Mathematics. — *On the mutual inductance of two parallel coaxial circles of circular cross-section.* By C. J. BOUWKAMP. (Communicated by Prof. H. B. G. CASIMIR.)

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1. *Introduction.*

As is well known, the mutual inductance of two parallel coaxial circles of vanishing cross-section is expressible in terms of complete elliptic integrals. However, in practice the cross-sectional dimensions of a conductor are finite, and the question arises how to calculate the effect of a non-vanishing section. The solution of this problem is of value in standard measurements. A second possible application concerns the calculation of the mutual inductance of two coils both consisting of closely spaced parallel windings, it being understood that the transverse dimensions of the coils shall be small compared to the distance between their centres, while the currents flowing through the coils in the longitudinal direction are considered, to a first approximation, as being uniformly distributed over the respective cross-sections.

Numerous expressions have been derived for the inductances of circular coils of rectangular and square sections, for a survey of which the reader is referred to the publications of the Bureau of Standards (see refs 1—4 at the end of the paper). Surprisingly, however, expressions for the mutual inductance of two circular wires of *circular* cross-section do not seem to have ever been published. This may partly be due to the fact that practical coils are likely to have rectangular sections. Yet a natural extension of the problem of two circles of vanishing section is to assume these sections circular, because the wires used are mostly of that type. Therefore, in this paper we shall derive an expression for the mutual inductance of two parallel coaxial toroids, both of circular section, on the assumption that the current density in either ring is constant over the corresponding section, which implies that the skin effect is ignored.

Let R_1 and R_2 denote the radii of the toroids (see fig. 1), and let a_1, a_2 be the radii of the corresponding cross-sections, D the axial distance between the centres of the toroids. Moreover, let $\varepsilon_1 = a_1/R_1$, $\varepsilon_2 = a_2/R_2$.

In practical cases the parameters ε_1 and ε_2 are very small compared to 1, and at any rate not greater than 1. It will, therefore, be adequate to expand the mutual inductance, M, of the toroids into a double power series of the variables ε_1 and ε_2. As we shall see, only even powers occur in such an expansion, so that one may write

$$M = \sum_{n,m=0}^{\infty} M_{n,m}\, \varepsilon_1^{2n}\, \varepsilon_2^{2m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $M_{n,m}$ is independent of a_1 and a_2. The object of this paper, then,

is to give an explicit expression for the coefficients in their dependence on D, R_1, R_2, n and m. These coefficients appear to be expressible in terms

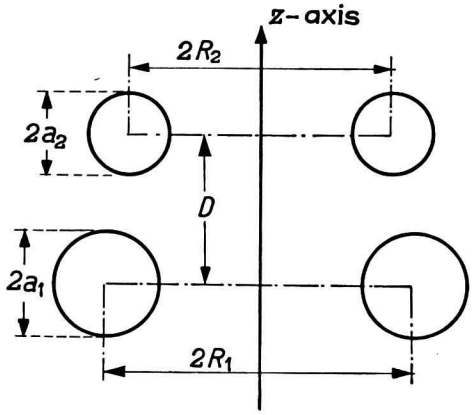

Fig. 1. Cross-sectional view of the system of coaxial toroids.

of a simple integral involving Bessel functions, which integral can be readily evaluated, at least for small values of $n + m$, leading to complete elliptic integrals of the first and second kinds.

The Bessel-function representation in question is a generalization of a known expression for $M_{0,0}$ due to HAVELOCK (ref. 5), viz.,

$$M_{0,0} = 4\pi^2 R_1 R_2 \int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, dt. \quad . \quad . \quad . \quad . \quad (2)$$

On the other hand, $M_{0,0}$ is known (since MAXWELL) in terms of elliptic integrals. For instance [1]),

$$M_{0,0} = 4\pi \sqrt{R_1 R_2} \left\{ \left(\frac{2}{k} - k \right) K - \frac{2}{k} E \right\}. \quad . \quad . \quad . \quad . \quad (3)$$

in which K and E are the complete elliptic integrals of the first and second kinds to modulus k, where

$$k^2 = \frac{4 R_1 R_2}{D^2 + (R_1 + R_2)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equivalent expressions are obtained if Landen's transformation of elliptic integrals is applied to (3); the simplest of them have been listed by GROVER (ref. 4).

