Mathematics. — *Sequences of points on a circle.* By N. G. DE BRUIJN and P. ERDÖS. (Communicated by Prof. W. VAN DER WOUDE).

1. Introduction. We consider sequences $\{a\}$ of points $a_1, a_2, a_3, \ldots$ on a circle with radius $1/2\pi$, in other words numbers mod 1. The numbers $a_1, a_2, \ldots, a_n$ define n intervals with total length 1; denote by $M_n^1(a)$ and $m_n^1(a)$ the largest and the smallest length. Clearly

$$n M_n^1(a) \geqslant 1 \geqslant n m_n^1(a).$$

Analogously $M_n^r(a)$ and $m_n^r(a)$ denote the maximum and minimum length of the sum of r consecutive intervals, so that $n M_n^r(a) \geqslant r \geqslant n m_n^r(a)$. We put

$$\limsup_{n \to \infty} n M_n^r(a) = \varLambda_r(a)$$

$$\liminf_{n \to \infty} n m_n^r(a) = \lambda_r(a)$$

$$\limsup_{n \to \infty} M_n^r(a) / m_n^r(a) = \mu_r(a)$$

and

$$\varLambda_r = \text{g.l.b. } \varLambda_r(a) \quad, \quad \lambda_r = \text{l.u.b. } \lambda_r(a) \quad, \quad \mu_r = \text{g.l.b. } \mu_r(a).$$

We are able to determine

$$\varLambda_1 = 1/\log 2 \quad, \quad \lambda_1 = 1/\log 4 \quad, \quad \mu_1 = 2.$$

The problem of $\varLambda_r, \lambda_r, \mu_r$ is closely related to a problem concerning "just distributions" solved by Mrs VAN AARDENNE-EHRENFEST [1]). All we can prove is that $\mu_r \geqslant 1 + 1/r$ (and analogus inequalities for $\varLambda_r$ and λ_r); we conjecture that $r(\mu_r - 1)$ is unbounded. From this the theorem of Mrs VAN AARDENNE-EHRENFEST would follow.

2. A sequence which gives the best possible values of $\varLambda_1(a)$, $\lambda_1(a)$, $\mu_1(a)$. Take $a_k = {}^2\log(2k-1)$, reduced mod 1. We show that $a_1, \ldots, a_n$ occur in the following order

$${}^2\log n, {}^2\log(n+1), \ldots, {}^2\log(2n-1). \quad \ldots \quad (2.1)$$

Namely, no two of the a_k's and no two of the numbers (2.1) are congruent mod 1, but each number in (2.1) is congruent to just one a_k.

It follows from (2.1) that the lengths of the intervals defined by $a_1, \ldots, a_n$ are

$${}^2\log \frac{n+1}{n}, {}^2\log \frac{n+2}{n+1}, \ldots, {}^2\log \frac{2n-1}{2n-2}, {}^2\log \frac{2n}{2n-1},$$

[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam 48, 266—271 (1945) = Indagationes Mathematicae, 7, 71—76 (1946).

and so

$$n\,M_n^1(a) = \frac{n\log\left(1+\dfrac{1}{n}\right)}{\log 2}, \quad n\,m_n^1(a)\,\frac{n\log\left(1-\dfrac{1}{2n}\right)^{-1}}{\log 2}.$$

For $n\to\infty$, $n\,M_n^1(a)$ increases to the limit $1/\log 2$; $n\,m_n^1(a)$ decreases to the limit $1/\log 4$; $M_n^1(a)/m_n^1(a)$ increases to the limit 2. It follows that $\varLambda_1(a)=1/\log 2$, $\lambda_1(a)=1/\log 4$, $\mu_1(a)=2$.

3. Lower bound for $\varLambda_r(a)$.

Let $\{a\}$ be a sequence, n a natural number, and suppose that ϱ is such that

$$k\,M_n^1(a) < \varrho. \qquad (n\leqslant k < 2n) \quad . \quad . \quad . \quad . \quad (3.1)$$

Let the intervals determined by $a_1,\ldots,a_n$ be $I_1,\ldots,I_n$, arranged in descending order of length. Denote the length of I_j by a_j; so that

$$a_1\geqslant a_2\geqslant\ldots\geqslant a_n; \quad a_1+\ldots+a_n=1. \quad . \quad . \quad (3.2)$$

