of medicina forensis henceforth maintain its position by the side of the precipitation in vitro as a valuable method.

Among the many questions that show themselves in the study of our subject, there was also the ,wing: what happens to the injected horse-serum during the anaphylactic shock? If there were only a minimal quantity free and unchanged in the circulation of the intoxicated animal, it ought to be possible that with its blood a normal animal could be sensitized. Now it has appeared to me that this is never crowned with success. From this it may be inferred that all the antigen taken up in the blood-circulation is at once fixed by the cells of the hypersensitive organism, resp. deprived of its specific character at the same time.

Mathematics. — *"On the structure of perfect sets of points"*. By Dr. L. E. J. Brouwer. (Communicated by Prof. Korteweg).

(Communicated in the meeting of March 26, 1909).

§ 1.

Sets of points and sets of pieces.

The sets of points discussed in the following lines are supposed to be lying within a finite domain of a Sp_n.

By a *piece* of a closed set of points μ we understand a *single point or closed coherent set of points,* belonging to μ, and not contained in an other closed coherent set of points belonging to μ.

We can regard as elements of μ its pieces as well as its points, in other words we can consider μ on one hand as a *set of points,* on the other hand as a *set of pieces.*

Let us choose among the pieces of μ a fundamental series $S_1, S_2, S_3, \ldots$, then to μ belong one or more pieces ${}_1S_\omega$, ${}_2S_\omega, \ldots$ with the property that S_n lies entirely within a for indefinitely increasing n indefinitely decreasing distance ε_n from one of the pieces ${}_\alpha S_\omega$. These parts ${}_\alpha S_\omega$ we shall call the *limiting pieces* of the fundamental series $S_1, S_2, S_3, \ldots$

As thus the set μ possesses to each of its fundamental series of pieces at least one limiting piece, *a closed set of points is likewise closed as set of pieces.*

By an *isolated piece* of μ we understand a piece having from its rest set in μ a finite distance, in other words a piece, the rest set of which is closed.

THEOREM 1. *Each piece of μ is either a limiting piece, or an isolated piece.*

Let namely S be a non-isolated piece, then there exists in μ a fundamental series of points $t_1, t_2, t_3, \ldots$ not belonging to S, converging to a single point t of S. If t_1 lies on S_1, then S_1 has a certain distance ε_1 from S. There is then certainly a point t'_2 of the fundamental series possessing a distance $< \varepsilon_1$ from S, lying therefore not on S_1 but on an other piece S_2. Let ε_2 be the distance of S_2 from S, then there is certainly a point t'_3 of the fundamental series possessing a distance $< \varepsilon_2$ from S, lying thus neither on S_1, nor on S_2, but on a third piece S_3. Continuing in this manner we determine a fundamental series of pieces $S_1, S_2, S_3, \ldots$, containing consecutively the points $t_1, t'_2, t'_3, \ldots$ converging to t. So the pieces $S_1, S_2, S_3, \ldots$ converge to a single limiting piece which can be no other than S.

By a *perfect set of pieces* we understand a closed set, of which each piece is a limiting piece.

A perfect set of pieces is also perfect as set of points; but the inverse does not hold. For, a perfect set of points can very well contain isolated pieces.

We shall say that two sets of pieces *possess the same geometric type of order*, when they can be brought piece by piece into such a one-one correspondence, that to a limiting piece of a fundamental series in one set corresponds a limiting piece of the corresponding fundamental series in the other set. So in general a closed set considered as a set of pieces possesses not the same geometric type of order as when considered as a set of points.

A closed set we shall call *punctual*, when it does not contain a coherent part, in other words when all its pieces are points.

§ 2.

Cantor's fundamental theorem and its extensions.

The fundamental theorem of the theory of sets of points runs as follows:

If we destroy in a closed set an isolated point, in the rest set again an isolated point, and so on transfinitely, this process leads after a denumerable number of steps to an end.

