Mathematics. — *On the dissection of rectangles into squares.* (Third communication.) By C. J. BOUWKAMP. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

7. *Some special squared rectangles.*

Two differently squared rectangles are called *congruent* if their reduced sides are two by two equal. They are called *conformal* if their reduced sides are proportional. From the list in section 6 we can easily obtain all cases of conformal or congruent rectangles of order less than 14.

The simplest example of two conformal rectangles is provided by IX, 130, *c* and XII, 585, *f*, in such a sense that they have together 21 elements, 21 being the least possible number of elements totally involved. This example was already given in A (in that paper one should read $cf = 55$, instead of 15, p. 327). It may be noticed that the 12-th order squaring above shows two horizontal line segments at a same level, corresponding to a pair of equipotential vertices in the network from which it was derived. When this pair of conformal rectangles is made equal in size, they will show two elements in common. The imperfect squaring XIII, 1040, *f'* is also conformal with either of the squarings above.

Next there are two pairs of conformal rectangles containing 10 and 13 elements. The first pair is provided by X, 224, *a*, and XIII, 1008, *b*, whilst the second pair is formed by X, 224, *b* and XIII, 1008, *e*. Either of the 13-th order squarings contains two horizontal line segments at a same level. Upon transformation on the same size, four common elements are found in both cases.

Furthermore, the 13-th order squarings XIII, 1060, *e* and *f* are conformal; they show five common elements after transformation on equal size.

The following cases of conformal rectangles all include imperfections: XIII, 1088, *k'*, *l'*, *m'*, two of which are even congruent, and XIII, 1088, *g'*, *h'*, *i'*, *j'*, three of which are congruent.

The remaining conformal rectangles are all of the congruent type. Two congruent rectangles of different full sides are XII, 615, *d* and XIII, 1025, *g*; they have one element in common. Also congruent and of different full sides are XIII, 935, *b* and XIII, 1122, *a*. It is interesting to note that they contain the same set of elements. *It is the simplest example of building up a rectangle with the same set of elements in two different ways*, as is seen by inspection of the list in section 6. This example is not new, as it was already derived in paper A, from certain general considerations, starting from networks with a pair of equipotential vertices.

All remaining cases of congruence, as far as the list of section 6 is concerned, are such that they give pairs of squarings of the same order and the same full sides. The simplest example is provided by XII, 608, *f* and *g*. They have four elements in common [1]).

A pair of congruent simple squarings of order 13, one of which is imperfect, is formed by XIII, 1025, *b'* and *c*.

Five common elements occur in each of the following pairs of congruent perfect squarings:

XIII, 928, *h* and *i*; XIII, 992, *f* and *g*; XIII, 1088, *o* and *p*.

Four common elements in

XIII, 1015, *j* and *k*; XIII, 1088, *e* and *f*; XIII, 1115, *j* and *k*.

Three common elements in

XIII, 1073, *k* and *l*.

One common element in

XIII, 1015, *d* and *e*; XIII, 1073, *b* and *c*.

Last but not least, there is in our list a pair of congruent squarings with no common elements, namely XIII, 1015, *g* and *h*. *In fact it is the simplest example of a rectangle of given order that can be dissected into two*

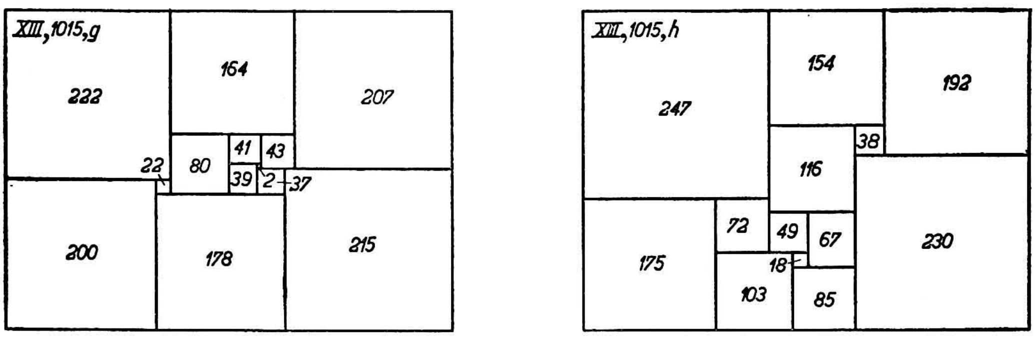

Fig. 9. The simplest example of congruent squarings with completely different sets of elements.

