Mathematics. — *Negationless intuitionistic mathematics.* By G. F. C. GRISS. (Communicated by Prof. L. E. J. BROUWER.)

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Introduction.

In these proceedings I already gave a sketch of some parts of *negationless intuitionistic mathematics* [1]). In the following I will treat of this subject more fully and systematically.

On philosophic grounds I think the use of the negation in intuitionistic mathematics has to be rejected. Proving that something is not right, i.e. proving the incorrectness of a supposition, is no intuitive method. For one cannot have a clear conception of a supposition that eventually proves to be a mistake. Only construction without the use of negation has some sense in intuitionistic mathematics.

But I am not going any further into the philosophical side of this question. *In intuitionistic mathematics one can consider the construction of negationless intuitionistic mathematics as a pure mathematical problem.*

To elucidate my intention I'll first give some examples, which will show what questions arise.

1. In elementary geometry there are often given proofs with the aid of the negation, whereas a proof without the use of the negation is simpler, e.g.

If a line in a triangle cuts off from two sides parts that are proportional to those sides, that line is parallel with the third side.

Let be given in $\triangle ABC$

$$CA : CB = CD : CE,$$

then we must prove that DE is parallel with AB.

Proof with negation:

If DE was not parallel with AB, we could draw a line, different from DE, that would cut BC in a point F different from E. Then we should have

$$CA : CB = CD : CF.$$

So CE and CF would be equal and E and F would coincide, which is impossible. The supposition, that DE would not be parallel with AB, is absurd, consequently $DE \parallel AB$.

Proof without negation:

Draw a line through D parallel with AB, that cuts BC in E'; then

$$CA : CB = CD : CE'.$$

So $CE = CE'$, so that E and E' coincide. So DE coincides with DE', whereas $DE' \parallel AB$, so that $DE \parallel AB$.

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[1]) Vol. 53 (1944).

The second demonstration is really simpler than the first. If, however, one wishes to avoid negation consistently, one must give a positive definition of parallel lines. Instead of saying: parallel lines (in a plane) are lines which do not intersect, one must define: parallel lines are such lines, that any point of one of them differs from any point of the other one. *And this, again, presupposes a positive definition* (i.e. a definition without negation) *of difference relative to points.*

If one wants to draw the line DE' in the second demonstration, one most probably draws a line different from DE, though DE' and DE prove to coincide. The figure has perhaps been one of the causes of the use of negation.

Another cause is the practice of questioning. One asks for instance: Which rational numbers satisfy $x^2 - 2 = 0$? The answer must be: No rational number satisfies. The question has been put in the wrong way. The fact is that $x^2 - 2$ differs positively from zero for every rational number.

2. Has the equation $ax + by = 0$ a solution for x and y, different from zero, i.e. a solution with at least x or y different from zero? The letters represent real numbers.

In intuitionistic mathematics they make a distinction between positively and negatively different with regard to real numbers. Two real numbers differ positively, if there can be indicated two approximating intervals which lie outside one another; they differ negatively, if it is impossible that they are equal; you can only divide by a real number if it differs positively from zero. In negationless mathematics the idea negatively different is, of course, omitted. Therefore we mean henceforth by different positively different.

If now in the equation $ax + by = 0$ a is different from zero, we can divide by a and find $x = -\frac{b}{a}y$, so among others $x = -\frac{b}{a}$ and $y = 1$ is a solution different from zero. If b is different from zero $x = 1$ and $y = -\frac{a}{b}$ is a solution different from zero. If a and b are both zero, then $x = 1$ and $y = 1$ is a solution different from zero. The result is:

$ax + by = 0$ has a solution different from zero, if at least one of the coefficients a and b differs from zero or if both are zero.

By using the negation this positive result can be formulated negatively: *It is impossible that no solution different from zero exists.* If namely there were no solution different from zero, a would be zero, for if a differs from zero, there would be a solution different from zero; likewise b would be zero; but then a and b were both zero, so there would still be a solution different from zero. This is impossible according to our supposition, so it is impossible that no solution different from zero exists.

