Astronomy. — *"On the periodic solutions of a special case of the problem of four bodies".* By Prof. W. DE SITTER. (Communicated by Prof. E. F. VAN DE SANDE BAKHUYZEN).

The special case considered in this paper is that of a central body and three planets, or satellites, whose masses are small compared with the mass of the central body, and whose orbits are all situated in one and the same plane, the mean motions (in longitude) being *roughly* proportional to the numbers 4, 2 and 1. This special case is realised in nature by the three inner Galilean satellites of Jupiter, if the inclinations, the influence of the sun and of the fourth satellite, and the compression of the planet are neglected. This latter restriction is not essential, since the compression does not disturb the periodicity, provided only the motions take place exclusively in the plane of the planet's equator.

Neglecting at first the relation between the mean motions, we will consider the periodic solutions of the problem thus generalized for the case that the masses of the satellites are zero, i.e. for the unperturbed problem. These may be divided into two kinds, analogous to POINCARÉ's well known classification of the periodic solution of the problem of three bodies. In the solutions of the first kind (*sorte première* of POINCARÉ) the (unperturbed) orbits of the satellites are circles, in those of the second kind they are Keplerian ellipses with arbitrary excentricities.

The solutions of the first kind exist, if the *differences* of the mean motions are commensurable, thus:

$$\nu_1 - \nu_3 = p\nu, \qquad \nu_2 - \nu_3 = q\nu,$$

p and q being integers, mutually prime. This condition can also be expressed by saying that the mean motions must satisfy a linear equation of the form

$$\alpha\nu_1 + \beta\nu_2 + \gamma\nu_3 = 0,$$

where α, β and γ are mutually prime whole numbers, satisfying the relation

$$\alpha + \beta + \gamma = 0.$$

The mean motions can then be expressed thus:

$$\nu_1 = c_1\,\nu - \varkappa \qquad \nu_2 = c_2\,\nu - \varkappa \qquad \nu_3 = c_3\,\nu - \varkappa,$$

where c_1, c_2, c_3 are again whole numbers. We have then:

$$\alpha = c_2 - c_3 \qquad \beta = c_3 - c_1 \qquad \gamma = c_1 - c_2$$

$$p = c_1 - c_3 \qquad q = c_2 - c_3$$

Then, if we put

$$\nu t = \tau \qquad \varkappa t = v,$$

and if we count the time from the instant of a conjunction of II and III, and the longitudes from the common longitude of these satellites for that instant, we have

$$\lambda_3 = c_3\tau - v$$
$$\lambda_2 = c_2\tau - v$$
$$\lambda_1 = c_1\tau - v + K$$
$$\alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 = K.$$

After the lapse of the period

$$T = \frac{2\pi}{v}$$

the relative positions of the four bodies are the same as for the instant $t = 0$, the whole system being rotated in a retrograde direction through the angle $\varkappa T$.

By a reasoning entirely similar to that used by POINCARÉ[1]) for the solutions of the first kind of the problem of three bodies, we find easily that the condition, that these solutions shall remain periodic if the masses have small finite values, is

$$K = 0° \text{ or } 180°.$$

In other words, there must be a symmetrical conjunction or opposition of the three satellites at the beginning of the period.[2])

The reasoning by which the existence of these solutions for small values of the masses is proved, fails in only one case, viz. when

$$\frac{\varkappa}{v} = 0 \text{ or a whole number.}$$

This exceptional case is analogous to the well known exceptional case for the periodic solutions of the first kind of the problem of three bodies.

For the special case of Jupiter's satellites we have

$$\alpha = 1,\ \beta = -3,\ \gamma = 2,\ K = 180°$$
$$\lambda_1 = 4\tau - v + 180°$$
$$\lambda_2 = 2\tau - v$$
$$\lambda_3 = \tau - v$$
$$\tau = \lambda_2 - \lambda_3 \qquad v = \lambda_2 - 2\lambda_3$$

In the system of Jupiter we find that v is small compared with τ. We have roughly (in degrees per day):

$$\nu = 51°.0571$$
$$\varkappa = 0.7395.$$

1) *Les méthodes nouvelles de la mécanique céleste*, tome I, § 40.

2) See also *Les méthodes nouvelles*, t. I § 50.

It is owing to this particular circumstance, that the motion of the satellites can also be considered as a periodic solution of the second kind, as will now be shown.

In the periodic solutions of the second kind of the unperturbed problem the excentricities are arbitrary, and the mean motions (not only their differences) are mutually commensurable. In other words we have here $\varkappa = 0$.

If the masses are not zero, these solutions may also remain periodic. In the perturbed motion we must then distinguish the mean motions in *longitude* and those in *anomaly*. Let

$$l_i = n_i t + l_{io} \text{ be the mean anomaly}$$
$$\lambda_i = \nu_i t + \lambda_i, \quad \text{,, ,, ,, longitude,}$$

if then π_i be the longitude of the pericentre, we have

$$\lambda_i = l_i + \pi_i$$

$$\nu_i = n_i + \frac{d\pi_i}{dt} \qquad n_i = c_i \, \nu$$

Inquiring into the conditions that these solutions shall remain periodic for small finite values of the masses, we find again that there must be a symmetrical conjunction at the beginning of the period, i. e. for $t = 0$. The angles

$$\pi_{10} - \pi_{20} \qquad \pi_{20} - \pi_{30}$$
$$l_{10} \qquad l_{20} \qquad l_{30}$$

must all be 0° or 180°. One of the angles l_{io} (e.g. l_{30}) can always be made identically zero (or 180°) by a convenient choice of the zero epoch. There thus remain 4 angles, each of which can have one of two values. We have thus 16 combinations which may *a priori* be expected to give rise to periodic solutions.

