Mathematics. — *Line geometry and Quantum mechanics.* By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

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§ 1. *Introduction.*

The essential difference between classical physics and quantum physics lies in the fact, that classical physics specifies a physical state by a *finite* number of coordinates, whereas quantum physics uses in principle an infinite number of coordinates. The series of coordinates e.g.

$$a_1, a_2, a_3, \ldots a_n, \ldots$$

was represented in quantum physics (using some system of orthogonal functions) by a function of an auxiliary variable ξ, the wave function, Transition to a different ordertype of the coordinates gives us another representation of the same state: e.g. the ordertype $\omega\, 2$

$$a_1, a_3, a_5, \ldots \quad ; \quad a_2, a_4, a_6, \ldots$$

specifies the states by means of *two* functions of one variable. An ordertype $\omega^n . \varkappa$ would specify the states by means of $\varkappa$ functions of n variables. *So neither the number of functions, nor the number of variables used is of fundamental importance.*

On the other hand the structure of Dirac equations, being formally that of the incidence of a point and a line in threedimensional projective space G_4, suggests from the mathematical point of view the use of the theory of projective invariants, of complex symbols rather than a tensor calculus using special metrical tensors. Also the Poisson brackets of classical physics correspond to algebraical expressions in quantum mechanics! In the following we show that indeed the formalism of a geometry of second order, that of line geometry in G_4 is extremely simple used for quantum-physical purposes.

§ 2. *Generalisation of Dirac equations.*

a. A linecomplex q^2 in G_4:

$$q_{12}\,\pi_{34} + q_{13}\,\pi_{42} + \ldots = 0\,,$$

can be represented symbolically by $(q^2\,\pi^2) = 0$ and has only one invariant

$$(q^2\,q_1^2) = \Omega\,(q^2) = q_{12}\,q_{34} + q_{13}\,q_{42} + q_{14}\,q_{23}\,,$$

from which, as is well known, follows an isomorphism of the projective group in G_4 and the sexternary group leaving invariant $\Omega(q^2)$. The relation

can be transformed by one of FELIX KLEIN's substitutions in the sum of six squares e.g.

$$Q_4 + i Q_6 = q_{12} \quad , \quad Q_1 - i Q_2 = q_{13} \quad , \quad -Q_3 - i Q_5 = q_{14}$$
$$Q_4 - i Q_6 = q_{34} \quad , \quad Q_1 + i Q_2 = q_{42} \quad , \quad -Q_3 + i Q_5 = q_{23} .$$

With these coordinates corresponds a differentialoperator in six-dimensional Euclidean space E_6:

$$\Delta_4' + i \Delta_6' = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial i x_6} = \partial_{12}' = -\partial_{21}' = \partial_{34} = -\partial_{43} , \quad \text{etc.}$$

which allows to introduce an alternating differentialoperator of order ½ in threedimensional space described by complexsymbols transforming cogredient to the points. We have then immediately the relations between the sixdimensional linear form $(\Delta' Q)$ and the quarternary bracket $(\partial^2 q^2)$

$$(\Delta' Q) \equiv 2 \operatorname{div} Q \equiv (\partial^2 q^2)$$
$$(\partial^2 \partial_1^2) \equiv \Delta_6 \equiv \text{Laplaceoperator.}$$

An alternating 2-tensor in E_6 corresponds with a pencilcomplex in G_4; a symmetrical 2-tensor with a quadratic linecomplex.

Rot P' can e.g. be represented by

$$(\Delta' \pi)(\pi P') \quad \text{in} \quad E_6 , \quad \pi_i \pi_k = -\pi_k \pi_i ,$$

and pushing down into G_4 we obtain

$$(\partial^2 \pi^2)(\varrho^2 p^2) \quad , \quad \pi^2 \cdot \varrho^2 = -\varrho^2 \cdot \pi^2 .$$

The components of Rot P' are thus expressed by linear combinations of the 15 coefficients of a pencilcomplex in G_4.

The symbolical methods for the study of these forms have allready been discussed by R. WEITZENBÖCK, Komplexsymbolik (1908).

b. The necessary and sufficient condition for the possibility of splitting up

$$q_{ik} = (fg)_{ik} = \begin{vmatrix} f_i & g_i \\ f_k & g_k \end{vmatrix} ,$$

in which f, g are transforming cogredient to the points of G_4 is, that the linecomplex q^2 be a special linecomplex i.e. $(q^2 q_1{}^2) = 0$.