2. *Generalization of* HAVELOCK'*s integral.*

First of all we calculate the mutual inductance of a circle of radius r of vanishing cross-section and the toroid (R_1, a_1), z being the axial distance

[1]) If R_1 and R_2 are the numerical values of the respective lengths measured in centimetres, then $M_{0,0}$ is the numerical value of the inductance also measured in centimetres. To make eq. (3) consistent in the rationalized m.k.s. units, the right-hand member has to be multiplied by 10^{-7} (empty space).

between their centres (see fig. 2). To that end we have to integrate (2), with $R_1 = r_1$, $R_2 = r$, $D = |z - z_1|$, over the cross-section of the toroid.

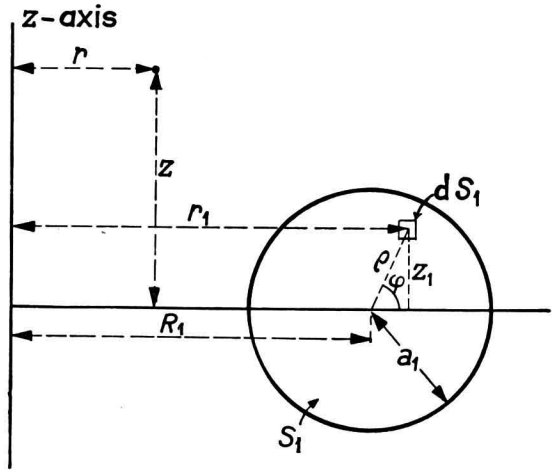

Fig. 2. Illustrating the integration over the cross-section of the first toroid.

Thus, if $|z_1| \leq a_1 \leq z$,

$$M' = 4\pi^2 r \int_0^\infty e^{-zt} J_1(rt)\,dt \; \frac{1}{\pi a_1^2} \iint_{S_1} e^{z_1 t} J_1(r_1 t)\, r_1\, dS_1. \quad . \quad . \quad . \quad (5)$$

In order to obtain for M' an expansion in powers of ε_1 analogous to (1), we must expand the surface integral, S, occurring in (5) in that manner. This is most easily achieved by introducing polar coordinates ϱ, φ around the centre of the cross-section, by means of which the surface integral is transformed into

$$S = \frac{1}{\pi a_1^2} \int_0^{a_1} \varrho\, d\varrho \int_{-\pi}^{\pi} (R_1 + \varrho \cos\varphi)\, J_1(R_1 t + \varrho t \cos\varphi)\, e^{\varrho t \sin\varphi}\, d\varphi \quad . \quad . \quad (6)$$

However, to make S tractable, we first replace the Bessel function in the integrand according to the integral representation

$$J_1(x) = \frac{i}{2\pi} \int_{-\pi}^{\pi} e^{-ix\sin\theta} \sin\theta\, d\theta,$$

which leads to

$$S = \frac{i}{2\pi^2 a_1^2} \int_{-\pi}^{\pi} e^{-iR_1 t \sin\theta} \sin\theta\, d\theta \int_0^{a_1} \varrho\, d\varrho \int_{-\pi}^{\pi} (R_1 + \varrho\cos\varphi)\, e^{\varrho t(\sin\varphi - i\sin\theta\cos\varphi)}\, d\varphi. \quad (7)$$

The integrations with respect to φ and ϱ can be carried out, giving simple Bessel functions, as will now be shown. For that purpose, let us consider the integral

$$F = \int_{-\pi}^{\pi} e^{x(\sin\varphi + u\cos\varphi)}\, d\varphi.$$

For positive values of u one has

$$\sin\varphi + u\cos\varphi = \sqrt{1+u^2}\,\sin(\varphi + \arctan u).$$

Now, since the integration over φ has to be performed over a full period of $\sin\varphi$, the phase component $\arctan u$ is of no consequence. That is to say,

$$F = \int_{-\pi}^{\pi} e^{x\sqrt{1+u^2}\sin\varphi}\,d\varphi = 2\pi I_0(x\sqrt{1+u^2}),$$

in which I denotes the modified Bessel function of imaginary argument. The function F is analytic in u, in spite of the radical. On account of the principle of analytic continuation we may apply our result to $u = i\sin\theta$; thus

$$\int_{-\pi}^{\pi} e^{x(\sin\varphi - i\sin\theta\cos\varphi)}\,d\varphi = 2\pi I_0(x\cos\theta). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