Now if $\frac{d\pi_i}{dt}$ were zero, then at the end of the period $T = \frac{2\pi}{\nu}$ the configuration for $t = 0$ would be exactly restored, as it is in the unperturbed problem. It is, however, sufficient to insure the periodicity of the solution, that the value of $\frac{d\pi_i}{dt}$ integrated over a complete period shall be the same for the three satellites. In addition to the conditions of symmetry we have therefore the conditions

$$\int_0^T \frac{d\pi_1}{dt}\, dt = \int_0^T \frac{d\pi_2}{dt}\, dt = \int_0^T \frac{d\pi_3}{dt}\, dt = -\varkappa T \quad . \quad . \quad (1)$$

After the completion of the period the whole system is then rotated through the angle $-\varkappa T$, as in the solutions of the first kind.

The mean motions in *longitude* are the same as in the solutions of the first kind, viz.:

$$\nu_i = c_i \, \nu - \varkappa,$$

The mean motions in *anomaly* remain rigorously commensurable.[1])

I will now restrict the discussion to the special case represented in the system of Jupiter, viz.:

$$c_1 = 4 \quad , \quad c_2 = 2 \quad , \quad c_3 = 1.$$

For the general case similar results will be found, which I do not however at present propose to investigate.

Moreover I will limit myself to the consideration of small excentricities, which is the only case that is of immediate practical value. Whether the conditions (1) do also admit solutions with large excentricities, is a question which can only be answered by a special investigation.

Under these restrictions we find that out of the 16 combinations satisfying the conditions of symmetry, there are only 4 which also satisfy the conditions (1). For two of these $\varkappa$ is positive, and for the two others it is negative. Further, if the quantity

$$\lambda_1 - 3\,\lambda_2 + 2\,\lambda_3 = K$$

is formed for each of these solutions, it will be found that one of the solutions with a positive $\varkappa$ has $K = 0^\circ$ and the other has $K = 180^\circ$, and similarly for the solutions with a negative $\varkappa$. Of these four solutions that with $K = 180^\circ$ and $\varkappa$ positive (the case of nature) is the only stable one.

These solutions of the second kind thus appear, on both sides of the exceptional point $\varkappa = 0$, as the natural continuations of the two possible solutions of the first kind ($K = 0^\circ$ and $K = 180^\circ$). In the solutions of the first kind the unperturbed orbit is circular, the perturbed orbit is affected by a "great inequality", with the argument $c_i \tau$. In the solutions of the second kind this inequality appears as an equation of the centre [2]). In the solutions of the first kind we have the condition that the unperturbed excentricity must be zero; corresponding to this the excentricities in the solutions of the second

1) These solutions are based on the same principle as those investigated by SCHWARZSCHILD (Astr. Nachr. 3506). SCHWARZSCHILD, however, only considers the case of two planets, one of which has an excentricity, and at the same time an infinitely small mass. Consequently the orbit of the other planet, which is a circle, is not perturbed.

2) In the integration by the usual method, this inequality presents itself as a perturbation of the excentricities and pericentres.

Besides this "great" inequality there are, of course, a number of others, whose arguments are multiples of τ, which are the same in the two solutions kind are not arbitrary, but must be determined from the equations (1). When the value of $\varkappa$ is the same for both cases, the two solutions are entirely equivalent.

In order now to investigate these solutions according to the theory of POINCARÉ, we must write down the conditions of periodicity

$$\psi_i = \int_0^T \frac{dE_i}{dt}\,dt = 0$$

where for E_i we must take successively each of the elements of the system. If further β_i be the small correction to be applied to the value of E_i (for $t = 0$) in the unperturbed orbit, in order to retain the periodicity in the perturbed orbit, then the stability of the solution depends on the roots of the equation

$$\Delta(s) = \begin{vmatrix} \frac{\partial\psi_1}{\partial\beta_1} + 1 - s & \frac{\partial\psi_1}{\partial\beta_2} & \cdots & \frac{\partial\psi_1}{\partial\beta_n} \\ \frac{\partial\psi_2}{\partial\beta_1} & \frac{\partial\psi_2}{\partial\beta_2} + 1 - s & \cdots & \frac{\partial\psi_2}{\partial\beta_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\psi_n}{\partial\beta_1} & \frac{\partial\psi_n}{\partial\beta_2} & \cdots & \frac{\partial\psi_n}{\partial\beta_n} + 1 - s \end{vmatrix} = 0 \quad . \; . \; . \quad (2)$$

If we put $s = e^{\alpha T}$ (or, approximately, $1 - s = -\alpha T$), then the condition that the orbit shall be stable, is that all the values of α^2 are real and negative (with the exception of one or more, which may be identically zero).

I will introduce the elements

$$L_i, \quad \Pi_i, \quad l_i, \quad \pi_i,$$

of which the meaning is

$$L_i = m_i \sqrt{a_i} \qquad \Pi_i = L_i \sqrt{1 - e_i^2}$$

l_i = mean anomaly

π_i = longitude of pericentre.