The Moebius correlation with regard to a general linecomplex is given by $(q^2 xy) = 0$. There exist points with an undetermined conjugated plane only for special complexes and for the points x, on the axis of the complex, holds

$$(q^2 xy) \equiv 0 \{y\} .$$

The equations of motion of the free Dirac electrons are given by the condition that f lies on the axis of the special linecomplex ∂^2.

The conditions

$$(\partial^2 fg) \equiv 0 \{g\}$$

read, written in full

$$\left.\begin{aligned}\partial_{34} f_2 + \partial_{42} f_3 + \partial_{23} f_4 = 0\\ \partial_{34} f_1 + \partial_{13} f_4 + \partial_{41} f_3 = 0\\ \partial_{21} f_4 + \partial_{42} f_1 + \partial_{14} f_2 = 0\\ \partial_{12} f_3 + \partial_{23} f_1 + \partial_{31} f_2 = 0\end{aligned}\right\}$$

Introducing the operators

$$p_k = \frac{\partial}{\partial x_k}$$

and the matrices

$$\alpha_1 = \begin{Vmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{Vmatrix}, \quad \alpha_2 = -i \begin{Vmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&1&0&0\\ -1&0&0&0 \end{Vmatrix}, \quad \alpha_3 = \begin{Vmatrix} 0&0&1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&-1&0&0 \end{Vmatrix},$$

$$\alpha_4 = \begin{Vmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{Vmatrix}, \quad \alpha_5 = i \begin{Vmatrix} 0&0&-1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&1&0&0 \end{Vmatrix}, \quad \alpha_6 = -i \begin{Vmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{Vmatrix}$$

this system reads

$$(\sum_k \alpha_k p_k) \cdot f = 0. \quad \ldots\ldots\ldots \quad (1)$$

A quantity $f_i \not\equiv 0$ can only exist if the symbolical invariant of the operator ∂^2 vanishes i.e.

$$(\partial^2 \partial_1^2) \cdot f \equiv \Delta_6 \cdot f = 0. \quad \ldots\ldots\ldots \quad (2)$$

From the fact that the p_k transform contragredient to the points of E_6 and from the invariance of equation (1) it follows that the α_k must transform cogredient to the points of E_6. (see also Remark I).

To a physical state we make correspond a special symbolical linecomplex. The equations (1), (2) then contain a generalisation of the Dirac equations of motion of the free electron, and are easily identified with those if we, e.g., specialising, put

$$f_i = e^{\Delta x_4} f_i^* (x_1, x_2, x_3, x_6).$$

c. General equations of motion for non-free particles lie before hand: they can be obtained from the, only possible, projective invariant form

$$(\partial^2 fg) + (c^2 fg) \equiv 0 \{g\}. \quad \ldots\ldots\ldots \quad (3)$$

If we put, specialising,

$$\frac{\partial f_i}{\partial x_4} = (m_0 c - \Delta) f_i \quad , \quad \frac{\partial f_i}{\partial x_5} = S f_i$$

and apart from numerical constants

$$C = (\mathfrak{a}_x, \mathfrak{a}_y, \mathfrak{a}_z, \Lambda, S, i\varphi)$$

we obtain the well known Dirac equations for the electrons in an arbitrary field. The postulate of Dirac

$$p_k \rightarrow p_k + c_k$$

corresponds to the postulate of projective invariance in G_4.

That the corresponding wave equation of the general equations of motion contains 15 additional spin terms is evident from the multiplication into $(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 + \alpha_5 p_5 - \alpha_6 p_6)$ of (3).

d. We can also specialise the equations (3) by supposing a relation between the f_i, which can be interpreted as a *structure of the particle.* If e.g. we put

$$\frac{\partial f_n}{\partial x_6} = \sum_k A_{nk} f_k$$

the equation (3) becomes with 5 variables

$$(\Sigma \alpha_\mu \frac{\partial}{\partial x_\mu} + \alpha_\mu L_\mu + \alpha_\mu \alpha_\nu \Lambda_{\mu\nu} + M \| 1 \|) \cdot f = 0$$

$$\Lambda_{\mu\nu} = - \Lambda_{\nu\mu} \qquad (\mu, \nu = 1, 2, 3, 4, 5)$$

which are *Møller's meson equations* [1]). The matrix $\|A\|$ can be a general matrix; without a relation between the f_i we cannot obtain terms $\alpha_\mu \alpha_\nu$ in an invariant way as a matrix that anticommutes with $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ must vanish.