The discoverers of this theorem, CANTOR [1]) and BENDIXSON [2]) proved

[1]) Mathem. Annalen 23, p. 459—471.

[2]) Acta Mathematica 2, p. 419—427.

it with the aid of the notion of the *second transfinite cardinal* Ω, which is however not recognised by all mathematicians. LINDELÖF [1]) gave a proof independent of this notion, where, however, the process of destruction itself remaining non-considered, the result is more or less obtained by surprise.

Only for linear sets there have been given proofs of the fundamental theorem, which at the same time follow the process of destruction and are independent of Ω [2]).

The rest set which remains after completion of the process of destruction and which we may call the *Cantor residue*, is after CANTOR [3]) *a perfect set of points*, however of the most general kind, thus *in general not a perfect set of pieces.*

An extension of the fundamental theorem, enunciated by SCHOENFLIES[4]) and proved by me [5]), can be formulated as follows:

If we destroy in a closed set an isolated piece, in the rest set again an isolated piece, and so on transfinitely, this process leads after a denumerable number of steps to an end.

My proof given formerly for this theorem was a generalisation of LINDELÖF's method, but at the same time I announced a proof which follows the process of destruction, and which I give now here; in it is contained a proof of the fundamental theorem, which in simplicity surpasses by far the existing ones, is independent of Ω, and follows the process of destruction:

By means of Sp_{n-1}'s belonging to an orthogonal system of directions we divide the Sp_n into n-dimensional cubes with edge a, each of these cubes into 2^n cubes with edge $\frac{1}{2}a$, each of the latter into 2^n cubes with edge $\frac{1}{4}a$, etc.

All cubes constructed in this way form together a denumerable set of cubes K.

Let now μ be the given closed set, then K possesses as a part a likewise denumerable set K_1 consisting of those cubes which contain in their interior or on their boundary points of μ.

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1) Acta Mathematica 29, p. 183—190.

2) SCHOENFLIES, Bericht über die Mengenlehre I, p. 80, 81; Gött. Nachr. 1903, p. 21—31; HARDY, Mess. of Mathematics 33, p. 67—69; YOUNG, Proceedings of the London Math. Soc. (2) 1, p. 230—246.

3) l. c. p. 465.

4) Mathem. Annalen 59; the proof given there p. 141—145, and Bericht über die Mengenlehre II, p. 131—135 does not hold.

5) Mathem. Annalen 68, p. 429.

To each destruction of an isolated point or isolated piece in μ now answers a destruction of at least *one*[4]) cube in K_1; but of the latter destructions only a denumerable number is possible, thus also of the former, with which CANTOR's theorem and SCHOENFLIES's theorem are proved both together.

Let us call the rest set, which remains after destruction of all isolable pieces, the *Schoenflies residue*, then on the ground of theorem 1 we can formulate:

THEOREM 2. *A Schoenflies residue is a perfect set of pieces.*

§ 3.

The structure of perfect sets of pieces.

Let S_1 and S_2 be two pieces of a perfect set of pieces μ. Let it be possible to place a finite number of pieces of μ into a row having S_1 as its first element and S_2 as its last element in such a way, that the distance between two consecutive pieces of that row *is smaller than a*. Then we say, *that S_2 belongs to the a-group of S_1*.

If S_2 and S_3 both belong to the a-group of S_1, then S_3 belongs also to the a-group of S_2, so that μ breaks up into a certain number of "a-groups". This number is finite, because the distance of two different a-groups cannot be smaller than a.

If $a_1 < a_2$, and if an a_1-group and an a_2-group of μ are given, then these are either entirely separated or the a_1-group is contained in the a_2-group.

If two pieces S_1 and S_2 of μ are given, then there is a certain maximum value of a, for which S_1 and S_2 lie in different a-groups of μ. That value we shall call the *separating bound of S_1 and S_2 in μ*, and we shall represent it by $\sigma_\mu (S_1, S_2)$.

