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A Recursive Algorithm for Lossless Embedding of Non-Passive systems

ABSTRACT

The lossless embedding problem for the class of real and non-contractive scattering matrices, and hence for non-passive systems is addressed. The approach is based on a modified LIS model. A recursive Schur-type algorithm will be devised, which enables the computation of the lossless embedding for arbitrary scattering matrices including the boundary case of scattering matrices with singular values equal to one.

1. Introduction

Due to its vast field of potential applications in network and system theory, signal processing and control, Schur analysis and its various directions of generalizations is currently a growing field of intensive research among mathematicians, especially among operator theorists. With view to concrete computations, it seems worthwhile to focus on a rather unpretentious point within this powerful framework: the development of extended Schur methods for real and constant matrices. It is mainly a synthesis problem for linear and resistive $n$-ports, which motivates such a restricted investigation. Consider a discrete-time, linear state space system with input-output transfer function $t(z)$, given by the linear fractional transformation $F$

$$t(z) = F[S](z) := J + H(z1_n - F)^{-1}G,$$

where

$$S = \begin{bmatrix} F & G \\ H & J \end{bmatrix}.$$

According to the reactance extraction approach to linear systems, first introduced by Youla and Tissi [8], the real and constant matrix $S$ may be regarded as the scattering domain description of a linear and resistive (non-dynamic) multiport. It is well known that the system to realize $t(z)$ is passive, if $S$ may be chosen to be a contractive matrix, that is, all its singular values are less than one. However, for example in control applications, one will more likely be confronted with the task of modelling a given, possibly active, or even unstable system using a network model, instead of implementing a given transfer function by a passive realization. In this case, $S$ no
longer is a contraction, but may exhibit arbitrary singular values. The same situation occurs in model reduction for time-varying systems, a theory recently developed by Dewilde and van der Veen [7] that extends classical norm approximation results in the vein of Adamjan, Arov and Krein to the non-stationary or non-Toeplitz case. It is the aim of the present work to devise systematic means in order to incorporate such active (or non-passive) network models. The modified LIS model embodies the representation of real and constant matrices as description for the physical quantities at the input ports of a lossless resistive multiport, whose termination at the output ports is chosen appropriately (non-passive).

2. Multiport Models for Matrices

The following definitions will serve to supply terms necessary for a geometric vector space representation for the technical notion of an $n$-port. For the rest of the paper, we assume all $n$-ports to be linear and resistive.

**Definition 1** Let $a, b \in \mathbb{R}^n$ denote vector-valued input and output signals of a linear $n$-port, respectively. The behaviour of an $n$-port is defined as the set of all admissible signal pairs $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$.

The behaviour completely determines the input-output characteristics of an $n$-port in the black-box sense. Since we will assume all multiports to be linear, the behaviour has the structure of an $n$-dimensional linear subspace of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, which determines the $n$-port uniquely. In this perspective, the set of all linear $n$-ports is equivalent to the Grassmannian manifold of all linear $n$-dimensional subspaces of $\mathbb{R}^{2n}$, which means that any $n$-port may be uniquely identified by one point in $\text{Grass}(n, 2n, \mathbb{R})$.

In order to enable any concrete investigation of a given $n$-port, an explicit representation for the behaviour is needed, and therefore, a basis is introduced.

**Definition 2** Let $x = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{2n}$ denote the vector of an admissible signal pair. A basis matrix is defined as a matrix being composed of $n$ linear independent column vectors of admissible signal pairs

\[ X := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{2n \times n}. \]

Therefore, the behaviour of the $n$-port is given by the column space of a basis matrix

\[ B := \text{im}(X) = \left\{ x \in \mathbb{R}^{2n} | \exists m \in \mathbb{R}^n : x = Xm \right\}. \]

We may think of the behaviour as being the graph of the linear transformation $S \in \mathbb{R}^{n \times n}$, which will be interpreted as the scattering matrix of an $n$-port. The scattering matrix performs the mapping of an input signal $a$ to its corresponding output signal $b$ via $b = Sa$, such that
Based on an obvious partitioning of the basis matrix, we may write

\[
X = \begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
1_n \\
S
\end{bmatrix} A, \quad S = B A^{-1}.
\]

Hence, the use of the scattering matrix formalism amounts to a very specific choice of affine coordinates in Grass\((n, 2n, \mathbb{R})\) and degenerates whenever \(A \in \text{GL}(n)\). The basis matrix will never exhibit any degeneracies and is amenable for describing active multiports that possibly involve unbounded, but closed scattering operators, that do not possess a matrix representation due to the occurrence of infinite singular values.