§ 3. *Densities in quantum mechanics.*

a. Let us consider the quantities

$$j_k = \varphi' \alpha_k \varphi,$$

the sixth component is apart from a constant the quarternary linear form $I = (\varphi' \varphi)$. As is easily controlled we have

$$\sum_{k=1}^{6} j_k^2 = \sum_{k=1}^{6} (\varphi' \alpha_k \varphi)^2 = 0.$$

The vector j_k in E_6 is an isotrope vector.

b. Between the α_k, $k = 1, 2, 3, 4, 5$ we have the relations

$$\alpha_i \alpha_k = - \alpha_k \alpha_i$$

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4 = \alpha_5 .$$

If now we form the quantities

$$P_k = \varphi' \alpha_k \varphi \; ; \; p_{ik} = \varphi' \alpha_i \alpha_k \varphi \; ; \; q_{ikl} = \varphi' \alpha_i \alpha_k \alpha_l \varphi \; ; \; a'_n = \varphi' \alpha_i \alpha_k \alpha_l \alpha_m \varphi$$

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[1]) C. MØLLER, D. Danske Vid. Selsk. math.-fys. Medd., 18 (1941).

we can represent P, p^2, q^3, a' by a point P, a planecomplex p^2, a linecomplex q^3 and a G_4 in G_5.

Because of the commutationrules between the α_k we have

$$\alpha_i\, \alpha_k\, \alpha_l = -\,\text{signum}\,(i\,k\,l\,m\,n)\, \alpha_m\, \alpha_n$$

$$\alpha_i\, \alpha_k\, \alpha_l\, \alpha_m = +\,\text{signum}\,(i\,k\,l\,m\,n)\, \alpha_n$$

or

$$P_k = a'_k \qquad p_{ik} = -\,q'_{ik}\,,$$

q'_{ik} being the space coordinates of q^3. If there did not exist relations of the form

$$(p_1^2\, p_2^2\, x) = \text{factor}\,(a'\, x) \quad ; \quad (p^2\, P\, \pi^2) = \text{factor}\,(q^3\, \pi^2)$$

the physical system would apart from the linearform $(\varphi'\, \varphi)$ have other invariants, independent of this one, as $(p^2 p_1{}^2\, P)$, $(p^2 q^3)$,...
We have to control the projective invariant relations between P, p^2, q^3, a'.

I. We consider the linecomplex $(p^2\, P \pi^2)$ and calculating the 15 relations we obtain

$$\sum_{\text{cycl } i,k,l} p_{ik}\, P_l = I\, q_{ikl}$$

or

$$\underline{(p^2\, P\, \pi^2) \equiv I\,(q^3\, \pi^2)} \qquad \{\pi^2\}\,.$$

II. The space $(p^2 p_1{}^2\, x)$ leads to calculate the 5 forms

$$p_{ik}\, p_{lm} + p_{il}\, p_{mk} + p_{im}\, p_{lk} = I\, a'_n$$

or

$$\underline{(p^2\, p_1^2\, x) \equiv I\,(a'\, x)} \qquad \{x\}\,.$$

So there is only one invariant I, $(a'P) = I^2$ and the relations between the "densities" are in geometrical form

I. q^3 is the compound of p^2 and P.

II. a' is the focalspace of the planecomplex p^2;
P is the focus of the linecomplex q^3.

c. In sixdimensional space we can build a G_4-complex with

$$p_{ik} = \varphi'\, \alpha_i\, \alpha_k\, \varphi \qquad (i, k = 1, 2, 3, 4, 5)$$

$$p_{k6} = -\,p_{6k} = \varphi'\, \alpha_k\, \varphi$$

We, then, have firstly to consider the linecomplex

$$(p^2\, p_1^2\, \pi^2) = (s^4\, \pi^2).$$

From b we see immediately

$$s_{iklm} = (p^2\, p_1^2)_{iklm} = I\, P_n = I\, p_{n6}$$

$$s_{ikl6} = p_{ik}\, p_{l6} + p_{il}\, p_{6k} + p_{ik}\, p_{kl} = p_{ik}\, P_l + p_{li}\, P_k + p_{kl}\, P_i = I\, q_{ikl} = I\, p_{mn}.$$