If fartheron we represent the *distance of S_1 and S_2* by $\alpha (S_1, S_2)$, then $\sigma_\mu (S_1, S_2)$ converges with $\alpha (S_1, S_2)$ to zero, but also inversely $\alpha (S_1, S_2)$ with $\sigma_\mu (S_1, S_2)$. For otherwise convergency of $\sigma_\mu (S_1, S_2)$ to zero would involve the existence of a coherent part of μ, in which two different pieces of μ were contained, which is impossible.

The maximum value of a for which μ breaks up into different a-groups we shall call the *width of dispersion* of μ, and shall represent it by $\delta(\mu)$. This width of dispersion of μ is at the same time the greatest value which $\sigma_\mu (S_1, S_2)$ can reach for two pieces S_1 and S_2 of μ.

[4]) Even of an infinite number.

The maximum value of a, for which μ breaks up into *at least n* different a-groups we shall call the *n-partite width of dispersion* of μ, and shall represent it by $\delta_n(\mu)$. Clearly $\delta_n(\mu)$ is $\leqq \delta(\mu)$.

For μ exists furthermore a series of inc1easing positive integers $n_1(\mu)$, $n_2(\mu)$, $n_3(\mu)$, ... in such a way that $\delta_n(\mu)$ for n between $n_{k-1}(\mu)$ and $n_k(\mu)$ is equal to $\delta_{n_k(\mu)}(\mu)$. This quantity $\delta_{n_k(\mu)}(\mu)$ we call the k^{th} *width of dispersion* of μ and as such we represent it by $\delta^{(k)}(\mu)$.

We now assert that it is always possible to break up μ into m_1 perfect sets of pieces $\mu_1, \dots \mu_{m_1}$ so as to have $\delta(\mu_h) \leqq \delta_{m_1}(\mu)$ and $\alpha(\mu_{h_1}, \mu_{h_2}) \geqq \delta_{m_1}(\mu)$.

Let namely be $\delta_{m_1}(\mu) = \delta^{(k)}(\mu)$; we can then obtain the required number m_1 by composing each μ_h of a certain number of $\delta^{(k)}(\mu)$-groups belonging to a same $\delta^{(k-1)}(\mu)$-group. We are then also sure of having satisfied the condition $\alpha(\mu_{h_1}, \mu_{h_2}) \geqq \delta_{m_1}(\mu)$.

Fartheron we can place the $\delta^{(k)}(\mu)$-groups of a same $\delta^{(k-1)}(\mu)$-group into such a row that the distance between two consecutive ones is equal to $\delta^{(k)}(\mu)$. If we take care that each μ_h consists of a non-interrupted segment of such a row, then the condition $\delta(\mu_h) \leqq \delta_{m_1}(\mu)$ is also satisfied.

Let us now break up in the same way each μ_h into m_2 perfect sets of pieces $\mu_{h1}, \dots \mu_{hm_2}$ in such a way that $\delta(\mu_{hi}) \leqq \delta_{m_2}(\mu_h)$ and $\alpha(\mu_{hi_1}, \mu_{hi_2}) \geqq \delta_{m_2}(\mu_h)$, and let us continue this process indefinitely.

If then we represent by F_ν an arbitrary row of ν indices, then we shall always find

$$\delta_{m_\nu}(\mu_{F_{\nu-1}}) \leqq \delta_{m_1+m_2+\dots+m_\nu+1-\nu}(\mu) \quad . \quad . \quad . \quad . \quad (A)$$

As μ is a perfect set of pieces, the width of dispersion $\delta(\mu_{F_\nu})$ can converge to zero only for indefinite increase of ν; out of the formula (A) follows, however, that for indefinite increase of ν that convergency to zero always takes place and, indeed, uniformly for all ν^{th} elements of decomposition together.

At the same time the separating bound of every two pieces lying in one and the same ν^{th} element of decomposition converges uniformly to zero; so these elements of decomposition converge themselves uniformly each to a single piece.

If finally a variable pair of pieces of μ is given, then their distance can converge to zero only when the order of the smallest element of decomposition, in which both are contained, increases indefinitely.