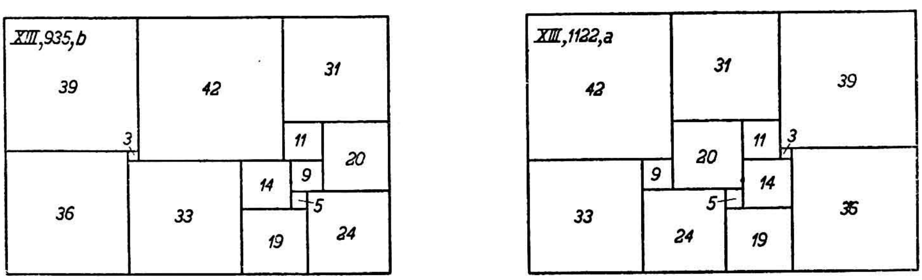

Fig. 10. The simplest example of congruent squarings with just the same set of elements differently arranged.

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1) See also paper A, p. 330; the concept of "equivalence" in A is completely different from that defined in section 3. In the sense of paper A, equivalent squarings are conformal and have the same full sides.

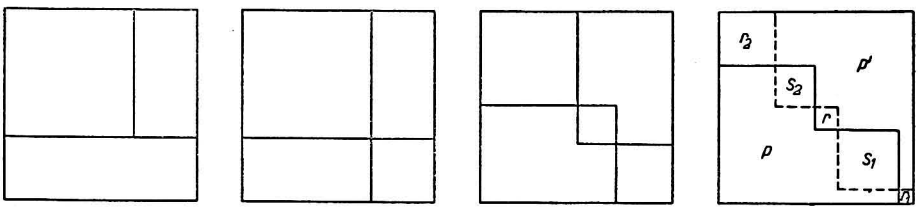
completely different sets of squares. It may be remarked that they originate from essentially different networks, though their complexity is the same.

Figs 9, 10 show the simplest examples of congruent rectangles that contain either a completely different set of elements, or just the same elements differently arranged.

8. *Perfect squares.*

We have already given the simplest example of a simple squared square; cf. fig. 1. It is not at all perfect, however, as it contains double elements. One naturally asks whether a simple *perfect* square can be constructed. The authors of paper A have given a device how this may be done. They pretend to have constructed "a simple 'uncrossed' perfect square of order 55, which, when drawn out, disguises its symmetrical origin very skilfully" (A, p. 334). In this section, however, we shall prove that actually the suggested construction *fails,* and so we are led to the conclusion that, unfortunately, there is no simple perfect squared square known at present.

We first show how *compound* perfect squares can be obtained. Three different methods were developed in paper A.



α β γ δ

Fig. 11. Several methods for the construction of perfect squared squares.

1. Fig. 11, a shows a compound square, consisting of one square and two rectangles which in their turn are supposed to be dissected into squares.

From careful examination of the list given in section 6 it appears that only *three* different solutions exist if both rectangles are subject to the condition that their orders do not exceed 13. *Two* solutions are non-trivially imperfect, as for each of them both rectangles are non-trivially imperfect themselves. One solution contains a square of side 16, together with the squared rectangles XIII, 962, a' and XII, 585, a'. The other consists of a square 22, together with the squarings XII, 663, b' and XIII, 1040, a'.

Only the third solution happens to be perfect. It contains a square of side 231, together with the perfect squarings XII, 608, a and XIII, 985, a. Thus we have obtained a perfect square of order 26. Presumably *it is the simplest* (that means of minimum order) *perfect squared square known at present.* It was already given in paper A; its code has been given in section 1 of the present paper.

2. Fig. 11, β shows a compound square, consisting of two squares and two rectangles of equal size which in their turn may be dissected into squares.

Obviously, it is not difficult to construct non-trivially imperfect squares of this character: we merely take twice any rectangle of our list in section 6. On the other hand, there is in our list only one couple of squarings leading to a perfect square. This square is of the order 28, containing two squares of sides 422, 583, respectively, together with the pair of totally different congruent squarings of fig. 9.

3. Cf. fig. 11, γ.

For this purpose two perfect squarings of equal size are required, having only one element in common, subject to the condition that this common element is corner element in either of the two rectangles. In paper A a device was given how to construct a pair of such special rectangles. One of the simplest examples of perfect squares, constructed in this way, is of the order 39. Its code reads as follows:

(900, 393, 520) (263, 130) (3, 224, 293) (133) (244, 152) (155, 69) (61, 91) (362) (31, 30) (276) (476, 205, 219, 275) (191, 14) (177, 56) (113, 218, 638) (8, 105) (80, 199, 97) (437, 119) (420) (318).