We can therefore sum up all relations given by I, II in

$$\underline{(p^2\, p_1^2\, \pi^2) \equiv I\,(r^4\, \pi^2)} \qquad (\pi^2)$$

$$r'_{ik} = p_{ik} \qquad i, k = 1, 2, 3, 4, 5, 6.$$

The invariant of the complex is now evidently

$$(p^2\, p_1^2\, p_2^2) \equiv I\, \Sigma\, p_{ik}^2 \equiv 2\, I^3 \quad \text{as} \quad \Sigma\, p_{ik}^2 = (p^2\, q^3) + I^2.$$

Remark I.

If we transform (f_1, f_2, f_3, f_4) by a general projective transformation S of G_4

$$\bar{f} = S \cdot f$$

the bracket $(\partial^2 f\, g)$ will be transformed, apart from the transformation modulus, the determinant of S, in $(\bar{\partial}^2 \bar{f}\, \bar{g})$. The generalised Dirac equations will then be transformed in

$$(\Sigma\, a_k\, \bar{p}_k)\, \bar{f} = (\Sigma\, a_k\, \bar{p}_k) \cdot S f.$$

As we remarked already in § 2 the fact that p_k transform contragredient to the points in E_6 in the orthogonal transformation corresponding to the projective transformation S in G_4 we can multiplying at the left side by an operator T restore the original equations whilst

$$f_i\,(x_1, \ldots x_6) = \bar{f}_i\,(\bar{x}_1, \ldots \bar{x}_6).$$

The a_k are then transformed cogredient to the points of E_6. We will deduce the explicite form of T.

a. The general projective transformation S of G_4 can be represented by the symbolical product [2])

$$S: \ (a'\, x)\,(\alpha\, u') \quad \text{which reads} \quad \tilde{x}_i = \sum_k a_i\, a'_k\, x_k.$$

The determinant of the transformation is $D = (a'\, b'\, c'\, d')\ (\alpha\, \beta\, \gamma\, \delta)$, apart from a constant. The inverse transformation is

$$S^{-1}: \ (a'\, b'\, c'\, u')\,(\alpha\, \beta\, \gamma\, x) = (\alpha'\, x)\,(a\, u')$$

Indeed

$$S^{-1}\, S: \ (a'\, x)\,(\alpha\, \alpha')\,(a\, u') = \tfrac{1}{4}\, D \cdot (u'\, x)$$

as we obtain splitting up $a\ \ \alpha'$ in $(b'\, c'\, d')$, $(\beta\, \gamma\, \delta)$ and transforming the a' into the last bracket.

The transformation of the line coordinates in G can be easily found: intersecting

$$(a'\, x)\,(\alpha\, u') = 0 \qquad (b'\, y)\,(\beta\, u') = 0\,, \quad (x\, y)_{ik} = p_{ik}$$

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[2]) The projective invariants of S are given by the chains, the first of which is the diagonal sum

$$I_1 = (a'\, \alpha)\,, \quad I_2 = (a'\, \beta)\,(b'\, \alpha)\,, \quad I_3 = (a'\, \beta)\,(b'\, \gamma)\,(c'\, \alpha)\,, \quad I_4 = (a'\, \beta)\,(b'\, \gamma)\,(c'\, \delta)\,(d'\, \alpha)\,.$$

we have

$$(a' x)(b' y)(\alpha \beta \pi^2) \equiv \tfrac{1}{2}(a' p)(b' p)(\alpha \beta \pi^2)$$

or

$$\tilde{p}_{ik} = \{(\alpha\beta)_{ik} \; \Sigma \; (a' b')_{\lambda\mu} p_{\lambda\mu} = (\alpha\beta)_{ik} L$$

where, introducing the sixdimensional coordinates $X_1, X_2, \ldots, X_6$ we have

