Mathematics. — *On the dissection of rectangles into squares.* (First communication.) By C. J. BOUWKAMP. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 30, 1946.)

1. *Introduction.*

Many mathematical problems originate from early puzzling and recreation. For instance, the problem of magic squares. Less known is the puzzle problem treated below, namely that of the dissection of a rectangle of commensurable sides into a finite number of non-overlapping unequal squares, by means of straight line segments drawn parallel to the sides of the rectangle. The latter problem is, unlike that of magic squares, of very recent times only.

The first example of a rectangle, divided into (nine) incongruent squares, was apparently given by MOROń in 1925. It was also published in various books on amusement-mathematics, such as KRAITCHIK's "La Mathématique des Jeux".

From the present author's own experience it may be concluded that skilful puzzlers do not encounter great difficulties in constructing such a squared rectangle, containing a small number (10, say) of squares. The question to construct all the possible rectangles with 10 squares, however, will not be so easily answered. In this connection it may be remarked that another nine-squares-solution was published in 1940 only [1]).

Already DEHN remarked that the difficulty of the problem is the semi-topological one of characterizing how the various squares fit together. This difficulty has been completely overcome by the authors of paper A. Curiously they succeeded to associate a squared rectangle with the flowing of electric currents in certain networks. This provides a typical example of how to overcome mathematical difficulties by adequate physical reasoning.

The afore-mentioned authors have proved that 9 is the minimum number of two by two unequal squares that can completely build up a rectangle without overlapping one another. Moreover they have shown that there exist two ninth-order [2]) solutions, apart from those obtained by trivial transformations, such as reflections, rotations. Furthermore they found

---

[1]) R. L. BROOKS, C. A. B. SMITH, A. H. STONE and W. T. TUTTE, The dissection of rectangles into squares, Duke Math. J. **7**, 312—340 (1940).

Henceforth this fundamental paper will be quoted by the letter A; special attention may be drawn to the bibliography at the end of A.

[2]) The finite number of squares in a dissection is called the *order* of the squaring.

6 solutions of order 10, 22 of order 11, and 67 of order 12. Only for squared rectangles of order 9 and 10 were the elements [3]) of the dissection explicitly given, whilst for squarings of order 11 only the "full" sides were specified in paper A.

Basing ourselves on the fundamental paper A, we have constructed the squarings of order less than 14. We found their total number amounting to 311; 214 of them are of order 13, the remainder being as specified above.

The following definitions are frequently used in A as well as in the present article. A squaring is called *perfect* if all the squares of the dissection are two by two unequal. Otherwise the squaring is called *imperfect*. We are mostly interested in perfect squared rectangles, though a certain type of imperfect rectangles will be considered too, namely the so-called *non-trivially imperfect* ones. The latter may contain equal elements; only in such a sense, however, that equal elements have never a side in common; nor must it be possible to get them in such a position by a trivial displacement of some of the elements. The remaining imperfect squarings, called *trivially imperfect*, are not investigated here.

A squaring is called *compound* if it is possible, by suitably omitting of some (not all, but at least one) dissecting line segments, to get the original rectangle dissected (by the remaining segments) into rectangles, not necessarily squares. If this is not possible, the squaring is called *simple*.

Furthermore, a compound squaring is called *trivially compound* if one of the elements has a side equal to one of the sides of the squared rectangle under consideration; when this element is omitted, a squaring of order one less can be obtained. Conversely, once a squaring of order n is given, one can readily obtain two different trivially compound squarings of order $n+1$, by merely introducing an extra $(n+1)$-th square whose side equals one of the two sides of the given n-th order rectangle. Compound squarings that are not trivially compound are called *non-trivially compound*.

It will be clear that in case of simple squarings the character of imperfection is only non-trivial.