All squares so far obtained are compound. We now investigate in detail the method that (according to the authors of A) would lead to simple perfect squared squares.

Consider the "rotor" network of fig. 12, with terminals A_1, A_2, A_3. Let its wires have unit conductance, and let currents $87a$, $87b$, leave the network at A_2, A_3, respectively. The current entering at A_1 must then be $87\ (a + b)$. The complete set of currents is uniquely determined, and is shown in fig. 12. The currents are integral linear combination of a and b. Without lack of generality, we may suppose a and b to be integers, subject to $0 < b < a$.

This network is a generalization of the "polar" networks treated before, in so far that now more than two terminals are present. It corresponds to a *squared polygon* of angles $\pi/2$ and $3\pi/2$. For example, the rotor network of fig. 12, in action, corresponds to a squared polygon P the dimensions of which are shown in fig. 13.

The typical corner elements C_1, C_2 (shaded in fig. 13) have sides $27a - 8b$, $8a + 35b$, respectively. It must be noted that the situation of fig. 13 is possible only if $27a - 8b < 49\ (a - b)$ and $8a + 35b < 87b$; thus $41/22 < a/b < 13/2$. Otherwise at least one of the corner elements is too large. If the inequality above is not fulfilled, it is impossible to draw in fig. 13 the rectangle R which is important in the further construction.

The vertical left side of the polygon may be considered as the terminal A_1, and the remaining vertical boundaries at the right correspond to A_3, A_2. The current flows horizontally from left to right. The ingoing current $87(a + b)$ is equal to the left vertical side, the two outgoing currents $87a$, $87b$, are equal to the other vertical boundaries.

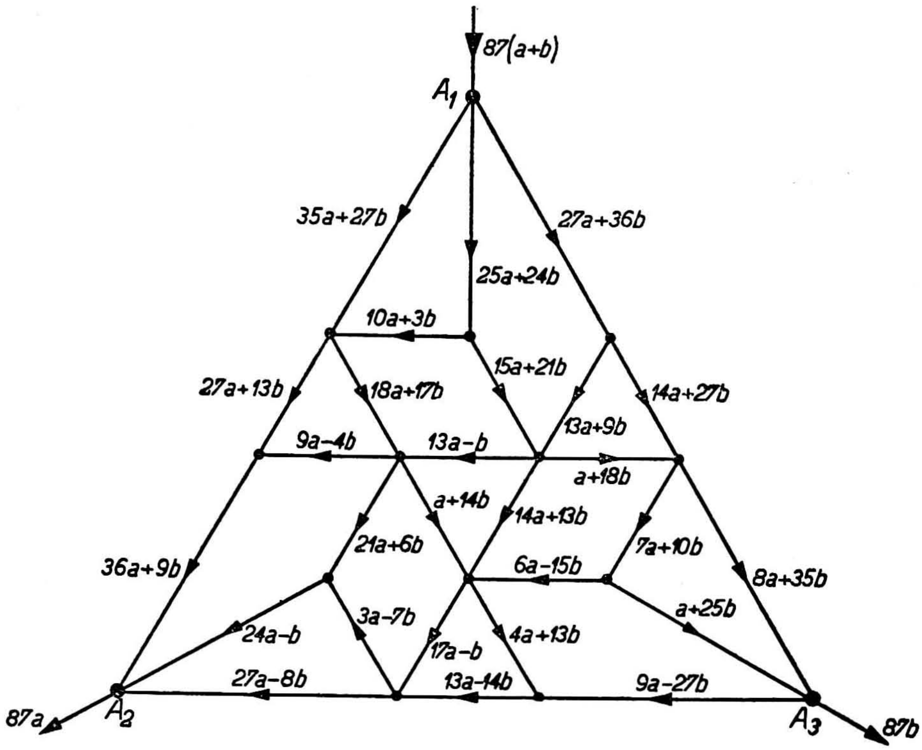


Fig. 12. Currents in a typical rotor network.

