KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXII

N°. 6.

Translated from: „Verslag van de gewone vergaderingen der Wis- en Natuur kundige Afdeeling" Vol. XXVII and XXVIII.

AMSTERDAM
JOHANNES MULLER.
1920.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXII

N°. 6.

President: Prof. H. A. LORENTZ.
Secretary: Prof. P. ZEEMAN.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vol. XXVII and XXVIII).

CONTENTS.

Mathematics. — "*On an involution among the rays of space*". By Prof. JAN DE VRIES.

(Communicated at the meeting of September 29, 1918).

1. Among the rays of space four arbitrary plane pencils of rays determine an involution of pairs of rays; each pair consists of the two transversals t,t' of four rays a,b,c,d appertaining one to each of the four pencils. The pencil (a) we shall also denote by (A,α); A is the vertex, α the plane of (a). Similarly the other pencils are denoted by (B,β), (C,γ), (D,δ).

A straight line t determines four rays a,b,c,d, which, in general, have still another transversal t'. If a,b,c,d appertain to a quadratic regulus, then each line of the complementary, regulus is conjugated to every other line of this complementary regulus.

If t describes the pencil (T,τ), t' engenders a ruled surface which we shall denote by $(t')^x$, where x is the degree of this surface. By the rays of (T,τ) the four pencils (a), (b), (c), (d) are rendered projective.

2. We now suppose that in some way a projective correspondence is established between these pencils and consider the ruled surface T engendered by the transversals t,t' of four corresponding rays.

From A the pencils of rays $(b),(c),(d)$ are projected by three projective pencils of planes; hence three transversals of corresponding rays b,c,d exist which intersect the corresponding ray a at A, so that A,B,C,D are triple points of T. Similarly $\alpha, \beta, \gamma, \delta$ are triple tangent planes, since they contain three lines t.

The intersection of T and α thus consists of three straight lines and a curve which has a triple point at A. Since every ray a contains the points of transit of a pair of transversals t,t', the said curve is of *the fifth degree.* Hence T is a *ruled surface of the eighth degree.*

Eight tangents of the curve α^5, which T has in common with α, pass through A; it follows from this that T contains *eight* rays t, which coincide each with its corresponding ray t'.

3. If t is made to describe the pencil of lines (T,τ), the ruled surface T breaks up into the pencil (t) and a ruled surface $(t')^7$. Hence *the transformation (t,t') converts a plane pencil of lines into a ruled surface of the seventh degree.*

On the line $\alpha\beta$ the pencils $(a), (b)$, as soon as they have become projective, determine two projective point-ranges. One of the united points (coincidences) thereof is the point $\alpha\beta\tau$; through the other passes a line t'. Besides this ray t' the line $\alpha\beta$ also meets the three rays t', which lie in α and the three rays t' in β.

By α the surface $(t')^7$ is intersected along a curve α^4 with a triple point at A. Through every point of intersection of α^4 and τ passes a line t' which coincides with its conjugated line t. Hence *the double-rays of the involution* (t, t') *constitute a line-complex of the fourth order.*

4. In order to make sure of this by another reasoning we consider the threefold infinity of ruled quadrics (bcd). Through three points, arbitrarily chosen on the lines $\alpha\beta, \alpha\gamma, \alpha\delta$, there passes *one* ruled surface (bcd). Hence the system is linear (*complex*) and intersects α along a complex of conics α^2. The conics which touch a ray a at a point R, constitute a system of single infinity. The ruled surfaces of which they are the curves of transit, also constitute a single infinity; the base-curve ϱ^4 is at R tangent to a. Every chord t of ϱ^4 passing through R lies on a ruled surface (bcd), which is at R tangent to the ray AR; the two transversals of the quadruplet a, b, c, d thus coincide in t. Hence the cubic cone which projects ϱ^4 from R, consists of double-rays $t \equiv t'$.

In addition to these the point R carries a plane pencil of double-rays, lying in the plane α. In order to understand this we observe that every line t of α belongs to a pair of lines of the complex $\{\alpha^2\}$. Let S be the point of intersection of t and the second line of this pair. Since AS is at S tangent to the corresponding hyperboloid (bcd), t represents the two transversals of a ray-quadruplet a, b, c, d and is therefore a double-ray of the involution (t, t').

In this way we find again that *the double-rays constitute a line-complex of order four.*

Simultaneously it has become evident that this complex has four *principal planes* $\alpha, \beta, \gamma, \delta$ and, by analogy, four *principal points* A, B, C, D.

5. We now consider three rays b, c, d meeting a line t_A, which passes through A. Each line t which intersects b, c, d, meets in α a certain ray a and is therefore conjugated to t_A. Hence every ray t_A corresponds to ∞^1 rays t'; otherwise: *the rays of the sheaf of lines* $[A]$ *are singular.*

By the rays t_A a trilinear correspondence is established between the pencils $(b), (c), (d)$; in fact, two arbitrary rays b, c determine a transversal t_A, which defines in its turn the corresponding ray d.

32*

In any arbitrary plane pencil of rays (l) a trilinear correspondence is similarly originated. Each of the three double-rays (coincidences) is a transversal of three corresponding rays b, c, d, and therefore a line t' of a ruled surface (bcd), which passes through A and is consequently conjugated to a ray t_A.

Hence the rays which in the correspondence (t, t') are conjugated to the rays through A form a *cubic complex.*

Similarly every ray t_α (in the plane α) is also conjugated to a regulus; so *the rays of the plane α, denoted as a whole by $[\alpha]$, too are singular.*

6. We now consider a ray t of the sheaf of lines which has $D^* \equiv \alpha\beta\gamma$ for its vertex; let d be the ray of (D, δ) which meets t. To t now are conjugated all the ray's t' of the plane pencil which has D^* for its centre and is situated in the plane $D^* d$. Hence this pencil consists of *singular rays* which are each conjugated to every ray of the pencil. The sheaf $[D^*]$ contains ∞^1 of such singular pencils of rays, the planes of which pass through the line $\Delta\Delta^*$.

By an analogous reasoning it is found that the rays in the plane $\delta^* \equiv ABC$ are singular and constitute ∞^1 plane pencils conjugated to each other and having their centres on the line $\delta\delta^*$.

Hence the involution (t, t') contains *eight sheaves and eight planes of singular rays.*

The ray AB meets two definite rays c, d, but *all* rays a, b, and is therefore conjugated to all transversals of these rays c, d. Similarly the ray $\alpha\beta$ is conjugated to ∞^2 rays t'. Thus there are *twelve principal rays,* each of which is conjugated to all rays of a *bilinear congruence.*

7. The hyperboloids $(b\,c\,d)$ which pass through A, constitute evidently a double infinity. Thus through each ray a passes one of these $(b\,c\,d)$. Hence every ray a forms with three definite rays b, c, d a quadruplet belonging to one and the same hyperboloid. The transversals t of this quadruplet form a regulus of which any two lines in the involution (t, t') are reciprocally conjugated. We shall call such a regulus *singular.*

Thus the pencils $(a), (b), (c), (d)$ can be made projective in such a way that every four corresponding rays are directrices of a singular regulus.

We now consider the congruence which contains the ∞^1 singular reguli.

In any plane φ the projective pencils $(a), (b), (c)$ determine three projective point-ranges; hence in φ there lie three lines t, each intersecting four corresponding rays a, b, c, d. Similarly any arbitrary point carries three lines of the congruence. Hence the *singular reguli form a congruence* (3,3).

The pencils of planes, projecting the projective line-pencils (b), (c) from A, engender a quadric cone $(t_A)^2$, so A is a singular point of the congruence. Similarly the rays of the (3,3) which lie in the plane α, envelop a conic. Thus the congruence has *four singular points of the second order* (A, B, C, D) and *four singular planes of the second order* $(\alpha, \beta, \gamma, \delta)$.

It also contains singular plane pencils of rays. If the lines a, b of a quadruplet lying on a hyperboloid meet, c and d intersect also. On $\alpha\beta$ are situated evidently two points $a\,b$ (coincidences of two projective ranges). Each of these is conjugated to one of the two points cd lying on $\gamma\delta$. The transversals of four thus associated rays a, b, c, d form two plane pencils: one lying in the plane $a\,b$ with centre at $c\,d$, the other in the plane $c\,d$ with $a\,b$ for its centre. Hence the congruence (3,3) has *twelve singular plane pencils of rays.*[1])

8. If a pencil (t) contains a ray of the (3,3), the ruled surface $(t')^7$ breaks up into the singular regulus, to which the ray t belongs, and a ruled surface $(t')^5$ having A, B, C, D as double-points and $\alpha, \beta, \gamma, \delta$ as bitangent planes.

If (t) contains two rays of the (3,3), the locus of t' consists of two singular reguli and a ruled surface $(t')^3$, which has a conic and a line t' in common with α. The intersection of the aforesaid with τ consists of two rays $t' \equiv t$ (passing through T) and a third line e, which constitutes the single directrix of $(t')^3$; the double directrix passes through T.

9. To the rays t resting on a line l correspond the rays t' of a *complex of the seventh order.* In fact, l meets seven rays of the ruled surface $(t)^7$, which is conjugated to a plane pencil of rays (t'); this pencil therefore contains seven rays t' of the complex into which the particular linear complex with directrix l is transformed.

The complex $\{t'\}^7$ has principal points at the vertices of the eight sheaves, principal planes in the eight planes of singular rays of the involution (t, t'). For, in fact, l meets one ray of each singular pencil of the sheaf $[A^*]$ and two rays of a quadratic regulus $(b\,c\,d)$ passing through A; thus in the last case the corresponding ray t_A is a *double ray* of the complex $\{t'\}^7$.

---

[1]) The here considered (3,3) is a particular species of the congruence composed of the transversals of the corresponding rays of *three* projective plane pencils. This more general (3,3) too has 12 singular line pencils; contrarily, however, it has only 3 singular points and 3 singular planes of the second order.

Mathematics. — *"On an involution among the rays of space, which is determined by two* REYE *congruences"*. By Prof. JAN DE VRIES.

(Communicated at the meeting of September 29, 1918).

1. REYE's congruence consists of the ∞^2 twisted cubics α^3 which can be laid through five points A_k. It is bilinear: for through any arbitrary point passes one curve only and an arbitrary line is twice intersected by one curve only [1]).

Let $[\beta^3]$ denote a second REYE congruence with principal points B_k. An α^3 and a β^3 have *ten* bisecants in common; the ∞^4 sets of ten rays determined by $[\alpha^3]$ and $[\beta^3]$ together fill up the entire ray-space and constitute therein an involution I^{10}. An arbitrary straight line t is bisecant to one α^3 and to one β^3 and therefore conjugated to *nine* lines t'.

If α^3 is composed of a conic α^2 (in the plane α) and a line a (intersecting the conic at A), then the set of rays in I^{10} which is determined by this α^3 and an arbitrary β^3, consists of the *three* chords of β^3 in α and the *seven* transversals of a and α^2. For the chords of β^3 which rest on a form a ruled surface of the fourth degree having with α^2, in addition to the point A, seven points in common.

If β^3 too is composed of a curve β^2 and a line b (intersecting β^2 at B) then the corresponding group in I^{10} consists of the line $\alpha\beta$, the two lines in α joining the transit of b with the points of transit of β^2, the two joins in β of the transits of a and α^2, and lastly five transversals of a, b, α^2, β^2.

If α^3 and β^3 have a point S in common, then four of the common chords pass through S; they constitute the common edges of the cones which project α^3 and β^3 from S.

2. Every line a_k through the principal point A_k is *singular* with respect to the involution; for it is a chord of ∞^1 curves α^3 and belongs therefore to ∞^1 groups of the I^{10}.

The congruence $[\alpha^3]$ can be generated by two systems of quadric cones each having a principal point for its vertex. If, for instance

1) REYE's congruence is treated elaborately in the first chapter of J. DE VRIES' Thesis (*Bilineare congruenties van kubische ruimtekrommen,* Utrecht 1917).

$A_1A_2, A_1A_3, A_1A_4, A_1A_5$ and $A_2A_1, A_2A_3, A_2A_4, A_2A_5$ are taken for basal edges, then the curves α^3 having a line a_1 drawn through A_1 for a chord will all lie on the cone α_1^2, which is determined by a_1.

At each of the two points of intersection of a_1 and the curve β_1^3 which has a_1 for a chord, three rays t' meet which, in two differents groups of the I^{10}, are conjugated to $t \equiv a_1$. It follows from this that:

Each singular ray passing through one of the ten principal points (A_k, B_k) *is by the transformation* (t, t') *converted into a ruled surface of the sixth degree.*

The curve β_1^3, which has a_1 for a chord, has also a chord a_2 passing through A_2; the latter meets the cone α_1^2 at A_2 and at a point P. Through P passes a curve α^3 having with β_1^3 the chords a_1 and a_2 in common; hence a_2 lies on the ruled surface α_1^6, corresponding to a_1.

The plane B_1a_1 intersects the cone α_1^2 also along an edge a_1^1. On a_1 and a_1^1 the curves α^3 of this cone determine two projective point-ranges; hence through B_1 pass two chords of curves α^3 lying on α_1^2. The chords of the curves α^3 meeting a_1 therefore constitute a *quadratic complex*.

The cone of the complex at the point B_k has four edges in common with the cone which projects β_1^3 from B_k. Hence the ruled surface α_1^6 has a quadruple point at each of the principal points B_k.

The line $a_{12} \equiv A_1A_2$ is a *principal ray* of the I^{10}. For the curve β_{12}^3 which has a_{12} for one of its chords, determines a group of the I^{10} with each of the curves α^3; a_{12} therefore belongs to ∞^2 groups and the rays t' which are conjugated to a_{12} constitute a *congruence*. The chord t' of β_{12}^3 which passes through a point P, is at the same time chord to an α^3 and therefore conjugated to $t \equiv a_{12}$. In a plane Φ lie three chords t' of β_{12}^3, each of which is at the same time chord to a curve α^3. Hence the ray $t \equiv a_{12}$ is conjugated to the rays t' of a congruence (1,3).

3. The ray A_1B_1 is a common chord to ∞^2 pairs α^3, β^3 and is therefore a *principal ray* of I^{10} and conjugated to the rays t' of a line-congruence $[t']$

Each point S of A_1B_1 carries one α^3 and one β^3, and consequently three rays t'; hence the *order* of $[t']$ is *three*.

The curves α^3 which have A_1B_1 among their chords lie on a cone α_1^2 and determine an I^3 on the conic which α_1^2 has in common with a plane Φ. The chords of the α^3 which lie in Φ therefore envelop another conic. Similarly Φ contains also a conic which is enveloped by the chords of the curves β^3 which have A_1B_1 among their chords. Hence in Φ lie four chords t' conjugated to A_1B_1, so that the *class* of $[t']$ is *four*.

Through A_1 there passes a curve β_2^3; the quadric cone projecting this curve from A_1 is intersected by the projecting cones of the curves α^3 which have A_1B_1 among their chords along sets of three edges t'. It follows from this that the congruence $[t']$ has *singular points of the second order* at A_1 and B_1.

Every β^3 of which A_1B_1 is a chord has a chord a_2 passing through A_2; the latter chord intersects the cone a_1^2 determined by A_1B_1 at a point P (in addition to A_2). The α^3 passing through P has the chord a_2 in common with β^3. It appears from this that A_2 is a singular point of the congruence (3, 4).

A plane Φ through A_2 intersects the quadratic cone b_1^2 determined by A_1B_1 along a conic, on which the curves β^3 cut out an involution I^2. Hence through A_2 pass two of the chords in the plane Φ belonging to the curves β^3. Thus we find that the principal points A_k and B_k $(k \neq 1)$ are *singular points of the second order* with respect to the congruence (3, 4) conjugated to A_1B_1.

The line A_1B_1 is a chord of the figure α^3 composed of $a_{12} \equiv A_1A_2$ and an α^2. The cone β_1^2 which contains the curves β^3 of which A_1B_1 is a chord determines an involution I^3 in the plane $\alpha_{345} \equiv A_3A_4A_5$, the sets of which involution consist of the transits of the curves β^3. The chords t' of the congruence $[t']$ which lie in α_{345} therefore envelop a conic. It follows from this that the planes α_{klm} and β_{klm} are singular planes of the second order with respect to the congruence (3,4) conjugated to A_1B_1.

4. Every ray t in the plane α_{345} is *singular* since it is a chord to all conics α^2 lying in α_{345}. Together with the curve β^3, which has t among its chords, the ∞^1 curves α^3 evidently determine ∞^1 groups of rays t'; each group consists of the two additional chords of β^3 which lie in α_{345} and of seven rays t' resting on a_{12}. All these sets of seven belong to the ruled surface $(t')^4$ engendered by the chords of β^3 which rest on a_{12}; they constitute an I^7 on this surface.

To the *rays* $[t]$ *of the plane* α_{345} as a whole are conjugated the rays $\{t'\}$ of the *special linear complex* which has a_{12} for directrix. These rays form ∞^3 groups of seven rays.

Every ray $t \equiv \alpha_{klm}\beta_{klm}$ is a principal ray of the I^{10}; for, in fact, it is a chord to ∞^1 conics α^2 and to ∞^1 conics β^2, and therefore conjugated to ∞^2 groups of rays t'.

Now consider for example the line $t \equiv \alpha_{345}, \beta_{345}$ as a common chord of two definite conics α^2 and β^2. Of the corresponding lines t' two lie in α_{345}; they join the transit of b_{12} with the two transits of β^2. Similarly two rays t' lie in β_{345}; the remaining five are transversals to a_{12} and b_{12}.

Hence the rays of the bilinear congruence which has a_{12} and b_{12} for directrices, can be arranged into ∞^2 groups of five lines t'.

5. If the ray t describes a pencil (T, τ) the nine corresponding rays t' remain on a ruled surface (t'); the degree of this surface we can determine by investigating the number of lines t' resting on $A_1 A_2$.

Consider therefore, to begin with, the rays t' meeting a_{12} outside A_1 and A_2. Such a line t' can be chord to an α^3 composed of a_{12} and a conic α^2 in the plane α_{345}. Of the pencil (t) the particular ray which rests on a_{12} meets on the intersection of the planes α_{345} and τ on α^2 and is therefore a chord to a composite α^3. Together with the β^3 which has t among its chords this α^3 determines *six* lines t' resting on a_{12} and, in addition to this, three lines t' in the plane α_{345}.

Similarly the plane α_{125} contains *three* lines t'; these are common chords to an α^3 and a β^3, the first composed of a_{34} and a conic in α_{125}, the latter having among its chords the ray of the pencil (T,τ) which meets a_{34}.

In the same way the planes α_{123} and α_{124} too contain *three* rays t' each of the ruled surface conjugated to (t).

6. In order to determine the number of rays t' passing through A_1 we consider the surface Δ engendered by the curves α^3 having each one ray of the pencil (T, τ) among their chords.

Let d denote the line of transit of α_{123} in the plane τ, D the point of transit of a_{34}. The ray of (T,τ) which meets a_{34} determines on Δ a conic, which lies in α_{123} and in combination with a_{34} constitutes an α^3 belonging to the above-mentioned surface Δ. It follows from this that Δ passes through the ten lines a_{kl} and contains ten conics, one in each of the planes α_{pqr}. Hence the intersection of Δ and α_{123} consists of the lines a_{12}, a_{13}, a_{23} and a conic passing through A_1, A_2, A_3, so that Δ is a surface of degree five.[1])

An arbitrary plane Φ is intersected by the curves α^3 in the triplets X_1, X_2, X_3 of an involution $(X)^3$. Since through any two curves α^3 a quadric surface can be laid, it is possible to join any two triplets of the aforesaid involution by a conic. Hence a point

[1]) Δ^5 evidently has a triple point at each of the five points A_k. The locus of the pairs of points at which each α^3 of Δ^5 meets the corresponding ray t is a curve c^4 having a double point at T. Hence Δ^5 is intersected by τ along an additional line l. It follows from this that Δ^5 is also the locus of the curves α^3 which meet l.

Y_3 lying on the straight line X_1X_2 determines a triplet, the other two members Y_1 and Y_2 of which are collinear with X_3. If Y_3 describes the line X_1X_2 the line $y_3 \equiv Y_1Y_2$ revolves about the fixed point X_3. The triplets of the involution therefore determine polar triangles with respect to a conic[1]).

7. The pencil (T,τ) determines in the congruences $[\alpha^3]$ and $[\beta^3]$ two surfaces Δ_A and Δ_B (§ 6) and conjugates to each curve α^3 of Δ_A a curve β^3 of Δ_B.

Now consider the cone generated by the chords which can be drawn from A_1 to the curves β^3 lying on Δ_B.

A plane Φ through A_1 intersects Δ_B along a curve c^6 which contains ∞^1 triplets of the involution $(X)^3$. If the point X_1 describes the curve c^6 the line X_2X_3 envelops a curve of the fifth class, every tangent of which is a chord to one of the curves β^3 lying on Δ_B. Hence through A_1 pass five of these chords, whence it follows that the just mentioned cone of chords through A_1 is of degree five; it will be denoted by K^5.

Now consider a ray t_α of (T,τ) as a chord of an α^3. The quadric cone which projects this curve from A_1 has ten edges a_1 in common with K^5. Each of these rays a_1 is a chord to a β^3 which has also one of the rays, t_β, of the pencil (t) among its chords; to each ray t_α therefore are conjugated ten rays t_β.

A ray t_β is a chord to a definite curve β^3; this curve sends a chord a_1 through A_1. The curves α^3 which have a_1 among their chords, form a cone $a_1{}^2$; this cone intersects τ along a conic on which the points of transit of the curves α^3 determine an involution I^3. The triplets of chords joining these transits two and two envelop a second conic. Hence two of these chords pass through T. To a ray t_β therefore correspond two rays t_α.

(T,τ) evidently contains twelve rays $t_\alpha \equiv t_\beta$. To each of these the involution I^{10} conjugates a ray a_1; on the ruled surface (t') corresponding to the pencil (t) A_1 is therefore a twelve-fold point. Hence the points of intersection of the ruled surface (t') and the line a_{12} lie on 12 lines a_1, 12 lines a_2, 3 lines of α_{123}, 3 of α_{124}, 3 of α_{125} and on 6 lines which do not lie in any of these planes and do not pass through A_1 or A_2.

A plane pencil of lines is by the transformation (t,t') converted into a ruled surface of degree 39.

This ruled surface has 10 *twelve-fold points* (A_k, B_k).

The surfaces Δ_A and Δ_B each have a curve of the fourth degree

---

[1]) This conic is the locus of the points whereat Φ is touched by the curves α^3.

in common with τ. These curves each have a double point at T and therefore intersect at 12 additional points S. Through each of these points pass 3 lines t' of the ruled surface (t'). Hence $(t')^{39}$ has *twelve triple points* in the plane τ.

The 20 planes α_{klm} and β_{klm} are *triple tangent planes* for each of them contains three rays t'.

Mathematics. — *"On an involution among the rays of space"*. By G. SCHAAKE. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of January 25, 1919).

In the Proceedings of this Academy of the méeting of September 29, 1918, XXVII, p. 256 a communication of Prof. JAN DE VRIES is to be found, dealing with an involution of rays determined by four plane pencils of rays which are arbitrarily chosen in space. In the sequel this investigation will be completed on certain points.

1. In § 4 of the communication referred to above a system of ∞^3 ruled quadrics is mentioned of which each regulus has one generator in each of the three line-pencils (b), (c) and (d). The number of these surfaces passing through three points is there determined by choosing one point in each of the planes β, γ and δ containing the pencils. Then in the first place the rays b, c and d, which pass through these points, certainly furnish a ruled surface $(b\,c\,d)$ which satisfies the condition. In addition to this, however, it is possible to construct for instance a surface of which the generators passing through the points chosen in the tangent planes β and γ do not belong to the system of rays b, c, d, whilst the third generator passing through the point in the plane δ is one of the system (d). Hence the system of quadrics is *not linear*.

Now consider the quadrics of the system which pass through an arbitrary point P. On each of these surfaces lies a line, passing through P, which meets the lines b, c and d, by which the surface is defined.

The rays passing through P determine a trilinear correspondence T_P between the pencils (b), (c) and (d). Each triplet furnishes a surface of our system of quadrics passing through P. Hence, if we want to know how many of the ruled quadrics pass through three given points P, Q and R, we must discover how many triplets of rays the three trilinear correspondences T_P, T_Q and T_R have in common.

Such a trilinear correspondence, however, is represented by an equation between the direction-parameters of b, c and d. If we regard these quantities as coordinates with respect to a Cartesian system of axes in space, the equation represents a cubic surface, for convenience sake here referred to as the "correlating surface", which has conical points at the points at infinity of the axes of coordinates and intersects the plane at infinity along the lines at infinity of the three planes of coordinates. It follows from this that two such surfaces have a twisted sextic k^6 in common, which has three double-points, one at the point at infinity of each axis. Three surfaces therefore have six points in common with finite coordinates.

We conclude from this that the correspondences T_P, T_Q and T_R have six triplets of rays in common. One of these consists of the lines b, c and d which pass through the point of intersection A^* of β, γ and δ. The other triplets furnish each a ruled surface passing through P, Q and R. Hence *through any three points there pass five surfaces of the system.*

Moreover, if it is taken into consideration, that by the rays of a plane too a trilinear correspondence is established between the pencils (b), (c) and (d), it follows that there are five surfaces which are tangent to three given planes, six passing through two given points and tangent to a given plane, and six which pass through a given point and touch two given planes.

2. If the curve of intersection k^6 of two "correlating surfaces" is projected on a plane of coordinates, the result is a plane curve λ^4 with double points at the points at infinity of the two axes of coordinates. It follows from this that the surfaces $(b\,c\,d)$, which pass through two given points P and Q, determine a (2,2)-correspondence between the pencils (b) and (c), obtained by conjugating the line b of such a surface to its line c. The same holds for the surfaces which pass through two infinitely near points and thus are tangent to a line l at a point S. If we project (b) and (c) from S, we obtain two pencils of planes between which a (2,2)-correspondence exists. In this correspondence the plane which contains the axes SB and SC of these two pencils, and which belongs to both pencils, is conjugated to itself. For, in fact, this plane intersects β and γ along two lines b and c which meet at a point T lying on the line of intersection of β and γ. If to these lines we conjugate the ray of (d) which passes through the point of intersection of ST and δ, the corresponding $(b\,c\,d)$ breaks up into two planes, the line of intersection of which contains S. The here considered lines b and c are therefore generators of a ruled surface $(b\,c\,d)$, which is at S tangent to l, from which it follows indeed, that the plane passing through SB and SC is conjugated to itself in the (2,2)-correspondence. Each pair of planes of this correspondence furnishes by its line of intersection a generator of a ruled surface which is tangent to l at S. Hence the locus of these generators is a cubic cone, so that the deduction of the order of the complex of double rays, as enunciated by Prof. DE VRIES in § 4 of his communication, holds good.

3. The number of surfaces $(b\,c\,d)$ which pass through a straight line a, is evidently equal to the number of those which pass through three points, *i. e.* to five. If, however, only those ruled surfaces are considered on which a belongs to the same regulus as b, c and d, then we must exclude the regulus determined by the lines b, c and d which pass through the points of intersection of a with β, γ and δ resp. Furthermore, for instance, the plane $(a\,D)$ and the plane passing through BC and through the point of intersection of $(a\,D)$ and $\beta\gamma$ constitute a degenerated hyperboloid of our system, which passes through a. Of this kind two more specimens can be pointed out. Hence *through* a *there passes only one non-degenerated ruled quadric* $(b\,c\,d)$, on which a, b, c and d belong to the same regulus. The deduction of the congruence (3,3) of the singular reguli, enunciated in § 7 of the above-mentioned communication therefore remains valid.

4. Now consider a ray t which meets the lines AB and $\gamma\delta$. The corresponding rays a and b lie in the plane φ passing through AB and t and intersect on the line $\alpha\beta$; the lines c and d determined by t pass through the point of intersection of φ and $\gamma\delta$. Hence the conjugated lines t' form a pencil with its vertex on $\gamma\delta$, in a plane passing through AB.

Hence *the involution* (t,t') *contains six additional bilinear congruences of singular rays.*

We now consider the line AA^*. The corresponding lines b, c and d pass through A^*; a is indefinite. To the line $t = AA^*$ are therefore conjugated all the rays of the sheaf A^*. Similarly the whole system of rays of the plane α^* as a whole is conjugated to the line $t = \alpha\alpha^*$.

Hence *there are eight additional principal rays,* four of which are conjugated to the rays of a sheaf, the remaining four to those of a plane. Thus *the total number of principal rays is twenty* (*vide* § 6 of Prof. DE VRIES' communication).

5. To the rays t of a sheaf S are conjugated in the involution (t,t') the lines of a congruence Σ. We shall now discover the number of lines of Σ which pass through an arbitrary point P. To a line u passing through P we conjugate the line t which passes through S and meets the same rays a and b as u. If u describes a plane pencil with vertex P, then t engenders a quadric cone. If, furthermore, we associate to a line t the ray u' which passes through P and meets the same lines c and d, then the correspondence (t,u') too is quadratic. The correspondence (u,u') therefore is of the fourth

degree, moreover birational, so that it contains six double rays. One of these is the line PS, in which a ray u and a line t coincide. The other five double rays meet the same rays of (a), (b), (c) and (d) as the corresponding line t and are therefore each conjugated as a line t' to rays t of Σ. Hence the order of Σ is *five*.

The number of lines of Σ in a plane W is found by conjugating to a ray w of W the line t, which passes through S and meets the same rays a and b, and to t the line w', which meets the same lines c and d as t. The correspondence (w, w') in W is again birational and of degree four, and so has six double rays. Here there is no line w coinciding with a line t; the class of Σ is therefore *six*.

Hence to a sheaf is conjugated a congruence (5,6).

To the four lines SA^* etc. and to the transversals through S which meet the six pairs $(AB, \gamma\delta)$ correspond plane pencils of lines. Hence *each congruence Σ contains ten singular pencils of lines.*

In the same way it is proved that *to a plane field of rays V corresponds a congruence Φ* (6,5).

If S, or V, contains one or more principal rays, this reduces the order of Σ, or Φ, which reduction is easy to calculate in each case.

6. Two complexes $\{t'\}^7$ (*vide* § 7 of Prof. DE VRIES' communication), which are conjugated to special linear complexes of lines t, with axes l and m, have a congruence C (49,49) in common, which we are going to investigate.

In the first place C contains the congruence Λ conjugated to the bilinear congruence L which has l and m for its directrices. Now L (1,1) has with a congruence Σ (6,5) eleven rays in common just as with a Φ (5,6), from which it follows that Λ is a congruence (11, 11).

Moreover each $\{t'\}^7$ contains as double rays the lines of the sheaves A, B, C, D, those of the planes $\alpha, \beta, \gamma, \delta$, and the lines of the congruence (3,3), constituted by the singular reguli. For, in fact, the line l meets two generators of every singular regulus, each generator corresponding to the entire regulus. In the intersection of the two complexes corresponding to l and m, each of the nine above-mentioned complexes is to be reckoned for four. Together they therefore count for a congruence (28, 28).

Furthermore each $\{t'\}^7$ contains as single rays the lines of the four sheaves A^*, etc.. those of the four planes α^*, etc. and the lines of the six bilinear congruences $(AB, \gamma\delta)$, etc. (§ 4). These form together a congruence (10, 10).

By the foregoing investigation the congruence (49, 49) is completely accounted for and the completeness of the discovered system of singular rays is controlled.

7. By virtue of the HALPHEN theorem the congruences Σ, which are conjugated to two sheaves P and Q have 61 rays in common. To these belongs in the first place the line t' corresponding to the line $t \equiv PQ$.

Each congruence has among its generators the principal rays AB etc. (6 in number), $\alpha\beta$ etc. (6), AA^* etc. (4). For, in fact, to each of the twelve former lines corresponds a bilinear congruence, to the latter four a sheaf and all these have one ray which passes through an arbitrary point. This accounts for 16 of the common rays.

The remaining 44 are found as follows. According to § 1 there exist reguli (bcd) which pass through A and also through P and Q. As follows from § 5 of Prof. DE VRIES' communication a regulus (bcd) through A is conjugated to a ray passing through A. Hence the two congruences Σ have in common five rays passing through A and as many passing through B, C and D resp.

Furthermore there are six surfaces (bcd) which are tangent to α and pass through P and Q. A surface (bcd) which is tangent to α, is conjugated to a line lying in α. Hence the two congruences Σ have in common six rays lying in α and six in β, γ and δ each.

So in this way we find indeed $4 \times 5 + 4 \times 6 = 44$ additional common rays.

The investigation of the intersection of two congruences Φ is quite analogous.

Slightly different is the case of the common rays of the congruences Σ and Φ which are conjugated to a sheaf P and a plane of rays V, which by virtue of the HALPHEN theorem have 60 rays in common.

To these common rays belong the principal rays AB (6), $\alpha\beta$ etc. (6), 12 lines in all.

There are six surfaces (bcd) passing through A and P and simultaneously tangent to V. This furnishes six lines through A, common to Σ and Φ. The same holds for B, C and D, so we have 24 lines in all.

The six surfaces (bcd), which are tangent to α, pass through P and are also tangent to V, furnish six common rays lying in α. Just as many we find in the planes β, γ and δ, together 24.

By the foregoing enumeration the 60 common rays are indeed accounted for.

This § constitutes a proof of the completeness of the system of principal rays.

Mathematics. — "*On an involution among the rays of space, which is determined by a bilinear congruence of twisted elliptical quartics*". By Prof. JAN DE VRIES.

(Communicated at the meeting of March 29, 1919).

§ 1. The twisted quartics (first species) which intersect each of two fixed curves of the same species at eight points form a *bilinear congruence* [1]). An arbitrary line t is bisecant to one of these curves ϱ^4; at its point of transit, P, through a fixed plane α the line is intersected by one other bisecant t' of the said ϱ^4. The lines t and t' constitute one pair of an *involution of rays* which will be investigated in the sequel.

Every bisecant b of the fixed curve β^4 is singular with respect to the congruence $[\varrho^4]$. For, in fact, this congruence is generated by two systems, (β^2) and (γ^2), of quadrics of which the fixed curves β^4 and γ^4 are the base-curves; the hyperboloid β^2, which contains b, is intersected by the surfaces of the system (γ^2) along curves ϱ^4 each having the line b for a chord. The second line b^* which this hyperboloid sends through the point $b\alpha$, is a common chord to the same curves ϱ^4. Hence the bisecants of the base-curves β^4 and γ^4 are *not* singular with respect to the involution (t, t').

§ 2. The congruence $[\varrho^4]$ however contains more singular bisecants s; on each line s the systems of quadrics determine the same involution; the lines s constitute a line-congruence (7,3) [2]).

By the involution determined on s by (β^2) and (γ^2), these systems of quadrics are rendered projective. The curves σ^4, which are generated by two homologous hyperboloids, lie on a quartic surface, which contains the line s. With a plane σ through s this surface has a curve σ^3 in common, the locus of the point-couples which the curves σ^4 have in common with σ outside s. The involution of these point-couples is central *i.e.* the chords t' which carry the couples, meet at a point S of σ^3. The chords t' of any σ^4 which meet s constitute a regulus; the hyperboloids containing these reguli constitute a system, of which s and σ^3 form the base. Since σ^3 has two points

[1]) *Vide* my communication: "*A bilinear congruence of quartic twisted curves of the first species*". (These Proceedings vol. XIV p. 255).

[2]) *l. c.*, p. 257.

in common with s, the centre S of the involution will, if σ is made to revolve about s, twice occupy a position on the line s at each complete revolution. Hence the chords t' meeting s at a point P constitute a *cubic cone* with s for a double edge. The singular lines s are therefore also *singular with respect to the involution* (t, t').

§ 3. We shall now consider the locus Ψ of the curves ϱ^4, each of which furnishes one ray of the plane pencil (T,τ) by one of its chords. The curves $\varrho_B{}^4$ which intersect β^4 at B, lie on the hyperboloid $\gamma_B{}^2$ which passes through B. This hyperboloid intersects τ along a conic, on which the curves $\varrho_B{}^4$ determine an involution I^4; the corresponding directing curve is of the third class. Hence the pencil (t) contains three chords of curves $\varrho_B{}^4$ and from this it follows that the base-curves β^4 and γ^4 are *triple curves* on the surface Ψ.

The locus of the point-couples Q,Q' at which the rays t of the pencil (T,τ) are twice intersected by curves ϱ^4 is a curve τ^5 with a triple point T, which curve contains the points of transit B_k and C_k $(k = 1, 2, 3, 4)$ of the base-curves β^4 and γ^4.[1])

A hyperboloid β^2 has, in addition to the four points B, three point-couples Q,Q' in common with τ^5. Hence the rays t establish a (3,3)-correspondence between the quadrics of the systems (β^2) and (γ^2); in consequence the locus Ψ is a surface of degree *twelve*.

Let (L,λ) be another pencil of lines, λ^5 the corresponding curve (analogous to τ^5). Of the points of intersection of λ^5 and Ψ^{12} $8 \times 3 = 24$ coincide with the points B_k, C_k; the remaining 36 form 18 couples of points, each couple common to a curve ϱ^4 and one of its chords. It follows from this that *the bisecants of the curves ϱ^4 of each of which a given pencil of lines contains one chord, constitute a line-complex of order* 18.

§ 4. The curve ϱ^4 passing through a point P, assumed in α, is projected from P by a cubic cone, the edges s^* of which are *singular rays* of the involution (t,t'). Each edge is conjugated to every other generator of this cone and is therefore at the same time a ray of coincidence. The curve τ^5 determined by a pencil of lines (T,τ) has five points P in common with the line $\alpha\tau$; hence each of these points furnishes one singular ray s^* in the pencil. The rays s^* therefore constitute a *complex of the fifth order*.

In general a line t_α in the plane α is chord to one ϱ^4. All chords of this ϱ^4 which meet t_α, are in the involution (t,t') conjugated to t_α. To this *singular ray* t_α correspond therefore the rays of a quadric regulus and the rays of the two cubic cones which project ϱ^4 from its points of intersection with t_α.

1) l.c. p. 256.

We shall now consider a pencil of lines (T, τ). The curve ϱ^4 which has a ray t of this pencil for one of its chords, has six chords in α; the points of intersection Q of these six lines t_α with τ we conjugate to the point $P \equiv (t, \alpha)$. To the pencil (Q, α) corresponds (§ 3) a line-complex of order 18, constituted by the bisecants of the curves ϱ^4 which meet the rays of (Q, α) twice; this complex has 18 rays t which belong to (T, τ), so that to Q are conjugated 18 points P. Since P thus coincides 24 times with Q, (T, τ) contains 24 chords of curves ϱ^4, each of which chords meets one of the chords of the same ϱ^4, which lie in α. The curve τ^5, corresponding to (T, τ) (§ 3) determines on $\alpha\tau$ five points P_0; the curve ϱ^4 passing through one of these points conjugates three rays t_α to the ray TP_0; hence TP_0 is to be counted thrice in the above-mentioned group of 24 rays t. It follows from this that by the transformation (t, t') a *plane pencil of lines* is transformed into a combination of *five cubic cones* and a *regulus of degree nine.*

This regulus $(t')^9$ has the line $\alpha\tau$ for its directrix. The ruled surface on which it lies has, in addition to the line $\alpha\tau$ and the five rays TP_0, three more lines t' in common with the plane τ of the pencil (T, τ). A confirmation of the foregoing result may be obtained as follows. In the plane τ each curve ϱ^4 determines four points R_k; the chords $u \equiv R_1 R_2$ and $v \equiv (R_3, R_4)$ are reciprocally conjugated by a quadratic transformation [1]). If u describes the pencil (T, τ), v envelops a conic; the point of intersection of u and v therefore will thrice reach a position on the line $\alpha\tau$. Hence there are three rays t of the pencil, of which the corresponding ray t lies in τ (and does not coincide with t).

§ 5. A sheaf of lines with vertex M is transformed by (t, t') into a congruence. A curve ϱ^4 of which one chord t belongs to $[M]$, has two chords u passing through the point N. To the point of intersection, P, of t and α we conjugate the points of transit, Q_1 and Q_2, of the chords u_1 and u_2. Similarly to each point Q correspond two points P. If P moves along a line t describes a plane pencil; in the complex $\{u\}^{18}$, determined by this pencil, u will then describe a cone of degree 18, Q a curve α^{18}. Hence P and Q are correlated in a (2,2) correspondence of degree 18. Since this correspondence in general contains 22 coincidences, the *order* of the congruence which corresponds to $[M]$ is 22.

The curves ϱ^4 which possess a chord passing through M, constitute a surface of degree five. Hence in α there lies a curve α^5 each

[1]) *Vide* my communication: "A quadruple involution in the plane and a triple involution connected with it." (These Proceedings vol. XIII, p. 86).

33*

point of which radiates a cubic cone of singular rays s^*. Since these rays are at the same time coincidences $t \equiv t'$, the image of the sheaf $[M]$ has at M a *singular point of order five*.

Let μ be an arbitrary plane, Φ the plane through M and $\alpha\mu$. To each ray t of (M, Φ) correspond in μ six chords of the particular curve ϱ^4 which meets t twice; their points of intersection Q with $\alpha\mu$ we conjugate to the point of intersection, P, of t and $\alpha\mu$. The line-complex $\{u\}^{18}$ which is conjugated to the pencil (Q, μ) has 18 rays t which belong to the pencil (M, Φ) and therefore determines 18 points P. Since Q coincides 24 times with P, μ contains an equal number of rays t' which are in (t, t') conjugated to rays of $[M]$. Hence the *class* of the congruence which constitutes the image of a sheaf of lines is 24; it is a congruence (22, 24).

The total of the rays $[\mu]$ of a plane is transformed into a con gruence of which the order evidently is 24. In order to find its class we have to bear in mind that a plane Φ can only contain such rays t' as pass through the point of intersection, Q, of the planes α, μ and Φ. The cone with vertex Q of the complex $\{u\}^{18}$ which corresponds to the pencil (Q, μ), breaks up in the pencil (Q,μ), the cubic cone which projects the curve ϱ^4 passing through Q, and a cone of degree 14. The last mentioned cone contains the additional chords which are sent through Q by the curves ϱ^4 meeting rays of (Q, μ) twice. The three rays t' in Φ which are furnished by the cubic cone, are each conjugated to each of the three generators lying in μ, and are therefore to be counted thrice. It follows from this that the *class* of the congruence is 23. So *the image of the total of lines of a plane is a line-congruence* (24, 23).

Chemistry. — *"The trimorphism of allocinnamic acid."* By Dr. A. W. K. DE JONG, Buitenzorg. (Communicated by Prof. P. VAN ROMBURGH).

(Communicated at the meeting of March 29, 1919).

As has already been pointed out in a previous communication, the formation of the same double acid of normal- and allo-cinnamic acid with the different forms of allo-cinnamic acid is in conflict with the view that these forms are chemical isomerides.

STOBBE, the most zealous exponent of this view, has undertaken, in conjunction with SCHÖNBURG[1]), a detailed investigation in which, according to them, it is clearly shown that the allocinnamic acids are chemical isomerides. Their first series of experiments[2]) with the two modifications melting at 68° and 42° respectively led to the following conclusion: "Aus den in diesem Abschnitte beschriebenen 70 Einzelversuchen geht hervor, dass die stabile 68° - Säure und die metastabile 42° - Säure bei Abwesenheit von Keimen unverändert umzukrystallisieren sind; die erstere aber mit Sicherheit nur dann, wenn bei dem Lösungsakte und bei den späteren Vorgängen die Temperatur ihres Schmelzpunktes bzw. des durch das Lösungsmittel erniedrigten Schmelzpunktes nicht erreicht wird, wenn also ein Schmelzen der 68° - Säurekrystalle vermieden wird. Tritt dieses ein, so kann 42° - Säure als Krystallisationsproduct auftreten. Hiernach bewahren also die beiden Säuren in ihren Lösungen bei genau bekannten Bedingungen ihre Individualität. Die Lösungen beider Säuren sind trotz der gleichen Lichtabsorption und trotz der gleichen Leitfähigkeit doch verschieden. Die Allozimtsäure (68°) und die Isozimtsäure (42°) sind zwei chemisch isomere Verbindungen". On page 233 of the same communication this conclusion is extended to the case of the acid melting at 58°.

It appeared to me to be not impossible that the above mentioned investigators had been led astray by the presence of crystal nuclei which, as is only too well known, play a prominent part in the

[1]) Annalen, 402, 187 (1914).

[2]) loc. cit. p. 199.

case of these acid forms. This was the more probable, since it is clear from their communication that they had formed no adequate idea of the nuclei in question. Although they mention the "beim Einfüllen der Lösung etwa an den Wandungen entstandenen Keime" [1]), they neglect to give sufficient attention to the nuclei which can occur in the solution, and to those which are notoriously present in the air.

Before describing how the experiments of STOBBE and SCHÖNBURG were repeated, it is desirable to discuss the considerations which form the basis of the experimental method adopted.

By "nuclei" are to be understood molecular complexes which remain over from the solid state after solution, and which can be formed in the liquid as a preliminary to crystallisation.

According to the solvent used, nuclei and single molecules, or nuclei, double molecules and single molecules can occur in the solution.

If a solution of one of the forms of allocinnamic acid is prepared, a complete or an incomplete dissociation into single molecules results, according to the temperature and the concentration. The higher the temperature and the smaller the concentration, the more complete is the dissociation. It is thus possible to prepare two kinds of solutions, namely, those which contain only single molecules, and those which, in addition, contain also nuclei. In solvents in which double molecules can occur, a third kind of solution is also possible containing single and double molecules, while a solution with nuclei may also at the same time contain single and double molecules.

While within the solutions equilibria between the different kinds of molecules are established, for which, of course, a longer time is required according as the molecules are more strongly held together, the concentration of the solution is greater, and the temperature lower; equilibria are also established above the solutions between the nuclei, the double-molecules and the single molecules. These equilibria are dependent on the composition of the solution.

It is now sufficiently known from experimental investigation, that the atmospheric nuclei of the forms of allocinnamic acid are very persistent. It follows therefore from this that the equilibrium in the air lags behind variations in the solution. As a consequence nuclei are often still present in the air when the solution consists of single molecules only.

---

1) loc. cit. p. 198.

In the experiments it is therefore especially necessary to exclude the dangerous air-nuclei, and, at the same time, to give the solutions sufficient opportunity to reach the equilibrium condition.

The experiments of STOBBE and SCHÖNBURG were repeated with due regard to these considerations in the following manner. The solutions were prepared some time before the distillation and were kept during this time in the dark. At the same time the solutions were several times transferred to another flask. During this process the air nuclei were got rid of by transferring the solution first of all to a small flask which was filled to the brim, blowing away the air above it, and then pouring it into the new flask. Before use any nuclei still remaining in the air above the solution were removed by filling outside the laboratory a small measuring cylinder to the brim with the solution and blowing away the air over it. The solution was then poured from the cylinder through a glass funnel into a distillation flask (50 c.c. to 100 c.c.). The latter was closed by means of a cork through which passed a glass tube reaching to the middle of the bulb of the flask. A plug of cotton wool was inserted into the glass tube, while a larger plug was tied round the tube externally, fitting into the neck of the flask above the side-tube. Both the flask and the tube with the plugs were heated for several hours beforehand in a steam-heated air oven. In fitting them together care was taken that the fingers did not come into contact with any interior part.

In order to exclude the possibility of accidental inoculation, a rapid current of air saturated with the solvent was drawn through the flask for about five minutes. The air was led in through the tube, so that any atmospheric nuclei present could be carried off through the side-tube. The cork was now raised a little, and, by means of a copper wire, which had been heated red-hot, the larger of the cotton wool plugs was pushed below the side-tube of the flask. The latter was then closed again by the cork.

Since STOBBE and SCHÖNBURG observed that the melting of the 68°- and the 58°-acid, which takes place in petroleum ether and in water at 50° and 40°—42° respectively, must be avoided, as this gives rise to the formation of 42°-acid, it was necessary to drive off the solvent at a temperature not exceeding 35°. At this temperature they found no transformation of the two higher melting forms into the 42°-acid.

When the concentration of the solutions was small, the distillation was generally effected in a partial vacuum, while with more concentrated solutions the solvent was evaporated by means of a current of dry air at ordinary temperatures (25°—30°). After the distillation and after the removal of the remainder of the solvent by a dry air current, the flask was closed and placed in ice, whereupon after a longer or shorter time crystallisation began.

The preparation of the allocinnamic acid by subjecting an aqueous solution of sodium or potassium cinnamate (containing about 1 °/₀ acid) to light was somewhat modified, so that a more rapid transformation was attained, and at the same time the unaltered cinnamic acid could easily be used again.

Flat tin-plate vessels were used. In these the solution was set out in the daylight, and the water lost by evaporation was made up daily. After about eight to fourteen days, exposure the solutions were worked up.

In order to separate out the allocinnamic acid the solution was evaporated to about one tenth of its volume and acidified with sulphuric acid when cold. After cooling, the normal cinnamic acid was filtered off, washed, and immediately after drying was used again for the preparation of a new solution. If this acid melts on the water-bath, then allocinnamic acid is still present. This can be extracted by means of hot ligroin. The filtrate was neutralised with alkali, and the solution then evaporated, until crystals began to form. After cooling sulphuric acid was added, which caused the allocinnamic acid to separate out as an oil. This is dissolved in ligroin, and the solution is allowed to crystallise quietly after "seeding" with the 68°- or the 58°-acid as required.

These forms crystallise in large crystals, which even by their appearance and also by their melting points are easily distinguished from the crystals of the double acid of normal and allocinnamic acids, which occurs only in small quantity. In this way almost perfectly colourless crystals are obtained at first which, after recrystallising once from petroleum ether solution, are quite pure. Repetition of the above procedure yields crystals with a pale yellow colour. These may be decolourised in alcoholic solution by means of animal charcoal.

The water solution can afterwards be extracted with ether and yields a further small quantity of impure allocinnamic acid.

The transformation of the 68°-acid into the 58°-acid is easily brought about by boiling the crystals with a little water for a quarter of an hour. The flask is then closed by means of cotton-wool, and the boiling continued. On cooling the allocinnamic acid separates out in oily drops which are transformed into the 42°-acid in ice. If the solution with the oily drops is "seeded" with a trace of the 58°-acid, beautiful needle crystals of this acid are formed on standing.

Experiment gave the following result:

Solutions of the 58°-acid and the 68°-acid in petroleum ether, saturated at 25° (about 0.26 grm. and 0.17 grm. in 5 c.c. respectively) were allowed to stand for eight days in the dark at 25° to 30°. *Without the previous presence of crystals* the solutions gave, on evaporation of the solvent at the ordinary temperature by means ot a current of air, a residue which in ice was transformed into the 42°-acid.

After having stood for eight days in the dark, less concentrated solutions of the two acids gave always the 42°-acid. This 42°-acid remained unchanged during the whole period of observation, about one month.

If, however, the solvent was distilled off immediately after the preparation of the solution, it was found impossible to effect the transformation even of a solution containing only 0.05 grm. of either of the two acid forms in 5 c.c., into the 42°-acid. Solutions in petroleum ether which were kept for eight days before distillation *in contact with crystals of one or other of the two higher melting forms,* gave, either during or immediately after the distillation of the solvent, crystals of the acid from which the solution was made.

Liebermann and Trucksäss [1]) succeeded frequently in excluding nuclei of the higher melting acids by filtering the petroleum ether solution and afterwards heating it on a water bath at 35°. In six out of ten experiments with 68°-acid the transformation into the 42°-acid was effected. The same result was obtained with the 58°-acid in two out of four cases.

Experiments were carried out to ascertain if it were not perhaps possible to remove the nuclei more quickly than by a complete dissociation at ordinary temperature. The same procedure was adopted as before to exclude atmospheric nuclei. Heating at 35° was however, omitted. It was found that when solutions, almost saturated at the ordinary temperature and prepared immediately beforehand, were filtered through cotton-wool, ordinary filterpaper, or even through quantitative filterpaper, they yielded the original acids. If the solutions, even those containing crystals, were filtered after standing for twenty four hours, a residue was frequently, though not always, obtained, which yielded crystals of the 42°-acid. The reason for this may be ascribed to an alteration in the size

---

[1]) Ber. **43**, 411 (1910).

and also in the number of the nuclei, or possibly in both causes together. The transformation of the nuclei of the 68°- and the 58°-acid in ethyl ether and benzene solutions, saturated at 25°, without crystals, did not take place even after they had stood in the dark for fourteen days. This is very probably due to the great concentration of the solutions. At 25° about 4.4 grms. of 58°-acid is soluble in 1.6 grms. of ethylether, and about 4.6 grms. in 2.3 grms. of benzene. The 68°-acid dissolves to the extent of about 6.6 grms. in 3.2 grms. of ethylether and about 4 grms. in 3 grms. of benzene.

An ether solution containing 2 grms. of 68°-acid in 5 c.c. and a benzene solution containing 2.1 grms. of acid in 3.6 grms. of solvent, gave a residue after standing for eight days in the dark which yielded the 42°-acid on crystallisation. After two months, standing in the dark a solution containing 2 grms. of the 58°-acid in 1.6 grms. of ether and a benzene solution with 1.8 grms. of acid in 2.5 grms. of solvent gave also a residue which crystallised out as 42°-acid. An attempt to remove the nuclei from ethylether and benzene solutions, saturated at 25°, by filtration through cotton wool or filter paper was not successful, even when the solutions had been kept for more than ten days free from crystals.

From the foregoing it appears that the transition of the 58°-acid and the 68°-acid into the 42°-form in solution can take place independently of the melting of these forms, and that at 25°—30° they can form solutions which, apart from differences of concentration, are identical, provided that the nuclei are afforded an opportunity for transformation, and that atmospheric nuclei are excluded. When, however, the concentration of the solution is high, as may be the case with ether and benzene solutions, then it is not possible to break up the nuclei, or to remove them by filtration. In this case there exists most probably an equilibrium between the nuclei and the other molecules.

One of the principal arguments of STOBBE and SCHÖNBURG for the chemical isomerism of these acid forms is thus rendered ineffective, while the results are in complete agreement with the assumption of the trimorphism of the allo-cinnamic acids.

In connection with what STOBBE and SCHÖNBURG have communicated regarding the transformation of the 42°-acid and the 58°-acid into the 68°-acid at —14° (ice and salt), it was of importance to investigate if the same change also took place in solution.

Various solvents were used. The most important results were obtained with water, so that these may be detailed first.

An experiment was made with 68°-acid which had been freed from 68°-acid nuclei by boiling with water. The solution was evaporated down until the acid separated out as an oil at ordinary temperature. This solution gives crystals of 42°-acid on cooling to 0°. The presence of a small quantity of liquid acid indicates at once the occurrence of undesired inoculation.

Small quantities of this solution were introduced into the flasks with the cotton wool plugs.

After filling and drawing air through the flask, the solution was again boiled. After cooling the flasks were placed in a freezing mixture. The temperature of the mixture was in some experiments about —10°, in others about —16°. After twenty-four hours these temperatures were 0° and 5° respectively. (The mixtures were kept in a box packed with hay).

The cooling was continued until transformation had taken place. This point is easily recognised from the more copious crystallisation and also from the form of the crystals. After the ice had melted, the flask was opened, the tube with the plug withdrawn, and the solution carefully poured out so that the crystals as far as possible remained in the flask. The flask was then closed in the usual way, and the few drops of water were removed by means of a stream of dry air at the ordinary temperature. The melting point was then determined.

It was found that, whenever the initial temperature of the mixture was —10°, the 58°-acid was always formed, while, when the initial temperature was —16°, the 58°-acid was formed in nearly as many cases as the 68°-acid. (58°-acid in five experiments and 68°-acid in seven). There is apparently a range of temperature within which the 58°-acid is formed, while at lower temperatures the 68°-acid is obtained.

The transformation at —10° sometimes requires several days; at —16° it is complete after a few hours. In this way a method is given by which one or other of these acid forms may be prepared. Experiments with a solution in petroleum ether of low boiling-point were carried out as follows. A dilute solution of the 68°-acid in petroleum ether was prepared and freed from nuclei at the ordinary temperature. Portions of this solution were introduced into several flasks through which a rapid stream of air was drawn. After the cotton-wool plug had been pushed below the side tube, a large portion of the solvent was distilled off. The solution was then cooled. It was found that, both when the initial temperature was —10° and when it was —16°, the 58°-acid was obtained in several cases, but generally the 68°-acid was formed.

Solutions in benzene prepared in a similar way showed at —10° a transformation only in exceptional cases. In these cases the 68°-acid was obtained. At —16° after twenty four hours the 68°-acid was always formed.

Further experiments were made by adding a few drops of the solvent to crystals of the 42°-, 58°-, and the-68°-acid contained in the flasks with the cotton wool plugs, in such a way that crystals still remained in the solution. The solvent was introduced into the flasks through the glass tube with the cotton wool plug, the part of the tube projecting beyond the cork having been previously heated in order to prevent infection. It was found that the 58°- and the 68°-acid were unchanged. The 42°-acid was, however, transformed, the same changes being observed as with the solutions.

STOBBE and SCHÖNBURG [1]) assert that the 42°-acid and also the 58°-acid in the solid state are transformed into the 68°-acid on cooling in ice and salt. As the results obtained by me would seem to cast doubt on the correctness of this assertion, the solid substances, after having been carefully dried, were cooled for six days in ice and salt in the flasks with the cotton wool plugs. In the case of the 42°-acid the drying was effected by heating the flask to 80°—90°. It was observed that none of the three acid forms was altered by cooling. As STOBBE and SCHÖNBURG always worked with capillary tubes, a small quantity of the dry acid was introduced into a capillary from the flask. After six days cooling this also showed no change in the melting point. The solid substances are thus unchanged by cooling in this way. If, however, the 42°-acid is moist, transformation can take place. To this fact the changes of this acid observed by the above investigators are probably due.

It is difficult to understand the transformation of 58°-acid into 68°-acid on cooling, as observed by STOBBE and SCHÖNBURG, unless one assumes that they used only ten capillaries for the cooling of the acid, in which after determined intervals of time the melting-point was taken. In that case the possibility is always present that 42°-acid is formed by the melting, as this takes place more easily in capillaries than would appear from the results of these investigators (p. 239). From the 42°-acid in a moist state the 68°-acid would then be formed.

As a further result of the investigation may be deduced, that there is a great difference between double molecules and nuclei. The transformations which have been described take place only in solu-

[1]) loc. cit., p. 218, 236.

tions of suitable concentration, while even a chloroform solution which contained 35.4 % by weight of allocinnamic acid, and in which, therefore, double molecules were certainly present, no transformation was observed after six days' cooling. In my opinion the explanation of this is that only solutions containing nuclei can be transformed, and that there exists only one kind of double molecule.

VAN DER WAALS' supposition that double molecules consist merely in the temporary association of the single molecules, is thus rendered more probable. Further investigation is required to clear up this point.

In the nuclei we have thus a determinate arrangement of the molecules. They represent the smallest particles of the substance in the solid condition, the simplest nucleus consisting of two molecules. It is not necessary to assume that the molecules are united at the carboxyl groups in order to explain the existence of different isomerides, since, as I have found, coumarine, which has no carboxyl group in the molecule, occurs also in a metastable form. It appears to me more probable that the reason for the occurrence of isomerides must be sought in the double bond. I hope shortly to return to this point.

STOBBE and SCHÖNBURG (p. 200) consider the occurrence of "Lösungsgemische" of the 42°- and the 68°-acid as an argument for the isomerism of these acids. It is clear from what has been said above, that the nuclei in the solution play an important part. The authors have however, interpreted their experiments in a different sense. They supposed that they had found that "In jedem Einzelfalle als Verdampfungsrückstand 68°-Säure erhalten wird nach einem Gesamtzusatz von 2.9—4.1 Proz. 68°-Säure, d. h. waren weniger als 2.9 Proz. 68°-Säure zur 42°-Säurelösung zugesetzt, so schmolz der Verdampfungsrückstand bei 42°; betrug der Zusatz mehr als 4.1° Proz. 68°-Säure, so zeigte der Verdampfungsrückstand den Schmelzp. 68°". They therefore conclude that the distillation residues are to be regarded as mixtures of the 42°-acid and the 68°-acid. On p. 204 they state: "Wenn, wie oben gezeigt worden, die Lösung von der 68°-Säure verschieden von der Lösung der 42°-Säure ist, und wenn nach Zusatz von wenig 68°-Säurelösung zur 42°-Säurelösung ein bei 42° schmelzender Verdampfungsrückstand erhalten wird, so kann dieser nicht reine 42°-Säure (fest) sein. Es muss vielmehr ein Gemisch der beiden isomeren Säuren sein". It is then assumed that solid solutions of these forms exist, and that the crystals are mixed crystals of the 42°-acid and the 68°-acid. It is obvious from what has been said above that this hypothesis is devoid of foundation.

Also on theoretical grounds, from the point of view of the authors, this hypothesis is untenable.

If the 68°-acid and the 42°-acid are *chemically* different, *the "seeding" power of the former must be a property resident in the molecule.* It is therefore impossible to explain why a nucleus of 68°-acid or a trace of this substance is capable in a short time of converting a large quantity of 42°-acid, while, when the molecules of the 68°-acid were distributed in a regular manner among those of the 42°-acid, as must be the case with mixed crystals, no transformation took place until 2.9 % of 68°-acid was present.

Fusion experiments have also led these investigators to assume the existence of solid solutions or mixed crystals. On p. 213 they write: "Ein Teil der eben besprochenen, bei 42° schmelzenden Erstarrungsproducte bleibt jahrelang unverandert, ein anderer Teil verwandelt sich bei Zimmertemperatur, zuweilen schon nach Minuten oder Stunden ohne erkennbare Ursache in 68°-Saure. Diese erstarrten Schmelzen sind also unter einander nicht gleich; sie sind ebenso wie die aus den Lösungsgemischen erhaltenen Verdampfungsruckstände, feste Lösungen oder Mischkristalle mit wechselnden Anteilen 68°-Säure und 42°-Säure [1]). It is not altogether improbable that these transformations could be brought about by one or more atmospheric nuclei which had not been broken up, in cases where no care had been taken to ensure the removal of these A single nucleus remaining in the melt is sufficient to cause transformations of this kind.

On the assumption of the trimorphic nature of allocinnamic acid it might be expected that, when the different forms were melted, the dissociation into single molecules would be more complete according as the time of heating is longer and the temperature higher. This was confirmed by STOBBE and SCHONBURG for the 58°-acid (p. 239) and the 68°-acid (p. 211). In these experiments only 5—7 mgr. was introduced into each capillary. Experiments with 10—50 mgr. of the 68°-acid in larger capillaries showed that even heating for twenty-five minutes at 70° was not sufficient, even in a single case, to bring about a permanent change into the 42°-acid, while on heating 5—7 mgr. of the acid for ten minutes the transformation was effected in four out of ten experiments. The same thing has already been stated by other observers, namely, that large quantities are more difficultly transformable than small quantities.

---

[1]) The opinion of STOBBE that a solid solution is the same thing as nuclei in a melt, is certainly by no means always correct. Nuclei are molecule-complexes, and the molecules of the nuclei in different parts of the melt with nuclei are not necessarily uniformly distributed. On the other hand it is precisely in the case of a solid solution or of mixed crystals that we have a uniform distribution of the molecules among themselves.

The presence of nuclei affords an easy explanation of this. Let us suppose, for example, that the dissociation of the 68°-acid is allowed to proceed so far that there are now only two nuclei remaining in 20 mgr. of the substance.

Suppose also that the substance may now be divided into four equal parts. In two parts there are now at the most one nucleus each; in the other two parts there is no nucleus In this way we have, using portions of 5 mgr., 50 % of the substance transformed into 42 -acid under the applied conditions of temperature and heating. If the substance had not been subdivided, then the 20 mgr. with the two nuclei would have been transformed, either immediately or after several hours, into 68°-acid. [1]) *It is now easy to see that*, if the probability of transformation for 5 mgr. is 50 %, it is 25 % for 10 mgr., 12.5 % for 20 mgr., 6.25 % for 40 mgr., etc., that is, the probability of complete transformation with a given temperature and time of heating becomes smaller and smaller as greater quantities of substance are used.

For each experiment STOBBE and SCHÖNBURG heated only ten tubes, so that the figures obtained by them are certainly not to be used as mean values. They found, for example, on heating 5—7 mgr. of 68°-acid for 10 minutes at 70° that four tubes out of the ten were transformed. (In the case that one part melted at 42° and another at 68°, complete transformation was not obtained). At 100° also only four of the ten showed a transformation into the 42°-acid. At 70° the mean value was probably somewhat lower. If we assume that the mean for ten minutes' heating at 70° was 40 %, then the probability of the conversion of 0.05 gr. in ten minutes is only 5 %, that is, the conversion should take place in one experiment out of twenty.

In order to investigate if perhaps atmospheric nuclei were in part responsible for the difficulty of the transformation, quantities of the 68°-acid were melted in *U*-tubes. One limb of these *U*-tubes was provided with a plug of cottonwool. Through this limb a rapid current of air at 70° was drawn during the time the *U*-tube was being heated in a water-bath at 70°, in order to drive out the air nuclei through the other limb.

After the heating the other limb was closed by means of a cotton wool plug. Working in this way I was as unsuccessful as STOBBE and SCHÖNBURG in transforming 0.05 gr. of 68°-acid permanently into 42°-acid by heating for twenty minutes at 70°.

[1]) This is the reason for the phenomenon, observed by STOBBE and SCHÖNBURG, that in the capillary tubes some portions melted at 42° and others at 68° or 58°.

In this experiment it frequently happened that the melt first crystallised to the 42°-acid and afterwards was transformed back into the 68°-acid. This was also noticed by STOBBE and SCHÖNBURG. This phenomenon is therefore not due to atmospheric nuclei, but to nuclei in the liquid.

Although for a permanent transformation into the 42°-acid a complete absence of 68°-acid nuclei is essential, it is, of course, to be understood that a single nucleus of this acid, remaining over on melting the acid, is not necessarily sufficient to cause an immediate transformation of the melt. This may only take place after the lapse of several hours. Moreover, the nuclei vary in size, and the effect produced by the smallest, which is probably built up of only two molecules of allocinnamic acid, is presumably less effective and less rapid than that of nuclei consisting of several molecules.

From the foregoing it appears that the arguments used by STOBBE and SCHÖNBURG in support of the chemical isomerism of the allocinnamic acids are fallacious, while the experimental results obtained admit of satisfactory explanation on the assumption of the trimorphism of these acids.

Short summary of the results obtained.

1. An experimental method was worked out and applied by which it is possible to avoid inoculation, to exclude atmospheric nuclei, and to bring about the complete dissociation of those nuclei present in the liquid.

2. With this method it was shown that dilute solutions of the 58°-acid and the 68°-acid give, after removal of the solvent by distillation at ordinary temperature, a residue which is transformed into the 42°-acid.

3. When the concentration of the solutions of the 58°-acid and the 68°-acid is great, as may occur with ether and benzene as solvents, these acids are not transformed into the 42°-acid at ordinary temperatures.

4. The solutions may be transformed by cooling in ice and salt. In this way an aqueous solution containing no nuclei of the 58°-acid or the 68°-acid gives at —10° the 58°-acid and at —16° the 58°-acid or the 68°-acid.

5. In the solid state the acid forms appear to be unaltered after cooling for six days in ice and salt.

6. The arguments of STOBBE and SCHÖNBURG in support of the chemical isomerism of these acids are shown to be fallacious. All the results obtained are completely explicable on the assumption that the allocinnamic acids are trimorphous.

Buitenzorg, January 1919.

Chemistry. — A. W. K. de JONG. "*The Truxillic Acids*". (Communicated by Prof. VAN ROMBURGH.)

(Communicated at the meeting of May 3, 1919).

The separation of truxillic acids from cinnamic acid. Since mixtures of cinnamic acid and the truxillic acids are obtained by the action of light on the salts of the former, it was necessary to be in possession of a good method of separation.

Although various attempts have been made to carry out the separation in the wet way, these have so far failed to yield quantitative results. This is partly due to the increased solubility of the truxillic acids in presence of other truxillic acids and especially of cinnamic acid; and partly to the smallness of the quantity of the truxillic acids compared with the cinnamic acid present. With petroleum ether, for example, undoubtedly one of the best solvents for the purpose, no quantitative separation is obtained, since β-cocaic acid and δ-truxillic acid are very appreciably soluble in proportion to the amount of cinnamic acid present. The same is true in a less degree of the other acids.

The attempt was also made to effect the separation by means of the acid potassium salt, which is difficultly soluble in alcohol. This, however, appeared to be impracticable, since some of the truxillic acids were also precipitated to some extent.

For the present there remains only the sublimation method. This, however, with RIIBER's apparatus proceeds very slowly. For this reason the sublimation was carried out at ordinary pressure in a current of air at 130°. The substance was placed in a little boat, which in its turn was placed in a glass tube. The whole was heated in a sand bath at 130°. Sublimation was continued until the weight of the residue became constant.

Separation of the truxillic acids from each other. The acids were dissolved in the calculated quantity of $N/_{10}$ potassium hydroxide solution on heating. To the solution anhydrous calcium chloride was added, 1.5 grm. for each 10 c. c. of solution. After twenty-four hours the precipitate, which may contain the calcium salts of β-, δ- and ε-truxillic acid, was filtered off and washed with calcium chloride solution (1.5 grm. per 10 c.c.). The acids in the filtrate were separated by means of hydrochloric acid and ether and then weighed.

They were once more dissolved in the calculated quantity of $N/_{10}$ potassium hydroxide, and calcium chloride (1.5 grm. for each 10 c. c.) again added. After twenty-four hours the precipitate was filtered and washed with a little calcium chloride solution (1.5 grm. in 10 c. c.). The precipitate is added to that first obtained.

Separation of β-, δ-, and ε-truxillic acids. The calcium salts are treated with hydrochloric acid and ether, and the acids dissolved in the calculated quantity of N/10 potassium hydroxide. Twice the volume of water is then added, and as much N/10 barium chloride solution as was used of N/10 potassium hydroxide. After twenty-four hours the precipitate is filtered and washed with water. It consists of the barium salts of β- and ε-truxillic acid. The acids are extracted from the filtrate by means of hydrochloric acid and ether. They are redissolved in N/10 potassium hydroxide. Twice the volume of water is added, and as much N/10 barium chloride as was used of N/10 potassium hydroxide. In this way a little more β- and ε-truxillic acids are obtained as barium salts. The filtrate now obtained yields δ-truxillic acid with hydrochloric acid and ether, which, if necessary, can be purified by recrystallisation from boiling water.

The precipitated barium salts are boiled with water, cooled, and filtered. Hydrochloric acid is added to the filtrate. If a precipitate is formed, the above treatment is repeated until no precipitate is obtained. The filtrates yield ε-truxillic acid on treatment with hydrochlorid acid and ether. This may be purified, if necessary, by recrystallisation from boiling water. The undissolved barium salt gives β-truxillic acid with hydrochloric acid and ether.

Separation of α-, γ-truxillic acids and β-cocaic acid. To the filtrate from the precipitated calcium salts 8.5 grms of anhydrous calcium chloride per 10 c.c. is added. The precipitate is filtered after twenty-four hours and washed with a solution of calcium chloride prepared by dissolving in water as much calcium chloride in grams as there are c.c.'s of water. The acids are extracted from the filtrate, and these are subjected to a similar procedure in order to separate a small quantity of β-cocaic acid as calcium salt. The precipitated calcium salt gives β-cocaic acid, when treated with hydrochloric acid and ether. This may be recrystallised from boiling water if necessary.

The filtrate from the precipitated calcium salt gives α- and γ-truxillic acid with hydrochloric acid and ether. In order to separate these the acid mixture is boiled with water (25 c.c. per 0.1 grm.) with a reflux condenser for half an hour and is then filtered hot.

The residue consists of α-truxillic acid. On cooling the filtrate yields γ-truxillic acid which, if necessary, may be recrystallised from boiling water.

In order to test the effectiveness of the method of separation a mixture of the six truxillic acids was subjected to the treatment above described with the following result.

	Quantity used grm.	Quantity found grm.	Melting point.	Melting point after recrystallisation from water.
α-truxillic acid	0.119	0.086	270°	—
β „ „	0.100	0.096	202° - 204°	—
γ „ „	0.134	0.099	200° — 215°	220° — 226°
δ „ „	0.106	0.132	gummy	172° — 174°
ε „ „	0.078	0.079	208° — 220°	230°
β-cocaic „	0.106	0.120	165° — 175°	189° — 190°
Total . .	0.643	0.612		

For the sum of the α- and the γ-acid 0.224 grm. was found.

The method is therefore sufficient for the detection of the truxillic acids in presence of each other. If there are only two truxillic acids in the mixture an almost quantitative separation may be effected.

From the above separation several properties of the truxillic acids may be noted. The following may be added.

β-cocaic acid[1]) forms with cinnamic acid a well crystallised double acid with equal proportions of the components. This is obtained by boiling a petroleum ether solution of cinnamic acid, saturated at the ordinary temperature, with a little β-cocaic acid until the latter is dissolved (0.1 grm β-cocaic acid in 500 c.c.). On cooling the double acid separates out, frequently only after several days, in long needles, which melt at 139°. The filtrate gives a fresh quantity of double acid whenever 0.1 grm. of each of the acids is dissolved in it by boiling. The composition is determined by sublimation at 130°—140°. The solubility of γ-truxillic acid in chloroform is increased in a remarkable degree by the presence of β-cocaic acid.

The ammonium salts of the truxillic acids slowly lose their ammonia when their aqueous solutions are evaporated on a water bath and are transformed into the free acids. The ammonium salt of cinnamic acid also possesses this property.

[1]) The acid (m.p. 190°) formerly separated from the acids derived from the coca-alkaloids appears to be β-cocaic acid.

Physics. — "*The Propagation of Light in Moving, Transparent, Solid Substances.* II. *Measurements on the* FIZEAU-*Effect in Quartz*". By Prof. P. ZEEMAN and Miss A. SNETHLAGE.

(Communicated at the meeting of May 3, 1919)

1. In communication I the apparatus has been described that has proved suitable for the investigation of the FIZEAU effect in solid substances.

We have now carried out experiments with quartz, which was traversed by beams of light in the direction of the optical axis. We were led to the choice of this substance by the consideration that in general, crystals are the most homogeneous bodies that we know, and the scattering of light in a crystal must be exceedingly slight on account of its regular structure [1]).

It appeared to us later that the best optical glass, for our purpose, can be compared in some respects with quartz, in others it is even preferable.

In some series of experiments 10 quartz rods were used, supplied by the firm of STEEG and REUTER, with endplanes normal to the optical axis, and of the dimensions $10 \times 1.5 \times 1.5$ cm. Later on four similar rods supplied by the firm of A. HILGER, Ltd. were added to them. For a series of experiments the rods were joined together to form a column of a length of 100 cm.; in a second series of 140 cm. They were placed one behind another in a groove which was milled in a wooden beam, fastened to the driving apparatus by means of four solid screws. The different rods are separated from each other by rubber discs with round apertures of a diameter of about 13 mm. Each quartz rod rests in a groove 13 mm. deep, and is pressed down by two brass plates, fastened with screws in the upper surface of the wooden beam, a thin piece of cork being placed under the plates. The space remaining at the ends of the groove is filled up by a piece of brass tubing. Solid brass plates, which clasp the beam, shut off the ends of the groove.

[1]) LORENTZ. Théories statistiques en thermodynamique, p. 42.

2. In order to place the quartz rods in the groove, we proceeded in the following way. After the interference-bands had been produced with great distinctness, and the beam had been placed on the apparatus, one quartz rod was put in the groove, and if necessary the interference-lines were made distinct anew. It was then ascertained which of the four positions obtained by rotating the rod round its longitudinal axis, gives bands that change least, when the machine is made to assume different positions. Then the second rod is placed behind the first, likewise in four positions etc., till all the rods have been arranged. In order to prevent reflected light from entering the interferometer, each of the rods is placed in a somewhat sloping position by putting a piece of thin cardboard at one end. The rods are put in one by one. After each addition it is tried, whether the correct position has been obtained.

We may still remark in this connection that the glass cylinders with which we have made experiments (see the following communication) have been manufactured so exceedingly well by the firm of ZEISS, that on rotation about the longitudinal axis in a cylindrical groove there does not appear an appreciable change of the interference bands. Hence the optical control becomes a great deal simpler than for quartz. The interference bands finally photographed through the quartz column are decidedly less distinct than the interference bands that are observed when the column has been removed. The lines have become slightly diffuse. This is not the case when the glass cylinders of ZEISS have been introduced. The diameter amounted to 25 mm. with a length of 20 cm. As there were used six cylinders, there were twelve reflecting planes for a total length of glass of 120 cm. In the experiments with the quartz column of 140 cm. length the number of reflecting planes amounted to twenty-eight. Though this great number of reflecting planes must have an unfavourable influence on the distinctness of the system of fringes, yet it was beyond all doubt that it was not owing to this cause that the quartz column had a more unfavourable influence than the glass column. We might still have eliminated the reflections on the interfaces by introducing a liquid of the mean index of refraction of quartz between the successive rods. The complication of the apparatus, which would ensue from this, and the unfavourable experience which we had with moving liquids, made us resolve to put up with the reflections.

3. As source of light a 12 Ampères arc-lamp was used, the light of which was made sufficiently monochromatic by means of filters.

Experiments were carried out with three different colours, the effective wave-lengths of which amounted to 6510, 5380, 4750 Å.U.

4. When white light traverses the apparatus, we easily distinguish the central band. Its centre is the point whose displacement we should wish to measure in an experiment with white light. Also for incident monochromatic light we can speak of the centre of the central band. It is the point that remains fixed when the interference lines rotate, or become narrower or wider through any cause that does not depend on the Fizeau effect. The position of the centre can be determined by means of the horizontal and movable vertical cross-wires in the telescope, by subjecting the interference bands to some modification with the compensator, thus causing the centre to be observed clearly.

When the centre has been determined, the movable vertical wire is displaced over a few bands, so that this wire can have no disturbing influence on the measurement on the photo.

A series of photos is then taken on one photographic plate, in which the directions of the movement alternated.

The observed effect is derived from the displacement of the centre. Of course plates on which a notable rotation of the interference bands has occurred, are rejected.

5. The following table may serve as an example of the results obtained by measurement of a plate taken with:

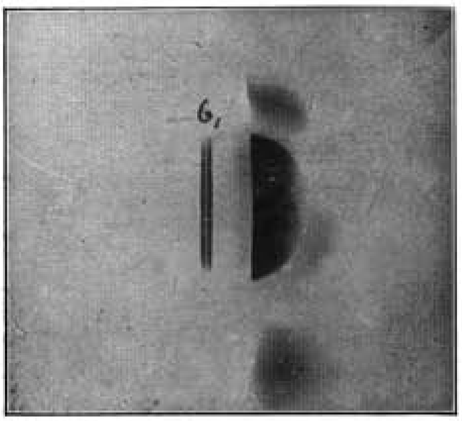
Green light $\lambda = 5380$ Å.U.

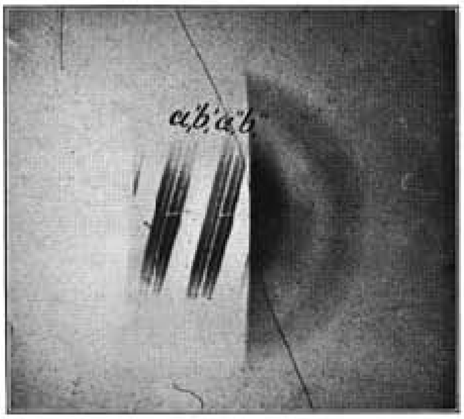
Number of the plate	Maximum velocity in cm. per sec.	Length of column of quartz in cm.	Observed effect	Effect reduced to 1 m. of quartz and max. velocity 10 meters	Mean value for the plate
48	750	100	92	123	
			137	183	
			118	157	152
			99	132	
			125	167	

The effects are given in thousandths of the distance of the fringes.

Altogether photos have been taken on eleven plates with green light $\lambda = 5380$ Å. U. In all fifty-one values have been obtained in this way

P. ZEEMAN and A. SNETHLAGE: "The Propagation of Light in Moving Transparent Solid Substances. II. Measurements of the Fizeau-Effect in Quartz".





Proceedings Royal Acad. Amsterdam. Vol. XXII.

for the observed effect, but not the same number of photos have been taken on all plates as on Nr. 48. The velocities used ranged between 750 and 950 cm./sec. the length of the quartz column being 100 cm. for nine plates, and 140 cm. for two. The obtained values may now be used in two ways to derive a final result from them. For each plate a mean value can be derived, and the arithmetical mean may be taken of the eleven values thus obtained. In this way:

$$0{,}146 \pm 0{,}012,$$

as final result of the effect reduced to a velocity of 1000 cm./sec. and a length of quartz of 100 cm., the mean error being recorded after the $\pm$ sign.

Another way in which the values can be combined is by taking the arithmetical mean of the fifty-one values. Thus we find:

$$0{,}148 \pm 0{,}006.$$

From formula (4), which was given in our communication I, and will be proved presently, follows for the theoretical value of the effect:

$$0{,}143.$$

6. With red light $\lambda = 6510$ Å. U. twenty-seven values have been obtained for the effect on six plates. To eight of them corresponds a quartz column of 140 cm., to nineteen one of 100 cm. The velocities range between 750 and 960 cm./sec.

The result, when the mean values of the different plates are combined, is:

$$0{,}123 \pm 0.014.$$

The arithmetical mean of the twenty-seven separate values yields:

$$0{,}125 \pm 0.007.$$

The calculation gives for the expected effect:

$$0.115.$$

7. The results with violet light $\lambda = 4750$ Å. U. should be received with some diffidence, as it appeared afterwards that the violet filter transmitted some red light, which had not been detected at first. Hence it is possible that this cause slightly vitiated the later series. It must be said, however, that no trace of change could be ascertained in the values of the later series.

When the eight mean values of the different plates are combined, the result becomes:

$$0,156 \pm 0.008$$

The arithmetical mean of the thirty-one separate values differs very little from this:

$$0,156 \pm 0.007$$

The theoretical value is 0.166.

8. We collect the results in the following table.

λ	Δ_w	Δ_{th}
4750	0.156 ± 0.007	0.166
	0.156 ± 0.008	
5380	0.148 ± 0.006	0.143
	0.148 ± 0.012	
6510	$0.125 \pm 0\ 007$	0.115
	0.123 ± 0.014	

The observed displacement of the bands is indicated under Δ_w. The mean error has been calculated in two ways, as was discussed above. The second values are those derived from the average of the mean values of the individual plates.

Under Δ_{th} the theoretical value is given calculated by the aid of the data for the index of refraction for the ordinary ray in quartz, taken from KOHLRAUSCH's data.

It is not to be denied that taking the particular difficulties of the experiments into consideration, the agreement between theory and observation is very satisfactory.

The change of the effect with wave-length as well as the absolute value of the effect are represented very well. In the discussion of the experiments with glass, for which the dispersion is greater than for quartz, we shall have an opportunity to point out the very pronounced influence of the dispersion term.

9. *The formula for the optical effect.* We consider two of the rays which bring about the interference phenomenon, and which have passed over opposite paths. We shall denote quantities which refer to the first ray, by one accent, and those belonging to the second ray by a double accent. Each of the paths traversed, consists of three parts: 1 a path 1 in the air, 2 a path 2 in the quartz column, 3 a path 3 in the air.

The times expressed in seconds, which the light requires to pass over each of the parts we call resp. t'_1, t'_2, t'_3, t''_3, etc.

If the quartz is at rest, of course $t'_1 = t''_2$ etc. If, however, the quartz moves with the velocity w, the time required to traverse the quartz column (length l) in the direction from 1 to 2

$$t'_2 = \frac{l}{\frac{c}{\mu'} + \left(1 - \frac{1}{\mu'^2}\right) w - w} = \frac{l}{\frac{c}{\mu'} - \frac{w}{\mu'^2}} \quad \ldots \ldots \quad (1)$$

in which the difference between the velocity of the light in quartz, and of that of the column itself must be taken into account. While the light is passing through the quartz, the quartz moves on, hence t_2 is changed by an amount:

$$-\frac{lw}{\frac{c}{\mu'} - \frac{w}{\mu'^2}} \cdot \frac{1}{c} \quad \ldots \ldots \ldots \ldots \quad (2)$$

We get for the ray in the opposed direction:

$$t''_2 = \frac{l}{\frac{c}{\mu''} - \frac{w}{\mu''^2}} \quad \ldots \ldots \ldots \ldots \quad (3)$$

and for the other quantity:

$$+\frac{lw}{\frac{c}{\mu''} + \frac{w}{\mu''^2}} \cdot \frac{1}{c} \quad \ldots \ldots \ldots \ldots \quad (4)$$

For the first ray the entire difference of time becomes, therefore:

$$\frac{l}{\frac{c}{\mu'} - \frac{w}{\mu'^2}} - \frac{lw}{\frac{c}{\mu'} - \frac{w}{\mu'^2}} \cdot \frac{1}{c}$$

or expressed in periods for one ray

$$\frac{l}{\lambda}\left(\mu' + \frac{w}{c} - \frac{w}{c}\mu'\right) \quad \ldots \ldots \ldots \quad (5)$$

and for the other

$$\frac{l}{\lambda}\left(\mu'' - \frac{w}{c} + \frac{w}{c}\mu''\right) \quad \ldots \ldots \ldots \quad (6)$$

When we now consider that:

$$\mu' = \mu + \lambda \frac{d\mu}{d\lambda} \frac{w}{c} \quad \ldots \ldots \ldots \quad (7)$$

and

$$\mu'' = \mu - \lambda \frac{d\mu}{d\lambda} \frac{w}{c} \quad \ldots \ldots \ldots \quad (8)$$

we find, after substitution of this in the formulae (5) and (6), and after subtraction for the entire phase difference of the two rays that: —

$$\Delta = \frac{2lw}{c\lambda}\left(-\lambda \frac{d\mu}{d\lambda} - 1 + \mu\right)$$

or, on reversal of the direction of motion, an optical effect: —

$$\Delta = \frac{4lw}{c\lambda}\left(\mu - 1 - \lambda \frac{d\mu}{d\lambda}\right) \quad \ldots \ldots \ldots \quad (9)$$

With regard to the dispersion term it is still noteworthy that in FIZEAU's experiment with water the light is transferred from standing *water* to moving *water*, and $\frac{w}{c/\mu}$ must be written in the formulae instead of $\frac{w}{c}$.

10. *Derivation of formula* (9) *from the theory of relativity.*

After we had communicated formula (9) to Prof. LORENTZ, he had the kindness to give us a derivation strictly from the theory of relativity, which will follow here.

Let x', t' be a system of coordinates, in which the rod AB is at rest; length of the rod l'.

Light motion on the lefthand side of A:

$$a_1 \cos n'\left(t' - \frac{x'}{c} + p'_1\right) \quad \ldots \ldots \ldots \quad (1)$$

In the rod:

$$a_2 \cos n'\left(t' - \frac{x'}{v'} + p'_2\right)$$

On the righthand side of B:

$$a_2 \cos n'\left(t' - \frac{x'}{c} + p'_2\right) \quad \ldots \ldots \ldots \quad (2)$$

v' velocity of propagation belonging to n'.

We easily find:

$$n'(p'_2 - p'_1) = n' l'\left(\frac{1}{c} - \frac{1}{v'}\right) \quad \ldots \ldots \ldots \quad (3)$$

Through the relativity transformation:

$$x' = ax - bct \quad , \quad t' = at - \frac{b}{c}x \qquad a^2 - b^2 = 1 \quad . \quad . \quad (4)$$

and

$$x = ax' + bct' \quad , \quad t = at' + \frac{b}{c}x'$$

we may pass to a system, in which the rod moves with a velocity:

$$w = \frac{bc}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (2) and (3) we derive

$$\alpha_1 \cos\left[n\left(t - \frac{x}{c}\right) + n'p'_1\right] \text{ and } \alpha_2 \cos\left[n\left(t - \frac{x}{c}\right) + n'p'_2\right] \quad . \quad (6)$$

$$n = (a + b)\, n' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The phase difference between (1) and (2), i.e. the change of phase brought about by the presence of the rod, is given by (3) in angular measure, and this same difference of phase still exists between the expressions (6).

Expressed in wave-lengths or periods, it is:

$$\Delta = \frac{n'l'}{2\pi}\left(\frac{1}{c} - \frac{1}{v'}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Here, however, we must express n' and v' in the n and v corresponding to it. When we neglect the terms of the second order,

$$a = 1 \; , \; b = \frac{w}{c} \; ,$$

follows from (4) and (5), hence according to (7):

$$n = \left(1 + \frac{w}{c}\right)n' \; , \; n' = \left(1 - \frac{w}{c}\right)n$$

$$v' = v + (n' - n)\frac{dv}{dn} = v \quad \frac{w}{c}\, n\, \frac{dv}{dn}$$

$$l' = l$$

After substitution in (8) we find for the part of Δ that depends on w:

$$\frac{nl}{2\pi} \cdot \frac{w}{c} \left\{ \frac{1}{v} - \frac{1}{c} - \frac{n}{v^2} \frac{dv}{dn} \right\},$$

or after introduction of $n = \frac{2\pi c}{\lambda}$, $v = \frac{c}{\mu}$

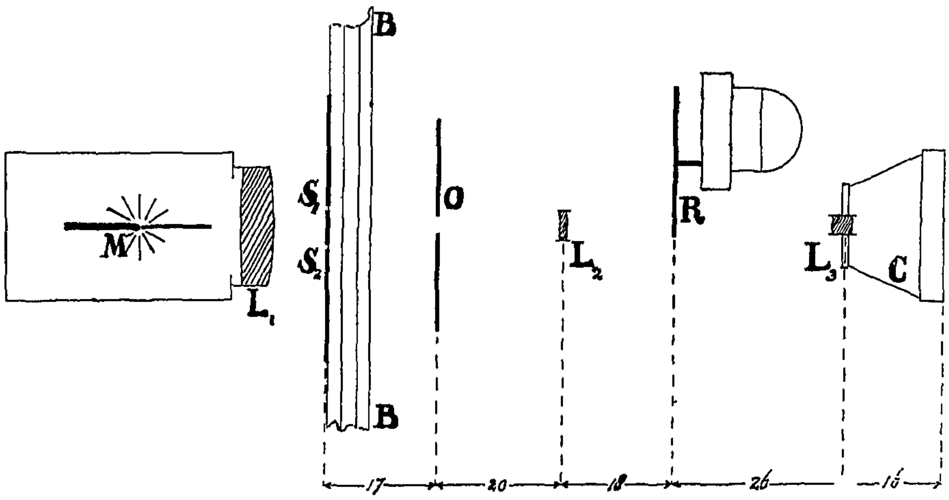
$$\frac{l}{\lambda c}\left(\mu - 1 - \lambda \frac{d\mu}{d\lambda}\right) w$$

by the aid of which (9) of § 9 follows immediately.

11. *Direct determination of the velocity of the beam.*

In I § 3 the method has been indicated by which the maximum velocity was determined. We have, however, also measured it directly by the following method, which was, not applied until the experiments with glass were undertaken, but which is described here, because it has confirmed the velocity determination on the supposition of a fly-wheel revolving with a constant angular velocity.

To the beam BB is attached a black screen with two slits S_1 and S_2, across which threads are stretched for accurate refer-



ence. Consider only the slit S_1, which moves in the field of light of the condenser L_1. By the aid of the achromatic lens L_2 a sharp image of S_1 is projected on the plane R of the circular plate of a light interruptor used for a large galvanometer. This latter apparatus, to which our attention was drawn by Mr. WERTHEIM SALOMONSON, and which was put at our disposal by him, consists of an electromotor with a centrifugal speed indicator. One of the five axes of the apparatus revolves 25 times per second. The aluminium plate R (diameter 30 cm) has 40 slits in its circumference, each about 1 mm. wide, so that 1000 flashes arise per second. The distance between two slits is about 23 mm., so that light is allowed to pass during 1/20000 second.

An image of the stationary slit S_1, projected on the photographic plate of the camera C, when the disc rotates, is reproduced in b, on the Plate, fig. 1: the two cross-wires are seen, and on the righthand side of the slit the rim of the circular plate R, the centre of which lies on the left side. The dark circles are caused by reflections, but are of no consequence.

When S_1 moves and the disc revolves, after every thousandth of a second the light is let through the slit in R, and S_1 is photographed by means of the lens L_2.

The images of S_1 assume an oblique position, because during the displacement of the beam, the slit in the disc gets continually higher in its movement.

It is easy to see that the slope of the image is determined by the ratio of the velocity of the slit to that of the beam, or rather to that of the image on the disc. From the distances of S_1 and R to L_2, and from L_2 to R and the plate, the reduction which the velocity of the beam undergoes in the image, can be immediately estimated, or it can be directly measured by photographing a divided scale in the plane S_1 S_2. The amount of obliqueness of the slit image (Plate fig. 2) immediately gives an approximate value of the velocity of the beam, which can, indeed, be found in a still simpler way from the distance of corresponding points of b'_1 b''_1.

For a more accurate determination of the velocity the second slit S_2, which is at 4,15 cm. distance from S_1, can be of service. Let us suppose that about 1000 cm./sec. have been found for the approximate value of the velocity of the beam, then S_1 has shifted about 1 cm. after every thousandth of a second, and S_2 gets about to the first position of S_1 after four thousandths of a second. Hence between the images of S_1 an image of S_2, viz a_1'', will appear in general on the photo, from which the position of the slit images can be accurately derived, and then we know that in order for the beam to move a distance 4,15 cm., 4 + a fraction thousandths of seconds are required, which can be measured from the relative positions. In order to distinguish the slits, a cross-wire has been stretched only over S_1.

We only get two images of each of the slits on the photo, because it is only possible that images are formed by the different lenses within a limited cone.

As it is our intention to determine the velocity of the beam with a definite direction of motion, the shutter (I § 4), which as a rule is before the objective of the telescope, has been placed at 0 (cf. the above figure). This ensures that the light passes through the apparatus at the right moment. The experimental error of the determination of the velocity appears to amount to at most 0.75 %.

We still wish to draw attention to a peculiarity of the described method of the velocity determination, which seems interesting from a theoretical, though not from a practical, point of view.

The measurement actually takes place with a *moving measuring rod* (length at rest $= S_1 S_2$), a peculiarity which we do not remember having seen used in practice with any other method.

We are indebted to Messrs. W. DE GROOT and G. C. DIBBETZ JR. for their assistance in the theoretical and experimental work, and to Mr. J. VAN DER ZWAAL for his help in the difficult adjustment of the apparatus and the manufacture of the auxiliary appliances.

Chemistry. — *"On some nitro-derivatives of dimethylaniline".* By M. J. SMIT. (Communicated by Prof. VAN ROMBURGH).

(Communicated at the meeting of June 28 1919).

In 1914 a paper was published by VAN ROMBURGH and Miss WENSINK [1]) in which the action of ammonia and methylamine on 1, 2, 3, 4. trinitrodimethylaniline was described. The remarkable phenomenon was here described that, besides the nitro-group in the 3-position, also the dimethylamino-group was replaced by the amino- or the methylamino-group as the case might be. At the instigation of Prof. VAN ROMBURGH I have undertaken a more extensive investigation into the behaviour of amines and, in general, of compounds containing an amino-group, with respect to the trinitro-derivatives of dimethylaniline, in particular the 1.3.4.6. isomeride. These investigations are, however, still in the initial stage, and will be described in due course in a thesis.

Before undertaking the research above indicated, it appeared to me desirable to study somewhat more accurately the reaction in which the two isomeric trinitro derivatives of dimethylaniline are produced. In the nitration of 1.3.4 dinitromethylaniline [2]) the principal product is always the 1.3.4.6. derivative; the isomeric 1.2.3.4. compound is only formed in small quantities. Since, however, it is precisely the latter, in consideration of the position of the nitro-groups, which reacts the most easily, it seemed to me to be worth while to try to establish the conditions in which a better yield of this substance could be obtained.

In spite of numerous attempts in which the experimental conditions were varied between the widest limits, I was not successful in so modifying the conditions that the yield of 1.2.3.4 trinitro-dimethylamine was increased to any appreciable extent. In these experiments the great influence which the presence of nitrous acid exerts on the reaction velocity, was again most distinctly apparent; a phenomenon which has already been repeatedly observed in nitration experiments,

---

[1]) These Proceedings, XVII, 1034 (1915).

[2]) VAN ROMBURGH, Verslagen Kon. Akad. v. Wet. 23 Febr. 1895, III, 257.

and which was also noticed in the nitration of 1.3.6 dinitro dimethylaniline. (See below). Moreover it appeared to be important not to allow the action of the nitric acid to be unduly prolonged.

If for the nitration nitric acid is used which is completely free from nitrous acid, it is necessary to use a fairly concentrated acid (Sp. Gr. 1.37—1.40) in order to avoid prolonging the reaction unduly. To prevent excessive rise of temperature a moderate excess of nitric acid should be used.

As a rule 10—12 c.c. of nitric acid was added to 1 gram of the amine. If necessary the reaction may be accelerated by the addition of a small portion of sodium nitrite. In these conditions the two isomeric trinitro-derivatives are probably the only products.

If, however, nitric acid is used containing much nitrous acid, or if the reaction is allowed to proceed for a long time, or, again, if the temperature rises appreciably above 20° C., then the action of the nitrous acid becomes manifest, and nitroso-compounds are easily formed. The isolation of the required isomerides is thus rendered extremely difficult, and, of course, the yield is reduced. In these conditions two light yellow substances of melting points 108°—109° C. and 201° C. respectively, were isolated. Both could be crystallised from alcohol and are obtained in the form of fine light yellow needles with a greenish reflection. The first can be obtained, as appeared later, by the action of nitrous acid on 1.3.4 dinitrodimethylaniline and is transformed on treatment with nitric acid (Sp. Gr. 1.41) into the other. The latter is obtained by the action of nitrous acid on 1.3.4.6 trinitrodimethylaniline.

In all probability these substances are therefore 1.3.4 dinitrophenylmethylnitrosamine (M.p. 108°—109° C.) and 1 3.4.6 trinitrophenylmethylnitrosamine (M.p. 201° C).

The presence of the product of the reaction of nitrous acid on the 1.2.3.4 trinitro-compound, also a yellow substance with greenish reflection, which melts at 96°—97° C. after recrystallisation from alcohol, could not be detected. This is not surprising seeing that only a minute quantity of the 1.2.3.4 isomeride is produced in the reaction.

Direct nitration of *m*-nitrodimethylaniline (M.p. 60° C.) with nitric acid (Sp. Gr. 1.4) did not lead to a better result. Here also the chief product was the 1.3.4.6 compound.

The nitration of the 1.3.6 dinitrodimethylaniline which, like the corresponding 1.3.4 isomeride, is formed from dimethylaniline in presence of excess of sulphuric acid[1]), was also investigated.

[1]) Rec. VI, 253 [1887].

Van Romburgh had already found[1]) that the principal product of the reaction is 1.3.4.6 trinitrodimethylaniline.

Besides this compound there is also formed in small quantity a substance which crystallises from ethyl acetate in beautiful yellow crystals. The substance melted with slight decomposition at 132° forming a pale yellow liquid.

At first I was of opinion that I had obtained a new trinitro-isomeride, probably the 1.2.3.6 compound. On varying the conditions however, with the object of increasing the yield, when a smaller quantity of nitric acid was used for the nitration, a rise in temperature to 40° C. was observed, and a copious evolution of brown fumes took place. I was surprised to find that more of the substance had been formed. It occurred to me that the compound might be a product of the action of nitrous acid. This assumption appeared to be correct. If the 1.3.6 compound is dissolved in dilute sulphuric acid (1 : 1) and sodium nitrite is added, an almost quantitative yield of the above substance is obtained. On treatment with nitric acid (Sp. Gr. 1.4) the substance is transformed into the nitrosamine (M.p. 201°) already described. It must therefore be considered as the nitrosamine of the 1.3.6 compound.

In the nitration of the 1.3.6 dinitro-derivative the effect of nitrous acid on the reaction velocity is extraordinarily great. By the addition of urea it is possible to stop the reaction altogether. If the temperature of the reaction is perceptibly higher than room temperature the only final product obtained is the substance melting at 201°. Both the substance melting at 132° and the 1.3.4.6 trinitro-derivative are transformed into the above nitrosamine.

Repeated attempts to obtain an isomeric trinitro-derivative were all unsuccessful, the only product obtained being the nitrosamine.

This research is being continued, and the results will be described in greater detail later.

Bergen op Zoom, June 1919.

---

1) These Proceedings, III, 258.

Chemistry. — "*Pressure- and temperature-coefficients, volume- and heat-effects in bivariant systems.*" By P. H. J. HOENEN, S.J. (Communicated by Prof. SCHREINEMAKERS.)

(Communicated at the meeting of Sept. 27, 1919).

In a previous communication [1]) we developed a general law (of which the so-called BRAUN's law is a particular case) giving a relation between the pressure- and temperature-coefficients of the solubility of several solid substances, with which a solvent is saturated, and the heat of solution and volume increase accompanying the solution of these substances. In the present communication we shall attempt to find a similar relation for arbitrary bivariant systems.

I. *Heterogeneous Equilibria.*

1. With n components we have a bivariant system in the usual sense of the term, when there are n coexisting phases. In this case there are two independent variables, e.g., pressure and temperature. We can, however, even when there are fewer than n phases present, retain only these two as independent variables, if we subject all variations in the system to the condition that the composition of the whole remains constant. Then, no matter how many phases we have, provided the number is not more than n, pressure and temperature alone remain the independent variables.

There must thus be a relation among all the systems. Such systems differ greatly from bivariant systems in the ordinary sense of the term, i.e. from systems with n phases, in that in the latter case the composition of the phases is separately independent of the composition of the system as a whole. This is not the case with the systems which are only "bivariant with constant total composition."

We shall illustrate the above by a consideration of the equilibrium equations. We assume that we have n components in l phases.

Let the composition of the phases be as follows:

$$\begin{array}{ll} 1^{st} \text{ phase}: & x_1,\ y_1,\ z_1, \ldots \\ 2^{nd} \quad ,, & x_2,\ y_2,\ z_2, \ldots \\ \ldots & \ldots \\ l^{th} \quad ,, & x_l,\ y_l,\ z_l, \ldots \end{array}$$

1) See the preceding communication in these Proceedings.

The composition of each phase is given in terms of the absolute quantity in mols of each component. Between these quantities the following relations subsist:

$$\left.\begin{array}{l} x_1 + x_2 + \dots\dots x_l = X \\ y_1 + y_2 + \dots\dots y_l = Y \\ z_1 + z_2 + \dots\dots z_l = S \\ \dots\dots\dots\dots \end{array}\right\} \quad \dots\dots \quad (1)$$

The number of equations in (1) is n The quantities $X, Y, S\dots$, which determine the total composition, are to be considered constant.

If we represent the ζ-functions of the separate phases by $Z_1, Z_2, \dots Z_l$, the equilibrium conditions are:

$$\left.\begin{array}{l} \frac{\partial Z_2}{\partial x_2} - \frac{\partial Z_1}{\partial x_1} = 0 \;\; ; \;\; \frac{\partial Z_3}{\partial x_3} - \frac{\partial Z_1}{\partial x_1} = 0 \;\; ; \dots\dots \;\; \frac{\partial Z_l}{\partial x_l} - \frac{\partial Z_1}{\partial x_1} = 0\,; \\ \frac{\partial Z_2}{d y_2} - \frac{\partial Z_1}{\partial y_1} = 0 \;\; ; \;\; \frac{\partial Z_3}{\partial y_3} - \frac{\partial Z_1}{\partial y_1} = 0 \;\; ; \dots\dots \;\; \frac{\partial Z_l}{\partial y_l} - \frac{\partial Z_1}{\partial y_1} = 0\,; \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \quad \dots \quad (2)$$

The number of equations (2) is $n\,(l-1)$.

With regard to the form of these equations it may be noted that the expressions on the left are homogeneous functions of degree 0 with respect to the variables $x_1, y_1, \dots$, and are thus only dependent on the ratios of these variables to each other $\left(\text{e.g.}, \frac{y_1}{x_1}, \frac{z_1}{x_1}, \text{etc.}\right)$ and not on the absolute values. Besides p and T we have therefore only $l\,(n-1)$ unknowns or variables, since in each phase there are only $n-1$ ratios which determine the composition.

If $l = n$, we have $n\,(n-1)$ equations (2) with $n\,(n-1)$ unknowns (besides p and T). For given values of p and T the composition is thus completely determined by these equations and is thus independent of the total composition $X, Y, \dots$. Equations (1) serve only for the calculation of the absolute values of x_1, etc.

If $l < n$, we have fewer equations (2) than unknowns which determine the composition of each phase. In this case for the calculation of the composition of the phases we must make use of the equations (1), so that the composition of each phase is dependent on the total composition.

We have, however, always a sufficient number of equations for the calculation of the composition of each phase for a given value of p and T, for we have $n + n\,(l-1) = nl$ equations in nl unknown quantities $x_1, y_1, \dots x_2, y_2, \dots$ in which p and T can be considered as the independent variables.

35*

We have therefore, independently of the number of phases, provided $l < n$, a "bivariant system with constant total composition".

2. We shall now investigate for a bivariant system consisting of n components in l phases ($l \gtreqless n$) a relation between the pressure- and temperature-coefficients for the transition of the components from one phase to another, and the heat effects and volume changes which accompany this transition.

The composition of the different phases may be represented as before. The ζ-function of the system is represented by Z, the entropy by H, and the volume by V. For the separate phases these quantities are represented by Z_1, H_1, V_1, etc.

We have then:

$$Z = Z_1 + Z_2 \ldots\ldots + Z_l.$$
$$V = V_1 + V_2 \ldots\ldots + V_l.$$
$$H = H_1 + H_2 \ldots\ldots + H_l.$$

These quantities are given as functions of p and T and also of $x_1, y_1, \ldots, x_2, y_2 \ldots$, etc. in which p and T are the only independent variables. With regard to notation, the following may be remarked. Partial differentiation with respect to one independent variable, the other independent variable alone being kept constant, (i.e., in a state of equilibrium), is indicated by a stroke above the differential coefficient; partial differentiation with respect to one variable, all other variables being considered constant, (in this case heterogeneous equilibrium is not necessarily present) is indicated by the absence of the stroke.

We can establish the desired relations by the method described in a previous communication for an analogous case. We differentiate the equations (2) partially, first with respect to p, and then with respect to T. After multiplication by suitably chosen factors the equations are added together. The following, however, is a shorter and, in my opinion, a more elegant method.

We begin with the simple, purely analytical equation:

$$\frac{\overline{\partial^2 Z}}{\partial T \partial p} = \frac{\overline{\partial^2 Z}}{\partial p \partial T},$$

This may be written:

$$\frac{\overline{\partial V}}{\partial T} = -\frac{\overline{\partial H}}{\partial p} \text{ or } \frac{\overline{\partial V}}{\partial T} + \frac{\overline{\partial H}}{\partial p} = 0 \; . \; . \; . \; . \; . \; . \quad (3)$$

But

$$\frac{\overline{\partial V}}{\partial T} = \frac{\partial V}{\partial T} + \frac{\partial V}{\partial x_1}\overline{\left(\frac{\partial x_1}{\partial T}\right)} + \frac{\partial V}{\partial x_2}\overline{\left(\frac{\partial x_2}{\partial T}\right)} \cdots + \frac{\partial V}{\partial y_1}\overline{\left(\frac{\partial y_1}{\partial T}\right)} + \frac{\partial V}{\partial y_2}\overline{\left(\frac{\partial y_2}{\partial T}\right)} + \cdots \quad (4)$$

and

$$\overline{\frac{\partial H}{\partial p}}=\frac{\partial H}{\partial p}+\frac{\partial H}{\partial x_1}\overline{\left(\frac{\partial x_1}{\partial p}\right)}+\frac{\partial H}{\partial x_2}\overline{\left(\frac{\partial x_2}{\partial p}\right)}+\dots\frac{\partial H}{\partial y_1}\overline{\left(\frac{\partial y_1}{\partial p}\right)}+\frac{\partial H}{\partial y_2}\overline{\left(\frac{\partial y_2}{\partial p}\right)}+\dots \quad (5)$$

Also

$$\frac{\partial V}{\partial T}=\frac{\partial V_1}{\partial T}+\frac{\partial V_2}{\partial T}+\dots\frac{\partial V_l}{\partial T}$$

and

$$\frac{\partial H}{\partial p}=\frac{\partial H_1}{\partial p}+\frac{\partial H_2}{\partial p}+\dots\frac{\partial H_l}{\partial p}$$

As now for a given phase (k)

$$\frac{\partial V_k}{\partial T}+\frac{\partial H_k}{\partial p}=0\,,$$

since

$$\frac{\partial V_k}{\partial T}=\frac{\partial^2 Z_k}{\partial T\partial p} \text{ and } \frac{\partial H_k}{\partial p}=-\frac{\partial^2 Z_k}{\partial p\partial T},$$

we have also

$$\frac{\partial V}{\partial T}+\frac{\partial H}{\partial p}=0.$$

If we add (4) and (5) and take (3) into consideration, we have as a result:

$$\left.\begin{aligned}&\frac{\partial V}{\partial x_1}\overline{\left(\frac{\partial x_1}{\partial T}\right)}+\frac{\partial H}{\partial x_1}\overline{\left(\frac{\partial x_1}{\partial p}\right)}+\frac{\partial V}{\partial x_2}\overline{\left(\frac{\partial x_2}{\partial T}\right)}+\frac{\partial H}{\partial x_2}\overline{\left(\frac{\partial x_2}{\partial p}\right)}+\dots.\\&\dots.+\frac{\partial V}{\partial y_1}\overline{\left(\frac{\partial y_1}{\partial T}\right)}+\frac{\partial H}{\partial y_1}\overline{\left(\frac{\partial y_1}{\partial p}\right)}+\frac{\partial V}{\partial y_2}\overline{\left(\frac{\partial y_2}{\partial T}\right)}+\frac{\partial H}{\partial y_2}\overline{\left(\frac{\partial y_2}{\partial p}\right)}+\dots=0\end{aligned}\right\} \quad (6)$$

From equations (1) we have also

$$\overline{\frac{\partial x_1}{\partial T}}=-\overline{\frac{\partial x_2}{\partial T}}-\overline{\frac{\partial x_3}{\partial T}}\dots-\overline{\frac{\partial x_l}{\partial T}}$$

$$\overline{\frac{\partial x_1}{\partial p}}=-\overline{\frac{\partial x_2}{\partial p}}-\overline{\frac{\partial x_3}{\partial p}}\dots-\overline{\frac{\partial x_l}{\partial p}},$$

Similar expressions may be deduced for the quantities y_1, etc.

On substituting these expressions in (6) we obtain:

$$\left(\frac{\partial V}{\partial x_2}-\frac{\partial V}{\partial x_1}\right)\overline{\frac{\partial x_2}{\partial T}}+\left(\frac{\partial H}{\partial x_2}-\frac{\partial H}{\partial x_1}\right)\overline{\frac{\partial x_2}{\partial p}}+\dots$$

$$\dots.+\left(\frac{\partial V}{\partial y_2}-\frac{\partial V}{\partial y_1}\right)\overline{\frac{\partial y_2}{\partial T}}+\left(\frac{\partial H}{\partial y_2}-\frac{\partial H}{\partial y_1}\right)\overline{\frac{\partial y_2}{\partial p}}+\dots.\quad=0$$

In this equation the expression $\frac{\partial V}{\partial x_2}-\frac{\partial V}{\partial x_1}$, $\left(\text{which is the same as } \frac{\partial V_2}{\partial x_2}-\frac{\partial V_1}{\partial x_1}\right)$ represents the volume increment associated with the transition of one mol of the component (x) from an infinitely large quantity of the first phase into an infinitely large quantity of the second phase, the variables p, T, and the other components remaining constant. This volume increase may be denoted by V_{x12}. The expression $\frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial x_1}$ represents the heat absorbed in the same operation divided by T. This heat effect may be denoted by Q_{x12}. For the corresponding differences for the other phases and components analogous symbols may be used. We have now:

$$\left.\begin{array}{l} V_{x12}\,\overline{\dfrac{\partial x_2}{\partial T}} + \dfrac{Q_{x12}}{T}\,\overline{\dfrac{\partial x_2}{\partial p}} + V_{x13}\,\overline{\dfrac{\partial x_3}{\partial T}} + \dfrac{Q_{x13}}{T}\,\overline{\dfrac{\partial x_3}{\partial p}} + \ldots \\ \qquad + \ldots\; V_{y12}\,\overline{\dfrac{\partial y_2}{\partial T}} + \dfrac{Q_{y12}}{T}\,\overline{\dfrac{\partial y_2}{\partial p}} + \ldots \quad = 0. \end{array}\right\} \quad . \quad . \quad (7)$$

This is one of the relations which it was our object to establish. From (6) other l—1 similar relations may be derived, in which the pressure and temperature coefficients of the components of one of the l—1 other phases do not occur. We obtain other less symmetrical relations, when we eliminate for the one component the coefficients for one of the phases, for a second component its coefficients for another phase. If a component is absent in one of the phases, the corresponding coefficients vanish.

Note I. If one of the phases consists of all the components, and the other phases are all pure components, then we have the case for which in the previous communication the "generalised BRAUN's law" was established. If these conditions are introduced into equation (6), an expression of this law results. The verification of this may be left to the reader.

Note II. If there are n components in n phases, the heat effects and the volume increments occurring in (7) have values which are independent of the total composition. When the number of phases is less than n, that is, when the equilibrium is merely "bivariant with constant total composition" then the values are functions of the total composition.

Note III. In our discussion we have nowhere made use of any explicit relation connecting Z with the composition. The results are therefore valid also in the case of reacting components.

Note IV. The line of argument adopted leads to a similar formula in the case of homogeneous equilibria. This will be discussed in a future communication.

Katwijk a. d. Rijn, August 1919.

Chemistry. — "*Extension of the law of Braun*". By P. H. J. HOENEN S.J. (Communicated by Prof. SCHREINEMAKERS).

(Communicated at the meeting of September 27, 1919).

Through the brilliant researches which have been carried out in recent years in the VAN 'T HOFF-Laboratory, attention has again been directed to the so-called BRAUN's law. At present this is generally expressed by the formula

$$\left(\frac{\partial x}{\partial p}\right)_T : \left(\frac{\partial x}{\partial T}\right)_p = -\frac{T \Delta V}{Q}, \quad \ldots \ldots \quad (1)$$

in which, for the equilibrium solid-liquid, $\left(\frac{\partial x}{\partial p}\right)_T$ represents the pressure coefficient of the solubility, $\left(\frac{\partial x}{\partial T}\right)_p$ the temperature coefficient, Q the differential heat of solution, ΔV the differential increase in volume.

This law of BRAUN is a particular case of a general law which we shall proceed to develop.

1. Let us suppose we have a solution saturated with respect to n solid substances. Let the quantity of solvent be one mol, and let the amount of the dissolved substances, which are present in the saturated solution at pressure p and temperature T, be $x, y, z, \ldots$ mols. Then in this case the following relation holds:

$$\left(\frac{\partial x}{\partial p}\right)_T \times \frac{Q_x}{T} + \left(\frac{\partial x}{\partial T}\right)_p \times \Delta V_x + \left(\frac{\partial y}{\partial p}\right)_T \times \frac{Q_y}{T} + \left(\frac{\partial y}{\partial T}\right)_p \times \Delta V_y + \ldots = 0 \quad (2)$$

Here Q_x represents the heat necessary for the solution of one mol of the first component in an infinitely large quantity of solvent of the given composition at constant pressure p and temperature T. It is therefore the differential molecular heat of solution of this component. ΔV_x is the corresponding volume increase, i.e., the differential molecular volume increment. The other symbols require no further explanation. If we are dealing with one substance only, (2) becomes:

$$\left(\frac{\partial x}{\partial p}\right)_T \times \frac{Q_x}{T} + \left(\frac{\partial x}{\partial T}\right)_p \times \Delta V_x = 0,$$

This equation is the same as (1). It is therefore obvious that BRAUN's law is a particular case of (2).

2. We shall first of all establish (2) for the simple case that the solution is saturated with respect to two solid substances only; and we assume further that these two substances together with the solvent are independent components in the sense of the phase rule, so that two degrees of freedom are at our disposal, namely, pressure and temperature. The ζ-function of the liquid phase is represented by Z, those of the solid phases (per mol) by ζ_x and ζ_y. The equilibrium conditions are:

$$\frac{\partial Z}{\partial x} - \zeta_x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{\partial Z}{\partial y} - \zeta_y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The left hand side of these equations are functions of x, y, p, and T, of which. however, only two may be varied independently.

By differentiating these equations with respect to p with T constant there results:

$$\frac{\partial^2 Z}{\partial x^2}\frac{\partial x}{\partial p} + \frac{\partial^2 Z}{\partial x \partial y}\frac{\partial y}{\partial p} = -\left(\frac{\partial V}{\partial x} - v_x\right) \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

$$\frac{\partial^2 Z}{\partial x \partial y}\frac{\partial x}{\partial p} + \frac{\partial^2 Z}{\partial y^2}\frac{\partial y}{\partial p} = -\left(\frac{\partial V}{\partial y} - v_y\right) \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

The notation here requires no explanation. It may be remarked that the right-hand side of $(3a) = -\Delta V_x$ and that of $(4a) = -\Delta V_y$.

If we differentiate the two equations with respect to T, keeping p constant, we obtain:

$$\frac{\partial^2 Z}{\partial x^2}\frac{\partial x}{\partial T} + \frac{\partial^2 Z}{\partial x \partial y}\frac{\partial y}{\partial T} = \left(\frac{\partial H}{\partial x} - \eta_x\right) \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

$$\frac{\partial^2 Z}{\partial x \partial y}\frac{\partial x}{\partial T} + \frac{\partial^2 Z}{\partial y^2}\frac{\partial y}{\partial T} = \left(\frac{\partial H}{\partial y} - \eta_y\right) \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

The right-hand sides of these equations (3b) and (4b) are equal to $\frac{Q_x}{T}$ and $\frac{Q_y}{T}$ respectively.

From these four equations the following relation is derived:

$$\left(\frac{\partial x}{\partial p}\right) \times \frac{Q_x}{T} + \left(\frac{\partial x}{\partial T}\right) \Delta V_x + \left(\frac{\partial y}{\partial p}\right) \frac{Q_y}{T} + \left(\frac{\partial y}{\partial T}\right) \Delta V_y = 0 \quad . \quad (5)$$

Proof: From (3a) and (4a) follows, on substituting $\frac{\partial^2 Z}{\partial x^2} = r$, $\frac{\partial^2 Z}{\partial x \partial y} = s$, $\frac{\partial^2 Z}{\partial y^2} = t$,

$$\frac{\partial x}{\partial p}=\frac{s\Delta V_y-t\Delta V_x}{rt-s^2}\quad;\quad\frac{\partial y}{\partial p}=\frac{s\Delta V_x-r\Delta V_y}{rt-s^2}\;;$$

From (3*b*) and (4*b*) follows:

$$T\frac{\partial x}{\partial T}=\frac{tQ_x-sQ_y}{rt-s^2}\quad;\quad T\frac{\partial y}{\partial T}=\frac{rQ_y-sQ_x}{rt-s^2}\;.$$

Substitution of these values in the left hand side of (5) gives a fraction of which the numerator $=0$, while the denominator >0, if the equilibrium is stable. The equation (5) is thus established.

We may remark that the separate sums, as $\frac{\partial x}{\partial p}\times\frac{Q_x}{T}+\frac{\partial x}{\partial T}\Delta V_x$, are not zero except in the special case when $\frac{Q_x}{Q_y}=\frac{\Delta V_x}{\Delta V_y}$.

3. We shall now attempt to establish the general equation (2). We assume that we have a liquid phase consisting of one mol of solvent and $x, y, z, \ldots$ mols of the dissolved substances. At pressure p and temperature T the solution is saturated with respect to these substance. We have thus $n+1$ components in as many phases and have therefore two degrees of freedom at our disposal.

The equilibrium conditions are (for the notation see above):

$$\left.\begin{aligned}&\frac{\partial Z}{\partial x}-\zeta_x=0\\&\frac{\partial Z}{\partial y}-\zeta_y=0\\&\frac{\partial Z}{\partial z}-\zeta_z=0\\&\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(6)$$

The expressions on the left-hand side of these n equations are again functions of $x, y, z, \ldots p$, and T. The last two we consider as independent variables. If we differentiate, first with respect to p alone and then with respect to T alone, we obtain the two sets of equations:

$$\left.\begin{aligned}&\frac{\partial^2 Z}{\partial x^2}\frac{\partial x}{\partial p}+\frac{\partial^2 Z}{\partial x\partial y}\frac{\partial y}{\partial p}+\frac{\partial^2 Z}{\partial x\partial z}\frac{\partial z}{\partial p}+\ldots\ldots=-\Delta V_x\\&\frac{\partial^2 Z}{\partial x\partial y}\frac{\partial x}{\partial p}+\frac{\partial^2 Z}{\partial y^2}\frac{\partial y}{\partial p}+\frac{\partial^2 Z}{\partial x\partial z}\frac{\partial z}{\partial p}+\ldots\ldots=-\Delta V_y\\&\frac{\partial^2 Z}{\partial x\partial z}\frac{\partial x}{\partial p}+\frac{\partial^2 Z}{\partial y\partial z}\frac{\partial y}{\partial p}+\frac{\partial^2 Z}{\partial z^2}\frac{\partial z}{\partial p}+\ldots\ldots=-\Delta V_z.\\&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}\right\}\quad\ldots\quad(6a)$$

$$\left.\begin{array}{ll}
\frac{\partial^2 Z}{\partial x^2}\frac{\partial x}{\partial T}+\frac{\partial^2 Z}{\partial x \partial y}\frac{\partial y}{\partial T}+\frac{\partial^2 Z}{\partial x \partial z}\frac{\partial z}{\partial T}+\ldots\ldots & =\frac{Q_x}{T} \\
\frac{\partial^2 Z}{\partial x \partial y}\frac{\partial x}{\partial T}+\frac{\partial^2 Z}{\partial y^2}\frac{\partial y}{\partial T}+\frac{\partial^2 Z}{\partial x \partial z}\frac{\partial z}{\partial T}+\ldots\ldots & =\frac{Q_y}{T} \\
\frac{\partial^2 Z}{\partial x \partial z}\frac{\partial x}{\partial T}+\frac{\partial^2 Z}{\partial y \partial z}\frac{\partial y}{\partial T}+\frac{\partial^2 Z}{\partial z^2}\frac{\partial z}{\partial T}+\ldots\ldots & =\frac{Q_z}{T} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &
\end{array}\right\} \quad . \quad . \quad (6b)$$

We have again written ΔV_x, for $\frac{\partial V}{\partial x} - v_x$, etc., and $\frac{Q_x}{T}$ for $\frac{\partial H}{\partial x} - \eta_x$, etc.

If we multiply the first of the equations (6*a*) by $-\frac{\partial x}{\partial T}$, the second by $-\frac{\partial y}{\partial T}$, etc., the first of equations (6*b*) by $\frac{\partial x}{\partial p}$, the second by $\frac{\partial y}{\partial p}$, etc. and add together the $2n$ equations, we obtain an equation the right hand side of which is:

$$\frac{\partial x}{\partial p} \times \frac{Q_x}{T} + \frac{\partial x}{\partial T} \Delta V_x + \frac{\partial y}{\partial p} \times \frac{Q_y}{T} + \frac{\partial y}{\partial T} \times \Delta V_y + \ldots\ldots$$

The left hand side of the resultant equation is zero. This may be shown as follows. Each term of the left hand side contains one of the "unknowns" $\frac{\partial x}{\partial p}$, $\frac{\partial y}{\partial p}$, etc. from the equations (6*a*). Let us consider the terms which contain one of these unknowns, e.g., $\frac{\partial x}{\partial p}$. In the summation these terms are contributed (1) by the first terms of the equations (6*a*) and by no other terms of these equations, (2) by the complete left hand side of the first equation (6*b*), which was multiplied throughout by $\frac{\partial x}{\partial p}$, and by no other equation of (6*b*).

The terms involving $\frac{\partial x}{\partial p}$ are therefore:

$$-\frac{\partial x}{\partial T}\frac{\partial^2 Z}{\partial x^2}, -\frac{\partial y}{\partial T}\frac{\partial^2 Z}{\partial x \partial y}, -\frac{\partial z}{\partial T}\frac{\partial^2 Z}{\partial x \partial z}, \ . \ . \ \text{from (6}a\text{) and}$$

$$\frac{\partial^2 Z}{\partial x^2}\frac{\partial x}{\partial T}, \frac{\partial^2 Z}{\partial x \partial y}\frac{\partial y}{\partial T}, \frac{\partial^2 Z}{\partial x \partial z}\frac{\partial z}{\partial T}, \ . \ . \ . \ . \ \text{from (6}b\text{),}$$

all terms being multiplied by $\frac{\partial x}{\partial p}$.

From the structure of equations (6*a*) and (6*b*) it appears that the sum of the factors by which $\frac{\partial x}{\partial p}$ is multiplied is zero, and that the same holds for each of the "unknowns". The left hand side of the resultant equation is therefore shown to be zero. We have then as a result:

$$\frac{\partial x}{\partial p} \times \frac{Qx}{T} + \frac{\partial x}{\partial T} \Delta V_x + \frac{\partial y}{\partial p} \times \frac{Qy}{T} + \frac{\partial y}{\partial T} \Delta V_y + \ldots\ldots \quad = 0,$$

that is, equation (2) results. This is the equation which we set out to establish as an extension of BRAUN's law.

Note I. It is not necessary that the solvent should be a pure substance. It may be a mixture of different substances of which, however, none occurs in the solid state. With this assumption the above method of proof remains exactly the same, and the validity of the result is unaffected. The quantities Q, etc., have, of course in general different values when the "solvent" is differently constituted.

Note II. In the above treatment we have nowhere made use of any explicit relation connecting Z and the composition. It follows from this that the results are valid both for constant and for reacting components. The only assumption made was that the components were independent in the sense of the phase theory.

Note III. For the general case we can give a demonstration on the lines of that given for the simple case of three components. It would then be seen that we must deal with a state of stable equilibrium. Since the proof is more involved than that given above we do not reproduce it here. In a later communication dealing with a more general problem another proof will be found.

Katwijk a. d. Rijn, August 1919.

Physics. — "*On the rings of connecting-electrons in* BRAGG's *model of the diamondcrystal.*" By D. COSTER. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of October 25, 1919).

The beautiful investigations of the two BRAGGS[1]) have given us a clear insight in the structure of the diamondcrystal. As is known according to these investigators the structure of this crystal may be represented by the following scheme: a set of cubes, where the *C*-atoms are situated in the corners and in the centres of the side-planes; in which another set of identical cubes, which may be obtained from the first by translating it parallel to itself in the direction of one of the cube-diagonals over a quarter of this diagonal (see fig. 1, where only those atoms are represented, which are situated within a fundamental cube). If we assume, that the valency of the atoms also have a principal meaning in the crystal, this system is of a perfect symmetry. Every *C*-atom namely has in its neighbourhood four other atoms at the same distance and symmetrically situated. (The lines which join each atom with its 4 neighbour-atoms form the diagonals of a cube). In this way the four valencies of the *C*-atoms are satisfied. Now we may assume, that the "bonds" between the atoms are formed by rings of electrons as it is the case in BOHR's model of the hydrogen-molecule. DEBYE and SCHERRER[2]) for instance suggest a model, where each carbon-atom should part with four electrons, one for each valency, for which consequently two electrons should be available. These should revolve about the connecting-axis of two nuclei in a plane perpendicular to this axis and half-way the distance between the nuclei. So the nucleus itself should still retain two electrons and behave at a distance as a four-fold charge. If once we have admitted, that the "bonds" are formed by rings of electrons, from the point of view of symmetry there is much to be said in favour of this model[3]).

DEBIJE and SCHERRER however arrive at the conclusion, that such a model is inconsistent with the experimental data of the two

[1]) Proc. Roy. Soc. Londen (1914) A 89, p. 277.
See also: BRAGG. X-rays and crystalstructure.

[2]) Phys. Z. S. (1918) XIX, p. 476.

[3]) Of course many difficulties yet remain, e.g.: how is the direction of rotation in the orbits. We can also say but little about form and magnitude of the orbit.

BRAGGS (and also with the data, they have obtained with their own method of crystal-photography). In my opinion however they neglect an important element in their reasoning and in this state of things nothing can be said about the existence or non-existence of such rings of electrons on account of data about scattering of Röntgen-rays. This I hope to prove in the following. To this purpose I intend to follow the clear method in which BRAGG has treated the subject.

We consider the octahedronplanes (the planes (111) in the usual notation), which contain the C-atoms, e.g. the plane A, B, C, F, G, H (see fig. 1), a second plane contains D, a third E. All these planes contain an equal number of atoms, their mutual distance is alternately $\frac{1}{4}\ d$ and $\frac{3}{4}\ d$, as represented by fig. 2. If we only regard the reflection by the planes a, according to the ordinary suppositions we shall have a maximum intensity in the reflected beam for

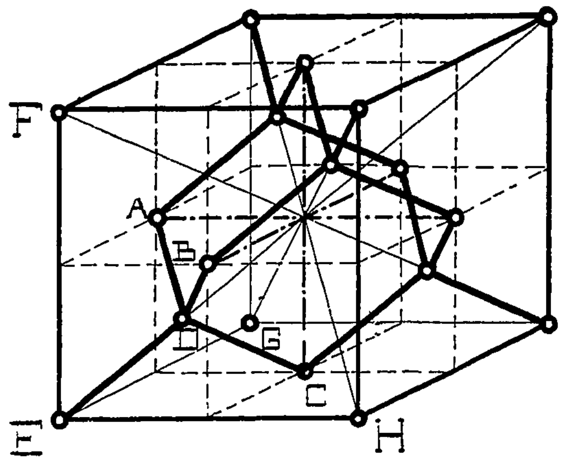


Fig. 1.

$$2d \sin \varphi = n\lambda,$$

here n has the values 1, 2, 3 etc. Regarding also the planes a' we see that the spectrum of the 2^{nd} order ($n = 2$) disappears, because the planes a' give half a wave-length phase-difference with the planes a. For the same reason the spectrum of the 6^{th} order would disappear. The BRAGGS have observed with the use of $Rh.$-K-rays spectra as far as and including that of the 5^{th} order; of the spectrum of the 2^{nd} order nothing could be detected. This very result has given them one of their strongest arguments in favour of the crystalmodel they suggested. With the model of DEBIJE and SCHERRER it is another case. In the usual way they assume, that the scattering is only caused by the electrons and may be calculated in the classical manner. In their calculations they

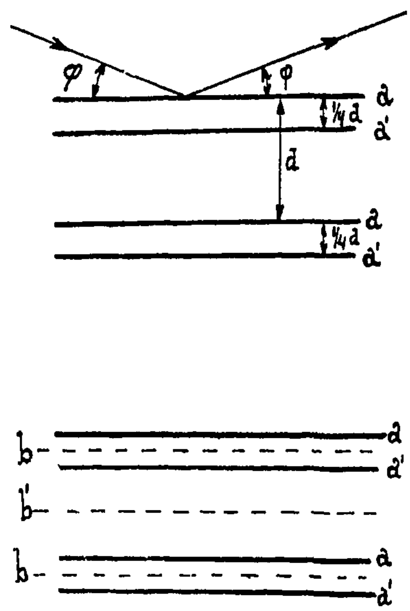


Fig. 2 and 3.

suppose, that the connecting-electrons may be placed in their common centre of gravity. The octahedron-planes are situated as represented by fig. 3. In b and b' we now have the connecting-electrons, in b three times as many as in b'. In this case the nucleus-electrons give also no contribution to the spectrum of 2nd order, the connecting-electrons however should give an intensive spectrum; whereas, as has been said before, the experiment does not give the slightest indication of it, therefore DEBIJE and SCHERRER reject this crystalmodel.

Regarding however a definite octahedron-plane (for instance that with *positive* indices 111), we see, that only $\frac{1}{4}$ of the orbits of the connecting-electrons coincide with those planes (i.e. those belonging to b fig. 3). The other orbits form angles of about 70° with these planes. From the following calculation it may be concluded that it is not admissible to assume, as in fact is done by DEBIJE and SCHERRER, that the electrons of these orbits always remain in the same octahedron-plane. For the sake of simplicity we assume the connecting-electrons moving uniformly in a circular orbit. Suppose $b\,b$ (fig. 4) to be the considered octahedron-plane, $c\,c$ the plane of the orbit, both perpendicular to the plane of the paper. The different phases of the beams reflected in the ordinary way by the electrons of the plane $b\,b$ are only determined by the distance h of the electron to $b\,b$. To calculate the total reflected beam we are to multiply the separate beam from each electron by the phasefactor $e^{-i\varkappa h}$, where $\varkappa = \frac{4\pi \sin\varphi}{\lambda}$ (φ is the complement of the angle of incidence). If we assume the electrons distributed at random in their orbits, then the probability that an electron is at a distance $h \rightarrow h + dh$, is

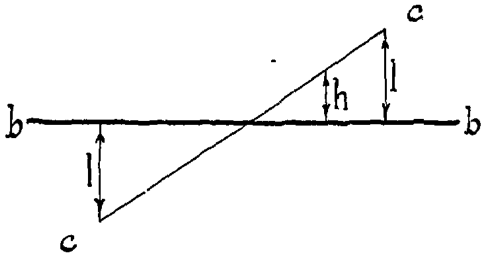


Fig. 4.

$$\frac{dh}{\pi\sqrt{l^2-h^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore the total amplitude of the reflected beam is to be multiplied by

$$\int_{-l}^{+l} dh \frac{e^{-i\varkappa h}}{\pi\sqrt{l^2-h^2}} = -\int_{-\pi}^{0} \frac{1}{\pi} e^{-il\varkappa\cos\omega}\, d\omega = J_0(l\varkappa) \quad . \quad . \quad . \quad (2)$$

here J_0 is the BESSELIAN function of order zero.

Taking into consideration, that $\varkappa = \frac{4\pi \sin\varphi}{\lambda}$, we find, if BRAGG's relation

$$2d \sin\varphi = n\lambda$$

is satisfied,

$$\varkappa = \frac{2\pi n}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As is known the function J_0 is real for real values of the argument and oscillates between decreasing positive and negative limits and so behaves like a "damped" sinefunction. Here this means, that the phase-difference between the resultant beam and a beam reflected by the plane bb is zero or 180°. The absolute value of (2) is always less than 1 except for the argument 0; the motion of the electrons therefore implies a decreasing of the intensity of the reflected Röntgenbeam. The experiment requires, that the spectrum of the second order by reflection from b and b' disappears. This happens strictly if

$$3\, J_0\, (l\varkappa) + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

since the plane b contains thrice as many electrons as b'.

The smallest value of l, which satisfies (4) and (3) for $n = 2$ is 0,258 d. If we assume according to BRAGG $d = 0{,}203.10^{-8}$ c.m., then $l = 0{,}524.10^{-8}$ c.m., which should give for the radius of the orbit of the electron $r = 0{,}56.10^{-8}$ c.m., which value cannot be excluded for being impossible[1]). Here it is of importance that the relation (4) holds independently of the wave-length of the Röntgenrays. Now I do not intend to attach high value to this calculation of the radius of the orbit. Firstly because my supposition (uniform circular motion of the electrons) is too schematic, secondly it is not probable, that DEBYE and SCHERRER should have been able to ascertain an intensity which should remain for instance below a 100th of that of the spectrum of the first order. This gives in the above case for r all values between about 0,52 and $0{,}62.10^{-8}$ and also between 0,70 and $0{,}81.10^{-8}$ c.m. Greater values of r are a priori improbable.

Now the question arises if the existence or non-existence of the rings of connecting-electrons yet may be proved in the manner suggested by DEBYE and SCHERRER. The spectra of higher order obtained by reflection from the octahedron-plane are not adapted for the purpose. Thus the spectrum of the 6th order should give a difference between the model with the connecting-electrons and that without.

---

1) If we only take account of the change of the two nuclei concerned (as a fourfold charge) and neglect all the disturbances, then according to BOHR an orbit of one quantum and two electrons has a radius of about 0,75 10^{-8} c.m.

First however the intensity decreases in general with the order of the spectrum [1]); secondly the intensity which we should expect according to (2) is very small, because $J_0(l\varkappa)$ is again negative for $n=6$ for the considered value of r (about $0{,}55.10^{-8}$ c.m.).

Now it is interesting to consider the reflection by the other crystal-planes. Here we shall follow the method also used by DEBYE and SCHERRER. When we have a regular crystal, then the intensity of a beam reflected according to the relation of BRAGG is proportional to the square of the so-called structure-factor [2]) S, which is given by

$$S = n\,\Sigma\,A_n\,e^{i2\pi(p_n h_1 + q_n h_2 + r_n h_3)} \quad . \quad . \quad . \quad . \quad (5)$$

Here A_n is proportional to the amplitude of the beam radiated by the n^{th} centre of the fundamental cube, $p_n\,q_n\,r_n$ are the ordinates of this centre in the cube, whose edge is 1; $h_1\,h_2\,h_3$ are the indices of the considered crystalplane. These may have a common divisor. If they are for instance 024, then the spectrum of the 2^{nd} order of the plane 012 in the ordinary notation is meant.

For BRAGG's crystalmodel this factor is:

$$S_B = 6\left(1+e^{i\frac{\pi}{2}(h_1+h_2+h_3)}\right)\left\{1+e^{i\pi(h_1+h_2)}+e^{i\pi(h_2+h_3)}+e^{i\pi(h_3+h_1)}\right\} \quad . \quad . \quad (6)$$

DEBYE and SCHERRER assume that the connecting-ring scatters in the same way as the nucleus-electrons. Also for their model we may put all A_n's $=1$.

Therefore they obtain:

$$\left.\begin{aligned} S_D = 2\Big[\Big(1+e^{i\frac{\pi}{2}(h_1+h_2+h_3)} &+ e^{i\frac{\pi}{2}(h_1+h_2+h_3)}\Big(1+e^{i\frac{\pi}{2}(h_1+h_2)}+ \\ &+ e^{i\frac{\pi}{2}(h_2+h_3)}+e^{i\frac{\pi}{2}(h_3+h_1)}\Big)\Big] \\ \Big\{1+e^{i\pi(h_1+h_2)} &+ e^{i\pi(h_2+h_3)}+e^{i\pi(h_3+h_1)}\Big\} \end{aligned}\right\} \quad . \quad . \quad (7)$$

Taking into consideration the position of the orbits of the ring-electrons in the above-given way, we get for the structure factor:

$$\left.\begin{aligned} S = 2\Big[1+e^{i\frac{\pi}{2}(h_1+h_2+h_3)}&+e^{i\frac{\pi}{2}(h_1+h_2+h_3)}\Big(J_0(l_1\varkappa)+ \\ +J_0(l_2\varkappa)e^{i\frac{\pi}{2}(h_1+h_2)}&+J_0(l_3\varkappa)e^{i\frac{\pi}{2}(h_2+h_3)}+J_0(l_4\varkappa)e^{i\frac{\pi}{2}(h_1+h_3)}\Big)\Big] \\ \Big\{1+e^{i\pi(h_1+h_2)}&+e^{i\pi(h_2+h_3)}+e^{i\pi(h_3+h_1)}\Big\} \end{aligned}\right\} \quad . \quad . \quad (8)$$

Here l and $\varkappa$ have the same signification as in (2) and (3); the indices at the different magnitudes l refer to the four different angles

[1]) See e.g. BRAGG. Proc. Roy. Soc. A. 89, p. 279, fig. 2.

[2]) See D. and SCH. Phys. Z. S. (1916), p. 279.
For the meaning of this factor see: MARX. Handb. d. Rad. Bd. V. p 581.

which the orbits of the electrons can make with the crystallografic plane under discussion.

The annexed table gives $\frac{|S^2|}{64}$ calculated[1]) for the three cases; in the last case once for a value $r = 0{,}56.10^{-8}$ and once for $r = 0{,}81.10^{-8}$ c.m. Here the ratio between the numbers standing in the same column is only of importance. We have to remark that the spectra (002) and (024) disappear independently of the assumed value of r. Only to make also the spectrum (222) disappear we are bound to certain limits in the choice of r.

Indices.	Br.	D. and Sch.	$r = 0.56.10^{-8}$	$r = 0.81.10^{-8}$
(111)	18	11.6	2.9	5.8
(002)	0	0	0	0
(022)	36	4	0.61	7.8
(113)	18	0.34	1.64	3.55
(222)	0	16	0	0.038
(004)	36	4	11.1	9.0
(133)	18	2	2.1	2.42
(024)	0	0	0	0

In calculating this table no account has been taken of the different factors[2]) that strongly affect the intensity of the expected spectra (mostly those of higher order). Because as yet all is quite uncertain and the foregoing speculations are very schematic, I thought it unnecessary to involve them in the calculations. The table however shows that especially the numbers of the fourth column do not more contradict the experimental data than those of the first[3]). From which we may conclude that for the present it will not be possible to draw a conclusion from the experimental data concerning the existence or non-existence of the connecting-rings. Perhaps here the study of the crystals of homologous elements (*Si*, *Ge*)[4]) may bring a decision.

[1]) The first two columns are taken from D. and SCH.

[2]) e.g. LORENTZ- and DEBIJE factor, see MARX Handbuch V, p. 581 a.f.

[3]) See BRAGG l.c. and DEBIJE and SCHERRER l.c.

[4]) *Si* seems to behave completely as diamond, cf. DEBIJE and SCH. Phys. Z. S. (1916) p. 282.

With *Ge* the number of connecting-electrons is already small compared with that of the nucleus-electrons.

Chemistry. — "*In-, mono- and divariant equilibria*". XX. By Prof. SCHREINEMAKERS.

(Communicated at the meeting of November 29, 1919).

Equilibria of n components in n phases, in which the quantity of one of the components approaches to zero; the influence of a new substance on an invariant (P or T) equilibrium.

In the communications XVI, XVII and XVIII we have seen that a region is two-leafed in the vicinity of a turning-line and one-leafed in the vicinity of a limit-line [e.g. curve *ab* or *cd* in fig. 1 (XVI)]. We shall consider the latter case more in detail.

We take the equilibrium $E = F_1 + F_2 \ldots + F_n$ of n components in n phases under constant pressure. This equilibrium is (Comm. XVII) monovariant (P); viz. it has one freedom under constant pressure.

The equations (2) and (3) (XVII) are true for this equilibrium; on change of one of the variables e.g. of x_1 this equilibrium traces in the P,T-diagram a straight line parallel to the T-axis.

In the vicinity of a limit-line of a region e.g. in the vicinity of curve *ab* or *cd* in fig. 1 (XVI), the quantity of one of the components approaches to zero. When this is the case with the component X, viz. with that component, the quantities of which are indicated in the different phases by $x_1\, x_2 \ldots x_n$, then in (2) and (3) (XVII):

$$\frac{\partial Z_1}{\partial x_1}, \quad \frac{\partial Z_2}{\partial x_2} \quad \ldots \frac{\partial Z_n}{\partial x_n}$$

become infinitely large, viz in Z_1 the term $x_1 \log x_1$ is found, in Z_2 the term $x_2 \log x_2$ etc.

Now we write:

$$Z_1 = Z_1' + RT\, x_1 \log x_1 \qquad Z_2 = Z_2' + RT\, x_2 \log x_2 \quad \ldots \quad (1)$$

Herein $Z_1'\, Z_2' \ldots$ and their differential quotients remain always finite also for $x_1 = 0,\ x_2 = 0 \ldots$ It follows from (1):

$$\left.\begin{aligned} \frac{\partial Z_1}{\partial x_1} &= \frac{\partial Z_1'}{\partial x_1} + RT\,(1 + \log x_1) \\ \frac{\partial Z_2}{\partial x_2} &= \frac{\partial Z_2'}{\partial x_2} + RT\,(1 + \log x_2) \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (2)$$

etc. The n equations (2) (XVII) now pass into:

$$\left.\begin{aligned} Z_1' - RT\,x_1 - x_1 \frac{\partial Z_1'}{\partial x_1} - y_1 \frac{\partial Z_1'}{\partial y_1} \ldots\ldots = K \\ Z_2' - RT\,x_2 - x_2 \frac{\partial Z_2'}{\partial x_2} - y_2 \frac{\partial Z_2'}{\partial y_2} \ldots\ldots = K \end{aligned}\right\} \quad \ldots \quad (3)$$

etc. The first series of the equations (3) (XVII) passes into:

$$\frac{\partial Z_1'}{\partial x_1} + RT \log x_1 = \frac{\partial Z_2'}{\partial x_2} + RT \log x_2 = \ldots\ldots = K_x - RT \quad . \, (4)$$

The following series of the equations (3) (XVII) become:

$$\frac{\partial Z_1'}{\partial y_1} = \frac{\partial Z_2'}{\partial y_2} = \ldots\ldots = \frac{\partial Z_n'}{\partial y_n} = K_y \quad \ldots\ldots \quad (5)$$

etc. It follows from (4):

$$\left.\begin{aligned} RT \log \frac{x_2}{x_1} = \frac{\partial Z_1'}{\partial x_1} - \frac{\partial Z_2'}{\partial x_2} \\ RT \log \frac{x_3}{x_1} = \frac{\partial Z_1'}{\partial x_1} - \frac{\partial Z_3'}{\partial x_3} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

or

$$x_2 = \mu_2\, x_1 \qquad x_3 = \mu_3\, x_1 \;\ldots\ldots\; x_n = \mu_n\, x_1 \;.\quad \ldots \quad (7)$$

in which $\mu_2\, \mu_3 \ldots$ are defined by (6).

For values infinitely small of $x_1\, x_2 \ldots$ the ratios between $x_1\, x_2 \ldots x_n$ are consequently defined by (7).

Now we give the increments: $dT . x_1\, x_2 \ldots dy_1 \ldots dy_2$ etc., to the variables $T\, x_1\, x_2 \ldots y_1\, y_2 \ldots$ etc., in which we put $x_1 = 0$ $x_2 = 0 \ldots$

Now it follows from (3):

$$\left.\begin{aligned} H_1\, dT + RT\, x_1 + y_1\, d \frac{\partial Z_1'}{\partial y_1} + \ldots\ldots = -dK \\ H_2\, dT + RT\, x_2 + y_2\, d \frac{\partial Z_2'}{\partial y_2} + \ldots\ldots = -dK \end{aligned}\right\} \quad \ldots \quad (8)$$

etc. in which the sign d indicates that we have to differentiate according to all variables.

Now we add the n equations (8) after having multiplied the first by λ_1, the second by λ_2 etc. Then we obtain, when we use the relations which follow from (5):

$$\Sigma\,(\lambda\, H)\;\; dT + RT\, \Sigma\,(\lambda\, x) + \Sigma\,(\lambda\, y)\, d \left(\frac{\partial Z_1'}{\partial y_1}\right) + \ldots = -\,\Sigma\,(\lambda)\;\; dK \;(9)$$

Now we define $\lambda_1\, \lambda_2 \ldots$ in such a way that they satisfy the n—1 equations (10)

36*

$$\left.\begin{aligned} \Sigma(\lambda) &= \lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n = 0 \\ \Sigma(\lambda y) &= \lambda_1 y_1 + \lambda_2 y_2 + \ldots\ldots + \lambda_n y_n = 0 \\ \Sigma(\lambda z) &= \lambda_1 z_1 + \lambda_2 z_2 + \ldots\ldots + \lambda_n z_n = 0 \end{aligned}\right\} \quad . \quad . \quad (10)$$

etc. By this the n—1 ratios between the coefficients $\lambda_1\ \lambda_2 \ldots$ are defined. As

$$\left.\begin{aligned} \Sigma(\lambda x) &= \lambda_1 x_1 + \lambda_2 x_2 + \ldots\ldots + \lambda_n x_n \\ \Sigma(\lambda H) &= \lambda_1 H_1 + \lambda_2 H_2 + \ldots\ldots + \lambda_n H_n \end{aligned}\right\} \quad . \quad . \quad (11)$$

the ratio $\Sigma(\lambda x) : \Sigma(\lambda H)$ is also defined. Now it follows from (9)[1]:

$$(dT)_P = -\frac{RT\,\Sigma(\lambda x)}{\Sigma(\lambda H)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The value of dT in (12) depends on $\Sigma(\lambda x)$, consequently on the n increments $x_1\ x_2 \ldots x_n$. We may express them, however, in one of those increments e.g. in x_1. With the aid of (7) we obtain then:

$$(dT)_P = -\frac{RT\,x_1\,\Sigma(\lambda \mu)}{\Sigma(\lambda H)} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

wherein:

$$\Sigma(\lambda \mu) = \lambda_1 \mu_1 + \lambda_2 \mu_2 + \ldots\ldots \lambda_n \mu_n \quad . \quad . \quad . \quad . \quad (14)$$

When we take the equilibrium $E = F_1 + F_2 + \ldots + F_n$ of n components in n phases at constant temperature, then it is monovariant (T). In the same way as above we find now:

$$(dP)_T = \frac{RT\,\Sigma(\lambda x)}{\Sigma(\lambda V)} = \frac{RT\,x_1\,\Sigma(\lambda \mu)}{\Sigma(\lambda V)} \quad . \quad . \quad . \quad (15)$$

Herein $\lambda_1\ \lambda_2 \ldots$ have again the values, which are defined by (10) $\Sigma(\lambda x)$ has also the same value of (11) viz.:

$$\left.\begin{aligned} \Sigma(\lambda x) &= \lambda_1 x_1 + \lambda_2 x_2 + \ldots\ldots + \lambda_n x_n \\ \text{while} \qquad \Sigma(\lambda V) &= \lambda_1 V_1 + \lambda_2 V_2 + \ldots\ldots + \lambda_n V_n \end{aligned}\right\} \quad . \quad . \quad (16)$$

$\Sigma(\lambda\mu)$ has again the same value as in (14).