As already stated, our investigation is not restricted to *perfect* squarings. We rather consider all the *simple* ones, whether they are perfect or imperfect. In addition to the numbers of squarings given previously, which are in fact the *simple perfect* ones, there are 43 *simple imperfect* squarings, namely 1, 0, 0, 9, 33, of order 9, 10, 11, 12, 13, respectively.

Our main aim is then to classify those 354 simple squarings of order less than 14.

Of course, there still remain some other perfect rectangles, namely the compound ones. It can be shown that all the compound perfect squarings of order less than 14 are trivially compound, with only one exception. The

---

[3]) The squares of a squared rectangle are called *elements* of the squaring.

exceptional case, which is thus non-trivially compound, consists of a rectangle dissected into 4 squares and 1 rectangle, the latter in its turn being dissected into 9 unequal squares (details will be given in section 5). The trivially compound perfect squarings can be derived in a trivial manner from the simple perfect ones of lower order, namely by introduction of a new element whose side is equal to one of the sides of the original simple perfect squaring. There are 4, 16, 60, 195, compound perfect squarings of order 10, 11, 12, 13, respectively.

In addition to the number of 275 compound perfect squarings above, there are similarly 28 compound non-trivially imperfect ones, namely 2, 2, 2, 22, of order 10, 11, 12, 13, respectively. They can be derived in a trivial manner from the simple imperfect squarings of order less than 13, with only two exceptions which will be given in section 5.

Only the trivially imperfect (which are compound too) squarings are not yet enumerated. Unfortunately these squarings cannot be found in some trivial manner. Furthermore, without the restriction on imperfection, the total number of squared rectangles would become too large for adequate classification. For instance, there are already 1, 1, 2, 5, 11, 29, trivially imperfect squared rectangles of order 1, 2, 3, 4, 5, 6, respectively. These are the reasons why we omit the trivially imperfect rectangles in our further investigation.

A complete specification of the various types of squared rectangles of order not exceeding 13 is given in table I.

TABLE I.

Numbers of squared rectangles of different type, and of order less than 14. Only the trivial imperfections are excluded (the latter show equal elements lying aside, and thus belong to the compound type).

Type \ Order	9	10	11	12	13	Total
Simple, perfect	2	6	22	67	214	311
Simple, imperfect	1	0	0	9	33	43
Trivially compound, perfect	0	4	16	60	194	274
Non-trivially compound, perfect	0	0	0	0	1	1
Trivially comp., non-trivially imperfect	0	2	2	2	20	26
Non-triv. compound, non-triv. imperf.	0	0	0	0	2	2
Perfect	2	10	38	127	409	586
Non-trivially imperfect	1	2	2	11	55	71
Total	3	12	40	138	464	657

It is interesting to note that there is only one *square* amongst the total number of 657 squared rectangles so far obtained. It is of the order 13;

although it is simple, it is to a high degree imperfect, as it contains 5 pairs of equal elements. *It provides the simplest example of a simple squared square,* in so far that 13 is the minimum possible order of such a square. This remarkable square is drawn out in fig. 1; the various numbers correspond to the relative linear sizes of the elements.

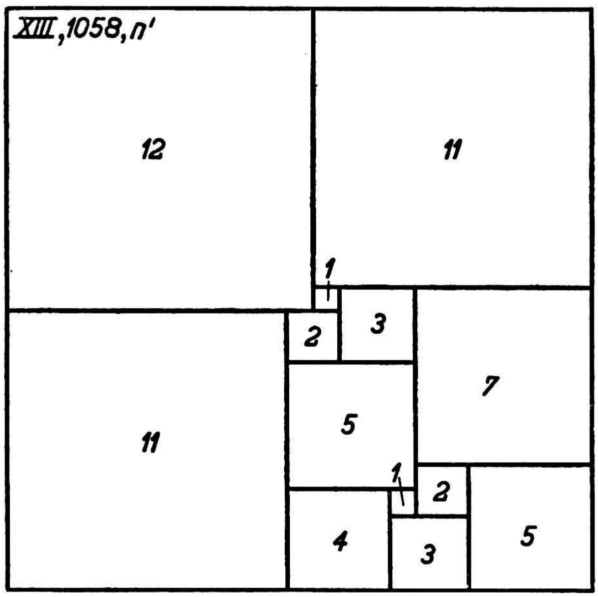

Fig. 1. The simplest example of a simple squared square.