Supposing the units to be so chosen that the constant of GAUSS and the mass of the central body are unity, the equations of motion are:

$$\frac{dL_i}{dt} = -\frac{dF}{\partial l_i} \qquad \frac{d\Pi_i}{dt} = -\frac{\partial F}{\partial \pi_i}$$

$$\frac{dl_i}{dt} = \frac{\partial F}{\partial L_i} \qquad \frac{d\pi_i}{dt} = \frac{\partial F}{\partial \Pi_i}$$

$$F = F_0 - R$$

$$F_0 = -\frac{m_1^3}{2L_1^2} - \frac{m_2^3}{2L_2^2} - \frac{m_3^3}{2L_3^2}$$

In the unperturbed motion we have

$$l_i = n_i t + l_{i0}, \qquad n_i = \frac{m_i^3}{L_i^3} = \frac{1}{a_i^{3/2}},$$

and the constants a_i must be such, that

$$n_1 = 4\nu \qquad n_2 = 2\nu \qquad n_3 = \nu.$$

The integral of areas is

$$\Phi = \Pi_1 + \Pi_2 + \Pi_3 = \text{const.}$$

By means of this integral we can eliminate Π_3, and diminish the number of degrees of freedom by one. For this purpose we introduce

$$G_1 = \Pi_1 \qquad G_2 = \Pi_2$$

$$g_1 = \pi_1 - \pi_3 \qquad g_2 = \pi_2 - \pi_3$$

The equations then preserve the canonic form. [1])

In forming the equations $\psi_i = 0$ we need only those terms of R, whose integral over a complete period does not vanish, i.e. those in whose arguments the mean anomalies do either not occur, or occur only in the combinations

$$l = l_1 - 2\,l_2 \qquad l' = l_2 - 2\,l_3,$$

of which the mean motions are zero. The constant term will not be required in what follows. Of the others, we only require the terms of the lowest degree in the excentricities. Thus, introducing the further notations

$$\pi_1 - \pi_2 = g_1 - g_2 = \omega$$

$$\pi_2 - \pi_3 = \quad g_2 \quad = \omega',$$

we find that R can be replaced by

$$R' = \frac{m_1 m_2}{a_2}\{-A\,\varepsilon_1 \cos(l + 2\omega) + B\varepsilon_2 \cos(l + \omega)\} +$$

$$+ \frac{m_2 m_3}{a_3}\{-A\varepsilon_2 \cos(l' + 2\omega') + B\varepsilon_3 \cos(l' + \omega')\} \quad . \quad . \quad . \quad (3)$$

where

$$A = a'\,(4\,A^{(2)} + A_1^{(2)})$$

$$B = a'\,(3\,A^{(1)} + A_1^{(1)} - \frac{2}{\sqrt{aa'}}),$$

The symbols $A_p^{(i)}$ have the usual meaning (LEVERRIER, Annales de Paris, tome 1, p. 260, 262), and must be computed for the value

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[1]) The integral of areas still exists, if the compression of the planet is taken into account, provided only the motion takes place in the plane of the equator. Also those terms of the perturbing function which are here used, remain the same. The conclusions reached below, thus can be applied unaltered to the case of a compressed planet.

$$\frac{a}{a'}=\left(\frac{n_2}{n_1}\right)^{2/3}=\left(\frac{n_3}{n_2}\right)^{2/3}=\left(\frac{1}{2}\right)^{2/3}=0.630.$$

The coefficients A and B then are pure numbers. Their values are

$$A=+2.381$$
$$B=+0.964.$$

The meaning of the symbols ε_i occurring in R' is

$$\varepsilon_i=\sqrt{\frac{L_i-\Pi_i}{2L_i}}=\sin\frac{1}{2}\varphi_i,$$

or approximately

$$\varepsilon_i=\frac{1}{2}e_i$$

The expressions of the various differential coefficients of R' are:

$$\frac{\partial R'}{\partial L_i}=\frac{2\sqrt{a_i}}{m_i}\frac{\partial R'}{\partial a_i}+\frac{\cos\varphi_i}{4m_i\sqrt{a_i}}\frac{1}{\varepsilon_i}\frac{\partial R'}{\partial\varepsilon_i}$$

$$\frac{\partial R'}{\partial\Pi_i}=\frac{1}{4m_i\sqrt{a_i}}\frac{1}{\varepsilon_i}\frac{\partial R'}{\partial\varepsilon_i}$$

$$\frac{\partial R'}{\partial G_1}=\frac{\partial R'}{\partial\Pi_1}-\frac{\partial R'}{\partial\Pi_3}\qquad\frac{\partial R'}{\partial G_2}=\frac{\partial R'}{\partial\Pi_2}-\frac{\partial R'}{\partial\Pi_3}$$

$$\frac{\partial R'}{dl_1}=\frac{\partial R'}{dl}\qquad\frac{\partial R'}{\partial l_2}=-2\frac{\partial R'}{dl}+\frac{\partial R'}{\partial l'}\qquad\frac{\partial R'}{\partial l_3}=-2\frac{\partial R'}{\partial l'}$$

$$\frac{\partial R'}{\partial g_1}=\frac{\partial R'}{d\omega}\qquad\frac{\partial R'}{\partial g_2}=-\frac{\partial R'}{\partial\omega}+\frac{\partial R'}{\partial\omega'}.$$

The quantities β_i and ψ_i will be supposed to be correlated to the different elements as follows:

To:	L_1	L_2	L_3	G_1	G_2	l_1	l_2	l_3	g_1	g_2
correspond:	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
and:	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}.

If in R' and its differential coefficients the elements are replaced by their unperturbed values, these funtions become constant. Consequently the first terms of the developments of the functions ψ_i in powers of the masses are of the form

$$\int_0^T D\,dt=T\,.\,D,$$

where D represents a differential coefficient of R' in which, after the differentiation, the unperturbed values of the elements have been substituted.