The simplest mode in which this process of decomposition can be executed is by taking all m_k's equal to 2. If then we represent the two elements of decomposition of the first order by μ_0 and μ_2, those of the second order by $\mu_{00}, \mu_{02}, \mu_{20}, \mu_{22}$, and so on, then in this way the different pieces of μ are brought into a one-one correspondence with the different fundamental series consisting of figures 0 and 2. And two pieces converge to each other then, and only then, when the commencing segment which is common to their fundamental series, increases indefinitely.

Let us consider on the other hand, in the linear continuum of real numbers between 0 and 1, the perfect punctual set π of those numbers which can be represented in the triadic system by an infinite number of figures 0 and 2. The geometric type of order of π we shall represent by ζ.

Two numbers of π converge to each other then and only then, when the commencing segment which is common to their series of figures, increases indefinitely.

So, if we realize such a *one-one* correspondence between the pieces of μ and the numbers of π, that for each piece of μ the series of indices is equal to the series of figures of the corresponding number of π, then to a limiting piece of a fundamental series of pieces of μ corresponds a limiting number of the corresponding series of numbers in π, so that we can formulate:

THEOREM 3. *Each perfect set of pieces possesses the geometric type of order* ζ.

For the case that the set under discussion *is punctual and lies in a plane,* this theorem ensues immediately from the following well-known property:

Through each plane closed punctual set we can bring an arc of simple curve.

Combining SCHOENFLIES's theorem mentioned in § 2 with theorem 3 we can say:

THEOREM 4. *Each closed set consists of two sets of pieces; one of them possesses, if it does not vanish, the geometric type of order* ζ, *and the other is denumerable.*

§ 4.

The groups which transform the geometric type of order ζ *in itself.*

Just as spaces admit of groups of continuous one-one transformations, whose geometric types of order [1]) are again spaces, namely

[1]) In this special case formerly called by me "Parametermannigfaltigkeiten" Comp. Mathem. Annalen 67, p. 247.

the finite continuous groups of LIE, the geometric type of order ζ admits of groups of continuous one-one transformations, which possess likewise the geometric type of order ζ.

In order to construct such groups we start from a decomposition according to § 3 of the set μ into m_1 "parts of the first order" $\mu_1, \mu_2, \ldots \mu_{m_1}$, of each of these parts of the first order into m_2 "parts of the second order" $\mu_{h1}, \mu_{h2}, \mu_{h3}, \ldots \mu_{hm_2}$, etc.

The parts of the first order we submit to an arbitrary transitive substitution group of m_1 elements, of which we represent the order by p_1, and which we represent itself by g_1.

After this we submit the parts of the second order to a transitive substitution group g_2 of $m_1 m_2$ elements which possesses the parts of the first order as systems of imprimitivity and g_1 as substitution group of those systems into each other. We can then represent the order of g_2 by $p_1 p_2$.

The simplest way to construct such a group g_2, is to choose it as the direct product of g_1 and a substitution group γ_2, which of the parts of the second order leaves the first index unchanged and transforms the second index according to a single transitive substitution group of m_2 elements.

We then submit the parts of the third order to a transitive substitution group g_3 of $m_1 m_2 m_3$ elements which possesses the parts of the second order as systems of imprimitivity and g_2 as substitution group of those systems into each other. We can represent the order of g_3 by $p_1 p_2 p_3$.

In this way we construct a fundamental series of substitution groups $g_1, g_2, g_3, \ldots$

Let τ_1 be an arbitrary substitution of g_1; τ_2 a substitution of g_2 having on the first index of the parts of the second order the same influence as τ_1; τ_3 a substitution of g_3 having on the first two indices of the parts of the third order the same influence as τ_2; and so on.

The whole of the substitutions τ_n then determines a substitution of the different fundamental series of indices into each other, in other words *a transformation τ of the pieces of μ into each other*.