It is of course impossible to draw out in this journal all the 657 squared rectangles as specified in table I. Even the total number of 354 simple squarings would require too much space. The difficulty, however, can be easily overcome by a suitable *coding system.*

First we suppose the rectangle to be drawn out in such a manner that its largest sides are horizontal. Then, the element in the upper-left corner should not be smaller than the three remaining corner elements. Now, for orders not exceeding 13, the geometrical configuration of the simple squarings appears to be unique except in the case of the square of fig. 1. Henceforth we will always "orient" a squared rectangle in the above sense before the proper coding starts, though this is not necessary.

Now the given oriented rectangle is squared by horizontal and vertical line segments. Consider the group of elements with their upper horizontal sides in a common horizontal segment. The individual elements of this group are conveniently ordered by a reading from left to right. The various groups themselves are ordered according to upwards-downwards-reading, starting with the upper horizontal side of the given rectangle. If necessary, line segments at the same horizontal level are ordered from left to right too. In the written code the various groups are separated by parentheses, the elements of a group by commas. Thus the code of the squared square of fig. 1 reads as follows

$$(12, 11)\ (1, 3, 7)\ (11, 2)\ (5)\ (2, 5)\ (4, 1)\ (3).$$

This system of coding provides us with a very simple method to pass over from code to figure, and vice versa. So MOROŃ's solution (cf. fig. 4 γ) is coded as

$$(18, 15)\ (7, 8)\ (14, 4)\ (10, 1)\ (9),$$

and the other perfect nine-squares-rectangle is (see fig. 4 β)

$$(36, 33)\ (5, 28)\ (25, 9, 2)\ (7)\ (16).$$

The coding of compound squarings is similar. As an example we may give the code of a *perfect squared square* of order 26, published in paper A, fig. 9, p. 333, namely

$$(231, 136, 123, 118)\ (5, 113)\ (20, 108)\ (95, 34, 7)\ (27, 61)$$
$$(209, 205, 194)\ (11, 183)\ (44, 172)\ (168, 41)\ (1, 43)\ (42, 85).$$

This squared square is compound, consisting of one square and two rectangles which in their turn are dissected into 12 and 13 squares, respectively. Probably it is the simplest (that means of minimum order) perfect squared square, known at present.

The few examples above clearly demonstrate the usefulness of our coding system.

2. *Squared rectangles from an electrical point of view* [4]).

Let us suppose an oriented squared rectangle to be drawn upon a thin metal plate. The upper and lower side of the rectangle be electrodes of infinitely conducting material. Next a downwards flow I of electric current may exist through this plate, and due to a potential difference V across the electrodes. By a suitable choice of units we can suppose the number I of the total current to be equal to the value of the horizontal side of the rectangle. In the same manner V may be taken equal to the height of the rectangle. The flow is homogeneous; stream lines are vertical, equipotential lines are horizontal. Let us now make infinitely thin cuts along the *vertical* line segments; this does not influence the flowing. The various squares still remain connected to one another by means of the *horizontal* line segments.

We have now obtained an electrical "network", in which the current I is streaming through "wires", each of which is a thin metal plate. Every such "wire" is flown through by a certain current i, while its ends show a potential difference v. On account of our choise of units, the ratio v/i is

[4]) We are indebted to Prof. VAN DER POL and Mr VAN DER MARK for most valuable discussions.

equal to 1 for all the "wires"; that means their ohmic resistances are all equal to 1.