Now we have

$$\frac{dL_i}{dt} = \frac{\partial R'_i}{\partial l_i} \quad , \quad \frac{dG_i}{dt} = \frac{\partial R'_i}{\partial g_i}.$$

The functions $\psi_1, \psi_2, \psi_3, \psi_4$ and ψ_5 therefore contain only sines of linear combinations of the angles l, l', ω, ω'. Further we have

$$4\frac{\partial R'}{\partial l_1} + 2\frac{\partial R}{\partial l_2} + \frac{\partial R'}{\partial l_3} = 0\,,$$

and therefore the equation $\psi_1 = 0$ is a necessary consequence of $\psi_2 = 0$ and $\psi_3 = 0$. The four remaining mutually independent equations

$$\psi_2 = \psi_3 = \psi_4 = \psi_5 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

correspond to what has in the beginning of this paper been called the "conditions of symmetry". If we put

$$l_0 = (l_1 - 2\,l_2)_0 = \alpha \qquad l'_0 = (l_2 - 2\,l_3)_0 = \alpha'$$
$$\omega_0 = (\pi_1 - \pi_2)_0 = \beta \qquad \omega'_0 = (\pi_2 - \pi_3)_0 = \beta',$$

(the subscript 0 indicating the unperturbed value, or the average value over a complete period in the periodic solution), then the equations (4) are satisfied, if each of the angles

$$\alpha \;,\; \alpha' \;,\; \beta \;,\; \beta'$$

is either 0° or 180°.

These conditions being satisfied, we can, in the differential coefficients of R' (after the differentation) replace the angles l, l', ω, ω' by $\alpha, \alpha', \beta, \beta'$.

The developments of the functions $\psi_1, \ldots \psi_5$ in powers of β_i are

$$\psi_1 = T\left\{\beta_6\frac{\partial^2 R'}{\partial l_1^2} + \beta_7\frac{\partial^2 R'}{\partial l_1 \partial l_2} + \ldots + \beta_{10}\frac{\partial^2 R'}{\partial l_1 \partial g_2}\right\} + \begin{array}{c}\text{terms of}\\ \text{higher orders}\end{array}$$

and similar formulae for ψ_2, ψ_3, ψ_4 and ψ_5. Then we find easily

$$\psi_6 = T\,\beta_1\frac{\partial^2 F_0}{\partial L_1^2} + \text{ terms of higher order,}$$

and similarly ψ_7 and ψ_8. These equations give $\beta_1 = \beta_2 = \beta_3 = 0$, in other words the mean motions in *anomaly* (n_i) are not changed. Finally we have

$$\psi_9 = -\int_0^T \frac{\partial R'}{\partial G_1}dt = -\,T\frac{\partial R'}{\partial \Pi_1} + T\frac{\partial R'}{\partial \Pi_3}$$

$$\psi_{10} = -\,T\frac{\partial R'}{\partial \Pi_2} + T\frac{\partial R'}{\partial \Pi_3}$$

The equations $\psi_9 = \psi_{10} = 0$ are thus found to represent the conditions (1), since

$$\int_0^T \frac{d\pi_1}{dt}\, dt = -T \frac{\partial R'}{\partial \Pi_1}$$

From the value (3) of R' we find easily (remembering that $a_i^{-3/2} = n_i = c_i\nu$ ($c_1 = 4$, $c_2 = 2$, $c_3 = 1$) and $\nu T = 2\pi$)

$$\Omega_1 = \frac{1}{2\pi}\int_0^T \frac{d\pi_1}{dt}\, dt = -m_2 \frac{a_1}{a_2}\frac{A}{\varepsilon_1} \cos \alpha,$$

$$\Omega_2 = \frac{1}{2\pi}\int_0^T \frac{d\pi_2}{dt}\, dt = \frac{1}{2}\left\{ m_1 \frac{B}{\varepsilon_2} \cos(\alpha + \beta) - m_3 \frac{a_2}{a_3}\frac{A}{\varepsilon_3} \cos \alpha' \right\},$$

$$\Omega_3 = \frac{1}{2\pi}\int_0^T \frac{d\pi_3}{dt}\, dt = \frac{1}{4} m_2 \frac{B}{\varepsilon_3} \cos(\alpha' + \beta').$$

The conditions (1) can thus be written

$$\Omega_1 = \Omega_2 = \Omega_3 = -\frac{\varkappa T}{2\pi} = -\frac{\varkappa}{\nu}.$$

Collecting now the 16 different possible combinations of values

	α	β	α'	β'	Ω_1	Ω_2	Ω_3	
(1)	0°	0°	0°	0°	—	0	+	*impossible.*
(2)	0	0	0	180	—	0	—	conditional.
(3)	0	0	180	0	—	+	—	*impossible.*
(4)	0	0	180	180	—	+	+	*impossible.*
(5)	0	180	0	0	—	—	+	*impossible.*
(6)	0	180	0	180	—	—	—	**possible.**
(7)	0	180	180	0	—	0	—	conditional.
(8)	0	180	180	180	—	0	+	*impossible.*
(9)	180	0	0	0	+	—	+	*impossible.*
(10)	180	0	0	180	+	—	—	*impossible.*
(11)	180	0	180	0	+	0	—	*impossible.*
(12)	180	0	180	180	+	0	+	conditional.
(13)	180	180	0	0	+	0	+	conditional.
(14)	180	180	0	180	+	0	—	*impossible.*
(15)	180	180	180	0	+	+	—	*impossible.*
(16)	180	180	180	180	+	+	+	**possible.**

of a, a', β and β', we find the following summary. Only those combinations can give rise to periodic solutions, in which Ω_1, Ω_2 and Ω_3 are of the same sign. The letter O stands for *undetermined.*

Out of these 16 combinations there are only two, (6) and (16), for which the perturbed orbit can remain periodic for all values of the masses. There are four: (2), (7), (12) and (13), for which the periodicity is only possible if a certain condition involving the masses is satisfied.