$$\begin{aligned} L = \{ & (a' b')_{13} + (a' b')_{42} \} X_1 - i \{(a' b')_{13} - (a' b')_{42}\} X_2 + \\ \{ & -(a' b')_{14} - (a' b')_{23} \} X_3 - i \{(a' b')_{14} - (a' b')_{23}\} X_5 + \\ \{ & (a' b')_{12} + (a' b')_{34} \} X_3 + i \{(a' b')_{12} - (a' b')_{34}\} X_6 . \end{aligned}$$

or calculating the $\widetilde{X}_1, \ldots, \widetilde{X}_6$ from the $\tilde{p}_{ik}$

$$\widetilde{X}_1 = \frac{(\alpha\beta)_{13} + (\alpha\beta)_{42}}{2} L, \; \widetilde{X}_2 = i \frac{(\alpha\beta)_{13} - (\alpha\beta)_{42}}{2} L, \; \widetilde{X}_3 = -\frac{(\alpha\beta)_{14} + (\alpha\beta)_{23}}{2} L$$

$$\widetilde{X}_4 = \frac{(\alpha\beta_{12}) + (\alpha\beta)_{34}}{2} L, \; \widetilde{X}_5 = +i \frac{(\alpha\beta)_{14} - (\alpha\beta)_{23}}{2} L, \; \widetilde{X}_6 = -i \frac{(\alpha\beta)_{12} - (\alpha\beta)_{34}}{2} L .$$

As this transformation is an orthogonal transformation in E_6 we obtain the inverse transformation

$$X_i = \sum_k O_{ik} \widetilde{X}_k$$

by simply interchanging the rows and columns, apart from a numerical constant.

On the other hand

$$\sum_k c_k a_k = \begin{Vmatrix} c_4 - i c_6 , & 0 , & c_3 - i c_5 , & c_1 - i c_2 \\ 0 , & c_4 - i c_6 , & c_1 + i c_2 , & -c_3 - i c_5 \\ c_3 + i c_5 , & c_1 - i c_2 , & -c_4 - i c_6 , & 0 \\ c_1 + i c_2 , & -c_3 + i c_5 , & 0 , & -c_4 - i c_6 \end{Vmatrix}$$

so we can easily express

$$\sum_k O_{ik} a_k = [a', b']_i \begin{Vmatrix} (\alpha\beta)_{34} & 0 & -(\alpha\beta)_{14} & (\alpha\beta)_{13} \\ 0 & (\alpha\beta)_{34} & (\alpha\beta)_{42} & (\alpha\beta)_{23} \\ -(\alpha\beta)_{23} & (\alpha\beta)_{13} & -(\alpha\beta)_{12} & 0 \\ (\alpha\beta)_{42} & (\alpha\beta)_{14} & 0 & -(\alpha\beta)_{12} \end{Vmatrix}$$

where $[a', b']_i$ is the symbolical coefficient of X_i in L.

Multiplying on the right with S we have as e.g.

$$(\alpha\beta)_{34} \gamma_1 c_i' - (\alpha\beta)_{24} \gamma_3 c_i' + (\alpha\beta)_{23} \gamma_4 c_i' = (\alpha\beta\gamma)_{341} c_i'$$

$$\sum_k 0_{ik} a_k \cdot S = [a', b']_i \begin{Vmatrix} \ldots, (\alpha\beta\gamma)_{341} \; c_\lambda', \ldots \\ \ldots, (\alpha\beta\gamma)_{342} \; c_\lambda', \ldots \\ \ldots, (\alpha\beta\gamma)_{231} \; c_\lambda', \ldots \\ \ldots, (\alpha\beta\gamma)_{421} \; c_\lambda', \ldots \end{Vmatrix}$$

from which introducing the a, a' by using cyclical symmetry is obtained

$$\sum_k O_{ik}\, a_k \cdot S = T\, a_i$$

$$T = \begin{Vmatrix} T_{22} & -T_{21} & -T_{24} & T_{23} \\ -T_{12} & T_{11} & T_{14} & -T_{13} \\ -T_{42} & T_{41} & T_{44} & -T_{43} \\ T_{32} & -T_{31} & -T_{34} & T_{33} \end{Vmatrix} \qquad T_{ik} = (S^{-1})_{ik}.$$