The negative formulation is shorter, but distorted, and *the details of the positive result are lost.*

In non-intuitionistic mathematics $ax + by = 0$ has always a resolution different from zero. In this formulation the positive result has vanished entirely.

3. If $ax + b \neq 0$ for each value of x, then $a = 0$.

Proof without negation: Take $c \neq 0$ (this means c is different from zero) and determine x_1, in such a way that $cx_1 + b = 0$.

As $ax_1 + b \neq 0$, it follows that $(a - c)\ x_1 \neq 0$, which gives $a \neq c$. So $a \neq c$ for each value $c \neq 0$, so $a = 0$.

In this proof we used among others the proposition: If a differs from c for each value c that differs from b, then $a = b$.

This proposition replaces the negative proposition: If a does not differ from b, then $a = b$.

The positive proposition has to be proved for the different sorts of numbers, to begin with the natural numbers. But therefore again it proves to be necessary *to construct the whole of negationless intuitionistic mathematics from the beginning.*

4. We prove first:

Two triangles are congruent, if they have equal two pair of sides and the angles opposite the first pair, while the sum of the angles opposite the other pair differs from 180°.

Proof: Let be given in $\triangle ABC$ and $\triangle A'B'C'$: $AB = A'B'$, $BC = B'C'$, $\angle C = \angle C'$ and $\angle A + \angle A' \neq 180°$.

According to the sinus-rule $\sin A = \sin A'$, so that

$$2 \sin \tfrac{1}{2}(A - A') \cos \tfrac{1}{2}(A + A') = 0.$$

The latter factor differs from zero, so $\sin \frac{1}{2}(A - A') = 0$, so that $\angle A = \angle A'$. So $\triangle ABC \cong \triangle A'B'C'$.

I remark, that by using the negation $\angle A + \angle A' \neq 180°$ can be replaced by $\angle A + \angle A'$ is not equal to zero. However, no example is known where the second formulation could be applied but not the first, for no real number is known, of which is shown, that it is not possibly equal to zero, whereas it is not shown, that it differs (positively) from zero. It seems as if in intuitionistic mathematics the only way to show that two real numbers are not equal consists in showing that the numbers are (positively) different.

Two triangles are congruent, if they have equal one side, the angle opposite that side and the sum of the two other sides, while of one of the adjacent angles is known that they are either equal or different.

We need only prove the case in which the adjacent angles differ.

Let be given in $\triangle ABC$ and $\triangle A'B'C'$: $AC = A'C'$; $\angle B = \angle B'$ $AB + BC = A'B' + B'C'$, while $A \neq C'$.

Produce AB with $BD = BC$ and $A'B'$ with $B'D' = B'C'$, then $\triangle ACD = \triangle A'C'D'$, for $AD = A'D'$, $AC = A'C'$, $\angle D = \angle D'$, while $\angle ACD + \angle A'C'D' = (\angle C + \frac{1}{2}\angle B) + (\angle C' + \frac{1}{2}\angle B') =$

$= \angle C + \angle B + \angle C' \neq \angle C + \angle B + \angle A$, so $\angle ACD + \angle A'C'D' \neq 180^\circ$. From $\triangle ACD \cong \triangle A'C'D'$ it follows immediately $\triangle ABC \cong \triangle A'B'C'$.

Bij using negations one finds:

If two triangles have equal one side, the angle opposite that side and the sum of the two other sides, it is impossible, that they are not congruent.

Compare this with the second example.

We recapitulate the results we have obtained with these examples. In some cases it is simpler to avoid the use of the negation (ex. 1). Positive properties can sometimes be formulated more briefly in a negative way, but details get lost (ex. 2). The parts of intuitionistic mathematics which in a positive construction are disposed of are less important, for probably examples cannot be constructed for which a negative property could be applied and a corresponding positive property could not (ex. 4). To construct negationless mathematics one must begin with the elements and a positive definition of difference must be given instead of a negative one (ex. 1 and 3).

