Mechanics. — *"Further remarks on the solutions of the field-equations of* EINSTEIN'S *theory of gravitation".* By Prof. W. DE SITTER.

(Communicated in the meeting of April 26, 1917).

1. EINSTEIN has recently [1]) enounced the postulate, that a solution of the field-equations

$$G_{\mu\nu} - \tfrac{1}{2}\lambda g_{\mu\nu} = -\varkappa T_{\mu\nu} + \tfrac{1}{2}\varkappa g_{\mu\nu} T \quad \ldots \ldots \quad (1)$$

in order to be admissible for the actual physical world, must have no discontinuities "at finite distances". In particular the determinant g must for all points at finite distances be different from zero. This postulate is not fulfilled by my solution B, as EINSTEIN very correctly points out, and as is also shown very clearly in my communications. This postulate, however, in the form in which it is enounced by EINSTEIN, is a *philosophical,* or metaphysical, postulate. To make it a *physical* one, the words: "all points *at finite distances*" must be replaced by. "all *physically accessible* points". And if the postulate is thus formulated, my solution B does fulfil it. For the discontinuity arises for

$$r = r_1 = \tfrac{1}{2}\pi R.$$

This is at a finite distance in space, but it is physically inaccessible, as I have already pointed out [2]). The time needed by a ray of light, and a fortiori by a moving material point, to travel from any point r, ψ, ϑ to a point r_1, ψ_1, ϑ_1 (ψ_1 and ϑ_1 being arbitrary) is infinite. The singularity at $r = r_1$ can thus never affect any physical experiment, or as I expressed it l.c., the paradoxical phenomena, or rather absence of phenomena, resulting from this singularity, can only happen before the beginning, or after the end of eternity.

2. A similar remark has been made by Prof. FELIX KLEIN, in a letter to the present writer dated 1918 April 19. He writes:

[1]) *Kritisches zu einer von Herrn* DE SITTER *gegebenen Lösung der Gravitationsgleichungen*, Sitzungsber. Berlin, 7 March 1918, page 270.

[2]) *On Einstein's theory of gravitation and its astronomical consequences, third paper*, Monthly Notices of the R. A. S. Vol. LXXVIII, page 17—18.

On the curvature of space, these Proceedings, Vol XX, p. 229.

"Denken Sie sich die ganze vierdimensionale Welt von Weltlinien durchfurcht. Nun scheint es doch bei allen Ansätzen im EINSTEIN'schen Sinne eine notwendige physikalische Voraussetzung zu sein, dass man diese Linien, so wie sie sich kontinuierlich an einander reihen, mit einem positiven Richtungssinn versehen kann (der von der "Vergangenheit" zur "Zukunft" führt). Dies ist nun im Falle B nicht möglich. Lege ich nämlich einer ersten Linie nach Belieben einen positiven Sinn bei und übertrage diesen unter Beachtung der Kontinuität auf die Nachbarlinien, so komme ich schliesslich, wegen der Zusammenhangsverhältnisse des Elliptischen Raumes, zur Ausgangslinie mit umgekehrtem Sinn zurück. Es entspricht das dem Umstande, dasz die Ebene der elliptischen (wie der projektivischen) Geometrie eine *einseitige* Fläche ist, bei der sich die Indicatrix ↻, die ich um irgend einen Punkt der Ebene herum legen mag, wenn ich sie längs einer durch den Punkt laufenden Geraden verschiebe, bei Rückkehr zum Ausgangspunkte umgekehrt hat: ↺. Meine Bemerkung in Math. Annalen 37, p. 557—58: dass die Uebertragung der Schering'schen Potentialtheorie auf den Fall der elliptischen Ebene unstatthaft ist, ruht genau auf demselben Umstande".

Prof. KLEIN's remark is undoubtedy correct: we return to the starting point with the positive direction reversed, but only if we have travelled *along a straight line*, or at least along a line *which intersects the polar line of the starting point.* This "motion", though mathematically thinkable, is *physically impossible,* for the same reason as above. If we travel along an arbitrary closed curve, which does not intersect the polar line of any of its points, i.e. if we describe a physically possible circuit, then we shall, on returning to the starting point, find the positive direction unaltered.

