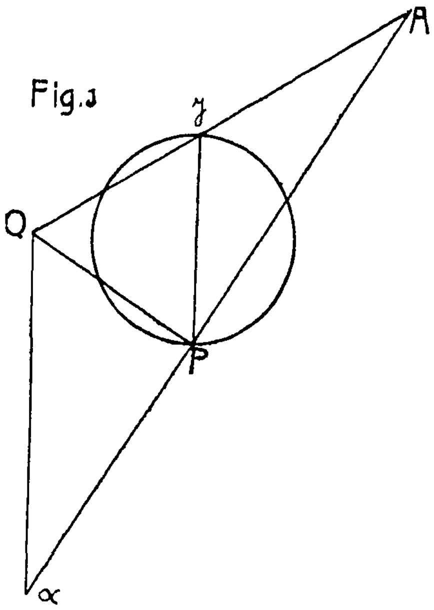


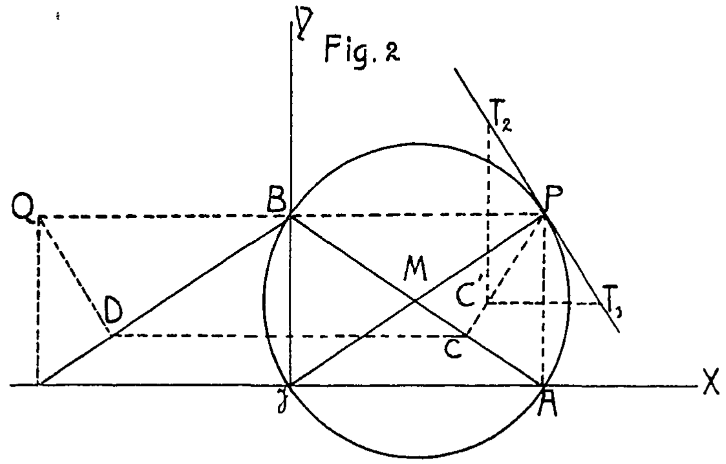
Fig. 1

b. Let P be the pole (fig. 1), I the inflectional pole (common point of the tangents in the inflectional points); so the inflectional circle is known. Let A be a moving point, then α is determined as follows: Draw AI and AP; determine the point of intersection Q of AI with the normal through P on AP. Draw out of Q the parallel to IP, which cuts P in α.

2. Application to the elliptic motion (fig. 2). Let AB (l) be the right line gliding with its points A and B along the rectangular axes IX and IY and let the demand be to construct the conic λ^2 corresponding to l.

The circumscribed circle (M) of $\Delta\, ABI$ is the inflectional circle; P is also directly known; the centre of curvature belonging to a point of AB can be constructed according to (1*b*) and so each point of λ^2 can be determined. However, some points of λ^2 are immediately known. The centre M of AB describes a circle having I as centre, the centres of curvature α and β belonging to A and B lie at infinite distance in the directions IB and IA; λ^2 furthermore touches circle (M) in P; so λ^2 is an equilateral hyperbola passing through I with directions of asymptotes IB and IA and touching circle (M) in P.

The centre C of λ^2 can be determined in the following manner.



We suppose λ^2 to be constructed by means of the projective pencils formed by rays parallel to IX and IY. If the two points united in the point of contact P were separated then the two pairs of parallel rays through these points would determine two points A_1 and A_2 on IX and two points B_1 and B_2 on IY and the centre would be the point of intersection of $A_1 B_1$ and $A_2 B_2$. It is true A_1 and A_2 coincide in A, and B_1 and B_2 in B; but from the preceding follows that the centre C lies on AB. If in P we draw the tangent to λ^2 perpendicular to the normal PI, then a point of each asymptote lies at equal distance from P. So we measure $PT_1 = PT_2$ and we draw $T_1C' // IX$, $T_2C' // IY$; C' would be the centre of λ^2 if C' were situated on AB. However, out of the figure is evident that C' lies on a right line symmetric to T_1T_2 with respect to PA, and therefore perpendicular to AB. So the centre C of λ_2 is the footpoint of the normal let down out of P on AB.

If we consider different positions of AB and if we construct the successive positions of the point C, then the locus is an astroid on the axes IA and IB. The hyperbola λ^2 keeps touching the invariable circle with IP as radius; so the astroid is the locus of the centres of the equilateral hyperbolae with asymptote directions IA and IB passing through I and touching the last-mentioned circle.

The two diameters IA and IB of circle I form with the right line at infinity a polar triangle of the circle; so the points C have the signification of poles of one of the sides of that polar triangle.

It is to be proved that the locus of the poles of the two other sides with respect to λ^2 is likewise the astroid just found. To that end we construct the pole of IX.

If we take as centres of the projective pencils of rays generating λ^2 point I and the point of IX at infinity, then on account of the former reasoning the pole of IX lies on the parallel through B to IP; at the same time this pole lies on a parallel drawn through C to IX; so the point of intersection D of the latter right lines is the demanded pole. As D is symmetric to C with respect to IY, it also belongs to the astroid. In the same way we can prove that the pole of IY is likewise a point of the astroid.

By projective transformation the above problem can be put as follows:

Given a conic and a polar triangle of it. To determine the locus of the poles of the sides of that triangle with respect to the system of conics passing through the vertices and touching the orginal conic.

If we regard this as a problem by itself we arrive at the following algebraic solution:

Take the polar triangle as triangle of coordinates; then for the equation of the given conic can be written:

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for that of the conic described about that polar triangle:

$$p_1x_2x_3 + p_2x_3x_1 + p_3x_1x_2 = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we introduce the condition that (2) touches (1) then the coefficients of the latter satisfy the relation:

$$(a_1p_1^2 + a_2p_2^2 + a_3p_3^2)^3 = 27a_1a_2a_3p_1^2p_2^2p_3^2.$$

The pole of one of the fundamental sides, e.g. of $x_3 = 0$, is found by substitution of

$$p_1 = -p_3\frac{x_1}{x_3}\ ;\ p_2 = -p_3\frac{x_2}{x_3}\,.$$

By this the equation of the locus of these poles becomes:

$$(a_1x_1^2 + a_2x_2^2 + a_3x_3^2)^3 = 27a_1a_2a_3x_1^2x_2^2x_3^2\,,$$

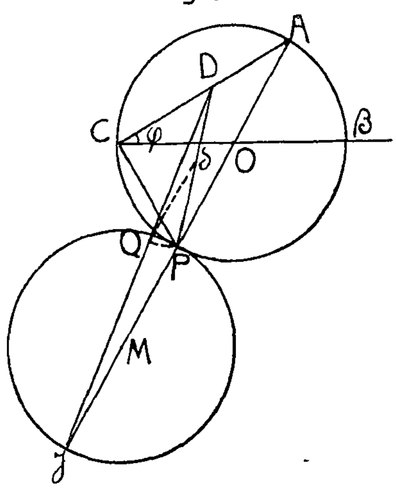
which can also be written in the form:

$$a_1^{\frac{1}{3}}x_1^{\frac{2}{3}} + a_2^{\frac{1}{3}}x_2^{\frac{2}{3}} + a_3^{\frac{1}{3}}x_3^{\frac{2}{3}} = 0\,.$$

We recognize in this the form of the astroid on oblique coordinates; the curve itself is a projective transformation of the common astroid. The locus of the poles of the other sides gives the same result.

3. Application to the cardioid motion. Let AC be the right line l, which passing through the fixed point C of circle (O) glides with one of its points A along the circumference; let now too the demand be to construct the conic λ^2 corresponding to l (fig. 3).

Fig. 3



Circle (O) is the cuspidal circle; the pole P lies diametrically opposite to A and the inflectional circle (M) is symmetric to (O) with respect to the tangent in P. Now λ^2, too, is to be constructed according to the preceding point by point; this takes place in the following way:

Let D be a point of l, draw DP and DI; the normal in P on DP intersects DI in Q, the parallel to PI out of Q intersects DP in δ.

Just as with the elliptic motion we can again construct some particular points. If we apply the general construction to point C, it is evident that γ lies halfway CP; O is evidently a point of λ^2 and β is the centre of curvature of the point at infinity on l; so the conic λ^2 passes through γ, O, β, and osculates circle (O) in P.

Whilst thus the construction of λ^2 offers no difficulties, the generated system of conics is more intricate than the preceding.

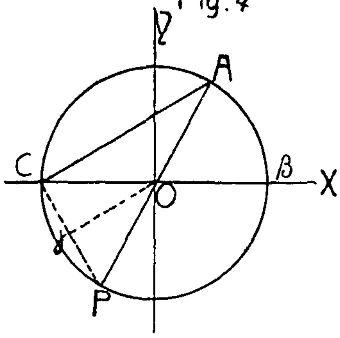
Some properties are to be found geometrically; thus it is soon evident that the system contains two parabolae.

For a parabola is necessary that AC be a tangent to the inflectional circle (M). Let us imagine the two touching circles (O) and (M) and if we draw from the endpoint A of the common diameter the tangents to circle (M), we see that we can give two positions to (M) so that one of the tangents passes through C, so there are two parabolae belonging to the system. From the figure is evident:

$$sin\ \varphi = \frac{1}{3}.$$

From this ensues: For all values of $\angle ACO$, which are lying between the values $\varphi = sin^{-1}\frac{1}{3}$ on one side as well as on the other of CO, λ^2 becomes a hyperbola, for all values outside those limits λ^2 becomes an ellipse, the transition between the ellipses and the hyperbolae is formed by two parabolae.

The locus of the centres of this system of conics does not become a simple curve. Simpler are the loci of the poles of the diameter $O\beta$, and the normal directed on to it out of O which we can take as axes for the calculation.



Let therefore (fig. 4) $O\beta$ be the X-axis, the normal OY on it the Y-axis, then we find the equation of λ^2 as follows:

Let $OA = a$, $\angle AC\beta = \varphi$, thus $\angle AO\beta = 2\varphi$. As λ^2 passes through O, β and touches circle (O) in P, its equation can be written:

$$y(x \cos 2\varphi + y \sin 2\varphi + a) + m\,(y - x \operatorname{tg} 2\varphi)\,(y - x \operatorname{tg} \varphi + a \operatorname{tg} \varphi) = 0\,;$$

the coefficient m is determined by the condition that the point $\gamma\,(-a \cos^2 \varphi, -a \cos \varphi \sin \varphi)$ lies on λ^2. By substitution of the coordinates of γ for x and y and after reduction we get:

$$m = \cos \varphi \cos 2\varphi \sin \varphi$$

and the equation of λ^2 becomes:

$$\sin^2 \varphi \sin 2\varphi \,.\, x^2 + (\cos 2\varphi - \cos \varphi \sin \varphi \sin 2\varphi - \sin^2 \varphi \cos 2\varphi)\, xy +$$
$$(\sin 2\varphi + \cos \varphi \sin \varphi \cos 2\varphi)\, y^2 + a\,(1 + \sin^2 \varphi \cos 2\varphi)\, y - a \sin^2 \varphi \sin 2\varphi \,.\, x = 0$$

or shorter:

$$2 \sin^3 \varphi \,.\, x^2 + \cos \varphi\,(4 \cos^2 \varphi - 3)\, xy +$$
$$\sin \varphi\,(3 - 2 \sin^2 \varphi)\, y^2 - 2a \sin^3 \varphi \,.\, x + a \cos \varphi\,(3 - 2 \cos^2 \varphi)\, y = 0$$

The three derivatives become:

$$4 \sin^3 \varphi \,.\, x + \cos \varphi\,(4 \cos^2 \varphi - 3)\, y - 2a \sin^3 \varphi = 0. \quad . \quad . \quad (1)$$

$$\cos \varphi\,(4 \cos^2 \varphi - 3)\, x + 2 \sin \varphi\,(3 - 2 \sin^2 \varphi)\, y + a \cos \varphi\,(3 - 2 \cos^2 \varphi) = 0. \ (2)$$

$$2 \sin^3 \varphi \,.\, x - \cos \varphi\,(3 - 2 \cos^2 \varphi)\, y = 0. \quad . \quad . \quad . \quad . \quad (3)$$

If we eliminate out of these equations two by two the value φ, we get the three loci.

Finally we shall deduce the simplest of these loci, namely the locus of the poles of the axis OX which is obtained by eliminating φ out of (1) and (3).

From (1) and (3) we deduce by subtraction the following two simpler equations:

$$3\, y \cos \varphi = 2a \sin^3 \varphi\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$2 \sin^3 \varphi \,.\, x - \cos \varphi\,(3 - 2 \cos^2 \varphi)\, y = 0; \quad . \quad . \quad . \quad . \quad (5)$$

after substitution of the value $\sin^3 \varphi$ out of (4) into (5) we find:

$$\cos^2 \varphi = \frac{3(a-x)}{2a}$$

and from this finally for the locus

$$27\, y^2 (a - x) = (3\, x - a)^2 \, ;$$

so this is a cissoid, whose cusp lies at a distance $x = \frac{1}{3} a$ from point O and whose asymptote passes through β.

OBSERVATION. The systems of conics treated in these two cases are simply infinite systems, where more than one conic pass through a point and more than one conic touches a right line; so they are distinguished from the ordinary pencils and the tangential ones.

Thus for the first mentioned system six conics pass through a point and twelve conics touch a right line.

Astronomy. — *"On the masses and elements of Jupiter's Satellites, and the mass of the system"*, by Dr. W. DE SITTER. (Communicated by Prof. J. C. KAPTEYN).

(Communicated in the meeting of February 29, 1908).

The determination of the elements and masses of the satellites of Jupiter and of the mass and the dynamical compression of the planet, which is communicated in the following pages, is based almost exclusively on heliometric and photographic observations made at the observatories at the Cape of Good Hope, Pulkowa and Helsingfors, in the years 1891 to 1904.

In addition to these I have also included in the discussion the observations made by BESSEL with the heliometer at Königsberg in 1832 to 1839, and the values of the node of the second and the perijove of the fourth satellite in 1750.0 (for which DELAMBRE's values were adopted). These latter have however, as will appear later on, only a very slight effect on the final results. The determination of all masses and elements is thus practically independent of observations of eclipses.

Previous to 1891 no series of observations of the satellites except of the eclipses had been made with the purpose of determining the elements of the orbits. Such series of observations as had been executed in the first half of the nineteenth century by BESSEL, AIRY and others, were avowedly intended only to determine the mass of Jupiter. Accordingly the satellites were by these astronomers, so far as possible, observed only near elongation. The series of observations made by GILL at the Cape in 1891, and the series of photographs taken in the same year by DONNER (on the suggestion of BACKLUND) and continued by him and by KOSTINSKY to 1898, are the first series of observations of the satellites in every point of their orbits. The discussion of the Cape observations of 1891 by me then suggested the desirability of further observations, which were executed by COOKSON with the Cape heliometer in 1901 and 1902, while photographic observations were made at the Cape in 1902, 1903 and 1904.

It will be conducive to a good understanding of what follows if I collect here at once all the notations used.

The theory, which was compared with the observations, is SOUILLART'S[1]). This theory gives the longitudes and latitudes of the satellites, referred to the plane of Jupiter's orbit of 1850·0. As I have explained in *Cape XII.* 3[2]) page 96, the orbit of 1900·0 has been used in its place.

The radii-vectores and the longitudes of the satellites in their orbits are given by the formulas:

$$r_i = a_i \varrho_i$$
$$v_i = l_i + \vartheta_i + \text{inequalities}$$
$$l_i = n_i t + \varepsilon_i .$$

We have the rigorous equations:

$$\varepsilon_1 - 3\ \varepsilon_2 + 2\ \varepsilon_3 = 180^\circ$$
$$n_1 - 3\ n_2 + 2\ n_3 = 0.$$

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1) *Théorie analytique des mouvements des satellites de Jupiter*, par M. SOUILLART, Memoirs R. A. S. XLV, 1880.

Théorie analytique des mouvements des satellites de Jupiter, seconde partie, par M. SOUILLART, Mémoires des savants étrangers, XXX, 1887.

2) *Annals of the Royal Observatory at the Cape of Good Hope*, under the direction of Sir DAVID GILL, K. C. B. Volume XII:

Part I. (Not published).

Part II. *Determination of the mass of Jupiter and orbits of the satellites*, by BRYAN COOKSON M. A. (1906).

Part III. *A determination of the inclinations and nodes of the orbits of Jupiter's satellites*, by Dr. W. DE SITTER. (1906).

Part IV. *Determination of the elements of the orbits of Jupiter's satellites*, by BRYAN COOKSON. (1907).

The titles of these papers, which I shall often have to quote, are referred to by the abbreviations used in the text above. I shall also often quote:

Publications of the Astronomical Laboratory at Groningen, N'. 17. *On the Libration of the three inner satellites of Jupiter*, by W. DE SITTER, Sc. D. (1907), which is referred to as: *Gron. Publ. 17.*

The quantities ϑ_i are the libration, which is determined by the formulas:

$$\vartheta = l_1 - 3l_2 + 2l_3 - 180^\circ = k \sin \frac{t-t_0}{T} = k \sin \beta (t-t_0)$$

$$\vartheta_i = \frac{Q_i}{\beta^2} \vartheta \quad , \quad \beta^2 = Q_1 - 3Q_2 + 2Q_3.$$

The quantities Q_i (and therefore β^2 and T) depend on the masses, and have been given in *Gron. Publ.* 17, Art. 18, up to terms of the third order.

The inequalities can be divided into three groups, according to their periods, of which the first group may be divided into three subdivisions. These are:

Ia. Equations of the centre. The osculating excentricities and perijoves — excluding their periodic perturbations (which are taken into account separately as inequalities of the longitudes and radii-vectores) — are determined by the formulas:

$$h_i = 2E_i \sin \Omega_i = 2\Sigma_j \tau_{ij} e_j \sin \tilde{\omega}_j$$
$$k_i = 2E_i \cos \Omega_i = 2\Sigma_j \tau_{ij} e_j \cos \tilde{\omega}_j.$$

Here e_i and $\tilde{\omega}_i$ are the *own* excentricities and perijoves of the four satellites. The angles $\tilde{\omega}_i$ vary proportionally to the time, and the coefficients τ_{ij} depend on the masses, τ_{ii} being unity. We have then

$$\delta v_i = -\cos l_i \, h_i + \sin l_i \, k_i$$
$$\delta \varrho_i = -\tfrac{1}{2} \sin 1^\circ (\sin l_i \, h_i + \cos l_i \, k_i).$$

The squares of E are negligible, except for the fourth satellite. The corresponding term is mentioned under *Ic.*

Ib. The great inequalities. These arise (as perturbations in h_i and k_i) through the commensurability of the mean motions of the three inner satellites. They are:

$$\delta v_1 = x_1 \sin 2(l_1 - l_2) \qquad \delta \varrho_1 = -\tfrac{1}{2} \sin 1^\circ \, x_1 \cos 2(l_1 - l_2)$$
$$\delta v_2 = -x_2 \sin (l_1 - l_2) \qquad \delta \varrho_2 = \tfrac{1}{2} \sin 1^\circ \, x_2 \cos (l_1 - l_2)$$
$$\delta v_3 = -x_3 \sin (l_2 - l_3) \qquad \delta \varrho_3 = \tfrac{1}{2} \sin 1^\circ \, x_3 \cos (l_2 - l_3)$$

Ic. Minor inequalities of short periods. The periods of all the inequalities of group *I* are short (not exceeding 17 days).

II. Inequalities arising through the commensurability of the mean motions, and having periods between 400 and 500 days. These only exist for the satellites *I*, *II* and *III*. In the radii-vectores they are negligible. Their expressions are:

$$\delta v_i = \Sigma_j \varkappa_{ij} \sin \varphi_j$$
$$\varphi_i = l_2 - 2l_3 + \tilde{\omega}_i .$$

The coefficients $\varkappa_{ij}$ are proportional to e_j, and also depend on the masses.

III. Inequalities with very long periods (exceeding 12 years). These also are negligible in the radii-vectores.

The latitudes of the satellites over the plane of Jupiter's orbit are given by the formula:

$$s_i = I_i \sin (v_i - N_i).$$

The inclinations and nodes[1]) are in SOUILLART's theory determined by the formulas:

$$I_i \sin N_i = \Sigma_j \sigma_{ij} \gamma_j \sin \theta_j + \mu_i \omega \sin \theta + \text{periodic terms}$$
$$I_i \cos N_i = \Sigma_j \sigma_{ij} \gamma_j \cos \theta_j + \mu_i \omega \cos \theta + \text{periodic terms}$$

Mr. COOKSON and I have in all our work on the satellites referred the latitudes to a fundamental plane, which is defined by its inclination and node referred to the ecliptic and mean equinox of date. For these MARTH's values have been adopted, which are for 1900.0:

$$\mathcal{I} = 2^\circ\, 9'\, 3''.94 \quad \mathcal{N} = 336^\circ\, 21'\, 28'',4$$

The longitude of the node of this plane on LEVERRIER's orbit of Jupiter, counted *in the orbit*, and the inclination on that orbit are:

$$\theta_0 = 315^\circ\, 25'\, 48''4. \quad \omega_0 = 3^\circ\, 4'\, 4''.75.$$

The longitude of the node of the orbital plane on the fundamental plane, counted *in the fundamental plane*, is therefore

$$\theta'_0 = 135^\circ\, 24'\, 34''.3.$$

The longitudes in the fundamental plane have been counted from the point O, of which the longitude is[2])

$$O = 135^\circ\, 27'\, 2''.5.$$

If the inclination and *descending* node of the fundamental plane on the orbit of 1850.0 are represented by ω_0 and ψ_0, (thus $\psi_0 = \theta_0 + 180^\circ$) and if the longitudes of the nodes Ω_i are reckoned from this descending node, we have:

$$p_i = - i_i \sin \Omega_i = - I_i \sin (N_i - \psi_0)$$
$$q_i = \quad i_i \cos \Omega_i = \quad I_i \cos (N_i - \psi_0) + \omega_0$$

from which

$$p_i = \Sigma_j \sigma_{ij} \gamma_j \sin (\psi_0 - \theta_j) + \mu_i \omega \sin (\psi_0 - \theta) + \text{periodic terms}$$
$$q_i = \Sigma_j \sigma_{ij} \gamma_j \cos (\psi_0 - \theta_j) + \mu_i \omega \cos (\psi_0 - \theta) + \omega_0 + \text{periodic terms.}$$

[1]) By node I always mean ascending node, unless otherwise stated.

[2]) MARTH evidently intended to adopt $O = \theta'_0$. Probably he has applied the correction, needed to derive $'$ from $\theta + 180^\circ$, with the wrong sign.

Here γ_i and θ_i are the *own* inclinations and nodes. The angles θ_i vary proportionally with the time and the coefficients σ_{ij} depend on the masses. We have again $\sigma_{ii} = 1$. ω and θ are the inclination and node of the mean equator of the planet on the orbital plane for 1900.0. In the discussions we have used the abbreviations:

$$x_0 = -\omega \sin(\psi_0 - \theta)$$
$$y_0 = -\omega \cos(\psi_0 - \theta) - \omega_0.$$

x_0 and y_0 thus determine the position of the equator. The adopted fundamental plane nearly co-incides with the equator, and the node ψ_0 has nearly the theoretical motion of the node θ.[1] The angle $\psi_0 - \theta$ is therefore constant and very nearly equal to 180°.

In *Gron, Publ.* 17, Chapter IV, I have given the quantities Q_i, β^2, τ_{ij} $\frac{d\tilde{\omega}_i}{dt}$, x_i, $\varkappa_{ij}$, σ_{ij}, μ_i, $\frac{d\theta_i}{dt}$ as functions of the masses, or rather, as functions of the small quantities $\varkappa'$ and ν_i, which are defined by

$$Jb^2 = 0.0219087 \quad (1 + \varkappa) \qquad (b = 1 \text{ for } d = 39''.0)$$
$$= 0.0000\,0000\,530042\,(1 + \varkappa) \qquad \text{(astronomical units).}$$
$$\varkappa' = \varkappa + 0.055$$
$$m_1 = 0.0000\,40\,(1 + \nu_1) \qquad m_3 = 0.0000\,80 \quad (1 + \nu_3)$$
$$m_2 = 0.0000\,22\,(1 + \nu_2) \qquad m_4 = 0.0000\,424\,751\,(1 + \nu_4).$$

Of $\varkappa'$, ν_2 and ν_3 only the first power was kept, of ν_1 and ν_4 all powers, which could be derived from SOUILLART's formulas were taken into account.

For the reciprocal of the mass of the system the value

$$\mathfrak{M} = 1047.40.$$

was adopted.

The data to be derived from the observations can be divided into three groups.

A. The inclinations and nodes, represented by the quantities p and q_i, i.e. the quantities determining the latitudes.

B. The data determining the longitudes and radii-vectores. These are the mean longitudes, the equations of the centre and the coefficients of the great inequalities of the three inner satellites (l_i, h_i, k_i, x_i)

C. The mean distances a_i.

A. In determining the elements from eclipse-observations, the elements of group *B* are derived from the observed epochs of the middle of the eclipse, those of group *A* from the duration of the

[1] The motion of ψ_0 actually used by MARTH is not exactly the theoretical motion of θ. The difference is however negligible.

eclipse. This duration depends not only on those elements, but as well on the form of the shadow-cone, i.e. on the geometrical compression of the planet. This latter not being known with sufficient accuracy, it is impossible to determine the latitudes from observations of eclipses. The elements of the first group must therefore be derived exclusively from heliometric or photographic extra-eclipse-observations of the satellites.

B. For the determination of the elements of group *B*, however, the eclipses are very valuable. One eclipse-observation, which is a determination of *time*, provides a much more accurate determination of the longitude of the satellite than one extra-eclipse-observation. On the other hand the latter can be repeated as often as the weather and the available time of the observer permit, while eclipses only occur in a limited number. Another advantage of eclipse-observations is that their accuracy is independent of the distance of Jupiter from the earth, while the accuracy of extra-eclipse-observations is inversely proportional to that distance. Extra-eclipse-observations are thus generally combined in series extending over a few months on both sides of the epoch of opposition. It must not be forgotten, however, that away from the opposition the time during which Jupiter is above the horizon, and therefore the number of observable eclipses, diminishes rapidly.

For the first satellite, where eclipses are numerous, and micrometrical observations least accurate, the advantage is very probably on the side of the eclipse-observations; for the fourth, of which eclipses are rare and extra-eclipse-observations are most accurate,[1]) this ratio is reversed. So long as nothing is known about the results derived from the series of photometric eclipse-observations made at the observatory of Harvard College in the years 1878 to 1903, it is not possible to form a definite judgment regarding the relative value of the two kinds of observations. Anyhow the attempt is justified to derive also the elements of group *B* exclusively from extra-eclipse-observations.

C. The four mean distances represent only one unknown quantity, since the determination of their ratios from the mean motions (also taking into consideration the uncertainty of the perturbations which must be applied) is very much more accurate than the direct deter-

[1]) My meaning is, of course, that the determination of the jovicentric place of the satellite from extra-eclipse-observations is most accurate for IV. This is due only to the large mean distance, not to the observation itself. This latter, i.e. the determination of the relative geocentric place, seems to be slightly more accurate for II and III than for I and IV.

mination of these ratios from the observations. This one unknown — the scale-value of the system — from which the mass of the planet is derived, can naturally only be determined from extra-eclipse-observations. It has already been remarked that all series of such observations made before 1891, were made with a view to this determination.

The number of unknowns of the problem is thus 32, viz:

A.	the "own" inclinations and nodes	γ_i, θ_i . . .	8	unknowns
	the position of the equator	ω, θ . . .	2	,,
	the dynamical compression	Jb^2 . . .	1	,,
B.	mean longitudes with one rigorous condition	ε_i . . .	3	,,
	,, motions ,, ,, ,, ,,	n_i . . .	3	,,
	the amplitude and phase of the libration	k, t_0 . . .	2	,,
	the own excentricities and perijoves	$e_i, \tilde{\omega}_i$ . . .	8	,,
	the mass of each satellite	m_i . . .	4	,,
C.	the reciprocal of the mass of the system	$\mathfrak{M}$ . . .	1	,,
			32	

The observations which have been used in the derivation of the results to be communicated below are the following:

1. Heliometer-observations made in 1891 at the Royal Observatory, Cape of Good Hope, by GILL and FINLAY. These have been reduced by me and were published in my inaugural dissertation [1]. After the publication some mistakes and errors of computation have been found, which have been already corrected in the results used here. The corrected results will soon be published in Cape Annals, Vol XII, Part. I.

The high order of accuracy of this series is due to the principle, introduced by GILL, to measure only distances and position-angles of the satellites relatively to each other, and not relatively to the planet [2]. Thus large systematic errors are avoided.

2. Heliometer-observations made in 1901 and 1902 at the Cape Observatory by BRYAN COOKSON, M. A., reduced by himself, and published in Cape XII.2. Corrections to the values of the unknowns as published there were afterwards given in Cape XII.4, *Appendix*. In these series

[1]) Reduction of Heliometer-observations of Jupiter's satellites, made by Sir DAVID GILL, K. C. B. and W. H. FINLAY, M. A., by W. DE SITTER. Groningen, J. B. WOLTERS 1901.

[2]) HERMANN STRUVE had previously used the same method in his observations of the satellites of Saturn.

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also only relative positions of the satellites *inter se* were determined.

From these three series all elements were derived, and all have been used in the final discussion, the values being taken unaltered from the definitive publications already quoted. The only exception is the position of the fundamental plane for 1902, the inclination of which on the ecliptic is 2°8′38″, instead of 2°11′38″, as printed in Cape XII.2 page 191 [1]).

3. Photographic plates, taken at the Cape Observatory in 1891 and 1903, measured and reduced by me, and published in Cape XII.3. The quantities p_i and q_i alone were derived for each epoch. These have been taken unaltered from Cape XII.3.

4. Photographic plates, taken at the Cape in 1904, measured and reduced by me. From these plates were derived p_i and q_i, which are published in Cape XII.3, and l_1, l_2, l_3, which are published in *Gron. Publ.* 17. The published results have been adopted unaltered.

In these last three series also only coordinates of the satellites relatively to each other were used. The planet was not measured by me.

5. Photographic plates, taken at the Cape in 1902, measured and reduced by Cookson, and published in Cape XII.4. This series requires a closer consideration.

Mr. Cookson has measured on the plates differences of *RA* and decl. of the four satellites *and Jupiter*. The pointing on the planet is, according to his own statement, "not very accurate" (Cape XII.4, p. 24). But, according to the author, high accuracy is not required, since it is eliminated in the reductions. This elimination, however, is very incomplete.

It is effected as follows. From the measured differences of *R. A.* $x_i - x_p$ a preliminary solution is made, the resulting values of the unknowns are substituted in the equations of condition, and residuals are formed, which may be called δx. The mean of these residuals for any one plate, say δx_0, is then considered to be the correction δx_p to be applied to x_p, i. e. the error in the pointing on the planet with reversed sign. This mean is therefore subtracted from the observed co-ordinates $x_i - x_p$. Now this method only eliminates the *accidental* part of the correction δx_p. The systematic part is already in the first approximation included in the values of the

1) The inclination and node referred to the equator are correct as printed, in the reduction to the ecliptic some mistake must have occurred. The consequence of this is that the inclination ω_0 of the fundamental plane on the orbit of Jupiter requires a correction of $-0°.0092$, instead of $+0°.0375$, as would appear from the printed data of Cape XII.2.

unknowns Δh_i, Δk_i, Δx_i, and is not removed from them by the subsequent approximations. The coefficients of these unknowns consist of a constant and a periodic part, of which the former amounts on, an average to three times the latter. (See e.g. COOKSON, Cape XII.4, p. 102). If this periodic part is neglected, the three unknowns cannot be separated, and they represent together only one unknown, which I have called F_i (see my dissertation, p. 69), for each satellite. Thus, if the systematic part of δx_p had been introduced as an unknown the equations of condition would have been :

$$\frac{dx}{dF} F - \delta x_p + \quad \ldots = O - C.$$

Thus it would not be possible to separate F and δx_p. Whether the unknown δx_p is actually written down in the equations or not, does not affect the result; in any case the value which is found for F is not F itself, but $F - \delta x_p \Big/ \frac{dx}{dF}$, and the residuals δx_i, and therefore also their mean δx_0, do *not* contain the systematic part of the error of pointing on the disc of the planet.

If we assume that the values of F found from the simultaneous heliometer observations (see above, sub 2), are the true ones, then the differences $P—H$, which are given by COOKSON in Cape XII. 4, page 102 (where for F_3 — 0.0295 should be read instead of —0.0395) are proportional to this systematic error, and we have $\delta x_p = \frac{dx}{dF}(P—H)$. We thus find for the four satellites :

$$\left.\begin{array}{rl} \delta x_p = & -\,0''.19 \pm 0''.04 \\ & -\,0\;.15 \pm \;\;.06 \\ & -\,0\;.17 \pm \;\;.05 \\ & -\,0\;.33 \pm \;\;.04 \end{array}\right\} \text{mean } -0''.21 = -0'.0035.$$

The agreement of the four values is remarkable. The probable errors, of course, would only be a true measure of the accuracy, if it could be assumed that the periodic parts of the coefficients of Δh_i etc. have been entirely without any influence on the final results, which is very far from being true, especially for the fourth satellite, of which only a small number of revolutions is included in the period of observations. The mean systematic error of pointing on the disc is of the same order of magnitude as the errors which I found to exist in the measures by RENZ (see below, sub 6). So there can be little doubt that this is the true explanation of the large and systematic differences between the results from the photographs and those from the heliometer in 1902. Accordingly I have rejected all the results from the photographic series, with the exception of p_i and q_i, which depend almost exclusively on differences of declination, in which the unknowns Δh_i, Δk_i, Δx_i have small and not constant coefficients, and the elimination of dx_p is therefore much more complete. I have adopted the values derived from the solution in which the orientation was determined from the *trails*. The reason why this is to be preferred to the orientation derived from the standard stars has been explained by me in Cape XII. 3, *Appendix*. The values of Δq_i and Δp_i have been adopted unaltered from Cape XII. 4.

6. Photographic plates taken at the observatories of Helsingsfors by Prof. DONNER and of Pulkowa by Dr. KOSTINSKY, measured by RENZ, and published in the Mémoires de St. Petersbourg, VIIIth series, Vol. VII, N°. 4 and Vol. XIII, N°. 1.

From the measures by RENZ I have derived corrections Δl_1, Δl_2, Δl_3 to the mean longitudes, which have been published in Gron. Publ. 17. The values found there have been adopted unaltered.

RENZ measured the positions of the satellites relatively to Jupiter. I have commenced my discussion of these measures by rigorously eliminating the pointing on the planet. It appears that these pointings are indeed subject to very large systematic errors (Gron. Publ. 17, art. 9*b*).

7. Heliometer observations made by BESSEL in Königsberg from 1832 to 1839, published by himself in "Astronomische Untersuchungen, Band II"; re-reduced by SCHUR and published in Nova Acta Acad. Leop. Carol., Vol. 44, pages 101—180. Only the values of h_3, k_3, h_4, k_4 are included in the discussion, and only h_4 and k_4 have contributed to the final result.

BESSEL has referred the satellites to the planet. His observations are affected by large systematic errors, as has been pointed out by SCHUR, in consequence of which their real accuracy cannot be assumed to be in accordance with the probable errors.

8. The values of the "own" nodes and perijoves in 1750. These have been determined by DELAMBRE and by DAMOISEAU. Regarding the accuracy of these determinations nothing definite is known. The agreement between the two results, which is very good, cannot be taken as a measure of the accuracy, since we do not know in how far DAMOISEAU is independent of his predecessor. It will be seen below that the *rôle* played by these data in the derivation of the final results is a very subordinate one.

If from a combination of the values found on different epochs for the osculating elements we wish to derive not only the values of these elements, but also of the masses, it is necessary that the expression of the perturbations as functions of the masses be known. The masses which form the basis of SOUILLART's theory probably require considerable corrections. In consequence of the mutual commensurability of the mean motions the perturbations of higher orders are very large —: in some cases larger than those of the first order. For these reasons the perturbations cannot be assumed to be linear functions of the masses. The formulas needed to compute the corrections to the perturbations corresponding to given corrections to the masses have been developed by me, on the basis of SOUILLART's numerical theory. They have been published in *Gron. Publ.* 17, *art.* 17.

The data required for the determination of the masses are:

I. The motions of the nodes, especially of θ_2. The inclination of satellite I is too small to allow the motion of its node to be determined with accuracy, and the motions of θ_3 and θ_4 are too slow to be of any importance for the determination of the masses, compared with θ_2. The motion of θ_2 is the datum from which the constant of compression Jb^2 must be derived.

II. The *motions of the perijoves, especially of* $\tilde{\omega}_4$. *The* excentricities of I and II are again too small to allow a determination of the motion of the perijove to be made. The motion of $\tilde{\omega}_3$ on the other hand, if it could be accurately determined, would be of little value for the determination of the masses on account of the small coefficients of these masses. The motion of $\tilde{\omega}_4$, which owing to the large excentricity of this satellite can be very accurately determined, is used for the derivation of the value of m_3.

III*a*. The great inequalities in the longitudes and radii-vectores of the first and third satellites. These depend chiefly on m_2, and serve to determine this mass.

III*b*. The great inequality of the second satellite. This furnishes an equation involving m_1 and m_3.

These data are those used by LAPLACE. To these I have added:

IV. The period of the libration. This depends on m_1, m_2 and m_3. Of these m_2 only has a small coefficient, consequently the observed period practically gives an equation between m_1 and m_3, from which combined with III*b* these two masses can be found [1]).

[1]) See "*Over de libratie der drie binnenste satellieten van Jupiter, en eene nieuwe methode ter bepaling van de massa van satelliet I*," door Dr. W. DE SITTER. Handelingen van het 10e Ned. Nat. en Geneesk. Congres, (Arnhem 1905), pages 125—128.

Finally I add for the sake of completeness:

V. The ratio of the two excentricities of III, from which m_4 must be determined. It has not been possible to determine this ratio from the data at my disposal, and I have therefore been compelled to leave m_4 uncorrected.

The investigation can thus be divided into the following parts, or subordinate investigations:

I. The determination of the *inclinations* and nodes on the different epochs, and of the motions of the nodes. This discussion must at the same time give the position of the mean equator, since the major part of the motions of the nodes is due to the compression of the planet, and consequently the plane of the equator is the one to which the theoretical motions are referred, and on which the own inclinations are constant. This discussion has been made with preliminary values of p_i and q_i in *Cape XII.* 3, *Chapters XV—XXI.*

II. The determination of the equations of the centre and of their secular variations. This was done in *Gron. Publ.* 17, Art. 19.

III. The determination of the great inequalities. These have been adopted unaltered from the heliometer observations of 1891, 1901 and 1902.

IV. The determination of the libration. This was carried out in *Gron. Publ.* 17.

The determination of the masses from the equations of condition furnished by these various subordinate investigations was effected in *Gron. Publ.* 17, so far as it was possible with the data which were then at my disposal. I there found the masses:

$$\left.\begin{array}{ll} \varkappa' = +\ 0.025 & \nu_2 = +\ 0.050 \\ \nu_1 = -\ 0.360 & \nu_3 = +\ 0.025 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (A)$$

The equations of condition from which corrections to these values were derived, will be communicated below. I will now first describe the various subordinate investigations I to IV, to which I add V: the determination of the mean motions, and VI: the determination of the mass of the system.

I. *Inclinations and Nodes.*

The data are the values of p_i and q_i for the five epochs 1891.75 1901.61, 1902.60, 1903.72, 1904.89. The unknowns are γ_i, Γ_{i0}, x_0, y_0 and the motions of the nodes [1]). These latter depend on $\varkappa'$ and ν_i, of which only $\varkappa'$ has been introduced as unknown. The

[1]) In this investigation we put for abbreviation $\Gamma_i = \psi_0 - \theta_i$.

discussion is carried out in Cape XII. 3, based on the masses of SOUILLART's theory. It must now be repeated with the masses (A). Further the following corrections must be applied.

a. The observed values of p_i and q_i must be reduced to one and the same fundamental plane for all epochs. At the time when the discussion of Cape XII. 3 was made, I had not at my disposal the data for carrying out this reduction for the epochs 1901 and 1902.

b. In the discussion of Cape XII. 3 I was compelled to reject the observations of the satellites III and IV in 1901 and 1902. COOKSON had found in the latitude of IV an empirical term, which had also influenced the results for III, and which could be demonstrated not to exist in the observations of 1891, 1903 and 1904. Mr. COOKSON has since then found the true explanation of this apparent periodic term, and has corrected his results accordingly. The corrected results must now be introduced into the discussion. It appears that now not only nothing must be rejected, but that also the representation of the observations generally is much improved.

c. The results of the photographs of 1902, which were not yet known when the discussion of Cape XII. 3 was made, must be taken into account.

It seems unnecessary to mention here all the details of the discussion. It will be published in Cape XII. 1, *Appendix*, and it will suffice here to state the results.

It may be remembered that in Cape XII. 3 two final solutions were made, of which Sol. VI was based exclusively on modern observations, while in Sol. VII the motion of θ_2 was derived from a comparison with DELAMBRE (1750), and the motions of the other nodes theoretically corresponding with this were adopted [1]). Thus $\varkappa'$ was not introduced as an unknown in this solution. The agreement of the solutions VI and VII was very good, with the exception of $\varkappa'$ and y_0. The values (A) of $\varkappa$ and ν_i are chosen so that the corresponding motions of the nodes are about the means of those found in Sol. VI and Sol. VII.

The corrections (a), (b) and (c) were now applied, the quantities σ_{ij} and μ_i, which are used in the solution were altered so as to correspond with the masses (A), and a new solution was made (Sol. VIII) in which, similarly to Sol. VI, the unknowns were γ_i, Γ_{i0}, x_0, y_0 and $\delta\varkappa'$. The method by which the solution was effected is the same as in Cape XII. 3, and has also been described

[1]) The correspondence was only approximate, the expressions of the motions of the nodes as functions of the masses (*Gron. Publ.* 17, art. 17), not yet being computed at that time.

in detail in these Proceedings (March 1906). The values found for γ_i and Γ_{i0} were very nearly equal to those found previously. The correction to $\varkappa'$ was very small, viz.:

$$\delta\varkappa' = +\ 0.0026 \pm .0058.$$

The masses now become

$$\left.\begin{array}{ll} \varkappa' = +\ 0.0276 & \nu_2 = +\ 0.050 \\ \nu_1 = -\ 0.360 & \nu_3 = +\ 0.025 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (B)$$

The motions of the nodes were now made to correspond with these masses, the values found for γ_i, Γ_{i0}, x_0 and y_0 were introduced into the equations of condition, and residuals $\Delta\gamma_i$ and $\sin\gamma_i\ \Delta\Gamma_i$ were formed. From these latter I then derived for each satellite separately a correction to the motion of the node. These corrections are given below sub I. The values of the nodes in 1750.0 were next computed and compared with those determined by DELAMBRE. This comparison gave the corrections II to the motions of the nodes.

Correction to the motion of	I (modern)	II (DELAMBRE)	Adopted
θ_1	$+ 0^\circ.0094 \pm .0029$		
θ_2	$- 0\ .00001 \pm 00009$	$- 0^\circ.00042 \pm ^\circ.00020$	$- 0^\circ.00010 \pm 00008$
θ_3	$- 0\ .00048 \pm \quad 23$	$- 0\ .00034 \pm \quad 20$	$- 0\ .00041 \pm \quad 15$
θ_4	$- 0\ .00013 \pm \quad 11$	$+ 0\ 00008 \pm \quad 50$	$- 0\ .00010 \pm \quad 10$

These corrections have been used as the right-hand members of equations of condition, from which, together with those derived from the other subordinate investigations, corrections to the values (B) of the masses have been determined. These equations will be given further on. It will be seen that the adopted values agree within the probable errors with those derived from the modern observations alone. If thus these latter were adopted, the final results could only be altered within their probable errors. The finally adopted masses are:

$$\left.\begin{array}{l} \varkappa' = +\ 0.0326 \ \pm\ .0075 \\ \nu_1 = -\ 0.350 \ \pm\ .030 \\ \nu_2 = +\ 0.050 \ \pm\ .050 \\ \nu_3 = +\ 0.005 \ \pm\ .020 \\ \nu_4 = \quad 0 \quad \pm\ 0.25 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

These were now introduced into the quantities σ_{ij} and μ_i and a new solution was made (Sol. IX), in which the motions of the nodes corresponding to the masses (C) were adopted, and accordingly $\delta\varkappa'$ was not introduced as unknown. The result is:

TABLE I INCLINATIONS AND NODES

Series	Observed correction	probable error	Residual	Observed correction	probable error	Residual	Observed correction	probable error	Residual	Observed correction	probable error	Residual
	p_1			p_2			p_3			p_4		
1891 *H*	+0°·0360	± °·0045	+ °·0023	+0°·0752	± °·0031	+ °·0025	—0°·0029	± °·0020	+ °·0032	+0°0630	± °·0010	+ °·0022
„ *P*	+ .0372	± 50	+ 35	+ 0733	± 36	+ 6	— .0024	± 19	+ 37	+ 0638	± 12	+ 30
1901	+ .0338	± 71	+ 20	+ ·1119	± 52	— 64	— 0105	± 33	+ 18	+ .0564	± 17	— 54
1902 *H*	+ .0091	± 65	+ 1	+ .0923	± 40	+ 19	— 0097	± 25	+ 18	+ .0636	± 13	+ 16
„ *P*	+ .0026	± 80	— 64	+ .0941	± 42	+ 37	— .0095	± 27	+ 21	+ .0612	± 14	— 8
1903	+ .0021	± 60	+ 75	+ .0526	± 33	— 33	— .0199	± 22	— 97	+ 0583	± 12	— 20
1904	— .0028	± 78	— 58	+ .0158	± 44	+ 8	— 0104	± 28	— 11	+ .0648	± 13	+ 27
	q_1			q_2			q_3			q_4		
1891 *H*	—0°·0273	± °0049	— °·0063	+0°·0867	± °0029	+ °0015	—0°·0681	± °·0017	+ °·0016	—0°·0137	± °·0010	— °·0005
„ *P*	— .0258	± 61	— 48	+ .0813	± 35	— 39	— 0748	± 23	— 51	— .0147	± 11	+ 15
1901	— .0793	± 79	— 57	— ·1658	± 45	+ 76	— .0369	± 30	+ 103	— .0191	± 16	0
1902 *H*	— .0769	± 59	— 37	— 1896	± 36	+ 41	— .0431	± 21	— 4	— .0172	± 13	+ 22
„ *P*	— .0820	± 62	— 88	— 1904	± 37	+ 33	— 0384	± 22	+ 43	— .0170	± 13	+ 24
1903	— .0597	± 48	— 58	— ·2120	± 32	— 29	— .0442	± 20	— 4	— .0209	± 11	— 6
1904	— .0336	± 77	— 34	— ·2253	± 48	— 90	— .0477	± 26	— 54	— .0210	± 17	0

Solution IX.

$$\gamma_1 = 0^\circ.0272 \pm {}^\circ.0028 \qquad \theta_1 = 60^\circ.2 \pm 7^\circ.0 - \{0^\circ.13614 \pm {}^\circ.00100\}\, t$$
$$\gamma_2 = 0.4683 \pm 16 \qquad \theta_2 = 293.18 \pm 0.19 - \{0.032335 \pm .000240\}\, t$$
$$\gamma_3 = 0.1839 \pm 26 \qquad \theta_3 = 319.73 \pm 0.52 - \{0.006854 \pm .000180\}\, t$$
$$\gamma_4 = 0.2536 \pm 23 \qquad \theta_4 = 11.98 \pm 0.67 - \{0.001772 \pm .000030\}\, t$$

The time t is counted in days from 1900 Jan. 0, mean Greenwich noon. The nodes are reckoned from the first point of Aries. The motions contain the precession, for which NEWCOMB's value was adopted. The probable errors of the motions of the nodes were computed from those of the masses (C). For the position of the mean equator referred to LEVERRIER's plane of Jupiter's orbit for 1900.0 I find

$$\omega = 3^\circ.1153 \pm 0^\circ.0014$$
$$\theta = 315.800 \pm 0.025 \quad (1900 \text{ Jan. } 0.0)$$

Table I contains the observed corrections to SOUILLART's theory, their probable errors derived from the discussion of each series separately, and the residuals which remain after the subtitution of the final values of γ_i, θ_i, ω and θ.

The probable error of weight unity, determined from these residuals is

$$\pm 0^\circ.0097.$$

Weights had originally been assigned, corresponding to a probable error of weight unity of

$$\pm 0^\circ.0100.$$

Comparing each residual with its probable error, we find the following distribution

	actual	theoretical
smaller than ϱ	$30\frac{1}{2}$	28.0
between ϱ and 2ϱ	$16\frac{1}{2}$	18.1
,, 2ϱ ,, 3ϱ	6	7.5
exceeding 3ϱ	3	2.4

Remembering that the corrections Δp_i and Δq_i are for each epoch the results of a series of observations, made for the different epochs by different observers and different instruments, and reduced absolutely independently of each other, we must consider this excellent agreement of the actual distribution with the ideal one according to the law of errors as a strong proof of the freedom of the observations from systematic errors. Accordingly the probable errors of the resulting inclinations and nodes can confidently be regarded as a correct measure of the accuracy. How much better the observations are represented by these values than by those adopted in SOUILLART's theory is evident at once by comparing the residuals with the observed corrections.

For 1750.0 I now find:

	Sol. IX.	DELAMBRE	DAMOISEAU
θ_2	$264^\circ.7 \pm 13^\circ.2$	$283^\circ.3$	$282^\circ.0$
θ_3	335.2 ± 10.0	352.5	353.5
θ_4	109.1 ± 1.8	105.0	98.3

The agreement with the values found by DELAMBRE and DAMOISEAU is now satisfactory. If the probable errors of θ_2 and θ_3 in 1750 are estimated at $\pm 5^\circ$ (see Cape XII. 3 page 111), the difference in both cases hardly exceeds the sum of the probable errors. As has been already said, I consider the probable errors of Solution IX as a true measure of the accuracy. This however they only remain for 1750 on the assumption that the theory, by means of which the elements have been carried back from 1900 to 1750, be correct. This, however, cannot be assumed without some qualification. It is well known that SOUILLART has integrated the equations of motion by two different methods. The difference between the motion of the node of II in 150 years according to the two methods is nearly $1^\circ.4$. It is thus quite possible that the terms of higher order in the masses, which are neglected in *both* methods, may also amount to a very appreciable quantity. In the interval of 150 years θ_2 has completed nearly five revolutions, while its motion is practically derived from the interval 1891—1904, during which the node has moved about 155° degrees. Remembering this, the agreement between the values carried back from 1900 to 1750, and those directly determined, is as good as could reasonably be expected.

In Cape XII. 3. I pointed out that the solutions VI (modern observations alone) and VII (motion of θ_2 from comparison with DELAMBRE) were in perfect agreement except for the motions of the nodes and for y_0. I then stated as my opinion, that the substitution of better masses for those of SOUILLART could be expected to reconcile the two solutions. This expectation has been entirely fulfilled. With regard to the motions of the nodes, (which are practically the same in Sol. VIII and Sol. IX) we have just seen that the agreement with 1750 is satisfactory. With regard to y_0 the following comparison of the different solutions shows that indeed the difference between Sol. VIII and IX is much smaller than between VI and VII, and now leaves nothing to be desired.

Values of y_0.

Sol. VI	$+ 0^\circ.0388 \pm {}^\circ.0044$	Sol. VIII	$+ 0^\circ.0454 \pm {}^\circ.0029$
Sol. VII	$+ .0490 \pm \phantom{{}^\circ.00}24$	Sol. IX	$+ .0473 \pm \phantom{{}^\circ.00}14.$

For the other unknowns the differences between the solutions VIII and IX are entirely negligible. In addition to the improvement of the masses, also the reduction to one and the same fundamental plane, and the corrections applied by COOKSON to the values for 1901 and 1902 are largely responsible for this improvement in the agreement of the two solutions.

II. *Equations of the centre.* The values of the own excentricities and perijoves were derived by me from the heliometer observations of 1891, 1901 and 1902, in *Gron. Publ.* 17, *Art.* 19. (See also these Proceedings, June 1907). The discussion was there carried out for two sets of coefficients τ_{ij}, the results agreeing within their probable errors. It is therefore unnecessary to repeat it here with the coefficients corresponding to the masses (C), which are intermediate between the two sets there used. The reasons why the photographic results of 1902 must be rejected, have already been given above. The finally adopted values are thus the same as in *Gron. Publ.* 17, with only a few unimportant alternations in the last decimal places, viz:

$$e_1 = 0^\circ.0031 \pm {}^\circ.0080 \quad \tilde{\omega}_1 = 155^\circ.5 \pm \infty + \{0^\circ.14703 \pm {}^\circ.00144\}\, t$$
$$e_2 = 0\ .0172 \pm 40 \quad \tilde{\omega}_2 = 62\ .7 \pm 10^\circ.0 + \{0\ .038955 \pm .000455\}\, t$$
$$e_3 = 0\ .0868 \pm 65 \quad \tilde{\omega}_3 = 338\ .3 \pm 3\ .0 + \{0.\ 007032 \pm .000180\}\, t$$
$$e_4 = 0\ .4264 \pm 20 \quad \tilde{\omega}_4 = 283\ .15 \pm 0\ .30 + \{0\ .001896 \pm .000021\}\, t$$

The probable errors depend on judgment, and are probably estimated rather too large. The values of e_1 and $\tilde{\omega}_1$ were not derived from the observed values of h_1 and k_1, but from the inequalities of group II, as will appear below when we treat of the libration. The adopted p. e. of e_1 is the largest value which can still be considered to be not improbable having regard to the observed values of h_1 and k_1. This p. e. being larger than the value of e_1 itself, the p. e. of $\tilde{\omega}_1$ cannot be stated.

The motions have been computed by the masses (C) and their probable errors correspond to the probable errors of these masses.

These values of e_i and $\tilde{\omega}_i$, and the values of τ_{ij} corresponding to the masses (C) give the residuals contained in Table II, together

TABLE II. EQUATIONS OF THE CENTRE.

Series	Observed correction	Probable error	Residual	Observed correction	Probable error	Residual	Observed correction	Probable error	Residual	Observed correction	Probable error	Residual
	h_1			h_2			h_3			h_4		
1891	+ 0°·031	± °·013	+ °·034	+ 0°·008	± °·008	+ °·001	+ 0°·060	± °·005	+ °·003	+ 0°·0617	± °·0026	+ °·0001
1901	+ ·091	± 28	+ 86	— ·015	± 19	+ 6	+ 007	± 14	+ 13	+ ·0653	± 51	— 108
1902	— ·008	± 34	— 41	— ·040	± 18	— 27	— ·022	± 11	— 13	+ ·0847	± 34	— 16
	k_1			k_2			k_3			k_4		
1891	— 0°·064	± °·021	— °·067	— 0°·055	± °·016	— °·005	— 0°·053	± °·008	+ °·003	+ 0°·0261	± °·0038	— °·0026
1901	— ·102	± 54	— 96	+ ·030	± 31	+ 1	— ·069	± 17	— 23	+ ·0390	± 59	+ 175
1902	— ·060	± 25	0	+ ·037	± 15	+ 4	— ·017	± 10	+ 28	+ ·0237	± 56	+ 35

with the observed corrections to SOUILLART's theory and their probable errors.[1]) The residuals are very satisfactory, especially so if satellite I is left out of account. (See also *Gron. Publ.* 17, pages 92 and 115).

From the values of $\tilde{\omega}_4$ in 1900, 1836 and 1750 I have already in Gron. Publ. 17 derived the motion of $\tilde{\omega}_4$. The value found there requires however a small correction. The values which BESSEL, and following his example SCHUR also, gives for $E_4 \sin \Omega_4$ and $E_4 \cos \Omega_4$, i. e. for h_4 and k_4, are in reality the values of $e_4 \sin \tilde{\omega}_4$ and $e_4 \cos \tilde{\omega}_4$. This was not noticed at first, and must now be corrected.

I now find for 1836

$$h_4 = - 0°.704 \qquad k_4 = - 0°.395.$$

Using, as before, the most probable values of e_3, $\tilde{\omega}_3$ and τ_{43}, we find from this:

$$e_4 \sin \tilde{\omega}_4 = - 0°.351 \quad e_4 \cos \tilde{\omega}_4 = - 0°.208$$
$$\tilde{\omega}_4 = 239°.4 \pm 0°.8.$$

We have now:

	$\tilde{\omega}_4$	*Residual*
1750.0	180°.4	+ 0°.1
1836.0	239. 4	0 .0
1900.0	283. 1	0 .0

from which:

$$\frac{d\tilde{\omega}_4}{dt} = 0°.001872 \pm 0°.000020 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

If the probable error were derived from the residuals, or from the probable errors for the separate epochs, we should find a much smaller value. The larger value has been adopted chiefly on account of the possibility of systematic errors of BESSEL, which will be mentioned below.

COOKSON has already (Cape XII. 2. page 197) derived the motion of $\tilde{\omega}_4$ from the observations of 1836, 1879 (SCHUR) 1891, 1901 and 1902. He finds:

$$\frac{d\tilde{\omega}_4}{dt} = 0°.001892 \pm 0°\,000024, \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

The values (α) and (β) agree within their probable errors. So, if (β) were adopted instead of (α), the final results could only be

[1]) In deriving these residuals the longitudes of the perijoves are, of course, counted from the point O, as was done in the tabular places.

altered within their probable errors. They would then be entirely independent of eclipse observations.

With the finally adopted elements we find for BESSEL the following residuals.

BESSEL 1836.0

	Observed	*Residual*		*Observed*	*Residual*
h_3	$-0^\circ.033 \pm {}^\circ.010$	$+{}^\circ.008$	h_4	$-0^\circ.704 \pm {}^\circ.007$	$+{}^\circ.028$
k_3	$-0\ .188 \pm\ \ 14$	$+\ .020$	k_4	$-\ \ .395 \pm\ \ 9$	$+\ .026$

It thus appears that, although $\tilde{\omega}_4$ is well represented, h_4 and k_4 leave large residuals. It is remarkable that all four residuals are positive. This must probably be ascribed to systematic errors in the observations, which have already been proved to exist by SCHUR's discussion, and which probably are not entirely eliminated by the empirical corrections applied by SCHUR.

The theoretical values of h_3 and k_3 are:

$$\tfrac{1}{2}\, h_3 = \tau_{31}\, e_1 \sin \tilde{\omega}_1 + \tau_{32}\, e_2 \sin \tilde{\omega}_2 + e_3 \sin \tilde{\omega}_3 + \tau_{34}\, e_4 \sin \tilde{\omega}_4$$

$$\tfrac{1}{2}\, k_3 = \tau_{31}\, e_1 \cos \tilde{\omega}_1 + \tau_{32}\, e_2 \cos \tilde{\omega}_2 + e_3 \cos \tilde{\omega}_3 + \tau_{34}\, e_4 \cos \tilde{\omega}_4 .$$

The two first terms are exceedingly small, but $\tau_{34}\ e_4$ is large, and this term has been used by LAPLACE to determine m_4, with which the coefficient τ_{34} is roughly proportional. An attempt to derive τ_{34} from a comparison of the equations of the centre in 1836 and 1900 had to be given up, as will be easily understood by considering the residuals and probable errors stated above. Also a comparison with 1750 is not possible, for DELAMBRE and DAMOISEAU both state nothing but the values of the coefficients and the arguments, and it is not possible to derive from these the values of h_i and k_i as found directly from the observations. I have thus been compelled to leave m_4 uncorrected.

The values of $\tilde{\omega}_3$ and $\tilde{\omega}_4$ for 1750, computed from the final values for 1900.0 and the final motions, are:

		DELAMBRE	DAMOISEAU
$\tilde{\omega}_3$	$313^\circ.0 \pm 10^\circ.3$	$309^\circ.4$	$315^\circ.0$
$\tilde{\omega}_4$	$179.3 \pm\ \ 1.2$	180.3	180.4

The agreement is excellent, in fact better than could have been expected.

(To be continued).

Botany. — "*Contribution N°.* 1 *to the knowledge of the Flora o Java*" by Dr. S. H. KOORDERS.

(Communicated in the meeting of February 29, 1908).

§ 1. On the oecological conditions, on the means of dissemination and o the geographical distribution of the species of Myricaceae, occurring wil in Java, especially in the higher mountains.

As has been shown by KOORDERS and VALETON in their critica systematic investigations of the *Myricacae* of Java, contributio N°. 9 to the knowledge of the Trees of Java, [in Mededeelinge uit 's Lands Plantentuin N°. LXI (1903) p. 99—105] there ar found on this island but two wild species, both of which generall become arborescent. These are:

1. *Myrica javanica* BL. (= *M. macrophylla* MIRB.) and
2. *Myrica longifolia* TEIJSM. & BINNENDIJK (= *M. integrifolia* ROXB = *M. Lobbii* TEIJSM. & BINN.).

The botanical investigations of the alpine flora of Java which I am now undertaking for the Dutch Ministry of the colonies in the Herbaria at Leiden and at Utrecht, often afford a not unwelcom opportunity of amplifying and correcting my previous publication on geographical distribution and oecology, which publi cations were mainly based on my botanical notes of numerou journeys in Java, Sumatra and North-East Celebes.

An example is presented by the order of the *Myricaceae*, sinc in a recent foreign publication [1]) observations appear to have bee overlooked, which had already been published, partly in 1903 i the above-mentioned Contribution N°. 9 to the knowledge of th Trees of Java, by KOORDERS & VALETON [2]), and partly in a still earlie small Dutch publication, [3]) which is very difficulty accessible abroad We are here concerned with some observations on the geographica distribution and especially on the means of dissemination of an alpin tree of Java, namely *Myrica javanica* BLUME.

1) ERNST, Prof. Dr. A., Die neue Flora der Vulkaninsel Krakatau mit 2 Karten skizzen und 9 Landschafts- und Vegetationsbildern. — Zürich 1907. — On p. 6 of this very interesting publication: "Auch bei früchtefressenden Tauben solle sich im Kropfe und Magen häufig Samen van ansehnlicher Grösse vorfinden un BECCARI gibt an, . . ." (l. c. p. 61).

2) KOORDERS & VALETON l.c. in Mededeelingen uit 's Lands Plantentuin LXI (1903 p. 102.

3) KOORDERS, Spontane en kunstmatige reboisatie op den Sendoro (Spontaneou and artificial reforestation on the Sendoro) in Tijdschr. v. Nijverheid en Landbouw van Nederl. Indie, Vol. 51 p. 241—287 (with a map).

§§ 1. MYRICA JAVANICA BL.

§§§ 1. GEOGRAPHICAL DISTRIBUTION AND OECOLOGY.

§§§§ 1. Distribution and oecology outside Java.

In 1895 I collected in the Sapoetan mountains of North East Celebes, at an altitude of 1400—1500 metres above sea-level, herbarium specimens of a small tree, which grew wild there. These specimens are now in the Mus. Botan. Hort. Bog. at Buitenzorg; I considered that they differed only so slightly from Javanese specimens of *Myrica javanica* BLUME, that I identified [1]) them with the latter and still regard them as conspecific. As is proved by a specimen which I saw in the Royal Botanical Museum at Berlin, this species was also collected by WARBURG in N. E. Celebes and was regarded by him as specifically different from the above-named Javanese species. Except from Celebes and from Java, no stations for *Myrica javanica* BLUME have been recorded in the literature. Nor have I seen any specimens collected outside Java, in the Herbaria at Leiden and at Utrecht.