**Definition 3** An **indefinite inner product** for vectors \(x, y \in \mathbb{R}^{2n}\) is defined by the bilinear form

\[
[x, y] := (x, J y) = y^T J x, \quad J := \begin{bmatrix}
1_n \\
-I_n
\end{bmatrix}
\]

with respect to the indefinite metric tensor \(J\).

Just as the standard inner scalar product, the indefinite inner product can be used to define such geometric concepts as 'length' and 'orthogonality' of vectors. For instance, two vectors are said to be \(J\)-orthogonal to each other, if their indefinite inner product vanishes, that is, \([x, y] = 0 \Leftrightarrow x[\perp]y\). A comprehensive treatment of the basic theory of vector spaces with an indefinite inner product can be found in [4].

An important criterion for the classification of systems is the notion of net power flow, which can be formulated efficiently by using the indefinite inner product.

**Definition 4** Let \(B\) denote the behaviour of a given \(n\)-port and let \(x = (a^T b^T)^T \in B\). The **squared \(J\)-length** of the vector \(x\) is defined as

\[
|x|^2 := [x, x] = x^T J x = a^T a - b^T b.
\]

Measuring the '\(J\)-length' of the vector \(x\) can be identified as measuring the net power flow into the \(n\)-port. However, since \(J\) is indefinite, the induced non-euclidean geometry gives rise to negative vectors or even non-zero vectors of vanishing '\(J\)-length'. Clearly, these cases are due to physical situations, where the \(n\)-port behaves as an active (reversed power flow direction) or lossless device, respectively. Vectors with a vanishing \(J\)-length will be called **isotropic vectors** (note that isotropic vectors are \(J\)-orthogonal to themselves).

By the classification of one individual basis vector with respect to the sign of its \(J\)-length it is not possible to draw any conclusions concerning the global energetic properties of a given multiport. It is rather necessary to consider the totality of all vectors in the behaviour.

**Definition 5** A subspace is called **positive (totally isotropic, negative)** if all vectors in this subspace are positive (isotropic, negative).
It is well known [4] that any subspace of an indefinite inner product space allows for a decomposition into the direct sum of mutually $J$-orthogonal positive, totally isotropic and negative subspaces $B^+ \oplus B^0 \oplus B^- = B$. Note that $B^0$ is sometimes called the radical of $B$, which is defined as $\text{rad}(B) := B \cap B^{\perp\perp}$, where $B^{\perp\perp}$ denotes the $J$-orthogonal complement of $B$ in $\mathbb{R}^{2n}$. While the dimensions of these subspaces are unique, the decomposition is not. In physical terms, this statement amounts to the characterization of an $n$-port by $n$ independent measurements, which are quite arbitrarily resulting in a basis set of $2n$-vectors. The system itself is fairly general, that is, it may be modeled by some interconnection of passive, active and lossless subsystems. The dimensions of these subsystems provide a natural and compact classification of $n$-ports, according to their energetic properties.

**Definition 6** Let the behaviour of an $n$-port be decomposed as $B = B^+ \oplus B^0 \oplus B^-$. The $J$-signature of the behaviour is defined as the triple of integers $(m, k, l)$, which satisfy $m+k+l = n$, where $m = \text{dim} B^+$, $k = \text{dim} B^0$, $l = \text{dim} B^-$. According to this definition, strictly passive $n$-ports have a $J$-signature $(n, 0, 0)$, whereas the $J$-signatures $(0, n, 0) \text{ or } (0, 0, n)$ represent lossless or totally active $n$-ports, respectively.

Since the behaviour is described in terms of a basis, the system property of being passive, lossless or active should be reflected somehow in the basis matrix.

**Definition 7** To each basis matrix $X \in \mathbb{R}^{2n \times n}$ of an $n$-port there exists a real $n \times n$-matrix, which will be called the $J$-Gramian of $X$ and which is defined as

$$\text{gram}(X, J) := X^T J X = A^T A - B^T B = A^T (1_n - S^T S) A. \quad (1)$$

Therefore the following lemma can be formulated, while dropping the obvious proof.

**Lemma 1** Let $X$ denote a basis matrix of a given $n$-port. The $n$-port is strictly passive (totally active) iff the $J$-Gramian of $X$ is positive (negative) definite. The $n$-port is lossless, iff the $J$-Gramian of $X$ is the zero matrix.

