Biographical report on

John Edensor Littlewood

(9 June 1885–6 September 1977)

by J. Korevaar.

John Edensor Littlewood was the most powerful British analyst since Newton. He is known for his profound contributions to concrete analysis and analytic number theory, and for a unique collaboration with an equally outstanding mathematician, G. H. Hardy. Their accomplishments brought England a leading role in analysis during the period 1910–40. For most of his life Littlewood was associated with Trinity College, Cambridge. He was a foreign member of our Academy since 1950.

Littlewood was born in Rochester (Kent), England. There were early Littlewoods among the archers that played a big role in the battle of Agincourt (1415). Littlewood's grandfather William Edensor Littlewood was a clergyman, his father Edward Thornton Littlewood a schoolmaster. His mother Sylvia Maud Ackland was a doctor's daughter; his brother Martin Wentworth Littlewood also became a medical man (another brother drowned as a child). Like his grandfather and father, Littlewood studied in Cambridge where he excelled: he was bracketed as Senior Wrangler (with J. Mercer) in the Mathematical Tripos examination, part I (his father had been ninth wrangler, his grandfather 35-th). He took part II of the Tripos, which dealt with genuine mathematics, in his third year and did so well that he received the B.A. degree a year early, upon special action by the University Senate. (As was usual in his time, Littlewood had arrived at the university with a very specialized secondary education. He had also had the privilege to have a real mathematician, F. S. Macauley, as a teacher.)

In 1906 he started research under E. W. Barnes (who later became bishop of Birmingham) on the subject of entire (analytic) functions. His own comment in retrospect: "I luckily struck oil at once" (quotation from his very interesting booklet A mathematician's miscellany, Methuen & Co., London, 1953). He had some difficulty getting this M. A. work accepted for publication by the London Mathematical Society, which would subsequently publish nearly half of his substantial output. Incidentally, in the next few years Littlewood would obtain more important results for entire functions, in particular on the question how small such functions can be on certain circles about the origin. His method to estimate the so-called minimum modulus has helped other mathematicians, notably Wiman and Valiron, to obtain striking results.

With the M. A. degree, Littlewood became a lecturer at Manchester. After three years he returned to Cambridge (1910), where he was Trinity college lecturer, Cayley university lecturer and then the first Rouse Ball professor (1928–50). As a Life fellow, he remained at Trinity after his retirement.

Upon his initial success with research, Barnes had proposed that Littlewood try to prove Riemann's hypothesis about the complex zeros of the so-called zeta function which plays an important role in number theory. According to this famous hypothesis, unproven (nor disproved) to this day, more than a hundred years after it was first introduced, all non-real zeros of the zeta function would lie on one line, the vertical line $x = \frac{1}{2}$. (It is known that they all lie between the vertical lines $x = 0$ and $x = 1$.) Although Littlewood was not able to carry out this heroic task, the assignment has led to some of his most important work. He was the first to explore the implications of the correctness of Riemann's hypothesis, in particular for the distribution of the prime numbers. Incidentally, the fact that the rather inexperienced Littlewood was given the Riemann hypothesis as a research problem says a good deal about the state of pure mathematics in Britain at that time: one had simply no idea how difficult the problem was. The British mathematicians were isolated, and far behind the developments in Germany and France. However, in the area of concrete analysis and number theory, the work of Hardy and Littlewood would soon transform the British position into a leading one.

Littlewood's first major achievement was his proof of a so-called Tauberian theorem for infinite series. In such theorems one considers those series which are summable by a certain method; the assertion is that under a suitable condition on the terms (the "Tauberian condition"), the series must actually be convergent. Littlewood considered the important case of Abel-summable series, and proved that such series must converge when the n-th term is bounded by a constant divided by n (The converse of Abel's theorem on power series. Proc. London Math. Soc. (2) 9 (1911) 434–448).

This Tauberian theorem which Hardy had conjectured led to the collaboration between Hardy and Littlewood that would continue until the former's death (1947). The collaboration became the most fruitful in the history of mathematics: it resulted in nearly a hundred joint publications, many of them of major importance. A considerable number of the papers dealt with Tauberian theorems, an area which Hardy and Littlewood developed and dominated until 1930. Then the leadership passed to Norbert Wiener in the United States, who introduced a more general viewpoint and introduced the powerful method of Fourier transforms.

