CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. I.

BY

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1. *Introduction.* [1])

In the following pages formulae will be derived concerning correlations in a particular type of solutions of the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} \tag{1}$$

We suppose the coefficient ν to be small. Before describing the solutions to be considered, a few general properties of eq. (1) will be mentioned.

If v is interpreted as a velocity, an "equation of momentum" can be formed by integrating (1) with respect to y. For a domain at the limits of which v vanishes and $\partial v/\partial y$ is of such magnitude that $\nu\,(\partial v/\partial y)$ can be neglected, we obtain:

$$\frac{d}{dt}\int v\,dy = 0 \tag{2}$$

An "equation of energy" can be formed by multiplying eq. (1) with v and integrating it with respect to y. For any domain at the ends of which v vanishes, we obtain

$$\frac{d}{dt}\int \frac{v^2}{2}\,dy = -\nu\int\left(\frac{\partial v}{\partial y}\right)^2 dy \tag{3}$$

When $v^2/2$ is considered as kinetic energy per unit length, eq. (3) shows a loss of energy through "internal friction", the amount of which per unit of length is determined by $\nu\,(\partial v/\partial y)^2$.

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[1]) This paper is a continuation of that on "The formation of vortex sheets in a simplified type of turbulent motion", these Proceedings **53**, 122 (1950) (Mededeling no. 64). Equation (1) has already been considered to some extent in the papers: "Application of a model system to illustrate some points of the statistical theory of free turbulence", these Proceedings **43**, 8 (1940); and "A mathematical model illustrating the theory of turbulence", Advances in Applied Mechanics **1**, 182 (1948). In those papers a coefficient 2 had been introduced before the term $v\,(dv/dy)$ of the equation; this factor has now been dropped in connection with the considerations of the paper on the formation of vortex sheets, which causes a slight difference in some of the formulae.

Equation (1) is invariant with respect to a shift over an arbitrary distance along the y-axis. We consider a domain of unlimited extent and assume that the initial course of v as function of y is such that mean values can be defined, *e.g.*:

$$\overline{v} = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} v \, dy \quad ; \quad \overline{v^2} = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} v^2 \, dy, \text{ etc.},$$

which for sufficiently large values of the interval $y_2 - y_1$ do not change when this interval is shifted along the y-axis. This property will remain valid throughout the development in the course of time and thus applies to $v(y)$ at any instant. We shall say that the state of our system [meaning the course of $v(y)$ at a given value of t] is *statistically homogeneous.*

The equation of energy (3) will then also apply to the mean values of both members. We write:

$$E = \tfrac{1}{2}\,\overline{v^2} \quad \text{(mean kinetic energy per unit length)}$$

$$\varepsilon = \nu\,\overline{(\partial v/\partial y)^2} \quad \text{(mean dissipation per unit length),}$$

and obtain:

(4) $$\varepsilon = -\,dE/dt$$

Equation (1) is moreover invariant with respect to a simultaneous change of sign of both v and y. When mean values calculated for the initial state of the system possess this property, it again will be preserved during the development of the system. The property implies that $\overline{v} = 0$.

2. *Correlation functions.* — Following the example of all modern authors on hydrodynamic turbulence, we introduce the quantities:

(5) $$\overline{v_1\, v_2} = \overline{v\,(y)\, v\,(y + \eta)} = \overline{v^2}\, f\,(\eta)$$

(6) $$\overline{v_1^2\, v_2} = \overline{v\,(y)^2\, v\,(y + \eta)} = (\overline{v^2})^{3/2}\, k\,(\eta)$$

(mean values to be taken with respect to y, with a fixed value of η, at a given instant of time). As is known, $f(\eta)$ is an even function, while $k(\eta)$ is an odd function, the development of which in the neighbourhood of $\eta = 0$ begins with a term in η^3.

If the development of $f(\eta)$ in the neighbourhood of $\eta = 0$ is written:

(7) $$f\,(\eta) = 1 - \eta^2/2\,\lambda^2 + \ldots$$

where λ is a quantity having the dimensions of a length, we obtain:

(8) $$\overline{(\partial v/\partial y)^2} = -\,\overline{v^2} \cdot f''_{\eta=0} = \overline{v^2}/\lambda^2$$

Hence (4) can be written:

(9) $$\varepsilon = -\,dE/dt = 2\,\nu\, E/\lambda^2$$

When the course of $v(y)$ presents a sufficiently random character, $f(\eta)$ and $k(\eta)$ will become zero for η increasing without limit. We assume that both are integrable and put:

$$L = \int_0^\infty f(\eta)\, d\eta \tag{10}$$

In this way L and λ are two linear dimensions connected with the course of $v(y)$. Both are functions of t. We introduce two dimensionless parameters, of the nature of REYNOLDS' numbers:

$$Re_I = (\overline{v^2})^{1/2} L/\nu \quad ; \quad Re_{II} = (\overline{v^2})^{1/2} \lambda/\nu \tag{11}$$

We suppose that ν is so small that Re_I is a large quantity.