But even from a general intuitionistic point of view a positive construction of the theory of natural numbers must be given: one cannot define 2 is not equal to 1 (i.e. it is impossible that 2 and 1 are equal), for from this one could never conclude that 2 and 1 differ positively. Conversely one could define in a positive way negation by means of difference, e.g. not equal means different, etc., but, for the present, this seems unfit.

We finally consider the property: If a and b are elements of the set of natural numbers, and if $a \neq b$, then $a < b$ or $a > b$ for each element a of the set. If we apply this property to $b = 1$, we get: For each element $a \neq 1$ of the set of natural numbers we have $a < 1$ or $a > 1$. $a < 1$, however, has not any sense in negationless mathematics. If we say: $a < b$ or $a > b$ for each a of a set, we mean 1) that for each a at least one of these conditions is fulfilled, 2) that conversively at least one element fulfils the condition $a < b$ and another one the condition $a > b$. Properties which do not hold for any element do not occur. The combination of two properties (a natural number is even, a natural number is odd) need not be a property. Suppositions, of which it is not certain that they can be realized by a mathematic system, are not made. In axiomatic mathematics we must also stick to this.

Negationless intuitionistic logic will differ much from the usual intuitionistic logic [2]) by the absence of the negation and the altered meaning of the disjunction. Yet I'll not begin by giving such a system: intuitionistic mathematics has also been studied before a formal system was given.

"Affirmative" mathematics [3]) is something quite different from the negationless intuitionistic mathematics I'll treat of.

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[2]) A. HEYTING, Die formalen Regeln der intuitionistischen Logik. Preuss. Akad. der Wissenschaften 1930.

[3]) D. VAN DANTZIG, On the principles of intuitionistic mathematics. Rev. hispan. 1941?

§ 1. *The natural number.*

1. *Construction of the natural numbers.*

Imagine an object, e.g. 1. It remains the same [4]), 1 is the same as 1, in formula $1 = 1$.

Imagine another object, remaining the same, and distinguishable [4]) from 1, e.g. 2; $2 = 2$; 1 and 2 are distinguishable (from one another), in formula $1 \neq 2$, $2 \neq 1$.

They form the set $\{1, 2\}$; 1 and 2 belong to the set. If conversely an object belongs to this set, it is 1 or 2. If it is distinguishable from 1, it is 2; if it is distinguishable from 2, it is 1.

Imagine again another object (element), remaining the same and distinguishable from 1 and from 2, e.g. 3; $3 = 3$; 1 and 3 are distinguishable, 2 and 3 too, in formula $1 \neq 3$, $3 \neq 1$, $2 \neq 3$, $3 \neq 2$.

They form the set $\{1, 2, 3\}$. If an element belongs to $\{1, 2, 3\}$, it belongs to 1, 2 or it is 3. If it is distinguishable from each element of $\{1, 2\}$, it is 3; if it is distinguishable from 3, it is an element of $\{1, 2\}$.

If, in this way, we have proceeded to $\{1, 2, \ldots, n\}$, we can, again, imagine an element n', *remaining the same*, $n' = n'$, *and distinguishable from each element* p *of* $\{1, 2, \ldots, n\}$, *in formula* $n' \neq p$, $p \neq n'$.

They form the set $\{1, 2, \ldots, n'\}$. If an element belongs to $\{1, 2, \ldots, n'\}$, it belongs to $\{1, 2, \ldots, n\}$ or it is n'. If it is distinguishable from each element of $\{1, 2, \ldots, n\}$, it is n'; if it is distinguishable from n', it is an element of $\{1, 2, \ldots, n\}$.

We can *cease* with the mth element and get a finite set $\{1, 2, \ldots, m\}$; we can also *proceed* unlimitedly and get the countably infinite set $\{1, 2, \ldots\}$.