The rightmost term in equation (1) makes evident the well-known fact that the $n$-port is strictly passive if its scattering matrix is strictly contractive and the $n$-port will be lossless if its scattering matrix is orthogonal. Combining this observation with Sylvester’s law of inertia, we can easily establish the coincidence of the $J$-signature of the behaviour and the inertia of the respective $J$-Gramian of any pertaining basis matrix. According to its $J$-signature or inertia $(m, k, l)$, an $n$-port is said to exhibit $m$ passive, $k$ lossless and $l$ active modes. Since we will deal in the following with the most general case of arbitrary inertia or $J$-signature, a warning seems to be appropriate: When a basis matrix for the behaviour contains isotropic vectors, then this in general does not imply any information concerning the dimension of the radical, even if the basis matrix exclusively consists of isotropic vectors. However, the radical of the behaviour is spanned by the isotropic vectors in a $J$-orthogonal basis.
3. Transformation of N-Ports

The modeling of matrices by lossless cascade transformations according to a Darlington-type realization approach for real matrices is based on the following stream of arguments. Note first that the notion of lossless cascade transformations induces immediately the question about the orthogonal group of an \( n \)-dimensional subspace in \( \mathbb{R}^{2n} \) and hence neatly amounts to Lie groups.

**Definition 8** The Lie group of \( J \)-orthogonal or lossless transformations is

\[
O(n, n) := \{ T \in \mathbb{R}^{2n \times 2n} \mid T^T J T = J \}.
\]

In the present context, a transformation \( T \in O(n, n) \) is interpreted as the chain scattering matrix of a lossless \( 2n \)-port. The mapping between the input scattering matrix \( S \) and the scattering matrix \( L \) of the output ports, performed by \( T \), is the linear fractional transformation

\[
S = (\gamma + \delta L)(\alpha + \beta L)^{-1}, \quad T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},
\]

whereas the corresponding mapping of the associated basis matrices is a linear one.

\[
\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} G \\ D \end{bmatrix} M, \quad S = BA^{-1}, \quad L = DG^{-1}, \quad M \in \text{GL}(n)
\]

This turns out to be a lot more convenient for the purpose of algorithmic investigations.

Since, by definition, the \( J \)-signature is invariant under the action of the Lie group \( O(n, n) \) on the set of all \( n \)-dimensional behaviour subspaces, the \( J \)-orthogonal mapping of basis matrices establishes an equivalence relation. The resulting equivalence classes are given by the \( \frac{1}{2}(n + 1)(n + 2) \) sets of behaviour subspaces with identical \( J \)-signature. Each individual class may be conceived as the \( O(n, n) \)-orbit of some distinguished element of the equivalence class, which should be "nice" in view of its role as a representer for the whole class.

**Definition 9** The representer matrix with \( J \)-signature \((m, k, l)\) is defined as the basis matrix

\[
L := \begin{bmatrix} G \\ D \end{bmatrix} := \begin{bmatrix} 1_m \\ 0_l \\ 0_m \\ 1_l \\ \pm 1_k \end{bmatrix}.
\]

The columns of the representer matrix are \( J \)-orthogonal as well as orthogonal and therefore, the \( J \)-Gramian of the representer matrix will be given by

\[
S := \text{gram}(L, J) = G^T G - D^T D = \begin{bmatrix} 1_m \\ -1_l \\ 0_k \end{bmatrix}.
\]
Note that the representer matrix can be interpreted as a basis matrix describing an \( n \)-port consisting of \( n \) decoupled 1-ports, each characterized by a scalar reflection coefficient \( s_i \in \{0, \infty, \pm 1\} \), \( i = 1, \ldots, n \). According to the Cayley transformation between scattering and impedance description \( Z = (1_n + S)(1_n - S)^{-1} \), this corresponds to \( m \) positive, \( l \) negative resistors and \( k \) open or short circuits. With these ingredients at hand, the following rather central theorem may be stated [1].

**Theorem 1 (Ball/Helton)** Let \( X \) denote a basis matrix for a given linear and resistive \( n \)-port, whose columns span the behaviour of an \( n \)-port. There exists a transformation \( T \in \mathcal{O}(n, n) \) and a representer matrix \( L \) such that

\[
X = TLM.
\]  
(2)

where the non-singular matrix \( M \) is any solution of

\[
\text{gram} (X, J) = M^TSM.
\]  
(3)

From a circuit-theoretic point of view, the decomposition (2) can be interpreted as the embedding of the \( n \)-port \( S \sim \{A, B\} \) into a lossless 2\( n \)-port \( T \), whose output-ports are terminated by a decoupled load \( L = \text{diag}(s_i) \sim \{G, D\} \).