If again we write v_1, v_2 for $v(y)$, $v(y+\eta)$ respectively, we can form the equation:

$$\frac{\partial}{\partial t}(v_1 v_2) = -v_1 v_2 \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial y}\right) + \nu \left(v_2 \frac{\partial^2 v_1}{\partial y^2} + v_1 \frac{\partial^2 v_2}{\partial y^2}\right).$$

Taking mean values from both sides with respect to y, we obtain, by a well known procedure, the fundamental equation:

$$\frac{\partial}{\partial t}\overline{(v_1 v_2)} = \frac{\partial}{\partial \eta}\overline{(v_1^2 v_2)} + 2\nu \frac{\partial^2}{\partial \eta^2}\overline{(v_1 v_2)} \tag{12}$$

If this equation is applied to $\eta = 0$, we come back to (4). On the other hand, since we have assumed that Re_I is large, it is possible that for values of η comparable with L or exceeding L, the second term on the right hand side of (12) can be discarded and:

$$\frac{\partial}{\partial t}\overline{(v_1 v_2)} \cong \frac{\partial}{\partial \eta}\overline{(v_1^2 v_2)} \tag{12a}$$

By integrating eq. (12) with respect to η from 0 to ∞, we find:

$$dJ_0/dt = 0 \tag{13}$$

where

$$J_0 = \int_0^\infty \overline{v_1 v_2}\, d\eta = \overline{v^2}\, L \tag{14}$$

This proves that J_0 is the analogue of LOITSIANSKY's invariant in hydrodynamic turbulence.

3. *Fourier analysis of* $\overline{v_1 v_2}$ *and* $\overline{v_1^2 v_2}$. — Following BATCHELOR we put:

$$\overline{v_1 v_2} = \int_0^\infty dn\, \Gamma(n) \cos n\eta \tag{15}$$

$$\overline{v_1^2 v_2} = \int_0^\infty dn\, \Psi(n) \sin n\eta \tag{16}$$

so that:

$$\pi \Gamma(n) = \int_{-\infty}^{+\infty} \overline{v_1 v_2} \cos n\eta \, d\eta \tag{17a}$$

$$\pi \Psi(n) = \int_{-\infty}^{+\infty} \overline{v_1^2 v_2} \sin n\eta \, d\eta \tag{17b}$$

The function Γ will be even in n and for small n we can write:

$$\Gamma = \Gamma_0 - \tfrac{1}{2} n^2 \Gamma_2 + \dots . \tag{18a}$$

From (14) we obtain $\Gamma_0 = 2 J_0/\pi$.

The function Ψ will be odd in n, so that for small n:

$$\Psi = n \Psi_1 - \tfrac{1}{6} n^3 \Psi_3 + \dots . \tag{18b}$$

The FOURIER transform of the fundamental equation (12) becomes:

$$\frac{\partial \Gamma}{\partial t} = n \Psi - 2 \nu n^2 \Gamma \tag{19}$$

This gives:

$$\left\{ \begin{aligned} d\Gamma_0/dt &= 0 \\ d\Gamma_2/dt &= -2 \Psi_1 + 4 \nu \Gamma_0 \\ d\Gamma_{2m}/dt &= -2 m \Psi_{2m-1} + 4 \nu m (2m - 1) \Gamma_{2m-2} \end{aligned} \right. \tag{20}$$

