Mathematics. — *The conformal* DIRAC *equation*. By J. HAANTJES. (Communicated by Prof. J. A. SCHOUTEN.)

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Introduction.

Several authors have tried to establish conformal wave equations [1]). By conformal wave equations is meant wave equations which depend only on $\mathfrak{G}_{hi} = g_{hi}(-\mathfrak{g})^{-\frac{1}{4}}$, ($\mathfrak{g} = \operatorname{Det} g_{hi}$), and which are therefore invariant under conformal transformations of the fundamental tensor: $g_{ih} \rightarrow \sigma^2 g_{ih}$. In one of his papers DIRAC has been lead to the following statement: "it seems that there is no simple way of getting a wave equation in conformal space corresponding to the ordinary wave equation for the electron" [2]). In the present paper we try to show that this statement has been somewhat too pessimistic. It will appear that the DIRAC equation in the unchanged form for particles without mass is a conformal wave equation and that the DIRAC equation for mass-particles is conformal invariant if we assume that the mass m becomes multiplied by σ^{-1} if g_{ih} is multiplied by σ^2. This means that a transformation of length with ϱ is attended with a transformation of mass with ϱ^{-1} or that $m\,(-\mathfrak{g})^{\frac{1}{8}}$ is invariant under conformal transformations. Under this assumption the physical dimension $[ML]$ is invariant as may also be presumed from PLANCK's constant h (Dimension $[ML^2\,T^{-1}]$). This transformation of m is in accordance with the results obtained in a paper by SCHOUTEN and HAANTJES [1]) as well as with the result of a recent paper [3]) in which a physical interpretation has been given of the invariance of the MAXWELL equations under conformal transformations. The results obtained in that paper will be used in the second section.

1. *The* DIRAC *equation.*

Let

$$ds^2 = g_{ih}\, dx^i\, dx^h, \qquad (h, i, \ldots = 1, 2, 3, 4), \quad \ldots \quad (1)$$

where the quadratic form is of signature $(- - - +)$, define the metric of the space-time V_4 of general relativity.

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1) A. M. DIRAC, Wave equations in conformal space, Annals of Math. **37**, 429—442 (1936). — O. VEBLEN, A conformal wave equation, Proc. Nat. Acad. of Sc. **21**, 484—487 (1935). — J. A. SCHOUTEN and J. HAANTJES, Ueber die konforminvariante Gestalt der relativistischen Bewegungsgleichungen. In this latter publication the definition of conformal invariant equations is somewhat different from our definition. Proc. Kon. Akad. v. Wetensch., Amsterdam **39**, 1059—1065 (1936).

2) L.c. p. 442.

3) Die Gleichberechtigung gleichförmig beschleunigter Beobachter für die elektromagnetischen Erscheinungen, Proc. Ned. Akad. v. Wetensch., Amsterdam **43**, 1288—1299 (1940).

In order to introduce spinors in our space we factorize the fundamental tensor g_{ih}

$$g_{ih} = a_{(i}\, a_{h)} = \tfrac{1}{2}\,(a_i\, a_h + a_h\, a_i), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a_i are non-commutative matrices of four rows and columns. The quantity a_i may be written as $a_{i\;.\;B}^{\;\cdot\;A}$ $(A, B, \ldots = 1, \ldots, 4)$ and represents a co-contravariant affinor of the second rank in the spin space of four dimensions. With each point of the space time is associated a spin-space E_4. The vectors of this space are called spinvectors and are denoted by v^A.

In spin space a covariant differentiation with parameters Λ_{jB}^{A} is introduced by the following conditions

$$\nabla_j\, a_{i\;.\;B}^{\;\cdot\;A} \equiv \partial_j\, a_{i\;.\;B}^{\;\cdot\;A} - \left\{ {h \atop ji} \right\} a_{h\;.\;B}^{\;\cdot\;A} + \Lambda_{jC}^{A}\, a_{i\;.\;B}^{\;\cdot\;C} - \Lambda_{jB}^{C}\, a_{i\;.\;C}^{\;\cdot\;A} = 0 \quad . \quad (3)$$

$$\Lambda_{jA}^{A} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The second condition is invariant because in the spin space we restrict ourselves to linear coordinate transformations of unit determinant.