Since I have not, at present at my disposal the specimens collected by me in Celebes and preserved in the Herbarium at Buitenzorg, I can give no further data regarding WARBURG's separation of his specimens from Celebes in connexion with the specific value of the differences, which I myself (*l. c.* p. 615) had already observed between the specimens of *Myrica javanica* BL., collected by me in Java and in Celebes. For the present I therefore continue to regard the arborescent *Myrica,* collected on the Sapoetan summit in N.E. Celebes, as identical with *Myrica javanica* BL. of the Javanese mountains.

§§§§ 2. Horizontal and vertical distribution and oecology outside Java.

As appears from Herb. Kds. in Mus. Hort. Bogor., I made in the years 1888—1903 the following observations, which in part have already been published in KOORDERS and VALETON, contribution to the knowledge of the Trees of Java, IX (1903) p. 102.

In West- and in Central Java above 1500 metres. In the Preanger on the Gede at 3000 m. near the summit, on the Galoenggoeng by the lake of Telagabodas at 1650 m. and 1700 m. In Tegal on the

[1]) KOORDERS, S. H., Report of an official botanical journey through the Mina hasa, being a first survey of the Flora of N. E. Celebes; with 10 maps and 3 plates. (In Mededeelingen uit 's Lands Plantentuin N⁰. XIX (1898). Batavia and The Hague p. 615).

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Slamat at 1800 m. and higher above Simpar. In the residency Banjoemas on the Prahoe and on the Dieng-plateau at 2500 m. In the residency Kedoe on the Merbaboe, Oengaran and Telemojo at 1800 m. and higher. Has hitherto not been found further East. Growing in Java mostly gregarious, and, together with about 15 other ever-green woody species, forming alpine forests. Proper to alpine regions, and found in lower regions only near *solfatare*, which are rich in mineral salts, hence exclusively on physiologically-dry soils. Prefers altitudes of 1800—3000 m. above sea-level; has not been observed in Java below 1500 m.

The following oecological conditions may be mentioned for *Myrica javanica*: the species withstands the low air-temperature, the intense insolation and low atmospheric humidity of JUNGHUHN's alpine zone, but does not appear to resist the over-dry climate of East Java.

The species occurs on very arid and stony soils, which are presumably poor in soluble mineral constituents, in consequence of long-continued washing out, and also grows luxuriously in physiologically-dry situations, near *solfatare*, etc. In the hot plain, the species is altogether wanting. It can fairly well resist much sunlight, as well as shade, and likewise is proof against strong winds.

I have not yet found Phanerogamic parasites on this alpine tree; on the other hand I have observed parasitic Fungi on it, for instance in the residency Kedoe the following: *Myxosporium candidissimum* RACIBORSKI, *Microcyclus Koordersii* HENNINGS and *Pestalozzia Myricae* KOORD. [1])

Even on the borders of the natural area of distribution of Myrica javanica, I have never observed these parasitic Fungi on the three in such quantity, that they alone would appear to limit the distribution of the adult-plants in Java. My experiments on infection with conidia of *Pestalozzia Myricae* do not, however, preclude the possibility, that this fungus imposes a natural limit on the development of the seedlings; the experiments showed that, with too much shade and too much moisture in the soil, most of the *Myrica* seedlings were killed off by the parasitic fungus, even in a district within the natural area of distribution.

At the same time these experiments showed, that *Myrica* seeds, when germinating in full sunlight (for instance on bare mountain-slopes, naked rocks and layers of rapilli), would not sustain serious injury from this fungus parasite which is often so harmful.

[1]) KOORDERS, Botanische Untersuchungen über einige in Java vorkommende Pilze, besonders über Blätter bewohnende parasitisch auftretende Arten [in Verhand. Koninkl. Akademie v. Wetenschappen, Deel XIII, Tweede Sectie (1907) No. 4, p. 183, 218 and 224].

A further study of parasitic fungi, especially of such as attack Phanerogam-seedlings, will probably afford a good explanation of some apparently insoluble problems of plant distribution, especially where other causes do not sufficiently account for the sudden absence of a species under apparently favourable oecological conditions.

Even in mountain regions, where the original vegetation has been completely destroyed by human interference, as for instance in central Java on the Sendoro, where in 1891 miles of country were laid waste by fire, extensive forests of *Myrica javanica* are formed by a natural process in a relatively short time. When I visited the volcano Sendoro in 1903, and therefore 12 years after the fire, the higher slopes, which in 1891 had been burned down to the rock, were covered with alpine bush, grown up naturally. These slopes, which were situated above the plantations of the Forestry Department, have been referred to in detail in a publication mentioned above. [1]) The woods extended for many thousands of acres, especially on the more humid S. W. side of the mountain, and in many places *Myrica javanica* was predominant to such an extent, that one might have spoken of almost pure Myrica forests. According to information obtained by me on the spot, there appeared in addition to Myrica javanica, as first tree-like pioneer on many of the most completely burned places, as soon as one year after the fire *Albizzia montana* BENTH, growing in groups in the midst of an extensive grass wilderness. This latter species has also been found by me repeatedly elsewhere, in the alpine regions of Java as one of the very first pioneers of the forest on the burned-down slopes of volcanoes.

Brief reference may here be made to some results of investigations, which I have carried out in various parts of Java, Sumatra and N. E. Celebes during many years, regarding the characteristics of the Phanerogam pioneers on volcano slopes after complete denudation by fire, and on other lands in the interior of Java, Sumatra and Celebes, after a similar loss of vegetation, due to other causes (e. g. deserted arable lands). It would appear that, without reference to the height above sea-level nor to the systematic position of the pioneers, the new vegetation is characterised by the following properties, which are related to the abovementioned peculiar oecological conditions:

1. Without exception the new plants are, by structure and distribution *xerophytes*, which remain alive under extraordinarily unfavourable conditions of water-supply and transpiration.

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[1]) Compare KOORDERS Spontaneous and artificial reforestation on the Sendoro l. c.

2. In addition to "poverty of water", nearly all these species can well resist direct sunlight, while many are not killed either by great poverty of light.

3. Nearly all grow rapidly or very rapidly, and all soon produce abundant seed. Many of the herbaceous pioneers of this vegetation already bear plentiful fruit within a few months, while several woody pioneers already flower and bear seed in two years.

4. The seeds are never large and are very easily distributed, either by wind or by animals (especially endozoically by birds).

5. The herbaceous species are mostly anemophorous, whilst the majority of the trees appear to be zoophorous. As might be expected *a priori*, species with seeds, which are only adapted for distribution by water, are altogether absent.

6. The new vegetation consists of herbaceous and erect woody plants; climbing plants only occur among the first pioneers exceptionally and in small numbers.

7. The woody pioneers, which in the first few months grow more slowly than the many herbaceous species (e. g. many gramineae and compositae), are nearly all characterized by a great power of resistance against shade, by an especially well developed root-system and by the possession of a foliage-crown, which by exclusion of light, causes the death of the herbaceous species beneath it, generally within one or two years of the closing of the crown of the young trees.

8. On account of their xerophytic nature, there need be no surprise, after what has been said above under no 1, that a few other plants may occasionally [1]) occur in Java among the first pioneers of vegetation. Such plants, which elsewhere, under different oecological conditions, are temporary or permanent epiphytes, occur on bare lava and on deserted stone buildings (for instance in the deserted fortress of Noesakambangan). Simularly a few land-halophytes grow more or less abundantly on the soil [e. g. *Dodonaea viscosa* (LINN). JACQ., in alpine districts of central and eastern Java].

[1]) Compare also l. c. p. 73 SCHIMPER Pflanzengeographie p. 90 and 102 and the literature cited in those publications.

§§§ 2. MEANS OF DISTRIBUTION.

The following observations were, in part, made by me in 1891, on a botanical journey in Central Java, and were written down in the same year, but were not published[1]) till some years later in a dutch article entitled: "dissemination of Myrica seeds by birds" in a memoir on spontaneous and artificial reforestation of the Sendoro.

"The above observations are of interest on account of the circumstance, that *Myrica javanica* REINW. (*Pitjisan, javanese*) has been found by experience to be one of the most important species of trees for the reforestation of the Sendoro, and especially on account of the fact, that as yet there are in the literature no numerical data as to the means and agents of distribution of this zoophorous species.

The drupaceous stone-fruits of *Myrica* trees being edible, and there always being a large number of birds, especially wild pigeons, in the *Myrica* reforestations, it seemed not improbable, that the fruits are eaten by the birds, and are disseminated by them.

The contents of the crop and gizzard of three birds, shot in the *Myrica* forests above Kledoeng (1450 m. above sea-level), were found to consist almost entirely of undamaged Myrica stones, some of which were still surrounded by the soft mesocarp; of these birds one green pigeon had 231, a second green pigeon 144, and one koetilan (Ixos haemorrhous) 4 seeds of *Myrica javanica*, still enclosed in the stone.

In the drop and gizzard of a single pigeon there were thus no fewer than 231 undamaged "seeds" of the Myrica tree. The gizzards of the two green pigeons contained only the red mesocarp with the seeds still enclosed in the stone, without other remnants of food, but the crop and the gizzard of the koetilan contained in addition some rests of insects.

Since all the stones which were examined, were quite undamaged and had only been freed from the surrounding fleshy portion of the fruit, it seems to me there is no doubt, that the strong endocarps, thrown out by the birds, will germinate extremely well, and it may be assumed, that the green pigeons especially contribute largely to the dissemination of the seeds of *Myrica javanica* on the G. Sendoro.

It may be added, that the above-mentioned stones, collected from the three bird's gizzards, were sown at my request by Mr. E. TOBI

[1]) KOORDERS, S. H, Spontaneous and artificial reforestation of the Sendoro in Java (in Tijdschrift van Nijverh. en Landb. v. Nederl. Indie, Dl 51, p. 241—287, with a map). — Compare also: VALETON, Th. The distribution of fruits by animals (in Teijsmannia IV, p. 219).

at Kledoeng, in order to see whether their germinative power had really not suffered by the sojourn in the crop and gizzard (KOORD. l. c. p. 45—47)".

To the above notes the following may now be added. When in 1903 I returned to the mountain in question, I learned verbally from the keeper, whom Mr. E. TOBI (now Chief Inspector, Head of the Forestry Service in the Dutch Indies) had kindly instructed at my request to sow the Myrica javanica-"seeds", that the "seeds" from bird's gizzards and crops, received by him, had all germinated excellently and had further developed well.

These observations may therefore be regarded as proving, that some birds, and especially a species of the large green wild pigeons referred to (probably *Vinago Capellei* or an allied species of the genus *Vinago*) may, in Java, contribute very largely to the dissemination of the alpine tree, here in question; they also show that in some cases 100°/₀ of the "seeds", which have been ingested, germinate well, and that the number of Myrica fruitstones found in the crop of a single bird may amount to 231.

As yet no data are known to me regarding the dissemination of Myrica seeds in Java by other animals.

§§ 2. MYRICA LONGIFOLIA TEIJSM. & BINNENDIJK.

§§§ 1. Distribution outside Java: unknown.

§§§ 2. Distribution and oecological conditions in Java. It is stated by MIQUEL (Flora Ind. Bat. I, 1, p. 872) that *Myrica longifolia* TEIJSM. & BINN., which, according to KOORDERS and VALETON, Bijdr. Booms. Java IX (1903) p. 104, is synonymous with *Myrica Lobbi* T. & B., had been found on the Megamendoeng in the Preanger. This has certainly never been confirmed, although I have repeatedly collected herbarium material in this district. Only in one single place have I found this species wild, namely in Central Java, in the residency Semarang on the G. Telemojo above Sepakoeng, at an altitude of 1700 m. above sea level. The tree grew sporadically in an evergreen, mixed forest on fairly dry soil of volcanic grit, on a lateral summit of the G. Telemojo, called G. Pendil. In the same forest there also occurred among other plants: *Weinmannia Blumei* PLANCH. (Saxifragaccae) and *Wendlandia Junghuhniana* MIQ. (Rubiaccae). At the original above-mentioned station of *Myrica longifolia* TEIJSM. & BINN., *Myrica javanica* BL. was however entirely absent.

§§§ 3. Means of distribution. The dissemination of this species very probably also takes place through the agency of birds, which eat the drupaceous stonefruits. Further data about the means of distribution of this rare species are wanting. The fruits resemble those of *Myrica javanica* BL., so closely however, that in any case I cannot consider the limited vertical and horizontal distribution of *Myrica longifolia* T. & B. as being due to difficulties of dissemination, but feel obliged to attribute it to other oecological conditions.

§ 2. On Oreiostachys, GAMBLE, a new genus of Gramineae-Bambuseae, collected by Dr. A. PULLE in Java at an altitude of 1600 m. above sea-level

When in 1907 I was engaged on the systematic and phytogeographical investigation of the Alpine flora of Java, and was working in the herbarium of the Royal Botanical Museum at Dahlem near Berlin [1]), Prof. Dr. F. A. F. C. WENT was so kind as to make me the offer, very highly appreciated by me, of giving me also for investigation a collection of Alpine plants, made in 1906 in Java by Dr. A. PULLE, and belonging to the Herbarium at Utrecht. In the preliminary naming of this collection, which proved to have been excellently collected, and preserved and labelled with great care, I found a fine flowering specimen of a *Bambusea,* the naming of which presented exceptionally great difficulties, here and at Berlin (*i. a.* by the absence of fruits, which are almost indispensable for the determination of *Bambuseae*). Thereupon I sent this herbarium

[1]) Most grateful mention should here be made of the assistance given me by Professor Dr. A. ENGLER, Director of the Royal Botanic Garden at Dahlem, near Berlin, and by Dr. D PRAIN, Director of the Royal Gardens, Kew, near London. Prof. Dr. A. ENGLER was so kind as to lend me the plants, which he had personally collected in Java on his last "Forschungsreise", especially in the highest mountain regions of Western Java, and also in the Tengger; this loan was not limited to my stay at Dahlem-Berlin, but continued after my return to Leiden.

Dr. PRAIN had the extreme courtesy to send me, for further study and in response to my request, made in 1907, while I was in Berlin, all the duplicate specimens of the Phanerogams, collected by me in 1899 on the Tengger mountains of Eastern Java, and described by me in Natuurk. Tijdschr. v. N. I (Dl. 60 p. 242—280 and 370—374, Dl. 62 p. 213—266), which specimens had been presented to the Royal Herbarium at Kew, by the Botanic Gardens of Buitenzorg; Dr. PRAIN also granted me the loan of these specimens for further study, while I was in Holland. When it is remembered, that all these specimens had already been incorporated in the herbarium according to their specific name, and were scattered among hundreds of thousands of other specimens, at the time when I made my request, it is evident, that this action implies extreme scientific liberality and readiness to help on the part of the Director of the Royal Botanic Gardens, and of the scientific staff of the Royal Herbarium at Kew.

These collections are still the subject of study.

specimen to Mr. J. S. GAMBLE, F. R. S. in England. This botanist, the author of the excellent monograph on Indian *Bambuseae*, (published in Vol. VII of the Annals of the Royal Gardens of Calcutta), succeeded in determining, that the plant belonged to a new genus of *Gramineae-Bambuseae-Arundinarieae,* near the genus *Sasa* SHIBATA; it might thus perhaps prove to be related to a plant, collected in Java by JUNGHUHN and only known to MIQUEL and to BÜSE in the sterile state. The latter plant was only briefly described by BÜSE sub n. 7 on p 393 of his *Plantae Junghuhnianae* and by MIQUEL sub n. 15 on p. 420 of Vol. III of his Flora Indiae Batavae under the name *Bambusacea spec. indet.* (without further addition). GAMBLE rightly supposed that the authentic specimen of JUNGHUHN's herbarium might be in the Herbarium at Leiden. A search, which I thereupon made in this herbarium, confirmed GAMBLE's surmise in a striking manner. For in the first place I succeeded in finding the authentic specimen of JUNGHUHN's plant, in a packet of JUNGHUHN's *Gramineae* of Java, returned long ago by BÜSE's heirs to the herbarium at Leiden; fortunately the specimen was not only in an excellent state of preservation and had the authentic determination label of BÜSE (1854), but also the original herbarium notes, which I presume to have been made in 1839 by JUNGHUHN, at the time of collection. In the second place I succeeded at Leiden, by a comparison of the leaves of the flowering branches of PULLE's herbarium specimen (which leaves were greatly reduced in size) with the sterile, normally developed leaves of JUNGHUHN's plant, in establishing the unspecific identity of the two plants an identity already surmised by GAMBLE. A comparison of the authentic labels of BÜSE and of JUNGHUHN, preserved at Leiden, further showed that a clerical error had arisen in BÜSE's text l. c. 393—394 (which MIQUEL l. c. 420 had cited without criticism and in an abbreviated form under "*Bambusacearum species dubiae*"). This error seems to me to have arisen from BÜSE's not having read with care JUNGHUHN's labels indicating the place of growth.

This was evident from the following:

1. On the above-mentioned labels of JUNGHUHN I could clearly read without difficulty: "J. Sunda-landschap. 3—6000′ Bambu-ö-o", while BÜSE l. c. 394 gives: *Habitat Javae sylvas intactas Pekalongan, altit.* 3—6000′. JUNGHUHN. — *Incolae hanc vocant Bambu oö, fide* JUNGHUHN: — *Species propria scandens aut ramis pendentibus?*" (BÜSE l. c.).

Pekalongan is however, not situated in the Sunda district ("Sunda-landschap = the present Preanger), but in Central Java.

2. Furthermore JUNGHUHN cannot have collected at Pekalongan at an altitude of 3—6000 feet, since Pekalongan lies in a low plain.

3. JUNGHUHN collected much herbarium material on and near the plateau of Pengalengan.

4. As is proved by authentic specimens preserved at Leiden, the plants collected on and near the Pengalengan plateau have often similar labels to the *Bambusacea spec. indet.* of JUNGHUHN, here in question.

5. The herbarium specimen of Dr. PULLE, no. 3173 was also collected near the Pengalengan plateau in the Preanger, and also at an altitude of 1600 m.; the species is not yet known from any other locality. Taking this into account, there seems little doubt, that MIQUEL's words "bij Pekalongan" etc., under JUNGHUHN's Bambusacea, should be deleted and should be replaced by: in the Preanger, at 3000—6000 feet (1000—2000 m.) probably on or near the plateau of Pengalengan discovered in 1839 by JUNGHUHN, and called there according to JUNGHUHN *Bambu-ŏ-ŏ* (Sundanese).

According to verbal information, which Dr. PULLE was kind enough to give me, his *Bambusacea* no. 3173 was called by the Sundanese *Awi-eueul.* Here I adopt the same transliteration as was used by the late Dr. J. BRANDES and most other authors, for the Sundanese *eu*-sound.

I now consider the word *Bambu*-ŏ-ŏ- a less correct transcription [1])

[1]) In Java all the species of Bamboe are invariably referred by the natives to one of the following genera: *Bambu* (Malay) = *Awi* (Sundanese); *Pring* (Low Javanese = *Dĕling* (High Javanese).

HASSKARL writes *Awi-ülül* (Sundanese) for his *Bambusa elegantissima*, HASSK. Plant. Jav. rar. (1848) 42; Miq. Fl. Ind. Bat. III (1885) 420 = *Beesha elegantissima* (HASSK.) Kurz = *Melocanna elegantissima* (HASSK.) Kurz. = *Schizostachyum elegantissimum* (HASSK.) Kurz in Indian Forests I (1876) 347. — About an authentic specimen of *Schizostachyum elegantissimum* (HASSK.) Kurz Mr. J. S. GAMBLE was so kind as to give me the following valuable information: "I have here, belonging to my Herbarium and probably a duplicate from Calcutta, authentic pseudophylls — large things 12 inch long and 6 inch broad and very hairy, marked in Kurz' own handwriting "*Melocanna elegantissima*". They do not look at all as if they could belong to Dr. PULLE's plant (GAMBLE msc.). Lateron, when the additional from his plant asked for from Java by Dr. PULLE of one of his friends in the Preanger will be received here, I hope there will be an opportunity for giving more information on this interesting species.

for Awi-eueul, because among the Sundanese bamboo is not called *"bambu"* but *"awi"*, and because the "specific" name *ö-ö* would probably be more correctly written *eu-eu* or perhaps better still *eueul.*

It is a remarkable fact, that during the course of more than half a century the natives have evidently used practically the same "specific" name *eueul* or *eueu* to designate a species, which for so long a time remained imperfectly known to science. This fact is also of importance, in that it will be possible, by means of this constant native name, to find the species on the spot, for further study and in order to obtain the seeds und fruits, as yet unknown to science.

The specimens of JUNGHUHN's collecting number 143 are now registered in the Herb. Lugd. Bat. as H. L. B. *n. 901*, 7—617—618—619—620.

According to the description [1]) of his journey, JUNGHUHN botanized for some time in October 1839 in the same high mountain regions, where Dr. PULLE found his flowering *Oreiostachys*. He does not, however, mention in that publication the plant which was later referred to by BÜSE l. c. and by MIQUEL l. c. as *Bambusacea spec. indet.* In any case, I have been unable to find anything in the description of the journey, which would give certainty regarding my surmise, mentioned above and still regarded as probable, that JUNGHUHN's sterile specimen was collected in 1839.

Dr. JONGMANS. 2nd Conservator at the Herbarium at Leiden, was so courteous as to allow me to lend to Mr. GAMBLE a small fragment of JUNGHUHN's sterile herbariummaterial of MIQUEL's *Bambusacea spec. indet.* n. 15, which material I had examined.

Mr. GAMBLE was thus enabled to supplement his diagnosis of the leaves, and further to confirm the conspecific identity found by myself, of Herb. JUNGHUHN n. 143 with Herb. PULLE n. 3173.

I here add without alteration the generic and specific diagnoses of *Oreiostachys Pullei* GAMBLE, which Mr. J. S. GAMBLE F. R. S. has kindly placed at my disposal, and beg to thank him heartily for his disinterested help, so highly appreciated by myself. A brief résumé as to locality, native, names etc. is appended.

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[1]) JUNGHUHN. Uitstapje naar de bosschen van de gebergten Malabar, Wajang en Tiloe op Java, (in Tijdschr. voor Natuurl. Geschied. en Physiol. VIII (1841) 349—412.

[2]) Mr. A. H. BERKHOUT in Wageningen, late of Dutch East Indian Forest Department, was so kind as to inform me, that he observed *Awi-eueul* (which he kept for Bambusa elegantissima) growing common on the slopes of the mountains Malabar and Patoeha, but never in flower. HASSKARL (l. c. 42) gives for his Bambusa elegantissima: "inter montes Tilu et Malabar provinciae Bandong in terra Preangeieana copiosissime obviam venit". To HASSKARL the flowers were also unknown.

According to a letter received by me, the flowers and flowering branches, referred to in the following diagnoses, have been described by GAMBLE after the Utrecht herbariumspecimen (PULLE n. 3173) and the leaves after the above-mentioned fragment of the Leiden specimen (JUNGHUHN n. 143 = *H. L. B.* n. 901, 7—617—618—619—620).

"**Oreiostachys,** GAMBLE *gen. nov.* Spiculae 1 *florae, ovato-oblongae, secus ramos paniculae in racemis brevibus dispositae ; floribus hermaphroditis.* Glumae *subcoriaceae, mucronato-acuminatae, multinerves, dorso apicem versus pallide villosae ; 4—6 vacuae inferiores, ab imo gradatim auctae ; florens vacuis simillima ; palea etiam glumis similis sed bimucronata, ecarinata, dorso interdum corrugata, quam gluma villosior, dorso basi interdum rachilla terminali munita.* Lodiculae 3, *breves, nunc obtusae, nunc spathulatae, pilis longis sericeis ciliatae.* Stamina 6 ; *filamenta longissima glabra ; antherae elongatae, loculis inferne acutis.* Ovarium *glabrum, ovoideum vel cylindricum, apice incrassatum ; stylus basi 3-fidus, stigmatibus plumosis.* Caryopsis *non visa.*"

"*Gramina suffruticosa, culmis maxime fistulosis ; pseudophyllis scabris, apice fimbriatis, apiculo brevi.* Folia *petiolata, cum vaginis articulata, nervulis transversis nullis vel obscuris.* Inflorescentia *in culmis aphyllis ; ramis longis vel brevibus verticillatim dispositis, sed vaginis et pseudophyllis munitis, quam maxime decomposita.*" (J. S. GAMBLE *msc.* 28. *I.* 1908 *in Herb. Acad. Rheno-Traject et Herb. Acad. Lugd. Bat.*)

"**Oreiostachys Pullei,** GAMBLE *spec. nov.* Suffrutex *scandens* 10 *m. altus* (*fide cl.* PULLE) ; *culmi inter nodos maxime fistulosi ; nodi annulati ; pseudophylla straminea, scabra, ore longe fimbriata, apiculo brevi.* Folia *tenuiter membranacea, lineari-lanceolata, apice longe setaceo-acuminata, basi inaequaliter cuneata, margine et apice scabra, supra laevia infra paullo asperula et ad costam prope basin villosula,* 12—20 *cm. longa,* 1—3 *cm. lata ; costa subtus lucens, nervi utrinque* 5—8 *haud conspicui, nervulis transversis perobscuris ; vaginae glabrae striatae, apice ciliis paucis albis rigidis munitae ; ligula longiuscula puberula serrata.* Inflorescentia *in culmis florentibus ; paniculae ad nodos verticillatae, ramis longis vel brevibus, ad* 10—12 *cm. longae ; spiculae in ramulis racemosis pauciflorae, acutae,* 10—15 *cm. longae,* 2—3 *mm. latae,* 1-*florae ; rhachis angulata, sinuata, scabra ; bracteae multe foliosae foliis similes sed vaginis majoribus.* Glumae *vacuae* 4—6 *subcoriaceae, ovatae, longe mucronatae,* 5—15-*nerves,* *nervulis transversis obliquis frequentioribus, dorso scabrae et sub apice albo-villosae; gluma florens vacuis simillima,* 12 *mm. longa* 5 *mm. lata; palea etiam florenti similis et aequilonga, bimucronata, dorso rotundata, ecarinata.* Lodiculae 3, 1—1.5 *mm. longae, basi cuneatae, apice obtusae, longe albo-fimbriatae; interdum elongatae, spathulatae (an in floribus morbosis?)* 7—8 *mm. longae, longe ciliatae.* Stamina 6*; filamenta longissima; antherae* 7—7.5 *mm. longae.* Ovarium 1—2 *mm. longum, glabrum; stigmatibus plumosis plerumque plus minus coalitis.* Caryopsis *non visa.* (J. S. GAMBLE *msc* 28. *I.* 1908 *in Herb. Acad. Rheno-Traject. et Herb. Acad. Lugd. Bat.*).

Geographical distribution: Outside Java: unknown. In Java: Only in Western Java, in the Preanger, at an altitude of 1000—2000 m. Here only represented by two collecting numbers 1) in virgin forest, probably in 1839 on or near the plateau of Pengalengan (JUNGHUHN n. 143, in Herb. Lugd. Bat. sub n. H. L. B. 908, 7—617—618—619—620. — branches which only bear leaves) and 2) in the Wajang-Windoe mountains, near the tea-plantation Malabar in virgin forest, at a height of 1600 m. above sea-level (A. PULLE n. 3173 in Herb. Rheno-Traject. Suffruticose, climbing, 10 m. high. Spikes dark violet. Flowering without leaves. This species of Bamboo has not flowered for a very long time, according to verbal information obtained locally. Flowering branches collected on June 25, 1906). — Native name: *Awi-eueu or Awi-eueul* (Sundanese) near Pengalengan.

This species is phyto-geographically very interesting, because 1) this monotype represents a genus, which according to GAMBLE (see above), is more nearly related to the genus *Sasa* SHIBATA, occurring in Japan, but absent from the Malay Archipelago, than to the other genera of *Gramineae-Bambuseae,* represented in the Archipelago by numerous wild species; 2) because in Java, according to my numerous journeys, nearly all wild-growing Bambusaceae only occur below 1600 m., while *Oreiostachys Pullei* GAMBLE, was found growing wild by PULLE 1600 m. above sea-level, 3) because this species appears to be endemic in Java, and seems there to be localized in a few mountain regions of the Preanger.

Histology. — *"A method of cold injection of organs for histological purposes"*. By Prof. H. J. HAMBURGER.

For a considerable time the want has been felt of replacing at injections for histological purposes the warm substance, for which as a rule stained gelatine was taken, by a cold one; not only because when using a warm mass the technical difficulties, which are great already, are rendered more complicated still, owing to the care necessary to keep organ and mass at bodily temperature, but also because in a warm waterbath the structure of the tissues is frequently impaired. Therefore TAGUCHI proposed in 1888 [1]) to use for this purpose a suspension of Japanese Indian ink in water, but GROSSER [2]) pointed out as a drawback that on further treatment of the sections the isolated grains not seldom drop, if not out of the smaller yet out of the larger vessels; whilst already at the cutting they are not seldom dispersed over the surface of the section. He therefore tried to find a fluid which could easily be solidified after the injection and found that the white of a hen's egg cut and afterwards filtrated answered this purpose very well.

When we too wished to apply this method the difficulty made itself felt, that in this manner we could not obtain the mass in a sufficiently fluid state. When according to the prescription we rubbed the piece of Indian ink over the plate of ground glass a membrane was always formed. Moreover it was found that the suspension thus prepared, when kept in a bottle, had become a solid mass after 24 hours, although evaporation was out of the question.

Probably this had to be attributed to the Indian ink of which, as is wellknown, many kinds are found in the trade. But we did not succeed in getting a better one.

We then tried to obviate this difficulty by mixing the egg white solution with *liquid Indian ink* as is to be obtained in the trade under the name of GÜNTHER-WAGNER'sche flüssige Perltusche, in the volumetric proportion of 1 to 1. The result was a thin liquid mass, which, when examined under the microscope, contained only extremely small particles, which were in BROWN's molecular motion.

After injection with this fluid the organ was fixed in sublimateformol by which the injected egg white could be precipitated. After the usual washing with water containing iodine, pieces of the organs were stained with alumcochineal, and afterwards embedded in paraffin

1) Archiv. f. Mikrosk. Anatomie, **31**, p. 565, 1888.

2) Zeitschr. f. Wissenschaftliche Mikroskopie, **17**, p. 187, 1900.

in the usual manner. On microscopic examination the blood-vessel were found to be filled now *with a perfectly homogeneous black mass*

This method has an advantage over that of GROSSER in the fac that we need not fear the injection-fluid becoming solidified befor the injection; besides the preparation of the suspension requires *muc less time.*

In another direction too we have simplified the method, viz. by substituting blood-serum for egg white. A mixture of 3 parts o blood-serum with 2 parts of the above named Indian ink gave excellent results.

The blood-serum need not be derived from the same species of animal. For injections of caviae or rabbits we got good results by using horse-serum or cow-serum, fluids that are easily obtained.

Here too fixation was brought about by means of sublimate-formol.

As yet kidneys and liver were microscopically examined. But the injection fluid also penetrated skin, muscles and brain.

An attempt to prepare suspensions of carmine grains in serum suggested itself now, but these experiments failed as the carmine particles conglomerated. Perhaps, however, mixtures of dissolved carmine or of colloidal fluids may be prepared with serum, giving good results.

The above mentioned experiments were made in cooperation with Mr. A. F. DE BOER and Mr. G. A. KALVERKAMP, medical students.

Groningen, March 1908.

Mathematics. — "*The sections of the net of measure-polytopes M_n of space Sp_n with a space Sp_{n-1} normal to a diagonal.*" By Prof. P. H. SCHOUTE.

1. In the first part of a communication on fourdimensional nets and their sections by spaces (*Proceedings*, Febr. 1908) we have i.a. transformed the net (C_8) into a net (C_{16}) and a net (C_{24}); so here the regular simplex, the fivecell C_5, was not considered. Whereas the regular simplex of Sp_2, the equilateral triangle, furnishes a plane-filling all by itself as well as in connection with some other regular polygons, and the regular simplex of Sp_3, the tetrahedron, can fill the space in combination with the octahedron, it is impossible, as was shown in the quoted paper, to find for the regular simplex C_5 of Sp_4 other regular cells, which can together fill the space of Sp_4. This leads us gradually to the question, whether it is not possible

to point out one or more polytopes — if not quite regular ones — which with C_5 fill the fourdimensional space. We have here in view to give to this question an answer, emanating from the connection of a few results formerly arrived at.

2. We consider the net (M_5) of the measure-polytopes M_5 of space Sp_5 and cut this by a space Sp_4 normal to a diagonal. This work breaks immediately up into two parts. First the section of space Sp_4 with a definite measure-polytope M_5 must be found, e. g. with the one, the centre of which has been taken for origin of a rectangular system of coordinates with axes parallel to the edges; we must next investigate how we can prove from this section in which way the intersecting space Sp_4 affects the other measure-polytopes of the net.

The answer to the first part of this question can be found by means of one of the two diagrams 1 and 2, which we shall therefore discuss successively. Of these diagram 1 is what we arrive at when we project

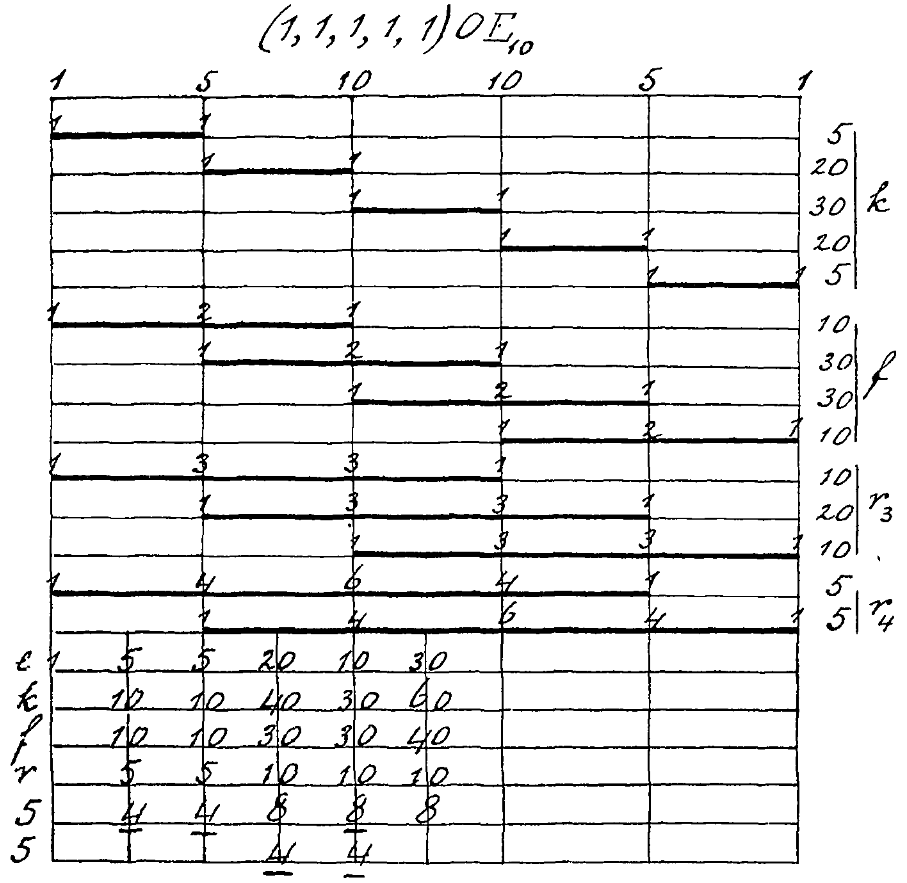

Fig. 1.

the bounding elements of M_5 on the diagonal; it is an extension of the second diagram $n=5$ of the plate, added to the communication on the section of the measure-polytope M_n of the space S_n with a central space Sp_{n-1} normal to a-diagonal (*Proceedings*, Jan. 1908). Here, too, we restrict ourselves to a few sections, viz. to the transition forms and to those intermediary forms which bisect the distance of two adjacent transition forms; according to the notation introduced there, we distinguish the transition forms by the symbols $\frac{1}{5}M_5$, $\frac{2}{5}M_5, \frac{3}{5}M_5, \frac{4}{5}M_5$, the intermediary forms by the symbols $\frac{1}{10}M_5, \frac{3}{10}M_5$, $\frac{5}{10}M_5, \frac{7}{10}M_5, \frac{9}{10}M_5$. As these sections have been incidentally already found in the last quoted paper, we can suffice here by a mere enumeration; to be able to indicate relations in measure we again assume that we have taken half the edge of M_5 as measure-unit.

Transition forms. As two sections pM_5 and qM_5 of which the fractional symbols p and q complete each other to unity, form two oppositely orientated positions of the same polytope, we have here to deal with but two transition forms, viz. $\frac{1}{5}M_5 = -\frac{4}{5}M_5$ and $\frac{2}{5}M_5 = -\frac{3}{5}M_5$. Of these $\frac{1}{5}M_5$ is a regular fivecell $C_5^{(2\sqrt{2})}$, whilst $\frac{2}{5}M_5$ is formed (see *Proceedings*, page 488 under $n=6$) by truncating a fivecell $C_5^{(4\sqrt{2})}$ at the vertices as far as halfway the edges and hence transforming it into a polytope (10, 30, 30, 10) with edges $2\sqrt{2}$; for the last form *Proceedings*, page 503, can be compared.

Intermediary forms. Of the three intermediary forms $\frac{1}{10}M_5 = -\frac{9}{10}M_5$, $\frac{3}{10}M_5 = -\frac{7}{10}M_5$, $\frac{5}{10}M_5$ the first is a $C_5^{(\sqrt{2})}$, the second (*Proceedings*, page 488 under $n=7$) a fivecell $C_5^{(3\sqrt{2})}$ truncated as far as a third of the edges, passing by this proceeding into a polytope (20, 40, 30, 10) with edges $\sqrt{2}$, the third (*Proceedings*, page 487 under $n=5$) a $C_5^{(5\sqrt{2})}$ truncated as far as three fifths of the edges, which has on account of this passed into a polytope (30, 60, 40, 10) with edges $\sqrt{2}$.

We shall now pass to diagram 2 where the plane through two opposite edges PQ, $P'Q'$ intersecting the diagonal $P'Q$ has been taken as plane of projection. The projection of $M_5^{(2)}$ on this plane is the

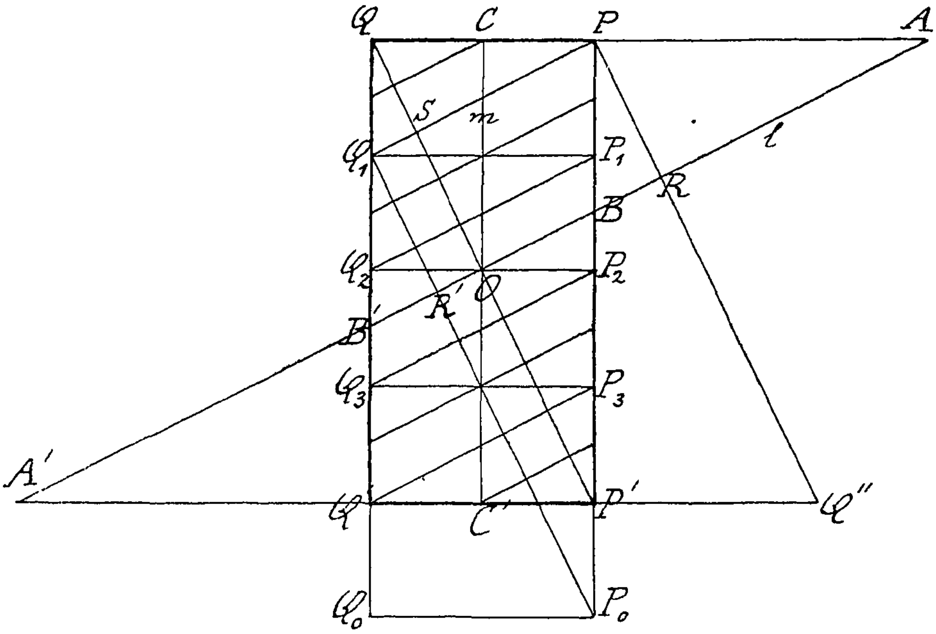

Fig. 2.

rectangle $PQQ'P'$ with edges $PQ = 2$, $PP' = 4$, which is divided by three lines parallel to PQ into four equal rectangles. The intersecting space Sp_4 passing through the centre O stands according to the perpendicular l, erected in O on the diagonal $P'Q$, normal to the plane of projection. If we suppose (*Proceedings*, page 491) a few measure-polytopes $M_5^{(2)}$, which are laid against each other in the direction of the edge PQ on either side, to be united to a prism of which the basis is an $M_4^{(2)}$ and the edges normal to OA have the direction PQ, then the section of the space Sp_4 through O with this prism is a rhombotope Rh_4 of which AA' — with a length of $4\sqrt{5}$ — represents the axis with the period 4. Comparison of this rhombotope with the measure polytope $M_4^{(2)}$ of $M_5^{(2)}$ lying in the space Sp_4 perpendicular according to m on the plane of projection shows us that the rhombotope can be obtained by stretching this polytope $M_4^{(2)}$ in the direction of the diagonal CC' to an amount of $OA : OC = \sqrt{5}$.

This rhombotope is truncated perpendicularly by the spaces Sp_3 projecting themselves in the points of intersection B, B' of the axis AA' with the sides PP', QQ' of the rectangle. If again we make

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use of the annotation $a\,(p, q)$ formerly introduced (*Verhandelingen*, vol. IX, N°. 7, page 17) then the central section is a polytope $4\sqrt{5}\left(\frac{3}{8}, \frac{5}{8}\right)$ and we find, omitting the length of axis $4\sqrt{5}$ alike for all sections, for the transition forms and the intermediary forms described above the following rhombotope symbols:

$$\frac{1}{10} M_5 = \left(0, \frac{1}{8}\right), \quad \frac{3}{10} M_5 = \left(\frac{1}{8}, \frac{3}{8}\right), \quad \frac{5}{10} M_5 = \left(\frac{3}{8}, \frac{5}{8}\right), \quad \frac{7}{10} M_5 = \left(\frac{5}{8}, \frac{7}{8}\right), \quad \frac{9}{10} M_5 = \left(\frac{7}{8}, \frac{8}{8}\right),$$

$$\frac{1}{5} M_5 = \left(0, \frac{1}{4}\right), \quad \frac{2}{5} M_5 = \left(\frac{1}{4}, \frac{2}{4}\right), \quad \frac{3}{5} M_5 = \left(\frac{2}{4}, \frac{3}{4}\right), \quad \frac{4}{5} M_5 = \left(\frac{3}{4}, \frac{4}{4}\right).$$

3. The second part of the question, viz. how the intersecting space Sp_4 affects the other measure-polytopes can now be answered by means of analytical geometry as well as by descriptive geometry.

With reference to the system of coordinates assumed above the centres and vertices of all cells $M_5^{(2)}$ of the net have all nothing but integers as coordinates, the centres only even integers, the vertices only odd ones. From this follows in general that the distances from the centres to the central space $\sum_1^5 x_i = 0$ are multiples of fifth parts of the diagonal, those of the vertices to the same space odd multiples of tenth parts of the diagonal. In this way a space of intersection $\sum_1^5 x_i = p$ in general furnishes five different sections of which the fractions placed before M_5 differ respectively $\frac{1}{5}$. If the space of intersection passes through a vertex we find the transition sections; if it passes though a centre we find the intermediary forms.

We arrive at the same result by diagram 2. If we allow the same space Sp_4 bisecting perpendicularly the diagonal $P'Q$ of the central cell to intersect the right adjacent cell with the diagonal P_1Q'', then the segment QO cut from the diagonal of the central cell passes into PR, which means a decrease of $QS = \frac{1}{5} QP'$, and this is repeated every time a cell is taken further to the right. If we exchange the central cell by an other one of which the projection $P_0 P_1 Q_1 Q_0$ covers for three fourths that of the central one, then QO passes into $Q_1 R'$, again a decrease of $\frac{1}{5}$, and this too is repeated every time the projection moves onward in the direction PP' to an amount of PP_1. So here too we find five different symbols pM_5, of which the fractions gradually increase with $\frac{1}{5}$. With the aid of the above table this result of the notation pM_5 can be transformed into that of the rhombotope symbols.

We have now answered the question put at the commencement. If we wish to fill Sp_4 with C_5 and a single other groundform, then the form (10, 30, 30, 10) with the same length of edges can do service; both forms appear then in two oppositely orientated positions. If by the side of C_5 we allow two other groundforms to fill Sp_4, we can make use of the forms (20, 40, 30, 10) and (30, 60, 40, 10) of the same length of edges, if we take into consideration difference in orientation, then this space-filling demands five forms. And if one does not object to connecting more than two really different groundforms we can take the five forms

$$\frac{1}{20} M_5, \frac{5}{20} M_5, \frac{9}{20} M_5, \frac{13}{20} M_5, \frac{17}{20} M_5,$$

i. e.

$$\left(0, \frac{1}{16}\right), \left(\frac{1}{16}, \frac{5}{16}\right), \left(\frac{5}{16}, \frac{9}{16}\right), \left(\frac{9}{16}, \frac{13}{16}\right), \left(\frac{13}{16}, 1\right),$$

of which the first is a $C_5^{(\frac{1}{4}\sqrt{2})}$; these appear in only one position.

4. Before passing on to the general case of Sp_n we indicate the shortest way, by which one can calculate the number of component parts when filling a fourdimensional block of one of the found forms but of k-times larger linear dimension. To prepare the general case of an arbitrary n we introduce a simpler notation. We distinguish the transition forms and the intermediary forms by the letters T and I and then indicate by exponent — this, to avoid rootsigns, in $\sqrt{2}$ as new unit — the size, by a footindex the place of the section. We then represent the polytope, formed by truncating regularly a regular fivecell with a length of edges $p\sqrt{2}$ at the five corners to the fraction q of the edge by the symbol $qS^{(p)}$. Thus each of the five different forms is represented by four different signs as follows:

$$\frac{1}{10}M_5^{(2)}=\left(0\,,\frac{1}{8}\right)=I_1^{(1)}=\quad S_,^{(1)}$$

$$\frac{3}{10}M_5^{(2)}=\left(\frac{1}{8},\frac{3}{8}\right)=I_2^{(1)}=\frac{1}{3}S_,^{(3)}$$

$$\frac{5}{10}M_5^{(2)}=\left(\frac{3}{8},\frac{5}{8}\right)=I_3^{(1)}=\frac{3}{5}S_,^{(5)}$$

$$\frac{1}{5}M_5^{(2)}=\left(0\,,\frac{1}{4}\right)=T_1^{(2)}=\quad S_,^{(2)}$$

$$\frac{2}{5}M_5^{(2)}=\left(\frac{1}{4},\frac{2}{4}\right)=T_2^{(2)}=\frac{1}{2}S_,^{(4)}$$

whilst the forms appearing past the middle $\frac{7}{10}M_5^{(2)},\frac{9}{10}M_5^{(2)}$ and $\frac{3}{5}M_5^{(2)},\frac{4}{5}M_5^{(2)}$ of opposite orientation are indicated by $I_{-2}^{(1)}$, $I_{-1}^{(1)}$ and $T_{-2}^{(2)}$, $T_{-1}^{(2)}$.

By considering the truncated fivecells $qS_{(p)}$ we find immeditately:

$$\left.\begin{aligned} T_1^{(2k)} &= I_1^{(2k)} \\ I_2^{(k)} &= I_1^{(3k)} - 5I_1^{(k)} \\ T_2^{(2k)} &= I_1^{(4k)} - 5I_1^{(2k)} \\ I_3^{(k)} &= I_1^{(5k)} - 5I_1^{(3k)} + 10I_1^{(k)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Of these relations e. g. the last one is deduced in the following way: The form $I_3^{(k)}=\frac{3}{5}S^{(5k)}$ appears by truncating the fivecell $S^{(5k)}=I_1^{(5k)}$ to $\frac{3}{5}$ of the edges. As each two of the five polytopes $S^{(3k)}=I_1^{(3k)}$, which are taken off by the truncation, have an $S^{(k)}=I_1^{(k)}$ in common, we subtract when diminishing $I_1^{(5k)}$ by $5\,I_1^{(3k)}$ ten times $I_1^{(k)}$ too much.

Together the equations (1) lead to the relations of volume:

$$\frac{I_1^{(k)}}{1}=\frac{T_1^{(2k)}}{16}=\frac{I_2^{(k)}}{76}=\frac{T_2^{(2k)}}{176}=\frac{I_3^{(k)}}{230}=\frac{R^{(2k)}}{384},$$

where $R^{(2k)}$ is the rhombotope formed by the required stretching of an $M_4^{(2k)}$ in the direction of a diagonal. If the number 384 is deduced from the remark that $T_1^{(2k)}=\frac{1}{4!}R^{(2k)}$, then the two relations

$$2\,(16+176)=384 \quad , \quad 2\,(1+76)+230=384,$$

which express that $R^{(2k)}$ can be built up either out of the four forms $T_i^{(2k)}$ or out of the five forms $I_i^{(k)}$ can serve to control.

We shall now indicate at full length how the obtained relations will serve to get us over the entire difficulty of the determination of the demanded numbers. To this end we notice that the vertices of the k^5 measure polytopes $M_5^{(2)}$ forming together a block $M_5^{(2k)}$ project themselves on a diagonal of that block except in the ends in the $5k-1$ points dividing this diagonal into $5k$ equal parts. If we indicate (diagram 3) the $5k+1$ points obtained in this way on the diagonal by $A_0, A_1, A_2, \dots, A_{5k}$, then the segment A_0A_5 bears the projection of a single $M_5^{(2)}$, the segment A_1A_6 that of a group of five, the segment A_2A_7 that of a group of fifteen measure-polytopes,

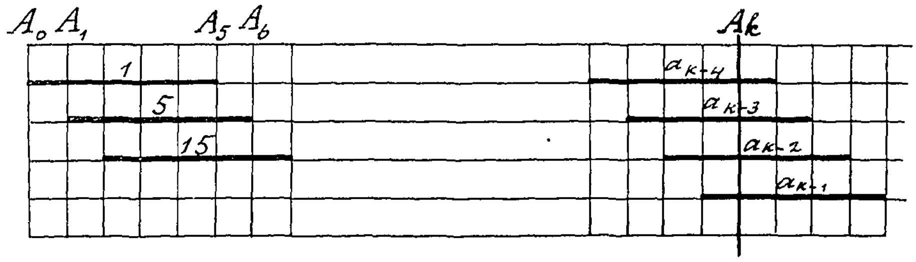

Fig. 3.

etc., where the numbers 1, 5, 15, etc. of the measure polytopes with the same projection are the coefficients a_p of the terms x^p in $(1 + x + x^2 + \dots + x^{k-1})^5$ for $p = 0, 1, 2$, etc. When determining the section $\frac{1}{5} M_5^{(2k)}$ we find that the intersecting space Sp_4 hits the diagonal of projection in the point of division A_k, from which ensues that the groups of polytopes $M_5^{(2)}$ corresponding to the coefficients $a_0, a_1, \dots a_{k-5}$ are not yet cut, the groups corresponding to the coefficients $a_k, a_{k+1}, \dots a_{5k-5}$ are no more cut, so that we have but to deal with the four groups shown to the right of the diagram:

$$a_{k-4} T_{-1}^{(2)}, \quad a_{k-3} T_{-2}^{(2)}, \quad a_{k-2} T_2^{(2)}, \quad a_{k-1} T_1^{(2)} .$$

Now for the coefficients a_p the particularity appears that for $p \leqq k$ they can be represented as binominal coefficients viz. by the equation

$$a_p = (p+4)_4,$$

whilst for greater values of p they are "gnawed" binominal coefficients. So we find here immediately

$$T_1^{(2k)} = I_1^{(2k)} = (k+3)_4 T_1^{(2)} + (k+2)_4 T_2^{(2)} + (k+1)_4 T_{-2}^{(2)} + (k)_4 T_{-1}^{(2)} \dots (2)$$

and in quite the same way

$$I_1^{(2k+1)} = (k+4)_4 I_1^{(1)} + (k+3)_4 I_2^{(1)} + (k+2)_4 I_3^{(1)} + (k+1)_4 I_{-2}^{(1)} + (k)_4 I_{-1}^{(1)} \ldots (3)$$

which two relations in connection with the ratios of volume lead back to the identities

$$\left.\begin{array}{l} k^4 = (k+3)_4 + 11\,(k+2)_4 + 11\,(k+1)_4 + (k)_4 \\ (2k+1)^4 = (k+4)_4 + 76\,(k+3)_4 + 230\,(k+2)_4 + 76\,(k+1)_4 + (k)_4 \end{array}\right\}.$$

From (1), (2), (3) we can now easily deduce all results. To prove this we mention for the two cases, in which the block consists of an even or of an odd number of measure-polytopes, the composition of the central section in the form

$$I_3^{(2k)} = \frac{5}{12} k^2 \left\{ (23k^2 - 11)\, T_1^{(2)} + (23k^2 + 1)\, T_2^{(2)} + \right.$$
$$\left. + (23k^2 - 1)\, T_{-2}^{(2)} + (23k^2 + 11)\, T_{-1}^{(2)} \right\},$$

$$I_3^{(2k+1)} = \frac{5}{12} k(k+1) \left\{ (23k^2 + 23k - 10)\left(I_1^{(1)} + I_{-1}^{(1)} \right) + \right.$$
$$\left. + (23k^2 + 23k + 8)\left(I_2^{(1)} + I_{-2}^{(1)} \right) \right\}$$
$$+ \frac{1}{12} (115k^4 + 230k^3 + 185k^2 + 70k + 12)\, I_3^{(1)}.$$

5. We shall now consider in the space Sp_n the net of measure-polytopes $M_n^{(2)}$ and shall discuss the transition sections and the intermediary forms situated in the middle between two adjacent transition sections furnished by spaces Sp_{n-1} perpendicular to a diagonal. We then find

$$\frac{1}{2n} M_n^{(2)} = \left(0, \frac{1}{2n-2}\right) = I_1^{(1)} = S^{(1)}, \qquad \frac{1}{n} M_n^{(2)} = \left(0, \frac{1}{n-1}\right) = T_1^{(2)} = S^{(2)},$$

$$\frac{3}{2n} M_n^{(2)} = \left(\frac{1}{2n-2}, \frac{3}{2n-2}\right) = I_2^{(1)} = \frac{1}{3} S^{(3)}, \qquad \frac{2}{n} M_n^{(1)} = \left(\frac{1}{n-1}, \frac{2}{n-1}\right) = T_2^{(2)} = \frac{1}{2} S_2^{(4)},$$

$$\frac{5}{2n} M_n^{(2)} = \left(\frac{3}{2n-2}, \frac{5}{2n-2}\right) = I_3^{(1)} = \frac{3}{5} S^{(5)}, \qquad \frac{3}{n} M_n^{(2)} = \left(\frac{2}{n-1}, \frac{3}{n-1}\right) = T_3^{(2)} = \frac{2}{3} S^{(6)},$$

$$\vdots \qquad\qquad\qquad \vdots$$

for n even

$$\frac{n-1}{2n} M_n^{(2)} = \left(\frac{n-3}{2n-2}, \frac{n-1}{2n-2}\right) = I_{\frac{1}{2}n}^{(1)} = \frac{n-3}{n-1} S^{(n-1)},$$

for n odd

$$\frac{1}{2} M_n^{(2)} = \left(\frac{n-2}{2n-2}, \frac{n}{2n-2}\right) = I_{\frac{1}{2}(n+1)}^{(1)} = \frac{n-2}{n} S^{(n)},$$

for n even

$$\frac{1}{2} M_n^{(2)} = \left(\frac{n-2}{2n-2}, \frac{n}{2n-2}\right) = T_{\frac{1}{2}n}^{(2)} = \frac{n-2}{n} S^{(n)},$$

for n odd

$$\frac{n-1}{2n} M_n^{(2)} = \left(\frac{n-3}{2n-2}, \frac{n-1}{2n-2}\right) = T_{\frac{1}{2}(n-1)}^{(2)} = \frac{n-3}{n-1} S^{(n-1)}.$$

If we restrict ourselves to these forms and if again we do not take the transition form consisting of a single vertex into consideration, we have in both cases to deal with n different forms, namely for n even with $\frac{1}{2}n$ transition and $\frac{1}{2}n$ intermediary forms, for n odd with $\frac{1}{2}(n-1)$ transition and $\frac{1}{2}(n+1)$ intermediary forms. Thus we get again in Sp_{n-1} two more or less regular space-fillings in which the regular simplex of that space shares.

In connection with the symbols ${}_qS^{(p)}$ the relations hold here

$$\begin{aligned}
I_2^{(1)} &= I_1^{(3)} - (n)_1\, I_1^{(1)}, \\
T_2^{(2)} &= I_1^{(4)} - (n)_1\, I_1^{(2)}, \\
I_3^{(1)} &= I_1^{(5)} - (n)_1\, I_1^{(3)} + (n)_2\, I_1^{(1)}, \\
T_3^{(2)} &= I_1^{(6)} - (n)_1\, I_1^{(4)} + (n)_2\, I_1^{(2)}, \\
&\vdots
\end{aligned}$$

which leads
for n even to

$$T_{\frac{1}{2}n}^{(2)} = I_1^{(n)} - (n)_1 I_1^{(n-2)} + (n)_2 I_1^{(n-4)} - \ldots + (-1)^{(\frac{1}{2}n-1)} (n)_{\frac{1}{2}n-1}\, I_1,$$

for n odd to

$$I_{\frac{1}{2}(n+1)}^{(1)} = I_1^{(n)} - (n)_1 I_1^{(n-2)} + (n)_2 I_1^{n-4} - \ldots + (-1)^{\frac{1}{2}(n-1)} (n)_{\frac{1}{2}(n-1)}\, I_1,$$

whilst the ratios of volume are determined by

$$\frac{I_1^{(1)}}{1} = \frac{T_1^{(2)}}{2^{n-1}} = \frac{I_2^{(1)}}{3^{n-1} - (n)_1} = \frac{T_2^{(2)}}{4^{n-1} - (n)_1 2^{n-1}} = \frac{I_3^{(1)}}{5^{n-1} - (n)_1 3^{n-1} + (\ )_2} = \text{etc.}$$

Farthermore the formulae of reduction hold:

$$\left.\begin{aligned}
I_1^{(2k)} &= (k+n-2)_{n-1} T_1^{(2)} + (k+n-3)_{n-1} T_2^{(2)} + \ldots + (k)_{n-1}\, T_{-1}^{(2)} \\
I_1^{(2k+1)} &= (k+n-1)_{n-1}\, I_1^{(1)} + (k+n-2)_{n-1}\, I_2^{(1)} + \ldots + (k)_{n-1}\, I_{-1}^{(1)}
\end{aligned}\right\}, \quad . \ . \ (1)$$

which enable us to calculate the number of the parts of different kinds, into which a block of $(2k)^n$ or $(2k+1)^n$ measure-polytopes $M_n^{(2)}$ can be cut up.

As an example, which gives something to calculate, we consider the case of the middle section perpendicular to the diagonal of a block of 10^{10} measure-polytopes $M_{10}^{(2)}$. We then find in connection with the relations

$$\frac{T_1}{1}=\frac{T_2}{502}=\frac{T_3}{14608}=\frac{T_4}{88234}=\frac{T_5}{156190}=\frac{R}{9\,!},$$

where R represents the rhombotope that is the sum of the nine forms

$$T_1\,,\quad T_2\,.\quad T_3\,,\quad T_4\,,\quad T_5\,,\quad T_{-4}\,,\quad T_{-3}\,,\quad T_{-2}\,,\quad T_{-1}\,,$$

starting from

$$\frac{1}{2}\,M_{10}^{(20)}=T_5^{(20)}=I_1^{(100)}-10\,I_1^{(80)}+45\,I_1^{(60)}-120\,I_1^{(40)}+210\,I_1^{(20)},$$

by applying

$$I_1^{(20k)}=(10\,k+8)_9\,T_1^{(2)}+(10\,k+7)_9\,T_2^{(2)}+\ldots.+(10\,k)_9\,T_{-1}^{(2)}$$

for $k=5,4,3,2,1$ after some calculation the result

$$394713550\,(T_1^{(2)}+T_{-1}^{(2)})+410820025\,(T_2^{(2)}+T_{-2}^{(2)})$$
$$+\,422709100\,(T_3^{(2)}+T_{-3}^{(2)})+430000450\,(T_4^{(2)}+T_{-4}^{(2)})$$
$$+\,432457640\,T_5^{(2)},$$

which after substitution of the relations given above leads back to the identity

$$T_5^{(20)}=10^9\,T_5^{(2)}.$$

Physiology. — *"The electric response of the eye to stimulation by light at various intensities"*. By W. Einthoven and W. A. Jolly. (Communication from the Physiological Laboratory of Leiden).

Although the electrical response of the eye to stimulation by light, which was discovered by Holmgren has since been studied by numerous observers, there has not so far been undertaken a systematic investigation of the electromotive changes which are caused by stimuli of very varying strength. Such an investigation, however, can as we hope to show, contribute not a little to our comprehension of the retinal processes.

We have in our work employed exclusively isolated frogs' eyes. We have been enabled on the one hand by means of the string galvanometer, which for the retinal currents may be regarded as the most sensitive instrument available, to record and measure very weak electromotive forces, such as are evoked by light of extremely low intensity; on the other hand we have tried by a suitable system of lenses to concentrate light of as great intensity as possible upon the retina of the eye under observation. The rays proceeding from the crater of an arc lamp, which passed through a collimator slit in close proximity were dispersed by a spectroscopic arrangement and from the spectrum so obtained any desired portion could be isolated by a simple device.

If we made use of rays lying between the wave lengths $\lambda = 0{,}590\,\mu$ and $\lambda = 0{,}497\,\mu$, whose green central part — about $\lambda = 0{,}544\,\mu$ — may be considered to have relatively a very strong effect on the eye [1]), we could by the aid of suitably chosen diaphragms vary the light intensities in the proportion of 1 to 10^{7}, and with the weakest intensity could obtain galvanometric deflections of several centimetres. The arrangement of our experiments did not permit of our easily diminishing the light further in an accurately measurable manner, but we hope later to be able to do so.

In some experiments white light has been used, which of course could be taken stronger than the spectral green. In this case all the rays of the visible spectrum lie at our disposal, and the light may be further increased by widening the slit or by replacing it with the crater itself. According to a rough calculation the intensity of the white light used by us, that is to say of the combined rays lying within the limits of the visible spectrum, is about 10 times greater than our maximum green. The intensities of the weakest green and of the white light are thus in the proportion of about [2]) 1 to 10^{10}.

If the isolated eye, which has not shortly before been exposed to strong light, be illuminated by rays of intermediate strength a form of curve is obtained similar to that recorded by previous observers [3]).

