Mathematics. — *On Plimpton 322. Pythagorean numbers in Babylonian mathematics.* By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

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NEUGEBAUER and SACHS gave an interpretation of the tablet Plimpton 322 (M.C.T. 1945) from the point of view, that the numbers on this tablet were obtained as a series solutions of the Pythagorean equation

$$d^2 = l^2 + b^2$$

in integers under the extra condition, that the proportion d/l changes from step to step by an, apparently, constant number. Their final result was, that the table was constructed by selecting numbers p/q and q/p from multiplicationtables so that

$$d/l = \tfrac{1}{2}\,(p/q + q/p)$$

satisfies the extra condition.

In our opinion a much simpler interpretation is possible, in which the production of Pythagorean numbers by using only one parameter — a method which seemed to be rejected according to NEUGEBAUER and SACHS — is the starting point.

The extra condition proves then to be absent, the "constant decrease" being merely accidental and the difficulties in terminology disappear.

Analysis.

1. If we put $l = 1$ the Pythagorean equation becomes

$$1 = d^2 - b^2 = (d + b)\,(d - b)$$

So when $d + b = \lambda$ we have $d - b = 1/\lambda$ and $d = \tfrac{1}{2}(\lambda + 1/\lambda)$, $b = \tfrac{1}{2}(\lambda - 1/\lambda)$, $l = 1$ are rational Pythagorean numbers, whenever λ is a rational number.

If therefore we form a table of sum and difference of the columns of a reciprocal table we have in

$$d = \lambda + 1/\lambda \qquad b = \lambda - 1/\lambda \qquad l = 2$$

Pythagorean triples.

e.g. 2, 24 0, 25 $d = 2,49$ $b = 1,59$ $(l = 2)$.

2. If we leave apart the constant l the numbers $\lambda + 1/\lambda$ and $\lambda - 1/\lambda$

are in general not relatively prime. Whether a *regular* common divisor (factors 2, 3 or 5) is present or not is evident from the last sexagesimal.

e.g. $\lambda = 2, 22, 13, 20$	$0, 25, 18, 45$	
$d = 2, 47, 32, 5$	$b = 1, 56, 54, 35$	both divisible by 5 (of the last sexagesimal),
$d = 33, 30, 25$	$b = 23, 22, 55$	both divisible by 5,
$d = 6, 42, 5$	$b = 4, 40, 35$	both divisible by 5,
$d = 1, 20, 25$	$b = 56, 7$	final numbers, reduced values;
Or $\lambda = 2, 8$	$0, 28, 7, 30$	
$d = 2, 36, 7, 30$	$b = 1, 39, 52, 30$	both divisible by 30, thus multiply by 2,
$d = 5, 12, 15$	$b = 3, 19, 45$	both divisible by 15, or multiply by 4
$d = 20, 49$	$b = 13, 19$	final numbers. **ib — si.**

3. NEUGEBAUER published M.K.T. I—16—23, discussing AO 6456, a complete six-place table of reciprocals. Taking all four-place values from this we find in the first and second column of Table I the reciprocals, the

TABLE I.

λ	$1/\lambda$	No.	d	b	Reduction
2, 24	0, 25	1	2, 49	1, 59	1
2, 22, 13, 20	0, 25, 18, 45	2*	*1, 20, 25*	56, 7	: 125
2, 20, 37, 30	0, 25, 36	3	1, 50, 49	1, 16, 41	×2 : 3
2, 18, 53, 20	0, 25, 55, 12	4	5, 9, 1	3, 31, 49	: 32
2, 18, 14, 24	0, 26, 2, 30	*	—	—	—
2, 15	0, 26, 40	5	1, 37	1, 5	×3 : 5
2, 13, 20	0, 27	6	*8, 1*	5, 19	×3
2, 10, 12, 30	0, 27, 38, 52, 48	—	—	—	—
2, 9, 36	0, 27, 46, 40	7	59, 1	38, 11	×3 : 8
2, 8	0, 28, 7, 30	8	20, 49	13, 19	×2 : 15
2, 6, 33, 45	0, 28, 26, 40	—	—	—	—
2, 5	0, 28, 48	9	12, 49	8, 1	×5
2, 2, 52, 48	0, 29, 17, 48, 45	*	—	—	—
2, 1, 30	0, 29, 37, 46, 40	10	2, 16, 1	1, 22, 41	×9 : 10
2	0, 30	11	*5* [1,15]	*3* [0,45]	×2 [: 2]
1, 57, 11, 15	0, 30, 43, 12	—	—	—	—
1, 55, 12	0, 31, 15	12	48, 49	27, 59	: 3
1, 53, 46, 40	0, 31, 38, 26, 15	—	—	—	—
1, 52, 30	0, 32	13	4, 49	2, 41	×2
1, 51, 6, 40	0, 32, 24	14	53, 49	29, 31	×3 : 8
1, 48	0, 33, 20	15	53	*28*	×3 : 8