Due to the invariance of the \( J \)-Gramian under the action of \( \mathcal{O}(n,n) \) the lossless embedding is intimately connected to the signed factorization of \( J \)-Gramians (see equ. (3)), where under a non-singularity condition on the leading principal submatrices of \( \text{gram} (X, J) \), the factor \( M \) may be chosen to be upper triangular.

In order to achieve an explicit expression for the \( J \)-orthogonal chain scattering matrix \( T \) of the lossless 2\( n \)-port, which accomplishes the mapping (2) basic properties of orthogonal transformations will be exploited.

**Lemma 2** Let \( S \in \mathbb{R}^{n \times n} \) be a given (scattering) matrix, such that \( X = (1_n S^T)^T \) is a basis matrix for the behaviour \( B = \text{graph} (S) \) of a pertaining \( n \)-port. Then \( \tilde{X} = (S 1_n)^T \) is a basis matrix for \( B^{[1]} = \text{graph} (S^{-T}) \).

Putting this lemma together with the fact that \( \text{rad} (B) = \{0\} \) iff \( 1_n - S^T S \) has full rank directly leads to

**Theorem 2** Let \( S \) be a scattering matrix which does not exhibit singular values equal to one. The \( J \)-orthogonal embedding for \( S \) is given by

\[
T = \begin{bmatrix}
1_n & S^T \\
S & 1_n
\end{bmatrix}
\begin{bmatrix}
M^{-1} & \\
& N^{-1}
\end{bmatrix}
\begin{bmatrix}
G & D^T \\
&D & G^T
\end{bmatrix},
\]

where the non-singular matrices \( M, N \) are any solutions to

\[
1_n - S^T S = M^T S M, \quad 1_n - SS^T = N^T S N.
\]
The proof may be viewed as a trivial consequence of Witt’s theorem [5] when applied to the mutually $J$-orthogonal subspaces $\text{graph}(S)$ and $\text{graph}(S^{-T})$.

It should be noted that this embedding formula is only valid, if the $n$-port to be embedded does not exhibit lossless modes, that is, the graph of $S$ has a signature $(m,0,l)$ or, in other words, $\text{rad}(\text{graph}(S)) = \{0\}$. This constraint is necessary, since the most right and most left factors involved in the embedding formula have full rank if and only if $1_n - S^T S$ has.

Note that for calculating the $J$-orthogonal transformation $T$ directly by use of the embedding formula, knowledge about the matrices $G$, $D$, $M$ and $N$ is required. In contrast to this, we do not assume to have any prior knowledge concerning the $J$-signature of the graph of $S$ or the inertia of $1_n - S^T S$. Additionally, no solution to the factorization problem, implicit in the lossless embedding problems, is assumed to be known. However, if we succeed in computing $T$, then a solution to the factorization problem (3) including the inertia for the $J$-Gramian is provided for free.

Finally, it should be pointed out that the solution to the embedding problem is not unique, since we allow the matrices $M$ and $N$ to be any solutions to the stated factorization problems. Having found one solution, the whole set of lossless embeddings for a given $n$-port may be parameterized in terms of a subgroup of $O(n,n)$. The elements of this subgroup leave a given load $L$ invariant or, in other words, the behaviour $L = \text{graph}(L)$ is an invariant subspace. This subgroup will be called the group of allpass transformations with respect to $L$ and is defined as

$$O_L(n, n) := \{ T \in O(n,n) \mid T|_L : L \rightarrow L \},$$

where $T|_L$ denotes the restriction of $T$ to the subspace $L$. Obviously, considering a given basis matrix $X$ for a behaviour $L = \text{im}(X)$, an allpass transformation $T_L$ must satisfy the equation

$$T_L X = XM_L, \quad T_L \in O_L(n,n), \quad M_L \in GL(n).$$

Hence, the effect of an allpass transformation $T_L$ applied to a particular basis matrix $X$ is a simple change of basis, represented by the matrix $M_L$, i.e. the outcome of the linear combination of the rows of $X$, performed by $T_L$ is identical to the result of combining the columns of $X$ according to $M_L$. It seems far from being trivial, to construct the allpass transformations for a given behaviour algebraically. We will come back to this problem in the next section.