If one continuously chooses an arbitrary symbol as a new object, this soon leads to difficulties. One can try to prevent this by proceeding systematically in the choice of new symbols. So you can take I, II, III, etc.; after proceeding to a certain symbol you get a new symbol by adding a I. This is not practical either. It is better to use a numerative system, though, in principle, the same difficulties arise.

2. *Properties of the relations "the same" and "different".*

Proposition: *Two elements of the set* $\{1, 2, \ldots, m\}$ *are the same or distinguishable.*

Proof: For $\{1, 2\}$ the proposition holds. Let the proof have proceeded to the set $\{1, 2, \ldots, n\}$. Denote two elements of $\{1, 2, \ldots, n'\}$ by a and b. a is an element of $\{1, 2, \ldots, n\}$ or it is n', likewise b. There are 4 possibilities: 1) $a = n'$ and $b = n'$, so $a = b$; 2) $a = n'$ and b belongs to $\{1, 2, \ldots, n\}$, so $a \neq b$; 3) a belongs to $\{1, 2, \ldots, n\}$ and $b = n'$, so $a \neq b$; 4) a and b belong to $\{1, 2, \ldots, n\}$, then $a = b$ or $a \neq b$.

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[4]) Here, again, I do not enter into philosophical questions. Cf. my philosophical sketch "Idealistische filosofie" (Van Loghum Slaterus, Arnhem, 1946).

Proposition: *If for two elements a and b of $\{1, 2, \ldots, m\}$ holds*: $a \neq c$ *for each* $c \neq b$, *then* $a = b$.

Proof: For $\{1, 2\}$ the proposition holds. Let the proof have proceeded to $\{1, 2, \ldots, n\}$. There are two possibilities: In $\{1, 2, \ldots, n'\}$ $b = n'$ or b belongs to $\{1, 2, \ldots, n\}$. 1) If $b = n'$, then a is distinguishable from each element of $\{1, 2, \ldots, n\}$, so $a = n'$ and $a = b$. 2) b belongs to $\{1, 2, \ldots, n\}$; take $c = n'$, then a also belongs to $\{1, 2, \ldots, n\}$, so $a = b$.

We can also formulate these propositions in the following way, though we anticipate the general theory of sets we hope to treat of in a following paragraph.

If we denote the complementary set of the element a of the set $\{1, 2, \ldots, m\}$ by A, the complement of A is a and the sum of a and A is $\{1, 2, \ldots, m\}$.

In this connection I mention the so-called main proposition of arithmetic.

If there is a one to one reciprocal correspondence between $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, p\}$, then $m = p$.

I do not repeat proofs which have been already given without using the negation.

For the elements of the set $\{1, 2, \ldots, m\}$ the following propositions hold now:

I $a = a$

II $a = b \rightarrow b = a$

III $a = b$ and $b = c \rightarrow a = c$

IV $a \neq b \rightarrow b \neq a$

V $a = b$ and $b \neq c \rightarrow a \neq c$

VI $a = b$ or $a \neq b$

VII $a \neq c$ for each $c \neq b \rightarrow a = b$.

VI replaces the negative proposition: Two natural numbers are the same or not, which in non-intuitionistic mathematics holds in virtue of the principium tertii exclusi, but which in intuitionistic mathematics must be proved.

VII runs with negation: If it is impossible, that a is not the same as b, then a is the same as b.

I briefly enumerate the negative propositions concerning the relations "the same" and "different" which have been replaced in a positive theory.

different $\rightleftarrows$ not the same.

the same $\rightleftarrows$ not different.

the same and different exclude one another.

two natural numbers are either the same or different.

3. *The order-relation.*

We define the relation *a precedes b, $a < b$*, which has the same meaning as *b follows a, $b > a$*, and the relation *a immediately precedes b* (*b immediately follows a*).