For all solutions we find from the equation $\psi_9 = 0$

$$\frac{\varepsilon_1}{\varepsilon_3} = 4\frac{a_1}{a_2}\frac{A}{B} = 6.225.$$

It needs hardly be pointed out, that this is only a rough approximation, the higher orders of ε_i having been neglected. In the system of Jupiter's satellites we find actually (see these Proceedings, March 1909 [1])) $\varepsilon_1/\varepsilon_3 = 6.77$.

Further if we put

$$\frac{m_1}{m_2} = \mu_1 \qquad \frac{m_3}{m_2} = \mu_3$$

$$P = \mu_3\frac{a_2}{a_3}A + \mu_1 B,$$

then we find, for the solutions (6) and (16), from $\psi_{10} = 0$

$$\frac{\varepsilon_2}{\varepsilon_3} = 2\frac{P}{B} \qquad \frac{\varepsilon_2}{\varepsilon_1} = \frac{1}{2}\frac{a_2}{a_1}\frac{P}{A}.$$

If the longitudes are counted from the apocentre of III, and the time from a passage of III through this apocentre, we have, for $t = 0$, $\pi_3 = 180°$, $l_3 = 180°$, therefore $\lambda_3 = 0°$. For the corresponding values for II and I we find, for $t = 0$, for solution (6)

$$\begin{array}{ll} \pi_2 = 0° & \pi_1 = 180° \\ l_2 = 0 & l_1 = 0 \\ \lambda_2 = 0 & \lambda_1 = 180 \end{array}$$

$$K = \lambda_1 - 3\lambda_2 + 2\lambda_3 = 180°$$

and $\varkappa$ is *positive*. the mean motion in anomaly exceeds the mean motion in longitude. This is the case of nature.

For the solution (16) we find:

$$\begin{array}{ll} \pi_2 = \ \ 0° & \pi_1 = 180° \\ l_2 = 180 & l_1 = 180 \\ \lambda_2 = 180 & \lambda_1 = \ \ 0 \\ K = 180° & \varkappa \textit{ negative.} \end{array}$$

[1]) The expressions there given are based on SOUILLART's theory. The quantities ε_i, which here appear as excentricities, are thus there considered as perturbations, and are called x_1, x_2, x_3.

The possibility of the solutions (2), (7), (12) and (13) depends on the sign of Ω_2. In all these cases $cos\,(\alpha+\beta)$ and $cos\,\alpha'$ are of the same sign. Thus if we put

$$Q=\mu_3\frac{a_2}{a_3}A-\mu_1 B,$$

we find that for positive values of Q the solutions (2) and (12) can exist, (7) and (13) being impossible; for negative values of Q (2) and (12) become impossible, but (7) and (13) are possible. We find for these solutions:

Solution (2)		Solution (12)	
$\pi_2=0°$	$\pi_1=0°$	$\pi_2=0°$	$\pi_1=0°$
$l_2=0$	$l_1=0$	$l_2=180$	$l_1=180$
$\lambda_2=0$	$\lambda_1=0$	$\lambda_2=180$	$\lambda_1=180$
$\varkappa$ *positive.*		$\varkappa$ *negative.*	

Solution (7)		Solution (13)	
$\pi_2=180°$	$\pi_1=0°$	$\pi_2=180°$	$\pi_1=0°$
$l_2=180$	$l_1=0$	$l_2=0$	$l_1=180$
$\lambda_2=0$	$\lambda_1=0$	$\lambda_2=180$	$\lambda_1=180$
$\varkappa$ *positive.*		$\varkappa$ *negative.*	

All four solutions have $K=0°$.

For the solutions (2) and (12) we find

$$\frac{\varepsilon_2}{\varepsilon_3}=2\frac{Q}{B} \qquad \frac{\varepsilon_2}{\varepsilon_1}=\frac{1}{2}\frac{a_2}{a_1}\frac{Q}{A}$$

and for (7) and (13)

$$\frac{\varepsilon_2}{\varepsilon_3}=-2\frac{Q}{B} \qquad \frac{\varepsilon_2}{\varepsilon_1}=-\frac{1}{2}\frac{a_2}{a_1}\frac{Q}{A}.$$

For $Q=0$ (or, if higher orders of ε_i are taken into account, for a value of $\frac{\mu_1}{\mu_3}$ in the neighbourhood of the value for which $Q=0$) we have $\varepsilon_2=0$. The solutions (2) and (7) then become identical, and similarly (12) and (13). We thus find that the two cases (2) and (7) form together one continuous family, which exists for all values of $\frac{m_1}{m_3}$. The same thing is true of (12) and (13).

Thus all that has been said above regarding the existence of the periodic solutions has now been proved. It remains to investigate their stability. For that purpose we must form the equation (2). We introduce the notations:

$$\frac{1-s}{T\sqrt{m_2}}=-\varrho, \qquad \frac{1}{m_2{}^2}\frac{\partial^2 R'}{\partial x\partial y}=(xy), \qquad m_2\frac{\partial^2 R'}{\partial p\partial q}=[pq]\,,$$

where x and y represent two of the variables l_i, g_i and p and q two of the variables L_i, G_i. The quantities (xy) are of the order zero in the masses, the quantities $[pq]$ are of the first order.