4. *Particular solution of eq.* (1). — We write: $y^* = (y - y_0)/(t - t_0)^{\frac{1}{2}}$ and assume: $v = V(y^*)/(t - t_0)^{\frac{1}{2}}$. Equation (1) transforms into:

$$\nu V'' - V V' + \tfrac{1}{2} y^* V' + \tfrac{1}{2} V = 0,$$

accents denoting derivatives with respect to y^*. Integration gives:

$$\nu V' - \tfrac{1}{2} V^2 + \tfrac{1}{2} y^* V = 0,$$

the integration constant having been adjusted so that V can vanish at infinity. We now put:

$$V = -2 \nu \, d\,(ln\, u)/dy^*$$

and obtain the following equation for u:

$$\frac{d^2 u}{dy^{*2}} = -\frac{y^*}{2\nu} \frac{du}{dy^*}.$$

This equation can be integrated. The resulting expression for V can be brought into the form:

$$V = \frac{2 \nu A \exp\{(A^2 - y^{*2})/4\nu\}}{4 \nu - A \int_A^{y^*} dy_1 \exp\{(A^2 - y_1^2)/4\nu\}} \tag{21}$$

where A is an integration constant.

The following approximations are valid when A^2/ν is large:

(*a*) when $y^* = A + \delta$, where $\delta \ll A$:

$$V \cong \tfrac{1}{2} A - \tfrac{1}{2} A \tanh (A\delta/4\nu);$$

(*b*) when $0 < y^* < A$ and y^* not too near to one of the endpoints:

$$V \cong y^*;$$

(*c*) when y^* is near zero or is negative:

$$V \cong 2 \sqrt{\frac{\nu}{\pi}} \frac{\exp(-y^{*2}/4\nu)}{1 - \mathrm{Erf}(y^*/2\sqrt{\nu})}.$$

With the aid of these approximations we can obtain a general picture of the course of V as a function of y^*, and thus of the course of v as a function of y and t (see fig. 1). The curve for v approaches to a rectangular triangle

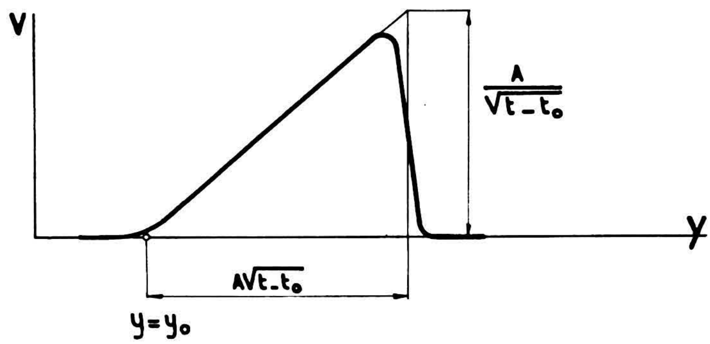

Fig. 1.

with base length $A(t - t_0)^{1/2}$ and height $A/(t - t_0)^{1/2}$. The area is equal to $\frac{1}{2}A^2$ and remains constant (conservation of momentum). The kinetic energy amounts to:

$$\tfrac{1}{6} (A \sqrt{t - t_0}) (A/\sqrt{t - t_0})^2 = \tfrac{1}{6} A^3 (t - t_0)^{-1/2},$$

while the value of the dissipation integral is found to be:

$$\nu \int (\partial v/\partial y)^2 \, dy = \tfrac{1}{12} A^3 (t - t_0)^{-3/2}.$$

In the approximation used here, only the steep part at the end of the curve contributes to the integral.

It should be noted that there is also a solution:

$$v = - V(y^*)/(t - t_0)^{1/2} \quad \text{with} \quad y^* = -(y - y_0)/(t - t_0)^{1/2}.$$

I have considered these results (and similar ones found in previous work) as an indication that an investigation of the behaviour of approximate solutions, formed out of almost rectilinear parts interspersed with nearly vertical jumps, will be sufficiently interesting to make it worth while. These approximate solutions belong to the category of those constructed in the preceding paper when we considered the properties of certain two-dimensional fields. An important property of these solutions is that they present features which can be easily *counted*; this is a distinct advantage when we have to do with statistical problems [2]).

5. The approximate method of solution is based upon the circumstance that in a domain where $\partial v/\partial y$ and $\partial^2 v/\partial y^2$ are of normal order of magnitude, we can simplify eq. (1) to:

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial y} = 0 \tag{22}$$

We will suppose that the initial course of v can be represented by a series of straight segments, forming a broken line. Consider a particular segment, given by:

$$v = \beta\,(y - \sigma),$$

where β and σ are functions of t. When this expression is substituted into (22), the equation is satisfied identically in y if:

$$\beta = 1/(t + t_0), \quad \text{and} \quad \sigma = \text{constant} \tag{23}$$

t_0 being another constant. Hence every straight segment will turn to the right, while its point of intersection with the horizontal axis (its "hinge point") remains unchanged, the angle α (compare fig. 2) increasing according to: $\tan\,\alpha = t +$ *constant.*