There has been much speculation about the way in which the collaboration between two such different people actually worked; Hardy and Littlewood themselves have never supplied much information on the point. Communication between them was mostly in writing, so that it made little difference that Littlewood spent the period 1914–18 doing ballistics in the Royal Artillery, and that Hardy was away from Cambridge 1919–31, when he held a chair at Oxford. It appears likely that Hardy frequently posed the problems and proposed principal directions of research, and that Littlewood mounted the first attack to crack a specific problem. Once the major difficulties had been overcome, Hardy would fill in details, polish the results and prepare them for publication, something Littlewood had little taste for.

Littlewood's most impressive result was perhaps the one of 1914 on the sign changes of the difference $\pi(x) - \mathrm{li}(x)$. Here $\pi(x)$ denotes the number of primes not exceeding x. The comparison function $\mathrm{li}(x)$ is the integral logarithm, that is, the (principal value) integral of 1 divided by the natural logarithm of t between the limits 0 and x. In first approximation, $\mathrm{li}(x)$ is equal to x divided by $\ln x$; the famous prime number theorem (proved just before 1900) shows that li(x) is a good analytic approximation to $\pi(x)$. Now all tables and calculations of prime numbers ever made showed that $\pi(x)$ is less than li(x) over the full range covered. Moreover, there was an approximate formula for $\pi(x)$ which strongly suggested that this inequality should be valid for all x. In the light of these facts, it came as a big surprise when Littewood proved that the difference $\pi(x)$ – li(x) must change sign infinitely often when x goes to infinity.

He himself calculated no explicit upper bound for the place of the first sign change; this was done later by his former student S. Skewes. After an initial publication on the subject in 1933 in which he assumed the truth of the Riemann hypothesis, Skewes proved in 1955 that 10 to the power 10 to the power 10 to the power 10^3 is such an upper bound. Skewes' upper bound has since been reduced to a number of „only" 1166 digits (R. S. Lehman, On the difference $\pi(x)$ – li(x). Acta Arithm. 11 (1966) 397–410). It was perhaps typical for Littlewood that he only published an outline of the method which led to the deep sign-change theorem; the details appeared later as part of a joint paper with Hardy (Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. Acta Math. 41 (1918) 119-196).

Other remarkable results of Littlewood concern schlicht functions, subharmonic functions and inequalities. Mathematical readers might enjoy the following unexpected theorem. For every bilinear form in infinitely many variables for which the rectangular partial sums remain uniformly bounded when the variables remain bounded by 1 in absolute value, the double series formed by the absolute values of the coefficients, raised to the power 4/3, must be convergent, and 4/3 is the smallest exponent that works! (On bounded bilinear forms in an infinite number of variables. Quart. Journal (Oxford) 1 (1930) 164-174.) The collaboration with Hardy also extended over a large variety of areas. Besides the subjects mentioned before, there was – in analysis – a great deal of joint work on Fourier series, conjugate functions and maximal functions; the latter play a big role in modern analysis. In number theory there was important work on the distribution of prime numbers, diophantine approximation and additive problems, notably Waring's problem.

The latter deals with representations of natural numbers as sums of squares, third powers, or more generally, k-th powers. For example, every natural number can be written as the sum of four squares of nonnegative integers. The famous German mathematician Hilbert had demonstrated that for every k, there must be a number s (depending on k), such that all natural numbers can be represented as the sum of s k-th powers, but his method provided no information on the size of s. Hardy and Littlewood gave explicit estimates for s with the aid of an analytic method, now known as their circle method. It had its origin in work of Hardy, and joint work of Hardy and the brilliant Indian mathematician S. Ramanujan, who died at an early age. The circle method was developed by Hardy and Littlewood in a famous series of articles entitled Some problems of 'Partitio Numerorum'. These appeared in the period 1920-28, spread over a number of different journals. (The last article, part VIII, with subtitle The number $\Gamma(k)$ in Waring's problem, appeared in Proc. London Math. Soc. (2) 28 (1928) 518-542.)