4. Cascade Decomposition

In the course of this section, we will introduce the basic transformation tools for a computational scheme, which allows for the recursive computation of the $J$-orthogonality $T$. For a rigorous geometric treatment of the action of the isometry $T$ two subspaces of $\mathbb{R}^{2n}$ can be associated to every $T$ [6].
Definition 10 The fix of the isometry $T \in O(n, n)$ is defined as the subspace of all vectors from $\mathbb{R}^{2n}$ that are fixed under the action of $T$, i.e.

$$\text{fix}(T) := \{ x \in \mathbb{R}^{2n} \mid x - Tx = 0 \} = \ker (1_{2n} - T).$$

The $J$-orthogonal complement of the fix in $\mathbb{R}^{2n}$ is called the path of $T$

$$\text{path}(T) := \{ x - Tx \mid x \in \mathbb{R}^{2n} \} = \text{im} (1_{2n} - T).$$

The path describes the motion in $\mathbb{R}^{2n}$ induced by $T$, and hence characterizes $T$ completely. These geometrical conceptions are intuitively clear. They immediately induce a compact characterization of the elements in the subgroup of allpass transformations with respect to a given behaviour.

Lemma 3 Let $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear transformation and $\mathcal{U} \subset \mathbb{R}^{2n}$ a linear subspace. $\mathcal{U}$ is an invariant subspace of $T$ if and only if the path of the restriction $T|_{\mathcal{U}}$ belongs to $\mathcal{U}$, i.e. $T|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ iff $\text{path}(T|_{\mathcal{U}}) \subset \mathcal{U}$.

Proof: To prove necessity, we assume $\mathcal{U}$ to be an invariant subspace of $T$. Then each $x \in \mathcal{U}$ trivially satisfies $x - Tx = (1_{2n} - T)x \in \mathcal{U}$. In view of sufficiency we assume $\text{path}(T) \subset \mathcal{U}$. Consequently for all $x \in \mathcal{U}$: $(1_{2n} - T)x = x - Tx \in \mathcal{U}$ and hence $Tx \in \mathcal{U}$.

Lemma 3 allows for an easy construction of allpass transformations especially when their path has dimension one. Moreover, the latter restriction is exactly what is already needed in order to accomplish a recursive computational scheme – to break up the $J$-orthogonal transformation into a product of 'simple' isometries [6].

Definition 11 An isometry $R \in O(n, n)$ is called simple if

$$\dim \text{path}(R) = \text{rank} (1_{2n} - R) = 1.$$ 

The set of simple isometries can be represented easily as the reflections

$$R = 1_{2n} - 2u(u^T J u)^{-1} u^T J, \quad \text{im}(u) = \text{path}(R), \quad |u|^2 \neq 0$$

such that

$$R : x \mapsto y = Rx, \quad |x|^2 = |y|^2, \quad u = x \pm y$$

holds. There exists a well known result of Scherk [5], [6] which relates the dimension of the path of $T$ to the number of simple isometries needed to generate any $T \in O(n, n)$ as $T = \prod_i R_i$. Using simple isometries in the cascade decomposition of $T$ results in a class of algorithms, which is sometimes referred to as "hyperbolic Householder" algorithms [2]. Besides the Householder algorithms, there exists the alternative class of algorithms, based on Givens transformations (e.g. for computing the QR-decomposition). A similar class of algorithms is induced by referring to the following definition.
Definition 12 An isometry is called elementary if the path vector \( u \in \mathbb{R}^{2n} \) of a simple isometry contains only two non-zero entries.

Depending on the index position of these two non-zero entries with respect to the indefinite metric tensor \( J \) we get the traditional coordinate dependent distinction between two different types of elementary isometries, which may be parametrized by the Schur parameter \( \varrho \).

1. Hyperbolic reflection

\[
(1 \leq i, j \leq n)
\]

\[
R_{i,n+j} = \frac{1}{\sqrt{1-\varrho^2}} \begin{bmatrix} 1 & -\varrho \\ \varrho & -1 \end{bmatrix}
\]

2. Euclidean reflection

\[
(1 \leq i, j \leq n \text{ or } n+1 \leq i, j \leq 2n)
\]

\[
R_{ij} = \begin{bmatrix} \sqrt{1-\varrho^2} & 0 \\ \varrho & -\sqrt{1-\varrho^2} \end{bmatrix}
\]