This result can immediately be applied to a series of segments. — The point of intersection of two consecutive segments moves along a horizontal line with a velocity equal to the height of the point above the

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[2]) It has been pointed out to the author by J. D. Cole and V. Bargmann at Pasadena, Cal., that the substitution

$$v = -\,2\nu\,\partial\,(ln\,u_1)/\partial y$$

can be used to transform the original equation (1) into a linear equation of the third order, which can immediately be integrated into:

$$\frac{\partial u_1}{\partial t} = \nu\,\frac{\partial^2 u_1}{\partial y^2} + u_1\,C\,(t),$$

$C(t)$ being an arbitrary function of the time. If we eliminate this function with the aid of the substitution: $u_1 = u_2 \cdot \exp\,\{\int C(t)\,dt\}$, we are left with:

$$\frac{\partial u_2}{\partial t} = \nu\,\frac{\partial^2 u_2}{\partial y^2}.$$

In this way it becomes possible to write down in explicit form solutions of eq. (1) starting from given initial conditions, *e.g.* from an arbitrarily distributed system of concentrated loads. However, the circumstance that ν is supposed to be very small, makes these solutions not always convenient in further deductions.

The case of a single concentrated load leads automatically to the particular solution considered in the text.

axis. If the point is below the axis, the velocity is negative and the point moves to the left. A graphical construction can be easily executed.

When the initial slope of the segments was positive, it will remain so for ever, the slope gradually decreasing to zero. If the initial slope is

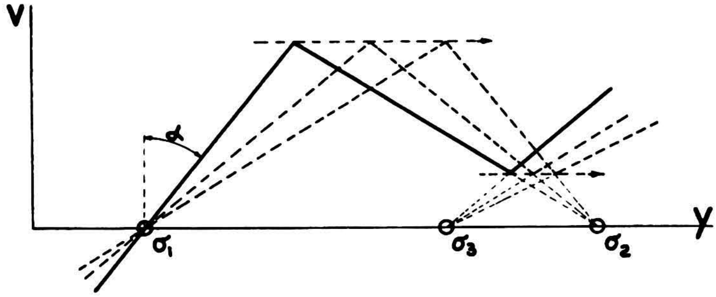

Fig. 2.

negative, the slope will increase in absolute measure and after a finite lapse of time the segment will approach to a vertical position. Equation (22) can then no longer be used and we must have recourse to an appropriate solution of the original equation. In the same way as in the preceding paper it is found that a better approximation to the course of v is given by the expression:

$$(24) \qquad v = \tfrac{1}{2}(v_- + v_+) - \tfrac{1}{2}(v_- - v_+) \tanh\{(v_- - v_+)(y - \xi)/4\nu\}$$

together with:

$$(25) \qquad d\xi/dt = \tfrac{1}{2}(v_- + v_+)$$

v_- being the limiting value of v on the left hand side and v_+ that on the right hand side of the vertical segment.

Two conclusions will be drawn from this result. One is, that the approximate picture in which the course of v is taken as a chain of rectilinear segments, can still be used when segments originally sloping downwards have acquired a vertical direction. From that instant onward they do not rotate any more, but move with the velocity $d\xi/dt$ given by (25), keeping their vertical direction. This feature can be introduced into the graphical construction alluded to before (see fig. 3). It must be observed that vertical segments following each other, can overtake one another. When this occurs, they combine to form a single segment, moving from then onward with a velocity again given by (25), v_- and v_+ now referring to the limiting velocities at the end points of the new segment.

The second conclusion is, that when greater accuracy is needed with regard to the "rounding off" at the ends of a vertical segment, we can use the hyperbolic tangent function indicated in (24) as a convenient approximation. In this way it is found that the dissipation integral referring to a single vertical segment has the value $\frac{1}{12}(v_- - v_+)^3$. — It should be kept in mind that $(v_{-}-v_{+})$ is positive for every vertical segment, as will be evident from the process by which these segments are generated.