In the previous considerations it is assumed that the quantity of the component X in the equilibrium $E = F_1 + F_2 + \ldots + F_n$ of n components in n phases is very small. When, however, this quantity becomes zero, then E passes into an equilibrium of n—1 components in n phases. This is monovariant and is represented in the P,T-diagram by a curve. Under constant pressure it is invariant (P), at constant temperature invariant (T). In this invariant (P or T) equilibrium between the phases $F_1 \ldots F_n$ may occur a

[1]) For another deduction see F. A. H. SCHREINEMAKERS, Die heterogenen Gleichgewichte von H. W. BAKHUIS ROOZEBOOM. III. 289.

reaction; the quantities $\lambda_1 \ldots \lambda_n$ of the phases participating in this reaction are defined by (10). The change in entropy occurring with this reaction $\Sigma(\lambda H)$ is defined by (11), the change in volume $\Sigma(\lambda V)$ is defined by (16).

Some of the coefficients $\lambda_1 \ldots \lambda_n$ are positive, other ones are negative. As long as we do not assume for this a definite rule, we may arbitrarily interchange positive and negative. We assume the following: The coefficients of the phases, which occur with a reaction, are taken positive; the coefficients of the phases which disappear with the reaction, are taken negative.

Now $\Sigma(\lambda)$ is the algebraical sum of the quantities of the phases which participate in the reaction, of course this is zero.

$\Sigma(\lambda y)$ is the algebraical sum of the quantity of the component Y which participates in the reaction; this is also zero. The same is true for the other components.

As the component X does not occur in the invariant (P or T) equilibrium, $\Sigma(\lambda x)$ has, therefore, another meaning. When we add, however, a little of the component X to this equilibrium, then it is divided between the n phases; this division is defined by (7), so that $x_1 \ldots x_n$ and consequently also $\Sigma(\lambda x)$ are defined.

Now we imagine a reaction in the invariant (P or T) equilibrium; $\lambda_1 \ldots \lambda_n$ represent, therefore, the quantities of the phases participating in the reaction. When those phases would contain the quantities $x_1 \ldots x_n$ of the new component, then $\Sigma(\lambda x)$ would be the algebraical sum of the quantity of the component X, which participates in this reaction. For this reason we shall call $\Sigma(\lambda x)$ "the fictitious quantity of reaction of the component X".

Now we take a point on the limit-curve of a region, e.g. point h on the limit-curve $a\,b$ in fig. 1. (XVI). This limit-curve represents an equilibrium of n—1 components (viz. the components Y, Z, U..) in n phases, consequently a monovariant equilibrium. In the point h itself $P T y_1 y_2 \ldots z_1 z_2 \ldots$ etc. have definite values; the same is true for the ratios of $\lambda_1 \ldots \lambda_n$ which are defined by (10) Now we add a little of the component X, this is divided over the n phases; this division is defined by (7). For a definite value of e.g. x_1 the ratios $\Sigma(\lambda x) : \Sigma(\lambda H)$ and $\Sigma(\lambda x) : \Sigma(\lambda V)$ are also defined. In accordance with (12) and (15) we know consequently also $(dT)_P$ and $(dP)_T$.

When $(dT)_P$ is positive, then the region E is situated at the right of the point h; we enter then the region, just as in fig. 1 (XVI) starting from h in the direction hl.

When $(dP)_T$ is negative, then the region E is situated below point h; then we enter the region, just as in fig. 1 (XVI) starting from h in the direction hm.

Consequently the region E is situated at the right and below the point h.

The direction of curve ab itself is defined in every point by:

$$\frac{dP}{dT}=\frac{\Sigma(\lambda H)}{\Sigma(\lambda V)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

It follows from our assumption over the sign of $(dT)_P$ and $(dP)_T$ that we have assumed $\Sigma(\lambda x):\Sigma(\lambda H)$ to be negative and $\Sigma(\lambda x):\Sigma(\lambda V)$ also to be negative. Then it follows from (17) that curve ab must be a curve, rising with the temperature, in the vicinity of point h, as is also drawn in fig. 1 (XVI).

In fig. 3 (XVI) $abchd$ represents a limit-curve which has a maximum of pressure in b and a maximum of temperature in c. It follows with the aid of (17) from the direction of branch ab that $\Sigma(\lambda H)$ and $\Sigma(\lambda V)$ have the same sign; we now choose the signs of $\lambda_1 \ldots \lambda_n$ in such a way that both are positive. Then it follows from the direction of the branches bc and cd with the aid of (17), which signs $\Sigma(\lambda H)$ and $\Sigma(\lambda V)$ must have on those branches. Then we have:

on branch ab	$\Sigma(\lambda H)>0$	$\Sigma(\lambda V)>0$
in b	$\Sigma(\lambda H)=0$	$\Sigma(\lambda V)>0$
on branch bc	$\Sigma(\lambda H)<0$	$\Sigma(\lambda V)>0$
in c	$\Sigma(\lambda H)<0$	$\Sigma(\lambda V)=0$
on branch cd	$\Sigma(\lambda H)<0$	$\Sigma(\lambda V)<0$

In each point of curve $abchd$ $\Sigma(\lambda x)$ has a definite sign; we are able to find this with the aid of (7) and (10).

When we assume that $\Sigma(\lambda x)$ is negative in each point of the curve, then it follows from (12) and (15) that the region E must be situated entirely within the limit-curve $abcd$ and consequently not, as in fig. 3 (XVI), where the part afe is situated outside.

When in each point of the curve $abcd$ $\Sigma(\lambda x)>0$, then it follows from (12) and (15) that the region must be situated at the left of and above branch ab, at the right of and above branch bc, at the right of and below branch cd. Then we have fig. 5 (XVI). [As it is apparent from the position of the letters, the printer has turned this figure; for this reason the reader has to place it in such a way that the tangent is horizontal in b and vertical again in c].

We may assume also that $\Sigma(\lambda x)$ is positive in the one part of the curve, negative in another part. We assume that $\Sigma(\lambda x)$ is positive in part $a\,b\,f$ of curve $a\,b\,c\,d$ fig. 3 (XVI) and negative in the part $f\,c\,d$. Then it follows from (12) and (15) that the region must be situated as is drawn in fig. 3 (XVI) viz. that a part $a\,f\,e$ of this region must be situated outside the limit-line and that this region must have a turning line $e\,f$.

It appears from the following that this point f must be a point of the turning-line. In this point $\Sigma(\lambda x) = 0$. As in this point also the equations (10) are valid, a phase reaction $\lambda_1 F_1 + \ldots + \lambda_n F_n = 0$ may occur between the n phases of the equilibrium $E = F_1 + \ldots + F_n$, in which an infinitely small quantity of the component X occurs now also.

Consequently when in a definite point f of curve $a\,b\,c\,d$ $\Sigma(\lambda x) = 0$, then f is a common point of turning- and limit-line; later we shall see that f is a point of contact. When $\Sigma(\lambda x)$ changes in sign in f, then f is a terminating point of the turning-line as in fig. 3 (XVI); when however $\Sigma(\lambda x)$ does not change its sign in f, then f is not a terminating point, but the curve proceeds further.

From (12), (15) and (17) follows the relation:

$$(dP)_T : (dT)_P = -\left(\frac{dP}{dT}\right)_{x=0} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The index $x = 0$ in the second part of (18) indicates that $\frac{dP}{dT}$ is true for the limit-curve, in which the component X is missing.

In order to comprehend the meaning of (18), we imagine the P,T-curve of the limit-equilibrium, to be drawn in which the component X does not occur, therefore. For this we take the curves $a\,b$ and $c\,d$ in the figures 1, 2 and 4 (XVI) and curve $a\,b\,c\,d$ in the figures 3 and 5 (XVI). [We have to place again the latter figure in the right position].

We shall call the branches on which the pressure increases with increase of T the "ascending" branches, e.g. the branches $a\,b$ and $c\,d$ in the figures 1, 2, 3, 4 and 5 (XVI). A branch like e.g. bc in figs. 3 and 5 (XVI), on which the pressure decreases at increase of T, is called a "descending" branch.

On an ascending branch $\left(\frac{dP}{dT}\right)_{x=0}$ is positive, then it follows from (18) that $(dP)_T$ and $(dT)_P$ have opposite signs. When $(dT)_P$ is positive and consequently $(dP)_T$ negative, then the region is situated at the right and below the branch; this is the case with respect to branch $a\,b$ in the figs. 1, 2 and 4 (XVI) and with respect to branch cd in the figs. 2, 4 and 5 (XVI). When $(dT)_P$ is negative and $(dP)_T$ consequently positive, then the region is situated at the left and above the branch; this is the case with respect to branch $a\,b$ in figs. 3 and 5 (XVI), and with respect to branch $c\,d$ in the figs. 1 and 3 (XVI).

Consequently we find: A region is situated always at the right and below or at the left and above the ascending branch of its limit-curve.

On the descending branch of a limit-curve $\left(\frac{dP}{dT}\right)_{x=0}$ is negative. It follows from (18) that then $(dP)_T$ and $(dT)_P$ have the same sign. When both are positive, then the region is situated, therefore, at the right and above the branch. When both are negative, then it is situated at the left and below the branch. In fig. 5 (XVI) the region is situated at the right and above branch bc; in fig. 3 (XVI) the region is situated at the right and above the part $b\,f$, and at the left and below the part $f\,c$ of branch $b\,c$.

Consequently we find:

a region is situated at the right and below, or at the left and above the ascending branch of its limit-curve; it is situated at the right and above, or at the left and below the descending branch of its limit-curve.

In Communication XI on: Equilibria in ternary systems, we have already deduced this same property for a special case viz. for the ternary region $F+L+G$, in which F represents a binary compound, with respect to its binary limit curve $F+L+G$.

Now it appears that this is true in general for each arbitrary region with respect to all its limit-curves.

We may express the results obtained above also in another way. The equilibrium $E=F_1+\ldots+F_n$ of $n-1$ components in n phases is monovariant or invariant (P or T). When we add a little of a new substance X, then a new equilibrium $E'=F'_1+\ldots+F'_n$ may arise. Herein the invariable phases have the same composition as in E; the variable phases (which of course not all need to contain the new substance X) differ still only very little from those in E.

We now may put the question:

how must the temperature change under constant P or: how must the pressure change at constant T in order that in both cases the equilibrium E passes into E'.

It is clear that both questions are only another form of the questions, treated above: how must the temperature be changed under constant P and the pressure at constant T in order to pass from a limit-curve into the corresponding region.

We take the equilibrium $E = L + F_2 + F_3 + \ldots$ of $n-1$ components in n phases (or of n components in $n+1$ phases). Herein $F_2 F_3 \ldots$ represent solid substances of invariable composition and L a liquid. On addition of a new substance X this occurs then only in the liquid.

When in this equilibrium E at constant T and under constant P there occurs the reaction:

$$\lambda_1 L + \lambda_2 F_2 + \lambda_3 F_3 + \ldots\ldots = 0 \quad \ldots\ldots \quad (19)$$

then $\Sigma(\lambda x) = \lambda_1 x$, when viz. x represents the concentration of the new substance in the liquid.

When we put $\lambda_1 = 1$, then $\Sigma(\lambda H)$ and $\Sigma(\lambda V)$ are the increases of entropy and volume, when one quantity of liquid is formed at the phase-reaction. We represent them by ΔH and ΔV.

(12) and (15) pass now into:

$$(dT)_P = -\frac{RTx}{\Delta H} \text{ and } (\Delta P)_T = \frac{RTx}{\Delta V}. \quad \ldots \quad (20)$$

When we represent by ΔW the quantity of heat which is to be added in order to form with the reaction one quantity of liquid, then (20) passes into:

$$(dT)_P = -\frac{RT^2 x}{\Delta W} \text{ and } (\Delta P)_T = \frac{RTx}{\Delta V} \quad \ldots \quad (21)$$

Reaction (19) may represent the common melting of the solid substances $F_2 F_3 \ldots$; this is the case when the reaction is of the form:

$$\lambda_2 F_2 + \lambda_3 F_3 + \ldots\ldots \rightleftarrows L \quad \ldots\ldots \quad (22)$$

and when $\lambda_2 \lambda_3$ are positive.

When the reaction is of the form:

$$\lambda_2 F_2 + \lambda_3 F_3 + \ldots\ldots \rightleftarrows L + \lambda_p F_p + \ldots\ldots \quad (23)$$

in which we take also positive all coefficients, then it represents the conversion of the liquids $F_2 F_3 \ldots$ into $F_p \ldots$ when simultaneously liquid is formed.

Now we assume that heat is to be added at the formation of liquid from solid substances, consequently at melting in accordance with (22) and at conversion in accordance with (23); then ΔW is positive; the change in volume at melting or conversion may be as well positive as negative. Now it follows from (21):

The common melting- or conversion temperature of one or more substances is lowered by addition of a new substance;

the common melting- or conversion-pressure of one or more substances is;

raised by a new substance, when the volume increases on melting or conversion;

lowered, when the volume decreases on melting or conversion.

This increase and decrease are at first approximation proportional to the quantity of the new substance.

When we apply those rules to the melting of a simple substance, then follows the known rule of the decrease of melting or freezing point; the first formula (21) is then the known formula of RAOULT-VAN 'T HOFF.

We may apply the previous deductions also when we substitute in (19) the liquid L by a gas G. In general ΔV is then positive and approximately equal to the volume V of the gas; by this we may give another form to the second formula (21) viz.

$$(dP)_T = Px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

We may deduce the previous rules also in the following way. We take the equilibrium $E = L + F_2 + F_3 + \ldots$, in which the new substance X is not yet present under constant pressure; then it is invariant (P) and it consists at a definite temperature, which we shall call T_0. When we assume that reaction (22) takes place from left to right at addition of heat, then it follows:

$$F_2 + F_3 + \ldots\ldots \rightleftarrows L \quad \Delta H > 0$$

(L)	$(F_2)\ (F_3)\ \ldots\ \ldots$
towards lower T	towards higher T.

Consequently the equilibrium $(L) = F_2 + F_3 + \ldots$ consists at temperatures lower than T_0. When we add the new substance X, then E passes into $E' = L' + F_2 + F_3 + \ldots$, in which L_1 differs from L; this equilibrium E' exists at a temperature T' which differs from T_0.

When we take away the liquid L' from E', then it passes into $F_2 + F_3 + \ldots$, consequently in the equilibrium (L) discussed above; as this exists at lower temperatures than T_0, it follows $T' < T_0$. On addition of the new substance the common melting-point must fall, therefore.

From reaction (23) we find the same for the common point of conversion. When we take at constant temperature the equilibrium $E = L + F_2 + F_3 + \ldots$, in which the new substance is not yet present, it is invariant (T); then it exists under a definite pressure P_0.

When reaction (22) takes place with increase of volume from left to right, then follows:

$$F_1 + F_2 + \ldots\ldots \rightleftarrows L \quad \Delta V > 0$$

(L)	$(F_1)\ (F_2)\ \ldots\ldots$
towards higher P	towards lower P

The equilibrium $(L) = F_1 + F_2 + \ldots\ldots$ exists, therefore, under pressures, larger than P_0. Hence is follows that the equilibrium E' occurs also under a pressure higher than P_0. Consequently when at common melting increase of volume takes place, then the melting pressure rises.

From reaction (23) the same follows for the common point of conversion.

When we assume that $\Delta V < 0$, then it follows that on addition of the new substance the pressure of melting or conversion falls.

Now we take the equilibrium:

$$E = L_1 + L_2 + \ldots\ldots + F_1 + F_2 + \ldots\ldots$$

of $n-1$ (or n) components in n (or $n+1$) phases. Again L_1 L_2 represent liquids, F_1 F_2 solid phases of constant composition. Formerly [1]) we have called the temperature at which this equilibrium occurs under constant pressure the "Schichtungstemperatur"; we may call it also the stratification-temperature. We write the reaction occurring in this equilibrium:

$$\lambda_1 L_1 + \lambda_2 L_2 + \ldots\ldots + \mu_1 F_1 + \mu_2 F_2 + \ldots\ldots = 0. \quad (25)$$

We may distinguish at this reaction the 2 main types:

$$\mu_1 F_1 + \mu_2 F_2 + \ldots\ldots \rightleftarrows \lambda_1 L_1 + \lambda_2 L_2 + \ldots\ldots + \mu_q F_q + \ldots\ldots \quad (26)$$

$$\lambda_1 L_1 + \ldots\ldots + \mu_1 F_1 + \mu_2 F_2 + \ldots\ldots \rightleftarrows \lambda_p L_p + \ldots\ldots + \mu_q F_q \ldots\ldots \quad (27)$$

in which we take all coefficients positive. In (26) the solid substances may be wanting on the right side, in (27) on the right or on the left side. Experimental examples of both types are known [2]). In order to express the difference between the two reactions we shall say: in (26) all liquids are situated in reaction-conjunction, in (27) two or more are situated in reaction-opposition [3]).

[1]) F. A. H. SCHREINEMAKERS, die heterogenen Gleichgewichte von H. W. BAKHUIS ROOZEBOOM III[2] 108.

[2]) F. A. H. SCHREINEMAKERS ibid. III[2] 106–113, 193–203.

[3]) In order to prevent confusion, see the following. In the books III[1] and III[2] mentioned above phases are many times spoken of which are situated in the diagram in conjunction or opposition. When we call this situation diagram conjunction and diagram opposition, then it appears that reaction-conjunction corresponds with diagram opposition and reaction-opposition with diagram-conjunction.

When we add a new substance, then this divides itself between the liquids; its concentrations $x_1\, x_2 \ldots$ are defined by (7). [It is apparent that the μ's in (7) have quite another meaning as in (25), (26) and (27)].

For reaction (26) $\Sigma(\lambda x) = \lambda_1 x_1 + \lambda_2 x_2 + \ldots$, in which occur only the λ's, not the μ's. As the λ's are all positive, $\Sigma(\lambda x)$ is also positive. With this we assume that heat must be added, in order that reaction (26) takes place from left to right, so that also $\Sigma(\lambda H)$ is positive. The sign of $\Sigma(\lambda V)$, however, is indefinite. [It is apparent that in $\Sigma(\lambda H)$ and $\Sigma(\lambda V)$ the μ's of (26) occur also].

Now it follows from (12) and (15)

$$(dT)_P < 0 \text{ and } (dP)_T \gtrless 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Hence it follows: when we have an invariant (P or T) equilibrium with 2 or more liquids, which are situated all in reaction-conjunction and when we add a new substance, then:

under constant P the stratification temperature is lowered;

at constant T the stratification-pressure is raised when the volume increases at the formation of the liquids;

lowered when the volume decreases at the formation of liquids.

For reaction (27) is

$$\Sigma(\lambda x) = \lambda_p x_p + \ldots\ldots - \lambda_1 x_1 - \lambda_2 x_2 - \ldots$$

so that $\Sigma(\lambda x)$ may be as well positive as negative. This depends on the partition of the new substance X between the different liquids.

In order to illustrate this further we consider a definite case, viz. the equilibrium

$$E = L_1 + L_2 + Z + U + \ldots\ldots + N. \quad . \quad . \quad . \quad (29)$$

between the n components $Y Z \ldots N$. Consequently in this equilibrium, all components, excepted Y, occur as solid phases. As there are, therefore, n—1 solid and 2 liquid phases, it is invariant (P or T). Now we represent the reaction by:

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 Z + \lambda_4 U + \ldots\ldots + \lambda_{n+1} N = 0 \quad . \quad (30)$$

so that $\Sigma(\lambda x) = \lambda_1 x_1 + \lambda_2 x_2$. For the definition of the relation between λ_1 and λ_2 we take from (10) the equation $\Sigma(\lambda y) = 0$. As the substance Y occurs only in the two liquids, it follows:

$$\Sigma(\lambda y) = \lambda_1 y_1 + \lambda_2 y_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Hence it appears that λ_1 and λ_2 have opposite signs, so that reaction (30) belongs to type (27). We write it in the form:

$$\lambda_1 L_1 + \lambda_3 Z + \lambda_4 U + \ldots\ldots \rightleftarrows L_2 \quad . \quad . \quad . \quad . \quad (32)$$

We have put, therefore $\lambda_2 = 1$, consequently λ_1 is positive; of course one or more of the coefficients $\lambda_3 \ldots$ may be negative.

Further we assume that L_2 is the liquid, which is formed on addition of heat. [When this should be the case with L_1, then we should have placed L_1 in the left part of (32)]. Now we have:

$$\Sigma(\lambda y) = y_2 - \lambda_1 y_1 \text{ and } \Sigma(\lambda x) = x_2 - \lambda_1 x_1.$$

Hence it follows:

$$\Sigma(\lambda x) = x_1 \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Now it follows from (12) and (15):

$$(dT)_P = -\frac{RT^2 x_1 \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right)}{\Delta W} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and

$$(dP)_T = \frac{RT x_1 \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right)}{\Delta V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Herein ΔW is the heat, wanted for forming one quantity of the liquid L_2; ΔV is the increase of volume occurring at this formation, which can be as well positive as negative.

Now we shall mean by partition-coefficient of a substance: the concentration of that substance in the liquid, which is formed on addition of heat, divided by the concentration of that substance in the other liquid. $x_2 : x_1$ is consequently the partition-coefficient of the new substance, $y_2 : y_1$ that of the component, which does not occur as solid phase.

Consequently we find:

when in an invariant (P or T) equilibrium with 2 liquids only components occur as solid phases, then both liquids are situated in reaction-opposition. The stratification-temperature under constant P by addition of a new substance:

is elevated (lowered) when the partition-coefficient of the new substance is smaller (larger) than that of the component which does not occur as solid phase [1]).

We may deduce from (35) similar rules for the influence of a new substance on the change in pressure at constant temperature.

We may also give a more simple form to (34) and (35). We have viz. expressed the concentrations of the components in the liquids in such a way that each liquid contains in all one molecule. We may, however, also mean by concentration the quantity of the

---

[1]) For some examples of the influence of a third substance on binary equilibria see F. A. H. SCHREINEMAKERS, Die heterogenen Gleichgewichte von H. W. BAKHUIS ROOZEBOOM III[2] 160.

components when the liquid contains one molecule of the component which does not occur as solid phase, consequently in our case of the component Y. As, therefore, y_1 and y_2 become $=1$, (34) and (35) pass into:

$$(dT)_P = -\frac{RT^2\,(x_2-x_1)}{\Delta W} \text{ and } (dP)_T = \frac{RT\,(x_2-x_1)}{\Delta V}. \quad . \quad (36)$$

Now we find:

the stratification-temperature is raised (lowered) under constant P on addition of a new substance, when the concentration of the new substance in the liquid, which is formed on addition of heat, is smaller (larger) than its concentration in the other liquid.

The first formula (36) has been deduced formerly for equilibria with two[1]) and more[2]) components.

To be continued.

Leiden, Inorg. Chem. Lab.

[1]) F. A. H. SCHREINEMAKERS. Zeitschr. f Phys. Chem. 25, 3?0 (1898).

[2]) H. A. LORENTZ ibid. 25. 332 (1898).

Mathematics. — "*On a formula of* SYLVESTER". By Prof. W. KAPTEYN.

(Communicated at the meeting of November 29, 1919).

In his paper "On the partition of numbers" Quart. Journ. of Math. I (1857) p. 141—152, SYLVESTER has given a general formula for the number of solutions in integers (zero included) of the equation

$$a_1 x_1 + a_2 x_2 + \ldots\ldots + a_r x_r = n. \quad \ldots\ldots \quad (1)$$

where n and a are given integers.

Applying this formula, which is given without proof, to a particular example, I found a fractional number. Of course this result is absurd. I therefore tried to construct a proof and found, as will be shown hereafter, that SYLVESTER's formula wants a slight correction.

If the fraction

$$\frac{1}{\Phi(x)} = \frac{1}{(1-x^{a_1})(1-x^{a_2})\,.\,.\,(1-x^{a_r})}. \quad \ldots\ldots \quad (2)$$

is developed in ascending powers of x, it is evident that the coefficient of x^n gives exactly the number of solutions in integers of the equation (1). We therefore proceed to reduce this fraction to its partial fractions and to develop every one of these in ascending powers of x. The denominator being a compound quantity, the first thing wanted is to determine its different factors.

Let $\underline{1-x^m} = 0$ denote the equation containing all the prime roots of the equation $1-x^m = 0$, then we have

$$1-x^m = \prod_{i=1}^{k} \underline{1-x^{d_i}}. \quad \ldots\ldots\ldots \quad (3)$$

where $d_1, d_2, \ldots d_k$ $(d_1 = 1, d_k = m)$ represent the different divisors of m.

To prove this theorem let $m = p^\alpha q^\beta \ldots t^\lambda$, $p, q, \ldots t$ being prime numbers, then the divisors of m are the several terms of the continued product

$$\left(1 + \sum_1^\alpha p^a\right)\left(1 + \sum_1^\beta q^b\right)\,.\,.\left(1 + \sum_1^\lambda t^l\right).$$

One of these being

$$d = p^a q^b \ldots t^l$$

we know that the number of prime roots of the corresponding equation $1-x^d = 0$ is

$$p^a q^b \, . \, . \, t^l \left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) . \, . \left(1-\frac{1}{t}\right)$$

The number of the prime roots corresponding to all the different divisors of m is therefore

$$\left[1+\left(1-\frac{1}{p}\right)\sum_1^{\alpha} p^a\right]\left[1+\left(1-\frac{1}{q}\right)\sum_1^{\beta} q^b\right] . \, .$$

$$. \, . \left[1+\left(1-\frac{1}{t}\right)\sum_1^{\lambda} t^l\right]=p^\alpha q^\beta \, . \, . \, t^\lambda = m.$$

Now these prime roots being all different, they must satisfy an equation of degree m, which, because every one of these roots is also a root of $1-x^m=0$, must coincide with $1-x^m=0$.

To illustrate this theorem, put $m=20=2^2 \, . \, 5$, then the divisors are

$$1, \quad 2, \quad 4, \quad 5, \quad 10, \quad 20$$

and the factors corresponding to the prime roots

$$1-x, \; 1+x, \; 1+x^2, \; 1+x+x^2+x^4, \; 1-x^2-x^3+x^4, \; 1-x^2+x^4-x^6+x^8$$

or

$$\underline{1-x}, \quad \underline{1-x^2}, \quad \underline{1-x^4}, \quad \underline{1-x^5}, \quad \underline{1-x^{10}}, \quad \underline{1-x^{20}}.$$

The continued product of these factors is evidently $1-x^{20}$, or

$$1-x^{20}=\prod_{i=1}^{6}(1-x^{di}).$$

Developing in the same way the several factors of $\Phi(x)$, we may write

$$\Phi(x)=\underline{(1-x^{\alpha_1})}^{r_1} \, . \, \underline{(1-x^{\alpha_2})}^{r_2} \, . \, . \, . \, \underline{(1-x^{\alpha_m})}^{r_m} \, . \quad . \quad . \quad . \qquad (4)$$

where the quantities α_i, ranged according to ascending magnitude, represent the different divisors of $a_1, a_2 \, . \, . \, a_r$, and r_i the numbers of the divisors α_i. We may remark here that $\alpha_1=1$ and $r_1=r$.

If, for instance

$$x_1+2\,x_2+5\,x_3+10\,x_4+20\,x_5=n$$

is the given equation, we have

$$\Phi(x)=(1-x)(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})$$

where

$$\begin{aligned}
1-x &= \underline{1-x} \\
1-x^2 &= \underline{1-x} \, . \, \underline{1-x^2} \\
1-x^5 &= \underline{1-x} \, . \, \underline{1-x^5} \\
1-x^{10} &= \underline{1-x} \, . \, \underline{1-x^2} \, . \, \underline{1-x^5} \, . \, \underline{1-x^{10}} \\
1-x^{20} &= \underline{1-x} \, . \, \underline{1-x^2} \, . \, \underline{1-x^4} \, . \, \underline{1-x^5} \, . \, \underline{1-x^{10}} \, . \, \underline{1-x^{20}}
\end{aligned}$$

hence

$$\Phi(x)=\underline{(1-x)}^5 \quad \underline{(1-x^2)}^3 \quad \underline{(1-x^4)} \quad \underline{(1-x^5)}^2 \quad \underline{(1-x^{10})}^2 \quad \underline{(1-x^{20})}$$

Proceeding now to determine the partial fractions of $\frac{1}{\Phi(x)}$, we know by CAUCHY's formula that

$$\frac{1}{\Phi(x)} = \mathcal{E} \frac{1}{((\Phi(z)))} \frac{1}{x-z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the double parentheses denote that the residues must be taken for all the roots of $\Phi(z) = 0$, viz. for all the roots of the equations

$$\underline{1-z^{\alpha_1}} = 0, \quad \underline{1-z^{\alpha_2}} = 0, \ldots \underline{1-z^{\alpha_m}} = 0.$$

By developing the factor

$$\frac{1}{x-z} = -\left(1 + \frac{x}{z} + \frac{x^2}{z^2} + \ldots + \frac{x^n}{z^n} + \ldots\right)$$

we get immediately for the required coefficient of x^n

$$P_n = -\mathcal{E} \frac{1}{z^{n+1}} \frac{1}{((\Phi(z)))} = \sum_{i=1}^{m} W_{\alpha_i}$$

where

$$W_{\alpha_i} = -\mathcal{E} \frac{1}{z^{n+1} (\underline{1-z^{\alpha_1}})^{r_1} (\underline{1-z^{\alpha_2}})^{r_2} \ldots ((\underline{1-z^{\alpha_i}}))^{r_i} \ldots (\underline{1-z^{\alpha_m}})^{r_m}}$$

or, restoring the original form of $\Phi(z)$

$$W_{\alpha_i} = -\mathcal{E} \frac{1}{z^{n+1} (1-z^{a_1})(1-z^{a_2}) \ldots (1-z^{a_r})} \frac{1-z^{\alpha_i}}{((\underline{1-z^{\alpha_i}}))}$$

where the residue is to be taken for all the roots of the equation

$$\underline{1-z^{\alpha_i}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Representing one of these roots by ϱ and putting

$$z = \varrho \, e^{-t}$$

the preceding value of W_{α_i} takes this form

$$W_{\alpha_i} = \Sigma \, \mathcal{E} \frac{\varrho^{-n} e^{nt}}{(1-\varrho^{a_1} e^{-a_1 t})(1-\varrho^{a_2} e^{-a_2 t}) \ldots (1-\varrho^{a_r} e^{-a_r t})} \frac{t}{((t))} \quad . \quad . \quad (7)$$

wherein the summation must be extended to all the roots ϱ of the equation (6).

The term W_1 may be further developed, for the corresponding equation (6)

$$\underline{1-z} = 0 \quad \text{or} \quad 1-z = 0$$

shows that the only root is $\varrho = 1$. Therefore W_1 reduces to

$$W_1 = \mathcal{E} \frac{e^{nt}}{(1-e^{-a_1 t})(1-e^{-a_2 t}) \ldots (1-e^{-a_r t})} \frac{t}{((t))}$$

37

This residue is the coefficient of $\frac{1}{t}$ in

$$X_1 = e^{nt} \quad lg\,(1-e^{-a_1 t}) \quad lg\,(1-e^{-a_2 t}) \,.\,.\, - lg\,(1-e^{-a_r t})$$

Now

$$lg\,(1-e^{-z}) = lg\,z - \tfrac{1}{2} z + \frac{B_1}{2\,!}\frac{z_2}{2} - \frac{B_2}{4\,!}\frac{z^4}{4} + \frac{B_3}{6\,!}\frac{z^6}{6} - \ldots \qquad (a)$$

as may be shown by integrating

$$\frac{1}{e^z-1} \quad \frac{1}{z} = -\tfrac{1}{2} + \frac{B_1}{2\,!}z \quad \frac{B_2}{4\,!}z^3 + \frac{B_3}{6\,!}z^5 - \,.\,. \qquad mod\ z < 2\pi$$

where

$$B_1 = \tfrac{1}{6}, \qquad B_2 = \tfrac{1}{30}, \qquad B_3 = \tfrac{1}{42} \,.\,.$$

between the limits o and z.

Substituting the values of $lg\,(1-e^{\alpha_i t})$, we obtain

$$X_1 = \frac{1}{a_1\, a_2 \,.\,.\, a_r\, t^r}\, e^{\,\nu t - \frac{B_1}{1\,!}\frac{s_2\, t^2}{2^2} + \frac{B_2}{3\,!}\frac{s_4\, t^4}{4^2} - \,.\,.}$$

where

$$\nu = n + \tfrac{1}{2} s_1 \quad \text{and} \quad s_i = a_1{}^i + a_2{}^i + \,.\,. + a_r{}^i$$

hence W_1 is the coefficient of t^{r-1} in the product

$$\frac{1}{a_1\, a_2 \,.\,.\, a_r}\left(1 + \nu t + \frac{1}{2\,!}\nu^2 t^2 + \frac{1}{3\,!}\nu^3 t^3 + \frac{1}{4\,!}\nu^4 t^4 + \,.\,.\right.$$

$$\left(1 - \frac{s_2}{24}t^2 + \frac{s_2{}^2}{1152}t^4 - \,.\,.\right)$$

$$\left(1 + \frac{s_4}{2880}t^4 \,.\,.\right)$$

. . . .

Applying the preceding to compute the number of solutions of the equation

$$x_1 + 2\,x_2 + 5\,x_3 + 10\,x_4 + 20\,x_5 = n.$$

we first observe that the different divisors of 1, 2, 5, 10, 20 are

1,	1,	1,	1,	1
	2,	5,	2,	2
			5,	4
			10,	5
				10
				20

thus

$$\Phi\,(x) = \underline{(1-x)^5}\cdot\underline{(1-x^2)^3}\ \underline{(1-x^4)}\ \underline{(1-x^5)^3}\ \underline{(1-x^{10})^2}\ \underline{(1-x^{20})}$$

The number of values α_i being six, we have six different terms W_{α_i}.

In this case, having

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 5, \quad \alpha_4 = 10, \quad \alpha_5 = 20$$
$$s_1 = 38, \quad s_2 = 530, \quad s_4 = 170642, \quad v = n + 19$$

we obtain

$$W_1 = \frac{1}{48000}\left[v^4 - 265\, v^2 + \frac{72741}{10}\right]$$

or

$$W_1 = \frac{1}{48000}\left[n^4 + 76\, n^3 + 1901\, n^2 + 17366\, n + 41930\frac{1}{10}\right].$$

For W_2 the equation (6)

$$\underline{1 - z^2} = 0$$

or

$$1 + z = 0.$$

shows, that also in this case there is only one root $\rho = -1$. Therefore W_2 reduces to

$$W_2 = \mathcal{E}\frac{(-1)^n\, e^{nt}}{(1+e^{-t})(1-e^{-2t})(1+e^{-5t})(1-e^{-10t})(1-e^{-20t})}\,\frac{t}{((t))}$$

or $(-1)^n$ multiplied by the coefficient of $\frac{1}{t}$ in

$$X_2 = e^{nt - lg(1+e^{-t}) - lg(1-e^{-2t}) - lg(1+e^{-5t}) - lg(1-e^{-10t}) - lg(1-e^{-20t})}.$$

Developing the logarithms in this expression by means of the equation (a) and

$$\left.\begin{aligned} lg\,(1+e^{-z}) &= lg\,(1-e^{-2z}) - lg\,(1-e^{-z}) = \\ &= lg\, z - \frac{z}{2} + \frac{3}{2}\cdot\frac{B_1}{2!}z^2 - \frac{15}{16}\cdot\frac{B_2}{4!}z^4 + \,.\,. \end{aligned}\right\} \quad (b)$$

we get

$$X_2 = \frac{1}{1600\, t^3}\left(1 + vt + \frac{v^2 t^2}{2!} + \,.\,.\right)\left(1 - \frac{97}{4}t^2 \ldots\right).$$

Hence

$$W_2 = \frac{(-1)^n}{48000}\,(15\, v^2 - 727\tfrac{1}{2})$$

or

$$W_2 = \frac{(-1)^n}{48\;00}\,(15\, n^2 + 570\, n + 4687\tfrac{1}{2}).$$

For W_4 the equation (6) is

$$\underline{1 - z^4} = 0$$

or

$$1 + z^2 = 0$$

37*

hence

$$W_4 = \Sigma \mathcal{E} \frac{\varrho^{-n} e^{nt}}{(1-\varrho e^{-t})(1-\varrho^2 e^{-2t})(1-\varrho^5 e^{-5t})(1-\varrho^{10} e^{-10t})(1-\varrho^{20} e^{-20t})} \frac{t}{((t))}$$

where the summation must be extended to both the roots ϱ of the equation $1 + z^2 = 0$ Writing $\varrho^2 = -1$, we obtain

$$W_4 = \Sigma \mathcal{E} \frac{\varrho^{-n} e^{nt}}{(1-\varrho e^{-t})(1+e^{-2t})(1-\varrho e^{-5t})(1+e^{-10t})(1-e^{-20t})} \frac{t}{((t))}$$

$$= \frac{1}{80} \Sigma \frac{\varrho^{-n}}{(1-\varrho)^2} = -\frac{1}{160} \Sigma \varrho^{-n-1}.$$

Denoting by Σ_p the sum of the p^{th} powers of the roots, and observing that $\varrho^4 = 1$, we know

$$\Sigma_0 = 2, \qquad \Sigma_1 = 0, \qquad \Sigma_2 = -2, \qquad \Sigma_3 = 0,$$

and generally, k being an integer number

$$\Sigma_{4k} = 2, \quad \Sigma_{4k+1} = 0, \quad \Sigma_{4k+2} = -2, \quad \Sigma_{4k+3} = 0.$$

Therefore

$$W_4 = -\frac{1}{160} \Sigma_{3n-1}$$

which gives different values for different values of n.

According to

$$n = 4p, \quad 4p+1, \quad 4p+2, \quad 4p+3$$

we get respectively the four values

$$W_4 = \frac{600}{48000} (0, 1, 0, -1)$$

For W_5 the equation (6) gives

$$\underline{1 - x^5} = 0$$

or

$$1 + x + x^2 + x^3 + x^4 = 0$$

thus

$$W_5 = \Sigma \mathcal{E} \frac{\varrho^{-n} e^{nt}}{(1-\varrho e^{-t})(1-\varrho^2 e^{-2t})(1-e^{-5t})(1-e^{-10t})(1-e^{-20t})} \frac{t}{((t))}.$$

Putting

$$X_5 = e^{nt - lg(1-\varrho e^{-t}) - lg(1-\varrho^2 e^{-2t}) - lg(1-e^{-5t}) - lg(1-e^{-10t}) - lg(1-e^{-20t})}$$

and reducing by means of the equations (a) and (b)

$$X_5 = \frac{e^{\left(n - \frac{\varrho}{1-\varrho} - \frac{2\varrho^2}{1-\varrho^2} + \frac{35}{2}\right)t + \left(\frac{\varrho}{2(1-\varrho)^2} + \frac{4\varrho^2}{2(1-\varrho^2)^2} - \frac{525}{24}\right)t^2}}{1000\,(1-\varrho)\,(1-\varrho^2)\,t^3}$$

we get

$$W_5 = \Sigma \frac{\varrho^{-n}}{2000\,(1-\varrho)\,(1-\varrho^2)} \left[n^2 + \left(35 - \frac{2\varrho + 6\varrho^2}{1-\varrho^2}\right) n + \right.$$
$$\left. + \frac{525}{2} - \frac{34\varrho + 98\varrho^2 - 42\varrho^3 - 114\varrho^4}{(1-\varrho^2)^2} \right]$$

or

$$W_5 = \frac{n^2 + 35n + \frac{525}{2}}{2000} \Sigma \frac{\varrho^{-n}}{(1-\varrho)(1-\varrho^2)}$$
$$- \frac{n}{2000} \Sigma \frac{\varrho^{-n}(2\varrho + 6\varrho^2)}{(1-\varrho)(1-\varrho^2)^2}$$
$$- \frac{1}{2000} \Sigma \frac{\varrho^{-n}(34\varrho + 98\varrho^2 - 42\varrho^3 - 114\varrho^4)}{(1-\varrho)(1-\varrho^2)^3}.$$

With the same notations as in the preceding case, we obtain

$$\Sigma_0 = 4, \quad \Sigma_1 = -1, \quad \Sigma_2 = -1, \quad \Sigma_3 = -1, \quad \Sigma_4 = -1,$$

and generally

$$\Sigma_{5k} = 4, \quad \Sigma_{5k+1} = -1, \quad \Sigma_{5k+2} = -1, \quad \Sigma_{5k+3} = -1, \quad \Sigma_{5k+4} = -1.$$

According to the values

$$n = 5p, \ 5p+1, \ 5p+2, \ 5p+3, \ 5p+4$$

we find

$$\Sigma \frac{\varrho^{-n}}{(1-\varrho)(1-\varrho^2)} = \frac{1}{5} \Sigma \varrho^{-n}(1-\varrho^3)(1-\varrho^4) = \frac{1}{5} \Sigma \varrho^{4n}(1+\varrho^2-\varrho^3-\varrho^4) =$$
$$= \frac{1}{5}(\Sigma_{4n} + \Sigma_{4n+2} - \Sigma_{4n+3} - \Sigma_{4n+4}) = (1 \ \ 0, \ 1, \ -1, \ -1)$$

$$\Sigma \frac{\varrho^{-n}(2\varrho + 6\varrho^2)}{(1-\varrho)(1-\varrho^2)^2} = \frac{1}{5}(2\,\Sigma_{4n+3} + 4\,\Sigma_{4n+4} - 6\,\Sigma_{4n}) = (-6, \ 0, \ 0, \ 2, \ 4)$$

$$\Sigma \frac{\varrho^{-n}(34\varrho + 98\varrho^2 - 42\varrho^3 - 114\varrho^4)}{(1-\varrho)(1-\varrho^2)^3} = \frac{1}{25}(-538\,\Sigma_{4n} - 8\,\Sigma_{4n+1} +$$
$$+ 32\,\Sigma_{4n+2} + 162\,\Sigma_{4n+3} + 352\,\Sigma_{4n+4}) = \frac{2}{5}(-269, \ -4, \ 16, \ 81, \ 176)$$

and therefore

$$W_5 = \frac{1}{48000}(24n^2 + 840n + 6300)(1, \ 0, \ 1, \ -1, -1)$$
$$+ \frac{1}{48000} \cdot 48n(3, \ 0, \ 0, \ -1, \ -2)$$
$$+ \frac{1}{48000}\frac{48}{5}(269, \ 4, \ -16, \ -81, \ -176).$$

In the same way we obtain, according to

$$n = 10p, \quad 10p+1, \quad 10p+2, \ldots \quad 10p+9$$

$$W_{10} = \frac{1}{48000}(120n + 1800)(-1, \ 0, \ 1, \ 3, \ 3, \ 1, \ 0, \ -1, \ -3, \ -3)$$
$$+ \frac{1}{48000} \cdot 60(5, \ 16, \ 21, \ 29, \ 19, \ -5, \ -16, \ -21, \ -29, \ -19)$$

and according to

$$n = 20p,\quad 20p+1, \ldots \quad 20p+19$$

$$W_{20} = \frac{1200}{48000}(-5,\ -8,\ -5,\ -7,\ -5,\ \ 3,\ 0,\ \ 3,\ \ 5,\ \ 7,$$
$$5,\ \ 8,\ \ 5,\ \ 7,\ \ 5,\ \ 3,\ 0,\ -3,\ -5,\ -7).$$

From the preceding formulae we may deduce the several results for $n = 10p + q$ $(q = 0, 1 \ldots 19)$

To illustrate this, taking $n = 10p + 7$, we obtain

$$W_1 = \frac{1}{48000}(n^4 + 76n^3 + 1901n^2 + 17366n + 41930\tfrac{1}{10})$$

$$W_2 = \frac{1}{48000}(\qquad - 15n^2 - 570n - 4687\tfrac{1}{2})$$

$$W_4 = \frac{1}{48000}(\qquad - 600)$$

$$W_5 = \frac{1}{48000}(\qquad 24n^2 + 840n + 6300)$$

$$+ \frac{1}{48000}(\qquad - 153\tfrac{3}{5})$$

$$W_{10} = \frac{1}{48000}(\qquad - 120n - 1800)$$

$$+ \frac{1}{48000}(\qquad - 1260)$$

$$W_{20} = \frac{1}{48000}(\qquad 3600)$$

and finally

$$\Sigma W_{\alpha_i} = \frac{1}{48000}(n^4 + 76n^3 + 1910n^2 + 17516n + 43329)$$

$$= \tfrac{1}{3}(p+1)(p+2)(10p^2 + 22p + 9).$$

That this value is an integer may be easily seen by writing

$$\Sigma W_{\alpha_1} = \tfrac{1}{3}p(p+1)(p+2)(10p+22) + 3(p+1)(p+2).$$

Comparing the general formula of Sylvester

$$W_{\alpha_i} = \Sigma \mathcal{E} \frac{\varrho^n e^{nt}}{(1-\varrho^{a_1} e^{-a_1 t})(1-\varrho^{a_2} e^{-a_2 t}) \ldots (1-\varrho^{a_r} e^{-a_r t})} \frac{t}{((t))}$$

with the formula (7), it is evident that in his formula ϱ^n ought to be replaced by ϱ^{-n}. This makes no difference in the values W_1 and W_2. In the example treated by Sylvester

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 = n$$

the values of W_3, W_4, W_5 and W_6 however want correction.

Physiology. — *"About the influence of radio-active elements on the development".* By Prof. A. J P. VAN DEN BROEK. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated at the meeting of November 29, 1919).

One of the elements, composing the living protoplasma, potassium, is radio-active. The investigations of ZWAARDEMAKER and his pupils about the signification of potassium in the organism have proved that, chiefly by streaming experiments of the isolated frog-heart, potassium can be substituted by an aequiradio-active quantity of any other radio-active element.

On account of this ZWAARDEMAKER concludes [1]: "die Radioaktivität und keine andere Eigenschaft der sich gegenseitig vertretenden Atome erfullt die fur die Automatie notwendige Bedingung" (l.c. pag. 49). Next to this substitute ZWAARDEMAKER has fixed the attention on a second fact, viz. an antagonism between different groups of radio-active elements.

The antagonism is expressed in the following scheme:

$$\left.\begin{array}{l}\text{Potassium}\\ \text{Rubidium}\\ \text{(Cäsium)}\end{array}\right\} \begin{array}{c}\rightarrow\\ \leftarrow\end{array} \left\{\begin{array}{l}\text{Uranium}\\ \text{Thorium}\\ \text{Radium}\\ \text{Ionium}\\ \text{Niton (Emanation)}\end{array}\right.$$

The uranium substitutes the potassium in certain experiments; but the elements together neutralize each others' effect.

These investigations raise the question if it were possible to substitute the potassium during the development by another radio-active element. I tried to obtain an answer on this question by experimental investigation. I will give a short account of the experiments taken and of the results which I obtained. The experiments were taken with frog-eggs and carried out in the following way.

After the fecundation (in the laboratory) the egglump was parted

[1] H ZWAARDEMAKER, Die Bedeutung des Kaliums im Organismus. Pflügers' Archiv Bd. 173.

immediately in equal quantities, these are placed in liquids containing potassium or in which different quantities of uranium-salt had been dissolved. There is practically no potassium found in the Utrecht water, thus, as much care as possible was taken, to bring up the uranium-tadpoles, with food containing no potassium, while the tadpoles that had been put in the liquid containing potassium, got as much ordinary (animal) food, as was possible. Rice, boiled in distilled water was given as food without potassium. In the first year the uranium-tadpoles were brought up in glass-bowls; to prevent the dissolving of potassium from the glass, the tadpoles were brought up in quartz bowls in the second and third years of the experiments.

I. A first series of experiments consisted in adding to the water a certain quantity of uranium-nitrate $UO_2(NO_3)_2$.

The potassium which was present in the eggs was compensated by giving $4\frac{1}{2}$ mgr. uranium-nitrate pro liter; moreover another 12,5; 25 mgr. (altogether 17 and $29\frac{1}{2}$ mgr.) and 50 mgr. pro liter.

At the same time as the tadpoles in these liquids, others were brought up in ordinary water, with piscidine[1]) and rice.

The eggs were laid on April 14th. On May 29th the piscidine-tadpoles are long 11—12 m.m.[2]) and have hind-limbs; on June 6th there are tadpoles of 14.2 m.m., which have hind- and front limbs; then metamorphosis and tailreduction regularly follow. On June 26th only a few tadpoles of 8—12 m.m. remain, these are not yet metamorphosed.

The rice-tadpoles develop far more slowly, and it now appears that at the same time the stage of development of the tadpoles differs considerably. On June 26th I found tadpoles of 6—12 m.m.; then the development slowly continues; on August 6th I found the first complete metamorphosis, (the tail has disappeared) the length being 15 m.m. On October 10th another tail-reduction takes place. Some tadpoles do not metamorphose, they become quite big animals, viz. 16—17 m.m., with only short hind-limbs.

As to the uranium-tadpoles they remain backwards and develop far more slowly, which are the most striking characteristics. The following table informs us about the size.

From this table it appears, that the development takes place considerably more slowly than with the animals under control, also the sizes are smaller. Although the differences seem little, the tad-

[1]) Piscidine is a preparation containing dried and powdered fish.

[2]) In these and all following measurements the length must be considered as taken from the top of the head to the beginning of the tail.

	$4^1/_2$ m.gr. uranium-nitrate pro L.	17 m.gr. uranium-nitrate pro L.	$29^1/_2$ m.gr. uranium-nitrate pro L.
June 26th	5.5—10 m.m.	5— 9.8 m.m.	5 —7.5 m.m.
July 10th	6 —11 m.m.	5—10 m.m.	4.5—9 m.m.
July 30th	7 —11 m.m.	6— 9.5 m.m.	5 —10 . m.m.
August 14th	9.5 m.m.		
August 24th	10.5—11.5 m.m.		
September 3	12 m.m. } front-limbs beginning of tail-reduction	10.2 m.m. } front-limbs beginning of tail-reduction	
September 14th	11 m.m.	8.4 m.m.	8.5—9.6 m.m. (frontl.)
November 27th	10 m.m.	11 m.m.	8—11.8 m.m.

poles are all the smaller in proportion as the quantity uranium-nitrate is greater. The following gives a more detailed account.

a. $4^1/_2$ mgr. uranyl-nitrate pro Liter.

On June 26th and July 10th the measurements are as mentioned; the biggest tadpoles only have an indication of hind limbs. On July 30th the biggest tadpoles have hind limbs which do not lie any more straight along the tail, but which are abducted. On August 14th it was the first time that front limbs broke through with one tadpole of 9.5 m m ; on Sept 3rd. and Sept. 14th the first tail-reductions were observed. On November 27th the last tadpole of 10 mm. with small hind limbs, showing signs of diminishing vitality, was killed and fixed.

b. 17 mgr. uranyl nitrate pro Liter.

From April 14th till July 10th the growth makes very little progress, the maximum length only being 10 mm ; tiny little points of hind limbs are present, and it is not before July 30th that one tadpole has hind limbs in abduction; on September 3rd. the first tail-reduction is observed. On November 27th the last living tadpole has a length of 11 mm.

c. $29^1/_2$ mgr. uranyl-nitrate pro Liter.

The growth still goes more slowly than the preceding ones. On July 30th I see tadpoles with points of hind limbs The beginning of tail-reduction was observed for the first time on September 7th; after that regularly; the size of the tadpoles then being 9,6—11,5 mm. On November 27th the last two tadpoles of 8.5 and 11 mm. were fixed.

II. In different periods of the development tadpoles were taken out of the uranium solutions and put into ordinary water. It then appeared that almost immediately the development took place much more quickly than with the tadpoles that remained in the uranium-solution; it must be well understood that the piscidine-tadpoles grew more quickly than the rice-tadpoles.

III. A number of tadpoles being in a young stage of development (on April 30th) were put into a liquid, containing a quantity of uranium-salt and an aequiradio-active quantity of potassium. In this liquid the tadpoles grew almost as quickly as the animals under control; on July 15th I already found one tadpole with front- and hind limbs; several other tadpoles soon get them as well; the smallest ones are 8—9 m.m. So these tadpoles grew more quickly than the uranium-tadpoles.

It may be said, that as a general result of the experiments made, tadpoles in a medium containing a radio-active substance antagonistic to potassium, grow and metamorphose less quickly than in a medium only containing potassium. The food given to the tadpoles, is not the cause of this lingering, for in regard to the tadpoles in ordinary water, fed with rice, the uranium-tadpoles also remain backward in development.

The first question that can be put is, if the tadpoles have taken uranium next to or instead of potassium.

This question cannot yet be affirmed. From an investigation voluntarily undertaken by Prof. RINGER, it appeared that no uranium could be demonstrated in the tadpoles.

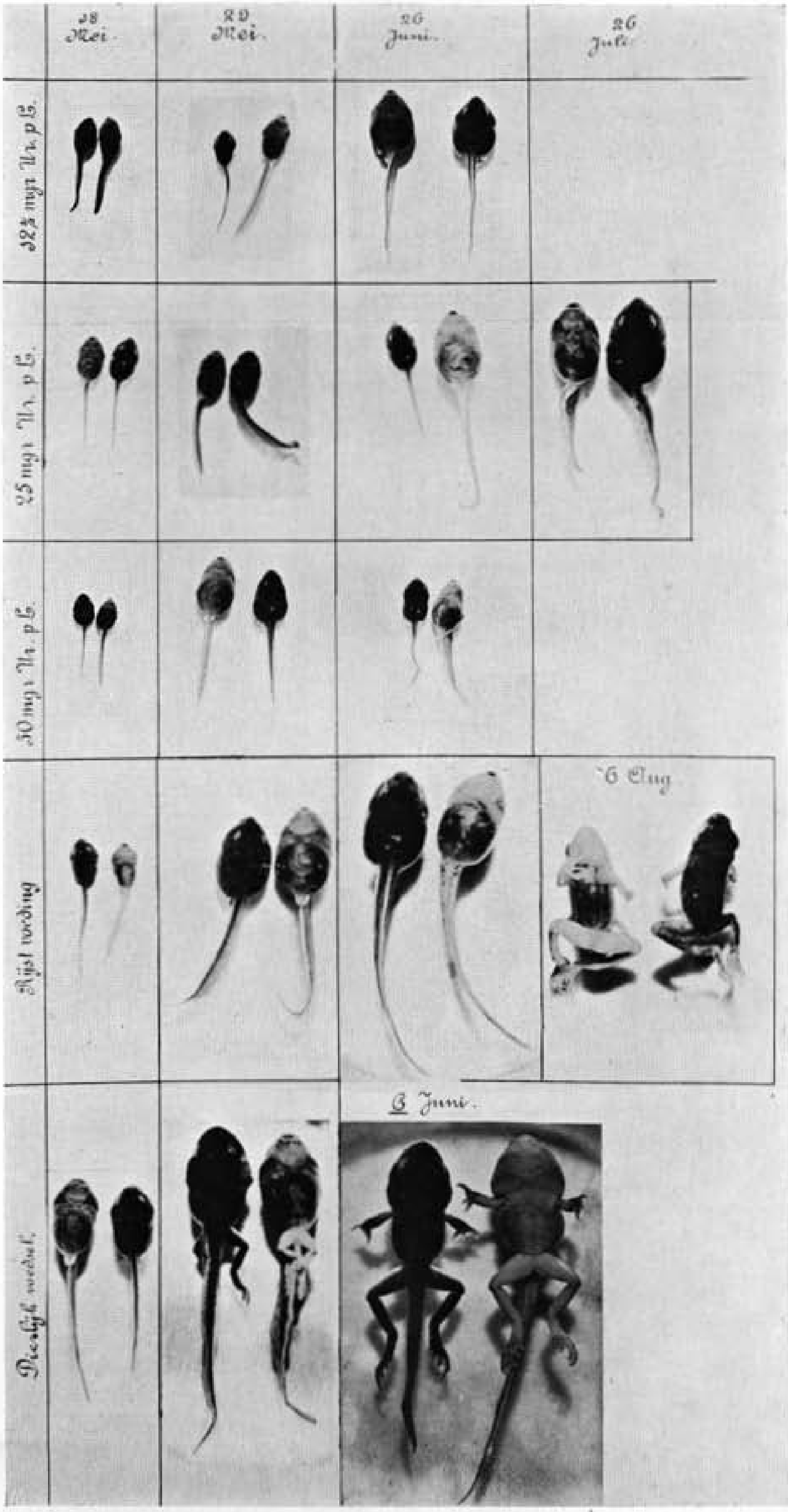
In another respect this experiment had a very important result though the percentage of potassium found in the tadpoles fed with piscidine, was in two cases 0.76 and 0.82 °/₀ from the dried tadpoles. The uranium tadpoles had evidently taken the minimal potassium quantity and even 0.91 °/₀ kalium was found. The potassium-percentage was only 0 49 °/₀ of those tadpoles, which had been in water containing 17½ mgr. $UO_2(NO_3)_2$ and 50 mgr. KCl pro Liter (being aequiradio-active quantities). It here makes the impression that the presence of ± aequiradio-active quantities of the antagonistic substances has thrown obstacles in the way of taking in potassium. Quite remarkable it is though that the concerned tadpoles should only stay little behind in growth to those, which had been normally brought up. [1])

The remaining backward in growth might be imputed to a possible poisoning caused by the uranium salt.

In 1919 I have made a single experiment on this subject. On May 6th I had five bowls containing 4 L. water each, and resp. 0, 2½, 5, 7½ and 10 mgr. uranium-salt; every single bowl contained 75 tadpoles as well; on June 19th resp. 51, 38, 24, 20 and 20 tadpoles

[1]) Prof. RINGER fixes the attention on the fact, that the chemical investigation, viz. the investigation of the radio activity of the dried tadpoles, took place with so small a quantity of the dried tadpoles, that it is desirable this investigation should be repeated, namely about the absence of uranium.

A. J. P. v. d. BROEK: "On the influence of radio-active elements on the development".



On this plate are shown the minimum- and maximum-length of tadpoles from different solutions of uraniumsalt at various data. The fourth row from above are tadpoles fed with rice, the fifth (last) row are tadpoles fed with piscidine.

were still alive in these bowls; it thus seems as if the tadpoles die sooner and in greater number, while being in higher concentration. This result does not agree with HIRSCH's [1]) results, who supposes that the quickest development takes place in the concentration, nearest to the concentration in which life is impossible. In this case one should expect a quicker development in the higher concentrations than in the lower-ones, but this has not been proved.

Moreover HIRSCH has every time extended his investigation over a very short period (7 days) which does not seem desirable, if one takes into consideration the great variability in the development of tadpoles. Although apparently a poisoning in the solutions with a greater quantity of uranium-salt does not seem impossible, the fact that many tadpoles develop, and that they live quietly on, in much stronger uranium solutions, and the failing of uranium in the body, might plead against the poisoning of the uranium-salt as a cause of the more slow development.

Microscopic. investigation of series sections from some uranium-tadpoles, in comparison with normal tadpoles of the same size, has not yet shown differences in structure or in degree of development of certain organs, which should be of importance for the growth.

[1]) E. HIRSCH, Die biologische Wirkung einiger Salze Zool Jahrbücher. Band 34.

Mathematics. — "*Ueber die endlichen topologischen Gruppen der Kugelfläche*". By B. VON KERÉKJÁRTÓ. (Communicated by Prof. L. E. J BROUWER).

(Communicated at the meeting of November 29, 1919).

Die vorliegende Arbeit gibt eine neue Herleitung des Resultates, dass die endlichen topologischen Transformationsgruppen mit invarianter Indikatrix der Kugelfläche mit den Gruppen der regulären Körper identisch sind, was nach dem BROUWERschen Grundsatz [1]), laut dessen die topologischen Gruppen mit den konformen homöomorph sind, aus dem die konformen Transformationsgruppen der Kugelfläche betreffenden, bekannten Satze folgt.

Wir betrachten eine Gruppe G von n topologischen, die Indikatrix erhaltenden Transformationen der Kugelfläche in sich. Eine willkürliche Transformation t von G ist nach dem *Rotationssatz* [2]) eine ν-periodische Drehung, die also zwei Fixpunkte P und Q hat; die Anzahl der mit P bei G äquivalenten Punkte ist $\frac{n}{\nu}$. Wir verbinden P mit Q durch einen Weg b, der seine bei den Potenzen von t entstehenden Bilder ausser in P und Q nicht trifft. Sei von P aus R der erste solche Punkt von b, dass $\widehat{PR}$ eines seiner bei G entstehenden Bilder ausserhalb P trifft. Wenn $R = Q$, so ist G mit der zyklischen Rotationsgruppe $1, t, t^2, \ldots. t^{\nu-1}$ identisch. Wenn aber $R \neq Q$, so kann auf dem Bogen $\widehat{PR}$ höchstens ein mit R äquivalenter Punkt R' liegen. Wenn auf $\widehat{PR}$ kein mit R äquivalenter Punkt liegt, so ist R bei einer Transformation von G invariant, sodass der Bogen $\widehat{PR}$ ein zwei nicht-äquivalente Fixpunkte von G verbindender, seine Bilder ausserhalb der Endpunkte nicht treffender Bogen c ist. Wenn aber auf $\widehat{PR}$ ein mit R äquivalenter Punkt R' liegt, der bei keiner Transformation von G ausser der Identität invariant ist, so betrachte man das System der Bilder des Bogens $\widehat{R'R}$ von b; es besteht aus einander nicht treffenden Jordanschen Kurven, da R zu genau zwei solchen Bogen gehört; ferner ist dieses System bei G invariant.

1) Diese Proceedings XXI, S. 1143 (29. März 1919).

2) Math. Ann. Bd. 80, S. 36.

Sei γ eine der genannten Kurven; da das innere, d. h. keinen Bildpunkt von P enthaltende Gebiet von γ bei jeder es invariant lassenden Transformation von G einer Potenz derselben Drehung unterworfen ist, so kann man R' mit dem im Innern von γ existierenden einzigen Fixpunkt S (der nicht mit Q zusammenfallen kann) durch einen seine Bilder nicht treffenden Weg verbinden, welcher mit dem Bogen $\widehat{PR'}$ von b zusammen einen seine Bilder ausserhalb der Fixpunkte von G nicht treffenden und zwei Fixpunkte verbindenden Weg c bildet.

Die Bilder von c zerlegen die Kugelfläche in Elemente; falls eines dieser bei einer Transformation von G invariant, also einer Drehung unterworfen ist, so kann man einen Fixpunkt seiner Grenze mit dem in seinem Innern liegenden einzigen Fixpunkt T durch einen seine Bilder nicht treffenden Weg d verbinden. Die sämtlichen Bilder von c und d ergeben zusammen ein bei G invariantes System H von folgender Beschaffenheit: 1. H zerlegt die Kugelfläche in Elemente, von denen je zwei äquivalent sind und jedes nur bei der Identität invariant ist; 2. jeder Fixpunkt von G liegt auf H; 3. jeder Punkt von H, der zu mehr als zwei Bogen von H gehört, ist ein Fixpunkt von G. Die Anzahl der Elemente, in welche H die Kugelfläche zerlegt, ist n; die Anzahl der nicht äquivalenten Fixpunkte ist 3, ihre gesamte Anzahl ist also, wenn ν_1, ν_2, ν_3 ihre Multiplizitäten bezeichnen, gleich $n\left(\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}\right)$; die Anzahl der Kanten jedes Elementes ist 4, also die gesamte Anzahl der Kanten $2n$. Mithin besteht nach dem Eulerschen Polyedersatz die Formel

$$n + \Sigma \frac{n}{\nu_i} - 2n = 2, \quad \text{oder} \quad \Sigma \frac{1}{\nu_i} = 1 + \frac{2}{n},$$

woraus sich die bekannten Lösungen ergeben [1].

Mittels der gleichen Methode werde ich die Brouwerschen Resultate [2] in bezug auf die endlichen Gruppen von topologischen Transformationen des Torus herleiten.

[1] Klein, Vorlesungen über das Ikosaeder, S. 119.

[2] C. R. t. 168, S. 845 (28. April 1919).

Chemistry. — *"Catalysis"* — Part VII — *Notes on Catalysis in heterogeneous systems*. By NIL RATAN DHAR. (Communicated by Prof. ERNST COHEN).

(Communicated at the meeting of November 29, 1919).

1. It has been known for a long time that violet chromic chloride is practically insoluble in water, but in presence of reducing agents solution takes place due to the transformation into the soluble modification.

Anhydrous ferric sulphate dissolves slowly in water at the ordinary temperature, in other words, it may be said to have a small velocity of solution. I have found that reducing agents like stannous chloride, ferrous sulphate, sulphurous acid etc., markedly accelerate the velocity of solution of ferric sulphate in water.

In the case of chromic chloride, it is assumed that the reducing agent first reduces the insoluble chromic chloride to chromous chloride, and the original chromous chloride is transformed into soluble chromic chloride. The newly formed chromous chloride then acts on the insoluble chromic chloride as before.

The difficulty of an explanation like this is that we assume that the reducing agent acts on the solid chromic chloride and reduces it; from our experience in heterogeneous systems, we know how difficult it is to reduce a solid substance rapidly with a solution of a reducing agent.

Moreover, in the case of ferric sulphate we do not know two varieties of the salt as in the case of chromic chloride.

It is difficult to assume that ferrous sulphate would reduce ferric sulphate, for we know that a mixture of ferric and ferrous salts can exist unchanged for an indefinite period in absence of oxygen.

So we can say with OSTWALD (Grundlinien der anorganischen Chemie, Leipzig 1900; A. FINDLAY's trans. 603, 1902) that a sufficient explanation of these actions is still wanting.

2. The action of nitric acid on the metals generally, is somewhat complex, because the main reaction is complicated by side or concurrent, and by consecutive reactions. These again depend not only upon the particular metal under consideration, but also on the

concentration of the acid, the temperature and the concentration of the products of the reaction accumulating in the solution.

MILLON (Compt. rend. 1842, **14**, 904) and VELEY (Phil. Trans. 1891 A, **182**, 279) have shown that metals like copper, silver, mercury and bismuth have no action on cold dilute nitric acid unless a trace of nitrous acid is present. The nitrous acid may be present in the nitric acid as an impurity; it may be formed by the incipient decomposition of nitric acid when it is warmed.

According to VELEY, therefore, the dissolution of copper in nitric acid proceeding: $Cu + 3\ HNO_3 = Cu\ (NO_3)_2 + HNO_2 + H_2O$, is a resultant of a series of consecutive reactions: $Cu + 4\ HNO_2 = Cu\ (NO_2)_2 + 2\ H_2O + 2\ NO$; followed by $Cu\ (NO_2)_2 + 2\ HNO_3 = Cu\ (NO_3)_2 + 2\ HNO_2$.

The small trace of nitrous acid thus acts as a catalytic agent; nitrous acid is continuously produced and continuously decomposed according to the following equilibrium:

$$3\ HNO_2 \rightleftarrows HNO_3 + 2\ NO + H_2O.$$

Similar results have been obtained by RÂY (Trans. Chem. Soc. 1911, **99**, 1012) in the case of mercury and by STANSBIE (J. Soc. Chem. Ind. 1913, **32**, 311) in the case of silver.

Now MILLON (loc. cit.) and VELEY (loc. cit.) have pointed out that the presence of ferrous sulphate, "which removes the nitrous acid as fast as it might be formed" serves to prevent the chemical change between nitric acid and the metals.

But I have observed that ferrous sulphate exerts an accelerating influence on the complete dissolution of copper in 20°/$_0$ nitric acid at 18°. This result being different from those of previous investigators, I thought it worth while to observe the effect of both ferrous and ferric salts and various other substances on the complete dissolution of copper in excess of 20°/$_0$ nitric acid.

Equal lengths of copper wire of uniform sectional area were placed into test tubes and covered with an excess of 20°/$_0$ nitric acid. The mean temperature of the experiments was 18° and the tubes containing equal volumes of nitric acid and equal weights of copper wire were kept at rest. Weighed quantities of the solid substances used were added at the beginning of the experiments. The whole of the copper wire dissolved in about 30 minutes and the exact time of dissolution was noted. In order to get exactly comparable results one test tube was always set apart for a blank parallel experiment.

It has been found that the following substances exert an accelerating effect on the complete dissolution of copper in 20% nitric acid :

ferrous sulphate, ferrous chloride, ferric sulphate, ferric chloride, ferric nitrate, lead sulphate, lead nitrate. lead acetate, copper nitrate, copper chloride, barium nitrate, thallium nitrate, lithium nitrate, sodium nitrite, manganese chloride, chromic chloride, arsenious oxide, strychnine sulphate, ethylene bromide, carbon tetrachloride, hexachlorobenzene, phthalic anhydride etc.

On the other hand, the following substances have a retarding effect: hydrogen peroxide, potassium chlorate, potassium permanganate, chromic acid, sodium nitrate, ammonium nitrate, manganese nitrate, thorium nitrate, sodium sulphite, titanic acid, molybdic acid, ammonium persulphate, mangenese sulphate, cobalt chloride, copper acetate, copper sulphate, calcium nitrate, tartaric acid, ether, urea, acetic anhydride, benzoic anhydride etc.

In a foregoing paper of this series (Trans. Chem. Soc. 1917, **111**, 707) I have shown that sulphuric acid in small concentration is an accelerator, whilst in large concentrations it is a retarder in the oxidation of oxalic acid by chromic acid. Similar results have been obtained in the action of nitric acid on copper. The following substances in very small concentration exert a slight accelerating effect, whilst in large concentrations they have retarding effect:

Zinc chloride, nickel chloride, cobalt nitrate, aluminium nitrate, potassium chloride, strontium nitrate, cadmium nitrate, magnesium chloride etc.

When the concentration is very small, the effect of potassium nitrate, uranium nitrate, citric acid, potassium dichromate etc. is practically nil, but in concentrated solutions they are all retarders.

The effect of monochloracetic acid is very peculiar. In small concentrations, it is a feeble accelerator and in concentrated solutions it has a retarding effect, which instead of increasing, decreases with increase of concentration. A similar phenomenon has already been observed in the case of the oxidation of formic acid by chromic acid in presence of manganese chloride (*loc. cit.* p. 726).

It is practically impossible to give a complete explanation of these results, they being so diverse.

Ferrous sulphate and ferrous chloride behave as marked accelerators. It would appear that the acid nucleus in this particular instance, plays no part. A part of the ferrous ion reduces the nitric acid to nitric oxide and passes into the ferric state. The nitric oxide dissolves in the ferrous salt solution forming the unstable substance $FeNO^{\circ\circ}$. The dissolved nitric oxide then reduces a part of the nitric acid according to the following equation: $HNO_3 + 2\,NO + H_2O \rightleftarrows 3\,HNO_2$.

It is quite possible that some nitrous acid is produced by the direct reduction of nitric acid by ferrous ions. The formation of nitrous acid either by the direct reduction of nitric acid by ferrous salts or by the indirect reduction through the intervention of nitric oxide is proved by the following experiment. If nitric acid of the strength used in this research be taken in a test tube and a crystal of ferrous ammonium sulphate or ferrous sulphate be added to it, almost immediately the crystal is covered with the deep brown $FeNO^{\circ\circ}$ ion and a little nitric oxide also escapes. If urea crystals are now added, they are immediately oxidized with the evolution of carbon dioxide and nitrogen, indicating the presence of nitrous acid. So in presence of ferrous salts, nitrous acid, which is the active substance in the action of nitric acid on copper, is formed when we have an excess of nitric acid. This explains the accelerating influence of ferrous salts in the complete dissolution of copper in 20 % nitric acid.

As a matter of fact, the accelerating effect of ferrous salts is slightly greater than the accelerating effect of sodium nitrite on the dissolution of copper in nitric acid. The greater the concentration of the ferrous salt, the greater is the acceleration.

Ferric sulphate, ferric nitrate and ferric chloride exert a marked accelerating effect, though their activity is slightly less than that of sodium nitrite and the accelerating effect is proportional to the concentration of the ferric salt. It would appear that the acid nucleus in this case also, plays no part. The explanation of this activation seems to lie in the reduction of ferric salts by the nitric oxide which is a product of the chemical change between nitric acid and copper. The ferrous salt, which may thus be formed, will reduce a part of the nitric acid into nitrous acid, which activates the action of nitric acid on copper. It seems plausible that a part of the ferric salt would be reduced to the ferrous state by the metallic copper. It is well known that when a solution of a ferric salt is shaken with metallic copper, the ferric salt is partly reduced to the ferrous state and the copper is oxidized to the cupric salt and an equilibrium is set up: —

$$2\,FeCl_3 + Cu \rightleftarrows 2\,FeCl_2 + CuCl_2$$

The ferrous salt thus formed reduces the nitric acid to nitrous acid, which accelerates the action of nitric acid on copper.

In a similar way the accelerating effect of arsenious oxide, strychnine sulphate, phthalic anhydride etc. may be explained on the basis of the formation of nitrous acid by the action of these reducing agents on the nitric acid.

The retarding effect of the oxidizing agents like, H_2O_2, $KMnO_4$,

$H_2Cr_2O_7$, $KClO_3$, $(NH_4)_2S_2O_8$ etc. and of the reducing agents likeurea, sodium sulphite etc. is certainly due to the destruction of the nitrous acid as soon as it is formed. When the experiment was performed in such a condition so as to cover the copper wire with solid urea, the reaction became very slow, but it did not stop altogether.

It is very difficult to explain the difference in the behaviour of the nitrates on the oxidation of copper by nitric acid. Lithium nitrate is an accelerator, whilst sodium and potassium nitrates are retarders; from analogy we should expect calcium nitrate to have an accelerating effect, but as a matter of fact, both calcium and strontium nitrates are retarders, whilst barium nitrate is an accelerator.

Rennie and Cook (Trans. chem. Soc. 1911, **99**, 1035) have found that the accelerating or retarding effects of the nitrates of K, Rb, Cs were functions of the temperature and of the concentration of the acid.

Higley (Amer. chem. Jour. **17**, 18 (1895)) has shown that both NO_2 and N_2O_3 are the products of the reaction between copper and nitric acid. Evidently in the solution, we should consider the following equilibria:

$$3\,HNO_2 \rightleftarrows 2\,NO + HNO_3 + H_2O \quad . \; . \; . \; . \quad (1)$$

$$N_2O_4 + H_2O \rightleftarrows HNO_2 + HNO_3 \quad . \; . \; . \; . \; . \quad (2)$$

$$3\,NO_2 + H_2O \rightleftarrows 2\,HNO_3 + NO \quad . \; . \; . \; . \; . \quad (3)$$

Lewis and Edgar (J. Amer. chem. Soc. 1911, **33**, 292) have shown that in equilibrium (1) there is a change in the equilibrium constant with the concentration of nitric acid. It seems probable that nitrates may affect one or more of these equilibria and change the concentration of nitrous acid, which being the activating agent.

In this connection, it is interesting to observe that several reactions, in which nitric acid is the oxidizing agent, are autocatalytic. As for example, the actions of nitric acid on metals like Copper, Silver, Bismuth, Mercury etc., on starch, on sugar, on arsenious oxide, on hydrogen iodide (Eckstädt, Zeit. anorg. Chem. 1901, **29**, 51), on nitric oxide (Lewis and Edgar, *loc. cit.*) etc. become more pronounced as the chemical change proceeds.

The explanation is not far to seek. The nitrous acid is the active substance and its concentration and hence the reaction velocity increase with the progress of the chemical change. In all these cases I have found that the chemical change becomes more rapid when a nitrite is added at the commencement of the reaction.

It has been observed that the chemical change between nitric acid and copper may be practically stopped by agitating vigorously the tube containing copper and nitric acid, because the nitrous acid cannot accumulate round the copper.

Summary and Conclusion:

1. The velocity of solution of anhydrous ferric sulphate can be increased by the presence of sulphurous acid, stannous chloride, ferrous sulphate etc. No satisfactory explanation of reactions of this type is forthcoming.

2. The action of nitric acid (20 %) on copper has been studied in the presence of various substances and it has been observed that when the nitric acid is in excess and the whole of the copper is made to dissolve, ferrous and ferric salts exert a marked accelerating effect. In the light of the present investigation, the view hitherto accepted as regards the part played by ferrous salts in destroying nitrous acid, has to be modified. As a matter of fact, it has been proved that nitrous acid, which is the active substance in this reaction, is formed by the action of nitric acid on ferrous salts. Oxidizing agents like H_2O_2, $KMnO_4$, $H_2Cr_2O_7$ etc. destroy the nitrous acid and hence retard the change.

Out of the 56 substances, the effect of which was investigated, 22 act as accelerators and 22 exert a retarding influence in all concentrations; whilst 8 of them are slight accelerators in small concentrations and are retarders in concentrated solutions. Four of these 56 substances have been found to be neutral in small and retarders in large concentrations.

My best thanks are due to the VAN 'T HOFF Fund Committee for a grant for this research.

Chemical Laboratory, Muir Central College, Allahabad, India.

Chemistry. — *"Notes on Cobaltammines"*. By NIL RATAN DHAR. (Communicated by Prof. ERNST COHEN).

(Communicated at the meeting of November 29 1919).

In two previous investigations (Zeit. Anorg. Chem. 1913, **80**, 43; **84**, 224) I had occasion to study certain properties of the cobaltammines. This note is the result of the continuation of my previous work.

1. Let us consider the following series of compounds

$$[Co(NH_3)_6]Cl_3,\ \left[Co\begin{matrix}(NH_3)_5\\(NO_2)\end{matrix}\right]Cl_2,\ \left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]Cl,\ \left[Co\begin{matrix}(NH_3)_3\\(NO_2)_3\end{matrix}\right],$$

$K\left[Co\begin{matrix}(NH_3)_2\\(NO_2)_4\end{matrix}\right]$ and $K_3\,[Co(NO_2)_6]$. For the preparation of

$$\left[Co\begin{matrix}(NH_3)_5\\(NO_2)\end{matrix}\right]Cl_2,\ \left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]Cl,\ \left[Co\begin{matrix}(NH_3)_3\\(NO_2)_3\end{matrix}\right] \text{ and } K\left[Co\begin{matrix}(NH_3)_2\\(NO_2)_4\end{matrix}\right]$$

the general method of procedure is to mix a cobalt salt, ammonium chloride, ammonium hydroxide and a nitrite; by this a complex cobaltous compound is formed which is turned into the stable cobaltic compound by oxidation. The amount of a certain compound which will be formed, depends on the concentration of the reacting substances and on the solubility of the resulting complex compound. If the concentration of the nitrite in the solution is large in comparison with the concentrations of ammonium hydroxide and ammonium chloride, we should expect that several (NO_2) groups would enter the complex.

It has been known from a long time that aquopentammine salts can be converted into the corresponding hexammine salts by heating the aquo compound with ammonia in a sealed tube or in a bottle under pressure.

I found that if $\left[Co\begin{matrix}(NH_3)_5\\(NO_2)\end{matrix}\right]Cl_2$ is warmed with a dilute solution of sodium or potassium nitrite, we get mainly $\left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]Cl$, which could be purified by recrystallisation.

In a similar way croceo cobalt chloride $\left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]Cl$ can be converted into $\left[Co\begin{matrix}(NH_3)_3\\(NO_2)_3\end{matrix}\right]$ by warming it with a dilute solution of a nitrite, whilst $\left[Co\begin{matrix}(NH_3)_3\\(NO_2)_3\end{matrix}\right]$ can be converted into $K\left[Co\begin{matrix}(NH_3)_2\\(NO_2)_4\end{matrix}\right]$ by warming $\left[Co\begin{matrix}(NH_3)_3\\(NO_2)_3\end{matrix}\right]$ with a concentrated solution of potassium nitrite, ammonia escaping from the solution.

I tried to prepare the compound $K_2\left[Co\begin{matrix}(NH_3)_5\\(NO_2)\end{matrix}\right]$, which is still unknown, by warming $K\left[Co\begin{matrix}(NH_3)_2\\(NO_2)_4\end{matrix}\right]$ with potassium nitrite, but was unsuccessful.

On the other hand, one can convert $K\left[Co\begin{matrix}(NH_3)_2\\(NO_2)_4\end{matrix}\right]$ into $\left[Co\begin{matrix}(NH_3)_3\\(NO_2)_3\end{matrix}\right]$ and $\left[Co\begin{matrix}(NH_3)_3\\(NO_2)_3\end{matrix}\right]$ into $\left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]Cl$ by warming the compound in question with a mixture of ammonium chloride and ammonium hydroxide.

In all these cases, ammonium salts are used along with ammonium hydroxide, and their function is to suppress the ionisation of the base and form undissociated NH_4OH, which is in equilibrium with NH_3. The NH_3 then enters into the complex molecule.

2. If a fairly concentrated solution of aquopentammine cobaltic chloride $\left[Co\begin{matrix}(NH_3)_5\\(H_2O)\end{matrix}\right]Cl_3$ is left, it slowly gives a precipitate of the corresponding purpureo salt $\left[Co\begin{matrix}(NH_3)_5\\(Cl)\end{matrix}\right]Cl_2$

$$\left[Co\begin{matrix}(NH_3)_5\\(Cl)\end{matrix}\right]Cl_2 + H_2O \rightleftarrows \left[Co\begin{matrix}(NH_3)_5\\(H_2O)\end{matrix}\right]Cl_3.$$

This is a case of equilibrium in solution and the purpureo salt being much less soluble comes out as a precipitate.

If we start with a solution of purpureo cobalt chloride $\left[Co\begin{matrix}(NH_3)_5\\(Cl)\end{matrix}\right]Cl_2$ and add ammonium hydroxide and warm the mixture, we get the aquopentammine salt $\left[Co\begin{matrix}(NH_3)_5\\(H_2O)\end{matrix}\right].Cl_3$ in solution, and this is the usual method of preparation of the aquo salt.

I find that the ammonium hydroxide has only a catalytic effect on the hydrolysis of the purpureo salt into the aquopentammine salt. A solution of the purpureo chloride takes up a molecule of water and passes into the aquo salt very slowly even at the ordinary temperature. This hydrolysis is markedly accelerated by the presence of hydroxyl (OH′) ions. The greater the concentration of the hydroxide ions, the greater is the acceleration. The study of the reaction velocity of this hydrolysis may serve as a means of determining the concentration of hydroxide ions in a dilute solution of a base. Thus if we make a solution of the purpureo salt and add a few drops of a dilute solution of potassium hydroxide, the purple colour changes and becomes rose in a few minutes; but with a weak base like ammonium hydroxide the colour change takes a long time. This explanation may be true in the case of hydrolysis with the corresponding compounds of chromium and platinum. There is evidence to show that in some other cases of hydrolysis by alkali, the action of the hydroxide ions is catalytic. The decomposition of sodium chloracetate by alkali is a case in point (SENTER, Trans Chem. Soc. 1907, **91**, 473).

One can get the hydroxides of the cobaltammines in solution by treating the corresponding halide with moist silveroxide:

$$\left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]Cl + AgOH = \left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]OH + AgCl$$

The solution slowly decomposes even at the ordinary temperatures. The hydroxides of the other members of this series can also be prepared by this double decomposition. These hydroxides turn phenolphtalein pink and electric conductivity measurements show that they are strong bases of the type of sodium hydroxide.

But one cannot prepare the hydroxide from purpureo cobalt chloride $\left[Co\begin{matrix}(NH_3)_5\\Cl\end{matrix}\right]Cl_2$ by double decomposition with silver oxide. The explanation becomes simple on the light of the catalytic effect of hydroxide ions on the hydrolysis of purpureo salts into the aquo compounds. The hydroxide ions set free by the double decomposition act catalytically on the purpureo salt and actually one gets the aquopentammine hydroxide $\left[Co\begin{matrix}(NH_3)_5\\(H_2O)\end{matrix}\right](OH)_3$, which is stable in alkaline solution, (compare URBAIN et SÉNÉCHAL, Chimie des complexes, p. 280, "Les sels purpureo ne donnent pas une réaction de ce genre").

Summary and Conclusion.

1. The principle of the preparation of the cobaltammines is guided by the law of mass action and thus depends on the concentration

of the reacting substances. One can substitute a nitro (NO_2) group in a compound by the group (NH_3) on warming it with a mixture of ammonium hydroxide and a ammonium salt and on the other hand, NH_3 is replaced by NO_2 when the salt is warmed with a nitrite solution.

2. $$\left[Co\begin{matrix}(NH_3)_5\\Cl\end{matrix}\right]Cl_2 + H_2O \rightleftarrows \left[Co\begin{matrix}(NH_3)_5\\(H_2O)\end{matrix}\right]Cl_3$$

This hydrolysis reaction is catalytically accelerated by the presence of OH′ ions and the velocity is proportional to the concentration of hydroxide ions.

$$\left[Co\begin{matrix}(NH_3)_5\\(NO_2)\end{matrix}\right](OH)_2,\ \left[Co\begin{matrix}(NH_3)_4\\(NO_2)_2\end{matrix}\right]OH \text{ etc.}$$

are strong bases and can be prepared in solution. The base obtained from the purpureo cobalt chloride is the aquopentammine hydroxide $\left[Co\begin{matrix}(NH_3)_5\\H_2O\end{matrix}\right](OH)_3$.

Chemical Laboratory, Muir Central College, Allahabad, India.

Mathematics. — "*An involution of pairs of points and an involution of pairs of rays in space.*" By Dr. C. H. VAN OS. (Communicated by Prof. JAN DE VRIES.)

(Communicated at the meeting of September 29, 1918).

§ 1. *Introduction.* By several authors involutions have been treated, consisting of groups of points in the plane or in space. On the contrary involutions, consisting of groups of *straight lines*, do not seem to have been considered. In the following such an involution will be investigated. This involution is derived with the help of an involution of pairs of points, which is itself again connected with a certain *bilinear congruence of twisted cubics*.

The congruence in question $[\varrho^3]$ is formed by all the curves ϱ^3 which pass through two given points A_1 and A_2, and have three given straight lines a_3, a_4 and a_5 as bisecants. These curves are the moveable intersections of the quadratic surfaces out of two given pencils (ϱ^2_{34}) and (ϱ^2_{45}). The base-curve of the pencil (ϱ^2_{34}) consists of the lines a_3 and a_4 and the common transversals b_{134} and b_{234} which we can draw through the points A_1 and A_2 to these straight lines; that of the pencil (ϱ^2_{45}) consists of the lines a_4 and a_5 and their common transversals b_{145} and b_{245} passing through A_1 and A_2. [1])

Through a point P passes one ϱ^3 of the congruence; if we associate to P the point P', which on the curve ϱ^3 is harmonically separated from P by the points A_1 and A_2, we get an involution of *pairs of points* (P, P').

A straight line t is chord of one ϱ^3; let P and Q be its supporting points. Through the involution just found there are associated to the points P and Q two points P' and Q'. If we now associate the line t' connecting the points P' and Q', to t, we get an involution of *pairs of rays* (t, t').

§ 2. *Degenerate* ϱ^3 *of the involution.* We shall show that the

[1]) This congruence $[\rho^3]$ has been investigated by M. STUYVAERT (*Étude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre* Dissertation inaugurale Gand 1902) and by J. DE VRIES (*Bilineaire congruenties van kubische ruimtekrommen.* Proefschrift, Utrecht 1917).

congruence $[\varrho^3]$ contains *seven* systems of ∞^1 curves ϱ^3, each of which is degenerated into a conic k^2 and a straight line d.

In the first place the conic k^2 can pass through the points A_1 and A_2 and therefore lie in a plane π through the straight line $A_1 A_2$. Such a plane intersects the lines a_3, a_4 and a_5 in three points A_3, A_4 and A_5, which together with the points A_1 and A_2 define one conic k^2. The ruled surface ψ^2 formed by the common transversals of the lines a_3, a_4 and a_5, intersects this conic k^2 besides in the points A_3, A_4 and A_5 in one more point D; the transversal d passing through D. forms with k^2 a degenerate ϱ^3. The surface ψ^2 is intersected by the line $A_1 A_2$ in two points B_1 and B_2; the generatrices b_1 and b_2 of ψ^2 passing through these points, form each with the line $A_1 A_2$ a degenerate k^2 of the system just considered. The transversal d, which completes the degenerate k^2, formed by the lines $A_1 A_2$ and b_1 to a ϱ^3, is apparently no other than the line b_2. The three lines b_1, $A_1 A_2$ and b_2 form therefore together a degenerate ϱ^3.

It has just appeared that to every conic k^2 there belongs a definite transversal d; is the reverse also the case? In order to examine this we remark that the line $A_1 A_2$ is twice a component of a degenerate k^2, and is therefore *nodal line* of the surface formed by these conics k^2. A plane π through $A_1 A_2$ intersects this surface along the nodal line and along a conic k^2; it is therefore of order four. A transversal d intersects this surface besides in the lines a_3, a_4 and a_5 in one point D and so forms together with *one* conic k^2 a degenerate ϱ^3.

§ 3. In order to get a second series of degenerate ϱ^3, we draw the transversal b_{134} mentioned in § 1 and bring through the point A_2 and the line a_5 a plane α_{25}. This plane intersects the transversal b_{134} in a point D_1, the lines a_3 and a_4 in two points C_3 and C_4. The points A_2, D_1, C_3 and C_4 determine a pencil of conics each of which forms with the line b_{134} a degenerate ϱ^3.

As we can take one of the transversals b_{135}, b_{145}, b_{234}, b_{235}, b_{245}, instead of the transversal b_{134}, we get in all *six* pencils of conics degenerated in this way.

Each of the corresponding pencils of conics contains three pairs of lines; for the pencil lying in the plane α_{25} they are the pairs $(A_2 D_1,\ C_3 C_4)$ $(A_2 C_3,\ D_1 C_4)$ and $(A_2 C_4,\ D_1 C_3)$. Each of these pairs forms with the transversal b_{134} a ϱ^3, which has degenerated into *three straight lines*.

Lying in the plane α_{25} the line $A_2 C_3$ intersects the line a_5 and is therefore the transversal b_{235}; in the same way the line $A_2 C_4$ is the same as the transversal b_{245}. The curve $(b_{134}, A_2D_1, C_3C_4)$ belongs evidently only to the pencil of degenerate ϱ^3 which contain the line b_{134} as component; the curves $(b_{134}, b_{235}, D_1C_4)$ and $(b_{134}, b_{245}, D_1C_3)$ belong each to *two* pencils of degenerate ϱ^3. There are therefore six curves of the first and as many of the second kind. Hence together with the curve (b_1, A_1A_2, b_2) the congruence $[\varrho^3]$ contains *thirteen* ϱ^3 which have degenerated into three straight lines.

§ 4. *Singular points and bisecants of the congruence.* The points of the three lines a_3, a_4 and a_5 are *singular points* of the congruence.

Let us consider for instance a point A_3 of the line a_3. The surface φ^2_{45} through A_3 intersects an arbitrary surface φ^2_{34} along a curve ϱ^3, which passes through the point A_3. Through A_3 passes therefore a pencil of curves ϱ^3; all these curves pass also through the second point of intersection of the surface φ^2_{45} with the line a_3.

Also the points of the transversals b_{ikl} are *singular points*; for each of these transversals is component of a pencil of degenerate ϱ^3.

The straight lines through the points A_1 and A_2 are *singular bisecants*; for through any point of such a straight line there passes one ϱ^3 and as this passes also through the points A_1 and A_2, it has that straight line as bisecant.

In the second place the straight lines in the planes α_{13}, α_{14}, α_{15}, α_{23}, α_{24}, α_{25} brought through the points A_1 and A_2 and the lines a_3, a_4 and a_5 are *singular bisecants.* For each of these planes contains a pencil of conics k^2, each of which is a component of a degenerate ϱ^3, and a straight line in such a plāne is bisecant of all these conics.

In the third place the generatrices g_{34} of the surfaces φ^2_{34}, which cross the lines a_3 and a_4, are *singular bisecants* of the congruence. Such a line g_{34} is intersected by the surfaces φ^2_{45} in the pairs of points of a quadratic involution and the two points of such a pair are every time the supporting points of a curve ϱ^3. As the surfaces φ^2_{34} pass through the lines a_3, a_4, b_{134} and b_{234}, the lines g_{34} are the transversals of the lines b_{134} and b_{234}.

In the same way the transversals g_{45} of the lines b_{145} and b_{245} and the transversals g_{35} of the lines b_{135} and b_{235} are singular bisecants of the congruence.

The *singular bisecants* form therefore two sheaves, six fields and three bilinear congruences.

§ 5. *Pairs of points on a degenerate* ϱ^3. We now pass on to the consideration of the involution (P, P') and examine first what becomes of this correspondence, if the points P and P' lie on a degenerate ϱ^3.

With a view to this we remark, that the four harmonical points P, A_1, P', A_2 of a curve ϱ^3 from each of its chords s are projected by four harmonical planes. This must remain the case, if we let the ϱ^3 degenerate into a conic k^2 and a straight line d.

In the degeneration considered in § 2, the points A_1 and A_2 lie both on the conic k^2. The following two cases are now possible:

1. The point P lies also on the conic k^2. If we take as chord s a common secant of the conic k^2 and the line d, we see that also the point P' lies on k^2 and is harmonically separated from P by A_1 and A_2.

2. The point P lies on the line d. If we take the chord s in the same way, we see that the point P' lies on k^2 and by A_1 and A_2 is harmonically separated from the point of intersection D of the two components k^2 and d.

To the point D', which is harmonically separated from D, *all the points* of the line d are therefore associated; for the rest there belongs to every point P of the degenerate ϱ^3 one definite other point P'.

In the degeneration considered in § 3, the point A_1 lies on the line d, the point A_2 on the conic k^2 (or inversely). Two cases are again possible:

1. The point P lies on the conic k^2. If we again take as chord s a secant of k^2 and d, we see that the point P' lies also on the conic k^2 and is harmonically separated from P by the points A_2 and D.

2. The point P lies on the line d. If we take as chord s a straight line in the plane of the conic k^2, we see that the point P lies also on the line d and is harmonically separated from P by the points A_1 and D.

To each point of this degenerate ϱ^3 belongs consequently a definite other point. If P coincides with D, the point P' does the same.

If the ϱ^3 is degenerated into three straight lines, considerations of the same kind hold good.

§ 6. *Singular points of the involution* (P, P'). On every non degenerate ϱ^3 the points A_1 and A_2 are associated to themselves; it appears from the preceding § that this is also the case for the degenerate ϱ^3. These points are therefore *not* singular points of the involution. On the contrary the points of the lines a_3, a_4 and a_5 *are* singular points. Let us consider e.g. a point A_3 of the line a_3. In order to find the point A_3' associated to A_3 on a curve ϱ^3 passing through A_3, we must bring through the bisecant a_3 of this curve ϱ^3 a plane which by the planes α_{13} and α_{23} is harmonically separated from the plane which *touches* the curve ϱ^3 in the point A_3 and passes through the line a_3; this plane intersects the curve ϱ^3 in the point A'_3 in question.

As the plane (A'_3, a_3) passes through the line a_3, it is a tangent plane of the surface φ^2_{34} which contains the considered curve ϱ^3. Now the tangent planes of a ruled surface in the points of a generatrix are projectively associated to the points of contact; the point of contact B_3 of the plane (A_3, a) is therefore harmonically separated from the point A_3 by the points of contact of the planes a_{13} and a_{23}. As these two planes pass through the lines b_{134} and b_{234}, their points of contact B_{13} and B_{23} are the intersections of these transversals with the line a_3.

If the surface $\varphi_{34}{}^2$ describes the pencil $(\varphi_{34}{}^2)$, the plane $(A'_3\, a_3)$, which touches the surface φ^2_{34} in the point B_3, describes a pencil which is projectively associated to the pencil (φ^2_{34}). The figure produced by these projective pencils is a surface of the third order. To these planes of contact belong the planes α_{13} and α_{23} each of which is at the same time part of a degenerate surface φ^2_{34}; consequently these planes belong to the product and the rest is a *plane*.

The figure produced by two projective pencils passes through the base-curves of these pencils. The plane just found contains therefore the line a_4 as this is the case with neither of the planes α_{13} and α_{23}. It must also pass through the point B_3 as the intersection of a curve φ^2_{34} with its tangent plane in the point B consists besides of the line a_3, of a generatrix through the point B_3.

The locus of the points A'_3 which are associated to the point A_3 on the different curves ϱ^3 laid through the point A_3, belongs to the intersection of the plane (B_3, a_4) with the surface φ^2_{45}, on which all these curves ϱ^3 are situated and which passes through A_3. This intersection consists besides of the line a_4 of a *straight line* λ; this is the locus in question.

This line λ passes through the point of intersection of the plane (B_3, a_4) with the line a_5. Evidently this point of intersection is projectively associated to the point B_3, therefore also to the point A_3. The same must hold good for the intersection of the line λ with the line a_4. If the point A_3 describes the line a_3, the intersections of the line λ with the lines a_4 and a_5 describe two projective sequences of points. Consequently the line λ describes a *quadratic surface* ω^2, the locus of all the points associated to the points of the line a_3.

To each of the lines a_4 and a_5 belongs a similar surface ω^2.

§ 7. The points of the transversals b_{145} etc. are *not* singular points of the involution. For from the construction given in § 5 it follows that to every point of such a transversal a definite other point of the same transversal is associated, no matter of which degenerate ϱ^3 we consider the transversal to be a component.

From § 5 follows further that on each degenerate ϱ^3 of the first series there lies *one* singular point D'. We shall determine the locus of these singular points.

It appeared in § 2 that to this series belongs a ϱ^3 consisting of the straight line A_1A_2 and the two transversals b_1 and b_2 of the four lines $A_1 A_2$, a_3, a_4 and a_5. As we can combine each of the two transversals with the line A_1A_2 to a degenerate conic k^2, there lie on this conic *two* singular points D'_1 and D'_2.

A plane π through the line A_1A_2 contains *one* conic k^2 and consequently it intersects the locus in question, besides in the points D'_1 and D'_2, in one more point D'; the locus is therefore a *twisted cubic* σ^3.

A point D' is associated to a straight line d, which intersects the three lines a_3, a_4 and a_5. To the points associated to D' belong therefore three points which lie on the three lines mentioned; consequently the point D' must lie on the three surfaces ω^2 found in the preceding §. All these three surfaces pass therefore through the curve σ^3.

§ 8. If a point P describes a straight line l, the point P' associated to P describes a curve (l). As the line l has two points in common with each of the three surfaces ω^2, the curve (l) has two points in common with each of the three lines a_3, a_4 and a_5. A surface φ_{34}^2 intersects the line l in two points and contains both the points of the curve (l) associated to this line, so that in all this surface φ_{34}^2 has *six* points in common with the curve (l). For this reason (l) is a *twisted cubic*.

In general the line l and the curve $(l)^3$ have *no* points in common, for as a rule no two associated points of the involution (P, P') lie on l; for the rest this involution has only a finite number of coincidences, viz. the points A_1 and A_2 and the points D, found in § 5, in which the transversals b_{134} etc. intersect the corresponding planes α_{25} etc. As a rule therefore the line l does not contain any coincidences either.

If a point P describes a plane V, the point P' associated to P, describes a surface (V). In order to find the order of this surface, we draw in the plane V a straight line l. The curve $(l)^3$ associated to this line l, intersects the plane V in three points, each of which is associated to a point of l. The line l intersects therefore the locus of the pairs of points (P, P') lying in the plane V, in *three* points; consequently this locus is a curve of order three. The plane V containing as a rule no coincidences, this curve is the complete intersection of V with the surface (V), which for this reason is a surface of order *three*.

The surface $(V)^3$ contains the lines a_3, a_4 and a_5, for each point of one of these lines is associated to a line λ, which cuts the plane V in one point. In the same way the surface $(V)^3$ passes through the curve σ^3.

Let Q be a point of the plane V, l a straight line of V passing through Q. This line contains three points P, associated to a point P' in V. If we connect these points P' with Q and associate these lines of connection to the line l, we get in the pencil of the rays through Q a correspondence $(3, 3)$ with *six* coincidences. These must originate from the rays (P, P') passing through Q and each of these rays furnishes *two* coincidences, as the correspondence (P, P') is involutory. Through Q pass therefore *three* rays $P P'$ which lie in plane V and accordingly the lines $P P'$ as a rule form a *cubic line complex*.

§ 9. *Singular straight lines of the involution* (t, t'). We now proceed to the consideration of the involution (t, t') and first investigate its *singular rays*.

The line $A_1 A_2$ is bisecant of *all* the curves ϱ^3. On an arbitrary curve ϱ^3 each of the two supporting points A_1 and A_2 coincides with its associated point; in this case the line $A_1 A_2$ is associated to *itself*.

According to § 7 the line $A_1 A_2$ is a component of one degenerate ϱ^3 and as such contains two singular points D_1' and D_2'; to these points correspond all points of the two transversals b_1 and b_2 of the lines $A_1 A_2$, a_3, a_4 and a_5. If we consider the points D_1' and D_2' as supporting points, there is associated to the line $A_1 A_2$ a *bilinear congruence of rays* which has the lines b_1 and b_2 as directrices. If we consider one of the points D_1' and D_2' and one arbitrary other point of the line $A_1 A_2$ as supporting points, we find that to the line $A_1 A_2$ there are moreover associated two *fields of rays* lying in the planes which connect the lines b_1 and b_2 with the line $A_1 A_2$.

Also the line a_3 is bisecant of all curves ϱ^3. The supporting points E_3, F_3 are each time the two points of intersection of the line a_3 with a surface φ^2_{45}. The points E'_3 and F'_3 associated to these, lie on the generatrice λ and μ of the surface ω^2 corresponding to E_3 and F_3.

Through each pair of points (E_3, F_3) pass ∞^1 curves ϱ^3; the corresponding points E_3' and F_3' describe apparently two projective sequences of points. Moreover the pairs of points (E_3, F_3) form an involution on the line a_3; the pairs of generatrices (λ, μ) form therefore also an involution. Consequently the pairs of points (E_3', F_3') form an involution on the surface ω^2 and the lines connecting associated points of this involution are the rays associated to the line a_3.

We shall first demonstrate that each generatrix v of the surface ω^2 which belongs to the same system with the lines a_4 and a_5, contains one pair of points (E_3', F_3'). With a view to this we remark that two points E_3' and F_3' are situated on the same curve ϱ^3; this curve intersects the surface ω^2 besides in the supporting points of the bisecants a_4 and a_5. The congruence $[\varrho^3]$ being bilinear, each line v belongs as bisecant to one ϱ^3; the corresponding supporting points are the points in question E_3' and F_3'.

Through a point E_3 of the surface ω^2 there pass two rays of the congruence in question, viz. the line connecting E_3' with its associated point F_3', and the line v passing through the point E_3'; consequently the order of this congruence is *two.*

A tangent plane of the surface ω^2 contains one line v and one line λ. The straight line μ, associated to the line λ, cuts this tangent plane in a point F_3' and the line connecting this point with the associated point E_3' is a ray of the congruence in consideration, which together with the line v lies in this tangent plane. For this reason the *class* of the congruence is *two* as well.

Analogous considerations hold good for the lines a_4 and a_5. *Consequently to each of the lines a_3, a_4 and a_5 there corresponds a congruence* (2,2).

§ 10. A straight line *l through the point A_1* is bisecant of ∞^1 curves ϱ^3. The point A_1 corresponds to itself; the locus of the points P' corresponding to the points P of the line l is according to § 8 a curve $(l)^3$. This passes through the point A_1; for when P gets into A_1, P' coincides with P. The rays associated to the line l project the curve $(l)^3$ from the point A_1 and form therefore a *quadratic cone.*

The same holds good for a straight line through the point A_2. A straight line *l in the plane α_{23}* is bisecant of ∞^1 conics k^2. Let E and F be the points of intersection of the line l with such a conic. The points E' and F', associated to these points E and F, lie according to § 5 also on the conic k^2 and the straight line $E'F'$ is associated to the line l.

The locus of the points E' and F' is a conic k^2, for the line l

has one point in common with the line a_1. To this point corresponds a line λ, so that the curve $(l)^3$ which corresponds to the line l, must degenerate into this line λ and into a conic k^2, the locus of the pairs of points (E', F'). These pairs of points form an involution on the conic k^2; the line $e'F'$ passes therefore through a fixed point, so that to the line l a *plane pencil* of the plane α_{23} is associated.

The same holds good for a straight line in one of the planes α_{24}, α_{25}, α_{13}, α_{14} and α_{15}.

According to § 4 each transversal g_{34} of the lines b_{134} and b_{234} contains an involution of pairs of points (G,H) which are each time the supporting points of a curve ϱ^3. The associated points G' and H' lie on the curve (l^3), which through the involution (P,P') is associated to the line g_{34}. The pairs of points (G',H') form an involution on this line with two coincidences and the lines $G'H'$ determine a *quadratic ruled surface*, associated to the singular line g_{34}.

In the same way there corresponds to each of the lines g_{45} and g_{35} a *quadratic ruled surface.*

The straight lines which are associated to all the lines g_{34}, form together a line complex, the order of which we shall determine later on.

§ 11. It appeared in § 5 that on each degenerate ϱ^3 of the first system lies one singular point D' which is associated to all the points of the line d. A bisecant l of this ϱ^3 through the point D' corresponds therefore to a *plane pencil* which projects the line d from the point which is associated to the second supporting point of the bisecant.

These bisecants l form two plane pencils, which both have the point D' as base point; the first lies in the plane of the conic k^2, the second projects the line d from the point D'.

The plane of the conic k^2 passing through the line A_1A_2, the bisecants l of the first kind are the common secants of the line A_1A_2 and of the locus σ^3 of the points D'. As A_1A_2 and σ^3 have two points D'_1 and D'_2 in common, their common secants form *a congruence* (1,3).

A plane V intersects the curve σ^3 in three points; through each of these points passes one bisecant l of the second kind lying in the plane V; these bisecants form consequently a *congruence* of class *three.*

From a point P the curve σ^3 is projected by a cubic cone K^3. The planes which project the corresponding lines d from P, envelop a quadratic cone of which the tangent planes are projectively associated to the generatrices of the cone K^3; it happens *five* times that such a plane passes through the corresponding straight line, so that this line is a bisecant l of the second kind passing through P. Hence the *order* of the congruence formed by these bisecants is *five*.

To each ray l of one of the congruences (1, 3) and (5, 3) corresponds a plane pencil of straight lines l' which project a line d from a point of the corresponding conic k^2. For the lines l of the second kind this point coincides with D', so that the congruence (5, 3) is transformed *into itself*; for those of the first kind it is an arbitrary point of the conic k^2.

A plane V intersects the conics k^2 in the points of a curve c^4 that has a node in the intersection of the plane V with the line $A\,A_2$, and the lines d in the points of a conic c^2. Between the points of the curves c^4 and c^2 there evidently exists a correspondence (1, 2). The three points of intersection of these curves lying outside the intersections of the plane V with the lines a_3, a_4 and a_5 and with the two transversels b_1 and b_2 of the four lines A_1A_2, a_3, a_4 and a_5, are points D, hence coincidences of this correspondence. The lines connecting associated points of this correspondence, in other words the rays l' lying in the plane V, envelop therefore a curve of class five.

The rays l' corresponding to the rays l of the congruence (1, 3) *form consequently a line complex of order five.*

The degenerate curves ϱ^3 of the second series, found in § 3, do not contain any singular points.

§ 12. *Coincidences.* A line A produces a coincidence if its supporting points P and Q coincide with their associated points P' and Q'.

The involution (P, P') has a finite number of coincidences, viz. the points A_1, A_2 and the six points D found in § 5, in which the transversals b_{134} etc. cut the corresponding planes α_{25} etc. The line A_1A_2 and the lines connecting the points A_1 and A_2 with the points D are therefore rays of coincidence.

Let us further consider a line l through the intersection D_1 of the line b_{134} with the plane α_{25}. This line is bisecant of a degenerate ϱ^3 formed by the line b_{134} and a conic k^2 in the plane α_{25}; in the point D_1 this conic touches the plane brought through the lines l and b_{134}. For if we cause the two supporting points of a bisecant PQ of which the supporting point P lies on the line b_{135}, the supporting point Q on the conic k^2, to approach D_1, we get such a straight line l. The point P' associated to P lies on the line b_{134} and is harmonically separated from P by the points A_1 and D_1; it approaches therefore also to D_1 and in such a way that lim. $PD : P'D_1 = -1$. In the same way the point Q on the conic k^2 approaches to the point D_1. From this it is easily seen, that in the limit the line $P'Q'$ coincides with PQ so that the line l is a *ray of coincidence*.

Consequently the straight lines through these six points D are also rays of coincidence.

A line t is also a ray of coincidence, if P' coincides with Q and Q' with P, so that the supporting points P and Q are associated to each other in the involution (P, P'). According to § 8 these rays form a *cubic complex*.

§ 13. When a straight line t describes a *plane pencil*, the associated ray t' describes a *ruled surface* R, of which we shall determine the order.

Each ray is bisecant of one curve ϱ^3; the locus of the supporting points is a curve c; this has a node in the base point B of the plane pencil, for on the two rays t connecting B with the two other points of intersection of the ϱ^3 passing through B one of the two supporting points gets into B. Hence the curve c is of order four.

The curve c^4 has one point in common with each of the three lines a_3, a_4 and a_5; for if a ray t intersects one of these lines, one of the two supporting points gets into the point of intersection.

Through the involution (P,P') a curve $(l)^3$ is associated to a line l, hence to a curve of order four, in general one of order twelve. The curve ϱ^4 has one point in common with each of the straight lines a_3, a_4 and a_5 and to each of these points a line λ is associated, so that moreover a curve ϱ^9 is associated to the curve ϱ^4.

The pairs of supporting points form on the curve c^4 an involution with *six* coincidences; these are the points of contact of the six tangents which can be drawn from the node B at the curve c^4. The pairs of points of the curve ϱ^9, associated to them, form therefore also an involution with six coincidences. The lines connecting associated points of this involution form consequently a ruled surface of order *six*, which is the surface R.

We can also determine the order of R by trying to find the number of points of intersection of this surface with the line a_3. With a view to this we remark that to the line a_3 a surface ω^2 is associated, so that whenever one of the supporting points of a ray t lies on this surface ω^2, one of the supporting points of the associated ray t' lies on the line a_3. The surface ω^2 passes through the lines a_4 and a_5; the curve c^4 intersects this surface besides in the points it has in common with the lines a_4 and a_5, in *six* more points, so that the plane pencil in consideration contains *six* rays t of which one of the supporting points lies on the surface ω^3; consequently there are *six* rays t' intersecting the line a_3.

In the third place we can determine the order of R by trying to find the number of intersections with the line A_1A_2. For this purpose we remark that a ray t' intersecting the line A_1A_2, if it is not a singular ray, must be bisecant of a conic k^2. The two supporting points are associated to two points of the same conic, so that also the associated ray t intersects the line A_1A_2. The plane pencil contains one ray t intersecting the line A_1A_2; the associated ray t' rests also on the line A_1A_2.

According to § 11 there is a complex of order five consisting of rays t associated to singular rays t' which form a congruence (1,3) and each of which intersects the line A_1A_2. The plane pencil contains 5 rays of this complex, hence the surface R five rays t' of the (1,3).

In all the line A_1A_2 is intersected by six rays t', so that the surface R is of order *six*.

§ 14. We can now also determine the order of the line complex associated to the congruence of the singular rays g_{34} found in § 10.

A singular ray t', intersecting the line b_{134}, is bisecant of a degenerate ϱ^3 consisting of the line b_{134} and a conic k^2 of the plane α_{25}, passing through the point of intersection of this plane with the line t'. The supporting points of the associated ray t lie also on the line b_{134} and on the conic k^2. Now the plane pencil considered in the preceding § contains one ray t, which intersects the line b_{134}; hence there is one ray t', which intersects the line b_{134}.

The other five generatrices of the ruled surface R^6 intersecting the line b_{134} must be *singular* rays, therefore lines g_{34}. The plane pencil contains five rays associated to rays g_{34}; consequently these rays form a complex of order *five*.

§ 15. To a sheaf of rays corresponds a congruence $[t']$. In order to determine order and class of this congruence $[t']$, we take the base point B of the sheaf on the line A_1A_2.

It has been found already in § 13 that to a ray t intersecting the line A_1A_2 a ray t' is associated also intersecting A_1A_2. We shall now show that the rays t and t' intersect the line A_1A_2 in *the same* point.

Let k^2 be the conic which has the line t as bisecant, P and Q the corresponding supporting points, P' and Q' the points associated

39*

to these. Through a linear transformation of the plane π of the conic k^2 we can transform the points A_1 and A_2 into the circle points at infinity. If S be an arbitrary point of the conic k^2, the straight lines SP and SP' will be harmonically separated by SA_1 and SA_2, hence they will be perpendicular to each other after the transformation, so that PP' is a diameter of the circle k^2, the same as the line QQ'. The chords PQ and $P'Q'$ are therefore parallel and consequently intersect on the line A_1A_2.

To an arbitrary ray t through the point B corresponds, therefore a ray through the same point, so that to the congruence $[t']$ there belongs in the first place the sheaf itself.

To the line A_1A_2 corresponds a *bilinear congruence of rays*, also belonging to the congruence $[t']$, besides *two fields of rays*.

Through the point B passes a cubic cone of singular rays of the congruence (1,3) considered in § 11. To each of these rays corresponds a plane pencil which projects a line d from a point Q' of the corresponding conic k^2. The point Q' is associated to the second point of intersection Q of the ray with the conic k^2.

The cubic cone mentioned has the line A_1A_2 as nodal generatrix. The two generatrices coinciding with A_1A_2 belong to the two degenerate conics k^2 consisting of the line A_1A_2 and of one of the two lines b_1, b_2; hence the two leaves of the cone K^3, which pass through the line A_1A_2, touch at the planes of these degenerate conics consequently they also touch the two leaves both passing through A_1A_2, of the surface of order four, found in § 2, described by the conics k^2; the line A_1A_2 belongs therefore *six times* to the intersection of the cone K^3 with this surface. The rest of the intersection consists of the curve σ^3 projected by the cone K^3 and of the locus τ^3 of the points Q. The cone K^3 has *three* points in common with each of the lines a_3, a_4 and a_5 lying on the quartic surface mentioned; the curve σ^3 having these lines as bisecants, two of these points lie every time on the curve σ^3, while the third must lie on the curve τ^3.

It is further easily found that the curves σ^3 and τ^3 lying on one and the same cubic cone, have *three* points in common. In general through the involution (P,P'), to a cubic curve a curve of order nine is associated. However the curve τ^3 having one point in common with each of the lines a_3, a_4 and a_5, three straight lines λ belong to this associated curve and as it has three points in common with the curve σ^3 and for this reason contains three singular points D', three lines d belong to it. The complete locus of the points Q' is therefore a curve τ_1^3.

The rays in question, associated to the generatrices of the cone

K^4, project the lines d from the corresponding points Q' of the τ_1^4. In the same way as in § 11 we should therefore find that these rays form a congruence (5, 3). But it happens *three times* that the point Q' coincides with the point D and hence lies on the line d; these points are associated to the three points of intersection of the curves σ^3 and τ^3, for in them the point Q coincides with the point D'. In this case *all* rays through the point Q' intersect the line d. Accordingly, from the congruence (5, 3), which we should find in general, three sheaves are split off and we only find a congruence (2, 3).

To the sheaf of rays through a point B of the line A_1A_2 are associated one sheaf, two fields of rays, one bilinear congruence and one congruence (2, 3). In general there corresponds therefore to a sheaf of rays a congruence (4, 6).

§ 16. To a *field of rays* corresponds also a certain congruence. In order to investigate this, we consider the rays lying in a plane π through the line $A_1 A_2$.

A non singular ray of this field is bisecant of a conic k^2 in this plane π, hence associated to another bisecant of this conic. To the congruence in question belongs therefore in the first place the field of rays *itself*. To the line $A_1 A_2$ in the plane π correspond a bilinear congruence of rays and two fields of rays.

To an arbitrary straight line through the point A_1 corresponds a quadratic cone with the point A_1 as vertex. This intersects the plane π along two straight lines. The sheaf of the rays through the point A_1 belongs therefore also to the congruence in question and each of these rays must be counted *twice*, because it is associated to *two* rays of the plane π. The same holds good for the sheaf of the rays through the point A_2.

The plane π intersects the curve σ^3 besides in the points D'_1 and D'_2 in one more point; through this point passes a plane pencil of singular rays of the congruence (1, 3). To each of these rays corresponds a plane pencil, which projects the line d, belonging to the conic k^2, from a point of this conic; hence to the plane pencil mentioned corresponds the congruence of the lines resting on k^2 and d. As these have a point D in common, those lines of intersection form a congruence (1, 2). A *field of rays* is therefore transformed into a congruence (6, 6).

Mathematics. — "*On n-tuple orthogonal systems of n—1-dimensional manifolds in a general manifold of n dimensions.*" By Prof. J. A. SCHOUTEN and D. J. STRUIK. (Communicated by Prof. J. CARDINAAL).

(Communicated at the meeting of June 28, 1919).

I.

1. *Notations* [1]). A p-dimensional manifold may be denoted by V_p, a p-dimensional euclidean [2]) manifold by R_p. R_p may also denote an infinitesimal region, determined by p independent directions, in the vicinity of a point of V_n. As original variables in a V_n we use the systems x^λ and $y^j, \ldots$, with the corresponding covariant and contravariant vectors

$$\begin{matrix} \mathbf{e}_\lambda = \nabla x^\lambda; \ \mathbf{e}_\lambda' \\ \mathbf{s}_j = \nabla y^j; \ \mathbf{s}_j' \end{matrix} , \quad \ldots\ldots\ldots \quad (1)$$

which satisfy the conditions:

$$\begin{gathered} \mathbf{e}_\lambda' \cdot \mathbf{e}_\lambda' = \varepsilon_\lambda{}^2 \quad ; \quad \mathbf{e}_\lambda \cdot \mathbf{e}_\lambda = \varepsilon_\lambda{}^{-2} \\ \mathbf{s}_j' \cdot \mathbf{s}_j' = \sigma_j{}^2 \quad ; \quad \mathbf{s}_j \cdot \mathbf{s}_j = \sigma_j{}^{-2} \\ \mathbf{e}_\lambda \cdot \mathbf{e}_\mu' = \begin{cases} 0 & \text{when } \lambda \neq \mu \\ \varkappa = (-1)^{\frac{n(n-1)}{2}} & \text{when } \lambda = \mu \end{cases} \quad \ldots \quad (2) \\ \mathbf{s}_j \cdot \mathbf{s}_k' = \begin{cases} 0 & \text{when } j \neq k \\ \varkappa & \text{when } j = k. \end{cases} \end{gathered}$$

The fundamental tensor of this V_n may be written ${}'\mathbf{g}$:

$$\varkappa\,{}'\mathbf{g} = \sum_{\lambda,\mu}^{a_1,\ldots,a_n} g_{\lambda\mu}\,\mathbf{e}_\lambda\,\mathbf{e}_\mu = \sum_{\lambda,\mu}^{a_1,\ldots,a_n} g^{\lambda\mu}\,\mathbf{e}_\lambda'\,\mathbf{e}_\mu' = \sum_{j,k}^{1,\ldots,n} g_{jk}\,\mathbf{s}_j\,\mathbf{s}_k = \sum_{j,k}^{1,\ldots,n} g^{jk}\,\mathbf{s}_j'\,\mathbf{s}_k'. \quad (3)$$

We will choose the aequiscalar V_{n-1} belonging to x^λ and y^j in different ways according to the circumstances.

2. *Normal and V-creating fields.* In a manifold V_n may be given

[1]) For the notations used in this communication see also: J. A. SCHOUTEN, Die direkte Analysis zur neueren Relativitätstheorie, Verh. der Kon. Akad. v. Wetenschappen XII, 6 (1918), here further cited as A. R.

[2]) We will call a manifold euclidean when its Riemann-Christoffel affinor $\overset{4}{\mathbf{K}}$ is zero. Compare A. R. p. 58.

an R_p-field by the p-vector-field of the simple p-vectors [1] $_p\mathbf{v}$. Then any system of simple $(n-p)$-vectors, perfectly perpendicular to $_p\mathbf{v}$, determines an R_{n-p}-field. When the field $_{n-p}\mathbf{w}$ is everywhere perfectly perpendicular to a system of ∞^{n-q} manifolds $V_q, q \leqslant p$, the R_q-direction of V_q in each point lies in the region of $_p\mathbf{v}$. In the case that $_{n-p}\mathbf{w}$ is not also perpendicular to a system of ∞^{n-q-1} manifolds V_{q+1} we call $_{n-p}\mathbf{w}$ *V_q-normal* and $_p\mathbf{v}$ *V_q-creating.*

When $_p\mathbf{v} = \mathbf{v}_1 \frown \ldots \frown \mathbf{v}_p$, then $_p\mathbf{v}$ is V_q-creating, if the p equations:

$$\mathrm{v}_i \cdot \nabla f = 0 \quad , \quad i = 1, \ldots, p, \quad \ldots \ldots \quad (4)$$

have $n-q$ independent solutions. This condition is sufficient too, for the V_q can always be given as intersections of $n-q$ systems $V_{n-1} : f_1 = c_1, \ldots, f_{n-q} = c_{n-q}$, in which $c_1, \ldots, c_{n-q}$ are variable parameters. It is necessary that $q \leqslant p$, for in an R_n an $R_{n-p+\alpha}$ can only be perfectly perpendicular to an R_p, when $\alpha = 0$. Every solution f_k is also a solution of the equation:

$$(\mathrm{v}_j \cdot \nabla)(\mathrm{v}_i \cdot \nabla) f_k = (\mathrm{v}_j \overset{1}{\cdot} \nabla \mathrm{v}_i) \cdot \nabla f_k + \mathrm{v}_i \mathrm{v}_j \overset{2}{\cdot} \nabla \nabla f_k = 0, \; i, j = 1, \ldots, p, \quad (5)$$

hence also of:

$$(\mathrm{v}_j \overset{1}{\cdot} \nabla \mathrm{v}_i - \mathrm{v}_i \overset{1}{\cdot} \nabla \mathrm{v}_j) \cdot \nabla f_k = 0 \quad \ldots \ldots \quad (6)$$

Firstly we consider the case $q = p$. (In this case the system of differential equations (4) is a *complete system.*) If we write:

$$_{n-p}\mathrm{w} = \mathrm{w}_1 \frown \ldots \frown \mathrm{w}_{n-p} \,,$$

then we conclude from (6) that all vectors $\mathbf{v}_j \overset{1}{\cdot} \nabla \mathbf{v}_i - \mathbf{v}_i \overset{1}{\cdot} \nabla \mathbf{v}_j$ lie in the region of $_p\mathbf{v}$, which also can be written as follows:

$$(\mathrm{v}_j \overset{1}{\cdot} \nabla \mathrm{v}_i - \mathrm{v}_i \overset{1}{\cdot} \nabla \mathrm{v}_j) \cdot \mathrm{w}_k = 0 \quad \begin{matrix} i, j = 1, \ldots, p \\ k = 1, \ldots, n-p \end{matrix} \quad \ldots \quad (7)$$

If the equations (7) were not satisfied for some values of i and j, we should certainly have less than $n-p$ independent solutions, for it would be possible for these values of i and j to join the equations (6) to (4). In this way a system of r equations, $n \geqslant r \geqslant p$ might be obtained, which would be satisfied by all solutions of (4). But this system certainly has more than $n-r$ independent solutions. Hence the condition (7) is necessary and sufficient.

As:

$$2 \nabla \overset{1}{\cdot} (\mathrm{v}_i \frown \mathrm{v}_j) = (\nabla \cdot \mathrm{v}_i) \mathrm{v}_j - (\nabla \cdot \mathrm{v}_j) \mathrm{v}_i + (\mathrm{v}_i \cdot \nabla) \mathrm{v}_j - (\mathrm{v}_j \cdot \nabla) \mathrm{v}_i, \quad (8)$$

(7) is equivalent to

[1] A simple p-vector can be written as an alternating product of p non-ideal vectors and can be represented by a part of a definite R_p with definite volume, but of indefinite shape, together with a hyper screw-direction.

$$\mathrm{w}_k \cdot \{\nabla^1 (\mathrm{v}_i \frown \mathrm{v}_j)\} = \mathrm{w}_k \nabla \overset{2}{\cdot} (\mathrm{v}_i \frown \mathrm{v}_j) = 0, \quad \ldots \quad (9)$$

or also to

$$_{n-p}\mathrm{w} \nabla \overset{2}{\cdot} \mathrm{v}_i \frown \mathrm{v}_j = 0, \text{[1]} \quad \ldots \quad (10)$$

hence to

$$\boxed{_{n-p}\mathrm{w} \nabla \overset{2}{\cdot} {}_p\mathrm{v} = 0. \text{[2]}} \quad \ldots \quad (A)$$

In this equation the auxiliary vectors **v** and **w** occur no more. *It is the required condition that the* $_p\mathbf{v}$*-field may be* V_p*-creating.*

As:

$$\mathrm{w}_k \cdot \{\nabla \overset{1}{\cdot} (\mathrm{v}_i \frown \mathrm{v}_j)\} = \mathrm{w}_k \cdot (\mathrm{a} \cdot \nabla) \{\mathrm{a}^1 (\mathrm{v}_i \frown \mathrm{v}_j)\} \text{[3]} =$$

$$= (\mathrm{a} \cdot \nabla)\{\mathrm{w}_k \, \mathrm{a} \overset{2}{\cdot} (\mathrm{v}_i \frown \mathrm{v}_j)\}\text{[4]} - (\mathrm{a} \cdot \nabla) \mathrm{w}_k \overset{1}{\cdot} \{\mathrm{a} \overset{1}{\cdot} (\mathrm{v}_i \frown \mathrm{v}_j)\} = \quad \cdot \quad (11)$$

$$= (\mathrm{v}_i \frown \mathrm{v}_j) \overset{2}{\cdot} \nabla \mathrm{w}_k,$$

(A) is equivalent to

$$(\mathrm{v}_i \frown \mathrm{v}_j)^2 \; \nabla \frown \mathrm{w}_k = 0 \text{[1]}, \quad \ldots \quad (12)$$

and as

$$\nabla \frown {}_{n-p}\mathrm{w} = \sum_k^{1,..,n-p} \mathrm{a}\, \mathrm{w}_1 \ldots \mathrm{w}_{k-1} (\mathrm{a} \cdot \nabla) \mathrm{w}_k \mathrm{w}_{k+1} \ldots \mathrm{w}_{n-p}, \quad \ldots \quad (13)$$

also to

$$\boxed{{}_p\mathrm{v} \overset{2}{\cdot} \nabla \frown {}_{n-p}\mathrm{w} = 0 \text{[2]}} \quad \ldots \quad (B)$$

(B) can be deduced from (A) without returning to the auxiliary vectors $\mathbf{v}_i$ and $\mathbf{w}_k$. We can show also independently of (A) the necessity of (B). For, when $_{n-p}\mathbf{w}$ is V_p-normal, we always have

$$_{n-p}\mathrm{w} = \lambda \{(\nabla f_1) \overset{\frown}{\ldots} (\nabla f_{n-p})\}, \quad \ldots \quad (14)$$

in which λ is a function of the place. Hence

$$\nabla \frown {}_{n-p}\mathrm{w} = (\nabla \lambda)(\nabla f_1) \overset{\frown}{\ldots} (\nabla f_{n-p}), \quad \ldots \quad (15)$$

from which (B) is a direct result, because every **v** is $\perp \nabla f_j$.

When $p = n-1$, we see from (B), or clearer from (15), that $\nabla \frown \mathbf{w}$ is a simple bivector. From this we may deduce the following theorem. When a field **w** is V_{n-1}-normal and **w** is interpreted as

---

[1]) The forms (10) and (12) of the condition are identical with those occurring in E. VON WEBER, Vorlesungen uber das Pfaffsche Problem (TEUBNER, 1900) page 99 and 100.

[2]) (A) and (B) were already given without proof in J. A. SCHOUTEN, Over het aantal graden van vrijheid van het geodetisch meebewegende assenstelsel. Versl. der Kon. Ak. v. Wet 27 (18) 16—22.

[3]) The differentiating effect of a differential operator extends to the first coming *closing* bracket.

[4]) This term is zero, because $\mathrm{w}_k \perp \mathrm{v}_i$ and $\perp \mathrm{v}_j$.

the vector of velocity of a streaming liquid, in which case the component of rotation of the movement (with respect to a geodesically moving system) is given by $\nabla \frown \mathbf{w}$, in this rotation every point of the $R_{n-2} \perp \nabla \frown \mathbf{w}$ remains unaltered. [1]) Indeed, $d\mathbf{r} \overset{1}{} (\nabla \frown \mathbf{w})$ is a vector in the plane of $\nabla \frown \mathbf{w}$.

In the same way we can prove that, if ${}_p\mathbf{v}$ is V_q-creating and thus ${}_{n-p}\mathbf{w}$ is V_q-normal, the equations exist:

$$\boxed{{}_{n-q}\mathrm{w}\ \nabla \overset{2}{\cdot}\ {}_p\mathrm{v}=0}, \qquad (A')$$

$$\boxed{{}_q\mathrm{v} \overset{2}{\cdot}\ \nabla \frown {}_{n-p}\mathrm{w}=0}, \qquad (B')$$

in which ${}_q\mathbf{v}$ represents a q-vector in V_q and ${}_{n-q}\mathbf{w}$ an $n-q$-vector $\perp V_q$. For $p=n-1$ we see from (B') that for a V_q-normal vector-field $\mathbf{w}$ the component of $\nabla \mathbf{w}$ in V_q is a tensor.

3. *Canonical congruences.* A field of unit-vectors $\mathbf{i}_n$ determines a congruence [2]), $\mathbf{u}_n = \varkappa\, \mathbf{i}_n \overset{1}{\cdot} \nabla \mathbf{i}_n$ is the vector of curvature of the curves of this congruence and the modulus $u_n = (\mathbf{u}_n)_m$ is the geodesic curvature.

As

$$(\nabla \mathbf{i}_n)^1\, \mathbf{i}_n = \tfrac{1}{2} \nabla (\mathbf{i}_n \,.\, \mathbf{i}_n) = 0, \qquad (16)$$

the second ideal factor of $\nabla \mathbf{i}_n$ does not contain an index n. Hence $\nabla \mathbf{i}_n$ consists of two parts, a part $\overset{2}{\mathbf{h}}$ in the $R_{n-1} \perp \mathbf{i}_n$ and a part $\mathbf{i}_n\, \mathbf{i}_n \overset{1}{\cdot} \nabla \mathbf{i}_n = \varkappa \mathbf{i}_n\, \mathbf{u}_n$:

$$\nabla \mathbf{i}_n = \overset{2}{\mathbf{h}} + \mathbf{i}_n\, \mathbf{u}_n, \qquad (17)$$

In general $\overset{2}{\mathbf{h}}$ is the sum of a tensor ${}^s\mathbf{h}$ and a bivector ${}_s\mathbf{h}$. If $\mathbf{i}$ is a unit-vector in one of the $n-1$ mutually perpendicular principal directions of ${}^s\mathbf{h}$, we have

$$\varkappa\, {}^s\mathbf{h} \overset{1}{\cdot} \mathbf{i} = \lambda\, \mathbf{i}, \qquad (18)$$

and as

$$\nabla \smile \mathbf{i}_n = {}^s\mathbf{h} + \tfrac{1}{2} (\mathbf{i}_n\, \mathbf{u}_n + \mathbf{u}_n\, \mathbf{i}_n), \qquad (19)$$

we have

$$\varkappa(\nabla \smile \mathbf{i}_n)^1\, \mathbf{i} = \lambda\, \mathbf{i} + \mu\, \mathbf{i}_n, \qquad (20)$$

[1]) For R_n this is observed by A. SOMMERFELD. Geometrischer Beweis des DUPIN'schen Theorems und seiner Umkehrung, Jahresberichte der Deutsch. Math. Ver. 6 (99) 123—128, p. 128.

[2]) In A. R. p. 38 et seq. the word "Hyperkongruenz" is used. For the sake of simplicity we will use here the word congruence, in harmony with among others RICCI and LEVI-CIVITA.

or

$$(\varkappa\nabla \smile \mathbf{i}_n - \lambda\, {}^2\mathbf{g})^1\, \mathbf{i} = \mu\, \mathbf{i}_n, \quad \ldots \ldots \quad (21)$$

for

$${}^2\mathbf{g} \,!\, \mathbf{i} = \mathbf{i}. \quad \ldots \ldots \quad (22)$$

From (21) follows:

$$\mathbf{i}_n \,!\, (\varkappa\nabla \smile \mathbf{i}_n - \lambda\, {}^2\mathbf{g})^{-1} \,!\, \mathbf{i}_n = 0, \quad \ldots \ldots \quad (23)$$

or, when $X_{\lambda\mu}$ are the covariantive coordinates of $\nabla \smile \mathbf{i}_n$ and $i_{n\lambda}$ those of $\mathbf{i}_n$, in coordinates:

$$\begin{vmatrix} 0 & i_{n\,a_1} & \ldots\ldots\ldots & i_{n\,a_n} \\ i_{n\,a_1} & X_{a_1 a_1} - \lambda\, g_{a_1 a_1} & \ldots & X_{a_1 a_n} - \lambda\, g_{a_1 a_n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ i_{n\,a_n} & X_{a_n a_1} - \lambda\, g_{a_n a_1} & \ldots & X_{a_n a_n} - \lambda\, g_{a_n a_n} \end{vmatrix} = 0\,. \quad (24)$$

This equation of degree n—1 in λ is called by RICCI the *algebraic characteristic equation* of the congruence $\mathbf{i}_n$. [1]) Since the tensor ${}^2\mathbf{h}$ has, as is known, n—1 real principal directions, the equation (24) has n—1 real roots.[2]) When all roots are distinct (that is when no two roots are equal in *all* points of V_n, which does not exclude that they may be equal in some manifolds of less than n dimensions), we see from (21) that to a definite root λ_j belongs the direction:

$$\mathbf{i}_j = \mu_j\, (\varkappa\nabla \smile \mathbf{i}_n - \lambda_j\, {}^2\mathbf{g})^{-1} \,!\, \mathbf{i}_n\,. \quad \ldots \ldots \quad (25)$$

Two directions belonging to distinct roots are mutual perpendicular, because with regard to (20):

$$\lambda_j\, \mathbf{i}_j \,.\, \mathbf{i}_k = \varkappa \mathbf{i}_k{}^1\, (\nabla \smile \mathbf{i}_n) \,!\, \mathbf{i}_j = \lambda_k\, \mathbf{i}_k \,.\, \mathbf{i}_j = 0, \quad j \neq k \quad . \; . \quad (26)$$

A p-fold root determines a region R_p, perfectly perpendicular to the regions of the other roots, and in this region we may choose p arbitrary mutually perpendicular directions as principal directions.

In every case we can indicate to the given direction $\mathbf{i}_n$ *in every point* n—1 *mutually perpendicular principal directions that join to the congruence* $\mathbf{i}_n$ n—1 *mutually perpendicular congruences* $\mathbf{i}_j, j=1,2, \ldots, n-1$. RICCI calls these congruences the *orthogonal canonical congruences* belonging to $\mathbf{i}_n$. [3])

1) G. RICCI. Dei sistemi di congruenze ortogonali in una varietà qualunque, Memorie R. Acc. Lincei Ser. V 2 (95) 276—322, p. 301.

2) For a direct proof see e.g. G. RICCI, Sui sistemi di integrali independenti di una equatione lineare ed omogenea a derivate parziali di 1^0 ordine, Ann. di Mat. Ser. II 15 ($^{87}/_{88}$) 127—159, p. 134.

3) G. RICCI. Dei sistemi, p. 302. For the sake of brevity we will speak here of canonical congruences. See also G. RICCI and T. LEVI CIVITA, Méthodes de calcul differéntiel absolu, Math. Ann. 54 (01) 125—201; J. E. WRIGHT, Invariants of quadratic differential forms. Cambridge Tracts N^0. 9 (08), p. 73.

For the scalars λ and μ from (20) follows:

$$\lambda_j = \mathbf{i}_j \mathbf{i}_j \overset{2}{.} \nabla \mathbf{i}_n, \quad \ldots \ldots \ldots \quad (27)$$

$$\mu_j = \mathbf{i}_j \mathbf{i}_n \overset{2}{.} \nabla \smile \mathbf{i}_n = \tfrac{1}{2} \mathbf{i}_j . \mathbf{u}_n, \quad \ldots \ldots \quad (28)$$

and from (18) for $'\mathbf{h}$:

$$'\mathbf{h} = \sum_j^{1,\ldots,n-1} \lambda_j \mathbf{i}_j \mathbf{i}_j, \quad \ldots \ldots \ldots \quad (29)$$

and

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \, '\mathbf{h} = 0, \quad \ldots \ldots \ldots \quad (30)$$

or

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \smile \mathbf{i}_n = 0, \quad j \neq k. \quad \ldots \ldots \quad (31)$$

In the special case that $\mathbf{i}_n$ is V_{n-1}-normal, (B) gives:

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \frown \mathbf{i}_n = 0, \quad j \neq k, \quad \ldots \ldots \quad (32)$$

hence ${}_2\mathbf{h} = 0$. Instead of (17) then the equation holds:

$$\nabla \mathbf{i}_n = {}'\mathbf{h} + \mathbf{i}_n \mathbf{u}_n. \quad \ldots \ldots \ldots \quad (33)$$

By means of the idea of geodesic alteration, that is alteration with respect to a geodesically moving system, a simple geometrical interpretation can be given to the canonical directions. In consequence of (17) and (30):

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_n = - \mathbf{i}_k \mathbf{i}_j \overset{2}{.} \nabla \mathbf{i}_n, \quad j \neq k. \quad \ldots \ldots \quad (34)$$

Now $\varkappa \mathbf{i}_k \overset{1}{.} \nabla \mathbf{i}_n$ is the geodesic increment of $\mathbf{i}_n$ when moved in the field along $\mathbf{i}_k$ pro unit of length, and so $\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_n$ is the projection of this specific increment on the j-direction, i.e. the specific geodesic rotation in the $n \to j$ direction. Hence when ${}_2\mathbf{B}_k$ is the bivector of specific geodesic rotation of the system $\mathbf{i}_1, \ldots, \mathbf{i}_n$ when moved in the k-direction:

$$\mathbf{i}_k \overset{1}{.} \nabla \mathbf{i}_\alpha = {}_2\mathbf{B}_k \overset{1}{.} \mathbf{i}_\alpha, \quad \alpha = 1, 2, \ldots, n, \quad \ldots \ldots \quad (35)$$

then $\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_n$ is the nj-component of ${}_2\mathbf{B}_k$:

$$\mathbf{i}_n \mathbf{i}_j \overset{2}{.} \, {}_2\mathbf{B}_k = \mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_n, \quad j \neq k. \quad \ldots \ldots \quad (36)$$

So the nj-component of the rotation ${}_2\mathbf{B}_k$ is equal to the nk-component of the rotation ${}_2\mathbf{B}_j$. [1])

When $\mathbf{i}_n$ is V_{n-1}-normal, we get in consequence of (31) and (32):

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_n = 0, \quad j \neq k. \quad \ldots \ldots \quad (37)$$

Thus the nj-component of ${}_2\mathbf{B}_k$ is zero when $j \neq k$, or: the geodesic

[1]) RICCI, Dei sistemi, p. 303, gives another geometrical interpretation, in which he makes no use of the idea geodesically moving. Then however it is necessary to lay the V_n in a euclidean space of more than n dimensions.

rotation of $\mathbf{i}_n$ when moved in the k-direction occurs in the nk-plane. When we define the principal directions of curvature in a point of the V_{n-1} as the directions $\mathbf{i}_k$ in which the geodesic rotation of $\mathbf{i}_n$ occurs in the nk-plane[1]), we can conclude *that the canonical congruences are the principal directions of curvature of the* $V_{n-1} \perp \mathbf{i}_n$.

Still a consequence of (29) and (33) is

$$\varkappa \mathbf{i}_j \overset{1}{\cdot} \nabla \mathbf{i}_n = \lambda_j \mathbf{i}_j, \quad j = 1, 2, \ldots, n-1, \quad \ldots \quad (38)$$

equation equivalent to (37), which may also be considered as defining equation of the principal directions of curvature of the $V_{n-1} \perp \mathbf{i}_n$.

4. *The second fundamental tensor of the* $V_{n-1} \perp \mathbf{i}_n$. In order to make clearer the signification of ${}^{\prime}\mathbf{h}$, we choose the $V_{n-1} \perp \mathbf{i}_n$ as aequiscalar regions of the original variable x^{a_n} and $n-1$ arbitrary systems of V_{n-1} through $\mathbf{i}_n$ as aequiscalar regions of the original variables $x^\lambda, \lambda = a_1, \ldots a_{n-1}$. Then the directions of $\mathbf{e}_\lambda$ and $\mathbf{e}_\lambda{}'$ lay in the $V_{n-1} \perp \mathbf{i}_n$, while $\mathbf{e}_{a_n}$ and $\mathbf{e}'_{a_n}$ have the direction of $\mathbf{i}_n$:

$$\mathbf{i}_n = \varepsilon_n \mathbf{e}_{a_n} = \varepsilon_n \nabla x^{a_n} = \frac{1}{\varepsilon_n} \mathbf{e}'_{a_n} \quad \ldots \ldots \quad (39)$$

Then the contravariant $\lambda\mu$-characteristic number of ${}^{\prime}\mathbf{h}$ is:

$$ {}^{\prime}h^{\lambda\mu} = \mathbf{e}_\lambda \mathbf{e}_\mu \overset{2}{\cdot} {}^{\prime}\mathbf{h} = \tfrac{1}{2} \mathbf{e}_\lambda \mathbf{e}_\mu \overset{2}{\cdot} \nabla \mathbf{i}_n + \tfrac{1}{2} \mathbf{e}_\mu \mathbf{e}_\lambda \overset{2}{\cdot} \nabla \mathbf{i}_n, \quad \ldots \quad (40)$$

or, because $\mathbf{i}_n \perp \mathbf{e}_\lambda$ and $\perp \mathbf{e}_\mu$, and:

$$\nabla \frown \mathbf{e}_\nu = \nabla \frown \nabla x^\nu = 0, \quad \nu = a_1, \ldots, a_{n-1}, \quad \ldots \quad (41)$$

also:

$$\left.\begin{aligned} {}^{\prime}h^{\lambda\mu} &= -\tfrac{1}{2} \mathbf{i}_n \mathbf{e}_\mu \overset{2}{\cdot} \nabla \mathbf{e}_\lambda - \tfrac{1}{2} \mathbf{i}_n \mathbf{e}_\lambda \overset{2}{\cdot} \nabla \mathbf{e}_\mu = \\ &= -\tfrac{1}{2} \mathbf{e}_\mu \mathbf{i}_n \overset{2}{\cdot} \nabla \mathbf{e}_\lambda - \tfrac{1}{2} \mathbf{i}_\lambda \mathbf{e}_n \overset{2}{\cdot} \nabla \mathbf{e}_\iota = \\ &= -\frac{1}{2\varepsilon_n} (\mathbf{e}_\mu \mathbf{e}'_{a_n} \overset{2}{\cdot} \nabla \mathbf{e}_\lambda + \mathbf{e}_\lambda \mathbf{e}'_{a_n} \overset{2}{\cdot} \nabla \mathbf{e}_p) = \\ &= -\frac{1}{2\varepsilon_n} \left(\frac{\partial a^\lambda}{\partial x^{a_n}} a^\mu + \frac{\partial a^\mu}{\partial x^{a_n}} a^\lambda \right) = \\ &= -\frac{\varkappa}{2\varepsilon_n} (\mathbf{e}'_{a_n} \cdot \nabla) g^{\lambda\mu} = -\tfrac{1}{2} \varkappa (\mathbf{i}_n \cdot \nabla) g^{\lambda\mu}. \end{aligned}\right\} \quad \ldots \quad (42)$$

In the same way the covariant characteristic number of ${}^{\prime}\mathbf{h}$ is:

---

[1]) This definition is the natural extension of the definition of the lines of curvature on a V_2 in R_3 as the lines, along which the normals form a developable surface.

$$\left.\begin{aligned} {}^2h^{\lambda\mu} &= \mathbf{e}_\lambda{}' \, \mathbf{e}_\mu{}' \overset{2}{\;} {}^2\mathbf{h} = \tfrac{1}{2} \, \mathbf{e}_\lambda{}' \, \mathbf{e}_\mu{}' \overset{2}{.} \nabla \, \mathbf{i}_n + \tfrac{1}{2} \, \mathbf{e}_\mu{}' \, \mathbf{e}_\lambda{}' \overset{2}{.} \nabla \, \mathbf{i}_n = \\ &= \frac{1}{2\varepsilon_n} \mathbf{e}_\lambda{}' \, \mathbf{e}_\mu{}' \overset{2}{.} \nabla \, \mathbf{e}_{a_n} + \frac{1}{2\varepsilon_n} \mathbf{e}_\mu{}' \, \mathbf{e}_\lambda{}' \overset{2}{.} \nabla \, \mathbf{e}_{a_n} = \\ &= \frac{1}{2\varepsilon_n} \left(\frac{\partial a_{a_n}}{\partial x^\mu} a_\lambda + \frac{\partial a_{a_n}}{\partial x^\lambda} a_\mu \right) = \frac{1}{2\varepsilon_n} \left(\frac{\partial a_\mu}{\partial x^{a_n}} a_\lambda + \frac{\partial a_\lambda}{\partial x^{a_n}} a_\mu \right) \\ &= \frac{\varkappa}{2\varepsilon_n} (\mathbf{e}'_{a_n} \cdot \nabla) g_{\lambda\mu} = \tfrac{1}{2} \varkappa (\mathbf{i}_n \cdot \nabla) \, g_{\lambda\mu}. \end{aligned} \right\} \quad . \; . \; (43)$$

Hence ${}^2\mathbf{h}$ *is the second fundamental tensor of the* $V_{n-1} \perp \mathbf{i}_n$ [1]).

When $\mathbf{i}_n$ is geodesic without being V_{n-1}-normal, then we have $\mathbf{u}_n = 0$ and $\nabla \mathbf{i}_n$ lies totally in the region $\perp \mathbf{i}_n$:

$$\nabla \, \mathbf{i}_n = \overset{2}{\mathbf{h}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (44)$$

When $\mathbf{i}_n$ is normal too, $\nabla \, \mathbf{i}_n$ is symmetrical:

$$\nabla \, \mathbf{i}_n = {}^2\mathbf{h} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (45)$$

In this latter case choosing x^{a_n} as the length measured from a definite $V_{n-1} \perp \mathbf{i}_n$ along the curves of the congruence $\mathbf{i}_n$, we get:

$$\mathbf{e}_{a_n} = \mathbf{e}_{a_n}{}' = \mathbf{i}_n \quad . \; . \; . \; . \; . \; . \; . \; . \quad (46)$$

5. *Mutually orthogonal* V_{n-1}*-systems through a given congruence when the canonical congruences are singly determined.*

When given an $\mathbf{i}_n$, it is required to choose the original variables $y^1, \ldots y^n{}_{-1}$ in such a way that the corresponding aequiscalar V_{n-1} pass through $\mathbf{i}_n$ and that the vectors $\mathbf{s}_j = \nabla \, y^j$, $j = 1, \ldots, n-1$, are mutually perpendicular.

Hence the system of equations:

$$\mathbf{i}_n \cdot \nabla \, y^j = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (47)$$

$$\mathbf{s}_k \cdot \nabla \, y^j = 0, \quad k \neq j \quad . \; . \; . \; . \; . \; . \; . \quad (48)$$

must allow for every value of j $n-2$ independent solutions. The necessary and sufficient condition is, according to (7):

$$\mathbf{i}_n \overset{1}{.} \; \nabla \, \mathbf{s}_k - \mathbf{s}_k \overset{1}{\;} \nabla \, \mathbf{i}_n = \alpha_k \, \mathbf{s}_k + \alpha_n \, \mathbf{i}_n, \quad . \; . \; . \; . \quad (49)$$

in which α_k and α_n are arbitrary coefficients. Since $\mathbf{i}_n \perp \mathbf{s}_k$, and accordingly:

$$\mathbf{i}_n \overset{1}{.} \; \nabla \, \mathbf{s}_k = (\nabla \, \mathbf{s}_k) \overset{1}{.} \; \mathbf{i}_n = \nabla \, (\mathbf{s}_k \, . \, \mathbf{i}_n) - (\nabla \, \mathbf{i}_n) \overset{1}{\;} \mathbf{s}_k = -(\nabla \, \mathbf{i}_n) \overset{1}{\;} \mathbf{s}_k, \quad (50)$$

(49) is equivalent to:

---

1) Compare BIANCHI-LUKAT, Vorlesungen über Differentialgeometrie (1899) p. 601, form. (7). The principal directions of curvature may also be defined as the principal directions of the second fundamental tensor. So BIANCHI, p. 609, 618.

$$2\,(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{s}_k = -\,\alpha_k\,\mathbf{s}_k - \alpha_n\,\mathbf{i}_n \quad \ldots\ldots \quad (51)$$

This equation however is of the shape (20) and hence each of the desired vectors $\mathbf{s}_k$ forms one of the canonical congruences belonging to $\mathbf{i}_n$. At first we consider the case that the n—1 roots of (24) are all different. In this case every vector $\mathbf{s}_k$ must be equally directed with a definite $\mathbf{i}$:

$$\mathbf{i}_k = \sigma_k\,\mathbf{s}_k = \frac{1}{\sigma_k}\,\mathbf{s}_k' \quad \ldots\ldots \quad (52)$$

The n—1 canonical congruences belonging to $\mathbf{i}_n$ must therefore all be V_{n-1}-normal. In order that this may be so, $\mathbf{i}_n$ has to satisfy certain conditions that may be obtained as follows.

