Applied Mechanics. — The circular ring under the combined action of compressive and bending loads. By C. B. BIEZENO and J. J. KOCH.

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1. Introduction. In a previous paper 1), (in these lines to be quoted as ‘I’) it was stated that every equilibrium-system \((q, t)\) of radial and tangential forces, acting on a circular ring can be decomposed into a "compressive" system \(A(q, t)\) giving rise only to a normal force \(N\) in any cross-section of the ring (so that both the bending moment \(M\) and the shearing force \(D\) are zero) and a "bending" system \(B(q, t)\) characterized by \(N = 0\). The first system \(A\), if suitably magnified, leads to elastic instability of the ring. The required factors of magnification \(\lambda_1, \lambda_2\), arranged after their order of magnitude have been introduced in ‘I’ as the "characteristic numbers" of the buckling problem connected with the \(A\)-system and their numerical computation has been the object of that paper. Here it will be supposed, that the \(A\)-system is sufficiently small as to guarantee the elastic stability of the construction. It is obvious, that the deflections (and internal stresses) of the ring, due to the single action of the \(B\)-system, will be affected in a rather complicated way by the simultaneous action of the \(A\)-system, which — alone — would produce no deflections at all.

In this paper it will be shown how the deflection and internal stresses of the ring, under the combined action of the \(A\) and \(B\)-system can be derived from the corresponding quantities occurring with the \(B\)-system alone, with the aid of the characteristic numbers \(\lambda\) and the corresponding characteristic functions \(U\) (comp. ‘I’) of the \(A\)-system.

To prevent lengthy repetitions, the reader is supposed to be fully acquainted with paper ‘I’.

2. The method. If the ring is subjected to the \(k\)-th characteristic (or critical) load \(\lambda_k A\), the corresponding characteristic mode of distortion \(U_k\) is determined except for a factor of proportionality. To remove this ambiguity we restrict our attention to that deflection \(u_k\), which corresponds to the "normalized" characteristic function \(U_k = u_k + u_k = -\frac{r^2}{EI} M_k\) (comp. ‘I’, 3, and ‘I’, 4, 36). The same deflection \(u_k\) and consequently the same bending moment \(M_k = -\frac{EI}{r^2} U_k\) can likewise be sustained by a "bending"

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loadsystenm $B_k(q,t)$, which readily can be derived from the equations of equilibrium

\[
\begin{align*}
-D' &= qr \\
D &= -tr \\
M' &= -tr^2
\end{align*}
\]  

(comp. ("I", 2, 1), in which $N$ en $N'$ must be put equal to zero). From this it follows that the simultaneously acting loads $A$ and $xB_k$ ($x$ designing an arbitrary constant) will produce a bending moment $M = -\frac{EI}{r^2}(u'' + u)$, which for every value of $x$ is proportional to $M_k = -\frac{EI}{r^2} U_k$. If the particular value of $x$ is required for which the combined loads $A$ and $xB_k$ produce a bending moment not proportional but equal to $M_k = -\frac{EI}{r^2} U_k$ (and consequently produce the distortion $u_k$) it can be remarked, that the single compressive load-system $\lambda_kA_k$ is capable of sustaining the prescribed deflection $u_k$ and consequently to rouse the prescribed moment $M_k$. The load-system $A$ therefore stands up for the bending moment $\frac{1}{\lambda_k} M_k$, and the remaining part $\frac{\lambda_k - 1}{\lambda_k} M_k$ of the prescribed bending moment $M_k$ must be supplied by the bending load-system $xB_k$ and consequently $x = \frac{\lambda_k - 1}{\lambda_k}$.

We learn from these remarks that the effect of the "bending" load-system $B_k$ is magnified in the proportion $\frac{\lambda_k}{\lambda_k - 1}$ by the presence of the "compressive" load-system $A$.

The "reduced" moment $-\frac{r^2}{EI} M_B \equiv U_B$ belonging to any arbitrary $B$-system can be expanded into a series

\[
-\frac{r^2}{EI} M_B \equiv U_B = \sum_{k=1}^{\infty} b_k U_k . . . . . . (2)
\]

of the characteristic functions $U_k$ of the $A$-system under consideration (Comp. "I", 4).

