A RANK-INARIANT METHOD OF LINEAR AND POLYNOMIAL REGRESSION ANALYSIS

II 1)

BY

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2. CONFIDENCE REGIONS FOR THE PARAMETERS OF LINEAR REGRESSION EQUATIONS IN THREE AND MORE VARIABLES.

The probability set.

2.0. The probability set $I$ underlying the probability statements of this section is the $n(v + 2)$-dimensional Cartesian space $R_{n(v + 2)}$ with coordinates

$$u_1, \ldots, u_m, u_{v1}, \ldots, u_{vm}, v_1, \ldots, v_n, w_1, \ldots, w_n.$$  

Every random variable will be supposed to be defined on this probability set.

In this first place we consider $n(v + 2)$ random variables $u_{\lambda i}, v_i, w_i$ ($\lambda = 1, \ldots, v; i = 1, \ldots, n$). Furthermore we consider $(n + 1)v + 1$ parameters $a_0, a_\lambda, \xi_{\lambda i} (i = 1, \ldots, n; \lambda = 1, \ldots, v)$ and put:

$$\begin{align*}
(5) & \quad \theta_i = a_0 + \sum_{\lambda=1}^v a_\lambda \xi_{\lambda i} \\
(6) & \quad \eta_i = \theta_i + w_i \\
(7) & \quad x_{\lambda i} = \xi_{\lambda i} + u_{\lambda i} \\
(8) & \quad y_i = \eta_i + v_i
\end{align*}$$

So the variables $x_{\lambda i}$ and $y_i$ have a simultaneous distribution on $I$, and are therefore random variables.

We call $\xi_{\lambda i}$ the parameter values of the variable $\xi_{\lambda}$. The equation (5) is the multiple regression equation. The random variables $w_i$ are called “the true deviations from linearity”, while the random variables $u_{\lambda i}$ and $v_i$ are called “the errors of observation” of the values $\xi_{\lambda i}$ and $\eta_i$ respectively.

1) This paper is the second of a series of papers, the first of which appeared in these Proceedings, 53, 386–392 (1950).
Putting
\[ z_i = - \sum_{\lambda=1}^{\nu} a_\lambda u_{i\lambda} + v_i + w_i \]
we have
\[ y_i = a_0 + \sum_{\lambda=1}^{\nu} a_\lambda x_{i\lambda} + z_i, \]
the random variables \( z_i \) being called "the apparent deviations from linearity".

**Confidence regions for** \( a_0, a_1, \ldots, a_{\nu} \).

2.1. In order to give confidence regions for the \((\nu + 1)\) parameters \( a_0, a_1 (\lambda = 1, \ldots, \nu) \) we impose the following **conditions**:

**Condition I:** The \( n (\nu + 2)\)-uples \((u_{i1}, \ldots, u_{i\nu}, v_i, w_i)\) are stochastically independent.

**Condition II:**
1. Each of the errors \( u_{i\lambda} \) vanishes outside a finite interval \(|u_{i\lambda}| \leq g_{i\lambda} \).
2. For each \( i \neq j \) we have \(|\xi_{i\lambda} - \xi_{j\lambda}| > g_{i\lambda} + g_{j\lambda}\).
Furthermore we impose for the incomplete method to be mentioned:

**Condition III:**
\[ P[z_i < z_j] = P[z_i > z_j] = \frac{1}{2} \quad \text{for} \quad i \neq j \]
and for the complete method:

**Condition IIIa:** Each \( z_i \) has the same continuous distribution function.

2.2. Secondly we define the following quantities:
\[ G^{(i)}(\lambda) = y_i - \sum_{\lambda=1}^{\nu} a_\lambda x_{i\lambda} = a_0 + a_{i1} x_{i1} + z_i \quad (\lambda' = 1, \ldots, \nu; \ i = 1, \ldots, n). \]
Furthermore, after arranging the \( n \) observed points \((y_i, x_{i1}, \ldots, x_{i\nu})\) according to increasing values of \( x_{i\nu} \), (which, by condition II, is identical with the arrangement according to increasing values of \( x_{i\nu} \):
\[ x_{i1} < x_{i2} < \ldots < x_{in} \]
we define the quantities
\[ K^{(i)}(ij) = \frac{G^{(i)}(i) - G^{(i)}(j)}{x_{i1} - x_{i1}} = \]
\[ = \frac{y_i - y_j}{x_{i1} - x_{i1}} - \sum_{\lambda=1}^{\nu} a_\lambda \frac{x_{i\lambda} - x_{j\lambda}}{x_{i1} - x_{i1}} = \]
\[ = a_{i1} + \frac{z_i - z_j}{x_{i1} - x_{i1}} \quad (i = 1, \ldots, n-1; \ j = i+1, \ldots, n). \]
For any set of values \(a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_r\) we arrange the quantities \(K^{(l)}(i,j)\) according to increasing magnitude; we define \(K_1^{(l)}\) as the quantity with rank \(i\) in this arrangement:

\[
K_1^{(l)} < K_2^{(l)} < \cdots < K_{\binom{n}{2}}^{(l)}
\]

Finally we define the intervals \(I_{l'}(a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_r)\) as the intervals

\[
\left( K_q^{(l')}, K_{\binom{n}{2}}^{(l')} - q \right)
\]

with \(2q \leq \binom{n}{2}\); \(A_{l'}\) as the union of

\[
I_{l'}(a_1, a_2, a_{l'-1}, a_{l'+1}, \ldots, a_r) \text{ for all } a_1 (\lambda = 1, \ldots, v; \lambda \neq \lambda');
\]

and \(A\) as the union of all \(A_{l'}, (\lambda' = 1, \ldots, v)\).