By differentiating (8) with respect to θ we further obtain

$$\int_{-\pi}^{\pi} e^{x(\sin\varphi - i\sin\theta\cos\varphi)}\cos\varphi\,d\varphi = -2\pi i\tan\theta\, I_1(x\cos\theta). \quad . \quad . \quad (9)$$

The results (8) and (9) enable us to evaluate the φ-integral occurring in (7), viz.,

$$\int_{-\pi}^{\pi} (R_1 + \varrho\cos\varphi)\, e^{\varrho t(\sin\varphi - i\sin\theta\cos\varphi)}\,d\varphi = 2\pi\{R_1 I_0(\varrho t\cos\theta) - i\varrho\tan\theta\, I_1(\varrho t\cos\theta)\}.$$

The next step is to perform the integration with respect to ϱ. This can be accomplished in virtue of

$$\int_0^{a_1} \varrho\, I_0(\varrho t\cos\theta)\,d\varrho = \frac{a_1 I_1(a_1 t\cos\theta)}{t\cos\theta},$$

$$\int_0^{a_1} \varrho^2 I_1(\varrho t\cos\theta)\,d\varrho = \frac{a_1^2 I_2(a_1 t\cos\theta)}{t\cos\theta}.$$

The result of the combined integrations with respect to ϱ and φ in (7) then becomes

$$\int_0^{a_1} \varrho\,d\varrho \int_{-\pi}^{\pi} (R_1 + \varrho\cos\varphi)\, e^{\varrho t(\sin\varphi - i\sin\theta\cos\varphi)}\,d\varphi =$$
$$= \frac{2\pi a_1 R_1}{t\cos\theta}\{I_1(a_1 t\cos\theta) - i\varepsilon_1\tan\theta\, I_2(a_1 t\cos\theta)\}.$$

If this is substituted in (7) we obtain

$$S = \frac{i}{\pi\varepsilon_1 t}\int_{-\pi}^{\pi}\{I_1(\varepsilon_1 R_1 t\cos\theta) - i\varepsilon_1\tan\theta\, I_2(\varepsilon_1 R_1 t\cos\theta)\}\, e^{-iR_1 t\sin\theta}\tan\theta\,d\theta. \quad (10)$$

It does not seem possible to evaluate the integral (10) in finite terms. On the other hand, it possesses a remarkably simple Neumann-series expansion. This expansion may be obtained by first developing the integrand in powers of ε_1, which is easily done by means of the known power

series for the Bessel functions. This procedure leads to

$$S = \sum_{n=0}^{\infty} \varepsilon_1^{2n} \left[\frac{R_1}{2\pi} \frac{(R_1 t/2)^{2n-1}}{n!\,(n+1)!} \int_{-\pi}^{\pi} \{ n \sin^2\theta \cos^{2n-2}\theta + \right.$$
$$\left. + \frac{i}{2} R_1 t \sin\theta \cos^{2n}\theta \} \, e^{-i R_1 t \sin\theta} \, \mathrm{d}\theta \right].$$

Secondly, the term between brackets is developed according to ascending powers of $R_1 t$. The relevant coefficients are simple trigonometric integrals which can be expressed by means of gamma functions. When the terms are properly arranged it appears that the power series obtained is nothing more than that of the Bessel function $J_{n-1}\ (R_1 t)$, except for an elementary factor. Omitting further details, we obtain for the final result

$$S = \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(R_1/2)^{n+1}}{n!\,(n+1)!} \varepsilon_1^{2n} t^n J_{n-1}(R_1 t). \quad . \quad . \quad . \quad (11)$$

Substitution of (11) in (5) then leads to

$$M' = 4\pi^2 r \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(R_1/2)^{n+1}}{n!\,(n+1)!} \varepsilon_1^{2n} \int_0^{\infty} e^{-zt} J_1(rt) J_{n-1}(R_1 t)\, t^n \, \mathrm{d}t. \quad (12)$$

As to the convergence of the series (12), it should be remarked that, though we obtained (12) on the restriction $z > a_1$, the series is certainly convergent if $z > a_1{}^2/2R_1$ (this condition is less stringent than the former since $a_1/2R_1$ does not exceed $1/2$). To prove this we observe that the absolute magnitude of the integral in (12) is less than

$$\int_0^{\infty} e^{-zt} t^n \, \mathrm{d}t = n!\, z^{-n-1}.$$

Further, $\Gamma(n-\frac{1}{2})/(n+1)!$ is uniformly bounded for $n = 0, 1, 2, \ldots$. Therefore, a majorant of (12), except for a constant factor depending on R_1 and z, is given by the geometric series

$$\sum_{n=0}^{\infty} (a_1^2/2\,z\,R_1)^2,$$

which is convergent if $z > a_1{}^2/2R_1$. The actual domain of convergence of (12) may be wider still.