Therefore, an elementary isometry in \( O(n, n) \) is given as the appropriate embedding of these \( 2 \times 2 \)-matrices into the \( 2n \times 2n \) identity matrix. The Schur parameter \( \varrho \) must be strictly bounded by 1 in magnitude, since otherwise e.g. the hyperbolic reflection degenerates (\( |\varrho| > 1 \)) or even ceases to exist (\( |\varrho| = 1 \)). Therefore, due to this magnitude constraint, the following cases have to be distinguished with a view towards the elimination of vector entries using hyperbolic reflection operations:

1. \[ \frac{1}{\sqrt{1-\varrho^2}} \begin{bmatrix} 1 & -\varrho \\ \varrho & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{a^2-b^2} \\ 0 \end{bmatrix} \quad \text{only if} \quad |a| > |b|, \quad \text{with} \quad \varrho = b/a \]

2. \[ \frac{1}{\sqrt{1-\varrho^2}} \begin{bmatrix} 1 & -\varrho \\ \varrho & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{b^2-a^2} \end{bmatrix} \quad \text{only if} \quad |a| < |b|, \quad \text{with} \quad \varrho = a/b \]

3. if \( |a| = |b| \), then \( |\varrho| = 1 \) \( \rightarrow \) no elimination is possible

5. Recursive Algorithms

In this section we will discuss the derivation of a recursive Schur-type algorithm for computing the \( J \)-orthogonal transformation \( T \), the representer \( L \) and the factorization result \( M \), when an arbitrary \( n \)-port is given. For this purpose, a sequence of elementary isometries are applied to a pertaining basis matrix. This step aims to finally annihilate \( n \) of its rows in order to accomplish the mapping onto the respective representer matrix. The remaining \( n \) rows constitute the factorization result \( M \). To guarantee a recursive algorithm, the ordering of the elimination steps have to be organized in a way, as to ensure that zeros once generated, will not be destroyed by subsequent operations.

Let us assume that the input-output partitioning of ports is such that the \( n \)-port may be given by its scattering matrix \( S \). For convenience of presentation, it will be sufficient to consider an upper triangular \( 3 \times 3 \) matrix \( S \). Any arbitrarily structured scattering \( \hat{S} \) can be transformed easily to upper triangular form by use of an orthogonal matrix \( Q \), i.e. \( S = Q \hat{S} \), where \( Q \) is the
product of appropriately chosen euclidean elementary isometries. Therefore, the initial step of
the algorithm looks like

\[
\begin{bmatrix}
1_n \\
Q
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= 
\begin{bmatrix}
1_n \\
Q
\end{bmatrix}
\begin{bmatrix}
1_n \\
S
\end{bmatrix}
= 
\begin{bmatrix}
1_n \\
S
\end{bmatrix}.
\]

The following procedure employs a row-wise elimination strategy with basically two types of
actions at each stage \( k \).

**Step \( \alpha \)**: Using the \( k \)-th rows of \( A \) and \( B \) either entry \( a_{kk} \) or \( b_{kk} \) will be annihilated by use of a
hyperbolic elementary isometry. The magnitude constraint on the parameter \( g \) for the hyperbolic
reflection serves to decide, which of these two alternatives will be taken. For example, take
\( k = 1 \) and let assume \( |s_{11}| > 1 \). This results in the following scheme, where the symbolic entries
"\( m \)" and "\( * \)" denote already fixed elements of the matrix \( M \) and arbitrary matrix elements,
respectively.

\[
\begin{bmatrix}
1 \\
s_{11} \\
s_{12} \\
s_{13} \\
s_{21} \\
s_{22} \\
s_{23} \\
s_{33}
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
0 \\
1 \\
1 \\
m \\
m \\
s_{22} \\
s_{23} \\
s_{33}
\end{bmatrix}.
\]

**Step \( \beta \)**: A sequence of euclidean elementary isometries will complete the elimination of row
\( k \) of either \( A \) or \( B \), depending on the outcome of step \( \alpha \). Again, taking up the schematic
example above, the application of an euclidean elementary isometry to rows 1 and 2, followed
by another one to rows 1 and 3, finally gives

\[
\begin{bmatrix}
0 \\
1 \\
1 \\
m \\
m \\
s_{22} \\
s_{23} \\
s_{33}
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
0 \\
* \\
* \\
m \\
m \\
s_{22} \\
s_{23} \\
s_{33}
\end{bmatrix}.
\]

In this manner, one row of zeroes of the representer matrix as well as one row of the invertible
matrix \( M \) have been produced. Hence, this strategy provides the nice feature that after having
completed the \( k \)-th row elimination step, the problem size is reduced by one, that is, only the
remaining \( (n-k) \)-port has to be considered.