The current is led off from the cornea and the posterior surface of the bulbus. The current of rest is compensated in the usual way and the connections with the galvanometer are made in such a

[1]) Cf. F. Himstedt and W. A. Nagel. Die Verteilung der Reizwerte für die Froschnetzhaut im Dispersionsspectrum des Gaslichtes, mittels der Aktionsströme untersucht. Berichte der Naturforsch. Ges. zu Freiburg i. B., XI, 1901, p. 153.

[2]) The intensities of the light used will later be communicated in absolute measurement and at the same time the accurate proportion of the intensities of the green and white will be given.

[3]) Cf. for instance Francis Gotch, The Journal of Physiol. 29, p. 388, 1903. Ibid. 31, p. 1, 1904. Hans Piper, Engelmann's Arch. f. Physiol. Suppl. 1905, p. 133. E. Th. von Brücke u. S. Garten, Pflüger's Arch. f. d. ges. Physiol. 120, p. 290, 1907, the latter of whom give a critical review of the literature dealing with the subject. The observers mentioned have all made use of a quickly recording measuring instrument.

manner that a current passing from the cornea through the instrument to the posterior surface of the eye deflects the image of the string in an upward direction. An action current in this direction may be termed positive, and in the reverse direction negative.

On momentary illumination of the eye there is observed a small preliminary negative deflection which is immediately followed by an upward movement of the string. After a somewhat acute peak the curve sinks, at first rapidly then more gradually, but while still distant from the zero line it mounts again. This latter ascent begins a couple of seconds after the beginning of the illumination, and the second summit, which is reached much later, often considerably exceeds the peak in height. Finally the curve gradually regains the zero line.

If the illumination be continued for some time, a new elevation occurs at the moment of darkening whose height is greater the longer the illumination has endured.

The complicated form of these curves and the striking fact that a deflection in the same direction takes place both on illumination and on darkening suggest that there are in the eye two or more different processes occurring partly simultaneously partly successively whose fusion determines the form of the electric reaction.

Further investigation confirms this suggestion, and if recourse is had to very weak or very strong light it seems even to be possible to bring about a separation of the supposed processes. The phenomena are explained in the simplest manner by the assumption that the processes are three in number, whether they are together dependent upon the same substance or each upon a separate one. For the sake of convenience we shall speak of three substances and as we do not intend in the meantime to attempt to define them anatomically in the eye, we prefer to try to describe their characteristics and to mention the conditions, under which their effects appear as pure as possible.

The first substance.

The substance which we have termed "the first" reacts more quickly than the other two. On lighting it displaces the image of the string downwards, on darkening upwards. Its effect can with difficulty be obtained pure but nevertheless it is very marked in a light adapted eye, — which for the sake of brevity we may call a light eye [1]) — and the more so the stronger the illumination has been.

In the nature of the case the darkening stimulation can be taken

[1]) An eye which is dark adapted may be called a dark eye. Both terms are analogous to "Lichtfrosch" and "Dunkelfrosch" which are commonly used.

very strong in a light eye, and accordingly an eye which has been illuminated strongly develops on darkening a huge positive potential difference. The upward deflection so evoked can however not be of long duration, because by the darkening the light eye is beginning to be changed into a dark eye and therefore the effect of our first substance is no longer so clearly indicated.

Although in the light eye the conditions are less favourable for the lighting than for the darkening stimulus it is nevertheless possible to apply the former in either of two ways. In the first place we may suddenly increase the intensity of the light that is radiating on the eye, and secondly we may darken the light eye for a short period, so that it has not yet become a dark eye and then suddenly illuminate it.

The second method gives better results than the first and we possess numerous curves where after a short darkening of a light eye a strong light stimulus was applied. The "on effect" [1]) is a steep downward deflection and attains the considerable amount of 120 to 130 microvolts. It is true that it is followed immediately by an upstroke, the latter however is but small in comparison with the strong upstroke which under similar conditions is evoked in a dark eye.

The second substance.

The second substance reacts less quickly than the first. On lighting it moves the string with moderate velocity upwards, and on darkening slowly downwards, thus on applying stimuli of the same kind it develops potential differences which are opposed to those of the first substance. Its effect appears almost unmixed in a dark eye which is illuminated for a short time by weak light.

If when illuminating with light of very low intensity, the darkening follows rapidly upon the lighting, in a similar way as in a momentary illumination, there is recorded a curve of simple form, with a steeper anacrotic part which is evoked by the lighting and a less steep katacrotic part evoked by the darkening. The top of the curve lies, *within certain limits* higher the more the energy of the illumination is increased either by using greater intensity or longer duration of the light. These limits are determined by the functioning of the other two substances, which when their effects become perceptible influence the form of the curve and considerably complicate it. If a strong momentary illumination be applied there appears a short negative preliminary deflection by the function of the first substance

[1]) A convenient expression introduced by GOTCH.

and the very slow second elevation which follows must be attributed to the functioning of the third substance.

The third substance.

The third substance reacts in the same direction as the second substance but more slowly. On lighting it displaces the image of the string slowly upwards and on darkening still more slowly downwards. So much slower is the third substance than the other two that its effect in a recorded curve appears as a rule almost entirely isolated, and thus can be easily followed.

The effect of the third substance falls out under two conditions (1) In a fully light adapted eye and (2) in a dark eye submitted to very faint light for a short time.

Specially remarkable are the curves obtained if the duration of the lighting of a dark eye is systematically changed, and we wish to direct attention more particularly to the "off effect" in such cases. If the duration of the light is very short and the light is weak, then as already mentioned the effects of the second substance appear unmixed. The off effect here consists in the descent of the curve to the zero line.

If the duration of the light is taken a little longer, and the effect of the other two substances begin to become perceptible, the off effect is determined by the resultant of three forces: The first substance tends to displace the image of the string upwards. It is at first acting weakly but its strength increases regularly during illumination so that it soon surmounts the effect of the other substances. In the case of longer lighting the off effect therefore is always an upward movement which increases with the duration of the lighting.

The second substance tends to depress the image of the string, acts first with moderate strength but decreases gradually during lighting. As the second substance in particular is acting in a dark eye the conditions for its functioning grow during the illumination more unfavourable. A strong darkening effect can not be expected in a dark eye.

The third substance is so slow, that the darkening effects of the first and second take place usually at a moment when the third substance is still tending to displace the string upwards. The darkening effect of the third substance itself, consisting in a slow descent of the string, appears much later and fairly isolated.

The general result is that we can observe in a series of curves, — obtained from a dark eye where the light has been gradually lengthened in duration, — that the darkening effect, in the first curves a negative deflection, becomes in the later ones a positive deflection. The latter, on further lengthening of the duration of light, gradually increases in size. In the conflict between negative and positive deflections, there is sometimes seen an upward movement, which is immediately preceded by a small downward one.

Of the various particularities which occur in the course of the experiments we shall only briefly mention the latent period. The duration of this period is dependent to so high a degree upon the intensity of the illumination, that it is possible to some extent to judge of the intensities of light used by previous observers from the latent periods recorded by them. With very weak lighting there appear latent periods of the second substance which may exceed two seconds.

In opposition to GOTCH and GARTEN WALLER[1]) also mentions latent periods as large in amount as we have observed and others much larger, but as WALLER in his experiments made use of a slow THOMSON galvanometer, there remained the possibility that there were two opposite forces which at first neutralised one another and then after an interval one obtained the mastery. The forces assumed by WALLER agree with our first and second substances.

A more detailed description of our experiments accompanied by a reproduction of some of our curves will appear elsewhere.

Geophysics. — "*The height of the mean sea-level in the Y before Amsterdam from* 1700—1860". By Prof. H. G. VAN DE SANDE BAKHUYZEN.

Our section has been engaged in former years with an investigation of the subsidence of the land in the Netherlands, and it is especially to Dr. F. J. STAMKART, member of the committee for that investigation, that we owe several important communications on this subject.

Twenty years ago, when calculating the results of the precise levelling, I made some computations in order to determine the subsidence of the land but have not published them. The interesting paper on this subject of Mr. RAMAER, head-engineer, director of the hydrographic survey, has now induced me to re-examine my former notes and as they perhaps may contribute towards the solution of the problem, whether the land under Amsterdam has subsided since

[1]) AUGUSTUS D. WALLER, Philosoph. Transact. of the Royal Soc. of London, Ser. B, vol. 193, p. 123, 1900.

1700, I intend to publish them here as a continuation to the earlier reports of the "Comittee for the subsidence of the land."

My results have been chiefly derived from the heights of the water in the Y before Amsterdam, recorded from 1700 to 1860 at each hour of the day and at each half hour during the night in the town's tidal station situated at the present fishmarket near the "Nieuwe Markt." Part of them occur in two communications of STAMKART (Verslagen en mededeelingen der Kon. Akad. van Wetenschappen Afd. Natuurkunde, 15e deel 1863, p. 59—69 and 17e deel 1865, p. 261—303) and some in STAMKART's posthumous papers in keeping of the Academy.

The way in which these observations were made is described as follows by Dr. STAMKART in his paper in Vol. XVII p. 273. The tidal station was erected above the water; in the wooden floor of one of the rooms was a hole through which a gauging rod carrying a mark of the A.P. (zero of Amsterdam) was plunged vertically into the water so far until a notch of the rod caught on the wooden floor. The height to which the gauging rod was wetted showed the level of the water with regard to the zero on the gauge.

In order to draw reliable conclusions about the level of the North-sea on our coast based on the results of the water-level in the Y, it is necessary to investigate whether during the period under consideration variations have occurred in the influx and the outflow of the water of the Y before Amsterdam, owing to changes in the depth and width of the canals leading from the Northsea to the Y. It is very probable that these variations may produce opposed effects on the high and the low water and hence give rise to greater variations in the difference between high and low water than in the mean sea level. The variations of these differences in the succeeding years will therefore be a good standard of the changes in the canals.

From the tide tables of the town's tidal station we derive the following differences between high and low water during 58 years.

TABLE I.

Year	Difference between high and low water	Year	Difference between high and low water	Year	Difference between high and low water	Year	Difference between high and low water
1700	309 mm.	1715	222 mm.	1805	303 mm.	1847	303 mm.
1701	323 „	1716	299 „	1806	342 „	1848	309 „
1702	331 „	1717	342 „	1807	345 „	1849	318 „
1703	320 „	1725	327 „	1808	331 „	1850	322 „
1704	320 „	1749	318 „	1809	327 „	1851	324 „
1705	312 „	1775	328 „	1810	338 „	1852	329 „
1706	319 „	1796	318 „	1811	330 „	1853	319 „
1707	314 „	1797	313 „	1812	316 „	1854	320 „
1708	308 „	1798	302 „	1813	323 „	1855	287 „
1709	286 „	1799	287 „	1825	341 „	1856	314 „
1710	318 „	1800	288 „	1843	325 „	1857	317 „
1711	325 „	1801	327 „	1844	313 „	1858	295 „
1712	332 „	1802	321 „	1845	310 „	1859	328 „
1713	322 „	1803	284 „	1846	328 „	1860	332 „
1714	332 „	1804	323 „				

If we assume that in each of the 3 periods of 18 years this difference has been constant, we find for it:

in the 1st	period	1700—1717	319	mm.
„ „ 2d	„	1796—1813	318	„
„ „ 3d	„	1843—1860	316	„

The mean error of a yearly mean is then ± 14,8 mm.

If we suppose that from 1700—1860 the difference has remained constant, then the difference derived from all the 58 years amounts to 318 mm. and the mean error of a yearly mean to ± 14.4 mm. Therefore we are justified in assuming that the difference has remained constant during the whole period 1700—1860 and was 318 mm. ± 1,9 mm.

Moreover we derive from this table that between 1700 and 1860 no perceptible change has occurred in the influx and outflow of the water from the Northsea to the Y, no more than in the mean level of the Northsea with regard to the mean level of the Y.

As mean level of the Y before Amsterdam with regard to the zero adopted in the tidal station (zero of Amsterdam) we shall assume the half sum of the high and low water.

For the same 58 years we derive for that mean the following values.

TABLE II.

Year	Mean Sea level above A.P.	Year	Mean Sea level above A.P.	Year	Mean Sea level above A.P.	Year	Mean Sea level above A.P.
1700	— 172 mm.	1715	— 166 mm.	1805	— 105 mm.	1847	— 79 mm.
1701	„ 169 „	1716	„ 163 „	1806	„ 69 „	1848	„ 102 „
1702	„ 148 „	1717	„ 159 „	1807	„ 69 „	1849	„ 64 „
1703	„ 187 „	1725	„ 154 „	1808	„ 147 „	1850	„ 53 „
1704	„ 146 „	1749	„ 134 „	1809	„ 112 „	1851	„ 66 „
1705	„ 179 „	1775	„ 89 „	1810	„ 90 „	1852	„ 59 „
1706	„ 199 „	1796	„ 84 „	1811	„ 99 „	1853	„ 76 „
1707	„ 160 „	1797	„ 115 „	1812	„ 103 „	1854	„ 12 „
1708	„ 153 „	1798	„ 96 „	1813	„ 114 „	1855	„ 76 „
1709	„ 193 „	1799	„ 136 „	1825	„ 51 „	1856	„ 48 „
1710	„ 167 „	1800	„ 134 „	1843	„ 20 „	1857	„ 101 „
1711	„ 144 „	1801	„ 55 „	1844	„ 15 „	1858	„ 96 „
1712	„ 126 „	1802	„ 123 „	1845	„ 54 „	1859	„ 64 „
1713	„ 149 „	1803	„ 132 „	1846	„ 40 „	1860	„ 75 „
1714	„ 106 „	1804	„ 92 „				

These values show that the mean sea level has not remained unchanged with regard to the adopted zero of Amsterdam. This becomes still more evident if we form the means of the 3 periods of 18 years. We then obtain:

1708,5	—160,3 mm.	± 5,9 mm.
1725	—154 „	± 25,1 „
1749	—134 „	± 25,1 „
1775	— 89 „	± 25,1 „
1804,5	—104 „	± 5,9 „
1825	— 51 „	± 25,1 „
1851,5	— 61 „	± 5,9 „

If we suppose that during each of the periods of 18 years the mean sea level has remained unchanged we derive from the deviations of the yearly means from the mean of 18 years a mean error for each year of $\pm$ 25,1 mm. and in the mean of 18 years a mean error of $\pm$ 5,9 mm.

If on the contrary we suppose that during each of the periods of 18 years the mean sea level with regard to the adopted Amsterdam zero has varied proportionally to the time, we get for the mean error of the yearly mean $\pm$ 24,3 mm. and for the yearly variations:

from	1700—1717	+ 1,57 mm.	$\pm$ 1,10 mm.
„	1796—1813	+ 0,14 „	$\pm$ 1,10 „
„	1843—1860	— 2,30 „	$\pm$ 1,10 „ .

Hence in the 1st and 2nd periods, in agreement with the general variation of the mean sea levels from 1700—1826, the mean sea level has apparently come nearer to the adopted A.P., but has retired thence in the 3d period in agreement with the variation from 1825—1851,5. Nevertheless the mean errors of each of these yearly variations, $\pm$ 1,10 mm., are so large with regard to the variations themselves, that we attach only a very small weight to the values found; only to the yearly variation in the 3d period, more than twice the value of the mean error, we may attach a somewhat larger weight. If we adopt a uniform yearly variation between the years 1708,5 and 1804,5, this would amount to 0.58 mm.; in good harmony with this are the results for 1725 and 1749, but the result for 1775 shows a deviation of 32 mm.

We conclude that the elevation of the adopted A.P. above the mean sea level has gradually varied and that the variations can be considered as partly proportional to time; they cannot however be derived exactly from the observations.

The elevation of the A.P. in the tidal station above the mean sea level *in the first and the last year of the series of observations*, 1700 and 1860, are according to table II 162 mm. and 75 mm. each with a mean error of $\pm$ 25 mm. In order to obtain for these elevation values with a smaller mean error, we may use, upon supposition that no sudden variations have taken place in the zero of the gauging rod, the elevation observed in closely preceding or following years, which must be reduced to the year 1700 or to 1860 with an adopted yearly variation. Because the yearly variation is not known with great precision, as appeared above, it is desirable that these years should not be at a great distance from 1700 or from 1860; therefore

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I have confined myself to the mean of the 5 years 1700—1704 and 1856—1860. To these means we must add the variations during a period of two years, which are probably smaller than 3 mm. and 5 mm. the values which would follow from the periods of 18 years; instead of these I adopt 1 mm. and 4 mm. and consequently:

adopted A.P. above mean sea level in 1700 $= 164 + 1 = 165$ mm.
„ A.P. „ „ „ „ „ 1860 $= 76 + 4 = 80$ „

For the mean error of these values I have derived ± 12 mm.

As yet it remains undecided whether the variation from 165 mm. to 80 mm. is due to a slow variation in the mean level of the North sea on our coast, or to a variation of the adopted A.P. in the tidal station either caused by the sinking of the whole station or of the wooden floor, or by accidental or perhaps intentional changes in the height of the A.P. on the gauging rod which during the period from 1700 to 1860 has certainly been renewed several times.

Some data towards the solution of this dilemma may be borrowed from the elevations of the bench marks in the 5 sluices: Oude Haarlemmersluis, Nieuwebrugsluis, Kraansluis, Westindischesluis and Kolksluis; these bench marks have been established in 1682, and consist of grooves cut in stones indicating the elevation of the A.P. The good mutual agreement between the heights of the grooves in the year 1875 which appeared from the levelling made by our member Dr. Lely (the largest difference between them amounted to only 8 mm.) proves that those grooves have been placed with the greatest care, and makes us confident that in 1700, when the first observations in the tidal station were made, the zero on the gauging rod agreed well with that on the stones placed in the sluices some years earlier.

We are therefore entitled to assume with a high degree of probability that in 1700 the A.P. on the 5 sluices was 165 mm. above the mean level of the Y.

In 1860 Stamkart by a levelling has compared the height of the A.P. in the tidal station at that time with the heights of two bench marks in the tower of the St. Anthoniewaag. He found:

lower bench mark 3208,4 mm. above A.P. in the tidal station
higher „ „ 3705,4 mm. „ „ „ „ „ „

In the same year Dr. Stamkart and Mr. v. d. Sterr have also determined by means of levelling the difference in height between the higher bench mark in the St. Anthoniewaag and the grooves in the 5 sluices (Versl. en meded. XVII p. 277—284). From these observations we derive the following values for the height of the higher bench mark above the A.P. according to the mean of the 5 sluices in 1860:

Stamkart	v. d. Sterr	mean
3628 mm.	3624 mm.	3627 mm.

In the derivation of the mean value we have, with regard to the mean errors, accorded a greater weight to Stamkart's result.

In 1875 our colleague Dr. Lely by means of a still preciser levelling under direction of Cohen Stuart has derived 3622 mm. for the same difference in height.

The differences between the results of 1860 and those of 1875 may be explained very well by errors of observation, so that we may accept with a high degree of accuracy that the bench mark in the St. Anthoniewaag between 1860 and 1875 has not varied with regard to the 5 sluices and that in 1860 the height of the mark above the A.P. of the sluices was 3623 mm. with a mean error of $\pm$ 2 mm.

If from this value we subtract 3705, i. e. the height of the mark above the A.P. in the tidal station found by Dr. Stamkart in 1860 we find:

height of the A.P. according to the mean of the 5 sluices above the A.P. in the tidal station in $1860 = 82$ mm.

The mean error of this result is about $\pm$ 3 mm.

As in 1860 the height of the A.P. in the tidal station was elevated 80 mm. above the mean level of the Y, it follows that in 1860 the A.P. derived from the mean of the 5 sluices above the mean sea level is:

$$80 + 82 = 162 \text{ mm.} \pm 13 \text{ mm.}$$

If we compare this value with the corresponding value of the year 1700, i. e. 165, we may conclude that the height of the mean sea level in the Y, and hence the mean level of the Northsea on our coast has not perceptibly varied with regard to the ground in which the foundations of the 5 sluices are built.

The uncertainty of this conclusion may be expressed by a mean error of $\pm$ 18 mm.

The 5 sluices are not in close neighbourhood of each other, the extreme ones are separated by a distance of one kilometre; hence it is over a fairly extensive part of the ground on which Amsterdam is built that the level of the land with regard to the level of the Northsea has remained unchanged during more than one century and a half.

With the same degree of probability with which we have derived this invariability we may derive from the observations the subsider of the A.P. in the tidal station with regard to the A.P. derived frc the marks in the 5 sluices, amounting to 165—80 = 85 mm. betwe 1700 and 1860.

The method by which the height of the water in the tidal stati was obtained and the possible causes of the subsidence of the zero on t rod added to the invariability of the 5 grooves in the sluices and hence a fairly large part of the ground of Amsterdam with regard the sea, render the idea very probable that this subsidence has a pu local character and that we are not entitled to derive any results wi regard to the subsidence of a larger part of the ground of Amsterdai

It has often been asked what the Amsterdam zero represen Our colleague Dr. VAN DIESEN has devoted to this subject ε interesting study in which he has gathered from old documen everything which may help us to find how this zero has bee established. With certainty nothing can be derived from it. B the observations show: 1 that in 1700 the A.P. was 165 mm. : 12 mm. above the mean sea level in the Y, 2 that the height of tl mean high water was $\frac{318}{2} = 159$ mm. ± 1 mm. above the same mea sea level, and we conclude thence that both in 1700 and 1860 th A.P. within the limits of the errors of observation agreed with th mean high water in the Y.

Astronomy. — "*On the masses and elements of Jupiter's satellite and the mass of the system* (continued), by Dr. W. DE SITTEI (Communicated by Prof. J. C. KAPTEYN).

III. *The great inequalities.*

The values of these, derived from the heliometer-observations c 1891, 1901 and 1902, have been collected in Table III, togethe

TABLE III. GREAT INEQUALITIES.

Authority	x_1	x_2	x_3
1891	0°.509 ± 0°.018	1°.021 ± 0°.013	0°.059 ± 0°.007
1901	0·481 ± 47	1·089 ± 30	0·049 ± 20
1902	0 372 ± 34	1·171 ± 19	0·034 ± 12
DAMOISEAU	0·455	1·074	0·073
SOUILLART's theory	0·432	1·026	0·063
Masses (C)	0 430 ± ·020	0·988 ± ·017	0 064 ± 003

with their probable errors. The photographic determination of 1902 has been rejected for the reason which has already been explained.

From these values of x_i have been derived the equations of condition, which will be given below.

The arguments of these inequalities are $l_i + v$, where

$$v = l_2 - 2l_3 = l_1 - 2l_2 + 180°.$$

Their periods are thus nearly the same as those of the equations of the centre, and in a short series of observations, such as those used here, the great inequalities are not well separated from the equations of the centre. This is the reason of the bad agreement of the results from the three series of observations.

In the eclipses the period of the great inequalities is the same for the three satellites, viz: 438 days[1]). The periods of the equations of the centre in the eclipses have between 10 and 19 times this length, and the two classes of unknowns are thus well separable by eclipse observations. Here however, there arises a new complication, which did not exist in the case of extra-eclipse observations. The periods of the inequalities of group II, which are between 406 and 486 days, are nearly the same as the period of the great inequalities, and therefore the reliability of the determination of x_i from eclipse observations will depend in a large measure on the accuracy of our knowledge of the inequalities of group II. Thus *e. g.* with the masses (C) the coefficient of the inequality in the longitude of satellite II, which has a period of 463 days, is 0°.038. This inequality is entirely neglected by DAMOISEAU (being proportional to e_2), and it is probable that his value of x_2 — which, according to the introduction to his tables, was derived directly from the observations — will be more or less affected by this circumstance. The same thing is true in a somewhat lesser degree of the corresponding terms in the longitudes of I and III.

The uncertainty which still reigns supreme with regard to the values of the great inequalities, is disappointing. We may hope that the reduction of the photometric eclipse observations of the Harvard observatory will contribute to diminishing this uncertainty.

IV. *The Libration.*

The mean longitudes l_1, l_2, l_3, have been derived from the observations of 1891 (GILL, heliometer), 1892—93, 1893—94, 1894—95, 1895—96, 1897, 1898 (Helsingfors and Pulkowa, plates), 1901, 1902

[1]) See LAPLACE. *Mécanique Céleste,* Tome IV, Livre VIII, Chapitre II.

(Cookson, heliometer) and 1904 (Cape, plates). The reduction has been carried out in Gron. Publ. 17. The masses (A) are the result of this discussion. The period of the libration being independent of $\varkappa'$, it is the same for the masses (B) as for (A). Also the transition from (B) to (C) does not affect this period. It is thus only necessary to investigate in how far the change from (A) to (C) affects the inequalities of group II, and what is the effect of this on the libration. This effect was found to be so small that a new determination of the libration appeared superfluous. The finally adopted libration is thus the same as in Gron. Publ. 17, viz:

$$\vartheta = 0^{\circ}{\cdot}158 \, sin \frac{T - 1895.09}{7.00},$$

where the time T is expressed in years.

The probable error of the period corresponding to the adopted probable errors of the masses (C) is $\pm$ 0.13.

The corrections to the mean longitudes on 1900 Jan. 0.0 also have been adopted unaltered from Gron. Publ. 17.

Table IV contains the observed corrections to the mean longitudes, with their probable errors as derived directly from the observations, and the residuals remaining after subtitution of the final values of the inequalities of group II and the libration. The last two columns contain the p.e. of the quantity $\Delta l_1 - 3 \; \Delta l_2 + 2 \; \Delta l_3$, and the residuals for this same quantity.

In determining the libration from extra-eclipse observations we find the mean longitudes for epochs, which approximately co-incide with the epoch of opposition, and which therefore are on the average separated by intervals of 400 days. This interval differs but little from the periods of the inequalities of group II. These latter thus present themselves as inequalities with apparent periods between 6 and 8 years, and are therefore not well separable from the libration. In the eclipses this difficulty does not exist.

The method of successive approximations, which has been used in Gron. Publ. 17, to derive from the observations the most probable values of the libration and of the inequalities of group II, need not be explained here. It must suffice to refer the reader to that publication (see also these Proceedings, June 1907). The residuals of Table IV are practically the same as those found in Gron. Publ. 17, and they also need not be considered in detail here. Those of the satellites I and III are not very satisfactory, as has been pointed out there. On this point also the results derived from extra-

TABLE IV. MEAN LONGITUDES AND LIBRATION.

Series	Δl_1			Δl_2			Δl_3			ϑ	
	Observed correction	p. e.	Residual	Observed correction	p. e	Residual	Observed correction	p. e.	Residual	p. e.	Residual
1891	+ 0°·100	± °·006	— °·034	+ 0°·065	± °·003	— °·007	— 0°·031	± °·002	— °·013	± °·012	— °·039
1892—3	+ ·073	± 8	— 22	+ ·051	± 5	— 17	— 023	± 3	— 4	± 14	+ 21
'93—4	+ ·128	± 14	+ 31	+ ·019	± 9	— 26	— ·034	± 5	— 16	± 34	+ 76
'94—5	+ ·131	± 11	+ 11	— ·012	± 6	+ 9	— ·029	± 3	— 17	± 23	— 49
'95—6	+ ·152	± 6	— 8	— 026	± 4	0	— 004	± 2	+ 13	± 14	+ 18
'97	+ ·112	± 11	— 64	+ ·019	± 9	+ 6	— ·014	± 4	+ 9	± 30	— 59
'98	+ ·163	± 10	+ 2	+ ·120	± 5	+ 21	— 002	± 3	+ 26	± 19	— 10
1901	+ ·136	± 9	+ 23	+ 020	± 6	— 1	— ·037	± 4	— 26	± 23	— 24
'02	+ ·134	± 7	— 9	— ·025	± 4	+ 2	— ·023	± 3	— 7	± 17	— 29
'04	+ 231	± 12	+ 64	+ ·063	± 7	+ 15	— ·006	± 4	+ 26	± 27	+ 71

eclipse observations need confirmation from observations of eclipses.[1])

V. *Mean longitudes and mean motions.*

The corrections to the mean longitudes on 1900 Jan. 0.0 of the three inner satellites have been determined together with the libration, and the residuals have already been given in Table IV. For the fourth satellite the adopted correction is — 0°.030, and the residuals are given in Table V.

TABLE V Δl_4.

Epoch	Observed correction	p. e.	Residual
1891	— 0°0248	± °0010	+ °0035
1901	— ·0361	± 18	— 58
1902	— 0342	± 16	— 37

If the corrections are added to the values adopted in computing the tabular places, and then referred to the first point of Aries by adding the adopted longitude of the point O, we find for 1900 Jan. 0, mean Greenwich noon, the values which are given below, sub I.

In the introduction to his tables DAMOISEAU states the mean longitudes for 1750 Jan. 0.5, mean time of Paris. If we consider these as being derived directly from the observations, they require a small correction, since DAMOISEAU has used the value $493^s.2$ of the light-time, while in the reduction of the modern observations the value $498^s.46$ was adopted. If DAMOISEAU had adopted this latter value, he would have found the same longitudes for an epoch which is $5^s.26 \times \Delta$ earlier, Δ being the mean distance of Jupiter. The observed mean longitudes, in order to correspond correctly to the tabular epoch, therefore require the correction [2]) :

$$+ \frac{5.26}{86400} \cdot \Delta \cdot n_i = + 0.000317 \ n_i.$$

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1) It has also been pointed out in Gron. Publ. 17 that the series of extra-eclipse observations from which the libration was derived, not being made for this special purpose, does not in every respect fulfil the conditions necessary for a good determination of the libration.

2) In Gron. Publ. 17 I assumed, on the authority of COOKSON, Cape XII. 3, page 56, that MARTH's longitudes for 1750.0 were identical with DAMOISEAU's. This, however, they are not, MARTH having applied the correction for the change in the adopted constant of aberration with the wrong sign This was pointed out to me by Mr. BANACHIEWICZ.

Applying this correction, and carrying the longitudes forward to 1900 Jan. 0.0, Greenwich M. T., we find the values II below.

Mean longitudes for 1900 *Jan.* 0.0.

	I (*modern*)	*II* (DAMOISEAU)
$l_1 =$	$142°.604 \pm 0°.010$	$142°.645 \pm °.004$
$l_2 =$	$99\ .534 \pm\ .007$	$99\ .569 \pm\ .006$
$l_3 =$	$167\ .999 \pm\ .007$	$168\ .028 \pm\ .008$
$l_4 =$	$234\ .372 \pm\ .002^5$	$234\ .360 \pm\ .010.$

The estimated probable errors for DAMOISEAU do *not* contain the p. e. of the mean motions used for carrying the longitudes forward from 1750 to 1900. The uncertainty of DAMAUSEAU's mean motions has been estimated by the late Prof. OUDEMANS in these Proceedings (October 1906). He finds for the four mean motions, in units of the eighth decimal place:

$$\pm 73 \qquad \pm 55 \qquad \pm 37 \qquad \pm 24$$

Comparing the values I and II we find the following corrections to DAMOISEAU's mean motions:

$$dn_1 = -\ 0°.0000\,0075 \pm °.0000\,0020$$
$$dn_2 = -\ 0\ .0000\,0064 \pm \qquad 16^5$$
$$dn_3 = -\ 0\ .0000\,0053 \pm \qquad 20$$
$$dn_4 = +\ 0\ .0000\,0022 \pm \qquad 18$$

It is noticeable that these corrections are very nearly of the magnitude of the uncertainties estimated by OUDEMANS. If these corrections are applied, the resulting values do not satisfy the condition

$$n_1 - 3n_2 + 2n_3 = 0.$$

If, however, we apply the further corrections

$$dn_1 = -\ 2 \qquad dn_2 = +\ 3 \qquad dn_3 = -\ 3$$

to the eighth decimal place, then the condition is rigorously satisfied. The mean motions thus derived are those finally adopted. They are

$$n_1 = 203°.4889\,9261 \qquad n_3 = 50°.3176\,4587$$
$$n_2 = 101\ .3747\,6145 \qquad n_4 = 21\ .5711\,0965$$

These are the mean motions relatively to the point Aries. If the sidereal mean motions are required, they must be diminished by 0°.0000 3822.

VI. The mass of the system.

The determination of the mass of the system of Jupiter by NEWCOMB [1]),

[1]) Astronomical papers of the American Ephemeris, Vol. 5, Part. 5.

which has now become a classic in astronomy, was based on observations of satellites, on perturbations in the motion of comets, and of the planets Themis, Polyhymnia and Saturn. It seems to me advisable to retain of these only the determinations from the three planets. Of the older observations of the satellites the uncertainty of the scale-value (which is increased threefold in the mass of the planet) is such that their weight, compared with the modern observations, and with the determinations from the perturbations of planets, is absolutely negligible. Newcomb has also, for this same reason, assigned a very small weight to these observations of the satellites.

The use of observations of comets seems to me very dangerous. It is very uncertain, if not improbable, that the observed centre of light should retain the same relative position with respect to the centre of gravity throughout one apparition of the comet, and *a fortiori* in different apparitions. Newcomb also points out that the results based on observations of comets are unreliable for this reason. Nevertheless he assigns a large weight to the determination by von Haerdtl from Winnecke's comet, on the ground that the normal places of this comet are so well represented by von Haerdtl's results. It appears to me that this good representation does not diminish the stringency of the argument stated above, and in my opinion it is advisable to reject also this determination, together with those from other comets.

There remain the determinations from the three planets, which I adopt with the same weights assigned to them by Newcomb, and the modern observations of satellites, which were only made, or at least reduced, after Newcomb's discussion was published. For these latter the scale-value is determined in an entirely satisfactory manner by simultaneous observations of standard stars. Nevertheless I have assigned to these observations a relatively smaller weight than to the determinations from the planets, to allow for the possibility of small systematic errors in transferring the scale-value from the distance of the standard stars to the mutual distances of the satellites.

In my reduction of Gill's observations of 1891 I have included in the probable error of $\mathfrak{M}$ the effect of the uncertainty of the standard stars used for the determination of the scale-value. The probable errors stated by Cookson do not include this uncertainty. The distances of the stars used by Cookson are not so accurately known as of the stars used in 1891. I have for these reasons assigned a smaller weight to Cookson's two determinations than to Gill's. The several determinations and their probable errors and adopted weights are given in Table VI.

TABLE VI. RECIPROCAL OF THE MASS OF THE SYSTEM.

Authority	Observed values	Weight	Residual
KRÜGER, perturbations of Themis	1047·54 ± 0 19	5	+ 0·14
HILL, „ „ Saturn	38 ± ·12	7	— ·02
NEWCOMB, „ „ Polyhymnia	34 ± 06	20	— ·06
GILL—DE SITTER, Satellites, 1891	·50 ± ·06	10	+ ·10
COOKSON, „ 1901	·46 ± 09	4	+ ·06
COOKSON, „ 1902	·25 ± ·06	6	— 15

The mean by weights is 1047.394 ± .026. The simple mean is 1047.412. The mean of the determinations from the planets alone is 1047.380, and the mean of the determinations from the satellites is 1047.417. The value which I propose to adopt is

$$\mathfrak{M} = \mathbf{1047.40 \pm 0.03}.$$

The probable error was derived from the residuals. The distribution of these residuals, each compared with its own probable error as stated by the observers, is in excellent agreement with the theoretical distribution according to the law of errors. The adopted p.e. can therefore be considered to be a trustworthy measure of the real accuracy

I may be allowed to state as my conviction that it will not be possible in the near future materially to improve the value here adopted. In order to attain from observations of satellites a smaller probable error than ± 0.03, or $^1/_{35000}$, the scale-value must be known within less than $^1/_{130000}$. It thus appears useless to attempt a new determination of the mass from observations of the satellites, until we are in the possession of means as well of fixing the distance of a pair of standard-stars with this accuracy, as of transferring the scale-value determined therefrom to other (smaller) distances without the possibility of systematic errors. Investigations of modern heliometers point to the conclusion that the transferring of the scale-value from a distance of, say, 7000″ to one of 700″ is still subject to uncertainties, which may reach an amount equivalent to an error of 0″.1 in the larger distance, and which therefore may amount to $^1/_{70000}$ of the scale-value. On the other hand it seems a high demand on our present observational means to fix a distance of about 2° of two stars with an uncertainty smaller than 0″.07 = 0ˢ.005 [1]).

[1]) The accuracy of the distance of the standard stars used in 1891 was ± $^1/_{60000}$. (See my dissertation, page 8).

NEWCOMB has already pointed out that oppositions of Polyhymnia, as favourable as the one used in his work, will not recur till the end of the twentieth century, and a similar statement is true for Themis. HILL has pointed out [1]) that Jupiter produces in the motion of certain minor planets (those of the Hecuba type) perturbations of long periods, which amount to several degrees. Thus e. g. Freia is subject to a perturbation, whose geocentric amplitude is 12°.7 with a period of 121 years. The length of the period makes it impossible to derive an improved value of the mass by this method in the near future.

Derivation of the final masses.

The right-hand-members of the equations of condition, which have served to determine the corrections to the values (B) of the masses, have been derived, as explained above under I to IV, from:

I. the motions of the nodes θ_2 and θ_3 (those of θ_1 and θ_4 I leave out of consideration, as having too small weights),

II. the motion of the perijove $\tilde{\omega}_4$,

III. the great inequalities x_1, x_2, x_3,

IV. the period of the libration.

The equations are:

I.

$$\begin{array}{rrrrrcl} -.0266\,\delta\nu' & -.0030\,\delta\nu_1 & -.0001\,\delta\nu_2 & -\ 0040\,\delta\nu_3 & -.0002\,\nu_4 & = & -\ 000\,10 \pm .000\,08 \\ -.0051 & -.0003 & -.0007 & 0 & -.0007 & = & -.000\,41 \pm \quad 15 \end{array}$$

II.

$$+.00077\,\delta\nu' +.00004\,\delta\nu_1 +.00007\,\delta\nu_2 +.00082\,\delta\nu_3 -.00005\,\nu_4 = -.000036 \pm .000020$$

These three equations depend in part on the values of the elements in 1750, which were determined from eclipse-observations. It has already been pointed out above that practically the same results would be found from extra-eclipse observations alone.

III.

$$\begin{array}{rrrcrrrl} & & & & 1891 & 1901 & 1902 & \text{Adopted} \\ -.003\,\delta\nu_1 & +\ 403\,\delta\nu_2 & -.014\,\delta\nu_3 & = & +\ 080 & +\ 051 & -.058 & +\ 020 \pm\ 040 \\ +\ 195 & -\ 008 & +.816 & = & +.019 & +\ 087 & +\ 169 & +\ 050 \pm\ 40 \\ -.001 & +.060 & -.006 & = & -.004 & -\ 014 & -.020 & -.009 \pm\ 10 \end{array}$$

The probable errors of the separate determinations have been given in Table III. The p.e. of the adopted values were estimated according to the agreement of the separate values.

IV.

$$+2.40^5\,\delta\nu_1 +0.24^5\,\delta\nu_2 +1.35^5\,\delta\nu_3 = 0.000 \pm 0.18$$

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[1]) Collected works I, page 105.

If now we reduce all these equations to the same weight, so that the p. e. of their right-hand members becomes ± 0.10, we find, if also the signs of I are reversed:

	$d\varkappa'$	dv_1	dv_2	dv_3	v_4		*Res*	*Res.* Souill.
I.	$+33.3\,d\varkappa'$	$+3.8\,dv_1$	$+0.1\,dv_2$	$+5.0\,dv_3$	$+0.2\,v_4$	$= +0.12^5$	$+.02$	$-.78$
	$+3.4$	$+0.2$	$+0.4^5$	0	$+0.5$	$= +0.27$	$+.25$	$+.16$
II.	$+3.8^5$	$+0.2$	$+0.3^5$	$+4.1$	-0.2^5	$= -0.18$	$-.12$	$-.82$
III.		0	$+0.6$	0		$= +0.05$	$+.05$	$+.05^5$
		$+0.5$	0	$+2.0^5$		$= +0.12$	$+.16$	$+.06^5$
		0	$+0.1^5$	0		$= -0.09$	$-.09$	$-.09$
IV.		$+1.4$	$+0.1^5$	$+0.8$		$= 0.00$	$.00$	$+.\ 7$

The finally adopted corrections are.

$$d\varkappa' = +0.005 \pm .0075$$

$$dv_1 = +0.010 \pm .030 \qquad dv_2 = 0 \pm .050$$

$$dv_3 = -0.020 \pm .020 \qquad v_4 = 0 \pm 0.25$$

The corresponding values of the masses are.

$$\left.\begin{aligned} Jb^2 &= 0.0214\,180 \pm .0001543 && (b = 1 \text{ for } d = 39''.0) \\ &= 0.0000\,0000\,518169 \pm 3975 && \text{(astronomical units)} \\ m_1 &= 0.0000\,260 \pm .0000\,012 \\ m_2 &= 0.0000\,231 \pm 11 \\ m_3 &= 0.0000\,804 \pm 16 \\ m_4 &= 0.0000\,424\,751 \pm .0000\,106 \end{aligned}\right\} (C)$$

Substituting these corrections, there remain the residuals stated above. If SOUILLART's masses are substituted there remain the residuals given in the last column.

The equations II and III are contradicting each other II demands a negative value dv_3, III a positive value. On account of the bad agreement of the different determinations of x_3 I have assigned a very small weight to the equation III. It is to be noticed that the large negative correction dv_3 could have been partly avoided by assuming a large positive value of v_4, e. g. $v_4 = +0.5$. Even then, however, it would not be possible to bring about a satisfactory agreement of II and III without spoiling the representation of I and IV.

The probable errors stated for the corrections $d\varkappa'$ and dv_3 as well as the values of these corrections themselves, depend largely on judgment[1]). In estimating the probable errors I have taken into account as accurately as I could the imperfections as well of the theory on which the left-hand members of the equations of condition depend as of the observations from which the right-hand members are derived. It has been my aim to estimate true *probable* errors, i. e. the masses (C) are those which with our present knowledge of the system I consider the most probable, and I consider it equally probable that the deviation of the values (C) from the truth is smaller than the stated p. e., as that it exceeds this quantity.

The above contains all that can be derived from modern extra-eclipse observations. The resulting values of the inclinations and nodes, and of the mass of the system, i. e. the groups A and C of unknowns, must be considered as final, so far as the observational data at present available go. The results for the other unknowns (those of group B) cannot be accepted as final until they are confirmed by the reduction of the photometric eclipse observations of the Harvard observatory. With regard to the inclinations and nodes, I have already pointed out in Cape XII. 3 (page 121) that a new determination about the year 1920 is desirable. For the determination of m_4 it will be necessary, as was pointed out by me in my dissertation, p. 82 and 85, to supplement the modern observations by a determination of k_3 and k_2 about 1790 from a re-reduction of old eclipses. Of these an amply sufficient number exists. Between the years 1772 and 1799 I have found in the literature of the epoch records of 63 eclipses of which the immersion and emersion have been observed by the same person, and about one third of these have been observed by more than one observer.

In order to derive entirely satisfactory results it will also be necessary to revise SOUILLART's analytical theory, as pointed out by me in Gron. Publ. 17, page 118.

The masses and elements derived in the above, though not to be considered as final, still doubtlessly are much nearer to the truth than those used in SOUILLART's theory. It therefore seemed desirable to introduce them into the expressions for the latitudes, longitudes and radii-vectores as given by that theory. To take account of the uncertainties of the masses I give the coefficients as functions of the small quantities ϱ and λ_i, which are defined by

[1]) "The probable error arising from the uncertainty of such judgments must be included among the possible unavoidable sources of error." NEWCOMB, Astronomical Papers of the American Ephemeris, Vol. 5, Part 4, page 398.

[Note added in the English translation].

$$Jb^2 = (Jb^2)_0 (1 + \varrho)$$
$$m_i = (m_i)_0 (1 + \lambda_i),$$

where $(Jb^2)_0$ and $(m_i)_0$ represent the values (C). The squares and products of ϱ, λ_1, λ_2 and λ_3 will be neglected. These developments are based entirely on those of Gron. Publ. 17, and what was there said about their accuracy and reliability also applies here.

The semi-major-axes corresponding to the adopted mean motions and the adopted mass of the system have been computed by the formula [1]) :

$$n_i^2\, a_i^3 = f \frac{1 + m_i}{\mathfrak{M}(1 + \Sigma\, m_i)} \left(1 + \frac{Jb^2}{a_i^2}\right).$$

Their logarithms are

$$\begin{aligned} \log a_1 &= 7.450\ 1443 + 0\ 000\ 101\ \varrho \\ \log a_2 &= 7.651\ 8277 + 000\ 040\ \varrho \\ \log a_3 &= 7.854\ 6197 + 000\ 016\ \varrho \\ \log a_4 &= 8.099\ 8338 + .000\ 005\ \varrho \end{aligned}$$

The values of the coefficients τ_{ij}, which occur in the expressions for the equations of the centre, are

$$\begin{aligned}
\tau_{21} &= +0.0280 - 031\ \varrho + .027\ \lambda_1 - 002\ \lambda_2 + .055\ \lambda_3 \\
\tau_{31} &= -0.0053 - .003\ \varrho - .005\ \lambda_1 - .004\ \lambda_2 - .001\ \lambda_3 \\
\tau_{41} &= 0.0000
\end{aligned}$$

$$\begin{aligned}
\tau_{12} &= -0.0320 + .058\ \varrho + .027\ \lambda_1 - .011\ \lambda_2 - .061\ \lambda_3 \\
\tau_{32} &= -0.0447 + .022\ \varrho + .003\ \lambda_1 - .042\ \lambda_2 + .006\ \lambda_3 \\
\tau_{42} &= 0.0000
\end{aligned}$$

$$\begin{aligned}
\tau_{13} &= +0.0171 - .013\ \varrho + .002\ \lambda_1 + .014\ \lambda_2 + .015\ \lambda_3 \\
\tau_{23} &= +0.1619 - .098\ \varrho - .005\ \lambda_1 + .019\ \lambda_2 + .116\ \lambda_3 + .0019\ \lambda_4 \\
\tau_{43} &= -0.1173 + 112\ \varrho + .006\ \lambda_1 + 024\ \lambda_2 - .142\ \lambda_3 + .0163\ \lambda_4
\end{aligned}$$

$$\begin{aligned}
\tau_{14} &= +0.0016 - .002\ \varrho \qquad\qquad\quad + .001\ \lambda_2 + .001\ \lambda_3 + .0014\ \lambda_4 \\
\tau_{24} &= +0.0139 - .018\ \varrho - .001\ \lambda_1 - .001\ \lambda_2 + .010\ \lambda_3 + .0112\ \lambda_4 \\
\tau_{34} &= +0.0828 - 072\ \varrho - .001\ \lambda_1 - .017\ \lambda_2 + .009\ \lambda_3 + .0726\ \lambda_4
\end{aligned}$$

The daily motions of the own perijoves (referred to the first point of Aries) are :

$(\tilde{\omega}_1)$	$+0^\circ.14703$	$+^\circ.1295\ \varrho$	$+^\circ.0070\ \lambda_1$	$+^\circ.0166\ \lambda_2$	$+^\circ.0007\ \lambda_3$	$+^\circ.0001\ \lambda_4$
$(\tilde{\omega}_2)$	$+0.038955$	$+.02590$	$-\ 00371$	$+.00406$	$+.01974$	$+.00019$
$(\tilde{\omega}_3)$	$+0.007032$	$+.00530$	$+.00024$	$+\ 00100$		$+.00066$
$(\tilde{\omega}_4)$	$+0.001896$	$+.00075$	$+.00003$	$+.00007$	$+.00082$	$-.00005$

[1]) It will be seen that I adopt here LAPLACE's definition of the mean distances. All other constant terms of the radius-vector will be included in $\rho_i = r_i/a_i$. These ratios ρ_i must not be confounded with the small quantity ϱ representing a possible correction to the adopted value of Jb^2.

The great inequalities are:

$$x_1 = \overset{\circ}{0}.4303 - 0024\,\lambda_1 + .4228\,\lambda_2 - .0145\,\lambda_3$$
$$x_2 = 0.9875 + .1273 \quad - .0090 \quad + .8188$$
$$x_3 = 0.0636 - .0010 \quad + .0629 \quad - .0063$$

The coefficients of the inequalities of group II are:

$$\varkappa_{11} = \{-2.49 \quad - .04 \quad \varrho + .04 \quad \lambda_1 - 2.46\,\lambda_2 + .17\,\lambda_3\}\,e_1$$
$$\varkappa_{12} = \{+0.98 \quad - .19 \quad \varrho - .13 \quad \lambda_1 + \;.97\,\lambda_2 - .10\,\lambda_3\}\,e_2$$
$$\varkappa_{13} = \{+0.083 \quad - .03 \quad \varrho - 02 \quad \lambda_1 + .02\,\lambda_2\}\,e_3$$
$$\varkappa_{14} = \{+0.0062 - .003\,\varrho - .002\,\lambda_4\lambda_2 - .005\,\lambda_3 + .002\,\lambda_4\}\,e_4$$

$$\varkappa_{21} = \{+2.26 \quad - .05 \quad \varrho + 2.20\lambda_1 - .03\lambda_2 + \;.03\lambda_3\}\,e_1$$
$$\varkappa_{22} = \{+2.19 \quad + .16 \quad \varrho - \;.74\lambda_1 + .08\lambda_2 + 2.93\lambda_3\}\,e_2$$
$$\varkappa_{23} = \{-0.535 \quad - .27 \quad \varrho - \;.03\lambda_1 + .15\lambda_2 - \;.02\lambda_3\}\,e_3$$
$$\varkappa_{24} = \{-0.0368 + .005\varrho - (.002 - .017\lambda_4)\lambda_1 + .022\lambda_2 + .045\lambda_4\lambda_3 - .046\lambda_4\}\,e_4$$

$$\varkappa_{31} = \{-0\;01 - .01\,\lambda_1\}\,e_1$$
$$\varkappa_{32} = \{-0.67 - .65\,\lambda_2\}\,e_2$$
$$\varkappa_{33} = \{+0.109 + .07\varrho - 01\lambda_1 + .07\lambda_2\}\,e_3$$
$$\varkappa_{34} = \{+0.0078 + (\,00{:} + 009\lambda_4)\lambda_2 + .011\lambda_4\}\,e_4$$

The quantities determining the libration are:

$$Q_1 = + \{.003440 - 00022\lambda_1 - .00050\lambda_2 - 00142\lambda_3\}\,(1+\lambda_2)\,(1+\lambda_3)$$
$$Q_2 = - \{.005161 - .00021\lambda_1 - 00045\lambda_2 - 00135\lambda_3\}\,(1+\lambda_1)\,(1+\lambda_3)$$
$$Q_3 = + \{.000452 - 00002\lambda_3\}\,(1+\lambda_1)\,(1+\lambda_2)$$

$$\beta^2 = Q_1 - 3Q_2 + 2Q_3 \qquad \psi = \beta\,(t - t_0)$$
$$\vartheta = 0°.158\,\sin\psi$$
$$\vartheta_1 = + \;0.1735\;\vartheta \quad \vartheta_2 = - \;0\;2603\;\vartheta \quad \vartheta_3 = + \;0.0228\;\vartheta$$

The position of the orbital planes of the satellites is in SOUILLART's theory referred to the orbit of Jupiter, of which the inclination and node[1]) referred to the ecliptic and mean equinox are (according to LEVERRIER, but with NEWCOMB's precession):

$$\varphi = \;\; 1°\,18'\,31''.1 - \;0''.2051\;\mathrm{T}$$
$$\vartheta = 99 \;\; 26 \;\; 36 \quad + 36.396\;\mathrm{T},$$

where T is the time counted in tropical years from 1900 Jan. 0.0 Greenwich M. T.

It is preferable, however, to refer the latitudes of the satellites to the mean equator of the planet. The inclination and node of this mean equator referred to the orbit (the node being counted "in the orbit") are:

$$\omega = \quad 3° \;\; 6'\,55''.1 + \;\; 0''.0243\;\mathrm{T}$$
$$\theta = 315 \;\; 48 \;\; 0 \quad + 50\;.158 \;\; \mathrm{T}$$

The inclination and node of the mean equator referred to the ecliptic are thus:

[1]) Unless otherwise stated, node stands for ascending node.

$$= \quad 2^\circ 12' \; 8''7 + \; 0''.4231 \text{ T}$$
$$\mathfrak{N} = 336 \;\; 24 \;\; 24 \quad + 48 .916 \;\; \text{T}$$

The inclination and node of Jupiter's orbit referred to the mean equator are therefore (the node being counted "in the equator"):

$$\omega = \quad 3^\circ \;\; 6' \, 55''.1 + \;\; 0''.0243 \text{ T}$$
$$\theta' = 135 \;\; 46 \;\; 44 \quad + 50 .155 \;\; \text{T}$$

The position of the orbital planes of the satellites — excluding periodic, but including secular perturbations — referred to the mean equator, are given by the formulas:

$$i_i \sin(\theta' - \Omega_i) = p_i = \Sigma_j \, \sigma_{ij} \gamma_j \sin \Gamma_j$$
$$i_i \cos(\theta' - \Omega_i) = q_i = \Sigma_j \, \sigma_{ij} \gamma_j \cos \Gamma_j + (1 - \mu_i)\, \omega$$

Referred to the orbit of Jupiter they are[1])

$$I_i \sin N_i = \Sigma_j \, \sigma_{ij} \gamma_j \sin \theta_j + \mu_i \omega \sin \theta$$
$$I_i \cos N_i = \Sigma_j \, \sigma_{ij} \gamma_j \cos \theta_j + \mu_i \omega \cos \theta$$

where we have [2])

$$\Gamma_i = 180^\circ + \theta - \theta_i$$

If the periodic perturbations are represented by δp_i, δq_i, δs_i, we have for the latitude of the satellite referred to the mean equator

$$\beta_i = (q_i + \delta q_i) \sin(v_i - \theta') + (p_i + \delta p_i) \cos(v_i - \theta')$$

and referred to the orbital plane of Jupiter

$$s_i = I_i \sin(v_i - N_i) + \delta s_i .$$

Here v_i is the true orbit-longitude of the satellite. In both formulas quantities of the third order in the inclinations are neglected. The neglected terms in β_i are thus of the order of magnitude of 0°.00002 and in s_i of the order of 0°.01.

The values of the coefficients σ_{ij} and μ_i are:

$$\sigma_{21} = -\,0.019 + .012\,\varrho - .019\,\lambda_1$$
$$\sigma_{31} = -\,0.001 + .001\,\varrho - .001\,\lambda_1$$
$$\sigma_{41} = \quad 0.000$$
$$\sigma_{12} = +\,0.0203 - .020\,\varrho + .020\,\lambda_2$$
$$\sigma_{32} = -\,0.0347 + .028\,\varrho + .002\,\lambda_1 - .035\,\lambda_2 + .005\,\lambda_3 - .0005\,\lambda_4$$
$$\sigma_{42} = -\,0.0010 - .001\,\varrho - .001\,\lambda_2 + .001\,\lambda_3$$

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1) Rigorously these formulas are true with reference to the *fixed* orbit of Jupiter, and a correction must be applied to derive the latitude referred to the moving orbit. It is, however, sufficiently accurate to use the same formulas for the latitude referred to the moving orbit, provided we take for ω and θ the inclination and node of the mean equator referred to this same moving orbit (as was done here). For the motion of the node θ referred to the moving orbit I adopted $-$ 0''.0979 instead of $-$ 0''.0710 (SOUILLART II page 166). This in the value which results if SOUILLART's final value of b_4 is used instead of the approximate value used by SOUILLART himself.

2) The meaning of Γ_i is thus here slightly different from what it was in the subordinate investigation I.

$$\sigma_{13} = +\,0.0056 - .013\,\varrho + .003\,\lambda_2 + .010\,\lambda_3 - .0001\,\lambda_4$$
$$\sigma_{23} = +\,0.1488 + .132\,\varrho - .011\,\lambda_1 + .005\,\lambda_2 + .125\,\lambda_3 + .0026\,\lambda_4$$
$$\sigma_{43} = -\,0.1772 + .176\,\varrho + .008\,\lambda_1 + .028\,\lambda_2 - .211\,\lambda_3 + .0282\,\lambda_4$$

$$\sigma_{14} = -\,0.0018 - .003\,\varrho + .001\,\lambda_3 + .0018\,\lambda_4$$
$$\sigma_{24} = +\,0.0183 - .034\,\varrho - .002\,\lambda_1 - .002\,\lambda_2 + .017\,\lambda_3 + .0207\,\lambda_4$$
$$\sigma_{34} = +\,0.1203 - .110\,\varrho - .005\,\lambda_1 - .016\,\lambda_2 + .021\,\lambda_3 + .1064\,\lambda_4$$

$$\mu_1 = 0.99944 + .0009\,\varrho - .0002\,\lambda_3 - .0002\,\lambda_4$$
$$\mu_2 = 0.99428 + .0095\,\varrho + .0002\,\lambda_1 + .0001\,\lambda_2 - .0022\,\lambda_3 - .0023\,\lambda_4$$
$$\mu_3 = 0.97271 + .0294\,\varrho + .0012\,\lambda_1 + .0040\,\lambda_2 - .0010\,\lambda_3 - .0088\,\lambda_4$$
$$\mu_4 = 0.86245 + .0555\,\varrho + .0018\,\lambda_1 + .0045\,\lambda_2 + .0503\,\lambda_3 - .0056\,\lambda_4$$

The daily motions of the nodes θ_i are:

$$(\theta_1)\quad -0^\circ.13614 - .1327\,\rho - .0023\,\lambda_2 - .0010\,\lambda_3 - .00008\,\lambda_4$$
$$(\theta_2)\quad -0.032335 - .02602\,\rho - .00198\,\lambda_1 - .00013\,\lambda_2 - .00399\,\lambda_3 - .000191\,\lambda_4$$
$$(\theta_3)\quad -0.006854 - .00493 - .00021\,\lambda_1 - .00071\,\lambda_2 - .00004\,\lambda_3 - .000695\,\lambda_4$$
$$(\theta_4)\quad -0.001772 - .00077\,\rho - .00003\,\lambda_1 - .00007\,\lambda_2 - .00075\,\lambda_3 + .000098\,\lambda_4$$

and for the angles Γ_i we have:

$$\frac{d\Gamma_i}{dt} = 0^\circ.000038 - \frac{d\theta_i}{dt}.$$

The quantities p_i are thus:

$$\begin{array}{llll}
p_1 = +0.02720\,\sin\Gamma_1 & +0.00951\,\sin\Gamma_2 & +0.00103\,\sin\Gamma_3 & -0.00046\,\sin\Gamma_4 \\
p_2 = -\;.00052 & +\;.46830 & +\;.02734 & +\;.00464 \\
p_3 = -\;.00003 & -\;.01625 & +\;.18390 & +\;.03051 \\
p_4 = \;\;\;.00000 & -\;.00047 & -\;.03259 & +\;.25360
\end{array}$$

In q_i we have the same coefficients, and again in $I_i\,\sin N_i$ and $I_i\,\cos N_i$. The constant terms $(1-\mu_i)\,\omega$ of q_i and the coefficients of $\sin\theta$ and $\cos\theta$ in $I_i\,\sin N_i$ and $I_i\,\cos N_i$ respectively are:

$$\begin{array}{ll}
(1-\mu_1)\,\omega = 0^\circ.00174 & \mu_1\,\omega = 3^\circ.1136 \\
(1-\mu_2)\,\omega = 0.01792 & \mu_2\,\omega = 3.0974 \\
(1-\mu_3)\,\omega = 0.08502 & \mu_3\,\omega = 3.0303 \\
(1-\mu_4)\,\omega = 0.42851 & \mu_4\,\omega = 2.6868
\end{array}$$

The position of the true equator referred to the mean equator is defined by its inclination ω_1 and node ψ_1, which are determined by the formulas

$$\omega_1 \sin(\theta' - \psi_1) = \Sigma_j\,\sigma_{0j}\,\gamma_j\,\sin\Gamma_j$$
$$\omega_1 \cos(\theta' - \psi_1) = \Sigma_j\,\sigma_{0j}\,\gamma_j\,\cos\Gamma_j.$$

The inclination Ω and node Ψ of the true equator referred to the orbit of Jupiter are then:

$$\Omega\,\sin\Psi = \Sigma_j\,\sigma_{0j}\,\gamma_j\,\sin\theta_j + \omega\,\sin\theta$$
$$\Omega\,\cos\Psi = \Sigma_j\,\sigma_{0j}\,\gamma_j\,\cos\theta_j + \omega\,\cos\theta,$$

where we have:

$$\sigma_{01} = -0.00097(1+\lambda_1) \qquad \sigma_{01}\gamma_1 = -0^\circ.00003$$
$$\sigma_{02} = -0.00094(1+\lambda_2) \qquad \sigma_{02}\gamma_2 = -0.00044$$
$$\sigma_{03} = -0.00441(1+\lambda_3) \qquad \sigma_{03}\gamma_3 = -0.00081$$
$$\sigma_{04} = -0.00363(1+\lambda_4) \qquad \sigma_{04}\gamma_4 = -0.00092$$

Before giving the expressions for the perturbations I will first state the values of the arguments. For brevity I put

$$\tau = l_2 - l_3 \qquad v = l_2 - 2l_3 \qquad \varphi_i = v + \tilde{\omega}_i$$

$L =$ the mean longitude of Jupiter
$M =$ „ „ anomaly „ „
$W = 5l^{V} - 2l^{IV} - 16^\circ 31'$
$W_1 = l^{IV} - 2l^{V} - 1\ 30$ (in LEVERRIER's notation.)
$V = 2L - 2\theta' + 180^\circ$
$V' = 2L - \theta.$

The values of the arguments then are, if t is the time counted in days from 1900 Jan. 0, Mean Greenwich Noon (J. D. 2415020):

$$l_1 = 142^\circ.604 + 203^\circ.4889\,9261\,t$$
$$l_2 = 99.534 + 101.3747\,6145\,t$$
$$l_3 = 167.999 + 50.3176\,4587\,t$$
$$l_4 = 234.372 + 21.5711\,0965\,t$$

$$\tau = 291^\circ.535 + 51^\circ.0571166\,t \qquad v = 123^\circ.5 + 0^\circ.73947\,t$$
$$\psi = 252^\circ.4 + 0^\circ.14081\,t$$

$$\tilde{\omega}_1 = 155^\circ.5 + 0^\circ.14703\,t \qquad \varphi_1 = 279^\circ.0 + 0^\circ.88650\,t$$
$$\tilde{\omega}_2 = 62.7 + 0.03896\,t \qquad \varphi_2 = 186.2 + 0.77843\,t$$
$$\tilde{\omega}_3 = 338.3 + 0.00703\,t \qquad \varphi_3 = 101.8 + 0.74650\,t$$
$$\tilde{\omega}_4 = 283.15 + 0.001896\,t \qquad \varphi_4 = 46.7 + 0.74137\,t$$

$$\Gamma_1 = 75^\circ.6 + 0^\circ.136\,18\,t \qquad \theta_1 = 60^\circ.2 - 0^\circ.13614\,t$$
$$\Gamma_2 = 202.64 + 0.032\,373\,t \qquad \theta_2 = 293.16 - 0.032335\,t$$
$$\Gamma_3 = 176.09 + 0.006\,892\,t \qquad \theta_3 = 319.71 - 0.006854\,t$$
$$\Gamma_4 = 123.84 + 0.001\,810\,t \qquad \theta_4 = 11.96 - 0.001772\,t$$
$$\theta = 315^\circ.800 + 0^\circ.000\,0381\,t$$
$$\theta' = 135.779 + 0.000\,0381\,t$$

$$L = 238^\circ.0 + 0^\circ.08313\,t \qquad M = 225^\circ.3 + 0.08308\,t$$
$$W = 117.9 + 0.00112\,t \qquad W_1 = 64.2 + 0.01617\,t$$
$$V = 24.5 + 0.16608\,t \qquad V' = 160.3 + 0.16612\,t$$

The periodic perturbations in the latitudes are of the form:

$$\delta p_i = \varkappa \sin \alpha \qquad \delta s_i = \varkappa \sin (v_i - \alpha - \theta')$$
$$\delta q_i = \varkappa \cos \alpha$$

All coefficients being very small, we may in the arguments replace v_i by l_i, and neglect the difference of θ' and $180° + \theta$. The coefficients and arguments are:

	coefficient	*argument* $\delta p_i, \delta q_i$	*argument* δs_i
Sat. I	$+ 0^{\circ}.00042$	$\Gamma_2 - 2v - 2\theta'$	$l_1 + 2v + \theta_2$
	$+ 0.00025$	V	$l_1 - V'$
Sat. II	$- 0.00099$	$\Gamma_2 - 2v - 2\theta'$	$l_2 + 2v + \theta_2$
	$+ 0.00010$	$V + \Gamma_2$	$l_2 + \theta_2 - 2L$
	$+ 0.00078$	V	$l_2 - V'$
Sat. III	$+ 0.00010$	$V + \Gamma_3$	$l_3 + \theta_3 - 2L$
	$+ 0.00177$	V	$l_3 - V'$
Sat. IV	$+ 0.00032$	$V + \Gamma_4$	$l_4 + \theta_4 - 2L$
	$+ 0.00380$	V	$l_4 - V'$

The expressions for the longitudes and radii-vectores are given below. The inequalities are arranged in three groups, according to the periods, as explained in the beginning of this paper. Inequalities which are smaller than 1″ in longitude and 0.000005 in radius-vector have been neglected. The developments in powers of the small quantities ϱ and λ_i of the great inequalities (arguments 4τ, 2τ and τ for the satellites I, II, and III respectively), of the inequalities of group II and of the libration have already been given above, and only the values of the coefficients are repeated here. The more important of the smaller inequalities are here also given as functions of ϱ and λ_i. Where no development is given the coefficients were taken from SOUILLART's theory, corrected for the adopted values of the excentricities (and inclinations) but not for the masses. The multipliers of ϱ and λ_i are given in units of the last decimal place of the coefficients to which they belong.