It is now easy to obtain the Bessel-function representation of $M_{n,m}$ in question. To calculate M we simply have to integrate M' over the cross-section of the second toroid (R_2, a_2). Thus

$$M = 4\pi^2 \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(R_1/2)^{n+1}}{n!\,(n+1)!} \varepsilon_1^{2n} \int_0^{\infty} e^{-Dt} J_{n-1}(R_1 t)\, t^n \, \mathrm{d}t \frac{1}{\pi a_2^2} \iint_{S_2} e^{z_2 t} J_1(r_2 t)\, r_2 \, \mathrm{d}S_2.$$

The surface integral is of exactly the same form as before; consequently

$$\frac{1}{\pi a_2^2} \iint_{S_2} e^{z_2 t} J_1(r_2 t)\, r_2 \, \mathrm{d}S_2 = \sum_{m=0}^{\infty} \frac{\Gamma(m-\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(R_2/2)^{m+1}}{m!\,(m+1)!} \varepsilon_2^{2m} t^m J_{m-1}(R_2 t).$$

Inserting this in the preceding equation, we at once obtain the generalization

of HAVELOCK's expression (2), viz.,

$$M_{n,m}=4\pi\,\frac{\Gamma(n-\frac{1}{2})\,\Gamma(m-\frac{1}{2})}{n!\,m!\,(n+1)!\,(m+1)!}\left(\frac{R_1}{2}\right)^{n+1}\left(\frac{R_2}{2}\right)^{m+1}\int_0^\infty e^{-Dt}J_{n-1}(R_1 t)J_{m-1}(R_2 t)\,t^{n+m}\,\mathrm{d}t. \quad (13)$$

As was to be anticipated, $M_{n,m}$ is symmetric in the pairs of variables (n, R_1) and (m, R_2).

In deriving equation (1), with coefficients given by (13), we tacitly assumed $D > a_1 + a_2$. However, by using a similar argument as in the case of (12), we may readily show that the convergence of (1) with coefficients specified by (13) is guaranteed if $D > \frac{1}{2}\left(\frac{a_1^2}{R_1}+\frac{a_2^2}{R_2}\right)$. The actual domain of convergence may be larger, because the type of expansion (1) is expected to exist whenever the cross-sections of the toroids do not overlap and a_1 and a_2 are small enough, including $D = 0$ (see section 4).

3. *Evaluation of the Bessel-function integrals.*

We shall now express the coefficients $M_{n,m}$ in terms of elliptic integrals, at least for $n + m \leq 2$. For the sake of convenience we write down all coefficients of the order $(n + m)$ up to and including 2.

Order zero:

$$M_{0,0}=4\,\pi^2 R_1 R_2\int_0^\infty e^{-Dt}J_1(R_1 t)J_1(R_2 t)\,\mathrm{d}t;$$

Order one:

$$M_{1,0}=\tfrac{1}{2}\,\pi^2 R_1^2 R_2\int_0^\infty e^{-Dt}J_0(R_1 t)J_1(R_2 t)\,t\,\mathrm{d}t,$$

$$M_{0,1}=\tfrac{1}{2}\,\pi^2 R_1 R_2^2\int_0^\infty e^{-Dt}J_1(R_1 t)J_0(R_2 t)\,t\,\mathrm{d}t;$$

Order two:

$$M_{2,0}=\tfrac{1}{48}\,\pi^2 R_1^3 R_2\int_0^\infty e^{-Dt}J_1(R_1 t)J_1(R_2 t)\,t^2\,\mathrm{d}t,$$

$$M_{1,1}=\tfrac{1}{16}\,\pi^2 R_1^2 R_2^2\int_0^\infty e^{-Dt}J_0(R_1 t)J_0(R_2 t)\,t^2\,\mathrm{d}t,$$

$$M_{0,2}=\tfrac{1}{48}\,\pi^2 R_1 R_2^3\int_0^\infty e^{-Dt}J_1(R_1 t)J_1(R_2 t)\,t^2\,\mathrm{d}t.$$

