THEORY

SOME ANALOGUES OF THE SHEFFER STROKE FUNCTION IN n-VALUED LOGIC

BY

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The interpretation of the ordinary (two-valued) propositional logic in terms of a truth-table system with values "true" and "false" or, more abstractly, the numbers "1" and "2" has become customary. This has been useful in giving an algorithm for the concept "analytic", in giving an adequacy criterion for the definability of one function by another and it has made possible the proof of adequacy of a list of primitive terms for the definition of all truth-functions (functional completeness). If we generalize the concept of truth-function so as to allow for systems of functions of 3, 4, etc. values (preserving the "extensionality" requirement on functions examined) we obtain systems of functions of more than 2 values analogous to the truth-table interpretation of the usual propositional calculus. The problem of functional completeness (the term is due to Turquette) arises in each of the resulting systems. Strictly speaking, this problem is not closely connected with problems of deducibility but is rather a combinatorial question. It will be the purpose of this paper to examine the problem of functional completeness of functions in n-valued logic where by n-valued logic we mean that system of functions such that each function of the system determines, by substitution of an arbitrary numeral a for the symbol n in the definition of the n-valued function, a function in the truth table system of a values. A function is functionally complete in n-valued logic if for any natural number a, the substitution of a for n in the definition of the function will yield a functionally complete function for the system of truth tables (of the type described above) with a values.

In the two-valued propositional logic, the classical work was done with the use of the operations ¬, &, V, and → of which it was early discovered that ¬ and any of the other three will suffice to express anything desired. Formal proof of this has been provided by Post 1). Sheffer showed that the stroke function can define the above mentioned

functions and hence is functionally complete 2). In an extension to the \( n \)-valued case Post has shown that two functions, \( s(p) \), an analogue of negation, and \( pVq \), an analogue of disjunction, will suffice 3). Webb has proved the existence of a single function which will suffice 4). The first part of the proof of theorem 1 is equivalent to Post's proof although it differs in details as to method. In the following, we will use the expression "\( n \)-valued Sheffer function" to mean a two-place function which generate all the truth functions in the Logic of \( n \) truth-values 5).

**Theorem 1.** \( f_{1,1,1}(p, q) \) is an \( n \)-valued Sheffer function.

The following functions (and families of functions) will be used in the proof of the theorem:

<table>
<thead>
<tr>
<th>Function</th>
<th>Truth-value properties</th>
</tr>
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<tbody>
<tr>
<td>1. ( N_{m,x,y}(p, q) )</td>
<td>Takes the value ( m ) when ( p ) takes the value ( x ) and ( q, y ). Otherwise takes the value ( n ).</td>
</tr>
<tr>
<td>2. ( pVq )</td>
<td>Takes the minimum of the value of ( p ) and the value of ( q ).</td>
</tr>
<tr>
<td>3. ( \sum_{i=n}^{a+m} f_i(p) )</td>
<td>Takes the minimum value of ( f_a(p), f_{a+1}(p), \ldots f_{a+m}(p) ).</td>
</tr>
<tr>
<td>4. ( D_{m,i}(p) )</td>
<td>Takes the value ( m ) if ( p ) takes the value ( i ), otherwise takes ( n ).</td>
</tr>
<tr>
<td>5. ( p &amp; q )</td>
<td>Takes the maximum of the value of ( p ) and the value of ( q ).</td>
</tr>
<tr>
<td>6. ( \overline{p} )</td>
<td>Takes as value ( n + 1 ) minus the value of ( p ).</td>
</tr>
<tr>
<td>7. ( T_m(p) )</td>
<td>Takes the value ( m ) everywhere 8).</td>
</tr>
</tbody>
</table>

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2) H. M. Sheffer, "A Set of Five Independent Postulates for Boolean Algebras, with Application to Logical Constants", *Trans. Am. Math. Soc.* 14, 487–488 (1913). Properly speaking, Sheffer did not show that the stroke function is functionally complete, but only that it can define negation and disjunction. Since Post subsequently showed the latter to be functionally complete, the result follows from Sheffer's article.


5) In the remainder of this paper when a number is stated it is to be understood unless otherwise asserted as referring to that number congruent to the stated number modulo \( n \) which is greater than zero and less or equal, to \( n \). When truth-tables are refered to the following arrangement is presupposed (for one-place functions): The \( i \)th line of the table is that line where the argument takes the value \( i \), (for two-place functions) if the value of the first argument is \( x \) and of the second is \( y \) then the corresponding line of the truth-table is the \( i \)th line when \( i = (x - 1) n + y \).