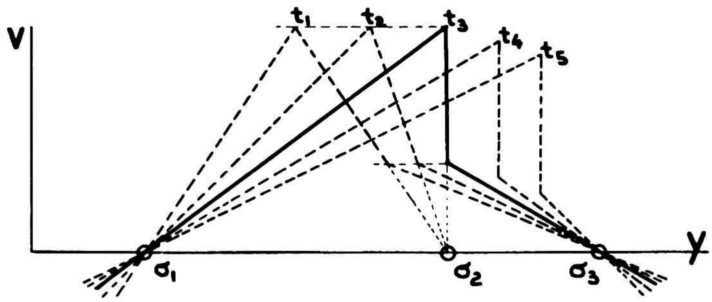

Fig. 3.

6. With a solution starting from a course of $v(y)$ given by a chain of straight segments, several periods can be distinguished in its history. There is first the period in which segments with a negative slope change into vertical segments, so that an increasing number of apparent discontinuities in the course of v is developed. When the number of vertical segments has sufficiently increased, there will arise an appreciable chance for segments to overtake each other and the number of vertical segments will again decrease. At the same time the slope of the positively inclined segments steadily becomes smaller, so that the average amplitude of the curve will become less and less. This may bring us to a third period, in which the REYNOLDS' number Re_I, defined in (11), will no longer be large enough for our approximation to remain valid. We should then have recourse to a more accurate solution of the original equation (see footnote 2).

In the following lines I will give attention to the *second* period, in which there is a gradual coalescence of vertical segments. In order to study this phenomenon in a pure form, not influenced by the formation of new vertical segments, it will be assumed that the initial state of the system is already given by a *series of parallel straight segments*, all having the same slope β upward to the right, separated from each other by vertical segments. During the development of the system in the course of time the slope of the inclined segments will decrease according to (23); their parallelism will be retained. To simplify the equations, we assume $\beta = 1/t$; the "initial state" of the system can be defined as the state for $t = 1$.

A description of the state of the system at a given instant of time is obtained (compare fig. 4) by stating

(1) the values of the ξ_i which define the positions of the vertical segments at that instant, and

(2) the values of the σ_i which define the locations of the "hinge points".

These data are subject to the evident conditions: $\xi_{i-1} < \xi_i$; $\sigma_{i-1} < \sigma_i$. We introduce the following derived quantities:

$$(26) \qquad \begin{cases} \lambda_i = \xi_i - \xi_{i-1} \quad ; \quad \tau_i = \sigma_i - \sigma_{i-1} \\ \zeta_i = \xi_i - \tfrac{1}{2}(\sigma_i + \sigma_{i-1}) \end{cases}$$

Thus λ_i is the length of a domain in which v changes linearly with y; $\beta\tau_i$ is the height of the vertical segment on the right hand side of λ_i; and

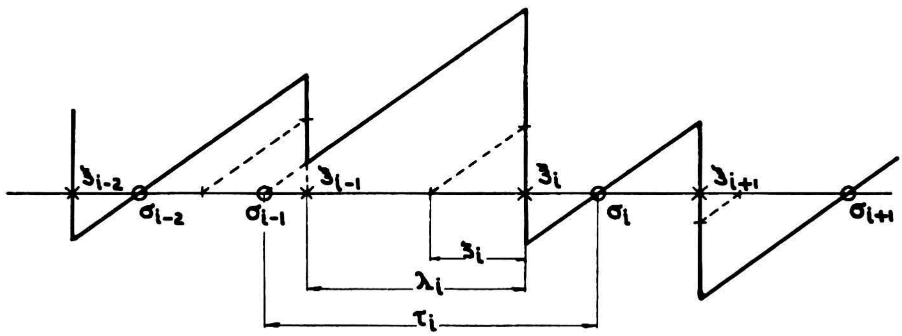

Fig. 4.

$\beta\zeta_i$ is the height above the y-axis of the point midway on this vertical segment. The quantities λ_i and τ_i are always positive; the ζ_i can be positive as well as negative.

7. In the present description the laws of motion of the system are:

(I) $\beta = 1/t$.

(II) $d\sigma_i/dt = 0$.

(III) $d\xi_i/dt = d\zeta_i/dt = \beta\zeta_i$.

(IV) When two consecutive ξ's (say ξ_{i-1} and ξ_i) come to coincidence, the corresponding vertical segments combine and the hinge point σ_{i-1} disappears (see fig. 5).