As we have seen, $M_{0,0}$ is expressible in terms of complete elliptic integrals. We shall now show that the same holds for the first-order coefficients. To start with we have

$$\int_0^\infty e^{-Dt}J_0(R_1 t)J_1(R_2 t)\,t\,\mathrm{d}t=-\frac{\partial}{\partial R_2}\int_0^\infty e^{-Dt}J_0(R_1 t)J_0(R_2 t)\,\mathrm{d}t=$$

$$=-\frac{\partial}{\partial R_2}\int_0^\infty e^{-Dt}\,\mathrm{d}t\,\frac{1}{2\pi}\int_0^\infty J_0(t\sqrt{R_1^2-2\,R_1\,R_2\cos\vartheta+R_2^2})\,\mathrm{d}\vartheta=$$

$$=-\frac{\partial}{\partial R_2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\vartheta}{\sqrt{D^2+R_1^2-2\,R_1\,R_2\cos\vartheta+R_2^2}}=-\frac{\partial}{\partial R_2}\left\{\frac{k\,K}{\pi\sqrt{R_1\,R_2}}\right\},$$

in which k is given by (4). In this we have used some well-known properties of Bessel functions, including the addition theorem, while the last integral has been transformed into the standard form of the elliptic integral of the first kind by choosing a new variable of integration, $\varphi = (\pi - \vartheta)/2$.

Now, by differentiating equation (4), we have

$$\frac{\partial}{\partial R_2} = \frac{k}{2R_2}\left\{1 - \frac{R_1 + R_2}{2R_1}k^2\right\}\frac{\partial}{\partial k}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Further, a known equation of the theory of elliptic integrals reads

$$\frac{\mathrm{d}K}{\mathrm{d}k} = \frac{E}{k(1-k^2)} - \frac{K}{k}.$$

With these auxiliary relations the differentiation with respect to R_2 can readily be performed; the result is found to be

$$\int_0^\infty e^{-Dt} J_0(R_1 t) J_1(R_2 t)\, t\, \mathrm{d}t = \frac{k}{2\pi R_2 \sqrt{R_1 R_2}}\left[K - E + \frac{\frac{1}{2}k^2 E}{1-k^2}\left(\frac{R_2}{R_1} - 1\right)\right]. \quad (15)$$

The corresponding expression for the first-order coefficient then is

$$M_{1,0} = \frac{\pi}{4}\frac{R_1}{R_2}\sqrt{R_1 R_2}\, k\left[K - E + \frac{\frac{1}{2}k^2 E}{1-k^2}\left(\frac{R_2}{R_1} - 1\right)\right], \quad . \quad . \quad (16)$$

while $M_{0,1}$ is obtained from (16) by interchanging R_1 and R_2.

We next turn to the second-order coefficients. By differentiating the identity [compare eqs (2) and (3)]

$$\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, \mathrm{d}t = \frac{1}{\pi\sqrt{R_1 R_2}}\left[\left(\frac{2}{k} - k\right)K - \frac{2}{k}E\right]. \quad (17)$$

with respect to D, we find that

$$\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, t\, \mathrm{d}t = -\frac{1}{\pi\sqrt{R_1 R_2}}\frac{\partial}{\partial D}\left[\left(\frac{2}{k} - k\right)K - \frac{2}{k}E\right].$$

Further, analogous to (14),

$$\frac{\partial}{\partial D} = -\frac{k^3 D}{4R_1 R_2}\frac{\partial}{\partial k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

while, in addition to the expression given for $\mathrm{d}K/\mathrm{d}k$, one has

$$\frac{\mathrm{d}E}{\mathrm{d}k} = \frac{E-K}{k}.$$

Carrying out all transformations, we finally arrive at

$$\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, t\, \mathrm{d}t = \frac{Dk}{2\pi(R_1 R_2)^{3/2}}\left[\frac{1 - \frac{1}{2}k^2}{1-k^2}E - K\right]. \quad . \quad (19)$$