6) The idea of this function is a generalisation of the \( T \)-function of Slupecki which together with the negation and implication of Lucasiewicz's system form a functionally complete set. (Cf. note 6), “Die Logik und das Grundlagenproblem” in Gonseth, *Les Entretiens de Zurich*, p. 97.
8. \( s(p) \) Takes as value the value of \( p \) plus one, 1 if \( p \) takes \( n \).

9. \( s^m(p) \) The \( m \)-fold application of \( s(p) \) — thus takes as value the value of \( p \) plus \( m \).

10. \( f_{1,1,1}(p, q) \) Takes as value 1 plus the minimum of the value of \( p \) and the value of \( q \), takes 1 if the value of \( p = \) the value of \( q = n \).

The family \( N_{m,z,y}(p, q) \), \( m = 1, 2, \ldots n - 1; \ x = 1, 2, \ldots n; \ y = 1, 2, \ldots n \) together with the function \( V \) is functionally complete since any function (except one) can be represented by a disjunction of members of the family, where \( N_{m,z,y}(p, q) \) takes the value \( m \) when \( p \) takes \( x \) and \( q \) takes \( y \); the function which takes \( n \) everywhere is the only exception and this is equivalent to

\[
N_{1,1,2}(N_{1,1,1}(p, q), N_{1,1,1}(p, q))
\]

\( N_{m,z,y}(p, q) \) for \( m > 1 \) can be defined in terms of \( N_{1,z,y}(p, q), \ = V \) and \( T_m(p) \) \([1 \leq m \leq n]\) as follows

\[
N_{m,z,y}(p, q) = T_{n-m+1}(p) V N_{1,z,y}(p, q)
\]

\( N_{1,z,y}(p, q) \) can in turn be defined in terms of \( D_{1,z}(p) \) \([1 \leq i \leq n]\) and \&:

\[
N_{1,z,y}(p, q) = D_{1,z}(p) \ & D_{1,y}(q)
\]

\& is to be defined by \( - \) and \( \vee \) by De Morgan’s law:

\[
p \ & \ q = \overline{p \vee q}
\]

Accordingly, the set \( - \), \( \vee \), with the families \( D_{1,z}(p) \) \([1 \leq i \leq n]\) and \( T_m(p) \) \([1 \leq m \leq n]\) is functionally complete.

Furthermore if we define the sum of a function \( \sum_{i=a}^{a+k} f_i(p) \) as:

\[
\begin{align*}
\sum_{i=a}^{a+k} f_i(p) &= f_a(p) \\
\sum_{i=a}^{a+k} f_i(p) &= f_{a+1+k}(p) \vee \sum_{i=a}^{a+k} f_i(p)
\end{align*}
\]

we can define \( - \) in terms of \( V \) and \( D_{m,i} \) \([1 \leq m \leq n, 1 \leq i \leq n]\)

\[
\overline{p} = \sum_{i=1}^{n} D_{n+1-i,i}(p).
\]

With the aid of the multiple application of \( s(p) \), \( s^*(p) \) (this can also be defined recursively, as follows \( s^0(p) = p, s^{k+1}(p) = s(s^k(p)) \) ), we can define \( D_{m,i}(p) \) \([m > 1]\) in terms of \( D_{1,i}(p), \ = V \), and \( s \):

\[
D_{m,i}(p) = s^{m-1}(T_{n-m+1}(p) V D_{1,i}(p))
\]
\( D_{1,1}(p) \) and \( T_m(p) \) are likewise defineable in terms of \( V \), and \( s \):

\[
D_{1,1}(p) = s^{n-1} \sum_{i=1}^{n-1} s^i(p)
\]

(for \( i \neq 1 \)) \( D_{1,i}(p) = s^{n-1}(\sum_{j=1}^{n-i} s^i(p) V \sum_{j=n-i+2}^{n} s^j(p)) \)

\[
T_1(p) = \sum_{i=1}^{n} s^i(p)
\]

(for \( m \neq 1 \))

\[
T_m(p) = s^{m-1} T_1(p).
\]

Accordingly, as shown by Post \( s(p) \) and \( V \) are functionally complete. Both of these can be defined in terms \( f_{1,1,1}(p, q) \)

\[
s(p) = f_{1,1,1}(p, p)
\]

\[
pVq = s^{n-1} f_{1,1,1}(p, q)
\]

We define \( f'_{n,1,1}(p, q) \) as the function which takes 1 if either \( p \) or \( q \) takes \( n \) and otherwise takes the maximum of the value of \( p \) and that of \( q \), plus one.