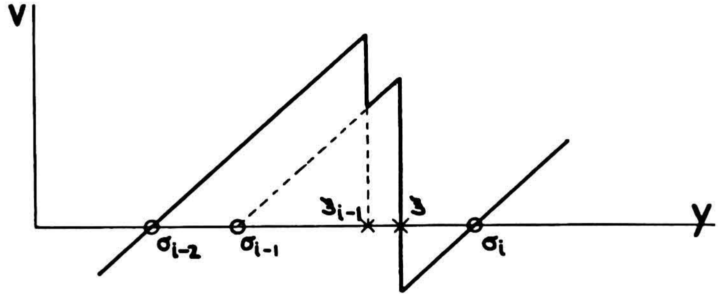

Fig. 5.

(V) The τ_i are constants, until two consecutive ξ_i coincide and a σ disappears. When σ_{i-1} disappears, τ_{i-1} and τ_i combine to form a single entity and the number of segments τ in a domain of the y-axis of length S decreases by one. The average length of these segments correspondingly increases.

(VI) The λ_i are functions of the time:

$$d\lambda_i/dt = \beta\{\lambda_i - \tfrac{1}{2}(\tau_i + \tau_{i-1})\} = \beta(\zeta_i - \zeta_{i-1}).$$

This quantity can be negative as well as positive. When λ_i is small, the chance for a negative value will be large; thus it is possible that λ_i decreases to zero and disappears from the system. Hence there is a progressive decrease of the number of segments λ, keeping exactly pace with the reduction of the number of segments τ. The average length of the λ_i correspondingly increases.

(VII) The ζ_i increase proportionally with the time, so that $\beta\zeta_i = \zeta_i/t$ remains constant, until two consecutive ξ_i come to coincidence. At the instant of coincidence of ξ_{i-1} and ξ_i we have $\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1} = \zeta_i + \tfrac{1}{2}\tau_i$; ζ_{i-1} and ζ_i disappear as such and are replaced by a single new value of ζ, equal to $\zeta^0 = \zeta_{i-1} - \tfrac{1}{2}\tau_i = \zeta_i + \tfrac{1}{2}\tau_{i-1}$ [3]).

8. We can consider various initial states of the system, that is, various initial arrangements of the successive values of λ_i and τ_i. Also one ζ must be given initially; the other ζ_i then follow from the λ_i and τ_i. We assume that the system is statistically homogeneous, so that mean values can be defined. Along with mean values taken over a certain length of the y-axis, it is convenient to introduce another type, obtained by summation with respect to i over a large number (N) of consecutive values, and division by N. Such mean values will be indicated by the sign $\overline{}$.

We introduce the assumption that the mean values of the λ_i and of the τ_i are equal:

$$\overline{\lambda_i} = \overline{\tau_i} = l \tag{27}$$

The number N_λ of segments λ in a certain large domain S of the y-axis

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[3]) The laws of motion and the tendency towards reduction of the number of segments can be illustrated with the aid of a molecular analogue, in which molecules colliding with each other immediately combine. We give to the molecules

coordinates	:	ξ_i
velocities	:	$d\xi_i/dt = \beta\zeta_i$
masses	:	τ_i
momenta	:	$\beta\tau_i\zeta_i$.

At every collision masses and momenta are added; kinetic energy is lost. The description of the process obtains slightly greater clarity, if instead of τ_i, ζ_i we introduce the notation $\tau_{i-1,i}$, $\zeta_{i-1,i}$. Then in a collision $\tau_{i-1,i}$ and $\tau_{i,i+1}$ combine to $\tau_{i-1,i+1}$, etc.

A simple graphical construction can be worked out, starting from the initial values of the $\xi_{i-1,i}$ and those of $\tfrac{1}{2}(\sigma_{i-1} + \sigma_i)$.

and the number N_τ of segments τ in this domain will then be practically the same:

$$N_\lambda = N_\tau = N = S/l \tag{27a}$$

When this relation is fulfilled in the initial state, it will be maintained in consequence of the laws of motion.

It follows that mean values obtained by summation with respect to i can be reduced to mean values defined with respect to unit length of the y-axis, by division through l.

The property of statistical homogeneity entails that in calculating mean values by summation, we may start from an arbitrary value of i. There is thus no difference between expressions like $\overline{\tau_i^2}$ and $\overline{\tau_{i+k}^2}$.