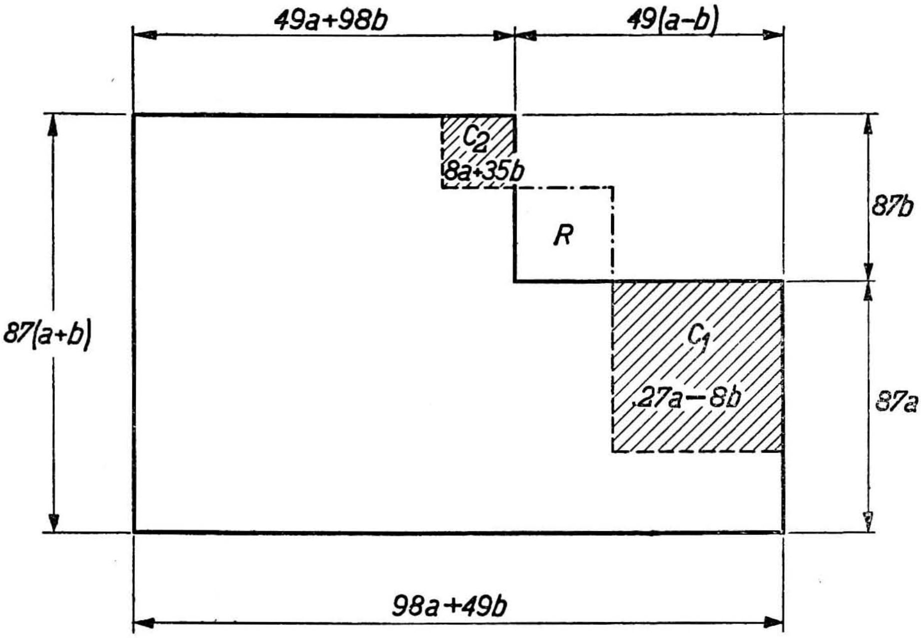


Fig. 13. Dimensions of the polygon corresponding to the rotor of fig. 12.

If the skew-symmetrical rotor network of fig. 12 is replaced by its reflection (leaving the currents at A_1, A_2, A_3, invariant), the new squared polygon P' will have the same shape as the old one P; this follows from the triad symmetry of the rotor. The set of currents in the reflected rotor are easily found from those in fig. 12. We also could have interchanged a and b, without reflecting the rotor; we prefer, however, the former method, in order to have always $a > b$ in the following.

The corner elements C'_1, C'_2 of P' have sides $27a - 9b$, $9a + 36b$, respectively. In order to be able to draw the analogous rectangle R', another condition has to be fulfilled, namely $40/22 < a/b < 17/3$. If both P and P' have existing rectangles R, R', we thus have the condition

$$41/22 < a/b < 17/3.$$

Let us now consider fig. 11, δ. It contains two congruent polygons; one of them (p) has full-drawn boundaries, the other (p') is partly dashed. Suppose (i) either of the two polygons is perfectly squared, (ii) the polygons p, p' have no common elements, except the pairs of typical corner elements $c_1 = c'_1 = S_1$, and $c_2 = c'_2 = S_2$, overlapping two by two. Then it clearly follows: r, r_1, r_2 are squares. Furthermore, if these squares are mutually unequal, and none of them is equal to some element of p, p', then fig. 11, δ obviously leads to a *simple perfect squared square*. The elements not drawn out in fig. 11, δ belong to p, p'.

This is in fact the construction proposed in paper A, in order to obtain simple perfect squares. For an actual example we only need two suitably chosen polygons p, p'. The authors of A suggest that the rotor network of fig. 12 leads to such a pair of polygons. This, however, is not true, as will be seen below.

It is important to note that the shape of P, P' can be varied by varying a and b. Obviously, there is only *one* degree of freedom, as only the ratio a/b is significant. It must be taken in mind that the elements of P, P' alter correspondingly, by variation of a/b. Now, P and P' can be used as p, p' respectively, if, and only if, $C_1 = C'_1$; $C_2 = C'_2$. That means

$$27a - 8b = 27a - 9b,$$
$$8a + 35b = 9a + 36b.$$

These independent conditions admit only $a = b = 0$. Hence *the construction of fig* 11, δ *fails for the rotor network of fig.* 12.

This can also be seen in a somewhat differing manner. If P is to be used as p, then the rectangle R in fig. 13 has to be a square, corresponding to r in fig. 11, δ. From this condition it follows $87b - (8a + 35b) = = 49(a - b) - (27a - 8b)$; thus $10a = 31b$. For the special values $a = 31$, $b = 10$ (the inequality above is fulfilled!), one can only *hope* that the now uniquely determined polygon P' can be used as the counter part p' of $p = P$. This, however, is not so. R' is even not a square, let alone it is equal to R, as is required. In fact, R' is a square, if, and only if, $a = 91$, $b = 31$.

Thus the construction fails. There is, consequently, no simple perfect squared square known at present. Moreover, it will be very unlikely, that for some other rotor network the above construction is effective.

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