The solution of the equations (3) and (4) can be obtained in a more simple form if we introduce a spinor $a_{5\;.\;B}^{\;\cdot\;A} = a^{5A}_{\;.\;.\;B}$, which satisfies the following equations

$$\left.\begin{array}{r} a_{5\;.\;B}^{\;\cdot\;A}\, a_{5\;.\;C}^{\;\cdot\;B} = a_C^A \\ a_5\, a_i + a_i\, a_5 = 0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where a_C^A is the unit spinor. It is known that such a spinor exists [4]. So we have

$$a_{(\varkappa}\, a_{\lambda)} = g_{\varkappa\lambda},\ (\varkappa, \lambda, \mu, \ldots = 1, \ldots .\, 5),\ g_{55} = 1,\ g_{h5} = 0. \quad . \quad . \quad (6)$$

Now every co-contravariant spinor $P^{A}_{\;.\;B}$ can be linearly expressed in terms of the following 16 spinors

$$a^{[\varkappa\lambda]\,A}_{\;.\;.\;\;.\;B} = \tfrac{1}{2}\,(a^{\varkappa}\, a^{\lambda} - a^{\lambda}\, a^{\varkappa}),\quad a^{\varkappa A}_{\;.\;.\;B},\quad a_B^A. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Using this theorem it follows from the equations (3) and (4) by a rather extensive calculation

$$\left.\begin{array}{l} \Lambda_{j\,B}^{A} = -\dfrac{1}{4} \left\{ {\varkappa \atop j\lambda} \right\} a^{\lambda\;.\;A}_{\;.\;\varkappa\;.\;B} - \dfrac{1}{16}\,(\partial_j\, a^{\varkappa C}_{\;.\;.\;D})\, a_{\lambda\;.\;C}^{\;\cdot\;D}\, a^{\lambda\;.\;A}_{\;.\;\varkappa\;.\;B} - \\ \qquad\qquad - \tfrac{1}{32}\,(\partial_j\, a^{\varkappa C}_{\;.\;.\;D})\, a_{\varkappa\lambda\;.\;C}^{\;\cdot\;\cdot\;D}\, a^{\lambda A}_{\;.\;.\;B},\ ^{5)} \end{array}\right\} \quad . \quad . \quad (8)$$

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[4]) Comp. f.i. J. HAANTJES, Die induzierte kovariante Ableitung für Spinoren. Proc. Kon. Akad. v. Wetensch., Amsterdam **41**, 155—160 (1938).

[5]) This equation has been taken from the paper referred to in footnote [4]), in which paper the calculation is carried out.

where $\left\{ {\varkappa \atop \mu\,\lambda} \right\}$ are the CHRISTOFFEL symbols of $g_{\lambda\varkappa}$. From (6) follows however the vanishing of $\left\{ {5 \atop \mu\,\lambda} \right\}$ and $\left\{ {\varkappa \atop \mu\,5} \right\}$, whereas the $\left\{ {h \atop i\,j} \right\}$ are indentical with the CHRISTOFFEL symbols derived from g_{ih}.

We now consider a conformal transformation of the fundamental tensor

$$g'_{hi} = \sigma^2\, g_{hi} \,. \qquad \text{(9)}$$

Then introducing the spinors

$$'\alpha_{\varkappa\,.\,B}^{\;\cdot\,A} = \sigma\, \alpha_{\varkappa\,.\,B}^{\;\cdot\,A}\,;\;\; '\alpha^{\varkappa} = \sigma^{-1}\,\alpha^{\varkappa}, \qquad \text{(10)}$$

we have

$$g'_{\varkappa\lambda} = {'\alpha_{(\varkappa}}\, {'\alpha_{\lambda)}}\,;\;\; g'_{h5} = 0,\; g'_{55} = \sigma^2, \qquad \text{(11)}$$

where σ is a function of x^h.