We want to have this "network" transformed into a more conventional one. Theretofore let us consider some "wire". We cut it along the upper and lower side, from left as well as from right, up to its middle vertical [5]). The freecoming left- and right-hand parts of the square plate are then rolled up towards the middle vertical, and afterwards melted together in the form of a true wire. The current through the "wire" is the same as that in the true wire. The same holds with regard to the potential difference v across it. The network still remains connected via the horizontal line segments, once the transformation is applied to all elements. Finally, we contract each infinitely conducting horizontal line segment to a single point. Especially, the horizontal sides of the rectangle are transformed into the terminals or *poles* of the network. Each element of the squared rectangle now corresponds to a *wire*, each horizontal line segment to a *vertex*, and finally each vertical line segment to a *mesh*, not containing other parts of the network in its interior.

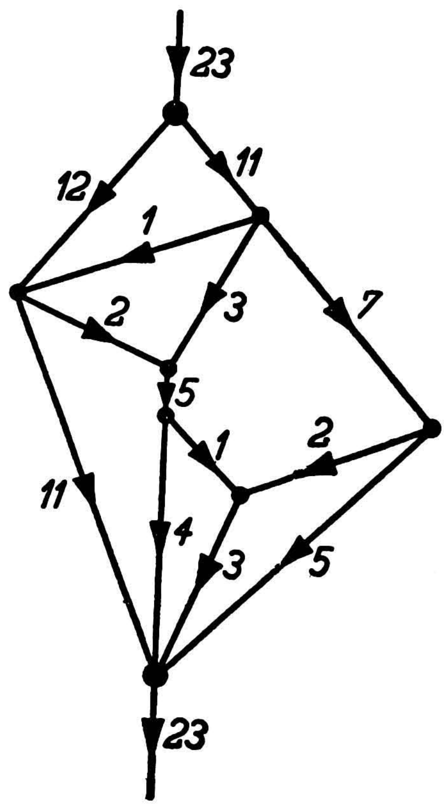

Fig. 2. Currents in the network corresponding to the squared square of fig. 1.

---

5) We suppose the horizontal line segments to be of perfectly conducting material.

The network corresponding to the squared square of fig. 1 is given in fig. 2. The arrows denote the directions of the currents, the numbers their relative magnitudes. If this network is realized by ohmic resistances of 1 ohm, then a potential difference of 23 volts between the poles causes a current of 23 amperes through the network. Therefore, the substitutional resistance of this special network is also equal to 1 ohm. In general the resulting resistance will be smaller, because for an oriented rectangle $I > V$. Hence, from an engineer's point of view, the network of fig. 2 is very remarkable: it provides us with the simplest planar [6]) network, without any series or parallel connection, that can be built up with equal resistances, all showing non-zero currents, such that the resulting resistance is equal to that of the individual wires.

It is evident that the energy $V.I$, delivered to the network by the source, is transformed into Joule heat $\Sigma\, vi$; indeed, the area of the rectangle is equal to the sum of the areas of the individual squares.

This association of linear graphs with squared rectangles was first derived in the quoted paper A, though not so physically as we did above. We showed that the squaring is a real network; the wires are only square metal plates.

It is convenient to join the poles of the network by an extra wire that contains the exterior source. The network obtained in this manner is planar, as is obvious from the construction above. A further simplification can be arranged by projecting the network upon a sphere because then all the wires, including that containing the source, are topologically equivalent as far as the concepts of "interior" and "exterior" are concerned.

Thus, starting with an oriented squared rectangle of order n, we obtain, in a unique way, a network on the sphere containing $n + 1$ wires which do not cross each other. It will be clear that a similar construction holds if one starts with a rectangle, dissected into rectangles that are not necessarily squares.

Conversely, starting with a planar network on the sphere, containing $n + 1$ wires (having general values of resistance), and after placing a source in one of them (no matter which one), we are led to an electrical network in which the currents and voltages are calculable by the laws of KIRCHHOFF. This network can be drawn in a plane in such a way that the poles lie at the outside, and are joined by the source-containing wire, which will be omitted further on. The resulting "polar" network, together with its currents and voltages, can be moulded into the unconventional form consisting of rectangular plates. The final result is therefore a rectangled rectangle.