This transformation is in the first place a *one-one transformation*; for, two different pieces of μ lie in two different parts of a certain, e.g. of the r^{th} order, and these are transformed by τ into again two different parts of the r^{th} order.

If fartheron $S_1, S_2, S_3, \ldots$ is a fundamental series of pieces, possessing S_ω as its only limiting piece, then, if $\lambda(n)$ is the lowest possible order with the property that S_n and S_ω lie in different parts of that order, $\lambda(n)$ must increase indefinitely with n.

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So by the transformation τ the fundamental series passes into a new fundamental series having as its only limiting piece the piece into which S_ω passes by τ.

As a set of pieces μ is thus *continuously* transformed by τ.

Let $\tau'_1, \tau'_2, \tau'_3, \ldots$ be a series of substitutions satisfying the same conditions as the series $\tau_1, \tau_2, \tau_3, \ldots$ If then $\tau_1 \tau'_1 = \tau''_1$; $\tau_2 \tau'_2 = \tau''_2$; etc., then the series $\tau''_1, \tau''_2, \tau''_3, \ldots$ likewise satisfies the same conditions.

If farthermore τ' and τ'' are defined analogously to τ, then $\tau \tau'$ is equal to τ''.

So the transformations satisfying the conditions put for τ form a group, which we shall represent by g.

To investigate the geometric type of order of this group, we decompose in the way indicated in § 3 a perfect set of pieces ϱ into p_1 parts of the first order $\varrho_1, \varrho_2, \ldots, \varrho_{p_1}$; each of these into p_2 parts of the second order $\varrho_{h1}, \varrho_{h2}, \ldots, \varrho_{hp_2}$; and so on.

The p_1 substitutions of g_1 we bring into a one-one correspondence to the parts of the first order of ϱ. Then the $p_1 p_2$ substitutions of g_2 into such a one-one correspondence to the parts of the second order of ϱ, that, if a substitution of g_2 and a substitution of g_1 have the same influence on the first index of the parts of the second order of μ, the part of the second order of ϱ corresponding to the former lies in the part of the first order of ϱ corresponding to the latter.

In like manner we bring the $p_1 \; p_2 \; p_3$ substitutions of g_3 into such a one-one correspondence to the parts of the third order of ϱ, that, if a substitution of g_3 and a substitution of g_2 have the same influence on the first two indices of the parts of the third order of μ, the part of the third order of ϱ corresponding to the former lies in the part of the second order of ϱ corresponding to the latter; and so on.

The parts of ϱ corresponding to a series $\tau_1, \tau_2, \tau_3, \ldots$ then converge to a single piece of ϱ, which we let answer to the transformation τ deduced from the series. Then also inversely to each piece of ϱ answers a transformation τ, and the correspondenceattained in this manner is *a one-one correspondence.*

Farthermore two transformations τ and τ' converge to each other then and only then, when their generating series $\tau_1, \tau_2, \tau_3, \ldots.$ and $\tau'_1, \tau'_2, \tau'_3, \ldots.$ have an indefinitely increasing commencing segment in common, in other words when the corresponding pieces of ϱ converge to each other. So the correspondence between the transformations τ and the pieces of ϱ is *continuous.*

The transformations τ, in other words the transformations of the group g, have thus been brought into a continuous one-one correspondence to the pieces of ϱ, *so that g possesses the geometric type of order ζ.*

If now we adjoin to each substitution group g_n a finite group g'_n of continuous one-one transformations of μ as a set of pieces in itself, tranforming of the pieces of μ the first n indices according to g_n, but leaving unchanged all their other indices, *then the fundamental series of the groups* $g'_1, g'_2, g'_3, \ldots$ *converges uniformly to the group* g.

The set whose elements are the groups g of the geometric type of order ζ constructable in the indicated manner possesses the cardinal number of the continuum. For, already the set of those series $m_1, m_2, m_3, \ldots$, which consist of prime numbers, possesses this cardinal number, and any two different series of this set give rise to different groups g.