The covariant differentiations in space-time and in spin space belonging to the metric g'_{hi} are defined by the vanishing of the covariant derivatives of g'_{hi} and $'\alpha_{h\,.\,B}^{\;\cdot\,A}$ respectively. This leads to parameters $\left\{ {h \atop i\,j} \right\}$ and $\Lambda'^{\,A}_{jB}$. We have

$$\left\{ {h \atop j\,i} \right\}' = \left\{ {h \atop j\,i} \right\} + A^h_j\,\sigma_i + A^h_i\,\sigma_j - g_{ij}\,\sigma^h,\; (A^h_i \overset{*}{=} \delta^h_i,\; \sigma_i = \partial_i \log \sigma), \qquad \text{(12)}$$

whereas $\Lambda'^{\,A}_{jB}$ is given by an expression, which is obtained from (8) by replacing $\left\{ {\varkappa \atop j\,\mu} \right\}$ by $\left\{ {\varkappa \atop j\,\mu} \right\}'$ and $\alpha_{\varkappa}$ by $'\alpha_{\varkappa} = \sigma\,\alpha_{\varkappa}$. Thus, taking in consideration that $\sigma_5 = 0$,

$$\left.\begin{aligned} \Lambda'^{\,A}_{jB} &= \Lambda^A_{jB} - \tfrac{1}{4}\,(A^{\varkappa}_j\,\sigma_\lambda + A^{\varkappa}_\lambda\,\sigma_j - g_{\lambda j}\,\sigma^{\varkappa})\,\alpha^{\lambda\,.\,A}_{\,.\,\varkappa\,.\,B} + \\ &\quad + \tfrac{1}{16}\,\sigma_j\,\alpha^{\varkappa\,.\,C}_{\,.\,\lambda\,.\,C}\,\alpha^{\lambda\,.\,A}_{\,.\,\varkappa\,.\,B} + \tfrac{1}{32}\,\sigma_j\,\alpha^{\varkappa C}_{\,.\,.\,D}\,\alpha^{\,.\,D}_{\varkappa\,.\,E}\,\alpha^{\,.\,E}_{\lambda\,.\,C}\,\alpha^{\lambda A}_{\,.\,.\,B} \\ &= \Lambda^A_{jB} - \tfrac{1}{4}\,\sigma_i\,\alpha^{i\,.\,A}_{\,.\,j\,.\,B} - \tfrac{5}{4}\,\sigma_j\,\alpha^A_B + \tfrac{1}{4}\,\sigma^i\,\alpha_{ji\,.\,B}^{\,.\,.\,A} + \tfrac{5}{4}\,\sigma_j\,\alpha^A_B \\ &= \Lambda^A_{jB} + \tfrac{1}{2}\,\sigma^i\,\alpha_{[ji]\,.\,B}^{\,.\,.\,A}\,. \end{aligned}\right\} \qquad \text{(13)}$$

Here we have used the following indentities

$$\left.\begin{aligned} &\alpha^{\varkappa\,.\,C}_{\,.\,\lambda\,.\,C} = 4\,A^{\varkappa}_{\lambda} \begin{cases} = 0, & \varkappa \neq \lambda \\ = 4, & \varkappa = \lambda \end{cases} \\ &\alpha_{\lambda\,.\,A}^{\,.\,A} = 0. \end{aligned}\right\} \qquad \text{(14)}$$