The true orbit-longitudes are:

$$
\begin{aligned}
v_1 &= l_1 + 0^{\circ}.0276 \sin \psi + \delta v_1 \\
v_2 &= l_2 - 0.0411 \sin \psi + \delta v_2 \\
v_3 &= l_3 + 0.0036 \sin \psi + \delta v_3 \\
v_4 &= l_4 + \delta v_4
\end{aligned}
$$

The radii-vectores are:

$$r_i = a_i \varrho_i$$

$$
\begin{aligned}
\varrho_1 &= 1.000\,0066 + \delta\varrho_1 \\
\varrho_2 &= 1.001\,0549 + .000\,014\,\lambda_1 + .000\,084\,\lambda_3 + \delta\varrho_2 \\
\varrho_3 &= 1.000\,0155 + .000\,009\,\lambda_1 + .000\,011\,\lambda_2 - .000\,002\,\lambda_4 + \delta\varrho_3 \\
\varrho_4 &= 1.000\,0755 + .000\,008\,\lambda_1 + .000\,008\,\lambda_2 + .000\,034\,\lambda_3 + \delta\varrho_4
\end{aligned}
$$

The inequalities δv_i and $\delta \varrho_i$ are:

Ia. Equations of the centre.

$$\delta v_i = a_{\,1} \sin (l_i - \tilde{\omega}_1) + a_{i2} \sin (l_i - \tilde{\omega}_2) + a_{i3} \sin (l_i - \tilde{\omega}_3) + a_{i4} \sin (l_i - \tilde{\omega}_4).$$

$a_{11} = + 0^\circ.0062$	$a_{12} + - 0^\circ.0011$	$a_{13} = + 0^\circ.0030$	$a_{14} = + 0^\circ.0014$
$a_{21} = - 0.0002$	$a_{22} = + 0.0344$	$a_{23} = + 0\ 0281$	$a_{24} = + 0.0118$
$a_{31} = \quad 0\ 0000$	$a_{32} = - 0\ 0015$	$a_{33} = + 0\ 1736$	$a_{34} = + 0.0706$
$a_{41} = \quad 0.0000$	$a_{42} = \quad 0.0000$	$a_{43} = - 0.0204$	$a_{44} = + 0.8528$

$$\delta \varrho_i = a'_{i1} \cos (l_i - \tilde{\omega}_1) + a'_{i2} \cos (l_i - \tilde{\omega}_2) + a'_{i3} \cos (l_i - \tilde{\omega}_3) + a'_{i4} \cos (l_i - \tilde{\omega}_4)$$

$a'_{11} = -.000054$	$a'_{12} = +.000010$	$a'_{13} = -\ 000026$	$a'_{14} = -.000012$
$a'_{21} = +.000002$	$a'_{22} = -.000300$	$a'_{23} = -.000245$	$a'_{24} = -.000103$
$a'_{31} = \quad .000000$	$a'_{32} = +.000013$	$a'_{33} = -.001516$	$a'_{34} = -.000616$
$a'_{41} = \quad .000000$	$a'_{42} = \quad .000000$	$a'_{43} = +.000178$	$a'_{44} = -.007445$

The inequalities of the groups I*b* and I*c* are of the form:

$$\delta v_i = \varkappa \sin \alpha \qquad \delta \varrho_i = \varkappa' \cos \alpha.$$

They are:

Argument	*coefficient in* δv_i	*coefficient in* $\delta \varrho_i$	
Satellite I.			
2τ	$+0^\circ.0034\,(1+\lambda_2)$	$-.000\,017\,(1+\lambda_2)$	
3τ	$+0.0016\,(1+\lambda_3)$	$-.000\,011\,(1+\lambda_3)$	
4τ	$+0.4303$	$-.003\,755$	(x_1, *see above*)
8τ	$+0.0014+23\lambda_2$	$-.000\,012-20\lambda_2$	
Satellite II.			
τ	$-0^\circ.0123\,(1+\lambda_3)$	$+.000\,061\,(1+\lambda_3)$	
2τ	$+0.9875$	$-.008\,617$	(x_2, *see above*)
3τ	$+0.0052\,(1+\lambda_3)$	$-.000\,058\,(1+\lambda_3)$	
4τ	$+0.0051+1\lambda_1-1\lambda_2+109\lambda_3$	$-.000\,034+8\lambda_1+1\lambda_2-83\lambda_3$	
5τ	$+0\ 0004\,(1+\lambda_3)$	$-.000\,006\,(1+\lambda_3)$	
6τ	$+0.0005+3\lambda_1+2\lambda_3$	$-.000\,008-5\lambda_1-2\lambda_3$	
l_2-l_4	$-0.0006\,(1+\lambda_4)$	$+.000\,004\,(1+\lambda_4)$	
$2(l_2-l_4)$	$+0.0005\,(1+\lambda_4)$	$-.000\,006\,(1+\lambda_4)$	
$\tau+\varphi_3$	-0.0005	$+.000\,002$	
$2\tau+\varphi_2$	-0.0003	$+.000\,006$	
$2\tau+\varphi_3$	$+0.0026$	$-.000\,021$	
$2\tau+\varphi_4$	$+0.0010$	$-.000\,008$	
$l_1-\tilde{\omega}_2$	-0.0004	$+.000\,003$	
$l_1-\tilde{\omega}_3$	-0.0004	$+.000\,002$	

Argument	coefficient in δv_i	coefficient in $\delta \varrho_i$	
Satellite III.			
τ	$-0\overset{\circ}{.}0636$	$+.000\,555$	(x_3, *see above*)
2τ	$-0.0011\,(1+\lambda_2)$	$+.000\,015\,(1+\lambda_2)$	
3τ	$-0.0008-6\lambda_1-2\lambda_2$	$-.000\,006-11\lambda_1+4\lambda_2$	
l_3-l_4	$-0.0041\,(1+\lambda_4)$	$+.000\,022\,(1+\lambda_4)$	
$2(l_3-l_4)$	$+0.0138\,(1+\lambda_4)$	$-.000\,132\,(1+\lambda_4)$	
$3(l_3-l_4)$	$+0.0010\,(1+\lambda_4)$	$-.000\,012\,(1+\lambda_4)$	
$\tau+\varphi_3$	-0.0008	$+.000\,007$	
$\tau+\varphi_4$	-0.0003	$+.000\,003$	
$l_3-2l_4+\tilde{\omega}_3$	$+0.0004$	$-.000\,000$	
$l_3-2l_4+\tilde{\omega}_4$	-0.0004	$+.000\,001$	
$2l_3-3l_4+\tilde{\omega}_4$	-0.0004	$+.000\,003$	
$l_3-2L+\tilde{\omega}_3$	$+0.0006$	$-.000\,005$	
Satellite IV.			
l_4-l_1	$-0\overset{\circ}{.}0003\,(1+\lambda_1)$	$+.000\,005\,(1+\lambda_1)$	
l_4-l_2	$-0.0005\,(1+\lambda_2)$	$+.000\,008\,(1+\lambda_2)$	
l_3-l_4	$-0.0023\,(1+\lambda_3)$	$+.000\,101\,(1+\lambda_3)$	
$2(l_3-l_4)$	$-0.0012\,(1+\lambda_3)$	$+.000\,018\,(1+\lambda_3)$	
$2l_4-2L$	$+0.0012$	$-.000\,015$	
$l_3-2l_4+\tilde{\omega}_3$	-0.0006	$+.000\,002$	
$l_3-2l_4+\tilde{\omega}_4$	$+0.0007$	$-.000\,006$	
$l_4-2L+\tilde{\omega}_4$	$+0.0064$	$-.000\,056$	
$2l_4-2\tilde{\omega}_4$	$+0.0040$	$-.000\,028$	

Inequalities of group II. (The expressions as functions of ϱ and λ_i have already been given above).

Argument	Coefficients in δv_i Sat. I	Sat. II	Sat. III
φ_1	$-0\overset{\circ}{.}0077$	$+0\overset{\circ}{.}0070$	$-0\overset{\circ}{.}0000$
φ_2	$+0.0169$	$+0.0377$	-0.0115
φ_3	$+0.0072$	-0.0464	$+0.0095$
φ_4	$+0.0026$	-0.0157	$+0.0033$

In the radii-vectores these inequalities can be neglected, with the exception of the following term in Satellite II:

$$\delta\varrho_2 = +\,.000\,006\ \cos\varphi_3.$$

Inequalities of group III. These also are negligible in the radii-vectores. The largest of them is:

$$\delta\varrho_3 = +\,.000\,001\ \cos(\tilde{\omega}_3-\tilde{\omega}_4).$$

In the longitudes we have:

Argument	*Coefficients in* δv_i			
	Sat I	*Sat. II*	*Sat. III*	*Sat IV*
M	$+0\overset{\circ}{.}0006$	$-0\overset{\circ}{.}0102$	$-0\overset{\circ}{.}0135$	$-0\overset{\circ}{.}0320$
W		-0.0008	-0.0012	-0.0029
W_1		-0.0001	-0.0001	-0.0003
Γ_2	$+0.0099$	$+0.0028$	$+0.0001$	
Γ_3		$+0.0016$	$+0.0024$	-0.0018
Γ_4	-0.0001	$+0.0019$	$+0.0029$	$+0.0010$
$\Gamma_2-\Gamma_3$	-0.0027	-0.0011	-0.0005	
$\Gamma_2-\Gamma_4$	-0.0005	-0.0002	-0.0001	
$\Gamma_3-\Gamma_4$	$+0.0011$	-0.0005	-0.0013	-0.0010
$\tilde{\omega}_3-\tilde{\omega}_4$	-0.0005	$+0.0002$	$+0.0009$	-0.0011

It should be kept in mind that all the above developments are based on SOUILLART's analytical theory. New values of the masses and elements were introduced into his formulas, and a few numerical mistakes were corrected, but the analytical formulas were not altered. The only exception is the expression for the period of the libration, which was computed to terms of the third order in the masses inclusive, while SOUILLART rested content with those of the second order (See Gron. Publ. 17, art. 18).

Physics. — *"An auto-collimating spectral apparatus of great luminous intensity"*, by Prof. H. E. J. G. DU BOIS, G. J. ELIAS and F. LÖWE. (Communication from the Bosscha-Laboratory).

In optical work an illuminator is often wanted, which combines great brightness with monochromatic purity of the light, if possible of the order 0,1 $\mu\mu$. In spite of the almost boundless variety of available spectral apparatus [1]), such an appliance is wanting. For this purpose WÜLFING [2]) it is true, constructed a monochromator and investigated the luminosities obtainable by means of different sources of light, but its aperture was only $^1/_8$, the dispersion also rather slight. An auto-collimator lately described by FABRY and JOBIN [3]) has an aperture of only $^1/_{10}$. Recently one of us has described three spectral apparatus with constant deviation (parallel or at right angle) [4]);

[1]) H. KAYSER, Handb. d. Spectroscopie **1**, p.p. 489 et seq gives a survey leading up to 1900.

[2]) E. A. WÜLFING, N. Jahrb. f. Mineralogie Beil. **12**, p. 343, 1898. — C. LEISS, Zeitschr. für Instr. kunde **13** p. 209, 1898. S. NAKAMURA, Ann. d. Phys. (4) **20** p. 811, 1906.

[3]) CH. FABRY & A. JOBIN, Journ. de Phys. (4) **3** p. 202, 1904.

[4]) F. LÖWE, Zeitschr. f. Instr. Kunde **26**, p. 330, 1906 and **27**, p. 271, 1907.

here however, it was not necessary either to pay attention to great brightness.

The combination of the last mentioned condition with a considerable value of dispersion naturally leads to the application of the principle of auto-collimation, preferably with 2 "halfprisms". In using the instrument as a secondary spectral source of light immobility of the entrance slit cannot be dispensed with, and for the exit slit it is also required — perhaps with the exception of small sources of light, easily moved. We are, however, aware of the drawbacks which attend this principle, viz. a higher degree of false reflexes, and the so-called "vignettation" of the beam of light due to the necessity of placing the slits at different heights.

The utmost brightness is specially required for experiments on polarisation, in which a nearly crossed position of the nicols allows only a very small fraction of the light to pass. It follows that in such cases polarisation by the apparatus itself does not give rise to difficulties; it may even prove advantageous that refraction should in each case take place at the angle of polarisation; for then there is absolutely no loss by reflection of light polarized parallel to the refracting edges of the prisms. So in its usual position the apparatus would allow light to pass which shows a strong partial polarisation along the vertical.

From BREWSTER's law it follows that the angle of refraction of the whole prisms must then amount to ($180^\circ — 2$ arc tg. n); we shall by preference choose prisms of 60° (resp. 30°), corresponding to $n = \sqrt{3} = 1,732$. For this case, with 2 whole and 2 half prims, the simple scheme of fig. 1 is naturally evolved, where evidently all the angles of incidence amount to 60°.

Now the glass must meet the following principal requirements: 1) index of refraction for a mean colour about 1,73; 2) no strong absorption of violet light; 3) homogeneity and absence of bubbles; 4) resistance against atmospheric influence; 5) sufficient dimensions of the rough blocks. In spite of the present ample choice it proved impossible as yet to satisfy all these 5 conditions to a sufficient degree In the instrument constructed in the spring of 1907 by C. ZEISS we therefore used heavy flint N°. 1771 of the firm of SCHOTT & Co. at Jena, for which $n_D = 1,794$; according to what precedes an angle of refraction of somewhat more than 58° (or 29°) corresponds to this. The value of dn between $C—F$ amounts to 0,0309; from this follows a dispersion for every whole prism of $4^\circ 4'$; hence for the whole course of the rays $2 \times (^1/_2 + 1 + 1 + ^1/_2) \times 4^\circ 4' = 25^\circ$ nearly.

In order to keep the system in the minimum of deviation, every

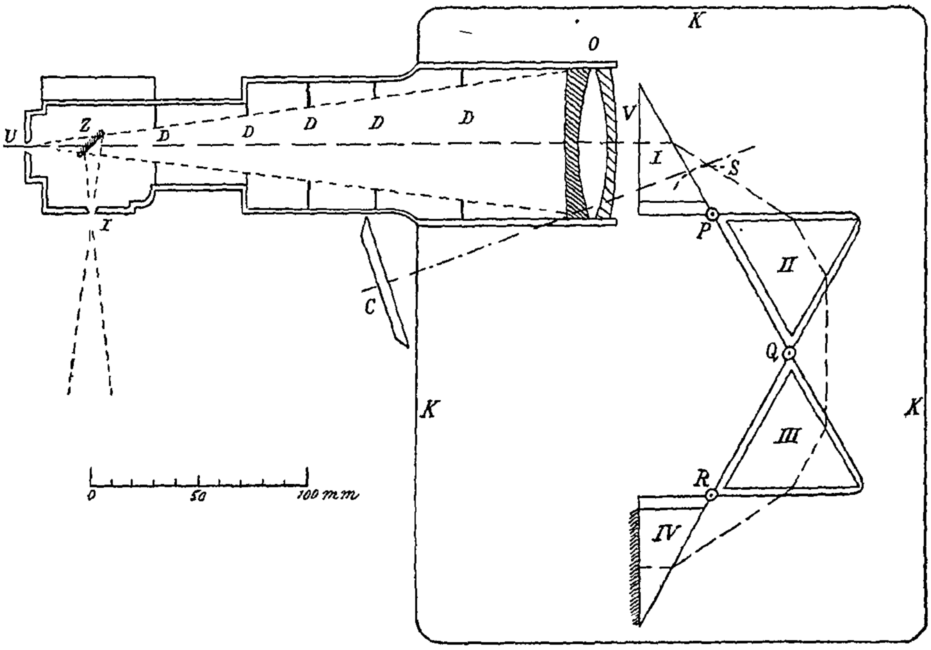

Fig. 1. — $^3/_{10}$ natural size.

prism must evidently be subjected to an equal rotation with respect to the preceding one, round the points R, Q and P. Prism I remains rigidly connected with the collimator tube; now let every point of II describe an arc α of a circle round P; then the points of III describe cycloids, those of IV higher cycloids, in which the total rotation of III and IV with respect to the ground plate amounts to 2α and 3α respectively, apart from their simultaneous translation.

In a similar case one of us *(F.L.)* successfully constructed a toothed-wheel mechanism for a quartz-monochromator years ago, which was now also chosen. The old arrangement for cutting the rather intricate forms of the teeth was now again adopted. An analogous mechanism was, moreover, lately described by HAMY [1]) and executed by JOBIN.

For the sake of simplicity prism II is primarily rotated by means of a worm-wheel arc S roughly represented in the diagram, which could be effected with a bamboo rod from the observer's place. The reading takes place on the circle C; the prism tables are provided with german silver feet, which slide on a glass plate; prism IV is

[1]) M. HAMY, Journ. de phys. (4) 7, p. 52, 1908; Zeitschr. f. Instr. Kunde **28**, p. 122 1908.

silvered on the back side. [1]) The angle of I amounts to 30°40′; the plane of entrance V — 52 × 52 mm. large — forms an angle of about 40′ with the wave-front, which causes the disturbing reflex to be thrown aside. The object-glass, consisting of three lenses, has a diameter of 67 mm., a focal distance of 260 mm., so that the aperture amounts to $^1/_4$. The lateral spherical aberration is according to calculation of the order 0,01 mm. A small part of the convex front surface has been blackened to prevent reflection; square diaphragms *D* have been placed in the collimator tube for the same purpose. The whole system of prisms is placed under a closed metal cover *K*; inside this the necessary chemical substances are supplied in order to protect the sensitive glass surfaces against the action of water-vapour, carbonic acid, hydrogen sulphide etc. Whether these measures will prove effectual remains, as a matter of course, to be seen after a considerable lapse of time.

At the end of the collimator tube the "slit holder" is arranged so as to rotate round its axis. The bilateral entrance slit I, which is provided with a prism for comparison, is 3.5 mm. long, it is slightly curved (radius of curvature 70 mm.), and can rotate a little so that the slope and curvature of the spectral lines is compensated for a mean colour; the slit is focussed by means of a spiral groove. A mirror silvered at the front side directs the rays towards the lens, which on their way back pass along its upper or lower side so as to reach the exit-slit *U*; the latter is also bilateral, 3,5 mm. long, but rectilinear. It may be exchanged for monocentric non-reflecting eye-pieces with a focal distance of 9 or 25 mm., or for a normal camera 60 × 90 mm., by means of which only a small spectral region can of course be photographed at the same time.

The whole apparatus is constructed without any iron, and mounted very compactly on a marble slab. The adjusting screws form a right-angled triangle, one of the catheti lying under the optical axis, whose height above the plane of the table is 125 mm.

For measurements in the ultraviolet the object-glass is replaced by a quartz-fluorite achromatic lens ($\varphi = 33$ mm., $f = 260$ mm., apert. $^1/_8$); it would of course be too expensive to fill the whole aperture; a couple of quartz-half prisms according to Cornu is also provided.

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[1]) In many respects it may be preferable to fix a metal mirror with glycerin to the back plane, for it is easy to remove it, and also to adapt the apparatus for transmitted light; in this case a telescope or a spectrograph with camera has to be added; the same mirror may be used, if necessary, to give the desired direction to the light. Besides, some alloys reflect considerably better than silver in the ultraviolet (about 320 $\mu\mu$).

Some time ago one of us [1]) applied to a spectrograph a peculiar graduation according to wave-lengths from 5 $\mu\mu$ to 5 $\mu\mu$, which proved very convenient. For instruments of great dispersive power, however, this principle hardly works well; more accurate results are obtained with a calibration curve, though this takes more time. For this calibration the lines of the gas spectra of hydrogen, helium, and those of a mercury arc lamp may be used; also those of the spark spectrum of copper and of the flame spectrum of potassium; in this way a sufficiently uniform distribution of lines is obtained between 410 and 770 $\mu\mu$. The accuracy of the readings is of the order 0.05 $\mu\mu$.

Though from the outset we had been intent upon preventing end play in the mechanism of motion, it proved as yet impossible to avoid this altogether, so that it was necessary for the readings to have the motion take place always in the same sense. We hope, however, to remedy this defect by further improvements.

With the apparatus used as a spectrometer a very satisfactory resolution of neighbouring spectral lines could be brought about, the theoretical dissolving power of the set of prisms in the usual sense amounting to 65000. Thus with the strong eye-piece the yellow helium line is seen resolved into its two components, whose distance apart amounts to about 0,035 $\mu\mu$.

FABRY and JOBIN (loc. cit. p. 208) give a comparative table of the breadth occupied in the spectrum by a wave-length interval of 1 $\mu\mu$ in the violet at about 434 $\mu\mu$; in the red the dispersion is of course much less:

APPARATUS	DISPERSION
BRUCE (Yerkes-Observatory)	1,4 mm per $\mu\mu$
MILLS (Lick-Observatory)	0,8 " " "
FABRY and JOBIN	2,0 " " "
DU BOIS, ELIAS and LÖWE	1,96 " " "
ROWLAND-grating of Berlin University, 1st order (5684 lines per cm; radius 390 cm),	2,18 " " "

FABRY and JOBIN's fourfold focal distance is therefore all but compensated by our greater dispersion.

Though particular care was taken to prevent reflexes, yet it proved

[1]) F. LÖWE, Zeitschr. f. Instr. Kunde **26**, p. 332, 1906.

impossible entirely to exclude diffuse light—probably due to the diffusion on the faces of and inside the prisms, so that we shall always have to take account of its presence, even though it be only to a very slight degree. In fact, we have not investigated any apparatus or prism, in which the disturbing influence of this phenomenon was not more or less felt. The question whether a certain diffusion still occurs with a really macro-homogeneous, optically "empty" refracting medium, is difficult to solve, and must for the present be considered a pending problem.[1])

The "vignettation" amounts on an average to 25 $^0/_0$, as may be observed by accommodating on the square objective diaphragm. When the apparatus was used as a monochromator the intensity of the light came up to what we expected; with sunlight it is still from 5 to 10 times higher (according to meteorologic circumstances) than with an arc-lamp crater projected on the entrance slit. Accordingly with monochromatic light of great purity even polarisation apparatus of very slight transmitivity may be used. When thus applying the instrument to illuminative purposes the entire path of the beam from the source of light on to the retina, and especially its divergence, ought to be carefully adapted to that part which lies within the apparatus, if all the possible benefit is really to be derived from it.

Physics. — "*The influence of temperature and magnetisation on selective absorption spectra*", II. By Prof. H. E. J. G. DU BOIS and G. J. ELIAS. (Communication from the Bosscha Laboratory).

§ 12. Since our former communication (These Proc. Febr. p. 578) the cryomagnetic arrangement was further improved in some respects in order to obtain a stronger field, and to diminish the inconvenient formation of rime. The truncated end-planes of the conic polar pieces had a diameter of 6 mm., the split cores [2]) a diameter of 3,5 mm.; the width of the slit at the end was from 0.4 to 0.6 mm., the slit being wedge-shaped so as to fit the convergence and divergence of the beam of rays between two lenses; it was arranged in such a way that the whole surface of the grating was illuminated, so that the theoretical dissolving power, — amounting to about 100.000 — had its full effect. Subsequent in the direction of the rays was a doubled quarter-wave plate with horizontal demarca-

1) C. A. LOBRY DE BRUYN and L K. WOLFF, Rec. d. Trav. Chim. **23**, p. 155, 1904; L. MANDELSTAM, Physik. Zeitschr. 8, p. 608, 1907; M. PLANCK, ibid. 8, p. 906, 1907.

2) H. DU BOIS, Zeitschr. fur Instr. Kunde **19** p. 360, 1899.

tion adjusted at the laboratory according to CORNU and W. KÖNIG [1]). On account of the considerable astigmatic difference in the images of horizontal and vertical lines formed by a concave grating, the plate was placed near the focus of a third lens in order to enable us to cancel this astigmatism for different parts of the spectrum by comparatively small displacements. The line of demarcation could then be adjusted sufficiently sharply in the spectrum, which KÖNIG had not succeeded in doing. A nicol followed the mica plate, and then came the principal slit. With this arrangement a normal doublet is known to appear in the spectrum as a broken line e. g. thus ; and on rotation through 90° of the nicol round the direction of the rays or of the $^1/_4$ plate round its *vertical* diameter at once appears.

§ 13. As a rule the samples were mounted in a copper frame-piece and clasped between the polar end-planes; it is desirable to have an airtight fitting so as to prevent cold currents of air with formation of rime. The level of the liquid air may now rise above the openings so that the sample is quite immersed. The air stagnating in the bores is effectually dried by the preliminary cooling with solid carbon dioxide. With thin samples we obtain in this way a field of 40 kilogauss, which is quite essential for the proper resolution of the quadruplets etc to be described later. With sunlight and a width of 0.05 mm. of the principal slit there was still plenty of light even in the violet; the FRAUNHOFER lines, however, proved so troublesome in many cases that the much weaker arc light had to be used. The spectrum was measured by means of a magnifying glass and a graduated glass scale, the divisions of which amounted to 0.225 mm., exactly corresponding to 0,1 $\mu\mu$ in the spectrum of the first order. The auto-collimator, which we also used has been described since our first communication (see the preceding paper).

All the following experiments were made with a longitudinal field, in other words with an axial direction of the rays; many new adjustments would be required after turning round the heavy electro-magnet, so that we hope to extend the observations to an equatorial direction of rays later on.

§ 14. Third series. Of the large number of coloured com-

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1) A. CORNU, Compt. Rend. **125** p. 555, 1897. — W. KÖNIG, Wied. Ann. **62** p. 242, 1897. We found it safer not to place this arrangement at the end of the beam near the magnifying glass, on account of polarisation by the grating; cf. P. ZEEMAN, These Proc, Oct. 1907.

pounds of trivalent titanium and vanadium we investigated some without, however, having found anything noteworthy as yet. The selective properties in this series culminate for chromium; we shall therefore restrict ourselves to a closer investigation of some chromic compounds already discussed in our former paper.

Chromium alum.

From the well-known regular crystals plates of a thickness of about 2 and 3 m.m. were cut. At 18° a rather intense band 669,8—671,6 is seen in the red; at —193° it becomes considerably narrower, viz.: 668,6 —669,4, the centre shifting 1.7 $\mu\mu$ towards the violet; moreover another rather strong line 670,2 appears; between 619 and 716 no less than 21 fainter and sharper bands and lines are actually visible.

In a field of 34 kilogauss the two principal lines appeared broken; the horizontal distance of the corresponding edges of their upper and lower halves, henceforth briefly called the *break*, amounted to about 0,10 $\mu\mu$; the sense was *opposite* [1]). Band 668.6– 669.4 shows one fine narrow satellite on the red side, towards the violet two of them; the former disappeared in the field; the two latter ones became very vague, and seemed, as seen with sunlight, to join in the break of the principal band.

Ruby.

§ 15. With the square plate (7 × 7 × 3 m.m.) mentioned in our preceding paper a long edge contained the optical axis. From the same ruby cone a small quadratic prism (1,5 × 1,5 × 4 m.m.) was now ground, the axis being parallel to a short edge. With the slight thickness of 1,5 m.m. sufficient absorption is shown even with grating dispersion. We must now distinguish the cases that the optical axis is || or ⊥ with respect to the direction of the field.

I. Optical axis || direction of field:

A. Pair of bands in the blue at —193°. Besides the two bands in the red already described, a pair in the blue are rather striking among the other 8; we shall briefly call these B_1 and B_2. At —193° their situation is: $B_2 = 474{,}2—474{,}9$, and $B_1 = 476{,}1—476{,}5$ (at 18° they lie 474,9—475,7 and 476,5—477,1, more towards the red). The distance of the central lines measured in the grating spectrum, amounted to 1,63 $\mu\mu$. In a field of 36 kgs (= kilogauss) the break for B_1 amounted to 0,04 $\mu\mu$ and for B_2 to 0.055 $\mu\mu$, the sense being

[1]) I. e. with respect to that which has been found up to now for all vapours; such an opposite sense was also observed by J. BECQUEREL in most cases.

opposite; an asymmetry in the break of the bands towards both sides — with respect to their position with field off — appeared to exist, but could not be measured with sufficient certainty. At a temperature considerably exceeding that of liquid air, the blue bands are no longer to be determined in the grating spectrum.

§ 16. *B. Pair of bands in the red*; we call them R_1 and R_2.

1) At $-193°$ we have $R_2 = 691{,}7$ and $R_1 = 693{,}1$, the distance measured in the grating spectrum being 1,38 $\mu\mu$.

Line R_1: Width with field off 0,065 $\mu\mu$. With 23 kgs. a *triplet* begins to appear, which is not yet clearly visible with 18 kilogauss; lefthand line (red side) not sharply divided from middle line, forming together a strong line, 0,10 $\mu\mu$ wide; righthand line (violet side) divided from middle line at a distance of 0,09 $\mu\mu$. With 26,5 kgs. the triplet further resolves, the distance on either side becoming 0,11 $\mu\mu$.

With 36 kgs. the lefthand line is strong, the middle line perhaps stronger still, not sharply divided, distance 0,165 $\mu\mu$; the righthand line faint, at a distance of 0,14 $\mu\mu$ from the middle line.

Line R_2: Width with field off 0,055 $\mu\mu$. With 23 kgs. triplet: lefthand line not separated from middle line, forming together broad line 0,075 $\mu\mu$ wide; righthand line separated from middle line at a distance of 0,07 $\mu\mu$. With 26 kgs. the triplet further resolves; distance 0,08 and 0,09 $\mu\mu$ respectively.

With 36 kgs. the lefthand line is rather strong, not quite detached from the middle line, at a distance of 0,115 $\mu\mu$; the righthand line faint, more clearly separated from the middle line, at a distance of 0,15 $\mu\mu$.

In all these cases the lateral components were circularly polarised in the opposite sense; as the middle line vanished at neither of the two positions of the $\lambda/_4$ plate, it could not be circularly polarized; linear polarisation was not observed and quite excluded on account of axial field-symmetry. It is not yet the moment here to enter into an explanation of this highly remarkable phenomenon; it may perhaps simply be due to imperfect resolution of the inner lines of a quadruplet [1]). A magnetic displacement of the middle line with respect to its position with field off [2]) could not be ascertained; at all events it never amounted to more than 1 or 2 hundredths of $\mu\mu$.

There is no reason in this case to doubt of the proportionality of the resolution with the intensity of the field.

1) Cf. P. Zeeman, These Proc. Febr. 1908.

2) Cf. H. Kayser, Handb. d. Spectroscopie, 2 p. 655, Fig. 52. Something similar was also sometimes observed for the sextuplet of D_2.

2) At $-79°$ the bands were already considerably widened and faded so that the thicker ruby plate had to be investigated through which the light proceeded 7 mm. in the direction of the axis.

Heating from $-193°$ to $-79°$ displaced R_1 by 0,62 $\mu\mu$, R_2 0,58 $\mu\mu$ towards the red so that their distance now became 1,42 $\mu\mu$. In a field of 18,5 kgs. R_1 exhibited a lefthand break of 0,12, $\mu\mu$, a righthand one of 0,065 $\mu\mu$, and R_2 deviated 0,04 on the left, 0,07 $\mu\mu$ on the right.

3) At $+18°$ and a field of 18,5 kgs. R_1 exhibited a break of 0,07 $\mu\mu$ towards both sides, R_2 one of 0,055 $\mu\mu$. Heating from $-193°$ to $+18°$ shifted R_1 0,76 $\mu\mu$, R_2 0,69 $\mu\mu$ towards the red, so that their distance now became 1,45 $\mu\mu$ [1]).

4) At $+200°$ the phenomenon was rather vague. By estimation the two lines showed a symmetrical break of 0,04 $\mu\mu$ with 18,5 kgs. Heating from 18° to 200° moved both R_1 and R_2 1,1 $\mu\mu$ towards the red, their distance therefore not being changed. As yet we have not heated the ruby any higher.

In general we may perhaps conclude from the rather intricate course of the phenomenon that the influence of magnetisation slightly decreases with increase of temperature. The distance between R_1 and R_2, on the other hand, seems to become a little larger.

§ 17. We now proceed to the second case:

II. Optical axis $\perp$ direction of field, where we must distinguish the ordinary and the extraordinary spectrum. In this case only the nicol, no longer the double $\lambda/4$ plate was used, because circular polarisation does not come in here.

1. *Ordinary spectrum*; plane of polarisation horizontal:

A. Pair of bands in the blue at $-193°$. The width with field off amounted to 0,17 for B_1, to 0,14 $\mu\mu$ for B_2, the distance of the central lines being 1,68 $\mu\mu$; the lines looked about equal: In a field of 36 kgs. the width increased to 0,26 $\mu\mu$ for the two lines; half the increase in width amounted therefore for B_1 to 0,045, for B_2 to 0,06 $\mu\mu$.

B. Pair of bands in the red at $-193°$. We have (cf. § 7) $R_2 = 691{,}8$ and $R_1 = 693{,}2$. The width with field off amounted to 0,08 for R_1, to 0,07 $\mu\mu$ for R_2, their distance in the grating spectrum being 1,41 $\mu\mu$.

With a field of 20 kgs. R_1 became widened, and seemed shaded

[1]) We gave up the idea of reproducing a photograph, because the reproduction in our former paper is greatly inferior in distinctness to our own prints. Moreover, where measurement proves possible, reproduction appears almost superfluous.

in the middle, R_2 showed a doublet at a distance of 0,3 $\mu\mu$; so the aspect was about the same as that preliminarily sketched in § 7 [1].

With a field of 23 kgs. R_1 showed a *quadruplet*, the four components of which had about the same intensity, and the distances of which seemed slightly to decrease towards the violet. The intervals were now as bright as the spectral background; only between the pair lying on the violet side the interval seemed slightly darker; the distance of the outer lines was 0,28 $\mu\mu$, and the mutual distances differ little from a third of this. R_2 also gave a quadruplet, the two inner lines of which are much fainter than the outer ones, and symmetrically distributed (without careful focussing of the magnifying glass one may therefore feel inclined to see a doublet); the distance of the outer lines was 0,285 $\mu\mu$ respectively.

With 30 kgs. the phenomenon was exactly equal, with distances 0,39 and 0,38 $\mu\mu$.

With 36 kgs. R_1 exhibited a quadruplet as above, further resolved. From red to violet

the distances of the middle lines
amounted to 0,15 0,145 0,14, total 0,435 $\mu\mu$
the widths of the lines „ 0,07 0,025 0,035 0,065 $\mu\mu$

The distance between the extreme limits amounted to 0,50 $\mu\mu$; the middle between them appeared to be displaced 0,04 $\mu\mu$ towards the violet with respect to the position with field off.

For R_2 on the other hand we obtained values for

the distances of the middle lines 0,15 0,20 0,085, total 0,435 $\mu\mu$
the widths of the lines 0,055 very narrow 0,045 $\mu\mu$

The distance of the extreme limits amounted to 0,47 $\mu\mu$; the displacement of the middle with respect to the position with field off was less than for R_1 and could not be ascertained by measurement.

Let δR_1 and δR_2 stand for the total distances between the outer quadruplet lines and $\mathfrak{H}$ for the intensity of the field, then we have

$\mathfrak{H}$ (*Kilogauss*)	δR_1 ($\mu\mu$)	$\frac{1000\,\delta R_1}{\mathfrak{H}}$	δR_2 ($\mu\mu$)	$\frac{1000\,\delta R_1}{\mathfrak{H}}$
23	0,28	12,2	0,285	12,4
30	0,39	13,0	0 38	12,6
36	0,435	12,1	0,435	12,1

[1]) As was observed in § 7 the numerical determinations there bear a strictly preliminary character; through a misprint the estimation of the intensity of the field was 30 kilogauss: this should be 20 kgs. The data now given possess already greater reliability; they were each derived from 2 to 5 readings.

The sufficiently good agreement of the ratios proves the proportionality of the resolution with the intensity of the field, at least as a first approximation; it is rather improbable that weaker fields should exhibit any deviations from this proportionality.

§ 18. Almost analogously behave the lines in the

2. *Extraordinary spectrum*; plane of polarisation vertical.

A. Pair of bands in the blue at —193°. The width with field off amounted to 0,10 for B_1, to 0,15 $\mu\mu$ for B_2, the distance of the middle lines was 1,70 $\mu\mu$; the lines appear somewhat displaced compared with the ordinary spectrum, viz. B_1 0,025 $\mu\mu$ towards red, and B_2 0,007 towards violet; moreover B_2 was vaguer and paler than B_1.

In a field of 36 kgs. the widths became 0,18 and 0,22 $\mu\mu$; so for B_1 and B_2 respectively half the increase in width amounted to 0,04.

B. Pair of bands in the red at —193°. The width with field off amounted to 0,07 for R_1, to 0,06 $\mu\mu$ for R_2, their distance being 1,41 $\mu\mu$.

They both seem to have shifted 0,02 $\mu\mu$ towards the violet, compared with their position in the ordinary spectrum; R_1 is fainter.

With 36 kgs. R_1 exhibits a quadruplet of 4 lines about equally strong, at apparently equal distances, too indistinct, however, to be measured; distance of the extreme limits 0,49 $\mu\mu$; the middle appeared to have moved 0,02 $\mu\mu$ towards the violet with respect to the position with field off.

For R_2 the inner lines of the 4 were probably slightly stronger than the outer ones; the determinations were rather uncertain; the distance of the limits about 0,4 $\mu\mu$.

§ 19. Fifth series. Of this we now investigated a few sulphates of the material used in 1899, which crystallise monoclinically as octohydrates; they do so in plates containing both optical axes. As a matter of course no circular polarisation occurs; in this respect uniaxial and even more so cubic crystals, e. g. chromium alum, are to be preferred.

Neodymium sulphate [Nd_2 $(So_4)_3$. 8 H_2O]. Rosy-red plate 0,8 mm. thick at —193°. Two narrow bands in the yellow, and three in the green exhibited an increase in width of from 0,05 to 0,08 $\mu\mu$ in a field of 40 kgs; two of the last mentioned became brighter in the middle, and so began to look like doublets.

Samarium sulphate [Sm_2 $(So_4)_3$. 8 H_2O]. Light yellow semi-transparent plate of crystal, 2,8 mm thick at —193°. Two narrow bands in the yellow-green exhibited an increase in width in a field of 28 kgs. the amount of which ought to be determined with a sample of better transparency.

Physics. — "*Isotherms of monatomic substances and their binary mixtures. II. Isotherms of helium at — 253° C. and — 259° C.*", by Prof. H. KAMERLINGH ONNES. Communication N°. 102[c] from the Physical Laboratory at Leiden.

§ 1. *Survey of the determinations.* The measurements were made in the same way as those of Comm. N°. 102[a] (Dec. '07). The whole of the piezometer had a four times larger content, viz. about 2 liters, the piezometer reservoir on the other hand was more than four times smaller, it was, namely, somewhat more than 2 cm^3. Accordingly the densities to which the measurements refer, are considerably larger, and lie between 591 and 794 times the normal one. The temperatures at which the determinations were made, are measured on the hydrogen thermometer of Comm. N°. 95[c].

$$t = -252°.84 \text{ C. and } t = -258°.94 \text{ C.}$$

from which by extrapolation by means of table XXV of Comm. N°. 101[b] (Dec. '07) see § 3 of Comm. N°. 102[b] follows for the temperatures below 0° C. measured on the absolute scale

$$\theta = -252°.84 + 0°.12 = -252°.72$$

$$\text{and } \theta = -258\ .94 + 0°.12 = -258°.82$$

The determination of the mean temperature of the gas in the capillary stem of the piezometer reservoir, with regard to the part that extends above the bath in the cryostate, required here greater accuracy than before, because compared with the quantity of the gas in the smaller reservoir that in the stem was of more importance. With a view to the determination of this mean temperature a cylindric reservoir of the same height as the capillary was placed by the side of and on a level with the capillary, which reservoir was filled with helium, and provided with an appliance to read the pressure in it [1]). By means of this pressure it is easy to derive with the required accuracy what mean density for the gas in the capillary of the piezometer must be taken. At 0° the pressure in this auxiliary apparatus was 118.3 cm. of mercury. With the measurement at — 253° C. it varied between 33.1 and 51.1 cm., at — 259° C. between 31.8 and 48.1 cm.

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[1]) A similar contrivance has been applied by different observers in the determination of the mean temperature of the capillary of a gas thermometer (TRAVERS, SENTER and JACQUEROD, Ph. Tr. Royal Soc. London Ser. A. vol. 200 p. 143 (1902)).

§ 2. *Results for* pv_A.

The subjoined table contains the results of the determinations in the same way as table I of Comm. N°. 102^a.

TABLE I. Helium. Values of pv_A.				
N°.	θ	p	pv_A	d_A
1	— 252°.72	53.848	0.09120	591 53
2		60.716	0 09533	626 92
3		65.997	0 09867	668.87
4	— 258°.82	40 012	0.06150	650.65
5		46.222	0.06559	704 71
6		53.326	0.07063	754.97
7		59.797	0.07531	794.00

The corresponding values for $pv_{A,d=0}$ are:

for — 252°.72 $\qquad pv_A = 0.07455$

for — 258°.82 $\qquad pv_A = 0.05222$

§ 3. *Further results.* The number of points on every isotherm is too small, and the densities are too large to allow already now the derivation of the first individual virial coefficients of the polynomial of state (cf. § 4 of Comm. N°. 102^a). If, however, we give a graphical representation, it shows that the isotherm pv_A for — 259° must exhibit a minimum and hence B_A must be negative at this temperature. Further follows from the isotherm of — 253°, that the intersection of this line with the axis $d = 0$ lies near the BOYLE-point. Probably B_A is also already negative at — 253° though only slightly. All this agrees very well with what was derived in § 5 of Comm. N°. 102^a, and speaks for the validity of the extrapolation applied there with a view to the calculation of the critical temperature of the helium.

In conclusion I gladly express my thanks to Mr. C. BRAAK for his assistance in this investigation.

Physics. — "*On the measurement of very low temperatures.* XX. *Influence of the deviations from the law of* BOYLE-CHARLES *on the temperature measured on the scale of the gas-thermometer of constant volume according to observations with this apparatus.*" By Prof. H. KAMERLINGH ONNES and C. BRAAK. Comm. N°. 102^d from the Physical Laboratory at Leiden.

§ 1. In Comm. N°. 97^b (Jan. '07) under XV the formula of CHAPPUIS (see Comm. N°. 95^e (Oct. '06) form. (3)) for the calculation of the temperatures according to the hydrogen thermometer of constant volume was compared with formula (6) of XIV of the same Communication, in which formula attention has been paid to the deviations from the law of BOYLE, whereas they are neglected in CHAPPUIS' formula. As the result of this comparison we stated there that for a dead space of $^1/_{100}$ the mean relative coefficient of pressure between 0° and 100 is to be increased with 2 units of the 7th decimal, and the coefficient of pressure of the hydrogen thermometer at 1090 mm. zero point pressure was, therefore, to be put at 0,0036629 instead of at 0,0036627, a modification which is, however, so slight, that it just coincides with the limit of the errors of observation. We have just found out that for this calculation inaccurate values of $B^{(p)}_{0^\circ}$ and $B^{(p)}_{100}$ have been used. New calculations have revealed that the difference is much smaller than was stated just now, so that it is to be taken into account only for much higher values of the dead space and, with the exception of carbonic acid, has no influence even on CHAPPUIS' last decimal (the 8th). That the use of the incorrect $B^{(p)}$ was not detected, was due to the fact that the calculation of neglections indicated in XV had accidentally led to the same result, here, however, because the four corrections, as has been mentioned in XV, had been erroneously taken with the same sign, whereas they almost entirely cancel each other. We shall therefore in future keep to the unchanged coefficient of pressure 0.0036627.

A consequence of the improved calculation is also that table XVIII of Comm. N°, 97^b (Jan. '07) can be dispensed with. The first two corrections derived in XIV § 3 of the Communication mentioned, now become so small that they fall outside the region of observation. The correction calculated at the end of § 3 becomes somewhat smaller for CHAPPUIS' carbonic acid thermometer than has been given there, viz. -0.22×10^{-6}, to which another correction of -0.8×10^{-7} is to be added, if also the expansion by the pressure of the gas is to be taken into consideration.

§ 2. The restoration of our former value 0.0036627 further involves the following modifications, which are all of no importance as they do not exceed the errors of observation, but should be applied to make the agreement in the calculations complete:

1. that in table XVI of Comm. N°. 97^b (Jan. '07) in the first column the values of table XII are restored, and so all the numbers in the last decimal are increased by a unit. The latter holds also for the values of the second column of table XVI,

2. that in table XVII of the same Communication the values of the first column, except the last two, are increased by a unit in the last decimal,

3. that no further corrections are required for the temperatures in table XVI of Comm. N°. 99^a (June '07) and table XX of Comm. N°. 100^a (Dec. '07) (see conclusion of § 14 of Comm. N°. 99^a and of § 18 of Comm. N°. 100^a).

4. that in § 3 of Comm. N°. 100^b (Dec. '07) the value for $pv_{A100°.2}$ and the corresponding virial coefficients are subjected to small changes, which, however, are of no importance,

5. that the last line of Comm. N°. 101^a (Dec. '07) must be left out,

6. that in § 1 of Comm. N°. 101^b (Dec. '07) $\alpha_{AV} = 0{,}0036619$ changes into 0,0036617, and $T_{0°C.} = 273°.08$ into 273°.10, while $T_{0°C} = 273°.07$ of note [1]) in the § mentioned changes into $T_{0°C.} =$ 273°.09 and that in § 2 $t = -273°.08$ C. becomes —273°.10 C., the changes in B'_{100} and in the values of table XXV being imperceptible,

7. that the numerical values in §§ 1 and 3 of Comm. N°. 102^b (Dec. '07) require the emendations which have been applied in the translation in the Proceedings (Febr. 29 '08) (See footnote 1 there).

Physics. — "*On the condensation of helium.*" By Prof. H. KAMERLINGH ONNES. Communication N°. 105 of the Physical Laboratory at Leiden.

(Not communicated here, see next communication).

Physics. — "*Experiments on the condensation of helium by expansion.*" By Prof. H. KAMERLINGH ONNES. Communication N°. 105 of the Physical Laboratory at Leiden.

In the last session I communicated what I had observed in expanding helium, which at a temperature of — 259° C. had been strongly compressed. I made the experiment in consequence of my determinations of the isotherms of helium at different temperatures i. a. also at — 253° C. and — 259° C., from which I had calculated nearly 5° K. for the critical temperature of helium [1]). It thence followed that it would be possible by rapid expansion of helium compressed at 100 atm. at the meltingpoint of hydrogen to pass below the critical temperature and to cause a mist to appear in the gas [2]). It was to put this conclusion to the test, that I compressed nearly 7 liters of helium, purified by burning with copperoxyde and leading over charcoal at the temperature of liquid hydrogen (so that I could trust to have a gas with only very small admixtures) in a thick walled tube placed in a non silvered vacuum glass with liquid hydrogen, and provided with a stopcock through which the helium could be let off from the tube into a gasholder, a gasbag or a vacuum. The liquid hydrogen round the tube was exhausted at such a pressure that hydrogen crystals just appeared at the surface of the liquid. The vacuumglass with hydrogen was surrounded by a second non silvered vacuumglass with liquid air. In the thickwalled tube, leaving only a small clearance, there was placed an extremely thin walled beaker [3]) for protecting the gas which was cooled by expansion against conduction of heat from the walls, the layer of gas, between the beaker and the walls of the tube, though it was very thin, being a bad conductor.

At the expansion of the helium a dense gray cloud appeared from which separated out solid masses floating in the gaseous helium, resembling partly cotton wool, partly also denser masses, as if floating in a syrupy liquid, adhering to the walls and sliding downward while at the same time vanishing rapidly (20″). There was no trace of melting.

As far as I could judge then from the experiments I considered it probable that this solid substance was for the greater part helium.

If helium passed immediately to the solid state then the position of the vapour line in respect to the adiabatics would be more favourable for condensation than was to be expected according to the formula of VAN DER WAALS. The voluminous aspect of the solid mass was in harmony with this. By the above and also by other observations

[1]) OLSZEWSKI from expansion experiments has deduced that the critical temperature of helium lies below 2° K. DEWAR estimates the *boiling* point according to the absorption in charcoal at higher than 5° K. (This would agree with a critical temperature of 8 K. Note added in the translation).

[2]) Liquefaction by making use of the JOULE KELVIN process would also be possible. (Note added in the translation).

[3]) This device has been used by OLSZEWSKI in his experiments on the expansion of hydrogen (Note added in the translation).

which afterwards gave rise to doubt or proved incorrect, I had for some time the conviction that I had seen solid helium rapidly giving off vapours of the pressure shown by the gas (once more than 15 atm. was observed).

The continuation of my experiments has shown that they must be explained in quite a different way. By a not sufficiently explained cause the gas proved to be not so pure as was to be expected considering the method of purification. In analysing what was absorbed by charcoal at the temperature of boiling hydrogen till the charcoal did no more absorb hydrogen, (so that the gas could only contain traces of hydrogen) it could be proved that in one case the gas has contained only 0.45 and in another only 0.37 volume percents of hydrogen at most [1]). But this small admixture must have had a very great influence.

For at a repetition of the experiment with the helium subjected to the new treatment no cloud at all was observed. The experiment is not decisive as the velocity of expansion had been too small, but it is difficult before further investigation to find in the difference of velocity of expansion the cause that the helium in the tube remained now perfectly clear.

The explication of the previous observations is to be found in solution phenomena of solid hydrogen in gaseous helium. The phenomena which made the impression of being the giving off of vapour had been the solution of deposited solid hydrogen in the gaseous helium, the latter rapidly returning from the lower temperature to that of melting hydrogen, and the pressure increasing in consequence. Helium at the temperatures, that come into account here can according to the theory of mixtures take up at every temperature a percentage of hydrogen determined by that temperature in such a way that it is not deposited at any pressure. On plausible suppositions one can deduce that at temperatures above the melting point of hydrogen this percentage can be considerable and that at this melting point itself it can be more than one percent. From mixtures with smaller percentage the hydrogen is only deposited at lower temperatures e.g. by expansion. By the smallness of the quantity of hydrogen present it is also explained that after prolonged blowing off of the helium no solid hydrogen was left. For the quantity left was so small that it could evaporate in the space which it found at its disposal.

It remains remarkable that as small a quantity of admixture as the gas contained has been able to give the total phenomenon of a

[1]) About a small possible quantity of neon I could not yet be certain.

substance condensing to a solid and reevaporating, though the rapid evaporation, in which even denser masses were seen to be blown away sometimes, is in harmony with the smallness of this quantity of substance. There cannot have been much more than 1 mgr. or 15 cubic millimetres of hydrogen in round numbers in the tube — probably there was less in it — and yet the tube of nearly 7 cubic centimetres was over its whole length for almost a quarter filled with a dense flaky substance.

As far as the experiments on the expansion of helium at the melting point of hydrogen are now advanced they show the curious forms that the solution phenomena of a solid in a gas take in the case of helium and hydrogen. They further point to the possibility of realising with mixtures of hydrogen and helium the rising or falling of the solid substance according to the pressure exerted on the gas, the barotropic phenomenon for a solid and a gas. But the question of condensing helium is to be considered yet as an open one, which will ask an extensive investigation.

POSTSCRIPTUM.

I have had the occasion to repeat the experiment with the gas that remained perfectly clear in the last expansion experiment; and which also according to the spectroscopic test contained only traces of hydrogen. I now used a greater velocity of expansion. A thin cloud appeared and vanished extremely rapidly (in 1″ nearly). The mist now had another aspect.

It is possible that the traces of hydrogen left in the gas will prove sufficient to cause this mist. But it is also possible that the mist has been a liquid cloud and the changed aspect seemed to point to this. If this might prove to be the case then the critical point would be nearly as I calculated it from the isothermals and helium would follow tolerably well the laws of VAN DER WAALS. The tube broke and I could not attain more certainty about the nature of the cloud.

The preceding experiments show very strikingly how careful one has to be in making conclusions from the appearing or not appearing of a cloud by expansion. A decision about the critical temperature of helium is therefore only to be obtained by a prolonged systematical investigation which will take much time.

(April, 24, 1908).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Friday April 24, 1908.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Vrijdag 24 April 1908, Dl. XVI).

CONTENTS.

Mathematics. — "*On the cyclic minimal surface*". By Prof. J. C. KLUYVER.

(Communicated in the meeting of January 25, 1908).

ENNEPER (Zeitschr Math. Phys. 14) pointed to the existence of a minimal surface containing a system of circles lying in parallel planes, with centres situated on a plane curve. Let us suppose that this curve passes through the origin of the rectangular coordinates, that it is situated in the XZ-plane and that the variable circle with the centre $(\xi, 0, \zeta)$ and the radius R, generating the surface, lies always in a plane parallel to the XY-plane.

The rectangular coordinates x, y, z of a point of the surface are given by the equations

$$x = \xi + R \cos \alpha \quad , \quad y = R \sin \alpha \quad , \quad z = \zeta,$$

so that they are expressed in the two parameters α and ζ. We find that the differential equation of the minimal surfaces is satisfied when

$$R^2 (\xi'' R \cos \alpha + RR'') - R^2 (1 + \xi'^2 + R'^2 + R'^2 + 2 \xi' R' \cos \alpha) = 0,$$

in which equation the dashes denote the differentiations with regard to ζ.

The equation breaks up into

$$\xi'' R = 2 \xi' R'$$

and into

$$RR'' = 1 + \xi'^2 + R'^2.$$

The first equation furnishes

$$\xi' = \frac{AR^2}{b^2},$$

where A denotes a positive constant and b the minimum value of R.

The second equation now passes into

$$\frac{d}{d\zeta}\left(\frac{R'}{R}\right) = \frac{1}{R^2} + \frac{A^2 R^2}{b^4}$$

and the integration furnishes

$$R'^2 = \frac{1}{b^2}(R^2 - b^2)\left(1 + \frac{A^2 R^2}{b^2}\right),$$

so that finally we can express ξ en ζ in R by means of elliptic integrals.

We find

$$\xi = \frac{A}{B}\int_b^R \frac{dR}{\sqrt{(R^2-b^2)\left(1+\frac{A^2R^2}{b^2}\right)}} \quad , \quad \zeta = b\int_b^R \frac{R^2 dR}{\sqrt{(R^2-b^2)\left(1+\frac{A^2R^2}{b^2}\right)}}.$$

Here an elliptic argument can be introduced. We put

$$R = \frac{b}{cn\,u},$$

$$k = \sin\theta = \frac{1}{\sqrt{1+A^2}},$$

and we find

$$\xi = bk' \int_0^u \frac{dw}{cn^2 w} \quad , \quad \zeta = bku.$$

By allowing u to vary from $-K$ to $+K$ the centre M with the coordinates ξ, ζ in the XZ-plane describes completely the locus of the centres and the equation

$$R = \frac{b}{cn u}$$

indicates how the radius of the circle changes during the motion.

We notice that the minimal surface depends on two constants b and k, that the smallest circle ($u = 0$) is found in the XY-plane, that with respect to the origin there is symmetry, and that for $u = K$, $\zeta = bkK$ the radius R has become infinite whilst at the same time the centre M is at infinite distance.

As however

$$\lim_{u=K} (\xi - R) = b \lim_{u=K} \left[k' \int_0^u \frac{dw}{cn^2 w} - \frac{1}{cn\,u} \right] = \frac{b}{k'} (k'^2 K - E)$$

and $\xi - R$ retains therefore a finite value the surface contains two right lines

$$z = \pm\, bk\,K,$$

$$x = \pm \frac{b}{k'} (k'^2 K - E).$$

For $k = 1$ the eliptic integrals degenerate. We have

$$\xi = 0, \quad \zeta = bu, \quad R = b\,Ch\,u,$$

and the surface has passed into a catenoid. The smaller k is, the more the surface deviates from the catenoid and the more oblique it becomes. For, we find for the coefficient of direction of the tangent to the locus of the centres M:

$$\frac{d\zeta}{d\xi} = \frac{k\,cn^2\,u}{k'}$$

and the greatest value of this coefficient $k : k'$, which is arrived at in the origin, tends to zero when k tends to zero. The surface is then altogether in the XY-plane.

I shall now endeavour first to investigate in the following wh
it is possible to bring through two equal circles placed in parall
planes a cyclic minimal surface and then to calculate the part of tl
minimal surface extended between those circles.

When for both circles the radius R is taken equal to 1, the centr
$M(\xi, \zeta)$ and $M'(-\xi, -\zeta)$ are situated in the XZ-plane symmetrical
with respect to the origin and their planes are parallel to the X
plane, the question is whether the two equations:

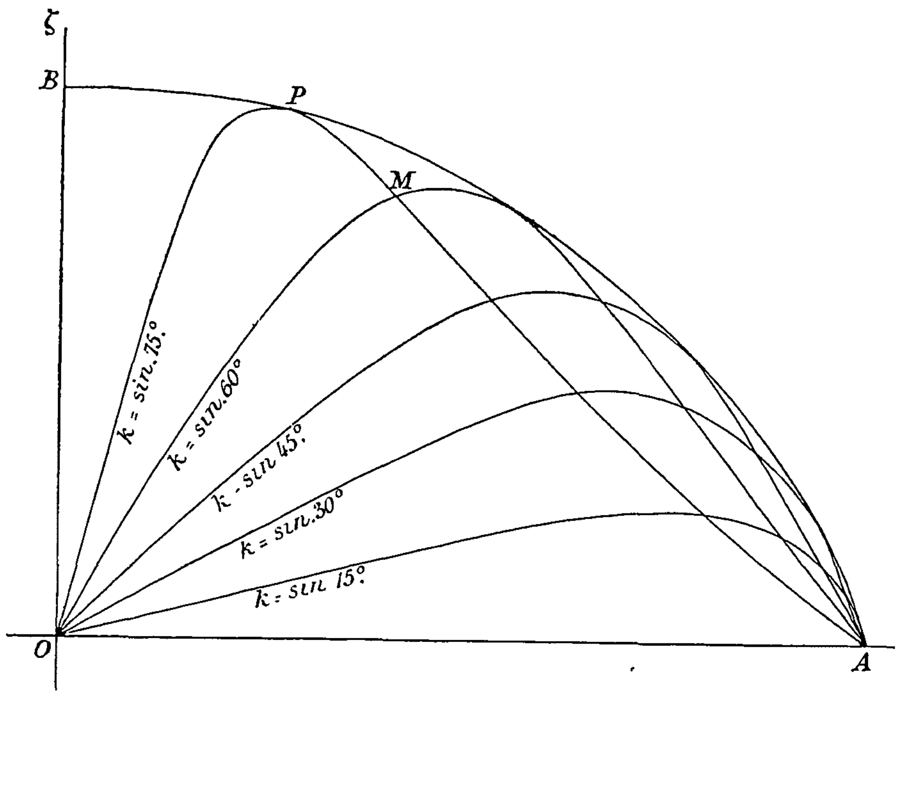
$$\xi = k' \, cn\, u \int_0^u \frac{dw}{cn^2 w}, \quad \zeta = ku \, cn\, u$$

admit of suitable solutions for k and u. If these are found, we ha
$b = cn\, u$ and both parameters b and k of the minimal surface a
known.

In order to investigate the indicated equations we regard for tl
present in the $\xi\zeta$-plane ξ and ζ as variables and we consider tl
curve which is described by point (ξ, ζ), when for constant k tl
variable u describes the range of values from 0 to K. We have:

$$\xi(0) = 0 \quad , \quad \zeta(0) = 0,$$
$$\xi(K) = 1 \quad , \quad \zeta(K) = 0.$$

So for all values of k the curve will run from the origin O
point A on the ξ-axis (see the diagram).



Farther we have:

$$\xi = k' \, cn\, u \int_0^u \frac{d\,(tn\, w)}{dn\, w} < k' \, cn\, u \int_0^u \frac{d\,(tn\, w)}{dn\, u},$$

$$\xi < \frac{k' \, sn\, u}{dn\, u},$$

so that from:

$$\frac{d\xi}{du} = \frac{1}{cn\, u}\,(k' - sn\, u\; dn\, u\; \xi)$$

follows:

$$\frac{d\xi}{du} > k' \, cn\, u.$$

We conclude that for increasing u the variable ξ grows regularly from 0 to 1. So the curve OA is intersected but once by a line $\xi =$ constant.

At the same time:

$$\frac{d\zeta}{du} = k\,(cn\, u - u\, sn\, u\; dn\, u) = k\, cn\, u\left(1 - u\,\frac{sn\, u}{sn\,(u+K)}\right).$$

For small u we find $\frac{d\zeta}{du}$ to be positive, it keeps on decreasing, becomes one time zero and is then negative. So the variable ζ reaches somewhere a maximum and the curve OA is either not cut by a line $\zeta =$ constant or in two points The form of the curve $k =$ constant is therefore as is indicated schematically in the diagram. In order to be able to compare the curves belonging to different values of k we can determine the values which the differentialquotient $\frac{d\zeta}{d\xi}$ assumes in the points O and A.