**Theorem 2.** \( f'_{n,1,1}(p, q) \) is an \( n \)-valued Sheffer function.

This can be shown by the following series of definitions: \( E_{m,1}(p) \) is the function, which takes \( m \) when \( p \) takes 1, otherwise 1).

\[
s(p) = f'_{n,1,1}(p, p)
\]

\[
s^m(p) = \begin{cases} s^0(p) = p \\ s^{k+1}(p) = s(s^k(p)) \end{cases}
\]

\[
p \& q = s^{n-1} f'_{n,1,1}(p, q)
\]

\[
\prod_{i=m}^{m+k} f_i(p) = f_m(p)
\]

\[
\prod_{i=m}^{m+k+1} f_i(p) = f_{m+k+1}(p) \text{ and } \prod_{i=m}^{m+k} f_i(p)
\]

\[
T_n(p) = \prod_{i=1}^{n} s^i(p)
\]

\(* \) As in corollary to Theorem 1, the Lucasiewicz–Śłupecki primitives can be easily shown to be functionally complete, by the following definitions:

\[
s(p) = NCC TpNpNpNCpNp
\]

\[
N_{1,1,1}(p, q) = NCCpNqNCpNq
\]

\[
N_{2,1,1}(p, q) = s(NCTpNN_{1,1,1}(p, q))
\]

\[
A'pq = CNpq
\]

\[
f_{1,1,1}(p, q) = A'A'A'A'N_{2,1,1}(p, q) N_{2,1,1}(s(p), q)
\]

\[
N_{2,1,1}(s^a(p), q) N_{2,1,1}(p, s(q)) N_{2,1,1}(p, s^2(q)) N_{1,1,1}(s(p), s(q))
\]
for \( m \neq n \), \( T_m (p) = s^m T_n (p) \)

\[
E_{n,1} (p) = s \prod_{i=1}^{n-1} s_i^{-1} (p)
\]

for \( m \neq 1 \), \( E_{n,m} (p) = s \left[ \prod_{i=1}^{n-m-1} s_i (p) \quad \& \quad \prod_{i=n-m+1}^{n} s_i (p) \right] \)

\[
E_{k,m} (p) = s^k (T_{n+1-k} (p) \quad \& \quad E_{n,m} (p))
\]

\[
\overline{p} = \prod_{i=1}^{n} E_{n-i+1,1} (p)
\]

\[
pVq = \overline{p} \quad \& \quad \overline{q}
\]

\[
 f_{1,1,1} (p, q) = s (pVq).
\]

On the basis of the isomorphism of \( s (p) \) in \( n \)-valued logic to the “successor” function for congruence classes modulo \( n \), we can easily obtain a considerably larger number of SHEFFER functions.

**Theorem 3.** If \( a \) and \( n \) are relatively prime, \( f_{1,1,a} (p, q) \) and \( f'_{n,1,a} (p, q) \) are \( n \)-valued SHEFFER functions.

\( f_{1,1,a} (p, q) \) is the function which takes as value the minimum of the values of \( p \) and \( q \), plus \( a \) (\( n + a \) being considered as equal to \( a \)). \( f'_{n,1,a} (p, q) \) takes the maximum of the values, plus \( a \).

\( s_a (p) = f_{1,1,a} (p, p) \). By EULER'S theorem \( S_a^{s_a^{-1}} (p) \) takes as value the value of \( p \) plus one. Hence

\[
s (p) = S_a^{s_a^{-1}} (p). \quad \text{Thus} \quad f_{1,1,1} (p, q) = s^{n-a+1} f_{1,1,a} (p, q).
\]

Analogously for \( f'_{n,1,a} (p, q) \).

\( f_{1,1,a} (p, q) \) is the function which takes \( i + a \) if \( p \) or \( q \) takes \( i \) and takes \( i + a + b \) if \( p \) and \( q \) do not take any \( j \) \( [i \leq j < i + b \leq i - 1] \) and \( p \) or \( q \) takes \( i + b \) (where \( n + c \) is taken as equal to \( c \)). \( f'_{1,1,a} (p, q) \) is the function which takes \( i + a \) if \( p \) or \( q \) takes \( i \) and takes \( i + a - b \) if \( p \) and \( q \) do not take any \( j \) \( [i + 1 \leq i - b < j \leq i] \) and \( p \) or \( q \) take \( i - b \) (where \( -c \) is taken as equal to \( n - c \)).

**Theorem 4.** If \( a \) and \( n \) are relatively prime, \( f_{1,1,a} (p, q) \) and \( f'_{i,1,a} (p, q) \) are \( n \)-valued SHEFFER functions.