Application of ∇ to (31) gives:

$$(\nabla\,\mathbf{i}_j)\overset{1}{\cdot}(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{i}_k + (\nabla\,\mathbf{i}_k)\overset{1}{\cdot}(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{i}_j + \{\nabla(\nabla \smile \mathbf{i}_n)\}\overset{2}{\cdot}\,\mathbf{i}_j\mathbf{i}_k = 0, j \neq k, (53)$$

and transvection with $\mathbf{i}_n$:

$$\mathbf{i}_n\overset{1}{\cdot}(\nabla\mathbf{i}_j)\overset{1}{\cdot}(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{i}_k + \mathbf{i}_n\overset{1}{\cdot}(\nabla\,\mathbf{i}_k)\overset{1}{\cdot}(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{i}_j + \mathbf{i}_j\,\mathbf{i}_k\,\mathbf{i}_n\overset{3}{\cdot}\,\nabla(\nabla\mathbf{i}_n) = 0. (54)$$

In consequence of (46) and (51) $(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{i}_k$ contains only $\mathbf{i}_k$ and $\mathbf{i}_n$. Further $\mathbf{i}_j$ is V_{n-1}-normal, so that according to (B):

$$\mathbf{i}_n\overset{1}{\cdot}(\nabla\,\mathbf{i}_j)\overset{1}{\cdot}\,\mathbf{i}_k = \mathbf{i}_k\overset{1}{\cdot}(\nabla\,\mathbf{i}_j)\overset{1}{\cdot}\,\mathbf{i}_n \quad \ldots\ldots \quad (55)$$

Hence (54) is equivalent to:

$$-\mathbf{i}_k\overset{1}{\cdot}(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}(\nabla\mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{i}_j - \mathbf{i}_j\overset{1}{\cdot}(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}(\nabla\mathbf{i}_n)\overset{1}{\cdot}\,\mathbf{i}_k + \mathbf{i}_j\,\mathbf{i}_k\,\mathbf{i}_n\overset{3}{\cdot}\,\nabla(\nabla \smile \mathbf{i}_n) = 0. \quad (56)$$

If now

$$'\mathbf{g}_n = \mathbf{a}_n\,\mathbf{a}_n = \mathbf{b}_n\,\mathbf{b}_n = \varkappa \sum_j^{1,\ldots,n-1} \mathbf{i}_j\,\mathbf{i}_j \quad \ldots\ldots \quad (57)$$

we have a quantity:

$$\overset{4}{\mathbf{g}}_n = \mathbf{a}_n\,\mathbf{b}_n\,\mathbf{b}_n\,\mathbf{a}_n. \quad \ldots\ldots \quad (58)$$

which when transvected twice with an arbitrary affinor of second degree gives the component of this affinor in the $R_{n-1} \perp \mathbf{i}_n$. Introducing the tensor:

$$'\mathbf{p} = \overset{4}{\mathbf{g}}_n\overset{2}{\cdot}\{(\mathbf{i}_n\,.\,\nabla)(\nabla \smile \mathbf{i}_n) - 2\,T\,(\nabla \smile \mathbf{i}_n)\overset{1}{\cdot}(\nabla\,\mathbf{i}_n)\}, \quad \ldots \quad (59)$$

we get from (56):

$$\boxed{\mathbf{i}_j\,\mathbf{i}_k\overset{2}{\cdot}\,'\mathbf{p} = 0, \quad j \neq k, \quad j, k = 1, 2, \ldots, n-1} \quad \ldots \quad (C')$$

Hence the first condition is that the tensor $'\mathbf{p}$ *has the same principal directions as* $'\mathbf{h}$.

Since on account of (19):

$$\overset{4}{\mathbf{g}}_n\overset{2}{\cdot}(\mathbf{i}_n\,.\,\nabla)(\nabla \smile \mathbf{i}_n) = \overset{4}{\mathbf{g}}_n\overset{2}{\cdot}(\mathbf{i}_n\,.\,\nabla)\,'\mathbf{h} + \varkappa\mathbf{u}_n\,\mathbf{u}_n, \quad \ldots \quad (60)$$

and on account of (19) and (30):

$$\overset{4}{\mathrm{g}}_n \overset{2}{\cdot} 2\,T(\nabla \smile \mathrm{i}_n) \overset{1}{\cdot} (\nabla\, \mathrm{i}_n) = 2\,{}^2\mathrm{h} \overset{1}{\cdot} {}^2\mathrm{h} + \varkappa \mathrm{u}_n \mathrm{u}_n + \overset{4}{\mathrm{g}}_n \overset{2}{\cdot} 2\,T\,{}^2\mathrm{h} \overset{1}{\cdot} {}_2\mathrm{h}, \quad (61)$$

we have:

$$ {}^2\mathrm{p} = \overset{4}{\mathrm{g}}_n \overset{2}{\cdot} (\mathrm{i}_n \cdot \nabla)\, {}^2\mathrm{h} - 2\,T\,{}^2\mathrm{h} \overset{1}{\cdot} {}_2\mathrm{h} - 2\,{}^2\mathrm{h} \overset{1}{\cdot} {}^2\mathrm{h}. \quad . \quad . \quad . \quad (62)$$

Since ${}^2\mathrm{h} \overset{1}{\cdot} {}^2\mathrm{h}$ has the same principal directions as ${}^2\mathbf{h}$, *we may express the first condition also in another way, viz. that*

$$\overset{4}{\mathrm{g}}_n \overset{2}{\cdot} \{(\mathrm{i}_n \cdot \nabla)\, {}^2\mathrm{h} - 2\,T\,{}^2\mathrm{h} \overset{1}{\cdot} {}_2\mathrm{h}\}$$

has the same principal directions as ${}^2\mathbf{h}$:

$$\boxed{\mathrm{i}_j\, \mathrm{i}_k \overset{2}{\cdot} \{(\mathrm{i}_n \cdot \nabla)\, {}^2\mathrm{h} - 2\,T\,{}^2\mathrm{h} \overset{1}{\cdot} {}_2\mathrm{h}\} = 0,\ j \neq k,\ j,k = 1, 2, .., n-1\,.} \quad (C)$$

(C) can also directly be found when we start from (30) and reason in the same way as we did when deriving (C').

In order to get a second condition we apply ∇ to (30) and after that we transvect with i_n. This gives:

$$\{(\mathrm{i}_l \cdot \nabla)\, \mathrm{i}_j\}\, \mathrm{i}_k \overset{2}{\cdot} {}^2\mathrm{h} + \{(\mathrm{i}_l \cdot \nabla)\, \mathrm{i}_k\}\, \mathrm{i}_j \overset{2}{\cdot} {}^2\mathrm{h} + \mathrm{i}_j\, \mathrm{i}_k \overset{2}{\cdot} (\mathrm{i}_l \cdot \nabla)\, {}^2\mathrm{h} = 0 \quad . \quad (63)$$

or, according to (29):

$$\lambda_k\, \mathrm{i}_k\, \mathrm{i}_l \overset{2}{\cdot} \nabla\, \mathrm{i}_j + \lambda_j\, \mathrm{i}_j\, \mathrm{i}_l \overset{2}{\cdot} \nabla\, \mathrm{i}_k + \varkappa\, \mathrm{i}_j \mathrm{i}_k \overset{2}{\cdot} (\mathrm{i}_l \cdot \nabla)\, {}^2\mathrm{h} = 0 \quad . \quad . \quad (64)$$

or:

$$(\lambda_k - \lambda_j)\, \mathrm{i}_k\, \mathrm{i}_l \overset{2}{\cdot} \nabla\, \mathrm{i}_j + \varkappa\, \mathrm{i}_j\, \mathrm{i}_k \overset{2}{\cdot} (\mathrm{i}_l \cdot \nabla)\, {}^2\mathrm{h} = 0 \quad . \quad . \quad . \quad . \quad (65)$$

The vectors i_j, i_k and i_l being all V_{n-1}-normal and mutually perpendicular, so that

$$\left.\begin{aligned} &\mathrm{i}_k\, \mathrm{i}_l \overset{2}{\cdot} \nabla\, \mathrm{i}_j = \mathrm{i}_l\, \mathrm{i}_k \overset{2}{\cdot} \nabla\, \mathrm{i}_j = -\, \mathrm{i}_j\, \mathrm{i}_k \overset{2}{\cdot} \nabla\, \mathrm{i}_l = -\, \mathrm{i}_k\, \mathrm{i}_j \overset{2}{\cdot} \nabla\, \mathrm{i}_l = \\ &= \mathrm{i}_l\, \mathrm{i}_j \overset{2}{\cdot} \nabla\, \mathrm{i}_k = \mathrm{i}_j\, \mathrm{i}_l \overset{2}{\cdot} \nabla\, \mathrm{i}_k = -\, \mathrm{i}_k\, \mathrm{i}_l \overset{2}{\cdot} \nabla\, \mathrm{i}_j, \end{aligned}\right\} \quad (66)$$

or:

$$\mathrm{i}_k\, \mathrm{i}_l \overset{2}{\cdot} \nabla\, \mathrm{i}_j = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

the equation (65) is equivalent to

$$\boxed{\mathrm{i}_j\, \mathrm{i}_k \overset{2}{\cdot} (\mathrm{i}_l \cdot \nabla)\, {}^2\mathrm{h} = 0, \quad j \neq k, k \neq l, l \neq j, \quad j, k, l = 1, 2, .., n-1\,.} \quad (D)$$

Since $(\nabla \mathbf{i}_j) \overset{1}{\cdot} \mathbf{i}_j$ and in consequence $(\mathbf{i}_l . \nabla)\, \nabla\, (\mathbf{i}_j . \mathbf{i}_j)$ is zero, so that:

$$\mathrm{i}_j\, \mathrm{i}_j \overset{2}{\cdot} (\mathrm{i}_l \cdot \nabla)\, {}^2\mathrm{h} = (\mathrm{i}_l \cdot \nabla)\, (\mathrm{i}_j\, \mathrm{i}_j \overset{2}{\cdot} {}^2\mathrm{h}), \quad . \quad . \quad . \quad . \quad (68)$$

also the principal directions of ${}^2\mathbf{h}$ are singly determined.

The tensor $\varkappa ds\, \mathbf{i}_l \overset{1}{\cdot} \nabla\, {}^2\mathbf{h}$ being the geodesic differential of ${}^2\mathbf{h}$, when moved over ds in the direction of $\mathbf{i}_l$, the second condition (D) expresses *that by an infinitesimal translation in a direction perpendicular to* $\mathbf{i}_n$ *and perpendicular to* $m \leqq n-2$ *of the canonical directions belonging to* $\mathbf{i}_n$, *the component of the geodesic differential of* ${}^2\mathbf{h}$ *in the* R_m, *determined by these* m *directions, has principal directions coinciding with* m *of the principal directions of* ${}^2\mathbf{h}$.

The two conditions (C) and (D) are not only necessary, but also sufficient. Indeed, from (C) and (C'), which are perfectly equivalent, we conclude (56) and from (30) we conclude (54). Comparing (54) and (56), we get (55). From (D) we get, when comparing with (65), which results from (30), the equation (67). But (55) and (67) show that congruence $\mathbf{i}_j$ is V_{n-1}-normal.

Finally we have the result:

The necessary and sufficient conditions that we can bring $n-1$ mutually orthogonal V_{n-1} through the congruence $\mathbf{i}_n$, whose corresponding algebraic characteristic equation has but unequal roots, are that $\mathbf{i}_n$ satisfies the equations (C) and (D) [1].

The number of the equations (C) is $\frac{(n-1)(n-2)}{2}$, the number of the equations (D) is $\frac{(n-1)(n-2)(n-3)}{2}$, according to the fact that j and k may be interchanged without creating a new equation, but not j and l [2]. Considered as differential equations in the characteristic numbers of $\mathbf{i}_n$ both systems (C) and (D) are of second order.

6. *Simplifications for the case that the given congruence is V_{n-1}-normal.* If $\mathbf{i}_n$ is V_{n-1}-normal, then ${}_2\mathbf{h} = 0$, and ${}^2\mathbf{g}_n$ is the first and ${}^2\mathbf{h}$ the second fundamental tensor of the $V_{n-1} \perp \mathbf{i}_n$. Its principal directions determine the directions of principal curvature. (61) changes into:

$$\overset{4}{\mathbf{g}}_n \overset{2}{\cdot} 2\, T(\nabla \smile \mathbf{i}_n) \overset{1}{} (\nabla \mathbf{i}_n) = 2\, {}^2\mathbf{h} \overset{1}{\cdot} {}^2\mathbf{h} + \varkappa \mathbf{u}_n \mathbf{u}_n, \quad \ldots \quad (69)$$

and (62) into:

$${}^2\mathbf{p} = \overset{4}{\mathbf{g}}_n \overset{2}{\cdot} (\mathbf{i}_n \,.\, \nabla)\, {}^2\mathbf{h} - 2\, {}^2\mathbf{h} \overset{1}{\cdot} {}^2\mathbf{h} \quad \ldots \quad (70)$$

(C) changes into:

$$\boxed{\mathbf{i}_j \mathbf{i}_k \overset{2}{\cdot} (\mathbf{i}_n \,.\, \nabla)\, {}^2\mathbf{h} = 0,} \quad \ldots \quad (C_1)$$

and gets with this the same shape as (D).

In the same way as with (68) we see here:

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{\cdot} (\mathbf{i}_n \,.\, \nabla)\, {}^2\mathbf{h} = (\mathbf{i}_n \,.\, \nabla)(\mathbf{i}_j \mathbf{i}_k \overset{2}{\cdot} {}^2\mathbf{h}), \quad \ldots \quad (71)$$

hence also the principal directions of $\varkappa(\mathbf{i}_n \,.\, \nabla)\, {}^2\mathbf{h}$ are singly determined.

---

1) (C') is deduced for the first time by Ricci, Dei sistemi, vergel. (A), p. 309. (D) has in his paper a less simple shape, in our notation:

$$\mathbf{i}_j \mathbf{i}_k \mathbf{i}_l \overset{3}{\cdot} \nabla (\nabla \smile \mathbf{i}_n) = \tfrac{1}{2} (\mathbf{i}_k \,.\, \mathbf{u}_n)\, \mathbf{i}_j \mathbf{i}_l \overset{2}{\cdot} \nabla \mathbf{i}_n + \tfrac{1}{2} (\mathbf{i}_j \,.\, \mathbf{u}_n)\, \mathbf{i}_k \mathbf{i}_l \overset{2}{\cdot} \nabla \mathbf{i}_n, \qquad (D')$$

and is denoted equation (B), p. 309. (D') results when we apply the operation $(\mathbf{i}_l \,.\, \nabla)$ to (31). (C) and (D) are consequences of (30), (C') and (D') of (31). Here we have first deduced (C'), because the condition in this shape is identical with the condition given for R_3 by Lilienthal, which is very important for the problem, as may be seen in the second part of this paper.

2) G. Ricci, Sui sistemi, p. 152.

In connection with the already given geometrical interpretation of (D) we get the following theorem:

I. *A system of* ∞^1 V_{n-1} *in a* V_n, *whose second fundamental tensor* $^2\mathbf{h}$ *has* $n-1$ *principal directions that are singly determined except on determined* V_r, $r < n-1$, *belongs then and only then to an n-tuple orthogonal system, if the component of the geodesic differential of* $^2\mathbf{h}$, *when moved perpendicular to m of the principal directions of* $^2\mathbf{h}$, *has in the* R_m *determined by these m directions principal directions coinciding with the denoted m principal directions of* $^2\mathbf{h}$.

This theorem is given for a system of V_2 in R_3 by MAURICE LEVY [1]).

When the principal directions of $^2\mathbf{h}$ for a certain point P are not singly determined, we conclude from (68) that also the principal directions of $\varkappa(\mathbf{i}_n \, . \, \nabla)\, ^2\mathbf{h}$, and hence those of $^2\mathbf{h}$, for all points of a curve of the congruence $\mathbf{i}_n$ through P are undetermined in the same way. From this we see:

II. *In a system of* ∞^1 V_{n-1} *in a* V_n *belonging to an n-tuple orthogonal system all points in which all or some directions of principal curvature are not singly determined (umbilics in a wider sense) are arranged on loci consisting of curves of the congruence orthogonal on the* V_{n-1}.

Also this theorem is first given for a system of V_2 in R_3 by M. LÉVY. [2])

Since the characteristic numbers of $\mathbf{i}_n$ may be expressed in the first differential quotients of the parameter determining the system of the V_{n-1}, the equations (C) and (D) are partial differential equations of the *third* order in this parameter. [3])

---

[1]) M. LÉVY. Mémoire sur les coördonnées curvilignes orthogonales. Journal de l'Ecole Imp. Polytechnique 26 (70) 157—200, p. 159.

[2]) M. LÉVY. Mémoire etc. p. 174.

[3]) For a short survey and a discussion of the literature of this differential equation of the third order for a system of V_2 in R_3 [for here (D) disappears and the system (C) reduces to one equation] see e.g. LUCIEN LÉVY, Sur les systèmes de surfaces triplement orthogonaux. Mém. couronnés et Mém. des savants étrangers, Bruxelles 54 (96), 89 p., p. 5 and following pages.

Geology. — "*On Foraminifera-bearing Rocks from the Basin of the Lorentz River (Southwest Dutch New-Guinea)*". By Dr. L. RUTTEN, correspondent of the Academy.

(Communicated at the meeting of October 25, 1919).

The Dutch South-New-Guinea Expeditions of 1907 and 1909 made a pretty large collection of rocks from the basin of the river Lorentz (North-River). Many specimens of rocks bore Foraminifera of Tertiary age. They will be briefly described in the present paper.

After the latest résumé of the Foraminefera literature [1]) of New-Guinea, only one more publication was brought forward by R. BULLEN NEWTON [2]), in which are described some Lepidocyclina-bearing limestones, found near the snowline on the summit of mount Carstensz. For the literature we, therefore, refer to this résumé.

The rocks of the above-mentioned expeditions were taken from a zone, of which K. MARTIN [3]) described numerous fossils as early as 1911; in this zone, extending from the 137th degree to 141th degree longitude, the basin of the Lorentz-River covers only a small area. It could, therefore, be anticipated that the collection — with regard to the boulders it contained — would not open up many new viewpoints. Of course the fragments struck from the solid rock were of greater interest; they are however few in number.

For geological purposes, therefore, we do not feel called upon to give a detailed discussion of the collection. Neither did paleontological considerations require more than a short description of the material. The vast majority of the fragments are limestones, in which Lepidocyclinae and Nummulites predominate. These fossils cannot be removed from the rock and must, therefore, be examined from microscopical sections. True, our knowledge of the systematic arrangement of the Indian Lepidocyclinae has been somewhat clarified, but the various species can with difficulty be

[1]) L. RUTTEN. In Nova Guinea, VI. 2. 1914, p. 22—25.

[2]) R. BULLEN NEWTON. Organic Limestones etc. from Dutch New Guinea. Reports on the collections made by the British Ornithological Union Expedition and the Wollaston Expedition in Dutch New Guinea, 1910—1913, Vol. II. Report 20, 1916. For British New-Guinea see also R. BULLEN NEWTON, Geol. Mag. (6) V. 1918, p 203—212.

[3]) K. MARTIN. Samml. Geol. Reichsmus. Leiden (1). IX. 1911, p. 84 e.v.

distinguished, even when separate individuals are examined. A minute study of thin sections is, therefore, most often disappointing. It is even more difficult in the case of Nummulites: the specific differences of the Indian species being very little known, so that even the determination of isolated individuals is often difficult.

Both for geological and for paleontological reasons we shall, therefore, confine ourselves to a brief description.

1). *The Lepidocyclina limestones.* Boulders with Lepidocyclinae have been found in the Lorentz-River near Sabang and Geitenkamp, in the Bibis-River (Van der Sande-River) and where the Koekoek-River, flows into the Reiger-River[1]). Solid rock of Lepidocyclina limestone was detected in the Resi-chain, the Went-mountains, on Mount Permadi and near the Perameles-bivouac. The limestones belong to various types.

Most numerous are pure gray to brownish-gray limestones, invariably distinguished by the occurrence of large Lepidocyclinae. In addition mostly Heterosteginae and often minute Nummulites occur. (Boulders: Sabang no. 85. 1907, 111a. 1907, Geitenkamp no. 195. 1907, Bibisriver no. 544. 1909, Koekoekriver no. 385. 1907; Solid Rock: Went-mountains no. 631. 1909, and Perameles Bivouac nos 629 and 930. 1909). The diameter of the Lepidocyclinae is mostly over 15 mm., in one boulder (544. 1909) even more than 40. They are macrospherical and the first chamber is completely invested by the second. The fossils are slightly lenticular and do not possess a distinct median tubercle. Columns of an intermediate skeleton are sometimes absent; they occur, however, in most fossils and are distributed irregularly over the whole test. They are clearly coniform, their diameter is mostly smaller than that of the lateral chambers, sometimes they become bigger and invest the lateral chambers in the tangential section. These Lepidocyclinae belong to the group of the Lepidocyclina insulaenatalis J. a. Ch.

The Heterosteginae can hardly be distinguished from the recent H. depressa d'Orb.; in one fragment (544. 1909) the nummuliform portion is strongly developed, so that the fossils resemble H. margaritata SCHLUMBERGER[2]).

The Nummulites belong to the minor forms intermediate between Nummulites and Operculina, of which i. a. VERBEEK has described N. Niasi II and N. Dungbrubusi. Their diameter is only 2 m.m., the number of whorls is 3 to 4. They resemble Nummulites in

[1]) For topographical details see the sketch-map in Bulletin 64 of the "Maatschappij ter Bevordering van het Natuurkundig Onderzoek der Ned. Koloniën", 1910.

[2]) C. SCHLUMBERGER. Samml. Geol. Reichsmus. Leiden (1). VI. 1902, p. 250—252.

40*

the slow increase of the height of the whorls, they are like Operculina with regard to the inconsiderable investment and the texture of the wall. They are of no stratigraphical significance whatever.

Besides the fossils recorded also the following occur: small Lepidocyclinae (629. 1909), Cycloclypeus (85. 1907),? Polystomella (85. 1907), Gypsina (630 and 631, 1909), Rotalidae (544. 1909), and? Orbitolites (85. 1907). The occurrence of Alveolinae with only one layer of chambers per whorl (85. 1907) strikes us as very remarkable. Also in West New-Guinea near Karas traces of these primitive Alveolinae were found in a Lepidocyclina limestone [1])

It appears from the frequent occurrence of large Lepidocyclinae and of primitive Alveolinae, that the rocks described belong to the oldest Miocene or to the Oligocene.

A boulder from the slope of mount Permadi (452 m.) (no. 353 1907) differs from the above rocks only in that among the fossils small Lepidocyclinae prevail. Very much like it again is a gray limestone from the Went-chain, which contains besides very few large Lepidocyclinae and numerous Heterosteginae, also Gypsina, Alveolina s.str., Amphistegina,? Calcarina, Rotalidae and Lithothamnium (N°. 623, 1909).

A very striking difference exists between the rocks thus far described, and a boulder from Sabang (N°. 84. 1907), which abounds in Corals and Lithothamnia, but contains only traces of Lepidocyclinae.

A boulder from the Koekoek-River (N°. 328, 1907) contains Cycloclypeus annulatus Martin, small Lepidocyclinae with very thick tubercles, Globigerinae and Corals.

Four rocks from the Resi-mountains N°. 310, 311, 312, 361. 1907) and a boulder from Geitenkamp (N°. 170, 1907), make up the rear in the series of Lepidocyclina-rocks. They are all grayish-white, rather crystalline limestones, which no doubt belong together, though not all of them are characterised by typical fossils. Small Lepidocyclinae occur in 170 and 310. 1907, badly preserved Nummulinidae in 310, 311, 361. 1907, other badly preserved Foraminiferae in 170, 310, 311. 1907; Lithothamnium in 170 and 311. 1907 and badly preserved Corals in 170, 310, 361. 1907.

It has thus been proved that in the basin of the Bibis-, Lorentz-, and Reiger-Rivers neogenic Lepidocyclina-bearing rocks must occur. Of the two first-named basins also the solid rock is known: *in the region between the Resi-mountains in the South and Mount*

---

[1]) L. Rutten. Nova Guinea, l. c., p. 38.

Permadi in the North old Miocene to Oligocene deposits are spread over a vast area.

All Lepidocyclina limestones are very pure.

II. *The Operculina and Heterostegina-limestones.* Three more limestones originate from the neighbourhood of the Went-mountains, which differ rather much from the preceding. Two boulders, each of which bear the numbers 542. 1909 were found in the Bibis-River. The one is a rather crystalline, reddish-gray limestone with scanty quartz-splinters. It contains many badly preserved Heterosteginae. The other is a glauconitic limesandstone, in which besides numerous grains of quartz many Operculinae and traces of Heterosteginae are seen.

A whitish-gray, sugar-grained limestone from mount Permadi (N°. 349. 1907) contains besides numerous quartz-grains badly preserved small Rotalidae, Textularidae, Miliolidae and other small Foraminifera-remains, also numerous small Operculinae.

A glauconitic, quartz-rich limestone from the Went-mountains (N°. 616. 1919) contains besides occasional Operculinae and Miliolidae, also very numerous small Heterostegina. They are small (horizontal section 2—3 mm., vertical section 1 mm.) knob-shaped fossils; built up almost entirely of spirals, which embrace each other. The surface is covered with numerous thick tubercles, which constitute the basal part of conic columns of intermediate skeleton. These tubercles are connected by thin, irregular bands. In the transverse section these fossils bear a strong likeness to small Nummulites, in the median section, however, we see that they are small Heterosteginae, which nearly always lack the peripheral, evolute skeleton part common among this genus. They are individuals of Heterostegina (Spiroclypeus) pleurocentralis Carter.

Nothing can be said for certain about the age of the discussed Operculina- and Heterostegina-limestones, but it is very probable they belong to the same formation as the Lepidocyclina-limestones. It is remarkable that all the Operculina-Heterostegina-limestones contain quartz-grains or quartz-splinters, that even some are true lime-sandstones.

III. *Crystalline limestones from the Hellwig chain.* Highly crystalline sugargrained, gray to grayish-red limestones without recognizable fossil-remains were collected in the Hellwig-mountains (N°. 337, 1907; 650, 652, 666, 1909) and on mount Kristal (N°. 342, 1907). We must admit complete ignorance of their age.

IV. *The Alveolina- and the Lacazina-limestones.* A number of Eocene limestones, in which Alveolinae or Lacazinae predominated alternately, were collected as boulders or were found as solid rock. They are invariably blackish gray rocks, in which the fossils are to be recognized as white spots.

Five boulders (Sabang N°. 117, 1907. Alkmaar N°. 286a. 1907, and Geitenkamp N°. 187*a* 1907 Bibisriver N°. 527. 1907, affluent of the Bibisriver N°. 738. 1909) belong to the Alveolina-limestones. They are all compact, gray limestones with occasional quartz-splinters. They abound in Alveolinae, which we refer to the primitive type. Some are pointed and spindle-shaped and must be referred to A. Wichmanni Rutten, others are rather ellipsoid and may be classed among A. Javana Verbeek. Besides these, small Rotalidae and Miliolidae present themselves. Furthermore there are numerous small Nummulinidae, more than 5 mm. in length, which belong to Operculina or to the Nummulites. They bear a strong resemblance to the forms intermediate between these two genera, which are known to us from the Eocene Alveolina-limestone of Tandjung Seilor (East Borneo)[1]).

Another Alveolina-limestone, probably also a boulder, was found on the northern slope of Geluksheuvel (N°. 320. 1907). The limestone is considerably discoloured; it contains the same petrifications as the boulders described above. We are struck with the irregular forms presented by many Alveolinae. They are most likely the consequence of stunted growth.

Three grayish-white limestones, resembling the preceding specimens very much, were collected on the solid rock of Wilhelminatop (N^rs 707, 709, 712, 1909). Besides very numerous individuals of Lacazina Wichmanni Schl. also rests of Alveolina Wichmanni and Miliolidae, Rotalidae and small Nummulinidae occur in this rock. Although the limestones of Wilhelminatop belong to the same type as the boulders from the Lorentz- and Bibis-rivers, the latter must take their origin from another source, as the distribution of the fossils in the rocks is different; the boulders do not contain any Lacazinae, which on the other hand predominate on Wilhelminatop.

The Alveolina- and Lacazina-limestones decidedly belong to the Old Tertiary.

The purity of these limestones is remarkable, they seldom present quartz-splinters.

V. *Nummulina-Alveolina limestone.*

A blackish-gray limestone from a boulder bank near Sabang looks at first sight very much like the Alveolina-limestone described (N^r 144. 1907). We see, however, in the sections that Alveolinae

1) L. RUTTEN. Samml. Geol. Reichsmuseum Leiden (1). X. 1915, p. 10.

occur very rarely in the rock, and that the Nummulites are far superior in number. The Alveolinae are very remarkable. They must undoubtedly be called a plain type. Their shape is quite irregular, it adapts itself entirely to the interspaces left by the Nummulites in the rock; the Alveolinae fill up the spaces between the "idiomorphous" Nummulites, so to say, in an "allotriomorphous" shape. (Fig. 1 and 2). Since the latter never present any marked deformations, it is not admissible to assume that the deformations of Alveolinae have originated through mountain-pressure, after the animals died off. Most likely these deformations are the result of a stunted growth; the Alveolinae grew at the bottom among dead Nummulitic shells, and had to conform their shapes to the surroundings.

Nummulites seem to be of various species. For the greater part they are small species (horizontal section 4—5 mm., vertical section 1½—2 mm.), which have only about 6 whorls. Skeleton-columns are not numerous; where they do present themselves, they are always conical. Sometimes we note a central tubercle. The septal bands seem to be radial, wavy lines.

These Nummulites seem to belong to the group of N. Bagelensis

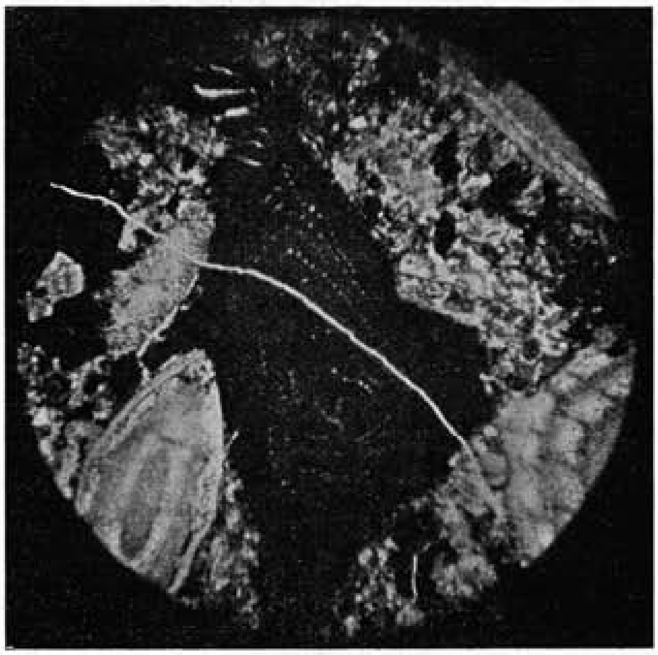


Fig. 1

VERBEEK. Besides these also larger forms occur (diameter more than 10 mm.), whose septal bands are sparingly covered with tubercles.

This limestone contains but few quartz-splinters.

VI. *Nummulitic Limestones.* A great number of Nummulitic limestones were found as boulders. Probably of one finding place we also possess the solid rock. On the Northwestern and Southwestern

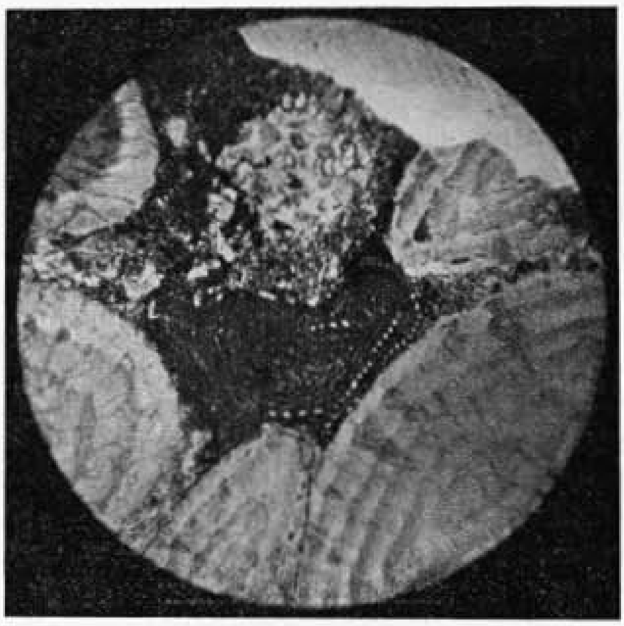


Fig. 2.

slope of Mount Permadi two Nummulitic limestones were found, one of which is probably a boulder (N°. 350. 1907), while the other may have arisen from the solid rock (N°. 347, 1907). Both are rich in fossils and have only few quartz-splinters. They are faintly glauconitic. By far the greater number of fossils are Nummulites, whose size seldom exceeds 15 mm.; moreover Lithothamnium occurs.

A boulder of red Nummulitic lime, enclosing many small quartz-splinters, was found in the Bibisriver (N°. 559. 1909). It contains many small and large Nummulites with retiform septalbands and a few Operculinae.

In the Lorentz-River boulders of Nummulitic lime were found near Sabang (N°. 111. 1907), Geitenkamp (N°. 196. 1907) and Alkmaar (N°. 243, 286. 1907).

N°. 111. 1907 is a grayish-brown Nummulitic Breccia with a little glauconite and traces of quartz. Besides large and small Nummulites, also a very few Lithothamnia occur. N°. 196. 1907 is also a Nummulitic breccia with traces of Alveolina. The Nummulites with a diameter not larger than 15 mm. possess maeandriform septalbands. Nummulites with such septalbands also occur in N°. 243, 1907. They are located together with rare Alveolinae in a red ground-mass of limestone. The limestone N°. 286. 1907 rich in quartz-splinters bears half-sized Nummulites and some Alveolinae.

Three boulders of Nummulitic limestones were found in the Schultz-River, an eastern affluent of the Lorentz-River (N^rs^ 796, 797, 798. 1909). They are all of them gray limestones, rich in splinters and rounded quartz-grains, in which many large and small Nummulites present themselves. In part the Nummulites possess reticulated septalbands. By the side of Nummulina, Miliola occurs and perhaps also Operculina.

Finally at the place where the Koekoek River flows into the Reiger River a limestone with scarce quartzsplinters was found (N^r^ 388. 1907), which contains besides dubious Heterosteginae, numerous Nummulites (largest size 15 m.m.) with retiform septalbands.

Whereas the Lepidocyclina-limestones described at the outset, and probably also the Operculina-Heterostegina rocks, belong to the Neogene, the Alveolina-Lacazina-rocks and the Nummulitic limes are unquestionably to be referred to the Eogene. The absence of Assilinae and Orthophragminae and the frequent occurrence of Nummulites with retiform septalbands, point to the fact that we have to do with the younger parts of the Eogene.

Some very young clayey, Foraminifera-bearing rocks, easily decomposing in water, were found south of the mountain-zone. In clay from Kruisheuvel near Sabang (N^r^ 107. 1907), which hill has partaken of the latest mountainfolding, numerous Rotalidae were found. Many Polystomellae occur in clay that occurs as a solid rock on Zuilenheuvel, not far from Geitenkamp, (N^r^ 177*a*. 1907). A sort of clay, found near Alkmaar, is rich in several stratigraphically insignificant Textularidae and Miliolidae (N^r^ 258*a*. 1907). The three clays recorded here very much resemble the young tertiary marls, known from Timenâ in Northern New-Guinea. [1]) Probably also their ages are about the same.

Finally we have still to record some non-typical Foraminifera-bearing rocks. A boulder from Alkmaar (N°. 224. 1907) is a glauconitic, dark gray lime with Lithothamnium, Corals, ? Bryozoa, ? Orbitolites and a very small Nummulinida, probably Polystomella of undoubted Tertiary age. In the Bibis River a solid limestone was found, containing besides numerous Lamellibranchiatae and Corals also small individuals of Polystomella cf. craticulata F. and M. This rock is also Tertiary.

[1]) L. Rutten. Nova Guinea, l. c., p. 34.

Besides the rocks described thus far, a number of other limestones and limesandstones from the basin of the Lorentz River were examined, in which however no Foraminifera appeared to occur.

It is remarkable that in none of the rocks examined, which in part were rather coarse-clastic, volcanic materials occur. In this respect, then, there is a sharp contrast between North- and South-New-Guinea.

Buitenzorg, November 1916.

Physics. — *"On the Effective Temperature of the Sun."* (2nd Communication). By H. GROOT. (Communicated by Prof. H. W. JULIUS).

(Communicated at the meeting of September 27, 1919).

In a previous article of March 1919 it was demonstrated that the determination of the effective solar temperature by the application of PLANCK's radiation formula to the data of ABBOT, does not lead to a same temperature, independent of the considered kind of light, as estimated by A. DEFANT, but on the contrary that the value of T determined in this way varies systematically as λ.

The meaning of the results so found will be examined in this article.

It is necessary beforehand to define as strictly as possible what we mean by the term "effective temperature", as the same meaning is not always attached to this expression.

The reason that we cannot simply speak of the sun's temperature is, first, that the sun has not the same temperature at all depths (thermodynamics show that for an extensive gas-mass — we must consider the sun as such — the temperature varies from layer to layer), and secondly that we cannot either indicate the temperature of a definite layer nor know the way in which the temperature depends on the distance from the centre of the sun.

We *can* however find what temperature we should have to assign to the sun, so that, if it were an absolutely black body, it would behave *in a definite respect* exactly in the same way as we observe in reality.

We may ask for example, what temperature an "absolutely black sun" must have if the position of maximum intensity in its spectrum is to be the same as in the real spectrum; or if the solar constant is to have the same value as the constant that has been determined experimentally. The first question may be answered by the aid of WIEN's law; the second question by the application of the formula of STEFAN-BOLTZMANN.

The temperature thus found is called "effective" temperature.

Since, however, the sun is not an absolutely black body, we need not be surprised that the effective temperatures of the sun, which are found in these different ways, are not equal. It is, therefore, necessary when indicating the effective temperature of the sun, to state clearly beforehand from what condition imposed on the temperature of the absolutely black body which we think as taking the sun's place, it has been determined.

In what follows the condition has been chosen that the distribution of energy in the spectrum of the black body, calculated according to the law of PLANCK, will agree as closely as possible with that in the sun's spectrum, as has been derived by ABBOT from bolograms.

Accordingly, the temperature which we should have to assign to such a "black sun", if this condition is to be fulfilled, is the effective temperature, which will be discussed in this article, and which, we have found, appears to be dependent on the chosen λ.

The relation between T and λ is once more given below in table 1.

TABLE I.

λ_1	λ_2	T
0.4 m	0.5	(6400)
0.5	0.6	9000
0.6	0.7	10.000
0.7	0.8	9600
0.8	1.0	8000
1.0	1.2	5500
1.2	1.5	3800
1.5	1.8	(5400)
1.8	2.0	—

The way in which the values of T have been calculated is briefly as follows,

From PLANCK's formula:

$$f \cdot I_\lambda = \frac{7{,}211 \cdot 10^8}{\lambda^5\left(10^{\frac{2{,}1562 \times 2890}{\lambda T}} - 1\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is obtained for any value of T, and with any choice of the units (factor f), a definite curve $I_\lambda = \varphi(\lambda)$, which represents the distribution of intensity in the spectrum of the absolutely black body.

Conversely f and T can be found when I_λ is known for two values of λ.

The observed energy spectrum of the sun does not agree, however, with that of the black body, so that if we do apply PLANCK's formula for the calculation of f and T from the experimentally determined I_λ, the value of T will depend on the place where we choose I_λ.

In table I the 1st and 2nd columns give the values of λ from whose corresponding I_λ the T of the third column has been calculated. It seems to me that the application of PLANCK's formula to the experimentally determined energy spectrum of the sun's radiation has not much sense, unless we could really consider this as almost agreeing with the spectrum of an absolutely black body — the criterion of which would consist in finding the same T from arbitrarily chosen combinations of I_λ.

I wrote already in my previous article:

"The assumption that all kinds of light come to us from one photospheric surface, in other words that light of various wave-lengths should come from the same depth of the sun, appears more and more untenable.....

If, however, in reality light of different wave-lengths originates from different parts of the sun, it becomes very questionable whether we shall be allowed to apply PLANCK's formula, as we saw DEFANT do".

Instead of imagining one photospheric surface, as did DEFANT, we might try, what the supposition leads to that the sun is built up of a number of concentric "partial photospheres", each of them radiating as an absolutely black body, so that the total observed radiation is considered as built up of a number of partial radiations originating from different layers. In my previous article I pronounced the expectation that on this supposition, considering the fact that it seems to follow from the work of SPIJKERBOER and VAN CITTERT that in general we must look deeper into the sun for red light than for violet, the effective temperature would increase with the wave-length.

This expectation has proved erroneous. And on closer consideration it was, indeed, unfounded. The effective temperature of a layer can, in fact, only be derived from the distribution of energy in its spectrum — and the said result of SPIJKERBOER and VAN CITTERT teaches us nothing about this. It is, however, worth while to examine the hypothesis of the "partial photospheres", because this may, perhaps, make it clear how the effective temperatures determined according to PLANCK, are nothing but calculated quantities to which a physical sense can hardly be attached, and which certainly do not give an insight into the actual temperatures of the sun

If we possessed a means to consider exclusively light that reaches us from this photospheric scale, then, according to our supposition, every photosphere would possess its own energy spectrum, which would vary from photosphere to photosphere, and this for two reasons

1. The real temperature in the inner layers is different from that in the outer.

2. The radiation, reaching us from the inner layers has undergone a greater loss through absorption and scattering (and, so far as the latter cause is concerned, to a much greater degree for the shorter wave-lengths than for the longer), in consequence of which, even if the real temperature of the different layers were the same everywhere, the observed energy spectrum would still be different in the different layers.

What we do observe, however, is not the spectrum modified by scattering etc. of every layer separately, but the combination of all these spectra together.

To try and derive an effective temperature from this spectrum, which is far from "black" seems to be absolutely unpermissible; because the fundamental condition itself, that the energy spectrum used would in its main points resemble that of an absolutely black body, has not been fulfilled.

If, however, we do apply this procedure, it is not surprising that the found values of T appear in a high degree, to be dependent on λ.

The latter may be shown more clearly by the following method.

Let us imagine an energy spectrum formed by the superposition of only two spectra, originating from two really "black" bodies, which contribute about an equal amount to the total radiation, but whose temperatures differ greatly. Such a case is represented in figure 1.

Let the curve 1 correspond to the absolute temperature 3000°, the curve II to 1500°. Then the maxima lie respectively at $\lambda = 1\,\mu$ and $\lambda = 2\,\mu$. Summation of these yields the curve III, but by halving the ordinates curve IV has been derived from this, whose area is again equal to the area of each of the component curves, i.e. we reduce all the cases to *equal total radiation.*

It is now easy to see that when we derive the temperature from the shape of a small part (*ab* or *cd*) of the summation curve, at the same time considering this part as belonging to an energy curve of a black body, for *small* values of λ a temperature would

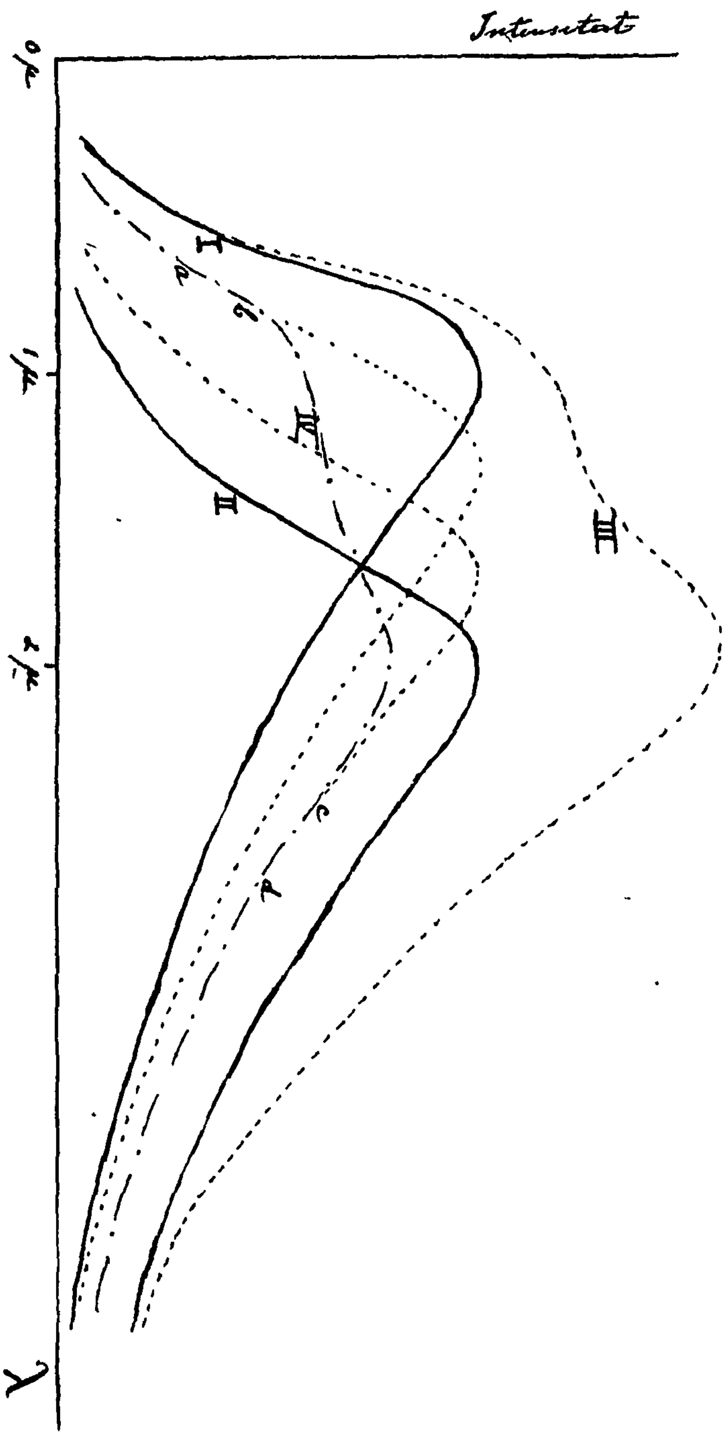


Fig. 1.

be found lying between 3000° and 1500°, but nearer 3000°; whereas on the other hand, for large values of λ, an intermediate temperature would be found lying more to the side of 1500°.

(The two imagined intermediate curves of radiation of black bodies have been drawn dotted, so that each of them again embraces the same area).

The temperatures calculated for different values of λ would have been still more divergent, if I had not been the original energy curve of a black body, but had presented a much greater slope towards the violet side on account of molecular scattering

Though for the sun everything is of course much more complicated than in these examples, the conclusion remains valid that T must be found dependent on λ, if our supposition should be justified that every layer radiates as a black body. But though this hypothesis accounts to a certain extent for the variation of the found values of T with λ, and is preferable in so far to the undoubtedly untenable supposition of DEFANT and others, that the radiation of the sun would issue from one single absolutely black photospheric surface — yet the hypothesis of the "partial photospheres" cannot be considered either as satisfactory.

Until by other means, some insight has been obtained into the power of emission of the successive layers of the sun's mass, and the degree in which they scatter and absorb the different kinds of light, hardly anything can be derived from the distribution of energy in the solar spectrum concerning temperatures on the sun.

The conception "effective temperature of the sun" has little value. This temperature varies in fact greatly according to the way in which it is defined, and none of the definitions warrant in any way, that by means of them an approximation is found of temperatures that actually prevail on the sun.

Mathematics. — "*Graphical determination of the moments of transition of an elastically supported, statically undeterminate beam.*" [1]) I. By C. B. BIEZENO. (Communicated by Prof. J. CARDINAAL).

(Communicated at the meeting of November 24, 1917).

1. Let a rectangular prismatic beam be charged by forces which cut its axis at right angles and which are parallel to one of the two other principal axes of its centre of gravity.

Its support, which is thought to be elastical, be applied in a number of points of support $A, B, C \ldots$ at the same level in such a way that the reactions of support $R_A, R_B, R_C \ldots$

1. are parallel to the lines of action of the charging forces.

2. are proportional to the local descents $y_A, y_B, y_C \ldots$ of the axis of the beam, so that $\alpha R_A = y_A, \beta R_B = y_B, \gamma R_C = y_C \ldots$.

It is required to define graphically the moments of transition in the beam.

2. In order gradually to conquer the difficulties which arise during the solution of the problem, the case of the beam on three, four and five points of support will successively be dealt with and that on the supposition, that the fieldlengths of the beam as well as the coefficients of stiffness of the elastic supports are equal. This restricting supposition can be introduced, because it does not influence the general construction, as will appear later.

When the case of the beam on five points of support has been treated, the general problem, which finds its analytical interpretation in the so called "theorem of five moments", must have been solved at the same time.

3. In fig. 1 for the beam ABC, supposed to be charged in the middle of each of its fields by a force of 1 ton, the lines

[1]) In the following treatise the reader is supposed to be thoroughly acquainted with the construction of the elastic link-polygon, which we owe to O. MOHR. (See for this construction: O MOHR, Abhandlungen aus dem Gebiete der technischen Mechanik, 2e Auflage S. 367; J. KLOPPER, Leerboek der toegepaste Mechanica, Deel III, p. 160).

$(l_A, l_I, l_{II}, l_{III}, l_B\, l_{IV}, l_V, l_{VI}, l_C)$ [1]) have been drawn, along which the "forces" are acting, which would play a part in the construction of the elastic link-polygon, if the beam lay on *fixed* points of support.

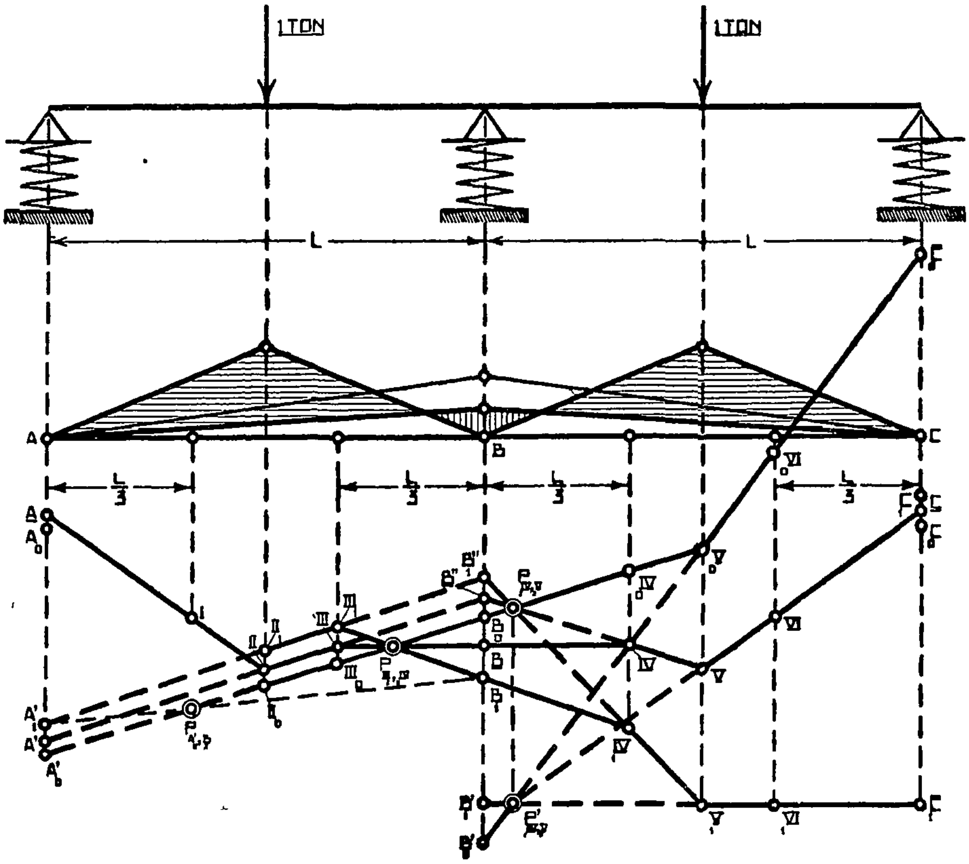


Fig. 1.

If now the descents $A\underline{A}$, $B\underline{B}$ and $C\underline{C}$ of the points of support A, B, C were known, it would be possible to construct the elastic link-polygon of the beam, because together with the point $\underline{A}$ also the point $\underline{A}'$ is fixed, which lies a known distance a under $\underline{A}$ and is the starting point of the construction of MOHR [2]).

The situation of $\underline{A}$, hence also that of $\underline{A}'$, in reality being unknown, we shall for the present try to find a solution of the problem by assigning to the moment of transition M_B a certain value, say x metre-ton. For in this way the reaction and therefore also the descents

[1]) By l_A, l_B, l_C the verticals passing through the points A, B, C are indicated; by l_I, l_{II} etc. the vertical lines on which the angles lie of the elastic link-polygons that will be drawn later on.

[2]) In the figure this point $\underline{A}'$ is by mistake indicated by A'.

AA_x, $\underset{x}{B}B$ and $C,{}_xC$ of the points of support A, B and C become known[1]).

Now the point A_x', belonging to A_x, is also fixed, so that the side $II_x\,III_x$ can at once be drawn in the right direction. For together with l_B this side must produce a point of intersection $\underset{x}{B''}$, the situation of which is known, because the sides II_x, III_x and III_x, ${}_xIV$ must cut from l_B a segment of definite length, representing the statical moment relative to B of the "force" acting along l_{III}. With the side $II_x\,III_x$ not only the side III_x, ${}_xIV$, but also the side ${}_xIV$, ${}_xV$ is determined, because the latter, on account of the equality of the fieldlengths AB and BC, must give a point of intersection with $II_x\,III_x$ on l_B.

Finally the side ${}_xV$, ${}_xVI$, ${}_xC$ can be drawn too, as this together with ${}_xIV$, ${}_xV$ must also cut a segment of known length from l_B.

If now the supposition, made with regard to M_B, had been right, the point of intersection ${}_x\overline{C}$ of ${}_xV$, ${}_xVI$ with l_C would coincide with the point ${}_xC$, which apparently does not happen.

The construction might however be repeated with judiciously chosen values of M_B, till the points ${}_x\overline{C}$ and ${}_xC$ quite or nearly coincide.

But these tentative attempts to find the real value of M_B are made superfluous by the construction to be given in the following paragraphs, from which the coincidence of the points ${}_x\overline{C}$ and ${}_xC$ ensues directly and exactly.

4. Closely following the line of thought developed in § 3 let us in the first place assign the value zero to the moment of transition $\underset{\bullet}{M_B}$.

In this case the beams AB and BC can be considered as two beams supported at their extremities, of which the reactions of support can be determined directly. The descents of the points A, B and C are also known; if μ represents the coefficient of stiffness of the springs they are:

$$AA_0 = \mu \,.\, \tfrac{1}{2}\,,\ \ \underset{0}{B}B = \mu \,.\, 2 \,.\, \tfrac{1}{2}\,,\ \ C\,{}_0C = \mu \,.\, \tfrac{1}{2}.$$

Through the point A'_0, which lies the known distance a, mentioned before, below A_0, the beam II_0, III_0 must now be drawn, which however, the "force" along l_{III} being zero, must act along the side III_0, ${}_0IV$, which itself passes through $\underset{0}{B}$.

---

[1]) An index placed under a letter denotes the value of the moment of transition belonging to the point of support indicated by the letter.

An index placed to the right or to the left of a figure or letter denotes the value of the moment of transition in the first point of support to the right or to the left.

The sides $II_0\,III_0$, III_0, ${}_0IV$ and ${}_0IV\,{}_0V$ coincide on the line $A^{0\prime}\,\underset{0}{B}$.

By finally measuring the known distance $\underset{0}{B}\underset{0}{B'} = b$, starting from $\underset{0}{B}$, it is possible to draw the coinciding sides $\underset{0}{B'}\,{}_0V\,{}_0VI$ and ${}_0VI\,{}_0\overline{C}$.

While the supposition $M_B = 0$ on the one hand causes a descent $C\,{}_0C$ of the point C, it leads on the other hand via the construction of the elastic link-polygon $A_0'\,I_0\,II_0\,III_0\,\underset{0}{B}\,{}_0IV\,{}_0V\,{}_0VI\,{}_0\overline{C}$ to an ascent $C\,{}_0\overline{C}$ of this point.

Let in the second place the value of one metreton be given to the moment of transition M_B[1]).

In this case the situations of the points of support are again known. The supposition $M_B = 1$ metreton namely gives rise, if the fieldlengths AB and BC in metres are indicated by L, to extra reactions of magnitudes: $-\frac{1}{L}$, $2\frac{1}{L}$ and $-\frac{1}{L}$ ton, to which correspond the extra descents $-\mu\frac{1}{L}$, $2\mu\frac{1}{L} - \mu\frac{1}{L}$, which can be drawn on the scale once introduced.

In the way indicated in § 3 there arises now a link-polygon $A_1'\,I_1\,II_1\,III_1\,\underset{1}{B}\,{}_1IV\,{}_1V\,{}_1VI\,{}_1\overline{C}$.

While as a result of the introduction of the moment of transition of one metreton the point ${}_0C$ has moved upward over the distance ${}_0C\,{}_1C$, the endpoint ${}_0\overline{C}$ of the elastic link-polygon $A_0'\,I_0\,II_0\,III_0\,\underset{0}{B}\,{}_0IV\,{}_0V\,{}_0VI\,{}_0\overline{C}$ has descended over the distance ${}_0\overline{C}\,{}_1\overline{C}$.

It will now be shown that on the introduction of a moment of transition of x metreton two points ${}_xC$ and ${}_x\overline{C}$ arise, the situations of which are defined by the equations:

$$({}_0C\,{}_xC) = x\,.\,({}_0C\,{}_1C),$$
$$({}_0\overline{C}\,{}_x\overline{C}) = x\,.\,({}_0\overline{C}\,{}_1\overline{C});$$

in other words it will be proved that the two series of points ${}_xC$ and ${}_x\overline{C}$ are similar.

5. If above the point of support B a moment of transition of x

---

1) The moment of bending, appearing in a cross-section of the beam, is called positive, when the right part of the beam exerts a dextro-rotatory couple on the left part.

metreton is introduced, the point A'_0 descends an amount $A_0' A_x' = - x\frac{\mu}{L}$, the point $\underset{0}{B}$ an amount $\underset{0}{B}\underset{x}{B} = x \cdot 2\frac{\mu}{L}$.

$\frac{A'_0 A'_x}{\underset{0}{B}\underset{x}{B}} = 2$ being constant, the series of points A'_x and $\underset{x}{B}$ are similar.

The lines $(A'_x \underset{x}{B})$ connecting their corresponding points, pass therefore through the fixed point $P_{A'_x \underset{x}{B}}$, which divides the distance of the lines l_A and l_B into parts which are to each other as $-1:2$.

The point $\underset{x}{B''}$, which belongs to the point $\underset{x}{B}$, lies at a distance $x \cdot \underset{1}{B''}\underset{1}{B}$ from this point, since the "force" falling along l_{III}, hence also the moment relative to B derived from this "force", increases linearly with the moment M_B.

As $\underset{x}{B}$ has descended over a distance $x \cdot \underset{0}{B}\underset{1}{B}$ relative to $\underset{0}{B}$, the point $\underset{x}{B''}$ lies $x \cdot \left\{\underset{1}{B}\underset{1}{B''} - \mu\frac{2}{L}\right\}$ above $\underset{0}{B}$.

The ratio $\frac{A'_x A'_0}{B''_x B_0}$ being constant, also the series of points A'_x and $\underset{x}{B''}$ are similar, so that also the lines $A'_x \underset{x}{B''}$ pass through one point $P_{A'_x \underset{x}{B''}}$, not indicated in the diagram.

The three angles of the variable triangle $A'_x \mathrm{III}_x \underset{x}{B}$, (of which $A'_1 \mathrm{III}_1 \underset{1}{B}$ gives one position) move in three straight lines l_A, l_{III} and l_B passing through one point, while two sides rotate round fixed points. Hence also the third side must rotate round a fixed point lying on the line connecting the centres of rotation of the two other sides.

If we further fix our attention on the variable triangle $\mathrm{III}_x \underset{x}{B''}{}_x\mathrm{IV}$, it appears that also the angles of this triangle move in three straight lines (l_{III}, l_B and l_{IV}) passing through one point, while two sides, viz. $\mathrm{III}_x\,{}_x\mathrm{IV}$ and $\mathrm{III}_x \underset{x}{B''}$, rotate round fixed points.

The third side rotates therefore also round a fixed point $P_{{}_x\mathrm{IV}_x\mathrm{V}}$ on $A'_0, \underset{0}{B}$.

But then the side ${}_x\mathrm{V}\,{}_x\mathrm{VI}$ too has a fixed centre of rotation $P'_{{}_x\mathrm{IV}_x\mathrm{V}}$. For the sides ${}_x\mathrm{IV}\,{}_x\mathrm{V}$ and ${}_x\mathrm{V}\,{}_x\mathrm{VI}$ cut from the line l_B, hence also from the vertical through $P_{{}_x\mathrm{IV}_x\mathrm{V}}$, a segment of constant length. As the point of intersection of the sides ${}_{x}IV\,{}_{x}V$ with this straight line is a fixed point, the point of intersection of the sides ${}_{x}V\,{}_{x}VI$ with the same straight line must also be invariable.

Consequently all the sides of the link-polygon $A_x'\,I_x\,II_x\,III_x\,B_x\,{}_{x}IV\,{}_{x}V\,{}_{x}VI\,{}_{x}\overline{C}$ rotate round a fixed point.

The series of points ${}_{x}\overline{C}$ is therefore similar to the series of points A'_x. But also the series of points ${}_{x}C$ is similar to this latter series. For this reason also the series of points ${}_{x}\overline{C}$ and ${}_{x}C$ are similar.

6. The double point $\underline{C}$ of these series at finite distance gives the real situation of the third point of support C of the beam, as it can on the one hand be considered as the point $\overline{C}$, through which the beam must pass on introduction of the moment of transition M_B belonging to $\underline{C}$ by reason of the construction of the elastic link-polygon, and on the other hand may be considered as the point C, which is found by the direct determination of the descents in consequence of the given charge and the moment of transition just mentioned.

When once this point $\underline{C}$ has been determined by the help of the proportion;

$$\frac{{}_{0}\overline{C}\,\underline{C}}{{}_{0}\overline{C}\,{}_{1}\overline{C}} = \frac{{}_{0}C\,\underline{C}}{{}_{0}C\,{}_{1}C}$$

the required link-polygon can be drawn completely, as $\underline{C}$ VI V must pass through $P'_{{}_{x}IV\,{}_{x}V}$, V IV through $P_{{}_{x}IV\,{}_{x}V}$, IV III through $P_{III_x\,{}_{x}IV}$, III II A' through the point of intersection $\underline{B}''$ of V IV and $\underline{l_B}$ and finally II I through the point A (lying at a distance a above A').

The magnitude of the required moment of transition M_B is determined by the segment $\underline{BB''}$.

7. Although in the preceding paragraphs the beam on three elastic supporting points has been fully discussed, we shall before proceeding to the beam on four points of support, make mention of one more theorem bearing upon the situations, considered in a horizontal sense, of the centres of rotation $P_{II_x\,III_x}$, $P_{A'_x B_x}$, $P_{III_x\,{}_{x}IV}$, $P_{{}_{x}IV\,{}_{x}V}$, $P'_{{}_{x}IV\,{}_{x}V}$.

It has already been pointed out in § 5, that the situation of $P_{A'_x B_x}$ is determined by the ratio. $\dfrac{A'_0\,A'_x}{B_0\,B_x}$, which is independent of the charge of the beam.

The ratio

$$\frac{III_x\, III_0}{\underset{x\,0}{BB}} = \frac{III_1\, III_0}{\underset{1\,0}{BB}} = \frac{\frac{1}{3}A_1'A_0' + \frac{2}{3}\underset{1\;0}{B''B}}{\underset{1\,0}{BB}} = \frac{1}{3}\frac{A_1'A_0'}{\underset{1\,0}{BB}} + \frac{2}{3}\frac{\underset{1\;1}{B''B} - \underset{1\;0}{BB}}{\underset{1\,0}{BB}} =$$

$$= \frac{1}{3}\frac{A_1'A_0'}{\underset{1\,0}{BB}} + \frac{2}{3}\frac{\underset{1\;1}{B''B}}{\underset{1\;0}{BB}} - \frac{2}{3}$$

by which the situation of $P_{\mathrm{III}_x\,{}_x\mathrm{IV}}$ is determined, appears to be also independent of the charge of the beam.

But then also the horizontal situations of the other centres of rotation $P_{\mathrm{II}_x\,\mathrm{III}_x}$, $P_{{}_x\mathrm{IV}\,{}_x\mathrm{V}}$, $P'_{{}_x\mathrm{IV}\,{}_x\mathrm{V}}$ are the same for all possible charges of the beam.

For if we consider the two triangles $A_1'\underset{1}{B}\,\mathrm{III}_1$ and $\overline{A}_1'\,\overline{B}_1\,\overline{\mathrm{III}}_1$ (the latter of which is supposed to bear upon an arbitrary charge differing from the one given), in these two affined figures the points $P_{A'_x\underset{x}{B}}$ and $\overline{P}_{A'_x\underset{x}{B}}$, $P_{\mathrm{III}_x\,{}_x\mathrm{IV}}$ and $\overline{P}_{\mathrm{III}_{xx}\mathrm{IV}}$ are homologous points.

From this it can immediately be derived, that also the points $P_{\mathrm{II}_x\,\mathrm{III}_x}$ and $\overline{P}_{\mathrm{II}_x\,\mathrm{III}_x}$, $P_{{}_x\mathrm{IV}\,{}_x\mathrm{V}}$ and $\overline{P}_{{}_x\mathrm{IV}\,{}_x\mathrm{V}}$ are corresponding points, so that the lines connecting them must pass through the pole of affinity, the point at infinity of the straight lines l.

$P_{\mathrm{II}_x\,\mathrm{III}_x}$ and $\overline{P}_{\mathrm{II}_x\,\mathrm{III}_x}$ as well as $\overline{P}_{{}_x\mathrm{IV}\,{}_x\mathrm{V}}$ and $P_{{}_x\mathrm{IV}\,{}_x\mathrm{V}}$ lie therefore perpendicularly above each other.

From this follows the theorem referred to in the beginning of this §:

The situation of the centres of rotation $P_{\mathrm{II}_x\,\mathrm{III}_x}$, $P_{A'_x\underset{x}{B}}$, $P_{\mathrm{III}_x\,{}_x\mathrm{IV}}$, $P_{{}_x\mathrm{IV}\,{}_x\mathrm{V}}$, relative to the lines l, is quite independent of the charge of the beam; it is exclusively connected with the stiffness of the beam and that of its supports.

8. Beam on four points of support.

When we have once made ourselves familiar with the line of thought, developed in the preceding paragraphs, it is rational to try and find a solution for the beam on four points of support according to the following program.

1. Cut the beam at the last point of support but one, and construct the situation of the point D in two ways. First by determining the reaction R_D of the beam CD, freely supported at its extremities, and secondly by drawing for the beam $ABCD$ the link-polygon belonging to $M_C = 0$. In this way two points

$._0D$ and $._0\overline{D}$,[1]) appear, which do not coincide, unless in reality there is no moment of transition above the point of support C.

2. Construct then in a similar way two points ${}_1D$ and ${}_1\overline{D}$ on the supposition that the moment of transition M_C is one metreton.

3. Prove, that the series of points $._yD$ and ${}_y\overline{D}$ arising on introduction of various moments of transition $M_C = y$ metreton, are similar. Then the double point $\underline{D}$ of these series, to be constructed by the help of ${}_0D$, $._0\overline{D}$ and $._1D$, $._1\overline{D}$, will indicate the real situation of the last point of support.

4. Starting from this point $\underline{D}$ construct the link-polygon in question $\underline{D}$ IX VIII VII $\underline{C}$ VI V IV $\underline{B}$ III II I $\underline{A}$.

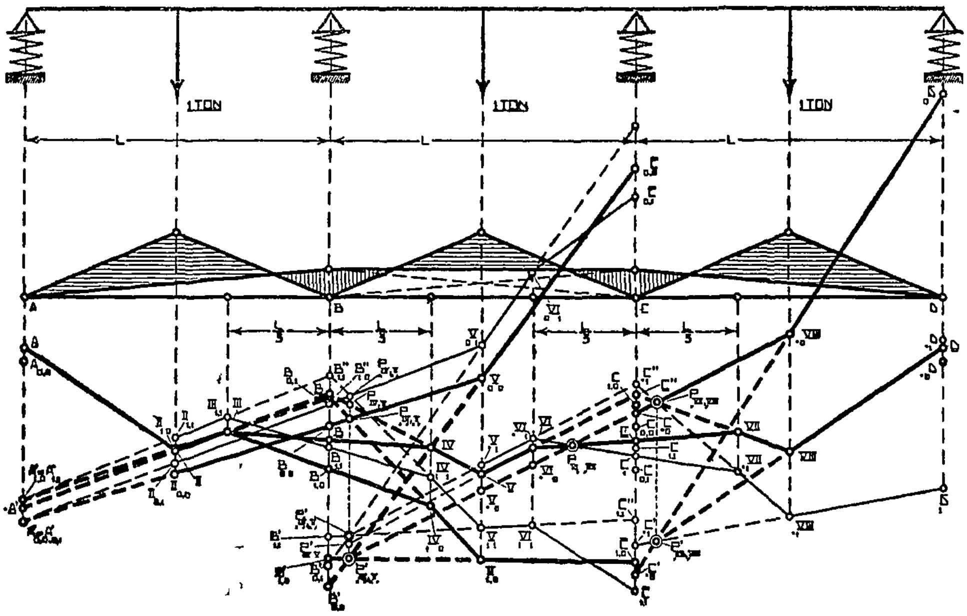


Fig. 2.

9. In fig. 2 the working program, developed to this end, has been put into execution on the supposition that each of the fields AB, BC and CD of the beam is charged in the middle by a force of one ton.

First the construction given in §§ 3—7 has been executed for the beam ABC, which besides by the two forces of one ton on each of the fields AB and BC is supposed to be charged at its extremity C by a force of $\frac{1}{2}$ ton (originating from the charge of the last field CD).

1) By the indices 0 and . added to the letters D, is indicated that the moment of transition in C is zero and that the moment of transition in B has the *right* value belonging to the supposition $M_C = 0$.

If we cut this beam at B the whole extra charge acts on the spring under C, so that in the determination of the points A_{00} [1]), B_{00}, C_{00} only the point C_{00} appears to have an extra descent.

Without any difficulty with the help of the link-polygons $A'_{00}\,\mathrm{II}_{00}\,B_{00}$, ${}_0\mathrm{V}_0\,\overline{C}_{00}$ and $A'_{1,0}\,\mathrm{II}_{1,0}$, B_{10}, ${}_0\mathrm{IV}_0$, ${}_1\mathrm{V}_0$, $\overline{C}_{10}$ the points $\overline{C}_{00}$ and $\overline{C}_{10}$ can then be constructed, which together with the point C_{00} and C_{10} determine the point $C_{.0}$ through which the beam ABC must pass at its extremity C, when besides the given charges it must bear in C a force of $\frac{1}{2}$ ton.

On the supposition $M_C = 0$ the side $P'_{\mathrm{x}}\mathrm{IV}_{0\,\mathrm{x}}\mathrm{V}_0$, $.\mathrm{V}_0$, $.\mathrm{VI}_0$, $C_{.0}$ can now be prolonged as far as l_{VIII}. After that from $C_{.0}$ a segment $C_{.0}\,C'_{.0} = c_0$ must be drawn in downward direction in order to make it possible to draw the side $C'_{.0}\,{}_{.0}\mathrm{VIII}\;{}_{.0}\overline{D}$. In this way, however, the point ${}_0\overline{D}$ is determined.

10. It is far more difficult to find the point $C_{.1}$, through which the beam ABC, considered as a whole, must pass if in C a moment of transition of one metreton is applied. For, when the connection of the beam above B is broken, this moment will, in opposition to the force just applied in C, besides on C also exert its influence on the point B.

The place of the point B_{00} is taken by a point B_{01} which lies $\frac{\mu}{L}$ higher. In the same way the point C_{01} lies a distance $2\frac{\mu}{L}$ below C_{00} because the couple of unity acting on the field BC as well as that, acting on the field CD, gives an extra descent $\frac{\mu}{L}$ to the spring under C.

In case the moment of transition in B is supposed to be zero we have therefore to do with the link-polygon A'_{01}, II_{01}, B_{01}, ${}_0\mathrm{V}_1$, ${}_0\mathrm{VI}_1$, $\overline{C}_{01}$ of which the two last sides now deviate and cut a segment of known length from l_C.

[1]) By a second index, placed to the right or to the left of a letter, the value of the moment of transition in the second point of support to the right or to the left is indicated, etc.

If in B a moment of one segmentton is introduced, the point $\underset{01}{C}$ is serment replaced by the point $\underset{11}{C}$ lying $\frac{\mu}{L}$ higher, while the construction of the elastic link-polygon A'_{11} II_{11}, III_{11}, $\underset{11}{B}$, ${}_1IV_1$, ${}_1V_1$, ${}_1VI_1$, $\underset{11}{\overline{C}}$ removes the point $\underset{01}{\overline{C}}$ to $\underset{11}{\overline{C}}$.

Instead of the earlier fixed centres of rotation $P_{III_{x,0}\ {}_xIV_0}$, $P_{{}_xIV_0\ {}_xV_0}$, $P'_{{}_xIV_0,\ {}_xV_0}$, other points $P_{III_{x,1}\ {}_xIV_1}$, $P_{{}_xIV_1,\ {}_xV_1}$, $P'_{{}_xIV_1,\ {}_xV_1}$, lying perpendicularly above them, appear; of these points for the present only the last is of importance.

For when the double point $\underset{.1}{C}$ of the series $\underset{x1}{C}$ and $\underset{x1}{\overline{C}}$ is constructed, also the point $\underset{.1}{C''}$ is known, through which the side $\underset{.1}{V}$, $\underset{.1}{VI}$ must pass. But this side must also contain the point $P'_{{}_xIV_1,\ {}_xV_1}$; it is therefore determined.

Consequently also the sides $VI_{.1}$, ${}_{.1}VII$, ${}_{.1}VII$, ${}_{.1}VIII$ and ${}_{.1}VIII$, ${}_{.1}D$ can be drawn, so that now ${}_{.1}D$ is determined. [1])

The construction of the points ${}_{.0}D$ and ${}_{.1}D$ conjugated to the points ${}_{.1}\overline{D}$ and ${}_{01}\overline{D}$ just found, does not present any difficulties.

11. According to the outline given in § 8 we must now investigate whether the series of points ${}_{.y}D$ and ${}_{.y}\overline{D}$, which appear on the introduction of various moments of transition $M_C = y$ metre-ton in the way described above, are similar.

To that purpose we consider in the first place the centres of rotation $P_{II_{x1}\ III_{x1}}$, $P_{III_{x1}\ {}_xIV_1}$, $P_{{}_xIV_1\ {}_xV_1}$ just mentioned, belonging to the moment of transition $M_C = 1$ metre-ton. These centres of rotation

[1]) Strictly speaking the construction of the link-polygon A'_{11}, II_{11}, .. $\underset{11}{B}$.... $\underset{11}{\overline{C}}$, mentioned in this § and drawn in fig. 2 for completeness' sake, is superfluous. For it serves exclusively for the determination of the ratio $\frac{\underset{01}{\overline{C}}\ \underset{11}{\overline{C}}}{\underset{01}{C}\ \underset{11}{C}}$, which only depends on the horizontal situation of the centres of rotation $P_{III_{x,1}\ {}_xIV_1}$, $P_{{}_xIV_1,\ {}_xV_1}$ etc. which corresponds to that of $P_{III_{x,0}\ {}_xVI_0}$, $P_{{}_xIV_0\ {}_xV_0}$ etc.. $\frac{\underset{01}{\overline{C}}\ \underset{11}{\overline{C}}}{\underset{01}{C}\ \underset{11}{C}}$ can therefore be put equal to the ratio $\frac{\underset{00}{\overline{C}}\ \underset{10}{\overline{C}}}{\underset{00}{C}\ \underset{10}{C}}$ already found.

lie perpendicularly above the centres of rotation $P_{\mathrm{II}_{x0}\,\mathrm{III}_{x0}}$, $P_{\mathrm{III}_{x0}\,{}_x\mathrm{IV}_0}$, $P_{{}_x\mathrm{IV}_0\,{}_x\mathrm{V}_0}$ on a straight line through A'_{00}, determined by the point $\underset{01}{B}$, lying $\frac{\mu}{L}$ above $\underset{00}{B}$.

As on introduction of the other moments of transition $M_C = y$ meterton there appear points $\underset{0y}{B}$, defined by $\underset{00}{B}\,\underset{0y}{B} = y.\ \underset{00}{B}\,\underset{01}{B}$, it is evident, that the centres of rotation mentioned, undergo vertical displacements, which are proportional to these moments.

Especially at the introduction of $M_C = y$ metreton the segment $P'_{{}_x\mathrm{IV}_0\,{}_x\mathrm{V}_0}, P'_{{}_x\mathrm{IV}_y\,{}_x\mathrm{V}_y}$ will be equal to y times the segment $P'_{{}_x\mathrm{IV}_0\,{}_x\mathrm{V}_0}, P'_{{}_x\mathrm{IV}_1\,{}_x\mathrm{V}_1}$.

On account of the law of superposition, on which the whole problem is founded, the descent $\underset{.0}{C}\,\underset{.1}{C}$ of the point C will increase in direct ratio to the value y of the moment of transition M_C.

The distance of the point $\underset{.y}{C''}$ to the point $\underset{.0}{C}$ can therefore be put equal to:

$$y\,.\,(\underset{1}{C''}\,\underset{1}{C} - \underset{0}{C}\,\underset{.1}{C}).$$

The lines $(P'_{{}_x\mathrm{IV}_y\,{}_x\mathrm{V}_y}, \underset{.y}{C''})$ connect therefore corresponding points of two similar series of points; they pass through one point.

As $.\mathrm{VI}_y\ .\mathrm{VI}_0$ can be linearly expressed in $\underset{.y}{C''}\ \underset{.0}{C}$ and $P'_{{}_x\mathrm{IV}_0\,{}_x\mathrm{V}_0}$, $P'_{{}_x\mathrm{IV}_y\,{}_x\mathrm{V}_y}$, the series of points $.\mathrm{VI}_y$ is also similar to the series $\underset{y}{C}$, so that the lines $.\mathrm{VI}_y, {}_y\mathrm{VII}_0$ too have a fixed centre of rotation $P_{\underset{y}{.\mathrm{VI}}\,\underset{y}{\mathrm{VII}}}$. But then also the sides $_{.y}\mathrm{VII}\ {}_y\mathrm{VIII}$ and $_{.y}\mathrm{VIII}\ {}_{.y}\overline{D}$ have fixed centres of rotation $P_{{}_y\mathrm{VII}\,{}_y\mathrm{VIII}}$ and $P'_{{}_y\mathrm{VII}\,{}_y\mathrm{VIII}}$.

The series of points $_{.y}\overline{D}$ is therefore similar to the series $\underset{.y}{C''}$, $\underset{.y}{C}$, $P'_{{}_x\mathrm{IV}_y\,{}_x\mathrm{V}_y}\ldots$ which in their turn are similar to the series $_yD$, for which holds good:

$$_yD\,{}_0D = y \times {}_1D\,{}_0D.$$

Hence the series of points $_y\overline{D}$ and $_yD$ are also similar. Their double point D at finite distance is the extreme point of the link-polygon in question for the beam on four points of support.

Now that this double point is known, the construction of the whole link-polygon no longer presents any difficulty.

For by the centre of rotation $P'_{\text{y VII y VIII}}$ the side $\underline{D}$ IX VIII is determined, by the centre of rotation $P_{\text{y VII y VIII}}$ the side VIII VII C'', by the centre of rotation $P_{\text{VI y y VII}}$ the side VII $\underline{C}$ VI.

If we furthermore draw $\underline{C}''$ VI in the first place the side VI V and in the second place the centre of rotation $P'_{\text{IV V}}$, hence also the point $P_{\text{IV V}}$, lying perpendicularly above it, through which V IV must pass, are fixed.

By means of $P_{\text{IV V}}$ we also find the line $A'_{00}\, P_{\text{IV V}}$, on which the centre of rotation of all the other sides must lie.

Now the link-polygon in question $\underline{D}$ IX VIII VII $\underline{C}$ VI V IV $\underline{B}$ III II I $\underline{A}$ can be completed.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXII

Nos. 7 and 8.

President: Prof. H. A. LORENTZ.

Secretary: Prof. P. ZEEMAN.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vols. XXVII and XXVIII).

CONTENTS.

Mathematics. — "*Quadratic involutions among the rays of space.*" By Prof. JAN DE VRIES.

(Communicated at the meeting of December 28, 1918).

In a communication which is to be found in part Vol. XXII, p. 478 of these *Proceedings* I have dealt with an involution, the pairs of which consist of the transversals to quadruplets of straight lines belonging one to each of four given arbitrary plane pencils of rays. In the sequel I shall consider a few involutions related to the above mentioned.

1. In the first place we assume two plane pencils of lines $(A, \alpha) \equiv (a)$, $(B, \beta) \equiv (b)$ and a quadric regulus $(c)^2$ *i.e.* one set of generators of an hyperboloid Γ^2. An arbitrary line t meets one ray a, one ray b and two rays c. If we conjugate to t the second transversal t' of these four lines, a *quadratic involution among the rays of space* is thereby defined.

If t describes a plane pencil, an involution is thereby determined in $(c)^2$, the pairs of which correspond projectively to the rays of the pencils (a) and (b).

Now consider the more general case where a quadratic involution *in* $(c)^2$ *is brought into a projective correspondence to the pencils* (a) and (b) in an arbitrary way. The transversals t, t' of the quadruplets of rays a, b, c, c' will constitute a ruled surface, the order of which we shall determine by an investigation after the number of lines t which rest on the line of intersection of the planes α and β.

On the line $\alpha\beta$ the projective pencils (a), (b) determine two projective point-ranges. Through each of the two united points (coincidences) passes a line t. The remaining rays t which meet $\alpha\beta$, lie in α or in β.

On the intersection of Γ^2 and α the points of transit of the pairs c, c' constitute an involution; the joins of the pairs of this involution form a pencil (C, α), which is projective to the point-range cut out on α by the pencil (b) and therefore also projective to the line-pencil which projects this point-range from C. Since each of the two united-rays (coincidences), rests on four corresponding rays a, b, c, c' there are in α (and in β too) two rays of (t). Hence *the ruled surface* (t) *is of degree six.*

The plane α intersects (t^2) still along an additional curve α^4, which must needs have a double point at A, as an arbitrary ray a is met by two transversals t, t' only. Since the line AB outside A and B meets two lines t [1]) and therefore at A has two points in common with $(t)^6$, it is necessary that A, and B too, is a *double point* of the ruled surface.

The curve α^4 has six tangents passing through A; hence $(t)^6$ contains six united-rays (double rays) of the involution (t, t').

The transversals of the pairs a, b form a quadratic line-complex; for, in an arbitrary plane (a) and (b) determine two projective point-ranges and the joins of corresponding points envelop a conic. This complex has four rays in common with the second regulus (set of generators) $(\gamma)^2$ of Γ^2. Each of these four rays meets two corresponding rays a, b and at the same time the rays c, c' conjugated thereto. Hence the ruled surface $(t)^6$ has four lines in common with the hyperboloid Γ^2.

2 If t is caused to describe the pencil (T, τ) the ruled surface $(t)^6$ breaks up into this pencil (t) and a ruled surface $(t')^5$. *Thus the transformation (t, t') converts a pencil into a ruled surface of degree five.*

Of the two united-points of the projective point-ranges on $\alpha\beta$ one now lies at $\alpha\beta\tau$; through the other passes a ray t'. Thus in α (and in β) there lie again two rays t'. The remainder of the intersection of $(t')^5$ and α is a nodal α^3 with double point at A. Each point of intersection of α^3 and τ is the transit of a ray t' which coincides with its conjugated ray t. Hence *the double rays of the involution (t, t') form a cubic complex.*

A confirmation of this enunciation can be obtained as follows. With Γ^2 $(t')^5$ has four rays γ in common (§ 1) and in addition thereto a twisted curve γ^6. At a point of intersection, C, of γ^6 and τ a ray t is intersected by the corresponding ray t'; hence C lies on a double ray $t \equiv t'$ and the second line of $(c)^2$ resting on this double ray meets τ in a point C', which must lie also on γ^6. Thus the six points of transit of γ^6 lie in pairs on three double rays belonging to (T, τ).

3. A ray t_A through A is intersected by a ray b and by two rays c, c' of $(c)^2$. Each ray t' which meets b, c and c' intersects on α a certain ray a and is therefore conjugated to t_A; hence the ray t_A is *singular.*

[1]) Lying in the united-planes (coincidences) of the projective pencils of planes which project (a) and (b) from AB.

42*

The tangent plane of the hyperboloid (b, c, c') at A intersects α along a line a which touches (b, c, c'). The transversals of the four rays a, b, c, c' therefore coincide. Hence every ray t_A is also to be regarded as a double ray; thus *the cubic complex of double rays* has *principal points* at A and B and, accordingly, α and β as *principal planes.*

It follows from the above, that the *sheaves of lines A* and *B* and *the planes α* and *β* consist of *singular rays of the involution* (t, t').

Together with A a ray b determines a pencil (t_A) and thereby at the same time a quadratic involution I^2 among the rays of the regulus $(c)^2$. Now, let there be given in the plane λ a pencil (l) with vertex L; then each point of $\beta\lambda$ determines, by means of I^2, an involution I^2 on the conic (Γ^2, λ). Through L therefore passes a ray l joining the points of transit of two rays c, c', which in combination with b determine a transversal t_A. If this ray l is conjugated to the ray l', which meets b, a projectivity is established in (l). Each of the two united-rays is then a ray t' which is conjugated to a ray t_A. It follows from this that the reguli $(t')^2$ which are conjugated to the *singular rays* t_A, constitute a *quadratic line-complex.*

Three other quadratic complexes $\{t'\}^2$ correspond to the sheaf of lines $[t_B]$ and to the plane systems of rays $[t_\alpha]$ and $[t_\beta]$.

The pencil (T, τ) contains two rays of each of these complexes; accordingly A, B, α, and β each carry two rays t' of the ruled surface $(t')^6$ into which (t) is transformed by the involution (t, t'). Thus it appears again that $(t')^6$ has A and B as double points, α and β as double tangent planes.

The ray AB meets two definite rays c, c', but *all* rays a and b. To $t \equiv AB$ therefore are conjugated all the rays of the bilinear congruence which has c and c' as directrices[1]). Similarly $t \equiv \alpha\beta$ is conjugated to ∞^2 rays t'. Thus the involution (t, t') has *two principal rays,* AB and $\alpha\beta$.

4. The lines of the regulus $(\gamma)^2$ too are *principal rays,* for a line γ meets two definite rays a and b, but *all* rays c; each transversal, t', of a and b rests on two rays c and is therefore conjugated to $t \equiv \gamma$.

The involution (t, t') has still other *singular rays.* If the point of intersection, S, of two rays a and c lies in the plane σ passing through two rays b and c', then the pencil (S, σ) consists of rays s

1) The congruence $[t']$, conjugated to AB belongs to the intersection of the line complexes which correspond to the sheaves $[t_A]$ and $[t_B]$.

each conjugated to all the other, hence of singular rays. Now a plane σ is intersected at two points S by the conic α^2, which Γ^2 has in common with α; every plane tangent to Γ^2 therefore contains two pencils (s).

In any arbitrary plane lie two points S, and therefore two rays s; through an arbitrary point pass two planes σ and consequently four rays s. Since a second system of singular rays is obtained by interchanging a and b in the foregoing reasoning *the pencils of singular rays form two congruences* (4,2).

The vertices of the pencils (s) lie on the conics α^2 and β^2, their planes envelop the hyperboloid Γ^2.

5. In order to obtain *another involution* among the rays of space, we consider two reguli $(c)^2$ and $(d)^2$, of the hyperboloids Γ^2 and Δ^2 respectively. Any two rays c, c' determine in combination with any two rays d, d' a pair of transversals (t, t') constituting one pair of the involution which will here be considered.

Now suppose that on Γ^2 an involution (c, c') be given which in some way is projectively related to an involution (d, d') assumed on Δ^2.

The transversals of the pairs d, d' form a linear line-complex, for, in a plane λ the points of transit D, D' of these pairs determine an involution on the transit (conic) of Δ^2, so that the joins of the point-couples D, D' form a pencil. This complex contains two lines l of the second regulus of Γ^2. There are therefore two transversals of pairs d, d' which meet *all* the rays c. In addition to these two an arbitrary ray c meets the two transversals of the pairs in $(c)^2$ and $(d)^2$ which are determined by c. Hence the transversals t, t' of the pairs c, c' and d, d' form a *ruled surface of degree four*, denoted by $(t)^4$.

Evidently $(t)^4$ contains also two rays of the second set of generators, $(\delta)^2$ of Δ^2.

6. Thus to the rays t of a pencil (T, τ) corresponds *a ruled surface* $(t')^3$, which contains two lines γ and two lines δ. This surface meets the intersection ϱ^4 of Γ^2 and Δ^2 at 12 points, eight of which lie on the last mentioned four lines; the remaining four carry each one ray c and one ray d intersecting τ at two points which are collinear with T.

This statement may be corroborated as follows. Through each point of ϱ^4 pass a line c and a line d. Their points of transit, C and D, through τ determine two point-ranges related by a 2,2-correspondence on the curves of transit γ^2 and δ^2 of Γ^2 and Δ^2.

The lines TC and TD are therefore reciprocally conjugated in a correspondence (4, 4). Of the 8 united-rays of this correspondence four pass through the points of intersection of γ^2 and δ^2; the remaining four each meet a pair c, d the point of intersection of which lies on ϱ^4 and therefore carries a ray t' conjugated to a ray t.

In addition to the two lines γ already mentioned the ruled surface $(t')^8$ has a twisted quartic γ^4 in common with Γ^2. This curve intersects τ at four points, which are two and two collinear with T (§ 2). It follows from this that *the double rays of the involution* (t, t') *form a quadratic complex.*

The single directrix of $(t')^8$ lies in τ, the double one passes through T.

7. The rays of the reguli $(\gamma)^2$ and $(\delta)^2$ are evidently (§ 4) *principal rays* of (t, t'). To each of these rays a *bilinear line-congruence* is conjugated having two lines c or two lines d for directrices. As each line c acts as directrix to two congruences (1,1), there emanate two pencils (t') from each of its points. The congruences (1,1) corresponding to the principal rays therefore constitute *two quadratic complexes.*

In a similar way as in § 4 we find a congruence of singular rays. Of the intersection ϱ^4 of the hyperboloids Γ^2 and Δ^2 each point is the vertex of a pencil (S, σ) consisting of rays s which are each conjugated to all the others, hence singular. For, in fact, the plane σ through the lines γ and δ, which are concurrent at S, intersects ϱ^4 still in the additional points C of γ, D of δ and E. Evidently CE belongs to $(c)^2$, DE to $(d)^2$. Each ray of (S,σ) meets two rays c', d' at S and intersects the lines $c \equiv CE$ and $d \equiv DE$; therefore (S,σ) consists of reciprocally conjugated rays s of the involution (t, t').

Since the vertices of the pencils S lie on ϱ^4 and the planes σ envelop a developable of the fourth class, *the pencils of singular rays form a congruence* (4,4).

8. Any three rays c of a cubic regulus $(c)^3$ determine in combination with each ray a of a pencil (A,a) two transversals, which form a pair of an involution of rays in space.

By the rays of a pencil (t) the rays of $(c)^3$ are ordered in an I^3, the sets of which are projectively correlated to the rays a. To begin with we again suppose that this correspondence is established in an arbitrary manner; then the transversals t, t' of the quadruplets of rays constitute a ruled surface which will here be investigated.

On the nodal curve γ^3, along which the ruled cubic Γ^3 is intersected by the plane α, the triplets of rays c determine an I^3. The conics joining two sets of this I^3 with the double point D and another point B of γ^3 have in addition to these points two points B', B'' in common, not lying on γ^3. The sets of the I^3 are therefore cut out on γ^3 by the system of conics with basal points D, B, B', B''. Only the pair of lines $DB, B'B''$ furnishes a set consisting of three collinear points. It appears from this that the plane α contains *one* line of the ruled surface (t), for the line $t \equiv B'B''$ does not rest on the three rays c of a triplet only, but also on the ray a conjugated thereto.

Through A passes similarly one ray of (t). Since a is still intersected by two additional transversals t, t', *the ruled surface* (t) *is of the fourth degree.*

The remaining curve α^3 which $(t)^4$ has in common with α, sends four tangents through A. Hence $(t)^4$ contains four double rays of the involution (t, t').

If t is caused to describe a pencil (T, τ) then $(t)^4$ breaks up into (t) and a cubic regulus $(t')^3$. Now again α contains one of the rays t'; the points of transit of the remaining lines t' constitute a conic α^2, which passes through A and intersects τ on the double rays which belong to the pencil. Hence *the double rays of the involution* (t, t') *form a quadratic complex.*

9. Let a_e be the particular ray of (A, α) which is intersected by the single directrix e of $(c)^3$. Every line t' which rests on a_e, is in (t, t') conjugated to e. To the line $t \equiv e$ therefore correspond all the rays of a *special linear complex.*

Similarly the double ray d of $(c)^3$ is conjugated to all the rays of the special linear complex having the ray a_d which rests on d for its axis.

In this involution (t, t') also the rays t_A through A are singular and each conjugated to the rays of a regulus having three lines c for its directrices and containing the lines d and e.

Similarly the rays t_α, lying in the plane α, are singular too and each correlated to the rays of a regulus which contains d and e.

Now consider the system of the hyperboloids (H), which are each determined by three lines c. The specimens which pass through a given point P arrange the lines c into the sets of a cubic involution of the second order. The involutions I^2_3 which thus belong to the points P, P', P'', have *one* set in common; the hyperboloids H therefore form a *complex* (triply infinite system). The hyperboloids corresponding to the rays t_A and therefore passing through A then constitute a *net* (twofold infinite system) all the specimens whereof have the lines d, e and the transversal t_0 through A of d and e in common. Through a point P therefore passes a single infinity of hyperboloids and these still have the transversal through P of d and e in common. Hence the lines t' through P which are conjugated to the rays of the sheaf $[A]$ form a pencil in the plane $(P\,t_0)$.

10. There are still other singular rays. Each plane ε through e contains two lines c. In ε lies a pencil of rays t, which has the point of intersection E of e and a_e for its vertex; these rays are *singular*, since they rest at E on a third line c and are therefore all conjugated to each other.

The sheaf $[E]$ *is therefore composed of* ∞ *pencils of singular rays.*

The plane δ passing through d and a_d contains a line c_0; through each point D of d pass two lines c, hence ∞^1 lines t, which rest at the same time on c_0 and a_d. It follows from this that *the plane of rays* $[\delta]$ *is composed of* ∞^1 *pencils of singular rays.* These have their vertices on the line d.

11. Lastly we consider a ruled surface Γ^4 with a double curve ϱ^3. The linear complex which can be laid through five generators c of Γ^4 contains all the lines c. The four rays c which rest on a line t meet besides the line t', which by the complex is conjugated to t. The involution (t, t') then consists of the pairs of conjugated directrices of a linear complex; its double rays are the rays of this complex.

Another well-known involution (t, t') is originated by the pairs of reciprocal polar lines of a hyperboloid. Its double rays are the two sets of generators of the hyperboloid.

Mathematics. — *"A Congruence of Conics"*. By Prof. JAN DE VRIES.

(Communicated at the meeting of January 31, 1920).

1. We shall suppose, that a trilinear correspondence [1]) exists between the ranges of points (A_1), (A_2), (A_3) lying on the crossing straight lines a_1, a_2, a_3. Through each triplet of corresponding points A_1, A_2, A_3 let a conic λ^2 be passed which intersects the fixed conic β^2 twice. The congruence $[\lambda^2]$, arising in this way, will be examined more closely; it passes into a congruence of circles, if β^2 becomes the imaginary circle at infinity.

2. PAIRS OF LINES. In four different ways λ^2 can degenerate into a pair of straight lines.

1. One of the lines, g, rests on a_1, a_2, a_3, the other, h, lies in the plane β of β^2.

If we keep the point A_1 fixed, A_2 and A_3 describe projective ranges, so that $g_{23} \equiv A_2 A_3$ describes a quadratic scroll. There are therefore two lines g_{23} resting on a_1; the two supporting points A'_1 will be associated to A_1. Each point A'_1 belongs to one point A_1; for the transversal through A'_1 of a_2 and a_3 determines two points A_2,A_3, hence one point A_1. Three times A_1 coincides with A'_1; there are therefore three lines g_{123}, each containing a group A_1, A_2, A_3. Each line h_{123} in β, intersecting g_{123}, forms together with this line a pair of lines belonging to the congruence. To group 1 belong accordingly *three systems*, each consisting of a fixed straight line and a ray of a plane pencil.

2. One of the lines, g_{23}, rests on a_2 and a_3, the other lies in β. To the intersection $A_1{}^*$ of a_1 with β a scroll $(g_{23}{}^*)$ is associated, which intersects β in a conic $\gamma_{23}{}^2$. Each ray of the plane pencil $(A_1{}^*, \beta)$ intersects on $\gamma_{23}{}^2$ two lines g_{23}, and forms with each of them a pair of lines. Group 2 contains therefore *three systems*, each consisting of a ray of a plane pencil and a straight line of a quadratic scroll.

3. Let us denote the point of a_3 associated to $A_2{}^*$, $A_3{}^*$, by $A_1{}^{**}$. Each line through $A_1{}^{**}$ resting on $A_2{}^* A_3{}^*$, forms with the latter a

[1]) R. STURM, Die Lehre von den geometrischen Verwandtschaften, I, 320.

pair of lines. Also here we find *three systems*, each consisting of a fixed line and a ray of a plane pencil.

4. The line g rests on a_2, a_3 and β^2; the line h cuts a_1 and β^2. Through the point B of β^2 passes one transversal $g_{23} \equiv A_2 A_3$; the corresponding point A_1 determines the plane of λ^2 and in this way the point B' of β^2; $h_1 \equiv A_1 B'$ forms with g_{23} the pair of lines. We find therefore *three systems* of pairs of lines in group 4.

Let us consider the correspondence (B,B'). Any ray h_1 of the plane pencil $(B'A_1)$ is cut by two rays g_{23} of the scroll corresponding to A_1; the transversal through B' of a_2 and a_3 is associated to a definite point of a_1, and intersects the corresponding ray h_1 in B'. Hence the ruled surface of the pairs of lines g_{23} which we have associated to the rays h_1, intersects the plane $(B'a_1)$ along a cubic passing through B'. But in this plane lies a line g_{23} connecting the points A_2,A_3 in $(B'a_1)$. The ruled surface (g_{23}) is therefore of order four; it intersects β^2 besides in B' in seven points B, which in the correspondence in question are associated to B'. Each of the eight coincidences is the double point D_1 of a pair of lines; the locus of D_1 is for this reason a twisted curve of order eight, δ_1^8.

The lines g_{23} form a ruled surface of order *four* with nodal lines a_2, a_3 and directrix β^2. To each point A_1 are associated four points D_1, while to a point D_1 there corresponds one point A_1. From this follows, that the order of the ruled surface (h_1) with director lines a_1 and δ_1^8, is *twelve*.

3. Order and class. With a view to defining the order of the congruence, we consider the conics λ^2 through a point P in β. To them belong in the first place the three pairs of lines of group 1, each formed by one of the lines g_{123} together with the line through P and the point (g_{123}, β). Further the six pairs of lines of group 2, defined by the three rays PA_k^*. As each of these three rays belongs to two pairs, we come to the conclusion, that the order of $[\lambda^2]$ is *nine*.

A plane through an arbitrary line k intersects a_1 and a_2 in the points A_1, A_2, and a_3 in a point A'_3, which we associate to the point A_3 corresponding to A_1,A_2. Of the scroll (g_{12}) defined by A_3, two lines rest on k; hence two points A'_3 are associated to A_3. As A'_3 coincides three times with A_3, three planes $A_1A_2A_3$ pass through k, which is consequently a chord of three conics λ^2. The class of $[\lambda^2]$ is therefore *three*.

4. Singular chords. According to a well known property of the

trilinear correspondence there are two neutral pairs A_1^n, A_2^n, which form a group with any point A_3. The line $A_1^n A_2^n$ is therefore a *singular chord.*

One of the conics λ^2 consists of this chord and the line in β resting on it and on a_3. From this follows, that the locus of the λ^2 which pass through A_1^n and A_2^n, is a *cubic dimonoid,* containing a_3.

The conical points of the six dimonoids can be indicated by $A_1^n, {}^nA_2, A_3^n, {}^nA_1, A_2^n, {}^nA_3$; in this order the six neutral chords are each time defined by two successive symbols. They form a hexagon, inscribed in a_1, a_2, a_3.

To the *singular chords* belong apparently also the three lines g_{123} and the three lines $A_k^* A_l^*$ in β.

Also the three lines a_k are *singular.* For each plane through a_1 contains the conic determined by the intersections with a_2 and a_3. Let us consider the intersection of the surface $\mathfrak{A}_1$, formed by these conics, with the plane β. To this belongs the conic β^2; the rest consists of straight lines. On a_1 rest two lines g_{23}; their intersections with β determine together with the point A_1^* two straight lines belonging to $\mathfrak{A}_1$.

The line $A_1^* A_2^*$ is cut by a line $A_1 A_3$ of the scroll corresponding to A_2^*; it lies therefore on $\mathfrak{A}_1$, as well as the line $A_1^* A_3^*$. Each of the three lines g_{123} forms a pair of lines with a straight line in β through A_1^*. The intersection of $\mathfrak{A}_1$ with β is therefore of order *nine.*

The locus of the conics λ^2 which intersect a_1 twice, is accordingly a surface $\mathfrak{A}_1^9$ with a sevenfold line a_1. containing the lines a_2, a_3 and the conic β^2.

5. Singular points. All points A_k of the lines a_k are *singular.* A straight line k through a point A_1 is intersected by two lines g_{23}, is therefore a chord of two λ^2 passing through A_1. The planes of the λ^2 through A_1 envelop consequently a quadratic cone; from this follows, that through any point of β^2 two of these λ^2 pass. Hence the locus of the λ^2 through A_1 is a surface $(A_1)^4$ with double curve β^2 and conical point A_1.

Also the points B of β^2 are *singular.* Through two points B, B' pass three λ^2; hence β^2 counts three times in the locus $\mathfrak{B}$ of the λ^2 through B. Moreover β and $\mathfrak{B}$ have in common the three lines through B meeting the lines g_{123}, and the lines joining B and the points A_k^*, which have to be counted twice. We conclude from this, that $\mathfrak{B}$ is a surface of order *fifteen* with threefold curve β^2 and three nodal lines a_k; the point B is twelvefold.

6. Surface of the conics resting on a given line l. Let us consider the intersection of this surface with the plane β. To this β^2 belongs *fifteen* times. Further three rays h_{123} which intersect l and each of which forms with one of the straight lines g_{123} a λ^2. Also the three lines joining the points A_k^* with the point (l,β) and each belonging to two pairs of lines. Then the two rays of the plane pencil (A_k^*, β), each forming a λ^2 with a straight line $A_l A_m$ resting on l; in all six rays. Finally the three lines $A_k^* A_l^*$, each of which belongs to a λ^2 of which the second component is the ray through A_m^{**} intersecting l. The complete intersection is therefore of *order* 48.

The surface in question is accordingly a Λ^{48} with fourfold lines a_1, a_2, a_3, fifteenfold curve β^2 and three double conics λ^2; these are the conics which have l for a chord and therefore intersect it twice.

Besides the lines mentioned lying in the plane β, Λ contains the three lines g_{123}, two lines g_{12}, two lines g_{23} and two lines g_{13}, all crossing the line l; further two lines g_{12}, two lines g_{23} and two lines g_{13}, intersecting l; then three lines resting on l, successively directed to the three points A_k^{**}; finally 3×16 pairs of lines, the components of which each contain one point of β^2; in 3×4 of them the line g_{kl} and in 3×12 the line h rests on l.

Mathematics. — "*On a Quartic Curve of Genus Two in which an Infinity of Configurations of* DESARGUES *can be Inscribed.*" By Prof. W. VAN DER WOUDE. (Communicated by Prof. J. C. KLUYVER).

(Communicated at the meeting of January 25, 1919).

In an article entitled "*The quartic Curve and its Inscribed Configurations*" H. BATEMAN [1]) comes to the conclusion that there exist quartic curves of genus two in which an infinity of configurations $(10_3, 10_3)$ of DESARGUES can be inscribed. BATEMAN makes only mention of the existence of these curves without entering more deeply into their properties.

Starting from considerations quite different from those of BATEMAN, I wish to indicate in this paper, what condition is *sufficient* for a uninodal quartic curve γ_4 being circumscribed to an infinity of these configurations. It will appear that each point of γ_4 is part of one of these configurations, and that we can construct each of them from one of its points if we consider γ_4 as being given. I shall first mention a few known properties of an arbitrary uninodal quartic.

1. Let for the present γ_4 represent a quartic which has a double point in O and is for the rest arbitrary. We denote the tangents at O by x and y, their equations are $x=0$ and $y=0$; each of these two lines meets γ_4 in one more point; the line joining these points is represented by $z=0$.

We can then represent γ_4 by:

$$\gamma_4 \equiv xy\,(x^2+mxy+y^2+z^2)+z\,(ax^3+bx^2y+cxy^2+dy^3)=0\,. \quad (1)$$

Out of O we can draw 6 tangents to γ_4 if we do not count the two at O; the points of contact are the intersections situated out of O, of γ_4 with the first polar curve of O, represented by

$$\pi \equiv 2\,xyz + ax^3 + bx^2y + cxy^2 + dy^3 = 0.$$

Hence

$$\gamma_4 + r\pi z = 0$$

indicates a pencil of quartics all of which possess a double point in O and touch x and y in that point; the other base points are

1) *American Journal of Mathematics (36).* H. BATEMAN. *The quartic curve and its inscribed configurations.*

the points of contact of the 6 tangents drawn out of O to γ_4, and the 4 points of intersection of z with γ_4.

If we now put

$$r = -1,$$

we choose out of the pencil a curve which has degenerated into the two lines x and y and a conic β with the equation

$$\beta \equiv x^2 + mxy + y^2 - z^2 = 0.$$

β is called "the conic of BERTINI".

On the conic β are situated the points of contact of the 6 *tangents drawn out of O to γ_4, and the two points of intersection of γ_4 with z.*

If we further draw through O an arbitrary line

$$y - lx = 0,$$

its points of intersection with γ_4 are found from

$$l\{x^2(1 + ml + l^2) + z^2\} + xz(a + bl + cl^2 + dl^3) = 0,$$

those with β from

$$x^2(1 + ml + l^2) - z^2 = 0,$$

from which appears at once:

Any line l through O intersects γ_4 besides in O in 2 *more points which are harmonically separated by the points of intersection of l with β.*

2. The curve in question is now obtained by putting in (I) the coefficient m equal to zero. The geometrical significance of this is the following:

The curve γ_4 considered is cut by the line z joining the two other points in which γ_4 is intersected by its tangents at the node, in two more points, harmonically separated by the former two.

For convenience' sake I shall speak in the future of the "harmonical uninodal curve γ_4".

Its equation is:

$$\gamma_4 \equiv xy(x^2 + y^2 + z^2) + z(ax^3 + bx^2y + cxy^2 + dy^3) = 0 \quad . \quad (1)$$

If we now put

$$\varphi \equiv (1+C)x^2 + (1-C)y^2 + 2z\left[x\left(b-d\,\frac{1-C}{1+C}\right) + y\left(c-a\,\frac{1+C}{1-C}\right) - z\right]$$

and

$$\psi \equiv 2xy + \frac{4a}{1-C}xz + \frac{4d}{1+C}yz,$$

in which C is an arbitrary constant, we get:

$$4\gamma_4 \equiv 2xy\varphi + \{(1-C)x^2 + (1+C)y^2\}\psi \quad . \quad . \quad . \quad . \quad (2)$$

For this reason we can produce γ_4 as the locus of the intersections of corresponding curves of 2 projective pencils of conics:

$$\varphi + \lambda \psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{and } (1-C)\,x^2 + (1+C)\,y^2 - 2\,\lambda xy = 0, \quad . \quad . \quad . \quad . \quad (4)$$

where the pencil (4) *consists of an involution of rays with O as centre.*

3. We will now first give a geometrical interpretation of the way in which the projectivity between (3) and (4) has been fixed.

By putting $\lambda = \lambda_1$ we choose an arbitrary conic from (3) intersecting z in the points A and A' defined by:

$$(1+C)\,x^2 + 2\,\lambda_1 xy + (1-C)\,y^2 = 0.$$

We put moreover

$$(1+C)\,x^2 + 2\,\lambda_1 xy + (1-C)\,y^2 \equiv (1-C)\,(y+px)\,(y+p^1x),$$

so that

$$1 + C = (1-C)\,pp^1 \quad \text{and} \quad 2\lambda_1 = (1-C)\,(p + p^1).$$

The points B and B', harmonically separated from A and A' by the points of intersection of z and β are then respectively found from:

$$x - py = 0$$

$$\text{and } x - p^1y = 0$$

Hence the pair of rays which project the points B and B' out of O, have for equation:

$$(x - py)\,(x - p^1y) = 0$$

or

$$(1-C)\,x^2 + (1+C)\,y^2 - 2\,\lambda_1 xy = 0.$$

For this reason the projectivity between (3) *and* (4) *is fixed in this way: a pair of corresponding conics of* (3) *and* (4) *always intersect z in pairs of points (A, A') and (B, B') so that A and B and also A' and B' are harmonically separated by β.*

4. Concerning the pencil (3) we remark that both φ and ψ are harmonically circumscribed to β, i.e. circumscribed to an infinity of polar triangles of β, as appears directly from this that one of the simultaneous invariants, — generally called Θ —, formed out of the coefficients of β and φ (or ψ), becomes zero.

From this follows:

1. *each conic of* (3) *is harmonically circumscribed to β;*

2. *the base points of the pencil* (3) *form a polar quadrangle of β, i.e. a quadrangle of which each side passes through the pole of the opposite side with regard to β.*

We call the base points of (3) S_{15}, S_{25}. S_{35}, S_{45}, the join of S_{15} and S_{25} is called s_{34}, that of S_{35} and S_{45} be s_{12}. To the pencil (3) belongs a conic which has degenerated into (s_{12}, s_{34}); let z be intersected by the first of these lines in A_{12}, by the second in A_{34}; the point of z harmonically separated from A_{12} by β, is B_{12}; also A_{34} and B_{34} are harmonically separated by β. The conic (pair of lines) of (4) corresponding to this degenerate conic of (3), is therefore formed by OB_{12} and OB_{34}. Let the intersection of OB_{12} with s_{34} be called S_{12}, that with s_{12} be T_{12}; S_{12} and T_{12} are points of γ_4.

Now O is the pole of z with regard to β; OB_{12} is therefore the polar line of A_{12}; for this reason the pole of s_{12} lies on OB_{12}. We knew already that the latter point also lies on s_{34}; hence S_{12} is the pole of s_{12} with regard to β. In this way we find that the pole of each side of the quadrangle $S_{15}\,S_{25}\,S_{35}\,S_{45}$ lies on γ_4. Now the four corners of a polar quadrangle form with the six poles of the sides a configuration $(10_3, 10_3)$ of DESARGUES; hence all the corners of this configuration lie on γ_4.

It is noteworthy that the points S_{ij}, S_{jk}, S_{ki} always lie on a line s_{lm} of which S_{lm} is the pole. Each of these lines has a fourth point of intersection with γ_4; if we choose e.g. s_{12}, it cuts γ_4 besides in S_{34}, S_{45}, S_{53} in one more point T_{12}, which also lies on OS_{12}.

By giving to the equation of γ_4 the form

$$\gamma_4 \equiv 2\,xy\varphi + \{(2-C)\,x^2 + (1+C)\,y^2\}\,\psi = 0 \quad . \quad . \quad . \quad (5)$$

we have been able to show that in it a Cf. $(10_3, 10_3)$ of DESARGUES is inscribed.

But the equation (5) also contains the entirely arbitrary constant C; by varying this we shall find an infinity of pencils (3) and (4) and an infinity of configurations. Hence:

In γ an infinity of configurations $(10_3, 10_3)$ of DESARGUES *can be inscribed; each configuration is self-polar with regard to β.*

5. In (5) φ and ψ are functions of x, y, z and C. Let $P(x', y', z')$ be an arbitrary point of γ_4, so that

$$\gamma_4\,(x', y', z') = 0.$$

Let us then determine C, so that

$$\varphi\,(x', y', z'\ C) = 0.$$

We can find two values of C satisfying this condition; then also according to (5)

$$\psi\,(x', y', z', C) = 0.$$

If we therefore consider C as a variable parameter, each point of γ is twice a base point of a pencil (3). On the other hand it is clear from what precedes that every point e.g. S_{12}, can only be part of one configuration, which we can easily construct, starting from that point.

With a view to this we join S_{12} O; the intersection of this line with γ_4 is T_{12}; we draw s_{12}, the polar line of S_{12} with regard to β, which cuts γ_4 besides in T_{12} in S_{34}, S_{45} and S_{53}; the polar lines of these 3 points cut γ_4 in the 6 other points of the configuration and T_{34}, T_{45}, T_{53}. Yet each point, e.g. S_{12}, though belonging to only one configuration, is twice a base point of the pencil (3); for we can as well produce this pencil and also γ by starting from a pencil (3) with the base points $S_{12}, S_{13}, S_{14}, S_{15}$ as from one with the base points $S_{12}, S_{23}, S_{24}, S_{25}$.

Any point of a harmonical curve γ_4 *belongs to one configuration* $(10_3, 10_3)$ *of* DESARGUES; *if we consider* γ_4 *as given we can easily construct these configurations from one of their points.*

Moreover it has appeared from the way in which we produced γ_4 that:

If a configuration $(10_3, 10_3)$ *of* DESARGUES *and a point O are given, we can produce a harmonical curve* γ_4 *circumscribed to this configuration, with its node in O, while the conic of* BERTINI *connected to it coincides with the conic relative to which the configuration is self-conjugated.*

Mathematics. — "*On the quasi-uniform convergence*". By Prof. J. WOLFF. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of February 22, 1919).

I.

We consider within an interval $a \leq x \leq b$ a convergent series of continuous functions $f(x) = f_1(x) + f_2(x) + \ldots$.

As ARZELA has shewn a necessary and sufficient condition that $f(x)$ should be continuous is the "*quasi-uniform convergence*" of the series [1]). This serves to express that, if two positive numbers ε and N are assumed, ε as small as we please and N as large as we like, there exists a number $N' > N$ such that for each x of the interval a number n_x of terms of the series can be determined between N and N', the sum of which, $S_{n_x}(x)$, differs less than ε from $f(x)$.

This theorem constitutes, it is true, a complete solution of the problem: to replace the ordinary uniform convergence by another condition which is not only sufficient, but also necessary for the continuity of $f(x)$. However the quasi-uniform convergence can again be replaced by a wider condition by which a slight extension of ARZELA's theorem is obtained. We shall namely prove the following theorem:

1. *If the series is quasi-uniformly convergent at the points of a set E which is everywhere dense within the interval $a \leq x \leq b$, then $f(x)$ is continuous throughout this interval.*

According to this supposition there exists for every ε and every N a number $N' > N$ such that for every x of E an index n_x can be determined ($N < n_x < N'$) such that $|f(x) - S_{n_x}(x)| < \varepsilon$.

Now choose an arbitrary point x of the interval. In consequence of the convergence of the series a number N can be found such that, ε denoting an arbitrary positive number:

$$|f(x) - S_n(x)| < \tfrac{1}{3}\varepsilon, \quad \text{for} \quad n > N \quad . \quad . \quad . \quad . \quad (1)$$

For every n between N and the number N' conjugated to $\frac{1}{3}\varepsilon$, N, we can now in consequence of the continuity of $S_n(x)$ determine

---

[1]) *Mem. R. Acc. Bologna 1899.*

BOREL, *Leçons sur les Fonctions de variables réelles.*

an interval $(x - \delta, x + \delta)$ such that for every point ξ within it the inequality

$$| S_n(\xi) - S_n(x) | < \tfrac{1}{3} \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

holds good. Since the number of these indices n is finite, an interval I exists, with centre x such that for any ξ within I the relation (2) is satisfied by any n between N and N'. Thus, if ξ is a point of E lying within I, and if for n is chosen the index n_ξ corresponding to ξ, in the first place the relations (1) and (2) are satisfied and besides

$$| S_{n_\xi}(\xi) - f(\xi) | < \tfrac{1}{3} \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It follows from (1), (2), and (3) that

$$| f(x) - f(\xi) | < \varepsilon.$$

Hence we have $f(x) = \lim_{\xi \to x} f(\xi)$, where ξ coincides successively with all the points of the everywhere dense set E. If x' is a point of I not belonging to E, then

$$| f(x) - f(x') | = \lim_{\xi \to x'} | f(x) - f(\xi) | \leq \varepsilon.$$

Hereby the continuity of $f(x)$ is established.

2. In connection with ARZELA's theorem it appears thus that the quasi-uniform convergence at the points of a set E everywhere dense within $a \leq x \leq b$ involves quasi-uniform convergence throughout the whole interval.

3. From the fore-going it may be easily concluded that the quasi-uniform convergence in ARZELA's theorem can be replaced by the following criterion: "for every ε, N there exists an $N' > N$ and a set of points $B(\varepsilon, N)$, belonging to the first category of BAIRE, such that for every x of the interval *not belonging to* $B(\varepsilon, N)$ an index n_x $(N < n_x < N')$ can be determined which satisfies $| f(x) - S_{n_x}(x) | < \varepsilon$".

In order to establish this we take provisorily a fixed number N and a decreasing series of positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots.$ having zero for limit. Let $N'(\varepsilon_k, N)$ and $B(\varepsilon_k, N)$ correspond to ε_k with the above-mentioned meaning. Since $B(\varepsilon_k, N)$ consists of a countable set of nowhere dense sets of points, this is also the case with the set $B(\varepsilon_1, N) + B(\varepsilon_2, N) + \ldots = B(N)$, so that $B(N)$ also belongs to the first category of BAIRE. Now choose an increasing sequence of numbers $N_1, N_2 \ldots.$ tending to infinity and put $B(\varepsilon_1, N_1) +$

43*

$\dot{+} B(\varepsilon_2, N_i) \dot{+} \ldots = B(N_i)$, then the set $B(N_1) \dot{+} B(N_2) \dot{+} \ldots = B$ also belongs to this category, so that the complementary set $C(B)$ is everywhere dense. Now give ε, N arbitrarily. Let $\varepsilon > \varepsilon_k$ and $N < N_i$. All the points of $C(B)$ lie outside $B(\varepsilon_k, N_i)$, so that for every point of $C(B)$ between N_i and $N'(\varepsilon_k, N_i)$ an index n_x can be determined satisfying

$$|f(x) - S_{n_x}(x)| < \varepsilon_k < \varepsilon.$$

Hereby the proof is completed, since $C(B)$ is everywhere dense, so that the theorem of § 1 applies here.

4. We also obtain a sufficient condition by substituting in the fore-going for $B(\varepsilon N)$: a nullset. For, a set which consists of a countable number of such sets is a null set, so that its complementary set is everywhere dense and the fore-going reasoning applies again. It follows from this in particular that: *a convergent series of continuous functions represents a continuous function if the convergence is "almost everywhere" quasi-uniform.*

II.

5. In § 1 use has been made of the *convergence* of the series at the arbitrarily assumed point x, also in the case where x did not belong to the dense set E. The question may be put if it is necessary to suppose the series convergent throughout the whole interval.

If a series of functions which are continuous throughout the interval $a \leq x \leq b$ converges *uniformly* at the points of a set E which is everywhere dense within this interval, then this involves the uniform convergence of the series throughout the whole interval. By analogy we are led to the following question: If the terms of a series are continuous functions of x in the interval $a \leq x \leq b$, and if, besides, the series is quasi-uniformly convergent at the points of a set E which is everywhere dense within the said interval, is it then allowed to conclude to the *convergence* of the series throughout the whole interval and thereby to the continuity of the function represented by the series and thus to the quasi-uniformity of the convergence within $a \leq x \leq b$?

The answer is negative. In order to show this we consider the following series:

$$f(x) = x - x + x^2 - x^2 \ldots + x^n - x^n + \ldots.$$

This series converges quasi-uniformly to zero within the open interval $0 \leq x < 1$, which constitutes an everywhere dense set within the closed interval $0 \leq x \leq 1$. The convergence is quasi-uniform, since the sum of an even number of terms is always zero. For $x=1$ however the series is not convergent.

6. In this example $f(x)$ converges for $x=1$ to a limit $f(1-0)=0$. That this need not be the consequence of the quasi-uniform convergence within the interval $0 \leq x < 1$ is demonstrated by the following example:

Let $y=f(x)$ be represented by the zig-zag-line $A_1B_1A_2B_2\ldots$, where $A_1, A_2, A_3, \ldots$ are the points with $x=1, \frac{1}{2}, \frac{1}{4}, \ldots$ and $y=0$, $B_1, B_2, B_3, \ldots$ the points with $x=\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots$ and $y=1$. Now choose on the axis of y a countable set of points $P_1, P_2, P_3, \ldots$, everywhere dense in the interval (0,1). Let the function $y=S_n(x)$ be represented by the following line: from the right to the left first the zig-zag $A_1B_1A_2B_2\ldots A_nB_n$, then the segment of B_nA_{n+1} to the point of intersection, C_n, with the line $y=y_{P_n}$, last the line C_nP_n. Evidently $S_n(x)$ is continuous in the interval $0 \leq x \leq 1$. Also $\lim\limits_{n \to \infty} S_n(x)=f(x)$ for $0 < x \leq 1$, since from a certain value of n onwards $S_n(x)$ coincides with $f(x)$ for a thus situated point. Hence the series $S_1(x)+\{S_2(x)-S_1(x)\}+\ldots+\{S_n(x)-S_{n-1}(x)\}+\ldots$ converges in the interval $0 < x \leq 1$ to $f(x)$ and all the terms are continuous within $0 \leq x \leq 1$. *This convergence is quasi-uniform.*

In order to make this clear we choose ε and N arbitrarily. Since the set (P_i) is everywhere dense within (0,1), we can now choose a finite number of points $P_{n_1}, P_{n_2}, \ldots P_{n_k}$ between 0 and 1 of which the indices are $> N$ and which divide the interval into $k+1$ segments all $< \varepsilon$. Now, let x be an arbitrary value between 0 and 1.

If $f(x)$ coincides with one of the k values $S_{n_i}(x)$, then

$$|f(x)-S_{n_i}(x)|=0<\varepsilon.$$

If $f(x)$ does not coincide with any of these values, then

$$S_{n_i}(x)=y_{P_{n_i}}, \quad =1,2,\ldots k.$$

Since $0 \leq f(x) \leq 1$ one of the k indices satisfies

$$|f(x)-S_{n_i}(x)|<\varepsilon.$$

Hereby the quasi-uniform character of the convergence in the interval $0 < x \leq 1$ is established. At 0, however, $f(x)$ does not assume a limiting value, but oscillates between 0 and 1.

III.

7. Let $f_1(x), f_2(x)\ldots$ be functions which are continuous within the interval $a \leq x \leq b$ and let the series

$$f(x) = f_1(x) + f_2(x) + \dots$$

be quasi-uniformly convergent in the interval $a < x \leq b$.

$f(x)$ is then continuous in the latter open interval. Let M and m denote the maximum and minimum of $f(x)$ at a and let μ be an arbitrary number of the interval $m \leq \mu \leq M$. A set of points $x_1, x_2, x_3 \dots x_\nu, \dots$ can be constructed where $\lim\limits_{\nu \to \infty} x_\nu = a$ and $\lim\limits_{\nu \to \infty} f(x_\nu) = \mu$. From a limited number of indices it is possible to choose one for each point of this set such that $|S_{n_\nu}(x_\nu) - f(x_\nu)| < \varepsilon_1$, where ε_1 is an arbitrary positive number. There are therefore an infinite number of points x_ν where one and the same index can be used, which we call n_1. If x_ν tends to a, then $S_{n_1}(x_\nu)$ tends to $S_{n_1}(a)$ and $f(x_\nu)$ to μ; hence $|S_{n_1}(a) - \mu| \leq \varepsilon_1$. Let $\varepsilon_1, \varepsilon_2, \dots$ be a decreasing sequence of positive numbers having zero for limit. It is again possible to choose from a finite number of indices for every x_ν an index $n_\nu > n_1$ such that $|S_{n_\nu}(x_\nu) - f(x_\nu)| < \varepsilon_2$, hence there exists an index n_2 satisfying $|S_{n_2}(a) - \mu| \leq \varepsilon_2$.

Thus pursuing we find that there is a partial sequence of functions $S_{n_1}(x), S_{n_2}(x), \dots$ which at a converges to the value μ and for $a < x \leq b$ to $f(x)$. Hence:

If the series $f_1(x) + f_2(x) + \dots$ consists of terms which are continuous within $a \leq x \leq b$ and converges quasi-uniformly to $f(x)$ in $a \leq x \leq b$, and if μ is an arbitrary value lying between the maximum and the minimum of $f(x)$ at a, then the series can be transformed, by uniting the terms group-wise to one new term, into another series which converges in $a \leq x \leq b$, having μ for its limit at a and $f(x)$ at the other points.

In the example of § 5 we have $M = m = 0$. The series $(x - x) + (x^2 - x^2) + \dots$ is here a transformed series which converges everywhere to zero.

In the example of § 6 $M = 1$, $m = 0$. Choose a set $P_{n_1}, P_{n_2}, \dots$ having μ as limit. The partial sequence $S_{n_1}(x), S_{n_2}(x), \dots$ converges to $f(x)$ for $0 < x \leq 1$ and to μ at 0.

That the quasi-uniformity of the convergence in the open interval is no superfluous condition is illustrated by the series $1 - x + x^2 - x^3 + \dots$, which for $0 \leq x < 1$ represents $\dfrac{1}{1+x}$, so that for $x = 1$ we have $M = m = \frac{1}{2}$. In no way however the terms of the series $1 - 1 + 1 - 1 + \dots$ can be united to groups in order that the transformed series should converge to $\frac{1}{2}$.

8. The theorem of the preceding § can be reversed as follows:

Let a sequence of functions be given, $S_1(x), S_2(x), \ldots.,$ all continuous in $a \leq x \leq b$. Let the sequence converge in $a < x \leq b$ to a function $f(x)$ which is continuous in this interval, and let M denote the maximum of $f(x)$ at a, m the minimum. Now, if it is possible to conjugate to every number k between m and M a partial sequence of the given sequence which at a converges to k, then the convergence of the given sequence is quasi-uniform in $a < x \leq b$.

In order to prove this we give ε, N and on the line $x = a$ we choose the points $P_0, P_1, \ldots P_\nu$ such that

$$P_0 < m < P_1 < P_2 \ldots. < P_{\nu-1} < M < P_\nu$$

and at the same time

$$P_i P_{i+1} < \varepsilon \quad , \quad i = 0,\ 1, \ldots.\ \nu - 1.$$

It follows from the supposition that an index $n_i > N$ exists for which $S_{n_i}(a)$ lies between P_i and P_{i+1}. Thus we find ν indices. Since the functions $S_{n_0}(x), S_{n_1}(x) \ldots. S_{n_{\nu-1}}(x)$ are continuous at a, a number δ_1 can be found such that for $x—a < \delta_1$, $S_{n_i}(x)$ lies between P_i and P_{i+1}, where $i = 0, 1, \ldots. \nu—1$. Also a number δ_2 can be found such that for $x—a < \delta_2$ $f(x)$ lies between P_0 and P_ν, for, M and m are the maximum and the minimum of $f(x)$ at a. Let $\delta \leq \delta_1$ and $\delta \leq \delta_2$. For $x - a < \delta$, $f(x)$ belongs to one of the intervals $P_i P_{i+1}$. Hence it is possible for every x of the interval $a < x < a + \delta$ to choose from the ν obtained indices an index n_x such that $|f(x)—S_{n_x}(x)| < \varepsilon$.

In the same way it is possible to make for $a + \delta \leq x \leq b$ a similar choice from a finite number of indices $> N$, since the given sequence in consequence of the continuity of $f(x)$ converges quasi-uniformly in this interval. Hereby the theorem is established.

It is evident that in this theorem the words "*to every number k between M and N*" may be replaced by: "*to every number k of a set which is everywhere dense in the interval m, M*".

Mathematics. — "*Series of analytical functions*". By Prof. J. WOLFF. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of September 29, 1918).

OSGOOD's theorem: "*If the series $f_1 + f_2 + \dots$, all the terms whereof are analytical functions within a region T of the complex plane, is convergent at the points of a set (β) everywhere dense in T, and if besides, $| f_1 + f_2 + \dots + f_n | < G$ at every point of T (G being a constant), the series converges everywhere in T and there represents an analytical function*" [1]) has been again demonstrated by ARZELA [2]).

VITALI [3]) and PORTER [4]) have extended the theorem by proving that it is sufficient if only the set (β) of the points where the series converges has an internal point of T for a limiting point.

Of the thus extended theorem a simple demonstration shall be given in the sequel.

1. To this end we suppose that the f_i are analytical in T, that in T everywhere $| S_n | < G$ for every n, G being a constant, and that the series is convergent at the points $\beta_1, \beta_2, \dots$, having the internal point z_0 of T for limiting point, and we shall prove that the series converges uniformly in every region lying with its boundary within T.

Now describe a circle (R) with centre z_0 and radius R, lying in T. Let β_i be a point of (β) inside a circle $(\frac{1}{2} R)$ with centre z_0, and let $f(\beta_i)$ denote the sum of the series at (β_i), $S_n(\beta_i)$ the sum of n terms, then

$$S_n(\beta_i) = \frac{1}{2\pi i} \int_{(R)} \frac{S_n(t)dt}{t-\beta_i} \quad \text{and} \quad f(\beta_i) = \frac{1}{2\pi i} \lim_{n=\infty} \int_{(R)} \frac{S_n(t)dt}{t-\beta_i}$$

If β_k denotes another point of (β) inside $(\frac{1}{2} R)$, then

$$|S_n(\beta_i) - S_n(\beta_k)| < \frac{4G}{R} |\beta_i - \beta_k|,$$

---

1) W. F. OSGOOD. *Functions defined by infinite series*. Annals of Mathematics, Series 2, Vol. 3, Oct. 1901, p. 26.

2) V. C. ARZELA. Annals of Mathematics, Series 2, Vol. 5, 1904, p. 51.

3) G. VITALI. *Sopra le serie di funzioni analitiche*. Annali di Matematica, Serie 3a, tomo 10, 1904, p. 65.

4) M. B. PORTER. Annals of Math. Series 2, Vol. 6, 1904—5, p. 45 and p. 190.

for every n, so that

$$|f(\beta_i)-f(\beta_k)| \leq \varepsilon$$

as soon as

$$|\beta_i - \beta_k| < \frac{R\varepsilon}{4G},$$

whence it follows that $f(\beta_i)$ tends to a limit $f(z_0)$ as β_i tends to z_0.

Let η denote an arbitrary positive number. For all sufficiently small values of $|\beta_i - z_0|$ we then have:

$$|f(\beta_i) - f(z_0)| < \frac{\eta}{3}.$$

Also $|S_n(\beta_i) - S_n(z_0)| < \frac{2G}{R} |\beta_i - z_0|$ for every n, so that for all sufficiently small values of $|\beta_i - z_0|$ the relation

$$|S_n(\beta_i) - S_n(z_0)| < \frac{\eta}{3}$$

holds, where n is arbitrary.

Now choose a β_i satisfying these two conditions, then from a certain n onwards we have:

$$|S_n(\beta_i) - f(\beta_i)| < \frac{\eta}{3}.$$

It follows thence that from this value of n onwards we have continually:

$$|S_n(z_0) - f(z_0)| < \eta,$$

from which we conclude to the convergence of the series at z_0. The sum there is $f(z_0)$.

At the same time it has become evident that

$$\lim_{\beta_i = z_0} f(\beta_i) = f(z_0)$$

2. Provisorily we consider n constant. Then for $|z - z_0| < \frac{1}{2} R$ we have:

$$S'_n(z_0) = \frac{S_n(z) - S_n(z_0)}{z - z_0} + \psi_n(z),$$

where $\lim\limits_{z=z_0} \psi_n(z) = 0$. The function $\frac{S_n(z) - S_n(z_0)}{z - z_0}$ is analytical inside $(\frac{1}{2} R)$, its absolute value being less than $\frac{2G}{R}$. If n now is made to increase infinitely, this function tends to a limiting value at the points β_i and therefore, according to § 1, also at z_0. At z_0 it has for every n the value $S_n'(z_0)$. It follows thence that $S_n'(z_0)$ tends to a limit $f^{(1)}(z_0)$.

Similarly

$$S''_n(z_0) = 2!\,\frac{S_n(z) - S_n(z_0) - \frac{z-z_0}{1} S'_n(z_0)}{(z-z_0)^2} + \Phi_n(z),$$

where $\lim\limits_{z=z_0} \Phi_n(z) = 0$. The first term of the right-hand side is analytical inside $(\frac{1}{2}R)$, its absolute value being less than $\frac{4G}{R^2}$, since $\left|\frac{S_n^{(k)}(z_0)}{k!}\right|$, the absolute value of the coefficient of $(z-z_0)^k$ in the development $S_n(z) = S_n(z_0) + \sum\limits_1^\infty a_k (z-z_0)^k$, is less than $\frac{G}{R^k}$. As n increases infinitely this function tends to a limit at the points β_i and therefore at z_0 too, from which it follows that a limit

$$\lim_{n=\infty} S''_n(z_0) = f^{(2)}(z_0)$$

exists.

Thus pursuing we find that for every k a limit

$$\lim_{n=\infty} S^{(k)}(z_0) = f^{(k)}(z_0).$$

exists.

3. For an arbitrary z inside $(\frac{1}{2}R)$ we have:

$$S_n(z) = S_n(z_0) + (z-z_0)\, S'_n(z_0) + \ldots\ldots + \frac{(z-z_0)^k}{k!} S_n^{(k)}(z_0) + \ldots \qquad (1)$$

For a fixed value of z the terms of this series, if n increases infinitely, tend to those of the series

$$f(z) = f(z_0) + (z-z_0) f^{(1)}(z_0) + \ldots + \frac{(z-z_0)^k}{k!} f^{(k)}(z_0) + \ldots\ldots \qquad (2)$$

which represents a function, analytical in $(\frac{1}{2}R)$, since $\left|\frac{f^{(k)}(z)}{k!}\right| \leq \frac{G}{R^k}$.

From $\left|\frac{S_n^{(k)}(z_0)}{k!}\right| < \frac{G}{R^k}$ it follows that the series (1) converges *uniformly*, if the terms are considered as functions of the two independent variables z and n, at the points of the set $|z-z_0| \leq \frac{1}{2}R, n = 1, 2, \ldots$

It follows from this that S_n converges uniformly to f inside $(\frac{1}{2}R)$.

4. Instead of $\frac{1}{2}R$ as well λR could have been chosen, where λ is an arbitrary positive number < 1. Hence S_n converges uniformly to an analytical function in the interior of every circle (R) lying wholly inside T. Let z be an arbitrary point in T, then z can be

enclosed within the last of a chain of circles, all lying within T, the first being (R) with centre z_0 and every circle having its centre within the preceding one. Since the points where S_n converges condense towards the second centre, S_n converges uniformly throughout the second circle; similarly within all the following, hence in any circle with centre z which lies in T. Every region lying with its boundary within T can be covered by a *finite* number of such circles which involves the uniform convergence of S_n to an analytical function throughout τ.

5. Lastly we shall give a simple proof of OSGOOD's original theorem. According to § 1, if S_n is convergent at the points of the set (β) which is everywhere dense in T, it converges everywhere throughout T and the limiting function f is continuous in T, whilst $|f| \leq G$. Now draw a circle (R) with centre z_0, lying altogether in T. If $|z-z_0|$ is again $< \frac{1}{2} R$, then

$$f(z) = \frac{1}{2\pi i} \lim_{n=\infty} \int_{(R)} \frac{S_n(t)dt}{t-z} = \frac{R}{2\pi} \lim_{n=\infty} \int_0^{2\pi} \frac{S_n(t)\, e^{\theta i}\, d\theta}{t-z}, \quad t = z_0 + Re^{\theta i}.$$

We now make use of the self-evident extension to complex functions of the following theorem of OSGOOD [1]):

If a function $\varphi_n(\theta)$, continuous in the interval $a \leq \theta \leq b$ for every n, converges to a function $\varphi(\theta)$ which is continuous throughout this interval, and if, besides, $|\varphi_n(\theta)| < G$ throughout the interval and for every n, G being a constant, then

$$\lim_{n=\infty} \int_a^b \varphi_n(\theta)\, d\theta = \int_a^b \varphi(\theta)\, d\theta.$$

If we put $\varphi_n(\theta) = \dfrac{S_n(t)\, e^{\theta i}}{t-z}$, then φ_n is continous in $0 \leq \theta \leq 2\pi$, $|\varphi_n| < \dfrac{2G}{R}$, and $\varphi(\theta) = \dfrac{f(t)\, e^{\theta i}}{t-z}$ is continous in $0 \leq \theta \leq 2\pi$. Hence

$$f(z) = \frac{1}{2\pi i} \int_{(R)} \frac{f(t)\, dt}{t-z}.$$

Since f is continuous on (R), it follows from this that f is analytical inside $(\frac{1}{2} R)$.

The same lemma can be used to prove in a simple way that S_n converges uniformly to f inside $(\frac{1}{2} R)$.

1) W. F. OSGOOD. *On the non uniform convergence.* Am. Journal of Math. 1897. For an extension *vide* H. LEBESGUE, *Leçons sur l'Intégration,* p. 114.

For $|z-z_0| \leq \frac{1}{2}R$ *viz.* we have

$$|f(z)-S_n(z)| \leq \frac{1}{\pi}\int_0^{2\pi} |f(t)-S_n(t)|\,d\theta, \quad t = z_0 + Re^{\theta i}.$$

Here $\varphi_n = |f(t) - S_n(t)|$ is continuous in $0 \leq \theta \leq 2\pi$ and $2G$; φ is zero. Hence

$$\lim_{n=\infty} \int_0^{2\pi} |f(t)-S_n(t)|\,d\theta = 0.$$

For all sufficiently large values of n therefore $|f-S_n|$ is less than a given number everywhere inside $(\frac{1}{2}R)$, that is to say, S_n converges uniformly to f inside $(\frac{1}{2}R)$.

By virtue of the same lemma we have

$$\lim_{n=\infty} S_n^{(k)}(z) = \frac{k!}{2\pi i} \lim_{n=\infty} \int_{(R)} \frac{S_n(t)dt}{(t-z)^{k+1}} = \frac{k!}{2\pi i}\int_{(R)} \frac{f(t)dt}{(t-z)^{k+1}} = f^{(k)}(z),$$

whence it follows that the series may be differentiated termwise infinitely often.

Let C be a regular curve in T then, again by virtue of the above-mentioned lemma, we have

$$\int_C f(z)dz = \lim_{n=\infty} \int_C S_n(z)dz.$$

Hence, if C is closed and its points, as well as the points enclosed, are all internal points of T, then

$$\int_C f(z)dz = 0.$$

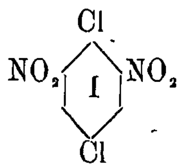
From this also it may be concluded, according to a theorem enunciated by MORERA [1]), that f is analytical throughout T.

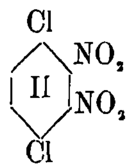
[1]) Reale Istituto Lombardo di Sc e lettere, Rendic., 2nd series, Vol. 19, 1886.

Chemistry. — *"On the Nitration-Products of p-Dichlor-Benzene".* By Dr. A. J. DEN HOLLANDER and Dr. F. E. VAN HAEFTEN. (Communicated by Prof. A. F. HOLLEMAN).

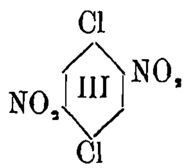
(Communicated at the meeting of November 29, 1919).

In 1868 JUNGFLEISCH [1]) nitrated p-dichlor-benzene by boiling it with a mixture of fuming nitric acid and sulphuric acid for some hours. In this way he obtained a mixture of dinitro-p-dichlor-benzenes, about whose constitution opinions are still divided. Theoretically three isomers are possible, viz.:





and

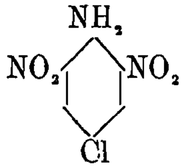


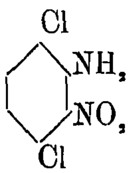
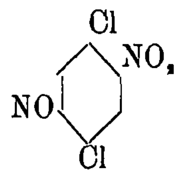
JUNGFLEISCH himself isolated two compounds out of it, which he denoted by α and β, but of which he did not determine the structure. For the α-compound he gave 87° as melting-point, for the β-compound 107°. There is formed much more of the former compound according to him than of the latter. KÖRNER [2]), and much later ULLMANN and SAKE [3]), proved that structure I applies to this chief product; the melting-point was, however, found at 105° by them. ENGELHARDT and LATSCHINOFF [4]) had also observed this higher melting-point, and could isolate the β-compound also from the reaction-product; hence they confirmed JUNGFLEISCH's results in the main.

On the other hand MORGAN and NORMAN [5]) assert that compound III is formed as chief product; HARTLEY and COHEN [6]), who repeated the former's experiments, confirm this, and give as melting-point 105—106°. Two year ago, Miss EDITH NASON [7]) nitrated p-dichlor-benzene anew, and succeeded in also isolating III from the reaction mixture in a yield of 45,6 %, for which she, however, found the melting-point 81°.

1) A. ch. (4), **15** 259.
2) J. 1875, 324.
3) B. **44**, 3730 (1912).
4) Z. 1870, 234.
5) Soc. **81**, 1378, 1382 (1902).
6) Soc. **85**, 868 (1904).
7) Am. Soc. **40**, 1602 (1918).

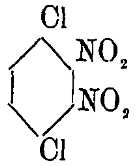
The structure proofs which these investigators bring forward for their supposed isomer III, are, however, quite insufficient. It, therefore, remained: 1st to decide what isomer or what isomers arise by the side of I (about which there is no difference of opinion); 2nd to furnish a conclusive structure proof for this isomer or these isomers.

In the first place we have now found that all three isomers are formed, I as chief product, II and III as bye-products. In this we have proceeded as follows: The crude reaction product is perfused with an excess of 4 N-alcoholic ammonia, 4 mol. NH_3 to 1 mol. dichlor-dinitro-benzene. After some stirring most of it goes into solution. It is then left standing for two days at the temperature of the room. Then I is converted into  2-6-dinitro-4-chloraniline, melting-point 145°, which is for the greater part deposited in fine needles and almost pure [1]). After filtration the alcoholic fitrate is distilled; then a residue is left, which consists of the isomers II and III, but also contains a certain quantity of the above mentioned chlor-dinitraniline. This residue is washed with water to remove Am-nitrite, then dried, and dissolved in about ½ liter of benzene (when 1 mol. of reaction product was started from). This solution is shaken out a few times with 20 cc. concentrated sulphuric acid, by which means the chlordinitraniline is removed. This is the case when the sulphuric acid gives no further colour.

Then the benzene is distilled off, after which a residue remains, consisting chiefly of the compounds II and III, as can be shown by treating it again with alcoholic ammonia, 100 cc. 4 N. NH_3 to 50 gr. mixture. Then the liquid is digested (temp. in the flask 80—85°) for 24 hours on a waterbath at a reflux condenser, and ammonia gas is led in a few times to compensate the loss. Then the liquid is neutralized, the greater part of the alcohol is distilled off, and the rest is poured into water. Through treatment of the reaction product with sulphuric acid, as given above, 2 nitro-3-6-dichloraniline  goes into solution and 1-4-dichlor-2-5-dinitro  benzene remains behind.

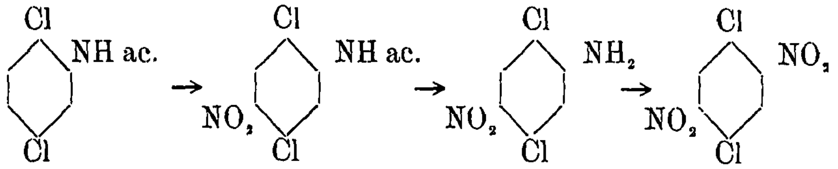
1) It contains still a certain quantity of the isomer III, from which it can be separated by sulphuric acid, see below.

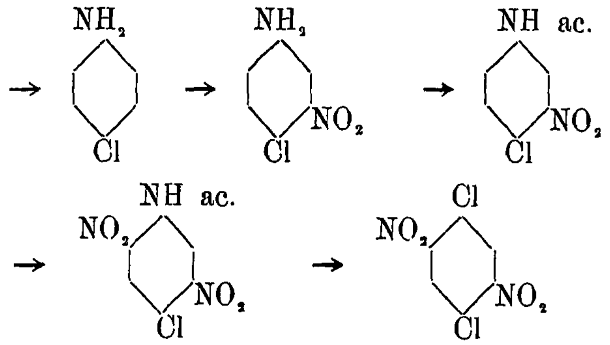
After recrystallisation from alcohol the latter is pure and presents the melting-point of 119°.

The formation of 2-nitro-3-6-dichlor-aniline, melting-point 68°, the structure of which has been ascertained through a research of BEILSTEIN and KURBATOW[1]), proves that in the crude reaction product the 1-4-dichlor-2-3-dinitrobenzene  must be present.

Through prolonged fractionated crystallisation from alcohol this could actually be separated out of it. It is probably JUNGFLEISCH's isomer β, and melts at 103°. The structure was proved by replacing the NH_2 group in 2-nitro-3-6-dichloraniline by NO_2 according to the method of KÖRNER and CONTARDI[2]).

Hence only the structure formula III remains for the third isomer of the melting-point 119°. This was, however, proved more closely in the two following ways:

1. 

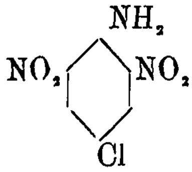
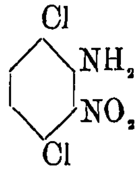
2. 

in which the structure of the intermediate products was every time determined.

When we try to separate the three components out of the crude nitration-product of JUNGFLEISCH, we succeed by means of very prolonged continued fractionated crystallisation from alcohol in obtaining the compounds I and II in pure state, but not III. We get at last a pretty considerable quantity of a cauliflowery mass, which is no longer liable to further separation by crystalli-

[1]) A. **196**, 221 (1879).
[2]) Atti (5) **22** II, 632 (1913).

sation, and which — as appears from the treatment with alcoholic ammonia applied to it as given above — consists chiefly of I, further of a little II, and pretty much III.

From the obtained compounds  and  the corresponding dichlordinitro benzenes can be easily regained through diazotation.

A full description of this investigation will shortly appear in the Recueil.

November 1919. *Org. chem. lab. of the Univ. of Amsterdam.*

Physiology. — "*The Quantitative Relations of the Nervous System Determined by the Mechanism of the Neurone.*" By Prof. EUG. DUBOIS.

(Communicated at the meeting of December 27, 1919).

For animal species with the same organisation of the nervous system (homoneuric species) the quantity of the neurone varies in the same way, relative to the body quantity, as that of the brain [1]), i.e. proportional to $P^{0.5\dot{5}}$ or $P^{5/9}$. The establishment of this fact may certainly be considered as a confirmation of the validity of our present views of the constitution of the nervous system, which, on solid grounds, was considered, also in its highest forms, as entirely consisting of series of structural elements, highly specialized cells, the neurones, which conduct processes of stimulation (impulsions) in definite directions, and preserve conditions of stimulation (impressions), directly or indirectly dependent on the sense organs and the muscles, and which are, accordingly, in mechanic dependence on the body. The other constituent parts of the nervous system were thought to be non-essential: they were supposed to serve only as a support and protection to the neurones.

From the parallelism of the relation of the *total* quantity of the brain with that of the neurone, *including the medullary sheath and the neurilemma,* it also appears that the same significance must be

[1]) We refer here to a *ratio* of two quantities expressed *in the same unit* with regard to a ratio of two other quantities which may be expressed in another unit, provided it be *the same for these latter two.* For two animal species whose body *weights* and brain *weights* are P and P_1, and E and E_1, and whose *volumes* of homologous neurones and their ganglion cells are N and N_1, C and C_1, the following equations may be put:

$$\left(\frac{P}{P_1}\right)^x = \frac{C}{C_1} \text{ and } \left(\frac{P}{P_1}\right)^v = \frac{N}{N_1} \text{ as well as } \left(\frac{P}{P_1}\right)^r = \frac{E}{E_1}.$$

Actually it appeared that:

$$v = 2x = r = 0.5\dot{5} \text{ or } {}^5/_9 \text{ and } \left(\frac{C}{C_1}\right)^2 = \frac{N}{N_1}.$$

The specific weight of the components of the nervous system (about equal to that of the body) lies, indeed, so little above 1 that, when the ratio of the *volumes* is taken, instead of that of the weights of these components to the body weight, the exponents are only reduced to 0.2754 and 0.5508 (instead of $0.2\dot{7}$ and $0.5\dot{5}$).

attached to that intermediate substance of the essential elementary components of the brain as to the coverings of the neuraxone. That *these* could not be entirely inactive, had been understood long ago. Just as the propagation of the process of stimulation in the axone has nothing in common with the conduction of the electric current through a wire, the comparison of the medullary sheath to the insulator of an electric cable undoubtedly represents this living substance as being much more passive than it is in reality, though it is true that it does not *directly* take part in the propagation of the impulses. Nor could the simple function of giving support and protection to the neurones be assigned with conviction to the neuroglia. No more does the equality of the volume of the medullary sheath with that of the axone, which DONALDSON and HOKE [1]) established for all classes of Vertebrates, fit in with the view that the former would only have the same significance as the insulating covering of the conductor in the cable.