We have

$$\left(\frac{d\xi}{du}\right)_{u=0} = k' \;,\; \left(\frac{d\zeta}{du}\right)_{u=0} = k,$$

$$\left(\frac{d\xi}{du}\right)_{u=K} = E - k'^2 K \;,\; \left(\frac{d\zeta}{du}\right)_{u=K} = -\, kk'K,$$

from which ensues

$$\left(\frac{d\zeta}{d\xi}\right)_{\xi=0} = \frac{k}{k'} \;,\quad \left(\frac{d\zeta}{d\xi}\right)_{\xi=1} = -\frac{kk'K}{E - k'^2 K} = -\frac{k'K}{k\int_0^K cn^2 w\, dw}. \quad (O)$$

From this is apparent that in O the value of $\frac{d\zeta}{d\xi}$ increases with k, that on the other hand the absolute value of $\frac{d\zeta}{d\xi}$ decreases in A for increasing k. For, if k becomes greater $k'K$ decreases, but the denominator $k\int\limits_0^K cn^2\, w\, dw$ increases.

Taking into consideration the form just sketched of a curve OA belonging to a definite value of k we find that a second suchlike curve belonging either to a larger value or to a smaller value of k will certainly intersect the first curve somewhere. So as soon as a cyclic minimal surface passes through two equal circles placed in parallel planes we shall be able to bring a second cyclic minimal-surface through these circles.

We must now investigate when the two cyclic minimal surfaces coincide, i. o. w. we must find the envelope of the curves OA.

If we put $c = k^2$, $c' = k'^2$, then the system of curves is given in the equations

$$\xi = \sqrt{c'}\, cn\, u \int\limits_0^u \frac{dw}{cn^2\, w} \ , \ \zeta = \sqrt{c}\, u\, cn\, u \ ;$$

we regard c as the parameter of the curve, $\varphi = am\, u$ as the parameter determining a point on a given curve, so that the coordinates (ξ, ζ) of a point of the envelope satisfy the condition

$$\frac{D(\xi, \zeta)}{D(c, \varphi)} = 0 \ .$$

If here and in future we put for shortness' sake

$$A(u) = \int\limits_0^u \frac{dw}{cn^2\, w} \quad , \quad B(u) = \int\limits_0^u \frac{dw}{dn^2\, w}$$

and we take into account that for constant $\varphi = am\, u$ we have

$$\frac{\partial u}{\partial c} = \frac{1}{2c}\left(B(u) - u\right),$$

we find

$$\frac{\partial \xi}{\partial c} = -\frac{1}{2\sqrt{c'}}\, cn\, u\, B(u) \quad , \quad \frac{\partial \zeta}{\partial c} = \frac{1}{2\sqrt{c}}\, cn\, u\, B(u) \ ,$$

$$\frac{\partial \xi}{\partial \varphi} = \sqrt{c'}\, sn\, u\left(c\, B(u) - Q(u)\right) \ , \ \frac{\partial \zeta}{\partial \varphi} = -\sqrt{c}\, sn\, u\left(c'\, B(u) + Q(u)\right),$$

where $Q(u)$ is given by the equations

$$Q(u) = u - E(u) - \frac{dn\, u\, cn\, u}{sn\, u} \ ,$$

$$= K - E - \int_u^K \frac{dw}{sn^2 w},$$

$$= \frac{1}{sn^2 u}\left(u - cn^2 u\, A(u) - dn^2 u\, B(u)\right),$$

$$= A(u) + k^2 B(u) - \frac{1}{sn\, u\, cn\, u\, dn\, u}.$$

From this ensues

$$\frac{D(\xi, \zeta)}{D(c, \varphi)} = - \frac{1}{2\sqrt{cc'}}\, cn\, u\, sn\, u\, B(u)\, Q(u),$$

and so the points of the envelope of the curves OA are determined by the equations

$$Q(u) = K - E - \int_u^K \frac{dw}{sn^2 w} = 0 \text{ [1]}.$$

As when c is given, the first member of the equation increases regularly from $-\infty$ for $u=0$ to $K-E$ for $u=K$, the equation $Q(u)=0$ admits of one solution u_0. By differentiating we find

$$\frac{du_0}{dc} = \frac{1}{2c}\int_0^{u_0} dw \left[\frac{dn^2 u_0}{dn^2 w} - 1\right],$$

i. e. a negative value; therefore the greater c is, the smaller is the argument u_0, which I call the critical argument. This argument moves finally between rather narrow limits. For $c=0$ we find $K = E = \frac{\pi}{2}$ and so also $u_0 = \frac{\pi}{2} = 1.5708$. For $c=1$ we find

$$Q(u) = u - E(u) - \frac{dn\, u\, cn\, u}{sn\, u} = u - \frac{1}{sn\, u} = u - \frac{Ch\, u}{Sh\, u}.$$

So the critical argument u_0 satisfies the equation

$$u_0 = \frac{Ch\, u_0}{Sh\, u_0}.$$

From this ensues

$$u_0 = 1.1997,$$
$$\varphi_0 = am\, u_0 = 56^\circ.28',$$
$$\cot \varphi_0 = u_0\, cn\, u_0 = 0.6627.$$

[1]) G. Juga. (Ueber die Constantenbestimmung bei einer cyklischen Minimalfläche, Math. Ann. Bd. 52) gives this equation in the form

$$cnu\, dnu + (E(u) - u)\, snu = 0.$$

For values of c between 0 and 1 it is easy to solve u_0 out of the equation

$$Q(u_0) = K - E - \int_{u_0}^{K} \frac{dw}{sn^2 w} = 0$$

by means of the tables of LEGENDRE. If u'_0 is an approximate value of the critical argument, the calculation of NEWTON furnishes

$$u'_0 - Q(u'_0)\, sn^2 u'_0$$

as following approximation. In this way the critical argument is calculated in the following table for some values of $k^2 = c$

$k = \sqrt{c}$	$\varphi_0 = am\, u_0$	u_0	$b = cn\, u_0$	ξ_0	ζ_0	φ'_0	ξ'_0	ζ'_0
sin 0°	90°	1.5708	0	1.	0.	90°	1.	0
15°	87° 1′	1.5442	0.0520	0.9966	0 0208	87° 0′	0.9954	0.0245
30°	79°17′	1.4701	0.1859	0.9498	0.1367	79°23′	0.9427	0 1423
45°	70° 3′	1.3708	0 3412	0.7930	0.3308	70°16′	0.7916	0.3325
60°	62°31′	1.2801	0.4614	0.5573	0.5116	62°35′	0 5549	0.5133
75°	57°57′	1.2198	0.5306	0 2813	0.6251	57°57′	0.2776	0.6265
90°	56°28′	1.1997	0.5524	0.	0.6627	56°28′	0.	0.6627

and moreover are indicated in it the coordinates ξ_0, ζ_0 of the point P, in which the curve OA belonging to each value of k touches the envelope of that system of curves.

By the equations

$$\xi_0 = \sqrt{c}\, cn\, u_0\, A(u_0) \quad , \quad \zeta_0 = \sqrt{c}\, x_0\, cn\, u_0$$

we now find in connection with the condition

$$Q(u_0) = 0$$

that ξ_0 and ζ_0 are given as functions of c only. We can deduce out of it

$$\frac{d\xi_0}{dc} = -\frac{1}{2\sqrt{c}}\, dn\, u_0\, B(u_0)\, [cn\, u_0\, dn\, u_0 + u_0\, c\, sn^3 u_0],$$

$$\frac{d\zeta_0}{dc} = \frac{1}{2\sqrt{c}}\, dn\, u_0\, B(u_0)\, [cn\, u_0\, dn\, u_0 + u_0\, c\, sn^3 u_0],$$

$$\frac{d\zeta_0}{d\xi_0} = -\frac{k'}{k}.$$

From this appears that for increasing k or $\sqrt{c}$ the coordinate ζ_0 decreases regularly and the coordinate ζ_0 increases regularly. In connection with the numbers inserted in the table it follows that the envelope of the curves OA has about the shape of a quadrant of ellipse BA of which half of the great axis $OA = 1$ and half of the small axis $OB = 0.6627$.

Moreover it is clear that the tangent to any curve $k =$ constant, in the point P where the latter touches the envelope, is normal to the tangent in the origin O drawn to this same curve. The preceding calculations now lead to the conclusion that through the two equal circles with radius $R = 1$ placed parallel and symmetrically with respect to the origin two cyclic minimal surfaces will pass, when the centre $M(\xi,\zeta)$ of the upper circle is situated inside the curve BA of the diagram, that the two surfaces coincide when M has arrived on the curve BA and that the circles cannot be connected by a minimal surface when M falls outside the curve BA.

If M lies inside the curve BA two curves OA pass through M. One of these touches the envelope in P, a point on curve OA between O and M. So the argument u belonging to M is greater than the critical argument u_0 in P and so the minimal surface belonging to it and extended between the circles M and M' would contain the two circles along which this minimal surface is cut by a second minimal surface with an infinitesimal slight difference. So this minimal surface is unstable. For the second minimal surface laid through the circles an argument u corresponds to M smaller than the critical argument u_0; this surface is therefore stable and can be realized in a proof of PLATEAU.

If two surfaces can be laid through the circles the most oblique surface (the surface belonging to the smaller value of k and with the greater value of the radius b of the mean section) is therefore always stable, the other is unstable.

It is worth mentioning that whilst here the quantities φ_0, ξ_0, ζ depend in rather an intricate way on $k = \sin\theta$, we can find by approximation out of simple formulae very accurate values for these quantities.

If we call the critital amplitude 56°28′ of the catenoid β, we shall be able to assume with great accuracy the following relations :

$$\cos\varphi_0 = \cos\beta \sin^2\theta \left(1 + \frac{4}{9}\cos^2\theta\right),$$

$$\xi_0^2 = 1 - \left(\frac{\cos\varphi_0}{\cos\beta}\right)^2,$$

$$\zeta_0 = \cot\beta \left(\frac{\cos\varphi_0}{\cos\beta}\right)^{\frac{7}{5}},$$

from which ensues for the equation of the envelope BA

$$\zeta_0^2 + \left(\frac{\zeta_0}{\cot\beta}\right)^{\frac{10}{7}} = 1.$$

In the table the values of φ_0, ξ_0 and ζ_0 calculated in this way are added in the three last columns, to be compared.

To conclude with we give a computation of a part of a given cyclic minimal surface with given parameters b and k, situated beween two equally large circles corresponding to the arguments $+u$ and $-u$.

The coordinates x, y, z, of a point of the surface are again determined by the equations:

$$x = bk' A(u) + \frac{b}{cn\,u}\cos\alpha, \quad y = \frac{b}{cn\,u}\sin\alpha, \quad z = bk\,u,$$

out of which we can find for the line-element on the surface the expression

$$\frac{ds^2}{b^2} = \frac{Pdu - i\,cn\,u\,d\alpha + i\,k'\sin\alpha\,du}{cn^2\,u} \times \frac{Pdu + i\,cn\,u\,d\alpha - i\,k'\sin\alpha\,du}{cn^2\,u},$$

in which P is determined by the equation

$$P^2 = (k'\cos\alpha + sn\,u\,dn\,u)^2 + k^2\,cn^4\,u.$$

We introduce for α an imaginary argument v.

We substitute

$$tg\,\tfrac{1}{2}\,\alpha = i\,\frac{tg\,\frac{1}{2}\,am\,v}{tg\,\frac{1}{2}\,am\,(u-K)}$$

and we find

$$\sin\alpha = \frac{i\,sn\,v\,sn\,(u-K)}{cn\,v - cn\,(u-K)},$$

$$\cos\alpha = \frac{1 - cn\,v\,cn\,(u-K)}{cn\,v - cn\,(u-K)},$$

$$\frac{d\alpha}{\sin\alpha} = \frac{dn\,v}{sn\,v}\,dv - \frac{dn\,(u-K)}{sn\,(u-K)}\,du,$$

$$P = \frac{cn^2\,u\,dn\,v\,dn\,(u-K)}{k'\,(cn\,v - cn\,(u-K))},$$

and finally

$$\frac{ds^2}{b^2} = \frac{dn^2\,v\,dn^2\,(u-K)}{k'^2(cn\,v - cn\,(u-K))^2}\,(du-dv)\,(du+dv).$$

From this ensues that $u+v$ and $u-v$ are the parameters of the lines of length zero, so that v is the parameter of the greatest incline.

According to the general properties of the minimal surfaces we have for the superficial element $d\Omega$ the expression

$$\frac{d\Omega}{b^2} = \frac{dn^2\, v\, dn^2\,(u-K)}{k'^2\,(cn\, v - cn\,(u-K))^2}\, du\, \frac{dv}{i},$$

and we find for that part of the surface limited by the two circles with the arguments $+u$ and $-u$:

$$\frac{\Omega}{4b^2} = \int_0^u du \int_0^{2iK'} \frac{dv}{i} \frac{dn^2\, v\, dn^2\,(u-K)}{k'^2\,(cn\, v - cn\,(u-K))^2}.$$

To perform the integration we start from the identity

$$f(u) = -\int_0^{2iK'} \frac{dv}{i} \frac{sn\,(u-K)\, dn\,(u-K)}{cn\, v - cn\,(u-K)} = 2K'\, Z(u-K) + \frac{\pi u}{K} =$$

$$= 2u\,(E'-K) + 2k'^2\, K'\, B\,(u),$$

which furnishes first

$$\int_0^{2iK'} \frac{dv}{i} \frac{dn^2\,(u-K)}{cn\, v - cn\,(u-K)} = \frac{k' f(u)}{cn\, u}$$

Moreover

$${}'\!\!\int^{2iK'} \frac{dv}{i} \frac{dn^2\, v - dn^2\,(u-K)}{cn\, v - cn\,(u-K)} = k^2\; {}'\!\!\int_0^{2iK'} \frac{dv}{i}\, cn\, v + 2\, k^2\, K'\, cn\,(u-K).$$

A dash before the integral sign indicates that the path of integration does not pass through point $v = iK'$.

Out of the two last equations follows by means of addition

$${}'\!\!\int_0^{2iK'} \frac{dv}{i} \cdot \frac{dn^2\, v}{cn\, v - cn\,(u-K)} = \frac{k' f(u)}{cn\, u} + k^2\; {}'\!\!\int_0^{2iK'} \frac{dv}{i}\, cn\, v + 2k^2 K'\, cn\,(u-K),$$

an equation which, if we differentiate with regard to u and then divide by $k'\, cn\, u$, passes into

$$\int_0^{2iK'} \frac{dv}{i} \cdot \frac{dn^2\, v\, dn^2\,(u-K)}{k'^2\,(cn\, v - cn\,(u-K))^2} = \frac{1}{cn\, u} \frac{d}{du}\left(\frac{f(u)}{cn\, u}\right) + \frac{2k^2 K'}{dn^2\, u}$$

$$= \frac{1}{2} \frac{d}{du}\left(\frac{f(u)}{cn^2\, u}\right) + \frac{2k^2 K'}{dn^2\, u} + \frac{k'^2 K'}{cn^2\, u\, dn^2\, u} + \frac{E'-K'}{cn^2\, u}.$$

Now integrating according to u between the limits 0 and u we find finally

$$\frac{\Omega}{4b^2} = \frac{u}{cn^2\, u}(E' - K') + E' A(u) + K' \frac{dn^2\, u}{cn^2\, u} B(u).$$

If the given circles have the radius $R = 1$ then b is equal to $cn\, u$ and we can write

$$\frac{\Omega}{4} = uE' + \xi\, cn\, u \frac{E' - K'}{k'} - sn^2\, uK'Q\, u),$$

where ξ again represents the x-coordinate of the centre M of the upper circle.

If this centre M moves on the envelope BA of the diagram, then u becomes equal to the critical argument u_0, $Q(u)$ equal to zero and we have obtained the greatest possible minimal surface Ω_0 for the given value of k. So

$$\frac{\Omega_0}{4} = u_0 E' + \xi_0\, cn\, u_0 \frac{E' - K'}{k'}.$$

We can now put the question where we have to put M on the envelope BA, that is what value must be given to k for Ω_0, to obtain the greatest possible value. To answer that question we substitute $c = k^2$ and $\varphi_0 = am\, u_0$, then Ω_0 is a function of c, whilst φ_0 and ξ_0 are connected with c by means of the equations

$$Q(u_0) = K - E - \int_{u_0}^{K} \frac{dw}{sn^2\, w} = 0,$$

$$\xi_0 = \sqrt{c'}\, cn u_0\, A\,(u_0).$$

By differentiation we find

$$\frac{d\varphi_0}{dc} = -\frac{1}{2} sn^2\, u_0\, dn\, u_0\, B\,(u_0),$$

$$\frac{du_0}{dc} = -\frac{1}{2c} cn^2\, u_0\, A\,(u_0),$$

$$\frac{d\xi_0}{dc} = -\frac{1}{2\sqrt{c'}} dn\, u_0\, B\,(u_0)\,(cn\, u_0\, dn\, u_0 + u_0\, c\, sn^2\, u_0),$$

and finally by means of these results

$$\frac{d}{dc}\left(\frac{\Omega_0}{4}\right) = \frac{K'-E'}{c'} cn\, u_0\, dn\, u_0\, B\,(u_0)\,(cn\, u_0\, dn\, u_0 + u_0\, c\, sn^3\, u_0).$$

As the right member of the last equation is always positive, Ω_0 always increases with c or with k. The greatest possible surface between the two circles is obtained by placing M in B; we have then a part of the catenoid, of which half the height is equal to $cot\, \beta = 0.6627$.

Now $\quad K' = E' = \frac{\pi}{2},$

$$\frac{\Omega_0}{2\pi} = u_0 = 1.1997.$$

The smallest value Ω_0 obtains for $k=0$. Then $\zeta_0=0$, $\xi_0=1$; the minimal surface consists only of the surface of the circles M and C' placed side by side in the XY-plane. We have

$$\frac{\Omega_0}{2\pi}=1.$$

So also the surface Ω_0 keeps moving between rather narrow limits. Although the value of Ω_0 depends again in rather an intricate way on k we can put pretty accurately, if once the critical argument u_0 or the amplitude φ_0 has been calculated,

$$\frac{\Omega_0}{2\pi}=\frac{1}{sn\, u_0}.$$

This is evident from the following table, in which have been inserted for some values of k the corresponding values of $\frac{\Omega_0}{2\pi}$ and of $\frac{1}{sn\, u_0}$.

k	$\frac{\Omega_0}{2\pi}$	$\frac{1}{sn\, u_0}$
Sin 0°	1.	1
15°	1 0002	1.0001
30°	1.0111	1 0176
45°	1 0556	1.0639
60°	1.1241	1 1271
75°	1.1795	1 1795
90°	1 1997	1 1997

As we have $b=cn\, u_0$, where b represents again the radius of the mean section we can in any case put with great approximation

$$\Omega_0=\frac{2\pi}{\sqrt{1-b^2}},$$

and in this way we obtain for the greatest possible just stable part of an arbitrary cyclic minimal surface that can be extended between two circles with radius $R=1$ the same expression as for the catenoid.

Botany. — "*Contribution N°. 1 to the knowledge of the Flora of Java.*" By Dr. S. H. KOORDERS. (Continuation [1])).

(Communicated in the meeting of March 28, 1908).

§ 3. On the geographical distribution, oecological conditions and means of dissemination of the Aceraceae, growing wild in the highest mountain regions of Java.

§§ 1. Synonyms and geographical distribution.

This order, which in BENTHAM and HOOKER's Genera Plantarum and in BOERLAGE Handleid. Flora N. I. forms part of the *Sapindaceae*, consists of two genera; only one of these (*Acer*, LINN.) occurs wild in Java. Of the genus *Acer* about 50 species are known; only one of these (*Acer niveum* BL.) belongs to the flora of Java, and has frequently been found there, growing wild in the higher mountain regions (up to 2550 m. above sealevel).

Some authors, e. g. PAX l. c., distinguish two varieties in Java, which were regarded by BLUME as species *Acer niveum* BL. *genuinum* PAX and *A. niveum* var. *cassiaefolia* (BL.) PAX. According to PAX l.c. the former of these has broad elliptical or ovate leaves with rounded base and a snowy white under surface, the latter oblong leaves with an acute base and a blue-grey under surface. The type is represented at Buitenzorg in Herb. Kds. by specimens from the G. Gedé (Herb. Kds. 12645 β) and the variety by specimens from Takóka (Herb. Kds. 7251 β). By far the greater number of specimens (e.g. many from the Gedé), however belong to neither of these two forms, as they combine various properties in a number of ways. We therefore consider the two varieties to be merely the extreme forms of one and the same, more or less varying [2]) type. Some specimens in Herb. Kds., should further be noted, in which the under surface of the leaf (in the dried state) appears to be green, e.g. Kds. 7265 β from the G. Slamat; by this character and also by the incipient serration of the leaf margin, these specimens approach to *A. laevigata* WALL. Kds. 7267 β from Pringombo should also be considered; the leaves, which, in the living state are pale blue-grey cannot be distinguished from those of *A. oblongum*. The colour of

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1) Continued from These Proc., Febr. 29th 1908 p. 687.

2) In his last monograph of the *Aceraceae* PAX l.c. (1902) 31 also, however, already says, that the variety *cassiaefolium* (Bl.) PAX, which he formerly separated off, scarcely differs from the type.

dried specimens in general, and of this species in particular, depends according to KOORDERS and VALETON Bijdr. Booms. Java IX (1903) p. 256), very largely on the manner and rate of drying of the herbarium.

Acer niveum BL. Rumphia III (1847) 193 t. 167 B. f. 1; HIERN in HOOK. Fl. Br. Ind. I, 693; PAX Monogr. d. Gattung Acer in ENGL. Botan. Jahrb. VII, 207; PAX in ENGLER Pflanzenreich Heft 8 IV, 163 (1902) 31; KOORD. et VALETON l.c. 254; — *A. laurinum* HASSK. in Tijdschr. v. Nat. Gesch. en Physiol. X (1843) 138 (nomen tantum); MIQ. Fl. Ind. Bat. I, 2 (1859) 582; — *A. javanicum* JUNGH. in Tijdschr. Nat. Gesch. en Physiol. VIII (1841) 391 (nomen tantum); — *A. cassiaefolium* BL. l. c. f. 2.

Geographical distribution outside Java: India or.: "Assam, hills of Martaban and Tenasserim" (BRANDIS, Indian Trees, 181). "Assam and Burma" (according to PAX l.c.) Malay Archipelago: Sumatra (JUNGH.! in Herb. Lugd. Bat.); in N. E. Celebes in the Minahasa on the Lolomboelang mountains (Herb. Kds in Mus. H. Hort. Bogor; comp. KOORD. Verslag botan. reis N. O. Celebes (1898) p. 409). Has also been collected in Celebes by WARBURG (comp. PAX l.c. 31).

Geographical distribution and oecological conditions in Java: Has been collected, according to Herb. Kds, in Western and Central Java, and also in Eastern Java, at an altitude of 700—2550 m. at the following points. Hitherto (according to Herb. Kds.) it has been found in the following places in Java: In the res. Bantĕn on the G. Karang at 1000 m. above Tjimanoek, and on the G. Poelasari at 1050 m. near bivouac Kihoedjan (both in the division Pandeglang). In the res. Preanger: 1) on the G. Gedé near and above Tjibodas at 1450 m., 1600 m., etc. and also at 2200 m. above sea level; 2) near Takoka at 1200 m. the Djampangs; 3) near Pangentjongan in the Galoenggoeng (in the div. Limbangan at 1250 m., 1400 m., and at 1800 m. above sea level); 4) near Tjigenteng in the Kendeng-Patoeha mountains at 1450 m. and 1600 m. above sea level. In the res. Tègal-Pèkalongan on the G. Slamat above Simpar at 1400 m. and above Soerdjå on the N.-W. Prahoe at 1400 m. In the res. Kĕdoe at 2200 m. on the G. Kĕmbang above Bĕdaka and at 2500 m. on the highest summit of the Prahoe-Diëng mountains. In the res. Banjoemas on the Midangan mountains near Pringåmbå 800 m. above sea level In the res. Sĕmarang on the G. Oengaran and the G. Tèlmâjâ at about 1400 m., e.g. above Sĕpakoeng In the res. Madioen on the Wilis-mountains above Ngĕbèl between 1400 m. and 2000 m. (not collected there at a greater height). In the res. Prâbâlingâ-Pasaroehan on the Tĕngger-mountains at 2000 m. near Ngadisari. In de res. Bĕsoeki on the Idjenplateau near bivouac Oengoep-oengoep at 1700 m. Up to the present this species is therefore known from the res. Bantĕn (in Western-Java) to the res. Bĕsoeki (in Eastern-Java) from 700 m. to 2550 m. above sealevel. — Occurrence: Does not grow socially in Java, but occurs fairly plentifully in some mountain forests e.g. in Western-Java on the G. Gĕdé. Oecological conditions: This species has not yet been observed by me in Java on soils, where there is a great, permanent dearth of water nor where there is physiological drought resulting from a large saline content, nor on soils rich in lime and common salt; neither does the species grow on soils which are periodically liable to strong dessication. It grows almost exclusively on permanently damp, fertile, volcanic soils, rich in humus, in close shady mountain forests of high trees and consisting of a great number of species. In the hot plain, even in permanently humid districts, the species does not occur. The lowest station is in a ravine in Eastern-Java at about 700 m., the highest is at nearly 2550 m. above sealevel in Central-Java. I feel obliged to consider the possibility of the occasional, be it very exceptional, occurrence of *Acer niveum* in physiologically dry, saline soils, in consequence of a herbarium note of JUNGHUHN, found by me in 's Rijks Herbarium, and referring to a specimen, collected by this naturalist on the Diëng-plateau at about 2000 m. near the Kawah-Tjondro-dimoeko. I have here as yet no other data at my disposal, which would show with certainty, whether this species does not only occur in Java "near", but also "on" such soils. — Leaffall: At the same moment there stood in the same locality (in the same forest, in close proximity to each other) two individuals of apparently the same age. On the 2nd of June 1898 one of these was in full (old) leaf, while the adjoining specimen was practically without leaves, except one branch which bore young foliage. On March 23rd 1893, near Takóka, one of the trees (of this species), which had been numbered for the purpose of the investigation, was completely without leaves, although it stood in the midst of tree species, which were then nearly all in full foliage. — Time of flowering and fruiting: Flowers were collected in June and in July, August, Sept. and Nov. — Habitus: A forest giant, which immediately reveals its presence, even in the thickest virgin forest, by its fallen leaves on the ground, and sometimes by its characteristic winged fruits; the leaves are noticeable on account of the colour of their lower surface, which remains greyish white for a fairly long time. This greyish white or bluish grey colour is also rather striking in the living plant. In the flowering period this giant of the forest further attracts attention by its almost leafless condition in the midst of evergreen trees. In alpine regions, at 2000 m. above sealevel, in Western and Central-Java, (e.g. Preanger, Bagèlen), this species stands out by its dimensions, which are rather considerable for a high altitude; so, for instance, at 2200 m. sealevel, on the G. Kembang near Bĕdaka, a specimen was 20 m. high, with a trunk ½ m. in diam. The above data about Java, relating to oecological conditions and geographical distrubution, have been taken from observations, made by me in Java 1888—1903, and mostly published in KOORDERS and VALETON l.c. 257—258. — In the National Herbaria at Leiden and at Utrecht I found with the specimens, collected in Java by JUNGHUHN, BLUME, REINWARDT, etc., and now examined by me, no special data about oecological conditions: in most cases there was only written on the labels "Java" without further indications.

§§ 2. Means of dissemination.

The only means of dissemination is the fruit, known as *samara*, which is primarily intended for distribution by wind, but which seems, in addition, to have a certain capacity for being transported by water, according to an experiment of mine. At least, if the fruits are quite dry, they remain floating for some days on a 3½ °/₀ solution of common salt. In this species the fruits are produced in Java, as far as is known, only once a year, but then mostly in great numbers. Although the winged fruits are fairly heavy (when dry they weigh about 100 milligrams, the wings, which are often 5 cm. long and 2 cm. broad, being included), and although I never found in Java any indication, that the fruits are distributed by animals, distribution must nevertheless take place easily, as is proved by the large number of localities, cited above, where the tree is found. As the occurrence of the species is limited to the higher regions of several active volcanoes, at places which are more than 40 kilometres apart, and which are separated by hot plains, in which the species has never been found wild in Java at the present time, it would appear, that the force of the wind on the higher mountains of Java is sufficient for transport over a distance of 40 kilometres, even of such large samarae as those of *Acer niveum*.

I think it however more probable, that in the case of this species, as in that of the next one, (*Dodonaea viscosa*) such large winged fruits have been and are still, only transported in stages. It may have been, that in former times other climatological conditions enabled these two species to grow wild in the 30—40 kilometres of intervening low lands, in such places where growth can no longer take place at the present time. It may also be that even under the present conditions of climate, isolated specimens have escaped notice and might be found between the two places so far apart. Finally we may suggest, that transport by wind does not primarily take place through the air direct, but chiefly in stages, in such a way, that the fruits remain for a longer or shorter time on the ground, or floating on the surface of water; in the latter case of course, till they are washed ashore and are then carried further by the wind.

The original occurrence across the sea of this *Acer* growing wild in Java and provided with fruits, which are apparently only adapted for wind transport, may, it seems to me, be readily explained by a combination of wind transport in stages with transport by water, but not exclusively by so called direct wind transport. This explanation possibly also applies to other species, growing in Java and belonging to other genera or orders, with physiologically similar fruits or seeds, which have hitherto only been regarded as anemophilous.

It should further be noted, that this species only bears fruit at an advanced age, when the crown has already attained a considerable height. This character is perhaps useful, since the tree generally occurs scattered in dense ever-green heterogeneous mountain forests, composed of high trees. For this species, which is obviously in the main dependent on wind distribution, the above-mentioned character is probably connected with the oecological conditions determining the original occurrence of the tree, and the character referred to, has arisen through natural selection. For in the damp Javanese mountain forests, which are generally very dense, only those species have a good chance of being disseminated by the wind, whose fruiting branches protrude above the dense leaf covering, formed by the crowns of the surrounding trees.

In connexion with the obvious relation between the conditions of growth, the fruiting period and the means of dissemination of *Acer niveum*, we may quote what has been said by VOGLER [1]) on a similar

[1]) VOGLER, P., Ueber die Verbreitungsmittel der Schweizerischen Alpenflanzen in Flora oder allg. botan. Zeitung 89 (1901) p. 2.

relation in the case of other species likewise having large winged fruits, such as those of *Acer*.

"... Derartige Arbeiten erhielten einen viel grösseren Werth, wenn sie einem Zusammenhang oder auch nur Paralellismus zwischen den ähnlichen Verbreitungsmitteln und anderen durchgehenden biologischen Verhaltnissen der betreffenden Arten nachgingen. Eine ganz kleine Untersuchung dieser Art bietet LUBBOCK [1]) in dem er nachweist, dass von 30 Gattungen, ""figured as having seeds or fruits with a long wing, known as a *Samara*""", alle zu den Bäumen oder Klettersträuchern gehören, keine einzige zu den niedrigen Kräutern" (VOGLER l.c.).

§ 4. On the geographical distribution, oecological conditions and means of dissemination of the Sapindaceae, growing wild in the highest mountain regions of Java.

The *Sapindaceae,* as defined by RADLKOFER in ENGLER and PRANTL's, Natürliche Pflanzenfamilien, consist of about 73 genera with over 600 species. Of these only a single species occurs in Java, growing wild in the highest mountain regions, namely *Dodonaea viscosa* (LINN) JACQ.

§§ 1. Synonyms.

Dodonaea viscosa (LINN.) JACQ. Enum. Pl. Carib. 19, *non* SIEBER, *non* Mart.; HIERN. in HOOK. Fl. Br. Ind. I, 697; KURZ For. Flora I, 287; BRANDIS Indian trees (1906) 186; HASSK. Pl. Jav. var. 292; KOORD. en VALETON Bijdr. Booms. Java IX (1903) 226; — *D. angustifolia* BLANCO Fl. Filip. ed. I, 312; — *D. angustifolia* LINN. F Suppl. 218; — *D. Burmanniana* DC. Prod. I, 616; — *D. Candollei* BL.! msc. in Herb. Lugd. Bat. = *D. Candoleana* BLUME! Rumphia III, 190; — *D. dioica* ROXB. Hort. Beng. (28); Fl. Ind. II, 256; — *D. Dombeyana* BL.! in Rumphia III, 189; — *D. ferrea* JUNGH.! msc. forma 1, 2 et 3 in Herb. Lugd. Bat.; — *D. jamaicensis* DC. Rod. I, 616; — *D. Kingii* G. DON, Syst. I, 674; — *D. latifolia* SALIB. Prod. 276; — *D. microcarpa* DC. Prod. I, 617; — *D. montana et littoralis* JUNGH. in Java I, ed. II 267; — *D. neriifolia* A. CUNN. ex A. GRAY Bot. U. St. Expl. Exped. I. 262; — *D. oblongifolia* LINK. Enum. Hort. Berol. I, 381; et in Bot. Reg. t. 1051; — *D. ovata* DUM.-COURS. Bot. Cult. ed. II, 7, p. 327; — *D. pallida* MIQ.! Anal. Bot. Ind. III, 7; — *D. pen-*

[1]) LUBBOCK, Flowers, fruits and leaves London (1886) p. 79 (quoted by VOGLER l.c.).

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tandra Griff. Notul. IV, 548; – *D. salicifolia* Dc. Prod. I, 617; — *D. Schiedeana* Schlecht. in Linnaea XVIII (1844) 33 (err. typ. 49); — *D. senegalensis* Blume ! nosc. in Herb. Lugd. Bat.; — *D. spatulata* Sm. in Rees Cycl. XII n. z.; — *D. triquetra* Jungh. in Natuurk. en Geneesk. Arch. Neêrl. Indië II (1845) 36; *non* Andr.; — *D. viscosa* Royen ex Blume !, Rumphia III, 191, – *D. Wightiana* Blume in Rumphia III, 189; — *D. Waitziana* Blume ! l. i.; — *D. Zollingeri* Turcz in Bull. Soc. Nat. Mosc. XXXVI (1863) I, p. 587; — *Caryophyllanthes littoreus* Rumphius Herb. Amb. IV, t. 50; — *Ptelea viscosa* Linn. Spec. ed. I, 108.

For the very numerous synonyms of this polymorphic species, which has extremely wide vertical and horizontal distribution, I have chiefly relied on the most recent literature as regards these species, which occur outside the Dutch East Indies, but have checked them as far as possible by the very rich material in the National Herbaria at Leiden and at Utrecht. The Dutch East-Indian synonyms are chiefly based on my own examination of the above collections, and on Koorders and Valeton Bijdr. Booms. IX l. c. From various facts it appears that this tree (at least the littoral form) was already known to Rumphius, and that it has been described as separate species by a large number of authors under more than 25 different specific names.

According to an unpublished note of Reinwardt, found by me with a herbarium specimen collected on the sandy beach of Ternate, this observer has the credit of having already realized, that the coast and the mountain forms of the specimens of *Dodonaea viscosa* from Malay Archipelago belong to one and the same species.

§§ 2. Geographical distribution and oecological conditions of Dodonaea viscosa outside Java.

According to the literature (e. g. Radlkofer) and the herbaria consulted by me at Leiden and at Utrecht, *Dodonaea viscosa* is generally distributed in tropical and subtropical regions of the whole world, and is known outside Java from sandy sea shores as well as from inland localities up to an altitude of 1400 meters. Brandis [Indian Trees (1906) 187] states: "Trans Indus, Afghanistan and Beluchistan. Common locally, often covering extensive tracts in the drier regions of North-West and Central India as well in the Deccan. Also on the seacoast" (Brandis l. c.). In the National Herbarium at Leiden I saw an authentic herbarium specimen of *Dodonaea arabica*

Hochst and Steud. According to the attached label, this specimen was collected on Dec. 8th, 1835 by W. Schimper (the father of the phytogeographer F. W. Schimper) at 4000 feet (1330 meters) above sea level on the summit of the mountain Kara in Hedschas (Arabia). According to Hooker Flora Brit. India l.c. this specimen is identical with the widely distributed *Dodonaea viscosa* (L.) Jacq. Hooker's view is undoubtedly correct. It seems to me that the occurrence of the littoral *D. viscosa* (L.) Jacq. on the above-mentioned mountain can easily be explained, by assuming that the locality, where Schimper collected his *Dodonaea*, was extremely poor in water. In 's Rijks Herbarium at Leiden I also saw a specimen of *Dodonaea viscosa* L. (det. P. Hennings) from Herb. Schlagintweit N°. 80846, which was collected in the Panjab in North-West India between November 15th to 28th, 1855 at 650—850 meters above sea-level, and finally a specimen from Herb. Fiebrig N°. 2501, correctly named *Dodonaea viscosa*, which was collected in 1903—1904 in Eastern Bolivia (South America) at a height of 1400 meters. As proved by a herbarium specimen from British India, due to Hooker and Thomson, and seen by me in Rijks Herb. at Leiden, *Dodonaea Burmanniana* D. C. which is synonymous with *D. viscosa*, grows there at a height of 0—600 meters above the sea. In 's Rijks Herbarium at Leiden I further saw a herbarium specimen, which according to the label, had been collected in 1841 by Forsten "on extensive beds of lava" in Ternate (Spice Islands); this specimen had been determined by Blume as *Dodonaea Candollei* Bl. var. *minor* Blume. In my opinion there is no doubt, that this is merely a form (from an arid locality) of the ordinary *Dodonaea viscosa* (L.) Jacq.

§§ 3. Geographical distribution and oecological conditions of Dodonaea viscosa in Java.

The following data regarding the vertical and horizontal distribution, and the oecological condition, of *Dodonaea viscosa* (Linn.) Jacq. which, in part have already been published in Koorders and Valeton l. c., can now be communicated; they are based on observations made by myself in Java 1885—1906, and on herbarium specimens collected by me.

In Western and Central Java, as well as in Eastern Java on sandy sea-shores, further in Central and in Eastern Java at 1450 m. above sea-level and higher, especially above 1800 m. and still at 2600 m. According to Herb. Kds. it has been collected in Java in the following localities: In Western Java: near Tjemara in S. W.

Banten, growing on the flat sandy beach. In the Southern Preanger near Palaboehanratoe, also on the sandy beach. In Central Java: on the G. Prahoe at 2000 m. on the Prahoe-Diëng mountains along the path from Soerdjâ to the Diëng plateau in the res. Tegal-Pekalongan. Near Sepakoeng (res. Semarang) on the G. Telemâjâ at about 1700 m. and also in the res. Semarang an the G. Merbaboeh above Andongtjemoro at about 1600 m. In the res. Kedoe on the G. Sendârâ near Kledoeng at about 1600 m. In the res. Madioen on the G, Wilis above Ngĕbĕl at 1450 m. and higher up the mountain to 2000 m. In the res. Pasoeroehan-Probolinggo on the G. Ardjoenâ above Malang at about 2100 m. and on the Tĕnggĕr mountains above Tosari and Ngadisari still at 2600 m. above sea-level. In the res. Besoeki on the Idjen plateau near the bivouac Oengoep-oengoep at 1700 m. and on the Kendèng ridge above Pantjoer at 1700 m.; also on the sandy beach of Gradjagan and on the sandy beach of Poeger (on the South coast of the divisions Banjoewangi and Djember respectively). Completely absent from the regions between the above alpine stations and those in the beach. On the other hand where this *Dodonaea* (*D. viscosa*) appears, it generally either grows socially forming smaller or larger woods, or it occurs at least in very large numbers. — Oecological conditions. It is completely restricted (at least when growing wild) to physiologically dry localities, namely either to the dry alpine regions of Central and Eastern Java above 1400 m. or on to the sea-beach, which is physiologically dry in consequence of its richness in salts. On the beach this species has been observed by me in W., as well as in Eastern Java. (Compare also under "Means of dissemination", and further K. & V. l. c. 229.

§§ 4. Means of dissemination of Dodonaea viscosa.

The inflated, thin-walled, light, winged fruits are not only eminently adapted for wind distribution, but (as has already been mentioned by some authors, and has been confirmed by me experimentally), they are also extremely well suited for transport by water. Of some fruits, which I placed in a $3\frac{1}{2}\,\%$ solution of common salt, $80\,\%$ still floated after 25 days.

In Java the plant bears a large number of fruits at an early age, e. g. before it is 2 years old.

As I have observed in Central Java, this species occurs wild on two volcanoes which are more than 40 kilometres apart, in a straight line, and on these only above an altitude of 1400 m., whereas it is completely wanting in the intervening plain, except on the sea-beach, 30 kilometres off. Since moreover no argument has been advanced in favour of dissemination by animals, it would appear, *that the winds of Central Java are capable of transporting the fruits of Dodonoea viscosa* over a distance of more than 30 kilometres although these fruits weigh 0,040 grams, and have a surface of 2½ square centimeters.

There is, however, scarcely need, to point out here, that great care is necessary [1]) in drawing conclusions as to transport by wind. I only refer to what has been said above, regarding the wind distribution of *Acer niveum.* Notwithstanding the apparent possibility of a direct transport by wind over large distances, I consider that also in the case of *Dodonaea* windtransport in stages is much more probable.

Its general occurrence on the tropical shores of the whole world is sufficient evidence of the extreme suitability for transport by water over very great distances, so that no more need be said on this point.

The extraordinary power of resistance, which I have repeatedly observed, against drought of the air and of the soil, against direct sunlight, against the saline contents of the soil and also against strong winds, together with the property of bearing numerous fruits at an early age, which fruits are well adapted to transport by wind and by water (also by sea water) — all these characteristics fully explain, why this tree appears in Java, as the pioneer of new vegetation not only in alpine regions, but also on sandy sea beaches.

According to what has been said above, the almost complete absence of the species from the broad belt between the beach und the mountains, is probably due, to the crowding out by other plants of such seedlings as may arise from fruits, which doubtless frequently fall in the intervening zone.

Summarising, it appears to me, that the apparently whimsical distribution of this characteristic Javanese Sapindacea can be readily deduced, with a large degree of probability, from the properties mentioned above, and especially from those properties, which are connected with the edaphic condition of the species.

§ 5. Note on some incompletely known species of Quercus, in 's Rijks Herbarium, at Leiden.

In KOORD. and VALETON Bijdr. Booms. Java X, 65 there are mentioned at the end of the description of 25 species af Quercus, growing

1) Compare also VOGLER in SCHROETER l c. 740.

wild in Java, five further species as "doubtful and incompletely known"; the latter were included on the authority of BLUME Mus. Lugd. Bat. I, 294—304; we were unable at the time at Buitenzorg to refer to the authentic specimens of these.

As I have now been able to examine the authentic specimens of BLUME in 's Rijks Herbarium at Leiden, I append my observations regarding these species.

1. *Quercus Pinanga* BLUME Mus. Lugd. Bat. I (1850) 303.

I completely agree with the view of KING, quoted in KOORD. and VALETON l. c. 65. The remark, published by BLUME l. c., that the above-mentioned species occurs "in Java in the mountain forest" must therefore be regarded as not wholly accurate because BLUME evidently prepared his diagnosis from a few leaves of *Quercus glabra* THUNB. (from Japan) an old tree of which was observed by me in a cultivated state in Hort. Bogor. as late as 1903.

Q. Pinanga BLUME should therefore be erased from the Flora of Java and be considered synonymous with *Q. glabra* THUNB.

2. *Quercus litoralis* BLUME l. c. 303.

On the authentic herbarium label there was written i. a.: "*Quercus litteralis* Bl., Java, leg. BLUME, Pasang-laut (Sund)".

Since the native name is Sundanese, this species cannot come from Eastern Java, as BLUME l.c. incorrectly remarks, but must come from Western Java, probably from the Preanger or Banten, where most of the specimens, collected by BLUME, were obtained.

The authentic specimen I regard as beyond doubt synonymous with *Quercus spicata* SM. var. *gracilipes* KING (comp. KOORD. and VALETON l c. 42). This species of BLUME's must also therefore be deleted.

3. *Quercus glutinosa* BLUME l.c. 304.

According to the authentic herbarium label of REINWARDT this species was named by REINWARDT in manuscript *Quercus micans* REINW., and was afterwards renamed by BLUME *Quercus glutinosa* BL. moreover, it was not collected "in the mountain forests of Western Java" but found by REINWARDT near Tondano in N.E. Celebes, in the year 1821. This species can therefore also be deleted from the flora of Java. It is not, as MIQUEL incorrectly thought, identical with *Quercus induta* BL., to which it shows a superficial resemblance; the species is specifically distinct from *Q. induta* BL., as was indeed already correctly surmised by DE CANDOLLE and by KING (comp. KOORD. and VALETON l.c. 65).

4. *Quercus sphacelata* BLUME l.c. 304.

The authentic specimen of this species consists of a branch with leaves, but without flowers. I consider it a large leaved shoot (for instance, from a latent bud of the trunk) of *Quercus spicata* SM. var. *gracilipes* KING.

On the authentic label is written: "*Quercus sphacelata* BL., Pasang, Java, in montanis Moeriah, Herb. Waitz."

5. *Quercus nitida* BL. l.c. 294.

The view, already expressed in KOORD. and VALETON l.c. 65, that this species, which so far has only been recorded with certainty from Sumatra, does not yet belong to the flora of Java, is confirmed in my opinion, by the material in 's Rijks Herbarium at Leiden.

Leiden, March 1908. (*To be continued*).

Geophysics. — "*The Starting Impulse of Magnetic Disturbances.*" By Dr. W. VAN BEMMELEN.

(Communicated in the Meeting of March 28, 1908).

Last year [1]) I communicated the compilation of a statistical list of the magnetic disturbances which the magnetograph at Batavia has recorded during the period 1880—1899. I drew the attention to the phenomenon of the starting impulse i. e the suddenly appearing change of the magnetic elements, which very often accompanies the beginning of a magnetic storm.

This phenomenon appearing in like manner at Batavia and at other places, I ventured a supposition on the manner in which we can represent to ourselves the appearance of magnetic disturbances. To obtain a closer knowledge of this in my opinion very instructive phenomenon, I requested at the end of 1906 all Magnetic Observatories to give me their data for a number of cases selected by myself.

With great readiness those data were forwarded to me from several observatories and it is an agreeable duty for me to express at this place my thanks for it.

Besides this material received from many sides I have worked out all cases registered at Batavia and at Buitenzorg and have also been able to watch the nature of the electric earth-current during the phenomenon. I wish to communicate here of the results of this material what is most important, commencing with Batavia.

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[1]) Proceedings 29 September 1906.
Also: Observations made at the R. Magn. and Met. Observatory of Batavia, Vol. XXVIII, App. III.

Batavia

Out of the diagrams obtained in the period 1882—1899, I measured for 131 cases the amount and the duration of the initial movements of the three components (Horizontal Intensity = H, Declination = D, Vertical Intensity = Z.

Direction of the initial movement.

$\Delta\, H$ was without exception positive.

$\Delta\, D$ was with a few exceptions West; but 12 % of the number of cases was introduced by a slight Easterly movement.

$\Delta\, Z$ was negative; but in 6 % of the number the movement was introduced by a slight positive movement.

Duration.

Here I have *not* taken into consideration the duration of the slight introductory movement.

124 cases furnished:

$$\Delta\, H = 4.5 \text{ min.} \quad \Delta\, D = 3.2 \text{ min.} \quad \Delta\, Z = 12.0 \text{ min.}$$

The duration of the Z movement is in general difficult to determine, as the decrease of the vertical force keeps on mostly much longer.

It is important to notice that the initial movement of D stops or is inverted, whilst of H the increasing movement keeps on.

Amount.

The average amplitude of the movement, arranged according to the different parts of the day in which it took place, expressed in 0.00001 C. G. S. (= γ):

h	$\Delta\, H$	$\Delta\, D$	$\Delta\, Z$
0—6 a.m.	+ 45	7 W	— 11
6—12 „	+ 41	10 „	— 16
0—6 p.m.	+ 52	7 „	— 16
6—12 „	+ 40	8 „	— 11

Of a characteristic inequality of the vector during the day little is noticeable. The amount $\Delta\, H$ arranged according to the duration of the movement is:

Duration	$\Delta\, H$	Number of cases
0— 2 min.	53 γ	15
2— 4 „	43	45
4— 6 „	42	35
6— 8 „	33	20
8- 15 „	46	6

So the amount of the movement is fairly well independent of the duration, from which results inversely that *the shorter the increase of H is, the quicker it is.*

Let us finally observe that the appearing of the slight easterly movement did not show any preference for certain times of day or year.

BUITENZORG.

Since May 1906 a TÖPFER-SCHULZE magnetograph has registered at Buitenzorg, which gives the curves of the three elements on the same registering-strip; this circumstance besides that of giving finer lines, greater sensibility and wider measure of time, is very suitable for the study of the initial-movement.

For the period May 1906 — Nov. 1907 I measured 29 cases and from that material it was clearly evident that in most cases the movements of the elements display a certain independence of each other and *do not always begin at the same moment.*

I calculated the azimuth of the horizontal component of the vector for the first part of the movement, so *before* the movement of D is inverted, and I found in 20 cases directions between the extremes N and N 58° W, an average of

$$N\ 21°\ W.$$

Vertical Intensity.

The results for ΔZ were surprising; for the vertical component showed, different from that of Batavia, an introductory positive movement followed by the slow negative movement known of Batavia.

Here no instrumental cause had anything to do with the matter: both magnetographs (of ADIE and SCHULZE) registered this premovement at one time at Batavia very rarely, but at Buitenzorg regularly. Luckily registrations have been made for more than a year at Batavia and at Buitenzorg at the same time and from those registrations it was evident that the introductory movement at Buitenzorg precedes that of Batavia. The introductory movement at Buitenzorg commences (according to the average of 29 cases) 0.3 minute after the H-movement begins and lasts about 1 to 3 minutes, after which the Z-lines of both places show simultaneously a decrease.

ANSWERS TO THE QUESTIONS.

The data on the initial movement were asked for in my letter for a number of cases, in which that movement had made its appearance under different circumstances at Batavia. The answers were of a very different nature and therefore I made of each case a summary diagram in which in an equal manner under each other was noted down the registration image of H, D and Z for each station. In many respects it would be useful to reproduce these diagrams, but the difficulties connected with it and the numerous imperfections of the material have made me set it aside.

These imperfections are chiefly caused by the measure of time of the diagrams of the various observatories not being taken ample enough to be able to fix the simultaneity of the different movements, which take place within a few minutes.

From the notation of the TOPFER-SCHULZE magnetograph at Buitenzorg where the circumstances were pretty favourable I could deduce with certainty that the commencement of the movement of the three components is often not simultaneous. Accuracy down to parts of minutes cannot be demanded of most magnetic diagrams, where 1 hour takes up 10, 15 or 20 mm. It was therefore impossible to draw trustworthy maps, on which the vector of disturbance during the initial movement was represented in its varying magnitude and direction, so I must restrict myself to the following.

In the following table the direction of the movement has been given for all the cases.

In the case of an introductory movement of slight amplitude this is indicated for H by $+/-$ or $-/+$, for D by w/E or e/W.

In some cases the introductory movement was of the same order of amplitude with the following movement and this is accordingly indicated by $\mathbf{+/-}$ or $\mathbf{-/+}$ and W/E or E/W.

We see in this table the movements of *one* or more elements for different constant direction. Further information is furnished by the annotation of the Rev. P. DE MOIDREY in the "Bulletin des Observations de l' Observatoire de Zie-Ka-Wei, T. XXXI" and the copies of disturbances for Greenwich, Parc st. Maur and Samoa. The former states for Zi-Ka-Wei the frequency of the positive H and Z movement at 95 pCt. and of the E movement of D at 90 pCt.; whilst out of the Samoa curves the H and Z movement proves to be $+$ in 9 cases.

Hence the following movements appear pretty constant:

Station	H	D	Z	Geogr. Latitude
Batavia	+	W	—	—6°
Buitenzorg	+	W	+/—	—7°
Manila	+		+	15°
Zi-Ka-Wei	+	E	—	31°
Samoa	+		+	—13°
Honolulu	+		no reg.	20°
San Antonio			+	29°
Coïmbra	+		no reg.	40°
Greenwich			+	51
St. Maur			—	49
Perpignan	+			43°
Bombay	+	no reg.	no reg.	19°
Mauritius	+		+	—20°
Melbourne			—	—38°

So it seems that the constancy in the appearance of a definite sense of movement decreases with the geographical latitude and that it is furthermore for H the greatest and for D the smallest, moreover for H always with a *positive* sense of movement.

EXTENSION OF THE MOVEMENT ABOUT THE EARTH.

The nature of the initial movements, which took place simultaneously on different points of the surface of the earth could be studied by means of the above mentioned summary diagrams.

It was quite evident that for H and D but not for Z places lying close to each other show about the same image, but that for places in other parts of the world this is often quite different.

The small number of stations only allowed a closer investigation of that difference for North-America and Europe. I therefore give below a survey of the movement for H and D in Europe and North-America, where however the latter for the years 1892—94 is represented only by two stations (Toronto in Canada and San Antonio in Texas).

Date			Europe		N.-America	
			ΔH	ΔD	ΔH	ΔD
May	18th	1892	+/—	*w/E*	—/+	*e/W*
July	16th	,,	—	*E* and *W*	are missing	
Aug.	12th	,,	—	*E* and *W*	+	*E* and *W*
,,	18th	1893	+	*e/W*	are missing	
Sept.	25th	,,	+	*W*	,,	,,
Jan.	2nd	1894	+	*w/E*	—/+	*E/W* (only San Antonio)
Febr.	20th	,,	—/+	*w/E*	—/+	*w/E*
July	20th	,,	+	*e/W*	+	*e/W*
Aug.	20th	,,	+	*e/W*	+	*w/E*
Nov.	13th	,,	—/+	*e/W*	+/—	*w/E*
Aug.	16th	1902	+	*e/W*	+	*w/E*
April	5th	1903	+	*e/W*	—/+	*w/E*
Febr.	3rd	1905	+	*e/W*	+	*w/E*

We see here repeatedly that the European group and the American one are the reverse of each other, and in Europe we find mostly a *N.W.* vector, in America a *N.E.* one, which points to a centre of disturbance near Greenland, thus situated near the magnetic pole and that of the Aurora-Borealis or pole of disturbance.

For some cases I have tried to obtain a survey by means of a map with the simultaneous vectors of disturbance, but here the almost insurmountable difficulty presents itself, that one cannot make up simultaneous values of ΔH and ΔD, which are really trustworthy. Above I pointed to the fact with reference to the measurements on the Buitenzorg magnetograms, that this requires an ample and trustworthy time-measurement. Though it was impossible for me to calculate exact values for the azimuth of those vectors, yet I could about fairly well find the direction.

For the initial movement of the disturbance on Febr. 3d, 1905 the vectors of the introductory movement pointed to a centre at the West coast of North America and that of the main movement to a centre near Greenland. For the disturbances on Aug. 16th, 1902 and April 5th, 1903 I found about the same [1]).

VERTICAL FORCE.

It is a striking fact, that places lying at a slight distance from each other show an opposite change in the vertical component. Above

[1]) Writing this I see from a paper by Dr. BRÜCKMANN (Meteor. Zeitschrift, 1907, No. 12) on the same subject, that he also arrives at a centre of disturbance appearing in the vicinity of the magnetic pole.

I already mentioned that Batavia and Buitenzorg are different in this respect and now I discovered that Greenwich and Paris (Parc St. Maur) show regularly an opposite movement. Out of the reproductions of diagrams of disturbances published for several years I made the following list.

St. Maur and Greenwich.

		St. Maur			Greenwich		
Date	Hour	ΔH	ΔD	ΔZ	ΔH	ΔD	ΔZ
1891 March 2	2	+	*e/W*	−	+	*e/W*	+
June 14	9	+	*E*	−	+/−	*e/W*	+/−
1892 January 4	19	+	*E*	−	+	*w/E*	-
February 13	5	+	*E W*	−	−/+	*E/W*	+
„ 20	19	+	*E/W*	−	+	*W*	+
March 11	23	+	*E/W*	−	+	*W*	+
May 16	22	+	*E*	−	+	*E*	+
„ 18	8	+	*w/E*	−	+/−	*w/E*	+/−
June 27	5	+	*E*	−	+	*W*	+
July 12	18	+	*W*	−	+	*W*	+
July 16	13	+	*e/W*	+/−	+	*E/W*	−/+
Aug. 3	14	+	*W*	−	+	*W*	+
Sept. 5	23	−/+	*E/W*	−	−/+	*E/W*	+
1893 March 25	4	+	*E*	−	+	*E*	+
April 26	16	−/+	*W*	−	−/+	*e/W*	−/+
June 9	13	+	*W*	−	−/+	*e/W*	−/+
August 6	4	+	*e/W*	−	+	*e/W*	+
September 8	1	+	*W*	−	−/+	*w/E*	+
1894 January 11	20	+	*W*	−	+	*W*	+/−
February 20	20	+	*E*	−	−/+	*W/E*	+
„ 22	22	+	*W*	+/−	+	*W*	+
„ 28	15	+	*W*	−	/+	*e/W*	−/+
July 20	6	+	*E*	−	+	*E*	+
1895 May 29	15	−/+	*e/W*	−	−/+	*e/W*	−/+
1896 July 23	18	−/+	*W*	−	−/+	*e/W*	+
Aug. 29	17	−/+	*W*	−	−/+	*e/W*	−/+
1898 March 15	1	+	*W*	−	+	*W*	+
„ „	13	+	*W*	−	/+	*W*	+
September 9	14	+	*W*	−	+	*W*	+ [1]
1899 January 28	19	+	*W*	−	+	*W*	+
June 28	22	+	*W*	−	+	*W*	+ [1]

[1]) During a disturbance.

It is seen that ΔH and ΔD generally correspond; only for H the introductory movement is more frequent at Greenwich; ΔZ is regularly opposite, for Greenwich positive and for Paris negative.

It is remarkable to notice out of the reproduced registering lines how the oscillations following upon the initial movement correspond again for the two places; a single striking quick movement amid the disturbance, as it were a new starting impulse, is then again opposite. This repetition seems to be a real phenomenon. Thus the initial shock on Oct. 30th, 1903 was a clear initial movement amid a disturbance going on already for hours. At places with higher latitude it lost itself in the oscillations of that older disturbance. The phenomenal violence of the second part of the disturbance is perhaps owing to two disturbances being placed one above the other. At the violent disturbances of Febr. 13th, March 6th and June 27, 1892, as well as of Aug. 6th, 1893 two initial impulses appeared.

The causes of the initial movement.

The remarkable inequality of the movement in the vertical force, so constant for places situated close to each other, offers us perhaps a means to clear up what is puzzling in this phenomenon.

If we attribute the appearance of those vectors of disturbance to that of electric currents, as is more than probable, then it is impossible to assume that the movement of electricity which generates these vectors would have in the free atmosphere such a distribution limited to the place. The cause of this must be in the appearance of the electric earth-current. We must assume that, when suddenly a disturbance arises, the earth-current then generated selects fixed paths through the earth-crust.

That the earth-current for different places of the earth situated close to each other may be different, is highly probable; at least for the surface-current I have found it lately[1]) for North- and South-Java. The inequality was, that as the corresponding magnetic variations became shorter the earth-current variations increased more in amplitude for the volcanic southern part than for the alluvial and diluvial northern part.

This great difference in earth-current must become much less for the deeper strata; proof of this is found in the equality of the magnetic variations at Batavia situated on the Northcoast of Java and at Buitenzorg on the edge of the volcanic part.

But the possibility for a difference when a current is suddenly

1) See the following paper: Earth-current registration at Batavia, 2nd communication.

SUMMARY TABLE OF THE DIRECTION OF THE INITIAL MOVEMENT.