\( s_a (p) = f_{i,1,a} (p, p) \). By EULER'S theorem \( S_a^{s_a^{-1}} (p) = s (p) \). Then

\[
f_{i,1,a} (s_i^{-1} (p), s_i^{-1} (q)) = f_{i,1,i+a-1} (p, q). \quad s^{n+2-(a+i)} f_{1,1,i+a-1} = f_{1,1,1} (p, q).
\]

Analogously for \( f'_{i,1,a} (p, q) \).

\( f_{1,1,a} (p, q) \) is the function which takes \( a + 1 \) if \( p \) or \( q \) takes \( 1 \); if not and \( p \) or \( q \) takes \( d + 1 \), \( f_{1,1,a} (p, q) \) takes \( a + d + 1 \); if not but either take \( 2d + 1 \), \( f_{1,1,a} (p, q) \) takes \( a + 2d + 1 \), etc. until \( md + 1 \) where \( md \) is the greatest multiple of \( d \) less than \( n \); if neither \( p \) nor \( q \) take any of these but one of them takes 2, \( f_{1,1,a} (p, q) \) takes \( a + 2 \); if not and either takes \( d + 2 \), \( f_{1,1,a} (p, q) \) takes \( a + d + 2 \) etc. until \( Xd + 2 \), where \( n - d < Xd + 2 \leq n \); if neither \( p \) nor \( q \) take any of these, but one of them takes 3.
$f_{1,d,a}(p, q)$ takes $a + 3$ etc. $f_{1,d,a}(p, q)$ is the result of $n + 1 - i$ applications of $s(p)$ to the arguments of $f_{1,d,a}(p, q)$.

If in the definition of $f_{1,d,a}(p, q)$ in place of starting with 1 and adding units of $d$ with $n$ as a limit, we start with $n$ and subtract units of $d$ with 1 as the limit, we obtain the function $f'_{n,d,a}(p, q)$. $f'_{n,d,a}(p, q)$ is the result of $n - i$ applications of $s(p)$ to the arguments of $f'_{n,d,a}(p, q)$.

Theorem 5. If $d$ is not a divisor of $n$ and $a$ is relatively prime to $n$, $f_{1,d,a}(p, q)$ and $f'_{1,d,a}(p, q)$ are $n$-valued Sheffer functions.

$s(p)$ can be defined as in theorem 4. Then $f_{1,d,a}(s^{i-1}(p), s^{i-1}(q)) = f_{1,d,a}(p, q)$. $f_{1,d,o}(p, q) = s^{n-(i+a)} f_{1,d,i+a}(p, q)$.

If $\Phi g_i(p)$ is defined recursively:

$$\Phi g_i(p) = \Phi g_i(p) = g_{m}(p)$$

then $G_{1,d}(p) = \Phi s^i(p)$ takes $d + 1$ on line 1 and 1 elsewhere, and $K(p) = G_{1,d}(G_{1,d}(p))$ which takes 1 on line 1 and $d + 1$ elsewhere.

(a) Case 1. There is an $x$ such that $n = xd - 1$.

$G_1(p) = (s^{n-d} G_{1,d}(p))$ takes 1 on line 1 and $n - d + 1$ elsewhere. Then $G_k(p) = s^{k-1} f_{1,d,o}(G_1(p), T_{n+2-(d+k)}(p))$ takes $k$ on line 1 and $n - d + 1$ elsewhere. Define $H_{k,m}(p)$ as follows:

$$H_{k,1}(p) = G_k(p)$$

Thus $H_{k,m}(p)$ takes $k$ if $p$ takes $m$ and takes $n - d + 1$ otherwise. Any one-place function can be defined by the $H$-functions combined with $f_{1,d,o}(p, q)$ as is obvious, since if $p$ takes $n - d + 1$ and $q$ takes $j$, $f_{1,d,o}(p, q)$ takes $j$ and $f_{1,d,o}$ is symmetrical. Hence also the function $W_1(p)$ which takes 1 on line 1, $d + 1$ on line 2, etc. and $W_2(p)$ the function such that $W_2(p) = G_1(p)$, can be defined.

Then $sW_2 f_{1,d,o}(W_1(p), W_1(q)) = f_{1,1,1}(p, q)$.

(b) Case 2. There are numbers $x$ and $y$ ($1 < y < d$) such that $n = xd - y$. $s^{n+y-(d+1)} K(p)$, takes $n - d + y$ on line 1 and $y$ elsewhere. $H_{n,1}(p) = s^{n-d} f_{1,d,o}(s^{n+y-(d+1)} K(p), T_{n+2-(d+k)}(p))$ takes $n$ on line 1 and $n - d + y$ elsewhere. $G_1(p) = s f_{1,d,o}(H_{n,1}(p), T_{n+2-(d+k)}(p))$ takes 1 on line 1 and $n - d + y$ elsewhere. The remainder of the proof is analogous to case 1 after the definition of $G_1$.