Taking this into consideration and in view of what physiological experiments have taught, the nervous system, hence the neurone, appears more and more in the light of a mechanism; though a stringent proof could not yet be furnished, chiefly because it is only in the last few years that a relation has been found between dimension and function of the nervous system. Thus doubt could still be entertained, chiefly as regards the organ of the brain, of the rational significance of the determined quantitative relations, which beyond any doubt point to the existence of a (as yet unknown) mechanical relation.

For — thus the reasoning ran — part of the brain must as "the organ of the mind" be as independent of the size of the body as the psychical processes that are enacted there; is it then possible to assume that the body mechanically determines the total quantity of the brain?

But the psychical processes, certainly, are not independent of the quantity of the brain and its parts. For Man is not only psychically superior to all the animals, he is also distinguished by the greatest true relative brain quantity, and by the extraordinary development, particularly of that part of the brain which performs the highest functions. We also meet with great differences in the well-calculated relative brain quantity (determined by the cephalisation

---

[1]) H. H. DONALDSON and G. H. HOKE, On the Areas of the Axis Cylinder and Medullary Sheath as seen in Cross Sections of the Spinal Nerves of Vertebrates. Journal of Comparative Neurology and Psychology. Vol. XV. Chicago 1905, p. 1—16.

of the nervous system) and in the relative development of the parts of the brain of higher and lower function between different animal species, apparently in accordance with their psychical capacities. With equal body weight as the Chimpanzee, Man has indeed three and a half times the brain weight of this most human-like animal species; the Chimpanzee, in its turn, has twice as much brain as would possess a Macaque, obtaining the same body weight, ten times as much brain as a Mouse or a Rat of the same body weight. Besides, the different Mammals differ very considerably with regard to the proportion of the more highly to the lower organized parts of the brain. It is beyond our power to estimate the amount of the psychical differences between the animal species, but these psychical differences are connected, so far as we can see, with the well-calculated relative brain quantity and the relative development of more highly organized parts of the brain.

Though, in view of those facts, we cannot reasonably assume the existence of essential, qualitative differences between the species, either in one or in the other respect, yet there is some difficulty in conceiving that even in case of the greatest quantitative differences, essentially equal "psychical powers", only differing in degree, correspond to this qualitative similarity of the nervous system. They think that a certain quantity of the brain, though it cannot be anatomically separated from the rest, would be mechanically independent of the body, and specially set apart for the intellect.

This view formulated in 1885 by MANOUVRIER [1]), though with a great deal of reservation and for want of something better, was refuted by LAPICQUE [2]) in 1907. The latter demonstrated that an equal brain quantity *i* can, indeed, be calculated every time for two psychically equivalent species, but not for three or more. Between the Lion and the Puma the calculation of the "brain quantity for the psychical functions" gives a value *four* times greater than between the Puma and the Cat. Yet these members of the Cat-tribe may be assumed to have quite the same organization of the brain.

I can now add a few more trios of other genera to these three species of my paper of 1897.

In the Dog-tribe, the value of *i* found between the Wolf and the

1) L. MANOUVRIER, Sur l'interprétation de la quantité dans l'encéphale. Mémoires de la Société d'Anthropologie de Paris. 1885. 2e Série, Tome 3me, p. 316 et seq.

2) L. LAPICQUE in Bulletins et Mémoires de la Société d'Anthropologie de Paris 1908, (Séance du 2 Mai 1907), p. 256 et seq.

44*

Jackal is *three* times that found between the Jackal and the Fennec. Likewise in Rodentia, the calculation of i between the large Malayan Squirrel and the Common Squirrel gives *double* the value obtained by comparison of this European species with the small Hudson Squirrel; between the Brown and the Black Rat i is found more than *three* times as much as between the Black Rat and the House Mouse [1]).

Nor can such a quantity, which would only serve for psychical functions and be independent of the mechanism of the body, be reasonally assumed to exist in the human brain. The hypothesis under consideration, which lacks anatomical or physiological foundation, must, therefore, be relinquished.

This can be stringently proved from the mechanism of the neurone. If there is no room for a non-mechanically determined quantity in the neurone, then this cannot either be the case in the complex of neurones, the nervous system.

The existence of fixed quantitative relations between the neurone and the body, and between the parts of the neurone inter se left hardly any room for doubt that these relations are determined mechanically; a closer consideration of them gives complete certainty on this head.

---

[1]) Let i be the hypothetical brain quantity (the weight), set apart for the "intellect", m a constant for the influence of the "masse organique" (MANOUVRIER) on the quantity (weight) of the brain, P and P_1 the body weights of two compared species, one of which is greater than the other, E and E_1 their brain weigths, then on the supposition,

$$E = mP + i \text{ and } E_1 = mP_1 + i$$

from which

$$m = \frac{E-E_1}{P-P_1}$$

and

$$i = E - \frac{P(E-E_1)}{P-P_1}.$$

In grams the weights of P and E are, for Canis lupus 37000 and 139, for Canis aureus 10000 and 73 (these two according to L. LAPICQUE in Bulletin du Muséum d'histoire naturelle. Paris 1912. N⁰. 1, p. 4), for Canis zerda 1500 and 25 (according to B. KLATT, in Sitzungsberichte der Gesellschaft naturforschender Freunde, Berlin 1918, p. 37), for Sciurus bicolor, Sciurus vulgaris, and Sciurus hudsonicus 1400 and 12, 323 and 6.1, 159 and 4.1, for Mus norvegicus, Mus rattus and Mus musculus 448 and 2.36, 200 and 1.59, 21 and 0.43 (according to my records in Zeitschrift für Morphologie und Anthropologie, 1914, p. 327, and my Paper in the Verhandelingen of this Academy of 1897), for Felis leo, Felis concolor and Felis domestica 119500 and 219, 44000 and 137.5, 3300 and 31 (according to records in the same Paper).

It has appeared in the first place that the volume of homologous, at the same time analogous ganglion cells (functioning in the same way) varies proportional as the power $0,2\dot{7}$ or $^5/_{18}$ of the body weight.

As it further appears from the found equality of the relations of different kinds of neurones and the brain, that the same mechanism must hold for all the neurones, we can study it by means of the neurones with peripheral nerve fibers, which are most accessible to investigation. In animal species having the same form the length of homologous nerve fibers varies necessarily in direct ratio to the length of the body, i.e. to $P^{0.3\dot{3}}$ or $P^{1/3}$, in animal species of dissimilar form the nerve lengths vary with a greater or smaller power of the body weight. In case of dissimilarity as well as in case of similarity in form the volumes of homologous, at the same time analogous neurones are however, found varying in proportion to $P^{0.5\dot{5}}$ or $P^{5/9}$. Hence the cross section of the nerve fiber must vary in inverse ratio to its length. As the nerve fiber constitutes by far the greater part of the volume of the neurone, the variation of the cross section may be put proportional as $P^{0.2\dot{2}}$ or $P^{2/9}$ for animal species having the same form, but different sizes.

Man and the Mouse are not species of similar form; the Mouse has relatively much shorter limbs, nevertheless homologous neurones which function in the same way, such as the motor neurones for the finger muscles, can very well be compared, which appears from the careful measurements by IRVING HARDESTY [1]). He found the nerve fiber of these neurones on an average 35 mm. long in full-grown mice, and the homologous nerve fiber of a man weighing 72 kg., i.e. 3600 times the weight of the mouse, 800 mm. long. These lengths are to each other as 22.86 : 1, i.e. as the power 0.3821 of the body weights, whereas in case of conformity the proportion would be as the power $0.3\dot{3}$ of the body weights or 15.32 : 1. The length of the nerve fiber of man ought then to be no more than 536 mm. From HARDESTY's measurements of the diameters of the axones, the nerve fiber of man appears, however, to be thinner in exactly the same ratio as it is longer. The area of the cross section varies proportionally as the power 0.1693 of the body weight, instead of as the power $0.2\dot{2}$ in case of similarity. Thus it is found that the *volumes* of these dissimilar neurones are exactly in the ratio of the power $0.5\dot{5}$ or $^5/_9$ of the body weights, and the square

1) IRVING HARDESTY, Observations on the Medulla Spinalis of the Elephant with some Comparative Studies of the Intumescentia Cervicalis and the Neurones of the Columna Anterior. Journal of Comparative Neurology, Vol. XII 1902, p. 171—172.

of the volumes of the nerve cells, as would be the case when the neurones were similar in form.

In other cases of dissimilarity in form the comparative length of homologous nerve fibers in the large species is *smaller* than in the case of conformity. But with all differences of the relative length and section of the nerve fibers the relation between the *volumes* of the neurones and their ganglion cells yet remains the same. We always find:

$$\left(\frac{C}{C_1}\right)^2 = \frac{N}{N_1}$$

and the volume of homologous ganglion cells always varies in proportion to the power $0.2\dot{7}$ or $^5/_{18}$ of the body weight:

$$\frac{C}{C_1} = \left(\frac{P}{P_1}\right)^{5/18}.$$

If this power were $0.3\dot{3}$ or $^6/_{18}$, its meaning would be clear at once. For the movements of animal species of the same form, but of different sizes, are slower, and the muscular contractions more prolonged, in proportion to the greater length of the body, the result being that large and small animals move along equally rapidly. The Tiger, e.g., moves at the same speed as the Cat, but with slower steps. It would be natural to conclude to a corresponding variation in the volume of the ganglion cells supplying the stimulation energy.

The proportion found departs little from $\left(\frac{P}{P_1}\right)^{6/18}$, but the deviation is constant, and has, accordingly, a rational significance. This opinion is supported by the fact that the area of the retina varies in the same way, in proportion to $\left(\frac{P}{P_1}\right)^{5/18}$. The comparison of the receptive cells of the retina with the ganglion cells of the brain is certainly reasonable, on account of the origin of this membrane as a bulging out of the primitive cerebral vesicle; the retina is actually a complex of neurones. But then it follows that the *area* of the retina must vary to the same degree as the nerve cell *volume*. In animal species of different sizes the impressions of the retina (images) vary with its area; those of the nerve cells of the spinal cord and the brain with the volumes of the cells. In the retina the area of the cross section and the volume of the receptive elements must vary to the same degree [1]).

[1]) The available data are not sufficient to allow us to judge about the variation of the area of the cross section of the receptive retina elements with the body weight of homoneuric animal species. GISA ALEXANDER SCHÄFER (Pflügers Archiv.

How can we account for this constant deviation from the simple proportion between the cell-volume and the length of the body?

We are put on the right track of this explanation by a dissimilarity of the cell in relation to its nerve fiber. We found that in similar animal species of different sizes, the area of the cross section of the nerve fiber varies in proportion to $P^{0.2\dot{2}}$ or $P^{4/18}$. With uniformity of the ganglion cell from which the nerve fiber proceeds, this section would have to vary with the $^{2}/_{3}$ power of the volume of this cell, so that the cell volume itself would then have to vary proportionally to $P^{0.3\dot{3}}$ or $P^{6/18}$. In reality the cell volume increases and decreases proportionally to $P^{5/18}$. Whence this unconformity in animal species of similar form?

The answer to this question is supplied by the closer examination of the cell volume. Only part of it, the plasm, is in direct relation to the nerve fiber; the axis cylinder arises in the plasm, its structure proceeds in it, passing by the nucleus.

It has already been known for some time, especially with regard to the large ganglion cells, that the size of the cell increases more than the size of the nucleus, hence the cytoplasm still more so [1]. The existence of a definite quantitative relation can here again be found by means of a power-equation. When the volumes of a large and a small cell C and C_1, and those of the nuclei K and K_1 are known, the value of the power k for the relation between the two relative volumes can be calculated by means of the equation:

$$\left(\frac{C}{C_1}\right)^k = \frac{K}{K_1}.$$

In table I I have collected the diameters of a number of nerve cells and their nuclei, borrowed from the works of GIUSEPPE LEVI (1906 and 1908) [1]. Of course only between cells having the same

Bd. 119 (1907), p. 574) gives 5.14 micra for the diameter in the Hare, and 4.6 micra for that in the Rabbit. The relative cross *section* 1.248 is here proportional to the $^{5}/_{18}$ power of the relative body weight.

[1]) GIUSEPPE LEVI, Studî sulla grandezza delle cellule. I. Ricerche comparative sulla grandezza delle cellule dei Mammiferi. Archivio Italiano di Anatomia e di Embriologia. Vol. V. Firenze 1906, p. 291—358. Cf. on the relation of nucleus and plasm of the ganglion cells, the tables and graphic figures, particularly the table of the spinal ganglion cells, p. 330, and the two figures XIX, and also the conclusions formulated p. 354.

Further data in: G. LEVI, I ganglî cerebrospinali. Studî di Istologia comparata e di Istogenesi. Supplemento al Vol. VII dell' Archivio Italiano di Anatomia e di Embriologia. Firenze 1908. 392 pp., 60 Tavole. (These papers will be referred to as 1906 and 1908).

TABLE I. — Diameter of ganglion cells and their nuclei, according to he measurements of GIUSEPPE LEVI (1906 and 1908), and linear dimension of the plasm of these cells corresponding to it. (Micra)

Animal species	Kind of the ganglion cell	Mean diameter of the cell	Mean diameter of the nucleus	Mean diameter of the plasm	Reference to the page of G. LEVI's papers
1. Bos taurus	Largest in spinal ganglia	110	25	85	1908.200
2. Bos taurus	id id.	104.3	24.1	80.2	1906.330
3. Tragulus kanchil	id. id.	59.5	15.5	44	" "
4. Lepus cuniculus	id. id.	56	18	38	1908.200
5. Cavia cobaia	id. id.	55	19	36	" "
6. Cavia cobaia	id. id.	49	16.8	32.2	1906 330
7. Mus decumanus	id. id.	46	16	30	1908 200
8. Mus musculus	id. id	37.2	14	23.2	" "
9. Arvicola arvalis	id. id.	25	11.2	13.8	1906 330
10. Felis domestica	Largest in ganglion spinale cerv. V	81	20	61	1908 200
11. Felis domestica	Larg. in gln. sp. cocc. I	63	17	46	" "
12. Python (species)	Largest in ganglia sp.	80	20	60	" 119
13. Varanus arenarius	id. id.	80	19	61	" 118
14. Seps chalcides	id. id.	29	11	18	" 120
15. Bos taurus	Largest cell. radic. ant. medull. spin. intum. lumb.	54.4	17.4	37	1906 334
16. Mus musculus	id id.	27.4	11.1	16.3	" "
17. Canis familiaris	Cells of Purkinje in cerebellum	31.1	11	20.1	" 335
18. Canis vulpes	id. id.	23	9	14	" "
19. Bos taurus	Larg. pyramidal cells of grey cortex	27.05	12.85	14.2	" 337
20. Tragulus kanchil	id. id.	17.75	9.95	7.8	" "
21. Canis familiaris	Ganglion cervicale superius n. sympathici	39.2	14.5	24.7	" "
22. Putorius putorius	id. id.	20.5	9.5	11	" "

shape can we derive the accurate relation of the volumes from the proportion between the lengths. To have a good chance in this respect I compared allied animal species, wherever obtainable, or at least those in which definite homologous cells may be considered as similar in form. Besides spinal ganglion cells were particularly chosen for the calculations, on account of their regular, round shape, which as such leads more to conformity. It is self-evident, that cells were compared which differed as much as possible in size; thus individual deviations, which would affect regular relations, as are supposed to exist between the volumes of the nucleus, the plasm, and the cell, are minimized and recede into the background. The linear dimension of the plasm (equal to the cube root of its volume) was taken as the difference between the diameter of the cell and that of the nucleus.

An exponent d can be calculated giving the relation between the volumes of the plasm D and D_1 and the volumes of the cells C and C_1, in the equation $\left(\frac{C}{C_1}\right)^d = \frac{D}{D_1}$; then the values recorded in the first column of figures in Table II are found. Most of these approach very closely 1.2 or $^6/_5$.

Besides it was found that $\left(\frac{P}{P_1}\right)^{0.2\dot{7}} = \frac{C}{C_1}$, so that $\left(\frac{P}{P_1}\right)^{0.2\dot{7}d} = \frac{D}{D_1}$. Thus the value of the plasm-exponent $\lambda\ (= 0.2\dot{7}\ d)$ can be calculated from the equation $\left(\frac{P}{P_1}\right)^{\lambda} = \frac{D}{D_1}$ with the value of d calculated for every pair of cells investigated. The results of these calculations are given in the second column of figures of Table II. Most of them differ but little from $^1/_3$.

Not being acquainted with the details of the data made use of in these calculations, we could not be certain beforehand, however guided in their choice by the principles stated, that they were indeed serviceable. In the case of Tragulus e g. there seems to have been something wrong with them. But finding for a fair number of cell-couples, chosen on those principles, such a striking conformity in the results of the calculations, we feel justified in admitting the real existence of regular proportions.

As was already stated couples of cells of similar form had to be chosen for the calculation to enable us to derive the volume from the diameter; but the general validity found earlier for the relation of volume between the ganglion cell and body, and considerations in connection with what follows, leave no room for doubt, that also between cells dissimilar in form, but functionally equal, the

TABLE II. — Calculated values of the exponents d, λ ($= 0.2\dot{7} d$) and k for the variation of the plasm-volume D with the cell-volume C and with the body-weight P, and of the nucleus-volume K with the cell-volume C

Animal species	Kind of the ganglion cell	d in $\left(\frac{C}{C_1}\right)^d = \frac{D}{D_1}$	λ in $\left(\frac{P}{P_1}\right)^\lambda = \frac{D}{D_1}$	k in $\left(\frac{C}{C_1}\right)^k = \frac{K}{K_1}$
1. Bos taurus (1) and Mus musculus (8)	From Gangl. spin.	1.198	0.3327	0.5348
2. Bos taurus (2) and Mus musculus (8)	id. id.	1.203	0.3342	0.5268
3. Bos taurus (2) and Tragulus kanchil (3)	id. id.	1.070	0.2971	0.7864
4. Lepus cuniculus (4) and Mus decumanus (7)	id. id.	1.202	0.3338	0.5987
5. Lepus cuniculus (4) and Mus musculus (8)	id. id.	1.206	0.3351	0.6143
6. Mus decumanus (7) and Mus musculus (8)	id. id.	1.210	0.3362	0.6288
7. Cavia cobaia (5) and Arvicola arvalis (9)	id. id.	1.216	0.3378	0.6703
8 Cavia cobaia (6) and Arvicola arvalis (9)	id. id.	1.259	0.3497	0.6025
9. Felis domestica (10 and 11, gln. cerv. V and cocc. I)	id id.	1.123	0.3119	0.6466
10. Python (species) (12) and Seps chalcides (14)	id. id.	1.187	0.3296	0.5892
11. Varanus arenarius (13) and Seps chalcides (14)	id. id.	1.203	0 3341	0.5386
12. Bos taurus (15) and Mus musculus (16)	Rad. ant. med. spin.	1.195	0.3320	0.6555
13. Canis familiaris (17) and Canis vulpes (18)	Purkinje cerebell.	1.199	0.3330	0.6651
14. Bos taurus (19) and Tragulus kanchil (20)	Gr. pyram. cerebr.	1.422	0.3950	0.6071
15. Canis familiaris (21) and Putorius putorius (22)	Ganglion cervic. sup. n. sympath.	1.248	0.3466	0.6523
Mean of 13 comparisons (without N°. 3 and N°. 14)		1.204	0.3344	0.6095

relation of *volume* of the plasm would remain the same. We arrive therefore at the following statement:

The plasm volume of the nerve cell varies in proportion to the cube root of the body weight, i.e. to the mean linear dimension of the body.

Hence we are justified in considering the plasm volume of the nerve cell as determined dynamically. The mean linear dimension of the body varies in inverse ratio to the rapidity of the movements, and in direct ratio to the duration of the muscle contractions (compare the Cat and the Tiger), because the muscular force, which is determined by the transverse section of the muscles, and the body weight are in this proportion.

The calculated values of the exponent k, in the equation for the variation of the nucleus-volume with the cell-volume, are given in the third column of Table II. It appears that most of these values, and their mean, are slightly below $^2/_3$, which value k ought to have, if the variation of the nucleus volume were proportional to the surface area of the cell. We may assume 0.6 or $^3/_5$ for the real value of k. This lies exactly halfway between that for proportionality with the area of surface of the cell and with the area of surface of the nucleus, that is the outer and the inner surface of the plasm. This leads to the conclusion that *the regulation of the metabolism of the plasm of the ganglion cell must be attributed to the nucleus.* Apparently we are, therefore, justified in considering the nucleus as the assimilator and dissimilator of the plasm — by catalysis or enzyme action, — and as the process of stimulation in the nerve fiber undoubtedly proceeds from (or ends in) the plasm, which is closely connected with the axone, the name of *neurokinete* may be applied to the nucleus. [1])

Further, it can be deduced from the value found for k, that the *volume of the nucleus varies proportionally as $P^{0.16}$ or $P^{1/6}$, and the square of the volume of the nucleus as the volume of the plasm of the cell.* Hence these two are in the same relation to each

[1]) The action of the same enzymes can give rise both to synthesis and analysis.

P. SCHIEFFERDECKER (Muskeln und Muskelkerne, Leipzig 1909, p. 150 et seq.) found that in the rabbit the relative nuclear volume of the red muscle fiber, which is rich in muscle haemoglobin, is much greater than that of the white muscle fiber, which is poor in muscle haemoglobin. In the red Soleus the relative nuclear volume is $2^1/_2$ times greater than in the white Gastrocnemius, which consists of muscle fibers of similar form, and which acts in conjunction with the Soleus. This, too, points to a catalytic relation between the nuclear quantity and the rapidity of the metabolism (oxygen consumption) of the cell.

other as the volume of the nerve cell to that of its nerve fiber. This is of great significance in the mechanism of the neurone.

Through the found proportionality of the volume of the plasm with $P^{1/3}$ *it now becomes clear that in animal species of the same form,* but of different sizes, the area of the cross section of the axis cylinder (and also of the nerve fiber) is in a relation of uniformity with the plasm of the nerve cell from which it proceeds. Hence this area varies as the $^2/_3$ power of the plasm-volume, i.e. by $P^{2/9}$ or $P^{0.2\dot{2}}$. The volumes of homologous nerve fibers of animal species having the same shape, the lengths of which are proportional as $P^{3/9}$ or $P^{0.3\dot{3}}$, must therefore vary propoctionally as $P^{5/9}$ or $P^{0.5\dot{5}}$. And as the ganglion cell constitutes only a very small part of the volume of the neurone (in the above described motor neurone of the finger muscles of man, e.g., the nerve fiber has 870 times larger volume than the ganglion cell), we may also say that the volume of the neurone, hence also the complex of neurones, which we call the brain, varies by the power $^5/_9$ or $0.5\dot{5}$ of the body weight. [1])

Thus the rational character of this apparently incomprehensible power is clearly shown.

At the same time, the mechanism of the neurone becomes more distinct.

Also for species of different shape, as Man and the Mouse, we saw the volume of homologous, also analogous ganglion cells (functioning in the same way) vary proportionally as $P^{0.2\dot{7}}$ or $P^{5/18}$, and we may, therefore, assume that the volume of the plasm of these cells varies proportionally as $P^{0.3\dot{3}}$ or $P^{1/3}$, hence as the *mean* linear dimension (for uniform species, as *every* homologous linear dimension) of the body. The proportionality with the *mean* linear dimension of the body is, indeed, a necessary condition for the cooperation of all the neurones in the nervous system, the nerve fibers of which inter se differ greatly in length. Consequently the relations of the elements of the neurocyte must be valid both for species of different shape and for species having the same shape.

The existence of these fixed relations of volume of the

---

[1]) The degree of accuracy of the data does not allow us to ascertain whether the volume of the neurone or only that of the nerve fiber varies proportionally as the square of the volume of the ganglion cell. Hence it may very well be that the ganglion cell, which besides being the station, is also the road for the impulsions, at least as regards its plasm, must also be included in the proportional section of the nerve fiber. Then the volume of the neurone is at least *almost exactly* proportional as $P^{5/9}$ or $P^{0.5\dot{5}}$.

neurone and its parts leave, indeed, hardly any room for doubt of the perfectly mechanical character of its arrangement. Though it is not possible to demonstrate this in details, because it is not yet fully known what takes place in the nerve fiber during the transmission of the stimulation process, the impulsions, from and to the ganglion cell, yet our present view of the nature of this mechanism can be tested by what we know about this arrangement.

What is transmitted in the nerve fiber as impulsions is, beyond doubt, a process of dissimilation, and it is highly probable that the colloids, contained in the living substance, play an important rôle. The plasm of the ganglion cell and the axone possess "Spumoidstruktur" (Rhumbler), is an emulsive foam mixture, consisting of two liquid phases. The denser of these colloid substances forms the walls of the minute spumoid compartments ("Schaumkämmerchen" of Rhumbler); so semipermeable membranes or osmotic films can act selectively on the ions liberated by the disaggregation of the molecules. Thus the anions diffuse in centrifugal or centripetal direction from one spumoid minute compartment to another [1]).

Material particles move in any case from one end of the nerve fiber to the other. Not the same particles: the movement is transmitted from one spumoid compartment to another lying in front of it, as it were from one transverse layer to another, but the work performed thus must be equal to that of a transverse layer of equal mass which moved from one end of the nerve fiber to the other.

But the process of stimulation is not transmitted in this way from one end to the other in the *whole* nerve fiber; this takes only place in the axone. We see the axone taking its central origin or terminating centrally *in* the plasm of the neurocyte, with gradual transition; it is enveloped with myeline only at some distance from this; the medullary sheath terminates at the muscle fiber, and the motor nerve end-plate is but a plate-like or antler-like extension of the axone. It also appears that at the peripheral ends of the afferent nerve fibers the axone is the real conductor. Though the medullary

---

[1]) Cf.: W. Nernst, Zur Theorie der elektrischen Reizung. Nachrichten von der Kön. Gesellschaft der Wissenschaften zu Göttingen. (Mathem. physik. Klasse). 1899, p. 104—108. — Max Verworn, Allgemeine Physiologie. Sechste Auflage. Jena 1915, p. 134 et seq., 319 etc. — O. Bütschli, Untersuchungen über mikroskopische Schäume und das Protoplasma. Leipzig 1892. — Hans Held, Beiträge zur Struktur der Nervenzellen und ihrer Fortsatze. (Zweite Abhandlung). Archiv für Anatomie und Entwickelungsgeschichte. Leipzig 1897, p. 204—289. — L. Rhumbler, Das Protoplasma als physikalisches System. Ergebnisse der Physiologie. (Asher und Spiro). Jahrgang XIV. Wiesbaden 1914, p. 484—617.

sheath has undoubtedly not the entirely passive significance of the insulator in the electric cable (just as the axone cannot be compared to the wire), it nevertheless certainly does not take part directly in the transmission of the process of stimulation. The axone, certainly, is directly involved in this transmission.

With the varying size of the animal species *the volume of the axone always varies proportionally to half the square of the volume of the ganglion cell.* The length l and the area of the cross section q of the axone (like that of the nerve fiber) indeed vary (as was discussed above with regard to the nerve fiber [1]) in inverse ratio to each other, so that their product $l\,q$ remains the same. Hence the product $l\,q$ varies proportionally as $\frac{1}{2}$ C^2.

Let us consider in this connection the propagation of the process of stimulation in the nerve fiber

On discharge of the ganglion cell (we shall confine ourselves to the efferent neurone; what follows holds inversely for the afferent neurone) potential energy of some form must certainly be consumed to supply motive energy in the nerve fiber. Let us suppose this to be performed by a layer of anions (or other material particles) placed in the cross sectional plane of the axone. It leaves the ganglion cell with a velocity v, travels the whole path to the other end of the axone, and assumes there a state of rest. This layer, whose mass is proportional to q, must have possessed a potential energy in the ganglion cell proportional to $l\,q$, and obtained a kinetic energy, equal to this, proportional to $\frac{1}{2}\,qv^2$.

We also found lq proportional to $\frac{1}{2}\,C^2$, from which follows that v is proportional to $\frac{C}{q^{1/2}}$. As for animal species having the same form q is proportional as $P^{0.2\dot{2}}$ or $P^{4/18}$ and C as $P^{0.2\dot{7}}$ or $P^{5/18}$, we find v proportional as $P^{0.1\dot{6}}$ or $P^{1/6}$. This is the same ratio in which the nucleus volume increases with increasing body weight. *The velocity of the metabolism of the cytoplasm and the velocity of the process of dissimilation in the axone varies proportionally with the increase of the nucleus volume*; we must therefore consider the movement in question as having begun in the cytoplasm with a velocity which

[1]) Comp. my communication "On the Relation between the Quantities of the Brain, the Neurone and its Parts, and the Size of the Body. These Proceedings Vol. XX, p. 1328—1337. The areas of the cross sections of analogous nerve fibers of Man and the Mouse referred to above, are based on direct measurements of the diameters of the *axones* by IRVING HARDESTY. I assumed double the cross-sectional areas of the axones for the nerve fibers, in virtue of researches by DONALDSON and HOKE and others.

is determined by the volume of the nucleus. The name of neurokinete may, indeed, very appropriately be given to the nucleus. We may once more point out the analogy to catalysis or enzyme action.

In dissimilar species (and neurones) the section of the axone varies proportionally as a smaller power of the body weight than $0.2\dot{2}$ or $^4/_{18}$ (between the Mouse and Man 0,1693) or as a greater power of the body weight. Therefore the velocity v, which a layer of anions (or other material particles) of the axone obtains in the ganglion cell, must in one case be greater (between the Mouse and Man v varies in the proportion of $P^{0.1931}$), in the other case smaller than with species (and neurones) of equal form.

The kinetic energy imparted to this layer by the ganglion cell, whose volume varies in the same relation with the body weight for species of dissimilar form as for species of equal form, continues to vary in the same way. For, the mass of the layer, determined by the cross section of the axone, varies *inversely as the length* of the axone (i.e. between the Mouse and Man in the ratio of $P^{0.3821}$), and the square of its velocity (between the Mouse and Man in the ratio of $P^{0.3862}$) varies inversely as the cross section, hence *directly as the length*. The energy remains proportional to $\frac{1}{2}\,qv^2$ and to lq.

In a previous communication I discussed the relation between the velocity of the propagation of the stimulation process, and the area of the cross section of the nerve fiber and the axone. [1]) It is self-evident that *per unit of length in the same time* the stimulation process performs more work in relation to the cross section of the axone. It corresponds morphologically that the joint area of the parts of the walls of the spumoid compartments that are placed transversely increases with the cross section of the axone, for in the same degree more anions diffuse in the unit of time. [2])

Thus our present views on the nature of the mechanism of the transmission of the stimulation process in the nerve fiber find full confirmation in the quantitative relations of the nervous system.

---

[1]) "The Significance of the Size of the Neurone and its Parts". These Proc. Vol. XXI, (1918) p. 711—729.

[2]) In connection with this the *chronaxy*, which expresses the time during which a nerve elaborates an electric stimulation, so that it reacts to it, is slight for thick nerve fibers, great for thin nerve fibers: L. LAPICQUE et R. LEGENDRE in Comptes rendus de l'Académie des Sciences. Paris 1913 Tome 157, p. 1163—1166. — L. LAPICQUE et R. LEGENDRE in Bulletin du Muséum d'histoire naturelle. Paris 1914, N°. 4, p. 248—252. — Cf. on chronaxy also J. K. A. WERTHEIM SALOMONSON, especially in "Nederlandsch Tijdschrift voor Geneeskunde. Jaargang 1919. 2e. Helft N°. 15.

Besides it appears that it is the nucleus of the ganglion cell, the neurokinete, that controls the mechanism of the neurone, of the nervous system, and indeed that of the entire animal organism. The sixth power of the volume of the nucleus varies then proportionally as the body weight, or the square of the nucleus-volume proportionally as the length of the body; thus also the sixth power and the square of the velocity of the movement of the anions in the cytoplasm and the axone.

AIRY [1]) knew only one instance in physical science in which the sixth power came really into application: if the velocity of a sea-current (or a river) be doubled, it will carry stones, pushing them on along the bottom, of sixty-four times the weight of those (having the same shape) carried before. In the animate world the sixth power plays a very important part. SUTHERLAND [2]) found that the body weights of all bird species are proportional to the sixth power of their brooding-times; those of mutually allied mammals to the sixth power of their times of gestation; thus the body lengths of the adult animals being proportional to the squares of these times. In all these cases, just as in the case treated here, we have to do with movement of, or relative to bodies having the same form.

Hence that cytoplastic metabolism, like this functional metabolism of the neurone, is mechanically determined, and the general occurrence of quantitative relations between nucleus and plasm [3]) is only the consequence of the mechanical relation in the dependence of these cell-elements on each other.

---

1) G. B. AIRY in Minutes of Proceedings of the Institution of Civil Engineers. Vol. XXIII. Session 1863—64. London 1864, p. 227.

2) ALEXANDER SUTHERLAND, Some Quantitative Laws of Incubation and Gestation. Proceedings of the Royal Society of Victoria. Vol. VII, (New Series). 1895, p. 270—286.

3) Compare the researches and studies of: J. J. GERASSIMOW in Zeitschrift fur allgemeine Physiologie. Bd 1. 1902, p. 220—258, — TH. BOVERI in Verhandlungen der Physikalisch-medicinischen Gesellschaft zu Würzburg. Neue Folge. Band 35. 1903, p. 67—88, — R. HERTWIG in Biologisches Centralblatt. Bd. 23, 1903, p. 49—62 and 108—119, — AUG. PÜTTER, Vergleichende Physiologie. Jena 1911, p. 32 et seq.

Mathematics. — "*Note on linear homogeneous sets of points*". By Dr. B. P. HAALMEIJER. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of October 25, 1919).

We shall call a linear set of points π *homogeneous* in the interval AB, if its subset, interior to an arbitrary sub-interval, allows of a uniformly continuous one-one representation on the subset of π interior to AB [1]).

If the set π is everywhere dense in the interval AB [2]), each of these representations determines a continuous one-one correspondence between the entire linesegments. As will be shown, we may in this case, assume the correspondences, postulated for a homogeneous set of points, to leave relations of order unaltered.

Let CD be a sub-interval of AB (possibly identical to AB) and E a point between C and D. We consider the following possibilities:

1. For every system of points C, D, and E the representation of the interval CD on CE leaves relations of order unaltered.
2. This is not the case.

First case. Suppose a representation of AB on FH has to be effected (order from left to right A, F, H, B). According to the assumption both AB and FH can be represented on AH with unaltered relations of order, hence AB on FH in the same way.

Second case. The assumption postulates the existence of an interval CD which can be represented on its sub-interval CE with inverted relations of order. Considering this representation is a continuous one-one correspondence between entire linesegments, it follows from the DEDEKIND axiom that a point P exists (not necessarily belonging to the set π), which corresponds to itself. This however establishes the possibility of representing the part of π interior to CD on itself with inversion of order-relations. It follows that the part of π interior to an arbitrary sub-interval of AB, possesses this same

[1]) An analogous definition has been given by HAUSDORFF for ordered sets, Grundz. der Mengenlehre p. 173. For linear sets of points BROUWER has introduced the following more extensive definition: a linear set of points π is homogeneous in the interval AB if for each couple PQ of its points interior to AB, there exists a continuous one-one transformation of the interval AB in itself, such that π passes into itself and the point P into the point Q. These *Proceedings* XX, p. 1194.

[2]) Which obviously is the case if π has any points inside AB.

property, hence all correspondences, postulated for the homogeneous set π, can be effected in such a way as to leave relations of order unaltered.

We now formulate the following theorem: *The linear continuum cannot be composed of two homogeneous sets of points, possessing the same geometric type.*

Our demonstration is to be an indirect one. Let the open line-segment AB consist of two sets of points π and π' of the kind mentioned. These sets π and π' possess the same geometric type, that is there exists uniformly continuous one-one correspondence between them. Evidently π and π' are both everywhere dense on AB.

To begin with, we assume that this correspondence inverts relations of order. Then π can be divided into two subsets π_1 and π_2 such that every point of π_1 is situated on the left, and every point of π_2 on the right of the corresponding point of π'. Besides, every point of π_1 lies on the left of every point of π_2. Hence, as $\pi_1 + \pi_2$ is everywhere dense, the DEDEKIND axiom postulates the existence of a separating point R. This point R however can belong to neither π_1 nor π_2. For instance let us assume R to be a point of π_1, then it is situated on the left of the corresponding point of π' and the continuity of the correspondence makes that this is also the case for all points of π inside a certain finite neighbourhood of R, which means a contradiction. Hence R belongs to π', but this also leads to a contradiction as the fact that R is situated either on the left or on the right of its corresponding point cannot be made to agree with the circumstance that all points of π' on the left (right) of R are also situated on the left (right) of their corresponding points.

We now come to the second possibility, namely that the correspondence between π and π' leaves relations of order unaltered. We distinguish two cases:

1. The set π contains both points situated on the left, and points situated on the right of the corresponding points or π'.

2. All points of π lie on the same side of the corresponding points.

First case. Let the point P_1 of π be situated on the left of its corresponding point P_1' and P_2 on the right of P_2'. The subset of π between P_1 and P_2, including the endpoints shall be called π_1. Let ${}_1\pi_1$ be the subset of π_1 consisting of those points, which, together with all points of π_1 situated more to the left, precede their corresponding points[1]), and let R be the last limiting point of ${}_1\pi_1$ on the right hand side. Then the assumption that R precedes its corresponding point, as well as the assumption according to which R follows on its corresponding point, leads immediately to a contradiction (we here consider the transformation of the *entire* segment AB in itself, which is determined by the correspondence between π and π'). Hence the point R must correspond to itself, but this is out of the question, both if R belongs to π or to π'.

Second case. All points of π lie on the left of the corresponding points. Let the points P_1' and P_2' of π' correspond to P_1 and P_2 of π respectively and let the order from left to right be P_1, P_1', P_2, P_2'. Of course, such a system of points can always be found.

We choose a point C of π' on the left of P_1 and we subject π' to a uniformly continuous one-one transformation in itself, such that P_1' passes into C and P_2' remains in its place. A transformation of this kind can certainly be found as π' is homogeneous. Let π'' be the transformed set, then a uniformly continuous one-one correspondence exists between π'' and π, such that π'' contains both points preceding and points coming after the corresponding points, and the reasoning used for the *first case* can now be applied.

To Prof. L. E J. Brouwer I am indebted for several remarks turned to advantage in the preceding note.

[1]) "Precede" here stands for "are situated on the left of".

45*

Mathematics. — "*On n-uple orthogonal systems of n—1-dimensional manifolds in a general manifold of n dimensions.*" By Prof. J. A. SCHOUTEN and D. J. STRUIK. (Communicated by Prof. J. CARDINAAL).

(Communicated at the meeting of October 25, 1919).

II.

7. DUPIN's *theorem and an inversion.* From theorem I we conclude that DUPIN's theorem also holds for a general manifold:

The V_{n-1} of an n-uple orthogonal system intersect along the lines of curvature.

This theorem may be inverted in the following way:

When $n-1$ mutually orthogonal V_{n-1}-systems, determined by the congruences $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ perpendicular to them, intersect along a congruence $\mathbf{i}_n$, and when we can choose the arrangement of the first congruences in such a way that the congruence $\mathbf{i}_n$ in each $V_{n-k+1} \perp \mathbf{i}_1, \ldots, \mathbf{i}_{k-1}$ is a congruence of lines of curvature for the V_{n-k} being the intersection of this V_{n-k+1} with the $V_{n-1} \perp \mathbf{i}_k$, $k = 1, \ldots, n-1$, then $\mathbf{i}_n$ is perpendicular to a V_{n-1}-system, orthogonal to the $n-1$ given systems, and $\mathbf{i}_1, \ldots, \mathbf{i}_n$ are the congruences of the lines of curvature for each of the n systems.

Proof. When the fundamental tensor $^2\mathbf{g}$ of the V_n is written:

$$^2\mathbf{g} = \mathbf{a}\,\mathbf{a} = \mathbf{b}\,\mathbf{b} = \ldots, \qquad (72)$$

then the ideal factor $\mathbf{a}$ can be decomposed as follows:

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}'', \qquad (73)$$

in which $\mathbf{a}'$ contains but $\mathbf{i}_k, \ldots, \mathbf{i}_n$, $\mathbf{a}''$ but $\mathbf{i}_1, \ldots, \mathbf{i}_{k-1}$.

$^2\mathbf{g}' = \mathbf{a}'\,\mathbf{a}' = \mathbf{b}'\mathbf{b}' = \ldots$ is the fundamental tensor of the $V_{n-k+1} \perp \mathbf{i}_1, \ldots, \mathbf{i}_{k-1}$ and the geodesic differentiation of a vector $\mathbf{v}$, which is wholly situated in this V_{n-k+1}, is determined by the equation:

$$\nabla' \mathbf{v} = {}^2\mathbf{g}' \overset{1}{\cdot} \nabla (\mathbf{a}' . \mathbf{v})\, \mathbf{a}' \qquad (74)$$

Hence for $\mathbf{i}_k$ we have:

$$\left.\begin{aligned} \mathbf{i}_n \overset{1}{\cdot} \nabla \mathbf{i}_k &= \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a})\, \mathbf{a} = \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a}')\, \mathbf{a}' + \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a}')\, \mathbf{a}'' = \\ &= \mathbf{i}_n \overset{1}{\cdot} \nabla' \mathbf{i}_k + \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a}')\, \mathbf{a}''. \end{aligned}\right\} \qquad (75)$$

According to the supposition $\mathbf{i}_n$ is a congruence of lines of curvature for the V_{n-k} being $\perp \mathbf{i}_k$ in the considered V_{n-k+1}, so that according to (38):

$$\varkappa\, \mathbf{i}_n \overset{1}{\cdot} \nabla' \mathbf{i}_k = \varrho_k \mathbf{i}_n \,, \quad \ldots \ldots \ldots \quad (76)$$

in which ϱ_k is a still unknown coefficient. Hence we conclude from (76):

$$\varkappa\, \mathbf{i}_n \overset{1}{\cdot} \nabla \mathbf{i}_k = \varrho_k \mathbf{i}_n + \sum_j^{1,\ldots,k-1} \mu_{kj} \mathbf{i}_j \,, \quad \ldots \ldots \quad (77)$$

in which μ_{kj} are still unknown coefficients. So it is supposed that it must be possible to arrange $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ in such a way that the equation (77) is satisfied in the same time for all values $k=1,\ldots,n-1$.

Since

$$\mathbf{i}_k \cdot \mathbf{i}_l = 0, \qquad k, l = 1, \ldots, n, \qquad k \neq l \quad \ldots \quad (78)$$

we find by application of $\mathbf{i}_n \cdot \nabla$:

$$\mathbf{i}_l\, \mathbf{i}_n \overset{2}{\cdot} \nabla \mathbf{i}_k = - \mathbf{i}_k\, \mathbf{i}_n \overset{2}{\cdot} \nabla \mathbf{i}_l \quad \ldots \ldots \ldots \quad (79)$$

For $k < l$ we have thus from (77), (78), and (79):

$$\mathbf{i}_k\, \mathbf{i}_n \overset{1}{\cdot} \nabla \mathbf{i}_l = 0, \qquad l = 1, \ldots, n-1 \,. \quad \ldots \quad (80)$$

hence:

$$\mu_{kj} = 0 \qquad \left.\begin{matrix} k = 1, \ldots, n-1 \\ j = 1, \ldots, n-2 \end{matrix}\right\} \quad \ldots \ldots \quad (81)$$

By this the equations (77) pass into:

$$\varkappa\, \mathbf{i}_n \overset{1}{\cdot} \nabla \mathbf{i}_k = \varrho_k \mathbf{i}_n \,, \qquad k = 1, \ldots, n-1 \quad \ldots \quad (82)$$

which can geometrically be interpreted in such a way that $\mathbf{i}_n$ is a congruence of lines of curvature in each of the $n-1$ given V_{n-1}-systems.

By application of $\mathbf{i}_k \cdot \nabla$ we conclude from (78):

$$\mathbf{i}_l\, \mathbf{i}_k \overset{2}{\cdot} \nabla \mathbf{i}_n = - \mathbf{i}_n\, \mathbf{i}_k \overset{2}{\cdot} \nabla \mathbf{i}_l, \quad k, l = 1, \ldots, n-1. \quad \ldots \quad (83)$$

Now $\mathbf{i}_l$ is V_{n-1}-normal, hence $\nabla \mathbf{i}_l$ is symmetrical in k and n, so that we have from (80) and (83):

$$\mathbf{i}_l\, \mathbf{i}_k \overset{2}{\cdot} \nabla \mathbf{i}_n = 0, \qquad k, l = 1, \ldots, n-1, \quad \ldots \quad (84)$$

hence $\mathbf{i}_n$ is V_{n-1}-normal and $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ are the congruences of the lines of curvature of the $V_{n-1} \perp \mathbf{i}_n$.

Since $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ are V_{n-1}-normal and mutually perpendicular, we have also from (67):

$$\mathbf{i}_j\, \mathbf{i}_k \overset{2}{\cdot} \nabla \mathbf{i}_l = 0, \qquad j, k, l = 1, \ldots, n-1. \quad \ldots \quad (85)$$

so that $\mathbf{i}_1, \ldots, \mathbf{i}_n$ are the congruences of the lines of curvature for each of the n systems $\perp \mathbf{i}_1, \ldots, \mathbf{i}_n$.

For a V_3' the proved theorem can be expressed in this way:

When two mutually orthogonal systems of surfaces intersect along a congruence of curves, which are the lines of curvature of one of *the two systems of surfaces, then there exists a system of surfaces orthogonal to the two given systems and the three systems intersect along their lines of curvature.*

For the R_3 this theorem has been first deduced by DARBOUX [1]).

8. LILIENTHAL'S *conditions.* We will now connect different shapes, in which the conditions occur in literature, for the case that $\mathbf{i}_n$ is V_{n-1}-normal, and inquire how far they remain valid, when more general manifolds are admitted.

In the same way as ${}^2\mathbf{h}$ the tensor ${}^2\mathbf{p}$ gets a simple significance when $\mathbf{i}_n$ is V_{n-1}-normal. Since on account of (19) and (42):

$$\nabla \smile \mathbf{i}_n = -\frac{\varkappa}{2} \sum_{\lambda\mu} \{(\mathbf{i}_n \cdot \nabla)\, g^{\lambda\mu}\}\, \mathbf{e}'_\lambda\, \mathbf{e}'_\mu + \mathbf{i}_n \smile \mathbf{u}_n, \quad . \quad . \quad . \tag{86}$$

the contravariant characteristic number of $\varkappa(\mathbf{i}_n \cdot \nabla)\, \nabla \smile \mathbf{i}_n$ is:

$$\left.\begin{aligned}
&\varkappa\, \mathbf{e}_\beta\, \mathbf{e}_\alpha \overset{2}{\cdot} (\mathbf{i}_n \cdot \nabla)(\nabla \smile \mathbf{i}_n) = -\tfrac{1}{2}\, \mathbf{e}_\beta\, \mathbf{e}_\alpha \overset{2}{\cdot} \sum_{\lambda\mu} [\mathbf{e}'_\lambda\, \mathbf{e}'_\mu\, (\mathbf{i}_n \cdot \nabla)^2 g^{\lambda\mu} + \\
&\qquad + \{(\mathbf{i}_n \cdot \nabla)\, \mathbf{e}'_\lambda\, \mathbf{e}'_\mu\}\, (\mathbf{i}_n \cdot \nabla)\, g^{\lambda\mu}] + \mathbf{e}_\beta\, \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n\, \mathbf{u}_n = \\
&= -\tfrac{1}{2}(\mathbf{i}_n \cdot \nabla)^2 g^{\alpha 3} + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \sum_{\lambda\mu} \{(\nabla \smile \mathbf{i}_n) \overset{2}{\cdot} \mathbf{e}_\lambda \mathbf{e}_\mu\, (\mathbf{i}_n \cdot \nabla)\, \mathbf{e}'_\lambda \mathbf{e}'_\mu\} + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n \mathbf{u}_n = \\
&= -\tfrac{1}{2}(\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} - \sum_{\lambda\mu} (\nabla \smile \mathbf{i}_n) \overset{2}{\cdot} \mathbf{e}_\lambda \mathbf{e}_\mu \{\mathbf{e}'_\lambda \mathbf{e}'_\mu \overset{2}{\cdot} (\mathbf{i}_n \cdot \nabla)\, \mathbf{e}_\beta \mathbf{e}_\alpha\} + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n \mathbf{u}_n = \\
&= -\tfrac{1}{2}(\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} + \sum_{\lambda\mu} (\nabla \smile \mathbf{i}_n) \overset{2}{\cdot} \mathbf{e}_\lambda \mathbf{e}_\mu \{\mathbf{e}'_\lambda \mathbf{e}'_\mu \overset{2}{\cdot} ((\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\beta \mathbf{e}_\alpha + \mathbf{e}_\beta (\nabla \mathbf{i}_n)^1\, \mathbf{e}_\alpha)\} + \\
&\qquad + \mathbf{e}_\beta\, \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n\, \mathbf{u}_n = \\
&= -\tfrac{1}{2}(\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} + \sum_{\mu} \mathbf{e}_\alpha \overset{1}{\cdot} (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\mu\, \mathbf{e}'_\mu \overset{1}{\cdot} (\nabla\, \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\beta + \\
&\qquad + \sum_{\lambda} \mathbf{e}_\beta{}^1 (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\lambda\, \mathbf{e}'_\lambda \overset{1}{\cdot} (\nabla\, \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\alpha + \\
&+ \mathbf{e}_\alpha \overset{1}{\cdot} (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n\, \mathbf{i}_n \overset{1}{\cdot} (\nabla\, \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\beta + \mathbf{e}_\beta \overset{1}{\cdot} (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n \mathbf{i}_n \overset{1}{\cdot} (\nabla\, \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\alpha = \\
&\qquad = -\tfrac{1}{2}(\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} + 2\, \mathbf{e}_\alpha\, \mathbf{e}_\beta \overset{2}{\cdot} T (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \nabla\, \mathbf{i}_n,
\end{aligned}\right\} \tag{87}$$

from which in connection with (59) we conclude:

$${}^2\mathbf{p} = -\frac{\varkappa}{2}\, \mathbf{e}'_\alpha\, \mathbf{e}'_\beta\, (\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} = \mathbf{e}'_\alpha\, \mathbf{e}'_\beta\, (\mathbf{i}_n \cdot \nabla)\, h^{\alpha\beta} \quad . \quad . \tag{88}$$

Hence the condition that ${}^2\mathbf{h}$ and ${}^2\mathbf{p}$ have the same principal directions, for the case $n = 3$, can be written in coordinates:

$$\boxed{\begin{vmatrix} g^{aa} & g^{ab} & g^{bb} \\ (\mathbf{i}_n \cdot \nabla)\, g^{aa} & (\mathbf{i}_n \cdot \nabla)\, g^{ab} & (\mathbf{i}_n \cdot \nabla)\, g^{bb} \\ (\mathbf{i}_n \cdot \nabla)^2 g^{aa} & (\mathbf{i}_n \cdot \nabla)^2 g^{ab} & (\mathbf{i}_n \cdot \nabla)^2 g^{bb} \end{vmatrix} = 0,} \tag{C_1}$$

1) G. DARBOUX, Sur les surfaces orthogonales. Annales sc. de l'Ecole Normale 3 (66) 97—141, p. 110.

and this is exactly the equation given for the first time for R_3 by LILIENTHAL [1]), and to which lately, also for R_3, WIERINGA [2]) has again drawn the attention. So this condition is a special case from RICCI's first. It remains also valid for an arbitrary linear element, and also for $n > 3$, then however it is no longer the only condition.

9. RICCI's *conditions.* Be $\mathfrak{i}_n$ again V_{n-1}-normal. Then we can choose an original variable y^n and vectors $\mathfrak{s}_n$ and $\mathfrak{s}'_n$, so that:

$$\mathfrak{i}_n = \sigma_n \mathfrak{s}_n = \frac{1}{\sigma_n} \mathfrak{s}_n' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

By means of this equation we can eliminate $\mathfrak{i}_n$ from (C) and (D) and substitute $\mathfrak{s}_n$ for it.

Since:

$$(\mathfrak{i}_n \cdot \nabla)(\nabla \smile \mathfrak{i}_n) = (\mathfrak{i}_n \cdot \nabla)\{\sigma_n \nabla \mathfrak{s}_n + \tfrac{1}{2}(\nabla \sigma_n)\mathfrak{s}_n + \tfrac{1}{2}\mathfrak{s}_n \nabla \sigma_n\}, \quad (90)$$

we have:

$$\left.\begin{array}{l}\overset{4}{\mathfrak{g}}_n \overset{2}{\cdot} (\mathfrak{i}_n \cdot \nabla)(\nabla \smile \mathfrak{i}_n) = \overset{4}{\mathfrak{g}}_n \overset{2}{\cdot} \{(\mathfrak{i}_n \cdot \nabla \sigma_n)\nabla \mathfrak{s}_n + \sigma_n \mathfrak{i}_n \overset{1}{\cdot} \nabla\nabla \mathfrak{s}_n + \\ \qquad + \tfrac{1}{2}(\nabla \sigma_n)\mathfrak{i}_n \overset{1}{\cdot} \nabla \mathfrak{s}_n + \tfrac{1}{2}(\mathfrak{i}_n \overset{1}{\cdot} \nabla \mathfrak{s}_n)\nabla \sigma_n\},\end{array}\right\} \quad (91)$$

or, since:

$$\nabla \sigma_n = \nabla (\varkappa \mathfrak{s}_n \cdot \mathfrak{s}_n)^{-\frac{1}{2}} = -\varkappa \sigma_n^3 (\nabla \mathfrak{s}_n) \overset{1}{\cdot} \mathfrak{s}_n = -\sigma_n \mathfrak{u}_n + \varkappa \sigma_n \mathfrak{s}_n \overset{1}{\cdot} (\nabla \sigma_n)\mathfrak{s}_n, \quad (92)$$

also:

$$\overset{4}{\mathfrak{g}}_n \overset{2}{\cdot} (\mathfrak{i}_n \cdot \nabla)(\nabla \smile \mathfrak{i}_n) = \overset{4}{\mathfrak{g}}_n \overset{2}{\cdot} \{(\mathfrak{i}_n \cdot \nabla \sigma_n)\nabla \mathfrak{s}_n + \sigma_n^2 \mathfrak{s}_n \overset{1}{\cdot} \nabla\nabla \mathfrak{s}_n - \varkappa \mathfrak{u}_n \mathfrak{u}_n\}. \quad (93)$$

Since on account of (31) and (69):

$$\mathfrak{i}_j \mathfrak{i}_k \overset{2}{\cdot} \{2\varkappa T(\nabla \smile \mathfrak{i}_n) \overset{1}{\cdot} \nabla \mathfrak{i}_n\} = \mathfrak{i}_j \mathfrak{i}_k \overset{2}{\cdot} \mathfrak{u}_n \mathfrak{u}_n, \quad . \quad . \quad . \quad (94)$$

the condition (C') gets the shape:

$$\boxed{\mathfrak{i}_j \mathfrak{i}_k \overset{2}{\cdot} \{\varkappa \sigma_n^2 \mathfrak{s}_n \overset{1}{\cdot} \nabla\nabla \mathfrak{s}_n - 2\,\mathfrak{u}_n \mathfrak{u}_n\} = 0}\ ^{3)}, \begin{array}{l} j \neq k \\ j, k = 1, 2, \ldots, n-1. \end{array} \quad (C_1)$$

Since:

$$\nabla \mathfrak{s}_n = \frac{1}{\sigma_n} \nabla \mathfrak{i}_n + \left(\nabla \frac{1}{\sigma_n}\right) \mathfrak{i}_n, \quad . \quad . \quad . \quad . \quad . \quad (95)$$

we further have, in connection with (30) and (33):

$$\mathfrak{i}_j \mathfrak{i}_k \overset{2}{\cdot} \nabla \mathfrak{s}_n = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

from which by application of $(\mathfrak{i}_k . \nabla)$ may be concluded:

$$\frac{1}{\sigma_n}(\mathfrak{i}_l \overset{1}{\cdot} \nabla \mathfrak{i}_j) \overset{1}{\cdot} (\nabla \mathfrak{i}_n) \overset{1}{\cdot} \mathfrak{i}_k + \frac{1}{\sigma_n}(\mathfrak{i}_l \overset{1}{\cdot} \nabla \mathfrak{i}_k) \overset{1}{\cdot} (\nabla \mathfrak{i}_n) \overset{1}{\cdot} \mathfrak{i}_j + \mathfrak{i}_j \mathfrak{i}_k \mathfrak{i}_l \overset{3}{\cdot} \nabla\nabla \mathfrak{s}_n = 0. \quad (97)$$

[1]) R. v. LILIENTHAL, Ueber die Bedingung, unter der eine Flächenschar einem dreifach orthogonalen Flächensystem angehört. Math. Annalen 44 (94), 449–457.

[2]) W. G. L. WIERINGA, Over drievoudig orthogonale oppervlakkensystemen. Diss. Groningen, (18) 59 pp., see p. 13.

[3]) See note [1]) of next page.

$(\nabla \mathbf{i}_n)^1_\cdot \mathbf{i}_k$ containing but $\mathbf{i}_k$ and $\mathbf{i}_n$ on account of (38), we find in connection with (67):

$$\boxed{\mathfrak{i}_j \mathfrak{i}_k \mathfrak{i}_l \,{}^3\, \nabla\nabla \, \mathfrak{s}_n = 0}, \; j \neq k, \; j \neq l, \; k \neq l, \; j, k, l = 1, 2, \ldots, n-1. \text{[1]} \qquad (D_1)$$

This equation (D_1) can be decomposed into:

$$\boxed{\mathfrak{i}_j \mathfrak{i}_k \mathfrak{i}_l \,{}^3_\cdot\, (\nabla \smile \nabla)\, \mathfrak{s}_n = 0}, \quad \ldots\ldots \quad (D'_1)$$

or:

$$\mathfrak{i}_j \mathfrak{i}_k \mathfrak{i}_l \,{}^3_\cdot\, (\nabla \smile \nabla)\, \nabla y^n = 0, \quad \ldots\ldots \quad (98)$$

and:

$$\mathfrak{i}_j \mathfrak{i}_k \mathfrak{i}_l \,{}^3_\cdot\, (\nabla \frown \nabla)\, \mathfrak{s}_n = 0. \quad \ldots\ldots \quad (99)$$

When $\overset{4}{\mathbf{K}}$ is the RIEMANN-CHRISTOFFEL-affinor of V_n, (99) can be written:[2])

$$\mathfrak{i}_j \mathfrak{i}_k \mathfrak{i}_l \,{}^3_\cdot\, \overset{4}{\mathbf{K}} \,{}^1_\cdot\, \nabla y^n = 0, \quad \ldots\ldots \quad (100)$$

or

$$\boxed{\mathfrak{i}_j \mathfrak{i}_k \mathfrak{i}_l \,{}^3_\cdot\, \overset{4}{\mathbf{K}} \,{}^1_\cdot\, \mathfrak{i}_n = 0.} \quad \ldots\ldots \quad (D_1'')$$

The equations (C_3), (D_1), (D'_1) and (100) are deduced by RICCI.[3]) The number of the equations (D'_1) is $\dfrac{(n-1)(n-2)(n-3)}{1.2.3}$, the number of the equations (D'') is $\dfrac{(n-1)(n-2)(n-3)}{3}$, because we may permutate not only j and k, but also k and l[4]). (D_1') contains third, (D_1'') only first differential quotients of y^n.

The conditions (D_1'') vanish identically, when the characteristic numbers $lkjn$ of $\overset{4}{\mathbf{K}}$ vanish. Since in a space of constant RIEMANN-curvature K_0:

$$\overset{4}{\mathbf{K}} = 2\, K_0\, (\mathfrak{a} \frown \mathfrak{b})\, (\mathfrak{a} \frown \mathfrak{b}) \text{ [5])}. \quad \ldots\ldots \quad (101)$$

the equation holds:

$$\mathfrak{i}_n \mathfrak{i}_l \mathfrak{i}_k \mathfrak{i}_j \,{}^4_\cdot\, \overset{4}{\mathbf{K}} = 0, \quad \ldots\ldots \quad (102)$$

so that the condition (D_1'') is an identity in such a space, and hence also in a euclidean space. Thus (D_1) reduces in this case to (D'_1). For

---

1) (C_3) can also be deduced from (84) in an analogous way as (D_1).

2) Comp. A. R. page 59.

3) G. RICCI, Dei sistemi etc., p. 314. Here the equations (C_3) and (D_1) are lettered (A_1) and (B_1). G. RICCI, Sui sistemi, p. 151.

4) Compare the observations of RICCI on occasion of a paper of DRACH, Comptes Rendus 125 (97) 598—601 and 810—811.

5) Compare Cf. BIANCHI-LUKAT, 1st. german edition, p. 574.

euclidean space the condition (D'_1) has been given by DARBOUX [1] [2]. The characteristic numbers $(lkjn)$ of $\overset{4}{\mathbf{K}}$ vanish too, when the $V_{n-1} \perp \mathbf{i}_n$ are geodesic.

10. LÉVY's, CAYLEY's *and* DARBOUX's *conditions.* Differentiating the relation:

$$\mathbf{i}_n = \sigma_n \mathbf{s}_n \quad \ldots \ldots \ldots \ldots \quad (103)$$

we get

$$\nabla \mathbf{i}_n = (\nabla \sigma_n) \mathbf{s}_n + \sigma_n \nabla \mathbf{s}_n \quad \ldots \ldots \ldots \quad (104)$$

Differentiating again, we get:

$$\nabla\nabla \mathbf{i}_n = (\nabla\nabla \sigma_n) \mathbf{s}_n + (\nabla \mathbf{s}_n) \overset{1}{\ } \mathbf{a} (\nabla \sigma_n) \mathbf{a} + (\nabla \sigma_n) \nabla \mathbf{s}_n + \sigma_n \nabla\nabla \mathbf{s}_n, \quad (105)$$

and from this and (104) we have for $\nabla\nabla \sigma_n$:

$$\varkappa \nabla\nabla \sigma_n = \sigma_n (\nabla\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n - \sigma_n^2 (\nabla\nabla \mathbf{s}_n) \overset{1}{\cdot} \mathbf{i}_n + \frac{2\varkappa}{\sigma_n} (\nabla \sigma_n)(\nabla \sigma_n). \quad (106)$$

Since:

$$(\nabla\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n = \nabla \{(\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n\} - (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{a} (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{a} = -{}^2\mathbf{h} \overset{1}{\cdot} {}^2\mathbf{h} - (\mathbf{u}_n . \mathbf{u}_n) \mathbf{i}_n \mathbf{i}_n, \quad (107)$$

we get, in connection with (92):

$$\overset{4}{\mathbf{g}}_n \overset{2}{\cdot} \nabla\nabla \sigma_n = -\varkappa\sigma_n {}^2\mathbf{h} \overset{1}{\ } {}^2\mathbf{h} - \varkappa\sigma_n^3 \overset{4}{\mathbf{g}}_n \overset{2}{\cdot} (\nabla\nabla \mathbf{s}_n) \overset{1}{\cdot} \mathbf{s}_n + 2 \sigma_n \mathbf{u}_n \mathbf{u}_n. \quad (108)$$

In connection with (C_3) this equation gives a new shape to the first condition:

---

[1]) G. DARBOUX, Leçons sur les systèmes orthogonaux et les coòrdonnées curvilignes I (98), p. 130, form. (35).

[2]) As a simple example for the application of (C_2) and (D'_1) for euclidean space, we can take the system $u = Y_1(y^1) + \ldots + Y_n(y^n)$, in which $y^1, \ldots, y^n$ are Cartesian coordinates. To calculate $g^{a_1 a_2}$ etc., it is necessary to find a system of n—1 V_{n-1} which determines in the V_{n-1} $u = \text{const.}$ a system of coordinates $\mathbf{e}_{a_1} \ldots$ Then $\varkappa \mathbf{e}_{a_1} . \mathbf{e}_{a_1} = g_{a_1 a_1}$, etc. For this purpose we must try to find n—1 independent solutions of the differential equations

$$\sum_i^{1,\ldots,n} \frac{\partial u}{\partial y^i} \frac{\partial \psi}{\partial y^i} = 0 \quad \text{or} \quad \sum_\iota^{1,\ldots,n} Y_\iota (y^\iota) \frac{\partial \psi}{\partial y^i} = 0$$

For the calculation compare e.g. WIERINGA, Diss. p. 21 and seq. Then we can see that the condition (D'_1) is identically satisfied, so that only LILIENTHAL's condition (C_2) remains, which can be written in this case:

$$\begin{vmatrix} \overset{1}{Y_\iota''} & \overset{1}{Y_l''} & \overset{1}{Y_k''} \\ Y_\iota' Y_i''' - 2\, Y_i''^2 & Y_l' Y_l''' - 2\, Y_l''^2 & Y_k' Y_k''' - 2\, Y_k''^2 \end{vmatrix} = 0,$$

or $Y_\iota' Y_\iota''' - 2\, Y_\iota''^2 = A Y_\iota'' + B$, in which A and B are constants.

This result has been deduced for $n = 3$ by SERRET, and for a general n by DARBOUX in another way as has been done here. Comp. DARBOUX, Leçons sur les systèmes orthogonaux etc., p. 140 and 141.

$$\mathbf{i}_j\,\mathbf{i}_k \overset{2}{\cdot}\, \nabla\nabla\, \sigma_n = 2\,\varkappa\sigma_n{}^{\prime}\, \mathbf{i}_j\,\mathbf{i}_k \overset{2}{\cdot} \{\mathbf{s}_n\, {}^1\, (\nabla \frown \nabla)\, \mathbf{s}_n\} = \varkappa\sigma_n\, \mathbf{i}_j\,\mathbf{i}_k\, {}^2\, (\mathbf{i}_n\, {}^1\, \overset{4}{\mathbf{K}}\, {}^1\, \mathbf{i}_n) \quad (109)$$

or

$$\boxed{\mathbf{i}_j\,\mathbf{i}_k \overset{2}{\cdot}\, \nabla\nabla\, \sigma_n = \varkappa\sigma_n\, \mathbf{i}_n\,\mathbf{i}_j\,\mathbf{i}_k\,\mathbf{i}_n \overset{4}{\cdot}\, \overset{4}{\mathbf{K}}.} \qquad (C_4)$$

Thus for a V_n, for which the characteristic numbers $(n\,k\,j\,n)$ of $\overset{4}{\mathbf{K}}$ vanish, this first condition can be written:

$$\boxed{\mathbf{i}_j\,\mathbf{i}_k \overset{2}{\cdot}\, \nabla\nabla\, \sigma_n = 0.} \qquad (C'_4)$$

This equation expresses that the tensor $\nabla\nabla\sigma_n$ *has the same principal directions as* $^2\mathbf{h}$. The geometrical signification of σ_n is that this quantity is proportional to the infinitesimal distance between succeeding $V_{n-1} \perp \mathbf{i}_n$ measured along $\mathbf{i}_n$.

In space of constant RIEMANN-curvature K_0 we have, in connection with (101):

$$\mathbf{i}_j\,\mathbf{i}_k\, {}^2\, \{\mathbf{i}_n\, {}^1\, \overset{4}{\mathbf{K}} \overset{1}{\cdot} \mathbf{i}_n\} = -\,K_0\, \mathbf{i}_j\,\mathbf{i}_k \overset{2}{\cdot} (\varkappa\, {}^2\mathbf{g} - \mathbf{i}_n\,\mathbf{i}_n) = 0, \quad . \quad . \quad (110)$$

from which we conclude that in this manifold the first condition has the shape (C'_4), hence also in euclidean space. In this latter case the condition is deduced for $n = 3$ by LEVY [1]), CAYLEY [2]), DARBOUX [3]), and for general values of n by DARBOUX [4]). Thus the necessary and sufficient conditions for manifolds of constant RIEMANN-curvature are (C'_4) and (D'_1).

11. WEINGARTEN's condition. We will try to find a shape of the conditions that only depends on $\mathbf{i}_n$ and no more on $\mathbf{i}_j, j=1, 2, \ldots, n-1$. When a tensor, whose principal directions do not coincide with those of $^2\mathbf{h}$, be transvected once with $^2\mathbf{h}$, an affinor arises whose alternating part is certainly not annihilated. Thus the condition that the principal directions coincide, is that the alternating part of the first transvection with $^2\mathbf{h}$ vanishes. Hence (109) is equivalent to:

$$\overset{4}{B}\,\mathbf{g} \overset{2}{\cdot} \{(\nabla\,\mathbf{i}_n)\, {}^1\, (\nabla\nabla\, \sigma_n) - \sigma_n\, (\nabla\,\mathbf{i}_n)\,\mathbf{i}_n\, {}^2\, \overset{4}{\mathbf{K}} \overset{1}{\cdot} \mathbf{i}_n\} = 0, \quad . \quad . \quad (111)$$

in which B may indicate that the bivector-part has to be taken.

---

1) M. LÉVY, Mémoire etc., p. 170.

2) A. CAYLEY, Sur la condition pour qu'une famille de surfaces fasse partie d'un système orthogonal, Comptes Rendus 75 (72), a series of articles.

3) G. DARBOUX, Sur l'équation du troisième ordre dont dépend le problème des surfaces orthogonales. Comptes Rendus 76 (73) 41—45, 83—86. See also e. g. BIANCHI-LUKAT 1st german edition.

4) G. DARBOUX, Leçons sur les systèmes orthogonaux etc., p. 128. His formula (32) is our formula (C'_4).

Since:

$$\nabla \{ (\nabla \mathbf{i}_n) \overset{1}{\cdot} (\nabla \sigma_n) \} = (\nabla \nabla \mathbf{i}_n)^1 \, \nabla \sigma_n + \mathbf{a} \, (\nabla \mathbf{i}_n) \overset{1}{\cdot} (\mathbf{a} \, . \, \nabla) \, \nabla \sigma_n, \quad (112)$$

we have:

$$\nabla \frown \{ (\nabla \mathbf{i}_n) \overset{1}{\cdot} \nabla \sigma_n \} = B \nabla \{ (\nabla \mathbf{i}_n)^1 \, \nabla \sigma_n \} = {}^1/_2 \, \overset{4}{\mathbf{K}} \overset{2}{\cdot} \mathbf{i}_n \nabla \sigma_n - B (\nabla \mathbf{i}_n)^1 \, \nabla \nabla \sigma_n, (113)$$

so that (111) is equivalent to:

$$\overset{4}{\mathbf{g}} \overset{2}{\cdot} \left[- \nabla \frown \{ (\nabla \mathbf{i}_n) \overset{1}{\cdot} \nabla \sigma_n \} + \frac{1}{2} \overset{4}{\mathbf{K}} \overset{2}{\cdot} \mathbf{i}_n \, \nabla \sigma_n - \sigma_n B (\nabla \mathbf{i}_n) \, \mathbf{i}_n \overset{2}{\cdot} \, \overset{4}{\mathbf{K}}{}^1 \, \mathbf{i}_n \right] = 0. \quad (114)$$

Since in a space of constant RIEMANN-curvature on account of (92) and (101):

$$\overset{4}{\mathbf{g}} \overset{2}{\cdot} \overset{4}{\mathbf{K}} \overset{2}{\cdot} \mathbf{i}_n \nabla \sigma_n = - \sigma_n \overset{4}{\mathbf{g}} \overset{2}{\cdot} \overset{4}{\mathbf{K}} \overset{2}{\cdot} \mathbf{i}_n \mathbf{u}_n = - 2 \sigma_n K_0 \overset{4}{\mathbf{g}}{}^2 \, \mathbf{i}_n \frown \mathbf{u}_n = 0, \quad (115)$$

the condition for such a manifold is, on account of (110), that the component of $\nabla \frown \{ (\nabla \mathbf{i}_n)^1_{\cdot} \nabla \sigma_n \}$ in the region $\perp \mathbf{i}_n$ vanishes. On account however of STOKES' law [1]), we have for each vector $\mathbf{v}$:

$$\int_s \mathbf{v} \, . \, d\mathbf{x} = - 2 \, \varkappa \int_\sigma {}_2\mathbf{f} \overset{2}{\cdot} (\nabla \frown \mathbf{v}) \, d\sigma, \quad . \quad . \quad . \quad . \quad (116)$$

in which s is a closed curve and ${}^2\mathbf{f} d\sigma$ the bivector of the surface-element of any surface σ bounded by this curve. From this we conclude *that in a space of constant* RIEMANN-*curvature we can also give as first condition that the linear integral of the vector* $(\nabla \mathbf{i}_n)^1_{\cdot} \nabla \sigma_n$ *along each closed curve in a* $V_{n-1} \perp \mathbf{i}_n$ *vanishes.* This condition is the only one for V_3. For an R_3 it has been first indicated by WEINGARTEN [2]) and RICCI [3]) has observed on occasion of WEINGARTEN's paper that the condition holds also for a V_3 of constant RIEMANN-curvature. From the above-mentioned we see that the condition, but no more as the only one, holds also for manifolds of constant RIEMANN-curvature, for which $n > 3$.

---

[1]) Comp. A. R., page 37 and 61.

[2]) WEINGARTEN, Ueber die Bedingung, unter welcher eine Flachenfamilie einem orthogonalen Flächensystem angehört. Crelle 83 (77), 1—12.

[3]) G. RICCI, Della equazione di condizione dei parametri dei sistemi di superficie, che appartengono ad un sistema triplo ortogonale. Rendiconti Acc Lincei Ser. V, III_2 (94) 93—96.

RICCI observes for the case $n = 3$ that WEINGARTEN's theorem remains also valid, when $\overset{4}{\mathbf{K}}$ has the shape:

$$\overset{4}{\mathbf{K}} = \mu \, (\mathbf{a} \frown \mathbf{b}) \, (\mathbf{a} \frown \mathbf{b}) + \nu \, (\mathbf{i}_1 \frown \mathbf{i}_2) \, (\mathbf{i}_1 \frown \mathbf{i}_2)$$

when ν is an arbitrary coefficient. This however holds also for general values of n.

12. *Mutually orthogonal V_{n-1}-systems through a given congruence, the canonical congruences being not singly determined.*

When the roots of (24) are not all different, these roots determine in general q mutually perpendicular regions $R_{p_1}, \dots R_{p_q}$. Within the region R_{p_α} every set of p_α mutually perpendicular directions satisfies the canonical conditions. The equations (47—51) teach us that it must be possible to choose the canonical directions in each of the regions R_{p_α} in such a way that they are V_{n-1}-normal, when through $\mathbf{i}_n$ there shall pass $n-1$ mutually orthogonal V_{n-1}-systems. Thus the conditions (C') and (D), depending on (55) resp. (67), i.e. of the being V_{n-1}-normal of *all* canonical congruences, will no more remain valid without any restriction.

When ${}_{p_1}\mathbf{i}, \dots {}_{p_q}\mathbf{i}$ are the unit-p-vectors belonging to the regions $R_{p_1}, \dots, R_{p_q}$, the equations:

$$\mathbf{i}_n \overset{1}{\cdot} \nabla y^z = 0 \quad \dots \dots \dots \quad (117)$$
$$_{p_\alpha}\mathbf{i} \overset{1}{\cdot} \nabla y^z = 0 \quad \alpha = 1, \dots, \beta-1, \beta+1, \dots, q$$

must be satisfied by p_β independent solutions. On account of (B) we thus have:

$$(\mathbf{i}_n \, {}_{p_1}\mathbf{i} \dots {}_{p_{\beta-1}}\widehat{\mathbf{i}} \, {}_{p_{\beta+1}}\mathbf{i} \dots {}_{p_q}\mathbf{i})^2 \;\nabla \frown {}_{p_\beta}\mathbf{i} = 0. \quad \dots \quad (118)$$

and from this we conclude:

$$\mathbf{i}_k \frown \mathbf{i}_n \overset{2}{\cdot} \nabla \mathbf{i}_j = 0, \quad \dots \dots \dots \quad (119)$$
$$\mathbf{i}_k \frown \mathbf{i}_l \overset{2}{\cdot} \nabla \mathbf{i}_j = 0, \quad \dots \dots \dots \quad (120)$$

in which $\mathbf{i}_j$ belongs to another region than $\mathbf{i}_k$ and $\mathbf{i}_l$, and for the rest the choice is arbitrary, provided $k \neq l$.

(119) has entirely the same form as (55) and from (120) follows for the special case that $\mathbf{i}_j$, $\mathbf{i}_k$, $\mathbf{i}_l$ each belong to different regions:

$$\mathbf{i}_k \mathbf{i}_l \overset{2}{\cdot} \nabla \mathbf{i}_j = 0, \quad \dots \dots \dots \quad (121)$$

an equation of the same form, and deduced in the same way as (67).

The equations (C') and (D) only remain valid under the abovementioned restricting conditions. They are besides no longer sufficient. A supplementary condition will be found in the following way:

The equation (65) shows:

$$\left.\begin{aligned}(\lambda_k - \lambda_j)\, \mathbf{i}_k \mathbf{i}_l \overset{2}{\cdot} \nabla \mathbf{i}_j + \varkappa \mathbf{i}_j \mathbf{i}_k \mathbf{i}_l \overset{3}{\cdot} \nabla\, {}'\mathbf{h} = 0,\\ (\lambda_l - \lambda_j)\, \mathbf{i}_l \mathbf{i}_k \overset{2}{\cdot} \nabla \mathbf{i}_j + \varkappa \mathbf{i}_j \mathbf{i}_l \mathbf{i}_k \overset{3}{\cdot} \nabla\, {}'\mathbf{h} = 0.\end{aligned}\right\} \quad \dots \quad (122)$$

valid for the case that $\mathbf{i}_k$ and $\mathbf{i}_l$ belong to the same region and $\mathbf{i}_j$ to another one. Then, subtracting the equations (122) one from the other we conclude, in connection with (121):

$$\boxed{\mathbf{i}_j (\mathbf{i}_k \frown \mathbf{i}_l)^2 \nabla {}^2\mathbf{h} = 0 \text{ [1]} \quad j \neq k,\ j \neq l,\ k \neq l,\ j,\ k = 1, 2, \ldots, n-1.} \qquad (E)$$

Under the mentioned conditions the equations (C'), (D) and (E) are not only necessary, but also sufficient. In fact, from (E) may be concluded, in connection with (122), since $\lambda_k = \lambda_l$, that $\nabla \mathbf{i}_j$ is symmetrical in l and k, when l and k belong to the same region, but j and l do not. From (D) we conclude, in analogical way as we have explained in the first part of this paper, that $\nabla \mathbf{i}_j$ is symmetrical in l and k, when l and k belong to different regions, different from j. (C') tells that $\nabla \mathbf{i}_j$ is symmetrical in n and k, when k differs from j. Hence these three conditions are sufficient to show that $\nabla \mathbf{i}_j$ is symmetrical in the region $\perp \mathbf{i}_j$, and thus that $\mathbf{i}_j$ is V_{n-1}-normal.

When we call [2]) $p_1, p_2 \ldots p_q$ the multiplicity of the roots of the algebraic characteristic equation (24), the number of equations (C') is the sum of the two-factorial products of the numbers $p_1, p_2, \ldots p_q$, and the number of the equations (D) is thrice the sum of the three-factorial products of these numbers. The number of the equations (E) is equal to the sum of the products of the form $p_k p_h \left(\frac{p_h + p_k}{2} - 1\right)$.

13. *Simplifications for the case that the given congruence is V_{n-1}-normal.*

When $\mathbf{i}_n$ is V_{n-1}-normal, (C) passes into (C_1) or (C_2), being valid for the case that $\mathbf{i}_j$ and $\mathbf{i}_k$ belong to different regions. (D) can also be brought into the form (D_1) and is then valid for the case that $\mathbf{i}_j$, $\mathbf{i}_k$ and $\mathbf{i}_l$ belong to different regions.

From (97) follows for the case that $\mathbf{i}_k$ and $\mathbf{i}_l$ belong to the same region and $\mathbf{i}_j$ to another:

$$\mathbf{i}_j (\mathbf{i}_k \frown \mathbf{i}_l)^3 \nabla\nabla \mathrm{s}_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

This equation can also be written in the form:

$$\boxed{\mathbf{i}_j \, \mathbf{i}_k \, \mathbf{i}_l \overset{3}{.} \overset{4}{\mathrm{K}} {}^1 \, \mathbf{i}_n = 0 \text{ [3]}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (E_1)$$

which has a formal analogy to (D_1''), but which is valid under different conditions. But the increment of the vector $\mathbf{i}_n$, when

[1]) (E) is the equation (C) of Ricci, Dei sistemi, page 312, but deduced from ${}^2\mathbf{h}$, and not from $\nabla \smile \mathbf{i}_n$.

[2]) Compare Ricci, Dei sistemi, p. 312.

[3]) (E_1) is (C_1) of Ricci, Dei sistemi, p. 314.

geodesically moved along the boundary of the surface-element $d\sigma$, is:[1])

$$D_{kl}\,\mathbf{i}_n = d\sigma\,\mathbf{i}_k\,\mathbf{i}_l\,{}^2\overset{4}{\mathbf{K}}\,{}^1_{\cdot}\,\mathbf{i}_n \quad . \; . \; . \; . \; . \; . \; . \; . \quad (124)$$

So (E_1) demands this increment to remain in the region formed by $\mathbf{i}_k$ and $\mathbf{i}_l$.[2])

Thus we have obtained the following theorem:

III. *A system of ∞^1 V_{n-1} in a V_n, whose second fundamental tensor, apart from determined V_r, $r < n$, has q singly determined principal regions $R_{p_1}, \ldots, R_{p_q}$, but within the regions of more than one dimension no singly determined principal directions, belongs then and only then to an n-uple orthogonal system, when by moving perpendicular to m of the principal regions of ${}^2\mathbf{h}$, the component of the geodesic differential of ${}^2\mathbf{h}$, in the manifold determined by these m regions, has principal regions that coincide with the m mentioned principal regions of ${}^2\mathbf{h}$, and when besides the increment of $\mathbf{i}_n$, when $\mathbf{i}_n$ is geodesically moved along the boundary of a surface-element in any principal region, remains entirely in this same principal region.*

14. *Necessary and sufficient conditions that a V_n may admit n-uple orthogonal V_{n-1}-systems.*

The condition (D_1'') is a condition for the V_n in which the n-uple orthogonal system exists. If we wish every system of n mutually perpendicular $(n-1)$-directions in each point of the V_n to belong to an n-uple orthogonal V_n-system, then (D_1'') must be valid for every set of four mutually perpendicular unit-vectors. It can be proved that $\overset{4}{\mathbf{K}}$ can then be written in the form:

$$\boxed{\overset{4}{\mathbf{K}} = (\mathbf{a} \frown \mathbf{z})\,(\mathbf{a} \frown \mathbf{z})} \qquad (F)$$

in which $\mathbf{z}^2$ is an arbitrary tensor. For $n = 3$ $\overset{4}{\mathbf{K}}$ can *always* get this shape and, as has been proved by Cotton[3]), every set of three mutually perpendicular directions in any point of a V_3 can belong to a triple orthogonal system. It can be proved that (F) is sufficient for $n > 3$ too.

---

1) A. R. p. 64.

2) An analogous geometrical interpretation can also be given to condition (D_1'').

3) E. Cotton Sur une généralisation du problème de la représentation conforme aux variétes à trois dimensions, Comptes Rendus 125 (97) 225—228, compare also E. Cotton, Annales de Toulouse 1 (99) 385—438, Chap. III.

15. *Addendum.*

In this paper the product $\mathbf{i}.\mathbf{i} = \varkappa$ of the system R_n^0 [1]) is used. $\varkappa$ can be found from the dualities existing in the orthogonal group, on which the identifications used in the system R_n^0 are founded. Now in investigations on differential geometry these identifications (e.g. of $\mathbf{i}_1$ and $\mathbf{i}_2 \frown \ldots \mathbf{i}_n$) are practically not used. In this case it is convenient to substitute $\varkappa$ by $+1$, then $\varkappa$ vanishes in all formulae, and the calculation grows much easier. It has however to be noted, that taking $+1$ for $\varkappa$ it is no longer permitted to make use of the identifications founded on the dualities of the orthogonal group.

[1]) J. A. SCHOUTEN, On the direct analyses of the linear quantities etc., These Proceedings 21 (17) 327—341; Die Zahlensysteme der geometrischen Groszen, Nieuw Archief (20) 141—156.

Palaeontology. — *"Ueber einige palaeozoische Seeigelstacheln (Timorocidaris gen. nov. und Bolboporites Pander)"*. By Prof. J. WANNER. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of January 31, 1920).

I. *Timorocidaris gen. nov.*

Die bis jetzt bekannten palaeozoischen Seeigelstacheln zeichnen sich im Vergleich zu den meso- und känozoischen durch eine bemerkenswerte Einförmigkeit aus. "As far as known, spines are very uniform in character within the species in the Palaeozoic, cases of marked deviation such as occur in some Cidaridae being almost unknown in these older types" sagt JACKSON [1]). Seeigelstacheln aus den permischen Ablagerungen der Insel Timor zeigen, dass dieser Satz für das jüngste Palaeozoikum nicht mehr als zutreffend gelten kann. Aber auch ausserhalb der einzelnen Art herrscht hier eine beträchtliche Mannigfaltigkeit der Typen, ähnlich wie bei den Krinoiden, Blastoiden und Korallen, die allerdings einen noch weit grösseren Reichtum an neuen und eigentümlichen Formen aufzuweisen haben.

Die Modifikationen, die bei den permischen Stacheln von Timor zu beobachten sind, fallen zum Teil in den Rahmen der Abänderungen, wie sie manche Cidariden zeigen. Wie dort erscheinen Gestalt und Skulptur der Stacheln innerhalb der gleichen Art in verschiedener Weise modifiziert, und wir wissen hauptsächlich durch Beobachtungen an rezenten Cidariden, dass diese Modifikationen im wesentlichen mit der Position der Stacheln an der Schale zusammenhängen. Fremdartiger ist die Modifikation des Gelenkes, das hier in einer Ausbildung erscheint, wie sie bisher noch bei keinem Seeigelstachel beobachtet worden ist.

Am bemerkenswertesten ist in dieser Hinsicht eine Seeigelart, deren Stacheln bei Basleo, der bekannten reichhaltigen Fundstätte permischer Versteinerungen auf Timor, zu den häufigsten und auffallendsten Fossilien gehören. Sie sind von mir und der Expedition MOLENGRAAFF's in mehreren Tausenden von Exemplaren gesammelt

[1]) JACKSON, R. T., Phylogeny of the Echini with a revision of palaeozoic species. Mem. of the Boston Soc. of Nat. Hist. Vol. 7, p. 78, Boston 1912.

worden, während von den hierzu gehörigen Asseln bis jetzt keine Spur entdeckt werden konnte. Ich schlage für dieselben den Namen *Timorocidaris sphaeracantha* vor.

Wie Tafelfig. 1*a*—*c* zeigt, handelt es sich zumeist um eigentümliche, knopfähnliche Stacheln mit einem annähernd halbkugelförmigen Körper, der auf der hemisphaeroidalen Oberseite gekörnt, auf der flachen Unterseite, die im allgemeinen senkrecht zur Längsachse des Stachels steht, glatt ist Auf der Unterseite wächst ein kürzerer oder längerer Stiel oder Hals heraus, der unten durch drei Flächen spitz zugeschnitten wird. Das Bemerkenswerte ist nun, dass diese drei Flächen als Gelenkfacetten mit ausgesprochenem Krinoiden-Charakter ausgebildet sind. Jede Fläche (Tafelfig. 2) besteht aus einer segmentartigen Ligamentfläche und einer im Umriss dreiseitigen Muskelfläche, die von der ersteren durch ein Querriff getrennt wird. Die Ligamentfläche liegt oben (distal) und zeichnet sich durch eine deutliche schlitzförmige Ligamentgrube aus, die Muskelfläche unten (proximal) und wird durch eine fast oder ganz bis an das Querriff reichende Medianfurche halbiert.

Dieser Typus zeigt nun in der Gestalt, in der Skulptur und auch im Gelenk mannigfaltige Abänderungen, von denen hier nur die wichtigsten kurz besprochen werden sollen.

Bei der Betrachtung der Gestalt der Stacheln fällt am meisten auf, dass bei vielen (Fig. 5*a*-*b*) der Körper im Querschnitt nicht kreisrund, sondern an einer oder an mehreren Stellen seitlich mehr oder weniger abgestutzt ist. Das trifft für ca. $^2/_3$ aller vorliegenden Stacheln zu. Die Körnelung der Oberseite ist bei diesen im Umriss häufig dreiseitigen, manchmal auch vierseitigen oder polygonalen Stacheln auf den sphaeroidalen Teil der Körperoberfläche beschränkt und auf den seitlichen abgestutzten Flächen entweder unvollkommen oder garnicht ausgebildet. Die meisten Stacheln haben sich also in ihrem freien Wachstum gegenseitig behindert und dürften somit eine fast geschlossene Decke, einen wahren Panzer über der Schale gebildet haben, vergleichbar mit dem Stachelpanzer, den der bekannte lebende *Colobocentrotus Mertensii* trägt. Diese Stacheln waren unbeweglich oder in ihrer Beweglichkeit zum mindesten sehr beschränkt.

Andere Stacheln (Fig. 6 und 12) sind mehr oder weniger birn- oder keulenförmig; bei manchen (Fig. 7) tritt über dem halbkugelförmigen Körper scharf von diesem abgesetzt ein zweiter ähnlich gestalteter auf; bei wieder anderen wächst der halbkugelförmige Körper am distalen Ende zu einem spitzen Kegel aus.

Die Mannigfaltigkeit der Skulptur kommt am augenfälligsten durch das Auftreten von gekörnten und völlig glatten Stacheln (Fig. 3 und 11) zum Ausdruck. Von 1768 näher untersuchten Stacheln erwiesen sich 1546 oder 87,4 % als gekörnt und 222 oder 12,6 % als glatt. Dass diese beiden Modifikationen keine verschiedenen Arten oder Varietäten repräsentieren, ist bei ihrer völligen Uebereinstimmung in allen übrigen Merkmalen als sicher anzunehmen. Es kann sich demnach bei den viel weniger häufigen glatten Stacheln, die mit den gekörnten auch durch Uebergänge verbunden sind, nur um eine durch die Position der Stacheln an der Schale bedingte Modifikation handeln. DÖDERLEIN [1]) hat gezeigt, dass bei der Mehrzahl der Cidariden auf den dem Buccalfelde zunächst stehenden Stacheln die Körnelung ihrer Oberfläche ganz allgemein mehr zurücktritt, sodass sie häufig ganz glatt werden. Es ist somit sehr wahrscheinlich, dass auch bei *Timorocidaris* die glatten Stacheln vorwiegend in der Umgebung des Buccalfeldes auftraten.

Eine andere skulpturelle Modifikation (Fig. 10*a b*) zeigt auf der Oberseite des Körpers anstatt einzelner Körner oder Pusteln ein unregelmässiges Netzwerk von Leisten, die rundliche oder verlängerte, grubenartige Vertiefungen umschliessen. Diese seltene Modifikation ist durch zahlreiche Uebergänge mit dem gekörnten Typus verbunden. Zu diesen Uebergangsformen gehört u.a. der in Fig. 13 abgebildete, auch gestaltlich modifizierte Stachel, der auch bemerkenswert ist, weil die Pusteln, kurzen Leisten und Vertiefungen der Oberseite auf der hohen becherartigen Seitenwand des Körpers von verschieden starken und langen Längsleisten und Furchen abgelöst werden.

Aehnliche, durch unregelmässige zellenartige Vertiefungen ausgezeichnete Skulpturen finden sich bei Seeigelstacheln nur selten, so z. Bsp. bei „*Cidaris*" *scrobiculata* BRAUN aus den St. Cassianerschichten. Man vergleiche insbesondere das von BATHER [2]) in Fig. 339 auf Taf. XI abgebildete Exemplar, von dem gesagt wird: „The surface ... is covered with small deep pits irregularly distributed and having a granular border apparently of fused pustules". Aus diesen Worten geht klar hervor, dass nach BATHER's Meinung die Wände, welche die Vertiefungen umgeben, durch eine *Verschmelzung* von Pusteln gebildet worden sind. Man kann jedoch auch annehmen, dass umgekehrt die Pusteln durch eine *Auflösung* der Wände in Pusteln entstanden sind und somit die zellige Skulptur die primäre und die körnige die sekundäre ist. Diese Auffassung dürfte vielleicht deshalb vorzuziehen sein, weil die zellige Skulptur bei keinem späteren Seeigelstachel mehr auftritt, wohl aber bei älteren Seeigeln, so

[1]) DÖDERLEIN, L., Die Japanischen Seeigel. I. Teil, 1887, p. 34.

[2]) BATHER, F. A., Triassic Echinoderms of Bakony, Budapest 1909, p. 183.

bei dem permischen *Timorocidaris* und, wie wir unten sehen werden, sogar schon bei Stacheln aus dem Unter-Silur.

Eine dritte sehr häufige Abänderung der Skulptur (Fig. 6 und 12) kommt durch eine mehr oder weniger ausgesprochene Anordnung der Körner in parallele, gerade oder gebogene Querreihen zustande.

Von den verschiedenen Modifikationen des Gelenkes schliesst sich eine sehr enge an die Stacheln mit drei krinoidenähnlichen Gelenkfacetten, die weitaus am häufigsten sind, an. Es sind Stacheln mit nur zwei krinoidenähnlichen Facetten. Die Lage und Beschaffenheit dieser Facetten ist genau dieselbe wie bei den dreifacettigen Stacheln. An der Stelle der fehlenden dritten Facette verlängert sich die Aussenseite des kreisrunden Stieles geradlinig nach unten. Diese Modifikation ist nicht allzu häufig und in ihrer Bedeutung nebensächlich. Interessanter ist eine weitere Abänderung. Von 2422 genauer untersuchten Stacheln zeichnen sich 2150 oder 88.7 % durch krinoidenähnliche Gelenkfacetten aus; die übrigen besitzen am unteren Ende des Stachelkopfes eine konkave, mehr oder weniger tiefe Aushöhlung ähnlich wie ein normaler Seeigelstachel. Die Aushöhlung ist entweder ziemlich regelmässig schüssel- oder trichterförmig (Fig. 4*b*) oder in unregelmässiger Weise von einigen Furchen und Wülsten durchzogen (Fig. 5*b*) im Umriss gewöhnlich kreisrund, gelegentlich gerundet dreiseitig und hierdurch an den Umriss der drei Facetten der häufigsten Stacheln erinnernd. Der stielförmige Hals selbst ist wie bei den Stacheln mit krinoidenähnlichen Gelenkfacetten bei verschiedenen Individuen kürzer (Fig. 4*b*) oder länger (Fig. 13); er kann auch ganz fehlen (Fig. 5*b*), sodass die konkave Gelenkfläche nur von einem niedrigen Wall umgeben wird, der sie von der übrigen Unterseite des Stachels trennt. Die Modifikation mit einfachem konkaven Gelenk ist durch Uebergänge mit der durch drei krinoidenähnlichen Facetten ausgezeichneten verbunden. Diese Uebergangsformen (Fig. 8 und 9 *a*—*c*) zeigen, dass das einfache Gelenk morphologisch nicht den drei krinoidenähnlichen Facetten zusammen, sondern nur einer einzigen entspricht. Es kann daher das konkave Gelenk nur aus *einer* krinoidenähnlichen Facette bei gleichzeitiger Reduktion der beiden andern oder, wenn man umgekehrt eine Entstehung des krinoidenartigen Gelenkes aus dem konkaven annimmt, aus der konkaven Gelenkfläche nur *eine* krinoidenähnliche Facette hervorgegangen sein, während für die beiden übrigen eine Neubildung anzunehmen ist.

Wie ist nun das Zusammenvorkommen dieser beiden Gelenktypen bei ein und derselben Art zu verstehen? Zunächst mag bemerkt werden, dass die Annahme, dass die Stacheln mit krinoidenähnlichem Gelenk einerseits und mit konkavem Gelenk andererseits verschie-

46*

denen Arten oder Varietäten angehören, als höchst unwahrscheinlich beiseite gelassen werden kann. Die Uebereinstimmung dieser beiden Gelenktypen in allen ihren übrigen Merkmalen, der Befund, dass auch die Abänderungen in der Gestalt und Skulptur bei beiden Gelenktypen genau dieselben sind, und schliesslich das schon erwähnte gelegentliche Vorkommen von Stacheln, die alle Uebergänge von dem einen Gelenktypus zum andern zeigen, sprechen bestimmt gegen eine solche Auffassung.

Die verschiedenartige Ausbildung des Gelenkes bei *Timorocidaris* lässt sich mit dem Zusammenvorkommen von gekerbten und glatten Hauptwarzen und dementsprechend mit dem Zusammenvorkommen von Stacheln mit gekerbtem und glattem Stachelkopf bei vielen fossilen Cidarisarten und bei manchen rezenten Arten von *Plegiocidaris, Tylocidaris, Dorocidaris* und *Leiocidaris* vergleichen. Nun ist nach DÖDERLEIN [1]) bei einer Reihe von *Plegiocidaris*arten, die sich durch gekerbte Hauptwarzen auszeichnen, das Vorkommen einer mehr oder weniger grossen Zahl von ungekerbten Hauptwarzen auf die obere Schalenhälfte beschränkt und auch von *Leiocidaris*, einer durch vorwiegend glatte Hauptwarzen ausgezeichneten Gattung, wird angegeben, dass es zumeist die dem Apicalfelde zunächst stehenden Hauptwarzen sind, die gekerbt sind. Man könnte daher daran denken, dass auch bei *Timorocidaris* die verschiedenen Gelenkmodifikationen mit der Stellung der Stacheln an der Schale in Beziehung zu bringen sind. Diese Auffassung scheint gestützt zu werden durch das Häufigkeitsverhältnis der beiden Gelenkmodifikationen. Denn von 2422 näher untersuchten Exemplaren besitzen 2150 oder 88,7 % krinoidenahnliche Gelenkflächen und 272 oder 11,3 % einfache konkave Gelenkflächen. Diese Zahlen stimmen auffallend genau mit denjenigen überein, die das Verhältnis zwischen gekörnten und glatten Stacheln bezeichnen. Gleichwohl ist diese Auffassung nicht haltbar, wie sich leicht feststellen lässt, wenn man die gekörnten und glatten Stacheln auf die Beschaffenheit ihres Gelenkes gesondert untersucht. Dabei ergibt sich nämlich, dass unter den gekörnten Stacheln 87,3 % und unter den glatten 87,8 % durch krinoidenähnliche Facetten und der Rest, also 12,7 bezw. 12,2 % durch konkave Gelenkflächen ausgezeichnet sind. Unter den glatten Stacheln aus der Umgebung des Buccalfeldes befindet sich also ein genau ebenso grosser Prozentsatz von Stacheln mit konkaver Gelenkfläche, wie unter den Stacheln der übrigen Schalenzonen.

Auch unter allen übrigen Modifikationen der *Timorocidaris*stacheln

1) l.c. p. 42 und 43.

ist keine einzige ausfindig zu machen, bei der nur einer der beiden Gelenktypen auftreten würde.

Es bleibt somit als wahrscheinlichste Annahme übrig, dass bei der gleichen *Timorocidaris*art in allen Schalenzonen unter einer überwiegenden Zahl von Stacheln mit krinoidenartigem Gelenk ein gewisser Prozentsatz von Stacheln mit konkavem Gelenk mehr oder weniger unregelmässig zerstreut auftritt. Die Art der Gelenkung ist labil.

Schwieriger ist es, zu ermitteln, welche Gelenkmodifikation wir als die primäre und welche als die sekundäre anzusehen haben. Ist das krinoidenähnliche Gelenk aus dem einfachen konkaven hervorgegangen oder umgekehrt?

Timorocidaris darf zweifellos als ein in sehr eigentümlicher Weise spezialisierter Typus gelten. Das Vorkommen einer überwiegenden Zahl seitlich abgestutzter Stacheln weist auf eine sehr geringe Beweglichkeit, wenn nicht Unbeweglichkeit dieser Stacheln hin. Diese Erscheinung dürfte wohl kaum als eine ursprüngliche gelten können, sie dürfte als eine Anpassung an die Lebensbedingungen auf einem Riff aufzufassen sein. Da nun auch die krinoidenähnliche Ausbildung des Gelenkes eine freie allseitige Beweglichkeit der Stacheln ausschliesst — je nach der Zahl der Facetten können sich die Stacheln in zwei oder drei Richtungen bewegen — so liegt es nahe, in der krinoidenähnlichen Ausbildung des Gelenkes und dem engen Zusammenschluss der Stacheln einen ursächlichen Zusammenhang zu sehen und anzunehmen, dass das krinoidenartige Gelenk aus einem einfachen konkaven entstanden ist, wobei die allseitige Beweglichkeit zugunsten einer besseren Verfestigungsmöglichkeit der schweren massiven Stacheln mit der Schale aufgegeben wurde. Wenn diese Auffassung die richtige ist, dann möchte man allerdings erwarten, dass die seitlich abgestutzten, garnicht oder nur äusserst wenig beweglichen Stacheln auch diejenigen sind, bei denen vorzugsweise das krinoidenartige Gelenk auftritt und dass umgekehrt die kreisrunden, in ihrem freiem Wachstum nach keiner Seite hin behinderten Stacheln, deren Gestalt auf keine Beschränkung der Beweglichkeit schliessen lässt, diejenigen sind, die hauptsächlich konkave Gelenkung zeigen. Wenn man das vorliegende Material daraufhin prüft, so ergibt sich, dass solche Beziehungen in keiner Weise bestehen. Bei den kreisrunden Stacheln sind beide Gelenkbildungen ebenso häufig wie bei den seitlich abgestutzten. So bietet das Material selbst allerdings keine Stütze für die Auffassung, dass die Ausbildung des krinoidenähnlichen Gelenkes bei *Timorocidaris* zu dem engen Zusammenschluss der Stacheln in Beziehung zu bringen ist. Auch beim lebenden *Colobocentrotus* ist es trotz des Zusammenschlusses der Stacheln zu einer panzerartigen Decke zu einer von der normalen konkaven abweichenden Ausbildung der Gelenkfläche nicht gekommen.

Zum weiteren Vergleich mag nochmals die Kerbung der Hauptwarzen herangezogen werden, der DÖDERLEIN in seinem anregenden Werke über die Japanischen Seeigel ein besonderes Kapitel gewidmet hat. Er sagt[1]): „Die Frage, ob die Cidariden mit glatten oder die mit gekerbten Hauptwarzen den ursprünglichen Zustand darstellen, lässt sich nicht mit Sicherheit beantworten", hält es jedoch für sehr wahrscheinlich, „dass unabhängig voneinander auf verschiedenen Linien aus Formen mit gekerbten Warzen solche mit ungekerbten allmählich sich herausgebildet haben", oder m. a. W., „dass die Kerbung der Hauptwarzen ein Charakter ist, der bei den Cidariden auf verschiedenen voneinander unabhangigen Entwickelungslinien allmählich verloren gegangen ist". Die kompliziertere Gelenkverbindung ware somit hier als der ursprüngliche Zustand, die einfachere als der spätere anzusehen. Ob wir diese Erfahrung auf die verschiedenen Ausbildungen des Timorocidarisgelenkes übertragen dürfen, lasst sich zur Zeit wohl nicht entscheiden. Immerhin scheint der Schluss, dass auch bei *Timorocidaris* die krinoidenartige Ausbildung des Gelenkes die primäre und die einfache konkave die sekundare ist, eine gewisse und vielleicht sogar ebenso grosse Berechtigung zu besitzen wie die umgekehrte Annahme.

Wie dem aber auch sein mag, das Bemerkenswerte bleibt jedenfalls das Vorkommen eines Gelenktypus bei palaeozoischen Echiniden, der mit demjenigen gewisser Krinoiden vollkommen ubereinstimmt. Es ist das ein neuer Konvergenzfall, der in die Reihe derjenigen Erscheinungen gestellt werden kann, die EIMER als „unabhangige Entwicklungsgleichheit" oder „Homöogenesis" bezeichnet. Denn der gleiche Charakter gelangt hier bei ganz verschiedenen Gruppen selbstandig zur Ausbildung.

Die Art der Verbindung und Befestigung der *Timorocidaris*stacheln mit der Schale ergibt sich aus der Beschaffenheit des Stachelkopfes und der Gelenkfacetten. Die nach unten zugespitzte Form des mit den Gelenkfacetten besetzten Stachelkopfes und das Vorhandensein eines Querriffes in jeder Facette, das ein Widerlager erfordert, lässt darauf schliessen, dass der seitlich facettierte Stachelkopf nicht *auf*, sondern *in* dem Warzenkopf gesessen hat, wie es Textfig. 1 schematisch veranschaulicht.

[1]) l.c., p. 37, 38.

Die zu diesen Stacheln gehörigen Warzenköpfe müssen demnach eine der Form des Stachelkopfes annähernd entsprechende Vertiefung

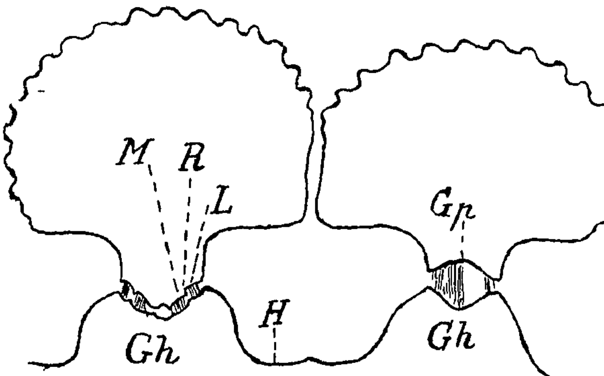


Fig. 1.

Fig. 1. Schematischer Längsschnitt durch zwei Stacheln von *Timorocidaris sphaeracantha* Wann. × 2. Links ein Stachel mit krinoidenartigen Gelenkfacetten, rechts ein Stachel mit einfachem konkaven Gelenk. *Gh*, Gelenkhöcker. *Gp*, Gelenk. *H*, Warzenhof. *M*, Muskelfläche mit der unteren Muskelschicht. *L*, Ligamentfläche mit der oberen Muskelschicht *R*, Querriff.

besessen haben und auch sehr gross gewesen sein. Zur Anheftung der Muskeln dienten ausser den Muskelflächen wahrscheinlich auch die Ligamentflächen. Nachdem jetzt von Hamann [1]) u.a. bei den Armgliedern rezenter Krinoiden auch die dorsalen Fasern als echte Muskelfasern gedeutet werden, scheint mir diese Annahme die zutreffendste zu sein. Es wäre demnach eine innere (untere) und äussere (obere) Muskelschicht und somit eine, allerdings nur äussere Aehnlichkeit mit der Muskulatur eines normalen Seeigelstachels vorhanden, wo diese bekanntlich gleichfalls aus einer äusseren und einer inneren Schicht besteht, die den Gelenkkopf wie ein doppelter Mantel umgibt. (Siehe Textfig. 2).

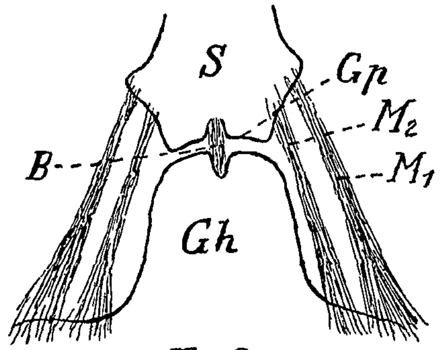


Fig. 2.

Fig. 2. Schematischer Längsschnitt durch eine Stachelbasis *S* und den Gelenkhöcker *Gh* eines normalen Seeigelstachels. M_1, M_2, äusserer und innerer Muskelmantel. *B*, Bindegewebesubstanz. Vereinfacht nach Ludwig-Hamann, Seeigel in Bronn's Klassen und Ordnungen des Tierreichs.

Auch bei der durch eine konkave Gelenkfläche ausgezeichneten Stachelmodifikation von *Timorocidaris* muss die Gelenkfläche zur

[1]) Bronn's Klassen und Ordnungen des Tier-Reichs. II. Bd. 3 Abtlg. Echinodermen, p. 1463. Leipzig 1905.

Anheftung der Muskulatur gedient haben. Denn ausserhalb des flachen Randes, der die Gelenkfläche umgibt, ist kein Platz für die Befestigung von Muskeln, wie die Oberflächenbeschaffenheit des Halses zeigt. Dementsprechend ist auch über diesem Rande keine Spur eines Ringes vorhanden. Diese konkave Gelenkfläche lässt sich daher nicht ohne weiteres vergleichen mit der Gelenkpfanne am unteren Ende eines normalen Seeigelstachels, wo sich die Stachelmuskulatur bekanntlich zwischen dem Ring und dem unteren Rande, der die vertiefte Gelenkfläche umgibt, anheftet. Der glatte Rand, der bei *Timorocidaris* die konkave Gelenkfläche umgibt, entspricht daher dem Ringe und nicht dem unteren Rande des normalen Seeigelstachels; er entspricht ferner dem oberen Rande der Ligamentfläche bei den *Timorocidaris*stacheln mit krinoidenähnlichen Facetten, wie die schon oben besprochenen Uebergangsformen zwischen den beiden Gelenkmodifikationen von *Timorocidaris* zeigen.

Die gleiche Gelenkausbildung wie bei der durch eine konkave Gelenkfläche ausgezeichneten Stachelmodifikation von *Timorocidaris* treffen wir u. a. bei den Stacheln von *Bothriocidaris* und auch bei mesozoischen Cidariden noch gelegentlich an. So sagt QUENSTEDT [1]) von den Stacheln von *Cidaris elegans* aus dem weissen Jura: „Einzelne Individuen (Taf. 62 Fig. 8) haben keine Spur eines Halsringes", und von *Cidaris coronatus*: „Besonders hervorzuheben sind die Gelenkgruben ohne Gelenkkopf, bloss mit scharfem Rande". Auch bei *Cidaris marginatus*, *florigemma* u. a. kommen nach QUENSTEDT Stacheln vor, die des Gelenkkopfes ermangeln.

Die starke Ausbreitung der glatten Unterseite der meisten *Timorocidaris*stacheln lässt auf grosse Warzenhöfe schliessen. Diese Warzenhöfe dürften zumeist ineinander geflossen und die Scrobicularringe, soweit solche überhaupt vorhanden waren, vielfach unterbrochen gewesen sein, da die meisten Stacheln sich gegenseitig berührten und zum Teil dicht aneinander geschlossen waren. Dazu kommt noch die schon erwähnte grosse Ausbildung der Hauptwarzen. Das sind alles Merkmale, durch die sich nach DÖDERLEIN [2]) die heute lebenden Cidariden in jugendlichem Alter auszeichnen; sie können deshalb als primitive Merkmale aufgefasst werden. Auch die knopfförmige Gestalt der Stacheln ist ein primitives Charaktermerkmal, wie die Ontogenie der Cidariden zeigt. „A young spine of a cidarid is short, broad, and distally rounded and reminds one of the character of the spines of Colobocentrotus", sagt JACKSON. [3]) Als weitere Merk-

1) QUENSTEDT, F. A., Petrefaktenkunde Deutschlands. Echiniden, p. 41 und 50.
2) l.c. p. 27.
3) l.c. p. 77.

male, die wahrscheinlich als ursprüngliche anzusehen sind, sind schliesslich noch die eigentümliche Ausbildung des Gelenkes, insbesondere auch das Fehlen eines Ringes und das gelegentliche Auftreten einer netzartigen Skulptur an Stelle der Körnelung zu nennen.

II. *Bolboporites Pander.*

Die *Timorocidaris*stacheln geben uns in mehrfacher Hinsicht den Schlüssel für das Verständnis der von PANDER [1]) schon 1830 unter dem Namen *Bolboporites* beschriebenen eigentümlichen Fossilien aus dem Unter-Silur von Russland, deren wahre Natur bis jetzt nicht richtig erkannt worden ist.

Die PANDER'schen Bolboporiten [2]) (Fig. 14 und 15) sind bekanntlich mehr oder weniger kreiselförmige, halbkugelförmige oder stark abgeplattet-kugelige, aus spätigem Kalk bestehende massive Körper. Die äussere Oberfläche dieser Körper ist mit eigentümlichen zelligen, unregelmässig eckigen oder abgerundeten, nur wenig tiefen Grübchen versehen, die umso länger und breiter werden, je mehr sie sich der Unterseite des Körpers nähern. Die Unterseite ist glatt und zeigt in der Mitte oder noch häufiger dem Rande genähert eine kleine Grube, zuweilen auch noch einige weitere unregelmässige Eindrücke.

Nach PANDER stehen die Bolboporiten am nächsten den Dactyloporen also den früher für Foraminiferen gehaltenen Kalkalgen. BRONN 1848 [3]) bezeichnete *Bolboporites* als „*Bryozoorum foss. gen. Calamoporae affine*". Die meisten späteren Autoren haben diese Gattung zu den tabulaten Korallen, zumeist in die Nähe von *Favosites* gestellt, so MILNE EDWARDS und HAIME 1852 [4]), DE FROMENTEL 1858 [5]), EICHWALD 1860 [6]), ZITTEL 1879 [7]) und QUENSTEDT 1881 [8]), letzterer allerdings mit grosser Reserve, da diese tabulate Koralle „aus spätigem Kalk besteht, wie wir es bei Stacheln von Echinodermen zu finden gewohnt sind." BRÖGGER 1882 [9]) hielt die Stellung dieses rätselhaften Gebildes

[1]) PANDER, Beiträge zur Geognosie Russlands 1830, p. 107.

[2]) Die besten Beschreibungen und Abbildungen finden sich bei PANDER (l.c.), EICHWALD (Lethaea rossica) und QUENSTEDT (Petrefaktenkunde Deutschlands, Echiniden).

[3]) BRONN, H. G., Index palaeontologicus, p. 170.

[4]) MILNE EDWARDS and J. HAIME, Monogr. des Polypiers fossiles des terrains paléozoiques, p. 246.

[5]) DE FROMENTEL, E., Introduction a l'étude des Polypiers fossiles, p. 269.

[6]) EICHWALD, Lethaea rossica, I, p. 495.

[7]) ZITTEL, K. A., Handbuch der Paläontologie I. Bd. 2. Lfg. p. 236.

[8]) QUENSTEDT, F. A., Petrefaktenkunde Deutschlands I. Abtlg. 6. Bd. Die Röhren- und Sternkorallen, p. 58.

[9]) BRÖGGER, W. C., Die silurischen Etagen 2 und 3 im Kristianiagebiet und auf Eker, p. 43.

für noch ganz unermittelt. Erst LINDSTRÖM ist 1883[1]) als erster mit Bestimmtheit für die Echinodermen-Natur dieses Fossils eingetreten, indem er sagt: „There can be no doubt left, that the fossils commonly named Bolboporites are neither corals nor bryozoa, but, as is evidently shown by their intimate structure, parts of the skeleton of some Echinodermatous animal, possibly some unknown starfish, amongst the recent ones of which blunt loosely affixed spines of nearly the same appearence often occur". 1888 spricht LINDSTRÖM[2]), worauf mich Herr G. HOLM in freundlicher Weise aufmerksam machte, nochmals von *Bolboporites* als „Fragments of some unknown Echinoderm, as shown by its intimate, characteristic structure". Auch JAEKEL und VON WOEHRMANN 1899[3]) sind sich über die Echinodermennatur dieses Fossils im Klaren gewesen. VON WOEHRMANN war jedoch merkwürdigerweise geneigt anzunehmen, dass die Bolboporiten im Innern der Theca zweier Chirocrinusarten ihren Platz hatten, wozu JAEKEL bemerkt: „Ich weiss nicht recht, welchen Platz und welche Funktion ein solcher massiger Körper im Innern der Theca gehabt haben soll. In Betracht könnte wohl nur die Möglichkeit kommen, dass diese Körper ursprünglich sehr porös als innere Madreporenfilter funktionierten. Die hufeisenförmige Narbe, die sich bei ihnen auf der einen flachen Seite findet, entspricht etwa in Form und Grösse der äusseren Oeffnung des primären Steinkanales und könnte diesem also innen angesessen haben als Eingangsoeffnung in den Filter".

Seitdem scheint dieses interessante und merkwürdige Fossil ganz in Vergessenheit geraten zu sein. Es wird in keinem neueren Lehrbuche der Palaeozoologie erwähnt. Da die Bolboporiten in den Sammlungen weit verbeitet sind, kann es sich nicht um sehr selten vorkommende Fossilien handeln. Nach BRÖGGER sind sie im Kristianiagebiet im oberen Teil des Expansus-Schiefers „überall ganz häufig". Massenhaft kommen sie indes, wie mir Herr G. HOLM in Stockholm auf meine Anfrage in freundlicher Weise mitgeteilt hat, nach seinen Erfahrungen wohl nirgends vor.

Die Bolboporiten liegen überall in Unter-Silur. Sie sind besonders in der Gegend von St. Petersburg bei Zarskoje, Pulkowa, Ropscha, am Lynnofluss bei Koltschanovo (Kalkschanovo) und an anderen Stellen im Orthocerenkalk und im oberen Teil des Glaukonitkalk-

---

[1]) LINDSTRÖM, G., Index to the generic names applied to the corals of the palaeozoic formations. Bihang till K. Svenska Vet.-Akad. Handl. Bd. 8, Nr. 9, p. 7.

[2]) LINDSTRÖM, G., List of the fossil faunas of Sweden. I. Cambrian and Lower Silurian, p. 10. Stockholm 1888.

[3]) JAEKEL, O., Stammesgeschichte der Pelmatozoen, p. 246. Fussnote.

steins verbreitet[1]). In Schweden finden sie sich nach LINDSTRÖM (1888, l.c.) im „Lower Gray Orthoceratite Limestone" und im „Chasmops Limestone", im Kristianiagebiet, wie erwähnt, im oberen Teil des vorwiegend kalkigen „Expansus-Schiefers". Auf Irland scheinen sie in der Landschaft Waterford im höchsten Teile der Etage 2 der „Tramore Limestone Series" vorzukommen[2]).

Ich glaube nun, diese Fossilien mit Bestimmtheit als Seeigelstacheln ansprechen zu können.

Unter den Merkmalen der Gattung *Bolboporites* ist die zellige Beschaffenheit der äusseren Oberfläche zweifellos dasjenige, das früher für die Deutung dieses Fossils als tabulate Koralle oder Bryozoe den Ausschlag gegeben hat. QUENSTEDT und besonders LINDSTRÖM haben erkannt, dass eine solche Deutung an der spätigen Natur der Bolboporiten ohne weiteres scheitern muss. Eine Erklärung für die eigentümliche Oberflächenstruktur haben sie indes nicht geben können. Durch das Auftreten einer ganz ähnlichen Skulptur bei manchen Stacheln von *Timorocidaris sphaeracantha* und bei den schon oben erwähnten Stacheln von *„Cidaris" scrobiculata* BRAUN ist das Vorkommen zelliger Skulpturen auch bei Seeigelstacheln erwiesen und steht also mit der Deutung der Bolboporiten als Seeigelstacheln nicht im Widerspruch. Der hauptsächlichste, aber unwesentliche Unterschied, der sich zwischen der Skulptur der Bolboporiten und derjenigen der genannten permischen und triadischen Stacheln ausfindig machen lässt, ist der, dass bei *Bolboporites* die Wände, welche die Vertiefungen umgeben, ganz glatt sind und abgesehen von den Stellen, wo sich die Wände benachbarter Zellen vereinigen, keine Körner oder Pusteln erkennen lassen. Das würde für die oben geäusserte Auffassung sprechen, dass die Netzskulptur ein ursprünglicher Zustand ist, und dass aus ihr die Körnelung durch Auflösung der Wände in Körner hervorgegangen ist. Vielleicht tritt die zellige Skulptur bei den ältesten Seeigelstacheln überhaupt häufiger auf, als es heute den Anschein hat. Es sei hier kurz auf die Fossilreste hingewiesen, die EICHWALD[3]) als Seeigelstacheln beschrieben und abgebildet hat mit dem Bemerken, dass sie wahrscheinlich zu seinem *Bothriocidaris globulus* gehören. Zwar hat später FR. SCHMIDT[4])

---

[1]) SCHMIDT, F., On the Silurian strata of the Baltic provinces of Russia. 1882. Quart. Journ. of the Geol. Soc. Vol. XXXVIII, p. 519.

[2]) REED, F. R. C., The lower palaeozoic bedded rocks of county Waterford. Quart. Journ. Geol. Soc. Vol. LV, p. 732. 1899.

[3]) EICHWALD, Lethaea rossica, Bd. 1 p. 655, Tab. XXXII, fig. 23 *a*, *b*. Stuttgart 1860.

[4]) SCHMIDT, FR., Ueber einige neue und wenig bekannte baltisch-silurische

gezeigt, dass die Stacheln von *Bothriocidaris globulus* eine ganz andere Beschaffenheit besitzen, und dass die von EICHWALD als Stacheln zu dieser Art citierten Stücke von Pulkowa nichts damit zu tun haben. Ob sie jedoch, wie SCHMIDT meint, eher als kleine Bryozoen anzusehen sind, scheint mir noch recht zweifelhaft zu sein. Nach der ausführlichen Beschreibung und den Abbildungen EICHWALD's scheinen alle Merkmale für einen Seeigelstachel mit zelliger Oberflächenstruktur zu sprechen. Leider fehlt mir das Material, um diese Frage entgiltig zu entscheiden.

Für das Verständnis der Bolboporitenskulptur als Seeigelstachelskulptur ist ferner von Belang, dass die einzelnen Zellen umso länger und grösser werden, je mehr sie sich der Unterseite nähern, eine Erscheinung, die, wie oben gezeigt wurde, in ganz ähnlicher Weise bei einigen Stacheln von *Timorocidaris* auftritt; ferner, dass „die Zellen häufig", wie schon PANDER bemerkt, „in einer gewissen Ordnung aneinandergereiht erscheinen, indem sie in konzentrischen einander berührenden Kreisen liegen", eine Anordnung, zu der auch die Körner bei vielen Seeigelstacheln neigen.

Schliesslich ist auch die glatte Beschaffenheit der Unterseite ein Merkmal, das, wie die glatte, flache Unterseite der *Timorocidaris*-Stacheln zeigt, gleichfalls für einen Seeigelstachel spricht. So lässt sich also die Oberfläche der Bolboporiten nach allen ihren Eigentümlichkeiten als Seeigelstacheloberfläche auffassen.

Noch wichtiger ist das Vorhandensein einer echten Gelenkfläche auf der Unterseite der Bolboporiten. Sie ist von den meisten Autoren zwar bemerkt, aber nur als „Grube" oder „Vertiefung" angesprochen worden. Selbst von einem so ausgezeichneten Beobachter wie QUENSTEDT, der übrigens, wie alle übrigen Forscher mit Ausnahme von PANDER die Bolboporiten mit ihrer Spitze nach unten abbildet, wird sie mit der Bemerkung abgetan: „Die Oberseite hat eine grosse, schwer zu reinigende Grube". Der richtigen Deutung am nächsten ist auch hier schon PANDER gekommen, indem er sagt, dass die Grube daraufhin zu weisen scheint, dass die Unterseite der Bolboporiten „vielleicht auf einem Stiele getragen wurde".

An mehreren gut erhaltenen Exemplaren (Fig. 14 *a*—*c*) des Bonner Museums vom Flusse Lynno bei Koltschanovo (Gouvernement St. Petersburg) auf die mich Herr Prof. STEINMANN in freundlicher Weise aufmerksam gemacht hat, zeigt diese Grube folgende Beschaffenheit: Im Umriss ist sie verlängert elliptisch, in der Mitte jedoch eingeschnürt, sodass sie in zwei mehr oder weniger gleiche

Petrefacten. Mém. de l'Acad. imp. des Sciences de St.-Pétersbourg. VII. sér. t. XXI N°. 11, p. 41. 1874.

Hälften zerfällt. Soweit die Stücke im Querschnitt nicht vollkommen kreisrund sind, ist die längere Achse der Grube zu dem grössten Querdurchmesser des Stachels annähernd parallel. Die Grube ist in der Regel von einem niedrigen Wall umgeben, ähnlich wie die konkave Gelenkfläche solcher *Timorocidaris*-Stacheln, bei denen der Hals und der Stiel sehr stark verkürzt ist. (Vgl. Fig. 4 *a. b*). Dieser Wall kann demnach als ein stark verkürzter oder noch unvollkommen entwickelter stielförmiger Hals aufgefasst werden. Auf einer Seite wird er da, wo er eingeschnürt ist, von einer schlitzartigen Furche unterbrochen; auf der entgegengesetzten Seite, die bei einer excentrischen Lage der Grube zugleich diejenige ist, die dem Rande der Unterseite genähert ist, zieht sich von der Einschnürung des Walles ein schwacher Rücken in die Tiefe der Grube hinab, ohne jedoch den Schlitz zu erreichen. Bei anderen Exemplaren (Fig. 15) fehlt der Wall, und der Schlitz liegt in der Tiefe der Grube. Es ist selbstverständlich, dass eine so ausgesprochen bilateral symmetrische Ausbildung des Gelenkes nur eine Bewegung in zwei diametral entgegengesetzten Richtungen erlaubte.

Die merkwürdige Beschaffenheit dieser Vertiefung konnte in der Tat kaum für einen Seeigelstachel sprechen, solange eine von der normalen wesentlich abweichende Ausbildung der Gelenkfläche von keinem Seeigel bekannt war. Zwar hat schon SCHULTZE 1866 [1]) seine Gattung *Xenocidaris* auf die abweichende Bildung der Gelenkfläche dieser in Eifeler Mittel-Devon vorkommenden Stacheln gegründet. Bei *Xenocidaris* zeigt sich „statt der knopfförmigen Verdickung der Basis eine concave perforierte Gelenkfläche, jedoch ist dieselbe nicht gleichmässig eingesenkt, sondern stark ausgekerbt, sodass der Stachel sattelartig auf dem ihm entsprechenden Tuberkel aufruht" Diese Gelenkbildung ist jedoch bei weitem nicht so aberrant wie diejenige von *Bolboporites*. Sie scheint mir zwischen dem Gelenktypus, wie ihn die *Timorocidaris*stacheln mit konkavem Gelenk zeigen, und demjenigen der normalen Seeigelstacheln zu stehen.

Die *Timorocidaris*stacheln zeigen nun zum erstenmal, dass bei palaeozoischen Seeigeln auch andere, von der normalen stark abweichende Gelenkbildungen möglich sind. Es liegt somit kein Grund mehr vor, der gegen die Deutung der Bolboporitengrube als Gelenkgrube sprechen könnte. Die richtige Deutung dieser Grube wurde vielleicht auch durch ihre wenig konstante Lage erschwert. Die Grube liegt nämlich bald in der Mitte der Unterseite, bald mehr oder weniger excentrisch dem Rande genähert. Bei genauerer Betrachtung zeigt sich

1) SCHULTZE, Monographie der Echinodermen des Eifler Kalkes, p. 14.

jedoch, dass die Lage der Grube auf der Unterseite keineswegs eine willkürliche ist. In der Mitte liegt sie bei den mehr oder weniger radialsymmetrischen Bolboporiten (*B. semiglobosa* und *B. mitralis*), exzentrisch stets bei den hornförmig gekrümmten (*B. uncinata*) und zwar so, dass sie sich stets nach derjenigen Richtung verschiebt, nach der sich die distale Spitze des Bolboporiten krümmt. Verbindet man die distale Spitze mit der Gelenkfläche durch eine Gerade, so steht diese letztere mehr oder weniger senkrecht auf der durch die Peripherie des Stachels gelegten Ebene.

Pander hat unter den ihm vorliegenden Bolboporiten auf Grund der äusseren Gestalt der Körper und der Grösse der Zellen vier Formen (*B. semiglobosa, triangularis, uncinata, mitralis*) unterschieden. Eichwald vereinigte diese in einer einzigen Art (*B. mitralis*). Dass er damit das Richtige getroffen hat, dürfte jetzt kaum mehr zweifelhaft sein, nachdem wir glauben, den Nachweis erbracht zu haben, dass es sich bei den Bolboporiten um Seeigelstacheln handelt. Es liegt jetzt nahe, die Pander'schen Formen als Stachelmodifikationen aufzufassen, die am gleichen Individuum in verschiedenen Schalenzonen auftraten und anzunehmen, dass der Pander'sche *B. mitralis* vielleicht vorwiegend auf die Umgebung des Apicalfeldes, die semiglobose Form auf die Umgebung des Buccalfeldes beschränkt war, während die beiden übrigen Formen (*B. uncinata* und *triangularis*) Stacheln der dazwischen liegenden Schalenzonen sind. Dass sich, wie Pander sagte, „nicht viele Uebergänge von der einen Form zur andern finden lassen", steht mit der Deutung dieser Formen als verschiedene Modifikationen derselben Stachelart nicht in Widerspruch. Der Eichwald'sche *B. stellifer* dürfte hingegen einer von *B. mitralis* verschiedenen Art angehören.

Die Tatsache, dass bis jetzt noch nie eine Assel der Bolboporitenschale gefunden wurde, kann selbstverständlich nicht als Einwand gegen die Deutung der Bolboporiten als Seeigelstacheln vorgebracht werden. Das Gleiche ist, wie oben bemerkt, bei den *Timorocidaris*- und vielen anderen Seeigelstacheln der Fall. Es sei nur an *Xenocidaris* aus dem Mittel-Devon der Eifel und an die zahlreichen Stacheln aus der oberen Trias von St. Cassian und vom Bakony erinnert.

Zusammenfassung.

Als wesentlichste Ergebnisse der vorangehenden Ausführungen sind hervorzuheben:

Seeigelstacheln sind im Palaeozoikum in einer grösseren Mannigfaltigkeit der Typen vertreten, als man dies bisher annehmen konnte. Auch innerhalb der Art treten im Palaeozoikum schon dieselben mannigfaltigen Modifikationen wie bei manchen späteren Cidariden auf.

Zu diesen Abänderungen der Gestalt und Skulptur gesellt sich bei *Timorocidaris sphaeracantha* gen. nov. et sp. nov. aus dem Perm von Timor eine sehr bemerkenswerte Modifikation des Gelenkes. Bei den meisten Stachelindividuen dieses neuen Typus besteht das Gelenk aus drei Facetten von ausgesprochenem Krinoiden-Charakter, eine Ausbildung, wie sie bisher von keinem andern Seeigel bekannt geworden ist; bei anderen, weniger häufigen Individuen ist eine einfache konkave Gelenkfläche vorhanden, die nicht mit der Gelenkpfanne am unteren Ende eines normalen Seeigelstachels verglichen werden kann.

Die unter dem Namen *Bolboporites* beschriebenen Fossilien aus dem Unter-Silur von Russland und Skandinavien sind als Seeigelstacheln zu deuten. Es sind somit die ältesten Echinidenreste, die wir kennen. Die auffallende zellige Oberflächenskulptur der Bolboporiten steht mit dieser Deutung durchaus im Einklang. Sie ist als altertümliche, auch noch in der Trias vorkommende Stachelskulptur anzusehen. Die Grube auf der Unterseite der Bolboporiten ist eine echte Gelenkgrube und als ein weiterer neuer Typus der Gelenkbildung bemerkenswert.

Bei den palaeozoischen Seeigelstacheln kommen somit verschiedene Gelenkbildungen vor. Als solche sind zu nennen: 1. Das Bolboporitengelenk; 2. das Krinoidengelenk (bei *Timorocidaris*); 3. das konkave Gelenk bei Stacheln ohne Ring und ohne Verdickung des Stachelkopfes (*Timorocidaris, Bothriocidaris* u.a.); 4. das Xenocidarisgelenk; 5. das konkave (normale) Gelenk mit oder ohne Kerbung des unteren Randes an Stacheln mit verdicktem Stachelkopf und mit Ring (*Archaeocidaris* u.a.). Für die beiden ersten Typen ist bezeichnend, dass sie nur eine beschränkte Beweglichkeit in wenigen Richtungen, für die übrigen, dass sie eine allseitige Beweglichkeit gestatten. Die Gelenkbildung war somit bei den palaeozoischen Seeigeln noch nicht so konsolidiert wie das bei den späteren Seeigeln der Fall ist. Die Natur hat ursprünglich auf verschiedene Weise versucht, die Stacheln mit der Schale zu verbinden, aber doch unfähig, die Art der Gelenke unbegrenzt abzuändern, hat sie in zwei verschiedenen Tiergruppen die gleiche Form der Gelenkflächen hervorgebracht.

TAFELERKLÄRUNG.

Fig. 1—13. *Timorocidaris sphaeracantha* gen. nov. et spec. nov. aus dem Perm von Basleo, Insel Timor.

Fig. 1. Häufigster Stacheltypus mit gekörnter Oberfläche und krinoidenartigen Gelenkfacetten. *a*. Von der Seite. *b*. Von der Unterseite. Nat. Gr. *c*. Skulptur der Körperoberfläche $\times 4$.

Fig. 2. Gelenkfacette $\times 5$.

Fig. 3. Glatter Stachel mit ausgehöhlter Unterseite und verkürztem Hals. *a*. Von der Seite. Nat. Gr. *b*. Von der Unterseite $\times 2^1/_2$.

Fig. 4. Stachel mit konkaver Gelenkfläche. *a*. Von der Seite. Nat. Gr. *b*. Von der Unterseite $\times 2^1/_2$

Fig. 5. Seitlich abgestutzter Stachel. Nat. Gr. *a*. Von der Seite. *b*. Von der Unterseite.

Fig. 6. Keulenförmiger Stachel. Körner in parallelen Querreihen angeordnet. Von der Seite. Nat. Gr.

Fig. 7. Fast glatter Stachel mit eingeschnürtem Körper. Von der Seite. Nat. Gr.

Fig. 8. Mittlere Partie eines Stachels von der Unterseite mit dem Gelenk $\times 4$ Zeigt vom Beschauer abgewandt eine grosse konkave Gelenkfläche, dem Beschauer zugewandt zwei unvollkommen krinoidenähnliche Facetten.

Fig. 9. Stachelkopf. $\times 4$. *a*. Von der Unterseite. Zeigt eine grosse konkave Gelenkfläche und zwei seitliche, unvollkommen krinoidenähnliche Facetten. *b* und c. Letztere von der Seite (spiegelbildlich) gesehen.

Fig. 10. Stachel mit zelliger Oberflächenskulptur. *a*. Von der Seite. N. Gr. *b*. Von oben $\times 2^1/_2$.

Fig. 11. Kleiner glatter Stachel mit dickem Hals.

Fig. 12. Birnförmiger Stachel mit $\pm$ parallelen Körnerreihen.

Fig. 13. Stachel mit seitlich stark verlängert-zelliger Oberflächenskulptur.

Fig. 14—15. *Bolboporites mitralis* Pander aus dem russischen Unter-Silur.

Fig. 14. Vom Lynnofluss bei Koltschanovo. *a*. Von der Seite. *b*. Von der Unterseite. Nat. Gr. *c*. Gelenk auf der Unterseite $\times 6$.

Fig. 15. Von Pulkowa. *a*. Von der Seite. *b*. Von der Unterseite. Nat. Gr. *c*. Gelenk auf der Unterseite $\times 6$.

Die Originale zu den Figuren 6 und 13 befinden sich in der Sammlung Molengraaff's in der Technischen Hochschule Delft, alle übrigen im geolog. pal. Museum der Universität Bonn.

Physiology. — "*Concerning Vestibular Eye-reflexes*. II. *The Genesis of cold-water nystagmus in rabbits*". By Dr. A. DE KLEYN and Dr. W. STORM VAN LEEUWEN. (Communicated by Prof. R. MAGNUS).

(Communicated at the meeting of January 31, 1920).

For an explanation of cold-water nystagmus we may have recourse to two theories. BARANY's theory is founded on the assumption of a stream of endolymph in one or more semicircular canals, brought about by local cooling of the labyrinthwall. This will cause also the endolymph, present there, to cool down and to flow off to the lowermost part of the semicircular canal. The ensuing lymph stream stimulates the sensory epithelium of the ampulla. In case the head of the animal is in a position in which the ampulla lies higher than the cooled part of the semicircular canal, the stream will be ampullofugal; if the reverse be the case an ampullopetal stream will result. The nystagmus elicited by each stream is of an opposite character.

BARTELS holds that by douching of the meatus with cold water the labyrinth would be eliminated, so that the nystagmus provoked would be like the spontaneous nystagmus after unilateral extirpation of the labyrinth. A warm water flow would be like stimulation of the N. vestibularis on the same side.

In a previous paper, issued from this institute, we have *disproved* BARTELS's *theory* [1]).

Moreover, it has already been contended by many other researchers. It was first of all pointed out that, if BARTELS's conception were correct, a cold-water nystagmus could not possibly be elicited from the unimpaired ear after unilateral extirpation of the labyrinth. HOFER [2]) has phrased it so well: "dieses tatsächliche Auftreten eines rotatorischen Nystagmus nach der operierten Seite wäre nach BARTELS'

---

1) A. DE KLEIJN and W. STORM VAN LEEUWEN. Ueber vestibuläre Augenreflexe I. Ueber die Entstehungsursache des kalorischen Nystagmus, nach Versuchen an Katzen und Kaninchen, Graefe's Arch. 5 Bd. 94 316, 1917.

A. DE KLEIJN and W. STORM VAN LEEUWEN. Over vestibulaire oogreflexen I Mededeeling. Kon. Acad. van Wetensch., Amsterdam. Wis- en Nat. Afd. Versl. Deel XXVI, 381, 1917.

2) J. HOFER. Untersuchungen über den calorischen Kaltwassernystagmus. Monatschr. f. Ohrenheilk. (1912) S. 1313.

Theorie, wie er ja selbst zugesteht, total unmöglich, weil eben das operierte Labyrinth fehlt und also nicht überwiegen kann über das gesunde, welches durch die kalte Ausspülung gelähmt werden soll; es sollte also nach BARTELS in so einem Fall gar kein Nystagmus auftreten, was aber den klinischen Tatsachen vollständig widerspricht" (S. 1317 und 1318). This argument, however, is not valid. BECHTEREW's [1]) wellknown experiments have shown us that when we extirpate a labyrinth and remove the other after some days, a nystagmus will occur again in the direction [2]) of the labyrinth that was removed first. So, if the cold-water-nystagmus were resulting from extirpation of the labyrinth on the douched side, we might also expect, some days after unilateral extirpation, on douching the unimpaired ear, a nystagmus towards the extirpated side. Indeed, BARTELS [3]) himself has suggested this interpretation. Another argument put forward by BARTELS [4]) against BARANY's theory, we do not quite understand. In a rabbit, with one octavus cut through, a cold-water or a warm water flow into the meatus of the unimpaired ear could provoke a nystagmus only towards the unimpaired ear. This finding of BARTELS's is not explained by BARANY's theory nor even by that of BARTELS. Neither were we ever confronted with this case in a prolonged series of experiments [5]). It is difficult to say what may have led to BARTELS's abnormal experience. It would be better perhaps in similar experiments to perform an extirpation of the labyrinth than a section of the octavus, since the latter operation may be attended with lesions of the central nerve-system.

Another cogent argument against the theory of BARTELS, put forward also by BARTELS himself, is that experimenters succeeded, by provoking a caloric nystagmus with various positions of the head in space, in obtaining now a nystagmus towards the non-douched ear, now again towards the douched one. This, indeed, is the main argument that turns up repeatedly in the literature. Still, it cannot be adduced against BARTELS's theory without also considering that, when examining

1) W. BECHTEREW. Ergebnisse der Durchschneidung des N. acusticus nebst Erörterung der Bedeutung der semizirkulären Kanäle für das Körpergleichgewicht. Pflüg. Arch. Bd. 30. (1883) S. 312.

2) In speaking about a nystagmus in a certain direction we always mean a nyst. with the quick component in that direction.

3) M. BARTELS. Ueber die vom Ohrapparat ausgelösten Augenbewegungen (Ophtalmostatik). Klin. Monatsbl. f. Augenh. Jhrg. 50. (1912) S. 200.

4) Discussion Verh. d. Otol. Gesellsch. Frankfurt. (1911) S. 214.

5) See F. QUIX. Ein Fall von translabyrintharisch operiertem Tumor acusticus. Verh. d. Otol. Gesellsch. Hannover (1912) S. 252

the caloric nystagmus, with various positions of the head in space, tonic reflexes of the eye-muscles may occur: the so-called compensatory eye-positions, which alter the position of the eye in the orbita. Therefore, it must be ascertained beforehand whether or no the spontaneous nystagmus occurring after unilateral extirpation of the labyrinth, alters its direction with different positions of the head in space.

Such experiments have been carried out, for aught we know, only by KUBO[1]). They will be briefly discussed here: KUBO severed one octavus. He does not tell us how he did it, nor whether he tried to ascertain by a subsequent control section if the process was successful. It would seem from the protocols that this is highly doubtful. Six of the experiments are reported in detail, of which a short description follows here:

Experiment 1, 4, and 5 will not receive consideration, because in them the nystagmus was not examined with different positions of the head.

Experiment 2.

In this experiment a nystagmus appeared with the quick component towards the operated side, after section of the right octavus had been performed. The nystagmus consequent on unilateral extirpation of the labyrinth, however, turns towards the unimpaired ear. KUBO adds only: "Diese Bewegungen bleiben unverändert, wenn man die Körperlage des Tieres ändert."

Experiment 3.

Left acusticus cut through. Subsequent vertical nystagmus-movements. After a couple of hours perfectly horizontal nystagmus with the quick component on the operated side towards the nose. Just as with the vertical nystagmus this direction is the same for any position of the animal. A flow of cold water into the right meatus is of no influence. After the semi-circular canal of the right ear has been laid bare, the experimenter states: "Nach Einspritzen von kaltem Wasser ändert sich die Richtung und es tritt eine rückweise Bewegung nach der Nase hin auf der operierten (linken) Seite auf."

This, however, was also the existing direction and opposite to the one we can look for in the case of cold-water nystagmus from the right ear. The vertical nystagmus also points to an imperfect section.

Experiment 6.

Section of left acusticus. First vertical, afterwards horizontal nystagmus (on the left with the quick component towards the nose.)

1) KUBO INO. Ueber die vom N. acusticus ausgelösten Augenbewegungen (besonders bei thermalen Reizungen.) Pflüg. Arch. 114. (1906) S. 143. 167.

47*

On the right a cold-water flow: Reversion of the nystagmus. In ventral position right eye with quick component towards the nose. In other position same direction. Here, then, in caloric examination no influence on the direction of the nystagmus through change of the position of the head in space. This, no doubt, is anomalous. Compensatory eye-positions are no longer distinct. This again indicates the deficiency of the experiment. Repeated application of cold water in the right ear yields on the right a nystagmus with the quick component towards that ear. This nystagmus is not affected by the position of the head in space.

The appearance of a nystagmus towards the douched ear on cold-water flow is the reverse of what is normally observed, and also the reverse of what was seen after the first washing. The imperfection of the experiment is also seen in the absence of any influence of the position of the head in space.

In our first communication it has been shown that in cats the spontaneous nystagmus after unilateral extirpation of the labyrinth, with different positions of the head in space, varies in nature and frequency, but not in direction. In our investigation of the cold-water nystagmus in normal animals and in animals after unilateral extirpation of the labyrinth, on the contrary, a considerable difference in the direction of the nystagmus with different positions of the head in space, has been demonstrated. It also appeared from subsequent experiments with rabbits that with them the case was fundamentally the same. Slight variations in the direction of the spontaneous nystagmus after unilateral extirpation of the labyrinth, however, do manifest themselves here, when the position of the head is varied, in consequence of the compensatory eye-positions, to be discussed later on, whereby the place of insertion of the eye-muscles in the orbita is altered. In the first communication evidence was also adduced to show that BARTELS's conception of the origin of the caloric nystagmus cannot be correct.

In the present investigation we purpose to ascertain whether additional data can be collected to support the theory of BÁRÁNY, who ascribes the caloric nystagmus to endolymph-streams. There are plenty of indications in the literature; to our knowledge an extensive experimental investigation has not been performed as yet.

Doubtless, the first question that arises is, whether douching of the meatus with cold-, resp. warm-water through the tympanum will engender such cooling down, resp. warming of the labyrinth-wall that endolymph streams are possible.

The result of a similar investigation carried on[1]) together with Prof. MAGNUS, was published in GRAEFE's Archiv., and led to the following conclusion:

"Bei Katzen, bei denen die Sympathicusbahnen zum Auge durch das Mittelohr verlaufen, tritt bei Ausspritzen des äusseren Gehörganges mit kaltem Wasser eine Sympathicuslähmung am Auge auf, die sich vor allem im Vortreten der Nickhaut äussert. Sie beruht auf einer Kälteparese der genannten Bahnen. Dadurch ist der Beweis geliefert dass beim Auslösen des kalorischen Nystagmus mit kaltem Wasser die Wand des Mittelohres über dem Labyrinth sich nachweisbar abkühlt."

We now pass on to report the results of our new experiments on the cold-water nystagmus in rabbits.

Our reason for selecting rabbits, while our previous experiments were chiefly carried out with cats, is the following:

First, in rabbits we seldom meet with rotatory nystagmus, of which the direction is always difficult to indicate. It is encountered in cats. The principal reason, however, is that in our experimentation we made use of an inquiry into the compensatory eye-positions, which have been carefully determined for the rabbit in conjunction with v. D. HOEVE[2]), but are difficult of determination for the cat.

Technique of our method.

A rabbit was suspended on an operation-board, and the head fixed firmly in a Czermak-clamp. Now in order to be able to bring the animal in any given position in space, the following contrivance was made (Fig. 1). The operation-board p-q-r-s is fixed to a wooden frame P-Q-R-S in such a way that the board p-q-r-s can rotate on the axis U-T, while the frame P-Q-R-S is again fixed to a second frame A-B-C-D, so that both P-Q-R-S and p-q-r-s can rotate on the axis V-W. A protractor is attached to P-Q-R-S, as well as to A-B-C-D, so that the degree of the rotation can be noted exactly in every direction. Now when the animal has been tied to the board in ventral position, a rotation on the axis V-W causes the animal to rotate on its bi-temporal axis. When moving the board round the axis U-T the animal turns on its occipito-caudal axis. When finally

[1]) A. DE KLEIJN und R. MAGNUS. Sympathicuslähmung durch Abkühlung des Mittelohres beim Ausspritzen des Gehörganges der Katze mit kaltem Wasser. Graefe's Archiv Bd. 96. (1918) S. 368.

[2]) J. v. D. HOEVE und A. DE KLEIJN. Tonische Labyrinthreflexe auf die Augen. Pflüg. Arch. Bd. 169. (1917) S. 241.

the board p-q-r-s is first revolved 90° about U-T, so that the animal is in lateral position, and when in this position the board is turned

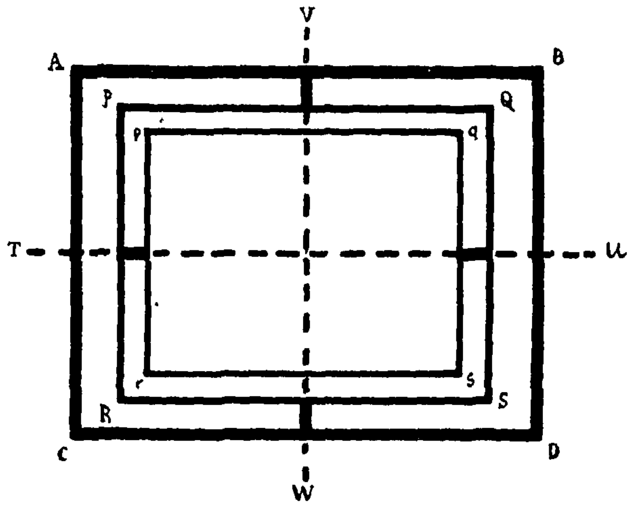


Fig. 1.

about the axis V-W, the animal will revolve about its dorso-ventral axis. A combination of rotations round the axes U-T and V-W enables us to bring the animal in any given position in space. In all of them the direction of the nystagmus consequent on a cold-water flow could be determined. In the following expositions the rotations in the different directions are described:

Rotation I.

Animal in ventral position, mouth-fissure horizontal. Rotation of the animal on its bi-temporal axis. Direction of rotation · head down, tail up.

Rotation II.

Animal in ventral position, mouthfissure horizontal. Rotation of the animal on its occipito-caudal axis. Direction of rotation: douched ear downwards.

Rotation III.

Animal in lateral position with irrigated ear downwards, mouth-fissure vertical. Direction of rotation: head down, tail up.

In these experiments the direction of the nystagmus consequent on a cold-water irrigation, was determined 37 times for every rotation of 360°. The first determination was always made at the normal position of that rotation; so e. g. at rotation I: the animal in ventral position, mouth-fissure horizontal. After this, while the ear was constantly being douched, the animal was moved every time 10° in the given direction and the direction of the nystagmus was noted. At the 37th determination the animal had come round again to its original position. Then the last determination served for a control-estimation. A short interval after every rotation of 10° was required before each reading, to preclude the possibility of a nystagmus, resp. deviation brought about by the *rotation* itself.

The irrigation of the *right* meatus took place from a height of 1,5 m., the cold-water used was of a temperature of ± 12° C. For every position in space, after it had continued for some time, the direction of the nystagmus was *valued* and the direction of the *rapid* component was marked down. (Figs 2, 3, and 4 not corrected).

This method does not yield perfectly reliable data; for a correct determination of the direction one might resort to cinematographic photos from which to decide on the direction. However, this was impracticable for a large number of determinations. Still, from what follows here we may infer that our method of valuation of the direction of the nystagmus yielded useful results.