A second differentiation with respect to D yields

$$\left.\begin{aligned}&\int_0^\infty e^{-Dt} J_1(R_1 t) J_1(R_2 t)\, t^2\, \mathrm{d}t = \\ &\frac{k^3}{4\pi(R_1 R_2)^{3/2}}\left[K - \frac{1-2k^2}{1-k^2}E - \frac{1}{2}\left\{\frac{1-k^2+k^4}{(1-k^2)^2}E - \frac{1-\frac{1}{2}k^2}{1-k^2}K\right\}\left\{\sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}}\right\}^2\right],\end{aligned}\right\} \quad (20)$$

in which use is made of the identity

$$\frac{k^2 D^2}{4 R_1 R_2} = 1 - k^2 - \tfrac{1}{4} k^2 \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2 . \quad . \quad . \quad . \quad . \quad (21)$$

Consequently,

$$M_{2,0} = \frac{\pi}{192} \frac{R_1}{R_2} \sqrt{R_1 R_2}\, k^3 \left[K - \frac{1-2k^2}{1-k^2} E + \tfrac{1}{2} \left\{ \frac{1-\frac{1}{2}k^2}{1-k^2} K - \frac{1-k^2+k^4}{(1-k^2)^2} \right\} \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2 \right] . \quad (22)$$

This equation might also have been obtained by differentiating (15) with respect to R_1. Again, $M_{0,2}$ is found from (22) by interchanging R_1 and R_2.

The remaining second-order coefficient, $M_{1,1}$, is determined in a similar way. Omitting details, we only mention the Bessel-function integrals occurring in the course of calculation:

$$\int_0^\infty e^{-Dt} J_0(R_1 t) J_0(R_2 t)\, t\, \mathrm{d}t = \frac{D}{4\pi (R_1 R_2)^{3/2}} \frac{k^3 E}{1-k^2} , \quad . \quad . \quad . \quad (23)$$

$$\left.\begin{aligned} &\int_0^\infty e^{-Dt} J_0(R_1 t) J_0(R_2 t)\, t^2\, \mathrm{d}t = \\ &= \frac{k^3}{4\pi (R_1 R_2)^{3/2}} \left[\frac{3-2k^2}{1-k^2} E - K + \frac{\frac{1}{4}k^2}{1-k^2} \left\{ K - \frac{4-2k^2}{1-k^2} E \right\} \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2 \right] \end{aligned}\right\} . \quad (24)$$

The last equation immediately leads to

$$M_{1,1} = \frac{\pi}{64} \sqrt{R_1 R_2}\, k^3 \left[\frac{3-2k^2}{1-k^2} E - K + \frac{\frac{1}{4}k^2}{1-k^2} \left\{ K - \frac{4-2k^2}{1-k^2} E \right\} \left\{ \sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right\}^2 \right] . \quad (25)$$

As might have been expected from the foregoing analysis, the coefficients of higher order are likewise expressible in terms of complete elliptic integrals, though it seems difficult to obtain the relevant explicit expressions. Therefore we shall not aim at giving these explicit expressions but shall indicate how the general coefficient $M_{n,m}$ may be obtained from $M_{1,0}$ and $M_{1,1}$ by a process of differentiation.

On account of symmetry we may assume that $n \geq m$. First of all we consider the case $m = 0$, $n = \nu + 1$, where $\nu = 1, 2, 3, \ldots$. Obviously $M_{\nu+1,0}$ is proportional to

$$\int_0^\infty e^{-Dt} J_\nu(R_1 t) J_1(R_2 t)\, t^{\nu+1}\, \mathrm{d}t .$$

Further, when $x = R_1 t$ is substituted in the known identity

$$\left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^\nu J_0(x) = (-1)^\nu \frac{J_\nu(x)}{x^\nu} ,$$

it is seen that

$$R_1^\nu \left(-\frac{1}{R_1} \frac{\partial}{\partial R_1} \right)^\nu J_0(R_1 t) = t^\nu J_\nu(R_1 t).$$

Consequently, $M_{\nu+1,0}$ is proportional to

$$R_1^\nu \left(-\frac{1}{R_1} \frac{\partial}{\partial R_1} \right)^\nu \int_0^\infty e^{-Dt} J_0(R_1 t) J_1(R_2 t)\, t\, \mathrm{d}t .$$