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Running this procedure for all \( k = 1, \ldots, n \) will produce the required result

\[
\begin{bmatrix}
0 & 0 & 0 \\
m & m & m \\
m & m & m \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
m & m & m \\
m & m & m \\
m & m & m
\end{bmatrix}
\]

as long as none of the basis vectors at hand of the remaining multiport is isotropic, i.e. that the situation \(|a_{kk}| = |b_{kk}|\) does not occur. In such a case, trying to apply the \(\alpha\)-elimination step above will directly lead to a parameter \(|\varrho| = 1\), which obviously violates the stringent magnitude constraint for the hyperbolic reflection operation. The reason for such singularities to happen may be characterized most easily using the \(J\)-Gramian. The singularities can then be assigned to singular leading principal submatrices of \(I_n - S^TS\). This also contains the special case of singular \(J\)-Gramians, which corresponds to scattering matrices \(S\) having singular values equal to one (boundary case), or in other words, that the radical of \(\text{graph}(S)\) is a non-trivial subspace.

In order to demonstrate the algorithmic treatment of singularities, consider the schematic example below, where the symbol "•" in the current basis vector denotes matrix elements of identical magnitude. Clearly, the \(\alpha\)-elimination step fails. The question how to proceed with the elimination cannot be decided upon the current basis vector alone. Information about the complete behaviour of the remaining multiport must be taken into account. Thus, if this happens, a transformation of the basis matrix from the right with the inverse of an upper triangular matrix will be performed. This upper triangular matrix is composed of the first \(k - 1\) \(m\)-rows and the last \(n - k + 1\) rows of \(A\). In the schematic example, this step looks like

\[
\begin{bmatrix}
0 & 0 & 0 \\
\ast & \ast & \ast \\
m & m & m \\
\ast & \ast & \ast
\end{bmatrix}
\rightarrow
\begin{bmatrix}
m & m & m \\
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}^{-1}
\begin{bmatrix}
0 & 0 & 0 \\
1 & 1 & 1 \\
\pm 1 & \delta_{23} & \delta_{33}
\end{bmatrix}
\]

The transformation from the right with a non-singular matrix simply constitutes a change of basis for the column-space of the basis matrix and, therefore, does not affect the behaviour. This can be considered equivalent to a return to a scattering matrix description \(\hat{S}\) for the remaining and yet unprocessed \((n-k)\)-port. Note, this step of considering properties of the entire space by changing the basis suitably cannot be inferred from analyzing the cascade model. A change of basis is merely a matter of mathematical representation and does not affect the actual multiport itself.

At this stage, we are not ready for continuing the regular elimination process, since the problem with the unit-modulus parameter \(\varrho\) obviously has not yet been resolved by the indicated change.
of basis. To proceed, the following cases for $k + 1 \leq j \leq n$ have to be distinguished

1.) $\delta_{kj} = 0$  2.) $\exists j : \delta_{kj} \neq 0$

In the first case, the $k$-th elimination step can be skipped. The rationale for this is that the respective basis vector for $graph(\hat{S})$ is isotropic, orthogonal and $J$-orthogonal to all other basis vectors. Hence, there is a non-trivial radical of $graph(\hat{S})$ and with a view towards parameterization, this implies an evident loss of parameters. In the second case, the respective basis vector is isotropic, but not $J$-orthogonal to the remaining basis vectors for $graph(\hat{S})$, which implies that it does not lie in the radical of the respective space. Therefore, a non-isotropic basis vector for replacing the isotropic one can be constructed by a further right transformation. For this purpose a linear combination of the $k$-th and the $j$-th column of the basis matrix is computed using an orthogonal transformation, where an elementary geometric reasoning shows that the outcome of this is not an isotropic vector again. To clarify this step, look at the schematic example with the assumption $\delta_{23} = 0$ and with values for $c$ and $s$ such that $\delta_{kk}$ is zeroed out. Clearly, by doing so, some fill-ins will be inserted inevitably below the main diagonals. These may be removed by a sequence of additional euclidean elementary isometries to restore the upper triangular shape of $A$ and $B$.