In figs. 2—4

$\rightarrow$ = Direction of the quick component of the nyst. towards the nose
$\leftarrow$ = ,, ,, ,, ,, ,, ,, ,, towards the temporal
$\uparrow$ = ,, ,, ,, ,, ,, ,, ,, upwards relative to the *orbita*
$\downarrow$ = ,, ,, ,, ,, ,, ,, ,, downwards.

Fig. 2—4 (not corrected) gives the mean of 5 experiments.

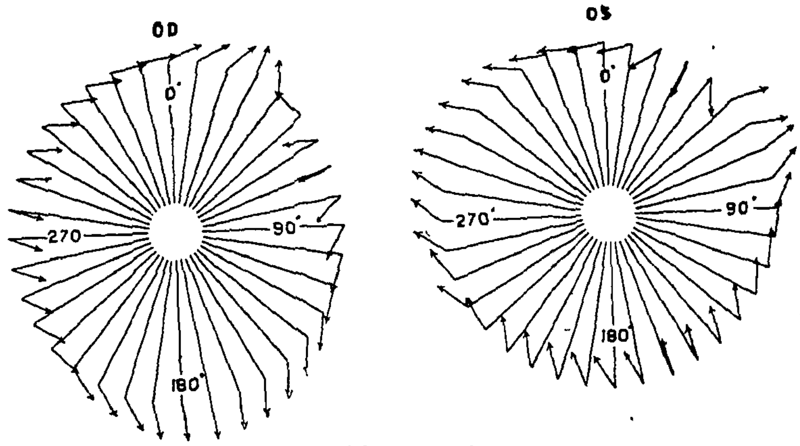
Now, however, the question rises: what influence is exerted on the nystagmus by the above-mentioned tonic eye-reflexes, occurring in the eye-muscles (compensatory eye-positions) with different positions of the head in space. On p. 246 of V. D. HOEVE's research, mentioned above, a curve is given of the rotatory movements.

With the aid of this curve the directions of the caloric nystagmus found, were now corrected as follows

We assume that douching with cold water, with the head in normal position, engenders an absolutely horizontal nystagmus with the quick component towards the nose. Now, when the position of the head changes from the normal into another position in space, so that a rotatory movement of the eyes ensues, e.g. of 45° with the upper cornea-pole towards the temporal, the insertion-points of the eye-muscles, notably of the Mm. internus and externus, will also be changed by this rotatory movement, and the same contractions and relaxations of these two eye-muscles, which caused with the normal position an horizontal nystagmus, will bring about a nystagmus of quite a different direction, viz. about 45° anteriorly upwards.

So for instance if an horizontal nystagmus appears at the normal

Rotation I (no correction).



with correction

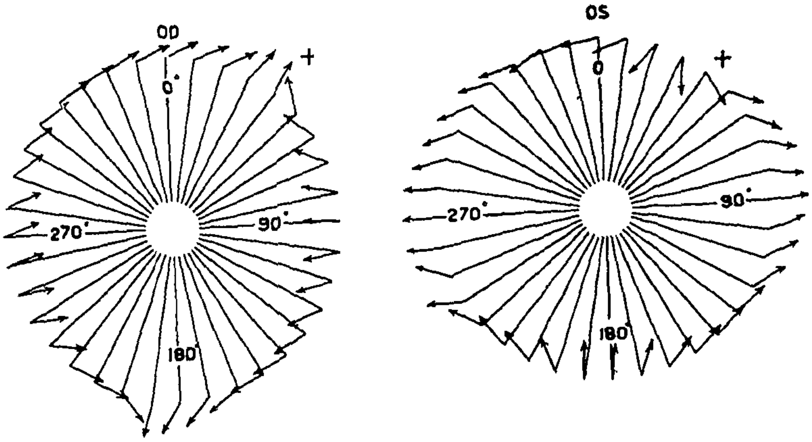
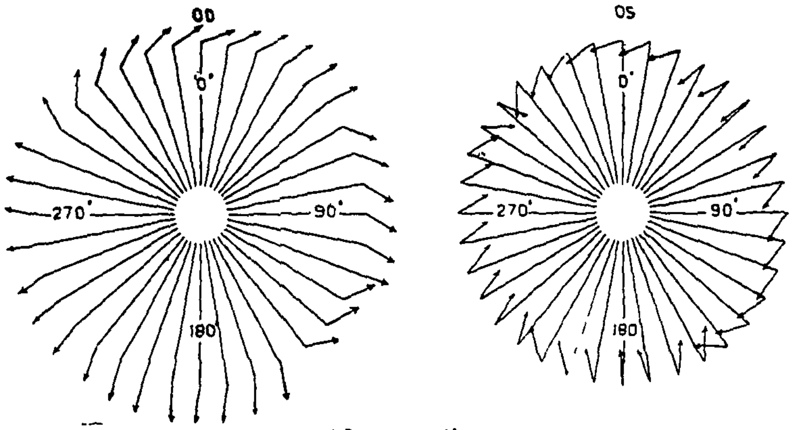


Fig. 2.

Rotation II (no correction).



with correction

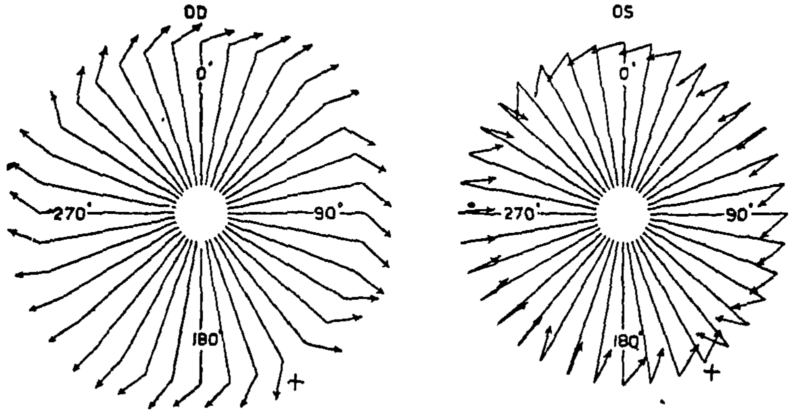


Fig. 3.

position and at another position of the head in space, with a rotatory

Rotation III (no correction).

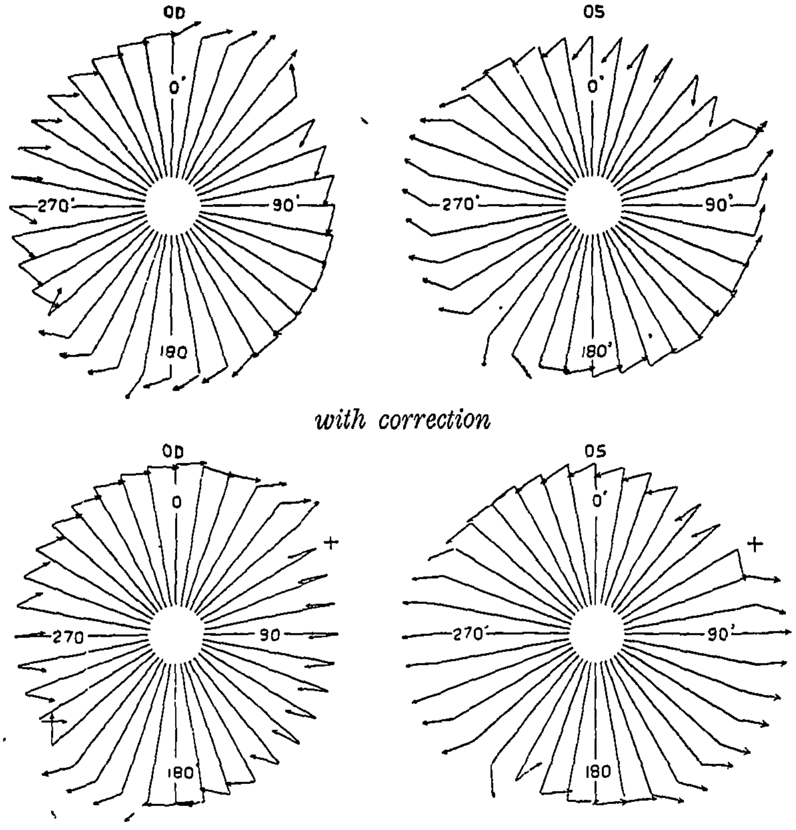


Fig. 4.

movement of 45° with the upper cornea-pole towards the temporal, a nystagmus of 75° anteriorly upwards, the correction is 75°—45°=30°.

The corrected direction, therefore, is that direction of the nystagmus that would be found, if the eyes were only under the influence of the labyrinth-stimulant consequent on the douching, and if there were no compensatory eye-positions.

Figs 2—4 illustrate our results before and after correction.

RESULTS.

As stated above, it had already been detected by BÁRÁNY, HOFER, and others that the direction of the nystagmus in man varies with different position of the head in space. This result was borne out by our experience.

When examining a rabbit, first in ventral position and subsequently with its head hanging downwards, we found a difference of 180° in the direction of the nystagmus.

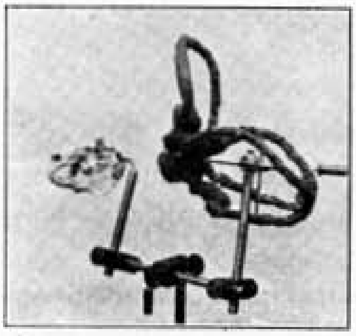
At first we supposed that, when e.g. the nystagmus of the left eye on douching the left meatus was directed anteriorly upwards in ventral position, and posteriorly-downwards with the head down, there would be an intermediate position in which there would be no nystagmus at all. In other words, if the nystagmus in ventral position is owing to an ampullo-fugal stream in the horizontal semicircular canal, and the nystagmus with the head down to an ampullo-petal stream, there would be no difference in the level of ampulla and of that portion of the semicircular canal that is cooled down by the douche and the nystagmus would consequently not appear. This proved not to be the case. True, in this reasoning the possibility has been eliminated of an influence of the cold water on the lymph-streams in the vertical semi-circular canals.

Considering that, although also the vertical canals may come into play, the horizontal canals are on account of their anatomic location, most exposed to the influence of the cold water, it could be anticipated on the ground of BARANY's theory that in the transition from ampullo-fugal to ampullo-petal stream in the horizontal canals, there would exist a short zone in which, with a slight variation in the position of the head, a marked change in the direction of the nystagmus would manifest itself abruptly. The critical point at which neither ampullo-fugal, nor ampullo-petal streams occur in the horizontal canals, so that only streams in the vertical canals can exert an influence here, receives a full discussion below.

Now when looking at the corrected figures, which illustrate the mean result of our experiments with the several rotations, the following observations can be made:

a. Rotation I. Douche of the right ear.

Observation right eye. With the animal in ventral position the nystagmus is anteriorly upwards. At 20° (i.e. head 20° below the



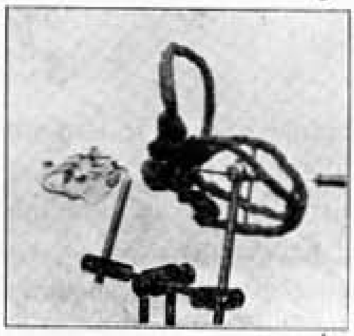


Fig. 5.

horizontal plane) the direction is still the same; at 30° a slight deviation begins; at 50° it is much more pronounced. At 80° the direction of the nystagmus deviates as much as 135° from the initial position and at 100° the change of direction of the nystagmus of 180° has been completed. Something like this occurs between the position of 170° and 270°.

Observation left eye. Fundamentally the same as right eye.

b. Rotation II. Douche of right ear.

Observation left eye: A sudden change in the direction of the nystagmus takes place here between 140° and 150°. While the nystagmus at 140° moves posteriorly-upwards, at 150° it is already anteriorly upwards. A similar marked change of direction is observed between 310° and 320°, the direction being respectively anteriorly downwards and downwards.

Observation right eye: Here it is less easy to say where the change of direction takes place. Presumably also between 140° and 150° and between 300° and 330°. That in this case the curve differs from all the others may be explained by the fact that the process of the experiments averagely represented by this curve, was very irregular in two out of five cases, which could not but be of great influence on the mean curve. There was no such irregularity with the left eye of these animals (which was examined on another day).

c. Rotation III. Douche of the right ear.

Observation left eye. Very great change of direction is found between 50° and 70° and a second change between 210° and 230°.

Observation right eye. Very great change between 40° and 60° and a second between 220° and 240°.

After the above facts had been ascertained, the critical point was

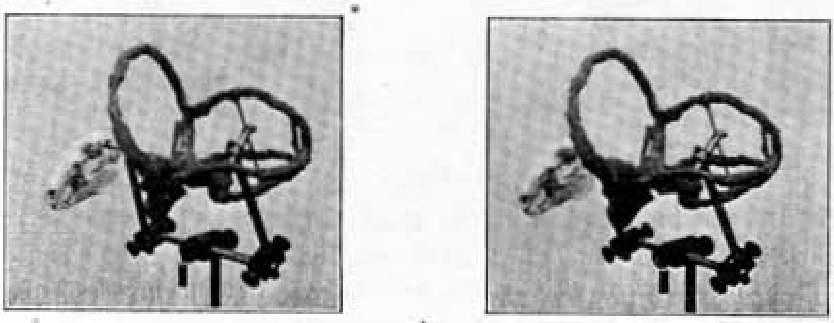


Fig. 6.

determined for the various rotations, i.e. the point at which the horizontal semicircular canal has reached its optimal horizontality and consequently no or hardly any streams can exist in this canal after douching the meatus. This determination was performed with the aid of a model in wax[1]) formerly made of the semicircular canals of a rabbit, which contrivance was arranged, after the indications of DE BURLET and KOSTER[2]), so as to afford an exact imitation of their natural position in the rabbit's skull.

This was to the following effect:

With the animal in ventral position with horizontal mouth-fissure (Fig. 5) the level of the ampulla of the horizontal semicircular canal is higher than the canal itself, so that an ampullo-fugal endolymph-stream will occur on a cold-water douche of the meatus.

With rotation I the horizontal canal is approximately horizontal at 40° (Fig. 6).

With rotation II the horizontal canal is approximately horizontal at 150° (Fig. 7).

With rotation III the horizontal canal is approximately horizontal at 57° (Fig. 8).

In figures 2—4 these points are indicated with crosses. At a glance it may be seen that *a marked change in the nystagmus occurs at the very place where the horizontal canal is approximately horizontal.*

When taking into account the considerable individual variations in the position of the semicircular canals in various animals of the same species, and when also considering the fact that our results are based upon the observation of five different animals, while the correction for the compensatory eye positions as well as the data

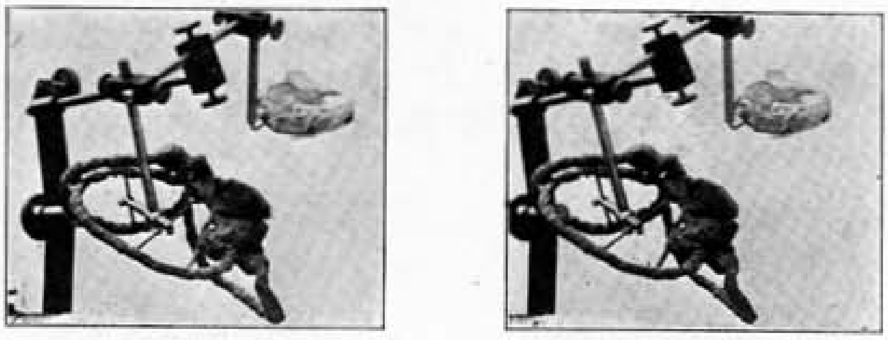


Fig. 7.

[1]) H. M. DE BURLET and A. DE KLEIJN. Ueber den Stand der Otolithenmembranen beim Kaninchen. Pflüg. Arch. Bd. 163. (1916) S. 321.

[2]) H. M. DE BURLET and J. J. J. KOSTER. Zur Bestimmung des Standes der Bogengänge und der acustica im Kaninchenschädel. Arch. f. Anatomie und Physiologie. Anatomische Abteilung. (1916) 59.

from the model in wax, refer to an animal that does not belong to this series, a striking resemblance can be stated between the changes of direction observed and those that could be anticipated with reference to the model.

The fact, however, is that with none of the rotations I—III does the horizontal semicircular canal attain horizontality. De Burlet and Koster's researches showed that the right horizontal semicircular canal is approximately horizontal when the animal turns from the ventral position about 30° round the bi-temporal axis with the head down and at the same time round the fronto-occipital axis about 7° to 8° with the left eye downwards.

We examined different animals in this position, from which it appeared that in most cases the nystagmus had not disappeared altogether and could neither be made to disappear by applying different variations in the rotation round the said axes. We observed, however, that the nystagmus-movements are very small in this position.

Only in two cases could the nystagmus be made to disappear completely, viz. with a rotation about the bi-temporal axis of 37° in the one and 30° in the other rabbit and combined with a rotation about the fronto-occipital axis of 5° in both animals.

This urges us to conclude that *the horizontal semicircular canal plays a principal part in caloric stimulation,* that, however, in most cases also the vertical canals exert some, though a small, influence. This influence, however, was not such as to enable us to make an accurate analysis of it from the curves.

For a positive solution of the problem it would be necessary to determine in one and the same rabbit the nystagmus in various positions of the head in space, as well as the compensatory eye-positions in the said positions and finally through microscopic examination of the labyrinth, to determine accurately the position of the semi-circular canals in that animal, after the method of De Burlet and Koster.

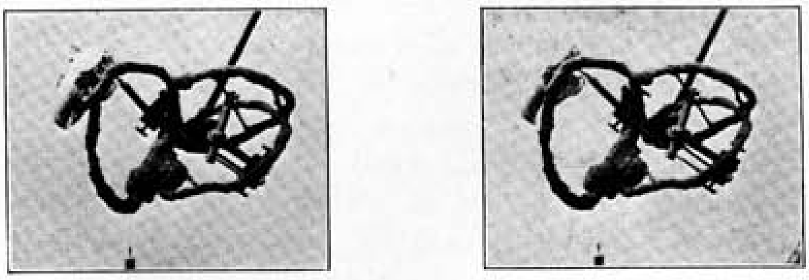


Fig. 8.

SUMMARY.

1. The above experimental results lend support to the theory of BÁRÁNY of the origin of cold-water nystagmus. The theory of BARTELS, on the other hand, conflicts with these results.

2. In the genesis of cold-water nystagmus the cooling down of the horizontal semi-circular canal plays the principal part; however, in the majority of cases some influence (though little) is also to be assigned to the vertical semicircular canals.

3. Earlier inquiries by MAGNUS and DE KLEYN have demonstrated a distinct cooling of the labyrinth-walls in cats, on douching the meatus with cold water.

4. Compensatory eye-positions should be taken into account when cold-water nystagmus with various positions of the head in space is observed.

Palaeontology. — "*Quelques insectes de l'Aquitanien* DE ROTT, *Sept-Monts* (*Prusse rhénane*)." By Dr. FERNAND MEUNIER. (Communicated by Prof. K. MARTIN).

(Communicated at the meeting of January 31, 1920).

La faunule entomologique décrite dans ce travail est assez variée. Elle fait suite à des travaux antérieurs, elle signale de nouvelles formes, complète ou rectifie, s'il y a lieu, les observations de HEYDEN, ou formule quelques remarques relatives aux anciennes descriptions de GERMAR.

Dans le monde des Coléoptères, relatons des empreintes bien conservées: Anomala tumulata HEYD., beau Melolonthidae et Stenus scribai HEYD., gracieux petit Staphilinidae. Une aile de Trichoptère ou Phryganien appartient au nouveau genre Ulmeriella. Parmi les insectes métaboles mentionnons la présence, à ROTT, d'intéressants hyménoptères Apides des genres Andrena et Eucera et de minuscules Terebrantia des genres Bracon et Cryptus. D'autres métaboles ne sont pas moins curieux à connaître. Citons d'abord l'empreinte et la contre-empreinte d'un frêle Mycetophilide, ou diptère fungicole Macquart, Boletina philhydra HEYD., espèce si soigneusement décrite par le paléontologiste rhénan; ensuite, un Empide, Empis melia HEYD., dont le dessin du réseau des veines des ailes (nervures) manque d'exactitude et nécessite un complément de diagnose.

Si les Bibionides sont fréquents sur les schistes DE ROTT, en revanche, leur état de conservation est souvent loin d'être parfaite.. Bien des formes de GERMAR et de HEYDEN resteront vraisemblablement toujours problématiques ou pour le moins douteuses. En effet, plusieurs des descriptions de ces paléontologistes manquent de précision et leurs dessins sont souvent imparfaits ou fantaisistes! Protomyia veterana HEYD. est une espèce bien critère, par sa petite taille et l'ensemble de ses caractères morphologiques. Bibio heydeni n. sp. (B. pannosus? HEYD.) et Bibio germari n. sp. (B. lignarius? GERM.) sont de si bonnes espèces, DE ROTT, qu'il est possible de les étudier très rigoureusement et de donner de bons dessins de leurs caractères les plus saillants. Les espèces de GERMAR, signalées dans son travail de 1837, ne sont données ici qu'à titre de curiosité, l'examen des insectes fossiles étant encore à cette époque tout-à-fait rudimentaire. On sait que leur étude n'a commencé à être basée sur des données rigoureuses, et n'a pris un réel essor, que depuis les remarquables travaux de feu S. H. SCUDDER.

Description des espèces.

1. Nevroptera.
Trichoptera.
Genre Ulmeriella nov. gen.
U. bauckhorni n. sp.

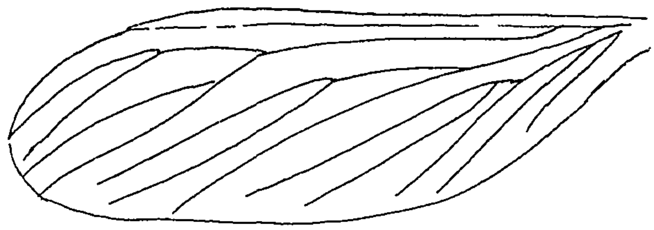


Fig. 1.

Dans des travaux antérieurs,[1]) j'ai décrit Phryganea ulmeri du Sannoisien d'Aix, en Provence, et Phryganea elegantula, de l'Aquitanien DE ROTT[2]). La collection de Monsieur BAUCKHORN, de Siegburg, renferme l'empreinte et la contre-empreinte d'un autre Trichoptère, à curieuse morphologie de la veination (nervation) des ailes.[3])

La nouvelle espèce, représentée seulement par une aile, mesure dix millimètres de longueur et 3 millimètres de largeur.

Nervure sous-costale anastomosée aux trois quarts de la longueur du bord antérieur de l'aile, Radius simple, puis offrant deux fourches; son secteur sortant au delà du milieu de la longueur de l'aile, fourche de ce secteur plus longue que la première fourche du radius; nervure médiane d'abord simple, à la base de l'aile, ensuite longuement fourchue: la branche supérieure de cette fourche l'est aussi, l'inférieure est simple. Trois nervures cubitales simples et deux[4]) nervures anales qui le sont aussi.

---

[1]) Entomolog. Mitteil. Bd. VII. N⁰. 10—12. S. 198—200 u. 3 Fig.; Berlin 1918.

[2]) Jahrb. d. preuss. geol Landesanstalt. Bd. XXXIX. S. 143. Taf. 10, fig. 1. Berlin 1918—19.

[3]) Le manque de réticulation du champ alaire et la conservation du fossile empêchent de décider avec quel genre de trichoptère le nouveau type de ROTT a le plus de rapports phylogéniques.

[4]) Cette partie de l'aile est un peu altérée par la fossilisation.

2. Coleoptera.
   Staphylinidae.
   Genre Stenus, Latr.
   Stenus scribai [1]) HEYD.
   (Palaeontograph. Bd. XV, S. 137; Taf. 22, fig. 13).

Ce Staphylien est une bonne espèce. Il a six millimètres de longueur. Tête arrondie, assez aplatie, moins large que le thorax, qui est aussi long que large. Elytres du tiers de la longueur de l'abdomen, ce dernier organe est composé de six segments. Fémurs renflés en massue, amincis à la base; tibias cylindriques, assez robustes.

Les antennes, les articles tarsaux et les ailes postérieures ne sont pas représentés sur le schiste.

Coll. BAUCKHORN. 1 spécimen.

Melolonthidae.

Rutelini.

Genre Anomala Samouell.

Anomala tumulata HEYDEN.

(Palaeontographica Bd. XV, S. 140; Taf. 23; fig. 18—19).

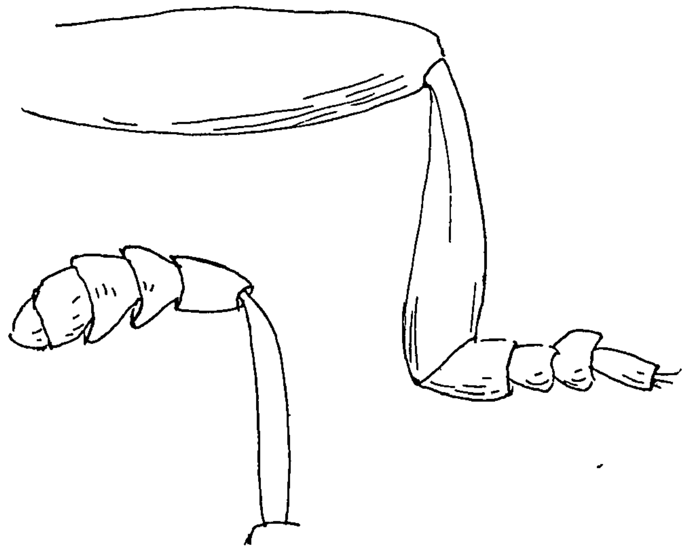


Fig. 2. Fig. 3.

Cette espèce a déjà été assez bien décrite par HEYDEN. Je complète ici la diagnose en l'accompagnant d'une reproduction phototypique, plus précise que celle de l'auteur allemand.

Tête petite et aussi large que le thorax. Antennes assez longues

1) Dans „Verhandelingen der K. Akademie van Wetenschappen van Amsterdam," p. 3. 1917, (du tiré à part) ce mot est erronément écrit comme Scribei. Cette espèce est dédiée à feu *Scriba*.

et composées de six articles: le scape cylindrique et plus long que les autres articles suivants réunis, qui sont plus larges que longs; le dernier article sub-ovoïde. Thorax (il devait être convexe) distinctement plus large que long; scutellum minuscule; élytres[1]) recouvrant les segments de l'abdomen, ovoïdes et ornés d'un sillon, très distinct, longeant parallèlement, à peu de distance, leur bord antérieur. Pattes robustes, fémurs assez dilatés et un peu plus longs que les tibias; articles tarsaux antérieurs composés de 4 articles; le 1er environ aussi long que les deux suivants réunis, le 4è plus long que le troisième; ongles des tarses courts, un peu robustes. Cavités des hanches bien développées.

Longueur du corps 6 mm.

Empreinte et contre-empreinte Coll. BAUCKHORN.

3. *Hymenoptera.*
Apidae.

Les Apides sont rarement conservés sur les schistes aquitaniens du Rhin. V. HEYDEN a signalé naguère deux espèces, assez frustes, Apis dormitans et Anthophora effosa. J'ai décrit, en 1915[2]), Apis oligocaenica du même gisement, dont il m'a été possible de donner tous les détails de la veination des ailes antérieures. L'espèce signalée brièvement, ci-dessous, me semble devoir se ranger avec les Andrénides du genre Andrena. On sait que chez les Halictes, le dernier segment dorsal de l'abdomen est orné d'un sillon longitudinal, très caractéristique, chez toutes les espèces de ce genre.

Genre Andrena Fabr.

Andrena tertiaria n.sp.

♀-Tête un peu plus large que le thorax. Antennes robustes, insérées en dessous du milieu de la face et composées de treize articles: le scape assez long, le funicule cylindrique et formé d'articles environ aussi longs que larges; le dernier article des antennes paraissant assez conique; mandibules robustes, larges, et échancrées à l'extrémité. Mésothorax convexe, scutellum semilunaire. Abdomen ovoïde, à premier segment plus développé que les suivants; le dernier assez conique au bout. Epines des tibias très appréciables; métatarse postérieur plus long que les articles 2—5 pris ensemble. Ailes aussi longues que l'abdomen, mais à veination très éffacée[3]) sur le schiste.

[1]) Ils étaient lisses, très vraisemblablement.

[2]) Zeitschrift d. deutschen geol. Gesellschaft. Bd. 67 S. 210 Taf. 21 Fig. 4; Berlin.

[3]) Elle devra être décrite après l'examen de spécimens, en meilleur état de conservation.

Longueur du corps 3 mm.

Observation: Au dire de Menge, le genre Andrena a été observé dans l'ambre de la Baltique; je ne l'ai jamais rencontré parmi plusieurs milliers d'inclusions d'insectes du succin:

Genre Eucera Latreille.

Eucera mortua u. sp.

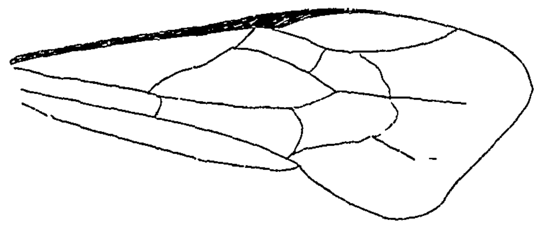


Fig. 4.

Parmi les Anthophorides fossiles, on ne connait que quelques formes tertiaires du genre Anthophora de l'Aquitanien de Rott. Von Heyden signale Anthophora effosa (Palaeontographica, Bd. X, S. 76; Taf. 10, Fig. 10). La description de cette espèce est peu précise. De Corent (France), feu E. Oustalet donne la diagnose de Anthophora gaudryi. O. Heer et d'autres paléontologistes citent plusieurs espèces des gisements d'Oeningen et de Radoboy. Le genre Anthophora a été observé dans le succin du Samland. A ma connaissance, le genre Eucera Latreille n'a jamais été remarqué sur les plaquettes de Rott. Eucera mortua est une des plus récentes trouvailles de M. l'Ingénieur Bauckhorn, de Siegburg.

Longueur de l'insecte 7 mm., longueur de l'aile 6 mm., largeur 3 mm.

L'insecte est fortement écrasé sur le schiste bitumineux. Toutefois, les caractères de la veination des ailes, des pattes et des organes copulateurs sont si bien conservés qu'il est aisé de ranger, avec certitude, cette nouvelle forme d'apide dans le genre Eucera.

♀-Tête robuste et paraissant aussi large que le thorax, qui était vraisemblablement entièrement ponctué. Pattes courtes et à tibias bien élargis pour la récolte du pollen et ornés, à leur extrémité postérieure, de calcars très distincts; articles tarsaux robustes, surtout le métatarse, qui est environ aussi long que les articles deux à cinq pris ensemble; ongles des tarses robustes, et paraissant unidentés. Abdomen ovoïde; organes copulateurs saillants et bifides à l'extrémité [1]). On sait que chez les Anthophora et les Eucera, les armures copulatrices des ♂ fournissent de bons caractères spécifiques pour le démembrement des espèces affines.

[1]) La fossilisation empêche de décrire le détail de leur structure morphologique.

48*

Ailes antérieures offrant une cellule radiale et deux cellules cubitales dont la deuxième reçoit les deux nervures recurrentes. Ailes postérieures non distinctes.

Terebrantia.

Braconidae.

Genre Bracon Fabr.

Bracon rottensis, Meun.:

Zeitschr. d. deutsch. Geol. Gesellsch. Bd. 67, S. 224—225, Taf. XXVII, fig. 2; Berlin 1915.

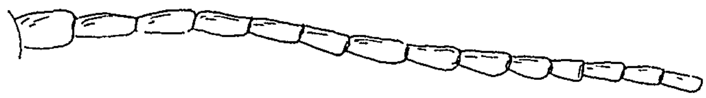


Fig. 5.

♀-Antennes assez longues, articles cylindriques et environ 3 fois aussi longs que larges. Tête un peu plus large que le thorax et tant soit peu aplatie. Scutellum du thorax bien développé. Abdomen ovoïde, les stylets de la tarière plus longs que cet organe. Pattes assez robustes (elles sont peu indiquées sur le schiste). Pour les autres caractères, voir la diagnose de 1915.

Coll. BAUCKHORN de Siegburg.

Observation. Ce Braconide s'observe, assez fréquemment, sur les plaquettes DE ROTT. La ponctuation du thorax semble avoir été comme chagrinée.

Cryptidae.

Genre Cryptus Fabr.

Cryptus sepultus n. sp.

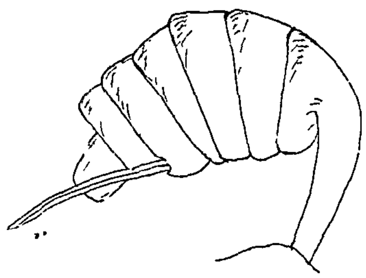


Fig. 6.

On ne connaît que peu les Cryptides fossiles. OSWALD HEER signale une espèce douteuse des schistes d'Oeningen; MENGE mentionne, sans les dérire, des Térébrants de ce genre de l'ambre de la Baltique; CHARLES BRUES a observé des Cryptines sur les plaquettes miocéniques de Florissant. Je n'en ai pas remarqué dans l'ambre sicilien ni dans le Copal subfossile de Zanzibar. L'espèce de la collection BAUCKHORN pourrait être mieux conservée; elle se classe cependant rigoureusement avec les Cryptus. Longueur de l'insecte 5 mm.

♀-Tête arrondie et aussi large que le thorax. Antennes cylindriques[1]) et paraissant être ornés d'articles rapprochés, comme c'est le cas chez les Cryptus GRAVENHORST. Dos du mésothorax et du métathorax gibbeux; ailes antérieures à nervation caractéristique des Cryptus, avec stigma très distinct et cellule radiale divisée; pas de cellule aréolaire? Ailes postérieures peu visibles. Abdomen composé de sept segments: le premier assez long, formant pétiole, un peu renflé après sa base, le deuxième segment cupuliforme; la tarière, qui est tigelliforme, sort du cinquième segment ventral; elle a environ la longueur des segments précédents. non compris le pétiole. Les fémurs et les tibias sont robustes, les articles tarsaux un peu grêles.

4. *Diptera.*

Empidae.

Genre Empis, LINNÉ.

Empis melia, HEYDEN.

(Palaeontographica, Bd XVII, S. 259—260; Taf. 45, fig. 27).

L'ambre renferme une intéressante faunule de diptères de la famille des Empidae, notamment des Empis et des Rhamphomyia. Ils doivent être rares sur les schistes de ROTT car VON HEYDEN ne décrit de ce gisement que Empis melia, espèce qui n'a que $2^1/_2$ lignes de longueur.

Le fossile, mentionné ici, a 10 millimètres de long, une longueur alaire de 8 mm. et une largeur de 3 millimètres. Je le considère comme la ♀ de cette espèce, l'exemplaire signalé par v. HEYDEN étant vraisemblablement le ♂. On sait que ches les Empis, les mâles ont la taille beaucoup moins grande que chez les femelles.

Le thorax et l'abdomen sont robustes. Les ailes offrent la veination si caractéristique des Empis mais imparfaitement figurée par VON HEYDEN. Pattes postérieures vigoureuses et courtement ciliées: les fémurs et les tibias d'égale longueur; métatarses environ aussi longs que les articles 2—5 réunis; le deuxième article à peu près aussi long que les articles trois et quatre pris ensemble, le cinquième plus court que le quatrième; ongles des tarses paraissant grêles.

Coll. BAUCKHORN. Empreinte et contre-empreinte.

Mycetophilidae.

Genre Boletina Staeger.

Boletina philhydra HEYD.

(Palaeontographica Bd. XVII, S. 246; Taf. 44, fig. 11.)

[1]) Elles sont trop altérées par la fossilisation pour décrire le détail de leur structure.

♀-Tête un peu aplatie et un peu plus large que le thorax. Ocelles indistincts. Antennes dépassant notablement la longueur du

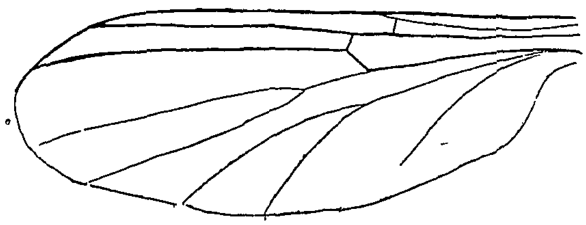


Fig. 7.

thorax; les deux premiers articles et ceux de l'extrémité peu appréciables[1]), les autres articles cylindriques et un tiers plus longs que larges. Palpes non représentés sur le schiste. Thorax un peu gibbeux et orné, aux côtés latéraux, de rares cils, écusson garni au bout de deux cils, assez longs. Le thorax devait être pourvu de trois bandes ou fascies de teinte plus sombre que le restant du thorax. Ailes plus longues que l'abdomen. Nervule assistante réunie au bord costal un peu avant le dessus de l'extrémité de la cellule humérale. Bord costal alaire peu prolongé après le cubitus (radius sec. Comstock and Needham). Pétiole de la fourche discoïdale (médiane) assez long, fourche posticale (Cubitale) distinctement plus longue que la discoïdale. Les deux nervures anales sont peu accusées. Abdomen de six? segments, finement ornés de cils courts et munis, à l'extrémité de chaque segment, d' une large bande de teinte sombre; bout de l'abdomen (oviducte) assez effilé[2]). Parties externes des tibias ornées de rares cils espacés; calcars assez longs, surtout les postérieurs. Articles tarsaux de la troisième paire de pattes longs, métatarse de cette paire plus long que les articles 2—5 réunis.

Coll. Bauckhorn. Empreinte et contre-empreinte.

♂ inconnu.

Bibionidae.

Genre Bibio Linné.

Bibio germari n. sp.

Bibio lignarius? Germar.

Bibio lignarius? Heyden.

L'espèce décrite par Germar, comme B. lignarius, est très problématique. Il en est de même de la fig. 23 des „Insecta carbonum

[1]) On ne peut compter exactement le nombre de leurs articles.

[2]) Les lamelles ne sont pas représentées sur le schiste.

fossilium" et de celle de Heyden „Palaeontographica Bd. VIII. S. 14 Taf. I fig. 4". La nouvelle forme, dont la diagnose suit, est représentée

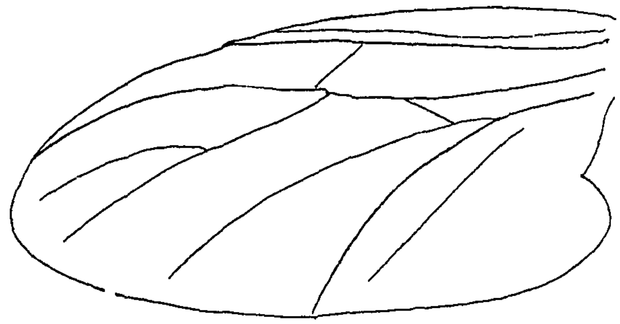


Fig. 8.

par l'empreinte et la contre-empreinte, d'une conservation remarquable.

Ce Bibionide mesure 12 mm. de longueur, l'aile a 9 mm. de long et $3^1/_2$ mm. de large.

♀-Tête assez grande, orbiculaire et un peu moins large que le thorax. Yeux bien séparés sur le front. Pipette robuste et ornée de cils courts. Cou très appréciable. Thorax assez gibbeux. Abdomen largement ovoïde, de sept segments, dont les côtés sont trés distinctement garnis de poils courts; dernier segment échancré à la partie centrale; lamelles de l'oviducte cylindriques. Ailes assez larges (elles devaient être assez enfumées[1]), la sous-costale plus rapprochée de la nervure radiale que du bord costal; le secteur du radius, qui part de la radiale avant le milieu de la longueur de l'aile, n'atteint pas l'apex de cet organe. Une petite nervule transversale relie le secteur du radius à la médiane. Cette dernière longuement fourchue; fourche de la nervure cubitale partant à peu de distance de la base de l'aile qui est pourvue de „Flügellappen" ou lobes alaires. L'aile paraît avoir deux nervures anales. Les pattes, peu représentées sur le schiste, sont ornées de courts cils.

Coll. Bauckhorn.

♂ Inconnu.

Bibio heydeni n.sp.

Cette espèce correspond peut-être à B. pannosus, forme de Bibionidae très imparfaitement décrite et figurée par Germar. L'exemplaire de la collection Bauckhorn, d'une conservation remarquable, permet d'en donner une diagnose plus précise.

[1]) Elles sont très foncées chez B. infumatus Meun. et aussi plus longues et plus larges.

♀. Ce Bibionide a 10 mm. de longueur, l'aile mesure 10 mm. de long et 3 mm. de large.

Les ailes sont un peu enfumées; la sous costale courtparalèlement à la costale et se réunit à cette dernière avant le milieu de la longueur de l'aile. Le radius s'anastomosant aussi au bord costal, à peu de distance de la sous-costale; secteur du radius un peu convexe et n'atteignant pas l'apex de l'aile; une nervule transversale oblique, dirigée vers le bout de l'aile, réunit la nervure radiale à son secteur; nervure médiane fourchue, la nervure cubitale a la base de la fourche rapprochée de la base de l'aile; une nervule transversale oblique, dirigée vers la base de l'aile, relie la base de la fourche médiane à la branche supérieure de la fourche cubitale; il y avait proba blement 2 nervures anales rapprochées. Abdomen cylindrique, assez large (il a 3 mm.), de sept segments bien arrondis aux bords latéraux et paraissant ne pas être ornés de cils, comme c'est le cas chez B. germari; Lamelles de l'oviducte petites, cylindriques.

♂ Inconnu.

Protomyia veterana HEYD. (Meun.)

(Palaeontographica Bd. XIV, S. 25, Taf. 8, fig. 4).

Par sa petite taille et sa forme trapue, cette espèce est bien reconnaissable.

Longueur de l'insecte $4^1/_2$ mm., l'aile mesure 5 mm. de long et 2 mm. de large.

♀-Tête arrondie et aussi large que le thorax. Yeux bien saillants; abdomen ovoïde et composé de sept segments. Ailes notablement plus longues que l'abdomen, assez larges; nervure sous-costale anastomosée au bord costal, un peu au delà du milieu de sa longueur. La distance entre Sc. et Ra. plus courte que celle entre Ra. et Ra_2. (secteur du radius). Ce dernier n'atteignant pas l'apex de l'aile. Pétiole de la fourche médiane environ aussi long, que la nervule unissant le secteur du radius à la nervure médiane (discoïdale). Fourche cubitale distinctement plus longue que la médiane. Dessous du dernier segment ventral comme incisé au centre. Pattes assez robustes.

Coll. BAUCKHORN. 1 spécimen

♂ Inconnu.

— *Quelques types de* GERMAR. —

L'Institut paléontologique de l'Université de Bonn possède quelques types du paléoentomologiste de Halle.

Ce sont les espèces suivantes: Buprestis carbonum, B. major, Ydsolophus insignis, Prionites umbrinus, Tenebrio effosus, Saperda lata, Silpha striatum, Alydus pristinus, Bibio xylophilus, Locusta exstincta. Sous l'influence des actions chimiques prolongées et de l'air, ces fossiles sont devenus trop frustes pour les décrire et pour en donner de bonnes reproductions phototypiques.

EXPLICATION DES FIGURES.[1] (Texte).

Fig 1. Aile antérieure de Ulmeriella bauckhorni nov. gen. n. sp.
Fig. 2. Antenne de Anomala tumulata Heyd.
Fig. 3. Articles tarsaux de ce Melolonthidae.
Fig. 4. Aile de Eucera mortua n. sp.
Fig. 5. Antenne de Bracon rottensis Meun.
Fig. 6. Abdomen de Cryptus sepultus n. sp.
Fig. 7. Aile de Boletina philhydra v. Heyd. (Meun.)
Fig 8. Aile de Bibio germari n. sp.

EXPLICATION DES PLANCHES.[2]

Fig. 1. Ulmeriella bauckhorni nov. gen. n. sp.
Fig. 2. Stenus scribai Heyden.
Fig. 3. Anomala tumulata Heyden.
Fig. 4. Andrena tertiaria n. sp.
Fig. 5. Eucera mortua n. sp.
Fig. 6. Bracon rottensis Meun. ♀
Fig. 7. Cryptus sepultus n. sp.
Fig. 8. Empis melia Heyden. ♀
Fig. 9. Boletina philhydra n. sp.
Fig. 10. Bibio germari n. sp.
Fig. 11. Bibio heydeni n. sp.
Fig. 12. Protomyia veterana Heyden.

[1]) Elles ont été faites par Mme F. MEUNIER.
[2]) Les clichés ont été exécutés, avec soin, par mon ami M. F. BASTIN d'Anvers.

Zoology. — "*The wing-design of Chaerocampinae*". By Prof. J. F. VAN BEMMELEN.

(Communicated at the meeting of October 25, 1919).

In a monograph, which is now being published as a supplement to Zeitschrift für Wissenschaftliche Insectenbiologie von CHR. SCHRÖDER in Husum, and of which I received the first part a few months ago, Dr. P. DENSO, the author of Palaearctic Sphingides in SEITZ' Macrolepidoptera, begins a description of the lepidopterous hybrids that have hitherto got known, with considerations on the wing-design of the species of *Celerio*. On page 1 he says about this: "Thorough investigations and theoretical considerations, which it would lead me too far astray to reconsider here, clearly show that the markings (and hues) of all *Celerio*-moths may easily and without constraint be derived from a primitive form, which only very slightly deviates from the pattern still found in the oldest species of Celerio, viz. *zygophylli* O., or likewise in *lineata*, when we only abstract from the white striation of the wing-veins. It must be mentioned here, that the original design of the species of *Celerio* is nearly related to that of the more closely-connected species of *Pergesa*".

I deplore that DENSO did not think fit to publish in detail his "thorough investigations and theoretical considerations" on the phylogenetic interrelations between the different species of *Celerio*.

For now we are obliged to deduce the grounds for his assertion "that *zygophylli* and *lineata* have to be considered as the (phylogenetically) oldest species" from a few remarks, which must be picked up here and there in the course of his paper.

Such being the case, I prefer first to expose my own views independently of DENSO's considerations and afterwards to discuss his deductions.

In my eyes the only way to acquire a trustworthy insight into the wing-design of *Celerio*-species, is to compare it with that of other genera of Sphingids, especially Chaerocampinae. When keeping this course, it becomes evident that their colour-pattern is a highly modified variation of the general ground-design of Heterocera-wings, due to reduction and obliteration of the general primitive set of seven transverse bars, by the influence of the *V*-diagonal-motive (this being the name which in my foregoing paper on the wing-pattern of Saturnidae I gave to the system of linear markings running obliquely across the wing from tip to root). Consequently in my opinion the most original pattern must be looked for in those Chaerocampinae that show the fewest traces of this influence of the *V*-diagonal on the transverse bars. Now it is evident that this does not at all occur in *zygophylli* and *lineata,* but on the contrary in *Pergesa* (*Deilephila, Metopsilus*) *porcellus,* and better still in *Berutana* (*Metopsilus*) *syriaca.* In this latter the forewing shows a set of transverse bars which remarkably agree with that of *Smerinthus populi,* though they do not to any notable degree pass over upon the hindwing.

The *V*-diagonal is only very slightly indicated at the apex of the wing in the shape of the foremost external triagonal spot, which extends from the wing-tip along the front-border, and shows the form of a dark-brown threesided blotch, growing fainter and of lighter hue from before backward. It remains separated from the convex blotch along the external margin by a narrow space, which is occupied by the well-known oblique white apical stripe, that is seen in so many different forms of Lepidoptera.

In the same way the posterior triagonal spot is well-developed, but remains separated from the anterior one by three internervural spaces, containing only faint traces of dark marginal spots.

Bar II is complete and well-marked, III on the contrary hardly visible, IV is a broad dark band, imperceptibly passing at its external side into the area where III would have occurred, had it been visible, but very sharply traced at its internal border. V is rather sharp, but does not reach the back-margin of the wing. VI is just indicated by a faint trace, VII on the contrary is invisible.

On the upper side of the hindwing a broad marginal seam and an obscuration of the root-field are the only traces of the pattern.

On the inferior surface the common heterocerous pattern occurs, viz. a design which is the same for front- and backwing, and betrays clear traces of reduction, when it is compared to that of the upper-side. For it only consists of a well-defined marginal range of (coalesced) spots, inwardly bordered by an irregular zig-zag-line, and moreover of the bars II and III, represented by brown lines on a lighter ground.

The wing-design of *porcellus* may rather easily be derived from that of *syriaca,* and this deduction presents a certain amount of probability, as the complete set of seven transverse bars is clearly discernible along the front-margin of the forewing. Three of these bars: the outward or distal ones (I, II and III) reach the hind-margin.

When I call this pattern an original one, this expression should not be taken in the absolute sense generally connected with it.

Precisely in the case of *porcellus*, it can be proved in a very striking way, that this would be inappropriate.

For not only the pattern of the seven transverse bars is represented on the fore-wing, but that of the *V*-diagonal as well. Or expressing it in other words, we may assert that the pattern of *porcellus* could be obtained by combination of that of *Smerinthus populi* with that of *Elpenor gallii* and *euphorbiae*, of course under omission or reduction of certain parts of each.

The best proof for this assertion can be given by superadding the wing-patterns of the above-mentioned species to that of *porcellus*, or, otherwise, by marking with a darker hue those elements of foreign wing-designs in the *porcellus*-pattern that can be discovered in it.

Bar I, otherwise called the marginal seam, shows in *porcellus* the usual type of an irregularly indentated, wine-red streak, which is characteristic of Chaerocampinae. It begins at the wing-tip with the above-mentioned oblique white stripe, which likewise is of so frequent occurrence among Sphingides, and can be considered as the outmost fragment of the *V*-diagonal.

Bar II begins with a tolerably distinct, rather dark blotch, in the wine-red streak along the front-margin, but gets much fainter as soon as it enters the yellow-brown central area of the wing, which it traverses in a well-marked inward curve.

Bar III likewise begins at the front-margin with a double-blotch, but becomes a single band when entering the yellow area, and at the same time gets into contact with the discoidal spot, which itself may be considered as a remnant of Bar IV. Furtheron B. III runs parallel to II, both being dislocated a little in the direction of the wing-root.

V and VI are represented by a pair of small, greenish-brown stripes in the red field, VII can confusedly be traced in the brownish root-area.

On the upper side of the hindwing the only point of similarity with the forewing is formed by the wine-red marginal seam, but the underside once more proves, that also in *porcellus* well-marked remnants of the transverse bars occur in the shape of dark spots and stripes on the light-yellow and rose-red fond. As in most other cases these are especially well-marked along the front-margin.

At the underside the similarity between fore- and hindwing is again much more pronounced than on the opposite surface, the design on the firstnamed wing being more reduced than on the last, especially as regards the root-field.

The same red, violet and greenish-golden-brown hues that decorate *porcellus*, are found back in *elpenor*, a great superficial similarity resulting from this, which finds its expression in the popular names.

But in the pattern an important difference prevails, for in *elpenor* the traces of transverse bars along the front-margin are almost completely absent, while on the contrary the *V*-diagonal-design is strongly expressed, though in fact it reaches the back-margin of the wing in its more distal part, and therefore deviates in a lower degree from the original transverse direction of the primary bars than is the case with other Chaerocampinae.

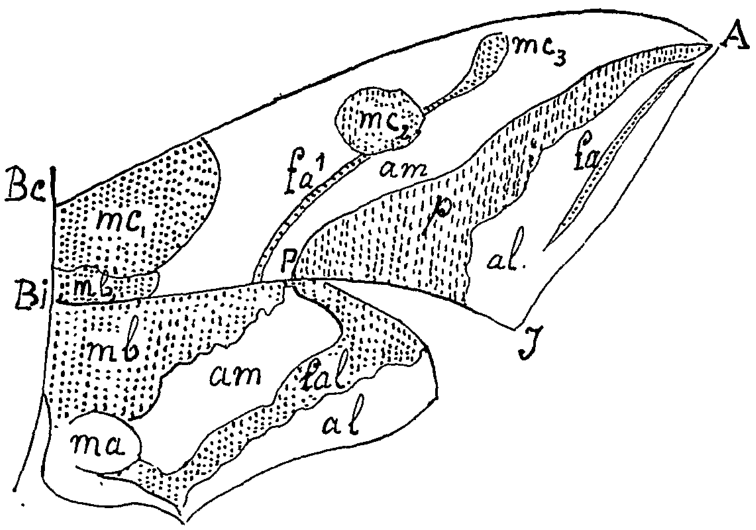


Fig. 1. (after Denso).

Denso pays special attention to this vicariating relation between the two parts of the back-margin of different species of Celerio, into which it is divided by the above-mentioned oblique line, which, starting from the wing-tip, forms the outer border of the light middle-field. He arranges these species in a series, beginning with *lineata*, where the meeting-point of this line with the back-margin lies farthest towards the proximal side, and ending with *nicaea*, which in its more distal position of this point more or less agrees with *elpenor*.

As far as I feel able to understand his views, he seems, for the just-mentioned reason, to consider *lineata* as more original than *nicaea*.

According to my conviction the relation between these two species is precisely the opposite one.

To me it just seems remarkable that Denso, when speaking of another detail of the wing-design, which he remarked in a few specimens of *nicaea* only, comes to a conclusion that exactly agrees with my views. For DENSO considers the occurrence of a dark line over the middle part of the wing, which appears from time to time (called by him fa_1 and running parallel to his median bar *am*) as an atavistic phenomenon. Now this line can scarcely be anything else than Bar III of *porcellus*, and therefore in my opinion may really be considered as the reappearance of an element of the original design.

In truth this unexplainable confusion and contradiction in his views can be remarked in different passages of DENSO's contentions: e.g. when he says in describing the *lineata*-design: "my investigations led me to assume, that *lineata* and certain specimens of *zygophylli* show a design, that very nearly approaches the original *Celerio*-pattern. In fact, when drawing the contours of the *lineata*-design, they completely include the elements of the pattern of all remaining species of *Celerio*, these latter therefore appearing to be due to the more or less far-reaching reduction of the original design. This may be demonstrated by Fig. 2".

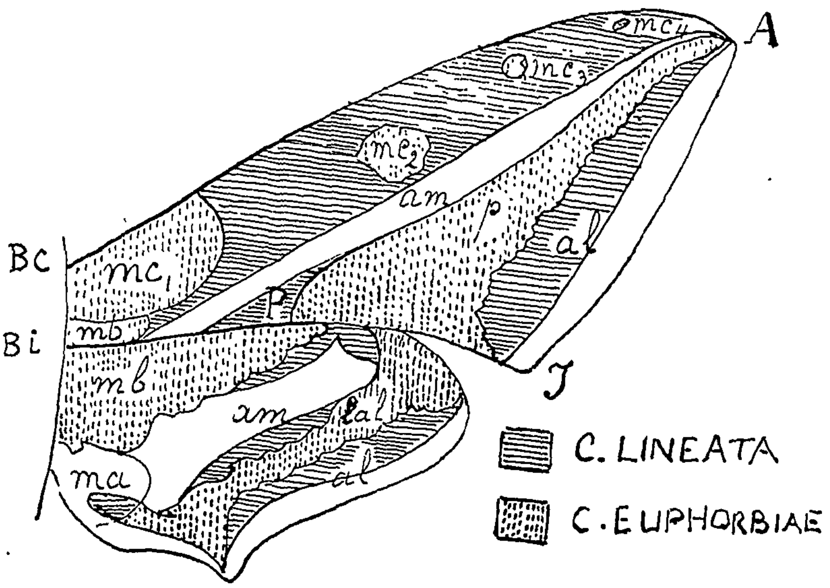


Fig. 2. (Copied after DENSO).

Now in this figure we remark, how *lineata*, besides showing the *V*-diagonal, possesses only a single vestige of a bar (l) along the external wing-margin, and therefore next to nothing of the original design, while in *euphorbiae* on the contrary the remains of at least four transversal bars are present along the front-margin, though in truth only in the shape of isolated blotches. Were we obliged to share DENSO's views, we should have to assume that self-colour is the more primitive condition of wing-coloration, all patterns taking their origin from it by dissociation of the homogeneous hue into spots and bars.

This really seems DENSO's opinion, notwithstanding a few lines before he asserts: "As to the underside of the wings, we also here find, that progression in phylogenetic development always goes hand in hand with an increasing loss of elements of the pattern. *Lineata* is richest in details, *gallii* less so, *zygophylli* the same, while *euphorbiae* and *nicaea* show the fewest components of the pattern".

I see no need here to remonstrate that this assertion can as well be applied to the upper side of forms like *C. lineata* and *D. elpenor,* in comparison respectively with *C. euphorbiae* and *D. porcellus.*

Neither can I agree with DENSO's contentions (p. 5) about the "manifes-

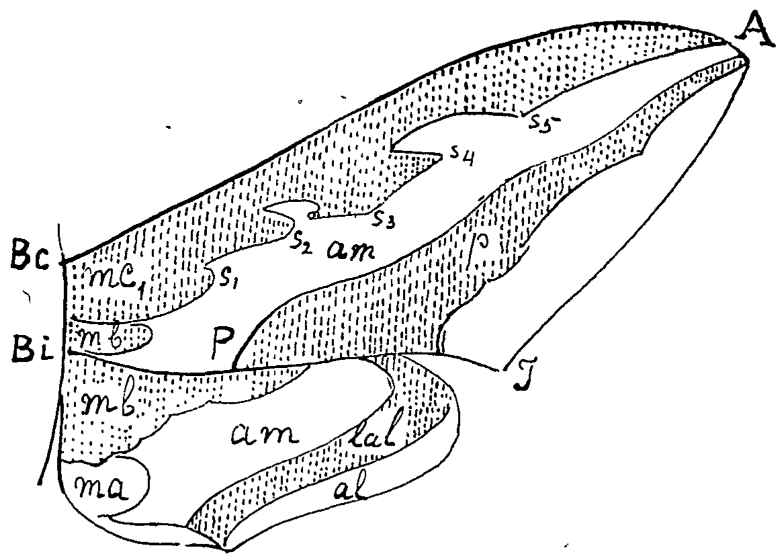


Fig. 3. (after DENSO).

tation of atavistic characters". He writes: "Very often we remark in pure species, e.g. *gallii* or *euphorbiae,* a dark, indistinct line, starting at the wing-tip near to the transverse bar *p,* and running parallel to the distal border, across the marginal field. In most cases this line is rather short, and disappears nearly halfway between apex and hind-corner; rarely it attains this corner and there joins with the bar *p.* It takes exactly the same course as does the distal bordering of bar *p* in *lineata.* Without doubt we here meet with an atavistic feature; it is nothing but the old borderline of the *p*-bar. In vain therefore should we look out for it in the *lineata*-group, while in *zygophylli* it will only occur on rare occasions and in a weak condition, as the regression of *p* has only just begun. *Gallii* often shows this line, *euphorbiae* more rarely, *nicaea* extremely seldom, which can be easily understood, as the two latter, being relatively young species, have already long since lost this borderline of *p*".

Comparing the wing-designs of *gallii, zygophylli* and *livornica* (as figured in SEITZ, comp. vol. II Taf. 41. d.) I come to the conclusion that in the second of these species the line in question, *fa*, is present without exception, but that it has been dislocated a little towards the internal side, and moreover that this line is also present in *Pergesa oldenhamii* and *japonica*, and likewise in *Celerio boisduvalii* and *minor*, though its shade may differ in saturation.

For the rest I cannot well understand, why precisely this line should be of special atavistic importance; though on the other side it is of course beyond discussion that it has originated by the coalescence of the external row of spots (Bar I), which runs parallel to and in the immediate neighbourhood of the external margin of the wings. Nor am I able to see why there still should exist differences in the degree of atavism between the several transverse striae, which, according to DENSO occur from time to time as variations in the different species. DENSO himself seems inclined to accept this difference, for he says:

"In contrast to the line *fa*, which forms a feature restricted to forms within the limits of the genus *Celerio*, another atavistic line goes back to a far wider plan, viz. to elements of design also appearing in the genus *Pergesa*. I mean a dark line fa_1, which, beginning at the costal spot mc_3, runs along the costal zone *ac* and parallel to the proximal margin of *p*, towards the posterior wing-border. Very often this line forms a connecting link between the spots mc_3 and mc_2. It only occurs in specimens, where the tendency to dissolution of the costal zone into separate costal-spots shows itself, or in which this dissolution has already been achieved, e.g. *zygophylli, vespertilio,* the *euphorbiae*-group and *nicaea*. Never on the contrary does it appear in *hippophaes*, *gallii* and *lineata*".

Judging from *vespertilio,* when compared with *askoldensis* and *mellus*, the line in discussion must be the one I designed as Bar III, but which here must have blended with IV, traces of this line being present not only in some, but in all specimens of *dahli* and *euphorbiae*, near to the posterior margin of the wing.

In accordance with these remarks, it is self-evident that my views about the wing-markings of the *euphorbiae*-group are absolutely in contradiction with those of DENSO. For this author says: "In *C. euphorbiae L.* the process of dissolution of the original Celerio-design has proceeded very far already".

I feel convinced that in this case a process of dissolution is out of the question, but that quite on the contrary we can still discover the last traces of the transversal bars along the front-margin of the wing, in the shape of isolated spots, the posterior and distal part of the wing meanwhile remaining under the dominion of the *V*-diagonal-pattern. Yet I am willing to admit, that the reduction of the transverse rows of spots to three or four irregular blotches along the front-margin (called by DENSO costal-spots) and the gradual diminution in size of these spots towards the wing-tip undoubtedly are in connection with the course of the *V*-diagonal, and that the entire set of these three or four blotches responds to the dark anterior marginal field of *C. lineata.* This latter area however I consider as a blending of those four blotches, i.e. as partial self-coloration, leading to uniformity of hue of the whole anterior marginal field. The justification for this way of regarding the question, I see in conditions as found in *C. gallii,* where the blotches, though in connection with each other, in such a way that the front border of the wing is entirely and uniformly dark-coloured, yet are perfectly distinct in their original extension by the occurrence of arcuate incisions from the side of the light diagonal middle *V*-bar.

That DENSO looks at this condition from an opposite point of view is revealed by his expression: "*Gallii* possesses a broad costal margin, in which the (light) groundcolour has *intruded,* (the italics are mine), especially from three points of the middle-area *am*".

The same considerations can be applied to the dark triangular area, which forms the postero-external border of the light diagonal bar, and which DENSO calls *p*. When speaking of *C. euphorbiae,* he remarks about this bar: "The proximal limit of the transversal bar *p*, in its hinder part, which touches the back margin of the wing, has been *removed* towards the posterior wing-angle". According to my view, it has remained at its original place.

Though his remarks about *gallii* are restricted to the words: "The bar *p* is broader than in *euphorbiae.* Its terminal point *P* is situated more towards the base of the wing", he declares in a preceding passage: "Starting from the distal border" (of the light median area) "the marginal coloration of *al* increases in extension at the cost of *p*, and moreover *am* broadens along the posterior wing-margin, thus causing the proximal limit of *p* to stand more perpendicularly to that margin".

Also in this regard therefore, DENSO's views are diametrically opposed to mine.

And yet I could see a possibility that DENSO's view of the matter might after all prove right. For this would be the case, when we had to surmise, that in *gallii,* and still more in *euphorbiae,* the presence of the spots along the anterior wing-margin was due to reversion of the archaic pattern, i.e. to atavism. We then should be obliged to imagine that in the pattern of *lineata,* the uniform dark anterior region of the diagonal-pattern, itself derived from the coalescence of the anterior parts of the original seven transversal rows of spots, had again been solved into a certain number of free blotches. The fact that this number is lower than seven, renders some probalility to the supposition that we have here to do with a secondary dissociation of an originally coherent longitudinal bar along the entire anterior wing-border. But according to my view the primary cause of this dissociation may be seen in the hereditary presence of the tendency to the formation of isolated marginal spots, belonging to the ancient pattern of transverse rows of maculae, which is common to all Heterocera.

When trying to analyse in this same way the complicated pattern of the upper-side of the forewings of *Deilephila* (*Daphnis*) *nerii,* we come to the conclusion that without constraint derivatives of all the seven transversal bands can be recognized in the alternately dark and light areas along the anterior wing-margin, but that only one of them, viz. V, runs on unbroken to the posterior margin, VI nearly doing as much, as it only becomes crossed by the white external seam of the dark root-field. The disturbances in the rest of the transverse bars may for the greater part be attributed to the well-known influence of the *V*- and the *Λ*-diagonal-design. The first manifests itself in the same manner as in *euphorbiae, gallii* etc., but in *nerii* only fragments of the light median bar of the remaining Deilephilas can be discovered. In the first place we remark the light apical marking, strongly contrasting to the extremely dark anterior segment of bar I. Then comes the white curved stripe in the middle of the posterior margin, abutting towards the median side against a peculiarly dark hinder part of a transverse bar (probably a fragment of IV) and which in its forward zig-zag-course gets twice abruptly broken. I presume that this characteristic white zig-zag-line represents part of the distal border of the triangular light central part, which broadens towards the hind margin and is so characteristic of Chaerocampinae.

For the rest this light central field is only represented next to the root-field by its most proximal part running along the posterior wing-border. This part narrows and describes a convex curve, thereby passing into the area of bar VI, and reaching the anterior margin. In the same way the light colour-party at the external border of V advances distally towards the anterior wing-margin, and so comes in contact with the fore-end of a still lighter bar, which begins in the area of III at the said margin, but takes such a sinuous course in a postero-external direction, that the dominion of III so to say curves up to that of II.

A similar feature can likewise be observed in another Sphingid, whose forewing-pattern agrees with that of *nerii* in a remarkable number of points, viz. *Dillina tiliae.* Here the feature in discussion is seen in the anterior part of the external border-line of the dark central field, by which the forewing is so characteristically divided into a proximal and a distal light area, and which itself is broken up (either completely or nearly so) into a larger anterior and a smaller posterior portion by a constriction along the course of the second cubital vein. This constriction corresponds in position and character to the above-mentioned white zig-zag-line of *nerii.*

That this explanation of the forewing-pattern of *nerii* is well founded, becomes especially evident when we compare it to that of nearly-related species, e.g. *hypothous* (MOORE, Lepidoptera Ceylon, Pl. 83; CRAMER, Pap. Exot. III pl. 285 D; Seitz, X 63a), *layardi* (MOORE, Pl. 81; Seitz 63a[3]), *protrudens* (Novara Exp. Zool. II, 2, Taf. LXXVI, 7; Seitz X, 63b[2]), *angustans* (Nov. Exp. Zool. Bd. II 2), *placida* (Seitz, X 63a[4]).

But as the most remarkable patterns in regard to this feature I consider those of *omissa* and its congeners (*miskini, anceus, sericeus, cunera*), because here parts of the *nerii*-pattern are so to say projected on that of *Smerinthus populi,* the latter appearing as if it were visible by transparency beneath the first.

Groningen, October 1919.

Physiology. — *"On Serum-lipochrome"*. (First part). By Prof. A. A. HIJMANS VAN DEN BERGH and Dr. P. MULLER.

(Communicated at the meeting at December 27, 1919).

In a previous investigation [1]) I gave evidence to show that the normal human blood-serum contains two pigments: bilirubin and a lipochrome. Prior to this, opinions about the materials that yield the colour of normal serum, were contradictory and confused. The French clinician GILBERT e.g. believed that the colour of the human bloodserum was due exclusively to bile-pigment and that it never contained lutein (lipochrome). The Italian researcher ZOJA, on the other hand, asserted that bilirubin is never present in the serum of normal man, but that the yellow colour is owing to lutein. We suspect these clashing opinions to have arisen from unsuitable methods of separating the pigments. It is especially the extraction of a protein-rich fluid like blood-serum, by shaking with ether and similar solvents, that yields unsatisfactory and differing results. When, however, we precipitate the serum by an appropriate amount of alcohol most of the bilirubin will pass over into this fluid, while from the ensuing protein-precipitate the lipochrome can be readily extracted with ether. In this way we are enabled to separate both pigments from the serum. After having watched the fate of bilirubin under various circumstances [2]), an inquiry on lipochrome naturally suggested itself to us.

Yellow pigments, which for the present may conveniently be termed "lipochromes", have until recently been investigated chiefly by botanists [3]). STOKES [4]) and SORBY [5]) discovered that in the green parts of plants besides chlorophyl numerous yellow pigments are to be found.

Prior to their findings carotin had already been separated from

[1]) HIJMANS VAN DEN BERGH u. SNAPPER. Deutsch. Arch. f. klin. Mediz. **110**, 540, 1913.

[2]) HIJMANS VAN DEN BERGH. Der Gallenfarbstoff im Blute. Leiden 1914.

[3]) For the literature see T. TAMMES. Flora **87**, 205, 1900, and C. v. WISSELINGH, Flora **107**, 371, 1915.

[4]) STOKES. Proc. Roy. Soc., **13**, 144, 1864.

[5]) SORBY. Ibid. **21**, 442, 1873.

Daucus carota, while ARNAUD showed in 1885 [1]) that a yellow pigment in the green parts of plants is identical with the carotin from carrots. ARNAUD made an extensive study of carotin and established that it is an unsaturated autoxydable carbohydrate. His analysis and further inquiries produced the empiric formula $C_{26}H_{38}$. Ever since many inquiries into these pigments have been undertaken. Latterly they have received WILLSTÄTTER's [2]) attention. He established in accordance with the assumption of previous inquirers that plants contain different pigments which — as BORODIN [3]) had observed — may be divided into two large groups. The pigments of the first group, to which carotin belongs, are rather easy to dissolve in benzene, hardly so in alcohol. The second group is represented by xanthophyll, which can readily be dissolved in alcohol, less readily in benzene. Either of these substances could be obtained in pure, crystalline condition. The elementary analysis, the determinations of molecular weight, and the analysis of iodine-addition products yielded the formula $C_{40}H_{56}$ for carotin and $C_{40}H_{56}O_2$ for xanthophyll. WILLSTÄTTER also corroborated that the two carbohydrates are highly unsaturated and autoxydable. They are very sensitive to acids, but are not attacked by alkali.

Also in animal products, particularly in the egg-yolk, in the serum of animals and men, former observers have found lipochrome pigments (KRUKENBERG, THUDICUM, SCHUNCK, KÜHNE) and have published interesting communications about them. They usually term them luteins. To WILLSTÄTTER and his co-workers we are indebted for considerable advance in this respect. It appeared that also the animal lipochromes or carotinoids may be divided into two groups according to their relative solubility in benzene and alcohol.

WILLSTÄTTER's pupil ESCHER managed to separate pure carotin from the corpus luteum of the cow. WILLSTÄTTER, in conjunction with ESCHER, has obtained lutein from the egg-yolk, and established that it is quite identical with the xanthophyll from plants, with a difference only in the melting-point.

Some years ago a series of papers appeared from the American Algricultural-Chemist PALMER [4]). We shall frequently refer to this work, but it may be expedient to state here what we deem to be the chief result of PALMER's work. The American researcher comes

---

1) A. ARNAUD. C. R. Ac. Sc. **100**, 751, 1885, and **102**, 1119, 1886.

2) WILLSTÄTTER u. STOLL, Untersuch. über Chlorophyll. Berlin 1913.

3) Quoted from WILLSTÄTTER.

4) PALMER, The Journ. of biolog. Chem. 1915—1919.

e.g. to the conclusion that the yellow pigment of the body-fat, milk-fat, and blood-serum of the cow is identical with carotin, whereas the yellow colouring matter of the egg-yolk, body-fat and blood-serum of fowls corresponds with xanthophyll. He also demonstrated that these pigments in animals are of alimentary origin. Finally that in the cow's intestine carotin is resorbed well nigh exclusively, whereas in the fowl's intestinal canal only xanthophyll is resorbed almost to the exclusion of other pigments.

Our prolonged investigation of lipochrome in the serum of the human blood led us to study some of its qualities more in detail.

As already stated in the paper referred to above[1]), we had observed that the lipochrome pigments behave differently in man and in the cow towards ethylalcohol. When we precipitate cow's serum with 2 vol. of alcohol and when centrifugalizing the precipitate, the lipochrome can be extracted from it with ether. The cow's pigment, then, is nearly *insoluble* in 64 perc. alcohol. When we submit human serum to the same process, we generally fail to extract pigment with ether from the protein precipitate; it can be obtained when we precipitate 1 vol. of human serum by an equal volume of alcohol. It appears then that the human lipochrome is often *soluble* in 64 %, but invariably insoluble in 48 perc. of alcohol.

This different behaviour of lipochromes towards 64 % alcohol we purposed to examine.

First of all we ascertained in which of the two groups of carotinoids, as established by WILLSTÄTTER, the pigments from carrots, egg-yolk, fowl's serum, cow's serum and human serum, have to be placed.

The materials to be examined are treated with 96 % ethylalcohol; subsequently with petroleum-ether. By adding an appropriate quantity of water all the lipochrome passes over to the petroleum-ether, which floats on the surface as a limpid, gold-yellow layer.

This layer is pipetted off. It contains besides the pigment, also fats, cholesterin and presumably still other substances. The fats are removed by saponification, the cholesterin is precipitated by digitonin. What remains is an incompletely purified solution of lipochrome in petroleum-ether. When adding to this fluid methylalcohol (90 % or stronger) the pigment will pass over to the lower methylalcohol layer, if we have to do with xanthophyll, to the benzene layer in the case of carotin. Following WILLSTÄTTER's example we always used methyl-alcohol for this process; ethylalcohol proved to be

[1]) Deutsch. Arch. f. klin. Mediz., loc. cit.

unserviceable. Neither did we deem it suitable to distinguish between the two groups of carotinoids by their spectroscopic properties. The same holds in TSWETT's method. He filters solutions of the pigment through a column of calcium-carbonate. The carotin will then pass through without being adsorbed, while the xanthophyll is left behind. TSWETT applies this method to separate the different sorts of xanthophyll, that according to him exist. However, since we only look for a separation between the two main groups, we have confined ourselves to the method of distribution between methylalcohol and benzene.

It was apparent from our results, as PALMER had already shown, that cow's serum contained only carotin [1]) egg-yolk and fowl's serum only xanthophyll. Human serum yielded results varying with the individual from which it was drawn. It usually contained a mixture of carotin and xanthophyll, carotin mostly preponderating. In only one case xanthophyll predominated slightly; in a few cases — very rare though — the amounts of xanthophyll and carotin were nearly equal. Not unfrequently did we find that along with carotin there was only very little xanthophyll.

In order to determine the solubility in 64 perc. alcohol we have extracted carrots, blood-serum and egg-yolk with ether after treatment with alcohol.

The ether was pipetted off and evaporated to dryness in fractionating flasks, in vacuo, at room-temperature (or gentle heating on a waterbath).

Subsequently 64 perc. ethylalcohol was added in each of the flasks and shaken up rapidly. The colour of the fluid was taken for the index of solubility.

It then appeared:

	Colour of 64 perc. alcohol.
Carrots	+ + + +
Eggyolk	0
Fowl's serum . .	+ + +
Cow's serum . . .	faint
Human serum . .	+ + or + + +

This makes it clear that whereas carotin obtained from carrots is easily soluble in 64 perc. ethylalcohol, cow's serum is almost insoluble.

The pigment from human serum (a mixture of carotin and xanthophyll, carotin most) is sparingly soluble.

---

1) PALMER rightly observes that also some very small quantities of xanthophyll may occur, which will come forth only when working with large quantities of serum.

What strikes us most is the different behaviour of the pigment of egg-yolk and of fowl's serum (xanthophyll), the former being insoluble, the latter readily soluble.

These properties are no doubt partly due to the presence of substances accompanying the pigments. When purifying the egg-yolk-xanthophyll, which is almost insoluble in 64 perc. alcohol, by saponifying and removing the fats in the ether-solution, the solubility increases. Something like it, but in a smaller degree, was witnessed in cow's serum. From this it is evident that some properties of lipochromes are markedly influenced by the presence of other substances. We have to keep this in mind when studying the lipochromes of blood-serum and other human products, since in this case it is impossible to examine them in a pure state. This will be easily understood when considering that in clinical inquiries the investigator has seldom more than a few cubic centimeters at his disposal, while WILLSTÄTTER and ESCHER used 6000 eggs to prepare 2.6 grms. of pure yolkpigment and ESCHER required 10.000 cow-ovaries to produce 0.45 grm. of carotin.

As has been stated above our first investigation showed us that in order to prepare lipochrome from bloodserum it is necessary to precipitate it with alcohol and after this to extract the precipitate with ether. Extraction by shaking the serum with different solvents yielded varying and generally bad results.

KRUKENBERG had also noticed that lipochrome can be extracted from cow's serum only with amylacohol. He insists that other means of extraction such as chloroform, ether, methyl-, ethylalcohol are not suitable.

A more extensive inquiry in this direction revealed that no trace of pigment could ever be obtained with petroleum ether from cow's serum, human serum or fowl's serum.

With ether we most often obtained no pigment from these three sera, at other times only little.

This result does not quite tally with the experience of PALMER, who also gave his attention to this point. He records that from cow's blood the pigment can never be extracted with ether, from fowl's blood always. To what this difference is due, we have not been able to make out. Anyhow, it is certain that 8 specimens of fowl's serum, examined by us, did not yield a pigment even after being rapidly shaken with pure ether; whereas from two other specimens, treated in a similar way, a rather considerable amount of pigment could be obtained.

When shaking cow's serum with amylalcohol, a trace of pigment is transmitted to the alcohol. At the same time, however, the lowermost layer is far more decolourized than the amount of pigment, which has passed over to the amylalcohol, could lead us to expect. Chloroform does not extract pigment from cow's serum.

On the other hand the pigment can be easily extracted from egg-yolk with ether. From which it may be concluded, that in fowl's blood-serum pigment occurs in a condition or in a combination different from the pigment in the egg.

To benzene again the egg-yolk yields nothing. When we boil the yolk, a considerable amount can be extracted with benzene.

From carrots a pigment can be readily obtained with ether, with benzene and likewise with alcohol.

Pigment may be extracted from finely ground maize with ether. More easily when the maize is boiled with alcohol. It is well to watch the behaviour of cow's serum towards ether and benzene. By shaking it with one of the two solvents, the pigment cannot be liberated. When, however, we add ethylalcohol to the serum and then ether, and subsequently separate it by a small quantity of water, the pigment will pass quantitatively to the upper (ether) layer. We have pointed out before that this behaviour of cow's serum had also struck PALMER. He believed (although we could not corroborate it) that fowl's serum invariably yields its pigment to ether. This induced PALMER to assume that fowl's lipochrome occurs in free state in the serum, whereas in cow's serum the lipochrome is present in combination with albumen. Of this compound termed by him caroto-albumen he has endeavoured to establish some properties.

We doubt whether the pigment occurs in cow's serum in combination with albumen. If this were so, we might expect that the pigment could be set free not only with alcohol, but also with other solvents that denature protein, so that it could be extracted with ether.

However, when salting out the albumen of the serum with ammonium-sulphate and subsequently treating it with ether or benzene, the pigment will not be taken up by it.

Neither is this the case when extracting with ether or benzene after precipitating it by boiling. Still the protein, as may be expected, is denatured more by boiling than by precipitation with alcohol, since the latter reaction is initially reversible, the former is not.

There is one more reason for our assertion that the "liberating" action of alcohol on lipochrome, which renders it fit to be extracted, rests upon something else then the decomposition of the protein-molecule. Whereas e.g. the denaturing of protein by alcohol is comparatively a slow process, the action on serum, which renders lipochrome accessible for ether or benzene, manifests itself instantly.

Lipochrome, then, is freed during the first phase of the action of alcohol on the protein (precipitation) and prior to the second phase (denaturation).

I doubt, therefore, whether the liberation of lipochrome by alcohol is due to a decomposition of a supposed albuminous compound. That this alcoholic action may occur, without any question about the destruction of an albuminous compound — as in lipochrome solutions which are free from protein — is borne out by the following experiment.

A concentrated solution of carotin is prepared by extracting finely rubbed carrots with a mixture of alcohol and ether. The ether is removed, after which a gold-yellow, alcoholic solution of carotin is left behind. When this solution is diluted with water, so that the alcohol-content in the mixture is very low, the carotin will not be precipitated, nevertheless the solution keeps clear. By evaporation in vacuo (if required with gentle heating in the waterbath) the rests of alcohol, still left behind, are removed as much as possible. The carotin does not precipitate then either, but a solution remains in which no solid particles are visible. It passes through the filter unchanged. Solutions of a higher concentration are opalescent, those of lower concentration are clear. The investigation Prof. KRUYT kindly performed confirms that the carotin in this solution is in a colloidal condition. It appears, then, that by this procedure we are able to solve carotin in water in a colloidal state, although under ordinary circumstances it is insoluble in water, as e.g. is also the case with mastic, cholesterin and many lipoids.

However, an attempt to extract this colloidal solution with ether fails. Even a two hours' rapid shaking in the shaking apparatus does not enable us to transfer the slightest trace of yellow pigment to the ether or the benzene. But as soon as we add to the mixture a small amount of alcohol, e.g. some drops to 5 cm^3 of colloidal carotin solution and 3 cm^3 of ether, the pigment will pass over instantly and quantitatively into the upper layer, whereas the lowermost layer is completely decolourized and generally becomes rather more opalescent. The best result is achieved, when first some drops of alcohol are added to the solution and after this the ether.

A similar result is obtained when, before carrying out these experiments, the fats and the cholesterin are removed from the alcoholic carotin solution respectively by saponification and by digitonin.

Similarly to the carrot-carotin, the carotin from cow's serum and human serum, and the xanthophyll from fowl's serum and egg-yolk can be obtained in an aqueous, colloidal solution. The latter are more opalescent than the carotin solution from carrots, the egg-yolk solution most of all. Nevertheless they pass through the filter unaltered, and the opalescence may be largely diminished by removal of the fats and of the cholesterin.

When extracting these aqueous colloidal solutions with ether, no trace of pigment passes over to the ether — as has been described for the carotin from Daucus carota. After the addition of a small amount of alcohol, the pigment will again pass over quantitatively into the ether layer.

It appears, therefore, that also in aqueous colloidal protein-free solutions alcohol plays a liberating influence upon lipochrome. Precisely the same influence is exerted by a small quantity of alkali: if to an aqueous carotin-solution ether is added, (which by itself does not extract any pigment from it) and subsequently some drops of 10 perc. NaOH, and this solution is shaken rapidly, all the pigment will pass over into the ether. We feel greatly indebted to Prof. KRUYT for investigating for us these colloidal solutions and for discussing the problem with us. He stated that the solutions presented TYNDALL's phenomenon, while under the ultramicroscope small particles are visible performing the Brownian movements. He also found that, besides by alcohol and 10 perc. NaOH, also all sorts of other salts, especially the bivalent and the trivalent metals exert the same liberating influence on lipochrome. For instance when we add to the colloidal carotin-solution some drops of aluminium-sol, and subsequently shake with ether, the lowermost layer will immediately be completely decolourized.

After having searched in vain for a similar phenomenon in the literature to which we had access, we learnt that WILLSTÄTTER had observed the same in aqueous colloidal solutions of chlorophyl. He prepared them — as we did the analogous lipochrome-solutions — by adding a large quantity of water to an alcoholic chlorophyl-solution and subsequently evaporating the alcohol in vacuo. He did not succeed now in extracting the chlorophyl from the aqueous solution with ether. Still the green pigment passed directly over into the ether, when he had added a small quantum of an electrolyte to the aqueous solution. WILLSTÄTTER did not ascertain whether some drops of alcohol had any "liberating" effect. He accounts for the action of the salt by assuming that the dispersed chlorophyl-particles cannot be reached by the ether. The addition of the electrolyte results in salting out the colloid whereby the particles are at first so small that they are invisible to the unaided eye and that the fluid seems to be perfectly clear. Gradually the flakes become visible. But even in the phase in which they are still invisible, the extremely small particles are accessible to the ether and are dissolved in it.

This interpretation also undoubtedly holds in the phenomenon observed by us with the lipochrome, when alkali or Al-sol, are added. However, it naturally does not account for the "liberating" action of some drops of alcohol, as in that case there is no question of precipitation. The alcohol, if added in adequate quantity, would rather change the colloidal solution into a true solution.

We are unable to account for this phenomenon, but we believe it to be analogous to another reaction previously observed by us. In studying the bile-pigments we had detected that bilirubin, as found in bile and in the blood-serum of patients with obstructive icterus, is directly and completely capable of combining with diazonium salts.

However, when the same reaction is performed in the blood-serum of patients suffering from what was formerly called hematogenous icterus, the reaction is retarded and incomplete. On addition of a small quantum of alcohol the reaction takes place directly and completely.

We cannot but assume that the bilirubin in the serum in the case of obstructive icterus, and in the bile from the gall-bladder, is in a different condition from the bilirubin in the serum of patients with hematogenous icterus. In the latter case it would seem that the bilirubin particles cannot get in touch with the diazonium-solution, except through the action of small quanta of alcohol. This behaviour of bilirubin in the serum from hematogenous icterus is similar to that of lipochrome in cow's serum.

The resemblance in the behaviour of an aqueous colloidal solution of carotin to that of the native cow's serum containing lipochrome, suggests the idea that, also in the serum the carotin occurs in a colloidal and analogous condition. However, this seems not to be the case, since in the aqueous colloidal solution the lipochrome is precipitated by the above-named substances (NaOH, Al-sol, etc.), so that it can be extracted with ether. On the other hand, when adding these reagents to the native serum, the pigment will not pass over to the ether.

Another remarkable difference between the native serum, containing lipochrome and the artificial colloidal solution, is the action of light. Earlier researchers detected already that carotin from

Daucus carota (subsequent researches proved the same to hold good for xanthophyll) exposed to the sunlight, is decoloured by the absorption of oxygen and after some time is completely decolourized. With our impure or incompletely purified solutions of pigments in alcohol and ether our results varied with the nature of the lipochrome and the solvent. After a close exposure to the quartzlamp the aqueous solutions are almost completely decolourized after 15—90 minutes, the interval varying with the nature of the lipochrome and of the solvent. Contrariwise the native substances (egg-yolk, carrots, cow's serum) are not decolourized under similar circumstances. It is evident, therefore, that with regard to sensitiveness to light, there is a difference between the native solutions of lipochromes and the colloidal aqueous solutions.

Chemistry. — "*The unsaturated alcohol of the essential oil of freshly fermented tea-leaves.*" By Prof. P. VAN ROMBURGH.

(Communicated at the meeting of May 31, 1919).

In 1895 in collaboration with my assistant at that time, Mr. C. E. J. LOHMANN, I investigated the ethereal oil from freshly fermented tea[1]), a small quantity of which we were successful in preparing with the cooperation of several tea-planters. The yield of this ethereal oil is extremely small, fifteen kilograms of the fresh leaves giving only one c.c.

We were able at that time to detect the presence in the oil of an unsaturated alcohol (b.p. 153°—155°) of the composition $C_6H_{12}O$, evidently a hexylene alcohol. From this by oxidation an acid could be obtained, smelling like rancid butter, the calcium salt of which gave on analysis a result which indicated the presence of butyric acid. Lack of material prevented us from determining whether the acid formed was the normal or the iso-butyric acid. Later, shortly before my departure from Java, I had the opportunity of obtaining a larger quantity of the ethereal tea-oil (about 120 c.c.), which enabled me to resume the research and to investigate more in detail whether by the oxidation of the unsaturated alcohol one or other of the butyric acids is really formed. A knowledge of the nature of the acid is of course of primary importance for the elucidation of the structure of this acid.

After treatment with alkali in order to saponify the methyl salicylate[2]) (the presence of which we had demonstrated in 1896) and other esters[3]) possibly present, the crude oil was fractionated several times. The largest fractions boiled between 154° and 156° and between 156° and 158°. These were mixed and distilled in vacuo; the principal fraction boiled at 75°—80° at 28—30 mm. pressure.

The sp. gr. at 15° was 0.8465; $n_{D^{19}}$ 1.43756.

Elementary analysis gave 71.17 °/₀ C. and 12.74 °/₀ H. The formula $C_6H_{12}O$ requires 71.91 °/₀ C. and 12.10 °/₀ H.

1) Verslag omtrent den staat van 's Lands Plantentuin te Buitenzorg for the year 1895, p. 119.

2) The same for the year 1896, p. 168.

3) The salicylic acid isolated was not odourless. The smell resembled that of phenyl acetic acid.

The liquid was now treated with anhydrous sodium sulphate and again distilled in vacuo. Further analysis of the product gave, however, no better results. (71.08 % C. and 12.59 % H).

The unsaturated tea-alcohol forms with avidity an addition compound with bromine, as was previously shown. The quantity of bromine added, however, was smaller than is to be expected from a substance of the formula $C_6H_{12}O$, being only 87 % of that amount.

Two fresh determinations gave the following results:

I. 1.017 grm. of the alcohol in chloroform solution cooled in ice-water add 1.363 grm. bromine.

II. 0.529 grm. add 0.707 grm.

From these result it appears that only 83.2 % and 83.5 % respectively of the calculated quantity of bromine is added.

As I suspected that the unsaturated alcohol perhaps contained a hexyl alcohol as impurity, I attempted to purify a larger quantity of the bromine addition compound [1]) from this by heating in vacuo at 100°. A subsequent treatment with zinc dust should give the hexylene alcohol in a pure state. Since, however, the bromine addition product gave hydrobromic acid, I was unable to carry out this intention.

Treatment of the unsaturated alcohol with phenylisocyanate gave no crystallised product. On the other hand α-naphthylisocyanate gave an α-naphthylurethane (m. p. 76°), the melting point of which could be raised to 80° after repeated recrystallisation from petroleum ether.

On treatment with phthalic anhydride, the tea alcohol gave a liquid acid ester of which the silver salt melted at 140°.

Oxidation of the tea alcohol with potassium permanganate in neutral as well as in alkaline sodium carbonate solution, proceeds very smoothly. About 3 c.c. of acid were obtained from 11.5 grm. on treatment with 50 grm. potassium permanganate in 4 % solution. This acid, as before, had a smell resembling that of butyric acid. On distillation of the acid, however, the principal fraction, besides a small first fraction in which formic acid could be detected, was a liquid boiling between 125° and 145°, while the residue in the flask consisted of a liquid of higher boiling point with a smell of perspiration. The principal fraction, on redistillation, gave a liquid of boiling point 140°—145°; which on being boiled with water and calcium carbonate was transformed into a calcium salt which was found on analysis to contain 21.2 % Ca. This result in conjunction with the boiling point of the acid obtained, show that the latter

[1]) This does not solidify in liquid ammonia

consists of propionic acid, the calcium salt of which contains 21.5 % Ca. On heating the ammonium salt an amide with a melting point of 78° was obtained which, on mixing with propion-amide, produced no depression of the melting point.

The hexylene alcohol obtained from tea-oil might therefore be hexene-3-ol-6 of the formula $CH_3 . CH_2 . CH : CH . CH_2 . CH_2OH$. A hexylene alcohol has been obtained by H. WALBAUM [1]) from Japanese peppermint oil to which after investigation he attributes the structure of a β-γ-hexenol. This alcohol is presumably the same as that extracted from tea oil.

On oxidation with potassium permanganate the β-γ-hexenol gives propionic acid as principal product. With chromic acid a hexylene acid is obtained. The α-naphthylurethane prepared from the alcohol melts at 80°, while the melting point of the silver salt of the acid phthalic acid ester melts at 126°. On treatment with bromine only 70 % of the quantity required by theory is absorbed.

It is true that the melting points of WALBAUM's silver salt and my own do not agree, but the other properties of the tea-alcohol justify the assumption that the latter to a great extent consists of β-γ-hexenol. I am, however, for the moment unable to explain why, on oxidation with potassium permanganate, an acid was obtained previously, the calcium salt of which contained only 18.6 % of calcium. The acid on that occasion was *not* distilled, as the quantity available was too small, and may have contained, for example, hexylic acid, by which the calcium content of the propionic acid formed would be lowered. Finally it may be possible that the heating of the crude oil with alkali in order to remove the methyl salicylate, has caused a shifting of the double bond.

This research, as well as the investigation of the other constituents of the tea-oil, is being continued [2]).

Postscript.

Since the above paper was communicated, the firm Messrs. SCHIMMEL and Co. of Leipsic sent me at my request a small quantity of the unsaturated alcohol prepared from Japanese peppermint oil, for which I desire herewith to express my thanks. The α-naph-

---

[1]) Journ. f. prakt. Chemie, **96**, 254 (1917).

[2]) The ethereal oil of tea was some time ago the subject of an investigation by Dr. DEUSS (Mededeelingen van het Proefstation voor Thee, XLII, 21, 1917). This research merely confirmed our own observation that the oil contained an unsaturated alcohol together with methyl salicylate.

thylurethane obtained from this (m. p. 80°) when mixed with that prepared from the tea-oil caused no alteration of the melting point.

The acid phthalic acid ester prepared from it, gave a silver salt-melting at 128°. By recrystallisation from alcohol the melting point could be raised to 134° (not sharp). With the silver salt prepared from the tea-alcohol it gave a mixture melting at 138°.

The assumption is thus justified that the unsaturated alcohol prepared by me from the tea-oil is identical with the β-γ-hexenol, that is, with hexene-3-ol-6.

Utrecht. *Org. Chem. Laboratory of the University.*

Physiology. — "*A method for the determination of the ion concentration in ultra filtrates and other protein free solutions*". By Dr. R. BRINKMAN and Miss E. VAN DAM (Communicated by Prof. HAMBURGER).

(Communicated at the meeting of October 25, 1919).

A. Determination of the concentration of free calcium ions.

With regard to the biological actions of salts the actions of ions claim the first consideration. It is therefore desirable that we have at our disposal a method by which the ion concentrations are measured.

Up to this only the concentration of the free $H^{\cdot}$-ions have been measured directly; the concentrations of other, also physiologically important ions were not measured at all or determined only indirectly by calculation.

The concentration chain method can be applied only with great difficulty to the physiologically important metals owing to the disturbances brought about by the liberation of gas. DRUCKER [1]) has offered a method in which Ba-amalgam was used as an electrode. An analogous method can perhaps be worked out for the alkali metals. Such determinations have, however, not been made as yet

As an example of a case where it is necessary to know the ion concentration, we can point to the state in which the calcium occurs in the blood. It occurs there namely in three forms: as $Ca^{\cdot\cdot}$ ion, as undissociated calcium salt ($Ca(HCO_3)_2$) and as colloidal calcium-protein compound. More or less 25 % of the total quantity of calcium occurs in the latter state. According to RONA and TAKAHASHI [2]) the ion concentration of the calcium in the serum is determined by the equation

$$\frac{[Ca^{\cdot\cdot}].[HCO_3']}{[H^{\cdot}]} = k\,.\,(k = 350 \text{ on an average}).$$

For the serum which has the physiological $[H^{\cdot}]$ and carbonic acid tension, this means a $[Ca^{\cdot\cdot}]$ of 20—25 mgr. per L. Of the more or less 100 mgr. per L. of calcium which occurs in the serum, therefore, only $^1/_5$ part is present in the ion form. We learn from

1) Zeitschr. für Elektrochemie **19**, 804 (1913).

2) Biochem. Zeitschr. **49** p. 390.

the equation that this concentration of Ca ions is not directly dependent upon the total quantity of calcium; the concentration of the physiologically most important part of the plasma calcium is thus not governed by the amount of calcium salts present, but by the concentration of the hydrogen and bicarbonate ions.

By means of the method offered by us it is now possible in a simple way to measure directly the $Ca^{\cdot\cdot}$ ion concentration. In principle the method can equally well be applied to other ions.

We started with the determination of the concentration of $Ca^{\cdot\cdot}$ ions, because the results of the determination can in this case very easily be controlled by calculation.

I. *General principle of the Method. A few technical remarks.*

If in a binary electrolyte the concentration of the anion $= C_A$, that of the cation $= C_k$ and that of the undissociated salt $= C_n$, then, according to the law of mass action, the following relation exists

$$c_A \cdot c_k = k \cdot c_n, \text{ where } k \text{ is a constant.}$$

If the electrolyte is only slightly soluble the salt is practically completely dissociated and the concentration of the undissociated part may be neglected.

If now the solubility of the slightly soluble salt $= A$, then $C_A = C_k = A$, and the product $C_A \cdot C_k = A^2$ has a constant value (solubility product).

If this product and the concentration of one of the ions is known, the concentration of the other can therefore be calculated.

Supposing that the solution has a concentration of $Ca^{\cdot\cdot}$ ions $= C_{Ca}$ then the concentration of the C_2O_4 ions which can exist free beside these $Ca^{\cdot\cdot}$ ions, maximally $= \frac{P}{C_{Ca}}$, if P represents the solution product of CaC_2O_4. If now still more C_2O_4 be added, then the CaC_2O_4 will be precipitated or will remain in supersaturated solution.

If the formation of a supersaturated solution can be avoided, then it will be noticed, that, upon the gradual addition of C_2O_4'' ions to the solution containing $Ca^{\cdot\cdot}$ ions, at a certain moment a slight turbidity due to CaC_2O_4 results. At this stage the concentration of C_2O_4 ions has become so strong that the solubility product is just exceeded. The C_2O_4 ion concentration is then known, and also the solution product and the $Ca^{\cdot\cdot}$ ion concentration can thus be calculated. Vice versa, if we start with a known $[Ca^{\cdot\cdot}]$ we are able to determine the value and constancy of the solubility product.

Where this method is used therefore it is necessary to observe

50*

how great the $[C_2O_4'']$ is while only the merest sign of turbidity due to CaC_2O_4 can be detected. In general it can be done in the following way:

A series of small tubes, each containing 1 c.c. of a known $CaCl_2$ solution, was taken, and to the tubes in succession quantities of oxalate solution increasing gradually with each new tube were added with a capillary pipet divided into tenthousandths of c.c. The tubes were then left to themselves for from $^1/_2$ to 1 hour and consequently it was observed in which tube the first sign of turbidity due to CaC_2O_4 appeared.

It is clear that the formation of supersaturated CaC_2O_4 solutions has to be avoided.

In cases where the solutions held other salts besides (e.g. RINGER solution, ultra filtrate) we have never noticed supersaturation. As a matter of fact supersaturation occurred in the case of pure solutions of CaC_2O_4. This can be avoided by setting to work in the following way:

With a capillary pipet the desired quantities of a, say 0.05, N. strong oxalate solution is brought into the dry tubes. In a waterbath the tubes are evaporated down to dryness. After this the liquid containing the $Ca^{\cdot\cdot}$ is introduced into the tubes. In this way is prevented the formation of already supersaturated solutions.

For the determination of the calcium ion concentration it is moreover necessary to use tubes that are well closed with ground glass stoppers. This is necessary to keep the water free of carbonic acid or to keep a fixed carbonic acid tension constant.

It is necessary for the judging of the appearance or non-appearance of the CaC_2O_4 precipitate that the tubes should be cleaned as thoroughly as possible; this can be done in the usual way (chromic acid, ABEGG's steaming process etc.).

The best way for viewing the tubes is in a box with a slit in the bottom from which the light falls through the solution. Care should be taken that the light does not fall on the eye of the observer. The Tyndall phenomenon makes it possible to appreciate the slightest turbidity. Should the solution before the experiment already evince a slight opalescence (not due to CaC_2O_4) as is sometimes the case with serum and ultra filtrate, it is advisable to view the solutions by red light. The wavelength of this light being too great to cause refraction the opalescence is not apparent. The temperature during the experiment must of course remain constant. It is therefore best to work in a waterbath of constant temperature.

The results obtained by the above method can be controlled in another way still, viz. by measuring the electrical conductivity of the solutions.

If to a solution containing $C^{\cdot\cdot}$ and Cl' ions, C_2O_4'' and Na' ions be added, then the product $Ca^{\cdot\cdot} \times C_2O_4$ cannot exceed the square of the solubility of CaC_2O_4. If too many C_2O_4 ions have been added, then undissociated CaC_2O_4 must be formed. How much CaC_2O_4 will be formed, if the product is exceeded by a fixed quantity of C_2O_4?

To a binary electrolyte with a solubility A, a salt which has an anion in common with the first is added in a concentration x. Through this the solubility of the first salt is changed to A'. The total concentration of the anion then amounts to $A'+x$, that of the kation to A'. The solubility product is therefore; $A'(A'+x)$, and because this is constant we have:

$$A'\ (A' + x) = A^2$$

or

$$A' = \frac{-x \pm \sqrt{4A^2 + x^2}}{2}.$$

The quantity of undissociated salt which results when x Mol salt that has 1 ion in common with the first is added, therefore is:

$$A - \frac{-a \pm \sqrt{4A^2 + x^2}}{2}, \quad \ldots\ldots\ldots \quad (1)$$

if A represents the solubility of the first salt.

We have now e.g. 5 c.c. of an aqueous solution of $CaCl_2 \,.\, 6\,aq$, free of CO_2, containing per litre 0.56 millimol $Ca^{\cdot\cdot}$ and (2 Cl'). To this there is added several times successively 0.0050 c c. of a 0.05 N solution of $Na_2C_2O_4$. After every addition the conductivity is measured. The $Na_2C_2O_4$ may here be added in solution, for here the solution may be supersaturated.

By means of the first method the value now found for the solubility product is 0.055. From this it follows that a C_2O_4 concentration of a magnitude 1 millimol corresponds to the 0.56 millimol $Ca^{\cdot\cdot}$. Upon every addition of 0.0050 c.c. 0.05 N. $Na_2C_2O_4$ to 5 cc. of a solution of $CaCl_2$ 6 aq. the C_2O_4 concentration increases by 0.25 mm. After 4 additions therefore the solubility product is reached. What is the relation between the total concentrations of ions during these additions?

For the first addition the total ion concentration is

$$0{,}56 \text{ millimol } Ca^{\cdot\cdot} + 0{,}56 \text{ m.m. } (2\ Cl'') = 1{,}12 \text{ m.m.}$$

After the first addition of 0 025 mm. $Na_2C_2O_4$

$$0{,}56\ Ca^{\cdot\cdot} + 0{,}56\ (2\ Cl'') + 0{,}025\ C_2O_4 + 0{,}025\ (2\ Na'') = 1{,}17 \text{ m.m.}$$

Thus the total ion concentration after the 2nd addition is 1.22 m.m., after the 3rd 1.27 mm. and after the 4th 1.32 m.m. Upon the 5th addition the solubility product is exceeded. According to the deduced formula (1) the amount of undissociated CaC_2O_4 formed =

$$\sqrt{0{,}055} - \frac{-0{,}025 \pm \sqrt{4 \times 0{,}055 + 0.025^2}}{2} = 0{,}0115 \text{ mm.}$$

The total ion concentration becomes thus after the 5th addition

1.32 m.m. + 0.025 C_2O_4'' + 0.025 (2 Na$''$)—0.0115 $Ca^{\cdot\cdot}$—0.0115 C_2O_4'' = 1.347 m.m.

The total concentration of ions therefore does not increase by 0.05 m m but only by 0.027 mm.

With the 6th addition we get a value for the undissociated CaC_2O_4 of:

$$\sqrt{0{,}055 - \frac{0{,}050 + \sqrt{4 \times 0{,}55 + 0{,}05^2}}{2}} = 0{,}024 \text{ mm.}$$

The total concentration of ions after the 6th addition is 1.372 m.m., the increase is 0.025 mm

After the 7th addition it is found, calculated in the same way, that the total concentration of the ions is 1.402, the increase 0.03 m m,

It appears thus that the ion concentration with the first 4 additions increases regularly by 0.05 m.m. From the 5th addition onward, however, it increases only by 0.025 to 0.030 m.m. If now the electrical conductivity be examined after every addition it must appear to increase also in analogy with the increase of the total concentration of ions. Should it be found now that after the first 4 additions the conductivity increases only by half of the original value, then it is a proof that the true value has been found for the solubility product.

II. *Determination of the concentration of Calcium ions in a solution of pure $CaCl_2$ 6 aq.*

In 8 tubes with ground stoppers are brought respectively 0.0010, 0.0015, 0.0020, 0.0025, 0.0030, 0.0035, 0.0040, 0.0045 c.c. of a 0.05 N solution of $Na_2C_2O_4$. Consequently the tubes are placed in a waterbath for some time till the oxalate solutions are evaporated down to dryness. Hereupon into each tube there is introduced 1 c.c. of a $CaCl_3$ 6 aq. solution which contains 125 mgr. per L. After an hour the result is observed.

The solution of $CaCl_2$. 6 aq. was made from a chemically pure substance (The British Drug Houses); the strength of the solution was controlled by chlorine determination. The salt was dissolved in carefully boiled distilled water. All observations were made in small tubes of 2 cc. contents with ground stoppers.

The $Na_2C_2O_4$ solution was made from pure $Na_2C_2O_4$ after SÖRENSEN (KAHLBAUM). It contains no water of crystallisation, is not hygroscopic and is not affected by temperatures below 200°.

It appears that the first 6 tubes have remained perfectly clear but that the tubes with 0.040 and 0.045 c.c. oxalate solution show a faint turbidity.

The solubility product was reached thus, if, on an average, 0.0375 c.c. $\frac{N}{20}$ $Na_2C_2O_4$ solution was added to 1 c.c. of a solution of $CaCl_2$ 6 aq.;

the $CaCl_2$ 6 aq. solution contained 125 mgr. $CaCl_2$ 6 aq. or 0.57 millimol $Ca^{\cdot\cdot}$ per L.

The concentration of oxalate, therefore, was 0.095 m.m., the solubility product is found to be $0.095 \times 0.57 = 0.054$ mm. per L. The temperature during all the experiments was 20°. Table I gives the results of a. series of such experiments.

TABLE I.

Strength of $Ca^{\cdot\cdot}$ concentration.	Strenght of the C_2O_4'' conc. that had to be added to show just a precipitate.	Solubility product.
0.57 millimol	0.095 m.m.	0.054
0.55 „	0.095 „	0.052
0.38 „	0.145 „	0.055
0.37 „	0.15 „	0.054
0.28 „	0.20 „	0.056
0.28 „	0.20 „	0.056
0.10 „	0.54 „	0.054
1.00 „	0.056 „	0.056

From this table it appears thus that, with solutions of pure $CaCl_2$ 6 aq. of different strengths, a constant solubility product of CaC_2O_4 is found, *namely* 0.055 *mm. per L.*

Let this value now be controlled by the measurement of the electrical conductivity as it is described above.

The conductivity was determined in a "resistance vessel" according to HAMBURGER. The method is found described in Osmot. Druck u. Ionenlehre Bd. 1, pag. 98. The temperature was constant at 25°.

To 5 c.c. of a solution of $CaCl_2$ 6 aq. which contained 125 mgr. per L., repeated additions of a 0.05 N solution of $Na_2C_2O_4$ were made Subsequent to every addition the conductivity was measured after it had become constant.

The resistance of the pure $CaCl_2$ 6 aq. solution was

8.709×2000 C Ohm (C = capacity of the resistance vessel).

Table II gives the decrease in the resistance after every addition of oxalate solution. (See Table following page).

It is observed that after the 4th addition of oxalate the decrease in resistance is diminished to less than the half. With the 4th addition, therefore, the solubilityproduct was reached. The C_2O_4 concentration then was 0.1 millimol the $Ca^{\cdot\cdot}$ concentration 0.56 millimol and the product thus 0.056 millimol.

TABLE II.

Composition of solution.	Resistance.	Decrease in resistance.
5 cc. $CaCl_2$ 6 aq.	8.709 × 2000 c. Ohm	—
5 cc. $CaCl_2$ 6 aq. + 0.005 cc. $Na_2C_2O_4$	8.452 × 2000 c. Ohm	0.257 × 2000 c. Ohm
5 cc. $CaCl_2$ 6 aq. + 0.010 cc. $Na_2C_2O_4$	8.200 × 2000 c. Ohm	0.252 × 2000 c. Ohm
5 cc. $CaCl_2$ 6 aq. + 0.015 cc. $Na_2C_2O_4$	7.929 × 2000 c. Ohm	0.271 × 2000 c. Ohm
5 cc. $CaCl_2$ 6 aq. + 0.020 cc. $Na_2C_2O_4$	7.696 × 2000 c. Ohm	0 233 × 2000 c. Ohm
5 cc. $CaCl_2$ 6 aq. + 0.025 cc. $Na_2C_2O_4$	7.600 × 2000 c. Ohm	0.096 × 2000 c. Ohm
5 cc. $CaCl_2$ 6 aq. + 0.030 cc. $Na_2C_2O_4$	7.500 × 2000 c. Ohm	0.100 × 2000 c. Ohm
5 cc. $CaCl_2$ 6 aq. + 0.035 cc. $Na_2C_2O_4$	7.410 × 2000 c. Ohm	0.090 × 2000 c. Ohm

Subsequently more determinations of a similar kind were made by us, which always gave a result of 0.053—0.58, — a mean of **0.055** — for the solubility product.

Here it must be remarked still that it cannot be expected that the solubility product will just have been reached at the end of an addition; the mean value, therefore, has to be taken.

There is still the possibility that the decrease in resistance came about because the oxalate added in such large quantities did practically not dissociate completely; the way in which the decrease would take place then would *not* be such a sudden one. To control this the same quantities of oxalate were added to 5 c.c. of distilled water; the conductivity kept increasing proportionally to the quantities added.

We have now found by two methods which are independent of each other the constant value of **0.055** m.m. for the solubility product, when $Ca^{\cdot\cdot}$ and C_2O_4'' ions are added together.

The solubility of CaC_2O_4 has been found by KOHLRAUSCH to be $4.35 \cdot 10^{-5}$ Mol per L. (18°); the solubility product calculated from this is 0.0019 m.m. per L. and this is much below the value found by us.

KOHLRAUSCH measured the conductivity of a saturated solution of CaC_2O_4; he therefore did not start out from the individual ions.

HERZ u. MUHS[1]) found by adding together the ions a value of 0.034 gram per L. for the solubility of CaC_2O_4 from which follows

[1]) Ber. 36. 4, p. 3717.

a solubility product of **0.054**. This product thus agrees perfectly with ours.

The determination of HERZ U. MUHS *and our own determinations, by two methods, show thus conclusively that we have to reckon with a solubility product of* **0.055**.

III. *Determination of the concentration of $Ca^{\cdot\cdot}$ ions in solutions which hold other salts besides.*

1. *The concentration of $Ca^{\cdot\cdot}$ ions of* 0.02 °/₀ *$CaCl_2$ 6aq. in* 0.5 °/₀ *NaCl.*

For the system $CaCl_2 \rightleftarrows Ca^{\cdot\cdot} + 2\,Cl'$ the following also holds: $CaCl_2 \rightleftarrows K\,Ca^{\cdot\cdot}\,Cl'^2$. K can be found if the degree of ionisation α of a given $CaCl_2$ solution is known. For $Ca(NO_3)_2$ 0,1 % (= 6 m.M. per L.) α is 0,67 [1]).

We have therefore

$$[Ca(NO_3)_2] - \alpha\,[Ca(NO_3)_2] = K\,\alpha\,Ca^{\cdot\cdot}\,\alpha^2\,(NO_3)^2$$

or because

$$[CaNO_3] = [Ca^{\cdot\cdot}] = [NO_3] = 6 \text{ m.M. per L.}$$
$$1 - 0{,}67 = K \times 0{,}67^3\,(NO_3)^2$$
$$1 - 0{,}67 = K \times 0{,}67^3 \times 0{,}036$$
$$K = 30.$$

This is therefore the dissociation constant for $Ca(NO_3)_2$; that for CaCl will differ very slightly from it.

For 0,02 % $CaCl_2$ 6 aq. or 0,91 millimol per L., in 0,5 % NaCl also holds:

$$[CaCl_2] = K\,Ca^{\cdot\cdot}\,Cl'^2.$$

The conc. of Cl' is given by the dissociation of 0,5 % NaCl. Here $\alpha = 0{,}82$ (Osmot. Druck u. Ionenlehre, p. 53); $[Cl']$ thus becomes 7 m.M. In addition to this there is still $[Cl']$ of 0,91 m.M. $CaCl_2$, ± half of which we may consider to be dissociated without committing a large error. The total $[Cl']$ then becomes ± 8 m.M.

Thus

$$CaCl_2 = K\,(0{,}91 - CaCl_2)\,0{,}064. \quad K = 30. \quad CaCl_2 = 0{,}60 \text{ m.m.}$$

Of 0,91 m.m. $CaCl_2$ thus 0,06 m.m. is not dissociated while 0,31 m.m. is dissociated. The solution therefore contains **12,4** mgr. free $Ca^{\cdot\cdot}$ per L.

Experimentally it appears that an oxalate concentration of 0.18 m.m. is necessary before turbidity results in a solution of NaCl 0.5 °/₀ + $CaCl_2$ 6 aq. 0.02 °/₀. From this follows a $[Ca^{\cdot\cdot}]$ of — $0.055 : 0.18 = 0.30$ m.m., or 12 mgr. per L. This determination thus is perfectly in correspondence with the calculation.

2. *Determination of the concentration of Calcium ions in physiological salt solutions.*

In a solution of the composition: NaCl 0.7 °/₀, $NaHCO_3$ ± 0.18 °/₀, KCl 0.02 °/₀ and $CaCl_2$ 6 aq. 0.040 °/₀, with a certain carbonic acid

[1]) Osmot. Druck u. Ionenlehre. I, p. 53.

tension which was not exactly known, the concentration of hydrogen ions was $0.3.10^{-7}$ (determined with neutral red after SÖRENSEN) and the concentration of bicarbonate ions 0.02 N (determined by titration with 0.01 N . HCl and methyl-orange).

From this follows for the concentration of Calcium

$$[Ca^{\cdot\cdot}] = 350 \frac{0{,}3 \, . \, 10^{-7}}{0{,}02} = \mathbf{20 \text{ mg. per L.}}$$

Experimentally a CaC_2C_4 turbidity resulted with a concentration 0.1 millimol oxalate. From this follows a

$$[\,Ca^{\cdot}\,] \text{ of } 0{,}055 : 0{,}1 = 0{,}55 \text{ m.M.} = \mathbf{22 \text{ mgr. per L.}}$$

In a similar solution in which the $[H^{\cdot}]$ however was $0.45 \, . \, 10^{-7}$ and the $[HCO_3{}'] = 0{,}02$ N., the $Ca\,C_2\,O_4$ milkiness was seen with $[C_2O_4{}^{\cdot\cdot}] = 0{,}07$ m.m. Thus:

$$[Ca^{\cdot\cdot}] = 0.055 : 0.07 = 0{,}8 \text{ m.m.} = \mathbf{32 \text{ mgr. per L.}}$$

From the calculation $[Ca^{\cdot\cdot}] = 350 \dfrac{0{,}45 \, . \, 10^{-7}}{0{,}02}$ m.m. $=$ **30 mgr.** per L.

3. *Determination of the concentration of calcium in ultra filtrate.*

Human serum was centrifuged for 2 hours in ultra filters after DE WAARD [1]). CO_2 was passed through the ultra filtrate until $[H] = 0.3 \, . \, 10^{-7}$. (This was ensured by comparing the colour of neutral red in the ultra filtrate with neutral red in a phosphate mixture, according to SÖRENSEN, which had an $[H^{\cdot}] = 0.3.10^{-7}$).

A precipitate of CaC_2O_4 occurred with a $[C_2O_4{}'']$ of 0.1 m.m. per L. From this follows a $[Ca^{\cdot\cdot}]$ of 0.55 m.m. or **22** mgr. $[Ca^{\cdot}]$ ions per L., as has also been made probable by TAKAHASHI and RONA.

An attempt to apply these measurements directly in serum often fails because the turbidity due to CaC_2O_4 is very much less evident, and the opalescence which normally occurs so often in serum is a drawback.

In the limited number of instances where we could notice a definite turning point the same concentrations of $Ca^{\cdot\cdot}$ ions as in ultra filtrate were found.

As a rule, however, it is necessary where serum determinations have to be made to make ultra filtrate, which, after DE WAARD, is very simple.

[1]) Arch. Néérl. de phys. **2** 530 (1918).

Summary.

A simple method is described by which it is possible to measure the concentration of $Ca^{\cdot\cdot}$ ions in a solution of a mixture of salts e.g. ultra filtrate.

The method is based upon the following principle:

To a solution containing $Ca^{\cdot\cdot}$ there are added so many C_2O_4'' ions till the solubility product of $Ca_1C_2O_4$ is just reached. The juncture at which so many C_2O_4 ions are added that this product is just exceeded, is ascertained by the appearance of a slight milkiness due to CaC_2O_4. It does not matter whether in the mixture of salts there are present other ions still that can give a precipitate with oxalate. It is only necessary that CaC_2O_4 should be the most insoluble substance which can result in the solution.

The method is correct to 2—3 mgr $Ca^{\cdot\cdot}$ per L. The value of the solubility product was tested by the measurement of the electrical conductivity of the solution.

The principle of the method can likewise be applied for the determination of other ions. The only condition is the disposal of a reagent that gives a salt which is very slightly soluble with the ion whose concentration has to be measured.

Physiological laboratory of the
University of Groningen.

September 1919.

Geology. — *"On the Crustal Movements in the region of the curving rows of Islands in the Eastern part of the East-Indian Archipelago".* By Prof. H. A. BROUWER. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of November 25, 1916).

Of late years various explorers [1]) have pointed out a resemblance in the tectonic structure of the curving rows of islands of the Moluccas and that of many curving chains of Alpine structure. Large overthrusts formed down to the miocene, have been discovered in various islands and the zone, characterised by overthrusts, is bordered on the outside by a region, in which the mesozoic and tertiary deposits are slightly or more intensely folded, but no overthrusts occur. Actual facts seem to indicate that in the curving rows of islands of the Moluccas may be distinguished:

1. A zone characterised by overthrusts (Timor-Ceram row of islands).
2. A marginal zone without overthrust-tectonic (Sula-islands—Misool, Western New Guinea south of the Mac Cluer bay and probably also the Kei-islands).
3. An inner zone with the young active volcanoes.
4. A zone lying between 1 and 2 of older volcanic rocks (North coast of Netherlands-Timor, Wetter, Ambon, peninsula of Huamual in South-Western Ceram and Amblau).

We will now pass in review the features of these zones.

General situation and origin.

If the sea-level in the East-Indian archipelago were to subside 200 m., Sumatra, Java and Borneo would form one mass of land with the peninsula of Cambodja and Siam, just as Australia with the Aru-islands, the vast tract now occupied by the shallow Arafura-sea

[1]) J. WANNER. Geologie von West-Timor. Geol. Rundschau. Bd. IV. 1913. S. 136.
G. A. F. MOLENGRAAFF. Folded mountain chains, overthrust sheets and block-faulted mountains in the East Indian Archipelago. Compte Rendu du XIIe congrès géol. internat. Toronto 1913, p. 689.
H. A. BROUWER. On the Tectonics of the Eastern Moluccas. Proc. Kon. Ak. v. W. Amsterdam. Vol. XIX. N⁰. 2, p. 242—248.

and the bay of Carpentaria, New-Guinea and the islands of Misool, Waigeu, Batanta, Salawati, west of it.

Between those two landmasses lies an area in which deep sea-basins alternate with upheaved islands. The region of the curving rows of islands (the Timor-Ceram row and that of the young active volcanoes) considered by us, presents an aspect similar to that which parts of the geo-synclinal of the Mediterranean region must have presented in some part of the mesozoic period.

In the Jurassic period several geo-anticlines were formed in the latter region, which divided the original geo-syncline into a number of secondary geo-synclines and in connection with the parallelism between the direction of the (more recent) alpine mountain ranges and the axes of these mesozoic geo-synclinals, HAUG [1]) thinks it legitimate to assume that the formation of these mesozoic geo-synclines is due to beginning mountain-building movements. MOLENGRAAFF [2]) assumes on the ground of different features of the curving rows of upheaved islands of the Moluccas and of the adjacent deep sea-basins, that these islands have originated in the same way.

The outlying position of the Tenimber islands.

If we imagine the islands to the east of Timor (Letti, Moa, Lakor, Luang, Sermata and Babber) joined by a curve to the islands south-east of Ceram (Drie Gebroeders, Kur, Téor, Kasiwui, Gorong and Ceram Laut) the islands of the Tenimber group will be seen to lie outside this curve. This curve is e.g. also found on map N° 1 of VERBEEK's Molukken Verslag [3]), on which the Tenimber islands and the Kei-islands are lying outside his "belt of older rocks".

Now it is striking, that *in the Sahulbank, which constitutes the submarine continuation of the Australian block — i. e. the "Vorland"*

1) E. HAUG. Traité de Géologie. II, p. 1127.

2) G. A. F. MOLENGRAAFF. On recent crustal movements in the island of Timor and their bearing on the geological history of the East-Indian Archipelago. Proc. Kon. Ak. v. Wetensch. Amsterdam. June 1912.

3) R. D. M. VERBEEK. Molukken Verslag. Jaarb. v. h. Mijnwezen 1908. Wetensch. Ged. Atlas. Kaart 1.

See also: A. WICHMANN. Gesteine von Kisser. Jaarb. v. h. Mijnw. 1887, p. 120 and Samml. des Geol. Reichsmus. in Leiden. (The curving row of islands, separating the Banda Sea from the Arafura Sea is also here represented as a mountain range). Ibid. Der Wawani auf Amboina und seine angeblichen Ausbrüche. Ill. Tijdschr. Kon. Ned. Aardr. Gen. XVI. 1899, p. 109.

K. MARTIN. Die Kei Inseln und ihr Verhaltniss zur australisch-asiatischen Grenzlinie. Tijdschr. Kon. Ned. Aardr. Gen. VII. 1890, p. 241 ff.

against which the overthrust mountain chain is pushed up — a depression occurs just opposite the Tenimber islands.

We know that the shape of the folds of several mountain-chains is influenced by the resistance of the "Vorland". This also holds for the folds to which the formation of the uplifted curving rows of islands and the alternating deep ocean-basins have been ascribed above, and then we can compare the bending of the Timor-Ceram curve near the Tenimber-islands with the pushing forward of the Penninic overthrust sheets of the Alps in the lower parts of the hercynian mountains, against which they were pushed up (as between Mont Blanc and the Aar massif).

Behind the parts of greatest resistance of the "Vorland", the tectonic axes at a deeper level, and the islands at the surface will rise higher; this need not be, but may be, the reason why the Tenimber islands are not uplifted so high above the sea-level, as Timor is.

The Tenimber islands have been considered by us [1]) to belong to the overthrust mountain-range, and if the mountains on the South Coast of Timor, characterised by an imbricated structure with a uniform dip to the north-north-west, are autochtonous [2]), the overthrust mountain-range must in all likelihood also have been bent at the site of the Tenimber islands.

The outlying position of the Kei-islands.

The Kei-islands are like the Tenimber-islands situated opposite a depression in the region covered by the shallow Arafura Sea, and their outlying position can be explained in a similar way. Along the north coast of Groot-Kei the terraces of miocene limestone are surrounded by a younger and lower (probably quaternary) coral-terrace, while the terraces of miocene limestone in the southern part of the island are found down to the sea level. This points to an intenser uplift of the northern part of the island in post-tertiary time and this may, just as the outlying position, point to the persistence of crustal movements similar to those which gave rise to the overthrusts of the Timor-Ceram row of islands. The northern part of the island, namely, lies just opposite a protruding point of the depression in the region covered by the Arafura sea, and opposite this more resistant part of the "Vorland" the tectonic axes at a deeper level and the islands at the surface will be more elevated.

---

1) H A. BROUWER. loc. cit.

2) G. A. F. MOLENGRAAFF. Folded mountain chains etc., loc. cit., p. 691.

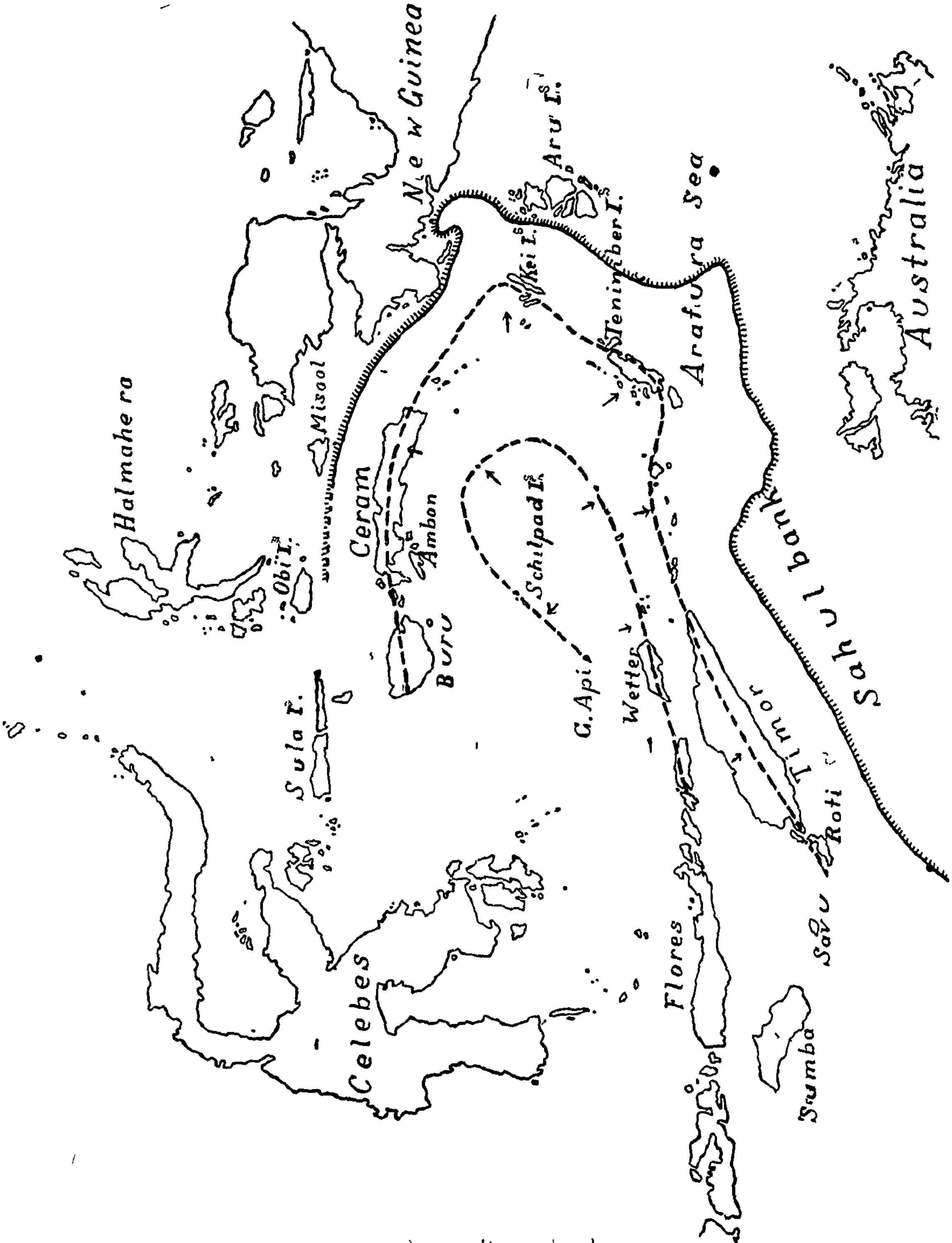


Fig. 1. The youngest crustal movements in the curving row of islands of the eastern Indian Archipelago.

– – – – – the two geoanticlines, rising and moving towards the "Vorland".

ııııııııııı approximate limit of the "Vorland".

The large island of Jamdena of the Tenimber-group consists — at all events in part — of mesozoic rocks. As regards the Kei-islands we found there only in one spot of small extent near the east coast amidst limestones of eocene age mica sandstone and ferriferous rocks, strongly resembling mesozoic rocks. [1])

The eocene in Groot-Kei is not folded intensely; the miocene is not folded at all. [2]) More to the west the strata seem to be folded more intensely, for in a new island near Ut (Klein-Kei-group) contorted, about vertical strata of probably eocene limestone were found. The tectonic relation of the above-mentioned mica sandstone and ferriferous rocks to the widely spread tertiary limestones and marls in Groot-Kei, has not been explained yet, neither is it possible yet to fix the eastern limit, once reached by the overthrusts of the Timor-Ceram curve near the Kei-islands. Perhaps here also the overthrust mountain-range has already made an outward bend; the old-miocene of Groot-Kei however lies about horizontal.

The Aru-islands.

These islands form a small elevation inside and near the border of the tract which we consider to be the "Vorland". They may perhaps also be considered as bulges similar to those which are elsewhere believed to result from the pressure to which the most exposed part of the "Vorland" is subjected.

On the occurrence of rocks older than Permian rocks.

In the Western Alps the central parts of the chain are formed by a series of autochtonous massifs (Mercantour, Montblanc, Aar Massif and others) which belong to the ancient hercynian mountains. Part of the overthrust sheets was pushed over these massifs and deposited north of them. For Australia the equivalents of the hercynian folding of Europe are known, but nothing is known for certain in this respect for the curving rows of islands under consideration. In the large island of Timor, which has been pretty well explored, we have no certainty about older rocks than those of Permian age, and MOLENGRAAFF says about this island: "The Fatu sheet is like the Tethyssheet, composed of rocks ranging in age from Permian to Eocene and probably to Miocene".

Indeed, so-called "old slate rocks" occur in numerous islands.

[1]) H. A. BROUWER. Geologische verkenningen in de oostelijke Molukken. Verh. Geol. Mijnb. Gen. 1916, p. 47.

[2]) R. D. M. VERBEEK. Molukken Verslag. loc. cit., p. 501.

VERBEEK supposes[1]) that Archean as well as old-Palaeozoic rocks occur among them. But, for reasons, which have been expounded in other papers we feel justified in assuming that these rocks — in part at least — are of younger date.[2])

When confining ourselves to the island of Timor, the available data seem to bear out that the older massifs, constituting the base of the Tethysgeosynclinal, have not, at all events not here, been raised sufficiently by the folding process to be denuded through the erosion, to which that portion of the overthrust mountain range that had been lifted up above the sea-level, was exposed for a long time.

This must have been the case also in an earlier stage in the region of the Alps, when in the middle-mesozoic period the Tethys-geosyncline was divided into different geosynclines and geoanticlines, with partial emersion of the latter.

Prolongation of the curve west of Timor and west of Ceram.

The island of Roti may be considered as a direct continuation of Timor; we find there rocks of the same kind and various facts point to a similarity in the tectonic structure[3]). Similar rocks are also found in Savu, but Sumba presents a totally different structure; not a vestige of the intense miocene foldings is found here. That the prolongation of the overthrust mountain-range does not proceed over Sumba is not surprising in connection with the contour of the "Vorland". South of Timor the limit of the Australian block bends southward, so that Sumba lies further from the "Vorland" and consequently assimilates itself more to the more northern row of the Sunda islands.

With respect to the prolongation of the curve west of Ceram MARTIN believes that vast overthrusts possibly also occur in Buru[4]).

The elliptical "belt of ancient rocks" indicated by VERBEEK on Plate I in his Molukken Verslag, diverges from Buru in south-

[1]) R. D. M. VERBEEK. Molukken Verslag, loc. cit., p. 738 Verslagen der Afd. Natuurk. Dl. XXV, 1916/17.

[2]) Comp. H. A. BROUWER. Geologisch Overzicht van het oostelijk gedeelte van den Oost-Indischen archipel. Jaarboek Mijnwezen in Ned.-Indië. 1917. Verh. II, p. 33—35.

Devonian rocks with *Spirifer Verneuili* occur in Celebes (H. A. BROUWER. Devonische afzettingen in den O.-I. archipel. De Ingenieur. 29 Nov. 1919).

[3]) H. A. BROUWER. Voorloopig Overzicht der geologie van het eiland Roti. Tijdschr. Kon. Ned. Aardr. Gen. XXI. 1914, p. 611.

[4]) Cf. G. A. F. MOLENGRAAFF. Verslag betreffende de wenschelijkheid etc. Tijdschr Kon. Ned. Aardr. Gen. XXXI. 1914, p. 369 ff.

western direction, but we cannot find sufficient evidence to look in this direction for the continuation of the Timor-Ceram row of islands. HOTZ [1]) reports the occurrence of rocks in the western part of the eastern peninsula of Celebes, which show a great resemblance to rocks, widely spread in Buru (MARTIN's Buru-limestones) while also the tectonic structure becomes more complicate than that of the eastern part of the east arm, where, as in the Sula islands, simpler tectonic relations prevail. This, however, does not convince us eventually of a prolongation of our overthrust mountain-range.

Curve with the young Volcanoes.

The young volcanoes of the Banda Sea are joined by VERBEEK by an ellipse of which only one half embraces volcanoes, no volcanoes being known on the northern half between Banda and the G^g Api, north of Wetter. This ellipse runs concentrically with VERBEEK's elliptical "belt of older rocks". In my opinion, we may as well assume that the volcanic islands rest upon a submarine ridge, which forms the continuation of the rows of islands to which Sumbawa and Flores belong, and which bends round considerably past Banda in the direction of the Siboga ridge with the Schildpad- and Lucipara islands and the G^g Api to the north of Wetter. On this supposition the Banda Sea would be encircled by two ridges, running concentrically wide apart, but the inner ridge bending sharply towards its termination.

Additionally we are able to record here, that between the Timor-Ceram row and the row of the young volcanoes, another zone seems to exist with a certain autonomy. We mean a zone of older volcanic rocks, having many features in common and occurring near the north coast of Dutch-Timor, in Wetter, in Ambon and in the peninsula of Huamual in South-West-Ceram. Then a very considerable portion of this zone would be covered by the sea. Among these volcanic rocks are serpentine breccias and serpentine conglomerates, tuffs, rhyolites, and andesites. Peculiar andesitic to basaltic rocks with glassy crusts, reminding us of the "pillowy lava" of Mullion Island and the upper-Devonian "Wulstdiabase" of the Westerwald occur in all localities. Their typical structure is indicative of submarine origin; the origin of such structures was observed by ANDERSON [2]) where the lava of the new volcano Matavanu in Savaii

[1]) W. HOTZ. Vorläufige Mitteilungen über geologische Beobachtungen in Ost-Celebes. Zeitschr. d. a. geol. Ges. LXV. 1913. Monatsber. N^o. 6, S. 329.

[2]) TEMPEST ANDERSON. Volcanic craters and explosions. The Geogr. Journ. Febr. 1912, p. 129.

(Samoa islands) reaches the sea, and also for the rocks of Mullion Island, which occur together with sediments with radiolaria TEALL[1]) assumes a submarine origin.

Comparisons with the Alps.

Although the geology of the region under discussion is as yet known only in broad outlines, it is permissible to conclude from the results of the inquiries of the last few years that the crustal movements bear some resemblance to those by which other curving alpine mountain ranges were built up, to witness the known overthrusts in an outward direction everywhere in the Timor-Ceram curve and the adaptation of the folds to the shapes of the "Vorland". Additional data that are being collected, prove this resemblance to be beyond dispute.

We know that the folded curves of mountains of the Mediterranean region correspond to the geosynclinals accumulated by bathyal sediments in the mesozoic and in the beginning of the tertiary period.

The jurassic and the cretaceous deposits reach a considerable thickness there, their horizontal extent is very large, fossils of the neritic zone are rare; all these characteristics are wanting in the generally little disturbed deposits of the same age outside the region of the alpine mountains. For the sake of comparison we point to the striking resemblance of the triassic to the jurassic and perhaps even younger deposits of the deep-sea, covering a vast extent in islands of the Timor-Ceram curve (Roti, Timor, Buru) which are situated far from each other, while different reasons justify the assumption that in that time an open sea connected the region of the East-Indian archipelago, the Himalaya and the Alps[2]). The investigation of the permian fauna of Timor also teaches us that the Tethys geosynclinal extended already in permian time from the Mediterranean Sea to the region of our Archipelago and a conformable succession of perm and trias seems to be the rule. The fact that permian deposits are as yet known only in the southern islands of the Timor-Ceram row of islands, goes to show that in that time the sea covered a smaller area in the eastern part of our Archipelago than in mesozoic time.

In the Mediterranean region the hercynian crustal movements were no longer distinctly perceptible already towards the end of

[1]) J. J. H. TEAL. On greenstones associated with radiolarian chert. Trans-Royal Geol. Soc of Cornwall 1894.

[2]) G. A. F. MOLENGRAAFF. L'expédition néerlandaise à Timor en 1910—1912. Arch. Néerl. des Sciences exactes et nat. 1915, p. 395 seqq.

the permian and in the triassic period this movement does not recur.

What we do observe at the site of the future intensive tertiary folds, is the formation of geosynclines, in which the bathyal trias is deposited. In the jurassic period different geosynclines and geoanticlines were formed whose course has been reconstrued by HAUG [1]) with the aid of stratigraphical data and by removing the deposits of the overthrust sheets to their original site. In the formation of these geoanticlines some parts may rise above the sea-level, which will cause rows of islands and also (under favourable circumstances) coralreefs to be formed, such as we know now in the eastern part of the East Indian archipelago. HAUG (loc. cit. p. 1126) says of the géanticlinal briançonnais: "La zone axiale du Briançonnais et la nappe supérieure des Préalpes, qui a sa racine dans son prolongement, sont caractérisées par un Lias corralligène ou tout au moins zoogène, faisant quelquefois défaut, par des couches à *Mytilus*, représentant le groupe Oolitique inférieur, et par du Tithonique corralligène. Ces formations néritiques indiquent la présence d'une crête sous-marine, voire d'un chapelet d'îles, correspondant à un nouveau géanticlinal". In the cretaceous period intensive crustal movements took place in most of the geanticlines, from which resulted partial upheaval above the sea-level, as is borne out by lacunae in the series of cretaceous deposits. Already in old-tertiary time real mountain ranges in the geographical sense were formed, while chiefly in the neogene the high mountain ranges arose, such as the Alps and the Himalaya.

We do not purpose to make a reconstruction of the aspect of the Tethys-geosyncline, as it was, during the mesozoic period, in the region of the East-Indian Archipelago. Such a reconstruction must be incomplete, since a considerable portion of the region is covered by the sea, so that our knowledge of it is little as yet. The Alpine geologist will in this respect always have the advantage not only in that the structure in the deep erosion valleys is much more denuded, but also because several continuous parts of the mountain range can be compared with each other.

On the other hand ARGAND [2]) has already pointed out, that the study of the rows of islands of Eastern Asia and Oceania teaches us what the condition may have been of Alpine mountain ranges with a similar distribution of land and water in earlier periods. We can compare the curving rows of islands of the Moluccas with the con-

[1]) E. HAUG. Traité de Géologie. II, p. 1125.

[2]) E. ARGAND. Sur l'arc des Alpes occidentales. Eclogae Geol. Helv. Vol. XIV. 1916, p. 179.

dition of the Western Alps in their development in the Jurassic period, as described by ARGAND [1]). Also here we see two geanticlines and a "Vorland" separated from each other by geosynclines. In the Lias the formation of the geosynclines and geanticlines is more accentuated, which continues down to the middle-jurassic, the geanticlines above the sealevel having disappeared. In the Upper-Jura this is followed by a moderate submersion, after which in cretaceous times the intense crustal movements begin, which reach their maximum in the tertiary period. The overthrust sheets moved in the direction of the "Vorland" and eventually were pushed over it; the sea-basins of the anticlines are moving down gradually and at last disappear altogether.

Oscillations, such as occurred in the jurassic period in the Alps and to which we have alluded above, are also known to us in the curving rows of islands in the Moluccas. The formation of the overthrusts was followed by a long period of denudation, then a submersion and deposition of sediments, which was followed again by upheaval above the sea-level.

SUMMARY.

The outwardly directed overthrusts to be observed everywhere in the Timor-Ceram curve mark the action of a tangential pressure, which caused the sediments, deposited in this region in mesozoic and tertiary until the beginning of miocene time, to be pushed in the direction of the "Vorland" and to be raised above the sea-level. The subsequent submersion may be accounted for by a temporary decrease of the intensity of the tangential pressure. The characteristics of the now appearing rows of rising islands and of the alternating sea-basins point to a recurrence of the crustal movements and do not clash with the assumption that these movements occur again in the direction of the "Vorland" and that consequently the rows of the uplifted islands indicate the spots where at greater depths the folding process continues with a tendency to form overthrusts. In this connection we refer once more to the outlying position of the Kei-, and Tenimber-Islands opposite the depressions of the "Vorland" and the stronger uplift of the northern part of Groot-Kei.

As the movements proceed the uplift of the rows of islands (with alternate intervals of temporary subsidence through decrease of the intensity of the tangential forces) will be accompanied by a shifting

[1]) E. ARGAND. La formation des Alpes occidentales. Eclogae Geol. Helv. Vol. XIV. 1916. Pl. 3.

in the direction of the "Vorland", the sea-basins will narrow and eventually the masses of the present rows of islands will be deposited on the site of the present Australian continent, a stage which e.g. was reached long before in the Alps. The bend in the inner curve i.e. that of the active volcanoes, as assumed by us, will widen and lengthen in consequence of the outward pressure in all directions. The same holds for the Timor-Ceram curve.

In conclusion we will compare the way in which the volcanic rocks of the inner curve of islands occur with that of the volcanic rocks encountered at the inner side of the Timor-Ceram row of islands.

A very considerable portion of the products of the young volcanoes is now deposited under the sea and we saw that part of the older volcanic rocks alluded to, evince characteristics indicative of a similar formation. The inner curve, less elevated than the outer one will rise higher above the sea-level as the crustal movements are prolonged. When the volcanic deposits, which at this day are still lying far below the sea, will be lifted up above the sea-level, they will perhaps have been folded already by the same crustal movements and will already have been uplifted or overthrust. When these deposits become visible at the coast, erosion has for a long period already been affecting the volcanic cones and the volcanic products lying far inland; they may even have disappeared completely through erosion. It appears, therefore, that the volcanic rocks will occur in the inner row of islands in the same way as now in the outer row.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXII

Nos. 9 and 10.

President: Prof. H. A. LORENTZ.

Secretary: Prof. P. ZEEMAN.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vol. XXVIII).

CONTENTS.

Mathematics — "*Zur Axiomatik der Mengenlehre*". By Prof. A. SCHOENFLIES, Frankfurt a. M. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meetings of February 28 and March 27, 1920).

Die Hilbertsche Grundlegung der Geometrie darf für alle analogen Untersuchungen als vorbildlich gelten. Zwei ihrer Eigenschaften sind es, auf die es hier ankommt. Erstens wird von allen sprachlichen Definitionen der Objecte, mit denen sie operiert, wie Punkt, Gerade, zwischen u.s.w. abgesehen; nur ihre gegenseitigen Beziehungen und deren Grundgesetze werden axiomatisch an die Spitze gestellt [1]). Zweitens werden die Axiome in verschiedene Gruppen gewisser Eigenart und Tragweite gespalten (die des Schneidens und Verbindens, die Axiome der Ordnung, der Kongruenz u.s.w.), und es ist eine wesentliche Aufgabe des axiomatischen Aufbaues, zu prüfen, bis zu welchen Resultaten eine einzelne oder mehrere dieser Gruppen für sich führen. Die gleiche Behandlung eignet sich für die Mengenlehre. Von sprachlicher Einführung der Begriffe Menge, Bereich u.s w. ist daher ebenso abzusehen, wie von der des Punktes oder Raumes. Ebenso kann man hier gewisse Axiomgruppen unterscheiden, die Axiome der Aequivalenz, die Axiome der Ordnung u.s.w. und kann die gleichen Fragen stellen, wie im Gebiet der Geometrie. Dies soll im Folgenden geschehen, und zwar für denjenigen Teil, der nur mit der Aequivalenz der Mengen, der Mengenteilung und Mengenverbindung, sowie der Mengenvergleichung operiert.

Will man die Probleme der Mengenlehre einer derartigen Behandlung unterwerfen, so ist es oberstes Erfordernis, die Begriffe der endlichen und der unendlichen Menge auf einer Grundlage einzuführen, die nur die ebengenannten Fundamentalbegriffe benutzt. Solche Definitionen sind ja in der Dedekindschen Begriffsbestimmung vorhanden: Eine Menge M heisst unendlich, wenn es eine (ächte) Teilmenge M' von M giebt, die aequivalent M ist; sie heisst endlich,

[1]) Der Euklidische Aufbau beginnt noch mit den Worten: Ein Punkt ist, was keine Teile hat. Eine Linie ist eine Länge ohne Breite usw. In dem Verzicht auf alle solchen sprachlichen Begriffsbestimmungen liegt einer der wesentlichen Hilbertschen, und durch ihn modern gewordenen Gedanken. Die Mengenlehre hat sich ihm bisher nicht erschlossen.

wenn es eine solche Teilmenge nicht giebt. *Sie* haben daher den *alleinigen Ausgangspunkt* zu bilden.

Die historische Entwickelung der Mengenlehre ist freilich wesentlich anders vor sich gegangen. Während vorstehend die *unendliche* Menge als das logisch *positiv* bestimmte Object erscheint, und die *endliche* Menge als ihr logisches *Gegenteil*, ist die historische Entwickelung umgekehrt von den endlichen Mengen als wohlbekannten mathematischen Objecten ausgegangen, und hat die unendlichen Mengen als Gegensatz der endlichen Mengen eingeführt. Der so benutzte Begriff der endlichen Mengen gehört aber bereits einem Gebiet an, das sich nicht mehr ausschliesslich an die Aequivalenzbeziehungen anschliesst. Der historisch überkommene Begriff der endlichen Menge ruht ja überhaupt nicht auf axiomatischer Grundlage. Mag man ihn sprachlich oder empirisch oder anschaulich auffassen, er war im wesentlichen an der Hand des Zahlbegriffs entstanden und ruht jedenfalls auf Voraussetzungen, in die auch die Ordnung als Grundbegriff eingeht. Diese gehört aber bereits einer Begriffsgruppe an, von der hier abzusehen ist. So laufen in der historischen Entwickelung der Mengenlehre zwei wesentlich verschiedene Bestimmungen der endlichen und unendlichen Mengen unvermittelt neben einander her und erschweren infolgedessen die Frage nach dem, was den einzelnen Sätzen axiomatisch zu Grunde liegt. Auch insofern ist eine Klärung des Sachverhalts wünschenswert.

Das Resultat erweist sich in zwei Punkten als durchaus eigenartig. Die Vergleichung der Mengen bezüglich ihres Grössencharacters ist nämlich nichts, was dem Mengenbegriff allein eigentümlich ist; sie betrifft allgemeiner *alle Objecte*, für die man das *Ganze* und den *Bestandteil* unterscheiden kann. Die Axiomatik, die hier entwickelt wird, ist also richtiger *eine Axiomatik der Gröszenlehre*, und zwar in dem besonderen Fall, dass es auch Grössen von *unendlichem* Character giebt. Dies bedingt, dass die *Elemente* der Mengen im Folgenden *gar nicht benutzt werden*; immer nur bilden die an sich möglichen Beziehungen zwischen den Ganzen und ihren Teilen den Gegenstand der Untersuchung. Deren auf axiomatischer Grundlage ruhende, umfassende Erörterung bildet den eigentlichen Inhalt der Arbeit. Ich habe aber doch die gewohnten Mengenbezeichnungen beibehalten. Für die *Elemente der Mengen* wird erst am Schluss eine auf den Begriff der Teilmenge sich stützende Einführungsmöglichkeit gezeigt. Sie erscheinen als solche Teilmengen, die selbst nicht weiter in Teilmengen zerlegbar sind (gleichsam als die Atome).

Eine zweite Eigenart der Untersuchung betrifft die logischen Notwendigkeiten, die der axiomatische Aufbau dieses besondern Gebietes

52*

verlangt. Ausser den selbstverständlichen axiomatischen Festsetzungen über die Regeln, nach denen man mit den Begriffen der Aequivalenz, der Teilmengen usw. zu operieren hat, treten auch noch Annahmen auf, die man wohl nicht erwarten mag. Bei ihrer Einführung handelt es sich aber — und darin besteht die genannte Eigenart — weniger um spezifisch mathematische Notwendigkeiten, als vielmehr um *rein logische*; also um Festsetzungen, die deshalb nötig sind, weil man ohne sie — um welches wissenschaftliche Gebiet es sich handeln mag — aus den in Frage stehenden Voraussetzungen *Schlüsse überhaupt nicht ableiten kann.* Ein allgemeiner Grundsatz der Logik lautet: E mere negativis nihil sequitur; d.h. aus lauter negativen Prämissen kann eine Folgerung nicht gezogen werden. Aus den Sätzen

kein $\mathfrak{A}$ ist ein $\mathfrak{B}$, kein $\mathfrak{B}$ ist ein $\mathfrak{C}$

lässt sich in der Tat eine Beziehung zwischen $\mathfrak{A}$ und $\mathfrak{C}$ nicht entnehmen; und ebensowenig gestatten die Sätze

kein $\mathfrak{B}$ ist ein $\mathfrak{A}$, kein $\mathfrak{C}$ ist ein $\mathfrak{A}$

eine Beziehung zwischen $\mathfrak{B}$ und $\mathfrak{C}$ [1]). Gerade solche Prämissen sind es aber, die uns bei den mengentheoretischen Problemen mehrfach begegnen, und deshalb der Einführung einer zwischen $\mathfrak{A}$ und $\mathfrak{C}$ oder zwischen $\mathfrak{B}$ und $\mathfrak{C}$ vorhandenen Beziehung den Stempel der axiomatischen Notwendigkeit aufdrücken.

§ 1. *Die Aequivalenz.*

Die mathematischen Objecte, von denen im Folgenden die Rede sein wird, heissen *Mengen* (Teilmengen, Verbindungsmengen). Alle sollen denselben Aequivalenzbeziehungen gehorchen, die wir als *Axiome der Aequivalenz* ($\sim$) einführen. Sie lauten: Sind M, N, P verschiedene Mengen, so gilt

I. Aus $M \sim N$ folgt $N \sim M$.

II. Aus $M \sim N$ und $N \sim P$ folgt $M \sim P$.

---

[1]) Aus den Vordersätzen

$\mathfrak{A}$ ist nicht $\mathfrak{B}$, $\mathfrak{A}$ ist nicht $\mathfrak{C}$

kann freilich in gewissen Fällen doch eine positive Folgerung gezogen werden und zwar für $\mathfrak{A}$ selbst. Nämlich dann, wenn man eine zwischen $\mathfrak{B}$ und $\mathfrak{C}$ bestehende positive Beziehung kennt. Aus den Sätzen:

Das Dreieck $\mathfrak{D}$ ist nicht spitzwinklig und
Das Dreieck $\mathfrak{D}$ ist nicht stumpfwinklig

folgt, dass $\mathfrak{D}$ rechtwinklig ist. Hier liegen nämlich nur scheinbar ausschliesslich negative Prämissen vor; zu ihnen kommt als positive der Satz: Jedes Dreieck ist entweder spitzwinklig oder stumpfwinklig oder rechtwinklig. Vgl. auch die Anmerkung auf S. 839.