In my former paper[1]) I pointed out that, in spherical space, the potential $g_{44}-1$ becomes infinite at the antipodal point. I concluded therefrom that, for the representation of the actual physical world, the elliptical space is to be preferred to the spherical. Prof. KLEIN has already made the same remark in his paper of 1890, quoted at the end of his letter. He points out, however, that in elliptical space the sign of the potential would be ambiguous. This would be the case if the above mentioned circuit were possible. Since it is impossible we can choose one of the two possible signs without the danger that any physical phenomena or experiments will ever lead to contradiction or indeterminateness.

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[1]) *On the curvature of space,* these Proceedings, Vol XX, p. **240.**

I use this occasion to point out that, as is well known, Prof. KLEIN was the first to call attention to the elliptical space and its relation to and difference from the spherical space, and generally to investigate and explain the different possibilities of non-Euclidean geometry [1]). In fact all geometrical concepts used in the different stages of the development of modern physical theory are contained in KLEIN's general scheme as given in the second of the papers quoted in the footnote.

3. If we start from the assumption, that the gravitational field is of such a nature that it is possible, by introducing a suitable system of coordinates, to bring the line-element into the form

$$ds^2 = -a\,dr^2 - b\,(d\psi^2 + \sin^2\psi\,d\vartheta^2) + f\,dt^2, \quad . \; . \; . \quad (2)$$

then we can call r the "radius-vector" and t the "time". If now we add the condition that a, b. f must be functions of r only, and not of t, ψ, ϑ, then these conditions may be briefly expressed by saying that the field is *static* and *isotropic*. Then the line-element of three-dimensional space is

$$d\sigma^2 = a\,dr^2 + b\,[d\psi^2 + \sin^2\psi\,d\vartheta^2] \quad . \; . \; . \; . \; . \quad (3)$$

and consequently we have

$$ds^2 = -d\sigma^2 + f\,dt^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2')$$

If now we add the hypothesis that $d\sigma^2$ shall be the line-element of a space of constant curvature, thus

$$\left.\begin{array}{l} r = R\,.\,\chi \\ d\sigma^2 = R^2\,\{\,d\chi^2 + \sin^2\chi\,[d\psi^2 + \sin^2\psi\,d\vartheta^2]\,\}, \end{array}\right\} \quad . \; . \; . \quad (3')$$

then the field-equations (1) reduce to one equation for f, of which the solutions A and B are

$$f = c^2 \quad . \; . \; . \; . \; . \quad . \; . \; . \quad (4A)$$

$$f = c^2 \cos^2\chi \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4B)$$

If we drop the condition of isotropy, then f may be a function of r, ψ, ϑ. For this case LEVI-CIVITA [2]) has given the general solution of the differential equation for f. He starts from EINSTEIN's original equation, i. e. the equation (1) with $\lambda = 0$. It is however not difficult to extend the proof to the general case. Then the equation (11′) of LEVI CIVITA [l. c. page 530] replaces (11) [p. 526],

[1]) *Ueber die sogenannte Nicht-Euclidische Geometrie*, Math. Annalen, Band 4 and 6 (1871 and 1872).

Programm zum Eintritt in die philosophische Facultät, Erlangen 1872, reprinted Math. Annalen, Band 43, p. 63.

[2]) *Realta fisica di alcuni spazi normali del Bianchi*, Rendiconti della R. Accad. dei Lincei, Vol. XXVI, p. 519 (May 1917).

and consequently we must use $K + \varkappa p - \lambda$ instead of $K + \varkappa p$. The equation (13) is not affected, but we now have $K^* = K - \frac{1}{2}(3K + \varkappa p - \lambda)$, and again $K^* = K$; therefore instead of (14) we find $3K + \varkappa p - \lambda = 0$, but the general solution of (13) remains the same. This solution, expressed in my notation, is

$$\sqrt{f} = a_0 \cos \chi + a_1 \sin \chi \sin \beta, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

β being the "latitude" referred to a plane of symmetry, whose inclination ε and node ϑ_0 on the plane $\psi = 0$ are given by

$$b_0 = -a_0, \qquad \begin{aligned} b_1 &= a_1 R \sin \varepsilon \sin \vartheta_0, \\ b_2 &= -a_1 R \sin \varepsilon \cos \vartheta_0, \\ b_3 &= a_1 R \cos \varepsilon. \end{aligned}$$

$b_0 \ldots b_3$ being the constants of integration introduced by LEVI-CIVITA.

The condition of isotropy now is $a_1 = 0$. If this is introduced (5) is reduced to (4 B), and it thus appears that my solution B is the *general solution* for the case of a static and isotropic gravitational field in the absence of matter.