Now put in the points $a_{n+1},\,a_{n+2},\ldots,\,a_{2n-1}$. Since any point "destroys" one I at most, there remains at least one interval of length $\geqslant a_p$ undisturbed after $a_{n+1},\ldots,\,a_{n+p-1}$ have been put in $(1\leqslant p\leqslant n)$. Hence

$$M_n^1(a)\geqslant a_1, \quad M_{n+1}^1(a)\geqslant a_2,\ldots, \quad M_{2n-1}^1(a)\geqslant a_n;$$

consequently, by (3.1) and (3.2),

$$\varrho\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n-1}\right)>1.$$

It follows that for at least one $k\,(n\leqslant k<2n)$ we have

$$k\,M_k^1(a)\geqslant\left(\frac{1}{n}+\ldots+\frac{1}{2n-1}\right)^{-1}=\sigma_n.$$

We have $\sigma_n < 1/\log 2$, $\sigma_n\to 1/\log 2$, and so $\varLambda_1(a)\geqslant 1/\log 2$. This holds for any $\{a\}$; the lower bound is attained for the sequence of section 2.

Similarly we can prove that for at least one $k\,(rn\leqslant k<(r+1)\,n)$ we have

$$k\,M_k^r(a)\geqslant\left(\frac{1}{rn}+\frac{1}{rn+1}+\ldots+\frac{1}{rn+n-1}\right)^{-1}$$

and so

$$\varLambda_r(a)\geqslant 1/\log\left(1+\frac{1}{r}\right)>r.$$

4. Upper bound for $\lambda_r(a)$ [2].

Let $\{a\}$ be a sequence, n a natural number, and suppose that ϱ is such that

$$k\,m_k^1(a)>\varrho \qquad (n<k\leqslant 2n). \quad . \quad . \quad . \quad . \quad (4.1)$$

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[2] The proof presented in this section was found by Mrs. VAN AARDENNE–EHRENFEST independently.

Let $a_{k_1}, a_{k_2}, \ldots, a_{k_{2n}}$ be the cyclic order of the points $a_1, \ldots, a_{2n}$ on the circle $(k_1, \ldots, k_{2n}$ is a permutation of $1, \ldots, 2n)$; put $k_{2n+1} = k_1$. If $k_i^* = \text{Max}(k_i, k_{i+1}, n+1)$, then the interval $a_{k_i}, a_{k_{i+1}}$ is one of the intervals determined by $a_1, \ldots, a_{k_i^*}$. It follows that its length is less than ϱ/k_i^*. Hence

$$1 > \varrho \sum_{i=1}^{2n} 1/k_i^*. \quad \ldots \ldots \ldots \quad (4.2)$$

We have $n < k_i^* \leqslant 2n$, and any $k \, (n+1 < k \leqslant 2n)$ occurs ε_k times as a k^*; $\varepsilon_k = 0, 1$ or 2. It follows that

$$\sum_{i=1}^{2n} 1/k_i^* = \sum_{n+1}^{2n} \frac{2}{k} + \sum_{n+2}^{2n} (2 - \varepsilon_k) \left\{ \frac{1}{n+1} - \frac{1}{k} \right\} \geqslant \sum_{n+1}^{2n} \frac{2}{k}.$$

Finally, by (4.1) and (4.2) we infer that at least for one $(n < k \leqslant 2n)$ we have

$$k \, m_k^1(a) \leqslant \left(\frac{2}{n+1} + \frac{2}{n+2} + \ldots + \frac{2}{2n} \right)^{-1} = \tau_n.$$

We have $\tau_n > 1/\log 4$, $\tau_n \to 1/\log 4$, and so $\lambda_1(a) \leqslant 1/\log 4$. The example of section 2 again shows that $1/\log 4$ is best possible.

Similarly we can show that for at least one $k \, (rn < k \leqslant (r+1)n)$ we have

$$k \, m_k^r(a) \leqslant r \left(\frac{r+1}{nr+1} + \ldots + \frac{r+1}{nr+n-1} \right)^{-1}$$

and so

$$\lambda_r(a) \leqslant \frac{r}{r+1} \bigg/ \log\left(1 + \frac{1}{r}\right) < r. \quad \ldots \ldots \quad (4.3)$$

5. Lower bound for μ_r.

Let $\{a\}$ be a sequence. We first prove that, for $r \geqslant 1$, $n \geqslant 1$ we have

$$M_n^r(a) / m_{n+1}^r(a) \geqslant 1 + \frac{1}{r}. \quad \ldots \ldots \ldots \quad (5.1)$$

We first suppose that $r > 1$. Let $I_1, I_2, \ldots, I_n$ be the intervals of the n-th stage, i.e. the intervals determined by $a_1, \ldots, a_n$. Let I_{k_0} be the one into which a_{n+1} falls, and let

$$I_{k_{-r+1}}, I_{k_{-r+2}}, \ldots, I_{k_0}, I_{k_1}, \ldots, I_{k_{r-1}} \quad \ldots \ldots \quad (5.2)$$

be consecutive on the circle [3]).