If the ring is subjected to the simultaneous action of the $B$- and $A$-system, the magnifying influence of the latter will exert itself to each component $B_k$ of the $B$-system, and in particular the resultant bending moment $M$, respectively the resultant function $U$, due to the joint systems $A$ and $B$, is represented by

\[
-\frac{r^2}{EI} M = U = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - 1} b_k U_k . . . . . . (3)
\]
The resultant distortion \( u \) of the ring is governed by the differential equation

\[
u'' + u = U.
\]  

(4)

Summarizing it can be stated, that the stress distribution and the distortion of a circular ring under the combined action of a \( A \)- and \( B \)-load can be calculated as soon as the characteristic numbers \( \lambda_k \) of the \( A \)-system and the corresponding characteristic functions \( U_k \) are known. The function

\[
U_B = -\frac{r^2}{E l} M_B,
\]

which has to be expanded into the series (2), must be computed by integrating the third eq. (1), and the coefficients \( b_k \) occurring in this series can easily be found by using the orthogonality of the functions \( U_k \).

3. Application. The ring compressed by two diametral forces \( P \).

With reference to ("I", 2) we replace the two compressing diametral forces \( P \) (whose points of attack may coincide with \( \varphi = 0 \) and \( \varphi = \pi \)) by their equivalent Fourier-series \( q \) and \( t \):

\[
q = a_0 + \sum_{k=1}^{\infty} a_k \cos k\varphi + \sum_{k=1}^{\infty} b_k \sin k\varphi; \quad t = c_0 + \sum_{k=1}^{\infty} c_k \cos k\varphi + \sum_{k=0}^{\infty} d_k \sin k\varphi
\]  

(5)

in which evidently all coefficients \( a_{2n+1}, b, c \) and \( d \) are zero and in which

\[
a_0 = -\frac{P}{\pi r} \quad a_{2n} = -\frac{2P}{\pi r}
\]

so that

\[
q = -\frac{P}{\pi r} - \frac{2P}{\pi r} \sum_{k=1}^{\infty} \cos 2k\varphi \quad t = 0. \quad \cdots \quad (6)
\]

From (6) it follows (comp. "I" 2, 18a, b) that the \( A \) and \( B \)-loads systems are represented by

\[
\begin{align*}
q^* &= -\frac{P}{\pi r} + \frac{2P}{\pi r} \sum_{k=1}^{\infty} \cos 2k\varphi \\
t^* &= \frac{2P}{\pi r} \sum_{k=1}^{\infty} \frac{2k \sin 2k\varphi}{4k^2 - 1}
\end{align*}
\]

(A) \quad \begin{align*}
q^{**} &= -\frac{2P}{\pi r} \sum_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} \cos 2k\varphi \\
t^{**} &= -\frac{2P}{\pi r} \sum_{k=1}^{\infty} \frac{2k}{4k^2 - 1} \sin 2k\varphi
\end{align*}

(B)  

(7)

The normal force \( N_0 \), acting upon the cross-section \( \varphi \) of the ring and caused by the \( A \)-system is given by

\[
N_0 = -\frac{P}{\pi} + \frac{2P}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2k\varphi}{4k^2 - 1} = -\frac{P}{\pi} \left[ 1 + \sum_{k=1}^{\infty} \varepsilon_{2k} \cos 2k\varphi \right] ; \quad \varepsilon_{2k} = -\frac{2}{4k^2 - 1}
\]  

(8)

and the bending moment \( M_B \) belonging to the \( B \)-system by

\[
M_B = -\frac{2Pr}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2k\varphi}{4k^2 - 1} = \cdots \quad \cdots \quad (9)
\]

\[
-\frac{2Pr}{\pi} [0.3333 \cos 2\varphi + 0.06667 \cos 4\varphi + 0.02857 \cos 6\varphi + 0.01587 \cos 8\varphi + 0.0101 \cos 10\varphi + 0.00699 \cos 12\varphi]
\]