We may write the DIRAC *equation* in general relativity in the following form

$$\alpha^{jA}_{\,.\,.\,C}\left(\frac{h}{i}\,\nabla_j\,\psi^C - \frac{e}{c}\,\varphi_j\,\psi^C\right) + m\,c\,\alpha^0\,\psi^A = 0, \qquad \text{(15)}$$

where φ_j represents the potential vector and a^0 stands for the spinor

$$a^0 = a^{[1}\, a^2\, a^3\, a^{4]}\, i_{1234} = \frac{1}{4!}\, a^h\, a^i\, a^j\, a^k\, i_{hijk} \quad . \quad . \quad . \quad . \qquad (16)$$

i_{hijk} represents the unit four-vector [6]). In order to investigate the conformal invariance (gauge invariance) of this wave equation we consider a conformal transformation of the fundamental tensor and compare (15) with the DIRAC equation belonging to g'_{ih}. This latter equation is obtained from (15) by replacing (comp. (10), (13) and (16)).

$$\left.\begin{array}{l} a^j \text{ by } \sigma^{-1}\, a^j \\ \Lambda_{jB}^{A} \text{ by } \Lambda_{jB}^{A} + \tfrac{1}{2}\, \sigma^i\, a_{[j}\, i_{i]\cdot B}^{\;A} \\ a^0 \text{ by } {}'a^{[1}\, {}'a^2\, {}'a^3\, {}'a^{4]}\, i'_{1234} = \sigma^{-4}\, a^{[1}\, a^2\, a^3\, a^{4]}\, (\sigma^4\, i_{1234}) = a^0. \end{array}\right\} . \qquad (17)$$

Moreover we write m' instead of m because there is a possibility that m changes under the gauge-transformation. Hence this wave equation becomes after multiplying by σ (ommitting the spin-indices)

$$\left.\begin{array}{r} a^j \left\{ \dfrac{h}{i}\, (\partial_j\, \psi' + \Lambda_j\, \psi') - \dfrac{e}{c}\, \varphi_j\, \psi' \right\} + \dfrac{h}{4\,i}\, a^j\, \sigma^i\, (a_j\, a_i - a_i\, a_j)\, \psi' + \\ + \sigma\, m'\, c\, a^0\, \psi' = 0. \end{array}\right\} \qquad (18)$$

Now

$$\frac{h}{4\,i}\, \sigma^i\, a^j\, (a_j\, a_i - a_i\, a_j) = \frac{h}{4\,i}\, \sigma^i\, (8\, a_i - 2\, g_{ij}\, a^j) = \frac{3}{2}\, \frac{h}{i}\, \sigma^i\, a_i . \quad . \qquad (19)$$

Thus the equation (18) can be written as follows

$$a^j \left\{ \frac{h}{i}\, \nabla_j\, \psi' - \frac{e}{c}\, \varphi_j\, \psi' + \frac{3}{2}\, \frac{h}{i}\, \sigma_j\, \psi' \right\} + \sigma\, m'\, c\, a^0\, \psi' = 0. \quad . \qquad (20)$$

We now assume

$$m' = \sigma^{-1}\, m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

which means that the mass alters under conformal transformations of the fundamental tensor in such a way that the mass m becomes multiplied by σ^{-1} if the fundamental tensor is multiplied by σ^{-1}. Under this assumption the physical dimension $[ML]$ is invariant. This is also in accordance with the dimension $[ML^2 T^{-1}]$ of the physical quantity h, which we have treated as a constant. (Comp. SCHOUTEN-HAANTJES l. c. [1]).

Under the assumption (21) it is easily seen that

$$\psi' = \sigma^{-\frac{3}{2}}\, \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

is a solution of (20) if ψ is a solution of (15). It would however not

[6]) For the definition of i_{hijk} compare J. A. SCHOUTEN and D. J. STRUIK, Einführung in die neueren Methoden der Differentialgeometrie I, p. 53.

be correct to conclude from this result that the DIRAC equation should not be invariant under conformal transformations. On the contrary in the following it will be shown that the wave function ψ' with g'_{ih} as fundamental tensor leads to the same physical results as the wave function ψ with g_{ih} as metric tensor.