Der Aequivalenzbegriff hat also sowohl *kommutativen*, wie auch *assoziativen* Character.

Aus diesen Axiomen folgt:

1. Aus $M \sim N$ und N *nicht* $\sim P$ folgt M *nicht* $\sim P$. Denn wäre $M \sim P$, so würde daraus in Verbindung mit $N \sim M$ gemäss I weiter $N \sim P$ folgen, im Gegensatz zur Voraussetzung.

Die Axiome I u. II zeigen, dass sie die Ausdehnung auf den Fall zulassen, dass M und N dieselbe Menge bedeuten. Wir fügen also als weiteres Axiom hinzu

III. Es ist $M \sim M$.

§ 2. *Teilmengen und Verbindungsmengen.*

Ist M' Teilmenge von M, so soll dies durch

$$M' \, t \, M$$

bezeichnet werden. Wir nehmen durchweg an, dass M' von M *verschieden* ist, und nennen insofern M' auch ächte oder eigentliche Teilmenge von M.

Für die Teilmengen sollen folgende Axiome gelten (*Axiome der Teilmengen*):

I. Aus $M' \, t \, M$ und $M'' \, t \, M'$ folgt $M'' \, t \, M$.

II. Jede Teilmenge M' von M bestimmt eindeutig eine zweite Teilmenge M_1 von M, die ihre *Komplementärmenge* bezüglich M heisst.

III. Die Komplementärmenge von M_1 ist wiederum M'.

Wir dürfen daher folgende Bezeichnungen einführen. Wir schreiben $M_1 \, k \, M'$ resp. $M' \, k \, M_1$ und setzen dem gemäss (III) in die Form

III'. Aus $M_1 \, k \, M'$ folgt $M' \, k \, M_1$.

Für die Beziehung von M_1 und M' zur Menge M selbst schreiben wir

$$M = (M_1, M') = (M', M_1),$$

und sagen, dass M in die Teilmengen M' und M_1 zerfällt. Zusammenfassend können wir also sagen:

Aus $M' \, t \, M$ folgt $M_1 \, t \, M$, $M_1 \, k \, M'$, $M' \, k \, M_1$, $M = (M', M_1)$.

Seien nun M und N zwei Mengen, so können bezüglich ihrer Teilmengen zwei Fälle eintreten. Entweder gibt es für M und N identische Teilmengen, oder es giebt keine solchen Teilmengen. In diesem Fall nennen wir die Mengen *fremd* zu einander, oder kurz fremd, und schreiben

$$M \, f \, N \text{ resp. } N \, f \, M.$$

Für fremde Mengen gilt der Satz:

1. Sind M und N fremde Mengen, so ist auch jede Teilmenge von M zu jeder Teilmenge von N fremd; d. h.

Aus $M f N$, $M' t M$, $N' t N$ folgt $M' f N'$.

Wären nämlich die Teilmengen M' und N' nicht fremd, und ist P eine in beiden enthaltene Teilmenge, so hätte man

$$P t M', \; M' t M \text{ und } P t N', \; N' t M,$$

und daher gemäss I auch

$$P t M \text{ und } P t N,$$

im Widerspruch mit der Voraussetzung.

1*a*. Der Satz gilt auch so, dass M' zu N selbst, und ebenso N' zu M fremd ist. Der Beweis ist derselbe.

Wir stellen weiter folgende Axiome auf:

IV. *Die beiden Komplementärmengen M' und M_1 einer Menge M sind fremde Mengen*; d. h.

Aus $M_1 k M'$ folgt $M_1 f M'$.

Diese Beziehung soll aber auch umgekehrt gelten; zu diesem Zweck führen wir folgendes weitere Axiom ein (*Axiom der Verbindungsmengen*).

V. *Zwei fremde Mengen N und P bestimmen eine und nur eine Menge M, deren Komplementärmengen sie sind*; d. h.

Aus $N f P$ folgt $N t M$, $P t M$ und $N k P$.

Die Axiome IV und V lassen sich also auch so auffassen, dass die Beziehungen $N k P$ und $N f P$ gleichwertig sind. Wir nennen die Menge (N, P) die *Verbindungsmenge* von N und P. Es folgt noch

2. Die Mengen N und P sind von ihrer Verbindungsmenge $M = (N, P)$ verschieden.

Denn da sie nach V Komplementärmengen von M sind, so ist jede eine ächte Teilmenge von M.

Die Menge (N, P) hat ausser N und P gemäss Axiom I auch jede Teilmenge N' und P' zu Teilmengen. Damit sind aber, wie wir durch ein weiteres Axiom festsetzen, nicht ihre sämtlichen Teilmengen erschöpft. Gemäss Satz (1) und (1*a*) ist auch N' zu P' fremd, ebenso N' zu P und N zu P'; nach Axiom V giebt es daher je eine Menge

$$(N', P'), \; (N, P') \text{ und } (N', P).$$

Für sie setzen wir nun fest:

VI. *Ist $M = (N,P)$ so sind auch die Mengen.*

$$(N', P'), \; (N', P), \; (N, P')$$

Teilmengen von M; es ist aber auch jede von N, N' P, P' verschiedene Teilmenge von dieser Form.

Wir folgern hieraus den Satz:

3. Ist $M = (N, P)$ und ist die Menge Q fremd zu N und fremd zu P, so ist sie auch fremd zu M; d.h.

Aus $Q f N$ und $Q f P$ folgt $Q f (N, P)$.

Wäre nämlich die Menge Q nicht fremd zu M, so gäbe es für sie und M eine identische Teilmenge; d.h. es gäbe eine Teilmenge Q', die gemäss VI eine der Formen

$$N, N', P, P', (N, P') \ (N', P) \ (N', P')$$

haben müsste. Diese Teilmenge Q' hätte also jedenfalls N oder P oder eine Teilmenge von N oder P als Teilmenge; d.h. es gäbe eine Teilmenge Q'' von Q', die mit N oder P oder einer Teilmenge von N oder P identisch wäre. Nun ist aber nach I Q'' auch Teilmenge von Q, und damit ergiebt sich ein Widerspruch gegen die Voraussetzung

Der Satz (3) lässt sich auch in die Form setzen:

3*a*. Ist die Menge Q nicht fremd zur Menge (N, P) aber fremd zu N, so ist sie nicht fremd zu P.

Will man den Begriff der Verbindungsmenge auf mehr als zwei Mengen ausdehnen, so hat man ein neues Axiom nötig. Es ist jedoch für das Folgende nicht erforderlich dies näher auszuführen.

§ 3. *Die Verknüpfung der Mengen.*

Die verschiedenen Beziehungen, die zwischen zwei Mengen M und N Platz greifen können, sind aus der folgenden von CANTOR angegebenen Aufzählung aller Möglichkeiten ersichtlich, die unsern Ausgangspunkt abgeben soll:

a. Es giebt ein $M' \sim N$, und ein $N' \sim M$.

b. Es giebt kein $M_1 \sim N$, aber ein $N' \sim M$.

c. Es giebt ein $M' \sim N$, aber kein $N_1 \sim M$.

d. Es giebt kein $M_1 \sim N$, und kein $N_1 \sim M$[1]).

Wir wollen diese vier Beziehungen durch

$$M a N, \ M b N, \ M c N, \ M d N \ldots\ldots \quad (A)$$

darstellen. Man erkennt zunächst unmittelbar:

1. Die Beziehungen (*a*) (*b*) (*c*) (*d*) schliessen einander gegenseitig aus.
2. Die Beziehungen $M a N$ und $N a M$, ebenso $M d N$ und $N d M$ sind identisch. Die Beziehung $M b N$ ist identisch mit $N c M$.

Wir erörtern sofort, welche dieser Beziehungen sich auf den Fall ausdehnen lassen, dass M und N dieselbe Menge bedeuten. Es findet sich

1) Die Anwendung oberer und unterer Indizes bei den Teilmengen im positiven und negativen Fall soll im Allgemeinen zur Erleichterung der Auffassung beibehalten werden.

2a. Die Beziehungen $M\,b\,M$ und $M\,c\,M$ sind widerspruchsvoll. Sie fordern namlich das gleichzeitige Bestehen von

$$M' \sim M \text{ und kein } M_1 \sim M.$$

Dagegen sind die Beziehungen $M\,a\,M$ und $M\,d\,M$ widerspruchsfrei. Uebrigens lasst sich dies auch als unmittelbare Folge von (1) und (2) auffassen.

Sei P eine weitere Menge, so besteht zwischen N und P ebenfalls eine der Beziehungen

$$N\,a\,P,\ N\,b\,P,\ N\,c\,P,\ N\,d\,P, \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

und es entsteht die Frage, welche Folgerung sich fur die Mengen M und P einstellt, wenn man eine Beziehung der Reihe A mit einer Beziehung der Reihe B kombiniert. Diese Aufgabe lasst sich ohne Einführung neuer Axiome nicht erledigen. Ein erstes, das den Begriff der Teilmenge mit dem der Aequivalenz verbindet, sei das folgende:

I. *Aus den Relationen*

$$M'\,t\,M,\ M \sim N$$

lassen sich die Relationen

$$N'\,t\,M,\ N' \sim M'$$

folgern; d.h. Ist $M \sim N$, so bedingt eine jede Teilmenge M' von M die Existenz einer Teilmenge N' von N, die zu M' aequivalent ist.

Vielleicht mag man erwarten, dass die Menge N' als diejenige wolbestimmte Menge eingefuhrt wird, die der Menge M' *gemäss der zwischen M und N bestehenden Aequivalenz* entspricht. Aber dies ist für den hier vorgenommenen Aufbau — jedenfalls an dieser Stelle — weder moglich noch notig. Es genugt, die *Existenz* einer Menge N' zu fordern; welches diese Menge ist, darf ganz offen bleiben. Es hangt dies damit zusammen, dass die Aequivalenz $M \sim N$ in ihrer besondern Eigenart hier nicht in Frage kommt; nur die Relationen, die die Eigenart des Aequivalenzbegriffs kennzeichnen, und für zwei Mengen und ihre Teilmengen bestehen, werden in Betracht gezogen.

Einen Teil der oben gestellten Frage hat bekanntlich schon CANTOR selbst beantwortet; man zeigt leicht

3. Aus $M\,a\,N$ und $N\,a\,P$ folgt $M\,a\,P$.
4. Aus $M\,a\,N$ und $N\,b\,P$ folgt $M\,b\,P$.
5. Aus $M\,a\,N$ und $N\,c\,P$ folgt $M\,c\,P$.
6. Aus $M\,b\,N$ und $N\,b\,P$ folgt $M\,b\,P$.
7. Aus $M\,c\,N$ und $N\,c\,P$ folgt $M\,c\,P$ [1]).

[1]) Diese Tatsachen entsprechen bekanntlich dem Umstand, dass wenn man den Fällen a, b, c die Beziehungen „gleich", „kleiner", gröszer" zuweist, die für

Die Beweise sind natürlich ausschliesslich auf die in a, b, c, d enthaltenen Beziehungen zu stutzen. Ein Beispiel möge zeigen, wie sie sich fuhren lassen. Um aus den Relationen

$$M\,b\,N \text{ und } N\,b\,P \text{ weiter } M\,b\,P$$

zu folgern, haben wir von

$$\text{kein } M_1 \smile N, \text{ ein } N' \smile M$$
$$\text{kein } N_1 \smile P, \text{ ein } P' \smile N$$

auszugehen, und daraus die Beziehungen

$$\text{kein } M_1 \smile P, \text{ ein } P' \smile M$$

abzuleiten. Wir beweisen zunächst den zweiten Teil. Wegen $P' \sim N$ giebt es nach I eine Teilmenge $P'' \sim N'$, und aus $N' \sim M$ folgt nun $P'' \sim M$. Die Richtigkeit der ersten Behauptung erweisen wir indirect. Wäre nämlich ein $M \sim P$, so folgte gemäss I aus $N' \sim M$ wiederum die Existenz einer Menge N'' von N', für die $N'' \sim M'$ sein müsste, und aus

$$M' \smile P,\ N'' \smile M' \text{ weiter } N'' \smile P,$$

während kein $N_1 \sim P$ sein kann.

Es bleibt noch übrig, das gleichzeitige Bestehen der Beziehungen

$$M\,b\,N \text{ und } N\,c\,P$$

zu untersuchen, sowie die Kombination von $M\ d\ N$ mit einer der Beziehungen

$$N\,a\,P,\ N\,b\,P,\ N\,c\,P,\ N\,d\,P.$$

Hier gilt zunächst, dass aus $M\,b\,N$ und $N\,c\,P$ eine bestimmte Beziehung zwischen M und P nicht folgt; d. h.

8. Mit $M\ b\ N$ und $N\,c\,P$ ist jede der vier Beziehungen $M\,a\,P$, $M\,b\,P$, $M\,c\,P$, $M\,d\,P$ verträglich.

Der Beweiss darf unterbleiben. Nur sei bemerkt dass dies dem realen Tatbestand entspricht, dessen axiomatische Grundlegung hier in Frage steht [1]).

Wir gehen nun zu dem Rest unseres Problems über und prüfen zunächst die Kombination von

$$M\,d\,N \text{ und } N\,d\,P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

Die Frage lautet auch hier, ob die Beziehungen (α) eine *bestimmte* Beziehung zwischen M und P bedingen und eventuell *welche*. Hier liegt der in der Einleitung genannte Fall vor, dass es sich um lauter negative Prämissen handelt. Diese Pramissen sind

---

diese Beziehungen geltenden assoziativen Gesetze erfüllt sind (z. B. aus $a = b$ und $b = c$ folgt $a = c$ usw.)

[1]) Für Mächtigkeiten würden die Relationen $m < n$ und $n > p$ bestehen; sie bedingen keine Grössenbeziehung zwischen m und p.

$$\left.\begin{array}{l}\text{kein } M_1 \frown N, \text{ kein } N_1 \frown M,\\ \text{kein } N_1 \frown P, \text{ kein } P_1 \frown N.\end{array}\right\} \quad \ldots\ldots \quad (\alpha')$$

Aus ihnen lasst sich auf directem Wege über die Beziehung von M zu P *nichts schliessen.* Teilweise gelingt es allerdings auf indirectem Wege; in einzelnen Fällen kommt namlich dadurch zu den obigen Pramissen eine neue Tatsache hinzu, die positiver Natur ist. Um die Untersuchung durchzuführen, hat man nämlich zu prüfen, ob die Annahme einer der Beziehungen

$$MaP,\ MbP,\ McP,\ MdP \quad \ldots\ldots \quad (\beta)$$

auf Grund der bisherigen axiomatischen Festsetzungen einen Widerspruch mit dem gleichzeitigen Bestehen der Beziehungen α bedingt, und zwar kommen naturgemass hier nur die Axiome von § 1, das obige Axiom § 3, *1* und der obige Satz 2 in Frage. Diese Prüfung haben wir ausführlich vorzunehmen [1]).

Zunachst sieht man leicht, dass die Beziehungen

$$MbP \text{ und ebenso } McP$$

als Folgen von (α) auszuschliessen sind. Wegen Satz (2) kann man nämlich die Beziehungen (α) auch in die Form

$$PdN \text{ und } NdM$$

setzen, und musste daher als Folgerung von (α) auch

$$PbM \text{ oder } PcM$$

erhalten. Aber MbP und PbM, und ebenso McP und PcM sind nach Satz (2) nicht identisch, womit die Behauptung erwiesen ist [2]).

---

[1]) In den Math. Ann. Bd. 72, S. 551 (1912) ist diese Untersuchung schon teilweise durchgeführt worden.

[2]) Die logische Eigenart des oben behandelten Problems entspricht also nicht ganz dem in der Einleitung genannten Tatbestand. Es lautet nämlich genauer so: Welche von vier möglichen Beziehungen wird durch die dem Problem eigentümlichen nur negativen Prämissen ausgeschlossen? Bei der Annahme, MbN oder McN seien die Folgen dieser negativen Prämissen, wird von selbst eine neue Tatsache eingeführt; die Symmetrie der Beziehungen MdN und NdP bezüglich M und P steht nämlich im Gegensatz zu der Unsymmetrie der Folgerungen MbP oder McP für M und P. Und daher ergab sich oben ein Resultat. Die Annahme, MaP oder MdP seien die Folgen der negativen Prämissen, liefert dagegen eine solche neue Tatsache nicht; es ergiebt sich daher, wie das obige weiter zeigt, ein Resultat nicht.

Allgemeiner gesprochen: Wenn die Prämissen: $\mathfrak{A}$ ist nicht $\mathfrak{B}$ und $\mathfrak{B}$ ist nicht $\mathfrak{C}$ sich auch in die Form setzen lassen $\mathfrak{C}$ ist nicht $\mathfrak{B}$ und $\mathfrak{B}$ ist nicht $\mathfrak{A}$, so kann damit nur eine solche Beziehung zwischen $\mathfrak{A}$ und $\mathfrak{C}$ vereinbar sein, die zugleich die *namliche* Beziehung zwischen $\mathfrak{C}$ und $\mathfrak{A}$ bedeutet. Eine genauere Analyse des hiermit mehrfach besprochenen logischen Problems von Seiten der Logiker wäre sehr erwünscht. Das letzte Wort soll mit dem vorstehenden nicht gesprochen sein.

Es ist weiter zu untersuchen, ob sich die Beziehung

$$M\,a\,P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\gamma)$$

als Folge von (α) einstellen kann. Hier ist ein Resultat, das dies unmöglich macht, nicht erhältlich. Die Beziehung $M\,a\,P$ bedeutet nämlich

$$\text{ein } M' \smile P, \text{ ein } P' \smile M \quad . \quad . \quad . \quad . \quad . \qquad (\gamma')$$

Die Verbindung mit (α) liefert gemäss § 1 die weiteren Relationen

$$\text{kein } N_1 \smile P', \text{ kein } N_1 \smile M'.$$

Genauer bedeutet dies: Es giebt eine Teilmenge P', der keine Teilmenge von N aequivalent ist, und es giebt auch eine Teilmenge M', der keine Teilmenge von N aequivalent ist. Dies stellt aber einen Widerspruch zu (α') oder zu (γ') nicht dar.

Es soll noch eine zweite Prufung vorgenommen werden; wir haben auch den assoziativen Character der Beziehungsregeln in Betracht zu ziehen. Ist $M\,a\,P$ das Resultat von (α), so heisst dies, dass das gleichzeitige Bestehen von

$$M\,d\,N, \quad N\,d\,P, \quad M\,a\,P$$

nicht widerspruchsvoll sein darf. Nun sollen aber zwei von diesen Beziehungen stets eine dritte bedingen, und daraus folgt, dass

$$\text{aus } M\,a\,P \text{ und } P\,d\,N \text{ wieder } M\,d\,N$$
$$\text{und aus } N\,d\,M \text{ und } M\,a\,P \text{ wieder } N\,d\,P$$

folgen muss. Es ist nun die Frage, ob diese Regeln einen widerspruchslosen Character haben. Dies ist in der Tat der Fall. Man sieht es am einfachsten daraus, dass man die assoziativen Gesetze, die die Beziehungen (a) und (d) mit einander verbinden, wenn man noch Satz (3) beachtet, in die einfache Form

$$(a\,a) = (d\,d) = a, \quad (a\,d) = (d\,a) = d$$

setzen kann; sie sind das genaue Analogen zu den Vorzeichenregeln

$$(+)(+) = (-)(-) = +\,; \quad (+)(-) = (-)(+) = -,$$

deren assoziativer Gesamtcharacter feststeht.

Wir haben endlich noch die Beziehung

$$M\,d\,P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\delta)$$

als mögliche Folge der Beziehungen (α) zu erörtern. Sie bedeutet

$$\text{kein } M_1 \smile P, \text{ kein } P_1 \smile M \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\delta')$$

Hier zeigt sich zunächst, dass sich aus ihr und den Relationen (α') weitere directe Folgerungen überhaupt nicht entnehmen lassen, da sie jetzt *samt und sonders* negativer Natur sind. Wir prüfen auch hier noch den assoziativen Gesamtcharacter. Ist $M\,d\,P$ das Resultat von $M\,d\,N$ und $N\,d\,P$, so bedingt es jetzt, dass

$$\text{aus } M\,d\,P \text{ und } P\,d\,N \text{ wieder } M\,d\,N$$
$$\text{und aus } M\,d\,M \text{ und } M\,d\,P \text{ wieder } M\,d\,P$$

folgt; hier aber ist der widerspruchsfreie Character evident. Also folgt:

9. *Mit den Beziehungen MdN und NdP kann sowol die Beziehung MaP, wie MdP zugleich bestehen.*

Keine der beiden Annahmen γ und δ führt also auf einen Widerspruch mit den in (α') enthaltenen Prämissen; wir können daher auf diesem Wege nicht zu einem Resultat über die vorliegende Frage gelangen. Man muss daher in der Tat die Folgerung, die sich aus MdN und NdP ergiebt, *axiomatisch einführen*; naturgemäss so, wie es durch den realen Tatbestand der Mengenlehre gefordert wird. Ihn aufzubauen ist ja einer der Zwecke dieser Darstellung. Wir setzen daher fest (*Axiom der Verknüpfung*)

II. *Aus $Md\,N$ und $N\,d\,P$ folgt $M\,d\,P$.*

Hieraus erhalten wir nun leicht die Antwort auf die noch ausstehenden Verknüpfungen für die Beziehungen (A) und (B). Zunächst beweist man

10. Aus $M\,b\,N$ und $N\,d\,P$ folgt $M\,b\,P$.

10a. Aus $M\,c\,N$ und $N\,d\,P$ folgt $M\,c\,P$.

Für den Beweis von (10) haben wir auszugehen von

$$\text{kein } M_1 \sim N, \text{ ein } N' \sim M,$$
$$\text{kein } N_1 \sim P, \text{ kein } P_1 \sim N,$$

und daraus die Beziehung $M\,b\,P$, also

$$\text{kein } M_1 \sim P, \text{ ein } P' \sim M$$

abzuleiten. Wir folgern zunächst, dass eine Beziehung

$$M'' \sim P$$

unmöglich ist. Aus $N' \sim M$ würde nämlich auf Grund dieser Annahme die Existenz einer Teilmenge N'' folgen, für die

$$N'' \sim M'' \sim P$$

wäre, im Widerspruch zu kein $N_1 \sim P$. Damit ist die Beziehung kein $M_1 \sim P$ erwiesen. Es ist jetzt noch zu zeigen, dass es ein $P' \sim M$ giebt. Wäre dies nicht der Fall, so bestände auf Grund des vorstehenden jetzt die Beziehung

$$\text{kein } M_1 \sim P, \text{ kein } P_1 \sim M,$$

also die Relation $M\,d\,P$, und zusammen mit der vorausgesetzten Beziehung $P\,d\,N$ folgte gemäss Axiom II die Beziehung $M\,d\,N$, im Widerspruch zu $M\,b\,N$. Damit ist der Beweis wieder geliefert. Ebenso wird der Beweis für $M\,c\,N$ und $N\,d\,P$ geführt, was einer ausführlichen Darstellung nicht bedarf.

Wir haben schliesslich noch die Kombination von

$$M\,a\,N \text{ und } N\,d\,P$$

zu erörtern. Wir folgern zunächst, dass diese beiden Relationen an sich nur die Folge

$$M\,d\,P$$

gestatten. Wir haben auszugehen von

$$M' \smile N,\ N' \smile M \text{ und}$$
$$\text{kein } N_1 \smile P, \text{ kein } P_1 \smile N,$$

und zeigen zunächst, dass hiermit nur

$$\text{kein } M_1 \smile P, \text{ kein } P_1 \smile M,$$

verträglich sind. Gäbe es nämlich eine Menge $M'' \sim P$, so folgerte man wie oben eine Relation

$$N'' \smile M'' \smile P$$

im Widerspruch mit der Voraussetzung: kein $N_1 \sim P$; ebenso folgt die Unmöglichkeit einer Beziehung $P'' \sim M$. Es kann also an sich nur die Relation

$$M\,d\,P$$

bestehen. Wiederum ist noch der assoziative Character des Resultats zu prüfen. Diese Prüfung führt hier auf einen Widerspruch. Aus $M\,d\,P$ und $N\,d\,P$ würde nämlich gemäss dem Axiom II $M\,d\,N$ folgen, im Widerspruch mit der Annahme $M\,a\,N$. Das gleichzeitige Bestehen von $M\,a\,N$ und $N\,d\,P$ führt also auf einen Widerspruch; d.h.

11. *Die Beziehungen $M\,a\,N$ und $N\,d\,P$ können nicht zugleich bestehen.*

Dagegen sei ausdrücklich festgestellt, dass die Sätze (10) und (10a) einen solchen Widerspruch nicht herbeiführen. Denn gemäss (2) ist $M\,b\,P$ mit $P\,c\,M$ identisch, und die beiden Beziehungen

$$P\,c\,M \text{ und } M\,b\,N$$

sind, wie wir oben unter (8) erwähnten, mit jeder der vier an sich möglichen Beziehungen zwischen N und P verträglich.

Damit ist unsere Untersuchung abgeschlossen; sie zeigt zugleich *die Widerspruchslosigkeit des Axioms II.* Wir ziehen aus ihm zunächst noch eine Folgerung; nämlich die, dass der Satz (11) auch in der Weise gilt, dass er das gleichzeitige Bestehen von

$$M\,a\,M \text{ und } M\,d\,N, \text{ sowie von } M\,a\,N \text{ und } N\,d\,N$$

ausschliesst. Aus $M\,a\,M$ folgt $M' \sim M$ und hieraus gemäss § 3, I

$$M'' \smile M' \smile M,$$

und daher besteht auch die Relation

$$M'\,a\,M;$$

diese kann aber nach Satz (11) nicht mit $M\,d\,N$ zugleich bestehen.

Weiter folgt aus $M\,a\,N$ zunächst

$$M' \smile N,\; N' \smile M,$$

also auch $N'' \sim M' \sim N$, während dagegen $N\,d\,N$ besagt, dass *kein* $N_1 \sim N$ ist. Also

11a. *Die Beziehungen* $M\,a\,M$ *und* $M\,d\,N$, *ebenso* $M\,a\,N$ *und* $N\,d\,N$ *schliessen einander aus.*

Es ergiebt sich damit das folgende Schlussresultat. *Mit den Beziehungen*

$$M\,d\,N \text{ und } N\,d\,P$$

erscheint sowol die Folgerung $M\,a\,P$, *wie auch die Folgerung* $M\,d\,P$ *verträglich.* Wird die Relation $M\,d\,P$ axiomatisch als Folgerung eingeführt, so bedingt dies, dass die Beziehungen $M\,a\,N$ und $N\,d\,P$ nicht zugleich bestehen können; würde man dagegen die Beziehung $M\,a\,P$ axiomatisch als Folgerung einführen, so ergiebt sich ein derartiges Resultat nicht. Trotzdem erfordert der Aufbau der Mengenlehre die Einführung der Folgerung $M\,d\,P$. Auf die Deutungsmöglichkeit der axiomatischen Annahme $M\,a\,P$ komme ich in § 7 zurück.

Für die Beziehungen $(a), (b), c, d$ gelten noch die folgenden besonderen Sätze:

12. Aus den Relationen

$$M\,a\,N,\; M\,b\,N,\; M\,c\,N,\; M\,d\,N$$

und

$$M \smile \mathfrak{M},\; N \smile \mathfrak{N}$$

folgt auch

$$\mathfrak{M}\,a\,N,\; \mathfrak{M}\,b\,N,\; \mathfrak{M}\,c\,N,\; \mathfrak{M}\,d\,N$$

und

$$M\,a\,\mathfrak{N},\; M\,b\,\mathfrak{N},\; M\,c\,\mathfrak{N},\; M\,d\,\mathfrak{N}$$

Für den Beweis mag ein Beispiel genügen. Werde von

$$M\,b\,N \text{ und } M \smile \mathfrak{M}$$

ausgegangen, so heisst dies

$$N' \smile M, \text{ jedes } M_1 \text{ nicht } \smile N.$$

Wir erhalten daher, falls $M_1 \sim \mathfrak{M}_1$ ist, gemäss § 1 sofort

$$N' \smile \mathfrak{M}, \text{ jedes } \mathfrak{M}_1 \text{ nicht } \smile N,$$

womit die Behauptung erwiesen ist.

13. *Aus* $M'\,t\,M$ *folgt* $M'\,a\,M$ *oder* $M'\,b\,M$; *d.h. Für jede Teilmenge* M' *gilt entweder* $M'\,a\,M$ *oder* $M'\,b\,M$.

Es giebt nämlich eine Teilmenge von M, die aequivalent M' ist, nämlich M' selbst, und daher ist die Beziehung (c) und (d) ausgeschlossen.

14. Aus $M' \mathfrak{t} M$ und $M \mathfrak{b} N$ folgt $M' \mathfrak{b} N$;
d.h. Besteht die Beziehung $M \mathfrak{b} N$, so besteht für jede Teilmenge M' von M die Beziehung $M' \mathfrak{b} N$.

Man hat nämlich gemäss (13) und nach Voraussetzung.

$$M' \mathfrak{a} M \text{ oder } M' \mathfrak{b} M \text{ und } M \mathfrak{b} N,$$

und damit gemäss Satz (4) und (6) die Behauptung.

15. Aus $M' \mathfrak{t} M$, $M'' \mathfrak{t} M'$, $M'' \mathfrak{b} M'$ folgt $M'' \mathfrak{b} M$;
d.h. Sind M' und M'' Teilmengen von M, für die die Beziehung $M'' \mathfrak{b} M'$ gilt, so ist auch $M'' \mathfrak{b} M$[1]).

Man hat nämlich wieder zugleich (nach 13)

$$M'' \mathfrak{b} M' \text{ und } M' \mathfrak{a} M \text{ oder } M' \mathfrak{b} M$$

und folgert daraus wie eben $M'' \mathfrak{b} M$.

§ 4. *Endliche und unendliche Mengen.*

Nach § 3, Satz (1) und (2) sind $M \mathfrak{a} M$ und $M \mathfrak{d} M$ die beiden einzigen der Beziehungen $(\mathfrak{a}), (\mathfrak{b}), (\mathfrak{c}), (\mathfrak{d})$, die eine Menge zu sich selbst haben kann; wir definiren nun: 1. Eine Menge heisst *unendlich*, wenn die Beziehung $M \mathfrak{a} M$ besteht; sie heisst *endlich*, wenn $M \mathfrak{d} M$ gilt. Man hat also im ersten oder zweiten Fall

$$\text{ein } M' \sim M; \text{ kein } M_1 \sim M,$$

und damit die *Dedekindsche Begriffsbestimmung*.

Wir folgern zunächst:

2. Aus $M \mathfrak{a} M$ oder $M \mathfrak{d} M$ und $M \sim \mathfrak{M}$ folgt $\mathfrak{M} \mathfrak{a} \mathfrak{M}$ und $\mathfrak{M} \mathfrak{d} \mathfrak{M}$.
Dies ist eine unmittelbare Folge von § 3, (12).

Für endliche und unendliche Mengen bestehen gewisse Sondersätze; diese sollen jetzt abgeleitet werden. Das Haupttheorem lautet:

3. *Für unendliche Mengen können nur die Beziehungen* $(\mathfrak{a}), (\mathfrak{b}), (\mathfrak{c})$ *bestehen; für endliche Mengen nur* $(\mathfrak{b}), (\mathfrak{c}), (\mathfrak{d})$.

Der Beweis ergiebt sich unmittelbar aus den in § 3 abgeleiteten Resultaten.

Sind nämlich M und N unendliche Mengen, und würde die Beziehung $M \mathfrak{d} N$ bestehen, so hätte man

$$M \mathfrak{a} M \text{ und } M \mathfrak{d} N,$$

und dies verstösst gegen den Satz (11a) von § 3.

Ebenso, wenn M und N endliche Mengen sind, so hätte man, falls sie die Beziehung $M \mathfrak{a} N$ gestatten,

$$N \mathfrak{a} M \text{ und } M \mathfrak{d} M,$$

und auch dies verstösst gegen Satz (11a) von § 3.

Damit ist der Satz (3) bewiesen. Er giebt zugleich den inneren

[1]) Dieser Satz berührt sich inhaltlich mit dem Satz 25 in Zermelos Grundlagen (Math. Ann. 65, S. 271).

Grund für die im Satz (11) von § 3 enthaltene Unvereinbarkeit von $M\,a\,N$ und $N\,d\,P$. Denn unserm Satz (3) gemäss besagt $M\,a\,N$, dass M und N unendliche Mengen sind, und $N\,d\,P$, dass N und P endliche Mengen sind. Beides schliesst sich aber aus.

4. *Für jede Teilmenge einer endlichen Menge besteht die Beziehung* $M'\,b\,M$; d.h.

Aus $M\,d\,M$ und $M'\,t\,M$ folgt $M'\,b\,M$.

Gemäss Satz (13) von § 3 gilt nämlich für jede Menge M und eine Teilmenge M' von ihr

$$M'\,a\,M \text{ oder } M'\,b\,M.$$

Hierzu kommt, da M eine endliche Menge ist, $M\,d\,M$. Diese Beziehung kann aber nach Satz (11) von § 3 mit $M'\,a\,M$ nicht zugleich bestehen; also muss es $M'\,b\,M$ sein.

Die weiteren noch abzuleitenden Sätze machen die Einführung eines neuen Axioms nötig, und zwar eines *Axioms über die Aequivalenz von Verbindungsmengen*. Es lautet:

I. Aus $M = (N,P)$, $N \sim \mathfrak{N}$, $P \sim \mathfrak{P}$, $\mathfrak{N}\,f\,\mathfrak{P}$ *folgt* $(N,P) \sim (\mathfrak{N}, \mathfrak{P})$; d.h. *werden in der Verbindungsmenge* (N, P) *die Mengen* N *und* P *durch die zu ihnen aequivalenten zu einander fremden Mengen* $\mathfrak{N}$ *und* $\mathfrak{P}$ *ersetzt, so ist die neue Menge der ursprünglichen aequivalent.*

Das Axiom gilt gemäss § 1, III auch für den Fall, dass nur eine Menge durch eine aequivalente ersetzt wird, d.h.

5. Aus $M = (N, P)$, $N \sim \mathfrak{N}$, $\mathfrak{N}\,f\,P$ folgt $(N, P) \sim (\mathfrak{N}, P)$.[1]

Wir beweisen nun der Reihe nach folgende Sätze:

6. *Jede Teilmenge einer endlichen Menge ist selbst eine endliche Menge*; d.h.

Aus $M\,d\,M$, $M'\,t\,M$ folgt $M'\,d\,M'$.

Wäre nämlich M' eine unendliche Menge, so müsste eine Beziehung

$$M'' \sim M'$$

bestehen. Setzt man nun

$$M = (M', M_1),$$

so ist gemäss § 2, VI auch

$$M''' = (M'', M_1)$$

eine Teilmenge von M_1 und aus Satz (5) folgte

$$M''' \sim M;$$

was einen Widerspruch gegen MdM darstellt.

7. *Ist M eine endliche, N eine unendliche Menge, so kann nur die Beziehung MbN bestehen*[2]; d.h. *Aus MdM und NaN folgt MbN.*

[1]) Es liegt nahe, Satz 5) als Axiom hinzustellen, und das Axiom als Folge. Der Beweis hätte aber die sachlich überflüssige Annahme $\mathfrak{N}\ f\ P$ nötig.

[2]) Auf diesen Satz wurde ich vor längerer Zeit von Herrn H. HAHN aufmerksam gemacht.

Der Beweis wird so geführt, dass die Unvereinbarkeit der Voraussetzungen mit MaN, McN, MdN gezeigt wird.

Würde zunächst die Beziehung MaN bestehen, so hätte man $M' \sim N$; und demgemäss erhielte man aus der Annahme MaN nach § 3 Satz 12 weiter auch

$$M\,a\,M' \text{ resp. } M'\,a\,M,$$

was aber, da M endliche Menge ist, gegen Satz (4) verstösst,

Wäre zweitens McN in Kraft, so folgte daraus $M' \sim N$, und nun hieraus und aus NaN weiter

$$M'\,a\,M',$$

was wiederum einen Widerspruch zum Satz (6) darstellt.

Endlich ist auch die Beziehung MdN unmöglich. Dennaus NaN folgt zunächst

$$N' \sim N;$$

hieraus und aus NaN und der angenommenen Relation MdN folgte dann weiter

$$N\,a\,N' \text{ und } M\,d\,N' \text{ resp. } N'\,d\,M.$$

Die Beziehungen NaN' und $N'dM$ sind aber gemäss § 3 Satz (11) nicht zugleich möglich. Also gilt in der Tat die Beziehung MbN.

8. Ist M eine unendliche Menge, so ist auch die Verbindungsmenge (M, N) eine unendliche Menge.

Der Beweis ist eine unmittelbare Folge des Axioms I. Denn

$$\text{aus } M' \sim M \text{ folgt } (M, N) \sim (M', N)$$

und damit ist der Satz, da (M', N) Teilmenge von (M, N) ist, bewiesen.

9. Eine Menge ist unendlich, wenn sie eine unendliche Teilmenge hat.

Ist nämlich M' diese Teilmenge, so ist

$$M = (M', M_1)$$

und daher gemäss Satz (8) auch M eine unendliche Menge.

Man kann diesen Satz auch noch so formulieren:

9'. Eine Menge ist endlich, wenn jede ihrer Teilmengen endlich ist.

10. *Ist M eine endliche Menge, so ist stets $M\,b$* (M, N); d. h. Aus $M\,d\,M$ folgt $M\,b$ (M, N).

Es ist nämlich M Teilmenge von (M, N). Ist nun (M, N) endlich, so folgt der Satz aus (6), ist aber (M, N) unendlich, so folgt er aus (7).

Zur Ableitung weiterer Sätze bedürfen wir neuer Axiome. Das Axiom I besagt, dass die Verbindungsmengen aequivalenter Mengen selbst aequivalent sind; wir haben jetzt noch zwei Axiome nötig, die die *Nichtaequivalenz der Verbindungsmengen nicht aequivalenter Mengen* betreffen.

II. *Sind M und N fremde Mengen, ist M_1 Teilmenge von M und N_1 Teilmenge von N, und ist M_1 nicht $\sim M$, N_1 nicht $\sim N$, so folgt daraus die Beziehung (M_1, N_1) nicht $\sim (M, N)$*; d. h.

Aus $M f N$, $M_1 t M$, $N_1 t N$, M_1 nicht $\sim M$, N_1 nicht N folgt (M_1, N_1) nicht $\sim (M, N)$.

Dieses Axiom soll für *alle* Mengen gelten. Für endliche Mengen reicht es aber noch nicht aus, und werde durch das folgende ersetzt und ergänzt:

III. *Sind M und N fremde und zugleich endliche Mengen, und ist M_1 Teilmenge von M, so soll stets (M_1, N) nicht $\sim (M, N)$ sein; d. h.*

Aus $M f N$, $M d M$, $N d N$, $M_1 t M$ folgt (M_1, N) nicht $\sim (M, N)$.

Für unendliche Mengen braucht dieses Axiom bekanntlich nicht erfüllt zu sein.

Auch die Voraussetzungen dieser Axiome besitzen durchaus den in der Einleitung genannten logischen Sondercharacter; sie sind sämtlich negativer Natur, soweit es sich um die hier allein in Frage stehenden Aequivalenzbeziehungen handelt. Man könnte freilich annehmen, dass in diesem Fall ein indirectes Beweisverfahren zum Ziele führen werde; die Annahme

$$(M_1, N_1) \sim (M, N) \text{ resp. } (M_1, N) \sim (M, N)$$

ist ja von positivem Character. Aber diese Vermutung trügt. Die Aequivalenz von Verbindungsmengen ist nämlich keineswegs nur so möglich, dass $M_1 \sim M$ und $N_1 \sim N$, ist sondern auch auf andere Weise; und daher kann aus der angenommenen Aequivalenzbeziehung ein Widerspruch mit den Voraussetzungen

$$M_1 \text{ nicht } \sim M, \quad N_1 \text{ nicht } \sim N$$

nicht abgeleitet worden.

Die negative Fassung unserer Axiome stellt uns zunächst vor die Aufgabe, die bestimmte Beziehung $(a), (b), (c), (d)$ aufzufinden, die zwischen (M, N) und den Mengen (M_1, N_1) und (M_1, N) besteht. Für das Axiom II kann es erst im nächsten Paragraphen geschehen; für das Axiom III soll es hier folgen.

Da (M_1, N) Teilmenge von (M, N) ist, so kann nach Satz 13 von § 3 nur die Beziehung (a) oder (b) realisirt sein. Aber der Fall (a) d. h.

$$(M_1, N)\, a\, (M, N)$$

ist unmöglich. Jede Teilmenge von (M_1, N) hat nämlich nach § 2, VI eine der Formen

$$M_1, M_2, N, N_1, (M_2, N), (M_1, N_1), (M_2, N_1),$$

wo M_2 eine Teilmenge von M_1 ist. Keine von ihnen kann aber zu (M, N) aequivalent sein. Da nämlich M und N endliche Mengen sind, so hat man für sie gemäss (10) die Relationen

$$Mb\,(M, N) \text{ und } Nb\,(M, N).$$

Gemäss Satz (4) hat man weiter

$$M_1\, b\, M,\ M_2\, b\, M,\ N_1\, b\, N$$

und damit folgt die Behauptung nach Satz (6) von § 3 bereits für M_1, M_2, N, N_1. Für die drei Verbindungsmengen folgt sie aus den Axiomen selbst; es ist ja, da M und N endliche Mengen sind,

$$M_1 \textit{ nicht } \sim M,\ M_2 \textit{ nicht } \sim M,\ N_1 \textit{ nicht } \sim N$$

und damit ist in der Tat die behauptete Nichtaequivalenz eine Folge von (II) und (III). Also

11. Für endliche (und fremde) Mengen M und N gilt die Beziehung

$$(M_1, N)\, b\, (M, N).$$

12. *Die Verbindungsmenge zweier endlichen Mengen ist selbst endlich*; d. h.

Aus $M\,d\,M$ und $N\,d\,N$ folgt $(M, N)\,d\,(M, N)$.

Wir haben nachzuweisen, dass die Beziehung

$$(M, N)\, a\, (M, N)$$

ausgeschlossen ist. Nun hat jede Teilmenge von (M, N) wieder eine der Formen

$$M, M_1,\ N, N_1,\ (M, N_1),\ (M_1, N),\ (M_1, N_1)$$

und wir beweisen, genau wie eben (vgl. auch § 5, 2), dass keine dieser Mengen zu (M, N) aequivalent ist. Damit ist der Satz bewiesen.

§ 5. *Das Aequivalenzproblem.*

Die wichtigste Aufgabe, die zu behandeln ist, betrifft den Nachweis, dass die Mengen M und N aequivalent sind, falls für sie die Beziehung

$$M\,a\,N \text{ oder } M\,d\,N$$

besteht; also der Satz (Aequivalenzsatz)

1. *Aus $M\,a\,N$ oder $M\,d\,N$ folgt $M \sim N$.*

Ehe der Beweis geführt wird, sollen die Aequivalenz-Relationen vorangestellt werden, die sich aus den vorstehenden Paragraphen unmittelbar ergeben:

2. *Aus $M\,b\,N$ und $M\,c\,N$ folgt M nicht $\sim N$.*

Wäre nämlich $M \sim N$, so hätte man auch (§ 3, 12)

$$N\,b\,N \text{ oder } N\,c\,N,$$

was aber gemäss § 3, 3 widerspruchsvoll ist. Hieraus folgt unmittelbar weiter

3. *Mit $M \sim N$ ist nur $M\,a\,N$ oder $M\,d\,N$ verträglich.*

Die Umkehrung dieses Satzes 3 ist es, die den eigentlichen Aequi-

53*

valenzsatz (1) bildet. Ist er bewiesen, so folgt endlich noch, als Umkehrung von (2)

4. *Aus* M *nicht* $\sim N$ *folgt* $M b N$ *oder* $M c N$.

Man kann diese vier Sätze auch folgendermassen zusammenfassen: *Die Beziehungen* (*a*) *und* (*d*) *sind hinreichende und notwendige Bedingungen für die Aequivalenz,* (*b*) *und* (*c*) *ebenso für die Nichtaequivalenz.*

Als Folge von (4) ergiebt sich, was in § 3 und 4 noch offen bleiben musste,

5. *Aus* $M_1 t M$ *und* M_1 *nicht* $\sim M$ *folgt* $M_1 b M$. d. h. *Besteht für die Teilmenge* M_1 *von* M *die Beziehung* M_1 *nicht* $\sim M$, *so gilt* $M_1 b M$.

Denn nach (4) gilt $M_1 b M$ oder $M_1 c M$; nach Satz (13) von § 3 nur $M_1 a M$ oder $M_1 b M$, also gilt $M_1 b M$.

Eine Anwendung hiervon giebt auch Antwort auf die bezüglich des Axioms II in § 4 gestellte Frage. Es folgt jetzt

6. *Sind* M_1 *und* N_1 *Teilmengen von* M *und* N, *und ist* M_1 *nicht* $\sim M$, N_1 *nicht* $\sim N$, *so folgt daraus stets* $(M_1, N_1) b (M, N)$.

Wir gehen nun zum Satz (1) über und beweisen zunächst den ersten Teil, also den eigentlichen Bernsteinschen Aequivalenzsatz. Sein Beweis folgt aus dem Axiom II von § 4 über die Nichtaequivalenz der Verbindungsmengen.

Aus der Voraussetzung $M a N$ folgt zunächst

$$\text{ein } M' \sim N, \text{ ein } N' \sim M.$$

Wäre nun M *nicht* $\sim N$, so hätte man nach § 1, 3

$$M \textit{ nicht } \sim M', \quad N' \textit{ nicht } \sim N.$$

Mit M und N sind aber auch M' und N' fremde Mengen (§ 2, 1); sie bestimmen daher eine Menge (M', N'), und für sie müsste gemäss Axiom II nunmehr

$$(M', N') \textit{ nicht } \sim (M, N)$$

folgen. Andererseits folgt aber aus den beiden ersten Relationen unmittelbar nach § 4, I

$$(M', N') \sim (M, N)$$

und damit ergiebt sich ein Widerspruch. Damit ist der Beweis bereits geliefert

Freilich beruht der Beweis auf einer gewissen Voraussetzung, die noch zu erörtern ist. Wir operieren mit der Verbindungsmenge von M und N und haben deshalb die Voraussetzung nötig, dass M und N fremde Mengen sind. Sind sie es nicht, so wird man am einfachsten so vorgehen, dass man folgendes neue Axiom zu Grunde legt: [1])

I. *Sind M und N keine fremden Mengen, so giebt es stets zwei ihnen aequivalente, zu einander fremde Mengen $\mathfrak{M}$ und $\mathfrak{N}$; so dass also*

$$\mathfrak{M} \sim M \text{ und } \mathfrak{N} \sim N, \text{ und } \mathfrak{M} f \mathfrak{N}.$$

Gemäss § 3, 12 besteht auch für sie die Beziehung

$$\mathfrak{M}\, a\, \mathfrak{N},$$

und auf sie lässt sich daher der obige Beweis übertragen. Aus $\mathfrak{M} \sim \mathfrak{N}$ folgt dann auch $M \sim N$.

Es handelt sich nun noch um den gleichen Nachweis für die Beziehung $M d N$. Ehe ich dazu übergehe, erinnere ich daran, dass die Eigenart der Beziehung $M d N$ in der Cantorschen Theorie offen geblieben war; für das durch sie bedingte Verhältnis von M zu N hatte sich ein Resultat nicht ableiten lassen. Das darf nicht Wunder nehmen; das hierin enthaltene Problem stellt nämlich wieder ein *logisch unlösbares* Problem, und damit eine illusorische Aufgabe dar. Wir haben ja als Prämissen zunächst nur die Aussagen

$$\text{kein } M_1 \sim N, \text{ kein } N_1 \sim M.$$

Dazu kommen, da M und N endliche Mengen sind,

$$\text{kein } M_1 \sim M, \text{ kein } N_1 \sim N,$$

also lauter Aussagen von negativem Character. Selbst der Weg des indirecten Beweises ändert daran in diesem Fall nichts; denn man müsste noch die Annahme

$$M \textit{ nicht} \sim N$$

hinzufügen. Nun wäre es ja möglich, dass die für den Beweis einzig in Frage kommenden Axiome II und III der Nichtaequivalenz von § 4 die Prämissen positiv beeinflussen könnten; aber auch das ist nicht der Fall. Denn diese Axiome lauten ja in ihrem Schlussteil übereinstimmend

$$(M_1, N_1) \textit{ nicht} \sim (M, N).$$

Wir müssen also von Prämissen ausgehen, die *samt und sonders* negativ sind, und kommen zu dem Schluss, dass sich die Aequivalenz $M \sim N$ im Fall endlicher Mengen ohne eine nochmalige neue axiomatische Festsetzung nicht folgern lässt. Das so gewonnene Resultat lässt sich auch in seiner allgemeinen Bedeutung leicht verstehen. Es läuft dem Tatbestand parallel, der uns aus der allgemeinen Theorie der endlichen Zahlgrössen geläufig ist. Dort muss die Festsetzung, *wann* zwei Grössen als gleich gelten sollen, erst frei —

---

[1]) Es entspricht dem von ZERMELO in seinen Grundlagen (Math. Ann. 65) enthaltenen Theorem 19.

natürlich zweckgemäss — geformt werden, ehe man die Frage, *ob* zwei gegebene Grössen als gleich zu gelten haben, in Betracht ziehen kann. Man denke z. B. an die Weierstrassische Theorie der Irrationalzahlen; sie setzt bekanntlich die Gleichheit zweier Zahlen a und b so fest, dass jeder Bestandteil von a kleiner ist als b und jeder Bestandteil von b kleiner als a. Eine solche axiomatische Festsetzung erweist sich also auch im Gebiet der endlichen Mengen, wenn man sie, wie hier, ausschliesslich auf die Mengenbeziehungen, d. h. auf die Nichtaequivalenz von Menge und Teilmenge gründet, als eine Notwendigkeit.

Es fragt sich nur, welche Festsetzung man zweckmässig zu Grunde legt. Beachtet man, dass es sich im Grunde um eine Axiomatik der Grössenlehre handelt, so liegt offenbar nichts näher, als die eben genannte Definition zu benutzen, und dies soll in der Tat geschehen. Wir setzen also fest (*Axiom der Aquivalenz endlicher Mengen*)

II. *Zwei endliche Mengen M und N sind aequivalent, wenn für jede Teilmenge M' und N' die Beziehung $M'\,b\,N$ resp. $N'\,b\,M$ besteht*; d.h.

Aus $M\,d\,M$, $N\,d\,N$, $M'\,b\,N$, $N'\,b\,M$ fur jedes M', N' folgt $M \sim N$.

Hieraus lässt sich der Satz, dass aus $M\,d\,N$ auch $M \sim N$ folgt, unmittelbar folgern. Ehe wir dazu übergehen, wollen wir noch die Berechtigung unseres Axioms und seine Stellung im gesamten Aufbau näher erörtern. Wir wollen zunächst nachweisen, dass von den vier Beziehungen

$$M\,a\,N,\quad M\,b\,N,\quad M\,c\,N,\quad M\,d\,N$$

nur die letzte mit dem Axiom verträglich ist.

Aus $M\,a\,N$ folgt

$$\text{ein } M' \sim N;$$

gemäss unserm Axiom ist aber für jedes M'

$$M'\,b\,N$$

und man erhielte also $N\,b\,N$, was aber nach § 3,3 widerspruchsvoll ist.

Aus $M\,b\,N$ folgt

$$\text{ein } N' \sim M;$$

was analog zur Relation $M\,b\,M$ führt, die ebenfalls widerspruchsvoll ist.

Endlich folgt aus $M\,c\,N$ genau wie eben die widerspruchsvolle Relation $N\,b\,N$.

Unser Axiom kann also in der Tat nur mit der Beziehung $M\,d\,N$ verträglich sein. Dies ist aber auch wirklich der Fall. Die Folgerungen, die sich aus

$$M'\,b\,N \text{ und } M\,d\,N, \text{ aus } N'\,b\,M \text{ und } M\,d\,N$$

ergeben, lauten gemäss § 3, 9, dass für *jedes* M' und N'

$$M' \, b \, M \text{ und } N' \, b \, N$$

ist; sie entsprechen der Endlichkeit von M und N und stellen die in § 4, 4 gefundene Eigenschaft der endlichen Mengen dar.

Zusammenfassend folgt also: Das Axiom II ist nur für endliche Mengen realisiert, und überdies weder im Fall $M \, b \, N$, noch $M \, c \, N$; damit ist aber der Beweis seiner Berechtigung geliefert. *Es ist für die endlichen Mengen und ihre Aequivalenz characteristisch.*

Der Beweis des Aequivalenzsatzes ergiebt sich nun folgendermassen. Gemäss § 4, Satz 4 ist für jedes M' und N'

$$M' \, b \, M \text{ und } N' \, b \, N;$$

ferner gilt nach Voraussetzung

$$M \, d \, N \text{ und } N \, d \, M,$$

und hieraus folgt nach § 3, 9 sofort

$$M' \, b \, N \text{ und } N' \, b \, M$$

und nunmehr nach unserm Axiom

$$M \sim N.$$

§ 6. *Sätze über Verbindungsmengen.*

Seien M und N einerseits, und $\mathfrak{M}$ und $\mathfrak{N}$ andrerseits fremde Mengen. Zwischen M und $\mathfrak{M}$, sowie zwischen N und $\mathfrak{N}$ besteht je eine der Beziehungen

$$M \, a \, \mathfrak{M}, \quad M \, b \, \mathfrak{M}, \quad M \, c \, \mathfrak{M}, \quad M \, d \, \mathfrak{M} \text{ und}$$
$$N \, a \, \mathfrak{N}, \quad N \, b \, \mathfrak{N}, \quad N \, c \, \mathfrak{N}, \quad N \, d \, \mathfrak{N}.$$

Es ist die Frage, welche Beziehung für

$$(M, N) \text{ und } (\mathfrak{M}, \mathfrak{N})$$

resultiert, wenn wir irgend eine Beziehung der ersten Zeile mit einer Beziehung der zweiten Zeile kombinieren.

Wir beweisen zunächst folgende Sätze

1. Aus $M \, a \, \mathfrak{M}$ und $N \, a \, \mathfrak{N}$ folgt $(M, N) \, a \, (\mathfrak{M}, \mathfrak{N})$.
2. Aus $M \, b \, \mathfrak{M}$ und $N \, b \, \mathfrak{N}$ folgt $(M, N) \, b \, (\mathfrak{M}, \mathfrak{N})$.
3. Aus $M \, c \, \mathfrak{M}$ und $N \, c \, \mathfrak{N}$ folgt $(M, N) \, c \, (\mathfrak{M}, \mathfrak{N})$.
4. Aus $M \, d \, \mathfrak{M}$ und $N \, d \, \mathfrak{N}$ folgt $(M, N) \, d \, (\mathfrak{M}, \mathfrak{N})$.
5. Aus $M \, a \, \mathfrak{M}$ und $N \, d \, \mathfrak{N}$ folgt $(M, N) \, a \, (\mathfrak{M}, \mathfrak{N})$.

Die Beweise von Satz (1), (4), (5) lassen sich folgendermassen zusammenfassen. Die Voraussetzungen lauten gemeinsam

$$M \sim \mathfrak{M} \text{ und } N \sim \mathfrak{N},$$

woraus gemäss Axiom I von § 4

$$(M, N) \sim (\mathfrak{M}, \mathfrak{N})$$

folgt. Im Fall (1) und (5) sind nun M und $\mathfrak{M}$ nach § 4, Satz 3 unendliche Mengen, also gilt dies nach § 4, 8 auch von (M, N) und $(\mathfrak{M}, \mathfrak{N})$ und daher ergiebt sich wieder

$$(M, N)\, a\, (\mathfrak{M}, \mathfrak{N}).$$

Im Fall (4) sind dagegen $M, N, \mathfrak{M}, \mathfrak{N}$ endliche Mengen, also auch (§ 4, 12) (M, N) und $(\mathfrak{M}, \mathfrak{N})$ und daher ist

$$(M, N)\, d\, (\mathfrak{M}, \mathfrak{N})$$

Wir beweisen nun den Satz (2)[1]. Dazu gehen wir von den Relationen

$$M\, b\, \mathfrak{M} \text{ und } N\, b\, \mathfrak{N}$$

aus, also von den Beziehungen

$$\text{kein } M_1 \sim \mathfrak{M} \quad \mathfrak{M}' \sim M,$$
$$\text{kein } N_1 \sim \mathfrak{N} \quad \mathfrak{N}' \sim N,$$

und erhalten zunächst

$$(\mathfrak{M}', \mathfrak{N}') \sim (M, N)$$

Wir folgern nun aus den gegebenen Relationen $M\, b\, \mathfrak{M}$ und $N b\, \mathfrak{N}$ mittels $M \sim \mathfrak{M}'$ und $N \sim \mathfrak{N}'$ weiter

$$\mathfrak{M}'\, b\, \mathfrak{M} \text{ und } \mathfrak{N}'\, b\, \mathfrak{N}$$

oder aber (§ 5, 2)

$$\mathfrak{M}' \textit{ nicht} \sim \mathfrak{M}, \quad \mathfrak{N}' \textit{ nicht} \sim \mathfrak{N}$$

und daraus endlich, gemäss Satz (6) von § 5

$$(\mathfrak{M}', \mathfrak{N}')\, b\, (\mathfrak{M}, \mathfrak{N}) \text{ oder}$$
$$(M, N)\, b\, (\mathfrak{M}, \mathfrak{N}).$$

In derselben Weise beweist man den Satz 3. Ein letzter Satz, der sich ableiten lässt, lautet:

6. Ist M eine endliche Menge, so folgt
aus $M\, b\, \mathfrak{M}$ und $N\, d\, \mathfrak{N}$ $\quad (M, N)\, b\, (\mathfrak{M}, \mathfrak{N})$.

---

[1]) Geht man zu Mächtigkeiten uber, so bezieht sich der obige Satz auf den Fall, dass

$$\mathfrak{m}_1 < \mathfrak{m}_2 \text{ und } \mathfrak{n}_1 < \mathfrak{n}_2$$

ist; er schliest daraus

$$\mathfrak{m}_1 + \mathfrak{n}_1 < \mathfrak{m}_2 + \mathfrak{n}_2.$$

In der allgemeinen Theorie fehlt noch heute ein Nachweis dieser Folgerung. Sie ist von F. BERNSTEIN unter der Annahme bewiesen worden, dass $\mathfrak{m}_2$ mit $\mathfrak{n}_1$ „vergleichbar" ist. (Math. Ann. 61 (1905) S. 129). Nun scheidet zwar in dem vorliegenden Aufbau die Vergleichbarkeit als offene Frage gemäss Satz 1 von § 4 aus, der Bernsteinsche Beweis stützt sich aber ausserdem auf den Aequivalenzsatz. Der obige Beweis stützt sich dagegen auf das Axiom II von § 4, das ja auch den Bernsteinschen Aequivalenzsatz zur Folge hat.

Wegen $M b \mathfrak{M}$ hat man nämlich

$$M \sim \mathfrak{M}',$$

wo mit M auch $\mathfrak{M}'$ eine endliche Menge ist. Hieraus und aus $N \sim \mathfrak{N}$ folgt weiter

$$(M, N) \sim (\mathfrak{M}', \mathfrak{N}).$$

Wir unterscheiden nun, ob $\mathfrak{M}$ eine endliche oder unendliche Menge ist. Im ersten Fall sind $(\mathfrak{M}', \mathfrak{N})$ und $(\mathfrak{M}, \mathfrak{N})$ endliche Mengen, ferner ist $(\mathfrak{M}', \mathfrak{N})$ Teilmenge von $(\mathfrak{M}, \mathfrak{N})$ und daher ist gemäss § 4, 4

$$(\mathfrak{M}' \mathfrak{N}) \, b \, (\mathfrak{M} \mathfrak{N}).$$

Ist aber $\mathfrak{M}$ eine unendliche Menge, so ist $(\mathfrak{M}, \mathfrak{N})$ nach § 4, 8 ebenfalls eine unendliche Menge; dagegen ist $(\mathfrak{M}', \mathfrak{N})$ nach § 4, 12 endlich und daher gilt ebenfalls (§ 4, 7)

$$(\mathfrak{M}', \mathfrak{N}) \, b \, (\mathfrak{M}, \mathfrak{N}).$$

Wegen $\mathfrak{M}' \sim M$, $\mathfrak{N} \sim N$ folgt daraus weiter

$$(M, N) \, b \, (\mathfrak{M}, \mathfrak{N}).$$

In den andern Fällen lassen sich eindeutige Folgerungen nicht entnehmen. Nur soviel sei bemerkt, dass mit den Relationen

$$M a \mathfrak{M} \text{ und } N b \mathfrak{N}$$

jede der beiden Beziehungen

$$(M, N) \, a \, (\mathfrak{M}, \mathfrak{N}) \text{ und } (M, N) \, b \, (\mathfrak{M}, \mathfrak{N})$$

verträglich ist.

§ 7. *Schlussbetrachtung.*

Die vorstehende Untersuchung liefert jedenfalls ein *hinreichendes* Axiomensystem für die Sätze, die die Aequivalenzprobleme der Mengen betreffen. Wird für den Augenblick noch die Bezeichnung $M e N$ für die Aequivalenz von M und N eingeführt, so handelt es sich genauer gesprochen, um die Kombination der Beziehungen, die durch

$$M a N, \ M b N, \ M c N \ \ M d N, \ M e N, \ M f N, \ M t N, (M, N)$$

dargestellt sind, und um die Art, wie sie assoziativ einander bedingen und sich mit einander verbinden. Ob die aufgestellten Axiome sämtlich notwendig sind oder auch entbehrliche Bestandteile enthalten, mag offen bleiben. Abgesehen von den Axiomen mehr formaler Bedeutung, wie die über $M e N$, $M f N$, $M t N$ sind es wesentlich die folgenden, die die materiellen Stützen des Aufbaues darstellen: Das Axiom der *Verknüpfung*, die Axiome über die *Aequivalenz der Teilmengen* und der *Verbindungsmengen*, die Axiome über die *Nichtaequivalenz* der Verbindungsmengen *nicht aequivalenter Mengen* und das Axiom über die *Aequivalenz endlicher Mengen.* Die Characterisirung, die in diesen Bezeichnungen enthalten ist, zeigt schon die Verschiedenheit der Gebiete, denen sie angehören, und zeigt auch ihre allgemeine Notwendigkeit für den Aufbau.

Wie bereits in der Einleitung erwähnt, ist die vorstehende Betrachtung zugleich eine Axiomatik der Grössenlehre; in der Tat ist ja von den Elementen der Menge nirgends die Rede. Dies ist auch die Tatsache, die dem in § 3 gefundenen Resultat seine Stellung im axiomatischen Aufbau anweist. Wir fanden dort, dass mit den Beziehungen $M\,d\,N$ und $N\,d\,P$ auch die Folgerung $M\,a\,P$ verträglich ist. Sie könnte deshalb an sich ebenfalls als axiomatische Festsetzung an Stelle des Axioms II eingeführt werden. Wie wir sahen, bewirkt sie als weitere Folgerung, dass aus $M\,a\,P$ und $P\,d\,N$ sich $M\,d\,N$ ergiebt, und liefert ebenfalls ein in sich widerspruchsfreies System von Beziehungen. Es liess sich durch die Formeln

$$(a\,a) = (d\,d) = a; \quad (a\,d) = (d\,a) = d$$

darstellen.

Dies wollen wir nun deuten. Zunächst ist zu beachten, dass in die vorstehenden Schlüsse die Beziehungen MbN und McN nicht eingehen, dass es sich bei ihnen vielmehr nur um $M\,a\,N$ und $M\,d\,N$ und deren Kombinationen handelt. Nur auf sie beziehen sich also die obigen Regeln und auf sie beschränke ich mich zunächst. Die Aufgabe ist dann, Objecte mit Grössencharacter zu finden, die sich diesen Regeln fügen. Die in § 3 erwähnte Analogie mit den Vorzeichenregeln macht dies leicht. Man erreicht es, indem man *entgegengesetzte Grössen* in Betracht zieht, deren Teile zum Ganzen in der durch (a) festgelegten Beziehung stehen, also der Dedekindschen Definition genügen; die Beziehung $M\,a\,N$ gilt dann für gleichartige, dagegen $M\,d\,N$ für entgegengesetzte Objecte. Einseitig begrenzte Geraden von unendlicher Länge aber entgegengesetzter Richtung bilden ein einfaches Beispiel, falls man als Teilmenge jeden ebenfalls unendlichen Bestandteil betrachtet und die Aequivalenz z. B. durch eineindeutige Aehnlichkeitsabbildung definirt. Für je zwei von ihnen besteht dann entweder die Relation (a) oder (d).

Man kann leicht erreichen, dass auch die Beziehungen (b) und (c) auftreten. Dies geschieht so, dass man auch *Paare* entgegengesetzt gerichteter Geraden als Objecte zulässt. Für je zwei solche Paare besteht dann die Beziehung (a), für jedes Paar und eine einzelne Gerade die Beziehung (b) oder (c), und für je zwei einzelne Geraden die Beziehung (a) oder (d). Die Gesetze

$$(a\,a) = (d\,d) = a, \quad (a\,d) = (d\,a) = d$$

bleiben offenbar bestehen. Beziehungen (bb), (bd), (dc), und (cc) sind unmöglich. Dagegen giebt es hier eine Regel für (bc); es kann sowol (a) wie (d) resultieren. Endlich ergeben die Beziehungen

$$(a\,b)\ (b\,a)\ (d\,b),\ (a\,c)\ (c\,a)\ (c\,d)$$

(b) oder (c) als Resultat.

Die Tatsache, dass die Cantorsche Theorie die Unvereinbarkeit der Annahme, M und N seien unendliche Mengen, mit der Beziehung $M\,d\,N$ des § 3 nicht nachtzuweisen vermochte, erfährt hierdurch neues Licht. Denn die Zulassung von Elementen von zweierlei Art, die einander entgegengesetzt sind, streitet weder gegen den Mengenbegriff als solchen, noch auch gegen die Dedekindsche Definition der unendlichen Mengen und die auf ihr ruhenden Eigenschaften. Für den so erweiterten Mengenbegriff kann aber, wie wir sahen, im Fall unendlicher Mengen auch die Beziehung $M\,d\,N$ realisiert sein. Wie weit sich auf solche Mengen die weiteren Begriffe und Sätze der Cantorschen Theorie übertragen lassen, mag an dieser Stelle auf sich beruhen bleiben.

Nur das sei noch erwähnt, dass die allgemeine Weiterführung der bisher gefundenen Resultate in erster Linie die Beziehung der Menge zu ihren Elementen, ferner den Ordnungsbegriff u.s.w. ins Auge zu fassen hat. Ich will noch kurz zeigen, wie man die Elemente der Menge auf der hier vorhandenen Grundlage einführen kann. Voranzustellen ist das folgende Axiom:

I. *Jede Menge enthält Teilmengen, die nicht mehr selbst in Teilmengen zerlegbar sind;* sie heissen unzerlegbare Teilmengen oder *Elemente.* Sie sollen durch

$$m\,T\,M \text{ oder kürzer durch } m$$

bezeichnet werden. Von ihnen gilt der Satz:

Ist $M \sim N$, so kann eine nicht zerlegbare Teilmenge von M keiner zerlegbaren Teilmenge von N aequivalent sein und umgekehrt.

Aus der Aequivalenz $M \sim N$ folgt nämlich nach Axiom I von § 3 zu jedem M' die Existenz einer Teilmenge N' von N, so dass

$$M' \sim N'$$

ist. Würde nun $m = M'$ ein zerlegbares N' bedingen und wäre N'' eine Teilmenge von N', so folgt aus $M' \sim N'$ gemäss demselben Axiom, dass N'' die Existenz einer Teilmenge von m bedingt, die zu N'' aequivalent ist; was aber einen Widerspruch darstellt.

Von diesem Tatbestand kann man nun wieder verlangen, dass er auch umgekehrt gilt; d. h. man kann fordern:

II. *Zwei Mengen M und N sind aequivalent, wenn jedem Element von M ein Element von N zugehört und umgekehrt.*

Dass diese Forderung an sich widerspruchsfrei ist, wurde ebengezeigt; dass sie auch den allgemeinen Axiomen genügt, die die Aequivalenzbeziehung regeln (§ 1, I und II, § 3, I, § 4, I), ist leicht zu sehen. Damit möge diese Betrachtung ihren Abschluss finden. Auf die Frage, wie mit der Einführung der Elemente und der neuen Aequivalenzbeziehung sich der axiomatische Aufbau ändern würde, soll hier nicht weiter eingegangen werden.

Jedenfalls entspricht die vorstehende Untersuchung den Forderungen, die im Anfang gestellt wurden. Sie sieht von allen Wortdefinitionen ab und benutzt ausschliesslich *Beziehungen* zwischen den Objecten, von denen sie handelt. Die Axiome liefern die Grundregeln für das Operieren mit ihnen. Gerade um dies deutlich hervortreten zu lassen, ist jedem Axiom und jedem Satz die ihm entsprechende formale Ausdrucksweise, also die Bindung, die die bezüglichen Beziehungen durch den Satz oder das Axiom erfahren, gegeben worden. Auch sind die einzelnen Axiome immer erst dann eingeführt worden, wenn sie für den Fortgang der Beweise nötig waren.

Mathematics. — "*Ueber eineindeutige, stetige Transformationen von Flächen in sich*". (Sechste Mitteilung [1]). By Prof. L. E. J. BROUWER.

(Communicated at the meeting of March 27, 1920).

Die in der fünften Mitteilung über diesen Gegenstand für die Kugel ausgeführte *Aufzählung aller Transformationsklassen* wird hier für die projektive Ebene erbracht werden.

Sei t eine eindeutige stetige Transformation der projektiven Ebene π in sich, k eine einseitige einfache geschlossene Kurve von π, h die Verdoppelung von k, G das in π von h umschlossene zweiseitige Gebiet, k' das Bild von k für t. Wir werden t *erster* oder *zweiter Art* nennen, je nachdem k' einseitig oder zweiseitig ist. Eine Transformation erster und eine zweiter Art können offenbar niemals derselben Klasse angehören.

§ 1. *Die Transformationsklassen erster Art.*

Sei T eine der beiden durch t bestimmten Abbildungen von $G+h$ auf die zweiseitige Verdoppelung β von π, G' bzw. h' das Bild von G bzw. h für T, I der Inhalt von β für eine bestimmte elliptische Massbestimmung in π. Alsdann ist, wenn wir G und β mit passenden Indikatrizen versehen, der Inhalt einer willkürlichen simplizialen Approximierung von G' gleich $\frac{2n+1}{2} I$, wo n eine nichtnegative ganze Zahl ist, welche wir den *Grad* von t nennen werden. Alle Transformationen erster Art, welche derselben Klasse angehören, besitzen offenbar denselben Grad.

Um auch die umgekehrte Eigenschaft zu beweisen, werden wir zwei Methoden angeben, von denen die erste vom Resultate der fünften Mitteilung über diesen Gegenstand Gebrauch macht, die zweite dem Beweisgange dieser Mitteilung parallel läuft.

Erste Methode. Wir konstruieren in G eine einfache geschlossene Kurve r_2 und eine innerhalb r_2 gelegene einfache geschlossene Kurve r_1, und bezeichnen das Innengebiet von r_1 met G_1, das Zwischengebiet von r_1 und r_2 mit G_2, das Zwischengebiet von r_2 und h mit

[1]) Vgl. diese Proceedings XI, S. 788; XII, S. 286; XIII, S. 767; XIV, S. 300; XV, S. 352 (1909—1912).

G_3. Wir werden t eine *gegen P reduzierte Transformation n^{ten} Grades* nennen, wenn T die Kurven h und r_1 je eineindeutig und beide mit dem gleichen Umlaufssinn auf den (β in zwei der Reihe nach mit den Graden n und $n+1$ überdeckte Hälften β_1 und β_2 zerlegenden) Grosskreis m, die Kurve r_2 auf den in β_2 gelegenen Pol P von m und die Gebiete G_2 und G_3 je eineindeutig auf das von P und m begrenzte Gebiet abbildet. Nach dem Resultate der fünften Mitteilung über diesen Gegenstand können wir zwei willkürliche gegen P reduzierte Transformationen n^{ten} Grades *unter Invarianz der durch dieselben bestimmten Abbildungen von r_2 und h* stetig ineinander überführen.

Hiermit ist aber unser Ziel erreicht: eine beliebige Transformation erster Art lässt sich nämlich durch stetige Modifizierung in eine gegen P reduzierte Transformation überführen, indem wir zunächst der Kurve h' die erforderliche Gestalt erteilen und sodann unter Invarianz aller Punkte von h' den Prozess zu Ende führen.

Zweite Methode. Wir werden t eine *normalisierte Transformation n^{ten} Grades* nennen, wenn *erstens* h' eine einfache geschlossene Kurve und eineindeutiges Bild von h ist (durch welches also β in zwei der Reihe nach mit den Graden n und $n+1$ überdeckte Hälften β_1 und β_2 zerlegt wird) und *zweitens* T eine einfach verzweigte Riemannsche Abbildung ist, deren Verzweigungspunkte alle in β_2 gelegen sind. In diesem Falle können wir in β_2 nach der LÜROTH-CLEBSCHschen Methode ein solches System von Verzweigungsschnitten mit dazu gehöriger Blätteranordnung anbringen, dass h' eine ganz im ersten Blatt gelegene Kurve wird. Aus dieser Bemerkung folgt unmittelbar, dass *alle normalisierten Transformationen n^{ten} Grades zur selben Klasse gehören.*

Wir werden t eine *kanonische Transformation n^{ten} Grades* nennen, wenn *erstens* h' ein Grosskreis und eineindeutiges Bild von h ist und *zweitens* n in G gelegene einander nicht treffende einfache geschlossene Kurven von T in solcher Weise in je einen einzigen Punkt von β übergeführt werden, dass die von diesen Kurven bestimmten Teilgebiete von G alle mit dem Grade $+1$ eineindeutig und stetig abgebildet werden, und zwar die nicht an h grenzenden auf die einfach oder mehrfach punktierte Kugel β, das an h grenzende auf eine von h' umschlossene, im allgemeinen ebenfalls punktierte Halbkugel. In diesem Falle können wir t *zunächst* mittels einer beliebig kleinen, alle Punkte von h' invariant lassenden stetigen Modifizierung in solcher Weise umformen, dass T eine einfach verzweigte Riemannsche Abbildung mit lauter, nicht nur in β, sondern auch in π, verschiedenen Verzweigungspunkten wird, und sodann mittels einer weiteren stetigen Abänderung in eine normalisierte Transformation überführen. *Mithin gehören auch alle kanonischen Transformationen n^{ten} Grades zur selben Klasse.*

Eine beliebige Transformation erster Art lässt sich aber durch stetige Modifizierung in die kanonische Form bringen: um dies zu bewerkstelligen, formen wir sie zunächst so um, dass h' ein Grosskreis und eineindeutiges Bild von h wird und wenden sodann unter Invarianz aller Punkte von h' die in der fünften Mitteilung über diesen Gegenstand erörterte Abänderungsmethode an, welche hier nur dahin zu ergänzen ist, dass a. a. O. S. 355 oben unter den Gebieten g_ν auch ein durch h begrenztes Gebiet g_{ν_h} auftritt, das für $a^{(\nu)}$ nicht nirgends dicht abgebildet wird, während wir mittels einer beliebig kleinen stetigen Modifizierung von $a^{(\nu)}$ erreichen können, dass kein weiterer Teil der Grenze von g_{ν_h} mit h zusammenhängt und dass das Bild von g_{ν_h} keine auf h' gelegenen Verzweigungspunkte aufweist; weiter tritt nebst den a.a.O. S. 359 und 360 unterschiedenen Gebieten erster, zweiter und dritter Art noch *ein einziges Gebiet vierter Art* auf, das eine der von h' umschlossenen Halbkugeln eineindeutig und stetig, entweder positiv oder negativ, überdeckt, während das a. a. O. im vierten Absatz von S. 359 angegebene Verfahren eventuell auch zu verwenden ist, um ein Gebiet zweiter bzw. dritter Art mit einem angrenzenden negativen bzw. positiven Gebiete vierter Art zu einem positiven bzw. negativen Gebiete vierter Art zu vereinigen. *Mithin gehören alle Transformationen erster Art n^{ten} Grades zur selben Klasse.*

§ 2. *Die Transformationsklassen zweiter Art.*

Sei wieder T eine der beiden durch t bestimmten Abbildungen von $G+h$ auf die zweiseitige Verdoppelung β von π und G' bzw. h' das Bild von G bzw. h für T, so wird β von einer willkürlichen simplizialen Approximierung von G' entweder überall mit einem geraden oder überall mit einem ungeraden Grade überdeckt. Im ersteren Falle werden wir t eine *gerade*, im letzteren Falle eine *ungerade Transformation zweiter Art* nennen. Die Transformationen einer Transformationsklasse zweiter Art sind offenbar entweder alle gerade, oder alle ungerade.

Sei θ die Fläche vom Zusammenhange der Kugel, welche aus π durch Identifizierung aller Punkte von k erhalten wird. Wir werden t eine *in k kontrahierte Transformation* nennen, wenn k' sich auf einen einzigen Punkt reduziert und zwar insbesondere eine *einfache in k kontrahierte Transformation,* wenn β für T von θ entweder mit dem Grade 0 oder mit dem Grade 1 überdeckt wird. Alsdann folgt aus dem Resultate der fünften Mitteilung über diesen Gegenstand unmittelbar, dass *alle einfachen in k kontrahierten Transformationen derselben Parität zur selben Klasse gehören.*

Eine beliebige Transformation t zweiter Art lässt sich aber durch stetige Modifizierung in die Form einer einfachen in k kontrahierten Transformation bringen: um dies zu bewerkstelligen, formen wir sie zunächst in eine in k kontrahierte Transformation um, wobei also h' sich auf einen einzigen Punkt P reduziert und β für T' von θ mit einem gewissen Grade m überdeckt wird, und modifizieren sodann t in solcher Weise weiter, dass h' der Reihe nach alle Lagen von zweimal durchlaufenen, durch P und den Gegenpunkt Q von P als Pole bestimmten Breitekreisen erhält, und sich schliesslich in Q zusammenzieht. In diesem Augenblicke wird β für T von θ entweder mit dem Grade $m+2$ oder mit dem Grade $m-2$ überdeckt: durch geeignete Einrichtung des Verfahrens können wir dafür sorgen, dass ein beliebig gewählter dieser beiden Werte erreicht wird. Hieraus folgt, dass wir durch passende Wiederholung desselben Prozesses t in eine einfache in k kontrahierte Transformation überführen können. *Mithin gehören alle Transformationen zweiter Art derselben Parität zur selben Klasse.*

§ 3. *Die Minimalzahlen der Fixpunkte.*

Weil einer eindeutigen stetigen Transformation von π in sich zwei eindeutige stetige Transformationen von β in sich entsprechen, welche nicht beide den Grad -1 besitzen, mithin nicht beide fixpunktfrei sein können[1]), so *besitzt eine eindeutige stetige Transformation der projektiven Ebene π in sich wenigstens einen Fixpunkt.*

Dass andrerseits *für keine Transformationsklasse von π die Minimalzahl der Fixpunkte mehr als 1 beträgt*[2]), erhellt aus der folgenden Transformation erster Art n^{ten} Grades:

$$\left\{\begin{aligned} tg\,\psi' &= tg\,\psi + \cos\varphi \\ \varphi' &= (2n+1)\,\varphi, \end{aligned}\right.$$

wo mit φ und ψ Länge und Breite auf β bezeichnet werden, und aus der folgenden geraden bzw. ungeraden Transformation zweiter Art:

$$\left\{\begin{aligned} \varphi' &= \varphi \\ \omega' &= 0 \text{ bzw. } \omega' = 2\omega, \end{aligned}\right.$$

wo mit φ und ω Länge und Polabstand auf β bezeichnet werden.

[1]) Vgl. Math. Annalen 71, S. 114.

[2]) Wegen der Beantwortung der analogen Frage für die Kugel und die beiden Ringflächen vgl. meine demnächst im Anschluss an einen Aufsatz von J. Nielsen in Math. Annalen 81 erscheinende Notiz: „*Ueber die Minimalzahl der Fixpunkte bei den Klassen von eindeutigen stetigen Transformationen der Ringflächen*".

Chemistry. — *"On the Symmetry of the* RÖNTGEN*patterns Obtained by means of Systems Composed of Crystalline Lamellae, and on the Structure of Pseudo-Symmetrical Crystals"*. By Prof. F. M. JAEGER.

(Communicated at the meeting of April 23, 1920)

§ 1. It is well known how SOHNCKE [1]) and MALLARD [2]), as a consequence of experiments formerly executed by NÖRREMBERG and VON REUSCH, have first tried to account for the optical properties of uniaxial, circularly-polarizing crystals, by the supposition that all such crystals are in reality only apparently higher-symmetrical intergrowths of very numerous, extremely thin, and often submicroscopical, crystalline lamellae of lower crystallographical symmetry. In many cases this supposition has afterwards been confirmed by experience; and just in the same way as in the experiment executed in 1869 by VON REUSCH, who demonstrated in a more or less perfect way the possibility of imitating the behaviour of uniaxial crystals endowed with rotatory power in the direction of their optical axis, by means of a number of *mica*-lamellae, regularly piled-up clock-wise or oppositely, while crossing under the same fixed angle, — thus the behaviour of such pseudo-symmetrical crystals, built up from microscopical lamellae, also appeared to approach the more closely to that of true tetragonal, trigonal, and hexagonal crystals, as the composing lamellae were thinner and more numerous. The complexes thus obtained are either dextro-, or laevo-gyratory, be it that the piling-up of the successive lamellae has occurred clockwise or in the opposed direction. During the investigations of HAGA and myself about the specific symmetry of the RÖNTGENpatterns obtained by diffraction of RÖNTGENrays in plane parallel plates of optically uniaxial crystals [3]), we had occasion to study also some pseudo-symmetrical crystals of this kind, which were characterised by more or less evident optical anomalies; and, while some of them, — e.g. the pseudo-tetragonal *strychnine-sulphate*, — gave RÖNTGENpatterns of so perfect a symmetry, that they could not be distinguished from those obtained with real tetragonal crystals, — we also found with some other crystals of this kind (racemic *triethylenediamine-cobalti-*

[1]) L. SOHNCKE, Zeits. f. Kryst. u. Miner, **19**, 529, (1899).
[2]) E. MALLARD, Ann. des Mines, (7), **19**, 256 (1881); *Traité de Cristallographie*, **II**. 262—304, (1881), H. POINCARÉ, *Théorie mathém. de la Lumière*, **II**, 275, (1892).
[3]) Cf. i.a.: H. HAGA and F. M. JAEGER, Proceed. Royal Acad. of Sciences, Amsterdam, **17**; 1204, (1915); **18**, 558, 1355, (1916).

bromide ($+ 3\ H_2O$), *benzile, apophyllite,* etc.), patterns, showing only a single plane of symmetry, and having, therefore, really the aspect of the patterns commonly obtained with monoclinic crystals, if cut parallel to a plane of their orthodiagonal-zône. On that occasion we also emphasized, that the cause of this abnormal behaviour might probably be ascribed to an imperfect orientation of the lamellae in one of the directions of intergrowths, these for the rest being equivalent. This imperfect orientation might then consist of either a slight rotation of the lamellae about one of their axes in the special direction mentioned above, or of a twinning of some of these lamellae. In all cases, however, it appeared necessary to make the supposition, that those particulars should have occurred more frequently in one of the directions of intergrowth, than in each of the others.

As it was our purpose to obtain a more exact insight into the real behaviour of such pseudo-symmetrical crystals composed of intergrown lamellae, with respect to the phenomenon of the diffraction of transmitted RÖNTGENrays, we have undertaken the study of the specific symmetry of the RÖNTGENpatterns, which could be obtained by means of systems of regularly piled-up *mica*-lamellae, in its dependency on the special structure of the used *mica*-complexes.

The results obtained, which are reviewed in the following pages, have in the first place shown some pecularities, pointing to a close analogy with the anomalies formerly found by us in the case of real pseudo-symmetrical crystals; on the other hand, however, the experience gained must necessarily lead to the conclusion that the views of SOHNCKE and MALLARD, — at least in so far, as *tetragonal* crystals endowed with circular polarisation be considered — cannot yet be considered to give a final explanation of the phenomena observed in these cases.

The RÖNTGENpatterns used here, have all been obtained in the Physical Laboratory of the University of Groningen by my colleague HAGA, to whom I wish once more here to offer my sincere thanks for his kind and expert help during this investigation.

§ 2. In these experiments, thin cleavage-lamellae of *muscovite*: $KH_2Al(SiO_4)_3$ were continually made use of. As is wellknown, this mineral has monoclinic-prismatic symmetry, with the parameters: $a:b:c: = 0.577 : 1 : 2,217$ and $= 84°55'$. This symmetry, however, very closely approaches a hexagonal one, the prism-angle of *muscovite* being 120°11′. A perfect cleavage occurs parallel to {001}, the preparation of very thin lamellae being thus extraordinarily facilitated. In this *mica*-species the optical axial plane is perpendicular to the crystallographical plane of symmetry (010); moreover, the first bisector is almost perpendicular to the plane of cleavage, while the dispersion ($\varrho > v$) also, differs only unappreciably from that of a rhombic crystal. The mineral is strongly birefringent (about: 0,038), with negative character.

In first instance now, the RÖNTGENpattern of a single lamella ($d = 0.32$ mm.) was obtained, as reproduced in fig 1 of Plate I; a stereographical projection of this beautiful and rich diffraction-image is given, moreover, in the diagram 1 of the text. It manifests the ordinary bilateral symmetry of the monoclinic crystals parallel to {001} or {100}; on more detailed examination, three directions may, moreover, be clearly discerned in it, which include angles of almost 60° with each other, and which are closely related to the hexagonal "radiation-figure" of this *mica*-species. obtained by pressure with a sharp object. The direction of the optical axial plane may also be discerned in it without much difficulty; it is indicated by a row of numerous smaller spots, situated perpendicularly to the plane of symmetry of the diffraction-pattern.

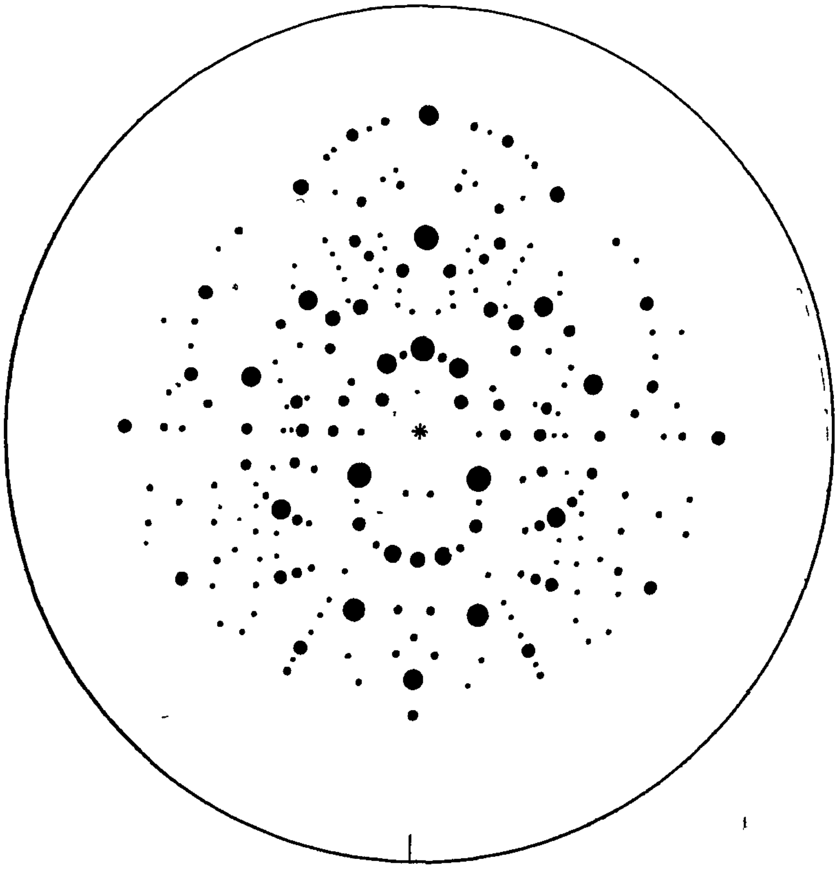


Fig. 1. Stereographical projection of the RÖNTGENpattern of a single *Muscovite*-lamella, parallel to {001}.

§ **3**. In the second place we investigated the behaviour of four preparations, these being dextro-, respectively laevogyratory *mica*-piles consisting of *muscovite*-lamellae crossing at 45° or 60°. The composing lamellae were cut from a *muscovite*-crystal in such a way, that their longer side was parallel to the optical axial plane of the mineral, their shorter edge thus being parallel to its plane of crystallographical symmetry. The central part of the complexes composed of hexagonally arranged lamellae, manifested in convergent polarized light between crossed nichols the almost perfect axial image of a uniaxial crystal endowed with circular polarisation; in the interference-image of the dextro- or laevogyratory complexes built up by lamellae crossing under 45°, there appeared only a single dark beam interrupted in the central part of the image, while also the coloured rings showed a somewhat elliptical distortion with a slight spiral constriction in the immediate vicinity of the dark beam. For the rest, the optical properties of the preparation appeared to vary quite continuously in all azimuths, being almost the same in all directions. The RÖNTGENpatterns obtained are reproduced in fig. 2 and 3 of Plate I, in the right position with respect to that

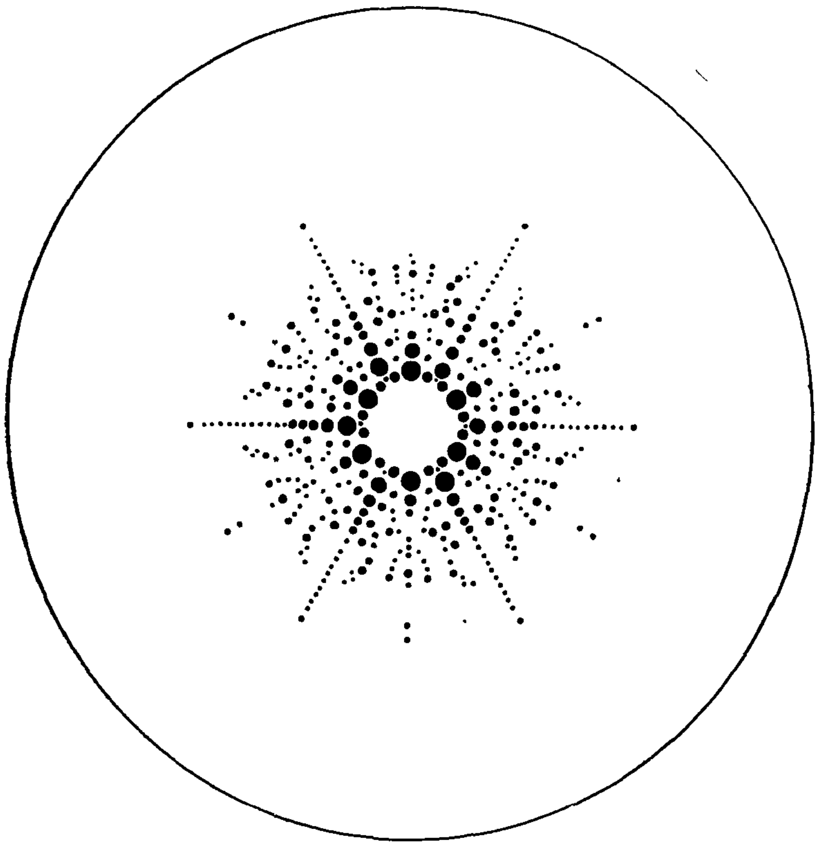


Fig. 2*a*. Stereographical Projection of the normal Diffraction-pattern of a dextro or laevogyratory Complex of *Muscovite*-lamellae, crossing at 60°.

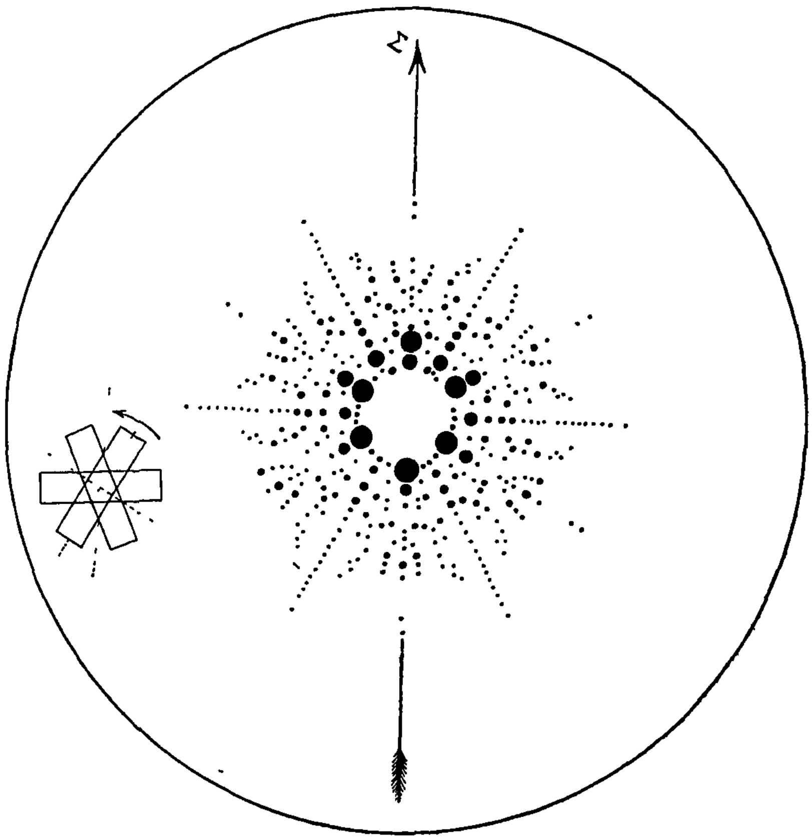


Fig. 2*b*. Stereographical Projection of the abnormal Diffraction-pattern of a dextro- or laevogyratory Complex of *Muscovite*-lamellae, crossing at 60°.

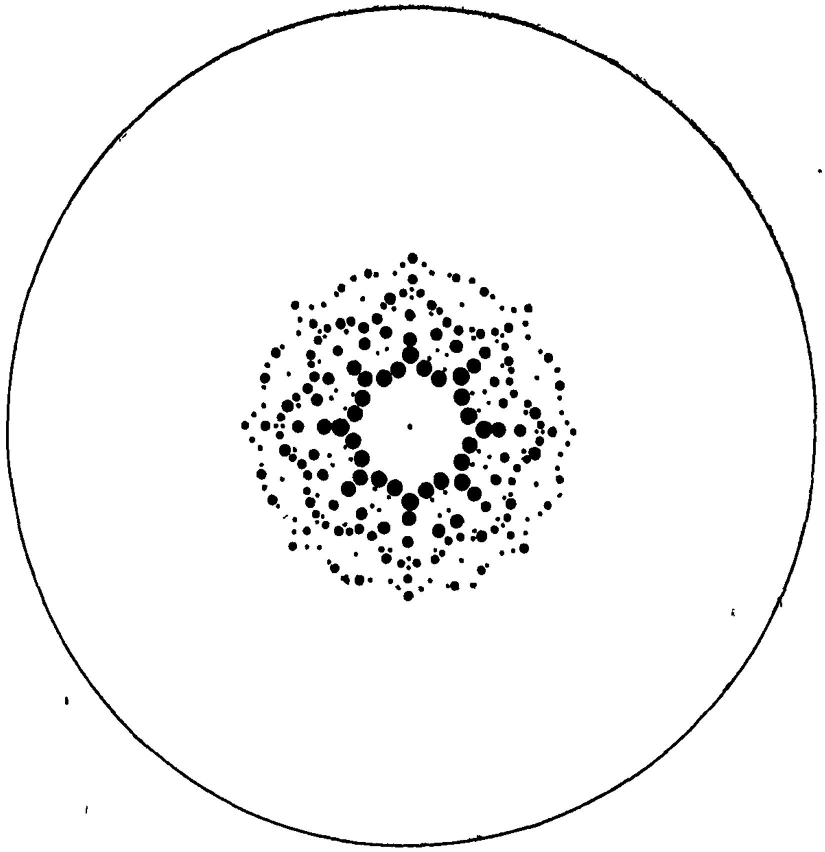


Fig. 3*a*. Stereographical Projection of the normal Diffraction-image of a dextro- or laevogyratory Complex of *Muscovite*-lamellae. crossing at 45°.

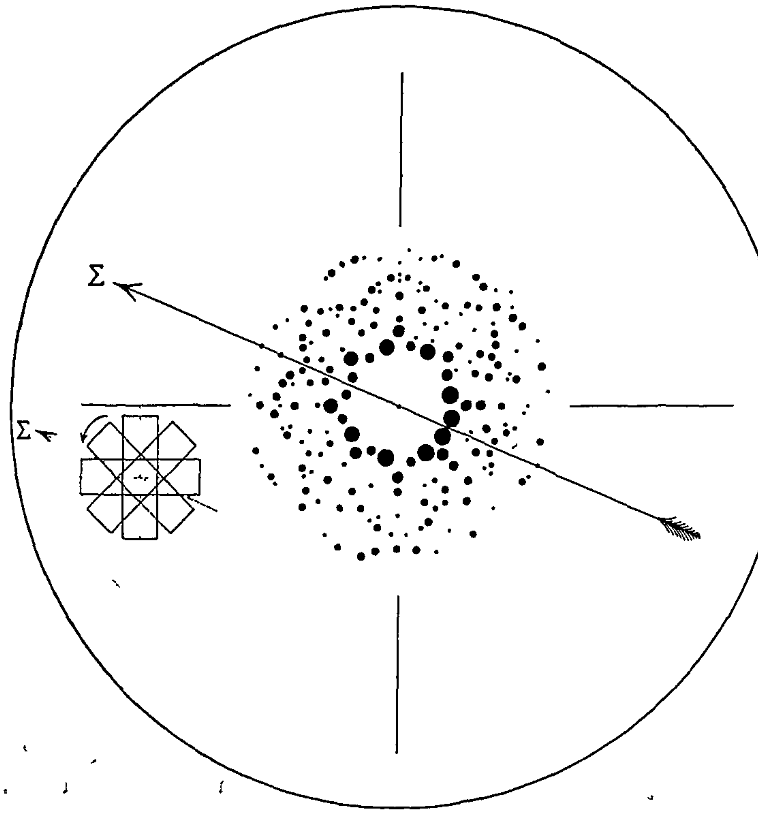


Fig. 3*b*. Stereographical Projection of the abnormal Diffraction-image of a dextro- or laevogyratory Complex of *Muscovite*-lamellae, crossing at 45°.

of fig. 1 of this plate, while the text-figures 2*a* and 2*b*, respectively 3*a* and 3*b*, represent stereographical projections, immediately relating to these diffraction-images. In all experiments the time of exposure of the photographic plates was two hours.

Secondly, we can remark, that the normal RONTGEN-images of fig. 2*a* and 3*a* show a perfect *hexagonal*, respectively *octogonal* symmetry, evidently consisting of a pattern repeated *six*, respectively *eight* times, the structure of which is, however, in the hexagonal image clearly *different* from that in the octogonal image, although the absorption of the RÖNTGENrays in these very thin lamellae plays only an insignificant part. Evidently the character of the composing patterns is here dependent in some way or other on the special way in which the secondary waves, emerging from the upper lamella, are modified by their passage through the next following lamella; and from this experience it seems, that this influence varies with the magnitude of the crossing-angle of two subsequent lamellae [1]).

[1]) On the modifications of a primary RÖNTGEN-pattern, if a secondary ray of it passes through a second and identically orientated crystalplate of the same sub-

The character of the whole pattern as that of an original figure repeated regularly a number of times, equal to the number of lamellae contained in a full turn of 360° (here, therefore 6 or 8), — was observed in all cases of normal diffraction-images; it can be considered as the *normal* character of the diffraction-patterns of such complex systems of lamellae.

Basing ourselves upon the experience gained in these and other cases, we may, therefore, safely enunciate as a general rule: *If the central part of a regular complex of crystalline lamellae, cut perpendicular to a plane of symmetry of the crystals, and crossing at angles* $\alpha = \frac{2\pi}{n}$, *be radiated through by* RÖNTGEN*rays, then the normal diffraction-pattern thus obtained, will exhibit an axis of n-fold symmetry, showing, therefore the image of an original pattern repeated n-times. The diffraction-image of the dextro- and laevogyratory complexes of this kind are always identical.*

§ 4. From what has been said, it must be concluded directly, that pseudo-*tetragonal*, circularly polarizing crystals can *not* be considered as built up in the way supposed by MALLARD, namely, if they do not consist of a substance, the molecules of which are themselves endowed with rotatory power. For it may be easily foreseen, that even in the case where the composing lamellae possessed *no* symmetry-plane whatever, the final diffraction-image will at least show an axis of *octogonal* symmetry, the eight planes of symmetry in fig. 3*a* then having disappeared. In this most general case of lamellae crossing at 45°, therefore, the pattern should all the same show an *octogonal* symmetry-axis, which, however, is *impossible* in crystallography, and which, in agreement with this fact, was never found by us in any diffraction-image of real or apparent tetragonal crystals. The RÖNTGENpatterns of any optically-inactive, pseudo-tetragonal crystal-species [1]), or those of optically-inactive, pseudo-tetragonal crystals

stance, conf. the paper of R. GLOCKER, Ann. der Physik, (4) **47**, 337, (1915). We have now started the systematical investigation of the phenomenon stated in the above, according to which the special character of the diffraction-image of such crossed lamellae varies with the angle φ, at which subsequent lamellae cross. From the fact, that the text figures 2—7 are drawn on the same scale as fig. 1, it will immediately be clear, that there can be *not* a mere superposition of images here, as e. g. a considerable number of the outer spots of fig. 1 have completely disappeared, even in so simple a case as that of fig. 4.

[1]) Conf. the pattern of *strychnine-sulphate*, in: F. M. JAEGER, *Lectures on the Principle of Symmetry and its Application in all natural Sciences*, 2nd Edition, Amsterdam, (1920), p. 194, 195. However, in this case the molecules of the substance have a rotatory power in solution also.

such as *potassium-ferrocyanide,* always manifested in their most complete and undisturbed form an axis of *four*fold symmetry at the highest, and the same appeared to be the case with all true tetragonal crystals hitherto investigated. But in such complexes of lamellae, an axis of *four*fold symmetry of the diffraction-image results only, when the composing subsequent lamellae include an angle of 90°, instead of 45°, as could be demonstrated e.g. by the pattern reproduced in fig. 4, obtained with a system of *muscovite*-lamellae, carefully arranged at 90°.

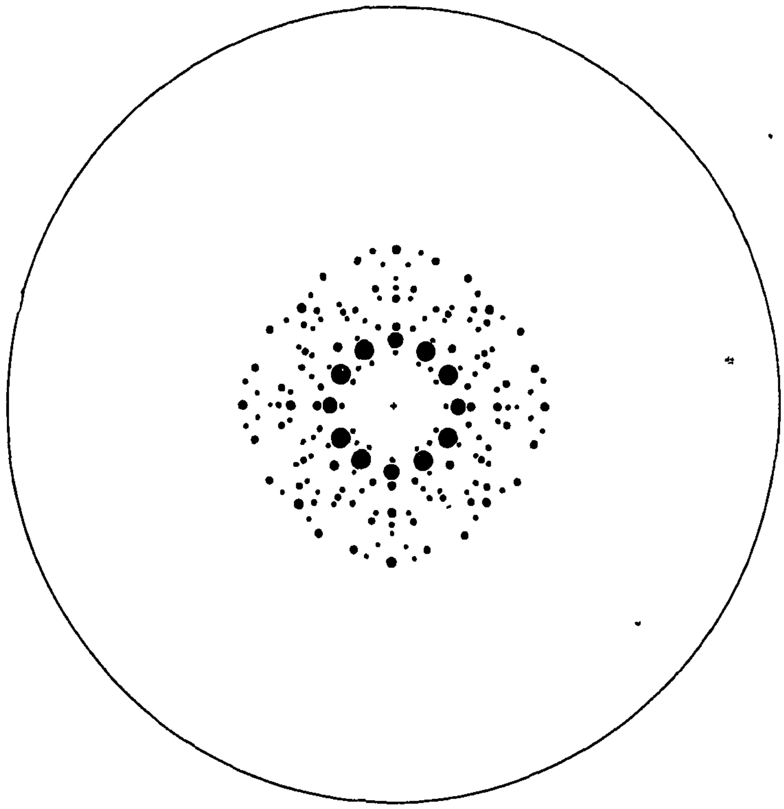


Fig. 4. Stereographical Projection of the RÖNTGEN-pattern of a Complex of *Muscovite*-lamellae crossing at 90°.

It must be concluded, therefore, that, if pseudo-tetragonal crystals be of the nature of polysynthetical intergrowths at all, the composing lamellae cannot cross at other angles than 90°. But from the mathematical theory of optical superposition [1]) it follows necessarily, — and the early experiments of NÖRREMBERG and others are in full agreement with this conclusion, — that such systems of lamellae

[1]) In 1906, at my request, professor LORENTZ was kind enough to develop once more the theory of the optical phenomena in systems of regularly piled-up lamellae. His results agree, although not quantitatively, yet in their principal features with those obtained by MALLARD and others.

crossing under angles of 90° will *never* manifest an optical rotation.

The supposition made by MALLARD is, therefore, only allowable for pseudo-tetragonal crystals *without* optical rotation, and there is no possibility to explain the special behaviour of real pseudo-tetragonal crystals endowed with rotatory power in this way, at least in those cases, where the molecules of the crystallised substance do not possess a molecular rotation of their own. It will be necessary to look for a special explanation in all cases concerning this kind of objects, as e.g. in that of the pseudo-tetragonal *ethylenediamine-sulphate,* etc.

§ 5. On closer examination of the original photographic plates of the patterns obtained with the hexagonal and octogonal complexes, which correspond with the projection-figures $2a$ and $3a$, it became evident that, although the situation of the spots on the plates completely agreed with that of the normal images in fig. $2a$ and $3a$, yet a distinct and rigorously determined abnormal distribution of their *intensities* was present, in such a way, that équivalent spots in the images did *not* possess the same intensity. Especially in the immediate vicinity of the centre, where very intensive spots were situated, the said phenomenon manifested itself most clearly. A more detailed study taught us, that this distribution of the intensities in the two images was, as drawn in the figures $2b$ and $3b$, i, e. *symmetrical with respect to only a single plane.* By special experiments it could be proved, that these anomalies did not depend on the position of the preparation with respect to the plane of the anticathode, or generally, to that of the luminous source: for on turning the preparation from its original position through 45° e.g., the plane of symmetry E in the images appeared to have turned also through the same angle on the new photograms. The cause of the said abnormalities must, therefore, be ascribed to the preparations themselves; and the close analogy of these anomalies with those formerly observed by us in real pseudo-symmetrical crystals, must be obvious, as also in those cases we observed a bilateral symmetry of the pattern, instead of the expected one, as was e.g. demonstrated with *rac. triethylenediamine-cobalti-bromide* and other preparations. The chief difference between these cases is, that in the preparations formerly studied, a number of spots were lacking altogether, their intensities being reduced to zero. Thus the bilateral symmetry of the patterns came there to expression in a higher degree, than was the case in our photograms which were obtained with objects, composed of a much smaller number of superimposed lamellae.

That it must be special properties of the preparations, that are the cause of such anomalies, becomes also evident from the fact, that the pseudo-symmetrical substances showing them, under favourable circumstances may occur in such well developed individuals, as to give perfectly *un*disturbed RÖNTGEN-patterns: thus e.g. with *potassium-ferrocyanide* in most cases certainly only bilaterally symmetrical images were obtained[1]), but occasionally there were found also perfect tetragonal patterns. And while *we* obtained an only bilaterally symmetrical RÖNTGEN-pattern with an apparently irreproachable individual of *benitoite* [2]) cut perpendicular to its optical axis, RINNE [3]) afterwards was able to obtain a quite normal trigonal diffraction-image of the same mineral.

Moreover, it was found that the direction of the single plane of symmetry in the RÖNTGENpattern was completely analogous in the two cases studied in the above: *its situation being in that of the hexagonal complex, as well as in that of the octogonal one, coinciding with the bisector of one of the angles of two subsequent lamellae of the mica-piles.* As the optical and microscopical investigation of the preparations did not reveal any abnormality in these directions, the only possible conclusion was, that the cause of this phenomenon must be attributed to some peculiarity in the lamellar arrangement.

In the case of the preparations with lamellae crossing at 60°, the explanation of the phenomenon may be given in the simplest way as follows.

The preparation of such *mica*-piles was hitherto executed only with the purpose of demonstrating the *optical* effects of such complexes: the apparent uniaxiality and the rotation of the plane of polarisation of the incident rays. Because of the fact that the optical orientation of each lamella does not differ appreciably from that of a rhombic crystal cut perpendicularly to its first bisector, it could be considered hitherto of no interest to the prepairer of such *mica*-piles, whether he piled up these lamellae in the same position as they were cut from the original crystal, or whether he turned them accidentally through 180° about an axis perpendicular to the plane of the lamella. For the final optical effect of the preparation will not be affected in the slightest degree by this turning. However, such a change of right and left, of the anterior or posterior part

1) The plane of symmetry being in these and other cases often parallel to the direction of the composing lamellae, contrary to what was observed here.

2) F. M. JAEGER and H. HAGA, Proceed. Acad. of Sciences Amsterdam. **17**, 1204 (1915).

3) F. RINNE, Miner. Centralblatt, (1919), p. 193.

of the lamellae, etc., is by no means indifferent any longer, if the distribution of the intensities of the spots in the diffraction-image by RÖNTGEN-rays be considered. For the *muscovite*-crystal has under all circumstances a true "monoclinic" molecular structure; the intensities of the diffraction-spots of a lamella parallel to {001} e.g., will, therefore, always be *different* to the left or to the right of the optical axial plane, while they will appear the same to the right or to the left of the plane of crystallographical symmetry. It must thus be evident, that the interchange of the two sides of a lamella in the way mentioned above, must really be of influence with respect to the special symmetry, which will be manifested in the distribution of the intensities of the diffraction-spots, as they will appear in the photographical image of the lamellar complex as a whole.

If the subsequent lamellae of a hexade are numbered 1 to 6, while the longer side of the lamellae, — as was really the case with our preparations, — is parallel to the direction of the optical axial plane in each lamella, then, if in the piling-up of the lamellae at angles of 60° first the lamellae 1 to 5 be taken in their right position, but N°. 6 be turned now through 180° in its own plane, the thus obtained hexade will give a diffraction-image, in which the intensities of the spots will be no longer distributed symmetrically with respect to six planes of symmetry, but in which *there is only a single plane* of this kind, exactly bisecting the angle between the superimposed pairs of lamellae: (1—4) and (2—5), and, therefore, being perpendicular to the pair: (3—6). By means of schematic figures, in which the distribution of the intensities of the spots, as effected by a single lamella is taken into account, it is possible to deduce systematically the general symmetry-character of the final distribution of intensities in the diffraction-image resulting from the complete hexade.

Undoubtedly such a reversion of a lamella will have accidentally occurred during the preparation of the *mica*-piles considered, just because there was no need for the prepairer to draw special attention to avoid such a reversion, and because with regard to his aim he was quite free to fix the subsequent lamellae in those positions, in which they accidentally were presented to him. Of course, there is a fair chance also, that during his work, he turned *two* or *three* lamellae in the way described; and it is necessary, therefore, also to consider the consequences of this for the final character of the diffraction-image, if all possible combinations of lamellae be in this way taken into account. In the case of such hexades, it appears unnecessary, however, to consider any combinations with a number of reversed lamellae greater than *three:* for it will be evident, that an accidental reversion of *four* lamellae, for example, will have the same effect as the turning through 180° of *two* lamellae, of *five* the same as if *one* were reversed, etc. These cases are, therefore, already contained amongst the possibilities formerly deduced in turning one, two, or three lamellae respectively.

A closer examination now taught us, that in the case of six *muscovite*-lamellae crossing at 60°, *three* kinds of diffraction-images might be produced: with respect to the intensities of the spots *normal* patterns (N); or such as are symmetrical with respect to a single plane bisecting the angle between two subsequent lamellae (diagonally-symmetrical; D); or finally such of the same symmetry, but in which the symmetry-plane now coincides with the direction of the lamellae themselves (lamellar-symmetrical; L). If *one* of the six lamellae be turned, there are *six* possibilities; if *two* be reversed, *fifteen* cases must be considered; and if *three* lamellae be turned through 180°, *twenty* possible combinations must be accounted for. In the first mentioned six cases only images with the bilateral symmetry D appear to be possible, as we found it just now in the case of fig. 2*b*; in the case of two reversed lamellae, we may find *three* combinations of pure *hexagonal*, normal symmetry, *six* combinations of diagonally-symmetrical character D, and *six* of lamellar symmetry L. In the last mentioned case of three reversed lamellae, we may find *two* possible combinations of normal character, here, however, not with hexagonal, but with *trigonal* symmetry; and *eighteen* combinations, corresponding to diagonally symmetrical diffraction-images D:

A Review of the Possible Types of Intensity-Distribution in the Diffraction-Patterns, Obtained by means of Mica-Complexes with Lamellae Crossing at 60°.

	If one lamella be turned:	If two lamellae be turned:	If three lamellae be turned:
Number of possible combinations:	6	15	20
Normal images: Asymmetrical Images: Diag. symm. Images: Lamell. symm. Images:	0. 0. 6. 0.	3 (hexag.). 0. 6. 6.	2 (trigon.). 0. 18. 0.

From this it appears, that a *mica*-complex piled-up arbitrarily and without special care, with lamellae crossing at 60°, will *never* produce a completely asymmetrical diffraction-pattern; and that there is an appreciably fair chance that the symmetry of it will be diagonally-symmetrical, as found in the case of fig. 2*b*; it is no wonder, that we just now met with *this* symmetry in the case of the preparation investigated in the above.

In the same way it is possible to deduce the possibilities to be expected, if the composing lamellae cross at 45°. However, because in such *mica*-piles there are always lamellae present perpendicular to, or coinciding with the geometrical symmetry-plane in one of the eight lamellae, the case of fig. 3*b* will never result from the reversion of a single lamella but only a lamellar symmetry *L* of the intensities can be produced thereby. A general review of the possible cases can be given as follows [1]):

A Review of the Possible Types of Intensity-Distribution in the Diffraction-Patterns, Obtained by Mica-Complexes with Lamellae Crossing at 45°.

	If one lamella be turned:	If two lamellae be turned:	If three lamellae be turned:	If four lamellae be turned:
Number of possible Combinations:	8	28	56	70
Normal images:	0.	4 (octogon.).	0.	6 (octogon.).
Asymmetrical Images:	0.	0.	16.	0.
Diag. Symm. Images:	0.	16.	0.	48.
Lamell. Symm. Images:	8.	8.	40.	16.

If only *two* lamellae be turned, there is an appreciable chance of a diagonally-symmetrical image, as found in fig. 3*b*; but if *four* lamellae be accidentally reversed, this chance is extremely great. For the rest, there are about equal probabilities for the bilaterally-symmetrical images *D* and *L*, both of which were observed formerly in the case of natural pseudo-symmetrical crystals.

§ **6**. A number of *mica*-piles were, moreover, prepared, in which the right orientation of the *muscovite*-lamellae was rigorously checked by comparison with their true position in a single *muscovite*-crystal [2]). First a dextro-, and a laevogyratory combination, in which the

[1]) My assistant Dr. A. Šimek was kind enough to check the number of these possible combinations systematically. I wish to express my best thanks to him here once more for the trouble he has given himself in this matter.

[2]) This crystal my colleague Prof. Bonnema most kindly gave me for this purpose from the mineralogical collection of the University.

lamellae were crossed at 120°, while attention was given to prevent a rotation of them through 180° about an axis perpendicular to the plane of cleavage. In these and the following preparations, the longer sides of the lamellae were always parallel to the geometrical plane of symmetry of the *muscovite*-crystal, contrary to what occurred in the *mica*-piles studied before. A normal image with a trigonal axis and three planes of symmetry passing through it, could be expected here beforehand. Because of the not wholly irreproachable material available, the patterns obtained were not suited for photographical reproduction; but notwithstanding this, it was possible to confirm the exactness of this prediction completely. A schematical projection of these patterns, which also in this case appeared to be identical for the dextro-, and laevogyratory complexes, is reproduced in fig. 5.

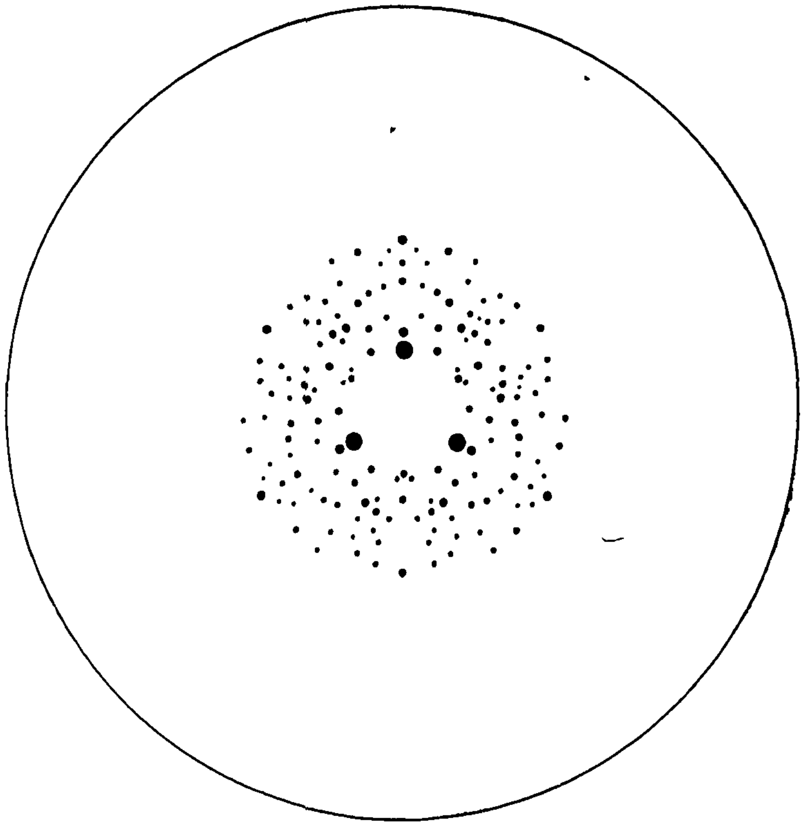


Fig. 5. Stereographical Projection (schematical) of the RÖNTGENpattern of dextro and laevogyratory *Mica*-piles, with Lamellae crossing at 120°.

Finally in fig. 6 the stereographical projection is reproduced (schematically) of two diffraction-images, obtained by two different *muscovite*-piles. In the first complex the lamellae crossed at 60°, and a rotation of the lamellae was carefully prevented; in the second preparation, however, the subsequent lamellae included angles of 120°, but each following lamella was turned with respect to the preceding through 180° about an axis perpendicular to its plane of cleavage. It will be easily understood that in this way the symmetry of the intensity-distribution in the two patterns must be essentially the same; it is remarkable, moreover, that also the patterns themselves appeared identical, notwithstanding the fact, that the sequence of

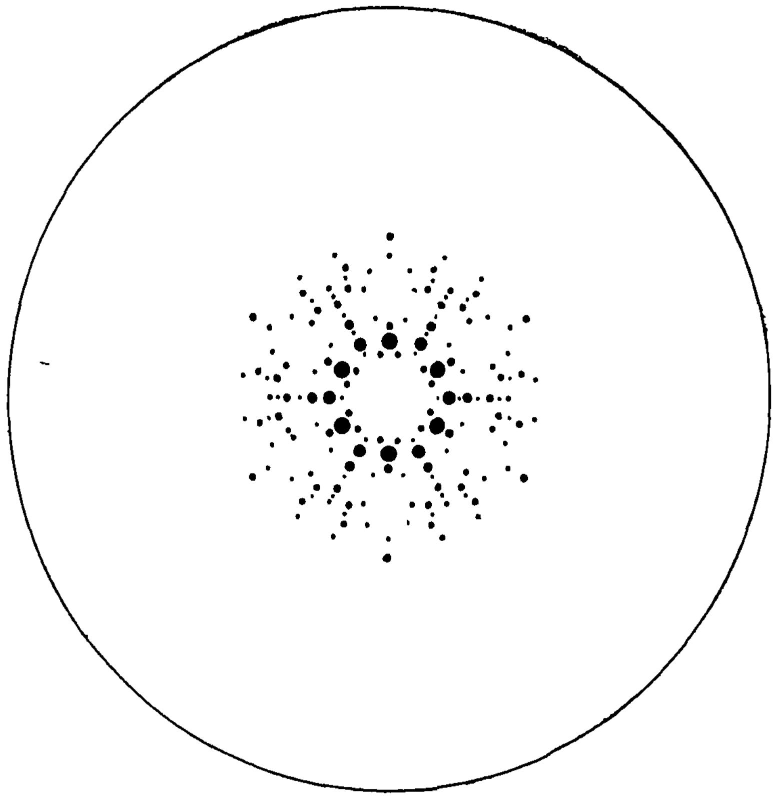


Fig. 6. Stereographical Projection (schematical) of the RÖNTGENpatterns of two *Mica* piles, the Lamellae of which crossed at angles of 60° and 120° respectively, with a partial reversion of some of them.

subsequent lamellae was different in the two *mica*-piles: both diffraction-images show a senary axis and six planes of symmetry passing through it.

§ 7. Regarding the results obtained in the above, hardly any doubt can remain as to the principal justification of our former view, according to which the observed abnormalities in the RÖNTGENpatterns of pseudo-symmetrical uniaxial crystals are in reality caused by a simple reversion of the position of the composing lamellae. Rotations of this kind may, for example, occur in some cases of *twin*-formation between those lamellae, if the axis of rotation or twinning be only no *real* symmetry-element of the crystallographical structure of the lamellae; at best it may be an axis of pseudo-symmetry of this structure. It is, therefore, by no means improbable, that finally *submicroscopical twinformation* between the lamellar units, composing the pseudo-symmetrical crystal, has to be considered as the primary cause of the anomalies formerly observed in the RÖNTGEN-diffraction-images of such crystals.

But then the question arises, why such a twinning-process happens oftener in one principal direction of intergrowth, than in the other equivalent ones? This question must arise, however, because in such crystals one has to deal *not* with a relatively *small* number of super-imposed lamellae in each direction, but with an *extremely great* number of them. It might be supposed that there were special influences during the growth of the crystal from its mother-liquor, which caused such a directing and preferential action in this respect: but it is at the moment difficult to guess, of what nature those influences really may be. Perhaps a factor of some importance therein may have been the *heat-effect* during the crystallisation, which causes convection-, and concentration-currents to appear in the environing liquor, corresponding in their turn to greater or smaller changes of the viscosity of the solution in those directions. It is well known, that the degree of viscosity of a medium plays an important rôle with respect to the occurrence of twins, and generally in such a way, that an increase of the viscosity appears most favourable to the occurrence of twin-formations. It is not improbable that influences of this kind may in the end appear to favour also the twinning of the very thin submicroscopical lamellae, of which the crystal is built up in one special direction.

Perhaps systematical investigations on the phenomena of crystallisation of such pseudo-symmetrical crystals under variable, but well-determined external circumstances, may in not too distant a future bring us better evidence on this subject.

Laboratory for Inorganic and Physical Chemistry of the University.

Groningen, April 1920.

Physiology. — "*On Adsorption of Poisons by Constituents of the Animal Body. I. The adsorbent power of serum and brain-substance for Cocain*". By L. EERLAND and W. STORM VAN LEEUWEN. (Communicated by Prof. R. MAGNUS).

(Communicated at the meeting of January 31, 1920).

In a previous paper[1]) STORM VAN LEEUWEN has shown that in the serum and the tissues of rabbits there are substances capable of inactivating pilocarpin. At the same time he was able to demonstrate that this does not happen by destroying pilocarpin, but through a physical adsorption of pilocarpin by certain components of the serum, whose nature could not be determined thus far. From quantitative investigations it also became evident that this physical adsorption proceeds according to the same laws that hold for the adsorption of dyes by animal charcoal.

In the paper alluded to just now, STORM VAN LEEUWEN has already pointed out that the adsorption of pilocarpin by rabbit's serum is not the only case of the kind, since many facts, described in the literature, render it highly probable that many similar adsorptions appear in the animal body. We know, for instance, that many poisons such as digitalis, atropin, cocain, strychnin etc. may be rendered inactive by animal tissue. This inactivation is commonly conceived to be a decomposition of the poison; we, however, believe that in many of those cases adsorption comes into play. True, in numerous cases poisons in the body are inactivated chemically, but we believe that this chemical action is in many cases preceded by a physical adsorption. The reason why we attach great importance to the question whether poisons are rendered inactive along the chemical path, or through adsorption, is that the great difference in the sensitivity of various individuals to poisons that bring about *a very quick, acute poisoning process,* can be accounted for by an adsorption, not by a chemical process.

The following example may serve to illustrate this:

---

[1]) W. STORM VAN LEEUWEN. Sur l'existence dans le corps des animaux de substances fixant les alcaloides Arch. Neerl. de Physiol. Tome 2 p. 650 1918.

It is well known that some people are less sensitive to the poisoning action of cocain than others. According to HATCHER and EGGLESTON [1]) cases are known in which 16 mgr. and 20 mgr. given subcutaneously were fatal, whereas in other cases 1.25 grms of cocain given subcutaneously had no effect whatever. HATCHER and EGGLESTON have proved conclusively that cocain, novocain and many other local anaesthetics become inactive very soon after being injected into an animal, while they have also demonstrated that various tissues, above all the liver, are able to decompose these poisons chemically. This, indeed was no novel experience, for BIER already found, when experimenting with rabbits, that cocain that has for some time been in contact with animal tissue, has thereby become less active, while SANO [2]) had come to the same conclusion for cocain with respect to brain-substance. BIER and SANO believed that this inactivation was caused by chemical decomposition.

HATCHER aud EGGLESTON's assertion that the liver can decompose cocain to a large extent, is incontestible. Still, this decomposing process cannot be so quick as to afford an explanation for the large differences in the sensitivity of different people. When after an injection of a few milligrammes of cocain the patient shows after a short time (a few minutes) serious symptoms of intoxication, the reason can *not* be that the cocain in his body is not decomposed quickly enough, for this decomposition cannot be so quick even with normal individuals. This, in fact, has also been pointed out by HATCHER and EGGLESTON themselves. Now it would seem to us that the abnormal sensitivity of some individuals to cocain might be explained as follows: When cocain is administered to a normal man or animal it will be used:

A in those places (i.a. the central and peripheral nervous system) where it exerts an influence.

B in other places (i.a. free chemoreceptors distributed in the blood). The sensitivity of a special individual to cocain will then be largely determined by the ratio between the number of the places of adsorption referred to under A and B. [3])

[1]) C. EGGLESTON and R. HATCHER. A further contribution to the pharmacology of the local anaesthetics. Journ. Pharm. and exp. Therap. vol. XIII. p. 433. 1919.

[2]) TORATA SANO. Ueber die Entgiftung von Strychnin und Kokain durch das Rückenmark. Ein Beitrag zur physiologischen Differenzierung der einzelnen Rückenmarks-abschnitte. Pflügers Arch. Bd. 120; p. 367. 1907.

TORATA SANO. Ueber das entgiftende Vermögen einzelner Gehirnabschnitte gegenüber dem Strychnin. Pflügers. Arch. Bd. 124, p. 369. 1908.

[3]) The places of adsorption sub *A* may be termed "*dominant* chemoreceptors", those sub *B* "*secondary* chemoreceptors."

In order to confirm this hypothesis it must first of all be ascertained whether the places mentioned sub B (i. e. the secondary chemoreceptors) really exist in the body.

In this paper we shall endeavour to settle this question with regard to cocain.

As already stated the researches of BIER, SANO, HATCHER and EGGLESTON, and others had already brought to light that cocain can be inactivated by animal tissue. It lay with us to show that this inactivation takes place through physical adsorption.

We had to proceed as follows:

1. We had to ascertain the action of a cocain solution of known strength on a special organ.

2. We had to show that the cocain solution became less active after the addition of animal tissue.

3. We had to demonstrate that the cocain was not decomposed in the less active mixture, so that all the active cocain could again be extracted from the mixture.

The effect of cocain upon the nervus Ischiadicus of the frog was taken as the index for cocain-action. We applied ZORN's [1]) method [2]), of which we give a brief description (see Fig. I).

The nerve of a nerve-muscle preparation is led through a small ebonite basin, which is to hold the cocain (and other liquids); on either side of the place where the nerve is in contact with the local anaesthetic, electrodes can be applied, which communicate with the secondary coil of an inductorium. By the aid of Pohl's commutator the nerve can be stimulated alternately by E′ and E″. First the position of the secondary coil is determined (to be read from S) at which the muscle can just be stimulated from E′ as well as from E″. Subsequently the liquid with the local anaesthetic is put in the basin, and after this we investigate how strong the solution must be in order to make the muscle after a certain time irresponsive to the stimulus from the electrode E′. The stimulus from E″ must retain its effect upon the muscle to make sure that during the experiment the excitability of the muscle itself is not diminished. We invariably experimented with a gastrocnemius-ischiadicus preparation of Rana esculenta. Due care was taken to keep the room-temperature constant. We made sure beforehand that the liquids used for

[1]) ZORN. Beiträge zur Pharmacologie der Mischnarcose. II. Zeitschr. f. exp. Path. und Ther. Bd. 12, p. 529 1913.

[2]) Cf. W. STORM VAN LEEUWEN. Physiologische waardebepalingen van geneesmiddelen. Wolters, Groningen, 1919.

55*

the cocain solution were in themselves indifferent to the nerve. This proved to be the case for 0.6 °/₀ Ringer, for 0.9 °/₀ Ringer, for serum as well as for an emulsion of brain-substance.

Experiment I. The liquid used was:

0.2 c.c. hydrochloras cocain (5 °/₀) + 4 c.c. 0.9 °/₀ Ringer's fluid, i.e. a solution of $^1/_4$ °/₀ cocain in 0.9 °/₀ Ringer.-

We found:

Accumulator 2 volts.

Stimulation at:	Reading taken of the inductorium with stimulation at (E')	Control (E'')	
3 00 h.	1.96	1.96	
3.02	1.96	1.96	
3.04	1.92	1.96	
3.06	1.92	1.96	
3.08	1.90	1.96	
3.10	1.90	1.96	
3.12	1.88	1.96	
3.14	1.86	1.96	
3.16	1.86	1.96	
3.18	1.86	1.96	
3.20	1.86	1.96	
3.24	1.82	1.96	
3.26	1.82	1.96	
3.28	1.80	1.96	
3 30	1.80	1.96	
3.32	1 78	1.96	
3.34	1.78	1.96	
3.36	1.74	1.96	
3.38	1.64	1.96	
3.40	1.40	1.96	
3.42	1.38	1.96	
3.44	1.34	1.96	
3.46	1 2	1.96	(muscle still responsive).

No contraction at the strongest current.

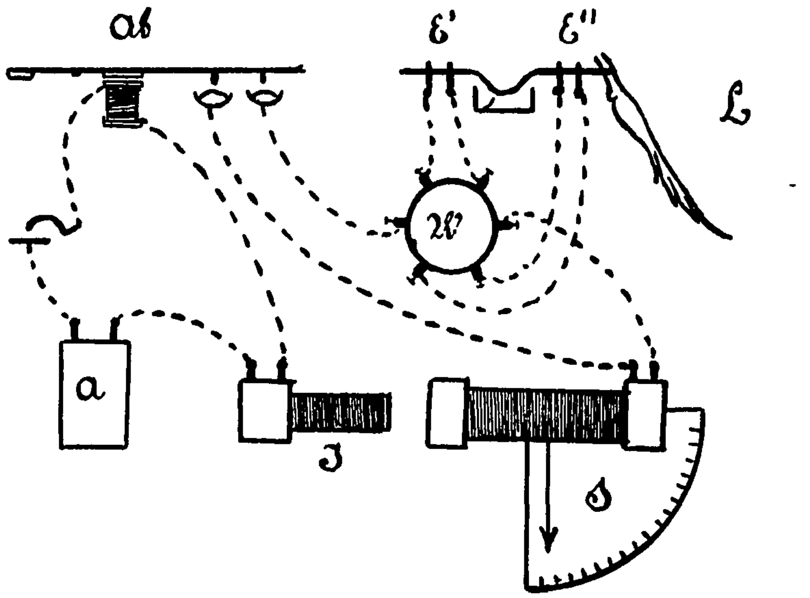


Fig. 1. Apparatus after ZORN (borrowed from a communication by ZORN).

So it appeared that the nerve had become irresponsive after 48 minutes by the effect of $^1/_4$ % cocain solution. The process of the experiment will be seen from the curve in Fig. 2.

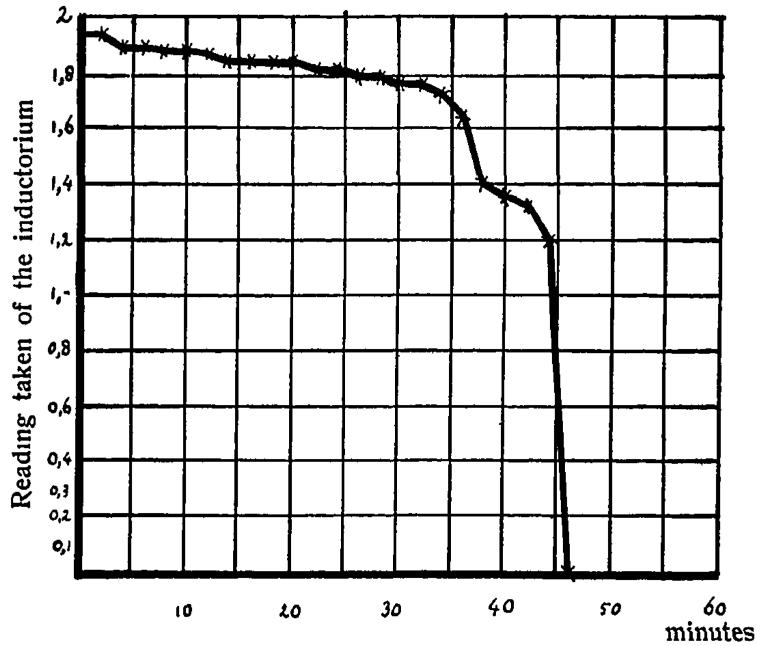


Fig. 2. Effect of $^1/_4$ % hydrochloras cocain upon the nervus ischiadicus of a muscle nerve preparation of Rana esculenta.

Abcissa: Time in minutes.

Ordinate: Stimulus required to make the muscle contract through indirect stimulation.

The same experiment was repeated several times to the following effect:

Exp. 2.	$^1/_4\,^0/_0$	cocain	solution	nerve	irresponsive	after	43	min.
,, 3.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	42	,,
,, 4.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	44	,,
,, 5.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	43	,,
,, 6.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	42	,,
,, 7.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	45	,,
,, 8.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	41	,,
,, 9.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	42	,,
,, 10.	$^1/_4\,^0/_0$	,,	,,	,,	,,	,,	43	,,

It follows, then, that on an average the nerve is irresponsive in $^1/_4\,^0/_0$ cocain solution in **43** minutes.

We now proceeded to ascertain the adsorbent power of human bloodserum.

Exp. 11. The liquid consisted of: 0,1 cc. 5 $^0/_0$ cocain solution + 1,9 cc. of serum, i.e. a concentration of $^1/_4\,^0/_0$ *cocain in serum.* In this case the muscle remained normally responsive for a whole hour, so that the effect of 5 mgr. cocain is eliminated by 2 cc. of human serum. The process of this experiment will be seen from the curve in Fig. 3.

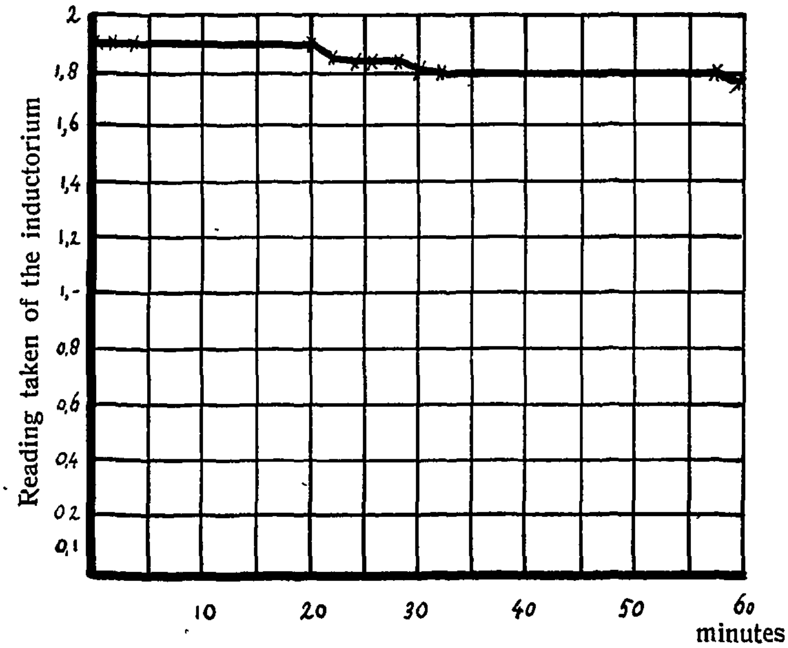


Fig. 3. Effect of $^1/_4\,^0/_0$ cocain in human serum on the nervus ischiadicus of a muscle nerve preparation of Rana esculenta.

After this Exp. 11 was repeated with dog's serum.

Exp. 12. The liquid used was 0.1 cc. 5 $^0/_0$ cocain + 2 cc. of dog's serum, that is about $^1/_4\,^0/_0$ cocain in serum: (see table p. 837. Here also the inhibiting influence of the serum can be seen distinctly. A similar result was obtained in *exp.* 13 with cat's serum and in *exp.* 14 with rabbit's serum. Hereafter we endeavoured to detach the cocain from the serum. To this end we used the liquid of exp. 12 and 13 to the following effect:

After 2 minutes' stimulation.	Reading taken of the inductorium (E')	Control (E'')
4	1.9	1.9
6	1.9	1.9
8	1.9	1.9
10	1.9	1.9
12	1.9	1.9
14	1.9	1.9
16	1.9	1.9
18	1.9	1.9
20	1.9	1.9
22	1.9	1.9
24	1.9	1.9
26	1.9	1.9
28	1.9	1.9
30	1.86	1.9
32	1.86	1.9
34	1.86	1.9
36	1.86	1.9
38	1.86	1.9
40	1.84	1.9
42	1.84	1.9
44	1.82	1.9
46	1.8	1.9
48	1.8	1.9
50	1.8	1.9
52	1.8	1.9
54	1.8	1.9
56	1.8	1.9
58	1.8	1.9
60	1.8	1.9

To 14 cc. of the liquid (serum + cocain) was added 1½ times the volume of alcohol 96 %, + 2 drops of HCL. This was centrifugalized and filtered, the filtrate was turbid. The precipitate was subsequently washed with alcohol and part of the alcohol was evaporated down in vacuo. After this the solution was acidified and shaken out twice with ether. The ether extract was then acidulated with $^1/_{10}$ N. HCL to get an aqueous cocain solution. This solution was again neutralized with bicarbonas natricus. With this liquid the experiment was repeated.

Exp. 15. We used the liquid of exp. 12 after extracting it with alcohol, the amount of cocain was calculated at about $^1/_4$ %.

Stimulation after:	Reading taken of the induct. (E')	Control (E'')
2 min.	1.96	1.96
4	1.96	1.96
6	1.9	1.96
8	1.74	1.96
10	1.68	1.96
12	1.68	1.96
14	1.68	1.96
16	1.66	1.96
18	1.6	1.96
20	1.56	1.96
22	1.52	1.96
24	1.5	1.96
26	1.4	1.96
28	1.38	1.96
30	1.32	1 96
32	1.26	1.96
34	1.2	1.96
36	1.18	1 96
38	1.1	1.96
40	(no longer any contraction).	1.96

So after 40 minutes the nerve was anaesthetic, from which it appears that through the treatment with acid and alcohol all the cocain adsorbed by the serum was detached. (Normal value for $^1/_4$ °/₀ cocain is 43 minutes).

Exp. 16. The liquid used is that of exp. 13 treated with alcohol and acid. Here also we found that after 40 minutes the muscle had lost its contractility, so that the result coincided with that of experiment 15.

In the following experiments we used a stronger solution of cocain, viz. $^1/_2$ °/₀ cocain.

Exp. 17. The liquid is 0.4 cc. 5 °/₀ cocain + 4 cc. Ringer 0.9 °/₀, consequently $^1/_2$ °/₀ cocain hydrochloricum.

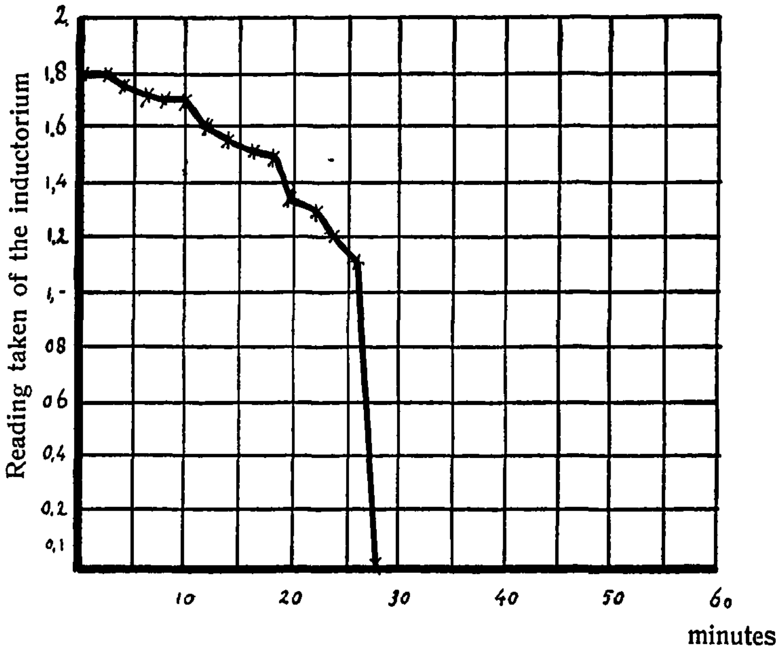


Fig. 4. Effect of $^1/_2$ °/₀ cocain hydrochloricum solution in Ringer's fluid (0 6 °/₀) on the nervus ischiadicus of a muscle nerve preparation of Rana esculenta.

The result of this experiment is represented by the curve in Fig. 4. After 28 minutes the nerve was irresponsive, while it appeared that, through stimulation with electrode E″, the muscle itself had remained responsive. Two other experiments yielded the same results.

Exp. 18. Liquid $^1/_2$ °/₀ cocain; nerve irresponsive after 30 min.

Exp. 19. Liquid $^1/_2$ °/₀ cocain; nerve irresponsive after 30 minutes. Average time in which the nerve becomes irresponsive with $^1/_2$ °/₀ cocain: 29½ min. When serum was added the adsorptive action revealed itself again distinctly.

Exp. 20. 0.4 cc. 5 °/₀ cocain + 4.5 cc. rabbit's serum, which is equal to ca. ½ °/₀ cocain hydrochloricum in serum.

Stimulation after:	Reading taken of the inductorium (E')	Control (E'')
2 min.	1.8	1.8
4	1.8	1.8
6	1.8	1.8
8	1.8	1.8
10	1.8	1.8
12	1.8	1.8
14	1.8	1.8
16	1.8	1.8
⋮	1.8	1.8
42	1.8	1.8
44	1.78	1.8
46	1.78	1.8
48	1.78	1.8
50	1.78	1.8
52	1.78	1.8
54	1.76	1.8
56	1.76	1.8
58	1.76	1.8
60	1.76	1.8

It wil be seen that we found distinct inhibition by serum also in this experiment, for after an hour the conductibility of the nerve had diminished only slightly.

This experiment was repeated (exp. 21), which again showed no anaesthesia of the nerve. Subsequently we inquired into the action of 1 °/₀ cocain.

Exp. 22. Liquid: 0.8 cc. 5 °/₀ cocain solution + 4.22 cc. Ringer 0.9 °/₀ equal to 1 °/₀ cocain hydrochloricum in Ringer's solution.

Stimulation after:	Reading taken of inductorium	Control
2 min.	1.9	1.9
4	1.9	1.9
6	1.88	1.9
8	1.84	1.9
10	1.8	1.9
12	1.7	1.9
14	1.6	1.9
16	1.4	1.9
18	1.3	1.9
20	1.2	1.9
22	(irresponsive)	1.9

Nerve does not respond any more after 22 min.

Exp. 23 1 °/₀ cocain irresponsive after 18 min.

Exp. 24 „ „ „ „ 22 „

Exp. 25 „ „ „ „ 20 „

From which we see that after about 20 minutes the conductibility of the nerve is eliminated by 1 °/₀ cocain hydrochloricum.

In experiment 26 and 27 we ascertained the influence of serum on the 1°/₀ cocain solution.

Exp. 26. Liquid: 1°/₀ cocain hydrochloricum in rabbit's serum:

Stimulation after:	Reading taken of inductorium E'	Control (E'')
2	1.8	1.8
4	1.8	1.8
6	1.78	1.8
8	1.78	1.8
10	1.74	1.8
12	1.74	1.8
14	1.74	1.8
16	1.72	1.8
18	1.66	1.8

(Table continued).

Stimulation after:	Reading taken of inductorium (E')	Control (E'')
20	1.64	1.8
22	1.6	1.8
24	1.56	1.8
26	1.5	1.8
28	1.46	1.8
30	1.4	1.8
32	1.2	1.8
34	1.16	1.8
36	1.1	1.8
38	—	1.8

After 38 minutes the nerve appears to be no longer responsive, so there must be distinct inhibition.

Exp. 27. Liquid: 0.8 cc. 5 % cocain solution + 4.2 cc. of cavia's serum i.e. to 1 % cocain hydrochloricum in cavia's serum.

Stimulation after:	Reading taken of inductorium (E')	Control (E'')
2	1.9	1.9
4	1.9	1.9
6	1.9	1.9
8	1.9	1.9
10	1.9	1.9
12	1.9	1.9
14	1.9	1.9
16	1.8	1.9
18	1.8	1.9
20	1.8	1.9
22	1.76	1.9
24	1.7	1.9

(Table continued).

Stimulating after:	Reading taken of inductorium (E')	Control (E'')
26	1.6	1.9
28	1.52	1.9
30	1.46	1.9
32	1.4	1.9
34	1.4	1.9
36	1.34	1.9
38	1.26	1.9
40	1.2	1 9
42	1	1.9
44	—	

The result of this exp. is similar to that of exp. 26, viz. only after 44 minutes irresponsiveness of the nerve.

The liquid of experiment 27 was treated with alcohol and acid as in experiment 15, and was used in *experiment* 28 (the cocain content was calculated at 1 °/₀). After 22 minutes the nerve was no longer responsive from which it appeared that (compare the results of experiments 23, 24, 25) through extraction with alcohol the cocain had been detached.

We now considered the question whether the behaviour of brain-substance toward cocain is similar to that of serum. To 5 grms of rabbit's brains was added 10 cc. of a 2 °/₀ solution of cocain in Ringer 0.6 °/₀. After standing for 30 minutes at room-temperature it was centrifugalized and the supernatant fluid was examined. A control experiment was made on 5 grammes of brains and 10 cc. of Ringer without cocain. The latter liquid proved to be indifferent to the nerve.

Exp. 29. Liquid: 5 grms of rabbit's brain-substance and 10 cc. 2 °/₀ cocain; contains 1,33 °/₀ cocain (see Table Exp. 27).

So it appears that after 50 min. the nerve has become anaesthetic. Since in the normal experiments with 1 °/₀ cocain anaesthesia appears after 20 minutes, we must conclude that also brain-substance inhibits cocain.

Exp. 30. Repetition of experiment 29 but with cat's brains.