With the aid of the values of ψ_i, which have been derived above, it is now easy to write down the determinant $\Delta(s)$. The differential coefficients such as $\frac{\partial\psi_6}{\partial\beta_1}=\frac{\partial^2F_0}{\partial L_1^2}$ will have m_2 in the denominator. To remove this, and to make all terms of the same type also of the same order in the masses, the five lower rows have been multiplied by m_2. Then the five upper rows, have been divided by $\sqrt{m_2}$, and the last five columns by $m_2\sqrt{m_2}$. Finally every term has been divided by T. The equation then becomes

$$\begin{vmatrix} -\rho & 0 & 0 & 0 & 0 & (l_1l_1) & (l_1l_2) & (l_1l_3) & (l_1g_1) & (l_1g_2) \\ 0 & -\rho & 0 & 0 & 0 & (l_1l_2) & (l_2l_2) & (l_2l_3) & (l_2g_1) & (l_2g_2) \\ 0 & 0 & -\rho & 0 & 0 & (l_1l_3) & (l_2l_3) & (l_3l_3) & (l_3g_1) & (l_3g_2) \\ 0 & 0 & 0 & -\rho & 0 & (l_1g_1) & (l_2g_1) & (l_3g_1) & (g_1g_1) & (g_1g_2) \\ 0 & 0 & 0 & 0 & -\rho & (l_1g_2) & (l_2g_2) & (l_3g_2) & (g_1g_2) & (g_2g_2) \\ K_1-[L_1L_1] & -[L_1L_2] & -[L_1L_3] & -[L_1G_1] & -[L_1G_2] & -\rho & 0 & 0 & 0 & 0 \\ -[L_1L_2] & K_2-[L_2L_2] & -[L_2L_3] & -[L_2G_1] & -[L_2G_2] & 0 & -\rho & 0 & 0 & 0 \\ -[L_1L_3] & -[L_2L_3] & K_3-[L_3L_3] & -[L_3G_1] & -[L_3G_2] & 0 & 0 & -\rho & 0 & 0 \\ -[L_1G_1] & -[L_2G_1] & -[L_3G_1] & -[G_1G_1] & -[G_1G_2] & 0 & 0 & 0 & -\rho & 0 \\ -[L_1G_2] & -[L_2G_2] & -[L_3G_2] & -[G_1G_2] & -[G_2G_2] & 0 & 0 & 0 & 0 & -\rho \end{vmatrix} = 0, \quad (5$$

For brevity we have put

$$K_1 = -\frac{3}{\mu_1 a_1^2}, \quad K_2 = -\frac{3}{a_2^2}, \quad K_3 = -\frac{3}{\mu_3 a_3^2}.$$

To simplify the determinant (5), we may use the relation, which has already been mentioned above,

$$4\,(l_1\,x) + 2\,(l_2\,x) + (l_3\,x) = 0,$$

where x represents an arbitrary element. We perform the following operations, which are here, in order to save space, only indicated (the ordinary figures refer to the columns, the roman figures to the rows):

To (8) add 4.(6) + 2.(7),	From	(*VI*)	subtract	4.(*VIII*)
	,,	(*VII*)	,,	2.(*VIII*),
,, (*III*) ,, 4 (*I*) + 2.(*II*),	,,	(1)	,,	4 (3),
	,,	(2)	,,	2.(3).

The determinant then becomes divisible by ρ^2, and the columns (3) and (8) and the rows (III) and (VIII) drop out. For the sake of clearness I will, however, continue to indicate the remaining columns by their original rotation-numbers.

Now we have[1])

$$(l_1 l_1) = (ll), \qquad (g_1 g_1) = (\omega\omega),$$
$$(l_1 l_2) = -2(ll), \qquad (g_1 g_2) = -(\omega\omega),$$
$$(l_2 l_2) = 4(ll) + (l'l'), \qquad (g_2 g_2) = (\omega\omega) + (\omega'\omega'),$$
$$(l_1 g_1) = (l\omega), \qquad (l_1 g_2) = -(l\omega),$$
$$(l_2 g_1) = -2(l\omega), \qquad (l_2 g_2) = 2(l\omega) + (l'\omega').$$

By means of these relations the determinant can be still further simplified. We perform the following operations

To (7)	add	2.(6),	From (*VI*)	subtract	2.(*VII*)
„ (*II*)	„	2.(*I*),	„ (1)	„	2.(2)
„ (10)	„	(9),	„ (*IX*)	„	(*X*)
„ (*V*)	„	(*IV*),	„ (4)	„	(5).

If now the remaining rows and columns are rearranged in the following order

1, 2, 6, 7, 4, 5, 9, 10,
I, II, VI, VII, IV, V, IX, X,

then the equation becomes

$$\Delta(\varrho) = \begin{vmatrix} -\varrho & 0 & (ll) & 0 & 0 & 0 & (l\omega) & 0 \\ 0 & -\varrho & 0 & (l'l') & 0 & 0 & 0 & (l'\omega') \\ A_{11} & A_{12} & -\varrho & 0 & A_{13} & A_{14} & 0 & 0 \\ A_{21} & A_{22} & 0 & -\varrho & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & (l\omega) & 0 & -\varrho & 0 & (\omega\omega) & 0 \\ 0 & 0 & 0 & (l'\omega') & 0 & -\varrho & 0 & (\omega'\omega') \\ A_{31} & A_{32} & 0 & 0 & A_{33} & A_{34} & -\varrho & 0 \\ A_{41} & A_{42} & 0 & 0 & A_{43} & A_{44} & 0 & -\varrho \end{vmatrix} = 0, \quad . \quad (6)$$

where the meaning of the coefficients is as follows (I mention only those coefficients that will be used below, those omitted all contain m_2 as a factor):

$$A_{11} = K_1 + 4K_2 \quad + \text{ terms of higher orders}$$
$$A_{12} = A_{21} = -2K_2 + \quad „ \quad „ \quad „ \quad „$$
$$A_{22} = K_2 + 4K_3 \quad + \quad „ \quad „ \quad „ \quad „$$
$$A_{33} = -[G_1G_1] + 2[G_1G_2] - [G_2G_2]$$
$$A_{34} = A_{43} = -[G_1G_2] + [G_2G_2]$$
$$A_{44} = -[G_2G_2]$$

[1]) These formulas suppose $(ll') = (\omega\omega') = (l'\omega) = (l\omega') = 0$. This is only true if the *third* and higher orders of ε_i are neglected.