In case of equal resistances, the rectangle becomes squared; its order is *n or less*. The latter addition is necessary because, accidently, some wires may be current-free, due, for instance, to properties of symmetry (in elec-

---

6) A network is called *planar* if it can be drawn in a plane without crossings.

tricity, wires with zero-currents are called "conjugate" to that containing the source). Obviously, a zero-current wire corresponds to a square of vanishing dimensions. This reduction of the order may also occur in case of unequal resistances, though in general n will be the actual number of elements in the rectangled rectangle.

It is clear that the difficulty to describe how the squares fit together has now been overcome completely. The question how to obtain the possible squarings of order n is reduced to a merely combinatorial problem: how to obtain the possible topologically different connected networks, involving $n + 1$ wires, that can be drawn on a sphere without crossings. Every such network will then give rise to a *class* of squared rectangles. The number in a class is at most $n + 1$, corresponding to as many places for the exterior source.

3. *Duality. Bounds for the numbers of vertices and meshes in terms of the numbers of wires.*

In electricity the interchanging of voltages and currents is governed by the *principle of duality*. Its interpretation in terms of rectangled rectangles is as follows. Instead of taking the electrodes along the *horizontal* sides of the rectangle, we may take them along the *vertical* ones. Accordingly, stream lines and equipotential lines have interchanged [7]); the same holds with regard to the vertices and meshes. The ohmic resistances of the respective polar networks are obviously reciprocal; so are the resistances of corresponding wires. Thus, these two polar networks are *electrically dual*. The corresponding "completed" (after joining the poles by the extra wire) networks N, N' on the sphere (we ignore the numerical values of the resistances) are said to be *topologically dual*. Unless otherwise stated, we use here the concept of duality in a topologically sense only.

Dual networks can be drawn in such a manner that the vertices of either of them lie inside the corresponding meshes of the other, whilst corresponding wires, and only these, cross each other [8]).

Roughly spoken, only one half of the total number of networks needs be investigated, as a pair of dual networks leads to one class of rectangled rectangles. Similarly, in case of self-dual networks, only half the number of wires needs insertion of an exterior source.

Our aim is to obtain the possible squarings up to a certain order with the only restriction that, in case of imperfection, equal elements do not lie aside, i.e. have no sides in common. Therefore, in the polar networks, simple

[7]) "Cross-points", which are common to four elements of the rectangling, should be first removed by suitable small displacements of some of the line segments; if not, there remains an ambiguity in the "cutting" process.

[8]) Cf. B. D. H. TELLEGEN, Geometrical configurations and duality of electrical networks, Philips tech. Rev. **5**, 324—330 (1940).

——————, Meetkundige configuraties en dualiteit van electrische netwerken, Tijdschr. Nederl. Radiogenootschap **9**, 37—60 (1941).

wires in series or parallel connection are forbidden. Furthermore, we may suppose that even the "completed" network on the sphere does not contain such connections because it would give either forbidden imperfect rectangles or trivially compound ones; in the latter case the largest element of the squaring would lie along the whole of one of the sides of the rectangles. As we already saw, the trivially compound squarings are not very interesting because they can be obtained in a trivial manner from squarings of lower order.

Let N be a planar network, containing T wires, K vertices, and M meshes. On account of EULER's polyhedron formula one has

$$K+M=T+2. \quad \ldots \ldots \ldots \quad (1)$$

As N does not contain any pair of wires in series, at least three wires come together in each vertex. Therefore, twice the number of wires cannot be less than three times the number of vertices:

$$K \leqslant 2T/3. \quad \ldots \ldots \ldots \quad (2)$$

Together with (1), this yields for the number of meshes

$$M \geqslant 2+T/3. \quad \ldots \ldots \ldots \quad (3)$$

Let N' be the dual network of N [9]). Its numbers of wires, vertices and meshes are $T'=T$, $K'=M$ and $M'=K$, respectively.