Similarly, the proof for $f'_{1,d,a}(p, q)$ is analogous to the proof for $f_{1,d,a}(p, q)$.

$b_{a,c}(p, q)$ is the function which takes the value $i + a$ if $p$ and $q$ take the value $i$ and otherwise takes the value $c$.

Theorem 6. If $a$ is relatively prime to $n$, $b_{a,c}(p, q)$ is an $n$-valued Sheffer function.
\[ s(p) = b_{1,n}(p, p). \quad T_n(p, q) = b_{1,n}(p, s(p)). \]
\[ N_{1,1,1}(p, q) = b_{1,n}(s^{n-1}b_{1,n}(p, q), T_n(p, q)). \]

Then \( N_{1,1,1}(p, q) \) can be defined as \( N_{1,1,1}(p, q) = N_{1,1,1}(s^{n+1-z}(p), s^{n+1-y}(q)). \)
\( N_{1,1,1} = b_{1,n}(s^{n-2}N_{1,1,1}(p, q), s^{n-1}T_n(p, q)). \) We define the following functions:
\[ A_0(p, q) = b_{1,n}(p, s^{n-1}(q)) \]
\[ A_1(p, q) = b_{1,n}(N_{1,n,n}(p, q), T_n(p, q)) \]
\[ (i \geq 2) \quad A_i(p, q) = A_{i-1}(p, q), s(M_{i-1}(p, q)). \]

where
\[ M_0(p, q) = T_n(p, q) \]
\[ M_{n+1}(p, q) = A_{n+1}(M_{n+1}(p, q), A_{n+1}(N_{1,n,n+1}(p, q), N_{1,n,n+1}(p, q))) \]
\[ L_n(p, q) = A_{n+1}(M_{n+1}(p, q), N_{1,n,n+1}(p, q)). \]

By induction for every \( i, [0 < i \leq n - 1] \) if \( p \) and \( q \) take \( n \) then \( A_i \) takes \( n \) and if \( p \) takes the value \( j \) \([0 < j \leq i]\) and \( q \), \( n \) or if \( p \) takes \( n \) and \( q \), \( j \) then \( A_i(p, q) \) takes the value \( j \). Then the remarks made in theorem 1 with respect to the completeness of the family \( N_{1,1,1}(p, q) \) together with \( V \), hold when \( A_{n-1} \) is substituted for V. Therefore \( b_{1,n}(p, q) \) is a SHEFFER function \(^8\). Then \( b_{i,c}(p, q) \) is a SHEFFER function for any \( c \) since \( s(p) = b_{1,c}(p, p) \) and \( b_{1,n}(p, q) = s^{n-c}b_{1,c}((s^c(p), s^c(q))). \) If \( a \) and \( n \) are relatively prime then \( s_a(p) = b_{a,c}(p, p) \) and by Euler's theorem there is an \( x \) such that \( s(p) = s_x(p). \)

Then \( s^{n+1-a}b_{a,c} = b_{1,c+n+1-a}(p, q) \) which is a member of the family \( b_{1,c}(p, q) \) and hence has been shown to be a SHEFFER function.

The present position of the investigation is such that the above theorems certainly do not exhaust all the SHEFFER functions. On the contrary, the present investigator is aware of many special cases which do not fall under the theorems, SWIFT has recently discovered \(^9\) in the 3-valued case while the above theorems give only 18 \(^9\). In the present state, the writer does not feel in a position even to conjecture as to what may be the necessary conditions (aside from the defining condition) for a function to be a SHEFFER function \(^{10}\).

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\(^8\) The part of the proof of theorem 6 above is due to WEBB, loc. cit.


\(^{10}\) It is to be noted that any function which can define all functions of two-arguments will also define any function of any number of arguments. For \( f_{1,1,1}(p, q) \) this can be done by constructing the appropriate analogues of the \( N \)-functions by taking the logical product of the appropriate \( D \)-functions for the number of arguments desired (e.g. the function which takes \( i \) if \( p, q \) and \( r \) take \( x, y \) respectively, and \( n \) otherwise can be defined as:
\[ D_{i,x}(p) \& D_{i,y}(q) \& D_{i,z}(r). \]

Since all other functions proved to define all two-place functions also define \( f_{1,1,1}(p, q) \), the result follows.
BIBLIOGRAPHY