The * denotes the reciprocals missing in *AO* 6456.
The *cursive* numbers are corrections necessary. No. 2, No. 6 are the same as those of NEUGEBAUER and SACHS, No. 15 differs from that in M. C. T: we divide 56 by 2 instead of multiplying 53 by 2.
The last column gives a process of reduction, to obtain d, b from $\lambda + 1/\lambda$ and $\lambda - 1/\lambda$. With the exception of No. 2 (and 11) all pairs d, b are relatively prime.

reduced values of $\lambda + {}^1/_\lambda$, $\lambda - {}^1/_\lambda$ forming the fourth and fifth column and numerating the reciprocals according to the third column, we see the columns IV, III, II of Plimpton 322. Apart from No. 11, which is given as 1, 15, 0, 45, *the values for* $l = 1$, and which hardly need a reduction, the proportion 5 : 3 being evident, we see some regular numbers "too much". These numbers are however of very high order. Writing $2^\alpha\, 3^\beta\, 5^\gamma$ (α, β, γ) we have

TABLE II.

*	2, 18, 14, 24	from (1, 1, 6)	or	(11, 5, 0)
	2, 10, 12, 30	from (1, 1, 7)	or	(13, 6, 0)
	2, 6, 33, 45	from (0, 6, 4)	or	(12, 0, 2)
*	2, 2, 52, 48	from (0, 4, 7)	or	(14, 3, 0)
	1, 57, 11, 15	from (0, 3, 6)	or	(12, 3, 0)
	1, 53, 46, 40	from (0, 7, 5)	or	(14, 0, 2).

It must be remarked, that the numbers with * are also missing in AO 6456. On the other hand the number missing in AO 6456: 2, 22, 13, 20 has its corresponding value on Plimpton 322, accepting the interpretation given here. This last number is obtainable from (0, 6, 3). The number of factors of all the others does also not exceed 13 and never more than three factors 5 occur:

(0, 0, 2); (0, 6, 3); (9, 1, 0); (7, 6, 0); (0, 3, 1); (0, 3, 0); (5, 5, 0); (7, 0, 0); (0, 0, 3); (1, 6, 1); (1, 0, 0); (8, 3, 0); (5, 0, 0); (3, 5, 0); (2, 3, 0).

As Plimpton 322 is about a millenium older than AO 6456 it is not unlikely that the high degree reciprocals were not yet contained in the table which the writer of the Plimtontablet had at his disposal. *But his table was complete under the conditions* $\alpha + \beta + \gamma \leq 13$; $\gamma \leq 3$.

Moreover the first column now contains the square of the diagonal minus unity for the case $l = 1$. *We thus see, that it is not necessary to complete the Plimpton tablet in column I*, as NEUGEBAUER and SACHS do, *by a unit which appears nowhere on the tablet*. Interpreting the first column as

$$\{\tfrac{1}{2}(\lambda + {}^1/_\lambda)\}^2 - 1$$

has the consequence that *takiltu* can be used in its ordinary meaning of square and the < mathematical > translation of the inscription above column I has to be: the square of the diagonal from which a unit has been substracted [1]).

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[1]) None of the missing numbers, Table II, is a divisor of 60^5. In addition to all divisors of 60^5 in the interval the table contains the three place reciprocals of 500.000, 512.000, 640.000 i.e. $\gamma \leq 3$.

P. VAN DER MEER discussing the inscription above column I indicated to me that the damaged sign between *i ... ú* could hardly be other than the polyvalent having among its values *seḫ*. So it is possible to read sag *iseḫú*, the side which fixes, determines the problem, which has been put equal to unity. This would give the key-stone of the interpretation.

Conclusion.

1. We have to read the columns from right to left.
2. The first column contains the ordinal numbers of reciprocals in a complete four-place table, not containing (α, β, γ) for $\alpha + \beta + \gamma > 13$, $\gamma > 3$, from 2, 24 — as near as possible to $1 + \sqrt{2}$ — to 1, 48.
3. The "solvent numbers" in the second and third column are the reduced values of diagonal and side obtained from $\lambda + {}^1/_\lambda$, $\lambda - {}^1/_\lambda$, by canceling common factors 2, 3, 5.
4. The first column contains the square of the diagonal diminished by unity if one of the sides is put equal to unity.
5. The constant decrease pro step of about one sixtieth in d/l is merely accidental. *There is no extra condition. The tablet contains everything that can be obtained from the four-place table mentioned above.*