Let $\omega_{\bar{A}B}$ represent the hermitean fundamental tensor in spin-space for which

$$\omega\, a^h = \bar{a}^h\, \omega \ ; \ \omega = \bar{\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where a barred letter means the complex conjugate quantity [7]). Then

$$\omega_{\bar{A}B}\, \bar{\psi}^{\bar{A}}\, \psi^B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

represents the probability function. In order to obtain the probability belonging to a 3-dimensional volume element the quantity (24) has to be multiplied by the volume $d\omega$ and it is this probability which has a physical meaning. If we now pass to the fundamental tensor g'_{hi} the expression (24) is multiplied by σ^{-3}, as a consequence of (22), whereas $d\omega$ becomes multiplied by σ^3, for every length is multiplied by σ. So the probability belonging to a 3-dimensional volume may as well be computed from ψ' and g'_{hi} as from ψ and g_{hi}.

Another result of (22) is that the current vector represented by

$$s^h = \bar{\psi}\, \omega\, a^h\, \psi \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

as a consequence of (10) and (22) is multiplied by σ^{-4} under the conformal transformation, as a current vector should do. For the current vector density [8])

$$\mathfrak{s}^h = s^h\, (-\mathfrak{g})^{\frac{1}{2}} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

of weight $+1$ has a physical meaning and therefore has to be invariant. Thus the current vector deduced from ψ' and g'_{hi} leads to the same results.

Therefore, an observer looking upon g'_{hi} as the metric tensor of space time and using the DIRAC equation belonging to this tensor as wave equation will find the same physical results. This states that:

The DIRAC *equations for particles without mass is a conformal wave equation, whereas the* DIRAC *equation for mass-particles is conformal invariant if we assume that the mass m becomes multiplied by σ^{-1} if the fundamental tensor is multiplied by σ^2.*

From the physical examples above we see that it is not ψ^A but the density

$$\chi^A = (-\mathfrak{g})^{\frac{3}{16}}\, \psi^A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

of weight $\frac{3}{8}$, which has a physical meaning and which is invariant

[7]) We do not prove here that such a quantity exist; it would carry us too far. Comp. f.i. the paper referred to in footnote [4]).

[8]) Comp. f.i. SCHOUTEN–HAANTJES l.c. [1]), p. 1059.

under conformal transformations. The probability density f.i. (comp. (24)) is represented by the function

$$\omega_{\bar{A}B}\, \bar{\chi}^{\bar{A}}\, \chi^{B}. \qquad (28)$$

From the DIRAC equation for ψ^A a conformal invariant differential equation for χ^A can be derived. But before doing this we have to introduce some conformal invariant quantities:

a. As the a^i are not conformal invariant we introduce the conformal invariant densities β_i and β^i of weight $-\frac{1}{4}$ and $+\frac{1}{4}$ respectively

$$\beta_i = (-\mathfrak{g})^{-\frac{1}{8}}\, a_i \ ; \ \beta^i = (-\mathfrak{g})^{+\frac{1}{8}}\, a^i \qquad (29)$$

which satisfy the equations

$$\tfrac{1}{2}(\beta_i\, \beta_j + \beta_j\, \beta_i) = \mathfrak{G}_{ij} = (-\mathfrak{g})^{-\frac{1}{4}}\, g_{ij}. \qquad (30)$$

This conformal tensor density $\mathfrak{G}_{ij}$ of weight $-\frac{1}{2}$ will be used for the raising and lowering of indices. We have f. i.

$$\beta^i = \mathfrak{G}^{ij}\, \beta_j. \qquad (31)$$

b. The quantity i_{hijk}, used for the definition of a^0 (16) is according to its definition equal to

$$i_{hijk} = (-\mathfrak{g})^{\frac{1}{2}}\, \mathfrak{n}_{hijk}, \qquad (32)$$

$\mathfrak{n}_{hijk}$ being the unit 4-vector density of weight -1 [9]). Thus a^0 may be expressed in terms of β_i in the following way

$$a^0 = \frac{1}{4\,!}\, a^h\, a^i\, a^j\, a^k\, i_{hijk} = \frac{1}{4\,!}\, \beta^h\, \beta^i\, \beta^j\, \beta^k\, \mathfrak{n}_{hijk} = \beta^0. \qquad (33)$$