The last integral, however, is proportional to $M_{1,0}$. Therefore, $M_{\nu+1,0}$ is obtained from $M_{1,0}$ by successive differentiations with respect to R_1. The corresponding explicit equation is easily found to be

$$M_{\nu+1,0} = \frac{2^{1-\nu}\,\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\,(\nu+1)!\,(\nu+2)!}\,R_1^{2\nu+2}\left(-\frac{1}{R_1}\frac{\partial}{\partial R_1}\right)^{\nu}\left(\frac{M_{1,0}}{R_1^2}\right). \quad . \quad (26)$$

The same holds for general values of n and m, in that $M_{\nu+1,\mu+1}$ ($\nu > 0$, $\mu \geq 0$) can be obtained from $M_{1,1}$ by means of successive differentiations with respect to R_1 and R_2, viz.,

$$\left.\begin{aligned} M_{\nu+1,\mu+1} = {} & \frac{2^{2-\nu-\mu}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\mu+\frac{1}{2})}{\pi\,(\nu+1)!\,(\mu+1)!\,(\nu+2)!\,(\mu+2)!}\,R_1^{2\nu+2}\,R_2^{2\mu+2}\,\cdot \\ & \cdot\left(-\frac{1}{R_1}\frac{\partial}{\partial R_1}\right)^{\nu}\left(-\frac{1}{R_2}\frac{\partial}{\partial R_2}\right)^{\mu}\left(\frac{M_{1,1}}{R_1^2\,R_2^2}\right). \end{aligned}\right\} \quad . \quad . \quad (27)$$

It will now be evident that $M_{n,m}$ is expressible in terms of elementary functions and complete elliptic integrals of the first and second kinds to modulus k [see eq. (4)] since $M_{1,0}$, $M_{1,1}$, $\mathrm{d}K/\mathrm{d}k$, $\mathrm{d}E/\mathrm{d}k$, $\partial k/\partial R_1$ and $\partial k/\partial R_2$ are all expressible in that way.

There exist numerous other relations between the M-functions. For instance,

$$M_{\nu+1,\mu+1} = \frac{(\nu-\frac{1}{2})\,(\mu-\frac{1}{2})}{4\,(\nu+1)\,(\nu+2)\,(\mu+1)\,(\mu+2)}\,R_1^{2\nu+1}\,R_2^{2\mu+1}\,\frac{\partial^2}{\partial R_1\,\partial R_2}\left\{R_1^{-2\nu}\,R_2^{-2\mu}\,M_{\nu,\mu}\right\}, \quad (28)$$

which is easy to derive from (27). We shall no longer deal with these questions, however.

Finally, it may be noted that the results (15), (17), (19), (20), (23), and (24) are not all new; some of these Bessel-function integrals have been studied by VAN WIJNGAARDEN (ref. 6), while others are to be found in WATSON's book (ref. 7) for $R_1 = R_2$.

4. *Approximate expressions for the mutual inductance.*

We shall now develop an approximate expression for M including terms of the fourth order in the radii of the cross-sections. The term independent of the radii a_1 and a_2 is $M_{0,0}$. The first-order correction includes terms with a_1^2 and a_2^2; it is represented by

$$M_{1,0}\,(a_1/R_1)^2 + M_{0,1}\,(a_2/R_2)^2,$$

while the next correction term is given by

$$M_{2,0}\,(a_1/R_1)^4 + M_{1,1}\,(a_1/R_1)^2\,(a_2/R_2)^2 + M_{0,2}\,(a_2/R_2)^4.$$

Inserting the values of the coefficients as found in the preceding section, we have [2])

$$M \approx 4\,\pi\,\sqrt{R_1\,R_2}\,\{F_0 + F_1 + F_2\}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

[2]) See preceding footnote.

in which

$$F_0 = \left(\frac{2}{k} - k\right) K - \frac{2}{k} E, \quad \ldots \quad (30)$$

$$F_1 = \frac{k}{16} \left\{ \frac{a_1^2 + a_2^2}{R_1 R_2} (K - E) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \left(\frac{a_1^2}{R_1} - \frac{a_2^2}{R_2}\right) \frac{\frac{1}{2} k^2 E}{1 - k^2} \right\}, \quad (31)$$

$$F_2 = \frac{k^3}{768} \left\{ A + B \left(\sqrt{\frac{R_1}{R_2}} - \sqrt{\frac{R_2}{R_1}} \right)^2 \right\}, \quad \ldots \quad (32)$$