For $\{1, 2\}$ we have $1 < 2$. If a and b are elements of $\{1, 2\}$ and if $a < b$, then $a = 1$ and $b = 2$. 1 immediately precedes 2.

For $\{1, 2, 3\}$ we have $2 < 3$ and $1 < 3$. If a and b are elements of $\{1, 2, 3\}$ and if $a < b$, then $b = 3$ and a belongs to $\{1, 2\}$ or a and b belong to $\{1, 2\}$. 2 immediately precedes 3.

If, in this way, we have proceeded to $\{1, 2, \dots, n\}$, for $\{1, 2, \dots, n'\}$ we have $p < n'$ for each p of $\{1, 2, \dots, n\}$. If a and b are elements of $\{1, 2, \dots, n'\}$ and if $a < b$, then $b = n'$ and a belongs to $\{1, 2, \dots, n\}$ or a and b belong to $\{1, 2, \dots, n\}$. n immediately precedes n'.

If for $\{1, 2, \dots, m\}$ $a < b$, *then* $a \neq b$.

Proof: For $\{1, 2\}$ the proposition holds. Let the proof have proceeded to $\{1, 2, \dots, n\}$. If a and b are elements of $\{1, 2, \dots, n'\}$ and if $a < b$, then $b = n'$ and a belongs to $\{1, 2, \dots, n\}$, so that $a \neq b$ or a and b belong to $\{1, 2, \dots, n\}$, so that $a \neq b$.

Property: If for $\{1, 2, \dots, m\}$ $(m > 2)$ $a < b$ and $b < c$, then $a < c$.

Proof: $\{1, 2, 3\}$: as $a < b$ b belongs to $\{2, 3\}$ and as $b < c$ b belongs to $\{1, 2\}$. So $b = 2$, $a = 1$ and $c = 3$, so that $a < c$. Let the proof has proceeded to $\{1, 2, \dots, n\}$. For the elements of $\{1, 2, \dots, n'\}$ is $a < b$ and $b < c$. $c = n'$ or c belongs to $\{1, 2, \dots, n\}$; in both cases b belongs to $\{1, 2, \dots, n\}$, so a too and $a < c$.

Property: If $a \neq 1$ is an element of $\{1, 2, \dots, m\}$, then $1 < a$.

Property: If $a \neq m$ is an element of $\{1, 2, \dots, m\}$, then $a < m$.

Property: If a and b ($b \neq 1$ and $b \neq m$) are elements of $\{1, 2, \dots, m\}$, for each element a that differs from b holds $a < b$ or $a > b$.

These properties, just as the following ones, can easily be proved by induction. The condition $b \neq 1$ (also $b \neq m$) is necessary in the last property, for if $b = 1$, there is no element a which would satisfy $a < b$. This cannot be allowed in negationless mathematics (Cf. Introduction). We return to this subject in the theory of sets.

If $a < b$ and if for each $c < b$ and $c \neq a$ $c < a$ holds, then b immediately follows a.

If $a < b$ and if for each $c > a$ and $c \neq b$ $c > b$ holds, then b immediately follows b.

If b immediately follows a $(a \neq 1)$, then for each $c < b$ and $c \neq a$ holds $c < a$.

If b immediately follows a $(b \neq m)$, then for each $c > a$ and $c \neq b$ holds $c > b$.

Finally:

$a \neq b$ and $a \neq c$ for each $c < b$ $(b \neq 1) \rightarrow a > b$.

$a \neq b$ and $a \neq c$ for each $c > b$ $(b \neq m) \rightarrow a < b$.

From the preceding properties follows, if we define

$$a \leqq b \text{ as } a = b \text{ or } a < b \text{ and likewise } a \geqq b.$$

$a \neq c$ for each $c < b$ $(b \neq 1) \rightarrow a \geqq b$.

$a \neq c$ for each $c > b$ $(b \neq m) \rightarrow a \leqq b$.

$a \geqq 1$ and $a \leqq m$.