From this equation it follows that a^0 is conformal invariant.

c. We have already seen that the mass m changes under conformal transformations. Therefore it is convenient to introduce

$$\mathfrak{m} = m\, (-\mathfrak{g})^{\frac{1}{8}} \qquad (34)$$

a quantity with the physical dimension $[M\,L]$ which is according to (21) invariant under conformal transformations of the fundamental tensor.

The differential equation for χ^A is obtained from the DIRAC equation (15) by multiplying with $(-\mathfrak{g})^{\frac{1}{8}}\,(-\mathfrak{g})^{\frac{3}{16}} = (-\mathfrak{g})^{\frac{5}{16}}$. Using

$$\nabla_j\, \mathfrak{g} = 0 \qquad (35)$$

we get

$$\beta^{jA}_{\cdot\cdot C}\left(\frac{h}{i}\, \nabla_j\, \chi^C - \frac{e}{c}\, \varphi_j\, \chi^C\right) + \mathfrak{m}\, c\, \beta^0\, \chi^A = 0, \qquad (36)$$

which equation will be called the DIRAC *equation* for χ^A. It is however

[9]) Comp. SCHOUTEN–STRUIK I, p. 53.

not at once clear from (36) that this equation is a conformal invariant one, because it involves the operator ∇_j. But as we will see $\beta^j \nabla_j \chi$ is invariant under the transformation $g_{hi} \rightarrow \sigma^2 g_{hi}$. We have

$$\beta^{j\,A}_{\cdot\,\cdot\,B} \nabla_j \chi^B = \beta^{j\,A}_{\cdot\,\cdot\,B} \left(\partial_j \chi^B - \frac{3}{8} \left\{ {h \atop j\,h} \right\} \chi^B + A^B_{j\,C}\, \chi^C \right). \quad . \quad . \quad (37)$$

Under a conformal transformation χ and β^j are invariant, whereas (comp. (12), (13) and (19))

$$\left. \begin{aligned} -\tfrac{3}{8}\beta^j \left\{ {h \atop j\,h} \right\} \rightarrow -\tfrac{3}{8}\beta^j \left\{ {h \atop j\,h} \right\}' &= -\tfrac{3}{8}\beta^j \left\{ {h \atop j\,h} \right\} - \tfrac{3}{2}\sigma_j \beta^j \\ \beta^j A_j \rightarrow \beta^j A'_j &= \beta^j A_j + \tfrac{1}{4}\beta^j (a_j\, a^i - a^i\, a_j)\, \sigma_i \\ &= \beta^j A_j + \tfrac{3}{2}\sigma_j \beta^j. \end{aligned} \right\} \quad . \quad . \quad (38)$$

From these transformations it is seen that (37) is invariant and so we have found (36) te be *the conformal invariant* DIRAC *equation for* χ.

All the physical quantities, deduced from ψ, may be expressed in terms of β^i and χ. The expression for the current vector density (26) of weight $+1$ f. i. is

$$\mathfrak{s}^h = \bar{\chi}\, \omega\, \beta^h \chi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

2. *The physical interpretation of the invariance under conformal transformation in special relativity.*

The physical interpretation of the conformal invariance has been given in a former paper [10]). We give here a brief summary of the results.