		Batavia			Manila			Zi-Ka-Wei			Tokio			Samoa			Honolulu		
		H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z
25 IV	1892	+	0	-	+	0	+	+	W										
18 V	"	+	e|W	+|-	+	0	+	+	W										
12 VII	"	+	W	-				+	0	-									
16 "	"	+	W	-				+	E	-									
12 VIII	"	+	W	-				+	E	-									
18 "	1893	+	W	-	0	0	+	+	E	-									
25 IX	"	+	W	-	+	0	0	+	0	0									
2 I	1894	+	E	-	+	W	+	+	E	-									
20 II	"	+	W	-	+	W	+	+	E|W	-									
20 VII	"	+	W	-	+	0	+	+	E	-									
20 VIII	"	+	W	-				+	E	-									
13 XI	"	+	W	-	+	0	+	+	E	-									
16 VIII	1902	Buitenzorg +	W	-				+	E		+	E	+				+	W	
26 I	1903	+	0	+|-				+	E		+	E	+				+		
5 IV	"	+	e|W	+							+	E	0				+	E	
3 II	1905	+	e|W	+				+	E	-	+	E	+	+	E	+	+	E	+
2 III	"	+	W	+				+	E	-	+	E	+	+	W	+	+	W	+

		Sitka			Baldwin			Toronto			Cheltenham			Vieques			San Antonio		
		H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z
25 IV	1892																		
18 V	"							-|+	E|W								+	W	+
12 VII	"																	w|E	
16 "	"																		
12 VIII	"							+|-	e|W								+	e|W	
18 "	1893																		
25 IX	"																		
2 I	1894																-|+	E	-|+
20 II	"							-|+	w|E								+	E	+
20 VII	"							-|+	e|W								+	W	+
20 VIII	"							+	w|E								+	w|E	+
13 XI	"							+|-	W								+|-	w|E	+
16 VIII	1902	-|+	e|W		+	e|W		+	e|W		+	e|W							
26 I	1903							-|+	0		+	W	0						
5 IV	"	-|+	E|W		-|+	W|E		-|+	w|E		-|+	w|E	+	+	W|E				
3 II	1905	-|+	e|W	-|+	-|+	w|E	-|+	+	W|E		+	W|E	+	+	w|E				
2 III	"				+	e|W	+|-	-	W		-	e|W	+|-	-	E				

		Coimbra			Ebro			Stoneyhurst			Kew			Perpignan			St. Maur		
		H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z
25 IV	1892	+	E					-|+	E					+	E	-			
18 V	"	+	e|W								+|-	w|E	+|-	+	E	-	+|-	w|E	-|+
12 VII	"	+	E|W					+	W		-|+	e|W	+	+	E	+	+	E	-
16 "	"										+	E|W	-|+	+		-|+	-|+	W|E	+|-
12 VIII	"							-|+	E|W		-|+	W|E	-|+	+	W|E	+|-	-|+	e|W	+|-
18 "	1893	+	W								+	e|W	+	+	W	+	+	e|W	-
25 IX	"	+	W					+	E|W		+	W	+	+	E	+	+	W	-
2 I	1894	+	w|E					+	E		+	w|E	+	+	E	+	+	w|E	-
20 II	"		e|W					+	E		+	w|E	+	+	E	+	+	E	-
20 VII	"	+	e|W					+|-	e|W		+|-	e|W	+|-	+|-	e|W	-|+	+|-	e|W	-|+
20 VIII	"	+	e|W					+	W		+	e|W	+	+	W	+	+	e|W	-
13 XI	"	+	e|W					+	W		-|+	e|W	-|+	+	W	+	+	e|W	-
16 VIII	1902	+	E								+	w|E		+			+	w|E	
26 I	1903	+	0																
5 IV	"	+	E|W					+	e|W		+	e|W	-|+				+	e|W	-
3 II	1905	+	e|W		+	e|W		+	e|W		+	e|W	+				+	e|W	+
2 III	"	-|+	e|W		-|+	e|W	+|-	-|+	W|E		-|+	w|E	+|-				-|+	E|W	+|-

		Uccle			Utrecht de Bilt			Copenhagen			Potsdam			Munich			Pola		
		H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z
25 IV	1892							-|+	e|W	-									
18 V	"	+|-	w|E	+|-	+			+|-	w|E	+|-									
12 VII	"	+		+	+	W	+	+	E	+									
16 "	"	-|+		-|+				-|+	E|W	-|+									
12 VIII	"	-|+	w|E	-|+				-|+	e|W	-|+									
18 "	1893	+	e|W		-|+	e|W													
25 IX	"	+	W		+	W	+	+	W	+									
2 I	1894	+	w|E		+	W|E		+	E	+									
20 II	"	+	w|E		-|+	W|E	-|+	-|+	w|E	-|+									
20 VII	"	+|-	e|W	+	+	e|W	+|-	+|-	e|W	+|-									
20 VIII	"	+	W	+	+	e|W	-|+	+|-	e|W	+|-									
13 XI	"	-|+	e|W		-|+	e|W	-|+												
16 VIII	1902	+		0	-|+	w|E	-|+							+	E		-|+	E	+
26 I	1903	-	W	+	-	W								+	0				
5 IV	"	+	W	+	-|+	e|W	+							+	e|W		+	e|W	+|-
3 II	1905	+	e|W	+	-|+	e|W					+	e|W	-	+	e|W		+	e|W	-
2 III	"	-|+		-|+	-|+	e|W					-|+	W	-	-|+	E|W		-|+	W	-

		Bochum			Jekaterinburg			Bombay			Dehra Dun			Barrackpore			Mauritius			Melbourne			Victorialand		
		H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z	H	D	Z
25 IV	1892							+																	
18 V	"							+									+	W	+	-|+	E|W	+|-			
12 VII	"							+									+	W	+	+	E	-			
16 "	"							+									+	W	+	+	w|E	-			
12 VIII	"							+									+	W	+	-|+	W|E	+|-			
18 "	1893							+												+|-	e|W	-			
25 IX	"							+									+			+	E	-			
2 I	1894							+									+	W	+						
20 II	"							+									+		+	+	w|E	-			
20 VII	"							+									+	e|W	-	+	e|W	-			
20 VIII	"							+									+	E	+	-|+	E|W	+			
13 XI	"							+									+		+	+	W	-			
16 VIII	1902		E					+				W					+	W	+	+|-	E	-	+	w|E	
26 I	1903		0					+			+														
5 IV	"		e|W					+			+	W					+	w|E	+	+|-	E	-	-|+	e|W	
3 II	1905		e|W		+	W	0	+			+	W		+	W		+	0	+	-|+	E|W	-			
2 III	"		W		+	E	+	+			+	E		+	E					+	e|W	-			

generated is made possible by these results for deeper strata too. On the diagrams obtained at Batavia when the earth-current is registered, there are a few cases of an initial impulse and as the time unit was very ample (1 millim. = 1 min.) and moreover as the magnetic component too was registered with great sensitiveness, these cases are very instructive.

Date and Hour.			Initial movement of the *E-W* earth-current commences before that of the magn. North-component.		
14 May 1906,	4^h *a.m.*	Bat. *T*	0.0 min.	Commenc.	gradual
30 July ,,	3 ,,	,,	0.6 ,,	,,	,,
3 September ,,	7 *p.m.*	,,	0.0 ,,	,,	pretty sudden
22 ,, ,,	8 ,,	,,	0.0 ,,	,,	sudden
10 November ,,	12 ,,	,,	0.0 ,,	,,	pretty sudden
26 ,, ,,	1 *am*	,,	0.0 ,,	,,	sudden
26 December ,,	11 *p.m.*	,,	0.0 ,,	,,	,,
8 January 1907	12 ,,	,,	0.0 ,,	,,	,,
15 ,, ,,	3 *am.*	,,	0.0 ,,	,,	pretty sudden
14 February ,,	3 ,,	,,	0.0 ,,	,,	sudden
27 January 1908	9 *p.m.*	,,	0.0 ,,	,,	,,

With the exception of *one* case, where indeed the determination of the time was less accurate on account of the gradual commencement of the initial movement, we thus find simultaneousness for the initial movement for earth-current and magnetic vector.

As has often happened, we must change a hypothesis of explanation formed on first getting acquainted with the facts, when later on we have arrived at a more extensive knowledge of the facts by extension of the material.

This is the case here too.

Though I at first thought to find the seat of the current of electricity which is supposed to generate the initial impulse in the highest layers of the atmosphere, the nature now revealed of the vertical component induces me to look for the seat rather in the earth itself.

At the outset the current must be in general an East-West current of positive or West-East of negative electricity, because everywhere the horizontal magnetic component increases The situation and form of that current seems to be variable and to undergo a great influence of a proper magnetism of the earth; it also seems to change during the increase in intensity and situation. For the magnetic disturbance itself following immediately upon the initial impulse we must assume that especially extra terrestrial currents are the cause; at least for the magnetic after-disturbance as well as for the part that shows a regular daily variation I have made this probable [1]). Moreover the Aurora Borealis points to this. The magnetic vector of after-disturbance is the mean vector of disturbance deprived of its greater and smaller oscillations during the disturbance. It increases rapidly after the initial impulse and then slowly decreases.

As here the horizontal intensity just decreases we must conclude to a likewise W-E. current of negative electricity in these higher atmospheric layers. It remains an open question why the intra-terrestrial current at the outset and the extra-terrestrial current during the further course of the disturbance have both a constant East-West direction.

Geophysics. — *"Registration of the earth current at Batavia."* 2nd part. By Dr. W. v. BEMMELEN.

In my first paper on the registration of electric earth-currents at Batavia, to investigate the connection between the oscillations in earth-current and magnetic force, I had to point to several unanswered questions.

First of all the fact that the earth-current between Anjer and Batavia is four times greater than the one between Batavia and Cheribon. I hope soon to be able to measure the current between Batavia and a place E. and S. of Anjer to try to shed light on this abnormality.

Further more it remained a mystery why that connection with the magnetic force showed such a characteristic difference for the current between Semarang and Batavia with that for the current between Batavia and places closer by. That difference consisted chiefly in the fact, that when the duration of a magnetic oscillation becomes shorter, the amplitude of the earth-current increased much more for the long line than for the short one.

I pointed out, that perhaps an influence of the distance might

[1]) Met. Zeitschrift 1895. p. 321. T. M. VIII p. 153.

have something to do with it and that a registration of the current at Semarang going over the distances Semarang—Cheribon and Semarang—Soerabaya would probably be able to enlighten us in that respect.

This idea I have, indeed, been able to realize by a visit to Semarang in the month of December 1907. At the Post- and Telegraph Office they kindly accommodated me for some days with a room where I could place the instruments used at Batavia. Though I had some delay by a slight accident, yet I could get excellent diagrams during two nights.

The result was definite, viz. *the current between Semarang—Soerabaja and Semarang—Cheribon corresponds in character and intensity to that between Batavia—Cheribon.*

Oscillation of the magnetic North component.		$\frac{\text{Amplitude of the earth current in Volt per K.M.}}{\text{Amplitude magnetic component in dynes.}}$			
		Sem.-Cheribon	Sem.-Soerabaja	Batavia-Cheribon	
Half oscillation	Amplitude			Found formerly.	Number of cases.
		18—19 December 1907.			
0.5 min.	1.2 γ	22 6	19.3	21.2	14
1.0 „	1.5 „	20.9	16.9	20.5	11
13.5 „	3.0 „	16.2	14.4	16.0	8
		19—20 December 1907.			
0.6 min.	0.5 γ	26.3	23.6	23.0	33
1.4 „	1.1 „	23.4	20.0	20.5	16
9.5 „	2.4 „	20.3	16.7	16.5	19

So:

Duration of half an oscillation		$\frac{\text{Earth-current Batavia-Cheribon}}{\text{Earth-current Semarang-Cheribon}}$	$\frac{\text{Earth-current Batavia-Cheribon}}{\text{Earth-current Semarang-Soerabaja}}$
$\frac{18-19}{\text{XII}}$ '07	0.5 min.	0.94	1.10
	1.0 „	0.98	1.21
	13.5 „	0.99	1.11
$\frac{19-20}{\text{XII}}$ 07	0.6 min.	0.87	0.97
	1.4 „	0.88	1.02
	9.5 „	0.81	0.99

53*

The registration of both nights together gives:

Duration of half an oscillation	Amplitude	Sem.-Cher.	Sem.-Soerabaja	Number of cases
0.36 min.	0.6	24.3	21.4	20
0.65 „	0.9	26.3	22.8	20
0.87 „	0 9	26.5	22.7	20
1 08 „	1.2	23.7	19.7	10
3.65 „	1.0	22.9	18.3	10
13 10 „	3.2	22.0	18.2	11

From these numbers we also find the initial increase of the earth-current amplitude for the (half) duration smaller than about 0.7 min. just as it was found for the currents Batavia—Cheribon and Batavia—Anjer.

For the difference in phase was found out of 14 cases, Semarang—Cheribon 17°.5, Semarang—Soerabaja 16°7, whilst for Batavia—Cheribon formerly 22° was found.

So the registration at Semarang furnished a highly important confirmation of the results found for Batavia—Cheribon. And yet no conclusion could as yet be drawn for an influence of the distance on the amplitude.

And indeed, new observations made at Batavia soon offered another view upon the subject. I heard that a connection with Semarang was possible at the same time along lines through the Northcoast plain and along the line already used round the South by the railroad.

Registration with these lines running between the *same* earthplates at Semarang and Batavia (observatory) gave the remarkable result that the current in both lines was unequal.

The Northline corresponded with results found before on the Batavia—Cheribonline, the Southline gave again the heightened increase of the earth-current amplitude when the duration of the oscillation decreased.

Duration of half an oscillation X	$\frac{\text{Amplitude Earth-current in Volt per K.M.}}{\text{Amplitude magnetic component in dynes}}$ Southline	Northline	Number of cases
0 50 min.	62	28	20
0.77	48	29	20
1.53	50	34	20
5.53	24	27	20
8.60	23	27	10
21.77	13	20	1

DIFFERENCE IN PHASE

Duration of half oscillation X	South-line	Number of cases	Duration of half oscillation X	Northline	Number of cases
0.8 min.	31°	39	1.3 min.	16°	68
6.8 „	31	18	6.5 „	18	40
16.4 „	31	20	17.0 „	26	23

So the difference in phase is different for the two lines. Extraordinary is here the increase of the difference in phase for the Northline, which is not found on the other lines.

Formerly was found for the difference in phase on the line Semarang—Batavia 36°. Whether the difference with the difference in phase now found of 31° is real must still be called doubtful.

These new results led to the conclusion that the difference in character found formerly might not be attributed to the greater distance, but to a peculiarity of the line itself. As the two lines round the South and the North were between the same earthplates and possessed about the same resistance, I had to conclude to an appearance of electromotive force in the line itself. There are two possibilities for this:

1st. induction immediately in the wire;

2nd. contact with the ground.

Now it is very well possible that both causes are very different in the diluvial and alluvial plain of the Northcoast and the volcanic Southern regions.

To separate these two causes I have taken the following double experiment.

Batavia and Cheribon are connected by two parallel brass wires of the intercommunal telephone; there are likewise two telegraph wires on the same poles between Batavia and Soerabaja. Such a double line I connected with my galvanometer and switched on between galvanometer and earth a resistance which was great compared to that of the wire. (For Cheribon that resistance was 5000 Ohm, for Soerabaja 40.000 Ohm). I then left both wires connected with the galvanometer for some hours and then broke off the connection with *one* of the wires.

After a few hours I switched this wire on again, but broke the connection with the other one, and then finally I connected both wires again.

If now the earth-current were only a current from groundplate to groundplate then during these changes it might change but slightly in intensity, as the total resistance changed so little.

On the other hand, if the earth-current were for a part not originating from the plate, but was immediately caused by induction or an other influence (e.g. the catching of electrons moving in the atmosphere) then that part when connected with *one* wire would be half of that when connected with two wires and so a considerable difference in intensity of the current would be noticeable.

It might be possible that this influence differed with the duration of the oscillation of the magnetic component and were different in the coastregion from that in higher volcanic regions in South-Java; in this way the difference in character found above might be explained.

Before mentioning the results of this experiments I wish to consider what the influence is of the loss by isolation.

The loss by isolation will chiefly take place along branches and poles accidentally touching the wire. The first influence will be irregular and in general for both lines alike.

With the second each telegraphpole will give an earth connection with great resistance for both wires at the same time, as the wires run across the same yokes.

Along this earth connection a current will run if the earth-potential at that place differs from that in the wire.

That current will then feel little influence of the fact whether *one* or both wires are connected with the galvanometer.

The result of the experiment for the lines to Cheribon as well as for those to Soerabaja was not ambiguous, as the figures below indicate.

Duration of half an oscillation		Amplitude earth-current in m.m. reading.		
		Amplitude X in m.m. reading.		
Both lines	One line	both lines	one line	
1.0 min.	1.2 min	2.5	2.5	Batavia-Cheribon
5.4 „	6.6 „	2.0	2.0	
0.6 min.	0.8 min.	6.1	5.7	Batavia-Soerabaja
8.3 „	19.3 „	1.4	1.5	

This simple experiment is in my opinion of fundamental importance, as it shows *that no electromotoric force is roused in the line itself,* a fact that a priori cannot be called so improbable.

Nothing remained now but to assume that the difference in character of the current in the North-line and in the South-line is caused by the fact that by the loss by isolation the current is partly taken up out of the ground over which the line runs and that that current was different in the Northcoastplain to that in the mountains. If this were so, then the earth-current in the Preanger country, where the Southline runs in a niveau of ± 600 M. between numerous volcanoes would have to show the same peculiarity. To prove this it was fortunately not necessary for me to remove with my instruments, but I could suffice by making the following connection.

Earth at Buitenzorg—Galvanometer—Observatory—Batavia—Buitenzorg—Tasikmalaja—Earth at that point. Buitenzorg is situated at the N.W.-foot of the mountains and Tasikmalaja at the East foot. Both places lie still at a height of ± 300 M. above the sea.

Loss by isolation along the poles on the there-and-back line Buitenzorg—Batavia—Buitenzorg could not bring the earth-current out of the plain between Buitenzorg and Batavia into the line, and could only cause a part of the Preangercurrent to flow away. That loss could thus not falsify the result.

The current between Buitenzorg and Tasikmalaja really proved to possess the above mentioned character, i.e. it showed a much stronger increase when the duration of the oscillation decreased than the

Northcoast lines Duration of half an oscillation	Batavia-Tasikmalaja Ampl. Earth-current in Volt per K.M. Amplitude X in dynes
0.4 min.	60
0.8 „	56
1.2 „	55
9 0 „	15

If loss by isolation is the cause of the inequality of the current between Batavia and Semarang round the North and the South, then that loss will be smaller in the dry season than in the wet one. And, indeed, I found that this was the case as the figures below will indicate.

18—21 Juni 1907 Duration of half an oscillation	18—21 Juni 1907 Ampl. Sem.-Bat. current in Volt. p. K.M. Ampl. X in dynes	December 1907 Duration of half an oscillation	December 1907 Ampl. Sem.-Bat. current in Volt. per K.M. Ampl. X in dynes
0.6 min.	43	0.5 min.	62
1.0 „	35	0.8 „	48
1.4 „	31	1.5 „	50
5.0 „	18	5.5 „	24
11 5 „	13	8.6 „	23
		21.8 „	13

These characteristic differences treated above can perhaps afford an occasion to find an explanation of the nature and the cause of the earth-currents, but more observations under other circumstances will undoubtedly be necessary.

It seemed important to me to investigate whether that great difference in earth-current is always incidental to difference in amplitude of the magnetic variations.

For, Buitenzorg lies on the edge of the volcanic Southernpart of Java, and Batavia lies in the Northcoastplain; moreover simultaneous registrations of the magnetic component are for both places available.

I have used the registration of the X-component at Batavia on the earth-current diagrams and of the TÖPFER-Unifilar of the X-component at Buitenzorg.

In January, February, March, July, August, September the magnetic variation-instrument registered on the earth-current diagrams the magnetic component perpendicular to the direction Batavia—Anjer, i. e. N4°E. The difference in direction with that on TÖPFER's instrument with which the X-component was registered, can be neglected.

On each diagram I compared the amplitude of a variation of short and of long duration, as much as possible at an equal distance from the basis. In this way I was independent of differences in values of the scale division and other differences.

I got as average case in 30 cases in the months of January—March '07 and 24 cases in the months of June—September '07 :

Average Amplitude of the			
Variations of short duration.		Variations of long duration.	
Buitenzorg	Batavia	Buitenzorg	Batavia
1.44 m.m.	13.21 m.m.	2.68 m.m.	21.27 m.m.
1.01 „	8.71 „	2.93 „	24.03 „
1.25 m.m.	11.21 m.m.	2 79 m.m.	22.49 m.m.

Whilst thus the longer variations give a proportion $\frac{22.49}{11.21} = 2.01$, the short pulsations give $\frac{2.79}{1.25} = 2.24$.

That difference of 10 % I believe must be ascribed to the following circumstance:

According to the image of the earth-current diagrams, on which the pulsations are large and easy to see, the points of reversion are pointed. On the Buitenzorgdiagrams on a ± ten times smaller scale those sharp points are blunted and we obtain a too small amplitude.

That shortening can be estimated at a tenth millimeter, i. e just 10 % of the amplitude.

With the oscillations of long duration that inaccuracy in the registration does of course not appear.

So we come to the conclusion *that the oscillations of short and of long duration of the magnetic force at Batavia and at Buitenzorg have the same ratio of amplitude and that they therefore cannot be caused, or only for a small part, by the current running through the outer crust of the earth.*

By far the greater part of the influence of the earth-currents must therefore come from currents at greater depths and of greater extension, and more equal in intensity.

Chemistry. — *"On the Tri-para-Halogen-Substitution-Products of Triphenylmethane and Triphenylcarbinol."* By Dr. F. M. JAEGER. (Communicated by Prof. A. P. N. FRANCHIMONT).

(Communicated in the meeting of March 28, 1908).

§ 1. Some years ago, I investigated [1]) crystals of *Tri-p-Chloro-Triphenylmethane,* from different preparations which had been obtained by Dr. P. J. MONTAGNE in two different ways, namely from *p*-leucaniline by diazotation and subsequent introduction of the three chlorine atoms and from tetrachlorobenzopinacoline by intramolecular rearrangement.

I then gave a detailed description of the remarkable optical behaviour of the compound in convergent polarised light and endeavoured to elucidate the same by a coloured figure.

Wishing to extend this research also to the other halogen-derivatives, I have first of all prepared the tribromoderivative of *p*-leucaniline by the method proposed by O. FISCHER and W. HESS. [2]) Afterwards I received from Prof. FISCHER a small quantity of each of the three halogen derivatives, which enabled me to prepare the three corresponding trihalogen-carbinols by oxydation with chromic acid in acetic acid solution, so that these three substances could be included also in this investigation. I will not omit to thank this savant once more for his kind assistance.

Of *Tri-p-Bromotriphenylmethane* [3]) I gave a description a short time ago in the Zeits. f. Kryst. **44,** 57—58. (1907). The habit of the crystals is quite analogous to that of the chloro-compound; they are more compact of form and generally much larger, but at the same time they cannot be measured so accurately, owing to a curving

[1]) Receuil **24.** 124, 131. (1905).

[2]) O. FISCHER und W. HESS, Berl. Ber. **38.** 336. (1905).

[3]) F. M. JAEGER, Zeits. f. Kryst. und Miner. Bd. **44.** 57. (1907).

of the planes. Nevertheless the complete isomorphism with the chloro-compound may be clearly shown, of course the differences are somewhat larger than in the case of isomorphous substitution products in which one atom only is replaced by another and not three at the same time, as is the case here.

§ 2. In the following the crystalforms of the diverse substitutionproducts are described.

Tri-p-Jodotriphenylmethane.

$(C_6H_4J)_3 : CH$; m. p. 132° C.

This compound was kindly presented to me for investigation by prof. O. FISCHER of *Erlangen*.

From ligroïn it crystallises in small, refractive, pale yellow needles which are readily measurable. From benzene, however, a *double compound* containing benzene crystallises in large transparent prisms. The chloro- and bromo-compounds, however, do *not* unite with benzene; from the benzene solution the crystals of the pure compounds are always deposited.

A. Tri-p-Jodotriphenylmethane, from ligroïn.

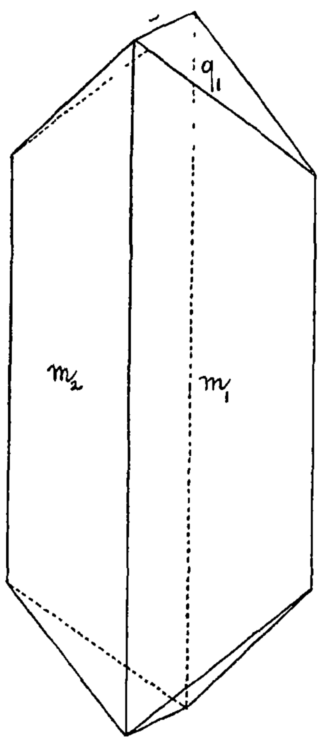

Fig. 1.
Tri-p-Jodotriphenylmethane.

The symmetry is *rhombic-bipyramidal*; the axial ratio is calculated as:

$$a : b . c = 0{,}5765 : 1 : 0{,}8798.$$

Evidently this substance is directly isomorphous with the *Cl* and the *Br*-compound although here the differences are again more considerable than usual on account of the simultaneous substitution of *three* isomorphogeneous atoms.

Forms observed : $m = \{110\}$, well developed and lustrous; $a = \{100\}$, very narrow and dim; $q = \{011\}$, yielding good reflexes; $p = \{130\}$, very narrow and dull.

Angular values :	Measured :	Calculated :
$m : m = (110) : (1\bar{1}0) =$	*59° 55½′	—
$m : q = (110) : (011) =$	70° 30½′	70° 44½′
$q : q = (011) : (011) =$	*82° 41′	—
$m : q = (110) : (130) =$	30° 15′	30° 30′

Distinctly cleavable along m.

The optical axial plane for *all* rays is {001}; the a-axis is the first bisectrix with positive character. Average strong, rhombic dispersion, with $\varrho > v$; the apparent axial angle in cedar-oil (1,54) is about 68°.

The sp. gr. of the crystals is 2,141 at 15°; the equivalent volume 290,64.

Topic parameters: $\chi : \psi : \omega = 4,7883 : 8,3061 . 7,3077$.

B. Tri-p-Jodotriphenylmethane + 1 Benzene.

Large, very lustrous and transparent crystals.

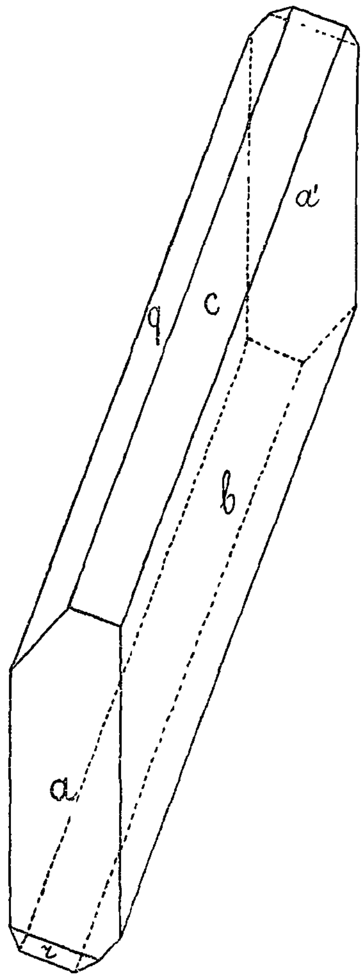

Fig. 2.
Tri-p-Jodo-Triphenylmethane + 1 Benzene.

When taken out of the motherliquor they keep transparent for a fairly long time but after a few hours they lose all their benzene while retaining their form; sometimes there is only a partial loss. It is not improbable that the amount of benzene varies with the temperature and pressure.

The symmetry is *triclino-pinacoidal.* Axial ratio:

$$a:b:c = 0{,}5719:1:1{,}4298.$$

$$A = 101°\,12' \qquad \alpha = 109°\ \ 8'$$
$$B = 123°\,15' \qquad \beta = 126°\,21'$$
$$C = \ \ 98°\ \ 5' \qquad \gamma = 107°\,32'$$

Forms observed: $b = \{010\}$, very predominant and lustrous; $c = \{001\}$ and $a = \{100\}$, well developed and yielding sharp reflexes; $q = \{0\bar{1}1\}$, also rather largely outgrown; $r = \{\bar{1}02\}$, narrow but readily measurable.

The habit is flattened towards {010} but elongated along the a-axis.

Perfect cleavage parallel {010}.

Angular values:	Measured:	Calculated:
$a:b = (100):(010) =$	*98° 5′	—
$c:b = (001):(010) =$	*78° 48′	—
$c:a = (001):(100) =$	*56° 45′	—
$a:r = (100):(10\bar{2}) =$	*50° 10′	—
$b:q = (010):(01\bar{1}) =$	*44° 4′	—
$a:q = (100):(01\bar{1}) =$	106° 8½′	

The ratio of the axes a and b in the two derivatives is quite analogous.

In accordance with the supposition of a varying benzene percentage the angular values of the individual crystals vary rather considerably.

§ 3. When we compare the three *para*-substituted trihalogen-compounds of triphenylmethane with each other, there can be no doubt as to the analogous molecular structure of the derivatives in the solid condition. Only in *optical* orientation the *chloro*-compound distinctly differs:

Tri-p-Chloro-compound:	*Tri-p-Bromo-compound*:	*Tri-p-Iodo-compour*
Rhombic-bipyramidal.	Rhombic-bipyramidal.	Rhombic-bipyramid.
Forms:	Forms:	Forms:
{110};{011};{010};{130};{012};{102}	{110} ; {011} ; {010} ; {102}	{110} ; {011} ; {13
$a:b:c = 0{,}5904:1:0{,}9261.$	$a:b:c = 0{,}5896:1:0{,}9003.$	$a:b:c = 0{,}5765:1:0{,}$
Cleavable towards {110}.	Cleavable towards {110}.	Probably cleavable wards {110}.
Thick-prismatic towards the c-axis.	Short-prism. tow. the c-axis	Elongated prisms tow the c-axis
$(110):(1\bar{1}0) = 61°\ 7'$	$(110):(1\bar{1}0) = 61°\ 3'$	$(110):(1\bar{1}0) = 59°\ 55$
$(110):(011) = 69°\ 47\tfrac{1}{2}'$	$(110):(011) = 70°\ 8'$	$(110):(011) = 70°\ 44$
$(011):(0\bar{1}1) = 85°\ 36'$	$(011):(0\bar{1}1) = 83°\ 59\tfrac{1}{2}'$	$(011):(0\bar{1}1) = 82°\ 41$
Sp. Gr. = 1,435;	Sp. Gr. = 1,752;	Sp. Gr. = 2,141;
Equiv. Vol. 242,16.	Equiv. Vol. 274,54.	Equiv. Vol.: 290,6

Chief dimensions of the crystal ructure. $: \psi : \omega = 4{,}5004 : 7{,}6225 : 7{,}0593.$	Chief dimensions of the crystal structure $\chi : \psi : \omega =$ $= 4{,}7327 : 8{,}0270 : 7{,}2267.$	Chief dimensions of the crystal structure $\chi : \psi : \omega =$ $4{,}7883 : 8{,}3061 : 7{,}3077.$
Optical orientation: The axial plane for *violet, blue* nd *green* is {001} but for the *range* and *red* rays it is, however)10}. The first diameter for all olours is the a-axis of — charac-r. The axial angle for *violet* is early 0°.	Optical orientation: For *all* colours the axial plane {001}. The first bisectrix is the a-axis of + character. Weak dispersion: $\varrho > v$. The apparent axial angle in cedar-oil is about 50°.	Optical orientation: For *all* colours the axial plane is {001} with the a-axis as first diameter of + character. Middlemost dispersion: $\varrho > v$. The apparent axial angle in cedar-oil is about 70°.

It should, however, be remarked that this *Tri-p-Chlorotriphenylmethane* exhibits also a very interesting optical variability as will be noticed from the subjoined observations:

a. Crystals from O. FISCHER; the compound is recrystallised from petroleum-ether (b.p. 40°—60°).

For *all* colours the optical axial plane was: {010}. Very strong dispersion: $\varrho > v$; the a-axis, is the 1st diameter and possesses a negative character. The apparent axial angle in olive-oil is very small and amounts to about 5°.

With other crystals, particularly the thicker prisms, I found that the axial plane for violet and blue rays is {001} but for *all* other colours {010}; the a-axis, is the first bisectrix, but now of a positive character; the very strong dispersion was: $\varrho >$ green.

Other little crystals only exhibited the *violet* in {001}, and the *blue, green, red, yellow* etc. in the plane {010}.

b. Crystals from the collection of the Organic Chemical Laboratory at *Leiden*, prepared by P. J. MONTAGNE. *They were optically perfectly identical with the crystals which I examined previously.* [1]) Some of the crystals had become opaque but had retained their form. This fact is already mentioned by MONTAGNE [2]) who observes also that the meltingpoint remains practically unaltered.

At my request Dr. MONTAGNE forwarded me some powder of *Tri-p-Chlorotriphenylmethane* from tetrachlorobenzopinacoline, which after recrystallisation from petroleum ether showed the following properties:

The axial plane for *all* colours is now {001}. Very strong dispersion: $\varrho > v$; the a-axis is the 1st bisectrix; the apparent axial angle

1) Zie Receuil d. Trav. d. Chim. d. Pays-Bas, **24**. 124, 131. (1905).

2. loco cit. p. 122.

in olive-oil is much larger than in the first case and amounts to about 10°.

Recrystallisation from petroleum-ether does not alter the properties of a definite crystal species; all preparations, however have the same meltingpoint and a complete identical crystalform.

We are therefore confronted with the fact that the compound $CH(C_6H_4Cl)_3$, m. p. 92° occurs, under varying circumstances, in forms which cannot be distinguished by chemical and crystallographical means, but whose *optical orientation is very different.* Sometimes, the crystals show a positive, sometimes a negative double refraction; one crystal shows a crossing of the axial planes for diverse colours, another for only a single colour; others again for no colour whatever, the axial plane then being either $\{001\}$ or $\{010\}$ whilst the dispersion is sometimes: $\varrho > v$, sometimes $\varrho < v$.

Of course, the possibility is not excluded that exceedingly small traces of foreign impurities cause this change of the so sensitive optical orientation. The result of the investigation of *Tri-p-Chlorocarbinol* showing its complete isomorphotropic relation to the said derivative, renders it not improbable that a trifling admixture of this oxidationproduct is the cause of the phenomenon.

In accordance with this is the fact, communicated to me privately by Dr. MONTAGNE, that a turbidity of the transparant crystals *never* occurs with the thin *rapidly* formed needles, but always with the thick and short crystals of *Tri-p-Chlorotriphenylmethane,* obtained by *slow* crystallisation.

But it is also conceivable that such large molecules as that of *Tri-p-Chlorotriphenylmethane* might in different circumstances suffer small deviations of their average atomistic configuration, which cannot be demonstrated chemically or crystallographically, but which can be shown optically.

Of late years numerous investigations have been carried out which must lead to the conclusion, that many properties of crystallised matter such as the growth- and cohesion-phenomena must be contributed to the regular molecular aggregation, whereas other ones such as the *optical* properties would have their origin, at least to a great extent, in the properties of the molecules themselves. This view is strengthened by different observations made with the so-called liquid crystals and doubly-refracting liquids; also by some experiments made by WALLÉRANT a.o. on the optical behaviour of deformed solid crystals. And phenomena like those observed here with *Tri-p-Chloro-Triphenylmethane* may show that it is possible that the spacial configuration of the chemical molecules is variable within narrow limits. I believe I have noticed something similar some time ago with a specimen of the *Dibromide* of 1-3-5 *Hexatriene* presented to me by Prof. VAN ROMBURGH[1]). Notwithstanding the identical crystal form the preparation made by addition of bromine to the hydrocarbon showed slight optical differences with that prepared from divinylglycol by means of PBr_3.

And although I will not as yet venture to give a decision one way or other, I fancy that on account of the phenomena described here the matter is of sufficient importance to be brought to the notice of chemists.

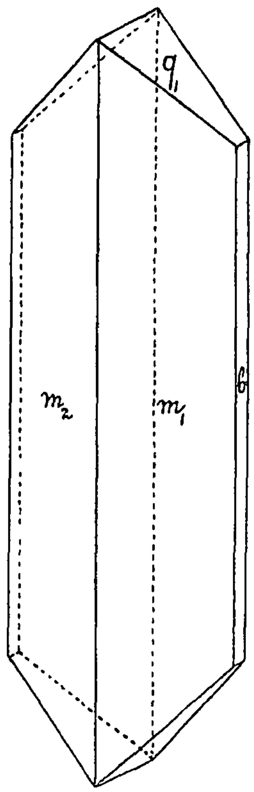

Fig. 3.
Tri p-Chloro-Triphenylcarbinol.

§ 4. **Tri-p-Chloro-Triphenylcarbinol.**

$(C_6H_4Cl)_3$: C-OH : m.p. 98° C.

Crystallises from ethyl alcohol in colourless, strongly refracting needles, also from ether + ligroïn. The crystals possess great lustre and are well constructed.

Rhombic-bipyramidal.

$a : b : c = 0,6009 : 1 : 0,9781.$

Forms observed: $m = \{110\}$, yields ideal reflexes; $q = \{011\}$, also yielding irreproachable images; $b = \{010\}$ and $p = \{210\}$, narrow but easily measurable. The habit of the crystals is elongated towards the c-axis. Crystals from ethylalcohol are short prismatic and still exhibit the forms $o = \{133\}$ and $s = \{102\}$, generally reflecting badly.

	Measured:	Calculated:
$m : q = (110) : (011) =$	*68° 53½′	—
$m : b = (110) : (010) =$	*59° 0′	—
$m : m = (110) : (\bar{1}\bar{1}0) =$	62° 1′	62° 0′
$q : b = (011) : (010) =$	45° 46½′	45° 38′
$q : q = (011) : (0\bar{1}1) =$	88° 36½′	88° 44′
$b : p = (010) : (210) =$	42° 27′	42° 16⅔
$p : m = (210) : (110) =$	16° 47½′	16° 43⅓.

[1]) Compare Trans. Chemic. Soc. (1908) p. 517—524.

No distinct cleavage was found.

The optical axial plane is {001}, with the a-axis as a first bisectrix of positive character. Weak dispersion: $\varrho > v$. The apparent axial angle in olive oil amounts to about 55°.

The sp. gr. of the crystals is: 1,423; the equivalent volume 255,44.

Topic parameters. $\chi : \psi : \omega = 4,5516 : 7,5748 : 7,4089$.

A comparison with *tri-p-chlorotriphenylmethane* shows that the morphotropic relations of both compounds are of such a nature that they border on *isomorphism*. In fact, both compounds form mixed crystals with each other.

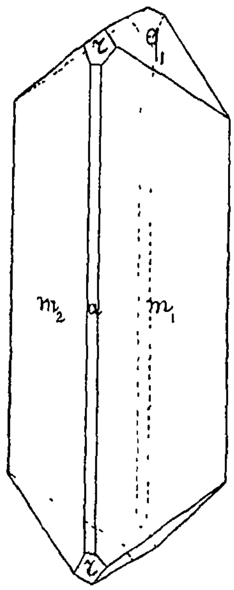

Fig 4.
Tri-p-Bromo-Triphenyl-carbinol.

§ 5. Tri-p-Bromo-Triphenylcarbinol.

$(C_6H_4Br)_3 : C.OH$; m.p. 133° C.

Crystallises from ethylalcohol in small colourless, clear crystals possessing a high lustre and a good geometrical construction.

Rhombic-bipyramidal.

$a : b : c = 0,8407 : 1 : 0,8081$.

Forms observed: $m = \{110\}$, predominant and yielding sharp reflexes; $a = \{100\}$ narrow but easily measurable; $q = \{011\}$, gives excellent reflexes and is well developed; $r = \{101\}$ small and somewhat dull. The crystals from alcohol are shown in fig. 4.

Angular Values:	Measured:	Calculated:
$m : a = (110) : (100) =$	*40° 3¹/₄′	—
$m : q = (\bar{1}\bar{1}0) : (0\bar{1}1) =$	*66° 8¹/₂′	—
$q : q = (011) : (0\bar{1}1) =$	77° 52′	77° 53′
$m : m = (110) : (1\bar{1}0) =$	80° 7¹/₂	80° 7¹/₂′
$m : r = (110) : (101) =$	57° 59′	57° 58′
$r : q = (101) : (0\bar{1}1) =$	55° 50¹/₂	55° 53¹/₂′
$r : r = (101) : (\bar{1}01) =$	87° 45′	87° 44′
$r : a = (101) : (100) =$	46° 7¹/₂′	46° 8′

No distinct cleavage.

The optical axial plane is {001}; the b-axis is the first bisectrix and of a negative character. The apparent axial angle in olive-oil is about 65°. No strong dispersion: $\varrho > v$.

The sp. gr. of the crystals is 1,847; the equivalent volume 269,08.

Topic parameters: $\chi : \psi : \omega = 6,1739 : 7,3439 : 5,9346$.

In contrast to what was found with both chloroderivatives, tribromocarbinol shows no distinct form-relationship with tribromotriphenylmethane [1]). The substitution of H by $-OH$, however, appears to exert an influence on the equivalent volume which is of a nature opposite to that which causes the same substitution in the chloro-derivative.

§ 6. Tri-p-Iodo-Triphenylcarbinol.

$(C_6H_4J)_3 : C—OH$; m.p.: 155° C.

Crystallises from ethylalcohol in fairly large yellowish crystals which, however, contain either no terminal planes at all or else strongly curved ones.

In any case the isomorphism with the previous compound may be easily proved.

Rhombic-bipyramidal.

$$a : b : c = 0{,}8543 : 1 : 0{,}817.$$

Forms observed: $m = \{110\}$, predominant and highly lustrous; $a = \{100\}$ narrow and generally absent but always giving a good reflexion; $q = \{011\}$ distinctly developed but in most cases curved and only approximately measurable; $r = \{101\}$ was observed once or twice.

	Measured:	Calculated:
$m : m = (110) : (1\bar{1}0) =$	*81° 1′	—
$q : q = (011) : (0\bar{1}1) =$	*78 29	—
$a : m = (100) : (110) =$	40° 36½	40° 30½′
$m : q = (110) : (011) =$	66 3½	65 44½
$m : r = (110) : (101) =$	58 10	58 18

No distinct cleavage.

The optical axial plane is $\{001\}$ with the b-axis as first bisectrix. Particularly large dispersion: $\rho > v$. The apparent axial angle in olive-oil amounts to about 80°.

The compound crystallises from benzene in combination with the solvent.

B. Tri-p-Iodo-Triphenylcarbinol + Benzene.

This occurs in large, yellowish needles having a strong lustre but generally possessing no terminal planes. In the case of one single individual however a few angles were measured. No trace of efflorescence was noticed in the crystals.

[1]) There is no question of a direct isomorphism. By exchanging the a- and b-axis we can find $a' : b' : c'$ 1,189 : 1 : 0,9612; which (with double a-axis) somewhat resembles the values for the bromoderivative.

Triclino-pinacoidal.

$a : b : c = 1,3991 : 1 : 1,6135$

$A = 94°\,12'$	$\alpha = 109°\,16'$
$B = 123°\,10'$	$\beta = 117°\,36'$
$C = 70°\,4^1/_2{}'$	$\gamma = 62°\,52'$

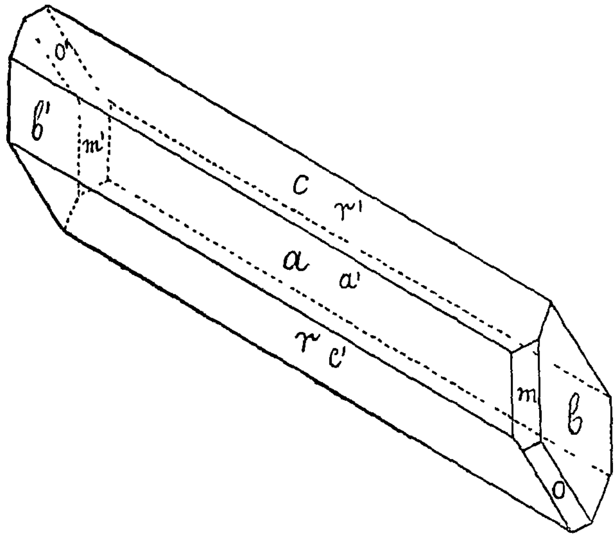

Fig. 5.
Tri-p-Jodo-Triphenylcarbinol + 1 Benzene.

Forms observed $c = \{001\}$ and $a = \{101\}$, equally strongly developed, $r = \{\bar{1}01\}$, broader than a and c and very lustrous; $b = \{010\}$, well developed; $o = \{11\bar{1}\}$ and $m = \{110\}$, about equally large and giving a good reflexion.

The habit is elongated towards the b-axis. The ratio $b : c$ is practically twice that of the *Tri-p-Jodotriphenolcarbinol* itself.

$a : c = (100) : (001) =$	* 56°.50′	—
$c : r = (001) : (\bar{1}01) =$	* 72 6	—
$c : b = (001) : (0\bar{1}0) =$	* 94 12	—
$a : b = (100) : (0\bar{1}0) =$	* 70 4¹/₂	—
$c : o = (001) : (1\bar{1}\bar{1}) =$	* 80 22	—
$m : o = (\bar{1}\bar{1}0) : (\bar{1}\bar{1}1) =$	39 22	39 .23¹/₂
$m : c = (\bar{1}\bar{1}0) : (00\bar{1}) =$	60 2	59 .45

No distinct cleavage.

On $\{\bar{1}01\}$ the extinction amounts to about $32^1/_2°$ in regard to the b-axis. Sp. Gz. 2,079 at 17°; Equiv. Vol. = 344,39.

Topic axes : $\chi : \psi : \omega = 8,4070 : 6,0090 : 9,6950$.

Groningen, March 1908.

Geophysics. — *"On the analysis of frequency curves according to a general method."* By Dr. J. P. VAN DER STOK.

§ 1. In working out meteorological data statistically (climatology), frequencies of all descriptions are found. No doubt the majority are between indefinite limits as most other frequencies of different origin, but it also happens that the limits are sharply defined as in the case of observations upon the degree of cloudiness, where they lie between 0 and 10.

An intermediate form is found in the frequencies of rain showers arranged according to duration or quantity; on the one hand they are rigidly limited by the zero value, on the other hand the heavy showers are without definite limits, so that the curve gradually approaches the axis of abscissae.

The elaboration of wind-observations requires the treatment of frequencies in two dimensions, and produces curves, which differ in character from other frequency curves according to the nature of their origin.

The development in series according to the formula of BRUNS [1]) and CHARLIER, appears to be the method indicated for frequencies with indefinite limits; but the deduction of this formula is based upon a generalisation in the use of definite integrals as already pointed out by BESSEL and therefore not quite free from premises, which may be applicable to the theory of probability but have no connection with the problem in question which may be defined as the analysis of an arbitrary function between given limits. Besides, this method of deduction can hardly be applied in the case of definite limitation.

The formulae of PEARSON, as also those of CHARLIER, are entirely based upon the premises of the theory of probability and, as they are not given in series form, they only contain a definite number of constants which, in some cases, is too limited to allow a complete characterisation of the curve, particularly in the working out of frequencies of the cloudiness, as will be shown in an example in another communication.

Besides, the constants, which partly appear in exponential form,

[1]) H. BRUNS. Wahrscheinlichkeitsrechnung und Kollektivmasslehre, Berlin, 1906.
Idem. Beitrage zur Quotenrechnung. Kon. Sachs. Gesellsch. d. Wiss. Bnd. 58. Leipzig, 1906.
C. V. L. CHARLIER. Researches into the theory of probability. Meddel. Lunds astr. observ. Ser. II. n°. 4. 1906.
Idem. Ueber das Fehlergesetz. Ark. for Matem. Astron. och. Fys. Bnd. 2. n°. 8, 1905.

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give no clear indication of the part they play in the construction of the curve, and it is not well possible to describe their function in a simple manner either verbally or graphically.

The object of this communication is to propose a general and simple method by which a curve may be found, which being integrated between certain limits, defined by the distribution of the data, will give the sums characteristic of this distribution, and that for frequencies of different kinds, as far as this is possible owing to the elements of uncertainty proceeding from the imperfection of the data which, of course, always remain.

This curve, representing the law which the phenomenon follows, should be called the frequency-curve; the curve of the aggregate values, obtained by grouping the original data within definite limits, may then be called the curve of distribution according to Bruns. Its form depends upon the degree of condensation of the original data (Abrundung after Bruns), but approximates more to that of the frequency curve as the condensation becomes less extensive and consequently the number of observations is greater.

Such a development of an arbitrary function can evidently be made in an infinite number of ways; it is therefore necessary to postulate some general principles.

The following premises apply to the method of development selected:

1. That the development takes place according to polynomia of an ascending degree.

2. that for the determination of the constants, the calculation of means of different orders is used, in relation to an origin favourably selected according to the requirements of the various cases.

The expression "moments" which is frequently employed, has been avoided as an unnecessary analogy with mechanical problems.

§ 2. Development between definite limits.

a. No given values of the function at the limits.

The polynomia, the degree of which is indicated by a suffix, are represented by Q_n, and the series by:

$$u = A_0 Q_0 + A_1 Q_1 + A_2 Q_2 + \dots \text{etc.} \quad \dots \quad (1)$$

The simplest form which can be given to the polynomia is:

$$Q_n = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots a_n$$

In this case the most practical choice for the origin of coordinates is evidently the mean between the limits as then, on integrating between the limits, all odd terms vanish; hence a separation between even and odd polynomia becomes necessary, and the general expression is:

$$Q_n = x^n + a_2 x^{n-2} + a_4 x^{n-4} + \ldots a_n \qquad n \text{ even}$$
$$= x^n + a_1 x^{n-2} + a_3 x^{n-4} + \ldots a_{n-2} \qquad n \text{ odd}$$

A simplification of the formulae can then be obtained by altering the scale value in such a way that the limits become ± 1, which is always possible; for the sake of convenience these limits have been omitted in the following expressions.

The means of different order are indicated by:

$$\mu_n = \int u x^n dx\,.$$

In order to enable us to calculate from the infinite series (1) the A-coeff. in a finite form, the unique and sufficient condition is that the a-coeff. be determined so that the condition:

$$\int Q_n x^m dx = 0 \quad \ldots \ldots \ldots \quad (2)$$

is satisfied for all values of $m < n$ as then all integrals beyond the $m + 1^{\text{th}}$ term vanish and, at the same time, the a-coeff. are entirely fixed, but for an arbitrary constant factor.

If this operation has been performed, it is at once evident from (2) that:

$$\int Q_m Q_n dx = 0$$

for all values of m different from n and, further, that:

$$A_n = \alpha \int u Q_n dx \quad \ldots \ldots \ldots \quad (3)$$

where:

$$\alpha^{-1} = \int Q_n Q_n dx = \int Q_n x^n dx\,.$$

The $^n/_2$ (n even) or $^{n-1}/_2$ (n odd) constants of the polynomium Q_n are calculated from the $^n/_2$ or $^{n-1}/_2$ equations:

$$\left.\begin{array}{l} \int Q_n dx = 0 \\ \int Q_n x^2 dx = 0 \\ \ldots\ldots\ldots \\ \int Q_n x^{n-2} dx = 0 \end{array}\right\} (n \text{ even}) \qquad \left.\begin{array}{l} \int Q_n x dx = 0 \\ \int Q_n x^3 dx = 0 \\ \ldots\ldots\ldots \\ \int Q_n x^{n-2} dx = 0 \end{array}\right\} (n \text{ odd})$$

or, for n even, from :

$$\frac{1}{n+1}+\frac{a_2}{n-1}+\frac{a_4}{n-3}+\ldots\ldots\frac{a_n}{1}=0$$

$$\frac{1}{n+3}+\frac{a_2}{n+1}+\frac{a_4}{n-1}+\ldots\ldots\frac{a_n}{3}=0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{2n-1}+\frac{a_2}{2n-3}+\frac{a_4}{2n-5}+\ldots\ldots\frac{a_2}{n-1}=0$$

for n odd, from :

$$\frac{1}{n+2}+\frac{a_1}{n}+\frac{a_3}{n-2}+\ldots\ldots\frac{a_{n-2}}{3}=0$$

$$\frac{1}{n+4}+\frac{a_1}{n+2}+\frac{a_3}{n}+\ldots\ldots\frac{a_{n-2}}{5}=0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{2n-1}+\frac{a_1}{2n-3}+\frac{a_3}{2n-5}+\ldots\ldots\frac{a_{n-2}}{n}=0$$

On eliminating successively from these equations $a_2, a_4 \ldots\ldots$ or $a_1, a_3 \ldots\ldots$ we find for the general expression of the polynomium:

$$Q_n = x^n - \frac{n(n-1)}{2\,.\,(2n-1)}\,x^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2\,.\,4\,.\,(2n-1)\,(2n-3)}\,x^{n-4} - \text{etc.} \qquad (4)$$

i. e., but for a constant factor, that of zonal harmonics, which we shall call P-functions.

This might have been expected as the condition (2), from which (4) arises, holds good also for the P-functions.

The Q-functions may, therefore, be considered as generalized P-functions, the latter presenting a special case of the former; if we write (2):

$$k_n \int Q_n x^m dx = 0\ ,$$

then :

$$k_n Q_n = P \quad \ldots\ldots\ldots\ldots \quad (5)$$

if k_n be defined so that:

$$k_n\, Q_n = 1 \ \text{for}\ x = 1.$$

The use of this constant no doubt offers advantages in treating problems relating to the potential theory, but for our purpose it would be of no importance and, in practice, entail superfluous work; some expressions certainly take a simpler form by its use, but what is thereby gained on the one hand is largely lost on the other as in calculating $u Q_n$ in (3), we have to deal with the unnecessary factor k_n.

However the relation (5), where:

$$k_n = \frac{(2n)!}{2^n \cdot n!\, n!}$$

so that:

$$Q_n = \frac{2^n \cdot n!\, n!}{(2n)!} P_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is useful in deriving from the well known properties of the P-functions those of the Q-functions.

They satisfy LEGENDRE's equation as well as the zonal harmonics:

$$(x^2 - 1) \frac{d^2 Q_n}{dx^2} + 2x \frac{dQ_n}{dx} - n(n+1) Q_n = 0.$$

The recurrent formula becomes:

$$Q_{n+1} - x Q_n + \frac{n^2}{(2n+1)(2n-1)} Q_{n-1} = 0,$$

and

$$Q_n = \frac{n!}{(2n)!} \frac{d^n (x^2-1)^n}{dx^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

Hence, we find:

$$\alpha^{-1} = \int Q_n Q_n \, dx = \frac{1}{k^2} \int P_n P_n \, dx = \frac{2}{k^2 (2n+1)} = \frac{2^{2n+1}\, n!\, n!\, n!\, n!}{(2n+1)!\,(2n)!}$$

and for A_n:

$$A_n = \alpha \left[\mu_n - \frac{n(n-1)}{2.(2n-1)} \mu_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} \mu_{n-4} - \text{ect.} \right] \quad (7)$$

b. Given $u = 0$ *for* $x = \pm 1$.

The case discussed sub a, where nothing is supposed to be known concerning the function to be developed, will seldom occur in practice and, as all adaptation is due to the accomodating power of the A-constants the application would, in such a case, necessitate the calculation of many terms and, therefore, hardly be profitable.

Now, in dealing with observations of the degree of cloudiness, the case presents itself, that a curve has to be found, which is characterized by the limiting values mentioned above.

The observations of serene sky (cloudiness zero) and of an entirely overcast sky (cloudiness ten) ought to be considered separately from the other observations as they constitute climatological factors of peculiar importance for the description of the climate (principally in northerly latitudes). Moreover they are to be regarded rather as discrete quantities, which do not show any continuous transition to a cloudiness resp. of degree 1 or 9.

The other degrees of cloudiness may then be regarded as observations of continuous quantities subject to the above mentioned conditions.

In this case we may easily cause all terms of the series (1) to suit these conditions by simply multiplying the series by a factor that vanishes for $x = \pm 1$ e. g. $x^2 - 1$, and then applying to the new functions, which we shall call R, the same reasonings as sub a.

The degree of the polynomia is then increased by two, so that we have to start with R_2.

The general expression becomes:

$$R_{n+2} = (x^2-1)\, R'_n = (x^2-1)\, [x^n + a_2\, x^{n-2} + \ldots a_n]\,,\ n \text{ even}$$

$$= (x^2-1)\, [x^n + a_1\, x^{n-2} + \ldots\ a_{n-2}],\ n \text{ odd}.$$

The result of this operation is evidently that the surface enclosed by the curve, as determined by the first term of the series, is not represented by a rectangle of base 2 and height 0.5 as in the case of the Q-functions, but by a parabola of base 2 and height 0.75, which makes again the surface equal to unity.

By alternately asymmetrical and symmetrical deformations the shape of this parabola is then altered by means of the next terms in such a manner as to make it approach more and more to the frequency curve corresponding to the given data.

It may be noticed here that in the case of fixed limits, there is no reason to choose for the origin of coordinates the point corresponding to the arithmetical mean; for logical and practical reasons the point intermediate between the limits is then indicated.

The condition, which has to be satisfied by the a-coeff. of the R-function, and by which they are fully determined, is now that:

$$\int R_{n+2}\, x^m\, dx = \int R'_n\, x^m\, (x^2 - 1)\, dx = 0, \quad m < n$$

or

$$\int x^{m+2}\, R'_n\, dx = \int x^m\, R'_n\, dx. \quad \ldots \ldots \quad (8)$$

The a-coeff. are calculated from the equations:

$$\left.\begin{aligned}&\frac{1}{(n+3)(n+1)}+\frac{a_2}{(n+1)(n-1)}+\frac{a_4}{(n-1)(n-3)}+\dots\frac{a_n}{3.1}=0\\&\frac{1}{(n+5)(n+3)}+\frac{a_2}{(n+3)(n+1)}+\frac{a_4}{(n+1)(n-1)}+\dots\frac{a_n}{5.3}=0\\&\dots\dots\dots\dots\dots\dots\dots\\&\frac{1}{(2n+1)(2n-1)}+\frac{a_2}{(2n-1)(2n-3)}+\frac{a_4}{(2n-3)(2n-5)}+\dots\frac{a_n}{(n+1)(n-1)}=0\end{aligned}\right\}\ n\ \text{even}$$

and

$$\left.\begin{aligned}&\frac{1}{(n+4)(n+2)}+\frac{a_1}{(n+2)(n)}+\frac{a_3}{(n)(n-2)}+\dots\frac{a_{n-2}}{5.3}=0\\&\frac{1}{(n+6)(n+4)}+\frac{a_1}{(n+4)(n+2)}+\frac{a_3}{(n+2)(n)}+\dots\frac{a_{n-2}}{7.5}=0\\&\dots\dots\dots\dots\dots\dots\dots\\&\frac{1}{(2n+1)(2n-1)}+\frac{a_1}{(2n-1)(2n-3)}+\frac{a_3}{(2n-3)(2n-5)}+\dots\frac{a_{n-2}}{(n+2)(n)}=0\end{aligned}\right\}\ n\ \text{odd}$$

By successive elimination of $a_2, a_4 \dots a_1, a_3 \dots$ we find from these equations for the general form of the R functions:

$$R_{n+2}=x^{n+2}-\frac{(n+2)(n+1)}{2\,.\,(2n+1)}x^n+\frac{(n+2)(n+1)(n)(n-1)}{2\,.\,4\,.\,(2n+1)(2n-1)}x^{n-2}-\text{etc.} \quad (9)$$

and from this expression by dividing it by x^2-1:

$$R'_n=x^n-\frac{n(n-1)}{2.(2n+1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n+1)(2n-1)}x^{n-4}-\text{etc.} \quad . \quad (10)$$

The recurrent formula for both R and R' is:

$$R'_{n+1}-x\,R'_n+\frac{n(n+2)}{(2n+3)(2n+1)}R'_{n-1}=0$$

and the functions are solutions of the diff. equations

$$(x^2-1)\frac{d^2R_{n+2}}{dx^2}-(n+2)(n+1)\,R_{n+2}=0$$

$$(x^2-1)\frac{d^2R'_n}{dx^2}+4x\frac{dR'_n}{dx}-(n+3)\,n\,R'_n=0.$$

On comparing the expression for R'_n with that for Q_n it is readily seen that the R' functions may be found by differentiation of the Q_{n+1}-function, so that:

$$R'=\frac{1}{n+1}\cdot\frac{dQ_{n+1}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This might have been expected as the value:

$$R' = k_n \frac{dQ_{n+1}}{dx}$$

satisfies the condition (8)

$$\int x^{m+2} \frac{dQ_{n+1}}{dx}\, dx = \int x^m \frac{dQ_{n+1}}{dx}\, dx, \quad m < n$$

which is easily proved by partial integration.

Therefore the series discussed here:

$$u = \Sigma\, A_n\, R_{n+2} \qquad n = 0\,.\,1\,.\,2\;\;.\;.\;.$$

might also (but for a constant factor) be written thus:

$$u = (x^2 - 1)\; \Sigma\, A_n \frac{dQ_{n+1}}{dx} \qquad n = 0\,.\,1\,.\,2\;\;.\;.\;.$$

The calculation of the A-constants is based upon the evident property of the R functions that:

$$\int R_{n+2}\, R'_m\, dx = 0, \qquad m \text{ different from } n$$

hence

$$A_n = \beta \int u\, R'_n\, dx$$

where:

$$\beta^{-1} = \int R_{n+2}\, R'_n\, dx = \int R_{n+2}\, x^n\, dx = \int x^n\, (x^2 - 1)\, R'_n\, dx = 0$$

or, by (11)

$$\beta^{-1} = \frac{1}{n+1} \int x^n\, (x^2 - 1) \frac{dQ_{n+1}}{dx}\, dx$$

From the diff. equation of the R-function follows:

$$\frac{d}{dx}\left[(x^2 - 1)\frac{dQ_{n+1}}{dv}\right] = (n + 2)\,(n + 1)\; Q_{n+1}$$

thence:

$$\beta^{-1} = \frac{n+2}{n+1} \int x^{n+1}\, Q_{n+1}\, dx$$

or by (8):

$$\beta^{-1} = -\frac{2^{2n+1}\,(n + 2)!\; n!\; n!\; n!}{(2n + 3)\,(2n + 1)!\,(2n + 1)!}$$

and A_n is calculated by the expression:

$$A_n = \beta \left[\mu^n - \frac{n(n-1)}{2.(2n+1)}\,\mu^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2\,.\,4\,.\,(2n+1)\,(2n-1)}\,\mu^{n-4} - \ldots \text{etc.}\right] \tag{12}$$

The negative sign of β is due to our having chosen as general factor $x^2 - 1$, a quantity which, by the definition of the limits, is always negative.

As well as the Q-functions, the R-functions might be multiplied by an arbitrary, constant factor, such that any peculiar development becomes possible or also with a view of simplifying some expressions. In our case e. g. k_n might be chosen so that $\beta = 1$; practically however this would hardly afford any advantage.

c. Given $u = \frac{u_1}{u_0}$ *for* $x = \pm 1$.

As has been remarked above, in working out observations of cloudiness the case presents itself that the frequencies for the extreme limits vanish; if, however, we have to deal, not with the original observations, but with average values as, e. g. daily means, the frequencies of serene and overcast sky, although still of peculiar interest for the knowledge of the climate, cannot be regarded as discrete values because, owing to the operation of taking the means, a continuous transition of these extreme values into the intermediate values must be assumed.

In this case, when the curves assume peculiar forms quite different from the well known curves generally met with, we can take care that the conditions for the extreme limits are bound to the first term of the series whilst all other terms remain as they are in the case discussed sub b.

Now the first term must contain three constants, two for the extreme values and one for the fixing of the area.

In the expression

$$a_0 + b_0 x + c_0 x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

the constants must satisfy the three conditions

$$u_1 = a_0 + b_0 + c_0$$
$$u_0 = a_0 - b_0 + c_0$$
$$2a_0 + \frac{2c_0}{3} = 1$$

hence:

$$4a_0 = 3 - (u_1 + u_0)$$
$$2b_0 = u_1 - u_0$$
$$4c_0 = 3(u_1 + u_0) - 3.$$

The reasoning as well as the application then remain the same as sub b; again

$$\int R_{n+2} R'_m \, dx = 0, \qquad m \text{ different from } n$$

with the exception however of the first term of the series which now assumes the form (13). In calculating A_n we have therefore to apply a correction to the expression for A_n which is easily found by remarking that:

$$(n+1)\int x^m R'_n\, dx = \int x^m \frac{dQ_{n+1}}{dx}\, dx$$
$$= \left(x^m Q_{n+1}\right)_{-1}^{+1} - m\int x^{m-1} Q_{n+1}\, dx\,.$$

For $m < n+2$ the last integral vanishes and, R_2 being of the second degree, we have to consider this case only.
We have, therefore:

$$(n+1)\int x^m R'_n\, dx = \left(x^m Q_{n+1}\right)_{-1}^{+1},\ m < 3$$

By (6) we find:

$$\left(Q_{n+1}\right)_{-1}^{+1} = \frac{2}{k_{n+1}} = \frac{2^n (n+2)!\, n!}{(2n+1)!}\ (n \text{ even})$$

whilst for n odd the expression vanishes.
Hence also:

$$\left(x^m Q_{n+1}\right)_{-1}^{+1} = \frac{2}{k_{n+1}} \qquad \left(m+n \text{ even}\right)$$

and equal to zero for $m+n$ odd; in calculating the constant A_n we have, therefore, only to apply a correction such that, instead of (12), now is used, for n odd:

$$A_n = \beta \int uR'_n\, dx - \frac{2^{n+1} b_0\, n!\, n!}{(2n+1)!} = \beta\int uR'_n\, dx - \frac{2^n(u_1-u_0) n!\, n!}{(2n+1)!} \qquad (14)$$

and for n even:

$$A_n = \beta \int uR'_n\, dx - \frac{2^{n+1}(a_0+c_0)\, n!\, n!}{(2n+1)!} = \beta\int uR'_n\, dx - \frac{2^n(u_1+u_0) n! n!}{(2n+1)!} \qquad (15)$$

This example of adaptation, of which many variants might be given, will suffice to demonstrate the applicability of the method to special cases.