Stimulation after:	Reading taken of inductorium	Control
2	1.9	1.9
4	1.9	1.9
6	1.9	1.9
8	1.9	1.9
10	1.9	1.9
12	1.9	1.9
14	1.9	1.9
16	1.8	1.9
18	1.7	1.9
20	1.7	1.9
22	1.7	1.9
24	1.7	1 9
26	1.7	1.9
28	1.7	1.9
30	1.7	1.9
32	1.7	1.9
34	1.7	1.9
36	1.6	1.9
38	1.5	1.9
40	1.5	1.9
42	1.5	1.9
44	1.4	1.9
46	1.26	1.9
48	1.2	1.9
50	—	1.9
52	—	1.9

Stimulation after:	Reading taken of inductorium	Control
2	1.8	1.8
4	1.8	1.8
6	1.8	1 8
8	1.8	1.8
10	1.8	1.8
12	1.8	1.8
14	1 8	1.8
16	1.8	1.8
18	1.8	1.8
20	1.8	1.8
22	1.7	1.8
24	1.6	1.8
26	1.5	1 8
28	1.5	1.8
30	1 5	1.8
32	1.4	1.8
34	1.3	1 8
36	1.3	1.8
38	1.3	1.8
40	1.3	1.8
42	1.3	1.8
44	1.3	1.8
46	1.3	1.8
48	1.3	1 8
50	1.3	1.8
52	1.2	1.8
54	1.1	1.8
55	—	1.8

From which we see that the nerve is irresponsive after 55 minutes. Here then there is also adsorption. In order to prove that the cocain is not decomposed, but adsorbed physically, brain-substance and cocain is treated with alcohol and acid as in experiment 15.

Exp. 31. Liquid: brain-substance + cocain after treatment with hydrochloric acid and alcohol, computed at 1 °/₀ cocain hydrochloricum.

Stimulation after:	Reading taken of inductorium	Control
2	1.8	1.8
4	1.8	1.8
6	1.7	1.8
8	1.68	1.8
10	1.6	1.8
12	1.5	1.8
14	1.44	1.8
16	1.4	1.8
18	1.4	1.8
20	1.3	1.8
22	1.1	1.8
24	—	1.8

Here, then, the cocain action manifests itself again, for the nerve is irresponsive after 24 minutes, so that no cocain has been decomposed by the brain-substance.

In order to show that from brain-substance, after extraction with hydrochloric acid and alcohol, no materials are abstracted which, of themselves, are deleterious to the nerve, so that thereby in experiment 31 the cocain action might have been intensified, we undertook a control exp. 32, in which a liquid was added to the nerve that was composed of 5 grms. of cat's brains and 10 c.c. RINGER 0.6 °/₀ and then extracted with hydrochloric acid and alcohol. This liquid again proved to be indifferent to the nerve, because within an hour the responsiveness had not diminished.

Exp. 33. This experiment is a repetition of exp. 31.

Liquid: Cat's brain-substance and cocain-solution equal to 1 °/₀ cocain hydrochloricum.

After 54 minutes the nerve is irresponsive, which again shows that the cocain action is inhibited by brain-substance.

Exp. 34. The liquid of exp. 33 was again treated with alcohol and hydrochloric acid.

Stimulation after:	Reading taken of inductorium	Control
2 min	1.9	1.9
4	1.8	1.9
6	1.7	1.9
8	1.68	1.9
10	1.66	1.9
12	1.64	1.9
14	1 64	1.9
16	1.62	1.9
18	1.6	1.9
20	1.56	1.9
22	1.4	1.9
24	1.2	1.9
26	1.1	1.9
28	—	1.9

We see from this that the cocain has again been detached. Of the cocain thus obtained. Dr. LE HEUX determined the melting point, which was 96,6° (uncorrected), which again proves that the cocain has not been decomposed (not even partially), but that only a physical adsorption has taken place. (Normal melting point of cocain hydrochl. 98°).

Since it had now become evident that brain-substance is capable of adsorbing cocain, we ascertained whether one of the familiar brain lipoids viz. lecithin [1]), could also exert this action.

Exp. 35. Liquid: 1 cc. 5 °/₀ lecithin solution + 1½ c.c. aqua distillata + 2½ c.c. RINGER (1.2 °/₀) without cocain. In this experiment the responsiveness of the nerve had hardly changed, from which we see that lecithin of itself does not injure the nerve.

Exp. 36. 1 cc. 5°/₀ lecithin solution + ½ cc. aq. dest. + 1 cc. 5°/₀ cocain + 2½ cc. Ringer (1.2°/₀), that is 1°/₀ cocain hydrochloricum in 1°/₀ lecithin.

[1]) The lecithin was supplied by MERCK.

Stimulation after:	Reading taken of inductorium (E')	Control (E'')
2 min.	1.9	1.9
4	1.9	1.9
6	1.9	1.9
8	1.9	1.9
10	1.9	1.9
12	1.9	1.9
14	1.9	1.9
16	1.9	1.9
18	1.9	1.9
20	1.9	1.9
22	1.9	1.9
24	1.9	1.9
26	1.9	1.9
28	1.9	1.9
30	1.9	1.9
32	1.9	1.9
34	1.8	1.9
36	1.72	1.9
38	1.6	1.9
40	1.5	1.9
42	1.3	1.9
44	1.12	1.9
46	—	1.9

The nerve is irresponsive after 46 minutes.

Exp. 36. Liquid: 3 cc. 2%$_0$ lecithin and 2 cc. Ringer (1.8%$_0$) and 1 cc. cocain solution 5%$_0$ is equal to 0.83%$_0$ cocain hydrochloricum in a 1%$_0$ lecithin solution.

Result: After 62 minutes the nerve is still responsive.

From experiments 35 and 36 it appears then that 50 mgrs of lecithin can inhibit the action of 50 mgrms of cocain considerably.

Exp. 37. Here we examined the influence of an ether extract of dried cat's brains. Of itself this extract is indifferent to the nerve, which after 60 minutes is still normally responsive.

Exp. 38. Liquid: 0.8 cc. 5 % cocain solution and 4.2 cc. extract of dried cat's brains, thus containing 1 % cocain.

Stimulation after:	Reading taken of inductorium (E')	Control (E'')
2 min.	1.8	1.8
4	—	—
⋮	—	—
46	1.8	1.8
48	1.7	1.8
50	1.7	1.8
52	1.68	1.8
54	1.6	1.8
56	1.5	1.8
58	1.42	1.8
60	1.3	1.8
62	1.1	1.8
64	—	1.8

Result: This extract proves to possess distinct inhibiting power, since only after 64 minutes the nerve becomes irresponsive (normally after 22 minutes).

CONCLUSIONS.

1. Our experiments produced evidence for our assertion that the action of cocain can be considerably inhibited by the addition of:
a. the serum of man, dog, rabbit and cavia;
b. the brain-substance of rabbit and cat;
c. ether-extract of dried cat's brains;
d. lecithin.

BIER's and SANO's experiments are hereby supported and extended.

2. This inhibition of cocain, is not brought about by a chemical decomposition of the cocain but by a physical adsorption; for, through extraction with hydrochloric acid and alcohol of a mixture with a reduced cocain action, all the cocain can be restored, which has still retained its activity. The melting point of this cocain also lies very near to normal values.

3. Serum, brain-substance and lecithin are of themselves not deleterious to the frog's nerve, nor when these materials (in control experiments) were extracted with hydrochloric acid and alcohol.

56*

Physics. — "*The Contributions from the Polarization and Magnetization Electrons to the Electric Current*". By Dr. A. D. FOKKER. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of June 27, 1919).

1. An important point in the theory of electrons is how to evaluate the electric current proceeding from the electrons which in their movements are bound to the atoms of matter. We require it for establishing the equations of the electromagnetic field in ponderable matter, and we know that it is responsible for the effects of polarization and magnetization.

Consider a stream of moving neutral atoms, and imagine them as consisting of a positive nucleus and one accompanying electron. The heavy nuclei will contain the centres of mass of the atoms their motion therefore will be identified with the motion of matter in bulk. The accompanying electrons will move round the nuclei or in their immediate neighbourhood. Now the stream of positive nuclei will form an electric current, and the stream of electrons of course will constitute another. For a great part these two currents will cancel one another, but not completely, as they would, if both motions were the same: the resulting current is clearly what arises from the intra-atomical motions of the bound electrons.

Obviously we shall know this current if, given the motion of a stream of particles, we can find the variation effected by displacing them slightly from their tracks, for it is by such small displacements that the motion of the electrons may be found from the motions of the nuclei. Our problem thus presents itself as a *variation problem*.

M. BORN has told us [1]) that the idea to put it thus is due to HERMANN MINKOWSKI. He has developed it after MINKOWSKI's death and compared his deductions with MINKOWSKI's posthumous notes. I venture to offer to the Academy a novel development of the same idea, which might claim a great simplicity and might be more exact in some points. In addition, a new second order contribution of the bound electrons is arrived at, which has been neglected until now, so far as I know (§§ 9 and 11).

---

[1]) H. MINKOWSKI—M. BORN, *Eine Ableitung der Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern vom Standpunkte der Elektronentheorie*, Math. Ann. 68, p. 526, 1910.

The Variational Displacements.

2. We consider a field of streaming discrete particles, the velocities being continuous functions of the coordinates and the time. We imagine a picture in a four-dimensional space-time-extension designing the motion-trails of the particles indicating their positions in successive instants. Now the displacements will consist of a shift in space and a shift in time, and we shall define these shifts with the aid of a field of a four-fold vector r^a, the components of which: $r^{(1)}, r^{(2)}, r^{(3)}$, being space-components and $r^{(4)}$ being the time-component, will be continuous functions of the coordinates and time x^a $(a = 1 \,..\, 4)$.

Mathematically, we define the shifts as the one-membered infinitesimal transformation group determined by the functions r^a $(a = 1 \,..\, 4)$, with parameter θ:

$$\Delta x^a = \theta r^a + \tfrac{1}{2}\theta^2 \sum_{1}^{4} (c)\, r^c \frac{\partial r^a}{\partial x^c} + \cdot \;\; \cdot \;\; \cdot$$

This will be clearer if we explain the nature of the r^a. If the variational parameter increases by an amount $d\theta$, then the particles are supposed as suffering an additional shift given by

$$r^a\, d\theta \quad (a = 1, 2, 3, 4),$$

the values of r^a being taken such as they are in the momentary point-instant occupied by the particle. Leaving out second order terms with θ^2, we at once see that the first approximation of the total shift will be

$$\theta\, r^a, \quad (a = 1 \,..\, 4),$$

and proceeding to second order terms we obviously get

$$\Delta x^a = \int_0^\theta \left\{ r^a + \Sigma\, (c) \frac{\partial r^a}{\partial x^c} r^c \vartheta \right\} d\vartheta,$$

$$\Delta x^a = \theta r^a + \tfrac{1}{2}\theta^2\, \Sigma\, (c)\, r^c \frac{\partial r^a}{\partial x^c},$$

where now the values of r^a and their derivatives have been taken in the point-instants of the particle's undisturbed motion.

The Variation of the Stream.

3. The following conception of the stream components will greatly facilitate our deductions.

Let N, a continuous function of space-time-coordinates, represent the density of the particles' distribution through space. At the instant

$x^{(4)}$, take an element of volume dV, situated at the point $x^{(1)}$, $x^{(2)}$, $x^{(3)}$. It will contain NdV particles. We assume that NdV is still a great number, notwithstanding dV being physically infinitesimal. Now, in our four-dimensional picture, consider the trails of these NdV particles, run through during an interval of time $dx^{(4)}$. These trails will cover an element of space-time-extension of magnitude $dVdx^{(4)}$. In the direction of the coordinate X^a the components will in the aggregate amount to

$$NdVdx^a.$$

It will readily be seen that the *streamcomponent in the direction of X^a is the aggregate of the X^a-components of the four-dimensional trails, run through by the particles per unit of volume per unit of time*:

$$\frac{N\,dV\,dx^a}{dV\,dx^{(4)}} = N\frac{dx^a}{dx^{(4)}} = Nw^a. \qquad (a = 1, 2, 3, 4).$$

We shall put $w^{(1)}$, $w^{(2)}$, $w^{(3)}$ for the components of the velocity: $dx^{(1)}/dx^{(4)}$, $dx^{(2)}/dx^{(4)}$, $dx^{(3)}/dx^{(4)}$. The fourth component equals unity: $w^{(4)} = dx^{(4)}/dx^{(4)}$, and accordingly the fourth streamcomponent $Nw^{(4)}$ is the number of particles per unit of volume.

It is obvious that the equation of continuity must be satisfied by these streamcomponents:

$$\Sigma\,(b)\,\frac{\partial Nw^b}{\partial x^b} = 0.$$

By the displacements the components will change to

$$Nw^a + \delta Nw^a + \tfrac{1}{2}\,\delta^2\,Nw^a,$$

where the first variation δNw^a is proportional to θ and the second variation $\delta^2 Nw^a$ will contain the second order terms with θ^2. It may be anticipated that the first variation will account for by far the greater part of the effects of polarization, whereas the second variation mainly gives the effects of magnetization.

4. We proceed to the evaluation of the *first variation*. Here we may consistently neglect θ^2.

The displacements will have changed the aggregate of the X^a-components of the trails under consideration: each dx^a passes into

$$dx^a + \Sigma\,(b)\,\frac{\partial\theta r^a}{\partial x^b}\,dx^b,$$

so that the aggregate becomes

$$N\,dV\left\{dx^a + \Sigma\,(b)\,\frac{\partial\theta r^a}{\partial x^b}\,dx^b\right\}.$$

On the other hand the four-dimensional extension covered by the trails has changed too: we find its magnitude by the aid of the JACOBIAN determinant:

$$(dV\,dx^{(4)})' = \begin{vmatrix} \dfrac{\partial(x^a+\Delta x^a)}{\partial x^a} & \dfrac{\partial(x^a+\Delta x^a)}{\partial x^b} & \cdot & \cdot \\ \dfrac{\partial(x^b+\Delta x^b)}{\partial x^a} & \dfrac{\partial(x^b+\Delta x^b)}{\partial x^b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} dV\,dx^{(4)},$$

$$= \begin{vmatrix} 1+\theta\dfrac{\partial r^a}{\partial x^a} & \theta\dfrac{\partial r^a}{\partial x^b} & \cdot & \cdot \\ \theta\dfrac{\partial r^b}{\partial x^a} & 1+\theta\dfrac{\partial r^b}{\partial x^b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} dV\,dx^{(4)} = \left[1+\Sigma(b)\,\theta\frac{\partial r^b}{\partial x^b}\right] dV\,dx^{(4)}$$

We must divide by this, and so when we follow the displacement, we may state a change of the streamcomponent into

$$Nw^a + \Delta\,Nw^a = \left[Nw^a + \Sigma(b)\,Nw^b\,\theta\frac{\partial r^a}{\partial x^b}\right].\left[1-\Sigma(b)\,\theta\frac{\partial r^b}{\partial x^b}\right].$$

But this is not the thing we want. This value is found in the point-instant $x^a + \Delta x^a$, after the displacement. We require the variation of the stream which we get if we stick to one and the same point-instant x^a both when the particles are shifted and when they are not. It is clear that the shifted particles which will by the displacement get to our point, had their starting-points elsewhere, in a point-instant which may be found if in the formula for Δx_a we change θ into $-\theta$.

So we have to correct the above expression by accounting for this different starting-point: instead of Nw^a we are to take

$$Nw^a - \Sigma(b)\frac{\partial Nw^a}{\partial x^b}\theta r^b,$$

and we get

$$Nw^a + \delta Nw^a = Nw^a + \Sigma(b)\left[-\frac{\partial Nw^a}{\partial x^b}\theta r^b - Nw^a\,\theta\frac{\partial r^b}{\partial x^b} + Nw^b\,\theta\frac{\partial r^a}{\partial x^b}\right].$$

Availing ourselves of the equation of continuity, we may put our result in the symmetrical form:

$$\delta Nw^a = \Sigma(b)\frac{\partial}{\partial x^b}\{\theta r^a\,Nw^b - \theta r^b\,Nw^a\}.$$

This formula is also given by BORN. It may be found, without deduction however, in a paper by LORENTZ [1]).

5. *The second variation* is easily found, without calculation, by submitting the first variation in turn to the operation which we had to apply to Nw^a in order to find δNw^a. So we get without difficulty

$$\delta.\delta\, Nw^a = \Sigma\,(b)\frac{\partial}{\partial x^b}\{\theta r^a\,\delta Nw^b - \theta r^b\,\delta Nw^a\},$$

$$\delta^2\, Nw^a = \Sigma\,(bc)\frac{\partial}{\partial x}\left\{\theta r^a\frac{\partial}{\partial x^c}[\theta r^b\, Nw^c - \theta r^c Nw^b] - \theta r^b\frac{\partial}{\partial x^c}[\theta r^a Nw^c - \theta r^c Nw^a]\right\}.$$

It is, however, important to remark that this formula implies the accurate definitions of the displacements as given in § 2. This can be verified by a direct deduction, following throughout the same line of argument as in the case of the first variation. We may refrain from reproducing the calculus, but it will be good to point out, that one has to develop the Jacobian with the required exactness up to the terms with θ^2, and, above all, that at the last step to be taken one has to be careful to choose the right starting-point from where the displacements will carry the particles to the point under consideration, viz.,

$$x^a - \theta r^a + \tfrac{1}{2}\,\theta^2\,\Sigma\,(c)\, r^c\frac{\partial r^a}{\partial x^c},$$

and *not* $x^a - \Delta x^a$, as we might be tempted to take.

Next, we have to give an interpretation of our mathematical result in physical terms such as polarization and magnetization.

The Simultaneous Displacements.

6.1. Before turning to the physical interpretation we must look closer into the nature of our displacement vector r^a and the underlying assumptions.

We are to assume, that the trails of the electrons can be found from the trails of the nuclei with the aid of the vectors r^a, in the indicated manner. First of all, we have taken these to be continuous functions of the coordinates. This implies that neighbouring atoms are supposed as having their electrons at similar distances in similar directions, the positions of the electrons relative to the nuclei varying but extremely slowly from one atom to the next. Of course this will not

[1]) H. A LORENTZ, HAMILTON's *Principle in* EINSTEIN's *theory of Gravitation*, Proc. R. Ac. of Amsterdam, 19, p. 751, 1915.

exactly or even nearly exactly correspond to reality. But we can commit no essential error by assuming the atoms as behaving in such a continuous way.

Secondly, we must observe, that the only reality we are concerned with is the aggregate of trails of nuclei and electrons, and that the choice of the vectors r^a is entirely arbitrary provided they furnish us with the right motion of the electrons relative to the nuclei. Obviously the choice can be made in a great many different ways. Sometimes it will be suitable to choose the r^a such that the time-component $r^{(4)}$ vanishes in all points where matter is in a stationary state. We need not specify a particular choice.

6.2. As yet the displacements considered have been accompanied by a shift in time. In view of the physical interpretation of the formulae obtained, it will however be necessary to realize the simultaneous positions of the electrons relative to the nuclei.

Now, in a first approximation, we find the electron belonging to the nucleus, which at the instant $x^{(4)}$ is in the point $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, shifted to the point

$$x^{(1)} + \theta r^{(1)},\ x^{(2)} + \theta r^{(2)},\ x^{(3)} + \theta r^{(3)}$$

at the instant

$$x^{(4)} + \theta r^{(4)}.$$

Thus we see that its position at the time $x^{(4)}$ will be given by

$$x^{(1)} + \rho^{(1)},\ x^{(2)} + \rho^{(2)},\ x^{(3)} + \rho^{(3)},$$

where

$$\rho^a = \theta r^a - w^a \theta r^{(4)}.$$

For an obvious reason $\rho^{(4)} = 0$.

Next, to obtain the second approximation, consider the nucleus at the instant

$$x^{(4)} - \theta r^{(4)} - \Sigma\,(c) \left\{ \tfrac{1}{2}\,\theta^2 r^c \frac{\partial r^{(4)}}{\partial x^c} - w^c\,\theta^2 r^{(4)} \frac{\partial r^{(4)}}{\partial x^c} \right\},$$

when its coordinates are

$$x^a - w^a \theta r^{(4)} - w^a\,\Sigma\,(c) \left\{ \tfrac{1}{2}\,\theta^2 r^c \frac{\partial r^{(4)}}{\partial x^c} - w^c\,\theta^2 r^{(4)} \frac{\partial r^{(4)}}{\partial x^c} \right\} + \tfrac{1}{2} \frac{dw^a}{dx^{(4)}}\,\theta^2\, r^{(4)}\, r^{(4)}.$$

This line implies the preceding as a special case, for $a = 4$.

Then the displacements of the electron will be

$$\theta\, r^a + \tfrac{1}{2}\,\theta^2\, \Sigma\,(c)\, r^c \frac{\partial r^a}{\partial x^c} - \theta^2\, \Sigma\,(c)\, w^c\, r^{(4)} \frac{\partial r^a}{\partial x^c}.$$

so that its actual position will be given by

$$x^a + \theta r^a - w^a \theta r^{(4)} + \tfrac{1}{2}\frac{dw^a}{dx^{(4)}}\theta^2 r^{(4)} r^{(4)} + \Sigma\,(c)\left\{\tfrac{1}{2}\theta^2 r^c\left(\frac{\partial r^a}{\partial x^c} - w^a\frac{\partial r^{(4)}}{\partial x^c}\right) - \right.$$
$$\left. - \theta^2 r^{(4)} w^c\left(\frac{\partial r^a}{\partial x^c} - w^a\frac{\partial r^{(4)}}{\partial x^c}\right)\right\}.$$

Taking $a = 4$, this formula yields the instant $x^{(4)}$, for $w^{(4)} = 1$, and all terms vanish except the first.

So we see that for $a = 1, 2, 3$ it gives the simultaneous displacements.

We can simplify considerably. Writing

$$\Sigma\,(c)\, w^c \frac{\partial}{\partial x^c} = \frac{d}{dx^{(4)}},$$

$$\Sigma\,(c)\, \theta r^c \frac{\partial}{\partial x^c} = \Sigma\,(c)\, \varrho^c \frac{\partial}{\partial x^c} + \theta r^{(4)} \frac{d}{dx^{(4)}},$$

we get for the simultaneous displacements:

$$s^a = \varrho^a - \tfrac{1}{2}\theta r^{(4)}\left[\frac{d\varrho^a}{dx^{(4)}} - \Sigma\,(c)\,\varrho^c\frac{\partial w^a}{\partial x^c}\right] + \tfrac{1}{2}\,\Sigma\,(c)\,\varrho^c\frac{\partial \varrho^a}{\partial x^c}. \qquad (6.2)$$

For $a = 4$ we have $s^{(4)} \equiv 0$.

6.3. Let us inquire what will be the polarization of matter, viz. the electrical moment per unit of volume. The electrical moment of one atom being es^a, where e is the charge of an electron, the answer is, in a first approximation, that the polarization has components

$$N e s^a, \qquad (a = 1, 2, 3).$$

Proceeding more carefully, we must take some closed surface, a sphere, say, sum up the electrical moments of the atoms within and divide by the volume. But what about the border atoms, which are intersected by the sphere? Must we leave them out, or must we reckon them as lying within the sphere? The difference will be of second order only, but it does make a difference.

A similar question has been raised by Lorentz in his Theory of Electrons (note 53). Lorentz decides himself to leave out the intersected atoms, and this is certainly right when we restrict ourselves to the first order terms, neglecting θ^2. But here we retain θ^2. Fortunately, our calculus leads us to the answer: it will show a correction to be made to the same effect as establishing the rule: the atoms are to be reckoned as lying within the surface, whenever more than half of the line joining nucleus and electron lies within the surface. This is a quite satisfactory rule.

Thus the polarization is:

$$N e s^a - \tfrac{1}{2}\,\Sigma\,(c)\,\frac{\partial\, N e s^a s^c}{\partial x^c}. \qquad (6.3)$$

6.4. The magnetic momentum of an atom has the components

$$\frac{1}{2c}e\left(s^a\frac{ds^b}{dx^{(4)}}-s^b\frac{ds^a}{dx^{(4)}}\right),$$

Hence the components of the magnetization are

$$cm^{ab}=\tfrac{1}{2}\,N\,e\left(s^a\frac{ds^b}{dx^{(4)}}-s^b\frac{ds^a}{dx^{(4)}}\right). \tag{6.4}$$

It is possible this ought to be corrected in the same way as shown for the polarization. The correction, however would be of the third order and contain θ^3; and this we drop throughout our investigation.

For this same reason we are justified in replacing s^a by ϱ^a in the expression for the magnetization.

Interpretation of the Variation of the Stream.

7. If e be the charge carried by an electron, then the current carried by the electrons is

$$e\,Nw^a+e\,\delta\,Nw^a+\tfrac{1}{2}\,e\,\delta^2\,Nw^a,\qquad (a=1,2,3,4).$$

Adding the current carried by the nuclei, viz. $-eNw^a$, we get for the resulting current:

$$e\,\delta\,Nw^a+\tfrac{1}{2}\,e\,\delta^2\,Nw^a.$$

Our results indicate that this can be written as a divergency of a skew-symmetrical tensor T^{ab}:

$$e\,\delta\,Nw^a+\tfrac{1}{2}\,e\,\delta^2\,Nw^a=\Sigma\,(b)\,\frac{\partial T^{ab}}{\partial x^b},$$

where T^{ab} is given by

$$T^{ab}=e\theta(r^a\,Nw^b-r^b\,Nw^a)+\tfrac{1}{2}\,e\theta^2\left\{r^a\,\Sigma\,(c)\frac{\partial}{\partial x^c}(r^b\,Nw^c-r^c\,Nw^b)-\right.$$
$$\left.-\,r^b\,\Sigma\frac{\partial}{\partial x^c}(r^a\,Nw^c-r^c\,Nw^a)\right\},$$

and

$$T^{ab}=-\,T^{ba}.$$

We shall see what this tensor contains. First writing

$$\frac{T^{ab}}{e}=(\varrho^aNw^b-\varrho^bNw^a)+$$

$$+\tfrac{1}{2}\,\theta r^{(4)}w^a\,\Sigma\frac{\partial}{\partial x^c}(\varrho^b\,Nw^c-\varrho^c\,Nw^b)-\tfrac{1}{2}\,\theta r^{(4)}\,w^b\,\Sigma\frac{\partial}{\partial x^c}(\varrho^a\,Nw^c-\varrho^c\,Nw^a)+$$

$$+\tfrac{1}{2}\,\varrho^a\,\Sigma\frac{\partial}{\partial x^c}(\varrho^b\,Nw^c-\varrho^c\,Nw^b)-\tfrac{1}{2}\,\varrho^b\,\Sigma\frac{\partial}{\partial x^c}(\varrho^a\,Nw^c-\varrho^c\,Nw^a),$$

we can arrange terms in such a way as to get

$$T^{ab} = w^b Ne\left[\varrho^a - \tfrac{1}{2}\theta r^{(4)}\,\Sigma\left\{w^c\frac{\partial\varrho^a}{\partial x^c} - \varrho^c\frac{\partial w^a}{\partial x^c}\right\} + \tfrac{1}{2}\Sigma\,\varrho^c\frac{\partial\varrho^a}{\partial x^c}\right] - \tfrac{1}{2}w^b\,\Sigma\,\frac{\partial Ne\varrho^a\varrho^c}{\partial x^c} -$$

$$- w^a Ne\left[\varrho^b - \tfrac{1}{2}\theta r^{(4)}\,\Sigma\left\{w^c\frac{\partial\varrho^b}{\partial x^c} - \varrho^c\frac{\partial w^b}{\partial x^c}\right\} + \tfrac{1}{2}\Sigma\,\varrho^c\frac{\partial\varrho^b}{\partial x^c}\right] + \tfrac{1}{2}w^a\,\Sigma\,\frac{\partial Ne\varrho^b\varrho^c}{\partial x^c} -$$

$$- \tfrac{1}{2}Ne\varrho^a\,\Sigma\,\varrho^c\frac{\partial w^b}{\partial x^c} + \tfrac{1}{2}Ne\varrho^b\,\Sigma\,\varrho^c\frac{\partial w^a}{\partial x^c} + \tfrac{1}{2}Ne\left[\varrho^a\,\Sigma\,w^c\frac{\partial\varrho}{\partial x^c} - \varrho^b\,\Sigma\,w^c\frac{\partial\varrho^a}{\partial x^c}\right].$$

We recognize the simultaneous displacements (6.2), and find

$$T^{ab} = w^b\left\{Nes^a - \tfrac{1}{2}\,\Sigma\,\frac{\partial Nes^a s^c}{\partial x^c}\right\} - w^a\left\{Nes^b - \tfrac{1}{2}\,\Sigma\,\frac{\partial Nes^b s^c}{\partial x^c}\right\} -$$

$$- \tfrac{1}{2}Nes^a\,\Sigma\,s^c\frac{\partial w^b}{\partial x^c} + \tfrac{1}{2}Nes^b\,\Sigma\,s^c\frac{\partial w^a}{\partial x^c} + \tfrac{1}{2}Ne\left(s^a\frac{ds^b}{dx^{(4)}} - s^b\frac{ds^a}{dx^{(4)}}\right).$$

8. Taking $b = 4$, some terms vanish, and we get

$$T^{a4} = Nes^a - \tfrac{1}{2}\,\Sigma\,(c)\,\frac{\partial Nes^a s^c}{\partial x^c}.$$

Remembering what has been found about the polarization in (6.3), we at once see that T^{a4} $(a = 1, 2, 3)$ are the *components of the polarization.* Thus the polarization is no 4-dimensional vector: its components are the space-time-components of a tensor.

When neither a nor b have the value 4, then the part of T^{ab} containing the polarization:

$$w^b\left\{Nes^a - \tfrac{1}{2}\,\Sigma\,\frac{\partial Nes^a s^c}{\partial x^c}\right\} - w^a\left\{Nes^b - \tfrac{1}{2}\,\Sigma\,\frac{\partial Nes^b s^c}{\partial x^c}\right\}$$

is nothing else but a component of the well known RÖNTGEN-*vector,* which in three-dimensional analysis is written [**p** . **w**], where **p** and **w** are the three-dimensional polarization and velocity vectors. We see that in our tensor the *components of polarization are always accompanied by the components of the corresponding* RÖNTGEN-*vector.*

9. In another part of T^{ab} $(a \neq 4,\ b \neq 4)$, viz.

$$cm^{ab} = \tfrac{1}{2}\,Ne\left(s^a\frac{ds^b}{dx^{(4)}} - s^b\frac{ds^a}{dx^{(4)}}\right),$$

we recognize the *components of magnetization.*

The remaining part however:

$$- \tfrac{1}{2}\,Nes^a\,\Sigma\,(c)\,s^c\frac{\partial w^b}{\partial x^c} + \tfrac{1}{2}\,Nes^b\,\Sigma\,(c)\,s^c\frac{\partial w^a}{\partial x^c},$$

indicates the existence of *a new effect.* It is of the second order and therefore has been neglected by LORENTZ[1]) and by CUNNINGHAM[2]). BORN does not separate it from the magnetization. But we can imagine an experiment (see below) where this effect will manifest itself apart from magnetism. So we shall keep these terms apart.

Here the quadratic electric moments of the atoms appear:

$$e s^a s^b,$$

the same quantities which occur in recent papers of DEBIJE and HOLTSMARK on the broadening of spectral lines from luminous gases under increased pressures.[3]) Half the sum of these quantities per unit of volume we shall call the *electrical extension* of matter, unless a better name be proposed. If an atom contains more than one electron, then we can have an electrical extension without polarization. We denote it by

$$K^{ab} = \tfrac{1}{2} N e s^a s^b.$$

and the corresponding part of the tensor can be written

$$k^{ab} = -\Sigma (c) \left\{ K^{ac} \frac{\partial w^b}{\partial x^c} - K^{bc} \frac{\partial w^a}{\partial x^c} \right\}.$$

10. In order to review the results reached thus far, let us gather them in a scheme, and let us for convenience' sake use rectangular coordinates x, y, z; t for the time, and three-dimensional notations for the (three-dimensional) vectors of polarization, magnetization, and velocity: $\mathbf{p}, \mathbf{m}$ ($\mathbf{m}_x = m^{23}$, etc.) and $\mathbf{w}$. In addition, write ${}^2\mathbf{K}$ for the three-dimensional extension tensor, and for the new vector $\mathbf{k}$:

$$\mathbf{k} = -[({}^2\mathbf{K} \cdot \nabla) \cdot \mathbf{w}],$$

where $({}^2\mathbf{K}.\nabla)$ is an operator having vector properties. Thus $\mathbf{k}_x = k^{23}$, etc. Then the contents of the tensor T^{ab} are:

$$\begin{array}{c|cccc} \overset{\to b}{\downarrow a} & & & & \\ & & c\mathbf{m}_z + \mathbf{k}_z + [\mathbf{p}.\mathbf{w}]_z & -c\mathbf{m}_y - \mathbf{k}_y - [\mathbf{p}.\mathbf{w}]_y & \mathbf{p}_x \\ & -c\mathbf{m}_z - \mathbf{k}_z - [\mathbf{p}.\mathbf{w}]_z & & c\mathbf{m}_x + \mathbf{k}_x + [\mathbf{p}.\mathbf{w}]_x & \mathbf{p}_y . \\ T^{ab}: & c\mathbf{m}_y + \mathbf{k}_y + [\mathbf{p}.\mathbf{w}]_y & -c\mathbf{m}_x - \mathbf{k}_x - [\mathbf{p}.\mathbf{w}]_x & & \mathbf{p}_z \\ & -\mathbf{p}_x & -\mathbf{p}_y & -\mathbf{p}_z & \end{array}$$

Applying the formula for the current from the bound electrons:

---

1) Encyclopaedie der Mathem. Wissenschaften.

2) The Principle of Relativity, Camb. Univ. Press.

3) P. DEBYE, *Das molekulare elektrische Feld in Gasen*, Phys. Ztschr. **20**, p. 160, 1919.

J. HOLTSMARK, *Ueber die Verbreiterung von Spektrallinien*, ib. p. 162.

See also P. DEBIJE, *Die* VAN DER WAALS*schen Kohasionskrafte*, Phys. Zschr. 21, p. 178, 1920.

$$\Sigma\,(b)\frac{\partial T^{ab}}{\partial x^c},$$

and putting it in the right hand members of the equations of the field, we get for the fundamental equations for moving non-conducting media, in three-dimensional vector notation:

$$rot\,\mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = rot\,\mathbf{m} + \frac{1}{c}\,rot\,\mathbf{k} + \frac{1}{c}\,rot\,[\mathbf{p}.\mathbf{w}] + \frac{1}{c}\dot{\mathbf{p}},$$

and

$$div\,\mathbf{E} = -\,div\,\mathbf{p}.$$

These are LORENTZ' equations with the addition of $rot\,\mathbf{k}$ to the current. We see a polarization current $\dot{\mathbf{p}}$, a RÖNTGENcurrent $rot\,[\mathbf{p}.\mathbf{w}]$ and the current of magnetization $rot\,c\mathbf{m}$.

A proposed Experiment.

11. Let us inquire further into the nature of the second order current

$$rot\,\mathbf{k}.$$

Referring to the definition:

$$\mathbf{k} = -\,[({}^{s}\mathbf{K}\,.\,\nabla)\,.\,\mathbf{w}],$$

we see that it is an effect due to the non-uniformity of motion in matter where the atomical charges lie outside one another. If these charges had fixed positions, i.e. if the electrons were rigidly fixed between the nuclei and if they therefore could be said to have exactly the motion of matter in bulk (i.e. of the nuclei, or rather motions interpolated between the nuclei) then our calculus indicates, that there would be no current resulting from the charges: the streams of positive and negative particles cancelling each other.

But in this case, the motion of matter being non-uniform, the electrons clearly would turn round the nuclei in an absolute sense, and the atoms would have a magnetic momentum. It is the part of $\mathbf{k}$ to counterbalance this slight magnetization, it then equals $c\mathbf{m}$ with opposite sign.

On the other hand, in case the electrons, instead of being rigidly fixed in the frame of the nuclei, always kept the same distance and in the same direction from the nuclei, not turning round in the rotating motion of matter, then $\mathbf{k}$ comes into play, not being balanced by a slight magnetization; so an induction field will be produced.

It should be possible to keep the electrons in the same direction from the nuclei by applying an electric field and maintaining a constant polarization. A rotatory motion then should produce an induction. We must be careful, however, to separate this from the

RÖNTGEN-effect, by eliminating the latter. This might be done in the following way:

Take a sphere of insulating material, which is mounted to perform rotatory oscillations round a vertical axis. Surround its equator by a circuit fixed in space. Apply an electric field of constant horizontal direction, and the oscillations of the sphere must induce an oscillating current in the circuit.

The effect will be small, but it should be detectable with the aid of the modern detectors of radiotelegraphy. It will be proportional to the square of the electric field applied.

It might be pointed out that a comparison of the effect with the produced polarization, would provide us with means to determine the number of electrons per atom, which are involved in the polarization, because, for a given polarization, the displacement **s** of the electrons is inversely proportional to the number n of displaced electrons per atom, and so the effect of **k** per electron is inversely to n^2. Materials with the same di-electric constant should show the effect to a degree inversely proportional to the number of polarizing electrons per atom.

Spontaneous Electric Polarization of Moving Magnets.

12. Though we have used in the title of this paper the denominations "Polarization and Magnetization Electrons", yet it is well known that it is impossible to make a rigorous distinction between the two. For even though there may be in some cases electrons which only produce polarization and no magnetization, there can be no electron which gives rise to a magnetization and never produces polarization.

In fact, whenever magnetized matter moves in a direction perpendicular to the magnetization, then it shows a polarization at right angles both to magnetization and motion.

The explanation runs as follows. A magnetic atom contains electrons sweeping round the nucleus, in circles, say, with uniform velocity, under the actions of electromagnetic forces. When the atom acquires a motion in the plane of the circling electrons, then the forces are modified in a way given by the theory of electrons and of relativity. The effect of this alteration of the forces will be that the orbit is no longer a circle, and becomes an ellipse, and that the velocity changes in such a way that the electrons during a longer time stay in one part of the ellipse near an end of the long axis than in the other. This clearly results into a polarization.

We shall call this the *polarization of moving magnetism.* It explains why no current is set up in a moving magnet on account of a motion perpendicular to its own internal induction field, so that with sliding contacts no current can be taken off. Thus, e. g., if we take a circular spring, the two ends pressing together, we can put a long magnet into it. Suppose that we can draw the magnet across the ring, the ends of the spring giving way and making a sliding contact: there will arise no current in the ring if we do it.

Again, this polarization is responsible for the electric force set up in a homogeneous magnetic field if the magnets producing the latter acquire a uniform motion at right angles to the field. The magnetic field may remain stationary and homogeneous: nevertheless an electric force will be induced by the motion of the magnets.

Afterwards these problems will be treated more adequately when we shall have explained the character of our deductions from the relativity point of view (see below § 20).

Then we shall also define a distinction between the di-electric polarization which is independent in itself, and the polarization of moving magnetism.

The Invariancy of the Results.

13. Thus far we did not want to refer to a single theorem of the theory of relativity to deduce our results. Nevertheless they possess the property of complete invariancy, not only in EINSTEIN-MINKOWSKI's theory of restricted relativity, but also in EINSTEIN's theory of general relativity. We proceed to show this.

This theory ascribes to a four-dimensional track the length ds:

$$ds^2 = \Sigma\,(ab)\,g_{ab}\,dx^a\,dx^b,$$

if dx^a $(a = 1 \,..\, 4)$ define the increments of the coordinates and time. The determinant of the g_{ab} is called g, and its minors divided by g are denoted g^{ab}.

What is the character of Nw^a? Remembering the definition (§ 3:)

$$Nw^a = \frac{\sqrt{g}\,N\,dV\,dx^a}{\sqrt{g}\,dV\,dx^{(4)}},$$

we notice that $N\,dV$ is a number, dx^a is a contravariant vector and $\sqrt{g}\,dV\,dx^{(4)}$ constitutes a scalar. Thus Nw^a is a contravariant vector multiplied by $\sqrt{g}$.

θr^a is a contravariant vector too, and so

$$\theta r^a\,Nw^b - \theta r^b\,Nw^a$$

is an skew-symmetrical contravariant tensor, multiplied by $\sqrt{g}$. (This is sometimes called a volume-tensor or a tensor-density, after WEYL). Then we know that

$$\delta\, Nw^a = \Sigma\,(b) \frac{\partial}{\partial x^b} \{\theta r^a\, Nw^b - \theta r^b\, \delta\, Nw^a\}$$

is the contravariant vector-divergency of this tensor, multiplied by $\sqrt{g}$, and thus of the same nature as Nw^a itself.

In like manner the second variation

$$\delta^2\, Nw = \Sigma\,(b) \frac{\partial}{\partial x^b} \{\theta r^a\, \delta Nw^b - \theta r^b\, \delta Nw^a\}$$

is a contravariant vector multiplied by $\sqrt{g}$ again.

It follows that our results are in complete accordance with relativity theory in the most general sense, and we are justified in applying any theorem of that theory.

Having thus recognized the true character of our tensor, we shall henceforth write $\sqrt{g}\ T^{ab}$ instead of T^{ab}

$$\sqrt{g}\ T^{ab} = e\theta r^a\, Nw^b - e\theta r^b\, Nw^a + \tfrac{1}{2}\, e\{\theta r^a\, \delta\, Nw^b - \theta r^b\, \delta\, Nw^a\}.$$

This will cause no confusion.

We must further keep in mind that w^a is no four-dimensional vector, but $w^a\, dx^{(4)}/ds$ is. We shall not introduce a new notation for this velocity vector.

The General Covariant Equations for the Field.

14. The covariant tensor of the field can be written as the rotation of the potential vector φ_a:

$$f_{ab} = \frac{\partial \varphi_b}{\partial x^a} - \frac{\partial \varphi_a}{\partial x^b}, \qquad (a = 1, 2, 3, 4;\ \ b = 1 \,..\, 4). \tag{14.1}$$

From these we get the contravariant components:

$$f^{ab} = \Sigma\,(cd)\, g^{ac}\, g^{bd}\, f_{cd},$$

and the fundamental equations of the theory of electrons are

$$\Sigma \frac{\partial}{\partial x^b} (\sqrt{g}\, f^{ab}) = \varrho v^a. \tag{14.2}$$

where ϱ is the density of the electric charges, and ϱv^a is a contravariant vector multiplied by $\sqrt{g}$.

From the relations (14.1) arises another equation. Multiply by the contravariant fourth rank tensor $\frac{1}{2}\delta^{abcd}/\sqrt{g}$, and contract twice. Here δ^{abcd} is 1 whenever the figures $abcd$ constitute an even permutation of 1234, and in other cases vanishes. Then we get the conjugate tensor f_*^{ab} [1]):

$$f_*^{ab} = \Sigma\,(cd) \frac{1}{2\sqrt{g}}\, \delta^{abcd} f_{cd}.$$

[1]) In order to get the covariant conjugate tensor components f^*_{ab} from the contravariant f^{cd}, multiply in the same way by the covariant tensor

$$\tfrac{1}{2}\sqrt{g}\ \delta_{abcd} \cdot (\delta_{abcd} = \delta^{abcd}).$$

If now we write

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}f_*{}^{ab}) = 0, \tag{14.3}$$

this must be an identity in virtue of (14.1).

The MINKOWSKIAN force acting on a moving charge e has the covariant components:

$$f_a = e\,\Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w^b f_{ab}.$$

These equations are supposed to hold within the finest structure of matter.

To obtain the equations of matter in bulk, we take the mean over a small region, containing a great many atoms. We define

$$F_{ab} = \frac{\int f_{ab}\sqrt{g}\,dx^{(1)}\,..\,dx^{(4)}}{\int \sqrt{g}\,dx^{(1)}\,..\,dx^{(4)}}, \qquad F^{ab} = \frac{\int f^{ab}\sqrt{g}\,dx^{(1)}\,..\,dx^{(4)}}{\int \sqrt{g}\,dx^{(1)}\,.\;dx^{(4)}};$$

It is readily seen that still $F^{ab} = \Sigma(cd)\,g^{ac}\,g^{bd}\,F_{cd}$.

The mean of the convection current ϱv^a, as produced by the bound electrons, we have just found, and so the equations for non-conducting matter are:

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,F^{ab}) = \Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,T^{ab}). \tag{14.41}$$

In conducting matter, the current from the conduction electrons $\sqrt{g}I^a$ must be added in the right hand member.

The other equations become

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,F_*{}^{ab}) = 0 \tag{14.42}$$

Now, we could try a solution $F^{ab} = T^{ab}$, and add a solution E^{ab} of the equations

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,E^{ab}) = 0 \tag{14.51}$$

and

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,E_*{}^{ab}) = -\,\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,T_*{}^{ab}), \tag{14.52}$$

$E_*{}^{ab}$ and $T_*{}^{ab}$ being the conjugate tensors of E^{ab} and T^{ab}. Then

$$F^{ab} = T^{ab} + E^{ab}$$

is a solution of equations (14.41) and (14.42). We shall call T^{ab} the *internal*, and E^{ab} the *external* field.

Separation of the Polarization and the Magnetization Tensor.

15.1. It has been remarked, that in our tensor $\sqrt{g}\,T^{ab}$ the polarization and magnetization are intimately interwoven. Indeed, it is not always easy to separate them.

We shall assume that our system of reference is such that g_{a4} vanishes for $a = 1, 2, 3$. This means, that in our system of reference the velocity of beams of light in opposite directions is the same. This also implies that whenever the first three contravariant components of velocity vanish, also the first three covariant components vanish.

Then we may in stationary points of matter (or in an arbitrary point after changing variables in a way that renders it stationary) separate T^{ab} into two tensors, one consisting of the $(a4)$-components, the other components vanishing, and vice versa. These tensors we may call the *polarization and magnetization tensors.* It should be understood, that a change of variables, or a motion of matter, cannot leave half of the components zero, as they are in stationary points: the magnetization tensor, e.g., completes itself with polarization components.

Write ${}_0T^{ab}$ for T^{ab} in a stationary point, and separate:

$${}_0T^{ab} = {}_0M^{ab} + {}_0P^{ab},$$

so that

$${}_0M^{ab} \,(=)\quad \begin{array}{cccc} & {}_0T^{12} & {}_0T^{13} & 0 \\ {}_0T^{21} & & {}_0T^{23} & 0 \\ {}_0T^{31} & {}_0T^{32} & & 0 \\ 0 & 0 & 0 & \end{array},$$

and

$${}_0P^{ab} \,(=)\quad \begin{array}{cccc} & 0 & 0 & {}_0T^{14} \\ 0 & & 0 & {}_0T^{24} \\ 0 & 0 & & {}_0T^{34} \\ {}_0T^{41} & {}_0T^{42} & {}_0T^{43} & \end{array}.$$

We could have taken the covariant components of the tensor T_{ab} and the separation would have had the same effect. This is due to the fact that g_{a4} vanishes for $a = 1, 2, 3$. If this is not the case, then we have first to change variables to make them vanish and afterwards make the separation.

15.2. We shall now briefly indicate what becomes of the constitutive relations between polarization and electric force, magnetization and magnetic force, and between the conduction current and electric force. We proceed in a quite formal way

First, to find the generalization of the equation $\mathbf{P} = (\varepsilon - 1)\,\mathbf{E}$, we form from the field-tensor a force-vector F^a :

$$F^a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w_b\,F^{ab},$$

and from the polarization tensor we form a vector P^a:

$$P^a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w_b\,P^{ab}.$$

and the required generalization will be

$$P^a = -(\varepsilon - 1)\,F^a.$$

Secondly, to generalize the relation $\mathbf{B} = \mu\mathbf{H}$, or rather

$$\mathbf{M} = \frac{\mu - 1}{\mu}\,\mathbf{B},$$

we proceed in a similar manner. From the conjugate field tensor we form a vector G_a:

$$G_a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w^{(b)}\,F_{*ab},$$

and from the conjugate magnetization tensor a vector Q_a:

$$Q_a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w^b\,M_{*ab}.$$

The generalized relation is

$$Q_a = -\frac{\mu - 1}{\mu}\,G_a.$$

The current of the free electrons is partly a convection current, partly a conduction current. The latter will be the component of the four-dimensional vector-density $\sqrt{g}I^a$ in a direction perpendicular to the four-dimensional velocity vector. The conduction vector thus is:

$$J^a = I^a - w^a \left\{\frac{dx^{(4)}}{ds}\right\}^2 \Sigma\,(b)\,w_b\,I^b.$$

This can be put otherwise, if we first form a skew-symmetrical tensor

$$I^{ab} = \frac{dx^{(4)}}{ds}\{I^a w^b - I^b w^a\}$$

and afterwards from this tensor form a vector again:

$$J^a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w_b\,I^{ab}.$$

The equation for the conduction current must be

$$J^a = -\lambda\,F^a.$$

We notice that in the common equation $\mathbf{J} = \sigma\mathbf{E}$, $\sigma = \lambda\sqrt{g}$.

16.1. Now take the contravariant tensor P^{ab} and form its conjugate:

$$P_{*ab} = \Sigma\,(cd)\,\tfrac{1}{2}\,\sqrt{g}\,\delta_{abcd}\,P^{cd}.$$

then we get the conjugate tensor with covariant components

$$P_{*ab} (=) \begin{array}{cccc} & \sqrt{g}P^{34} & -\sqrt{g}P^{24} & \sqrt{g}P^{23} \\ -\sqrt{g}P^{34} & & \sqrt{g}P^{14} & \sqrt{g}P^{31} \\ \sqrt{g}P^{24} & -\sqrt{g}P^{14} & & \sqrt{g}P^{12} \\ -\sqrt{g}P^{23} & -\sqrt{g}P^{31} & -\sqrt{g}P^{12} & \end{array}$$

By multiplying by the velocity vector and contracting,

$$\Sigma\,(b)\frac{dx^{(4)}}{ds} w^b P_{*ab},$$

we get a vector. This vector clearly vanishes in a stationary point, because $w^{(1)}$, $w^{(2)}$, $w^{(3)}$, and ${}_0P_{*a4}$ vanish, and it therefore *always* vanishes. Thus we conclude that we shall always have

$$0 = w^{(2)}\sqrt{g}P^{34} - w^{(3)}\sqrt{g}P^{24} + \sqrt{g}P^{23}, \tag{16.1}$$

and similar relations for cyclic permutations of the figures 123. It is thus confirmed that where $\sqrt{g}P^{a4}$ $(a = 1, 2, 3)$ are polarization components, the other components of this tensor consist of components of the corresponding RÖNTGEN-vector.

16.2. Apply a similar reasoning to the magnetization tensor. Multiply by the velocity vector and contract:

$$\Sigma\,(bc)\, g_{bc}\frac{dx^{(4)}}{ds} w^c M^{ab} = \Sigma(b)\frac{dx^{(4)}}{ds} w_b M^{ab},$$

This will be a vector vanishing in stationary points, since w_1, w_2, w_3, and ${}_0M^{a4}$ vanish. Therefore it will always vanish, and we shall have

$$0 = w_2 M^{12} + w_3 M^{13} + w_4 M^{14}. \quad (cycl.\ 123). \tag{16.2}$$

Here we meet the polarization of moving magnetism, $\sqrt{g}M^{a4}$, in terms of M^{ab}. We know from §§ 8, 9 that $\sqrt{g}M^{ab}$ must contain, besides the components of the magnetization and of k^{ab}, the components of the RÖNTGEN-vector corresponding to the polarization of moving magnetism also.

This will afford us means completely to express the polarization of moving magnetism in terms of the magnetization and **k** of moving matter (§ 19).

Comparison with Other Theories.

17. In constructing the polarization tensor EINSTEIN, following MINKOWSKI, starts from the vector P^a defined in § 15.2, and he puts for his tensor [1])

[1]) *Die formale Grundlage der allgemeinen Relativitatstheorie*, Berl. Sitz, **41**, p. 1065, 1914.

$$\frac{dx^{(4)}}{ds}\{P^a w^b - P^b w^a\}.$$

In order to show that this is the same as our tensor P^{ab}, take a special case, $a=1$, $b=2$ e.g., and write in full

$$\frac{dx^{(4)}}{ds}\{P^a w^b - P^b w^a\} = \left\{\frac{dx^{(4)}}{ds}\right\}^2 \left\{ w^{(2)} (w_2 P^{12} + w_3 P^{13} + w_4 P^{14}) - \right.$$

$$\left. - w^{(1)} (w_1 P^{21} + w_3 P^{23} + w_4 P^{24}) \right\}.$$

We can rearrange:

$$= \left\{\frac{dx^{(4)}}{ds}\right\}^2 \left\{ P^{12} (w^{(1)} w_1 + w^{(2)} w_2 + w^{(3)} w_3 + w^{(4)} w_4) + \right.$$

$$\left. + w_3 (w^{(1)} P^{32} + w^{(2)} P^{13} + w^{(3)} P^{21}) + w_4 (w^{(1)} P^{42} + w^{(2)} P^{14} + w^{(4)} P^{21}) \right\}.$$

and now we remark that the latter two bracket forms vanish in virtue of (16.1), for

$$\frac{dx^{(4)}}{ds} (w^{(1)} P^{32} + w^{(2)} P^{13} + w^{(3)} P^{21}) =$$

$$= \frac{1}{\sqrt{g}} \frac{dx^{(4)}}{ds} (w^{(1)} P_{*41} + w^{(2)} P_{*42} + w^{(3)} P_{*43}) = 0.$$

As

$$\left\{\frac{dx^{(4)}}{ds}\right\}^2 \Sigma w^b w_b = 1.$$

the required identity is shown to exist.

In the same way it can be shown that the magnetization tensor or rather its conjugate in the form

$$\frac{dx^{(4)}}{ds}\{Q_a w_b - Q_b w_a\}$$

agrees with our M_{*ab}.

18. Let us make the simplifying assumption of the absence of gravitation. Then the g_{ab} and g^{ab} have the values:

$$g_{ab} (=) \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & c^2 \end{matrix}, \quad g^{ab} (=) \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1/c^2 \end{matrix}, \quad g = -c^2 \text{ [1]}$$

If $\mathbf{A}$ and φ denote the common vector and scalar potentials, then the components φ_a are $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$ and $-c\varphi$. The components of the field are

[1] In order to avoid imaginaries, we shall everywhere in $\sqrt{g}$ take $|g|$.

$$F_{ab} \quad (=) \quad \begin{matrix} & \mathbf{B}_z & -\mathbf{B}_y & c\mathbf{E}_x \\ -\mathbf{B}_z & \cdot & \mathbf{B}_x & c\mathbf{E}_y \\ \mathbf{B}_y & -\mathbf{B}_x & & c\mathbf{E}_z \\ -c\mathbf{E}_x & -c\mathbf{E}_y & -c\mathbf{E}_z & \end{matrix},$$

$$F^{ab} \quad (=) \quad \begin{matrix} & \mathbf{B}_z & -\mathbf{B}_y & -\mathbf{E}_x/c \\ -\mathbf{B}_z & \cdot & \mathbf{B}_x & -\mathbf{E}_y/c \\ \mathbf{B}_y & -\mathbf{B} & & -\mathbf{E}_z/c \\ \mathbf{E}_x/c & \mathbf{E}_y/c & \mathbf{E}_z/c & \end{matrix}.$$

The equations for the field are (**14.41**)

$$\Sigma(b) \frac{\partial}{\partial x^b} (\sqrt{g}\, F^{ab}) = \Sigma(b) \frac{\partial}{\partial x^b} (\sqrt{g}\, P^{ab} + \sqrt{g}\, M^{ab}),$$

and we have, if **P** is the principal di-electric polarization:

$$\sqrt{g}\, P^{ab} \quad (=) \quad \begin{matrix} & [\mathbf{Pw}]_z & -[\mathbf{Pw}]_y & \mathbf{P}_x \\ -[\mathbf{Pw}]_z & & [\mathbf{Pw}]_x & \mathbf{P}_y \\ [\mathbf{Pw}]_y & -[\mathbf{Pw}]_x & & \mathbf{P}_z \\ -\mathbf{P}_x & -\mathbf{P}_y & -\mathbf{P}_z & \end{matrix}, \tag{18.1}$$

and

$$\sqrt{g} M^{ab} (=) \begin{matrix} & c\mathfrak{m}_z+\mathfrak{k}_z+[\mathfrak{n}.\mathbf{w}]_z & -c\mathfrak{m}_y-\mathfrak{k}_y-[\mathfrak{n}.\mathbf{w}]_y & \mathfrak{n}_x \\ -c\mathfrak{m}_z-\mathfrak{k}_z-[\mathfrak{n}.\mathbf{w}]_z & & c\mathfrak{m}_x+\mathfrak{k}_x+[\mathfrak{n}.\mathbf{w}]_x & \mathfrak{n}_y \\ c\mathfrak{m}_y+\mathfrak{k}_y+[\mathfrak{n}.\mathbf{w}]_y & -c\mathfrak{m}_x-\mathfrak{k}_x-[\mathfrak{n}.\mathbf{w}]_x & & \mathfrak{n}_z \\ -\mathfrak{n}_x & -\mathfrak{n}_y & -\mathfrak{n}_z & \end{matrix} \tag{18.2}$$

where $\mathfrak{n}$ denotes the (electric) polarization of moving magnetism. For the conjugate tensor of the field we have

$$F_*^{ab} \quad (=) \quad \begin{matrix} & \mathbf{E}_z & -\mathbf{E}_y & \mathbf{B}_x/c \\ -\mathbf{E}_z & & \mathbf{E}_x & \mathbf{B}_y/c \\ \mathbf{E}_y & -\mathbf{E}_x & & \mathbf{B}_z/c \\ -\mathbf{B}_x/c & -\mathbf{B}_y/c & -\mathbf{B}_z/c & \end{matrix}$$

We see that the equations (**14.42**) amount to

$$c\, rot\, \mathbf{E} + \dot{\mathbf{B}} = 0, \tag{18.31}$$

$$div\, \mathbf{B} = 0, \tag{18.32}$$

From the equations of the field we see that

$$div\, \mathbf{E} = -\, div\, (\mathbf{P} + \mathfrak{n}), \tag{18.41}$$

and

$$c\, rot\, \mathbf{B} - \dot{\mathbf{E}} = rot\, (c\mathfrak{m} + \mathfrak{k} + [\mathfrak{n}.\mathbf{w}] + [\mathbf{P}.\mathbf{w}]) + [\dot{\mathbf{P}} + \dot{\mathfrak{n}}]. \tag{18.42}$$

These are the equations we have met in § 10. Only we had not yet separated $\mathfrak{p} = \mathbf{P} + \mathfrak{n}$ there.

19. Let us solve $\mathfrak{n}$ in terms of $\mathfrak{m}$ and $\mathfrak{k}$. Referring to the equation of § 16 2 we must notice that

$$w_1 = -\mathfrak{w}_x, \qquad w_2 = -\mathfrak{w}_y, \qquad w_3 = -\mathfrak{w}_z, \qquad w_4 = g_{44} w^{(4)} = c^2,$$

and we get

$$c^2 \mathfrak{n}_x = \mathfrak{w}_y (c\mathfrak{m}_z + \mathfrak{k}_z + [\mathfrak{n} \cdot \mathfrak{w}]_z) - \mathfrak{w}_z (c\mathfrak{m}_y + \mathfrak{k}_y + [\mathfrak{n} \cdot \mathfrak{w}]_y)$$

or

$$c^2 \mathfrak{n} = [\mathfrak{w} \cdot (c\mathfrak{m} + \mathfrak{k} + [\mathfrak{n} \cdot \mathfrak{w}])]. \tag{19.1}$$

From this it is easily seen that

$$(\mathfrak{n} \cdot \mathfrak{w}) = 0,$$

and as

$$[\mathfrak{w} \cdot [\mathfrak{n} \cdot \mathfrak{w}]] = w^2 \mathfrak{n} - \mathfrak{w} (\mathfrak{n} \cdot \mathfrak{w}),$$

we get

$$\mathfrak{n}_x = \sqrt{g}\, M^{14} = \frac{[\mathfrak{w} \cdot (c\mathfrak{m} + \mathfrak{k})]_x}{c^2 \left(1 - \frac{w^2}{c^2}\right)}, \text{ a.so.} \tag{19.2}$$

and

$$\sqrt{g}\, M^{12} = \frac{c\mathfrak{m}_z + \mathfrak{k}_z}{1 - \frac{w^2}{c^2}} - \frac{\mathfrak{w}_z (\mathfrak{w} \cdot (c\mathfrak{m} + \mathfrak{k}))}{c^2 \left(1 - \frac{w^2}{c^2}\right)}, \text{ a.s.o.} \tag{19.3}$$

In this form our result for the magnetization tensor can be readily compared with the corresponding formulae of BORN [1]). He also points out the existence of the vector $\mathfrak{n}$ and states that it is the magnetic analogon to the RÖNTGEN-vector. We see that the factor $1/(1-w^2/c^2)$ disturbs the analogy. The difference in the appreciation of the result is this that BORN (apart from not separating $\mathfrak{k}$) takes the whole of the components $\sqrt{g}M^{23}$, $\sqrt{g}M^{31}$ and $\sqrt{g}M^{12}$ to be the components of magnetization and seems not to have become aware of the fact that they contain the RÖNTGEN-vector components belonging to $\mathfrak{n}$ as well as the magnetization components proper.

BORN emphasizes the complete symmetry of his electric and magnetic equations and certainly one can enjoy the mathematical beauty of the formulae thus written. It would, however, be erroneous to believe that the difference from LORENTZ' equations is more than a difference in form. Our investigation shows that the *physical contents* of BORN's equations is no other than what has been expressed by LORENTZ.

Action of Polarization of Moving Magnetism.

20. Let us illustrate some effects of $\mathfrak{n}$ by considering a long

[1]) l.c. form. 39 and 39′, pp. 546 and 547.

magnet moving at right angles to its magnetization. We shall follow the distinction of "internal" and "external" field at the end of § 14. The effect of this electric polarization **n**, called into existence by the motion of magnetized matter, is to produce an internal electric field (18.41):

$$\mathbf{E} = -\mathbf{n}.$$

This could be expected to act on free electrons, present in the magnet, and cause a conduction current. But these electrons are carried along with matter and therefore are moving with velocity **w** through the internal magnetic field where the induction vector is (see § 18.42):

$$c\mathbf{B} = c\mathbf{m} + \mathbf{k} + [\mathbf{n} \,.\, \mathbf{w}]$$

and, where the external field may be neglected [1]), they consequently are subjected to the NEWTONian force

$$e\left(\mathbf{E} + \frac{1}{c}\,[\mathbf{w} \,.\, \mathbf{B}]\right).$$

This expression vanishes according to the formulae of §§ 16.2 and 19, so that the free electrons moving along with the magnet are not driven sideways.

Therefore it is impossible with sliding contacts at the magnet's sides to get a current from it, and the experiment with the long magnet drawn across a circular spring is explained. (§ 12).

On the other hand, if we cut the magnet at right angles to the magnetization, and take out an infinitely thin lamella, so that a thin wire might be kept in the same place while the magnet is drawn across, then the "external" field in this split will simply be the continuation of the internal field. and the free electrons in the wire, not sharing the motion of the magnet, will be subjected to the electric force **E** only, so that an induction current will be set up in the wire.

Thus we see that it is *the polarization of moving magnetism that accounts for the inductive force,* when a magnetic pole moves across a wire, *in a case where the magnetic-field is homogeneous and stationary.*

Conclusive Remarks.

21. In conclusion we may remark that the result of the first variation is wholly incorporated in the polarization tensor. The

[1]) Suppose the magnetization as being homogeneous, and the free poles of the magnet as being at infinite distance.

greater part of the result of the second variation is represented in the magnetization tensor.

Consider once more the complete polarization (6.3 and 6.2):

$$N e \left[\varrho^a - \tfrac{1}{2} \theta r^{(4)} \left\{ \frac{d\varrho^a}{dx^{(4)}} - \Sigma \varrho^c \frac{\partial w^a}{\partial x^c} \right\} + \tfrac{1}{2} \Sigma \varrho^c \frac{\partial \varrho^a}{\partial x^c} \right] - \tfrac{1}{2} \Sigma \frac{\partial N e \varrho^a \varrho^c}{\partial x^c}.$$

Here $Ne\varrho^a$ is the term, by far the most important, which results from the first variation. It is difficult to tell in a few words, which part from the second order terms is exactly the polarization of moving magnetism. If the r^a are so chosen that $r^{(4)}$ vanishes in stationary points, then we can say that the greater part of

$$\tfrac{1}{2} Ne \, \Sigma \varrho^c \frac{\partial \varrho^a}{\partial x^c} - \tfrac{1}{2} \Sigma \frac{\partial N e \varrho^a \varrho^c}{\partial x^c} = \tfrac{1}{2} \delta N . e \varrho^a .$$

figures in the polarization tensor. A small fraction of it (in as much as δN is no scalar) appears, however, in the magnetization tensor, together with

$$- \tfrac{1}{2} N e \, \theta r^{(4)} \left\{ \frac{d\varrho^a}{dx^{(4)}} - \Sigma \varrho^c \frac{\partial w^a}{\partial x_c} \right\}$$

as the polarization of moving magnetism. But we refrain from entering into detail here.

Mathematics. — "*Ueber die Zerlegungsgesetze für die Primideale eines beliebigen algebraischen Zahlkörpers im Körper der l^h-ten Einheitswurzeln.*" By Dr. N. G. W. H. BEEGER. (Communicated by Prof. W. KAPTEYN).

(Communicated at the meeting of March 20, 1920).

Im letzten Hefte der "Mathematischen Zeitschrift" [1]) hat Herr T. RELLA die Zerlegungsgesetze für die Primideale eines beliebigen algebraischen Zahlkörpers im Körper der l-ten Einheitswurzeln dargestellt. l war dabei eine Primzahl. Im Folgenden werde ich zeigen dasz seine Methoden auch in dem Falle benutzt werden können wenn man statt des letztgenannten Körpers, den Körper der l^h-ten Einheitswurzeln nimmt. Man musz dann seinen Betrachtungen einige hinzufügen.

Es sei l eine ungerade Primzahl; $\zeta = e^{\frac{2\pi i}{l^h}}$; k ein Körper der mit $k(\zeta)$ einen Unterkörper vom Grade $m = a l^{h-h'-1}$ gemein hat, wo a ein Teiler von $l-1$ bedeutet. Der aus k und $k(\zeta)$ zusammengesetzte Körper (k, ζ) ist vom Relativgrad $\frac{\varphi}{m}$ über k und relativ-zyklisch. Wir setzen zur Abkürzung φ statt $\varphi(l^h)$. In (k, ζ) gelten folgende Zerlegungsgesetze:

1. Ist p eine von l verschiedene Primzahl, $\mathfrak{p}$ ein in p aufgehendes Primideal von k von Grade f. Gehört $p \ (mod\ l^h)$ zum Exponenten f_0 und ist ff' das kleinste gemeinschaftliche Vielfache von f und f_0 so zerfällt $\mathfrak{p}$ in (k, ζ) in z' Primideale vom Relativgrade f', wenn $\frac{\varphi}{m} = f'z'$.

2. Ist $\mathfrak{l}$ ein in l aufgehendes Primideal von k von Grade f und $l = \mathfrak{l}^e \mathfrak{a}$, $(\mathfrak{a}, \mathfrak{l}) = 1$; d der gröszte gemeinschaftliche Teiler von e und $\varphi(l^{h-h'})$; n die gröszte ganze Zahl für welche eine Kongruenz

$$-l \equiv \alpha^n \ (mod\ \mathfrak{l}^{e+1})$$

besteht, wobei α eine ganze Zahl von k bedeutet; d_1 der gröszte gemeinschaftliche Teiler von n und $\varphi(l^{h-h'})$, so ist m ein Teiler

[1]) Band 5. S. 11.

von d_1 und d_1 ein Teiler von d. Setzt man $d = f'd_1 = f'z'm$ so gilt in (k, ζ) die Zerlegung:

$$\mathfrak{l} = (\mathfrak{L}\,\mathfrak{L}_1 \ldots \mathfrak{L}_{z'-1})^{\frac{\varphi}{d}} \quad ; \quad N_k(\mathfrak{L}_i) = \mathfrak{l}^{f'}.$$

Der Beweis für 1. ist derselbe wie für den Satz. 1. des Herrn RELLA, wenn man darin l in l^h und $l-1$ in φ ändert.

Beweis für 2.

Wir setzen erst k zusammen uit $k\left(e^{\frac{2\pi i}{l^{h-h'}}}\right)$ zu einen Körper k_1. Dieser ist relativ-zyklisch vom Relativ-grade $\frac{l-1}{a}$ zu k. Der Relativgrad ist nicht teilbar durch l. Ist also in k_1:

$$\mathfrak{l} = (\mathfrak{L}'\,\mathfrak{L}'_1 \ldots \mathfrak{L}'_{z'-1})^{g'}$$

so hat das Primideal $\mathfrak{L}'$ keine Verzweigungsgruppe, weil der Grad dieser Gruppe eine Potenz von l sein musz und zugleich ein Teiler von $\frac{l-1}{a}$ [1].

Hieraus ergibt sich weiter dasz g' prim zu l ist, da die höchste Potenz von l_1 durch welche g' teilbar ist, dem Grade der Verzweigungsgruppe gleich ist. Man sieht leicht ein, dasz der Beweis des Herrn RELLA auch hier seine Gültigkeit behalt wenn man darin wiederum l in $l^{h-h'-1}$ und $l-1$ in $\varphi\,(l^{h-h'})$, ändert. Es ergibt sich dann die Beziehung:

$$g' = \frac{\varphi\,(l^{h-h'})}{d}\,e'$$

woraus folgt dasz e' nicht durch l teilbar ist, und weiter $e' = 1$ Dann hat man in k_1 die Zerlegung gefunden:

$$\mathfrak{l} = (\mathfrak{L}'\,\mathfrak{L}'_1 \ldots \mathfrak{L}'_{z'-1})^{\frac{\varphi\,(l^{h-h'})}{d}} \quad ; \quad N_k(\mathfrak{L}'_i) = \mathfrak{l}^{f'} \quad \ldots \quad (1)$$

Nun haben k_1 und $k\left(e^{\frac{2\pi i}{l^{h-h'+1}}}\right)$ den gemeinschaftlichen Unterkörper $k\left(e^{\frac{2\pi i}{l^{h-h'}}}\right)$. Wir setzen k_1 zusammen mit dem Körper $k\left(e^{\frac{2\pi i}{l^{h-h'+1}}}\right)$ zu einem neuen Körper k_2. Dieser Körper ist relativ-zyklisch von Relativgrade l in Bezug auf k_1. Und wir würden ebenso den Körper k_2 bekommen haben wenn wir gleich k mit dem zuletzt-genannten Kreiskörper zusammengesetzt hätten.

Ist ξ die den Körper k_1 bestimmende Zahl und $Z = e^{\frac{2\pi i}{l^{h-h'+1}}}$ so stellen die Zahlen

[1]) WEBER „Lehrbuch d. Algebra" II. S. 664 u. s. w.

$$\xi^i\, Z^j \begin{pmatrix} i = 0,\ 1, \ldots .\, g-1 \\ j = 0,\ 1, \ldots .\, \varphi\,(l^{h-h'+1}) - 1 \end{pmatrix}$$

eine Basis von k_2 dar, wenn g der Grad van k_1 ist. Die relativen Substitutionen von k_2 in bezug auf k_1 haben die Form $(Z : Z^b)$. Dazu gehört das Element

$$\mathfrak{E} = \{\ldots .,\, \xi^i\,(Z^j - Z^{jb}), \ldots .\}\ ^{1)}$$

Es ist hieraus ersichtlich dasz $\mathfrak{E}$ durch das Primideal $\mathfrak{l}_0 = (1-Z)$ teilbar ist. Und weil es ein Ideal von k_2 ist, ist es also teilbar durch $\mathfrak{L}''$ wenn dieses Ideal in k_2 auf l teilbar ist. Dann ist auch der Relativ discrimant von k_2 in bezug auf k_1 durch $\mathfrak{L}''$ teilbar. Also ist $\mathfrak{L}''$ ein ambiges Primideal[2]) und $\mathfrak{L}''^l$ ein Primideal von k_1. Es ist daher in k_2:

$$\mathfrak{L}'_i = \mathfrak{L}''^{\,l}_i \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad (2)$$

und

$$n_{k_1}(\mathfrak{L}''_i) = \mathfrak{L}'_i$$

In derselben Weise findet man, indem man wiederum k_2 zusammensetzt mit $k\left(e^{\frac{2\pi i}{l^{h-h'+2}}}\right)$ zu einem Körper k_3, dasz im Körper k_3 die Zerlegung

$$\mathfrak{L}'''_i = \mathfrak{L}''^{\,l}_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

gilt. Wenn man das Verfahren fortsetzt so findet man aus (1), (2), (3), ... den Beweis des zu erweisenden Satzes.

Schlieszlich bemerke ich dasz die Sätze ihre Gültigkeit behalten für $m = 1$.

---

1) BACHMANN, "Allgemeine Arithmetik der Zahlenkörper", S. 450.

2) HILBERT. "Bericht über die Th. d. a. Zahlkörper. Jahresb. d. D. M. V. Band IV. Satz. 93.

Chemistry. — "*The Electromotive Behaviour of Aluminium.*" I. By Prof. A. SMITS. (Communicated by Prof. H. A. LORENTZ.).

(Communicated at the meeting of February 28, 1920).

1. *Introduction.*

As early as 1914 [1]) we began to consider the behaviour of. aluminium from the point of view offered by the new theory of the electromotive equilibria.

As regards its electromotive behaviour aluminium is a most interesting metal. It has generally not been inserted in the electromotive series, because no certainty has been attained as yet about its place. In alkaline solutions aluminium precipitates the zinc, but it does not do so in neutral or acid solutions. To this is added the very remarkable fact that the amalgamated aluminium does precipitate the zinc from neutral solutions, and acts with violent decomposition on water, that it further rapidly oxidizes when exposed to the air, and exhibits a character that indicates that aluminium in this condition must be placed directly after the metals of the alkaline earths, thus: Mg—Al—Mn—Zn.

In connection with this the conclusion was obvious that commercial aluminium is in a noble, less active condition, or in other words that it is in a state of passivity. This was decidedly a step in the right direction, but an explanation of the behaviour of aluminium had not yet been given.

Most handbooks and publications state that commercial aluminium is covered with a coat of oxide, and that its passivity is owing to this.

Also the anodic polarisation of aluminium has made the peculiar character of this metal evident. It was found before, that when an $Al_2(SO_4)_3$-solution was used, the density of the current, *i/o*, on anodic polarisation continually decreased, whereas the electric potential rose, which may be seen from the following table, which has already been published before. [2]) Here the potential has been measured with respect to another aluminium rod as auxiliary electrode.

[1]) SMITS, ATEN, These Proc. **22**, 1133 (1914).

[2]) SMITS, ATEN, Loc. cit.

Al-electrode in 1/2 N $Al_2(SO_4)_3$

i/o	anode
0.8	+ 2.56
0.53	+ 3.48
0.46	+ 3.84
0.36	+ 4.13

It has been tried to account for this phenomenon by assuming the formation of an Al_2O_3-layer with great resistance, which supposition is, however, hardly tenable, for when the above-mentioned phenomenon presents itself, the aluminium-anode is *perfectly bright*. Besides when the current is reversed, the resistance has entirely disappeared.

When the tension is increased, there is actually formed a coat of Al_2O_3, $Al(OH)_3$ or of a basic salt. Then the density of the current is practically reduced to zero, but when the current is reversed, the potential of the aluminium-electrode is considerably smaller.

When the anode potential is carried up very high, e.g. to 200—500 V., the potential is reduced to from $^1/_{30}$ to $^1/_{50}$ on reversal of the current.

This property, the so-called *valve-action*, is used to transform an alternating current into a continuous one. With high current-densities the electric valve-action stops under ordinary circumstances owing to rise of temperature.

FISCHER [1]), therefore, used as anode an aluminium tube, through which water flowed, and in this way he succeeded in getting coats of oxide of a thickness of some tenths of millimeters.

The most extensive researches on the valve-action of aluminium have been performed by SCHULZE [2]). He assumes, that every newly-

1) Zeitschr. f phys. Chem. **48**, 177 1904.
2) Ann. der Phys. **21**, 929. 1906.
„ „ „ **22**, 543. 1907.
„ „ „ **23**, 226. 1907.
„ „ „ **24**, 43. 1907.
„ „ „ **25**, 775. 1908.
„ „ „ **28**, 787. 1909.
„ „ „ **34**, 657. 1911.
„ „ „ **41**, 593. 1913.
Zeitschr. f. Elektrochem. **20**, 307. 1914.
„ „ „ **20**, 592. 1914

formed aluminium-surface is immediately covered with a solid, not porous layer of oxide of molecular thickness. This layer, indeed, insulates, but according to him it can be pierced by the anions of the salt-solutions or by the O''-ion on anodic polarisation, and the oxygen formed then combines with the metal to Al_2O_3. The porous oxide layer offers an ever increasing resistance with increasing thickness, and at last the anions are almost exclusively discharged at the layer of oxide, and only very few succeed in traversing this layer, and reaching the metal, which he tries to prove by the fact that the quantity of generated oxygen is 96 °/₀ of the quantity of electricity transmitted.

When with a certain thickness of layer a definite potential gradient has been reached, sparking commences, which puts a stop to the increase of tension and the thickening of the oxide layer. This maximum tension is greatly dependent on the nature and the concentration of the anions; when this concentration increases, the maximum tension diminishes. It is remarkable that, when the current is reversed, no current passes below a certain potential, and that this minimum potential of the cathode is then many times smaller than the anodic-minimum-potential. Also the cathodic minimum potential depends greatly on the nature of the ions.

SCHULZE, and before him TAYLOR and INGLIS [1]) and GUTHE [2]), thought that they could find the explanation of this peculiar phenomenon by assigning to the gas layer that is formed in the pores of the $Al_2(OH)_6$, the property of allowing the anions to pass less easily than the cations.

It is clear that this explanation is not entirely satisfactory, the more so because there are still a great many other exceedingly remarkable phenomena on which it does not throw any light. Two of them may be mentioned here, first the phenomenon that amalgamated aluminium does not show valve-action, and secondly that a chlor-ion concentration in the electrolyte of 0.2 °/₀ renders the valve-action quite impossible.

2. When aluminium is considered from the point of view of the theory of the electromotive equilibria, the conclusion is readily reached that this theory is able to account for the above-mentioned remarkable behaviour of aluminium by means of the same principles as the polarisation-phenomena in the other metals.

In the first place it may be pointed out that it can easily be

[1]) Phil. Mag. **5**, 301, (1903).
[2]) Phil. Rev. **15**, 327, (1902).

demonstrated that it is erroneous to assert that commercial aluminium is covered with a coating of oxide. It was shown before that when the bottom of a vessel with an $Al_2(SO_4)_3$ solution is covered with a layer of mercury, and when through the solution an aluminium rod is immersed in the mercury layer, the aluminium rod immediately assumes the potential of the mercury, from which follows that the aluminium rod was not covered with an insulating layer of Al_2O_3, but was in direct contact with the mercury [1]).

Now that this fact has been established, and the initial condition is uncovered metal, it is clear that it must be explained why on anodic polarisation the potential of the metal becomes so strongly positive already with very small current densities that the tension of separation of the oxygen is reached. It is seen that here the same question presents itself as in the case of anodic polarisation of other inert metals. It was pointed out before that in the first place the most essential, the *primary* phenomenon, should be explained, viz. the change of the potential in noble direction; the oxygen, separation and the subsequent oxide formation are *secondary* phenomena.

The strong ennobling of the potential of aluminium on anodic polarisation must be explained by this, that while the withdrawal of electrons from the metal which is represented by

$$\overset{\uparrow}{3\,\theta_S}$$

is immediately followed by aluminium-ions going into sulution

$$\begin{array}{c} Al^{\cdots}_S \\ \downarrow \\ Al^{\cdots}_L \end{array}$$

because this heterogeneous equilibrium is instantaneously established, the homogeneous reaction

$$Al_S \rightarrow Al^{\cdots}_S + 3\,\theta_S$$

proceeds with very small velocity, so that the metal becomes poorer in ions and electrons. In consequence of this the potential of the metal becomes less negative or more positive, as appears from the equation:

$$E = -\frac{0{,}058}{v}\log\frac{L_M}{(M^{v\cdot}_L)} - 2{,}8$$

because in this case L_M becomes smaller.

This phenomenon is, therefore, primary, and if the metal is inert,

1) SMITS, ATEN. l.c.

as it is here, the potential of separation of the oxygen will soon be reached, and oxygen generation will set in, which under certain circumstances may lead to the formation of an adherent coating of oxide or hydroxide round the metal. Of course this coating gives rise to a certain resistance, which may rise to considerable amounts with increasing thickness. The assumption, however, that the resistance of such a coating should be different for different directions of current is not justifiable, so that there can be no doubt that the sudden decrease of the resistance on reversal of the current, must be owing to some other cause.

So far there is no reason to doubt that oxygen and hydrogen are negative catalysts for the setting in of the internal metal equilibrium. Hence the slight quantities of oxygen absorbed by the metal on anodic polarisation have a greatly retarding effect (Fe, Co, Ni).

Most probably this is likewise the case for aluminium, and to this it will have to be attributed that such a strong anodic polarisation has been observed in aluminium.

Accordingly this fact leads to the assumption that the metal is disturbed here to a great extent, i. e. that the metal surface becomes very poor in ions and electrons, or in other words, that the metal passes at its surface into a state which agrees with a metalloid in this that it possesses an exceedingly small electric conductivity.

On this strong anodic disturbance the aluminium surface becomes, therefore, a metal coating of great resistance, and this coating is in its turn surrounded by another of Al_2O_3.

As the study of the phenomenon of polarisation in other metals has taught, the disturbance that has arisen by anodic polarisation, stops immediately through reversal of the current. This behaviour must be explained by the fact that hydrogen, just as oxygen, though in a different degree, is a negative catalyst for the establishment of the internal metal equilibrium, and can yet apparently act positively catalytically, when it separates on a metal surface that has previously absorbed oxygen, as both negative catalysts then disappear amidst formation of water. The small quantity of oxygen absorbed then enhances the disturbance of the aluminium on anodic polarisation; hence the removal of this negative catalyst will immediately stop the disturbance, and the strongly metastable state of the aluminium surface will be transformed with great velocity in the direction of the state of internal equilibrium. This transformation changes the metal coating of great resistance suddenly into another of smaller resistance, which must, therefore, be attributed to the great velocity of the reaction:

$$Al_S \rightarrow Al_S^{\cdots} + 3\,\theta_S.$$

Hence the resistance that remains when aluminium is made from anode to cathode, is chiefly the resistance of the coating of Al_2O_3.

In the earlier paper cited here a great resistance was simply assigned to the solid solution of oxygen in aluminium, and too little stress was laid on the fact that the strong disturbance of the aluminium may lead to the formation of an aluminium surface that is very poor in ions and electrons.

Since 1914 researches on the electromotive behaviour of aluminium have been made in my laboratory, first by Miss RIWLIN, and afterwards continued by Mr. DE GRUYTER, both as regards commercial aluminium and amalgamated aluminium.

Working with very pure commercial aluminium, the polarisation and the curves of activation in different aluminium-salt solutions, have been determined, in which again, just as with iron, the strongly positively catalytic influence of halogen-ions came to light.

A closer examination of the valve action is in progress, and also the thermic and electromotive investigation of the system mercury-aluminium. In connection with this investigation there are other thermic and electromotive investigations being made on systems mercury-metal, as mercury-magnesium, mercury-tin etc. As a peculiarity it may be mentioned here that, as was already found by DE LEEUW[1]) in tin, mercury exerts an accelerating influence on enantiotropic conversions, so that, where a point of transition in the pure metal is not, or hardly, to be observed, this is generally very clearly seen on addition of a little mercury.

A good example of this is furnished by aluminium, for which a point of transition at $\pm$ 580° was found with great clearness from the investigations of amalgams rich in aluminium.

Also with a view to this the investigation mercury-metal with other important metals will be continued.

General Anorg. Chemical Laboratory
of the University.

Amsterdam, Jan. 30, 1920.

[1]) These Proc.

Mathematics. — "*Sur les ensembles clairsemés.*" By Prof. ARNAUD DENJOY. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of March 27, 1920).

Selon une définition que j'ai proposée, (Journal de Math. pures et appliquées, 1916), je dis qu'un ensemble est *clairsemé* quand il est *non dense sur tout ensemble parfait.*

Soit E un ensemble quelconque, E_1 l'ensemble des points de E qui sont limites à E. Soit α un nombre ordinal quelconque. Si α est de première espèce, soit E_α l'ensemble des points de $E_{\alpha-1}$ qui sont limites à $E_{\alpha-1}$. Si α est de seconde espèce, soit E_α l'ensemble des points communs à tous les E_α, de rang inférieur à α. Chacun des E_α contient tous les ensembles suivants. Je dis que tous les E_α sont nuls ou coïncident à partir d'un certain rang de α.

En effet $E_{\alpha+1}$ est l'ensemble commun à E_α et à son dérivé E'_α. Donc l'ensemble E'_α contenant $E_{\alpha+1}$, contient tous les ensembles E_λ d'indices λ supérieurs à α. Comme E'_α est fermé, E'_α contient tous les ensembles E'_λ si $\lambda > \alpha$. Donc, d'après un théorème connu, il existe un rang β tel que $E'_{\beta'} > E'_\beta$, si $\beta' < \beta$, et tel que $E'_\beta = E'_{\beta+1} = \ldots$ E_λ étant situé dans E'_α pour $\lambda > \alpha$, E'_λ est situé dans le dérivé E''_α de E'_α. Donc, si E'_β n'est pas nul, E'_β est parfait, puisqu'il coïncide avec un ensemble $E'_{\beta+1}$ contenu dans son dérivé E''_β. Dans ce cas, $E_{\beta+1}$, situé sur E'_β et ayant pour dérivé E'_β lui-même, $E_{\beta+1}$ est partout dense sur E'_β. $E_{\beta+1}$, que nous désignons par F, est *dense en lui-même* et a pour dérivé l'ensemble parfait E'_β ou P.

Si E'_β est nul, E_β a un nombre limité de points, ou est nul. En tous cas, $E_{\beta+1}$ est nul.

Soit P_0 un ensemble parfait sur lequel E est partout dense, et H l'ensemble commun à P_0 et à E. H est dans E_1 et, de proche en proche, dans E_α quelque soit α, donc dans $E_{\beta+1}$, donc, le dérivé de H, soit P_0, est dans P, dérivé de $E_{\beta+1}$. Si donc $E_{\beta+1}$ est nul, E est non dense sur tout ensemble parfait. Si $E_{\beta+1}$ n'est pas nul, soit G l'ensemble des points de E qui ne font pas partie de F. L'ensemble G_α est contenu dans E_α quelque soit α. Donc, $G_{\beta+1}$ est dans $E_{\beta+1}$, donc dans F, mais $G_{\beta+1}$ est aussi dans G. Comme G est distinct de F, $G_{\beta+1}$ est nul. Donc. G est *clairsemé*.

Tout ensemble est donc la réunion d'un ensemble dense en lui-même *et d'un ensemble clairsemé*, proposition dont on trouvera une autre démonstration dans le mémoire rappelé plus haut.

Il nous sera commode, avant d'aller pour loin, de considérer la famille d'ensembles fermés K_α ainsi définie. Si α est de première espèce, K_α est identique à $E'_{\alpha-1}$. Si α est de seconde espèce, K_α est l'ensemble commun à tous les ensembles $K_{\alpha'}$ d'indice α' inférieur à α.

Nous désignons la totalité de l'espace par K_0, et E facultativement par E_0. Je dis que E_α *est l'ensemble commun à* E *et à* K_α.

Pour $\alpha = 1$, K_1 est le dérivé de E_0, ensemble identique à E, et E_1 est bien l'ensemble commun à E et à K_1. Supposons la proposition vraie pour $\alpha' < \alpha$, et montrons-la pour α. Si α est de première espèce, alors par définition, d'une part K_α est le dérivé de $E_{\alpha-1}$, d'autre part, E_α est l'ensemble commun à $E_{\alpha-1}$ et à son dérivé, donc à $E_{\alpha-1}$ et à $K_{\alpha-1}$. Or, par hypothèse, $E_{\alpha-1}$ est l'ensemble commun à E et à $K_{\alpha-1}$. Donc, E_α est l'ensemble commun à E, à $K_{\alpha-1}$ et à K_α. K_α étant le dérivé de $E_{\alpha-1}$, contenu par hypothèse dans l'ensemble fermé $K_{\alpha-1}$, K_α est contenu dans $K_{\alpha-1}$. Donc, E_α est l'ensemble commun à E et à K_α.

Si α est de seconde espèce, E_α est par définition l'ensemble commun à tous les $E_{\alpha'}$ d'indices inférieurs à α, donc d'après notre hypothèse, E_α est l'ensemble commun à E et à tous les $K_{\alpha'}$; donc, à E et à K_α, si K_α est l'ensemble commun aux $K_{\alpha'}$. La propriété est donc démontrée dans tous les cas.

Dans le cas où $E'_{\beta'}$ existe, pour $\beta' < \beta$, avec $E'_\beta = 0$, alors $K_{\beta'+1}$ existe et $K_{\beta+1}$ est nul. Si β est de première espèce, faisons $\beta' = \beta - 1$. K_β existe. Si β est de seconde espèce, comme tous les $K_{\beta'}$ existent, il en est de même de K_β. Donc si E est clairsemé, il existe un nombre β tel que K_β existe, E possédant sur K_β un nombre fini ou nul de points.

Nous allons donner une propriété caractéristique des ensembles clairsemés, propriété qui montrera le parti qu'ils offrent dans les applications à la théorie des fonctions.

Théorème. — *La condition nécessaire et suffisante pour qu'il soit possible d'affecter à chaque point* M *d'un ensemble* E, *un ensemble propre* $I(M)$ *auquel* M *soit intérieur, de manière qu'aucun point de l'espace ne soit intérieur à une infinité d'ensembles* $I(M)$, *est que l'ensemble* E *soit clairsemé.*

1° La condition est nécessaire. En effet, si E n'est pas clairsemé, il contient un ensemble dense en lui-même F. Soit P le dérivé de F. P est parfait. Tout point M_0 de F est intérieur à un ensemble $I(M_0)$. On sait alors qu'il existe un ensemble R partout dense sur

P et dont chaque point est intérieur à une infinité de $I(M_0)$ (voir le mémoire cité plus haut). Le complémentaire de R relativement à P est formé par la réunion d'une infinité dénombrable d'ensembles non denses sur P. R est ce que j'ai proposé d'appeler un *résiduel* de P. La condition énoncée est donc nécessaire.

2°. La condition est suffisante. Supposons que E soit clairsemé. E est donc dénombrable. Car, l'ensemble Q des points au voisinage desquels un ensemble D est non dénombrable, est parfait, et D est partout dense sur Q. Cela posé, nous envisageons pour un point quelconque M de E, deux sortes de rangs. D'abord, E étant dénombrable, nous pouvons attribuer à M un rang entier propre n. D'autre part, dans la suite des ensembles E_α, formée comme il a été expliqué, considérons ceux de ces ensembles qui ne contiennent pas M. L'un d'eux a un rang inférieur à tous les autres, soit γ ce rang. γ ne peut pas être un nombre de seconde espèce. Car M, étant situé dans $E_{\gamma'}$, quelque soit $\gamma' < \gamma$, serait dans E_γ, si γ était de seconde espèce. On peut donc poser $\gamma = \delta + 1$. M est dans E_δ, mais non pas dans $E_{\delta+1}$. Donc, M est dans E_δ mais n'en est pas point limite. M est un point isolé de E_δ. Cela étant, $\varphi(n)$ étant une fonction quelconque de n tendant vers 0 quand n croît, nous prenons pour $I(M)$ un intervalle ou cercle ou sphère,... ayant pour centre M et un rayon $r(M)$ inférieur d'une part à $\varphi(n)$, d'autre part à la distance de M à $E'_\delta = E_{\delta+1}$.

Je dis qu'un point quelconque N de l'espace n'est intérieur qu'à un nombre limité d'ensembles $I(M)$. En effet, si N était intérieur à une infinité de tels ensembles $I(M)$, soient $M^{(1)}, M^{(2)}, \ldots, M^{(p)}, \ldots$ les centres de ces ensembles, $\delta_1, \delta_2, \ldots, \delta_p, \ldots$ les ordres analogues à δ correspondant à ces divers points, $n_1, n_2, \ldots n_p, \ldots$ leurs rangs dans le premier classement des M en série unilinéaire, et enfin r_p le rayon de $I[M^{(p)}]$. Puisque les n_p sont distincts, n_p croît indéfiniment avec p, donc $r_p < \varphi(n_p)$ tend vers 0, donc N est point limite des $M^{(p)}$.

Parmi les nombres transfinis δ_p, il y en a au moins un, soit δ, auquel nul autre n'est inférieur. On a $\delta_p \geqslant \delta$ pour toute valeur de p, l'égalité étant réalisée pour au moins une valeur de p. Donc, au moins un point M_δ de E_δ est dans la suite $M^{(p)}$. D'ailleurs E_δ contient E_{δ_p}, donc $M^{(p)}$, quelque soit p. Donc, N est un point limite de E_δ. Mais ceci est impossible, puisque $I(M_\delta)$ contiendrait N et que, par hypothèse $I(M_\delta)$ ne contient aucun point de E'_δ. La condition est donc suffisante.

Soit H un ensemble fermé. Supposons d'abord que H n'ait pas de point commun avec $K_1 = E'$. Alors, il n'y a évidemment qu'un nombre fini d'ensembles $I(M)$ contenant à leur intérieur au moins un point de H. On voit en effet comme ci-dessus, que si ces points étaient en infinité, chacun de leurs points limites serait sur H, puisqu'il n'y a qu'un nombre limité d'ensembles $I(M)$ dont le rayon surpasse un nombre positif donné. Si donc H est distinct de E' dérivé de E, nous aboutissons à une contradiction.

Plus généralement, *si l'ensemble fermé H est situé sur K_α et s'il n'a pas de point commun avec $K_{\alpha+1}$, il n'existe qu'un nombre limité d'ensembles $I(M)$ contenant à leur intérieur au moins un point de H.* En effet, si $\beta < \alpha$, à tout point M de E_β correspond un ensemble $I(M)$ sans points communs avec H, puisque $I(M)$ n'a pas de point commun avec $E'_\beta = K_{\beta+1}$, qui contient K_α et par suite H. Donc, les seuls points M dont les ensembles $I(M)$ peuvent contenir au moins un point de H sont les points M de E_α. Comme H est sans point commun avec $K_{\alpha+1}$ dérivé de E_α, nous sommes ramenés au premier cas. L'extension du théorème est démontrée.

Voici une application de la proposition ci-dessus à la théorie des fonctions. Désignons par $f(M)$ une fonction des coordonnées $x, y, \ldots u$ d'un point M de l'espace, et par $f(M - M_0)$ la fonction $f(x - x_0, y - y_0, \ldots, u - u_0)$. Soit $f_n(M)$ *une fonction bornée à l'extérieur de toute sphère ayant pour centre l'origine* 0 *des coordonnées, et telle que* $|f_n(M)|$ *croît indéfiniment quand M tend indifféremment vers* 0 (sans coïncider avec 0). Alors:

La condition nécessaire et suffisante pour qu'il existe des coefficients α_n indépendants de M et tels que la série $\alpha_n f_n(M - M_n)$ soit partout convergente, est que l'ensemble M_n soit clairsemé.

La condition est nécessaire. En effet, si l'ensemble E des points H_n n'est pas clairsemé, supposons donnée une suite quelconque de coefficients α_n. D'après $lim\,|f_n(M)| = \infty$ quand M tend indifféremment vers 0, n restant invariable, il existe une sphère ayant son centre à l'origine et en tout point de laquelle $|f_n(M)| > \dfrac{1}{|\alpha_n|}$. Soit r'_n le rayon de cette sphère. Entourons M_n d'une sphère I'_n de rayon r'_n. L'ensemble M_n n'étant pas clairsemé, il y a des points de l'espace intérieurs à une infinité de sphères I'_n. Pour chacun de ces points N, la série $\alpha_n f_n(N - M_n)$ est divergente comme ayant une infinité de termes supérieurs à 1 en valeur absolue.

La condition est suffisante. En effet, si E est clairsemé, nous pouvons autour de M_n décrire une sphère I_n de centre M_n et de rayon r_n telle que tout point de l'espace ne soit intérieur qu'à un nombre fini de sphères I_n. Soit, hors de la sphère de centre 0 et de rayon r_n, μ_n le maximum de $|f_n(M)|$. μ_n existe, puisque par hypothèse $|f_n(M)|$ est borné à l'extérieur de toute sphère ayant son centre à l'origine. Soit a_n un nombre quelconque inférieur en module à $\frac{1}{n^2\mu_n}$. La série $a_n f_n(M-M_n)$ converge en tout point M, comme n'ayant qu'un nombre limité de termes supérieurs en module aux termes de même rangs de la série $\frac{1}{n^2}$.

On montre aisément que *la série* $a_n f_n(M-M_n)$ *converge uniformément sur tout ensemble fermé* H *sans points communs avec* E' *ou plus généralement* sur tout ensemble fermé *contenu dans* K_α *et n'ayant aucun point commun avec* $K_{\alpha+1}$. En effet, il n'y a qu'un nombre limité d'ensembles I_n contenant des points d'un tel ensemble H. Donc, à partir d'un certain rang N, le n^e terme de la série est inférieur à $\frac{1}{n^2}$ en tous les points de H, quelque soit $n > N$.

Supposons que $f_n(M)$ *soit la somme d'une série*

$$u_{n\,1}(M) + u_{n\,2}(M) + \dots + u_{n\,p}(M) + \dots$$

uniformément convergente et à termes bornés (chacun séparément) à l'extérieur de toute sphère ayant son centre à l'origine. Alors, à l'extérieur d'une telle sphère ayant le rayon r_n défini plus haut, les sommes

$$u_{n\,p}(M) + u_{n\,p+1}(M) + \dots + u_{n.q}(M)$$

sont, indépendamment de p, de q et de M, bornées en module par un même nombre λ_n. (en particulier, avec $q=p$, $|u_{n.p}(M)| < \lambda_n$). Soit a_n un nombre de module inférieur à $\frac{1}{n^2\lambda_n}$. Je dis qu'*en ajoutant par colonnes les séries* $a_n f_n(M-M_n)$, *nous obtenons une série*

$$w_1(M) + w_2(M) + \dots + w_p(M) + \dots,$$

convergente en tout point M. En effet, on a:

$$w_p(M) = a_1 u_{1.p}(M-M_1) + a_2 u_{2.p}(M-M_2) + \dots + a_n u_{n.p}(M-M_n) + \dots$$

La série $w_p(M)$ est convergente puisque, M n'étant intérieur qu'à un nombre limité de sphères I_n, la série $w_p(M)$ n'a qu'un nombre limité de termes supérieurs en valeur absolue à l'inverse du carré de leur rang.

Soit ε un nombre positif. Nous voulons prouver que, M étant choisi, il est possible de déterminer N_0 de façon que $|w_{p+1}(M) + \dots + w_{p+q}(M)| < \varepsilon$ quelque soit $p > N_0$, et quelque soit q.

Cette relation s'écrit:

$$\left|\alpha_1 \sum_{m=p+1}^{m=p+q} u_{1\,m}(M-M_1) + \alpha_2 \sum_{m=p+1}^{m=p+q} u_{2\,m}(M-M_2) + \ldots + \alpha_n \sum_{m=p+1}^{m=p+q} u_{n\,m}(M-M_n) + \ldots\right| < \varepsilon.$$

Nous allons même montrer que l'on peut résoudre par $p > N_1$ inégalité

$$\left|\alpha_1 \sum_{m=p+1}^{m=p+q} u_{1,m}(M-M_1)\right| + \ldots + \left|\alpha_n \sum_{m=p+1}^{m=p+q} u_{n\,m}(M-M_n)\right| + \ldots < \varepsilon. \quad (1)$$

Nous divisons les termes de la série du premier membre de (1) en trois catégories.

1° M étant intérieur à un nombre limité (ou nul) de sphères $I(M_n)$, soient $M_{n_1}, M_{n_2}, \ldots, M_{n_h}$ les centres de ces sphères. Puisque les séries

$$\sum_{p=1}^{\infty} u_{n_1\,p}(M-M_{n_1}),\quad \sum_{p=1}^{\infty} u_{n_2.p}(M-M_{n_2}), \ldots, \sum_{p=1}^{\infty} u_{n_h\,p}(M-M_{n_h})$$

sont convergentes au point M, nous pouvons déterminer N_1 de façon que, si $p > N_1$,

$$|u_{n_i,p+1}(M-M_{n_i}) + u_{n_i,p+2}(M-M_{n_i}) + \ldots + u_{n_i,p+q}(M-M_{n_i})| < \frac{\varepsilon}{3h|\alpha_n|}$$

pour $i = 1, 2, \ldots, h$, quelque soit q. Les termes $\left|\alpha_{n_i} \sum_{n=p+1}^{i=p+q} u_{n_i\,m}(M-M_{n_i})\right|$ ont alors une somme inférieure à $\frac{\varepsilon}{3}$.

2° Soit N' un entier supérieur a $\frac{3}{\varepsilon}$. La série $\sum_{N'+1}^{\infty} \frac{1}{n^2}$ a une somme inférieure à $\frac{1}{N'}$, donc à $\frac{\varepsilon}{3}$. Tous les termes de la série (1) de rangs supérieurs à N' et différents des n_i, ont donc une somme inférieure à $\frac{\varepsilon}{3}$.

3° La série $\sum_{m=1}^{m\infty} u_{n\,m}(M)$ étant uniformément convergente pour n fixe et M variable avec dist. $OM > r_n$, nous pouvons déterminer un nombre $N_{2\,n}$ tel que, si

$$p > N_{2\,n}, \text{ on a } \left|\sum_{m=p+1}^{m=p+q} u_{n.m}(M)\right| < \frac{\varepsilon}{9}\,|\lambda_n|.$$

Donnons à n les valeurs $1, 2, \ldots, N'$ distinctes des n_i, les $N_{2\,n}$ ont une valeur maximum N_2. D'après $n \neq n_i$, M est extérieur à la sphère I_n de centre M_n et de rayon r_n. Si

$$p > N_2, \text{ on a } \left|\alpha_n \sum_{m=p+1}^{m=p+q} u_{n.m}(M-M_n)\right| < \frac{\varepsilon}{9n^2}.$$

Donc, la somme des termes correspondants à $n \neq n_1$, $n \leqslant N'$ est inférieure, si $p > N_2$, à $\frac{\varepsilon}{9}\sum_1^\infty \frac{1}{n^2} = \frac{\varepsilon}{9}\cdot\frac{\pi^2}{6} < \frac{\varepsilon}{3}$. Donc, si N_0 est le plus grand des deux nombres N_1 et N_2, la condition $p > N_0$ entraîne:

$$|w_{p+1}(M) + \ldots . + w_{p+q}(M)| < \varepsilon,$$

quelque soit q. La série $w_p(M)$ est donc partout convergente.

Application. $\varphi(n)$ *étant une fonction positive de l'entier* n, *jamais croissante,* la série

$$\varphi(1) \sin \theta + \varphi(2) \sin 2\theta + \ldots + \varphi(n) \sin n\theta + \ldots,$$

est convergente quelque soit θ. Soit $f(\theta)$ sa somme. $\theta f(\theta)$ tend vers 0 avec θ. Si la série $n\varphi(n)$ est divergente, $f(\theta)$ n'est pas sommable et $|f(\theta)|$ croit indéfiniment avec $\frac{1}{|\theta|}$. Soient θ_n une suite de valeurs de θ situées sur le segment $(-\pi, +\pi)$ et y formant un *ensemble clairsemé* quelconque.

Il existe alors une suite de nombres positifs ω_n *tels que, si* $|\alpha_n| < \omega_n$, *la série*

$$\varphi(1) \sum_{m=1}^{\infty} \alpha_m \sin(\theta - \theta_m) + \ldots . + \varphi(n) \sum_{m=1}^{\infty} \alpha_m \sin n(\theta - \theta_m) + \ldots .$$

est convergente quelque soit θ. Soit $\Gamma(\theta)$ sa somme.

J'ai défini sous le nom de totalisation un procédé d'intégration de certaines fonctions non sommables. La première condition remplie par les fonctions totalisables, — savoir que l'ensemble H des points d'un ensemble parfait P au voisinage desquels la fonction est non sommable sur P, H est non dense sur P, — cette condition est remplie par toutes les fonctions limites de fonctions continues, puisque, celles-ci étant ponctuellement discontinues, l'ensemble K des points de P au voisinage desquels l'une d'elles est non bornée sur P, K est non dense sur P. K contient évidemment H.

A toute fonction limite de fonctions continues, on peut donc faire correspondre une suite d'ensembles parfaits $P_1, P_2, \ldots, P_\alpha, \ldots$, correspondants aux divers nombres ordinaux des classes I et II. Par définition, si α est de première espèce, P_α est le noyau parfait de l'ensemble fermé constitué par les points de $P_{\alpha-1}$ au voisinage desquels f est non sommable sur $P_{\alpha-1}$. Si α est de seconde espèce, P_α est le plus grand ensemble parfait commun à tous les $P_{\alpha'}$ quand $\alpha' < \alpha$. Si Q_α est l'ensemble des points de P_α au voisinage desquels P_α est de mesure positive (ou épais), $P_{\alpha+1}$ est l'ensemble des points de Q_α au voisinage desquels f est non sommable sur Q_α. Donc, $P_{\alpha+1}$ est non dense sur Q_α et *a fortiori* sur P_α. Donc, tous les P_α sont nuls à partir d'un certain rang,

Etant donnée inversement une suite quelconque d'ensembles parfaits P_α, telle que 1°. P_α soit contenu dans l'ensemble Q_α des points où P_α est épais, et soit non dense sur Q_α, et que, 2°. si α est de seconde espèce, P_α soit le plus grand ensemble parfait commun à tous les $P_{\alpha'}$ si $\alpha' < \alpha$, il est curieux de constater qu'il est possible de former une série trigonométrique convergente $\Gamma(\theta)$ telle que l'ensemble des points de non sommabilité de $\Gamma(\theta)$ sur le continu ait pour dérivé d'ordre Ω, précisément P_1 et que la suite d'ensembles parfaits relative à $\Gamma(\theta)$ et déterminée par la première opération du calcul totalisant, soit précisément la suite P_α.

En effet, considérons l'ensemble E formé de la réunion des ensembles F_α suivants. F_1 est constitué par les milieux des intervalles contigus à P_1. Pour F_α, nous considérons les intervalles contigus à $P_{\alpha+1}$. Parmi ces intervalles contigus, désignons par i_α ceux qui contiennent des points de Q_α. Puisque $P_{\alpha+1}$, situé sur Q_α, est non dense sur Q_α, tout point de $P_{\alpha+1}$ est limite d'intervalles i_α. Or, sur chaque intervalle i_α, Q_α a une mesure positive, puisque Q_α possède cette propriété au voisinage de chacun de ses points, et qu'il en existe dans i_α. Soit, dans chaque i_α, un point N_α où l'épaisseur de Q_α est égale à 1. La réunion de tous les N_α, pour une valeur donnée de α est un ensemble F_α situé sur Q_α, et possédant un point et un seul dans chacun des contigus i_α de $P_{\alpha+1}$. F_α a pour dérivé $P_{\alpha+1}$. E sera par définition l'ensemble de tous les F_α.

Il est aisé de voir que l'ensemble E_α est formé par tous les F_λ de rangs supérieurs ou égal à α.

E est donc clairsemé puisque, P_α étant nul à partir d'une certaine valeur de α, il en est de même des F et par suite aussi de E_α.

Formons avec les points M_n ou θ_n de E la série trigonométrique $\Gamma(\theta)$ définie plus haut. Pour un segment σ_1 sans points communs avec $F_1 + P_1$, donc situé à une distance positive de E, il n'existe qu'un nombre limité d'intervalles I_n empiétant sur σ_1. La série $\alpha_n f(\theta-\theta_n)$ est donc uniformément convergente sur σ_1. Donc elle est continue et par suite sommable sur σ_1.

Si σ_1 contient un point N_1 et nul point de P_1, soit p le rang du point N_1 dans la suite θ_n. $\Gamma(\theta) - \alpha_p f(\theta-\theta_p)$, est continue sur σ_1. Comme $f(\theta-\theta_p)$ est non sommable autour de θ_p, Γ est non sommable sur σ_1. Donc, les N_1 sont les seuls points de non-sommabilité étrangers à P_1. Comme l'ensemble des points de non-sommabilité est fermé, et que le dérivé des N_1 est P_1, cet ensemble est $\Sigma N_1 + P_1$.

Donc, le premier ensemble parfait dont la considération s'introduit par la première opération du calcul totalisant est P_1.

De même sur tout portion ϖ_α de P_α intérieure à un contigu de $P_{\alpha+1}$, la série $a_n f(\theta-\theta_n)$ est uniformément convergente. ϖ_α est un ensemble parfait qui possède au plus un point de E. Toutes les fonctions $\alpha_n f(\theta-\theta_n)$ sauf une au plus sont sommables sur ϖ_α et leur série est uniformément convergente sur ϖ_α. Donc, f est sommable sur ϖ_α si ϖ_α ne contient aucun point N_α. Si ϖ_α contient un point N_α de E, f est ou non sommable sur ϖ_α en même temps que $f(\theta-\theta_p)$ si p est le rang de N_α dans la suite θ_n. L'épaisseur de Q_α en N_α est 1. Il en résultera généralement et en particulier si $\varphi(n) = \frac{1}{Ln}$, que $f(\theta-\theta_p)$ n'est pas sommable en θ_p. Donc, l'ensemble des points de non-sommabilité de f sur P_α est formé, en dehors de $P_{\alpha+1}$, par F_α. Comme le dérivé de F_α est $P_{\alpha+1}$, la suite d'ensembles parfaits P_α est donc celle que détermine la première opération de la totalisation pour la somme de la série trigonométrique $\Gamma(\theta)$.

Palaeontology. — "*Quelques insectes de l'Aquitainen* DE ROTT, *Sept-Monts* (*Prusse rhénane*)." By FERNAND MEUNIER. (Communicated by Prof. K. MARTIN).

(Communicated at the meeting of April 23, 1920).

On sait que GERMAR, HAGEN, les VON HEYDEN et SCHLECHTENDAL ont décrit naguère une série d'insectes des lignites DE ROTT. [1])

Dès 1894, je me suis occupé des articulés des Sept-Monts. Les espèces nouvelles signalées, jusqu'ici, se répartissent dans les ordres suivants:

1. *Névroptères.*

Phryganea elegantula MEUN. Ulmeriella bauckhorni MEUN.

2. *Hémiptères hétéroptères.*

Lygaeites mysteriosus MEUN.

3. *Coléoptères.*

Galerucella serrata MEUN.

4. *Hyménoptères.*

Apis oligocaenica MEUN. Nysson rottensis MEUNIER. Myrmica archaïca MEUN. Formica bauckhorni MEUN. Proctotrypites rottensis MEUN. Andrena tertiaria MEUN. Eucera mortua MEUN. Bracon rottensis MEUN. Cryptus sepultus MEUN. Ponera rhenana MEUN.

5. *Diptères.*

Tipula sp. Sciara heydeni MEUN. Protomyia sluiteri MEUN. Systropus rottensis MEUN. Brachypeza graciosa MEUN. Syntemna sepulta MEUN. Boletina sp? Neoglaphyroptera subvenusta MEUN. Pericoma minuta MEUN. Gymnopternus bauckhorni MEUN. Plecia superba MEUN. Helomyza bauckhorni MEUN. Bibio infumatus MEUN. Plecia pulchella MEUN. Anthomyia sp? Lasiosoma minutissima MEUN. Cyttaromyella bastini MEUN.

Dans le groupe des Trichoptères ou Phryganidae, j'ai observé l'empreinte et la contre-empreinte d'un insecte se distinguant des espèces déjà signalées de ce gisement.

---

[1]) SCUDDER, S. H. „Index to the known fossil Insects of the World including Myriapods and Arachnids." Bull. U. S. Geological Survey. N°. 71, Washington 1891 voir aussi le „Handbuch" de ANTON HANDLIRSCH.

Parmi les hémiptères hétéroptères, je complète la description de Pachymerus antiquus HEYDEN, très frustement figuré par cet auteur.

Les homoptères Cicadaires n'ont pas encore été rencontrés à ROTT. HEER signale une faunule d'Oeningen et de Radaboj. Les travaux entrepris, à ROTT, en vue de l'exploitation de la paraffine, m'ont fait découvrir une délicate empreinte de „Zirpe", se classant, irrécusablement, parmi les Jassidae Bythoscopinae du genre Agallia Curtis.

Au moyen d'un agrandissement photographique, fait avec soin par mon ami M. FERD. BASTIN d'Anvers, il m'est possible de donner un bon dessin restauré de l'élytre de cet Homoptère.

Parmi des Coléoptères Curculionidae mentionnons Rhynchites hageni HEYDEN. Monsieur BAUCKHORN m'a communiqué des exemplaires en parfait état de conservation.

Dans le monde des Hyménoptères Tenthredinidae (mouches à scie), M. BAUCKHORN a trouvé un fossile a veination des ailes enchevêtrée ce qui empêche de le placer, à coup sûr, parmi les Pinicolides et m'oblige, provisoirement, à le désigner sous le nom de Pinicolites graciosus, comme je l'ai fait d'ailleurs pour une autre bestiole des plaquettes du Sannoisien d'Aix en Provence (Tenthredinites bifasciatus).

Parmi les Diptères Mycetophilidae ou fungicoles, on a remarqué, à ROTT, une minuscule empreinte appartenant au genre Tetragoneura Winnertz. Ce genre, qui parait être rare sur les lignites de Sept-Monts, est représenté par plusieurs espèces critères dans l'ambre de la Baltique. Malgré de nombreuses recherches, je n'ai pu le découvrir dans le Copal sub-fossile et d'origine récente.

Les remarques concernant les Termitidae de ROTT sont peu précises. HAGEN et HEER ont bien mentionné, il est vrai, deux espèces des schistes ligniteux du Rhin, mais, leurs trouvailles nécessitent de nouvelles recherches, les dessins qu'ils en donnent ne pouvant guère nous satisfaire, au point de vue de la systématique moderne concernant le veination des ailes de ces Arthropodes.

M. BAUCKHORN qui collectionne depuis plus de dix ans, avec le plus grand zêle, les articulés de ROTT n'a rencontré aucun Termitidae. Ces archaïques formes d'insectes, assez abondantes dans le succin du Samland, s'observent aussi fréquemment incluses dans le Copal de diverses provenances africaines; elles appartiennent à plusieurs sous-genres de l'ancien genre Termes Linné.

Terminons en disant que dans son travail, si soigné, sur les Termitidae fossiles M. K. v. ROSEN, DE MUNICH, suggère l'idée que ces êtres sont à rapprocher,[1] phylogénétiquement parlant, des

[1]) Die fossilen Termiten. Eine kurze Zusammenfassung der bis jetzt bekannten Funde. Trans. of the second Entomological Congress, 1912

Blattidae Protoblattinae: Cette opinion, si intéressante qu'elle soit, demande encore à être étayée sur des bases plus irréfutables.

Description des espèces.

1. Névroptères.

Trichopteridae (Phryganidae).

L'aile décrite ci-dessous diffère notablement de Phryganea elegantula et de Ulmeriella bauckhorni. Elle présente les caractèrecs suivants. Longueur de l'aile 12 mill., largeur $3^3/_4$ mill.

Aile arrondie à l'extrémité. Nervure sous-costale rapprochée de la costale et anastomosée au delà du milieu du bord antérieur de l'aile. Nervure radiale longuement fourchue, son secteur part vers le milieu de sa longueur, il est aussi orné d'une fourche (ces deux fourches sont d'égale longueur). La médiane, d'abord simple à peu de distance de sa base, est ensuite longuement fourchue, son rameau supérieur l'est aussi, l'inférieur est aussi branchu. Le cubitus est fourchu peu après son point de départ, son rameau supérieur a aussi une fourche. Sur l'empreinte, on ne remarque qu'une nervure anale simple (il en existait vraisemblablement d'autres, mais, la presque totalité du champ anal est altéré par la fossilisation). La conservation (froissée) de l'empreinte et de la contre-empreinte ne permet pas d'établir les rapports phylogénétiques probables de ce fossile avec les espèces de la faune actuelle.

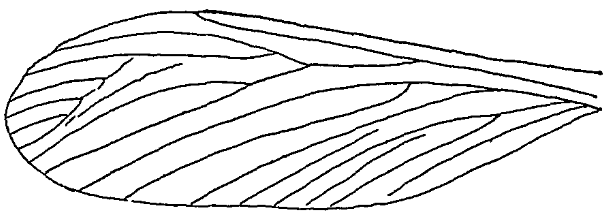


Fig. 1.

2. Hémiptères.

Hétéroptères.

Lygaeidae.

Genre Pachymerus Le Pelletier

Pachymerus antiquus Heyden.

Palaeontogr. t. VIII, p. 16—17, pl. 3, fig. 9.

Le genre Pachymerus est bien représenté dans les terrains tertiaires européens à Aix (Provence), à Oeningen, à Radoboj, à Brunnstatt (Alsace). Il ne semble pas être commun sur les couches aquitaniennes de ROTT. Lygaeites mysteriosus MEUN. est une intéressante trouvaille de ce gisement rhénan. Les espèces signalées d'Aix sont ordinairement frustes, ce qui empêche d'en faire une étude critère et de les comparer, avec soin, à celles des autres formations géologiques.

Longueur du corps $5^{1}/_{2}$ mill.

Pachymerus antiquus de Heyden a les antennes robustes et paraissant être composées de six articles, dont le premier est court, les autres articles cylindriques sont un peu élargis à l'extrémité. Les fémurs des trois paires de pattes sont dilatées. Les segments abdominaux sont très appréciables. Le type de v. Heyden est peu net, celui trouvé par M. Bauckhorn permet d'étudier la morphologie des antennes. De nouveaux documents s'imposent, avant de donner une rigoureuse description de cette espèce, l'insecte, figuré plus loin, étant couché sur le dos, ce qui ne permet pas de voir la veination des élytres le thorax, l'écusson et les autres organes.

Homoptères.

Genre Agallia Curtis.

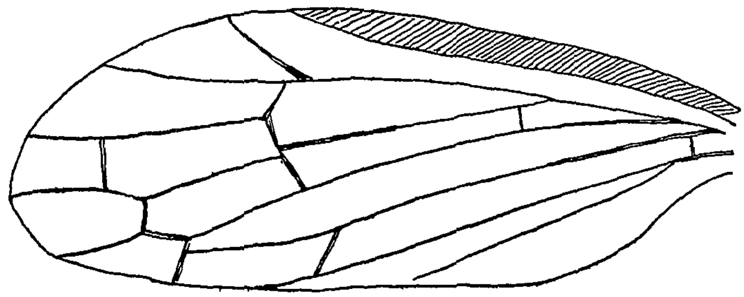


Fig. 2.

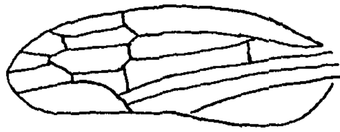


Fig. 2*a*.

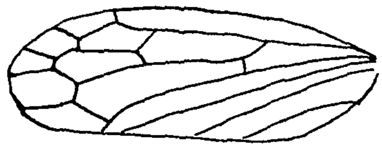


Fig. 2*b*.

Les genres européens Idiocerus Lew et Agallia Curtis différent des autres Bythoscopinae par les antennes qui sont insérées dans une cavité, peu profonde, de la face. Elle est très appréciable (tief.) chez les Pediopsis Burmeister, Macropis Lew et Bythoscopus Germar. La veination des élytres des Agallia et Bythoscopus s'éloigne par de menus détails des cellules apicales[1]).

Agallia sepulta n. sp.

Longueur du corps 4 mill.

[1]) Les caractères des antennes ne sont pas distincts.

Tête robuste et à peine plus large que le pronotum y compris les yeux qui semblent arrondis. Les ocelles (Nebenaugen) ne sont pas visibles. Elytres bien développés. Veination très nette: nervure cubitale[1]) réunie au bord costal à quelque distance de l'apex de l'élytre. Cellule basale (area basalis) très visible. Trois cellules discoïdales anté-apicales et trois cellules apicales; une nervure anale et une axillaire. Le réseau de la veination des ailes est peu précis et de plus enchevêtré. Les autres caractères ne sont pas conservés, ce qui empêche de comparer ce Cicadaire avec les autres Bythoscopinae et avec les Jassidae de la Sous-famille des Tettigonini, notamment avec le genre Tettigonia Olivier, déjà observé sur les schistes d'Aix, de Radoboj, d'Oeningen etc.

On sait que les Cicadaires ne sont pas très communs dans l'ambre de la Baltique, leur étude, déjà ancienne, demande à être entièrement revisée.

3. *Coléoptères.*

Curculionidae.

Genre Rhynchites. Hbst.

Rhynchites hageni HEYD, MEUN.

Palaeontographica t. XV, pl. 23. fig. 6 (1866).

Le rostre est un peu plus long que la tête, large (HEYDEN dit qu'il est deux fois aussi long que cet organe, ce que contredit son dessin). Les antennes, inséreés à la base du rostre, paraissent être composées de dix articles: le 1e cylindrique et distinctement plus long que les suivants; les articles 2—7 aussi cylindriques; les 3 derniers épaissis, cupuliformes et donnant à l'antenne l'aspect d'une massue. La partie antérieure du thorax est moins large que la postérieure, l'écusson est très petit. Les élytres ovoïdes, un peu allongés sont, comme le dit von HEYDEN, 2 fois plus longs que le thorax; ils semblent être ornés de huit stries longitudinales formées de points ciliés et rapprochés. Les pattes, assez robustes, (HEYDEN signale dans la diagnose qu'elles sont minces "dünn", le dessin les montre cependant assez vigoureuses); les articles tarsaux courts et trapus, comme l'indique exactement la figure de HEYDEN. L'abdomen paraît ovoïde et environ aussi long que les élytres. Longueur du corps 6 mill., longeur de l'élytre 3 mill., largueur $1^1/_4$ mill.

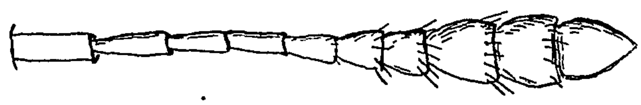


Fig. 3.

[1]) Classification de MELICHAR.

4. *Hymenoptères.*
Tenthredinidae (Chalastogastra).

Les Térébrants de cette famille sont peu représentés sur les couches Sannoissiennes d'Aix et les schistes Aquitaniens DE ROTT. Tenthredinites bifasciatus d'Aix avait la veination des ailes trop enchevêtrée pour décrire les menus détails de l'emplacement des veines. La forme de ROTT a des antennes rappelant, par leur curieux aspect, celle du genre Pinicola Brébisson. Le réseau des veines (nervures) du spécimen DE ROTT étant peu net, il est prudent d'assigner, pour le moment, à ce fossile le nom de Pinicolites (nov. gen.).

Pinicolites graciosus n. sp.

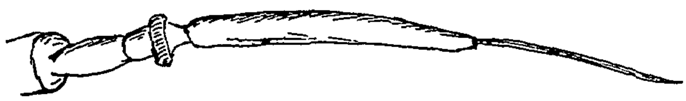


Fig. 4.

Longueur du corps $5^1/_2$ mill.

Les Térébrants Pinicolidae sont de bizarres mouches à scie par la morphologie, toute particulière, des antennes terminées en fouet. Les espèces actuelles ont 12 articles à ces organes. Chez le fossile, on ne peut déterminer exactement leur nombre. Les caractères, les plus appréciables, se résument comme suit : Tête robuste, un peu plus large que le thorax, yeux saillants. Antennes assez longues : le 1e article long, cylindrique ; le 2e court ; les articles suivants (assez indistincts, chez le fossile) sont épaissis ; le fouet de l'antenne, assez long, commence brusquement. La veination des ailes, très froissée sur le schiste, est indéchiffrable. La tarière est un peu allongée, assez grêle.

Ce Chalastogastra a été trouvé, à ROTT, par M. BAUCKHORN.

5. *Diptères.*
Mycetophilidae.
Genre Tetragoneura Winnertz.

Ce genre est bien représenté dans le succin de la Baltique. Il n'a pas été signalé des schistes d'Oeningen et de Radoboj. Je ne l'ai pas observé dans le Copal subfossile et d'origine recente. Le genre Tetragoneura semble être rare à ROTT. Tetragoneura sannoisiensis MEUN.[1]) vient des couches d'Aix en Provence.

[1]) Bull. de la Soc. Géol. de France, 4e serie, t XIV; année 1914, p. 197, fig. 11

Tetragoneura veterana n. sp.

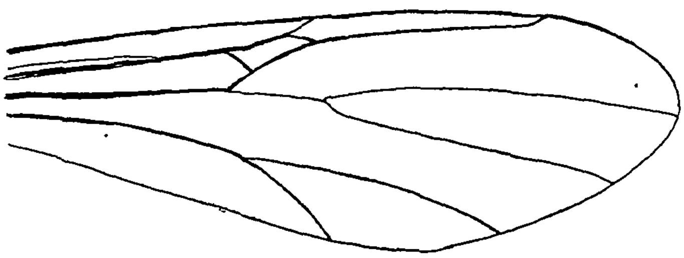


Fig. 5.

Longueur du corps $4^1/_2$ mill. Longueur de l'aile $2^1/_4$ mill.

Tête aussi large que le thorax, qui est un peu gibbeux. Ailes notablement moins longues que l'abdomen. Bord costal alaire longuement prolongé après le cubitus [1] (RADIUS sec. COMSTOCK et NEEDHAM). Nervule assistante anastomosée à la sous-costale à quelque distance de la cellule cubitale qui est subrectangulaire. Pétiole de la fourche discoïdale assez court, fourche posticale (5^e et 6^e nervures) plus longue que la discoïdale.

Le dessus des segments abdominaux devait être orné de bandes foncées. Les antennes, assez grêles, paraissent être pourvues d'articles subcylindriques et courts. Les autres caractères sont indistincts. [2])

BIBLIOGRAPHIE.

MEUNIER FERNAND. 1917. Sur quelques insectes de l'Aquitanien de ROTT (Sept-Montagnes, Prusse rhénane). Verhandelingen der Koninklijke Akademie van Wetenschappen, tweede Sectie, Deel XX, N°. 1, Amsterdam (Pour autre Bibliographie voir: page 16 et 17 de ce travail).

MEUNIER FERNAND. 1918 Neue Beiträge uber die fossilen Insekten aus der Braunkohle von ROTT (Aquitanien) am Siebengebirge (Rheinpreussen). Jahrb d Preuss. Geol. Landesanstalt, Bd. XXXIX, Teil 1, Heft 1, S. 141—153; Taf. 10 u. 11. Berlin, 1918 (1919).

FIGURES DU TEXTE.

[1]) Classification de WINNERTZ: Beitrag zu einer Monographie der Pilzmücken.

[2]) Par la forme de sa cellule cubitale et son radius arqué (cubitus sec. Winnertz) ce curieux Tetragoneura, nécessitera, peut-être, par la suite, la création du genre Paleotetragoneura.

59*

EXPLICATION DE LA PLANCHE[1]).

Fig. 1. Phryganidae gen.? *n. sp.* (Nevroptera).
Fig. 2. Pachymerus antiquus Heyden (MEUN.) Heteroptera.
Fig. 3. Agallia sepulta *n. sp.* (Homoptera).
Fig. 4. Rhynchites hageni Heyden (MEUN.) Coleoptera.
a. vu du dos.
b. vu de côté.
Fig. 5. Pinicolites graciosus *n. gen. n. sp.* (Hymenoptera).
Fig. 6. Tetragoneura veterana *n. sp.* (Diptera).

---

1) Les clichés ont été exécutés par mon distingué ami M. FERD. BASTIN d'Anvers. Les figures du texte sont de Mme F. MEUNIER.

Microbiology. — "*Chemosynthesis at denitrification with sulfur as source of energy.*" By Prof. M. W. BEIJERINCK.

(Communicated at the meeting of February 28, 1920).

In photosynthesis organic matter results from the reduction of carbonic acid by light as source of energy; the same takes place in chemosynthesis by chemical energy. Organisms with photo- or chemosynthesis are called autotrophes; those which feed on other organic substances are heterotrophes. The product of chemosynthesis is the body substance of the producers, always spore-free bacteria.

I described chemosynthesis at denitrification with sulfur as source of energy, on 16 April 1903 at the 9th Dutch Congress of Natural and Medical Science[1]). I then thought that in the process a facultatively anaerobic bacterium was concerned difficult to isolate by the plate-culture method. It was further presumed, that this species produced so much organic substance by chemosynthesis that the many directly visible bacteria, denitrifying with organic food, might live thereon. This supposition has proved to be erroneous; the latter themselves are in fact the operators of the sulfur denitrification as well as of the chemosynthesis. They are easily cultivated on broth-agar or broth-gelatin, but then they lose, and this is the new view, their autotrophy together with the power of sulfur denitrification, whilst preserving this power with organic food. The loss is caused by the growth with organic food and this loss being hereditarily constant, we have a case here similar to that which I described earlier for the nitrate ferment, and which I called "physiological species formation"[2]). Just as I then distinguished the oligotrophic from the polytrophic state we may in this case speak of the *autotrophic* and the *heterotrophic* condition of the operators[3]). The heterotrophic form is thus some common denitrifying bacterium.

On account of the little acquaintance with chemosynthesis acquired until now, I will begin with describing once more the original experiment[4]).

1) Phénomènes de réduction produits par les microbes. Archives Néerland. Sér. 2. T. 9, Pag. 153. 1904.

2) These Proceedings. Vol. 23, Pag 1163, March 28 (10 April) 1914.

3) As the existence of chemosynthesis is proved with certainty for the sulfur denitrification, but not for the nitration, the same nomenclature could not be followed in the two cases.

4) An enumeration of the chief processes accompanied with chemosynthesis is to be found in my paper: Bildung und Verbrauch von Stickstoffoxydul durch Bakteriën. Centralbl. f. Bakteriologie 2te Abt. Bd. 25, Pag. 30, 1910.

Arrangemeut and course of the experiment.

If a mixture of sulfur and chalk is introduced into a saltpetre solution with addition of some garden soil or canal mud, there will soon evolve at room temperature or at 25° to 30° C., a current of gas consisting of free nitrogen and carbonic acid. Thereby the saltpetre is denitrified, the sulfur is oxidised to sulfuric acid, found back as gypsum and potassiumsulfate, after the formula

$$6\,KNO_3 + 5\,S + 2\,CaCO_3 = 3\,K_2SO_4 + 2\,CaSO_4 + 2\,CO_2 + 3N_2$$

whereby per gram of decomposed nitrate about 1 cal. is produced. When after some days the process has become intense, the mud with the gas rises to the surface, and if the experiment is carried out in a flask, the contents can flow out with the gas as a slimy mass. This is bacterial slime, which keeps the sediment together.

If using distilled water with 10 °/₀ chalk, 10 °/₀ sulfur, 2 °/₀ potassium-saltpetre, 0.02 °/₀ bipotassium phosphate, 0.02 °/₀ magnesium chloride, and infecting with a small floccule from the said denitrification, we see after some days at 25° to 30° C. the very same phenomena as when using soil, only less intense; so the presence of soil is not necessary, but it clearly acts favourably. If the soil or mud is beforehand left a few days under a dilute saltpetre solution, so that all the organic substances fit for denitrification are removed, the soil remains quite as good for the sulfur-chalk experiment, hence the organic matter cannot be the cause of the favourable action on the process. It seems to result from the presence in the soil of colloidal silicic acid and aluminium silicate, which are to be considered as catalyzers that hasten the decomposition. So, in a thiosulfate denitrification the reaction goes on much swifter in presence of chalk and bolus (aluminium silicate) than with chalk only.

The saltpetre solution can be used in the most different concentrations. Even in 10 °/₀ solutions in tapwater made to a pap with sulfur and chalk, I saw at room temperature a spontaneous, intense gas production, with slime formation. The gas was nitrogen and carbonic acid; nitrogen oxydul seemed quite absent. The slime is bacterial slime, for the greater part consisting of different varieties of *Bacterium stutzeri* and *B. denitrificans*. It is so voluminous that its formation can only be explained by admitting that the said bacteria themselves produce this slime from the carbonic acid by chemosynthesis. With distilled water the result of the experiment is the same. In a closed bottle and with distilled water the process goes on as with accession of air, which proves convincingly, that presence of organic substance is not required for the development of the rich bacterial flora which encloses the chalk and sulfur, and where at last many infusoria and monads, that feed on the bacteria, may be observed. As said the organic matter of the bacterial bodies must here be formed from the carbonic acid, whilst the required chemical energy is produced by the oxidation of the sulfur. Consequently this is a case of chemosynthesis and no other analogous process is known which produces organic substance in a simpler and more profuse way. [1])

By decanting and renovating the saltpetre solution as soon as the evolution of gas diminishes, the activity returns. [2]) This being repeated a few times the precipitate changes into a slimy mass, so rich in slime-forming bacteria that at heating on a platinum plate in the BUNSEN burner carbon is separated. With concentrated sulfuric acid carbonisation is also easily demonstrated. As the rate of nitrogen of this slime is less than 3 %, it must chiefly consist of wall substance, which is evidently the chief product of the chemosynthesis. [3]) It results from the carbonic acid after the same formula as the starch in the chlorophyll granules by photosynthesis, thus

$$6\,CO_2 + 5\,H_2O = C_6H_{10}O_5 + 6O_2$$

so that oxygen is set free, which explains the ready course of the process in a closed bottle, when considering that all denitrifying bacteria require a little free oxygen.

Just as the organic denitrification, that with sulfur may as well take place in the dark as in the light. After pasteurisation no sulfur-denitrification or oxydation is observed.

The quantitative estimation of the carbon fixed by chemosynthesis was made as follows. The sediment was treated with hydrochloric acid and later with alkali to remove the chalk and the sulfur, whereby certainly a great portion of the organic substance is lost. In the remaining precipitate, which still contains gypsum, the organic matter was determined as carbonic acid after the method of HERZFELD-

---

1) It is true that chemosynthesis at the oxidation of hydrogen in presence of carbonic acid and soil, described by NIKLEWSKY and LEBEDEFF, is as productive in organic substance, but the experiment is less simple.

2) Addition of soda instead of decantation and renovation, also acts favourably. Evidently the dissolved sulfuric acid is difficultly neutralised by the chalk of the precipitate.

3) See also: A. J. LEBEDEFF, Ueber die Assimilation des Kohlenstoffs durch Wasserstoff-oxydierenden Bakteriën. Berichte d. Deutschen Botan. Gesellsch. Bd 27, Pag. 598, 1909. He says that the bacterium can oxidise hydrogen in absence of CO_2; this, however, is manifestly erroneous. Nor does he take into consideration the oxygen produced at the denitrification by the hydrogen of the saltpetre, used by him as source of nitrogen. His fear that by using ammonsalts nitrification would follow, is under these conditions unfounded.

WOLFF-DEGENER,[1]) by oxydation with bichromate and sulfuric acid. After a culture of about six weeks there was in this way found about 0.05 gram of carbonic acid per gram of oxidised sulfur, which corresponds to 0.013 gr. of organic carbon.[2]) This quantity, however, must certainly be doubled, for at the extraction of the chalk and sulfur at least half the weight of the bacterial substance is lost. I therefore esteem the production of organic carbon in relation to the oxidised sulfur at 2 $^0/_0$ in weight.

Old cultures containing much organic matter and in which the nitrate has disappeared produce H_2S, obviously in consequence of sulfate reduction, and perhaps, too, directly from the still present sulfur, whilst the hydrogen wanted for this originates from the organic material formed by chemosynthesis. Such liquids finally teem with infusoria and monads, and various other members of the so remarkable "sulfur-flora" and "-fauna".

The microscopical image during the period of chemosynthesis is that of very small, partly motile rodlets and micrococci. Spore-formers with chemosynthesis do not exist.

Plate culture.

The agents of the denitrification with sulfur were isolated on different solid media, but always with the result that the pure cultures, grown on organic media did not, or only feebly denitrify in the anorganic mixture; *only those of the silicic plates were but slightly enfeebled* in this function. The media used were: washed agar dissolved in distilled water, with salts; or tapwater-agar with $^1/_2$ $^0/_0$ thiosulfate, 0,1 $^0/_0$ saltpetre and 0,02 $^0/_0$ bipotassium fosfate; or silicic plates with the same mixture with or without addition of chalk, and finally broth-agar and broth-gelatin.

If on the media containing organic matter floccules of the sulfur denitrification are streaked off and cultivated at 30° C., there appear, already within 24 hours, denitrifying colonies which, especially on the broth plates grow with a remarkable rapidity. The two or three chief species recognisable among the denitrificators may be easily distinguished. On the media containing sulfur or thiosulfate and chalk, and on the silicic plates, the colonies remain small

[1]) F. TIEMANN und A. GÄRTNER. Die chemische, mikroskop. und bakteriol. Untersuchungen des Wassers. 3te Aufl. Pag. 247, 1889.

[2]) The quantity found by Mr. JACOBSEN at the direct oxidation of sulfur by bacteria was of the same order. (Die Oxydation des elementaren Schwefels durch Bakterien. Folia Microbiol. Jahrg. 1, Pag. 487, 1912).

and cannot be well recognised on account of the opaqueness of the medium. Yet I have further examined these colonies by making streaks of them on broth-agar plates, always finding that they more or less readily develop; colonies failing in this respect I did not find.

I have also tried to obtain anorganic denitrifications with those portions of the streaks on the sulfur- and thio-sulfate plates lying between the colonies, but as well in aërobic as in anaërobic condition always in vain. Neither microscopically nor by colouring, bacteria or microbes of other nature could be found in these parts.

Hence it follows with certainty that the agents of the anorganic denitrification grow to colonies both on the sulfur-chalk and the thio-sulfate plates and besides, as will be still further proved below, on the ordinary broth plates. The highly improbable hypothesis that they might be obligative anaërobes is disproved by these experiments, which are, however, well in accordance with the conception that by growth on organic matter their power of autotrophy gets lost.

To compare the broth with the thiosulfate medium I made the following experiment.

A platinum wire was bent so as to form at one end a loop, with which droplets of the same size could easily be taken up; the other end was curved to a circular base, which made it possible to place it on the balance and determine the weight of the droplet. Now drops of equal size were taken up with this loop from the anorganic denitrifications and transported for comparison to a thiosulfate- and to a broth-plate. The result was that the number as well as the species of the developing colonies were about the same. All the colonies grown on the thiosulfate plates, after being sown on broth-plates, developed very well, quite in accordance with what was observed already for the colonies grown on the sulfur-chalk plates.

So it is certain that the microbes causing the anorganic denitrification produce colonies on the organic plates.

This statement is of particular interest as the colonies, when again transferred to the anorganic sulfur-chalk mixture, do not, or only very feebly, denitrify, which means that they have almost or quite lost their power of chemosynthesis [1]).

This is not only true for the pure colonies separately, but likewise for the combinations that may be made of them. Even when the whole bacterial mixture on the plates is transported to the anorganic medium, only a slight or no chemosynthesis or denitrification

[1]) In "Untersuchungen über die Physiologie denitrifizirender Schwefelbakteriën, Sitzungsberichte Heidelberger Akademie. Biol. Abt. 1912", R. LIESKE has come to another result.

at all occurs. On the thiosulfate plates the germs preserve their autotrophy longer than on the broth plates, but there too, this power finally gets lost. The real cause of this loss is not yet quite explained. With certainty it can only be said to take place when the concerned germs *augment when fed with organic food.*

Especially on the broth plates at 30° C. the colonies develop rapidly. It seems that four or five species are thereby active. Three or four denitrify strongly in broth bouillon with 0.1 to 1 % potassium-nitrate, and they predominate so much that non-denitrifying species are not easily found. There is even no surer and easier method to obtain bacteria denitrifying with organic food than this anorganic denitrification, for although it is often difficult to isolate the active bacteria from the organic denitrifications, this is here by no means the case [1]).

Among the colonies obtained from the anorganic mixture there are, as said, some which do not denitrify with organic food. Probably they live in the sulfur-chalk cultures as saprophytes at the expense of the organic matter formed by the autotrophes.

On silicic-thiosulfate-nitrate-chalk plates develop, after two or three weeks, yellowish colonies of 1 to 1½ mm. in diameter and nearly 1 mm. high, evidently autotrophic. In the anorganic mixture, freed from air by boiling, they cause a vigorous denitrification after 24 hours at 28° C. already. When sown on broth-gelatin the colonies appear to consist of two soft varieties [2]) of *B. stutzeri*, which do not melt the gelatin and of which one shows the usual structure; the other, the commonest by far, lacks that structure completely, nevertheless it resembles *B. stutzeri* in the other cultural aspects. It consists of a white soft mass of extremely small rodlets. In broth nitrate both show strong denitrification, especially the soft form, so that it is one of the most intensely denitrifying bacteria I know. At re-inoculation from the organic into the anorganic food we also find here that the autotrophy and the power of anorganic denitrification are lost.

---

[1]) The most important denitrifying soil bacterium, the spore-forming *Bacillus nitroxus*, loses its denitrifying power quite or partly by growing on aërobic plates. Other species, such as *Bacterium pyocyaneum*, *B. stutzeri*, *B. denitrofluorescens* preserve, in aërobic plate cultures and in the collections, their denitrifying power unchanged for years.

[2]) In reality there are three varieties, but the third which shows the character of the ordinary tough, folded colonies of *B. stutzeri*, is rarer. — It must be admitted that the difference between the soft colonies and the typical *B. stutzeri* is, superficially, considerable, and I think that many other observers would bring them to distinct species.

The principal species.

The colonies from the sulfur-chalk denitrifications, which develop on the broth plates are for a part coloured yellow or reddish brown by carotin[1]), for the greater part, however, colourless. The brown species is a Micrococcus; it liquefies the gelatin and the micrococci differ much in size; the smaller ones are highly motile, but they lose their motility when transferred to broth-agar, whereby their denitrifying power, too, disappears. The yellow species is related to the brown and consists of small very motile rodlets. Here also the same variability.

The uncoloured colonies are of two types: soft, and tough or slimy.

All the soft ones liquefy the gelatin on which they grow intensely; sugars are not fermented, no fluorescence; they belong to three classes different by their size: 1. Extensive, rapidly growing, strongly denitrifying. 2. Middle sized, less rapidly growing, as strongly denitrifying. These two classes are allied by intermediate forms and may be brought to one single species, *Bacterium denitrificans*. 3. Very small and feebly growing, non-denitrifying bacteria, manifestly living at the expense of organic food produced by the other species through chemosynthesis.

With the pure cultures on an organic medium of the second form, I have succeeded in obtaining very feeble anorganic denitrifications, hence, chemosynthesis. This could, however, only be observed in the quite young cultures that had but for a short time grown on the broth medium. Cultures which have longer than two or three days been in contact with organic food and the air, can no more denitrify with sulfur and chalk, but still very well in saltpetre broth. For demonstrating the anorganic denitrification, test tubes are partly filled with mud, previously deprived of organic matter by keeping the mud under a saltpetre solution. To the mud sulfur and chalk are added and subsequently 1 $^0/_0$ saltpetre; the dissolved oxygen and the germs are removed by boiling; sterilisation is not wanted, as spore-formers with chemosynthesis do not exist.

Entrance of air is prevented by a hollow glass sphere, well fitting in the tube and floating on the liquid, but this precaution is not necessary.

With the pure cultures of the soft colonies I could not obtain any evolution of gas in this mixture, they manifestly lose their autotrophy still sooner than those of the second group.

The more or less tough, or slimy, or cartilaginous colonies belong

[1]) This pigment is soluble in CS_2 and turns blue or violet with concentrated sulfuric acid.

all to *Bacterium stutzeri*, if taking the conception of species in a broad sense; superficially there is a great difference between the colonies of this group. The usual form, which is very remarkable and easily recognisable by the shape of the colonies, has been described in these Proceedings by Professor VAN ITERSON [1]. Even in the smallest floccules of the sulfur denitrifications some form of *B. stutzeri* is found, although the soft colonies prevail. But besides, other varieties of *B. stutzeri* occur, for example such which slightly liquefy gelatin, or such which are light brown or rose-coloured, or whose colonies lack the so characteristic structure, and again others with that structure, but wanting the denitrifying power. There are, too, intermediate forms between the tough and the soft class, and I think it possible that they originate from each other by mutation.

That *Bacterium stutzeri* in the anorganic denitrifications possesses autotrophy, follows from the above described experiment with the silicic plates. But this may also be proved for colonies of "organic" origin, if only the right moment be chosen for experimenting with them. In the organic plate cultures the autotrophy of this species gets however rapidly lost. Only with quite fresh colonies, grown on thiosulfate-agar plates, and transferred to the anorganic medium, just at the time of their becoming visible a feeble but distinct anorganic denitrification could be obtained, which continued during several days with the same degree of intensity, only much feebler than the spontaneous denitrification. So it seems proved that the autotrophy does not disappear as an indivisable factor, but may get lost in parts.

That the autotrophy is really lost in the originally active colonies, is corroborated by the fact that not only the single colonies of the organic plates, but likewise the combinations of the colonies of the different species are quite inactive. Even all the colonies of broth-agar plates together, mixed with the undeveloped germs lying between them, do not produce any denitrification in the anorganic mixture. And this must be true for all the different species which produce anorganic denitrifications and evidently possess the power of chemosynthesis in their natural habitat.

This form of variability is obviously analogous to that of the nitrate ferment, which I formerly described [2]) and as said called physiological species-formation. In both cases a new elementary species is produced. It is remarkable that a number of species or varieties living under the same conditions are subject to this trans-

[1]) Ophoopingsproeven met denitrificeerende bakteriën. Acad. of sciences. Amsterdam, July 1902.

[2]) Ueber das Nitratferment und über physiologische Artbildung. Folia micro-

formation, and that between the principal form and the one that has completely lost its original character, some feebly denitrifying intermediate forms are found, which may be compared to subspecies.

Taking *B. denitrificans* as an example we can speak of *B. denitrificans autotrophus* and of *B. denitrificans heterotrophus*, the change being possible only in one direction, at least with our present knowledge.

This change is not a mutation in the accepted sense, as thereby the primitive stock continues to exist with the mutant under the same conditions under which the latter was formed. Here on the contrary all germs change simultaneously, so that in this case we have to do with a hereditarily constant modification, comparable to the pleomorphy of many Fungi, and to a certain extent, to alternation of generation. Comparable also to the production of somatic cells from germ cells during the ontogony of higher animals and plants, a fact certainly of general physiological signification. But modification and mutation are conceptions not sharply distinguishable and gradually related.

Another case of variability, similar to the loss of chemosynthesis by feeding with organic substances, I observed in various lower Algae respecting photosynthesis. For a long time I have been cultivating the gonidia of the lichen *Xanthorea parietina*, which are identical with the Protococcacee *Cystococcus humicola*. The first isolation was made by streaking off the said lichen, rubbed to a mash, on pure agar with salts and cultivating it in light. The thus obtained green, pure colonies, develop very readily as well in the light as in the dark on maltextract-agar and form large green masses, which, however, in course of time completely lose the power of photosynthesis, so that neither on agar with salts, nor in anorganic liquid media any growth takes place. Microscopically no difference is to be seen between the inactive chloroplasts of these cells and the active ones of normal *Cysto coccus* cells.

The very same I observed in cultures of *Pleurococcus vulgaris*, isolated from the bark of trees and long cultivated on maltextract-gelatin, on which it grows vigorously in the dark without losing the green colour. Hence it is clear that for photosynthesis the presence of chlorophyll in the living protoplasm is not sufficient, but the process requires still another factor, which may get lost through cultivation with organic food.

The greater part of the chlorella's of *Hydra viridis*, undoubtedly

biologica, 3e Jahrg. Heft. 2, Pag. 1, 1914. Recently I found that the ferment which produces nitrous acid from ammonium salts behaves in the same manner and changes, when fed with organic food into a saprophytous non-nitrifying form belonging to the so easily cultivable species *Chlorella vulgaris*, lose, when out of the *Hydra* body, howsoever fed, as well the power of photosynthesis as that of growth, so that it is very difficult to cultivate them. So, here is a case where change of food causes the loss as well of the function of photosynthesis as of that of growth.

CONCLUSION.

Some of the common denitrifying bacteria, such as *B, denitrificans* and *B. stutzeri* (these names taken in a broad sense), and probably some other species, may occur under two physiologically different modifications, which are hereditarily constant, when their feeding conditions remain unchanged. One form, the autotropic, is adapted to the anorganic medium (sulfur- or thiosulfate-chalk-nitrate) and shows chemosynthesis; the other, the heterotrophic form, requires organic food. They may be compared to the oligotrophic and the polytrophic condition of the nitrate ferment. Intermediate forms, feebly denitrifying in the anorganic medium, also occur, hence the autotrophy may be lost gradually.

The heterotrophic forms preserve the power of denitrification with organic food.

The nitrite ferments of the ammonium salts are also related to hereditary modifications with the character of saprophytes, living on organic food and unable to oxidise ammonium salts.

Great changes in the nature of the food may thus be the cause of hereditary modifications of certain factors, and this seems to throw some light on the causes which underlie ontogony.

Mathematics. — "*Complexes of Plane Cubics with Four Base Points.*" By Dr. K. W. RUTGERS. (Communicated by Prof. JAN DE VRIES.)

(Communicated at the meeting of January 31, 1920).

1. The image of surface Ψ_5 of the 5th order with a double curve E of the 5th order on a plane Π, is formed by the above mentioned complex S of plane cubics. By the aid of this surface Ψ_5 the following properties are derived [1]):

a. The triple point U of Ψ_5 is represented in Π by 3 points O_1, O_2, O_3, together these define a net out of S.

b. The double curve E corresponds in Π to a curve Θ of the 6th order with double points in the base points A_1, A_2, A_3, A_4 and in the points O_1, O_2, O_3. The points of Θ are associated to each other two for two; Θ is therefore hyperelliptic. On E lie 8 pinch-points, corresponding to 8 points ω on Θ.

c. The envelope of the joins of the associated points of Θ is a conic Δ, inscribed in the triangle O_1, O_2, O_3 and intersecting Θ in 6 points.

d. On Ψ_5 lie *five* systems of conics and *five* systems of plane cubics, which form together with the conics complete plane sections of Ψ_5. The curves of one of these latter systems are represented in Π by the straight lines joining two associated of Θ, hence by the tangents of Δ.

e. The bitangent planes (containing a conic and a plane cubic) of one system envelop a surface of class 3 and order 4. The contact curve of this is of order 7 and passes through the 8 pinch points of E. To this curve corresponds in Π a curve of the 5th order, c_5, the locus of the points of intersection of the tangents to Δ with the corresponding conic through the base points. The curve c_5 has double points in A_1, A_2, A_3, A_4 and passes through the 8 points ω.