Let O be an observer at rest with respect to a system of GALILEan coordinates belonging to the fundamental tensor g_{hi} and O' an observer, which has a constant acceleration $a c^2$ with respect to O. Then the system of GALILEan coordinates may be chosen in such a way that the worldlines of O and O' are

$$\left. \begin{aligned} &O: \;\; x^1 = x^2 = x^3 = 0 \\ &O': \;\; a\,\{(x^1)^2 - (x^4)^2\} + 2\,x^1 = 0,\; x^2 = x^3 = 0, \end{aligned} \right\} \quad . \quad . \quad . \quad (40)$$

and

$$g_{11} = g_{22} = g_{33} = -g_{44} = -1\,;\; g_{hi} = 0\; (h \neq i). \quad . \quad . \quad . \quad (41)$$

If we now pass to the coordinate system (h') by the transformation

$$\left. \begin{aligned} x^{1'} &= \frac{x^1 - \frac{1}{2}\, a\, g_{ij}\, x^i\, x^j}{1 + a x^1 - \frac{1}{4}\, a^2\, g_{ij}\, x^i\, x^j} \\ x^{\alpha'} &= \frac{x^\alpha}{1 + a x^1 - \frac{1}{4}\, a^2\, g_{ij}\, x^i\, x^j} \qquad (\alpha = 2, 3, 4) \end{aligned} \right\} \quad . \quad . \quad (42)$$

[10]) L.c. [3]).

the worldline of O' obtains the form

$$x^{1'} = x^{2'} = x^{3'} = 0. \quad \ldots \quad (43)$$

This equation shows that the observer O' is at rest with respect to (h'). The system (h') is of course not a GALILEan one with respect to g_{ih}, because g_{ih} has the components

$$\left.\begin{aligned} g_{1'1'} = g_{2'2'} = g_{3'3'} = -g_{4'4'} = -\sigma^{-2} = -(1 + a x^1 - \tfrac{1}{4} a^2 g_{ij} x^i x^j)^2 \\ g_{h'i'} = 0 \quad (h' \neq i') \end{aligned}\right\}. \quad (44)$$

But as this equation shows the system (h') is a GALILEan system with respect to the fundamental tensor $g'_{hi} = \sigma^2 g_{hi}$, where σ^2 is defined in (44).

If now a physical law is conformal invariant, it means that it is invariant under a transformation $g_{hi} \rightarrow g'_{hi}$. So the fundamental tensors g_{hi} and g'_{hi} and therefore the GALILEan systems belonging to g_{hi} and g'_{hi} are equivalent with regard to this law. This means however that the observers O and O' are equivalent. We have thus the result:

The conformal invariance of physical laws means the equivalence of observers, which have a constant acceleration with respect to each other.

The results of § 1 may now be brought in another form. The observer O considers g_{ih} as the metric tensor of space time. As wave equation he makes use of the DIRAC equation

$$\alpha^{jA}_{\cdot\cdot C} \left(\frac{h}{i} \partial_j \psi^C - \frac{e}{c} \varphi_j \psi^C \right) + m c \alpha^0 \psi^A = 0, \quad \ldots \quad (45)$$

where the α^i are chosen to be constant. One may take f. i.

$$\left.\begin{array}{lllll} \alpha^{1A}_{\cdot\cdot B}: & \alpha^{11}_{\cdot\cdot 3} = i, & \alpha^{12}_{\cdot\cdot 4} = -i, & \alpha^{13}_{\cdot\cdot 1} = i, & \alpha^{14}_{\cdot\cdot 2} = -i \\ \alpha^{2A}_{\cdot\cdot B}: & \alpha^{21}_{\cdot\cdot 3} = -1, & \alpha^{22}_{\cdot\cdot 4} = -1, & \alpha^{23}_{\cdot\cdot 1} = 1, & \alpha^{24}_{\cdot\cdot 2} = 1 \\ \alpha^{3A}_{\cdot\cdot B}: & \alpha^{31}_{\cdot\cdot 4} = i, & \alpha^{32}_{\cdot\cdot 3} = i, & \alpha^{33}_{\cdot\cdot 2} = i, & \alpha^{34}_{\cdot\cdot 1} = i \\ \alpha^{4A}_{\cdot\cdot B}: & \alpha^{41}_{\cdot\cdot 4} = -i, & \alpha^{42}_{\cdot\cdot 3} = i, & \alpha^{43}_{\cdot\cdot 2} = -i, & \alpha^{44}_{\cdot\cdot 1} = i \\ \alpha^{0A}_{\cdot\cdot B}: & \alpha^{01}_{\cdot\cdot 1} = -i, & \alpha^{02}_{\cdot\cdot 2} = -i, & \alpha^{03}_{\cdot\cdot 3} = i, & \alpha^{04}_{\cdot\cdot 4} = i \end{array}\right\} \quad . \quad . \quad (46)$$