2.3. We have the following theorem concerning the complete method for three and more variables:

**Theorem 4:** Under conditions I, II and III\(a\) the region \(A\) is a confidence region for the parameters \(a_1, \ldots, a_r\), the level of significance being \(\leq 2v. P[q - 1|n]^{\lambda}\).

**Proof:** If the set of assumed parameters values \(a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_r\) is the “true” set, it follows from the analysis in section 1.3., that \(I_{l'}(a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_r)\) is a confidence interval for \(a_\lambda\), to the level of significance \(2P[q - 1|n]\). Hence it follows that if \((a_1, \ldots, a_r)\) represents the “true” point in the \(a_1, \ldots, a_r\)-space, we have

\[
P[(a_1, \ldots, a_r) \in A_{l'\lambda}] = 1 - 2P[q - 1|n], \quad (\lambda' = 1, \ldots, v),
\]

which proves the theorem.

2.4. If condition III (but not necessarily III\(a\)) is fulfilled, the method mentioned above can be replaced by the following one. We replace the quantities

\[
K^{(l')}(i,j) \quad (\lambda' = 1, \ldots, v; \ i = 1, \ldots, n-1; \ j = i + 1, \ldots, n)
\]

by

\[
K^{(l')}(i, n_1 + i) \quad (\lambda' = 1, \ldots, v; \ i = 1, \ldots, n_1).
\]

The intervals \(I_{l'}^{(l')}(a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_r)\) are now defined as the intervals bounded by the values of \(K^{(l')}(i, n_1 + i)\) with rank \(r_1\) and \((n_1 - r_1 + 1)\) respectively, if they are arranged in ascending order; whereas the definitions of \(A_{l'}^{(l')}\) as the union of all \(I_{l'}^{(l')}\) and of \(A'\) as the union of all \(A_{l'}^{(l')}\).

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2) For the definition of \(P[q - 1|n]\) the reader is referred to section 1.3. (part I of this paper).

3) \(n_1 = \frac{1}{2} n\). Cf. section 1.2.
remain unchanged. The following theorem of the *incomplete method* for three and more variables will now be obvious from the analysis of section 1.1:

**Theorem 5.** Under conditions I, II and III the region $A'$ is a confidence region for the parameters $a_1, \ldots, a_n$, the level of significance being $\leq 2v. I_1(r_1, n_1 - r_1 + 1)$.

2.5. A confidence region for the parameters $a_0, a_1, \ldots, a_n$ can be constructed, if the median of $z_i$ is known, e.g. if the following condition is fulfilled:

*Condition IV:* The median of each $z_i$ is zero.

The method for the construction of this confidence region is analogous to the one given in section 1.2.

*A illustration for the special case $v = 2.*

2.6. The form of the region $A_3$ or $A'_3$ will now be indicated for the case of three variables:

$$y_i = a_0 + a_1 x_{1i} + a_2 x_{2i} + z_i.$$  

Using the incomplete method we find $n_1$ functions of $a_2$:

$$K^{(1)}(i, n_1 + i) = \frac{y_i - y_{n_1 + i}}{x_{1i} - x_{1, n_1 + i}} a_2 \frac{x_{2i} - x_{2, n_1 + i}}{x_{1i} - x_{1, n_1 + i}},$$

which are estimates of $a_1$, given $a_2$. They are represented by straight lines in the $a_1, a_2$-plane. For any value of $a_2$ we can arrange these quantities in ascending order. As long as (under continuous variation of $a_2$) the numbers $i_1$ and $i_2$ for which the statistics $K^{(1)}(i_1, n_1 + i_1)$ and $K^{(1)}(i_2, n_1 + i_2)$

![Fig. 1. $n_1 = 6, r_1 = 2.$](image-url)
have the $r_1$-th and $(n_1-r_1+1)$-th rank according to increasing order (with $r_1$ as defined in section 2.4.) remain constant, the extreme points of the confidence intervals vary along straight lines. If, when passing some value $a_2^*$ of $a_2$ either $i_1$ or $i_2$ changes, the corresponding straight line passes into another one, intersecting the first one in a point with $a_2 = a_2^*$.

So a diagram can be constructed, in which the $n_1$ straight lines are drawn in the $a_1', a_2'$-plane. This gives the stochastic region $A'_1$ depending on the given observations and bounded to the left and to the right by broken lines.

According to Theorem 5 it contains the true point $(a_1, a_2)$ with the probability

$$1 - 2 I_i(r_1, n_1-r_1 + 1).$$

The region $A'_2$, bounded above and below, can be constructed in a similar way; then the observed points must be arranged in ascending order of $x_2$.

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