The property of statistical invariance with respect to a simultaneous change of sign of v and y, now takes the form that any statistical quantity formed out of the λ_i, τ_i and ζ_i, will remain unchanged when the signs of all ζ_i are changed simultaneously with a change in the direction in which i is counted, no change of sign being made with respect to λ_i and τ_i. For instance:

$$\overline{\lambda_i\tau_i} = \overline{\lambda_i\tau_{i-1}} \;;\; \overline{\lambda_i\zeta_i} = -\overline{\lambda_i\zeta_{i-1}} \;;\; \overline{\tau_i\zeta_{i+k}} = -\overline{\tau_i\zeta_{i-k}} \quad (k = 0, 1, 2, \ldots).$$

This rule implies that:

$$\overline{\zeta_i} = 0 \quad ; \quad \overline{\tau_i^p\zeta_i} = 0 \quad (p = 1, 2, \ldots) \tag{28}$$

We cannot exclude the possibility that there may exist correlations between the values of consecutive λ_i or of consecutive τ_i, or between λ_i and τ_i of the same or only slightly differing indices. Such correlations may arise from the laws of motion. It is also to be observed that we require the property of statistical homogeneity to be valid with respect to expressions involving the ζ_i. The values of the ζ_i would become large, if over some large section of the y-axis the λ_i and τ_i should get more and more out of step; we assume that on the long run this is always corrected in such a way that it has a sense to speak of mean values like $\overline{\zeta_i}$ and $\overline{\zeta_i^2}$, etc.

We can neither assume that expressions of the type $\overline{\zeta_i\zeta_j}$ with $i \neq j$ will necessarily be zero in consequence of (28). However, it is possible to assume that such quantities will vanish when the difference between i and j increases without limit, in such a way that

$$\sum_{k=-\infty}^{k=+\infty} \overline{\zeta_i\zeta_{i+k}} \quad \text{converges} \tag{29}$$

This equation expresses the ultimate randomness of phase differences at large distances.

9. *Calculation of mean values.*

I. *Mean value of v.* — In order to show the method to be used in calculating mean values, we start with that of v, although it is evident that the result must be zero. We observe that over a segment λ_i the variable v changes linearly with y; hence for a single segment:

$$\int v\,dy = \frac{1}{\beta}\int v\,dv = \tfrac{1}{2}\beta\,\{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^2\}.$$

This expression must be summed with respect to i over all segments contained in a domain of length S; the sum must be divided by $S = Nl$. Having regard to the relations: $\overline{\zeta_i^2} = \overline{\zeta_{i-1}^2}$; $\overline{\tau_i\,\zeta_i} = 0$, etc., which follow from the rules of invariance mentioned in section **8**, it will immediately be found that $\overline{v} = 0$.

II. *Mean value of v^2.* — The same procedure is applied. Integration over the length of the segment λ_i gives:

$$\int v^2\,dy = \frac{1}{\beta}\int v^2\,dv = \tfrac{1}{3}\beta^2\,\{(\zeta_i + \tfrac{1}{2}\tau_i)^3 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^3\}.$$

Hence taking the mean value:

$$\overline{v^2} = (\beta^2/l)\,(\overline{\tau_i\,\zeta_i^2} + \tfrac{1}{12}\,\overline{\tau_i^3}) \tag{30}$$

We shall write:

$$\overline{\tau_i^2} = l^2\,(1+\omega) \ \ ; \ \ \overline{\tau_i^3} = l^3\,(1+\omega^*) \ \ ; \ \ \overline{\tau_i\,\zeta_i^2} = l^3\,\tilde{\omega} \tag{31}$$

Then:

$$E = \tfrac{1}{2}\,\overline{v^2} = \tfrac{1}{2}\,\beta^2\,l^2\,\{\tilde{\omega} + \tfrac{1}{12}\,(1+\omega^*)\} \tag{32}$$

III. *Value of ε.* — It has been mentioned at the end of section **5** that contributions to the dissipation integral exclusively derive from the vertical segments, each segment giving an amount which in the present notation is equal to $\tfrac{1}{12}\,\beta^3\,\tau_i^3$.

Hence:

$$\varepsilon = \tfrac{1}{12}\,(\beta^3/l)\,\overline{\tau_i^3} = \tfrac{1}{12}\,\beta^3\,l^2\,(1+\omega^*) \tag{33}$$

It follows that

$$\lambda^2 = \nu\,t\,\left\{1 + \frac{12\,\tilde{\omega}}{1+\omega^*}\right\} \tag{34}$$

10. To find the mean value of the correlation product $v_1 v_2$ an elaborate calculation is necessary, which will be given in sections **13** and **14**. However, it is a simple matter to find the value of J_0. It is convenient to write:

$$2\,J_0 = \lim \int_a^b \overline{v_1\,v_2}\,d\eta \quad \text{with } a \to -\infty,\ b \to +\infty.$$