The inequalities (2), (3) apply also to N'; therefore, interchanging of K, M is allowed. We thus obtain for K, M the same upper and lower bounds:

$$\frac{T+6}{3} \leqslant (K, M) \leqslant \frac{2T}{3}. \quad \ldots \ldots \ldots \quad (4)$$

The conditions (4) are only *necessary;* they do not guarantee the planarity of the network satisfying (4). Although we may certainly confine ourselves to networks fulfilling (4), many among them appear non-planar. Of course, the latter cannot be used in the construction of squared rectangles.

We would like to mention an interesting interpretation of EULER's polyhedron formula (1). If n_1, n_2, denote the numbers of the horizontal and vertical line segments, respectively, lying inside the rectangled rectangle of order n, then $n_1 = K-2$, $n_2 = M-2$, $n = T-1$, and hence $n = n_1 +$

---

[9]) Two polar networks N_1 (poles p_1, q_1) and N_2 (poles p_2, q_2) can be united to a "compound" network in two different ways by suitable coalescence of the respective poles, namely either by $p_1 \longleftrightarrow p_2$, $q_1 \longleftrightarrow q_2$ or by $p_1 \longleftrightarrow q_2$, $q_1 \longleftrightarrow p_2$. These two distict (in a topological sense) networks are *electrically equivalent*. If equivalent networks are identified, there is only one network N' dual to N, as was shown by WHITNEY. Cf. the second of TELLEGEN's papers quoted above. Squarings corresponding to equivalent networks are not essentially different as they can be transformed into one another by trivial geometrical displacements (reflection, rotation, translation) of some of the elements.

$n_2 + 1$. Therefore, the total number of elements in any dissection is one more than is the total number of dissecting line segments. Cross-points, however, must be first removed.

In table II the possible combinations of K, M, T are given as far as squarings of order less than 14 are concerned. It also shows the number of the vertical and horizontal line segments, which numbers are very characteristic for any given squared rectangle. On account of (4) they satisfy

$$\frac{n+1}{3} \leqslant (n_1, n_2) \leqslant \frac{2n-4}{3}. \quad \ldots\ldots \quad (5)$$

TABLE II.

Possible combinations of wires (T), vertices (K) and meshes (M) of the required networks. The same with regard to the order (n), horizontal (n_1) and vertical (n_2) line segments of the corresponding squarings.

T	K, M	n	n_1, n_2
6	4,4	5	2,2
7	—	6	—
8	5,5	7	3,3
9	5,6	8	3,4
10	6,6	9	4,4
11	6,7	10	4,5
12	6,8	11	4,6
12	7,7	11	5,5
13	7,8	12	5,6
14	7,9	13	5,7
14	8,8	13	6,6

4. *Construction of the required networks. Nine is the least possible order of a perfect squared rectangle.*

We will now briefly indicate how one can obtain all the required networks, such that certainly none of them has been overlooked. Again, we only consider networks without wires in series or parallel connection; the numbers of wires and vertices be T and K, respectively.

Henceforth a vertex is called a p_n if it is a junction of n wires ($n \geq 3$). The number of vertices p_n will be denoted by x_n (≥ 0). These numbers obviously satisfy

$$x_3 + x_4 + x_5 + \ldots = K,$$
$$3x_3 + 4x_4 + 5x_5 + \ldots = 2T.$$

For small values of K, T, the solutions of this diophantine system are easily obtained. Some simplification is gained by first eliminating x_3, viz.

$$x_4 + 2x_5 + 3x_6 + 4x_7 + \ldots = 2T - 3K. \quad \ldots\ldots \quad (6)$$

As an example, take $T = 13$, $K = 8$; then $2T - 3K = 2$. Obviously

x_6, x_7,... must be all zero. The only possible types of vertices are p_3, p_4, p_5. Moreover, there are either one p_5 or two p_4's, the remaining vertices being p_3's. Therefore, concerning the character of the vertices occurring in this example, only the following combinations are possible:

$$7p_3 + p_5 \; ; \; 6p_3 + 2p_4.$$

If, given T, the numbers K, M are unequal, we may confine ourselves to $K > M$, as follows from the principle of duality treated in the preceding section. Of course, one could also confine oneself to $K < M$. We, however, prefer the former restriction because then the right-hand side of (6) is smaller, and so is consequently the number of possible combinations of p_n's to be investigated.