$$A = \frac{1}{R_1 R_2} \left(\frac{a_1^4}{R_1^2} + \frac{a_2^4}{R_2^2}\right) \left(K - \frac{1 - 2k^2}{1 - k^2} E\right) + \frac{3 a_1^2 a_2^2}{R_1^2 R_2^2} \left(\frac{3 - 2k^2}{1 - k^2} E - K\right), \quad (33)$$

$$\left.\begin{aligned} B = \frac{1}{2 R_1 R_2} \left(\frac{a_1^4}{R_1^2} + \frac{a_2^4}{R_2^2}\right) \left(\frac{1 - \frac{1}{2} k^2}{1 - k^2} K - \frac{1 - k^2 + k^4}{(1 - k^2)^2} E\right) + \\ + \frac{3 a_1^2 a_2^2}{4 R_1^2 R_2^2} \frac{k^2}{1 - k^2} \left(K - \frac{4 - 2k^2}{1 - k^2} E\right). \end{aligned}\right\} \quad (34)$$

These expressions are simplified when the radii of the toroids are chosen equal, $R_1 = R_2 = R$. They become very simple if in addition the radii of the cross-sections are the same, $a_1 = a_2 = a$. In the latter case we obtain, when $D > 2a$,

$$\left.\begin{aligned} M^* = 4\pi R \left[\left(\frac{2}{k} - k\right) K - \frac{2}{k} E + \frac{k}{8} (K - E) \left(\frac{a}{R}\right)^2 + \right. \\ \left. + \frac{k^3}{768} \left(\frac{7 - 2k^2}{1 - k^2} E - K\right) \left(\frac{a}{R}\right)^4 + \ldots \right], \end{aligned}\right\} \quad (35)$$

where the modulus of the elliptic integrals now is given by

$$k = \{1 + (D/2R)^2\}^{-1/2}. \quad \ldots \quad (36)$$

Finally, the approximate expression (29), which has been derived on the assumption that $D > a_1 + a_2$, R_1 and R_2 arbitrary, remains valid if $D \geq 0$ provided a_1 and a_2 are small enough and $|R_1 - R_2| > a_1 + a_2$, because the type of expansion (1) exists whenever the toroids do not overlap and a_1, a_2 are sufficiently small. To realize this, we observe that the coefficients $M_{n,m}$ might have been obtained by expanding $M_{0,0}$ in a Taylor series of four variables (the coordinates pertaining to the cross-sections) and integrating over the cross-sections. The result then would have been a double power series in a_1 and a_2 with coefficients determined by partial derivatives of $M_{0,0}$ with respect to D, R_1 and R_2, and of non-vanishing domain of convergence.

The same conclusion is reached by observing that our analysis might also have been based on the left part of the identity

$$\frac{2}{\pi} \int_0^\infty \cos zt \, K_0(rt) \, dt = \int_0^\infty e^{-|z|t} J_0(rt) \, dt,$$

instead of on the right part, both integrals representing the function $(z^2 + r^2)^{-1/2}$. For example, the coefficient $M_{0,0}$ would become

$$M_{0,0} = \begin{cases} 8\pi R_1 R_2 \int_0^\infty \cos Dt\, I_1(R_1 t)\, K_1(R_2 t)\, dt, & (R_1 < R_2) \\ 8\pi R_1 R_2 \int_0^\infty \cos Dt\, K_1(R_1 t)\, I_1(R_2 t)\, dt. & (R_1 > R_2) \end{cases}$$

Similar expressions hold for $M_{n,m}$, which may be considered as analytic continuations of (13) to $D = 0$. Whether we use the first representation or the second, the final result is in any case that given by equations (29) through (34).

In concluding the paper, we remark that in the case of coplanar toroids ($D = 0$) expression (32) is simplified to

$$F_2 = \frac{k}{768}\left[\frac{1}{R_1 R_2}\left(\frac{a_1^4}{R_1^2} + \frac{a_2^4}{R_2^2}\right)\left(2K - \frac{2-k^2}{1-k^2}E\right) - \frac{3a_1^2 a_2^2}{R_1^2 R_2^2}\frac{k^2}{1-k^2}E\right], \quad (37)$$

in which now

$$k = \frac{2\sqrt{R_1 R_2}}{R_2 + R_2}. \quad \ldots\ldots\ldots \quad (38)$$

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