Since the quantity to be obtained depends on two integrations, one with respect to y to find the mean value of $v_1 v_2$, the other one with respect to η, we interchange the order of integration and first calculate:

$$v(y) \int v(y+\eta)\, d\eta$$

with a fixed value of y, which we suppose to be situated in the segment λ_i. The integral with respect to η, taken over the length of a segment λ_{i+k}, has the value:

$$\tfrac{1}{2}\beta\, v(y) \left\{ (\zeta_{i+k} + \tfrac{1}{2}\tau_{i+k})^2 - (\zeta_{i+k-1} - \tfrac{1}{2}\tau_{i+k-1})^2 \right\}.$$

We must take a sum of such expressions, from $k = -p$ until $k = +q$, p and q being large numbers. The domain of integration will not end precisely at vertical segments, and there will remain at each end an integral referring to a part of a segment only. We shall indicate the contributions from these parts by

$$\tfrac{1}{2}\beta\, v(y) \left\{ (\zeta_{i-p} + \tfrac{1}{2}\tau_{i-p})^2 - v_{\mathrm{I}}^2 \right\}$$

and

$$\tfrac{1}{2}\beta\, v(y) \left\{ v_{\mathrm{II}}^2 - (\zeta_{i+q} - \tfrac{1}{2}\tau_{i-q})^2 \right\}.$$

Summing up we obtain:

$$\beta\, v(y) \left\{ \sum_{-p}^{+q} \tau_{i+k}\, \zeta_{i+k} - \tfrac{1}{2} v_{\mathrm{I}}^2 + \tfrac{1}{2} v_{\mathrm{II}}^2 \right\}.$$

It should be observed that v_{I} and v_{II} are dependent on y.

We now integrate with respect to y over the length of the segment λ_i. This gives

$$\tfrac{1}{2}\beta^2 \left\{ (\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^2 \right\} \sum_{-p}^{+q} \tau_{i+k}\, \zeta_{i+k} - \delta_{\mathrm{I}} + \delta_{\mathrm{II}},$$

where δ_{I}, δ_{II} are the integrals of $\frac{1}{2}\beta\, v(y)\, v_{\mathrm{I}}^2$, $\frac{1}{2}\beta\, v(y)\, v_{\mathrm{II}}^2$ respectively.

When now the mean value is calculated by summation with respect to i, we can safely assume that the mean values of δ_{I} and δ_{II} cancel, since for large p and q there is no relation between $v(y)$ and either v_{I} or v_{II}. As regards the other terms, the mean value of quantities containing uneven powers of the ζ_i will either vanish or cancel on account of the second invariant property, so that there will remain:

$$\tfrac{1}{2}\beta^2 \sum_{k=-p}^{k=+q} \overline{(\tau_i\, \zeta_i + \tau_{i-1}\, \zeta_{i-1})\, \tau_{i+k}\, \zeta_{i+k}}.$$

In extension of (29) we assume that this sum converges. Hence we obtain:

$$J_0 = \frac{\beta^2}{2l} \sum_{k=-\infty}^{k=+\infty} \overline{\tau_i\, \tau_{i+k}\, \zeta_i\, \zeta_{i+k}} \tag{35}$$

Since this must be independent of the time, we must have:

$$(1/l) \sum \overline{\tau_i\, \tau_{i+k}\, \zeta_i\, \zeta_{i+k}} \sim t^2.$$

It can be proved in a direct way that this sum is not affected by the coalescence of segments and increases only in consequence of the relation $d\zeta_i/dt = \beta\zeta_i = \zeta_i/t$. We shall come to this in section **18.**

It is even true that

$$\beta\, \Sigma\, \tau_i\, \zeta_i$$

is not affected by the coalescence of segments (compare footnote 3, where this quantity occurs as the momentum of a system of combining molecules). This sum, however, is not convergent, while its mean value with respect to i is zero.

(To be continued).

Note. — The expression for J_0 can also be written:

$$J_0 = \frac{\beta^2}{l}\left\{\tfrac{1}{2}\,\overline{\tau_i^2\,\zeta_i^2} + \sum_{k=1}^{k=\infty} \overline{\tau_i\,\tau_{i+k}\,\zeta_i\,\zeta_{i+k}}\right\} \qquad (35a)$$

There may be cases where the terms with $k \neq 0$ all vanish.