One of the most successful collaborations included a third prominent analyst, G. Pólya. Together they wrote what is still the standard work on inequalities (Hardy, Littlewood and Pólya, Inequalities. University Press, Cambridge, 1934). By himself, Littlewood also wrote a few books, among them the rather original Lectures on the theory of functions, Oxford University Press, London, 1944. However, his books were not as outstanding as the renowned works of Hardy.

Littlewood has also collaborated with various other mathematicians. One was his very gifted student R. E. A. C. Paley, who died in a skiing accident at the age of 26. Together they introduced in Fourier analysis what is now known as the Littlewood-Paley function. (Although this notion has recently become very important, Paley is best known because of the Paley-Wiener theorem involving Fourier transforms in the complex domain, and because of the book on such Fourier transforms, popularly known as "Paley-Wiener".) Jointly with A. C. Offord, Littlewood wrote at length about the distribution of the zeros of a "random" entire function.

Much of Littlewood's later work has been of a more applied character. One thinks here first of his pioneering articles, jointly with (Dame) Mary L. Cartwright, on nonlinear differential equations, including Van der Pol's equation from the theory of vacuum tubes. In the course of these investigations they obtained important new fixed-point theorems (fixed-points of a transformation are points that are left in place). Such abstract "topological" theorems are used to prove the existence of periodic solutions of differential equations (Some fixed-point theorems. Annals of Math. (2) 54 (1951) 1–37). Other applied subjects that Littlewood dealt with include ballistics, n-body problems, the satellites of Jupiter, the Lorentz pendulum, general adiabatic invariants.

Like Hardy, Littlewood was a most extraordinary person of rare intellectual power. Both liked physical exercise: cricket and tennis were Hardy's sports; with Littlewood, it was gymnastics, cricket, hiking and later skiing and rock climbing. Both remained single, but there were very great differences. Where Hardy was aristocratic (although not by birth), tending to the ascetic, eccentric, flamboyantly intellectual, radical, anti-religious, pacifist, Littlewood was a much more normal man, earthy, without the need to impress an audience. The former liked to attend professional meetings and gladly served in a number of official mathematical functions, the latter avoided them. Both gave lively, stimulating lectures; Hardy's were highly polished, Littlewood's rather informal. After dinner in Trinity Hall, Hardy would typically retire to his room in order to write, whereas Littlewood enjoyed his nightly after-dinner drink in the Combination Room; his conversation was very entertaining. Hardy did not care for music, Littlewood was fond of it and liked to play the piano.

Littlewood had many distinguished research students, including (besides those mentioned earlier) A. O. L. Atkin, S. Chowla, H. Davenport, F. J. Dyson, T. M. Flett, A. E. Ingham, M. J. Lighthill, D. C. Spencer, H. P. F. Swinnerton-Dyer. For his „family", that is, students, former students and close collaborators, Littlewood maintained an outstanding collection of (mostly difficult) research problems. It was ultimately published in the United States (Some problems in real and complex analysis, Heath & Co., Lexington, 1968).

He has been honored both in Britain and outside. The London Mathematical

Society and the Royal Society awarded him medals and prizes; he received honorary doctorates from the universities of Liverpool, St. Andrews and Cambridge. He was elected to foreign membership in the Swedish and Danish Academies of Science as well as our own, and to corresponding membership of the Academies of Science in Göttingen and Paris. After retiring from his professorship at age 65, he made several trips to the United States where he was made welcome as the grand old man of analysis.

He remained active until a very high age. A few years ago he still produced a new proof of the prime number theorem. True to his style, he used the mightiest tools to bulldoze his way through in a little more than two pages (The quickest proof of the prime number theorem. Acta Arithm. 18 (1971) 83–86).

There can be no doubt that the work of Littlewood and Hardy-Littlewood is of lasting importance. Their impact is now visible throughout the world, especially in the United States and the Soviet Union where, since the 1930's, major developments in analysis have taken place. In The Netherlands, with its strength in analytic number theory, the Hardy-Littlewood influence has been felt very early.

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