With this in mind, the possible combinations of the various types of vertices for squarings up to the order 13 are easily derived. They are given in table III.

TABLE III.

Possible combinations of vertices as a function of the numbers of wires and vertices.

T	K	Combinations of vertices
6	4	$4p_3$
7	—	—
8	5	$4p_3 + p_4$
9	6	$6p_3$
10	6	$5p_3 + p_5$; $4p_3 + 2p_4$
11	7	$6p_3 + p_4$
12	7	$6p_3 + p_6$; $5p_3 + p_4 + p_5$; $4p_3 + 3p_4$
12	8	$8p_3$
13	8	$7p_3 + p_5$; $6p_3 + 2p_4$
14	8	$7p_3 + p_7$; $6p_3 + p_4 + p_6$; $6p_3 + 2p_5$; $5p_3 + 2p_4 + p_5$; $4p_3 + 4p_4$
14	9	$8p_3 + p_4$

The first combination of table III (i.e. $4p_3$, $T = 6$, $K = 4$) only admits of the tetrahedron. This network is self-dual; cf. fig. 3*a*. In case of $T = 8$ the only possible network is the self-dual four-sided pyramid of fig. 3*b*. For $T = 9$ one obtains the self-dual three-sided prism (fig. 3*c*) together with the simplest non-planar network (fig. 3*d*), which can be omitted further on. For $T = 10$ our "sieve" yields 4 networks, one of which is non-planar. The combination $5p_3 + p_5$ gives rise to the self-dual five-sided pyramid of fig. 3*e*. The combination $4p_3 + 2p_4$ admits 3 networks of which 2 are planar.

The latter combination will be treated in some detail, as it may give an idea of how the higher networks were actually obtained.

Two distinct cases may occur as to whether or not the pair of p_4's are connected by a wire. If they are not connected, any of the 4 remaining p_3's must be joined to both of them. The two remaining wires can yet be placed in one way only, and the final result is a parallel (or series) connection of two Wheatstone bridges; cf. fig. 3*f*. In the other case, where the vertices p_4 are connected to one another, either three or two of the p_3's must be connected to each of them. In the first case the yet unconnected p_3 must be joined to the other three p_3's, giving the non-planar network of fig.

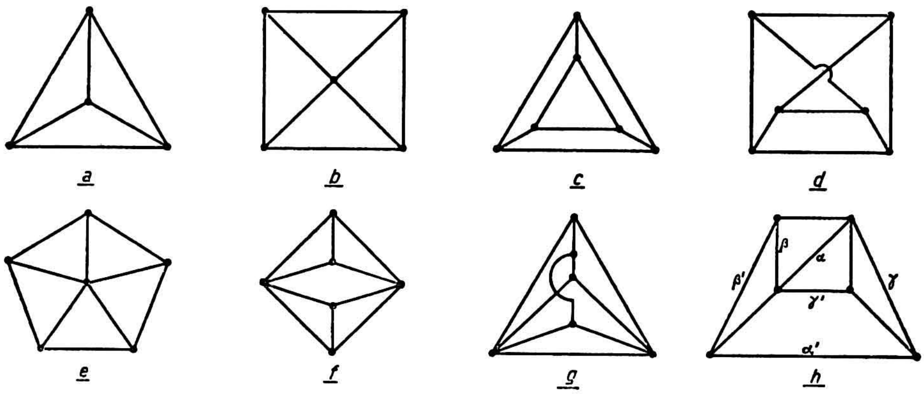

Fig. 3. The simplest networks up to $T = 10$ ($K \geq M$).