Six conics through A_1, A_2, A_3, A_4 touch the corresponding tangents of Δ; the points of contact are images of parabolical points of Ψ_5.

f. The curve Θ has 32 tangents in common with Δ; 8 of them

[1]) CAPORALI, *Sulla superficie del quinto ordine dotata di una curva doppia del quinto ordine*, Annali di Mat. (2), 7 or Memorie di geometria, p. 1.

are the tangents at the points ω; the others are tangents of Θ, containing two associated points.

g. The parabolic curve of Ψ_5 is of the order 20 and corresponds in Π to a curve of the 12^{th} order c_{12} with quadruple points in A_1, A_2, A_3, A_4; c_{12} and c_6 cut each other 28 times; 12 of the points of intersection are in the 6 points of contact of c_{12} and c_5; 16 lie in the 8 points ω, where c_{12} has double points.

2. Any point in Π is double point of one curve of S. The envelope of the double point tangents of the nodal curves which have their double points on a straight line l, is of the 7^{th} class [1]); it has 14 tangents in common with Δ, hence on l lie 14 points for which the nodal curve with a double point in one of these points, has a double point tangent touching Δ. It is easy to see, that to these 14 points also belong the 5 intersections of l and c_5. *The locus of the remaining points is therefore a curve* c_9 *of order* 9 with nodes in A_1, A_2, A_3, A_4; it is the image of the locus of the points of inflexion of the plane cubics on Ψ_5, represented by the tangents of Δ [2]). The locus is a curve ϱ_{19} of the 19^{th} order.

3. In connection with the last remark it ensues from this, that in the 8 points ω c_9 has tangents which are at the same time tangents of Δ. In these 8 points c_9 and Θ touch. The tangents in the nodes A_1, A_2, A_3, A_4 are the tangents from these points to Δ.

Each plane cubic of Ψ_5 has 3 points of inflexion B'_1, B'_2, B'_3, lying on one line b'. Each b' cuts Ψ_5 in 2 more points P'_1, P'_2, so that the ruled surface, formed by these straight lines, intersects the surface Ψ_5, besides in the locus of the points of inflexion, in a complementary curve ϱ, the locus of the points P'_1, P'_2.

In Π the images B_1, B_2, B_3, of the points B'_1, B'_2, B'_3 lie on one tangent to Δ, the images P_1, P_2 of the points P'_1, P'_2 on the corresponding conic through A_1, A_2, A_3, A_4. On any such a conic there cannot lie more than two points P. If the locus k of the points P is of order ξ and passes η times through each base point A_k, we have $2\xi - 4\eta = 2$ or $\xi - 2\eta = 1$ (1).

4. The curves c_9 and c_5 intersect, besides in the base points, in 29 points, to which the 8 points ω belong.

---

[1]) JAN DE VRIES, *Null Systems Determined by Linear Systems of Plane Algebraic Curves*. These Proc. Vol. XXII, No. 3, p. 156.

[2]) See my paper: Versl. Kon. Akad. v. Wet. XXVII, p. 791.

Now a conic k_2 and a plane cubic k_3 of Ψ_6 in a plane V intersect in 6 points, 4 of which belong to the double curve E; the other two are represented in Π by a pair of points Q_1, Q_2 of c_6. If now e.g. Q_1 is at the same time a point of c_9, it means, that on Ψ_6 the corresponding point Q'_1, intersection of k_2 and k_3, is either a point of inflexion of k_3 in V or a point of inflexion of the second cubic k'_3 (lying in a plane W), passing through Q_1. We can distinguish the following cases:

a. Q'_1 is a point of inflexion of k_3 in V. The three points of inflexion of k_3 lie on a straight line through Q'_1, which cuts k_2 in 1 more point. Q'_1 is therefore also a point of ϱ, in other words: there are a number of points in Π through which the curves, c_6, c_9 and k all pass.

b. Q'_1 is a point of inflexion of k'_3 in W. As V is tangent plane at Q'_1, the inflexional tangent of k'_3 must lie in V and there form one of the principal tangents; it is therefore a tangent either of k_3 or of k_2.

α. If the inflexional tangent is also tangent of k_3 at Q'_1, it intersects k_2 as well as k_3 in one more point. The conic in W must cut V in these two points, but this is impossible, because two conics of Ψ_6 do not intersect.

β. If the tangent at the point of inflexion Q'_1 is also tangent to k_2, k'_3 has in Q'_1 two points in common with k_2; that means that in Π the tangent at Q_1 to the conic through Q_1, A_k is again a tangent to Δ.

If we draw out of a point O the tangents to the conics of the pencil (A_1, A_2, A_3, A_4), the points of contact lie on a curve of the 3rd order, passing among others through the base points of the pencil. This cubic intersects c_6 in $15-4.2=7$ points; consequently the envelope of the tangents to the conics at their intersections with c_6 is of class 7 and has 14 tangents in common with Δ.

To them belong the 6 tangents in the 6 points where the conics and the corresponding tangents of Δ touch. (See 1st). There are therefore 8 points of intersection of c_9 and c_6 that are not points of k. Accordingly 13 of the 21 points of intersection also belong to k, while k can have no further intersections with c_6.

From this follows $5\xi-8\eta=13$ (2), which equation in combination with (1) gives $\xi=9$ and $\eta=4$. The curve ϱ is therefore represented by a curve k_9 of order 9 with quadruple points in A_k. It is itself of order 11.

5. The conics through A_k cut k_9 in a g'_2, so that k_9 (of genus 4)

is hyperelliptic; this is also the case with ϱ_{11} and the joins of the corresponding points form a ruled surface R, of which the plane sections are rational; for if we project ϱ_{11} out of an arbitrary point on a plane, there arises a hyperelliptic c_{11} of genus 4. The joins of the g'_2 on a hyperelliptic curve of order m, genus p, envelop a rational curve of class $m-p-1$, hence in our case a curve of class 6 [1]). The points of this envelope correspond one for one with the points of a plane section of R; these curves are therefore also rational. At the same time it appears from the projection that the order of R is *six*; R cuts Ψ_5 in ϱ_{11} and ϱ_{19}. The double curve D of R_6 is of the 10th order.

6. The intersection of Ψ_5 and R_6 has double points among others in the 30 points of intersection of E with R_6. Now ϱ_{19} has 8 double points in the pinch points of E; the remaining 22 are intersections of ϱ_{19} and ϱ_{11} on E; in Π c_9 and k_9 cut the curve Θ each in 22 points, which are associated points of Θ.

The intersection of Ψ_5 and R_6 has also double points in the 13 points which are represented as common points of c_5, c_9 and k_9. In these points the two surfaces touch. Finally double points arise in the 50 intersections of D with Ψ_5. Now c_9 and k_9 in Π, hence ϱ_{19} and ϱ_{11} on Ψ_5, intersect in 36 more points. The other 14 must be real double points of ϱ_{19}, therefore also of c_9. Through each of these latter points pass 2 straight lines of R_6; they are the points where the two cubics of Ψ_5 of the same system have a point of inflexion.

On the surface Ψ_5 for each system of cubics 14 points can be found where the two cubics through these points have points of inflexion; or:

There are 14 points where the principal tangents are the inflexional tangents of two cubics of the same system through that point; or:

In Π there are 14 points such that in the net of the curves out of S through one of these points X, the degenerations XA_1, XA_2, XA_3, XA_4, belong to the same system. [2])

---

[1]) BERTINI, *La geometria delle serie lineari sopra una curva piana secundo il metodo algebrico.* Annali di Mat., (2), XXII, p. 894, p. 1.

[2]) See my paper: Versl. Kon. Akad. v. Wet., XXVII, p. 797 and 798.

Physiology. — "*Experimental Influence on the Sensitivity of various animals and of Surviving Organs to Poisons*". 1st Part. By W. STORM VAN LEEUWEN and Miss C. VAN DEN BROEKE. (Communicated by Prof. R. MAGNUS.)

(Communicated at the meeting of January 31, 1920).

In a previous paper [1]) STORM VAN LEEUWEN has shown that in the serum and in the tissues of various animals there are substances — called by him free chemoreceptors — that are capable of adsorbing alkaloids. On the basis of these researches he came to the conclusion that the sensitivity of various animals to poisons — particularly alkaloids — does not depend only on the sensitivity of the organs acted upon by the poisons, but also on the number of "free chemoreceptors" [2]) present in the body.

In these experiments he had repeatedly to ascertain the influence of pilocarpin and of mixtures of pilocarpin and serum on surviving cat-guts. Thereby it appeared that, as a rule, a dosis of pilocarpin administered *after* previous treatment of the gut with serum exerted a stronger influence than before this treatment. In the experiments reported in this paper we have studied this problem more systematically, and in so doing we have arrived at the conclusion that this serum may have a twofold influence: when e.g. rabbit's serum is mixed with pilocarpin, this combination will affect the surviving gut much less than pilocarpin alone would do, because rabbit's serum, as mentioned before, contains substances that adsorb pilocarpin; if, however, we add to the solution in which the gut is suspended, first pilocarpin, subsequently serum and finally, after washing out the serum, again pilocarpin, the second quantum of pilocarpin will exert a stronger influence than the first. It follows, therefore, that besides the substances that can *adsorb* alkaloids, i.e. free or

1) W. STORM VAN LEEUWEN. Sur l'existence dans le corps des animaux, de substances fixant les alcaloïdes. Arch. Néerland. de Physiologie T. II, p. 650. 1918.

2) Afterwards we deemed it better to call these free chemoreceptors "*secondary* chemoreceptors" in contradistinction to "*dominant* chemoreceptors".

60*

secondary chemoreceptors, there must also be substances present in the serum, that *intensify* the action of poisons like pilocarpin on the surviving catgut (of course it is possible that these two substances are the same). We considered it of great importance to ascertain whether this phenomenon was an isolated instance or one out of many. In order to make this out we first of all ascertained whether, besides serum, there are other substances that exert a similar intensifying influence on pilocarpin and on other poisons, and secondly we did not confine ourselves to the study of the influence of poisons on surviving organs, but we have also examined the influence on the intact animal. In this first communication we will report only the results of our investigations of the surviving gut.

Influence of rabbit's serum on the sensitivity of cat-guts to pilocarpin.

In this set we experimented on the surviving cat-gut and always took — as in previous researches — strips of the small intestine of a cat, from which the mucosa had been removed. The intestine was opened along the place of insertion of the mesentery and after the mucosa had been removed a piece was cut off on either side of the place of insertion, so that only the contractions of the longitudinal fibres and not those of the circular fibres were registered. Guts treated in this way are peculiarly appropriate for a quantitative inquiry into the action of poisons. There is moreover the advantage that in the ice-box they keep good for days.

The sensitivity of guts treated in this way varies considerably. Sometimes at the first administration they react on the small quantum of 0,01 mgrms of pilocarpin, but at other times much larger quanta, up to 1 mgr. are required. However, after washing out the first doses and adding again pilocarpin every time, the sensitivity of the gut will, in most cases, augment considerably. We found that when this process of repeated washing and again administering the poison is prolonged sufficiently, the sensitivity of the gut will at length be such as to present a contraction with doses of about 0.01 mgrms of pilocarpin added to 75 ccm of Tyrode. When this condition is arrived at, the successive equal doses of pilocarpin—washed out after three minutes every time — produce an equal effect, and only then did we administer serum to examine its influence upon the sensitivity of the gut. It is essential to call particular attention to this proceeding, for it stands to reason that in the initial stage of the experiment, when the gut reacts only on large doses of pilocarpin, not on the small ones, the chance of augmenting the sensitivity by the aid of some substance or other, is much greater than at the terminal stage, when the gut has of itself reached its maximal sensitivity to pilocarpin for that day.

In twenty cases we added serum (either serum alone or serum and pilocarpin) to the gut, when it had obtained its constant sensitivity to pilocarpin. The serum and the pilocarpin was then removed by washing, after which the same dosis of pilocarpin that had previously yielded a constant result, was administered again. In 15 out of these twenty cases the pilocarpin action was stronger *after* the serum than before (four times the reaction was much stronger, five times distinctly stronger and six times only slightly so). Once the action remained the same and four times the second dosis had less effect than the first.

An instance of a considerable increase of the pilocarpin action through serum is shown in Fig. 1.

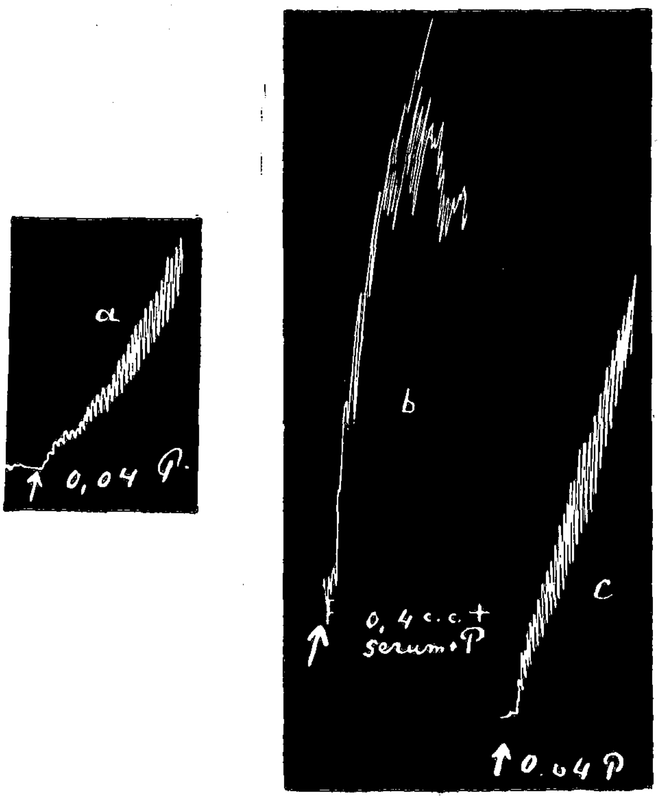


Fig. 1. Influence of rabbit's serum on the action of pilocarpin upon the surviving small intestine of the cat.

a. 0,04 mgrms of pilocarpin yields a moderate contraction.

b. 0,4 cc. of rabbit's serum + 0,4 mgrms of pilocarpin yields a strong contraction.

c. 0,04 mgrms of pilocarpin yields a stronger contraction than in *a*.

In this case the gut had after the administration of pilocarpin reached ultimately such sensitivity that 0.04 mgrms. of pilocarpin produced a distinct action within 3 minutes (*a*). After this the pilocarpin was washed out; then 0.4 ccm. of rabbit's serum was given to which a dosis of pilocarpin had been added (*b*). Thereupon the gut evinced large contractions (the magnitude of this deflection is immaterial to this investigation). After this serum and pilocarpin had been removed by washing, again 0.04 mgrms of pilocarpin was given (*c*), and as appears from the figure the action of this dosis was much stronger than before.

After it had thus been proved that serum is capable of increasing the action of pilocarpin (which in fact was known from our previous experiments) we have tried to find out which constituent of serum was responsible for this action. To this end we have investigated some substances.

Influence of cholesterin on the sensitivity of the cat-gut to pilocarpin.

In four cases we examined the action of a cholesterin-emulsion on the sensitivity to pilocarpin; in all of them the result was positive, twice the administration of cholesterin produced a much stronger pilocarpin-action than before; once it was distinctly stronger and once appreciably stronger. Figs 2 and 3 represent two instances of a distinctly increased action.

After repeated addition of pilocarpin to the solution in which the gut had been put (Fig. 2) and then washing it out again, the sensitivity of the gut had eventually become constant and the gut had reached a distinct contraction with 0,1 mgrm. of pilocarpin (*a*). After this pilocarpin had been washed out and the gut had been put back again in the vessel of 75 ccm. (which was always used for our experiments), 0,2 ccm. of cholesterin emulsion was added to this vessel (*b*); the gut did not show any reaction. This cholesterin was left in the vessel and then again 0.1 mgr. of pilocarpin was added (*c*). The pilocarpin action is then seen to increase considerably. In fig. 3 the course of the experiment was somewat different. Here the sensitivity of the fragment of the small intestine was such as to show a very distinct reaction on 0.03 mgr. of pilocarpin in 75 ccm. of Tyrode liquid (*a*).

It is our custom when the gut has been affected by a poison, to transmit it from the 75 cc. vessel to one of 150 ccm. capacity, in which the poison is then washed out.

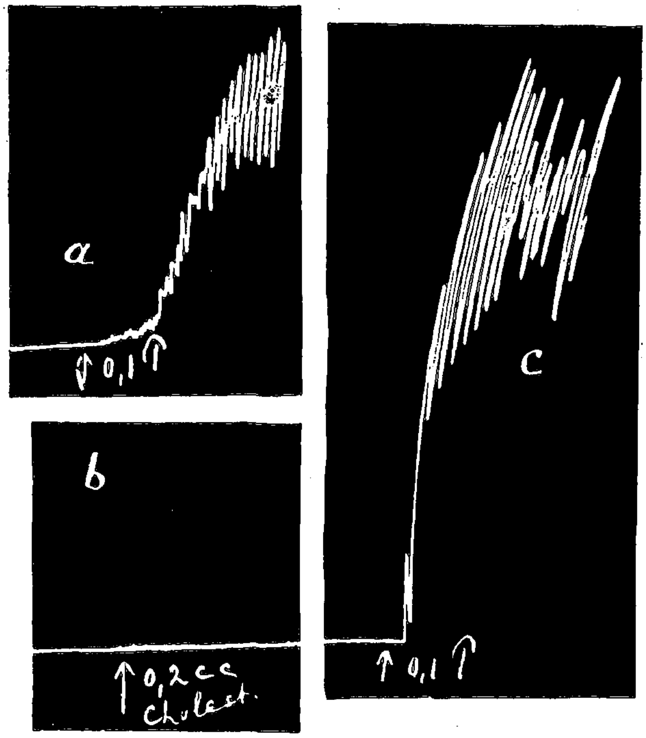


Fig 2. Influence of cholesterin on the pilocarpin-action.

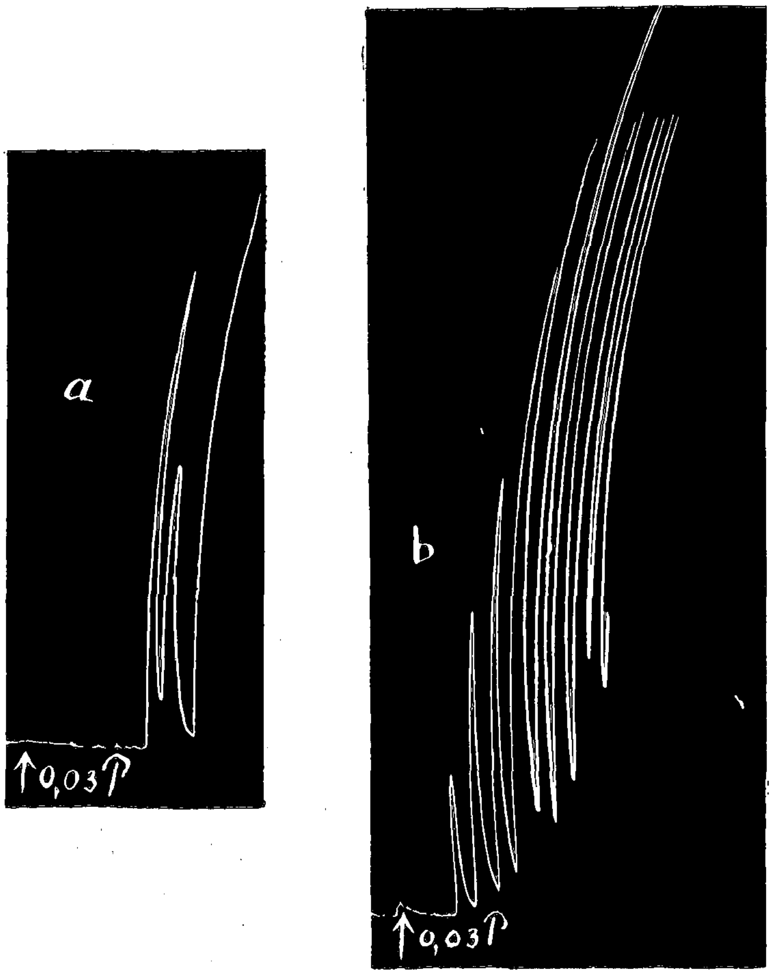


Fig. 3. Influence of cholesterin on the pilocarpin-action.

In the experiment we are going to describe now, 0,5 ccm. of a cholesterin emulsion was added in the 150 ccm. vessel, in which the gut remained in contact with the cholesterin-emulsion for 15 minutes and was then washed out. After 15 minutes the gut was transmitted to the vessel of 75 ccm., and after a few minutes again 0,03 mgr. of pilocarpin was poured into the vessel, as had been done before (*b*). From fig. 3 we again see clearly that the resulting pilocarpin action is much stronger than it had been before. On the basis of these experiments and of others proceeding in the same way, we must conclude that the cholesterin is capable of increasing the pilocarpin-action, as well when the cholesterin and the pilocarpin are added simultaneously to the gut, as when first the cholesterin is added, then the cholesterin is washed out, and subsequently pilocarpin is administered. This phenomenon cannot be considered as an instance of "potentiation" (BURGE), it being rather a sensibilization of the gut through cholesterin and serum.

Influence of lecithin on the sensitivity of the cat-gut to pilocarpin.

We now proceeded to find out the action of another constituent of serum, viz. lecithin.

In our first experiments it really appeared that lecithin as a rule had a reinforcing influence; afterwards, however, we were able to repeat this experiment with very pure lecithin [1]) prepared from brain-substance. It then became obvious that this very pure lecithin produced a less constant action than the impure substance we had used before. Fig. 4, however, shows again that pure lecithin can also intensify pilocarpin action. Here 0,01 mgr. of pilocarpin had only an inappreciable action (*a*). After this pilocarpin had been removed, again an equal quantum of pilocarpin was put in the vessel holding 75 ccm. of Tyrode solution (*b*) and after the pilocarpin had acted for about 4 minutes, a drop of a 5 % lecithin-emulsion was added, on which the gut immediately reacted by strong contractions. After this lecithin plus pilocarpin had been washed out again, a drop of lecithin was added to the gut to show that of itself this substance had no effect upon the movements of the gut (*c*); this lecithin was left in the vessel; again 0.01 mgr. of

[1]) We wish to acknowledge the kindness of Dr. LEVENE of the Rockefeller Institute in putting at our disposal this lecithin as well as quanta of cerebron and other substances to be mentioned lower down.

pilocarpin was added, again an increased action of the pilocarpin was the result. Fig. 4, however, does not illustrate a *typical* instance

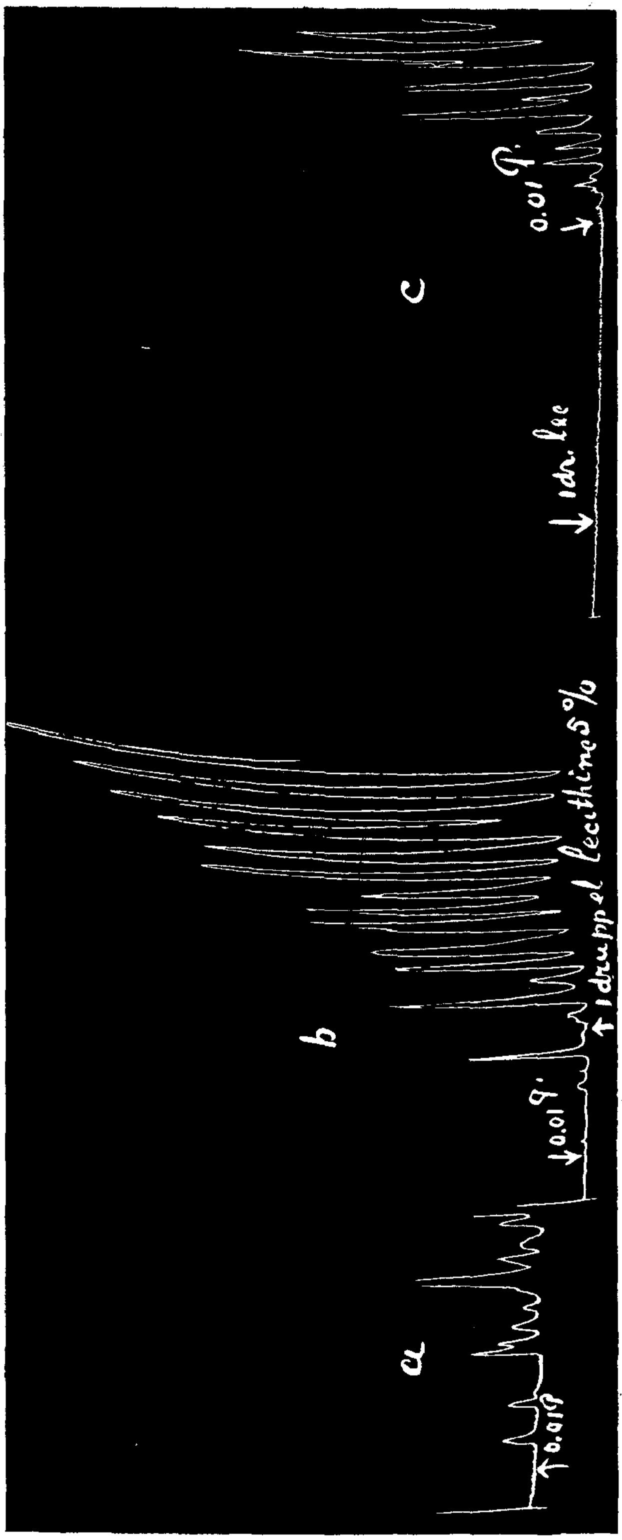


Fig. 4. Influence of lecithin on the pilocarpin-action.

of lecithin-action because, as already mentioned, in many cases the lecithin was inactive. We examined this in 13 experiments, 5 of which yielded positive results, i.e. 5 times the lecithin intensified the pilocarpin-action; 6 times, however, the result was negative, once it was doubtful and in one case the pilocarpin-action had weakened after the lecithin.

We see, then, that though lecithin undoubtedly intensifies the pilocarpin-action in some cases, this does not occur always. For the present we cannot state the reason. The explanation may lie in this that — as said above — we invariably added lecithin only when, after a repeated addition of pilocarpin, the gut had obtained its maximal sensitivity to this poison. Probably the lecithin would *intensify the pilocarpin-action with greater regularity, if it were* administered in an earlier stage, when the gut is still less sensitive to this poison. In that stage, however, it would be impossible to obtain accurate results.

Influence of cerebron on the sensitivity of the cat-gut to pilocarpin.

The influence of cerebron on the pilocarpin-action had to be examined in two directions, as was the case also with the influence of Witte's peptone, to be discussed lower down. STORM VAN LEEUWEN's inquiry, alluded to above, had shown that rabbit's serum, and also organs of the rabbit, contain substances capable of adsorbing pilocarpin physically.

Endeavours to determine the nature of these substances have failed up to now. It appeared from the inquiry referred to that cholesterin and lecithin are not the substances looked for.

It was necessary, therefore, to examine also cerebron (and Witte's peptone) in this direction. We proceeded as follows: we waited till the gut's sensitivity to pilocarpin had become constant; then pilocarpin was administered and subsequently pilocarpin + cerebron and finally again pilocarpin alone. In this way we could ascertain whether the addition of cerebron to the pilocarpin-solution (the cerebron was in contact with the pilocarpin for from $^1/_2$—2 hours, before it was added to the gut) lessens the action of it, and we could also ascertain whether, after the cerebron + pilocarpin had been washed out again, the following dosis of pilocarpin acted more forcibly than before, whereby it could be proved whether or no cerebron intensified the action of pilocarpin.

Because only a small quantum of cerebron was at our disposal, we undertook only three experiments. It appeared from them that cerebron is *not* capable of adsorbing pilocarpin, but that it has distinctly a slightly favourable influence upon the pilocarpin-action. This is shown in figs. 5 and 6.

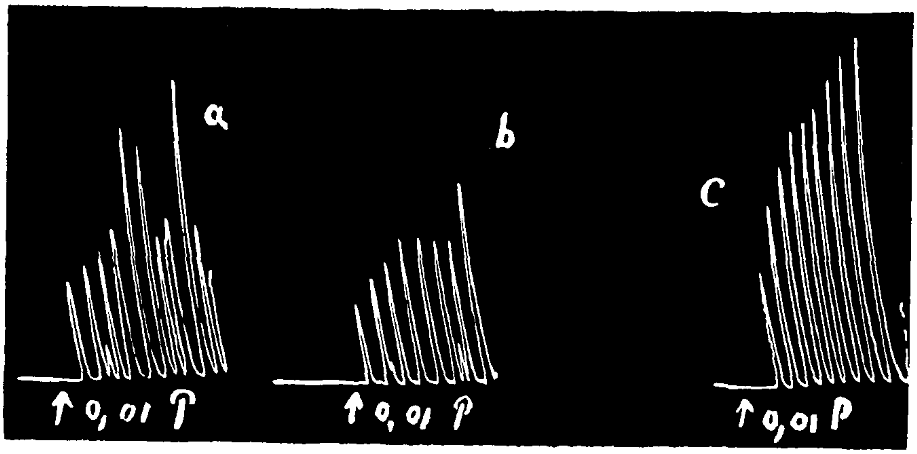


Fig 5. Influence of cerebron on the pilocarpin-action, between *b* and *c* the gut was washed out in a vessel of 150 cc. of Tyrode-liquid, which contained 1 cc. of 1% cerebron-emulsion.

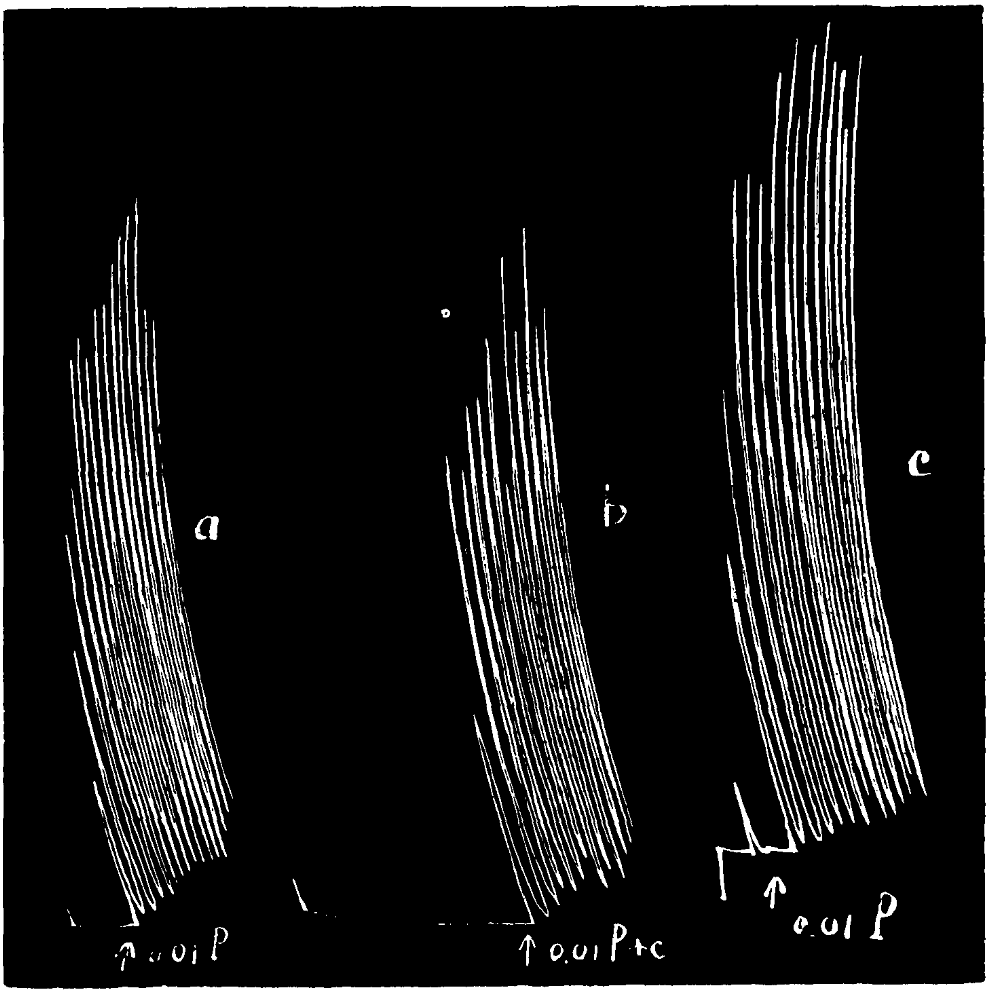


Fig. 6. Influence of cerebron on the pilocarpin-action.

In fig. 5 0.01 mgr. of pilocarpin had a distinct action (*a*), which became slightly less, when pilocarpin was given the second time (*b*). Then the gut was washed out in 150 cc. of Tyrode, to which 1 cc. of 1 °/₀ cerebron-emulsion was added. Then the gut was removed to the vessel of 75 cc. Tyrode and again 0.01 mgr. of pilocarpin was added (*c*); the subsequent contraction was obviously larger than before.

The process of the experiment of Fig. 6 was different; 0.01 mgr. of pilocarpin yielded a distinct action (*a*); after this dosis had been washed out 0.01 mgr. of pilocarpin was administered that had been dissolved for more than an hour in a 1 °/₀ cerebron emulsion, (*b*); the subsequent contraction of the gut was of precisely the same magnitude as before, which proves, therefore, that cerebron does not inhibit the action of pilocarpin. After the pilocarpin + cerebron had been washed out, again 0.01 mgr. of pilocarpin was given (*c*), which, as in Fig. 5, yielded a greater contraction than before. The same result was achieved in a third experiment with cerebron.

Influence of Witte's peptone on the action of pilocarpin upon the gut.

As in the case of cerebron two points had to be settled also regarding peptone viz. the capacity of adsorbing pilocarpin and of intensifying the action of pilocarpin. We deemed it possible that Witte's peptone (a mixture of albumoses) might adsorb pilocarpin, because ABEL [1]) has shown recently that albumoses occur in normal serum, which in themselves are not poisonous, but are capable of adsorbing poisonous substances.

The investigation of "peptone" proceeded in the same way as that of cerebron. The adsorptive property of peptone appeared to be very weak, but its intensifying effect on the pilocarpin-action is most strongly marked, as shown in fig. 7.

In the experiment illustrated by this figure 0.05 mgrm of pilocarpin was administered three times running (*a, b, c*), and the subsequent deflections of the gut were exactly the same in these three cases; after the pilocarpin had been washed out and the gut was again put back in the vessel of 75 ccm., $^1/_2$ cc. 1 °/₀ Witte's peptone was added to this vessel to demonstrate that this substance of itself

[1]) J. ABEL. On the presence of histamin (β-iminazolylethyl amin) in the hypophysis cerebri and other tissues of the body and its occurrence among the hydrolitic decomposition of proteins. Proc. Amer. Soc. for pharm. and exp. Therap. Journ. Pharm. and exp. Ther. vol. XIII, p. 511. 1919.

did not exert any influence upon the gut (*d*). Subsequently, again 0.05 mgr. of pilocarpin was added, and the following contraction is much larger than before the addition of peptone. After this had been washed out again 0.05 mgr. of pilocarpin was given. This, however, had been mixed an hour before with a 1 °/₀ peptone-solution, and the result was a weaker action of the pilocarpin than before (*f*), so that the peptone must be assumed to have adsorbed a small portion of the pilocarpin. After this had been washed out pilocarpin alone was given again twice (*g* and *h*). In both cases the action of pilocarpin was increasing, which proves conclusively that the peptone possesses in a marked degree the property of intensifying the pilocarpin-action. This experiment also gives evidence that peptone is capable of exerting this action 1 when both peptone and pilocarpin are in contact with the gut and 2 when the peptone has first been in contact with the gut, and is subsequently washed out.

In all we performed 16 experiments with peptone, in 5 of which the pilocarpin-action after the peptone was much stronger than before; 7 times the action was appreciably stronger; twice slightly stronger; twice equal and in only one case it was weaker. However in this case it got stronger again after adding pilocarpine a few times, so that in 14 out of 16 cases peptone had an intensifyng influence upon pilocarpin.

While this investigation was in progress, it appeared from other inquiries performed in this institute that peptone does not only intensify the pilocarpin-action on the gut but under certain conditions also intensifies the action of adrenalin upon the blood-pressure in the cat. In this inquiry we also detected that a similar effect was produced also by a dialysate of peptone. This induced us to investigate these dialysates also with respect to their effect upon the gut, which to our surprise proved to be different from that of peptone itself.

This effect of the "peptone" dialysate is manifest in fig. 8. In the experiment which it illustrates first 0.1 mgr. of pilocarpin is given some times running (*a*, *b*, *c*). The effect was the same every time. Then the gut was washed out not in a pure Tyrode-solution, but in one that contained a small quantum of dialysate, viz. a quantum that on analysis proved to contain 0,125 mgr. of nitrogen. In fig. 8 it may clearly be seen that the consequence of it was that the subsequent pilocarpin dosis had a weaker effect than before (*d*). After the action of the pilocarpin had continued for 3 minutes, the gut was again washed out in the vessel that contained besides the Tyrode also dialysate; the consequence was that the subsequent pilocarpin-action was less again (*e*); thereupon the gut was washed

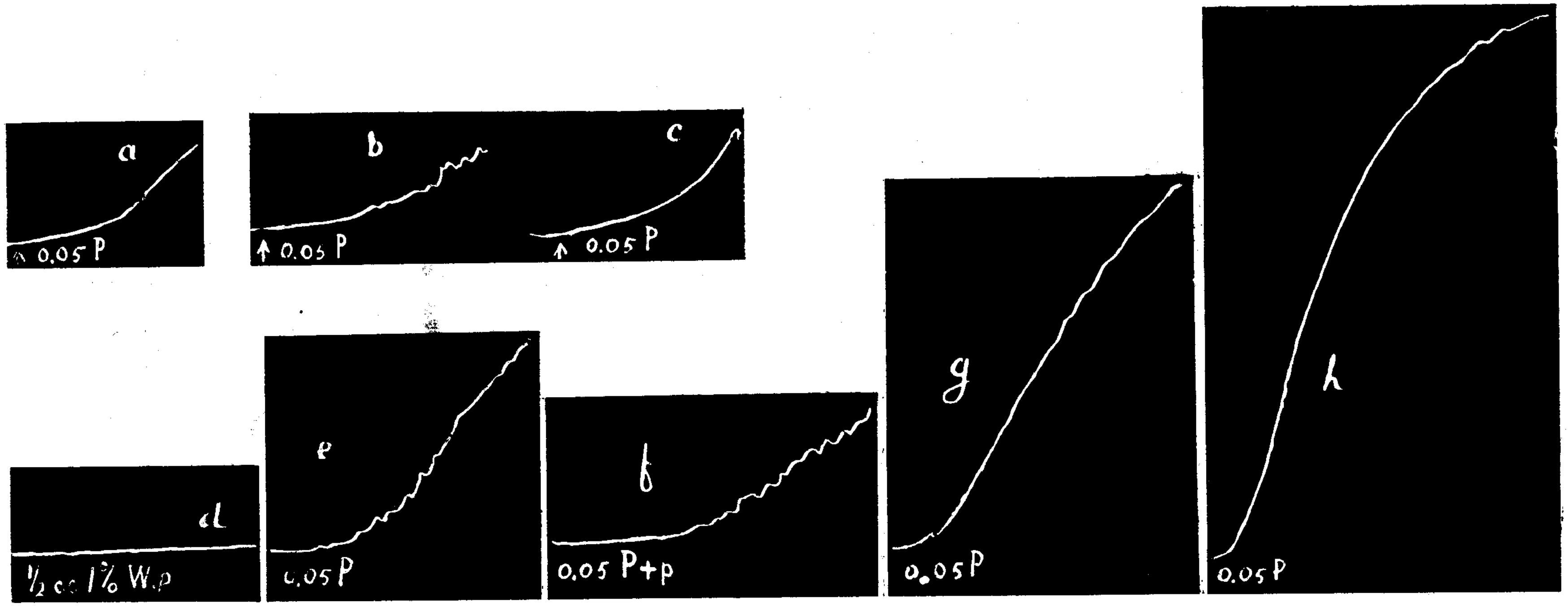


Fig. 7. Influence of Witte's peptone on the pilocarpin-action.

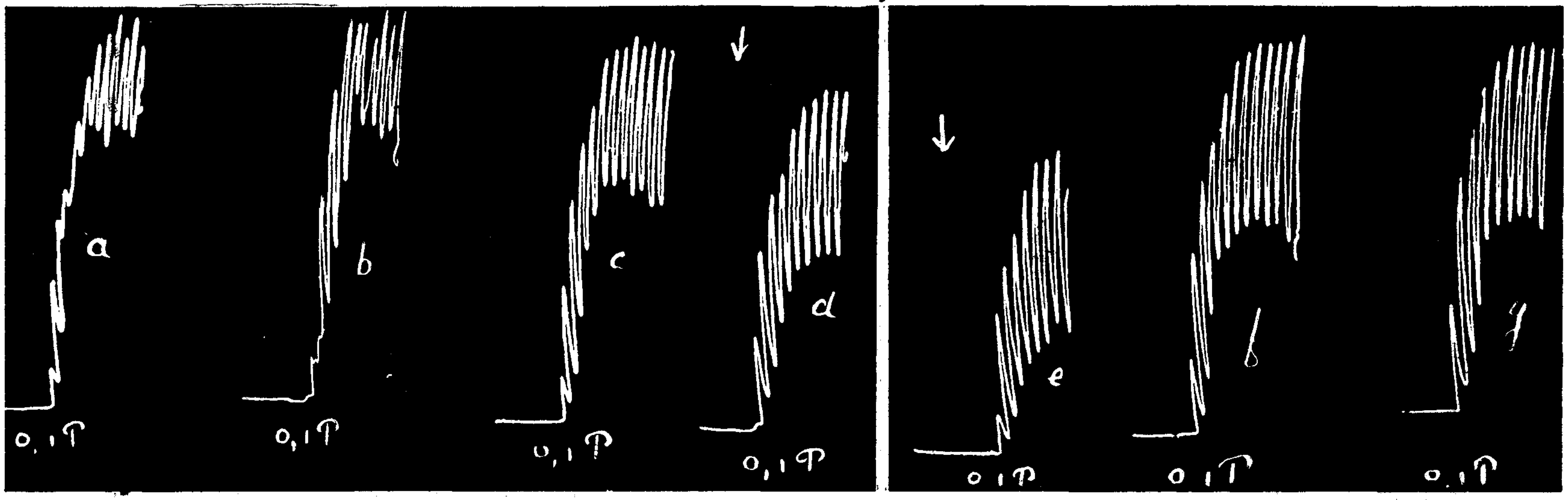


Fig. 8. Influence of a dialysate of Witte's peptone on the pilocarpin-action. The gut has been washed out between *c* and *d* and *d* and *e* in Tyrode-liquid, containing dialysate of Witte's peptone, between *e* and *f* washed out in ordinary Tyrode-liquid.

out in *pure* Tyrode solution, and the subsequent pilocarpin doses had about the same effect (f, g) as before the dialysate was added.

We call special attention to the circumstance that in this case the dialysate of WITTE's peptone had an opposite effect to that of peptone itself. This is the more remarkable since, in experiments on the blood-pressure in the cat to be reported afterwards, we found that the adrenalin-action is influenced in the same way by peptone and by dialysate.

We now proceeded to study the influence of Witte's peptone on another poison, viz. cholin. It appeared that the cholin-action was the same *before* and *after* the addition of peptone. It should be noted, however, that the curve showing the relation between the concentration and the action of cholin is not by far so steep as that of pilocarpin, which means that slight alterations in the dosis of cholin have not nearly so much influence upon the contraction of the gut as is the case with pilocarpin. It may be, then, that the peptone indeed exerts a slight influence in this respect, but that this influence does not manifest itself in consequence of the peculiarity of cholin just alluded to.

CONCLUSIONS.

Here then we have demonstrated that in the serum of various animals there occur substances, capable of intensifying the action of alkaloids — in this case pilocarpin. — on the surviving gut. We also found that cholesterin and cerebron also possess this property. With lecithin it was doubtful, while peptone acts very strongly in this respect and the effect of the peptone-dialysate was in an opposite direction.

When added to a pilocarpin-solution Witte's peptone appeared to inhibit the pilocarpin-action in a small measure, from which we may conclude that, like rabbit's serum, it contains substances that are capable of adsorbing pilocarpin. Cholesterin, lecithin and cerebron lack this property.

From the Pharmacological Institute of the University of Utrecht.

Utrecht, Jan. 1920.

Physiology. — "*Experimental Influence on the Sensitivity of various Animals and of Surviving Organs to Poisons*". (Part II). By W. STORM VAN LEEUWEN and Miss VAN DER MADE. (Communicated by Prof. R. MAGNUS).

(Communicated at the meeting at January 31, 1920).

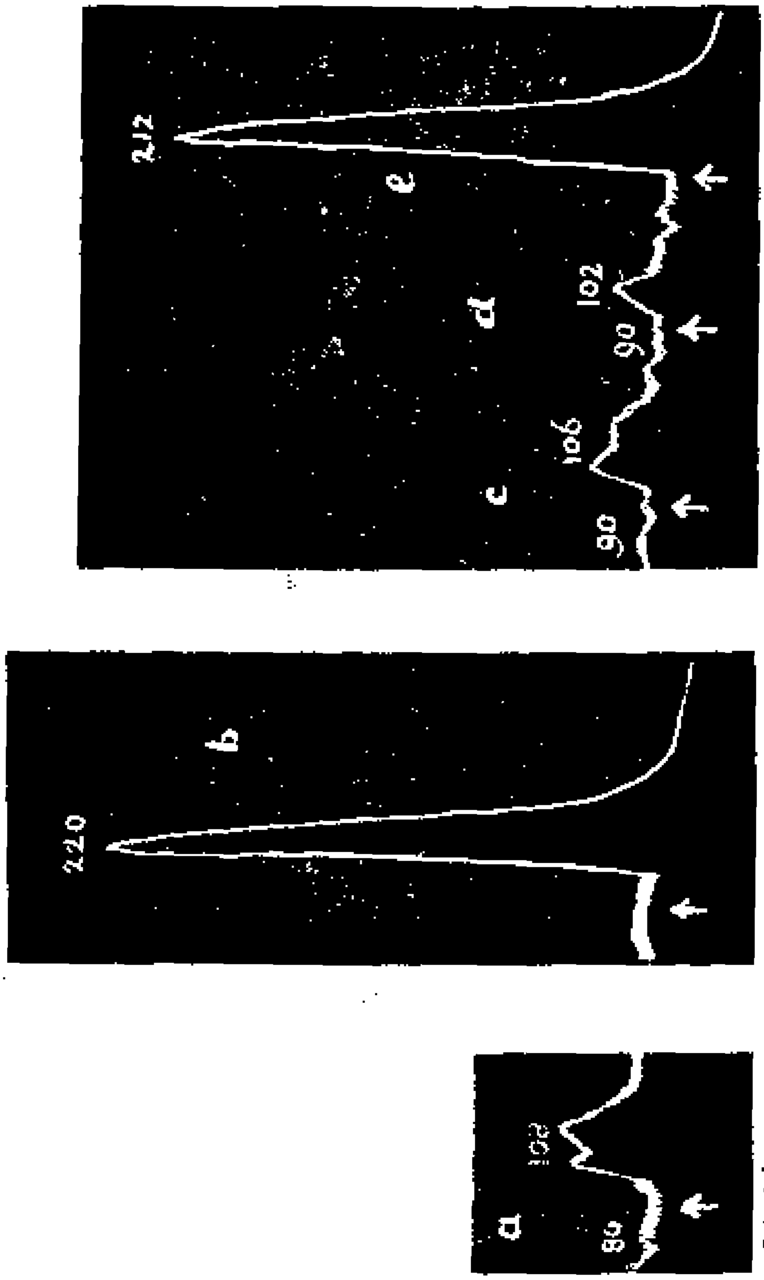
In the first part STORM VAN LEEUWEN and C. VAN DEN BROEKE have demonstrated that in the serum of various animals there are substances which intensify the action of an alkaloid (pilocarpin) on the surviving cat-gut. They also found a similar favourable influence to belong to cholesterin, to cerebron to Witte's peptone, and in certain cases also to lècithin. We considered it useful to ascertain whether similar favouring substances also come into play in the action of poisons on the unimpaired animal. We also wished to make out how far the presence of inhibiting substances could be demonstrated in that case. Our first object was to inquire into the action of adrenalin upon the bloodpressure in the cat and in the rabbit. We did this because, as indeed we know from the literature, successive adrenalin-injections into the cat and the rabbit bring about a rise of bloodpressure every time of the same extent, so that a quantitative investigation is very well possible here. Before describing the general results of this inquiry we wish to report the first experiment we made in this direction.

We wish to bring this experiment into prominence, since its process differed from all the others in a series of 50 inquiries, and because theoretically it seemed to us to be of some interest. In this experiment we ascertained the action of adrenalin on the decapitated cat. The smallest dosis that produced a distinct rise (of 14 mm. H.g.) in this animal, was 0·1 mgrm of adrenalin intravenously. Let it be observed here — we shall recur to it later on — that this is an extremely large dosis for the minimum action. As has been said, this 0.1 mgrm of adrenalin produced a rise of blood-pressure of 14 mm. Hg.; in other cases of 12 mm. Hg., 16 mm. Hg. and once as much as 28 mm. Hg. (fig. 1*a*).

After it had thus been shown that 0.1 mgr. of adrenalin — dissolved in 1 c.c. physiological water — was constant in its action, the animal was again given 0.1 arenalin of the same substance, but

61

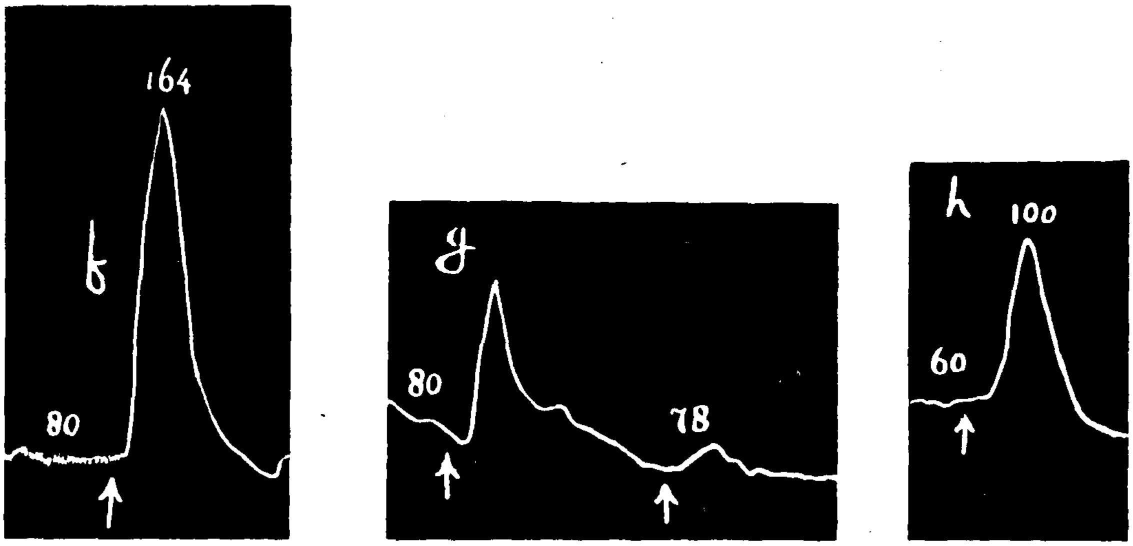
this quantum had previously been mixed with a small quantity (0.1 c.c.m.) of human serum. The consequence was (see fig. 1*b*) a



0.1 Adr in NaCl — 0.1 Adr in human serum (0.1) — 1 ccm human serum — 0.1 Adr in NaCl — 0.1 Adr in human serum.

Fig. 1 (*a*—*e*). Decapitated Cat. Bloodpressure. Abnormal reaction on adrenalin and on adrenalin + serum.

very strong rise of the blood-pressure, many times greater than that with 0.1 mgr. of adrenalin alone. With the subsequent injections the whole of a c.c. of human serum was first given alone, to show that of itself it had but little effect upon the rise of the bloodpressure (Fig. 1*c*) and after this again 0.1 mgr. of adrenalin, dissolved in physiological water, was injected. This yielded a rise of bloodpressure of only 12 m.m. mercury (Fig. 1*d*). Now again 0.1 adrenalin plus human serum was given, and again a considerable rise of the bloodpressure revealed itself (Fig. 1*e*). Then we tried 0,05 mgr. of adrenalin plus serum; also this still produced a marked rise of the bloodpressure (Fig. 1*f*) and at last it appeared that 0.01 mgr. of



0.05 Adr in human serum — 0.01 Adr in human serum — 0.1 Adr in NaCl — 0.1 Adr + serum Cat fresh.

Fig. 1 (*f*,*g*,*h*). Decapitated cat, bloodpressure, abnormal reaction on adrenalin and on adrenalin + serum.

adrenalin + serum (Fig. 1*g*) yielded a still larger rise of the bloodpressure than 0.1 mgr. of adrenalin did without serum. Since in other cases the minimum dosis adrenalin on which a decapitated cat reacts lies between 0,005 and 0,0005 mgrms, the sensitivity to adrenalin of this cat was brought to normal values by the addition of human serum. Fresh cat's serum produced a weaker effect than human serum (Fig. 1*h*) in this experiment.

In this first experiment it had thus been proved indubitably that in all probability human serum contains substances that intensify considerably the action of adrenalin on the decapitated cat.

Now it is remarkable that in other 50 experiments we never achieved the same result. It is difficult to account for this; in performing this first experiment we did not know of course that it was an exceptional case, so we have not paid special attention to the problem in what respect this cat differed from others, whether it was a female or perhaps a castrated male, nor did we examine in any way the organs of internal secretion. For the present this question must therefore remain unsettled, and we must confine ourselves to the statement that in 50 other cases we never observed a similair intense action. In order to make out whether perhaps this cat, which yielded such an abnormal reaction, was a castrated animal we first determined the sensitivity of a male cat to adrenalin (minimum active dosis 0.0005 mgr. of adrenalin); after this the animal was castrated and examined again three weeks later. The sensitivity to adrenalin was then the same as before.

In one respect the cat with its abnormal reaction to human serum decidedly differed from all the other animals examined, viz. the minimum dosis of adrenalin just sufficient in this animal to produce a distinct rise of the bloodpressure, was exceedingly large (0,1 mgr.); whereas in nearly all the other animals examined this minimum dosis lies between 0,005 and 0,0005 mgr. of adrenalin, i. e. from doses 20 to 200 times smaller. However this may be, it had appeared from this investigation that we may decide *on principle* that the influence of adrenalin on the bloodpressure in the animal does not depend only on the dosis of adrenalin and on the sensitivity of the specific organs, but also on the presence or the absence of substances in the serum of the animal that influence (in this case intensify) the adrenalin-action.

We also tried to find out whether the serum contains substances that inhibit the adrenalin action, thus far with negative result.

As stated above we never obtained a stronger action with a combination of serum and adrenalin than with adrenalin alone, except in the one case alluded to. However, the results published in the first part of this communication regarding the action of pilocarpin on the surviving gut and regarding the reinforcement of this action through Witte's peptone, induced us to examine also the action of peptone on the bloodpressure in the cat.

The influence of Witte's peptone itself on the bloodpressure in the cat and in the rabbit has been known for a long time. Many times already it has been shown in the literature that injections of rather large doses of peptone, from 300 to 500 mgr. per kg. animal, in cats and dogs, result in a marked fall of the bloodpressure with an ultimate standstill of the heart's action. The same action was noticed by us with larger doses of peptone. It is worthy of notice, however, that our experiments demonstrated that very small doses of peptone, sometimes from 10 to 100 times smaller than those which cause death, are capable of intensifying the adrenalin-action in the decapitated cat, as appears from fig. 2. Here 0.001 mgr. of adrenalin yielded twice running a rise in the bloodpressure respectively of 14 and 16 mm. Hg. (a, b); after an injection of 0.1 cc. of Witte's peptone $1^{\circ}/_{0}$ a similar quantum of adrenalin gave a rise of 18 to

22 mm. mercury (c, d) and after several more injections of peptone, 0.001 adrenalin gave a rise of the bloodpressure of 20, 20, 22, 20 and after this of 30 mm. Hg. ($e-k$). Though slight, this is nevertheless a clear increase in the rise of the bloodpressure and moreover the fall, which invariably followed after the rise with the first doses of adrenalin, had disappeared, as is the rule in such cases. In connection with a circumstance to be discussed lower down, it should here be pointed out directly that at the beginning of the experiment, i.e. when the adrenalin-action was still weak, the initial bloodpressure was 90—84 mm. Hg., while later on when the adrenalin action had augmented, the initial bloodpressure was higher viz. 106—100 mm. mercury. In five out of six cases we found an increase of the adrenalin-action by Witt's peptone. Besides the decapitated cat, also the narcotized rabbit and the decerebrated rabbit were examined on the action of Witte's peptone. In either case we indeed found a slight increase, but on the whole the influence of peptone on the adrenalin-action was very insignificant. Of course this concerns small doses only; when large doses of peptone are given, the result, in the cat as well as in the rabbit, is mostly first a phase in which the adrenalin has *less* effect than before, after this a phase in which small doses of adrenalin do not act at all, and finally a condition in which the bloodpressure of the animal is lowered in consequence of the peptone-injection and the cardiac action is arrested. Besides this action of peptone on the adrenalin-rise of the blood-pressure we also examined the effect of peptone-injections on cholin-action. As known, cholin in small doses has a lowering effect on the blood-pressure; in large doses it raises the bloodpressure after administration of atropin. We have noted the effect of peptone on the lowering influence on the blood-pressure of small doses of cholin in the cat and in the rabbit. We did not find any definite effect. It seemed, however, that after the administration of peptone or of the dialysate of peptone, the decrease in the bloodpressure caused by cholin, became less. Once a slight rise was noticed instead of a fall.

Because we consider the effect of peptone on the adrenalin-action in the decapitated cat as the most striking result, we took this action as the basis for a closer investigation.

First of all we have tried to ascertain whether this action of Witte's peptone is proper to all the component parts of this substance, or whether perhaps ingredients might be separated from the peptone, that are specially characterized by this property. This really proved to be the case. When we examined the influence of a Witte's peptone dialysate on the adrenalin-action in the decapitated cat, it

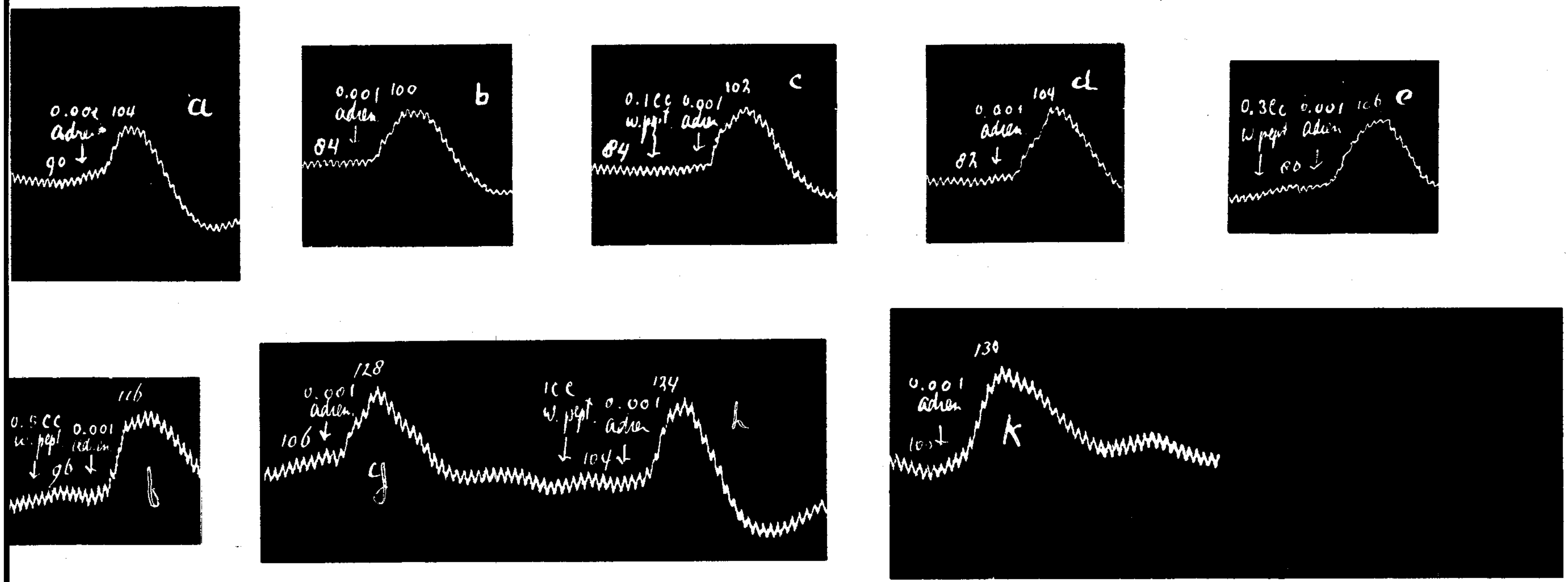


Fig. 2. Decapitated Cat. Bloodpressure. Influence of Witte's peptone on the bloodpressure. Rise in consequence of adrenalin.

appeared to be extremely intensifying as is shown in fig. 3. Here a dosis of 0.005 mgrm. of adrenalin injected several times intra-

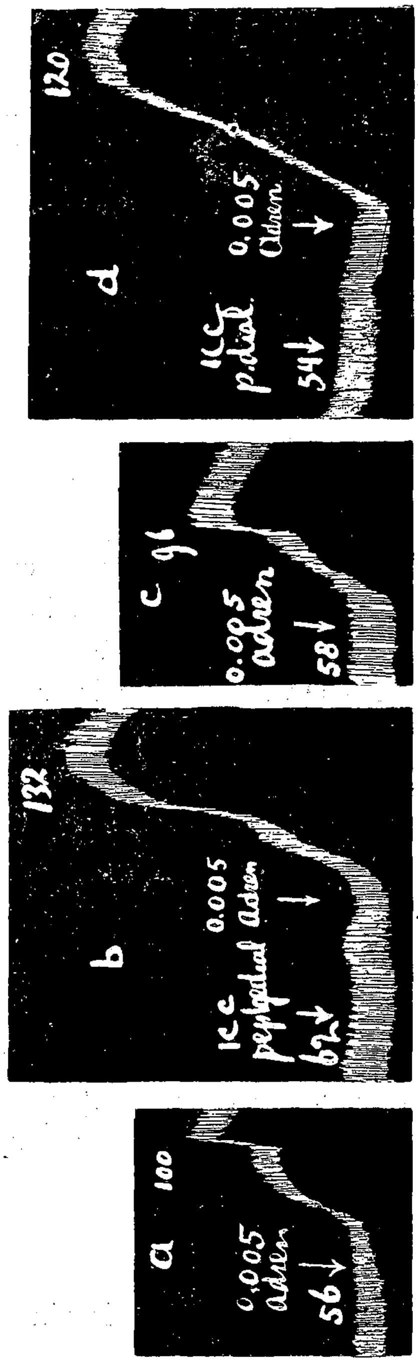


Fig. 3. Decapitated Cat. Bloodpressure. Influence of a peptone dialysate on the rise of bloodpressure caused by adrenalin.

venously, produced a rise of the bloodpressure respectively of 44, 32, 44 and 36 mm. Hg. (fig. 3*a*).

After 1 cc. of the dialysate of peptone, which contained per cc. about 1 mgr. of nitrogen, had been administered, an injection of an equal quantum of adrenalin (0.005 mgr.) produced a rise of the bloodpressure of 70 mm. Hg. (fig. 3*b*). Another injection given a short time after, caused again a rise of 38 (fig. 3*c*) as before the dialysate injection. But after injecting dialysate again the adrenalin yielded a rise of the bioodpressure of 66 mm. Hg. (fig. 3*d*); afterwards it fell again to the original value of 38.

In 7 experiments we have examined a decapitated cat for the effect of the dialysate[1]) of peptone on the intensifying action of adrenalin on the bloodpressure. In 6 of them we obtained a positive result, in one only a negative one.

As stated in our first communication, it had appeared that peptone exerts a different influence upon the surviving gut from that of the dialysate.

We deemed it useful, therefore, to examine also the action of the residue after dialyzation.

It is difficult, of course, to make minute quantitave observations of such a complicated substance as Witte's peptone.

Nevertheless our experiments have clearly demonstrated that the intensifying action on the rise of the bloodpressure caused by adrenalin is effected in the first place by the dialysate[1]), in a smaller measure by the residue and by Witte's peptone itself.

In the above experiments we have made use intentionally of decapitated and decerebrated animals to avoid narcosis. Our arguments for doing so were the following: First it is known that the symptoms produced by the anaphylactic shock are very similar to those which result from a peptone-injection. Secondly, it is also known that the symptoms of the anaphylactic shock decrease when the animal has first been narcotized. So we were right in using non-narcotized animals since e.g., as observed heretofore, Witte's peptone exerts a distinct effect on the blood-pressure-raising action of adrenalin in the decapitated cat, not, however, in the narcotized rabbit.

After we had discovered that the peptone-dialysate had a marked effect on the action of adrenalin in the decapitated cat, it was interesting to study this action also in the rabbit and more particularly in the decerebrated rabbit. It appeared, indeed, in a single case that the adrenalin-action is intensified by the dialysate, but this is not the rule. It is not easy to account for this phenomenon. We can

---

[1]) The dialysate used was obtained by dialysing $7\frac{1}{2}$ grms of Witte's peptone for three days in running water. The dialysate was ultimately evaporated to dryness, and was made up to 300 c.c. As much NaCl was added as would raise the percentage to 0,9. Afterwards we discovered that this method did not at all afford any dialysate of a constant action. Sometimes we obtained dialysates that were completely inactive. For the present we are not able to account for this phenomenon.

only call attention to the fact that, in the cat, small doses of adrenalin mostly produce a fall in the blood-pressure, and that peptone on the other hand would seem to hinder this fall. In the rabbit adrenalin has not such a lessening effect on the bloodpressure.

Before proceeding to the discussion of this phenomenon we must first state that an intensifying effect on the adrenalin-action has been described for other substances already some time ago. KRAUS and FRIEDENTHAL[1]) had shown that the effect on the rise of blood pressure of adrenalin is intensified by injection of thyreoidin-extract, which STORM VAN LEEUWEN has been able to corroborate in his personal investigations.[2])

FRÖHLICH and LOEWI[3]) had demonstrated that in the cat the adrenalin action can be intensified by previous cocain-injection, while CHIARI and FRÖHLICH[4]) found that substances that precipitate calcium (e. g. oxalic acid), also intensify the adrenalin-action. KEPINOW[5]) found an intensified adrenalin-action after injection of hypophysis-extract into the rabbit. These findings were corroborated by NECULESCU[6]), by AIRILA[7]), and by H. BÖRNER[8]).

H. BÖRNER believes that the influence of hypophysin on the adrenalin-action is not of necessity due to a sensibilization caused by hypophysin, but may be ascribed to the circumstance that hypophysin is deleterious to the cardiac action in the rabbit — and the rabbit is just the animal with which the phenomenon occurs — by which hurtful influence the velocity of the circulation of the blood is lessened, so that a quantum of adrenalin, given in a definite space of time, is diluted less and can exert a stronger action. In the cat hypophysin does not effect the adrenalin rise of the bloodpressure —

[1]) KRAUS and FRIEDENTHAL. Ueber die Wirkung der Schilddrüsenstoffe. Berl. Klin. Wochenschr. 1908. N^{0}. 38.

[2]) W. STORM VAN LEEUWEN. Physiologische waardebepalingen van geneesmiddelen. (Thesis 1919).

[3]) A. FROEHLICH and O. LOEWI. Ueber eine Steigerung der Adrenalinempfindlichkeit durch Kokain. Arch. f. exp. Path. und Pharm., Bd. 62, p. 59, 1910.

[4]) R. CHIARI and A. FROEHLICH. Erregbarkeitsveränderungen des vegetativen Nervensystems durch Kalkentziehung. Arch. f. exp. Path. u. Pharm. Bd. 64, p. 214, 1911.

[5]) KEPINOW. Ueber den Synergismus von Hypophysinextract und Adrenalin. Arch. f. exp. Path. und Pharm., Bd. 67, p. 247, 1912.

[6]) P. NICULESCU. Ueber die Beziehungen der physiologischen Wirkungen von Hypophysenextract, Adrenalin, sowie Mutterkornpräparaten und Imidozalyläthylamin. Zeitsch. f. exp. Path. und Ther. Bd. 15 p 1, 1914.

[7]) Y. AIRILA. Zur Kenntnis der Pituitrinwirkung. Skandinavisches Arch. f. Physiologie Bd. 31 p. 331 1914.

[8]) H. BÖRNER. Ursache der Steigerung der Adrenalinwirkung auf den Kaninchenblutdruck durch Hypophysenextracte. Arch. f. exp. Path. und Pharm. Bd. 79 p. 218. 1915.

as indeed the heart of this animal is not injured by hypophysin. It is unlikely that BÖRNER's interpretation applies to our case,

1. because peptone intensifies the adrenalin action in doses many times smaller than those which affect the circulation in the cat, and even a gradual rise of the blood-pressure occurs after the peptone-injection, (see also Fig. 1) and

2. because peptone, as recorded in our first communication, has also an intensifying influence upon the pilocarpin-action on surviving organs.

Another observation of ours may serve to explain why the dialysate influences the adrenalin-action on the rise of the bloodpressure in the decapitated cat and not in the rabbit. As stated before, in our investigations we hardly ever found in decerebrated rabbits an increase of the adrenalin-action through peptone or its dialysate. In one case however we detected a very marked action of the dialysate; this case has been illustrated in fig. 4.

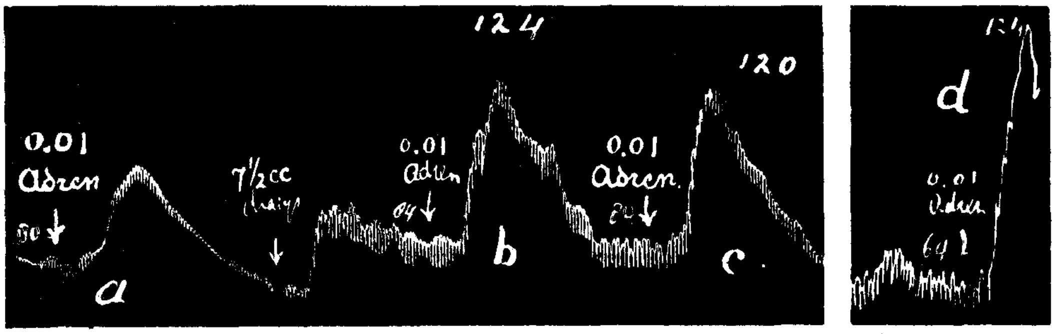


Fig. 4. Decerebrated rabbit. Bloodpressure Abnormal reaction to adrenalin and to adrenalin + dialysate of Witte's peptone.

Here the minimum quantity of adrenalin that produced a distinct rise of the bloodpressure, was rather large viz. 0.01 mgr. In two successive cases this quantity gave a rise of 16, respectively of 22 mm. Hg. (*a*). After injection of a small quantum of peptone-dialysate the action of the same amount of adrenalin on the rise of the bloodpressure increases largely, viz. 40 and 36 mm. mercury (*b*, *c*). After two more dialysate injections the rise of bloodpressure, brought about by 0.01 mgr. of adrenalin amounts to 60 mm. mercury, i.e. considerably higher than before (*d*).

In 11 experiments with the decerebrated rabbit this was the only case, in which the peptone-dialysate had an intensifying influence on the adrenalin, and it is rather remarkable — especially in connection with the experiment with a decapitated cat (fig. 1) described in the beginning of this communication, — that this particular rabbit, in which the dialysate positively exerted an influence, was much less sensitive to adrenalin than all the others. The dosis of adrenalin, capable of producing a distinct rise of the bloodpressure in the decerebrated rabbit, ranged in all other experiments from 0.0007 to 0.005, while in the one case, in which the dialysate *actually* intensified the action, the minimum quantity of adrenalin was considerably higher, viz. 0.01 mgr. It seems to us, that the reason why in most cases peptone, or its dialysate, does not intensify the adrenalin action in the decerebrated rabbit, is that in those cases the adrenalin, in consequence of the presence of certain substances in the serum, has already obtained the highest effect it can display under those circumstances (except in the experiment represented in fig. 4, of course); we believe that similar substances are not, or only to a less extent, to be found in the cat, so that in this animal peptone as a rule can play a part. Besides substances that exert an influence like peptone in our experiments, other substances must occur in the serum that are likewise very important, and that were wanting only in the cat of fig. 1, so that in this case the injection of serum resulted in this excessive increase of the adrenalin action.

Anyhow, we think that our experiments have evidenced that the intensity of the adrenalin action on the bloodpressure in the decapitated cat does not only depend on the dosis of adrenalin and on the sensitivity of the animal, but also in a high degree on the presence in the serum of certain substances that increase this action. If such substances are wanting entirely or nearly so, as in fig. 1, the administration of normal serum can largely intensify the adrenalin action. In case the substances are wanting only in part, a similar increase of the adrenalin action — in a much smaller degree though — can be effected by peptone or still more by dialysate. As a rule so many intensifying substances seem to be present in the rabbit, that adrenalin exerts the strongest action possible under those circumstances. In the experiment of fig. 4 this was not the case; here intensification of the action in the decerebrated rabbit could be reached with the dialysate. We are incompetent as yet to account for the fact that neither peptone nor its dialysate exerts the same influence in the *narcotized* cat as in the decapitated animal; there is, however, some analogy, viz. the symptoms of the anaphylactic shock are also less active, when the animal has been previously narcotized.

If we are right in our conception that the action of adrenalin on the bloodpressure depends in several animals on the presence in the serum of these animals of substances that can intensify the action, we are also entitled to expect that, if the adrenalin is to act upon an animal in which the serum is replaced by physiological salt-solution, the adrenalin will work less intensely. We have endeavoured to test this supposition experimentally.

For this purpose we performed in cats so-called plasmaphaeresis after ABEL [1]). These animals were bled and were given an injection of RINGER's solution to replace the deprived blood, first pure RINGER's solution and afterwards RINGER's solution to which had been added red bloodcells of the cat obtained from other cats. Before starting the plasmaphaeresis we have of course determined the sensitivity of the animal to adrenalin.

The experiments were conducted as follows:

Exp. 1. Cat 2,6 kg.; vagi severed; ether-narcosis. After injection of 0,005 mgr. of adrenalin fall of the bloodpressure from 76 to 54 mm. Hg., at another time from 92 to 76 mm. Hg. (Fig. 5*a* and *b*). The animal is now bled from the carotis as much as possible and simultaneously warm Ringer's solution (to which afterwards red blood cells are added) is injected into the vena femoralis. By this we manage to keep the bloodpressure up to the mark (96 mm. Hg.). Injection of 0,005 mgr. of adrenalin has now no effect at all (5*c*) 0,01 mgr. yields a slight rise.

Exp. 2. Cat 1,27 kg.: ether-narcosis. Injection of 0,005 mgr. of adrenalin causes a rise of the bloodpressure from 110 to 114 mm. Hg. The animal is bled and given Tyrode-solution + red bloodcells. The bloodpressure is lowered; 0,005 mgr. of adrenalin still produces a rise from 34 to 44; addition of a small dosis of cat's serum or peptone to the adrenalin-solution does not intensify the action. The blooding had not been sufficient in this case.

Exp. 3. Cat. Vagi cut; ether narcosis. Injection of 0,005 mgr. of adrenalin yields a rise of the bloodpressure from 168 to 178 mm. Hg. (Fig. 6*a*) Injection of adrenalin + $^1/_2$ c c. serum from another cat has the same effect. After blooding and injection of Tyrode solution + red bloodcells the bloodpressure falls to 78. Injection of 0,005 mgr. and of 0,01 mgr. of adrenalin no longer affects the bloodpressure (Fig. 6*b*). After injection of a fresh quantity of bloodcells the bloodpressure rises up to 110 mm. Hg. Injection of 0,01 mgr. of adrenalin has no action, little by little the bloodpressure decreases spontaneously; when it has reached 62 mm. Hg., again 0,01 mgr. of adrenalin is given. This is almost quite ineffectual. (Fig. 6*c*). By an injection of red bloodcells the bloodpressure is raised to 150 mm. Hg., 0,01 mgr. is inactive (Fig. 6*d*). An injection of 0,005 mgr. of adrenalin + $^1/_2$ c.c. serum from the same cat has no influence. The bloodpressure has meanwhile decreased to 78 mm. Hg, 0,01 mgr. of adrenalin has no effect. (Fig. 6*e*). By an injection of 10 c..c *serum* from another cat the bloodpressure is increased again to 100 mm. Hg., *now 0,01 mgr. of adrenalin gives a distinct rise of the bloodpressure* (Fig. 6*f*). After another injection of 8 c.c of cat's serum 0,01 mgr. of adrenalin still causes a distinct though insignificant rise.

Exp 4. Cat 3 kg. Vagi cut; ether-narcosis. 0,005 mgr. of adrenalin causes

[1]) J. ABEL, L. ROWNTREE and B. TURNER. Plasma removal with return of corpuscles. (Plasmaphaeresis) Journal of Pharm. and exp. Ther. Vol. 5 p. 625 1914.

the bloodpressure to fall from 152 to 130 mm. Hg., (after a preceding small rise), later on from 148 to 132 mm. Hg. After the animal has been bled repeatedly and has received several injections of Tyrode-solution + red bloodcells, the bloodpressure has fallen to 78 mm. Hg., 0,005 mgr. of adrenalin exhibits only a trifling action now (fall from 78 to 74 mm. Hg.), afterwards the bloodpressure has fallen

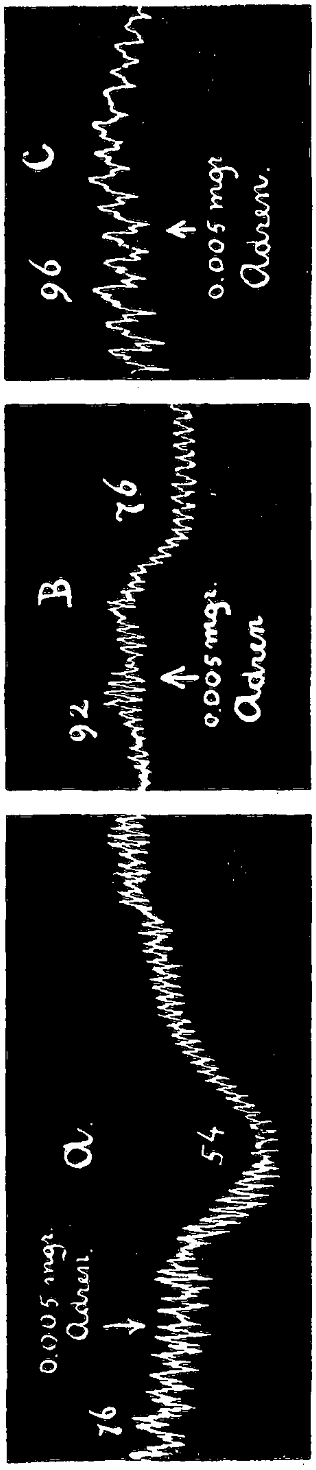


Fig. 5. Narcotized Cat. Bloodpressure. Plasmaphaeresis. Diminished action of adrenalin after removal of the plasma.

still lower and 0,005 mgr. of adrenalin is inert, so is 0,01 mgr. of adrenalin and so is even 0,02 mgr. of adrenalin. When in this phase through injection of red

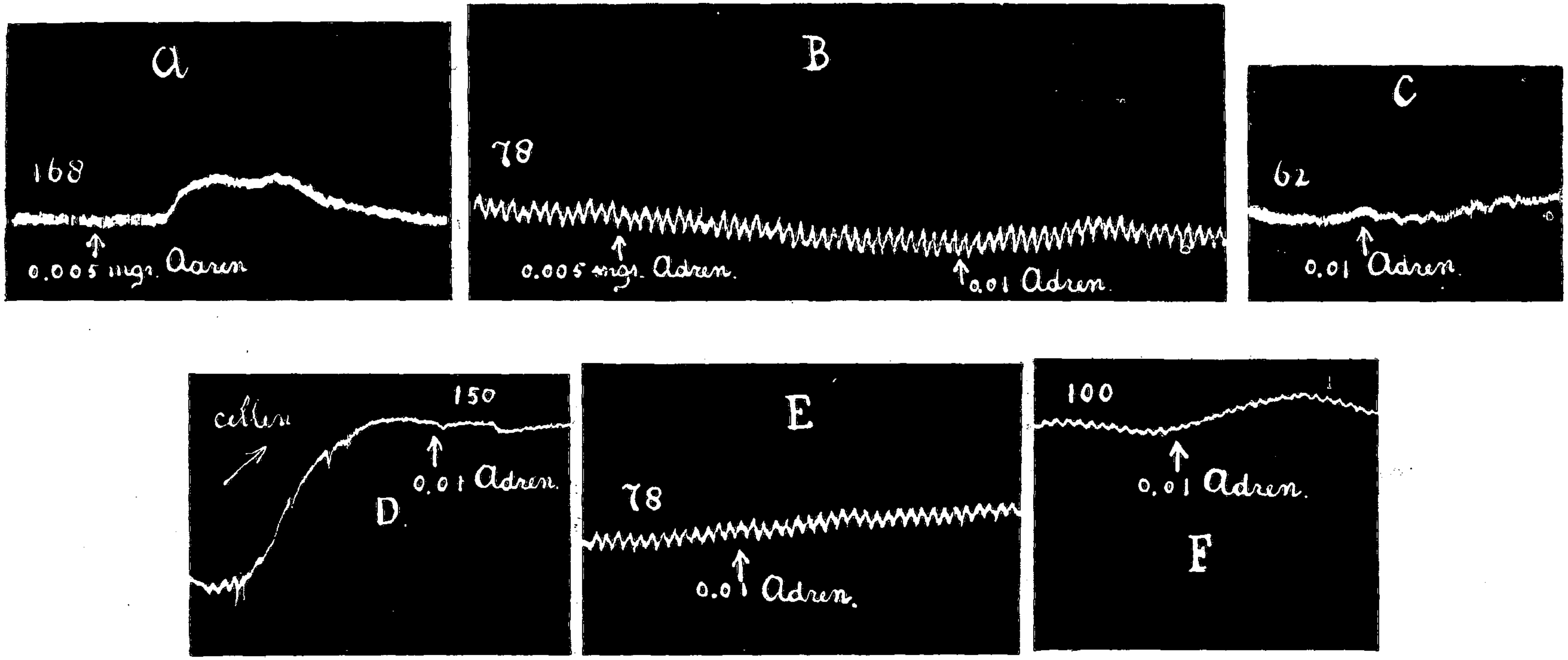


Fig. 6. Narcotized Cat. Bloodpressure. Plasmaphaeresis. Diminished action of adrenalin after removal of the plasma.

bloodcells the bloodpressure is heightened to 108 mm. Hg. again, 0,005 mgr. of adrenalin yields a slight rise of the bloodpressure and 0,02 mgr. of adrenalin a considerable one. The decreased adrenalin action after the blooding did, therefore, not depend only on the removal of plasma, but also on the fall of the bloodpressure, for after this fall had been checked, adrenalin acted better, though still less than at the commencement of the experiment.

These experiments lend support to our supposition that in serum substances occur that intensify the adrenalin action, when the serum has been substituted by Tyrodesolution + red bloodcells the adrenalin action diminishes, if namely a large amount of blood has been deprived in this way. This diminution of the adrenalin action manifested itself as well when the primary adrenalin action had decreased the blood pressure (exp. 1 and 4,) as when it had raised it (Exp. 3). The decrease of the adrenalin action appears, no matter whether the bloodpressure is lowered by the plasmaphaeresis or not. In exp. 3. 0,01 mgr. of adrenalin is inactive although the bloodpressure is as high as 150 mm. Hg, while previously with a bloodpressure that was only a few millimeters of mercury higher, 0,005 mgr. of adrenalin brought about a distinct increase.

It is essential to emphasize the fact that after removal of plasma adrenalin acts less strongly, *also when the bloodpressure is high*, because PEYTON ROUS and WILSON [1]) and BARBIER [2]) have recently demonstrated that in dogs and in cats the adrenalin action is weakened, when the bloodpressure becomes considerably lower in consequence of bleeding. We observed this phenomenon only once, namely in exp. 4, in which, with a bloodpressure of 78 mm. Hg., 0,005 mgr. of adrenalin was inactive, but after the bloodpressure was raised to 108 mm. Hg. through injection of red bloodcells with Tyrode, 0,005 mgr. of adrenalin produced an inconsiderable rise of bloodpressure. This, however, does not account for the insignificant adrenalin action after bleeding in our other experiments, because also when the bloodpressure after plasmaphaeresis amounted to 150 mm. Hg. (Exp. 3), the adrenalin action was lessened. In fact if a decreased adrenalin action is to be attained only by a low bloodpressure, this must be very low, in BARBIER's experiments e.g. 10 and 15 mm. Hg.

By the deprival of serum and the addition of Tyrode-solution + red bloodcells the viscosity of the blood is of course diminished In itself this would be a reason

[1]) PEYTON ROUS and G. WILSON. The influence of etheranesthesia of hemorrhage and of plethora from transfusion of the pressure effect of minute quantities of epinephrine. Journ. of exp. med. 1919, p. 173.

[2]) E. BARBIER. Hemorragie et adrenaline. Soc. de Biol. Tome 82, p 758, 1919.

for a diminished adrenalin action — after the mechanism indicated by H. Börner (see supra). — It seems to us unlikely, however, that this is the case in our experiments, since even a very considerable deprival of blood, by which after injection of Tyrode-solution the viscosity of the blood has much diminished, does not produce a weakened adrenalin action sometimes (Exp. 2), whereas conversely, when a phase has been reached in which the adrenalin acts no longer, 10 cc of serum, which will increase the viscosity only slightly, sometimes improve the adrenalin action. It appeared thus that considerable changes in the viscosity often involved no change in the adrenalin action, whereas such a change did occur in a case in which the viscosity could be influenced only in a small measure.

We presume to assert that especially this case, in which inactivity of the adrenalin resulted from the plasmaphaeresis and in which adrenalin action revived after the injection of serum, strongly favours our conception that substances occur in the serum which intensify the adrenalin action. It would be worth while to make more experiments on the action of adrenalin after plasmaphaeresis. Failing the opportunity of obtaining a large enough number of cats we have as yet not been able to do so. We purpose to extend our experimentation and, in connection with the researches of Kraus, Friedenthal, Kepinow etc., we also intend to study the influence exerted by the organs of internal secretion on the sensitivity of animals to adrenalin.

CONCLUSION.

The action of adrenalin on the bloodpressure does not only depend on the volume of the dosis, the velocity of the injection and the sensitiveness of the reacting organs, but is also influenced by substances in the blood that intensify this adrenalin action.

In some animals there is apparently a considerable deficiency of these substances (the cat of fig. 1). In them the sensitivity can be increased by injecting human serum or cat's serum. Other animals (most cats) possess a sufficient quantum of intensifying substances, but this quantum can be raised by injecting Witte's peptone or its dialysate. Most rabbits have so much of these substances in their blood, that peptone cannot increase the reaction to adrenalin. In one case only (Fig. 4), in which a rabbit was little sensitive to adrenalin, could this sensitiveness be increased by peptone.

Utrecht,
January 1920.

The Pharmacological Institute of the Utrecht University.

Mathematics. — "*A Congruence of Orthogonal Hyperbolas*". By Prof. JAN DE VRIES.

(Communicated at the meeting of February 28, 1920).

1. In any plane through the given point C lies one orthogonal hyperbola o^2, resting on the four crossing lines a_k. The congruence $[o^2]$ defined in this way will be examined here.

Any straight line k is a chord of one o^2. If however k passes through C, it is a chord of ∞^1 curves; it is in this case a *singular chord*.

Also the four lines a are *singular*; for the plane through C and a_1 contains a pencil (o^2), having for base-points the intersections of a_2, a_3, a_4 and the orthocentre of the triangle defined by them.

Finally also the two transversals b_{1234} of the lines a are *singular* chords, for in the plane Cb_{1234} any line cutting b_{1234} at right angles, forms with it a figure belonging to $[o^2]$.

2. To determine the order of the locus of the curves o^2 which have a straight line l through C as a chord, we first consider the surface formed by the orthogonal hyperbolas passing through two points P_1 and P_2 and resting on the lines a_1 and a_2.

The scroll which has a_1 and a_2 for directrices and a plane perpendicular to $l \equiv P_1P_2$ for director plane, contains two straight lines resting on l; for this reason l is a component of two figures o^2. From this it ensues, that the surface in question is a *dimonoid* O^4, with triple points P_1, P_2 and double torsal line l. Through P_1 and P_2 pass therefore *four* curves o^2 resting on a_1, a_2 and a_3.

Let us now consider the locus of the o^2 which have l as a chord, rest on a_1, a_2, a_3, and pass through P_1. There pass four curves o^2 through any other point of l; hence l is quadruple on the surface in question, which is for this reason a *monoid* O^6 with fivefold point P_1. From this appears, that the locus of the o^2 resting on a_1, a_2, a_3, a_4 and having a line l as a chord, is an *axial surface* O^8 with sixfold line l.

According to a wellknown property the axial surface O^8 contains *twenty* pairs of lines. To these belong the *eight* pairs each consisting of a transversal of l, a_k, a_l, a_m and the perpendicular to it inter-

62

secting a_n and l. Each of the other twelve pairs consists of a transversal of l, a_k, a_l and a transversal of l, a_m, a_n perpendicular to it.

3. Through any point P pass *six* curves of the congruence $[o^2]$. For the locus of the o^2 which have CP as a chord and which rest on the lines a, has CP as a sixfold straight line.

Any point A_k of the line a_k is *singular*. The curves o^2 through A_k form a *monoid* O^6 with vertex A_k and fourfold straight line A_kC. It contains *fourteen* pairs of lines arising in the following way. *Three* pairs consist each of a transversal through A_k to a_l, a_m and a straight line intersecting a_n and $l \equiv A_kC$. *Two* pairs consist each of a transversal of l, a_l, a_m, a_n and the perpendicular out of A_k to this transversal. In order to find the other pairs we consider the cone formed by the perpendiculars b_k out of A_k to the transversals of l, a_l, a_m. As two of these transversals are perpendicular to l, b_k coincides twice with l. The cone in question is therefore cubical and has l as a double generatrix. Consequently there are three orthogonal pairs of lines of which the line b_k passes through A_k. In this way the *nine* remaining pairs are found.

4. Also the point C is *singular*. The determination of the order of the surface Γ formed by the curves o^2 passing through C, comes to the determination of the number of orthogonal hyperbolas through C resting on five straight lines 1, 2, 3, 4, 5. Using the principle of the conservation of the number we can suppose the straight lines 1, 2, and 3 to lie in a plane φ. Through C and the point 12 pass *four* o^2, resting on 3, 4, and 5; analogously we find *four* of them through C and 23 and *four* through C and 13.

All the other figures satisfying the conditions are pairs of lines of which one line, s, lies in φ, while the other, t, passes through C. To these belongs in the first place the line s in φ intersecting 4 and 5, in combination with the perpendicular t out of C to s.

Let us now consider the plane pencil (s) in φ which has the intersection M of 4 for vertex. The perpendiculars out of C to the rays of (s) form a quadratic cone; the two generatrices t resting on 5, belong each to an orthogonal pair of lines (s, t). As we can interchange 4 and 5, the group considered contains *four* pairs (s, t).

Finally we find the figure formed by the transversal t through C to 4 and 5, combined with the line s in φ cutting it at right angles. In all we found $3 \times 4 + 1 + 2 \times 2 + 1 = 18$ figures o^2; the curves o^2 through C form consequently a surface Γ^{18}.

5. Any ray through C is a chord of *six* o^2, belonging to Γ; hence C is a *twelvefold* point.

The transversal b_{12} through C to a_1 and a_2 is cut at right angles by two transversals of a_3 and a_4; the *six* lines b_{kl} are accordingly *double lines* of Γ. To them 12 single lines are connected.

To each t_{123} of a_1, a_2, a_3 we draw the perpendicular b out of C and we consider the cone which has the straight lines b as generatrices. Let γ be a plane through C and a straight line c of the scroll to which a_1, a_2, a_3 belong. Through the intersection D of t_{123} we draw in γ the straight line d perpendicular to c. As c is cut at right angles by two lines t_{123}, d coincides twice with c, envelops consequently a curve of the third class with double tangent c. The three lines d meeting in C are generatrices of the cone (b); this is consequently cubical and there are three pairs of lines (b, t_{123}). In all we find *twelve* pairs of lines o^2 of which one of the lines rests on three straight lines a.

Finally there lie on Γ the two transversals b_{1234} each connected to a straight line through C.

Each of the four o^2 which have a line a as a chord, is a *double curve* of Γ.

6. To find the order of the surface Λ formed by the o^2 resting on a straight line l, we try to find the number of curves o^2, in planes through C, which rest on six straight lines 1, 2, 3, 4, 5, 6, and again suppose 1, 2, 3 to lie in a plane φ.

Through the point 12 pass *six* o^2 resting on 3, 4, 5, 6, while their planes pass through C. Analogously *six* pass through 23 and *six* through 13. All the other figures degenerate into a straight line s of φ and a line t cutting it at right angles.

The plane through C and the intersections of 4 and 5 with φ contains a figure (s, t) of which the line t rests on 6. We obtain here a group of three pairs (s, t).

If s is to pass through the point $D \equiv (4, \varphi)$, t must rest on 5, 6 and CD_4. The orthogonal projections t' of the straight lines of the scroll $(t)^2$ envelop a conic. Let the perpendicular r out of D_4 to t' be associated to the ray s joining D_4 with the intersection T of a line t; r being perpendicular to two lines t', hence associated to two rays s, there are three coincidences $r \equiv s$. We find therefore *three* pairs of lines (s, t) satisfying the given conditions; in all a group of 3×3 figures o^2.

From this it ensues at the same time, that the straight line r cutting the ray t in T at right angles, envelops a curve of the fourth

62*

class, for through D_4 passes also the line r at D_4 perpendicular to the ray t of which D_4 is the intersection.

Finally we have to consider the case that the line t rests on 4, 5 and 6. If we now also project the scroll $(t)^2$ orthogonally on φ and draw through the intersection T of t and φ the line r perpendicular to t, r envelops, as appeared above, a curve of the fourth class. From this follows, that also the plane (rt) envelops a curve of the fourth class, so that through C there pass *four* planes in each of which a transversal of 4, 5, 6 is cut perpendicularly by a transversal of 1, 2, 3.

In all we found $3 \times 6 + 3 + 3 \times 3 + 4 = 34$ figures o^2; the locus of the o^2 resting on a straight line l, is consequently a surface A^{34}.

The curve o^2 in the plane (Cl) is apparently a *double curve.* The four lines a are *sixfold* on A; for the curves o^2 through a point of a form a surface O^6.

7. The planes Ca_k may be called singular because they contain ∞^1 orthogonal hyperbolas. This will also be the case when a plane through C cuts the lines a_k in an orthocentrical group. Now the orthocentres of the triangles $A_1 A_2 A_3$ of which the planes pass through C, form a surface; there must therefore be a finite number of singular planes of the kind in question.

In order to determine this number, we first consider the locus of the orthocentre H of a triangle CA_1A_2, when A_1 lies on a_1, A_2 on a_2. The plane through a point A_1 perpendicular to the ray A_1C contains one point A_2, hence one triangle A_1A_2C of which H lies in A_1. Consequently the surface in question contains the straight lines a_1 and a_2.

In the plane Ca_1 lie ∞^1 triangles A_1A_2C; their orthocentres lie in a conic H^2 through C and the intersection D_2 of a_2. The intersection of the surface with Ca_1 consists of a_1 and H^2; we have therefore a surface H^3. Three times H lies on a_3, or, in other words, through C pass three planes in which the orthocentre of $A_1A_2A_3$ lies in C.

We consider now the surface formed by the orthocentres of the triangles $A_1A_2A_3$ of which the planes pass through C.

If H is to get on a_1, A_1A_2 must be perpendicular to A_1A_3. In each plane through a fixed straight line A_1C we draw through A_1 the line l perpendicular to A_1A_2. If this plane is perpendicular to A_1C, l coincides with A_1C; hence l describes a quadratic cone. Two of the generatrices intersect a_3; through A_1C pass consequently two planes in which H coincides with A_1. But then a_1 is a *double straight line* of the surface in question.

A line A_1C is cut at right angles by two lines A_2A_3; it contains therefore two points H, which as a rule lie neither on a_1 nor in C. It has appeared above, that there are three rays A_1C on each of which one of the points H lies in C; the pairs of points H form consequently a curve H_5 with triple point C.

Finally the plane Ca_1 contains a conic which is the locus of the orthocentre of a triangle $A_1D_2D_3$ (where D_3 is intersection of a_2 and Ca_1).

We may conclude, that the orthocentres of the *triangles* $A_1A_2A_3$ lie on a *surface* H^9 with *double lines* a_1, a_2, a_3 and *triple point* C.

From this it ensues, that there are *nine singular planes* in which the four points A_1, A_2, A_3, A_4 form an *orthocentrical group*.

Any straight line of such a plane is apparently singular.

Physiology. — *"On Optic "Stellreflexe" in the Dog and in the Cat"*. By Prof. R. MAGNUS and A. DE KLEYN.

(Communicated at the meeting of January 31, 1920).

In a series of researches carried out in the Pharmocological Institute of Utrecht various animals were examined on "Stellreflexe", i.e. those reflexes that make the animal resume its normal position, when it has been brought into an abnormal one.

In the first communication [1]) "Stellreflexe" in rabbits after extirpation of the cerebrum, were described and discussed in detail.

The ability of these animals to regain from any given position of the body their normal position finds its explanation in the cooperation of four different groups of reflexes whose centres lie in the *mesencephalon*.

These reflexes are:

1. "Stellreflexe" from the *labyrinths* towards the *head*, which make the head return from any given position to the normal one. They may be best seen when taking the animal by the pelvis and holding it up in the air in various positions.

2. "Stellreflexe" towards the *head*, provoked by *asymmetrical stimulation of the sensory nerves of the trunk*.

These reflexes may be best examined after bilateral extirpation of the labyrinths. When, after this operation the body lies in asymmetrical position on the ground, reflex action will cause the head to resume its normal position through asymmetrical stimulation of the nerves of the trunk.

3. "Stellreflexe" starting from the *neck*.

When the head has obtained its normal position through the above-named reflexes, but the body has not, a reflex is elicited by the abnormal position of the neck (rotation, flexion etc.) which makes the body resume its normal and symmetrical position with regard to the head.

4. "Stellreflexe" towards the *body through asymmetrical stimulation of the sensory nerves of the trunk*.

---

[1]) R. MAGNUS. Beiträge zum Problem der Körperstellung. I. Mitt. Stellreflexe beim Zwischenhirn- und Mittelhirnkaninchen. Pflügers Archiv. Bd. 163. S. 405. 1916.

Also when the head is not in its normal position, the body lying on the ground in asymmetrical position can be restored by reflex-action to the normal position through asymmetrical stimulation of the ground.

Optic "Stellreflexe" do not appear in rabbits after extirpation of the cerebrum.

From the 2^d^ communication [1]) it became evident that also *the normal rabbit with cerebrum* has only the above "Stellreflexe" at its disposal. *Optic* "Stellreflexe" could not be demonstrated in them either.

The 3^rd^ communication [2]) deals with observations made by Dr. DUSSER DE BARENNE in experimenting on *two cats and a dog after extirpation of the cerebrum.* It could be proved that also with these animals only the four above-named "Stellreflexe" play a part; *optic* "Stellreflexe" could not be demonstrated in them either.

Now, whereas with rabbits there is no difference between animals with and without cerebrum, this is altogether different with dogs and cats.

From this paper it will be seen that *normal dogs and cats, that is with a cerebrum, dispose of optic "Stellreflexe"*, and that with them the eyes may co-operate to enable the animals to retain their normal position. If one wishes to examine these optic reflexes, it is essential to hold the animals free in the air, for only then can the "Stellreflexe", resulting from asymmetrical stimulation of the nerves of the trunk on body and head, be eliminated.

Under these circumstances the animal depends for the time being only on its "Stellreflexe", emerging from the labyrinth, and after bilateral extirpation of the labyrinth not any "Stellreflex" can appear in dogs or cats that have been deprived of the cerebrum, and in rabbits with or without cerebrum, if the animals are held up free in the air. It now appeared that dogs and cats with cerebrum, but without labyrinth, still dispose of "Stellreflexe", which enable the animals to bring their head into the normal position.

These "Stellreflexe" are brought about by the eyes.

In order to demonstrate this we communicate the following results with a little dog:

---

1) R. MAGNUS. Beiträge zum Problem der Körperstellung. II. Mitteilung Stellreflexe beim Kaninchen nach einseitiger Labyrinthexstirpation. Pflügers Archiv. Bd. 174, bldz. 134.

2) J. G. DUSSER DE BARENNE u. R. MAGNUS: "Beiträge z. Probl. d. Körperstellung III. Die Stellreflexe bei der gross-hirnlosen Katze u. dem hirnlosen Hunde". To be published in Pflügers Archiv. 1920.

The normal animal was first examined, on "Labyrinth-Stellreflexe", while held up in the air, that is, before the bilateral labyrinthextirpation performed on it. In this experiment the eyes were blindfolded beforehand. The result was to the following effect:

Animal, held up free in the air by the pelvis.

Normal position of the pelvis: Head in normal position.

Held up on its right and left side: Head about normal (deviation $\pm$ 30′ from the normal position).

With its back placed in horizontal position: Head brought in normal position, either because the front of the body i.e. the neck and the upper part of the thorax is bent ventralward, or because the front of the body performs a spiral rotation of 180°.

Animal suspended with head downward: Head and muzzle are held vertically downward; the neck, however, is distinctly dorsi-flexed.

Animal suspended with head upwards: Head in normal position.

When carrying out the experiment without the eye-bandage, the result is precisely the same; only the head is brought into a perfectly normal position when the animal is held up free in the air, horizontally placed on its side.

On the 6th of June 1919 a bilateral extirpation of the labyrinth was performed. Some hours after the operation the animal keeps its head straight and no nystagmus is seen. Neither in this investigation nor in any of the following did the animal prove to possess any labyrinth-reflex.

June 7. 1919. When investigating in the air *without the eye-bandage* (so with *open* eyes) it appears that the animal *does not possess any* "Stellreflex" *in the air*.

Holding the animal in horizontal direction on its right or left side: Head falling to the right, resp. to the left side.

Holding the animal in horizontal direction on its back: Head falls on its back.

Suspended with head downwards: Head held as lying on its back.

Suspended with head upwards: Head in various positions (now latero-flexed to the right or retro-flexed).

From this investigation we conclude that *on the day after that of the operation (bilateral extirpation of the labyrinth) the animal, when held up in the air, does not dispose of "Stellreflexe" and the eyes do not act compensatively.*

An investigation of other dogs gave evidence that after bilateral extirpation of the labyrinth the animals *gradually* recover the ability of bringing their heads into the normal position again when they are held up in the air. It was also evident that the animals obtain this ability through the eyes and by fixing different objects around them. *When the eyes are blindfolded, the "Stellreflexe" will immediately disappear, so we have to do with optic "Stellreflexe".*

We have not made an inquiry of the successive appearance of the optic "Stellreflexe" in the above-named dog, since it was one of the first dogs, in which "Stellreflexe" were found and these had already been fully developed at that time.

July 1. 1919 we found:

Blindfolded the animal (in the air) has completely lost its sense of orientation.

Pelvis held on its right side: Head held on its right side. (Fig. 1).

Pelvis held on its left side: Head held on its left side.

Pelvis held on its back: Head held on its back.

Suspended with head downwards: Head held on its back.

Suspended with head upwards: Head retro- or latero-flexed.

Not blindfolded (*i.e. with eyes open*) the animal presents quite another image.

Pelvis held horizontally on its left or right side: Head in normal positions. (Fig. 2).

Pelvis held horizontally on its back: Head in normal position, the front of the body flexed ventralward, the animal fixing his surroundings with great interest.

Suspended with head downward: Considerable flexion of the head towards the back, muzzle upwards and head in normal position.

Suspended with head upwards: Head in normal position.

The above observations and other investigations of various dogs not reported here, tend to show that the dog, held up in the air, completely loses its sense of orientation directly after bilateral extirpation of the labyrinth, but also that after a few days it gradually learns by the aid of his eyes to bring its head into the normal position. Already after two or three days this ability begins to appear; it is almost complete after a week and quite accomplished after about a fortnight.

It is noteworthy that the development of optic "Stellreflexe" could be traced in a dog, of which on Dec. 4 1918 Dr. DUSSER DE BARENNE had removed the greater portion of the cerebellum, so that at the post-mortem only the frontal part of the vermis and small remnants of the cerebellum were found laterally from the medulla oblongata. When DE KLEYN on March 3rd 1919 had performed on this animal the bilateral extirpation of the labyrinth, no trace of "Stellreflexe" could be observed during an investigation on the 23rd of April and on the 26th of May, when the animal was held up free in the air with bandaged eyes. On the other hand the optic "Stellreflexe" were

Fig. 1.

quite distinct while the animal was investigated with open eyes. Lying on its sides, on its back and when the animal was suspended with the head upwards, the head was brought into its normal position. When hanging with the head downwards, the cervical vertebral column was flexed considerably. It is evident, therefore, that *also after removal of the greater portion of the cerebellum optic "Stellreflexe" still react.*

In cats the same optic "Stellreflexe" may be observed as in dogs. Young, tame cats are fittest for this purpose, as most full-grown cats are too wild when being examined in the air, and thereby hamper the experiment.

SUMMARY.

Cats and dogs deprived of their cerebrum possess the same four groups of "Stellreflexe", that have been described in a previous paper for rabbits.

In the air these animals depend on the "Labyrinth-Stellreflexe" towards the head and on the cervical "Stellreflexe" connected with them. When in such animals both labyrinths have been extirpated, they have completely lost their sense of orientation.

It is quite different with dogs and cats with cerebrum. When they are trying to find their orientation in space, they make use also of their eyes.

This may be demonstrated by examining them freely in the air after bilateral extirpation of the labyrinth.

Directly when the extirpation of the labyrinth has been carried out, dogs lose their sense of orientation almost completely, cats in a large degree. After a few days the animals have learned to use their eyes and sooner or later they are able, without labyrinths, to bring their heads into the normal position from the most varying positions in space. When watching the animals, it will be seen a

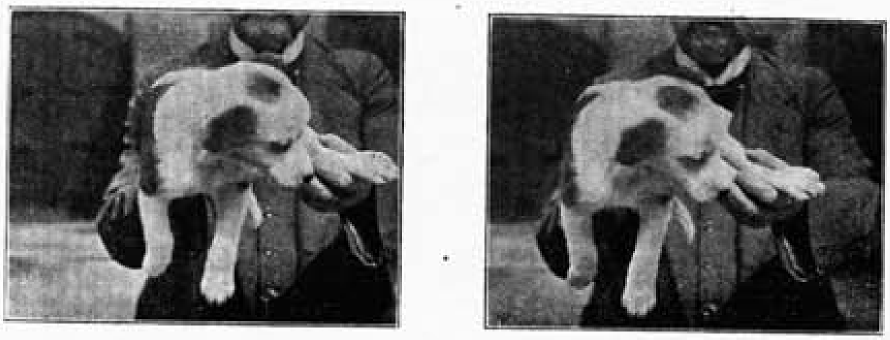


Fig. 2.

once that they make use of their eyes and that the optic "Stellreflexe" reveal themselves after the fixing of the surrounding objects. On examination of such animals without labyrinths, blindfolded or not, the "Stellreflexe" can be made to appear or to disappear at will.

The fact that the optic "Stellreflexe" react only in animals with unimpaired cerebrum, points to a correlation of the optic "Stellreflexe" with the presence of the cortex. This follows as a matter of fact, since dogs and cats deprived of their cerebrum, do not show optic reactions, except the pupillary reflex and the closing of the eyelids on exposure to light.

It is interesting to observe the contrast between the dog and the cat on the one side and the rabbit on the other. The normal rabbit with cerebrum has no optic "Stellreflexe", and as regards "Stellreflexe", does, therefore, not differ from a so-called Thalamus-rabbit. The apparatus essential in the rabbit for standing and for posture, is, indeed restricted to the brain-stem; in dogs and cats connections with the cortex, probably with the optic cortex, as the experiments have proved, also come into account. Special experiments are needed, of course, to ascertain whether the mere cirumstance of the optic cortex being intact, is sufficient for the optic "Stellreflexe" to present themselves.

The fact that dogs and cats, directly after extirpation of the labyrinths lose their sense of orientation more or less, leads to the conclusion that, in normal life, these animals use their labyrinths to obtain the orientation in space (in the air), and that for this purpose they use their eyes only when the labyrinths do not function properly.

Within the first few days after bilateral extirpation of the labyrinths it is easy to see how the animals gradually learn to use their eyes.

Zoology. — "*Rhythmical Skin-growth and Skin-design in Amphibians and Reptiles*". By Prof. C. PH. SLUITER.

(Communicated at the meeting of March 27, 1920).

VALENTIN HAECKER [1]) in a few very important communications and later on in a synthetic exposition has tried to give due value to a factor, until now little or not at all appreciated, in the explanation of the origin of the skin-design. Previous investigators, among whom I refer to HARRISON, ALLEN, TORNIER, GROSSER, ZENNECK and especially to v. RIJNBERK, chiefly tried to find a connection between the transversal stripes of the vertebrata and the segmental arrangement of other organs. ZENNECK, among others, found a connection between the appearance of pigment in the skin and the situation of blood-vessels in embryos of Tropidonotus natrix. The well-known researches of v. RIJNBERK, partly made in collaboration with WINKLER, to which in some respects the work of others (SHERRINGTON, BOLK, LANGELAAN, etc.) is connected, try to find the most important factor for the origin of the skin-design in the segmental innervation of the skin.

Thus the — in my opinion — ill-chosen term of "dermatome" has found its way into the scientific terminology. This term gives the impression as if the skin itself had a metameric structure, while the expressions of "overlapping of the dermatomes", "summation and interferential zones of the dermatomes" all reinforce this erroneous view.

Though some of the investigators have made it plausible, that in a number of cases the innervation and the design of the skin are correlated, yet one cannot derive from it a general guiding principle in explaining the design of the skin, and I think that we find in HAECKER's principle a wider base on which one might build with profit. This principle is described by HAECKER as follows: the skin-design of the vertebrata (and I prefer to add: also of the invertebrata) is dependent on the fact that the growth and the differentiation of the skin are clearly rhythmical. This rhythm is sometimes in correlation

[1]) V. HAECKER. Entwickelungsgeschichtliche Eigenschafts- oder Rassenanalyse Z.f. ind. Abstammungs- und Vererbungslehre. Vol. 14, p. 260, 1915.

Idem. Zur Eigenschaftsanalyse der Wirbeltierzeichnung. Biolog. Centralblatt Vol. 36, p. 448, 1916.

Idem. Entwickelungsgeschichtliche Eigenschaftsanalyse. Jena 1918.

with the metamerism of the body, but generally independent of it and in a high degree autonomic.

It is evident that it is easiest to trace the phenomena of this rhythmical growth in young animals and especially in quickly growing larvae or embryos. Thus HAECKER found this rhythmical growth for the first time confirmed in the larvae of Axolotl, as here the size of the cells, of which the epidermis is constructed and the fact that it has only two layers of cells, was very favourable for the research.

The large number of embryos and larvae of reptiles and of amphibians of which the Zoological Laboratory at Amsterdam disposes, led me to investigate the rhythmical growth in the skin of these young animals with a view to afford a further confirmation of HAECKER's supposition, that this rhytmical growth is the nearest cause of the skin-design.

As to the larvae of the amphibians, I examined the skin of the of *Megalobatrachus maximus*-larvae of the famous hatch of the Aquarium of the Society "Natura Artis Magistra". These larvae show very early a skin pigmentation, but I never found a metameric design, as HAECKER did in Axolotl. The pigmentation of the Megalobatrachus-larvae might rather be called diffuse, but not absolutely, as it is obvious on closer examination that the pigment, especially on the ventral side, is arranged more or less regularly in small groups.

The idea struck me directly, whether this were a case of a type of skin-growth which was indicated by HAECKER as the "chess-board-type", — theoretically possible, but not yet observed — and which was accepted by him as the possible original type of the skin-growth of the vertebrata.

I found on microscopical examination of the skin of the larvae of *Megalobatrachus maximus*, but especially in a young stage of 30 m.m. length, where the pigment formation was only in its first development, that the epidermic-cells were arranged very regularly indeed, into square fields in which, evidently the growth had proceeded centrifugally (Fig. 1). The cells lying in the middle of every field were separated more sharply by more strongly developed marginal zones than the younger ones lying against the edge. The very first pigment granules appear in the middle of these square fields. This agrees with the observations, made by GUSTAV TORNIER [1]), who found that no pigment appears in cells that are still in the dividing stage, but as a rule in those parts of the skin which are growing rapidly. These spots, arranged regularly on the ventral side of the larvae

[1]) G TORNIER. Experimentelles über Erythrose. Sitz.ber. Ges. Naturf. Freunde. Berlin 1907.

have their origin in these accumulations of pigment. On the dorsal side, this design passes gradually into a more or less diffuse pigmentation.

The pigmentation however is not regularly diffuse on the dorsal

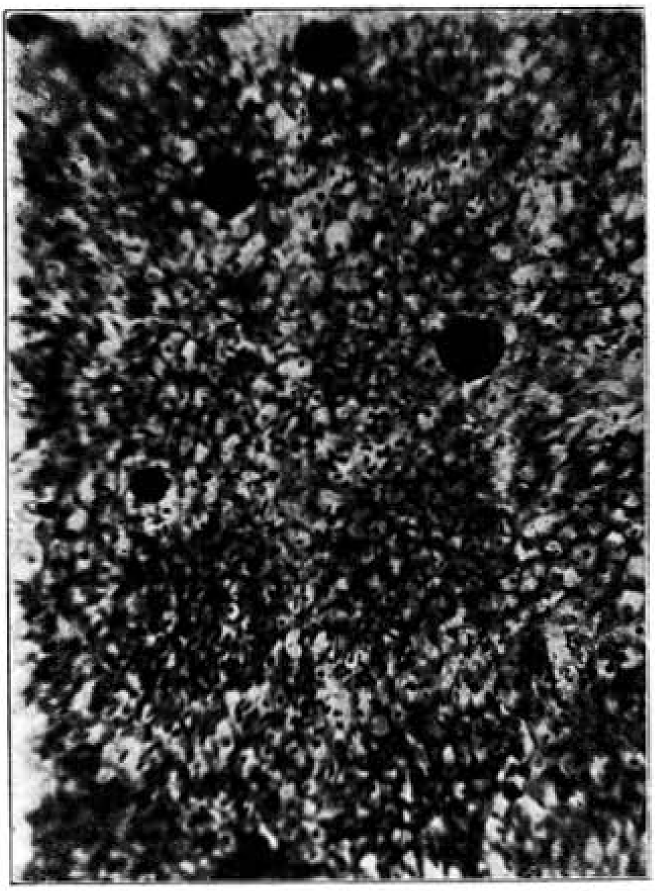


Fig. 1.

side, and especially not on the flanks of the bodies of the larvae. It is evident that the "cellstreams", described by HAECKER for the first time, are of importance for the pigmentation. The "cell-streams" were very obvious and strongly developed in the skin of the Megalobatrachus larvae, in a similar way as HAECKER found them in Axolotl. These cellstreams are series of cells, which, radiating from special centra, divide more quickly and shove in between other groups of cells. Thus they form regions of mòre intensely growing skin. (Fig. 2). This also seems to coincide with the distribution of pigment. I generally noticed that cellstreams radiated obliquely backwards from the well-known lateral sense organs. The pigment appears first close to the lateral sense-organs on the flanks of the body, and spreads from there along the cellstreams over the flanks of the body. However in the case of Megalobatrachus there is no obvious pattern at all, which is probably connected with the fact that

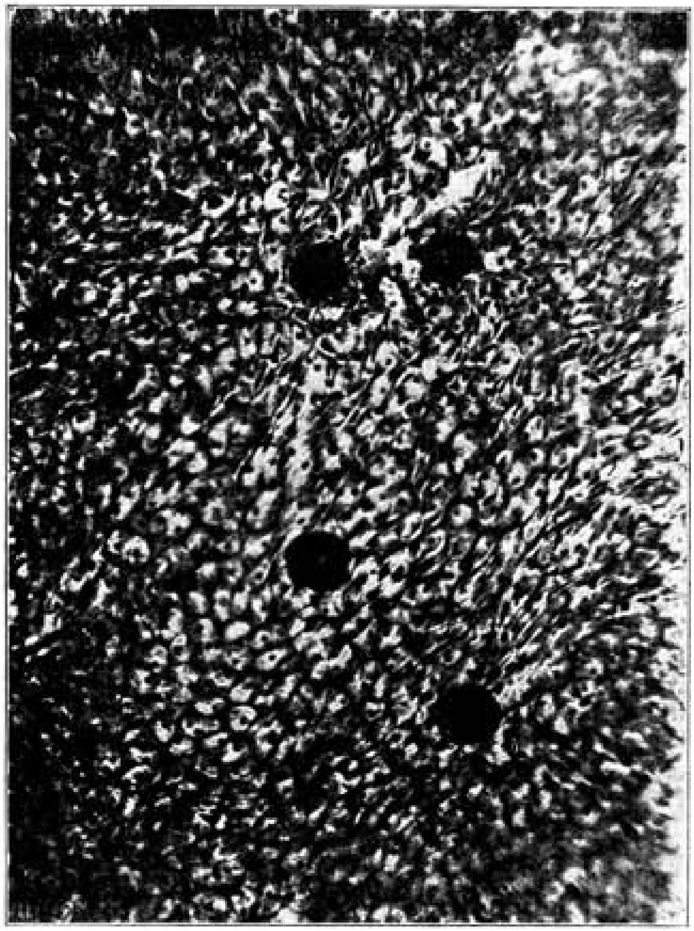


Fig. 2.

the lateral sense-organs are distributed over the skin in a strikingly irregular way. While in younger larvae the shades of colour in the pigmentation of the skin, owing to the cellstreams are obvious, they give place to a very regularly spread darker colour of skin in older larvae.

Once having found in Megalobatrachus, that the skin-growth, and in connection with this, the first appearance of the autochtone pigment, not only occurs in a similar way, as HAECKER found in Axolotl larvae, but that the "chess-board type" was evident in a specially young stage of development, which HAECKER supposed, but had not seen, — I tried to enlarge my investigation on the skin-growth in examining the embryos of different reptiles. The cytological examination, however, is much more difficult with these, as first of all the skin of the embryos of reptiles is not constructed of two layers of epidermic-cells only, but especially because these cells are much smaller, so that the directions of the cell grouping and the stages of division are much less visible than in the amphibian-skin with its large cells. Yet I succeeded at last in preparing large pieces of the skin of young embryos and in examining them entirely. Though my research on this subject is only in its first stage, I could state already that similar rhythmical growths appear in the skin of reptile-embryos, which coincide with the pigmentation. The development of the bodyform and with this the growth of the covering skin is much more complicated than in the generally simple cylindrical or barrel-shaped larvae of the amphibians. Many remarkable and important problems arise of this complication. It is evident that by preference those embryos are examined, that show peculiarities in their coat-patterns.

Thus I examined in the first place the embryos of Draco volans, of which a large number of very different ages were collected by Dr. L. DE BUSSY, at Medan, and presented to the Zoological Museum at Amsterdam. These embryos show a very clear and characteristic design, of which only a very indistinct image is left in the grown-up animals. First of all it may be stated that the design in question is quite independent of the metameric architecture of the rest of the body. This strikes us most in the design on the membrane between the prominent 5 or 6 ribs. In young embryos we see the appearance of 4 or 5 dark broad stripes which run obliquely across the ribs and thus cross the bloodvessels and nerves, which lie alongside the ribs (Fig. 3). We have probably to deal with a case of rhythmical, wavelike growth of the skin. Already at the first appearance these stripes appear as continuous pigment-zones in the intercostal membrane. A quicker growth and a coïnciding pigment formation take place there. The whole pattern on this membrane seems to me a typical illustration of the fact that this design is nothing but a consequence of the rhythmical skin-growth. Also on the rest of the body of young embryos of Draco volans, one finds a distinct connection between the first appearing design and the places of strongly developed growth. Large ribbed scales appear very early along the flanks of the body and they remain when the animal is adult. These scales, whose place does

Fig. 3.

not correspond to the metamerism of the rest of the body, at the same time indicate the place, where, for the first time pigmentation appears in the skin. We find the first trace of pigmentation in the beginning of the high ridge on the scales. Here of course is the place of most intense growth. From that point, the pigment spreads over the rest of the scale, to proceed from there gradually over the surrounding scales. Here also, I was able to observe cell-streams in many young embryos, where the large ribbed scales were only slightly to be distinguished. The design on the medial line of the back has likewise no connection whatever with the metamerism of the body. It forms crescent-shaped spots, with the opening turned backwards; they are placed at more or less regular distances and point also to a rhythmical skin-growth, though up to the present moment, I was not able to find any cell-streams.

At last I wish to point out some peculiarities which occur at the manifold transversal striping in the embryos of reptiles. It is well-known that EIMER accepted the longitudinal striped design as the original one and not only in the case of reptiles, but he made a general rule of this principle. Whether this theory is probable or not, may be left aside here; but in any case the fact remains difficult to explain, that in a great number of embryos of lizards, serpents and crocodiles very distinct cross-stripes appear first, even with forms, showing longitudinal stripes in the adult.

A number of embryos of *Lygosoma olivaceum*, collected partly by Dr. L. DE BUSSY, partly by myself, were at my disposal. The whole of the trunk and the tail shows sharply marked, broad, dark, nearly black stripes, which alternate with comparatively narrow, white stripes, without pigment in the skin. In the first place, not the slightest coincidence is to be found between the extension of this stripe design and the metamerism of the rest of the body. The left and the right side are not symmetrical, so that it often happens that a dark stripe meets a white one on the medial line of the back (Fig. 5). However it is well-known that this cannot form an argument against the metameric origin of the design, as the metameric spinal nerves and the bloodvessels to the left and the right are not always symmetrical. But also the number of the stripes is different on the two sides of the body and the following peculiarity of this difference is of great importance.

As is more than well-known, the embryos of all the reptiles lie more or less like a spiral in the egg and now one generally finds that the dark stripes become broader and split into two at their broadest point towards the convex side of the body, because at that point a white stripe appears. In this way a few more stripes occur on the convex side of the embryo, than on the concave side (Fig. 6).

Fig. 5.

Fig. 6.

I saw the same phenomenon take place wherever the transversal striated parts of an embryo lay in a sharp curve. This was very obvious in the tail of the embryos of *Gecko verticillatus*, the well-known Tokkè of the Dutch Indian Archipelago. The embryo's tail is curled like a spiral, turned a little dorsally. And here too we find that considerably more cross-stripes appear at the ventral convex side than at the concave dorsal side. (Fig. 7). The same phenomena appear on the tail and the trunk of crocodile embryos, as is generally known.

Fig. 7.

The question is raised now: what is the origin of this phenomenon? I think that we must look for the explanation in the rhytmical skin-growth. At the convex side the growth is certainly more energetic than at the concave side, where the body is compressed and the skin is not so tightly stretched and has even a few folds. For the present it may be left an open question whether the reason is an insufficient nourishment at the concave side, caused by pressure on the blood-vessels and probably on the nerves, as GUST. TORNIER [1]) supposes has taken place in an analogous case of snake embryos, deformed pathologically.

[1]) G. TORNIER, l.c. p. 1210.

Though the proof has not been given at present that this is a case of more or less quick rhythm in the division of the skin-cells — as is the case with amphibians — we cannot help thinking that we must look for the principal cause of the stripe design of the skin in rhythmical skin-growth. This may sometimes coincide with the rest of the metamerism of the body, but as a rule it is quite independent of it.

The future will teach us whether the conclusion to which TORNIER arrives is not too optimistic. He thinks that it will be possible to infer partly the conduct of every lizard or snake directly from the skin-design. In the first place it is necessary to trace, whether in the case of reptiles too, the stronger pigmented places of the skin of the embryos actually coincide with the places in the epidermis, where an intenser growth, and consequently a quicker division of cells occurs. I found that with Draco volans this investigation is much more difficult, but not impossible and I hope to be able to give further data in a following communication.

Physiology. — *"On the genetic relation between lymphocytes and granulated leucocytes."* By J. DE HAAN and K. J. FERINGA (Communicated by Prof. H. J. HAMBURGER).

(Communicated at the meeting of Februari 22, 1920).

There still exists a difference of opinion as regards the question to what extent the lymphocytes and granulated cells of the blood can be regarded as closely related cell kinds. It is agreed in general that the (neutrophile) granulated cells, when once formed, form a type by themselves for which there is no possibility of being transformed into other types of cells. Also with respect to the lymphocytes the majority of researchers would have them regarded as a type sui generis that under no circumstances can pass over into granulated cells. The possibility of forming the latter would in that case be reserved only for a particular cell kind, which, it is true, bears very much the appearance of the lymphocytes, from which, however, they should differ fundamentally. These cells are distinguished by the name of "myeloblasts". It is accepted that they occur normally only in the bone-marrow and only very exceptionally and in highly pathological states there can be formed in other places also, a kind of bone-marrow or myeloid tissue by "metaplasia" of cells present in the adventia of the vessels. This dualistic theory which was most zealously propagated by NÄGELI at present claims the greatest number of supporters; a lymphocyte never becomes a granulated cell; the latter can originate only from myeloblasts, by way of granulated large mononuclear myelocytes.

Besides this the monistic theory still finds numerous supporters of which WEIDENREICH and MAXIMOW claim the first place. They lean on this that embryonically in any case there is no question of a principial division between the granulated and the non granulated cells, that in lower organisms transitions between these two cell kinds have been ascertained, and, furthermore, on the fact that, up to this, attempts to attribute specific characteristics to myeloblasts which would principially distinguish them from other cells of the mononuclear type, have failed. Contrary to this the dualists hold that such a principial difference is afforded amongst others by the oxydase reaction which is positive for myeloblasts whereas it always turns out negative for lymphocytes.

The monistic theory gives more satisfaction for the reason that it is more in accordance with the fact that blood and connective tissue are by far the most genuine bearers of embryonic characteristics; it is the tissue from which every thing can originate under favourable conditions. This also accounts for the fact that very numerous forms of amoeboid cells, the so called wander cells have been described, under new names every time, in different parts of the body. It is difficult here to give definite characteristics because these cells remain rather loose from each other and different kinds of cells are found in one anothers immediate proximity.

For the rest the question should be a settled one for the unprejudiced researcher. Before long already it was found possible (amongst others by MAXIMOW by bringing corpora aliena into lymph glands) to cause typical myeloid tissue to be formed in atypical places. NÄGELI, who for such cases is dead against the metastasis of myeloid tissue from the bone-marrow, could still speak of metaplasia here although this appelation does not cover the dualistic theory, because after all, it still means that latent characteristics under special external influences make their appearance, so that the characteristics of the myeloblasts also are a product of external influences. Now however that MAXIMOW [1]) a short time back told of his success in cultivating myelocytes from lymph gland tissue of the rabbit in vitro with the aid of an extract of bone marrow, it seems to be high time to relinguish the idea that the myeloblasts have specific characteristics, and to accept the broader view that the unitarian tenet offers us. This is in accordance with the fact that all connective tissue elements are omnipotent up to a certain degree and that all these polentialities can be brought to light under different external conditions. All these possibilities have as result a very great variety of cell forms that can always be found in different tints and forms of protoplasm and nuclei as for instance, in young formative tissue. As long as it has not been possible to show why, f. i., the protoplasm of plasma-cells is basophile and why the one granulation has more acidic, the other more basic or neutral affinities, it is properly speaking not allowed to talk of specific characteristics of cells in such cases. Why, f. i., should a lymphocyte which does not give the oxydase reaction and does not exhibit typical granulation not have developed all these characteristics after

[1]) C. R. S. d. B.; 69, p. 225 and 235, 1917.

an hour under modified conditions? MAXIMOW thus demontraded that in the blood fluid with a slight change in properties the lymphocytes assume the granulation and all the characteristics of the granulated cells. The original account of MAXIMOW is too brief to allow of any further particulars being gleaned from it.

We believe that in the account which follows here we are offering a contribution by which likewise is shown, that in higher animals the blood lymphocytes can change into granulated cells. The results of this research may allow us to give some brief remarks on the biology of the white blood corpuscles in general.

The research upon which this communication bears, was carried on as a continuation of the work of one of us [1]) who has kept himself busy for a long time with the vital characteristics (glycogen concentration, phagocytosis, amoeboid movement etc.) of the exudation leucocytes of the rabbit. Detailed accounts about these will appear elsewhere. Only the method by which these leucocytes were obtained may be mentioned here. For this purpose 200 cc of NaCl 0,9 % or of a corresponding fluid were injected all at once intraperitoneally into rabbits. The following day this injection was repeated and a few hours later by means of a troicart, the canule of which had a number of holes in its side, the exudation was drained from the abdominal cavity. This exudation was mostly present in quantities varying between 50—100 cc. (if necessary we rinsed with NaCl 0,9) and contained always a large number of leucocytes. It appeared now, that this exudation followed practically without exception upon injection with all kinds of fluids: NaCl 0,9 %, RINGER-solution, ultrafiltrate of serum of different animals, ultrafiltrate or NaCl 0,9 % diluted with blood serum of the serum of the rabbit. Even the fact of working more or less with sterile precautions made little difference; an injection of sterile NaCl 0,9 %, namely gave an exudation as well, while an agar culture of the fluid tapped off proved to be sterile. From this it was concluded that every slight change in the tissues gives rise to the formation of an exudation and to emigration of cells from the blood without it being necessary to accept for that reason the presence of special chemotactic substances, like f.i. the researches of DOLD [2]) attempt to prove. Also the addition of a small quantity of starch to the injection fluid, as was done at first in imitation of the well-known method for causing sterile exudations, proved to be wholly unnecessary.

The exudation itself proved with the reagent of ESBACH contained, independant of a longer or shorter stay in the abdominal cavity, ± 1,2—1,5 % protein, and after it had been deproteinised it reduced FEHLINGS solution to a slight extent. The abdominal cavity of the rabbit normally contains fairly constantly a small quantity of fluid, at times a considerable amount. In this transsudate there are found only mononuclear cells, the so called macrophags, about the origin of which so many conflicting ideas existed and still exist.

Introduction of 0,9 % NaCl which up to this has been done as a rule only for the purpose of studying the resorptive functions of the endothelial cells of the internal cavities of the body, thus brings about a total change in the abdominal

[1]) Vide J. DE HAAN, Archiv. Néerl. de Physiol., tome II, 4, p. 674 (1918).
[2]) Vide Deutsch Arch. f. Klin. Med., **117**, p. 206, (1915).

cavity; besides the undeniable resorption, there is also a strong emigration. (Moreover it is also mentioned by WEIDENREICH [1]) that the injection with salt solution can cause emigration). How strong this emigration actually is, will appear from the fact that it is often possible after centrifugation of the fluid tapped off, to obtain 1—3 cc. of pure leucocytes as a reaction upon the double injection, i e. more white blood corpuscles than there are at any moment present together in the blood of the rabbit.

The cells of the exudation apart from variations of minor character, were constantly the same. The mononuclear transsudate cells were still in the majority but sparcely present if the fluid was drained off about 3 hours after the first injection. There were then hardly any granulated cells present; on the other hand the cell-type was totally reversed if the fluid was tapped off in the usual way after the second injection. The quantity of cells was then increased many fold and consisted almost exclusively ($\pm$ 95 %) of polynuclear granulated cells of the pseudo-eosinophile type such as are present in the blood of the rabbit.

On the ground of several indications it was concluded that the mononuclear which occur first, owed their origin to a local reaction by disquamation of epithelial cells, and, that only the granulated cells emigrated from the blood. Further it appeared that the duration of life of these granulated cells as such outside the blood could be only exceedingly short, and that in this case there is practically no question of any functions of living cells. This appeared f.i. from the fact that upon the introduction of starch grains or fat into the abdominal cavity where an exudation was already present, all cells were precipitated with the starch and without acting upon it were almost quantitatively destroyed and surrounded with formative tissne after a few days. If some days after the injection, the abdominal cavity is again rinsed, then these are again found principally mononuclear cells of which very many have taken up into their protoplasm one or more granulated cells. Thus to some extent a state of rest has set in again. The rabbit seems normal again; only on the first day it was a little out of sorts. This animal and also other kinds of animals used (goat guinea pig) responds as regards its health only very slightly; Even a prick in the bowels is endured without further effects.

Still the condition of the rabbit once used is not what is has been originally. This is apparent when after a week or a few weeks we inject it anew; already a few hours after the first injection we notice an exudation consisting nearly exclusively of polynuclear cells.

What was found during the progress of the exudation, was for the greater part in agreement with what other researchers held in this connection. It is known that the neutrophile granulated leucocytes when out of the blood tend to degenerate and the mononuclear to regeneration. But still as a rule the idea is entertained that, out of the blood the polynuclear cell carries out important functions as a living organism; This can be denied on the grounds of our investigations. We do not aim at belittling the very important functions of these cells as carriers of organic matter such as, for instance, ferments.

It was in continuation of these investigations, that we traced how the circulating blood of the rabbit responded to this emigration by milliards of the polynuclear cells. This appeared to be of such a

[1]) Die Leukozyten und verwandte Zellformen. Wiesbaden 1911, p. 385.

nature that there seemed to exist a flagrant contradiction between the course of the blood formula of the different types of white blood-corpuscles and the cells found in the exudation.

The numbers given by the different researchers as regards the percentage of the different kinds of white blood corpuscles in the blood of rabbit greatly diverge. Through the examination of a large number of rabbits we found the total number in 1 c.mm of a drop of the circulating blood as follows: total amount 7.500 — ± 12.000 of which, on an average 25—30 % were polynuclear cells and 70—75 % mononuclear cells, mostly smaller and larger lymphocytes. Only very exceptionally as much as 50 % polynuclear cells were found. This blood-formula in the rabbit is also, with respect to the proportion between the polynuclear and mononuclear cells, near enough just the reverse of what we find in man. More or less the same numbers are given by JÖRGENSEN [1]). One and the same rabbit exhibits on the whole oscillations of only minor importance.

After the injection of NaCl. 0.9 % however, the change is very marked. After the lapse of a few hours the blood corpuscles decreased in number to but 1,500—2,500 i.e. to about 1/5 the original number; during a short period the polynuclear cells had more or less disappeared from the blood, but also the lymphocytes decreased greatly in number. As an example I quote the following tabula:

Date.	Time.	Rabbit I.		Rabbit II.		REMARKS.
		Lymphocytes.	Polynucl. leucocytes.	Lymphocytes.	Polynucl. leucocytes.	
Jan. 12	10.30a.m.	10.175	2.500	7.650	2.075	
	12.30p.m.	—	—	—	—	Intraperitoneal injection of 200 cc. NaCl 0.9%.
	2 „	2.950	275	2.200	75	
	3 „	2.375	75	1.000	112	
	4 „	1.700	800	1.300	500	
Jan. 13	9.30a.m.	13.300	5.500	5.725	6.525	
	10.30 „	—	—	—	—	Intraperitoneal injection of 200 cc. NaCl 0.9%.
	11.30 „	7.950	5.800	4.050	1.850	
	2 p.m.	3.100	1.075	2.000	525	

We therefore see that the polynuclear cells rapidly diminish. The lymphocytes diminish more gradually. After 3 to 5 hours this decrease

[1]) Skand. Arch. f. Physiol., XXXII, 4/6, p. 253.

of both kinds of cells however ceases, and makes place for a slow and regular increase. If on the following day the examination is made anew, then mostly a slight (at times a considerable) leucocytosis is observed, with, perhaps, on an average a relative increase of the quantity of polynuclear leucocytes (e.g. rabbit II), but still mostly a very large majority of lymphocytes exists. Repeated injections always show the same phenomenon: Thus the local disturbance of equilibrium causes a reaction in the blood which lasts for a short time which reaction however is soon restored to equilibrium again. Also in the case of a many times repeated injection we always found this quick return to normal conditions.

Already in the contemplation of this reaction on the formula of the white blood corpuscles in the circulating blood we therefore come across difficulties if we adhere to a principial difference between lymphocytes and leucocytes; for blood consists mainly of lymphocytes; after an injection we see all the elements disappear simultaneously from the blood but still the result is an exudation which consists almost wholly of polynuclear cells.

To explain this, several things were possible.

Firstly: There is the possibility that the lymphocytes, unlike the polynuclear cells, do not permeate through the vessels but are only retained in the capillaries and afterwards return again to the circulating blood. It is not probable that this is exclusively the case because different researchers who have studied the formation of exudation saw besides the emigration of polynuclear cells also that of lymphocytes (SCHWARZ [1]), MAXIMOV [2]) and others).

Secondly: It is possible that the emigrated leucocytes remain somewhere in the tissues after the emigration. This is in accordance with the organisatory tendencies of the lymphoid cell type.

Thirdly. There is the possibility that all blood cells emigrate but that during that process the lymphocytes, to some extent at least, get converted into polynuclear cells. The last exposition thus brings with it at the same time the acceptance of the unitarian point of view. By accepting this possibility of transition it will become clearer that during the course of life the number of lymphocytes in the blood inspite of the continual influx undergo a percentile decrease, instead of still increasing, as would have to be expected, because there is hardly any destruction of lymphocytes to be seen, and emigration does not take place to the same extent as with the polynuclear cells. If however we accept that there is a continual tran-

1) Wien. Klin. Wochenschr., 1904, p, 1173.

2) ZIEGLER's Beiträge, 38, p. 301, 1905.

sition of the total number of lymphocytes formed in the body into polynuclear cells the maintenance of the equilibrium is wholly explained.

There was still to be explained why in the case of a repeated injection, inspite of a perfectly similar reaction of the blood, the polynuclear cells in the exudation occurred at a much quicker rate.

The microscopic examination of the tissues in and around the abdominal cavity, treated in the above way gave a satisfactory explanation of all the phenomena. The rabbit used for this purpose was treated in the following way.

Firstly an injection of 200 c.c. of NaCl 0.9 % at a time was given on two consecutive days; on the second day a suspension with 0.25 c.c. of leucocytes was drained off. After about 14 days the injection was repeated during three consecutive days and on the third day there were tapped of more than 1 milliard leucocytes. Six days later the injection was again repeated during 3 consecutive days and on the third day ± 4 milliard leucocytes were obtained. About 4 hours after a renewed injection on the 4th day the rabbit was killed.

The abdominal cavity contained more than 150 c.c. of fluid with an enormous quantity of leucocytes. All the abdominal organs showed a strong hyperaemia. Especially the omentum was rich in blood and swollen to a thick greyish-brown tissue. The omentum and two pieces of the mesentery with the portions of the small intestine fixed to them were hardened in sublimate and formol for further microscopic examination. Slides of the omentum and mesentery coloured after Giemsa showed that a large proportion of the white blood corpuscles which had disappeared from the blood had been deposited in the hyperaemic tissue. They were found in and around the blood vessels, sometimes in large conglomerates which consisted of lymphocytes of different seizes up to plasma cells. Between these groups there were fat cells and especially connective tissue which was of a pronounced formative type: Large fibroblasts, newly formed capillaries, plasma cells upto magrophags.

In some places several cells exhibited nuclear division. Furthermore the whole of the tissue was pervaded with a large number of leucocytes, in great numbers especially along the endothelial layer of the tissues. This was very distinct in a preparation of the surface-view of a thin part of the membrane; by far the greater number of the cells were polynuclear in this case.

A number of the mononuclear cells clearly show a transition into the polynuclear form. Here and there occurred larger and smaller groups of cells (sometimes also separate cells) consisting for the greater, or, in cases, also lesser part, of myelocytes — large cells with round or kidney-shaped nuclei and with a pronounced granulation in their protoplasm of the same character as that of the polynuclear cells; the fact that these cells together with other, greater and smaller nongranulated cells of the lymphoid type, and plasma cells, occurred together in one group, removed all doubt about these cells being evolved from lymphocytes. A number of these cells were rather small and of the lymphoid type, others were already large but not granulated. Many had a basophile protoplasm with pseudo-eosinophile granules. These groups lay mostly perivascular. A few myelocytes were noticed to be in different stages of cell division. Besides these there occurred conglomerates of almost fully developed polynuclear leucocytes. Slides of the normal omentum looked quite different: Resting endothelial cells with some resting connective tissue cells between them.

In this way there was formed, through a slight stimulus as an injection of NaCl 0.9 °/₀ in the abdominal cavity, repeated a few times, partly through local reaction, but mostly through the deposition of blood elements, a young tissue process of enormous extension. In this region there had been deposited the lymphocytes which had disappeared from the blood, which lymphocytes, together with the hyperaemic regeneration tissue, characterised the whole as myeloid tissue.

In this way the disappearance of the lymphocytes from the circulating blood is explained, and also the fact that. inspite of this disappearance, practically only polynuclear cells are met with in the exudation: the lymphocytes are partly at least converted into polynuclear cells.

This also explains why upon a later repeated injection the animal responds in such a way that there is a faster occurrence of polynuclear cells in the exudation. The myeloid tissue in the abdominal cavity has not quite come to rest again and responds to the new stimulus of NaCl solution again by an instantaneous conversion of the lymphocytes which were present, and those which were conveyed there, into granulated cells.

We can therefore conclude that the injection of NaCl 0.9 °/₀ forms a stimulus through which by far the greater number of the white blood cells present in the circulating blood accumulates in the capillaires of the abdomen; in addition there is also seen a reaction of the connective tissue elements through the same stimulus and out of these two reactions there results a tissue which is perfectly similar to myeloid tissue.

Again, from this it is evident once more, that the observed formula of the white blood corpuscles in the circulating blood gives but a very inaccurate idea of the manner in which they can occur in the whole of the vascular system. Only those white blood cells which are carried along with the blood, flow out of the finger and the veins of the ear. A considerable part will however be retained to a greater or lesser extent in the capillary region where there is a very great surface action.

A slight stimulus, such as the injection of a practically harmless fluid, in the abdominal cavity can cause a strong increase of that adhesion locally, and in this way, this adhesion after a few repetitions of the injection has the effect of sifting the blood *free from white blood corpuscles in the capillaries of the abdomen.* To a small degree this will undoubtedly always take place normally. In this local accumulation it comes to an increase of cells through division and besides, to a differentiation, in other words to a temporary, movable organ for the forming of different kinds of white blood corpuscles. This organ then owes its origin to the same extent to both blood and connective tissue. By looking at things in this light we could regard the bone-marrow not only as the factory of blood elements which are delivered into the blood; we would be equally justified in saying the reverse, namely that the elements of the blood are constantly attracted in the capillaries of the bone-marrow as a continuation of what takes place in the formation of bone in the embryo. In this way a tissue will arise that, in its turn, rejuvenates the blood, so that we have here the eternal process of reversion and equilibrium which we always find in the functions of the living cell.

We can expand the subject still further; from the moment that the blood is formed in the embryo there arises an indisputable separation between tissues and blood which makes itself evident by the difference in quantity of albumen, and the difference in the nature of the cells. As an utterance of this contrast we f.i. find that the polynuclear cells which belong normally to the blood cannot exist in the tissue fluid. But on the other hand there undoubtedly is an attraction between the two elements, and this reveals itself especially in the capillary areas; there the difference is less pronounced and there also cellular and dissolved substances are interchanged. We could say that blood and tissue have a separated existence, have so to say opposite charges.

Consequently the cells that wander in the blood always have an affinity for the tissues (capillary attraction) and the tissue elements in their turn for the blood. We can now further accept with reason that the tissue cells, the lymphocytes, as soon as they have entered the blood, will not retain their properties which they have in the tissues, unchanged; as all living matter under changed circumstances they will change their protoplasm, for all cell life is nothing but the interchange of protoplasm with the environment.

Where now the neutrophile cells can only exist in the blood it is obvious to reason that the origin of the neutrophile granulation, with all vital properties connected with it, must be an issue of the conditions of life made possible in the blood, whatever these conditions may be (e.g. can be mentioned the presence of red blood corpuscles, higher albumen percentage etc., etc.). The one affects the other and vice versa; for instance it could readily be supposed that the formation of red blood corpuscles and neutrophyles results from the higher albumen percentage, and again that the latter in its turn owes its origin to the said cell elements. It is just also possible that definite changes in the blood (slow current or modified reaction) form the stimulus which leads to the transformation of the protoplasm.

From this follows that by the continual supply of tissue cells to the blood a kind of equilibrium ensues. The constantly formed neutrophile cells which are the chief bearers of the blood characteristics will also have the greatest tendency to emigrate; since, however, their protoplasm is changed irreversibly they very soon perish in the tissues and that for the greater part lytically. This emigration will always take place if the contrast between blood and tissue becomes greater, and this will, inter alia, be the case in every abnormal function of the tissue which is the result of an injection. The increased contrast will lead to a more abundant emigration of polynuclear cells and to strong local attraction of the blood lymphocytes. By this accumulation into masses in the capillaries the contrast is, however, soon neutralised i.e. the difference between blood and tissue becomes less evident; so called myeloid tissue comes into being; on the one hand the endothelial cells form new capillaries and thereby come into contact with the tissue, on the other hand also lymphocytes, granulated cells, and blood albumen enter amongst the tissue elements. The embryonic state returns, the different types of cells found here, keep another in equilibrium and here we also notice the change of lymphocytes into granulated cells which now can emigrate further to the exudation, if the stimulation of the tissue still continues, or enter the blood. In agreement with this we very soon see that the equilibrium which was disturbed is restored again.

A similar local mingling of blood and tissue might be accepted for bone-marrow under the influence of a permanently existing stimulus which retains and rejuvenates the elements of the blood. Myeloblasts therefore in accordance with this conception could only be formed from so-called undifferentiated cells there where definite proporties of the blood are still present, therefore in the immediate vicinity of, perhaps also in the blood vessels; to all appearances the factors for this are more markedly present if many lymphoid cells accumulate locally combined with an abundant blood supply, and it does not happen at all there, where any important influence of the blood fails, as in lymph glands. In this way it can be explained why the metaplasia of the spleen into myeloid tissue is never observed in the Malpighian corpuscles but always in the pulpa which is rich in blood. In the pulpa the elements of blood and connective tissue are indiscriminately mished. On the ground of what has been mentioned above, it seems us better, in the case of the different cell types of mesenchymal origin, not to speak of strongly separated specific cell-types, for which there is no possibility of transition in one another; we should prefer to say, that all the different cell forms here are the result of the action of the varying conditions of the surrounding fluid, which on its turn is continually changed through the living cells; in the state of equilibrium issuing from this, perhaps one cell-type will prevail, but a modification of the fluid will at the same time cause a change in this equilibrium, so as to make that perhaps another form of cell will get the upper hand; only for the granulated cells, if once formed, there is no possibility for a reversal process. According to the presence of two chief liquids in the body, differing widely, it is reasonable, that we also met with two chief types of cells of which one prevails in one, the other in another medium; thus the bone-marrow shows normally a transformation into granulated cells, while the spleen consists principally of lymphoid tissue, although by no means exclusively. Under changes, due to pathological changes in the milieu, the development of the cells can be modified, as happens in the case of the different forms of leukaemia.

SUMMARY.

What has been discussed above can be summarised as follows: The total number of white blood corpuscles that occur in 1 c.m.m. of the blood which flows from a vein in the ear of a rabbit, amounts to about 7000—12000. Of these about 75 % are lymphocytes and about 25 % granulated cells. After an injection of NaCl 0.9 % in the abdominal cavity there is a very considerable decrease in the number of cells in the blood in which both kinds of cells take part in the same degree. This diminution reaches its minimum already after a few hours and then makes place for a gradual increase again.

The cells which have disappeared from the blood are met with again in the abdominal cavity; the fluid here rich in cells contains, however, almost exclusively polynuclear leucocytes. The lymphocytes take part in the formation of a young formative tissue which clearly bears the characteristics of myeloid tissue. Here in many places and accompanied by an increase in the number of cells lymphocytes are seen to pass over into myelocytes and polynuclear cells by the way of plasma cells, which polynuclear cells spread out from the centre where they originate. The result also of the created disturbance of equilibrium is in the first place direct emigration of polynuclear leucocytes and besides that the formation of myeloid tissue at the place where the disturbance of equilibrium was brought about. In the formation of this myeloid tissue the greater majority of the lymphocytes which have disappeared from the blood take part; by way of this myeloid tissue as intermediate station finally also the lymphocytes partly at least land in the exudation as polynuclear cells.

Groningen, Februari 1920. *Physiological Laboratory.*

ERRATUM.

In Professor BROUWER's paper: "*Ueber die Struktur der perfekten Punktmengen* (dritte Mitteilung)", p. 473 of this Vol., l. 1 from the top

for: mit ε gegen 0 konvergierendes
read: nur von ε abhängendes und mit ε gegen 0 konvergierendes