Put $M = M_n^r(a)$, $m = m_{n+1}^r(a)$ and denote by M_1 the maximum length of the sum of r consecutive intervals from the set (5.2). Denote the length of I_{k_i} by β_i. Let γ_1 and γ_2 be the lengths of the parts into which I_{k_0} is divided by a_{n+1}.

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[3]) If $2r - 1 > n$ the k_i are not all different.

Clearly at least one of the numbers $\beta_{-r+1}, \ldots, \beta_{-1}, \beta_1, \ldots, \beta_{r-1}$ (β_j say) is $\geqslant (M_1 - \beta_0)/(r-1)$; we may suppose that $j > 0$. Now we have

$$m \leqslant \beta_{j-r+1} + \ldots + \beta_{-1} + \gamma_1 + \gamma_2 + \ldots + \beta_{j-1}$$

and hence

$$m \leqslant M_1 - \beta_j \leqslant \frac{r-2}{r-1} M_1 + \frac{\beta_0}{r-1}. \quad \text{. . . . .} \quad (5.3)$$

On the other hand it follows from

$$m \leqslant \gamma_2 + \beta_1 + \ldots + \beta_{r-1} \leqslant M_1 - \gamma_1$$
$$m \leqslant \beta_{-r+1} + \ldots + \beta_{-1} + \gamma_1 \leqslant M_1 - \gamma_2$$

that

$$m \leqslant M_1 - \tfrac{1}{2} \beta_0. \quad \text{. . . . .} \quad (5.4)$$

Trivially we have $M_1 \leqslant M$. If $\beta_0 \leqslant 2 M_1 / (r+1)$ we infer $m \leqslant M_1 r/(1+r) \leqslant Mr/(1+r)$ from (5.3); if $\beta_0 \geqslant 2 M_1 /(r+1)$ we deduce the same result from (5.4). This proves (5.1) for $r > 1$.

If $r = 1$, (5.1) immediately follows from

$$m \leqslant \text{Min} (\gamma_1, \gamma_2) \leqslant \tfrac{1}{2} \beta_0 \leqslant \tfrac{1}{2} M.$$

Now suppose that n is a natural number and that for $nr \leqslant k \leqslant n(r+1)$ we have

$$M_k^r (a) / m_k^r (a) < \left(1 + \frac{1}{r}\right) \bigg/ \left(1 + \frac{1}{k}\right)^2. \quad \text{. . . . .} \quad (5.5)$$

It follows, by (5.1) that

$$m_{k+1}^r / m_k^r < k^2/(k+1)^2 \qquad (nr \leqslant k < nr + n)$$

and also

$$m_{rn+n}^r / m_{rn}^r < r^2/(1+r)^2. \quad \text{. . . . . .} \quad (5.6)$$

Trivially we have $m_{rn}^r \leqslant 1/n$; on the other hand, by (5.5)

$$m_{rn+n}^r > \frac{r}{1+r} M_{rn+n}^r \geqslant \frac{r}{1+r} \cdot \frac{r}{rn+n-1} \geqslant \frac{r^2}{(r+1)^2} \cdot \frac{1}{n}.$$

This contradicts (5.6). Hence for at least one k ($nr \leqslant k \leqslant nr + n$) (5.5) is not true. It follows that

$$\mu_r \geqslant 1 + \frac{1}{r}. \quad \text{. . . . .} \quad (5.7)$$

6. The inequalities (3.3), (4.3) and (5.7) are probably not best possible if $r \geqslant 2$. We conjecture that the expressions

$$r (\Lambda_r - 1) \quad , \quad r (1 - \lambda_r) \quad , \quad r (\mu_r - 1)$$

tend to infinity if $r \to \infty$.

We owe some useful remarks to Mrs. T. VAN AARDENNE-EHRENFEST and Mr. J. KOREVAAR with whom we first discussed the above problems.