§ 3. Development between definite limits on the one side and indefinite limits on the other.

a. No given value for the limit.

As has been noticed above, frequencies of duration and quantities of rainshowers lie between the asymmetrical limits: zero for the smallest and ∞ for the largest values.

Frequencies of this kind, therefore, offer an example of a transition between the case of fixed limits and infinite limits on both sides. As here there exists no symmetry in the limits, the zero-point cannot be chosen so that, on integrating, the odd functions vanish, hence a separation between even and odd functions would have no sense, and we are obliged to employ complete polynomia of ascending degree.

Here, as in the case discussed in § 2, there is no advantage in making the origin of coordinates coincide with the arithmetical mean and, from a logical as well as a practical standpoint, the zero-limit is indicated.

In order to develop the function between the limits ∞ and zero, the only thing to do is to multiply the series of polynomia with a suitable factor e. g. e^{-x}, so that the equation of the frequency curve becomes:

$$u = e^{-x}(A_0 S_0 + A_1 S_1 + \ldots \text{ etc.})$$
$$= A_0 \psi_0 + A_1 \psi_1 + \ldots \text{ etc.}$$

where:

$$S_n = x^n + a_1 x + a_2 x^{n-2} + \ldots a_n.$$

The conditions to be satisfied by the a coeff. are then:

$$\int_0^\infty e^{-x} S_n \, dx = 0 \ , \int_0^\infty e^{-x} x S_n \, dx = 0 \ldots \int_0^\infty e^{-x} x^{n-1} S_n \, dx = 0$$

and as:

$$\int_0^\infty e^{-x} x^n \, dx = n!,$$

the general conditional equations are:

$$n! + (n-1)!\, a_1 + (n-2)!\, a_2 + \ldots 1!\, a_{n-1} + 0!\, a_n = 0$$
$$(n+1)! + n!\, a_1 + (n-1)!\, a_2 + \ldots 2!\, a_{n-1} + 1!\, a_n = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(2n-1)! + (2n-2)!\, a_1 + (2n-3)!\, a_3 + \ldots n!\, a_{n-1} + (n-1)!\, a_n = 0$$

Hence we find for the general expression:

$$S_n = x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} - \ldots (-1)^n n! \quad . \quad (16)$$

The method of calculating A_n is the same as in the former cases as here too:

$$\int_0^\infty e^{-x} S_m S_n \, dx = \int_0^\infty \psi_m S_n \, dx = 0 \, , \qquad m < n$$

and

$$A_n = \gamma \int_0^\infty u \, S_n \, dx$$

where:

$$\gamma^{-1} = \int_0^\infty e^{-x} S_n S_n \, dx = \int_0^\infty \psi_n x^n \, dx$$

but:

$$\int_0^\infty \psi_n S_n \, dx = - (\psi_n S_n)_0^\infty + 2 \int_0^\infty \psi_n \frac{dS_n}{dx} \, dx$$

or, as the last integral vanishes according to the conditions:

$$\gamma^{-1} = - (\psi_n S_n)_0^\infty = n! \, n!$$

because, by (16), only the last term has to be taken into account.

The expression for A_n then becomes:

$$A_n = \frac{\mu_n}{n! \, n!} - \frac{n}{1!} \frac{\mu_{n-1}}{n! \, (n-1)!} + \frac{n(n-1)}{2!} \frac{\mu_{n-2}}{n! \, (n-2)!} - \dots \frac{(-1)^n}{n!} \quad . \quad (17)$$

by which the problem is solved.

The application to special cases will be simplified by a brief summary of the relations existing between the different quantities introduced which are analogous to those holding for zonal harmonics.

We remark that for S_n and ψ_n we can also write

$$S_n = (-1)^n \left(\frac{d}{dx} - 1 \right)^{(n)} x^n \quad , \quad \psi_n = (-1)^n \frac{d^n}{dx^n} (e^{-x} x^n) \quad . \quad (18)$$

hence:

$$S_n = - n S_{n-1} + \frac{x}{n} \frac{dS_n}{dx} \quad \text{and} \quad S_n = (x-n) \, S_{n-1} - x \frac{dS_{n-1}}{dx}$$

from which the recurrent formula:

$$S_{n+1} + (2n + 1 - x) \, S_n + n^2 S_{n-1} = 0 \, , \quad . \quad . \quad . \quad (19)$$

can be derived, wherein for S_n as well ψ_n may be written.

Further the functions satisfy the diff. equ.:

$$x \frac{d^2 S_n}{dx^2} + (1-x) \frac{dS_n}{dx} + n S_n = 0$$

$$x \frac{d^2 \psi_n}{dx^2} + (1+x) \frac{d\psi_n}{dx} + (n+1) \psi_n = 0.$$

b. Given $u = 0$ for $x = 0$.

In the same manner as the Q-series has been made to suit the zero-condition of the function at the limits, the ψ-series can be made fit for the case that the function assumes the zero value for the lowest limit by multiplication with x. This case presents itself e.g. for frequencies of wind-velocity, the curve of which originates at the zero-point as absolute calms do not occur.

By this operation the degree of the polynomia is increased by one and we can write down at once the new T function from (16) by multiplication with x and, at the same time, substituting $n+1$ for n except in the binomial factors which remain the same.

The condition for the determination of the a-coëff. is now:

$$\int_0^\infty e^{-x} x^m T_{n+1}\, dx = 0, \qquad m < n$$

and the general expression:

$$T_{n+1} = x^{n+1} - \frac{n}{1!} \cdot \frac{(n+1)!}{n!} x^n + \frac{n(n-1)}{2!} \frac{(n+1)!}{(n-1)!} x^{n-1} \ldots (-1)^n (n+1)! x \quad (20)$$

From this evidently:

$$S_n = \frac{1}{n+1} \cdot \frac{dT_{n+1}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

a similar relation as is shown by (11) between the Q and R functions.

Hence, if we put:

$$T_{n+1} = x T'_n$$

$$A_n = \gamma' \int_0^\infty u\, T'_n\, dx$$

where:

$$\gamma'^{-1} = \int_0^\infty e^{-x}\, T_{n+1}\, T'_n\, dx = \int_0^\infty e^{-x}\, x^n\, T_{n+1}\, dx =$$

$$\int_0^\infty e^{-x} x^n \frac{dT_{n+1}}{dx} = (n+1) \int_0^\infty e^{-x} x^n S_n\, dx = (n+1)!\, n!$$

so that:

$$A_n = \frac{\mu_n}{0!\,(n+1)!\,n!} - \frac{\mu_{n-1}}{1!\,n!\,(n-1)!} + \frac{\mu_{n-2}}{2!\,(n-1)!\,(n-2)!} - \ldots \frac{(-1)^n}{n!}. \quad (22)$$

If we call the series discussed here, the ψ'_{n+1} series, so that:

$$\psi'_{n+1} = e^{-x}\, T_{n+1} = e^{-x}\, x\, T'_n$$

we find the following relations:

$$\psi'_{n+1} = (-1)^n \frac{d^n}{dx^n}(e^{-x}x^{n+1}) = (-1)^{n+1} x \frac{d^{n+1}}{dx^{n+1}}(e^{-x}x^n) = -x\frac{d\psi_n}{dx}$$

$$x\frac{d^2T'_n}{dx^2} + (2-x)\frac{dT'_n}{dx} + nT'_n = 0$$

$$x\frac{d^2T_{n+1}}{dx^2} - x\frac{dT_{n+1}}{dx} + (n+1)\,T_{n+1} = 0$$

$$x\frac{d^2\psi'_{n+1}}{dx^2} + x\frac{d\psi'_{n+1}}{dx} + (n+1)\,\psi'_{n+1} = 0.$$

In exactly the same manner as the R-series could be expressed in diff. quot. of the Q series:

$$u_R = (x^2-1)\,\Sigma\, A_n\frac{dQ_{n+1}}{dx},$$

so the ψ' series might be expressed in diff. quot. of the ψ series:

$$u_{\psi'} = -x\,\Sigma\, A_n\,.\frac{d\psi_n}{dx}.$$

In dealing with this kind of frequency curves an alteration of the scale value offers great advantages as well as in the case of fixed limits.

In the case discussed in § 2 it was possible by this artifice to simplify the limits; here such an alteration has no influence upon the limits which remain 0 and ∞ if we write hx for x, but we are able by this means to accomodate the first term of the series, by which the area is determined, according to the form of the curve, so that the task of the A-coefficients is lightened.

By the factor h, which by its nature is a positive quantity, no complication in the calculation of the constants is introduced: the series is now:

$$u = e^{-hx}\,[A_0S_0(hx) + A_1S_1(hx) + \ldots \text{ etc.}]. \quad . \quad . \quad (23)$$

and, because:

$$\gamma^{-1} = \int_0^\infty e^{-hx}\,S_n(hx)\,S_n(hx)\,dx = \frac{1}{h}\int_0^\infty e^{-t}\,S_n(t)\,S_n(t)\,dt$$

$$A_n = h\left[\frac{h^n\mu_n}{n!n!} - \frac{n}{1!}\cdot\frac{h^{n-1}\mu_{n-1}}{n!(n-1)!} + \ldots \frac{(-1)^n}{n!}\right] \quad . \quad . \quad (24)$$

We might also omit the coeff. h in (24) and write (23):

$$u = he^{-hx}\,[A_0S_0(hx) + A_1S_1(hx) + \ldots \text{ etc.}] \quad . \quad . \quad (23^a)$$

The scale value h may, of course, be chosen quite arbitrarily; it is however desirable to do this in accordance with the nature of the curve and, therefore, to calculate it methodically from the given data.

This can be done by suppressing one of the A-constants in ($23a$) so that the mean value corresponding with this constant can be made use of to define h.

If then we put:

$$A_1 = 0$$

we find, as $A_0 = 1$,

$$\mu_1 = A_0 h \int_0^\infty e^{-hx}\, x\, dx = \frac{1}{h}.$$

§ 4. Development between the indefinite limits $\pm\infty$.

By reasons of symmetry, in this case it is logical to take e^{-x^2} for the factor by which the limits are determined and when, for the same reason, the arithmetical mean is chosen as the origin, the polynomium can, as in the case of fixed limits, be separated into even and odd functions, because then, on integrating between the limits, the odd functions vanish.

The series becomes then:

$$\begin{aligned} u &= e^{-x^2}\left[A_0 U_0 + A_2 U_2 + A_3 U_3 + \ldots \text{ etc.}\right. \\ &= A_0\varphi_0 + A_2\varphi_2 + A_3\varphi_3 + \ldots \text{ etc.} \end{aligned}$$

as, by the choice of the origin, the A_1-term has to be omitted. The conditional equation for the determination of the a constants is:

$$\int_{-\infty}^{\infty} x^m \varphi_n\, dx = 0 \qquad m < n \quad \ldots \ldots \quad (25)$$

or, generally, for n even:

$$[(n-1)(n-3)\ldots 1] + 2a_2[(n-3)(n-5)\ldots 1] + 2^2 a_4 [(n-5)(n-7)\ldots 1] + \\ + \ldots + 2^{n/2} a_n = 0$$

$$[(n+1)(n-1)\ldots 1] + 2a_2[(n-1)(n-3)\ldots 1] + 2^2 a_4 [(n-3)(n-5)\ldots 1] + \\ + \ldots + 2^{n-2/2} a_n = 0$$

. . . .

$$[(2n-3)(2n-5)\ldots 1] + 2a_2[(2n-5)(2n-7)\ldots 1] + \\ + 2^2 a_4 [(2n-7)(2n-9)\ldots 1] + \ldots + 2a_n[(n-3)(n-5)\ldots 1] = 0$$

and, for n odd:

$$[n(n-2)..\,1] + 2a_1\,[(n-2)\,(2-4)...1] + 2^2 a_2\,[(n-4)\,(n-6)\,..1] + \\ + 2^{n-1/2} a_{n-2} = 0$$

$$[(n+2)\,(n)..\,1] + 2a_1\,[(n)\,(n-2)\,..1] + 2^2 a_2\,[(n-2)\,(n-4) = ...1] + \\ + 2^{n-3/2} a_{n-2} = 0$$

. . . .

$$[(2n-3)\,(2n-5)...1] + 2a_1\,[(2n-5)\,(2n-7)\,..1] + \\ + 2^2 a_2\,[(2n-7)\,(2n-9)...1] + 2a_{n-2}\,[(n-2)\,(n-4).\,.1] = 0.$$

From this it follows that:

$$U_n = x^n - \frac{n(n-1)}{2^2\ 1!}x^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2^4\,.\,2!}x^{n-4} - \dots \text{ etc.} \quad (26)$$

from which we derive that U_n and φ_n are solutions of the equations:

$$\frac{d^2 U_n}{dx^2} - 2x\frac{dU_n}{dx} + 2n\,U_n = 0$$

$$\frac{\partial^2 \varphi_n}{dx^2} + 2x\frac{d\varphi_n}{dx} + 2(n+1)\,\varphi_n = 0$$

and the recurrent formula becomes:

$$2\,U_{n+1} - 2x\,U_n + n\,U_{n-1} = 0$$

The A-coefficients are determined in the same manner as in all former cases:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,U_m\,U_n\,dx = 0$$

for all values of m different from n so that:

$$A_n = \delta\int_{-\infty}^{+\infty} u\,U_n\,dx$$

where:

$$\delta^{-1} = \int_{-\infty}^{+\infty} e^{-x^2}\,U_n\,U_n\,dx = \int_{-\infty}^{\infty}\varphi_n\,x^n\,dx = \frac{n!}{2^n}\sqrt{\pi}$$

and

$$A_n = \frac{2^n}{\sqrt{\pi}}\left[\frac{\mu_n}{n!} - \frac{\mu_{n-2}}{2^2\,.\,1!\,(n-2)!} + \frac{\mu_{n-4}}{2^4\,.\,2!\,(n-4)!} - \text{etc.}\right]. \quad (27)$$

From the numerical values calculated by (26) and from the diff. equation it appears that, but for an arbitrary constant factor, the φ_n-functions are equal to the derivatives of the n^{th} order of φ_0 or e^{-x^2}; we might, therefore, write:

$$\varphi_n = k_n\,\frac{d^n\varphi_0}{dx^n},$$

this might have been expected as this value satisfies the condition (25), which can be easily proved by successive partial integration. If we put: $k_n = 1$,

$$\varphi_n = \frac{d^n e^{-x^2}}{dx^n} = (-2)^n\, U_n\, e^{-x^2}$$

and the expression for the A-coefficients becomes equal to that given by BRUNS.

Therefore $\frac{d^n \varphi_0}{dx^n}$ may be substituted for φ_n for the same reasons as, instead of the Q-functions, zonal harmonics might be employed; in practice however no labour is saved by this substitution as then the polynomia are charged with superfluous coefficients. After what has been said in § 3 about a change of the scale value, it will be sufficient to remark that in this case also the great advantage which can be derived from the introduction of a scale factor is the adaptation by means of the first term of the series to the shape of the curve, the surface remaining equal to unity.

The equation of the curve then becomes:

$$u = e^{-h^2x^2}\left[A_0 U_0(hx) + A_2 U_2(hx) + etc. \quad . \quad . \qquad (28)\right.$$

and:

$$A_n = \frac{2^n h}{\sqrt{\pi}}\left[\frac{h^n \mu_n}{n!} - \frac{h^{n-2}\mu_{n-2}}{2^2 \,.\, 1!(n-2)!} + etc.)\right]. \quad . \quad . \quad . \qquad (29)$$

The choice of the scale factor is of course quite arbitrary, but, in order to determine it in accordance with the nature of the curve, it is desirable to put $A_2 = 0$, then the average of the second order can be used for the definition of h and it is easily seen that:

$$\mu_2 = \frac{1}{2h^2}.$$

The coeff. of (29) in so far as they are independent of n may further be omitted and written before (28), then the equation of the curve becomes:

$$u = \frac{h}{\sqrt{\pi}}\, e^{-h^2x^2}\left[A_0 U_0 + A_3 U_3 + A_4 U_4 + etc.\right.$$

If we take into consideration only the first term in the development, we find the exponential law in its simplest form as $A_0 = 1$.

$$u = \frac{h}{\sqrt{\pi}}\, e^{-h^2x^2}.$$

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§ 5. Indefinite limits, two variables.

The treatment of wind observations now offers no difficulties as, in calculating the means of different-order, the two variables (projections upon two axes arbitrarily chosen) can always be separated and the method remains in all other respects quite the same. Only, instead of one mean of each order, we can now dispose of $p+1$ means of order p.

If by V_n be denoted the same function of y as U_n is of x, the equ. of the curve assumes, as $U_0 = V_0 = 1$, the form:

$$u(x,y) \doteq e^{-x^2-y^2}\,[A_0 + A_{1.0}\,U_1 + A_{0.1}\,V_1 + A_{2.0}\,U_2 + A_{1.1}\,U_1 V_1 + A_{0.2}\,V_2 + A_{3.0}\,U_3 + A_{2.1}\,U_2 V_1 + A_{1.2}\,U_1 V_2 + A_{0.3}\,V_3 + etc.] \quad . \quad (30)$$

The general expression for the polynomia is:

$$U_n\,V_m$$

and as, evidently:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}(U_n V_m)\,(U_p V_q)\,dxdy = 0$$

for all values of p different from n and of q different from m, we find for the A-coeff.:

$$A_{nm} = \varepsilon \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,u(U_n V_m)\,dxdy$$

where:

$$\varepsilon^{-1} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,(U_n V_m)^2\,dxdy = \frac{n!\,m!}{2^{n+m}}\,\pi \quad . \quad . \quad (31)$$

From the considerations of § 4 it follows that the function:

$$\Phi_{nm} = e^{-x^2-y^2}\,U_n V_m$$

may as well be given the form:

$$\Phi_{nm} = k_{nm}\,\frac{d^{n+m}}{dx^n dy^m}\,\Phi_0 = k_{nm}\,\frac{d^{n+m}}{dx^n dy^m}\,e^{-x^2-y^2}$$

as this satisfies the premised condition; then the series (30) assumes the form of a sum of diff. quot. like the series of Bruns and

$$\Phi_{nm} = (-2)^{n+m}\,U_n V_m\,e^{-x^2-y^2}$$

in accordance to which (31) has to be modified. If it is possible to remove the origin of coordinates to the arithmetical mean by a correction of the projections for their average value, then the terms with the coeff. $A_{1.0}$ and $A_{0.1}$ vanish from (30):

If we wish to alter the scale values according to the nature of the data, we have to write everywhere, hx and $h'y$, instead of x and y, whence

$$\varepsilon^{-1} = \frac{n!m!}{2^{n+m}} \frac{\pi}{hh'} .$$

The scale factors h and h' can then be determined by putting

$$A_{2.0} \text{ en } A_{0.2} = 0$$

and the two unmixed means of the second order can be disposed of for the determination of these constants:

$$\mu_2(x) = \frac{1}{2h^2} \text{ en } \mu_2(y) = \frac{1}{2h'^2}$$

If, further, we make the axes rotate about the origin so that they coincide with the principal axes of inertia, then also $A_{1.1}$ has to be put equal to zero and the corresponding mean

$$\mu_2 (x, y)$$

enables us to calculate the direction of the principal axes.

The series (30) then becomes:

$$u = e^{-x^2-y^2}[A_0 + A_{3.0}U_3 + A_{2.1}U_2V_1 + A_{1.2}U_1V_2 + A_{0.3}V_3 +$$
$$+ A_{4.0}U_4 + A_{3.1}U_3V_1 + A_{2.2}U_2V_2 + A_{1.3}U_1V_3 +$$
$$+ A_{0.4}V_4 + \text{enz.}$$

where all terms except the first represent the deviations from the normal exponential law, the terms of odd degree being a measure of the different kinds of skewness, the terms of even degree of the different kinds of symmetrical deviations.

Chemistry. — "*Equilibria in quaternary systems.*" By Prof. F. A. H. SCHREINEMAKERS.

Let us first take the system with the components: *water, ethyl alcohol, methyl alcohol* and *ammonium nitrate*; we then have at the ordinary temperature one solid substance and three solvents which are miscible in all proportions so that the resulting equilibria are very simple. The equilibria occurring in this system at 30° have been investigated and are represented in the usual manner in Fig. 1; the angular points W, M, A and Z of the tetrahedron indicate the components: water, methyl alcohol, ethyl alcohol and the salt: ammonium nitrate.

The curve wa situated on the side plane WAZ represents the solutions consisting of water and ethyl alcohol and saturated with solid salt; the curve wm represents the solutions of water and

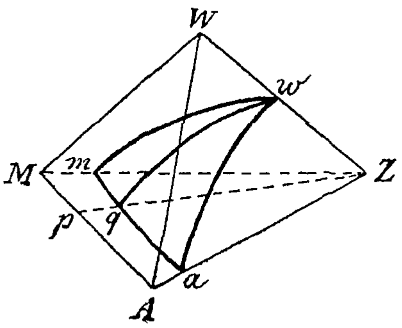

Fig. 1.

methyl alcohol mixtures saturated with solid salt, whilst *ma* indicates the solutions of mixtures of ethyl alcohol and methyl alcohol, also saturated with solid salt.

The quaternary equilibria, namely the solutions of mixtures of water, methyl alcohol and ethyl alcohol saturated with solid salt are represented by the surface *wma* which we may call the saturation surface of the solid salt *Z*.

If we introduce through one of the sides for instance through *WZ* a plane such as the plane *WZp* all points of that plane then represent phases containing the components *A* and *M* in the same proportion. This plane intersects the saturation surface along the curve *wq*; this, therefore, indicates solutions saturated with solid salt in which the relation between methyl alcohol and ethyl alcohol is constant.

The points of such a curve are easy to obtain; the two alcohols are first added together so as to yield a mixture represented by *p* for instance; on adding varying quantities of water we obtain the points of the line *pW* and on saturating these solutions with the salt the points of the curve *qw* are indicated.

In this manner different sections of the saturation surface with planes passing through the side *WZ* have been obtained.

In the system, water, methyl alcohol, ethyl alcohol and potassium nitrate perfectly analogous equilibria occur; the saturation surface for 30° in this system has been determined by Miss C. DE BAAT.

In the system: *water, ethyl alcohol, ammonium nitrate* and *silver nitrate* the relations are somewhat less simple, for at 30° we have two solid components and one double salt: $Ag\,NO_3 . NH_4\,NO_3$; the equilibria occurring at 30° are represented in Fig. 2. Whereas Fig. 1 is a perspective representation of the tetrahedron, Fig. 2 is a projection on a plane parallel to two sides crossing each other, in this case the sides: WA and AgN, so that in the projection, these stand perpendicular to each other and divide each other in two.

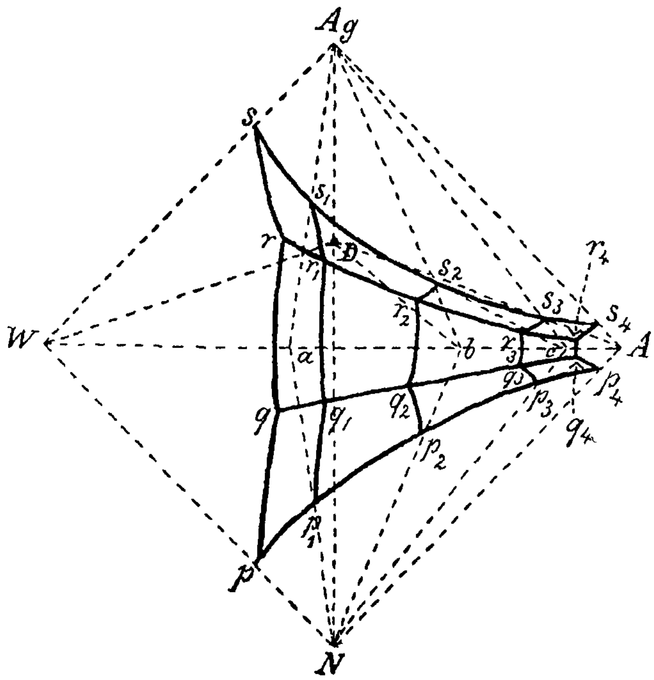

Fig. 2.

The angular points W, A, Ag and N indicate the components water, ethyl alcohol, silver nitrate and ammonium nitrate. The projection of an arbitary point within the tetrahedron on the projection plane is easily indicated. If we take the line WA as X-axis and the line NAg as Y-axis of a co-ordinate system and if we take as positive directions those towards A and Ag we find:

$$X = \frac{A - W}{2} \qquad Y = \frac{Ag - N}{2}$$

when A, W, Ag and N represent the quantities of alcohol, water, silver nitrate and ammonium nitrate indicated by the said point within the tetrahedron.

In this manner Fig. 2 has been deduced and it is readily noticed that the equilibria are represented by three surfaces, namely ss_4r_4r, rr_4q_4q and qq_4p_4p. The first surface is the saturation surface of silver nitrate, the second that of the double salt and the third that of the ammonium nitrate.

The double salt is represented in the figure by the point D which, of course, must be situated on the line $Ag\,N$. If the compositions of the phases were expressed in mol. $^0/_0$ D would fall in the origin of the co-ordinate system; this is however, not the case as the compositions are expressed in percentages by weight. The curve $s\,s_1\,s_2\,s_3\,s_4$, situated on the side surface $W\,Ag\,A$ is the saturation line of silver nitrate in water-alcohol mixtures; the solubility of this salt in water (point s) gradually becomes less on addition of alcohol; the solubility in absolute alcohol is represented by s_4.

The saturation line of ammonium nitrate in water-alcohol mixtures is represented by $p\,p_1\,p_2\,p_3\,p_4$. It will be noticed that the solubility of ammonium nitrate in water is much lessened by alcohol. The equilibria in the ternary system water, silver nitrate and ammonium nitrate are represented by the three saturation lines sr, rq and qp, situated on the side surface $W\,Ag\,N$; sr indicates the solutions saturated with silver nitrate, qp those saturated with ammonium nitrate and rq those saturated with the double salt. On drawing the line WD this will be seen to intersect the saturation line rq of the double salt; this is therefore soluble in water without decomposition.

In order to study the equilibria in the quaternary system I operated as follows. Instead of water, I took a water-alcohol mixture containing 41,8 $^0/_0$ of alcohol and in this determined the saturation lines or the silver nitrate, ammonium nitrate and the double salt. As the solutions all contained water and alcohol in constant proportion they must lie in a plane passing through the side $Ag\,N$ of the prism and intersecting WA in a point a indicating a 41,8 $^0/_0$ alcohol. In this manner I found the three saturation lines s_1r_1, r_1q_1 and q_1p_1 which therefore are all situated in the surface $a\,Ag\,N$: if the line aD is drawn it will be noticed that this branch intersects r_1q_1 showing that the double salt is also soluble without decomposition in dilute alcohol.

In a similar manner I determined the saturation line in water-alcohol mixtures containing 71,23 and 91,3 $^0/_0$ of alcohol; I always found three branches in the figure; they are represented by s_2r_2, r_2q_2 and q_2p_2 and by s_3r_3, r_3q_3 and q_3p_3.

As the line bD intersects the saturation line q_2r_2, the double salt is soluble without decomposition in 71,23 $^0/_0$ alcohol; with the line cD it is different; this no longer intersects the saturation line q_3r_3 of the double salt but only that of the silver nitrate r_3s_3 showing this is decomposed by 91,3 $^0/_0$ alcohol with separation of silver nitrate.

As the solubility of the components in absolute alcohol amounts to a few percent only, I have not investigated the ternary system alcohol — silver nitrate — ammonium nitrate but there is hardly any doubt that the solubility lines will give something as represented by $s_4 r_4 q_4 p_4$ and the double salt is bound to be decomposed by absolute alcohol with separation of silver nitrate.

From the preceding it is obvious that the following equilibria occur in the quaternary system:

$L + Ag\,NO_3$	of which L is represented by the surface: $s\,r\,r_4\,s_4$
$L + NH_4\,NO_3$	„ „ „ „ „ „ „ „ $q\,p\,p_4\,q_4$
$L + Ag\,NH_4\,(NO_3)_2$	„ „ „ „ „ „ „ „ $r\,q\,q_4\,r_4$
$L + Ag\,NO_3 + Ag\,NH_4\,(NO_3)_2$	„ „ „ „ „ „ „ curve: $r\,r_4$
$L + NH_4\,NO_3 + Ag\,NH_4\,(NO_3)_2$	„ „ „ „ „ „ „ „ $q\,q_4$

On looking at these equilibria several questions arise one of which I will mention. If, for instance we know that in the ternary system water, silver nitrate, ammonium nitrate, of which both salts are anhydrous, an anhydrous double salt occurs at 30° we may ask ourselves what equilibria will occur if the water is substituted by another solvent such as aqueous or absolute alcohol.

It is impracticable to answer this question in its entirety; if, however, we argue that no solid phases are formed which crystallise with the new solvent it becomes a fairly easy one. As a rule we can demonstrate that the same three saturation lines will occur also in the new solvent so that a solution saturated with the two components or solutions saturated with another double salt cannot be formed.

Therefore, although the same double salt must appear in both solvents, its behaviour in regard to the two pure solvents, may however, be quite different and various cases may occur; it may, for instance be soluble in both solvents without decomposition or it may be that, as in the case mentioned, it is soluble in the one solvent without and in the other with decomposition; or it may dissolve in both solvents with decomposition. In the latter event we may meet with two more cases; it may be that the same component is deposited from both solvents or it may be that one of the components is deposited from the one and the other from the other solvent.

Similar equilibria occur also at 30° in the systems:

water — alcohol — silver nitrate — potassium nitrate

and water — alcohol — benzoic acid — ammonium bezoate.

In the first system occurs a double salt of silver nitrate and potassium nitrate; in the latter, which is being investigated by Dr. H. Filippo, a combination of benzoic acid and ammonium benzoate is formed.

In the system: water, alcohol, ammonium sulphate and manganese sulphate quite different equilibria occur. The results of this investigation for 50° are represented in fig. 3; this is again the projection of the tetrahedron on a plane parallel to the sides *WA* and *MnN*. The angular points *W*, *A*, *N* and *Mn* indicate the components: water, alcohol, ammonium sulphate and manganese sulphate.

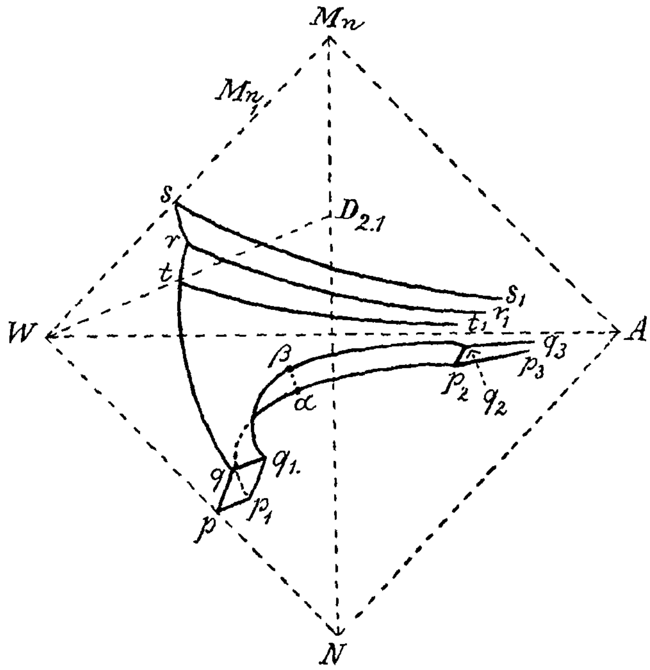

Fig. 3.

In this system an anhydrous compound $(Mn\,SO_4)_2\,(NH_4)_2\,SO_4$ occurs at 50° which is represented in the figure by the point D_2. The $Mn\,SO_4$ gives at that temperature the compound $Mn\,SO_4 \,.\, H_2O$, represented by the point Mn_1.

On the side plane *MnWN* we find the equilibria in the ternary system water, manganese sulphate, ammonium sulphate. Three saturation lines are found: *sr* is that of the $Mn\,SO_4 \,.\, H_2O$, *rtq* that of the double salt D_2, and *qp* that of the $Mn\,SO_4$. As the line WD_2, intersects the saturation line of the double salt in *t* this is soluble in water without decomposition; *t* represents this saturated solution.

The isotherm ss_1 which indicates the equilibria in the ternary system water, alcohol, manganese sulphate consists of two saturation lines of which only one ss_1 has been determined. This indicates the solution saturated with $Mn\,SO_4 \,.\, H_2O$; to this should join a saturation line with the anhydrous $Mn\,SO_4$ as solid phase which however, has not been determined.

The equilibria occurring in the ternary system: water, alcohol, ammonium sulphate are represented in the plane WAN by the isotherm $pp_1ap_2p_3$; this consists of the saturation lines pp_1 and p_2p_3 of the ammonium sulphate and of the branch p_1ap_2 of a binodal line with the critical liquid a. The points p_1 and p_2 therefore represent two ternary conjugated liquids saturated with ammonium sulphate.

The quaternary equilibria are represented by four surfaces:

srr_1s_1 is the saturation surface of the $Mn\,SO_4 \,.\, H_2O$

$rtqq_1\beta q_2q_3r_1$ is the saturation surface of the $D_{2.1}$

qpp_1q_1 and $q_2p_2p_3q_3$ are the saturation surface of the ammonium sulphate

$p_1ap_2q_2\beta q_1$ is the binodal surface.

The latter surface is divided in two parts by the critical line $a\beta$; with each point of the one part, a point of the other is conjugated; a similarly conjugated pair of points represents a pair of quaternarly conjugated liquid phases. The binodal surface, therefore, represents the equilibria liquid + liquid. The sections of the four saturation surfaces give three saturation lines.

rr_1 represents the solutions saturated with $Mn\,SO_4 \,.\, H_2O + D_{2.1}$

qq_1 and q_2q_3 represent „ „ „ „ $(NH_4)_2\,SO_4 + D_{2.1}$

$q_1\beta q_2$ represents the conjugated liquid pairs saturated with ammonium sulphate. The point β is the critical solution saturated with ammonium sulphate.

If a plane is brought through the side WA and the point $D_{2.1}$ this intersects as far as has been determined the saturation surface of $D_{2.1}$ in the curve tt_1; the double salt is therefore not only soluble in water but also in dilute alcohol without decomposition.

At 25° quite different equilibria occur in this system; on the side plane $W\,Mn\,A$ a new area of immiscibility is developed. At the same time the double salt $D_{2.1} = (Mn\,SO_4)_2\,(NH)_4)_2\,SO_4$ disappears in order to make room for the double salt $Mn\,SO_4 \,.\, (NH_4)_2\,SO_4 \,.\, 6H_2O$. The resulting equilibria much resemble those mentioned previously which occur at 30° in the system: water, alcohol, ammonium sulphate and lithium sulphate.

I will therefore not discuss these equilibria any further.

Botany. — "*The development of the ovule, embryo-sac and egg in Podostemaceae*". By Prof. F. A. F. C. WENT.

During my voyage to the West-Indies I had an opportunity of visiting in Surinam some of the rapids where *Podostemaceae* grow, namely the Armina falls of the Marowyne river. There I collected material of these remarkable plants, and at a later date I received an abundant supply obtained by the various expeditions, which of late years have investigated the interior of the colony. This material, preserved in alcohol, has suggested to me an investigation of the above order. I hope soon to publish the results *in extenso*, but wish in this place to deal briefly with one point, namely the development of the ovule, the embryo-sac and the egg.

As was mentioned above, the material was fixed in alcohol, but the fixation nevertheless proved to be good enough to allow of many cytological details being made out with a sufficient degree of certainty in stained preparations. In this preliminary communication I do not propose to discuss the method of treatment of the preparations, but merely record, that Messrs. A. H. BLAAUW and J. KUYPER have assisted me. A complete developmental series could only be obtained in the case of a few species, namely of *Oenone Imthurni Goebel* and *Mourera fluviatilis Aubl.* Of eight other species only a few stages of the development were examined, and of *Tristicha hypnoides Spr.* I only had the ripe seeds.

It soon became evident that the whole development of the ovule in this order departs very widely from the ordinary type of *Angiosperms*, but that within the limits of the order there is an extraordinary degree of uniformity, so that the differences between the species, which have been investigated, are so slight, that they may be passed over in silence in this preliminary notice. The description which follows, therefore applies to all the species.

The ovules are anatropous; in the youngest stage examined, the curvature had already taken place. In this stage the nucellus was still alone present and consisted of a central row of four cells surrounded by a single layer of peripheral cells. Of the central row the uppermost cell, which is therefore still surrounded by a cap of epidermal cells, becomes the spore mother-cell. Accordingly this cell is not only soon distinguished from all the other cells of the nucellus by its size, but also by its dense protoplasmic contents and by its large nucleus. The subsequent behaviour of this spore mother-cell will be further discussed below.

We may now consider how the integuments are formed. The outer one arises *first* and here we find the first deviation e normal course of development in Angiosperms. This integument simply arises as an annular fold on the nucellus, with which it remains connected by the chalaza, while for the rest it grows round pretty loosely. Finally there remains at the point, where its borders meet, a very narrow micropyle, which can only be seen properly in truly medial sections.

After the outer integument has already surrounded half the ovule, the inner one begins to develop. Cell divisions are seen to occur in a few epidermal cells of the nucellus, immediately above the point of attachment of the outer integument. These divisions take place in such a manner, that a wall arises in one of the basal cells of each longitudinal row of the epidermis; this wall forms an angle of 45° with the longitudinal direction of the ovule, so that each of the cells is divided into two. The upper half remains an epidermal cell of the nucellus, while the lower half develops to form the inner integument. In a transverse section the number of epidermal cells, counted on the periphery, is seen to be 5, occasionally 6 or 7. At first the inner integument will therefore show in transverse section an equal number of cells. Dividing walls soon arise, however, which make this inner integument two cells thick. More than two layers do not develop, as no further tangential walls are formed, but other walls, both radial and transverse to the long axis of the ovule are developed. Especially the number of radial walls is very different in the two cell layers; it is large in the outer layer, but on the other hand small in the inner layer. As a result, the number of cells of the inner layer of the inner integument is generally little more than five, when counted in transverse section. When afterwards the cells of the inner integument increase in size and often acquire dimensions, which make them very noticeable, it is the inner cells which are especially large. This growth is often accompanied by strong thickening of the walls.

The transverse walls, which arise in the cells of the inner integument, enable the latter to grow longitudinally. In this process the top of the nucellus remains free however, and is only surrounded by the outer integument, so that it lies in the endostomium; the strong longitudinal growth of the inner integument is chiefly directed downwards. At its base, near the chalaza, it of course remains connected with the nucellar tissue.

Now it is very remarkable, that the nucellar tissue does not participate by cell division in this strong longitudinal growth of the ovule. The portion of the nucellus, which projects beyond the inner integument, remains unaltered, except for certain changes, which the spore mother-cell undergoes, and which will be discussed below. We may however at once point out, that in the formation of embryo-sac and egg-cell, the whole apparatus remains in the same place, and is therefore never surrounded by the inner integument.

The portion of the nucellus lying below this, is now elongated by the extreme stretching of a single cell (or in some cases perhaps two cells) in the central and in each of the 5, 6 or 7 peripheral rows of cells, of which it consists. The nuclei often also assume an extended shape, so that one gets the impression that a passive stretching has taken place. At the same time a digestion of the longitudinal walls occurs, and finally the protoplasts also coalesce more or less. In this way a great cavity arises, containing protoplasm, often in a peripheral layer and with 6, 7 or 8 nuclei, perhaps sometimes even more in consequence of nuclear fragmentation, which seems to occur.

If an ovule is examined in this stage, without the history of its development having been traced, this cavity is inevitably regarded as the embryo-sac, and the real embryo-sac, which lies above it, is then taken for the egg-apparatus. It is in this way that WARMING, who, for want of the necessary material could only trace part of the development of the ovule, has regarded things.[1]) This pseudo-embryo-sac remains in existence during the further development of the ovule to the seed, and is only compressed more or less in some cases by the large increase in size of the cells of the inner integument, which has already been dealt with above. When the embryo begins to develop it grows out into this pseudo embryo-sac, in the same way as would happen with a true embryo-sac.

We may now pass on to consider the fate of the spore mother-cell. At a certain period its nucleus shows a clear synapsis stage. In the division, which follows this, the reduction of the number of chromosomes therefore probably takes place. The fixation was not sufficient to allow one to conclude with certainty that a hetero-typic division of the nucleus occurs (the nuclei are moreover extremely minute); such observations as were made, leave very little doubt, however, when considered in connection with the preceding synapsis, that the haploid generation begins here. This nuclear division is followed by a cell division and the formation of a dividing wall. The upper of the two cells, which are thus formed, gradually degenerates and becomes more and more flattened by compression; remnants of it

[1]) EUG. WARMING Familien Podostemaceae. II. Afhandling. Kgl. Danske Vidensk. Selsk. Skr. 6te Raekke, naturv. og math. Afd. 2det Bd. III. Kjöbenhavn, 1882. Compare e.g. p. 65 (107).

may nevertheless still be observed for a long time. In some cases the nucleus of this cell divides once more, in a plane perpendicular to that of the previous division, so that the equatorial plane of the second division is in the longitudinal direction, with respect to the ovule. Perhaps this division also takes place in other cases, in which the two nuclei cannot be seen on account of the unfavourable direction of the section and in consequence of the rapid degeneration of the cell. Only in a single instance I have thought that I observed a cell division following the division of the nucleus in the upper cell.

The lower of the above-mentioned two cells is the embryo-sac. Having regard to the size of the pseudo-embryo-sac, it is remarkable, that the real embryo-sac increases but little in size, and always remains situated in that upper part of the nucellus, which projects beyond the inner integument; it remains of course surrounded by the layer of epidermal cells, which later are only compressed and flattened more and more, so that they become scarcely visible.

The nucleus of the embryo-sac soon divides again. Only a single division was observed, and then the fixation did not allow many details to be made out; it can hardly be doubted, however, that this must be a homoiotypic division of the nucleus. The axis of this spindle is longitudinal with respect to the ovule and therefore also with respect to the embryo-sac. The lower of the two nuclei, which are formed, is seen to degenerate in the anaphases of the division, by a strong clumping of the chromatin masses, so that the latter come to lie at the base of the embryo-sac as a structureless chromatin-like clump, which stains deeply. This is evidently all, that can here be seen of the antipodal apparatus and of the lower polar nucleus. I shall call this nucleus the antipodal nucleus of the embryo-sac.

In contra-distinction to the last-named, the other nucleus assumes a normal shape and is prominent on account of its size. Soon afterwards there follows another division, of which I have been able to see the various stages. The axis of the spindle is this time also longitudinal to the embryo-sac and ovule. This division is not at first followed by a cell division, but afterwards each of the two daughter nuclei divides again. The actual process of division I have not observed, but have only found four nuclei; the second division evidently takes place very rapidly, for I have looked through hundreds of preparations of about this age, without getting the actual stage of division. This second division takes place in such a manner, that the axes of division are perpendicular to each other; for the upper pair of nuclei the axis is at right angles to the length of the embryo-sac, and for the lower pair it is parallel to it.

Before this last division has taken place, the embryo-sac is still seen to be a single cell, as was already stated above; after this division four cells, each with its nucleus, may be observed. It is of course possible, since I have not seen the actual nuclear division, that the latter is preceded by a cell-division, in such a way, that each cell contains a nucleus, and that afterwards each of these two cells divides again, after its nucleus has divided. However this may be, there are finally four cells, which, it should further be noticed, are not separated by cell-walls — four naked protoplasts therefore Of these four two, the synergids, lie at the top, next to each other; then follow the other two, one under the other, the upper one of the pair being the egg and the lower one all that remains of the embryo-sac with the upper polar nucleus.

Considering this lower cell first, we observe, that it remains small and that pretty soon its nucleus clumps to a little ball of chromatin, in which structure can no longer be discerned; often the antipodal nucleus may be seen at the same time. In other cases no remnants of it can be observed; I imagine that in such cases it has so far degenerated, that it can no longer be rendered visible. Yet another hypothesis might be suggested, namely, that these nuclei fuse like two polar nuclei. I regard this, however, as extremely improbable, for the very reason that the two nuclei are so clearly in a state of degeneration. Indeed, all the rest of the embryo-sac does not come to much; endosperm is not formed; the cell is still seen for some time, until it disappears with the developing embryo.

For some time the egg and the synergids undergo no further changes, and are ready for fertilisation. This process I have only been able to follow accurately in *Mourera fluviatilis Aubl.*; in a few other cases I found a young embryo, or sometimes pollen-grains, which had germinated on the stigma and had developed pollen-tubes. In a new species of *Apinagia,* still to be described, there occur, in addition to the normal hermaphrodite flowers, others, which have abortive stamens, and which remain inside the closed spathella, at least as far as I have been able to observe in the material at my disposal. Whether the latter flowers can also furnish ripe seeds, without fertilisation, I cannot say, as they had not developed beyond the stage, here described. In the numerous preparations of various *Podostemaceae* which I have examined, I found moreover many ovules, which were degenerating at the above-mentioned stage, evidently because no fertilisation had taken place. It seems to me, that the chance of regular pollination among these plants is probably not so very large, and that in consequence of this so many ovules ultimately abort.

I now pass on to describe what I have seen of the fertilisation itself, and must remark, that I have but rarely observed anything of the penetration of the pollen-tubes; to some extent this is probably a result of the process of fixation, during which such tender, thin structures readily shrivel up; at the same time the staining does not succeed well. In any case I can however state, that the pollen-tube penetrates through the micropyle, and then reaches the egg-apparatus by passing between two epidermal cells of the nucellus. In one case I observed two nuclei in the top of the pollen-tube, one of which appeared to be a generative and the other a tube-nucleus. In another case I saw a nucleus, which had a much elongated appearance, and was constricted in the middle, so that there might have equally well been two generative nuclei. Taking all the cases, which I have seen, into account, I am led to the view, that the conditions in the top of the pollen-tube are normal, so that there are two generative nuclei and one tube-nucleus. In the actual process of fertilisation, the top of the pollen-tube unites with one of the synergids; the synergid and especially also the egg undergo at the same time peculiar changes in shape, somewhat resembling amoeboid movements. What further happens in the synergid cannot readily be made out, because its contents stain very strongly and become highly refractive. I nevertheless also succeeded in this case in observing the main features of the process. At least one nucleus of the pollen-tube penetrates into the synergid and assumes, in so doing, a more or less vermiform shape. Thereupon a fusion of the synergid with the egg takes place, so that the protoplasts communicate with each other at least at one spot. This communication does not last long, but during it one of the generative nuclei evidently penetrates into the egg-cell; anyhow stages are found later, in which two nuclei lie close to each other in the egg. Still a little later these are found in contact, and afterwards they are found fused in such a manner, that the origin from two nuclei can still be seen.

The fertilized ovum now rapidly enlarges, while all other cells in its neighbourhood are crowded out. As the epidermal cells of the nucellus have generally aborted, this large cell lies more or less by itself in the endostomium, almost filling it up. By the first division wall there is formed a bladder-like basal cell, which remains in the cavity, and a smaller one, which is gradually pushed forward into the pseudo-embryosac. This cell now undergoes some divisions, in which the walls are formed perpendicular to the long axis of the young seed. When a row of four cells has thus arisen, the three which are turned towards the micropyle become a suspensor, while the fourth divides by a wall at right angles to the previous ones and becomes the embryo proper.

I have not traced the further development of the embryo, partly for want of sufficient material, but especially because WARMING has already furnished an excellent treatise dealing with this subject, and illustrated with figures. Considering the many new facts, which WILLIS has discovered about the germination of the *Podostemaceae* of Ceylon, an investigation of the American forms in this direction would certainly repay, since through Goebel we have only learned in detail of a single case. For this an investigation on the spot is necessary, and as will appear from the full paper, I have not been able to find much that is new in this direction.

What was hitherto known about the ovules of *Podostemaceae* we owe almost exclusively to WARMING. As was said above, this author described in detail the first development of the ovules of *Mniopsis Weddelliana Tul.*, and it was only owing to the want of the exact stages, that the meaning of certain organs did not become clear to him. The development proper of the embryo-sac was completely left out of account, but the development of the embryo of this plant, beginning with the two-celled stage, was treated very thoroughly. It is quite clear from his letter-press and from his figures, that the whole development takes place in the same way as in the species examined by myself. The same can be said of the other cases, in which he has stated or figured something regarding the ovules of *Podostemaceae* namely *Castelnavia princeps* TUL et WEDD.[1]) *Hydrobryum olivaceum* GARDN.[2]) and *Tristicha hypnoides* SPRENG[3]). On the last named CARIO[4]) had already made observations which seemed to indicate an agreement with the other *Podostemaceae* as regards the development of the ovule. This is of some little importance, because this plant deviates in the structure of its flowers from the majority of the species of the order. If the development of the ovule here corresponds to what I found in the species examined by me this agreement constitutes an additional reason for supposing, that the order is extremely uniform in its embryogeny, in which it differs so widely from the other Angiosperms. I have already

1) WARMING, l.c. Plate XIV. Fig. 9—21.

2) WARMING, Ibid. 6 Raekke, Nat. og math. Afd. VII. 4. 1891, p. 37, fig. 34.

3) WARMING, Ibid. 6 Raekke, Nat. og math. Afd. IX. 2. 1899. p. 113, fig. 6.

4) R. CARIO. Anatomische Untersuchung von *Tristicha hypnoides* Spreng. Botan Zeitung. 1881 S. 73, Taf. I. Fig, 20—24.

remarked, that much to my regret, I have only ripe seeds of *Tristicha,* but no younger stages. In the 78th Versammlung Deutscher Naturforscher und Aerzte in 1906 at Stuttgart R. VON WETTSTEIN made a communication : "Ueber Entwickelung der Samenanlagen und Befruchtung der Podostemonaceen". So far he has not published anything about this, however. I have indeed found an abstract of the communication in "Naturwissenschaftliche Rundschau" of 1906, Bd. XXI, p. 615, and in it several statements occur which agree completely with what I have observed, but in other respects there are such differences, that I must assume, that the reporter did not completely understand the meaning of the reader of the paper; I dare not therefore rely on this abstract.

The *Podostemaceae* differ on the following points from the ordinary arrangement in *Angiosperms,* as regards the development of the ovule: 1. The inner integument begins to develop after the outer; this is perhaps connected with the fact, that the top of the nucellus remains free in the endostomium, a phenomenon, which has been observed in other plants. 2. The peculiar development of a pseudo-embryosac by the stretching and dissolution of the cell-walls of a layer of the nucellus. I am not acquainted with anything in the vegetable kingdom corresponding to this. One could only point out, in explanation, that in many cases the developing embryo-sac exercises a solvent action on the surrounding tissue of the nucellus, and that in the present case a similar action is exerted on those cells of the nucellus which are turned towards the chalaza; these cells only disappear completely, when the embryo proceeds to develop there.

The phenomenon also suggests, that, to a certain extent, it is comparable to that of nucellar embryos. By this I mean, that these nucellar embryos prove the existence of causes, acting in the embryo-sac, which determine a developing cell to become an embryo. What these causes are, we do not know, but it is by no means inconceivable, that some day we may know them completely and even be able to imitate them, so that we may be able to produce an embryo at will. Similarly this phenomenon in *Podostemaceae* seems to me to prove, that there are causes acting in the ovule, which favour the development of such a large cavity as the embryo-sac, so that in those cases, in which the embryo-sac itself does not develop greatly, because it is enclosed and separated off in the upper part of the ovule, the cavity is formed by other cells, lying underneath the embryo-sac.

3. The development of the embryo-sac departs widely from the normal, in that no antipodal cells and no antipodal polar nucleus

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are formed, on account of the early degeneration of the nucleus, which, by its divisions should have given rise to these nuclei. Further more, after the egg-apparatus has been formed, the remaining portion of the embryo-sac is only very slightly developed, so that there is no question of the formation of endosperm (what happens to the second generative nucleus, if indeed present, I have not been able to make out). It is much clearer here than in most cases, that this portion of the embryo-sac and the egg-cell are sister-cells. This agrees with the view of PORSCH [1]), according to whom the egg-apparatus of the higher plants is a reduced archegonium, the synergids being the neck canal-cells and the upper part of the embryo-sac with the upper polar nucleus being the ventral canal-cell. The latter hypothesis is however specially difficult in this case, for here the positions of egg-cell and of ventral canal-cell would be exactly reversed. A reduction in the antipodal apparatus, similar to that which occurs here, is found in *Helosis guyanensis*, according to the investigations of CHODAT and BERNARD [2]), and a still further reduction exists in *Cypripedium*, where, according to the researches of Miss PACE [3]), the lower portion of the embryo-sac has not even been laid down at all. It need scarcely be argued, that we are here concerned with a progressive differentiation, and not with the recurrence of ancestral peculiarities. Perhaps it may not be amiss to point out, in conclusion that we cannot here fall back for "explanation" on a parasitic or saprophytic mode of life of *Podostemaceae*.

Mathematics. — *"On twisted curves of genus two"*. By Prof. J. DE VRIES.

1. A curve of genus *two* bears one and only one involution of pairs of points I^2. On the nodal biquadratic plane curve it is determined by a pencil of rays, having the node as vertex; its coincidences are then the points of contact of the six rays touching the curve. If we could arrange the points of the curve in a second I^2 then this I^2 would be projected out of the node by a system of rays with correspondence [2], and the above six tangents would furnish six rays of ramification whilst a [2] can have four only.

1) O. PORSCH. Versuch einer phylogenetischen Erklärung des Embryosackes und der doppelten Befruchtung der Angiospermen. Jena 1907.

2) R. CHODAT et C. BERNARD. Sur le sac embryonnaire de l'Helosis guyanensis. Journal de Botanique T. XIV. 1900. p, 72.

3) LULA PACE. Fertilization in Cypripedium. Botanical Gazette. XLIV. 1907. p. 353.

2 We shall now consider the fundamental involution of pairs of points, F^2, on a twisted curve ϱ^n of genus two. It can be generated by a pencil of cones of order $(n-3)$. For, through an arbitrary point P pass $\frac{1}{2}(n-1)(n-2)-2$ bisecants of ϱ^n, and the cones of order $(n-3)$ through these $\frac{1}{2}(n-3)n-1$ right lines intersect ϱ^n in two variable points more.

This F^2 arranges the planes through the arbitrary right line a in an $[n]$-correspondence. If a is cut by a bisecant b bearing a pair of F^2, then the plane ab is a double coincidence of $[n]$, for it corresponds to $(n-2)$ planes with which it does not coincide. On the other hand each coincidence of F^2 determines a single coincidence of $[n]$. The number of double coincidences amounts thus to $\frac{1}{2}(2n-6)=n-3$, so that a is cut by $(n-3)$ bisecants b. In other words:

(*The right lines bearing the pairs of the fundamental involution form a scroll of order* $(n-3)$).

To determine the genus of this scroll φ^{n-3} we make use of a wellknown formula of ZEUTHEN. When there is between the points of two curves c and c' such a relation that to a point P of c correspond $\varkappa'$ points P' of c' and to a point P' correspond $\varkappa$ points P, whilst it happens y' times that two points P' and y times that two points P coincide, then the genus p and the genus p' of the curves are connected with the numbers mentioned before by the equation [1])

$$2\varkappa'(p-1)-2\varkappa(p'-1)=y-y'.$$

If now the points P and P^* of a pair of F^2 correspond to the point of intersection P' of the line connecting them and a fixed plane, then $p=2$, $\varkappa'=1$, $\varkappa=2$, $y'=0$, $y=6$, so $2-4(p'-1)=6$ and $p'=0$.

So the scroll φ^{n-3} is of *genus zero* and possesses therefore a *nodal curve* of order $\frac{1}{2}(n-4)(n-5)$.

For a ϱ^6 this involutory scroll is quadratic, so it is a *hyperboloid* or a *cone*.

In the former case one of the systems of generatrices consists of trisecants, the other of the bisecants bearing the pairs of F^2. The points of support of the trisecants are then arranged in the triplets of an involution which is likewise fundamental (i. o. w. given with the curve). That the latter has eight coincidences is easy to see from the (2,3)-correspondence between the two systems of generatices.

By central projection we find a quadrinodal plane curve c^6, on

1) See ZEUTHEN, Math. Ann. III, 150. A simple proof has been given by KLUYVER (N. Archief v. W, XVII, 16).

which F^2 is cut by the conics containing the four nodes, whilst the lines connecting the pairs envelop a conic and at the same time bear the groups of a fundamental I^3 [1]).

If the involutory scroll of F^2 is a quadratic cone then every two pairs of F^2 lie in a plane through the vertex, which is at the same time a point of ϱ^5. This special ϱ^5 is evidently the section of a cubic surface and a quadratic cone, having a right line in common [2]).

4. We shall now consider a ϱ^6 of genus two. The involutory scroll of F^2 is now of order three (Φ^3). Let q be the double line, e the single director of Φ^3. As ϱ^6 lies on Φ^3 and a plane through q contains but one right line of Φ^3 which line bears a pair of F^2, we find that q has four points in common with ϱ^6, so it is a *quadrisecant.* So the fundamental involution is described by the pencil of planes having the quadrisecant as axis. From this is at the same time evident that ϱ^6 cannot have a second quadrisecant.

Each plane through e bears two pairs of F^2, so e is a chord of ϱ^6, and the pairs of F^2 are connected in pairs to form the groups of a particular I^4.

The planes connecting e with the two torsal right lines of Φ^3 are evidently double tangential planes of ϱ^6. On e therefore rest besides the tangents in the 6 coincidences of F^2 still the 4 tangents situated in those double tangential planes and the tangents to be counted double in the points of support of the chord e. The developable *surface of tangents* of ϱ^6 is therefore of order 14. This is evident also from the fact that the quadrisecant besides by the tangents in its points of support is intersected only by the six tangents of the coincidences.

By central projection out of a point of e we find a special c^6 with eight nodes of which the pairs of F^2 lie two by two on rays through a node which is at the same time the point of intersection of two nodal tangents.

5. The scroll β of the bisecants resting on a trisecant t is, like ϱ^6, of genus *two*. For, if the points B_1, B_2, B_3 of ϱ^6 lie with t in one plane then we can make each point B_k to correspond to the chord

1) A number of properties of c^5 are to be found in my paper: "Ueber Curven fünfter Ordnung mit vier Doppelpunkten" (Sitz. Ber. Akad. Wien, 1895, CIV, 46—59). The curves ϱ^5 and c^5 are treated by H. E. Timerding "Ueber eine Raumcurve fünfter Ordnung" (Journal f. d. r. u. a. Math., 1901, CXXIII, 284—311).

2) The central projection of this ϱ^5 has been treated in my paper quoted before page 63. It is generated by stating a projective correspondence between the rays of a pencil and the pairs of an involution, formed of the conics of a pencil.

$B_l B_m$, by which a (1, 1)-correspondence is determined between the points of ϱ^6 and the points of a plane section of the scroll β.

As each point of t evidently bears 5 bisecants, whilst a plane through t contains 3, the scroll β is a scroll of order 8. A plane section must now show singularities equivalent to 19 nodes. Now the intersection of t is a 5-fold point whilst the 6 intersections of ϱ^6 furnish as many nodes; the missing three nodes are evidently substituted by a threefold point which is the intersection of a trisecant resting on t.

So on the *scroll* τ *of the trisecants* these are arranged in pairs of an involution.

Furthermore follows from this that the scroll τ is of order 12. For, if x is the order of τ, then one of the $(x-1)$ points which t has in common with the remainder section in a plane laid through t is to be regarded as intersection of t; the remaining $(x-2)$ are derived from multiple curves. Now t is cut outside ϱ^6 by one trisecant and in each of its points of support by three trisecants, so $x-2=10$ and $x=12$.

6. Out of a point C of ϱ^6 we find F^2 projected on the curve in the triplets of an involution C^3.

For, if P is a point of ϱ^6 then the right line CP cuts the scroll Φ^3 in an other point F, and the plane through C, F and the point conjugate to it in F^2 determines on ϱ^6 two points P' and P'' more, forming with P an involutory group.

The planes $\pi \equiv PP'P''$ envelop a quadratic cone, namely the tangential cone of Φ^3 having C as vertex. A right line l through C is thus cut by two triplets of chords PP' situated in the two planes π through l; but moreover by the two chords connecting C with the two connecting points C' and C''. *The involutory scroll of* C^3 *is therefore of order eight.*

As we conjugate P to the chord $P'P''$ this scroll is also of genus *two*. In a plane section the point of intersection with ϱ^6 are nodes. From this ensues that there must be (see § 5) a *nodal curve* of order *thirteen*.

The central projection of ϱ^6 out of C is a quadrinodal c^5 upon which each group of C^3 is collinear to a pair of F^2. If we regard c^5 as central projection of a ϱ^5 then C^3 originates from the I^3 on the trisecants; consequently C^3 has like the last mentioned I^3 eight coincidences.

7. If we bring a cubic surface ψ^3 through 19 points of ϱ^6, this

curve lies on ψ^3, so it is the partial section of ψ^3 with the involutory scroll Φ^3. As q is nodal line of Φ^3 and single right line of ψ^3, the two surfaces have another line r in common. This r cannot coincide with the single directrix e, for then each right line of Φ^3 would have four points in common with ψ^3, viz: its points of intersection with ϱ^6, q and e; the surface ψ^3 would then however coincide with Φ^3.

Inversely we can regard ϱ^6 as section of a cubic scroll Φ^3 with nodal line q and a cubic surface ψ^3 having with Φ^3 the right line q in common and a right line r resting on the former one. A plane π through q cuts Φ^3 in a right line, ψ^3 in a conic, so it contains besides q two points of the curve of intersection, from which is evident that q is a quadrisecant; its points of support are coincidences of the (1,4)-correspondences between the points of contact of π with the two surfaces, one of the five coincidences is the point of intersection of q and r. That the single directrix of Φ^3 is a chord of ϱ^6, is evident f om the fact that it cuts ψ^3 on r, thus two times on ϱ^6.

8. If Φ^3 is replaced by a scroll of CAYLEY so that q is single directrix and at the same time generating line, then the conic of ψ^3, lying in the torsal tangential plane of Φ^3 determines on q two points each of which replaces in each plane π through q two points of intersection with ϱ^6; so they are *nodes* of ϱ^6. On this special curve the groups of F^2 are not arranged in pairs; for e coincides with q.

We obtain an other special ϱ^6 by taking instead of Φ^3 a cone with nodal edge q. The conics of ψ^3 situated in the planes touching Φ^3 along the nodal edge cut q in the points of support of the quadrisecant. Each edge of Φ^3 bears a pair of F^2, so that a plane through the vertex T contains three pairs.

The tangential cone out of T to ψ^3 has q and r as nodal edges; the six single edges which it has in common with Φ^3 are evidently tangents of ϱ^6 and contain the coincidences of F^2.

Through an arbitrary point O pass four tangential planes to Φ^3; the central projection of ϱ^6 furnishes a plane curve c^6 with four nodal tangents meeting in a single point C. The six single tangents out of C contain the coincidences of the fundamental involution, each ray of which through C bears three pairs. These are separated if we describe on F^2 a pencil of cubic curves having the eight nodes of c^6 as base-points.