\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 1 \\
1 & \delta_{23} \\
\pm 1 & \delta_{33}
\end{pmatrix}
\begin{pmatrix}
1 & c & s \\
1 & 0 & s \\
\delta_{23} & \delta_{33} & -c
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
* & * & *(c) \\
* & * & \quad 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
* & * & *(s) \\
* & * & \quad 1
\end{pmatrix}
\rightarrow
\text{regular recursion}
\]

After having completed this intermediate change of basis the regular recursion may continue until possibly another singularity is encountered. The $J$-orthogonal transformation $T$ is given in factored form as the product of all elementary isometries used during the elimination process. Furthermore, the matrix $M$ results as the product of all right transformations. Note, $M$ is not unique as soon as singularities have been encountered, due to either zeros on the diagonal of the signature matrix $S$ or non-upper triangular (orthogonal) changes of basis.

The row-wise elimination strategy provides an algorithm, which uses the minimum number of $n$ hyperbolic reflection operations. The three cases to be distinguished for each of these $n$ hyperbolic elimination steps correspond nicely with the three possible equivalence classes for each singular value $\sigma_i$ of $S$, i.e. $\sigma_i < 1, \sigma_i = 1, \sigma_i > 1$. 

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6. Numerical Example

Consider the upper triangular scattering matrix $S$ given, such that the algorithm is initialized with the basis matrix

$$
\begin{bmatrix}
A \\
B \\
S
\end{bmatrix} = \begin{bmatrix}
1.0000 & 0 & 0 \\
0 & 1.0000 & 0 \\
0 & 0 & 1.0000 \\
1.0000 & 0.5879 & 0.3063 \\
0 & 0.8089 & 0.0658 \\
0 & 0 & 0.9497
\end{bmatrix}
$$

and its associated $J$-Gramian

$$
\mathbf{1}_3 - S^T S = \begin{bmatrix}
0 & -0.5879 & -0.3063 \\
-0.5879 & 0 & -0.2333 \\
-0.3063 & -0.2333 & 0
\end{bmatrix}.
$$

It is obvious that the first basis vector is isotropic and just as well that the first principal leading submatrix of the $J$-Gramian is singular. Since the basis matrix already has the form $(\mathbf{1}_3 S^T)^T$, the change of basis involving the upper triangular matrix can be skipped. The process of changing the basis by an orthogonal matrix followed by the restoration of the upper triangular form is displayed by the snapshots of the basis matrix below

$$
\begin{bmatrix}
-0.5068 & 0.8621 & 0 \\
0.8621 & 0.5068 & 0 \\
0 & 0 & 1.0000 \\
0 & 0 & 1.1600 \\
0.6973 & 0.4100 & 0.0658 \\
0 & 0 & 0.9497
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1.0000 & 0 & 0 \\
0 & 1.0000 & 0 \\
0 & 0 & 1.0000 \\
0.6973 & 0.4100 & 0.0658 \\
0 & 1.1600 & 0.3063 \\
0 & 0 & 0.9497
\end{bmatrix}.
$$

Returning to the regular recursion we arrive at

$$
\begin{bmatrix}
0.7167 & -0.3989 & -0.0640 \\
0 & 1.0000 & 0 \\
0 & 0 & 1.0000 \\
0 & 0 & 0 \\
0.12934 & 0.3153 & 0 \\
0 & 0 & 0.9511
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1.0000 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1.0000 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
G \\
D
\end{bmatrix},
$$

where the factorization result and the signature matrix result as

$$
M = \begin{bmatrix}
-0.7071 & 0.4157 & -0.0640 \\
0.7071 & 0.4157 & 0.4972 \\
0 & 0 & 0.4931
\end{bmatrix},
\quad S = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$
7. Conclusion

We have presented a clear-cut geometric framework for the solution of various ‘singular’ problems encountered when computing lossless ($J$-orthogonal) transformations of linear systems in the context of LIS-modeling. These problems are related to the occurrence of isotropic vectors during a Schur-type recursion and hence, correspond to the vanishing or rank deficient invariant of the transformation. An appropriate extension of the classical (passive) LIS-model is found to be ideally suited to understand and overcome such difficulties.

The inherent degree of freedom in choosing the basis for a subspace turns out to be the essential step in devising new algorithms, which are able to overcome the problems mentioned. Although we have picked out one specific strategy for determining the change of basis, it should be noted that there exists quite a variety of possibilities in performing the detailed sequence of operations to find a suitable new basis. This gives rise to a whole family of different algorithms, which will be expected to exhibit different numerical properties. Concerning questions about the sensitivity of subspaces and their $J$-signature with respect to noisy data or finite precision computations, little seems to be known, except for some basic results concerning the numerical behaviour of hyperbolic transformations. However, delineation of well-posed algorithms or ill-conditioned problems related to isometries in $J$-spaces, is an open research problem.

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References


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