3*g*. In the second case the two points p_3, which are connected to both p_4's, cannot be connected to one another; the other pair of p_3's must do so, however. Furthermore, the two wires, not yet used, can be placed in one

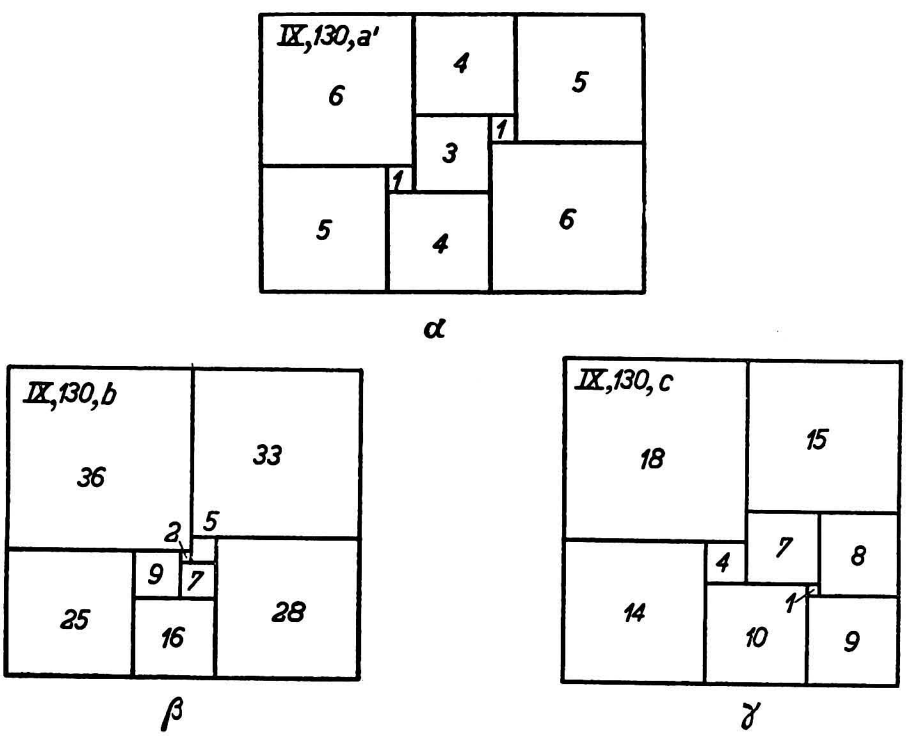

Fig. 4. There are three simple squarings of order 9, two of which are perfect.

way only, and the result is the self-dual network of fig. $3h$. Drawn in space, it is a three-sided prism with a diagonal in one of its uprising sides.

It is not difficult to see that out of the six planar networks in fig. 3 there is at most one ($3h$) that can be used in our problem because, on account of the large degree of symmetry, the other networks will show either (i) at least one zero-current or (ii) equal currents giving rise to trivial imperfection, after the insertion of a source in one of the wires (all having the same resistance).

Furthermore, in the network $3h$ only 6 of the total number of 10 wires need be broken up on account of symmetry properties. This number is again reduced to 3 because the network is self-dual. In fig. $3h$ the yet remaining wires are distinguished from each other by α, β, γ; their "duals" are primed correspondingly. When the source is inserted in α, the network gives a simple squaring of order 9. Although this squaring is imperfect, showing double elements, it is only non-trivially imperfect as equal elements do not lie aside; cf. fig. 4α. Two different perfect squarings of order 9 are obtained if the source is placed in β, γ, respectively. They have been drawn out in fig. 4β, γ.

We have thus shown in detail that 9 is the least possible order of a perfect squared rectangle; and there are only two distinct solutions involving nine elements.