The expressions $[pq]$ all contain m_2 as a factor. Thus, in order to derive the term independent of m_2 in the development of ϱ, we take all those expressions $= 0$. The determinant then becomes divisible by ϱ^4, and is reduced to its first four columns and rows. Four of the eight roots of our equation thus appear to be divisible by $\sqrt{m_2}$. The first terms of the other four are the roots of the equation:

$$\begin{vmatrix} -\varrho & 0 & s & 0 \\ 0 & -\varrho & 0 & s' \\ A_{11} & A_{12} & -\varrho & 0 \\ A_{21} & A_{22} & 0 & -\varrho \end{vmatrix} = 0,$$

or

$$\varrho^4 - (A_{11}\, s + A_{22}\, s')\, \varrho^2 + (A_{11}\, A_{22} - A^2_{12})\, ss' = 0 \,, \quad . \quad . \quad . \quad (7)$$

where we have put, for brevity:

$$(ll) = s\,, \quad (l'l') = s'.$$

The solution can only be stable, if the equation (7) has two real and negative roots. Now A_{11} and A_{22} are negative, and $A_{11}\, A_{22} - A^2_{12}$ is positive. The necessary and sufficient condition that the equation (7) shall have two real and negative roots is therefore, that both *s and s'* are *positive.* Now we have

$$\left.\begin{aligned} s &= \frac{\mu_1}{a_2} \left\{ A\, \varepsilon_1 \cos\alpha - B\, \varepsilon_2 \cos(\alpha + \beta) \right\} \\ s' &= \frac{\mu_3}{a_3} \left\{ A\, \varepsilon_2 \cos\alpha' - B\, \varepsilon_3 \cos(\alpha' + \beta') \right\} \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

For the six possible combinations we find the signs of s and s' as given below

	α	β	α'	β'	s	s'	
(6)	0°	180°	0°	180°	+	+	
(16)	180	180	180	180	—	—	*unstable.*
(2)	0	0	0	180	0	+	
(7)	0	180	18ı	0	+	0	
(12)	180	0	180	180	0	—	*unstable.*
(13)	180	180	0	0	—	0	*unstable.*

The solutions (16), (12) and (13), i.e. those with a negative value of $\varkappa$, are thus certainly unstable. For (2) s will be positive if

$$A\,\varepsilon_1 - B\,\varepsilon_2 > 0.$$

By using the value of $\frac{\varepsilon_2}{\varepsilon_1}$ found above, this leads to the condition

$$Q < 2\frac{a_1}{a_2}\frac{A^2}{B} = 7.41.$$

Similarly we find for (7) that s' will be positive if

$$Q > -\frac{1}{2}\frac{B^2}{A} = -0.46.$$

For the family consisting of the solutions (2) and (7) we thus find that s and s' are both positive for all values of Q between the limits -0.46 and $+7.41$. For the Jovian system we find $Q = +4.14$.

For the solution (6) both s and s' are always positive.

This is, however, not sufficient to prove the stability of these solutions. We must also consider the four remaining roots of our equation (6). To determine these I divide the last two rows and the 5th and 6th columns of that determinant by $\sqrt{m_2}$. Introducing then

$$\varrho = \varrho'\sqrt{m_2}, \qquad A_{ij} = m_2\,B_{ij},$$

the equation becomes

$$\Delta(\rho') = \begin{vmatrix} -\rho'\sqrt{m_2} & 0 & (ll) & 0 & 0 & 0 & (l\omega) & 0 \\ 0 & -\rho'\sqrt{m_2} & 0 & (l'l') & 0 & 0 & 0 & (l'\omega') \\ A_{11} & A_{12} & -\rho'\sqrt{m_2} & 0 & B_{13}\sqrt{m_2} & B_{14}\sqrt{m_2} & 0 & 0 \\ A_{21} & A_{22} & 0 & -\rho'\sqrt{m_2} & B_{23}\sqrt{m_2} & B_{24}\sqrt{m_2} & 0 & 0 \\ 0 & 0 & (l\omega) & 0 & -\rho' & 0 & (\omega\omega) & 0 \\ 0 & 0 & 0 & (l'\omega') & 0 & -\rho' & 0 & (\omega'\omega') \\ B_{31}\sqrt{m_2} & B_{32}\sqrt{m_2} & 0 & 0 & B_{33} & B_{34} & -\rho' & 0 \\ B_{41}\sqrt{m_2} & B_{42}\sqrt{m_2} & 0 & 0 & B_{43} & B_{44} & 0 & -\rho' \end{vmatrix} = 0.$$

If now again we neglect all terms which appear multiplied by $\sqrt{m_2}$, and if we perform the operations

From (7) subtract $\frac{(l\omega)}{(ll)}\cdot(3)$, from (8) subtract $\frac{(l'\omega')}{(l'l')}\cdot(4)$,

we find

$$\Delta(\varrho') = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \cdot ss' \cdot \begin{vmatrix} -\varrho' & 0 & \sigma & 0 \\ 0 & -\varrho' & 0 & \sigma' \\ B_{33} & B_{34} & -\varrho' & 0 \\ B_{43} & B_{44} & 0 & -\varrho' \end{vmatrix} = 0,$$

where we have put

$$\sigma = (\omega\omega) - \frac{(l\omega)^2}{(ll)}, \quad \sigma' = (\omega'\omega') - \frac{(l'\omega')^2}{(l'l')}.$$