We can sum up the preceding as follows:

THEOREM 5. *The geometric type of order* ζ *allows of an infinite number of groups consisting of a geometric type of order* ζ *of continuous one-one transformations and being uniformly approximated by a fundamental series of groups consisting each of a finite number of continuous one-one transformations.*

If in particular we consider those groups g for which each g_n is chosen in the way described at the commencement of this § as the direct product of g_{n-1} and a group γ_n, we can formulate in particular:

THEOREM 6. *The geometric of order* ζ *allows of an infinite number of groups consisting of a geometric type of order* ζ *of continuous one-one transformations and being uniformly convergent direct products each of a fundamental series of finite groups of continuous one-one transformations.*

§ 5.

The sham-addition in the geometric type of order ζ.

Let us choose the factor groups indicated in theorem 6 as simply as possible, namely g_1 as the group of cyclic displacements corresponding to a certain cyclic arrangement of the first indices, and likewise each γ_n as the group of cyclic displacements corresponding to a certain cyclic arrangement of the n^{th} indices; g is then commutative, and transitive in such a way that a transformation of g is determined uniformly by the position which it gives to one of the elements of μ.

Let us further choose an arbitrary piece of μ as *piece zero*. Let us represent this piece by S_0, and the transformation, which transfers S_0 into S_α and is thereby determined, by "$\mp S_\alpha$". That the piece S_β is transferred by this transformation into S_γ, we shall express by the formula

$$S_\beta \mp S_\alpha = S_\gamma,$$

which operation is associative and commutative.

Let us finally choose, in order to make the resemblance to ordinary ciphering as complete as possible, all m_n's equal to 10, let us take for each system of n^{th} indices the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in this order, and let us give to the piece zero only indices 0.

The different pieces of μ we can then represent biuniformly by the different infinite decimal fractions lying between 0 and 1, in such a way, however, that *finite* decimal fractions do not appear and that $\cdot 3\dot{0}$ is not equal to $\cdot 2\dot{9}$, whilst each group γ_n consists of the different ways in which one can add the same number to all n^{th} decimals, modulo 10.

Now according to the above we understand by $\cdot 5473 \ldots \mp \cdot 9566 \ldots$ the decimal fraction, into which $\cdot 5473 \ldots$ is transferred by the transformation which transfers $\cdot\dot{0}$ into $\cdot 9566 \ldots$, or, what comes to the same, the decimal fraction, into which $\cdot 9566 \ldots$ is transferred by the transformation which transfers $\cdot\dot{0}$ into $\cdot 5473 \ldots$

We shall call the operation furnishing this result, on the ground of its associativity and commutativity, the *"sham-addition"* of $\cdot 9566 \ldots$ to $\cdot 5473 \ldots$; it takes place just as ordinary addition, with this difference that in each decimal position the surplus beyond 10 is neglected, thus that different decimal positions do not influence each other. So we have:

$$\cdot 5473 \ldots \mp \cdot 9566 \ldots = \cdot 4939 \ldots$$

Let us understand analogously by $\cdot 5473 \ldots \frown \cdot 9566 \ldots$ the decimal fraction, into which $\cdot 5473 \ldots$ is transferred by the transformation which transfers $\cdot 9566 \ldots$ into $\cdot\dot{0}$, and let us call the operation furnishing this decimal fraction the *"sham-subtraction"* of $\cdot 9566 \ldots$ from $\cdot 5473 \ldots$; then this sham-subtraction is performed in the same way as ordinary subtraction with this difference, that "borrowing" does not take place at the cost of the preceding decimal positions, so that here again different decimal positions do not influence each other. So we have:

$$\cdot 5473 \ldots \frown \cdot 9566 \ldots = \cdot 6917 \ldots$$

By operating only with a finite number, great enough, of consecutive figures directly behind the decimal sign, sham-addition and sham-subtraction furnish in the type of order ζ a result agreeing with the exact one up to any desired degree of accuracy. In this too they behave like ordinary addition and subtraction of real numbers.