(comp. the third eq. (1) and the second eq. (7)).
With reference to ("I" 8), where an example has been treated analogous with the "A"-problem under consideration (viz. \( N = \lambda N_0 = \lambda (1 + \varepsilon_2 \cos 2\varphi + \varepsilon_4 \cos 4\varphi) \)), we restrict ourselves to the summing up of the following numerical results with respect to the first two characteristic numbers \( \lambda_1 \) and \( \lambda_2 \) and the two corresponding characteristic functions:

\[
\lambda_1 = 1.0336 \frac{3EI}{r^2} \pi \frac{\pi}{P} ; \quad \lambda_2 = 5.3628 \frac{3EI}{r^2} \pi /P
\]

\[
U_1/r \propto V_5^*/r = 0.8396 \cos 2\varphi - 0.0784 \cos 4\varphi - 0.0041 \cos 6\varphi - \\
-0.0014 \cos 8\varphi - 0.0005 \cos 10\varphi - 0.0003 \cos 12\varphi
\]

\[
-\frac{\pi}{P} N_0 V_5^* = -\frac{\pi}{P} W_5^* = +0.8123 \cos 2\varphi - 0.3796 \cos 4\varphi - \\
-0.0460 \cos 6\varphi - 0.0266 \cos 8\varphi - 0.0162 \cos 10\varphi - 0.0112 \cos 12\varphi
\]

\[
V_5/r = 6.9010 \cos 2\varphi + 15.3749 \cos 4\varphi - 4.9193 \cos 6\varphi + \\
+0.0784 \cos 8\varphi - 0.0651 \cos 10\varphi - 0.0308 \cos 12\varphi
\]

\[
-\frac{\pi}{P} N_0 V_{11}^* = -\frac{\pi}{P} W_{11}^* = 1.2819 \cos 2\varphi + 14.320 \cos 4\varphi - \\
-10.7560 \cos 6\varphi + 0.3678 \cos 8\varphi - 0.4107 \cos 10\varphi - 0.2678 \cos 12\varphi
\]

(in these latter series all terms following that with \( \cos 12\varphi \) are neglected).

In the expansion

\[
-\frac{M_B r^2}{EI} = b_1 U_1 + b_2 U_2 + \ldots \ldots \ldots (14)
\]

the coefficients \( b_1 \) and \( b_2 \) are defined by

\[
-\int_0^{2\pi} N_0 U_1 \frac{M_B r^2}{EI} d\varphi = b_1 \int_0^{2\pi} N_0 U_1^2 d\varphi \text{ resp.}
\]

\[
-\int_0^{2\pi} N_0 U_2 \frac{M_B r^2}{EI} = b_2 \int_0^{2\pi} N_0 U_2^2 d\varphi
\]

If we replace \( U_1 \) and \( U_2 \) by their approximate values \( V_5^* \) and \( \bar{V}_{11}^* \) we find:

\[
b_1 = -\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI} d\varphi : \int_0^{2\pi} V_5^* W_5^* d\varphi;
\]

\[
b_2 = -\int_0^{2\pi} \bar{W}_{11}^* \frac{M_B r^2}{EI} d\varphi : \int_0^{2\pi} \bar{V}_{11}^* \bar{W}_{11}^* d\varphi
\]
The computation of these coefficients only requires the evaluation of the integrals
\[
\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI} \, d\varphi, \int_0^{2\pi} V_5^* W_5^* \, d\varphi \text{ and the analogous ones for } b_2. \tag{17}
\]
If we write:
\[
M_B = -\frac{2Pr}{\pi} \sum_{k=1}^6 m_{2k} \cos 2k\varphi, \quad V_5^* = r \sum_{k=1}^6 v_{2k} \cos 2k\varphi,
\]
\[
W_5^* = -\frac{Pr}{\pi} \sum_1^6 w_{2k} \cos 2k\varphi \ldots \ldots \ldots \tag{18}
\]
(comp. 9 and 11) we find:
\[
\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI} \, d\varphi = \frac{2P^2 r^4}{\pi^2 EI} \sum_{k=1}^6 \int_0^{2\pi} w_{2k} m_{2k} \cos 2k\varphi \, d\varphi =
\]
\[
= \frac{2P^2 r^4}{\pi^2 EI} \sum_{k=1}^6 w_{2k} m_{2k} = 0.24348 \frac{2P^2 r^4}{\pi EI}
\]
\[
\int_0^{2\pi} V_5^* W_5^* \, d\varphi = -\frac{Pr^2}{\pi} \sum_{k=1}^6 \int_0^{2\pi} v_{2k} w_{2k} \cos 2k\varphi \, d\varphi = -Pr^2 \sum_{k=1}^6 v_{2k} w_{2k} =
\]
\[
= -0.71021 Pr^2
\]
whence
\[
b_1 = 0.34196 \frac{2Pr^2}{\pi EI} \ldots \ldots \ldots \tag{19}
\]
Analogously it is found that
\[
b_2 = 0.003822 \frac{2Pr^2}{\pi EI} \ldots \ldots \ldots \tag{20}
\]
The deflections \(u_1\) and \(u_2\) of the ring corresponding with the components
\[-b_1 \frac{EI}{r^2} U_1, -b_2 \frac{EI}{r^2} U_2 \ldots\] of the bending moment \(M_B\) can be calculated from the equations
\[
u'' + u_1 = b_1 U_1 \quad u'' + u_2 = b_2 U_2 \ldots \ldots \ldots \tag{21}
\]
resp.:
\[
u'' + u_1 = 0.28715 \cos 2\varphi - 0.02681 \cos 4\varphi - 0.00140 \cos 6\varphi -
\]
\[-0.00048 \cos 8\varphi - 0.00017 \cos 10\varphi - 0.00010 \cos 12\varphi
\]
\[
u'' + u_2 = 0.02638 \cos 2\varphi + 0.05876 \cos 4\varphi - 0.01880 \cos 6\varphi +
\]
\[+ 0.00030 \cos 8\varphi - 0.00025 \cos 10\varphi - 0.00012 \cos 12\varphi
\]
(comp. (11) and (19), resp. (12) and (20)), from which it follows:

\[
u_1 = \left[ -0.09572 \cos 2 \varphi + 0.00179 \cos 4 \varphi + 0.00004 \cos 6 \varphi + \right.
\]
\[
+ 0.00001 \cos 8 \varphi + \ldots \right] \frac{2 Pr^3}{EI} \]

\[
u_2 = \left[ -0.00879 \cos 2 \varphi - 0.00392 \cos 4 \varphi + 0.00054 \cos 6 \varphi + \ldots \right] \frac{2 Pr^3}{\pi EI} \]

The resultant distortion \( u^* \) of the ring, caused by the simultaneous action of the \( A \) and \( B \) load systems is given by

\[
u^* = \frac{\lambda_1}{\lambda_1 - 1} u_1 + \frac{\lambda_2}{\lambda_2 - 1} u_2 + \ldots = u_1 + u_2 + \ldots + \frac{1}{\lambda_1 - 1} u_1 + \frac{1}{\lambda_2 - 1} u_2 + \ldots \]

The series \( u_1 + u_2 + \ldots \) represents the deflection \( u \) of the ring which occurs if no secondary effect is present. This deflection can be calculated by one of the well-known elementary methods (i.e. with the aid of Castigliano's theorem). The series \( \frac{u_1}{\lambda_1 - 1} + \frac{u_2}{\lambda_2 - 2} + \ldots \) can readily be broken off with the third term in consequence of the fact, that \( u_k \) decreases and \( \lambda_k \) increases rather rapidly with increasing \( k \). We therefore write:

\[
u^* = u + \frac{u_1}{\lambda_1 - 1} + \frac{u_2}{\lambda_2 - 1} \ldots \ldots \ldots \]

At \( \varphi = 0 \) and \( \varphi = \frac{\pi}{2} \) the following results are obtained

\[
u_0^* = - \frac{2 Pr^3}{\pi EI} \left[ 0,11685 + \frac{0,09388}{1,0336 \frac{3\pi EI}{Pr^2} - 1} + \frac{0,01217}{5,3628 \frac{3\pi EI}{Pr^2} - 1} \right] \]

\[
u_{\pi/2}^* = + \frac{2 Pr^3}{\pi EI} \left[ 0,10730 + \frac{0,09748}{1,0336 \frac{3\pi EI}{Pr^2} - 1} + \frac{0,00433}{5,3628 \frac{3\pi EI}{Pr^2} - 1} \right] \]

The secondary effect to which the "compressive" \( A \)-system gives rise in influencing the deflections caused by the "bending" \( B \)-system as represented by the second and third terms in these expressions has been expressed in terms of the quotient \( \frac{3\pi EI}{Pr^2} \), viz. the quotient of the first critical all-sided pressure \( q_{cr} \) of the ring \( (q_{cr} = \frac{3EI}{r^2}) \) and the "mean normal pressure \( P/\pi \) belonging to the two forces \( P \).