We thus find that ϱ' is determined by an equation very similar to (7). For the coefficients B_{ij} we find easily

$$B_{33} = H_1 + H_2,$$
$$B_{34} = B_{43} = -H_2,$$
$$B_{44} = H_2 + H_3,$$

where

$$H_1 = -\frac{\partial^2 R'}{\partial \Pi^2_1} = -\frac{1}{16}\frac{1}{\mu_1 a_1 a_2}\frac{A}{\varepsilon^3_1}\cos\alpha,$$

$$H_2 = -\frac{\partial^2 R'}{\partial \Pi^2_2} = \frac{1}{16}\left\{\frac{\mu_1}{a^2_2}\frac{B}{\varepsilon^3_2}\cos(\alpha+\beta) - \frac{\mu_3}{a_2 a_3}\frac{A}{\varepsilon^3_2}\cos\alpha'\right\},$$

$$H_3 = -\frac{\partial^2 R'}{\partial \Pi^2_3} = \frac{1}{16}\frac{1}{\mu_3 a^2_3}\frac{B}{\varepsilon^3_3}\cos(\alpha'+\beta').$$

For the cases (6), (2) and (7), which are the only ones that we need investigate, all these expressions are negative. For H_1 and H_3 this is at once evident. For H_2 we find:

	Sol. (6)	Sol. (2)	Sol. (7)
$H_2 =$	$-\frac{P}{16a^2_2\varepsilon^3_2}$,	$-\frac{Q}{16a^2_2\varepsilon^3_2}$,	$\frac{Q}{16a^2_2\varepsilon^3_2}$,

which is also negative in all three cases. The equation determining the first term of ϱ' now becomes

$$\varrho'^4 - \{(H_1+H_2)\sigma + (H_2+H_3)\sigma'\}\varrho'^2 + \{H_1.H_2 + H_1.H_3 + H_2.H_3\}\sigma\sigma' = 0, \quad (9)$$

The condition that this equation shall have two real and negative roots is again that σ and σ' are both positive. Now we have

$$\sigma = (\omega\omega) - \frac{(l\omega)^2}{s}, \qquad \sigma' = (\omega'\omega') - \frac{(l'\omega')^2}{s'}.$$

It is only necessary to investigate those cases where s and s' are both positive. The conditions of stability thus become

$$s.(\omega\omega) > (l\omega)^2, \qquad s'.(\omega'\omega') > (l'\omega')^2.$$

The values of s and s' have already been given above (8). For the other quantities we find

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$$(\omega\omega) = \frac{\mu_1}{a_2} \left\{ 4\,A\,\varepsilon_1 \cos\alpha - B\,\varepsilon_2 \cos(\alpha + \beta) \right\},$$

$$(\omega'\omega') = \frac{\mu_3}{a_3} \left\{ 4\,A\,\varepsilon_2 \cos\alpha' - B\,\varepsilon_3 \cos(\alpha' + \beta') \right\},$$

$$(l\,\omega) = \frac{\mu_1}{a_2} \left\{ 2\,A\,\varepsilon_1 \cos\alpha - B\,\varepsilon_2 \cos(\alpha + \beta) \right\},$$

$$(l'\,\omega') = \frac{\mu_3}{a_3} \left\{ 2\,A\,\varepsilon_2 \cos\alpha' - B\,\varepsilon_3 \cos(\alpha' + \beta') \right\},$$

from which

$$s.\,(\omega\,\omega) = \frac{\mu_1^2}{a_2^2} \left\{ 4\,A^2 \varepsilon_1^2 + B^2 \varepsilon_2^2 - 5\,A B \varepsilon_1 \varepsilon_2 \cos\beta \right\},$$

$$(l\omega)^2 = \frac{\mu_1^2}{a_2^2} \left\{ 4\,A^2 \varepsilon_1^2 + B^2 \varepsilon_2^2 - 4\,A B \varepsilon_1 \varepsilon_2 \cos\beta \right\}.$$

Therefore

$$s.\,\sigma = -\frac{\mu^2_1}{a^2_2} A\,B\,\varepsilon_1\,\varepsilon_2 \cos\beta,$$

and similarly

$$s'.\,\sigma' = -\frac{\mu_3^2}{a_3^2} A\,B\,\varepsilon_2\,\varepsilon_3 \cos\beta'$$

The only stable solutions are thus those in which β and β' are both 180°, and the only solution which satisfies this condition is (6). This solution, i. e. the case actually occurring in nature, is thus found to be the *only* stable periodic solution.

It needs hardly be mentioned that all the proofs given above suppose, that the developments in powers of ε_i and m_i converge so rapidly, that the sign of the various quantities used is determined by their first term. What the upper limits of ε_i and m_i are satisfying this condition, cannot be stated without a special investigation, but nature teaches us, that for the values occurring in the system of Jupiter the solution (6) still exists as a stable solution.

Physics. — "*Contribution to the theory of binary mixtures,* XIII." By Prof. J. D. VAN DER WAALS.

We have considered the closed curve, discussed in the preceding Contributions, as the projection of the section of two surfaces, viz. $\frac{d^2\psi}{dx^2} = 0$, and $\frac{d^2\psi}{dv^2} = 0$, constructed on an x-axis, a v-axis and a T-axis. Let the x-axis be directed to the right, the v-axis to the front and the T-axis vertically. The projection of these sections on the other projection planes will now also be a closed curve, in general