The observer O' however looks upon g'_{hi} as the metric tensor of space time. He uses the wave equation (18)

$$'\alpha^{jA}_{\cdot\cdot B} \left(\frac{h}{i} \nabla'_j \psi'^C - \frac{e}{c} \varphi_j \psi'^C \right) + m' c \alpha^0 \psi'^A = 0. \quad \ldots \quad (47)$$

Here

$$'\alpha = \sigma^{-1} \alpha, \; m' = \sigma^{-1} m, \; \sigma = (1 + a x^1 - \tfrac{1}{4} a^2 g_{ij} x^i x^j)^{-1}, \quad . \quad (48)$$

whereas ∇'_j denotes the covariant derivative with respect to the metric tensor g'_{ih}. We now pass to the system (h'), a GALILEan system with respect to the fundamental tensor g'_{ih}. The components $\alpha'^{h'}$ will depend on x^h, but it is possible to choose a coordinate system (A') in the spin space such that the components $\alpha'^{h'A'}_{\cdot\cdot B'}$ and $\alpha^{0A'}_{\cdot\cdot B'}$ are the same as those of $\alpha^{hA}_{\cdot\cdot B}$ and $\alpha^{0A}_{\cdot\cdot B}$ with respect to (h) and (A). The transformation $(A) \rightarrow (A')$ is the linear transformation with coefficients

$$\alpha_A^{A'} : \left| \begin{array}{cccc} (1+\frac{1}{2}ax^1-\frac{1}{2}iax^2)\sigma^{\frac{1}{4}} & \frac{1}{2}a(x^3+x^4)\sigma^{\frac{1}{4}} & 0 & 0 \\ -\frac{1}{2}a(x^3-x^4)\sigma^{\frac{1}{4}} & (1+\frac{1}{2}ax^1+\frac{1}{2}iax^2)\sigma^{\frac{1}{4}} & 0 & 0 \\ 0 & 0 & (1+\frac{1}{2}ax^1+\frac{1}{2}iax^2)\sigma^{\frac{1}{4}} & \frac{1}{2}a(x^3+x^4)\sigma^{\frac{1}{4}} \\ 0 & 0 & -\frac{1}{2}a(x^3-x^4)\sigma^{\frac{1}{4}} & (1+\frac{1}{2}ax^1-\frac{1}{2}iax^2)\sigma^{\frac{1}{4}} \end{array} \right|$$

As g'_{hi} and $\alpha'_{i'}$ have constant components with respect to (h') and (A') the operator $\nabla'_{j'}$ in (47) written with respect to these systems may be replaced by $\partial_{j'}$. Thus using the systems (h') and (A') the observer O' may employ the same wave equation as the observer O. As has been pointed out in § 1, his solution $\psi'^{A'}$ leads to the same physical results and is connected with the solution ψ^A by the formel (comp. (22))

$$\psi'^{A'} = \sigma^{-3/2}\, \alpha_A^{A'}\, \psi^A \,. \quad \ldots \ldots \ldots \quad (49)$$

Recapitulating we have arrived at the following result:

The observer O' looking upon g'_{ih} as the metric tensor of space time may use with respect to his preferable systems (h') and (A') the same DIRAC *equation as the observer O with respect to the systems (h) and (A). The observers O and O' are physically equivalent.*