Mathematics. — "*On algebraic twisted curves on scrolls of order n with $(n-1)$-fold right line.*" By Prof. JAN DE VRIES.

1. If we intersect a cubic scroll Φ^3 by a pencil of planes having a generatrix a of Φ^3 as axis, we get a system of conics ϱ^2, all passing through the point O, where a meets the double right line d. If we take a (p, q)-correspondence between this pencil of planes and the pencil of planes with axis d, then in this way to each ϱ^2 are assigned p right lines r of Φ^3 and to each right line r evidently q conics ϱ^2. The locus of the points of intersection of the lines r and ϱ^2 corresponding to each other is a twisted curve of order $m=p+q$; for the points of the rational cubic curve which Φ^3 determines on an arbitrary plane are arranged in a (p, q)-correspondence, of which each coincidence is the point of intersection of a ϱ^2 with a right line r corresponding to it.

The twisted curve ϱ^m has the right lines r as q-fold secants, whilst it is intersected by each of the ∞^2 conics of Φ^3 in p points.

2. If Φ^3 is represented by central projection out of O on a plane τ cutting a and d in A and D, then the systems (r) and (ϱ^2) are transformed into the pencils (D) and (A) which are now likewise arranged in a (p, q)-correspondence. The curve c^m generated in this way has in D a p-fold point, in A a q-fold one. But it has moreover a q-fold point in the point of intersection B of the right line b of Φ^3, which still passes through O, for b is q-fold secant of ϱ^m. From this ensues that the correspondence (p, q) in τ cannot be taken arbitrarily.

The curve ϱ^m is completely determined by its central projection c^m. For, the cone projecting c^m out of O has a p-fold edge along d and q-fold edges along a and b, so its section with Φ^3 consists of $2p+2q=2m$ right lines and a twisted curve of order m having p points in common with d and q points with a.

As the singular points of c^m are equivalent to $\frac{1}{2}p(p-1)+q(q-1)$ nodes, the genus of c^m is indicated by

$$g=\frac{1}{2}(p+q-1)(p+q-2)-\frac{1}{2}p(p-1)-q(q-1)=$$

$$=(p-1)(q-1)-\frac{1}{2}q(q-1),$$

or by

$$g=(m-1)(q-1)-\frac{3}{2}q(q-1).$$

This is at the same time the genus of ϱ^m. It is evident that p may not be smaller than $(\frac{1}{2}q + 1)$. For the smallest values of p and q we have

m	p	q	g
2	1	1	0
3	2	1	0
4	3	1	0
4	2	2	0
5	4	1	0
5	3	2	1
6	5	1	0
6	4	2	2
6	3	3	1

The above considerations may be extended by taking instead of scroll Φ^3 a scroll Φ^n with $(n-1)$-fold right line d. Out of a point O of d now start $(n-1)$ right lines $a_1, a_2, \ldots a_{n-1}$. A (p, q)-correspondence between the pencils of planes (a_1) and (d) determines again a twisted curve of order $p + q = m$, having as central projection a c^m with p-fold point D and q-fold points in $A_1, A_2, \ldots A_{n-1}$; and inversely ϱ^m is again entirely determined by c^m. For the genus of ϱ^m (and c^m) we now find

$$g = \frac{1}{2}(p + q - 1)(p + q - 2) - \frac{1}{2}p(p-1) - \frac{1}{2}q(q-1)(n-1)$$

or

$$g = (q-1)(m-1) - \frac{1}{2}q(q-1)n.$$

To obtain general twisted curves we shall not be allowed to take p larger than 4, q larger than 3. For $n = 2$ we find evidently the well known considerations concerning curves on an hyperboloid.

5. If we substitute in τ for the curve c^n a curve passing p times through D, q times through A_k and moreover cutting the right line DA_k in s points, then this curve is evidently the central projection of a curve on Φ^n, having a multiple point in O. For, each of the $(n-1)$ tangential planes in O will now contain s right lines, touching the twisted curve in O.

Physics. — *"The influence of temperature and magnetisation on selective absorption spectra."* III. By Prof. H. E. J. G. du Bois and G. J. Elias. (Communication from the Bosscha-Laboratory).

§ 20. Since our former communication (These Proc. March p. 734) we have obtained a number of samples, the crystallisation of which in reasonable sizes from solutions in water or amylic acetate was brought about only after many failures and many weeks of patience. Notices concerning the influence of the anion or the temperature only on the absorption spectrum, must be laid aside as being too extensive, though incidentally some details may appear about it. With respect to the Zeeman-effect we shall also confine ourselves to a choice from the profuse material, which for the present can be little more than an enumeration of the many ways in which the influence of the magnetisation may manifest itself; it must be reserved for further investigation to impart more order and regularity to the present rather unsystematic series of results.

As a rule we worked again in the spectrum of the first order; in some cases we had recourse to the second order, in which some special effect may sometimes be better judged, at least from a qualitative point of view; for measurements the first order proved preferable on the whole. As we have never to do with very fine lines, too great a dispersion is of no use here, and certainly of much less importance than a strong magnetic resolving power. Very thin crystal chips — some tenths of a mm. thick — already exhibiting jet-black absorption bands, particularly for neodymium salts, we could use these, and expose them to very strong fields, mostly of 38—42 kilogauss. The distinctness of the spectral image depends to some extent on the choice of the proper thickness for every salt. The fields were measured with a bismuth spiral; the disturbance by the narrow slits is certainly less than with the usual round bores; the increase of the — saturated — magnetisation values of the polar end-pieces on cooling them down to -190° is probably slight; it would be desirable to obtain further information concerning these points. We consider, however, the accuracy of our field-measurements of the same order as that of the readings in the spectrum. We again preferred the latter to a photographic reproduction; for with visual observation the identification of the lines with field on and field off, especially with erbium compounds, proved to be decidedly easier.

§ 21. Third series. We have investigated a few organic double salts of chromium and potassium with a view to a possible

ZEEMAN-effect, the absorption spectra of which were fully described at ordinary temperature by LAPRAIK [1]). The so-called "blue" (dichroïtic red-blue) chromium-potassium oxalate $[Cr_2K_6(C_2O_4)_6 + 6\,H_2O]$ mentioned in our first paper exhibited in liquid air a strong band 696,4—701,4 (cf. § 5), evidently still too broad to be taken into account. This oxalate may not be mistaken for the so-called "red" compound:

Chromium-potassium oxalate $[Cr_2\,K_2\,(C_2O_4)_4 + x\,H_2O$; different authors consider $x = 8, 10, 12]$; this was obtained by CROFT in 1842, and its absorption-spectrum was investigated by BREWSTER [2]). Strongly dichroïtic (claret hue-bluish grey) probably monoclinic crystals. At $-190°$ a number of fine bands and lines in the red are seen with the spectrometer, the most striking of which are a rather strong band 680,0, and a strong band 692,5 between the red ruby bands $R_2 = 691,8$ and $R_1 = 693,2$ (comp. §§ 7, 17).

A plate, 1.5 m.m. thick had to be examined with sunlight on account of its strong absorption; for the same reason the crystallographic orientation could not be determined. At $-193°$ line 692,5 had a width with field off of 0,14 $\mu\mu$ with non-polarized light; in a field of 36,5 kgs. the widening amounted to about 0,05 $\mu\mu$.

Chromium-potassium malonate $[Cr_2K_6(C_3H_2O_4)_6 + 6H_2O]$, is evidently homologous with the "blue" oxalate. This could only be obtained as an interlaced dark crystal magma with irregular orientation, dichroïtic (grass green-sky blue). In the red at $-193°$ we find a strong band, the middle of which 693,3 coincides pretty nearly with the red ruby band $R_1 = 693,2$; and a broader rather faint band 698,3. A sample of a thickness of only 0,15 mm. exhibited band 693,3 with a width with field off of 0,8 $\mu\mu$ with unpolarized light; moreover it appears to have shifted 0,8 $\mu\mu$ towards the red with respect to the corresponding band of the oxalate above mentioned. In a field of more than 40 kgs. the band became distinctly vaguer and almost disappeared. We had no opportunity as yet to examine a malonate homologous with the "red" oxalate; perhaps the phenomenon would appear more clearly still in this case.

§ 22. Fifth series. We have now made a closer examination of some salts of the four metals *Pr*, *Nd*, *Sm* and *Er*, such as had been used in 1899.

Praseodymium sulphate $[Pr_2(SO_4)_3 .\ 8H_2O]$. Light green plate, containing both optical axes, 0,6 mm. thick. Exhibits several not

1) W. LAPRAIK, Journ. f. prakt. Chemie (2) **47** p. 307, 1893.

2) A. ROSENHEIM, Zeitschr. f. anorg. Chemie **11** p. 196, 1896; and **28** p. 337, 1901.

very narrow bands at — 193° in the violet and blue; in the orange some heavy broad bands, moreover a strong band 599,0—599,3, a pretty faint band 600,9—601,4. The plate was now investigated with the median line (dividing the acute angle formed by the axes into two equal parts) vertical in a field of 40 kgs.

With vertically polarized light band 599,0—599,3 appeared to be subject to a distinct widening of 0,1 $\mu\mu$; the other band also became wider and vaguer. With horizontally polarized light the phenomenon was analogous, but less distinctly to be seen; on the other hand some of the wider bands then show an unmistakable widening [1]).

Neodymium sulphate $[Nd_2(SO_4)_3 . 8H_2O]$.

§ 23. As a supplement to what was communicated in § 19 a number of plates of different thickness were more fully examined; they again contained both optical axes; the line dividing the acute angle was again placed in a vertical position.

Group of bands in the blue at — 193°; 8 of these bands were measured. For the sake of brevity we have been obliged to draw up the results in a table, where λ denotes the wave-length, β_0 the width with field off, β_x the width in a field of x kgs., $d\beta$ the widening; in case a multiplet is formed, the distance of the centre-lines of the extreme components is denoted by $d\lambda$; the value of $d\lambda/\lambda^2$ is expressed in cm^{-1}, as is now usually done.

41 Kilogauss — Plane of polarisation horizontal — Thickness 0.3 mm

	I	II	III	IV	V	VI	VII	VIII	
λ	469.5	472.8	474 0	474.5	475 3	476.2	477.0	477.4	$\mu\mu$
β_0	0.26	0.26	0.14	0.05	0.035	0.16	0.105	0.09	"
β_{41}	0.26	0.35	—	0 105	--	0 195	0.21	0.23	"
$d\beta$	0	0.09	—	0.055	—	0.035	0.105	0.14	"
$d\lambda$	—	—	0.22	—	0.18	—	—	—	"
$d\lambda/\lambda^2$	—	—	9.8	—	8.0	—	—	—	cm.$^{-1}$

1) From a copy that Prof. KAMERLINGH ONNES kindly sent us of the paper by himself and Mr. J. BECQUEREL (These Proc. X, p. 592) we now infer that the results given for the silicates of *Pr* and *Nd* really apply to the sulphates; we had then nearly finished our observations; as, moreover, these were made at — 193° instead of — 253° and in a much stronger field, the two series of results are not directly comparable; but they may serve to complete each other.

Here III formed an asymmetric doublet, of which the component on the red side was the narrower; V a faint doublet; for VI the middle became somewhat lighter.

With a vertical plane of polarisation all the bands became vaguer, most of them slightly shifting towards the red; in the field the bands widened or further faded away; now only III gave a symmetrical doublet.

Band in the blue-green at — 193°: with a horizontal plane of polarisation $\lambda = 511{,}9$, $\beta_0 = 0{,}13\ \mu\mu$; in a field of 42 kgs. a doublet appeared: width of the lefthand line 0,13, of the righthand line 0,18, of the light interval 0,09 $\mu\mu$; the whole made the impression of perhaps being a quadruplet. With a vertical plane of polarisation the phenomenon was analogous but less clear.

Group of bands in the green at — 193°: 6 bands were measured.

42 Kilogauss. — Plane of polarisation horizontal — Thickness 0,3 mm.

	I	II	III	IV	V	VI	
λ	521.2	523.0	523.9	525.3	526.0	527.5	$\mu\mu$
β_0	0.49	0.105	0.355	0.10	0.13	0.195	»
β_{42}	0.58	—	—	0.15	0.275	0.275	»
$d\beta$	0.09	—	—	0.05	0 145	0.08	»
$d\lambda$	—	0.26	0.29	—	—	—	»
$d\lambda/\lambda^2$	—	9.5	10.5	—	—	—	cm^{-1}.

Band II gave an ordinary doublet; that of III remained rather dark in the middle, so that we may infer a more complex structure in this case also.

With a vertical plane of polarisation all this was less clearly visible, band II. still gave a clear doublet, for III only a trace of this could be perceived.

Group of bands in the yellow at — 193°. Two rather sharply defined bands 576,0 ($\beta_0 = 0{,}3$) and 586,0 ($\beta_0 = 0{,}14$) exhibited a distinct widening of 0,05 $\mu\mu$ in a field of 42 kgs. The intermediate bands are too wide for this kind of observation.

Group of bands in the orange-red at — 193°: 5 bands were measured.

38 Kilogauss. — Plane of polarisation horizontal — Thickness 0,6 mm

	I	II	III	IV	V	
λ	623.2	624.1	625 6	627.2	628.3	$\mu\mu$
β_0	0.31	0.13	0.18	0.13	0.08	"
β_{38}	—	0.26	—	0 15	—	"
$d\beta$	—	0.13	—	0.02	—	"
$d\lambda$	0.22*	—	0.46*	—	0.34	"
$d\lambda/\lambda^2$	5 7	—	10.2	—	8.6	cm^{-1}.

Here I and III yielded very blurred doublets, of which the distances * of the extreme limits are given; V a distinct doublet with a shade between the components, perhaps a quadruplet.

With a vertical plane of polarisation the phenomenon was analogous and was confirmed with a thinner plate: I was a doublet, III was invisible here, with IV some light appeared in the middle with greater widening, structure probably complicated, V was again a very distinct doublet.

Group of bands in the red at —193°: Four bands, among which two rather sharp ones 674,4 and 676,2 showed a widening or a fading away in the field; the last-mentioned became a doublet, perhaps even a quadruplet, with plane of polarisation vertical.

Neodymium nitrate $[Nd(NO_3)_3 \,.\, 6\, H_2O]$.

§ 24. It appeared important to investigate crystals besides the amorphous nitrate (§ 9); with a much slighter thickness crystals show intense and narrow absorption bands; the natural monoclinic plates were directed perpendicularly to one of the optical axes, so that in this case a nicol could be done away with. The wave-lengths of the bands are on the whole slightly less — down to 3 $\mu\mu$ — for the nitrate than for the sulphate.

Band in the blue-green at —193°. The wave-length now was $\lambda = 511{,}3\ \mu\mu$; in a field of 41 kgs. a doublet appeared, the components of which had a distance of 0,22 $\mu\mu$.

Group of bands in the green at —193°. 5 bands were measured in the spectrum of the second order.

41 Kilogauss.				Thickness 0.2 mm.		
	I	II	III	IV	V	
λ	521.3	522 3	523.1	524 6	525.0	$\mu\mu$
β_0	0.155	0.11	0 18	0.09	0 045	"
β_{41}	0 265	0 18	0.22	—	0.11	"
$d\beta$	0 11	0.07	0.04	—	0.065	"
$d\lambda$	—	—	—	0.22	—	"
$d\lambda/\lambda^2$	—	—	—	8.0	—	cm^{-1}.

The doublet IV showed a shade in the middle.

Group of bands in the yellow at —193°. Two rather sharp bands 581,9 and 583,1 exhibited a widening of 0.05 $\mu\mu$ in a field of 42 kgs.; the others were too broad and too hazy.

Group of bands in the orange-red at —193°. 3 bands were measured.

40 Kilogauss.			Thickness 0.45 mm.		
	I	II	III S	III	
λ	624.2	625.2	(626.7)	626.9	$\mu\mu$
β_0	0.265	0.18	(0.05)	0.14	"
$d\lambda$	0 5	0 5	–	0.5	"
$d\lambda/\lambda^2$	12.8	12.8	—	12.8	cm^{-1}

Doublets I and III were normal, II on the other hand was asymmetric, the component on the red side being weaker; the satellite III S was no longer visible in the field.

Group of bands in the red at —193° : 8 bands were measured.

40 Kilogauss.							Thickness 0.45 mm.	
	I	II	III	IV	V	VI	VII	VIII	
	671.0	672.0	673 3	674.3	675.2	675 8	676.6	677.2	$\mu\mu$
β_0	0.31	0.26	0.25	0.22	0.25	0.26	0 22	0.31	"
β_{40}	v.	0.52	0.45	0.32	v.	v.	—	—	"
$d\beta$	—	0.26	0.20	0.10	—	—	—	—	"
$d\lambda$	—	—	—	—	—	—	0.80	0.90	"
$d\lambda/\lambda^2$	—	—	—	—	—	—	17.5	19.6	cm^{-1}

Band I, V and VI vanished in the field; with the field off II showed a dark core with shades on either side, which disappeared in the field. The two doublets VII and VIII gave the greatest resolution measured as yet — $1{,}5 \times (D_1 - D_2)$; with the field used the components facing each other happened to coincide, so that the pair of doublets looked like a very wide triplet with a heavy middle band. It ought to be possible to observe a phenomenon of this order of magnitude with every good spectroscope.

Finally we mention that the neodymium-magnesium nitrate of § 9 also occurs in hexagonal crystals; such optically uniaxial crystals are of great interest (§ 19); there also exists an isomorphous series of salts, which contain manganese, cobalt, nickel or zinc. Measurements on this subject have been made, partly they are still in preparation.

Samarium sulphate $[Sm_2(SO_4)_3 \,.\, 8\, H_2O]$.

§ 25. We now examined a more transparent sample (cf. § 19), which again contained both optical axes, and was placed like the other sulphates; 4 bands in the green were measured at —193°.

40 Kilogauss — Plane of Polarisation horizontal — Thickness 0.8 mm.

	I	II	III	IV	
λ	—	—	558.2	559.1	$\mu\mu$
β_0	0.09	0 09	0.105	0.18	"
β_{40}	0.18	0.11	0 18	0.31	"
$d\beta_{40}$	0.09	0.02	0 075	0.13	"

The effect was apparently small here; the widened bands were vague. With a vertical plane of polarisation the phenomenon scarcely changed.

Erbiumyttrium sulphate $[(Er, Y)_2\,(SO_4)_3]$.

§ 26. We also examined an impure product obtained by treatment of the original minerals with sulphuric acid, in which erbium and yttrium occur in variable percentages, and the latter preponderates: the crystals were monoclinic. The group of bands in the green, yellow-green, and red showed peculiar and intricate effects of mag-

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netisation; among others some bands which were hardly visible with field off were much more pronounced with the field on, in contrast to most of the other cases observed. Further measurements on this point are in progress.

It is quite probable that this product also contains other rare earth-metals (e.g. dysprosium and holmium); this is, moreover also rather likely for the other erbium salts.

Erbium nitrate [$Er(NO_3)_3 . 6H_2O$].

§ 27. Here too, besides the amorphous salts (§ 10) monoclinic crystal plates of an average thickness of 0.6 m.m. were examined, containing both optical axes. The bands were finer than for any sample examined before. On account of the very complicated resolutions it was often somewhat difficult to ascertain to what bands the different components belonged; on exciting the field a sudden confusion was observed from which the single bands slowly emerged again on breaking the current. These observations were all made with unpolarized light.

The results are best arranged in a table in a way somewhat different from the above.

Group of bands in the green at — 193°.

λ	β_0	Influence of a field of 39 Kilogauss.
516.4	.0.17	increases in width (not measured).
517 2 shade	0.13	gives a quadruplet, the outer lines of which are very fine, the middle ones (from violet to red) resp. 0.12 and 0.15 $\mu\mu$ wide, the distances of the middles amounting respectively to 0.08; 0.27; 0.08 $\mu\mu$, while the middle of the outer components seems to have shifted about 0.01 $\mu\mu$ towards the red with respect to the line with field off.
517.6	very narrow	no more visible.
518.0 518.3 518.6	0.06 0.11 0.07	give a very complex set of lines, which ought to be further investigated.

λ	β_0	Influence of a field of 39 Kilogauss.
519.1	0.10	gives a doublet (not measured).
519.7	0.16	gives an asymmetric quadruplet, the extreme lines of which are very fine; those on the violet side are very faint; the middle ones are resp. (from violet to red) 0.11 wide (this one very faint) and 0.135 $\mu\mu$; the distances of the middles are resp. 0.07, 0.255, 0.13 $\mu\mu$, the middle between the extreme components seeming to have been displaced 0.05 $\mu\mu$ towards violet with respect to the line with field off.
520.2	0.17	gives an asymmetric doublet, of which the component on the violet side is 0.05 $\mu\mu$ wide, the other very narrow; distance of the middles 0.175 $\mu\mu$; the mean of these middles is not sensibly displaced with respect to the line with field off.
520·7	0.14	gives an asymmetric triplet (doublet with satellite on violet side); not measured.
521.3	0.11	gives an asymmetric quadruplet; outer component on violet side rather strong, on red side feeble; middle components stronger.

§ 28. *Group of bands in the yellow-green at* — 193°

λ	β_0	Influence of a field of 39 Kilogauss.
534.8	0.13 (faint)	widens and fades away, not measurable.
535.5 shade[1)] 535.8	0.13 0.09	gives an asymmetric doublet, consisting of (from violet to red) first a shade of a width of 0.29 $\mu\mu$, then a strong band 0.18 $\mu\mu$ wide, then a shade, and at last a faint undefined band 0.26 $\mu\mu$ wide; the middle of the first component has shifted 0.17 $\mu\mu$ towards the violet side with respect to the original band 535.5, the place of the middle of the second component being 535.8.

1) With a somewhat thicker crystal they form together a heavy band.

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λ	β_0	Influence of a field of 39 Kilogauss.
536.95 537.15	narrow satellite 0.05	give together, seen in the first order, a triplet, the components of which (from violet to red) are resp. 0.10; 0.05; 0.07 $\mu\mu$, wide, the first very faint, the last two stronger, and connected by a shade; the situation of the middles of the first two is shifted resp. 0.44 and 0.14 $\mu\mu$ towards the violet, that of the last 0.05 $\mu\mu$ towards the red with respect to the band 537.15 with field off. Seen in the second order the line 536.95 becomes a symmetric doublet, about 0.44 $\mu\mu$ apart; the line 537.15 an asymmetric doublet, of which the component on the red side is very heavy.
537.35 537.6 537.8	narrow feeble satellite 0.06 narrow satellite somewhat stronger than the former	together give a triplet, the components of which (from violet to red) are resp. 0.08; 0.08; 0.10 $\mu\mu$, wide, the first two strong, and connected by a shade, the last very faint; the place of the middle of the first component has moved 0.02 $\mu\mu$ towards the violet, that of the two following ones resp. 0.07 and 0.34 $\mu\mu$ towards the red with respect to the original line 537.6.
538.5	narrow	gives a doublet (not measured).
539.15	0,09	gives a doublet, with components each 0.08 $\mu\mu$ wide, and distance of the middles 0.30 $\mu\mu$; on the violet side another shade is seen, where possibly a third component is found; the middle of the two lines of the doublet seems to have shifted 0.03 $\mu\mu$ towards the violet with respect to the original line 539.15.
539.7	0.08	gives an asymmetric sextuplet of which the four outer components are faint, and very narrow, the two middle ones heavy, and resp. (from violet to red) 0.06 and 0.03 $\mu\mu$ wide, the distances of the middles amounting resp. to: 0.045; 0.05; 0.11; 0.17; 0.035, total 0.41 $\mu\mu$. The middle of the two outer components coincides with the line with field off. The two com-

λ	β_0	Influence of a field of 39 Kilogauss.
		ponents on the violet side are connected by a shade.
540.3	0.07	gives a somewhat vague band, 0.22 $\mu\mu$ wide, in which lines could not be distinguished with certainty: it may, however, be a triplet. The middle seems to have been displaced 0.03 $\mu\mu$ towards the violet with respect to the line with field off.
540.8	0.07	gives an asymmetric quadruplet, of which the three components lying on the violet side are rather strong, the fourth weak; the mutual distances of the components are very nearly the same, and amount to 0.18 $\mu\mu$.

§ 29. *Group of bands in the red at* — 193°.

λ	β_0	Influence of a field of 40 Kilogauss.
640.3	not measured	gives an asymmetric doublet, (not measured).
640.9	not measured	gives an asymmetric doublet, (not measured).
642.2	0.09	gives a doublet, consisting of a faint, thin line on the violet side, and a strong line, 0.11 $\mu\mu$ wide, the middle of which is at a distance of 0.09 $\mu\mu$ from the other faint line, and has moved 0.18 $\mu\mu$ towards the red with respect to the line with field off. Probably on the red edge of this strong line another faint line is found, which is connected with it, so that the whole would form a triplet.
642.8	0.11	gives a triplet, the extreme components of which are weak, and very narrow, the middle ones strong, and 0.125 $\mu\mu$ wide; the distance of the middles (from violet towards red) amount resp. to 0.11 and 0.16 $\mu\mu$; the middle of the inner components has been displaced 0.035 $\mu\mu$ towards the red with respect to the line with field off.

λ	β_0	Influence of a field of 40 Kilogauss.
650.5	0.09 (somewhat vague)	is widened; the width amounts to 0.35 $\mu\mu$, the band on the violet side is darker than the one lying towards the red; (possibly a quadruplet is formed, which, however, is uncertain); a displacement, however, was not observed.
651.3	0.09 (somewhat vague)	increases in width and fades away; the width amounts to 0.40 $\mu\mu$; a shift towards violet seems to take place, but could not be ascertained.

§ 30. Seventh series. We now investigated.

Uranyl potassium sulphate $[UO_2 K_2 (SO_4)_2 + 2 H_2O]$.

A rhombic plate containing both axes, 0.7 m.m. thick, was examined in a field of about 40 kgs. with unpolarized light at —193°. The well-known bands appeared to be much more numerous and narrower than for uranyl nitrate (c.f. § 11), among others 487.8, 488.2, 488.8 and 490.5 in the blue. These bands seemed to fade somewhat in the field, but the phenomenon was uncertain here, and in any case the widening did not amount to more than 0.02 $\mu\mu$. For the many bands in the violet no action of the field could be perceived.

Physics. — "*The value of the self-induction according to the electron-theory.*" By Prof. J. D. VAN DER WAALS Jr. (Communicated by Prof. J. D. VAN DER WAALS).

Many physiciists refer to the existence of self-induction in order to make the existence of kinetic energy of electrons more intelligible. To a certain extent there is no objection to this, provided we keep in view, 1st that the kinetic energy consists for a large part electrical energy whereas for the calculation of the self-induction only the magnetical energy is taken into account, and 2nd that from a theoretical point of view it is the self-induction which is to be explained from the kinetic energy of the electrons, and not vice-versa. It is this second point which occasions me to make the following remarks.

Let us imagine a piece of metal which contains a great number of positively and negatively charged particles with a total charge zero, the positive charge of the positive particles being exactly equal to the negative charge of the negative ones. When we put this piece of metal into motion we must ascribe to it a certain amount of electromagnetic mass which is equal to the sum of the electromagnetic masses of the positive and negative particles. If, however, we send a current through the piece of metal, then the energy of this current is *not* equal to $\frac{1}{2}\Sigma\, mv^2$, m being the mass of a particle and v the mean velocity which is imparted to them by the electromotive force.

This difference can be explained in the following way. In the case that the piece of metal moves, the positive and negative particles move in the same direction and then in all points of space as well the electrical as the magnetical forces exerted by the different electrons have a different direction and nearly cancel each other, in such a way that we find only sensible forces in the points which are so near one of the electrons that the forces exercised by that electron strongly preponderate over those exercised by all the other electrons, and need only to be taken into account. In the case that a current passes through the metal on the other hand the magnetic forces of a great part of the electrons will act in the same direction, and in a point at some distance, where the force $\mathfrak{H}$ exerted by a *single* electron is negligibly small the magnetic force exercised by *all* the electrons contained in a unit of volume of the metal (which number we will call N) together will nearly amount to $N\mathfrak{H}$, in consequence of which the energy will be of the order $N^2\mathfrak{H}^2$. This energy proves not at all to be equal to the sum, but rather to N times the sum of the amounts of energy which the single electrons would occasion at that point. From this we may deduce that the energy of the current is much larger than $\frac{1}{2}\Sigma\, mv^2$.

Though it might be worth while to try and calculate the amount of this energy more accurately, it seems to me that there can be little doubt but we should find for it:

$$1/2\,(L + L')\,i^2$$

where L represents the coefficient of self-induction as it is usually calculated from the magnetic energy alone, and

$$1/2\,L'\,i^2 = 1/2\,\Sigma mv^2.$$

If we assume that the current is transferred by only one kind of electrons then we may write for $\Sigma \frac{1}{2}\, mv^2$ for each unit of volume $\frac{1}{2} N\, mv^2$.

If we now assume the "piece of metal" to be a circular circuit of metal wire with a radius R and the wire to have a circular section with a radius r, then we have $i = \pi r^2 N ev$ and

$$L' = \frac{\pi r^2 \, 2\pi R \, Nmv^2}{(\pi r^2 \, Nev)^2} = \frac{2R}{r^2} \, \frac{m}{Ne^2}.$$

Supposing r to be small compared with R we may calculate L for this circuit from the formula of KIRCHHOFF:

$$L = 2\,\pi\,R \left\{ l\left(\frac{8\,R}{r}\right) - 1{,}75 \right\}.$$

The number N being probably different for different metals, L' also appears to depend on the kind of metal of which the circuit consists, whereas L only depends upon the geometrical properties of the circuit. On the other hand we see that L' has a constant value for a given wire, independent of the way in which the wire is wound to a coil, whereas L depends in a high degree upon the way of coiling.

The ratio $\frac{L'}{L}$ will therefore in different cases have a very different value. We shall try to get an idea of what order this quantity can be, and inquire whether we are always justified in neglecting L' compared with L. To this purpose we can make use of the value of Ne, which has been calculated by J. J. THOMSON [1]) for bismuth. From the value of the resistance and from the variability of the resistance in the magnetic field THOMSON deduces that the value of Ne for bismuth amounts to about 0,11. If we put $\frac{e}{m} = 1{,}865.10^7$ we find

$$L' = \frac{R}{r^2} \, . \, 10^{-6} \, .$$

Metals with a greater conductivity will probably have a higher value for N. THOMSON estimates the value of N for copper or silver to be several thousands of times larger than for bismuth.

We obtain the same result by starting from the values

$$N_1 = 0{,}69 \, . \, 10^{19} \qquad N_2 = 0{,}46 \, . \, 10^{19}$$

for the numbers of positive and of negative particles per cm^3., which have been derived by DRUDE [2]) from the behaviour of bismuth in

1) J. J. THOMSON. Rapports présentés au congrès de physique à Paris, III. p. 145. 1900.

2) DRUDE. Ann. der Phys. IV Folge. 3. p. 388. 1900.

3) EDW. B. ROSA and LOUIS COHEN Bulletin of the Bureau of Standards. Vol. 4. No. 1. Reprint No. 75.

other respects. Putting $e = 10^{-20}$ we arrive at a value of Ne which does not deviate very much from that of THOMSON.

ROSA and COHEN [3]) calculate for the self-induction of a circle with $R = 25$ cm. and $r = 0{,}05$ cm.

$$L = 654{,}40496.$$

For this same circle we find

$$L' = 0{,}01,$$

so if we neglect L' we make *in this case only an error of* $\pm$ *0,002°/₀*. This value applies to bismuth, for other metals the correction is probably much smaller still. The correction is also relatively smaller when we have not one circle but many windings. On the other hand the correction is much greater if we take a thinner wire.

Notwithstanding the perfect agreement between the numbers of THOMSON and of DRUDE these values do not seem very reliable to me. It is therefore not superfluous to inquire whether we can find another way in which we might evaluate L'. Perhaps this might be done as follows.

HAGEN and RUBENS [1]) have shown that the reflective power of metals for infra-red light of large wave-length can be explained by ascribing to these metals the same conductivity for electric vibrations of the considered frequencies as for stationary currents. This seems to indicate that the mean free path between two collisions of an electron against the atoms of the metal is small compared with those wave-lengths [2]). As the same does not apply to light of a wave-length smaller than one micron, we should be inclined to deduce from these optical properties that the mean free path is not much smaller than one micron.

We cannot deny that this value of the free path is remarkably great, as we find a value for the free path for the molecules of the air at a pressure of one atmosphere, which is about 10 times smaller. But let us notwithstanding assume this value of the path-length to be correct, then it yields a new method to calculate L'. We find, namely, for the conductivity of a metal:

$$\sigma = \bar{n}\frac{2Ne^2l}{mu}$$

where u is the mean velocity of the heat-motion of the electrons,

[1]) HAGEN and RUBENS. Berl. Sitzungsber. 1903, p. 269. Ber. d. deutschen phys. Gesellsch. 1903, p. 145.

[2]) Comp. H. A LORENTZ. These Proceedings V, p. 666, 1903.

l the path-length, and n a constant, which according to DRUDE amounts to $^1/_4$, according to LORENTZ [1]) to $\sqrt{\frac{2}{3\pi}}$.

So we get:

$$L' = \frac{2R}{r^2}\frac{m}{Ne^2} = \frac{2R}{r^2}\frac{nl}{\sigma u}.$$

At $T = 300$ we may put $u = 1{,}75.10^6$. Moreover we have $\sigma = 6{,}14.10^{-4}$ for silver, and we shall assume $l = 10^{-4}$. This yields:

$$L' = \frac{R}{r^2}.5.10^{-8}.$$

This is about $^1/_{20}$ of the value found for bismuth, and not less than $^1/_{1000}$, as we expected. The error caused by neglecting L' amounts therefore for a circle of silver wire of $R = 25$ cm. and $r = 0{,}05$ cm. to $\pm$ 0,0001 %.

It seems to me as yet impossible to compute for L' a value which may be trusted to be accurate. Yet I think that the above calculations make it probable that for coils which are wound in such a manner that they have a large self-induction, the value of L' may be neglected compared with L; but that on the other hand L' may not always be neglected if the coil is wound in such a way that L is as small as possible. In the latter case it might perhaps be possible to determine L' experimentally, at least if L can then be calculated with a sufficient degree of accuracy. And if it should prove to be possible to determine the different values of L' for different metals, this would be a valuable datum for the extension of our knowledge of the motion of the electrons in metals.

Finally it may be remarked that we shall also find a considerable value for L' for a current which does not pass a metallic wire, but for instance a Röntgen-tube. The high value of the velocity of the electrons in this case, gives rise to a high amount of kinetic energy, and this "energy of the kathode-rays" will no doubt reveal itself as an increase of the self-induction of the circuit in which the tube has been inserted.

[3]) H. A. LORENTZ. These Proceedings VII, p. 448. 1904

Chemistry. — *"The action of concentrated sulphuric acid on glycerol esters of saturated monobasic fatty acids."* Preliminary communication. By B. W. van Eldik Thieme. (Communicated by Prof. S. A. Hoogewerff).

As is well known, the saponifications may generally be represented by the equation:

$$RCOOR_1 + H_2O \rightleftarrows RCOOH + R_1OH \quad . \quad . \quad . \quad . \quad (1)$$

that is we shall always obtain an equilibrium between the reacting molecules which is dependent on the temperature, on the medium and on the nature of the ester.

The velocity of the saponification is moreover very low and is vigorously accelerated by hydrogen ions; so long, however, as the quantity of the acid added does not considerably modify the nature of the medium the equilibrium will not be changed thereby.

In the technics of fat-saponification dilute sulphuric acid is used as catalyst, for instance in the Twitchell process; from the above it follows that we must not expect the process to complete its course, it is considered satisfactory when the fat is resolved to 94 à 96 $^0/_0$ of free fatty acids.

If we use a stronger acid the process becomes modified. Firstly, we are dealing with another medium (in practice where the quantity of acid is small the medium itself is changeable during the process), secondly we have besides the first process also the following:

$$RCOOR_1 + H_2SO_4 \rightleftarrows RCOOH + R_1OSO_3H \quad . \quad . \quad . \quad (2)$$

which means the expulsion of one of the acid residues by the other one.

Here also, however, we may expect the reaction to be reversible so that it will be completely to the right only when:

a. The sulphuric acid added, is anhydrous (100 $^0/_0$),

b. An excess of acid is added to dry fat,

c. The temperature at which the action takes place, is kept within definite limits. From this it follows, that the statement of Bünte [1]) that butterfat is completely saponified by sulphuric acid of sp. gr. 1.8355 (corresponding with 93.5 $^0/_0$ of H_2SO_4,) cannot possibly be correct. 5 grams of butterfat are heated in an Erlenmeijer flask of one litre capacity to 100°, 10 c.c. of 93.5 $^0/_0$ sulphuric acid are added and the whole heated for 10 minutes in a waterbath at 30—32°. 150 c.c. of water are added next.

[1]) Chem. Zeit. N⁰. 12 1894 pg. 204, also Kreis Chem. Zeit. N⁰. 76 1892 pg. 1394.

Moreover, the high temperature at which the action of the acid takes place is unsuitable for the purpose of a complete saponification, because, as will be seen, it is just the increase in temperature which causes the shifting of the equilibrium in equation (2) towards the left. On repeating BÜNTE's method I obtained the following figures:
With 93.5 % acid the butter fat was resolved to 81.0 % of free fatty acid
„ 98.5 % „ „ „ „ „ „ „ 89.7 % „ „ „ „
„ 100.0 % „ „ „ „ „ „ „ 92.2 % „ „ „ „

From these figures, the influence of the concentration of the sulphuric acid is very obvious; also the imperfection of the method so that it cannot be a matter of surprise that it has been entirely abandoned.

In order to get a better insight into the action of concentrated sulphuric acid on fats I chose as starting material pure trilaurin prepared from Tangkallak fat obtained from the fruits of Cylicodaphne Litsaea, a tree growing in West Java. The fat consists of trilaurin and triolein so that it is easy to prepare trilaurin from the same by recrystallisation from ether.

The sulphuric acid employed was 100.0 % as determined by titration. Experiment *a* took place at a temperature of 18°, experiment *b* and *c* at 1—2°. Time of action 30 minutes.

a 1 mol. of trilaurin to 6.5 mol. of H_2SO_4 gave 86.6 % free fatty acid
b 1 „ „ „ „ 26.0 „ „ H_2SO_4 „ 95.5 % „ „ „
c 1 „ „ „ „ 52.0 „ „ H_2SO_4 „ 100.0 % „ „ „

the reaction:

trilaurin + sulphuric acid = glyceroltrisulphuric acid + lauric acid seems, therefore, only practically complete with a very large excess of sulphuric acid and at a low temperature, for if experiment *c* is repeated and then again heated at 60° for 1½ hour, a shifting towards the left takes place and trilaurin is regenerated. The course of the investigation is briefly as follows:

The 100.0 % sulphuric acid is weighed in a flask which is then corked and placed in ice water. The weighed trilaurin is now added in small quantities. As by the action of the acid on trilaurin heat is generated no fresh portion of trilaurin must be added until the previous lot has dissolved [1]). When all the trilaurin has dissolved and the time of action has expired the contents of the flask are

[1]) From this evolution of heat with saturated compounds it follows that no undue importance should be attached to MAUMENÉ's experiment (Compt. rend. 1882. 35 pg 572) where this evolution of heat is made use of to detect unsaturated compounds.

poured on to pounded ice in order to prevent as much as possible a rise in temperature and consequent saponification. Sufficient alcohol is now added so as to obtain a 60% alcohol mixture and this is then shaken with a mixture of ether and petroleum ether. After washing with water the ether is evaporated. In experiment *c* a substance was left with ester number 0 and acid number 280.5, which points to pure lauric acid. This proves that all the trilaurin was decomposed.

On repeating experiment *c* with subsequent heating at 60° for $1^1/_2$ hour and removing the sulphuric acid in the manner described a substance was obtained with acid number 246.8 and saponification number 280.9; ester number 34.1. As trilaurin possesses an ester number 263.8, 12.9 % of trilaurin has been regenerated.

As regards the lauric acid which in the previous equation occurs together with glyceroltrisulphuric acid it must be remarked that this unites with H_2SO_4 to molecular compounds which are more or less soluble in benzene. Now if trilaurin is dissolved in 100 % acid (experiment *c*) and if this is shaken with dry benzene both lauric acid and sulphuric acid may be detected in that solvent. Compounds of a similar character have been described by HOOGEWERFF and VAN DORP [1]). In these additive compounds the oxygen is sometimes taken as quadrivalent such as:

$$RCOOH + H_2SO_4 = R - C \begin{matrix} \overset{\displaystyle H}{\overset{|}{=O}} - O - SO_3H \\ - O - H \end{matrix}$$

Others, H. MEIJER [2]), believe in the existence of a kind of mixed acid anhydrides:

$$R\,.\,COOH + H_2SO_4 = R - C \begin{matrix} =O \\ -O - SO_3H \end{matrix} + H_2O.$$

The latter is improbable as then we should want in all these compounds exactly 1 mol. of water of crystallisation for 1 mol. of the two acids. We already noticed in the saponification of butter fat that the concentration of the acid plays an important part; this is also the case with trilaurin. If now at a temperature of 1—2° we allow 52 mols. of 94.6%, sulphuric acid to act on 1 mol. of trilaurin for 30 minutes a substance was obtained, after removal of the sulphuric acid, consisting of 80% of lauric acid and 20% of undecomposed glyceride. This glyceride was separated from the

1) Recueil XVIII 1899 bl. 211.

2) Monatshefte für Chemie 24 p. 840.

lauric acid and its ester number found to be 244.0. The ester numbers of trilaurin, dilaurin and monolaurin are respectively 263.8, — 246.1 and 204.7 so that the separated glyceride is a mixture.

The mono- and dilaurin are probably formed here from compounds like C_3H_5 (OR) (O . $SO_3H)_2$, and C_3H_5 $(OR)_2$ (O . SO_3H) by decomposition with water $R = C_{11}H_{23}CO$.

Similar compounds are still under investigation. In the action of concentrated H_2SO_4 on nitroglycerol analogous reactions occur. NATHAN and RINTOUL[1]) in an article on: Nitro-glycerine und seine Darstellung write:

"Die Absorption des Nitroglycerins durch die Abfallsäure ist nicht nur ein Lösungsvorgang. Es findet noch eine zweite Reaktion statt, zwischen der Schwefelsäure und dem Nitroglycerine, unter Bildung von Sulfoglycerin und Salpetersäure. Diese umkehrbare Reaktion gelangt schnell in den Gleichgewichtszustand, so dass bei einer normalen Abfallsäure eine Hälfte des gesamten absorbierten Nitroglycerin als Sulfoglycerin vorhanden ist; während der Rest tatsächlich als Nitroglycerin in Lösung geht."

The reverse reaction (2) which still takes place at 60° even in the presence of a large excess of acid: glyceroltrisulphuric acid + lauric acid = trilaurin + sulphuric acid is to a certain extent comparable to the synthesis of glycerides according to GRÜN and SCHACHT[2]). They, however, write:

"Die Esterificirung des Glycerins durch Schwefelsäure bleibt — auch bei Anwendung von grossen Überschüssen an Säure — bei der quantitativen Bildung von Glycerindischwefelsäure (C_3H_5 (OH) $(0.SO_3H)_2$ stehen, dementsprechend treten auch bei der Einwirkung der organischen Säuren auf diese Verbindungen nur zwei Acyle in das Glycerinmolekül; man gelangt zur Diglyceriden."

"Die Bildung von Mono- und Triglyceriden konnte beim Einhalten der unten angegebenen Bedingungen nicht constatirt werden; ebenso wenig die Bildung anderer Nebenproducte."

It seems to me that this conclusion cannot conform to theory: it is also in conflict with my own observations. First of all, glyceroldisulphuric acid is never formed quantitatively in the esterification of glycerol by sulphuric acid, secondly byproducts are formed in their synthesis from diglycerides.

If one part of glycerol is dissolved in four parts of 98.3 °/₀ sulphuric acid there is formed chiefly a mixture of glyceroldi- and trisulphuric

[1]) Chemiker Zeitung No. 20. 1908. p. 246.

[2]) Berichte 38 p. 2284 (1905) see also Berichte 40 p. 1778 (1907).

acid, also a small proportion of the mono-acid. If to this mixture is added palmitic acid dissolved in H_2SO_4 a substance is obtained with an ester number of 205.1; the ester numbers of tripalmitin and dipalmitin are respectively 208.8 and 197.6. By one single recrystallisation from absolute alcohol, nearly chemically pure tripalmitin with an ester number of 208.1 and m.p. 64—65° could be isolated. Therefore, a mixture of dipalmitin and tripalmitin has been the main product. The barium salt prepared by me according to their method possesses another composition as stated by them; it should, however, be observed that $C_3H_6O_9S_2Ba + 2H_2O$ does *not* require 7.63 % of H_2O but 8.50%.

One part of chemically pure glycerol D 1.261 was dissolved in 4 parts of 98.5 % sulphuric acid. After 15 minutes an equal volume of water was added, the liquid was neutralised with barium carbonate and after removal of the barium sulphate by filtration the liquid was evaporated in vacuum. After adding a little alcohol it is again evaporated so as to get rid as much as possible of the water.

If now, an excess of absolute alcohol is added, a thick white precipitate of syrupy consistence is formed, which is shaken several times vigorously with alcohol to remove any free glycerol. The precipitate solidifies after a while and is then dried in vacuum over P_2O_5 to constant weight.

3.132 grams of the dried salt gave on evaporation with sulphuric acid 1.7380 grams of $BaSO_4$ = 55.49 % of sulphate or 32.65 % of barium 0.7740 grams gave 0.4295 grams of sulphate = 55.49 % or 32.65 % of barium.

Calculated for the Ba salt of the anhydrous di-acid 60.24 % $BaSO_4$.
" " " " " " " " mono-acid 48.67 % $BaSO_4$.

On heating the dried compound for 1½ hour at 105° in an air-bath it turns brown and evolves acrolein. In this operation 1.059 grams lost 0.011 grams or 1.03 %.

Therefore, a mixture of barium salts has formed which may be readily explained by the fact that on diluting the mixture of glycerol and sulphuric acid, the tri-acid already formed passes into lower acids.

CLAESSON [1]), who was the first to prepare glycerolsulphuric acid also observed this conversion of the tri-acid into the lower acids.

He prepared the tri-acid from anhydrous glycerol and chlorosulphonic acid; his statement that this tri-acid, on boiling with water or dilute acids, is readily and completely resolved into glycerol and sulphuric acid is, however, incorrect; at least after boiling for one

[1]) Journal für praktische Chemie [2] Bd. 20. p. 1. 1879.

hour there still remains a portion of the acid combined with glycerol in the form of a mono-acid.

Glyceroltrisulphuric acid was prepared by me according to CLAESSON from anhydrous glycerol and chlorosulphuric acid. 1.619 grams of the acid was dissolved in water and boiled for an hour, the solution was neutralised with barium hydroxide and the resulting barium sulphate weighed. If the sulphuric acid had been eliminated completely 3.411 grams of barium sulphate ought to have been formed but only 2.121 grams were found; therefore 0.542 grams of sulphuric acid was left in combination with glycerol.

The above experiments, therefore, throw a little more light on the sulphuric acid saponification of fats. Further communications will follow shortly.

Gouda, 5 April 1908. *Laboratory Candle Works.*

Palaeontology. — *"On Dulichium vespiforme sp. nov. from the brick-earth of Tegelen."* By Mr. CLEMEND REID F. R. S. and Mrs ELEANOR M. REID B. Sc. (Communicated by Prof. G. A. F. MOLENGRAAFF).

In our paper on the Fossil Flora of Tegelen published in 1907 [1]) we figured a fruit provisionally referred to *Rhynchospora*. though it did not possess the articulate beak of that genus. All the specimens then available were so much distorted and injured by germination that it was difficult to determine what the character of the perfect fruit would be. In addition to this, the most perfect specimen appeared to possess a quadrate base and 8 setae, characters unknown in *Dulichium*, to which genus the fruit was in other respects comparable.

Since the publication of our paper we have obtained more material, thanks to the kindness of Dr. LORIÉ and Baron L. GREINDL. This new material and a closer examination of the specimens before collected, enables us now to describe the fruit as a new species belonging to *Dulichium*, a genus now confined to America, though already recorded by Dr. N. HARTZ as occurring in an interglacial peat-moss in Denmark [2]). Dr. HARTZ's specimens are referred, we think correctly, to the only living species, *Dulichium spathaceum;* our fruits are very different.

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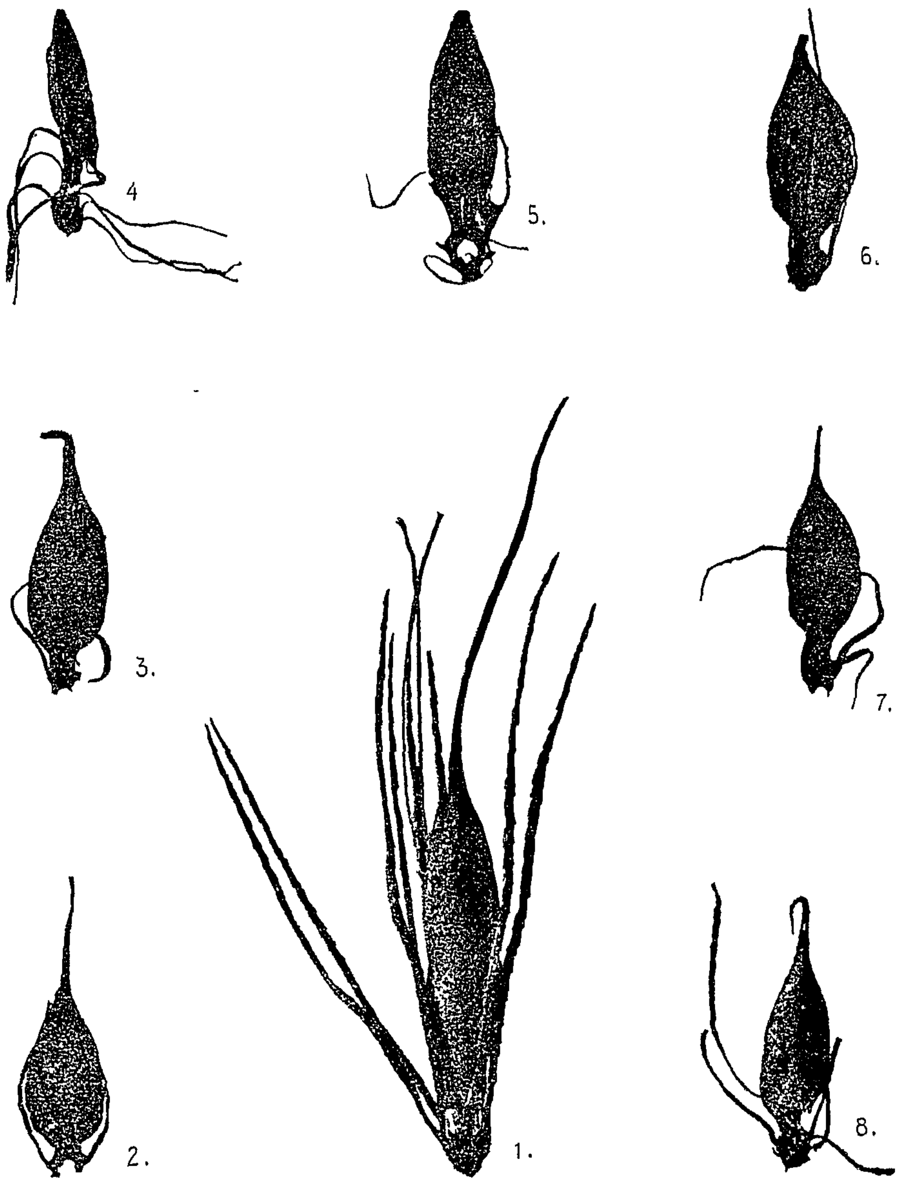
1) Verhand. Kon. Akad. Wetensch. (Tweede Sectie). Deel XIII, No. 6, fig. 105.
2) Dansk. geol. Forening 10, 1904, p. 13.

CLEMEND REID and Mrs ELEANOR M. REID. "On Dulichium vespiforme sp. nov. from the brick earth of Tegelen."



1. Dulichium spathaceum L. C. RICH. Recent, America. 2—8. Dulichium vespiforme *sp. nov.* Fossil, Tegelen. All the figures are magnified to the same scale — 12 diameters.

Fructus dimidio brevior eo *D. spathacei* tamen latior, long. (rostro incluso) circiter 3—5 mm.; setae 7 vel 8 (forsitan 9), longitudinaliter complanatae, canaliculatae, striatae; nux ovata subito in stipitem coarctata, in rostrum longum gracile attenuata, paulo triangularis vel plano-convexa, praesertim rostrum versus; superficies foveolata, multangula, eâ *D. spathacei* crassior; long. (rostro excluso) 2.0—2.5 mm., lat. 1 mm.

Figs. 1—8, photographed on the same scale, show the differences between the recent and fossil forms. In the living species the nut is oblong, not ovate, and is much narrower in proportion to its length. The long stalked nut with oblique attachment, inarticulate style, setae more than 6 with recurved hooks, are generic characters common to the two species. In section the nut of *D. vespiforme* is somewhat triangular near the base, becomes plano-convex in the middle, but loses its convexity as it passes into the beak; the beak at its base is flattened triangular, but becomes terete above.

As all the specimens we have yet seen (about 20) have apparently germinated, it is impossible to describe the exact shape of the nuts or the complete setae; we have therefore figured several actual specimens, without attempting to restore them. The setae are more or less broken, but we cannot find clear evidence of more than 8 in any of the specimens; they differ from those of *D. spathaceum* in their flattening, and they are longitudinally channelled instead of showing a midrib. The fruits undoubtedly belong to the genus *Dulichium*; and as the living species has a wide range in latitude we thought it possible that some form (several different ones have been recorded) might agree with our fossil. We find, however, that the fruits of the recent forms in the Kew herbarium vary only slightly; they are always much larger than our fossil, and the nut is long, narrow, and parallel-sided.

Whether the genus *Dulichium* originated in Europe or in North America there is nothing to show. It has now only one living species, confined to America; but this species has been found also in a fossil state in Denmark. Now, in an older deposit in the province of Limburg, we discover an extinct form. Very little is yet known as to the geological history of the Cyperaceae, and *Dulichium* will probably turn out to have been widely distributed and to have had many species. The genus is at present very isolated and the new fossil form makes no approach to any other genus.

Physics. — *"Change of wavelength of the middle line of triplets."* (Second Part). By Prof. P. ZEEMAN.

6. We will now return to the observations of § 4. Arranging these according to strength of field it appears that the distance $a'—a''$ changes considerably with increasing magnetic intensity. The displacement of line 5791 is not a linear function of the strength of the field but increases more rapidly than would follow from this simple relation. However it is impossible without further consideration to deduce the law of displacement, because, as remarked in § 4, the distance of the lines of comparison does not remain invariable. This is the reason why somewhat different values of $a'—a''$ are obtained, when these are calculated from the change of $a—a'$, than when the change of $b—a'$ is considered.

The direction however of the displacement of 5791 is easily determined. It is towards the *red* end of the spectrum. A shift towards the side of increasing wavelengths corresponds in the figure of § 3 to a displacement in the direction from a' towards a''. The less refrangible side of line 5791 is easily distinguished upon the negatives by the observation of the two weak less refrangible companion lines and the one weak more refrangible companion line [1]).

7. The shift of the middle line of the triplet may be demonstrated also by our method of the non-uniform field, if an echelon-spectroscope is made use of. A curvature of the middle line will be the immediate effect of the shift. If we use ROWLAND's grating such a curvature would be invisible nor have I observed it in that case.

The visibility of the curvature will be much increased by taking care that in the image points corresponding to very different intensities of field lie closely together. In order to attain this an eleven times reduced image of the vacuum tube, charged with mercury and placed into the field, was projected on the slit of the auxiliary spectroscope. The lens used was a photographic objective of 10 cm. focus.

The Plate gives somewhat enlarged reproductions of negatives relating to line 5791 resp. line 5770. The middle line is given in two succeeding orders. Between these the other components of the triplets are seen. With increasing magnetic force the components deviate further and further from their own middle line. In the central part of the field of view the maximum distance is reached.

[1]) JANICKI. Feinere Zerlegung der Spektrallinien von Quecksilber u.s.w. Inaugural. Diss. Halle a. S. 1905, Annalen der Physik. Bd. 19, 36. 1906.

The component towards the red in the figures is always at the left of its middle line, being concave to it in the central part; the second manifestly curved line is the component towards the violet belonging to the other order.

The curvature of the middle lines, the demonstration of which is the object of our present experiment, is undoubtedly visible in the figure for 5791. It is still more easily seen by comparison with a straight bit of paper.

In the figure for 5770 this kind of curvature is absent.

The asymmetry of the magnetic resolution of line 5791 is at once evident by the fact that one of the middle lines is approached more nearly by the outer component than the other.

If we denote by a_v and a_r the distances of the components to their middle lines, then what I called on a former occasion[1]) the amount of the asymmetry is equal to $a_v - a_r$. This difference is also equal to the difference of the distances separating the plainly curved lines from the middle lines to which they do not belong, and to which they are convex.

The two negatives were taken with the same field intensity of about 34000 Gauss.

The question now arises whether the difference $a_v - a_r$ is equal to twice the shift of the middle line or not. In the first case the asymmetry is brought about solely by the motion of the middle line towards the less refrangible wavelengths, the outer components having undergone a symmetrical displacement relatively to the unmodified line. The other, more general case one would rather expect without hypothesis or without the results of measurements.

8. In order to test the question by experiment, I have taken on the same negative as well the figures described in § 7 as the unmodified lines. It appeared however rather soon that, in the case of line 5791, only in the most intense fields the separation of the middle lines, taken with field on and with field off, was sufficient to allow measurements.

I therefore refrain from communicating these experiments. Only one detail of the vacuum tube, charged with mercury and used in all my experiments with strong fields, may perhaps be mentioned. This vacuum tube of the form indicated by PASCHEN, has a rather wide capillary. That part however of the capillary which is placed in the magnetic field is drawn out. Only over this short distance the

[1]) ZEEMAN, These Proceedings 30 November 1907.

capillary has a small diameter. Now the gap-width of the electromagnet may be considerably diminished; at the same time the electrical resistance of the vacuum tube is moderate.

9. Measurements were made in the following manner. The slit of the auxiliary spectroscope performing the preliminary analysis of the light, was widened in order to obtain in the echelon spectroscope light of the *two* yellow mercury lines simultaneously. The steps of the echelon were placed parallel to the slit of the auxiliary spectroscope. The image of the vacuum tube projected on the slit was now chosen in such a manner that only light from the uniform part of the field was analyzed. By means of a suitable small screen placed before the photographic plate its middle part could be exposed first to light under magnetic influence; then the unmodified lines were taken in the upper and lower parts of the plate. The plates taken confirmed the result obtained in § 6 as to the shift of line 5791 towards the red.

As to line 5770 the amount and even the existence of the shift is not quite certain at present [1]). All these measurements were not further pursued however, because after the publication of the first part of this paper [2]) and during my measurements there appeared a communication by GMELIN in the 1 April number of the physikalische Zeitschrift.

Independently of my paper and in another way our present subject was taken up by GMELIN. The enormous resolving power of the echelon spectroscope used by GMELIN apparently permits of greater accuracy in the measurements than would have been possible for me.

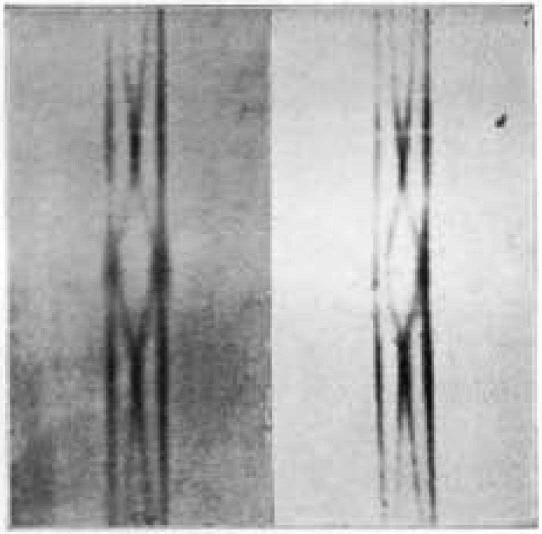
It may be remarked finally that, so far as the present results can settle the question, the observation of the asymmetry in a direction parallel to the lines of force, [3]) which first induced me to this investigation, but which was given with some reserve, must have been correct.

1) The first part of this communication contains an error, which I only noticed after the printing off. In the last division of § 4 the change of wavelength has been calculated from the measured displacement in the same manner as must be done, when the distance of two adjacent orders with the field off, is compared with the distance separating the components towards red and towards violet with the field on. Of course this proceeding is faulty in the case of § 4. Hence the last sentence of § 4 and the last column of the table in § 5 have lost significance. The further consideration of the peculiar change of the distance of orders noticed in § 5 must be reserved for a future paper.

2) ZEEMAN, These Proceedings 29 Febr. 1908.

3) ZEEMAN, § 7 in New Observations etc. These Proceedings 29 Febr. 1908.

P. ZEEMAN. "Change of wavelength of the middle line of triplets." (Second part).



Hg.	**5770**	**5791** **1 mm = 0.12** A.E.
resolution:	symmetrical.	asymmetrical.
middle lines:	straight.	curved.

ERRATA.

In the Proceedings of the meeting of December 1907:

In Pl. I belonging to the communication of Prof. H. KAMERLINGH ONNES and C. BRAAK (p. 413) the numbers I and II are to be interchanged.

p. 422 to footnote 1 add: In this communication the resistance thermometer of Comm. No. 95 *c* (Sept. '06), which is called Pt_I, was used.

p. 423 to footnote 1 add: The thermometer till now called Pt_I was named Pt'_I after the breaking of the wire.

p. 447 l. 20 from the top: for 79 *read* 78.

In the proceedings of the meeting of February 1908:

Pl. II belonging to the communication of JEAN BECQUEREL and H. KAMERLINGH ONNES in the subscript of Fig. 1 for 1.71 mM. in 1, 2, 3, 4 read 1.71 mM. in 1, 3, 4.

p. 597 l. 5 from the bottom: for we *read* they.
p. 604 l. 1 ,, ,, ,, ,, 106 and 107 *read* 147.
l. 15 ,, ,, ,, ,, on *read* of
p. 606 l. 19 ,, ,, ,, ,, observations *read* deviations
p. 610 l. 1 ,, ,, top, for down to *read* as far as.
l. 7 ,, ,, bottom: for 170 *read* 117.

In the proceedings of the meeting of Februari 1908.

p. 591 l. 14 from the bottom: for 0°.10 read 0°.06.
p. 522 l. 1 from the top: for 0.0000013 read 0.0000009.
l. 2 ,, ,, ,, : ,, 0.0036614 read 0.0036617.

In the paper by Dr. DE SITTER "On Jupiter's Satellites" (Meeting of March 28).

p. 721 the value of $log\ a_4$ should read

$$log\ a_4 = 8.0998360,$$

p. 727 the signs of a_{21} and a_{21}' should be inverted.

(May 26, 1908)