We have e.g. for $i = 1$ from

$$[(a'\,b')_{13} + (a'\,b')_{42}]\; c'_\lambda$$

$$\lambda = 1,\ (a'\,b'\,c')_{421}\ ;\ \lambda = 2,\ (a'\,b'\,c')_{132}\ ;\ \lambda = 3,\ (a'\,b'\,c')_{423}\ ;\ \lambda = 4,\ (a'\,b'\,c')_{134}$$

$$\sum_k O_{1k}\, a_k \cdot S = \begin{Vmatrix} a'_2 c_3\,, & -a'_2 c_4\,, & -a'_2 c_1\,, & a'_2 c_2 \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix} = T\, a_1\,.$$

Remark II.

The relations deduced in § 3 will be seen to correspond to the Pauli-identities. Recently G. PETIAU [3]) has deduced these relations using tensor calculus, introducing a special metrical tensor. It may be of use to show the identity of his final results with the relations obtained above.

A. The relation

$$f_{\alpha\beta}\, \omega_1 = f_{\alpha\beta\gamma}\, j^\gamma$$

reads in projective notation

$$\omega_1\, (p\,\pi')^2 \equiv (q\,\pi')^2\, (q\,j') \equiv (q'^{\,2}\, j'\, \pi'^{\,2}) \qquad \{\pi\}$$

or

$$(q'^{\,2}\, j')_{ikl} = \omega_1\, p'_{ikl}\,.$$

B. The relation

$$f_{\alpha\beta\gamma}\, f^{\gamma\delta} = \omega_1\, [g^\delta_\alpha\, j_\beta - g^\delta_\beta\, j_\alpha]$$

reads in projective notation

$$(q\,\pi')^2\, (q\,q')\, (q'\,x) \equiv \omega_1\, (g'\,g)\, (g'\,x)\, (g\,\pi')\, (j\,\pi')$$

which shows because of the fact that $(g'\,g)$ is a reducent on $(g'\,g)^2$ that this relation is completely independent of the metrical tensor used!! So it is superfluous to introduce a metrical tensor for these relations. We have

$$(q\,\pi')^2\, (q\,q')\, (q'\,x) \equiv \text{const}\, \omega_1\, (g'\,g)^2\, (x\,\pi')\, (j\,\pi')$$

and transforming to pointcoordinates we have

$$(\pi^3\, q^2)\, (q\, q_1^3\, x) \sim (q^3\, \pi^2)\, (\pi\, q_1^3\, x) \sim -\, (\pi^2\, q_1^2\, x)\, (\pi\, q_1\, q^3) \sim (q^3\, q_1^2)\, (q_1\, \pi^3\, x)$$

or

$$(j\,u') \equiv (q^3\, q_1^2)\, (q_1\, u')$$

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[3]) Revue scientifique **83**, 37 (1945).

which with the contravariant notation of q^3 results in

$$\omega_1 (j u') \equiv (p'^2 p_1'^2 u').$$

Conclusion:

If we represent a physical state by a symbolical special linecomplex

$$p_{ik} = (fg)_{ik} \quad \text{in} \quad G_4,$$

where f, g, are functions of six variables x_i, the generalised Dirac equations of motion correspond to the incidence of the point f with the axis of $\partial^2 + c^2$

$$(\partial^2 fg) + (c^2 fg) \equiv 0 \qquad \{g\}$$

which is of the form

$$\sum_k a_k \left(\frac{\partial}{\partial x_k} + c_k \right) \cdot f$$

in which the spin matrices a_k transform like the point coordinates of the orthogonal transformations of x_i. We can use general projective transformations for the f. The MØLLER meson equations can be obtained by relations between the f_i. The relations between the "densities"

$$\varphi' a_i a_k \varphi \quad , \quad \varphi' a_k \varphi$$

can be considered as those of the coordinates of a G_4-complex in E_6 P_{ik}, which, satisfies the relation

$$(P^2 P_1^2 \pi^2) \equiv (\varphi' \varphi) (R^4 \pi^2) \qquad \{\pi^2\}$$
$$R'_{ik} = P_{ik}.$$

The Dirac postulate for the equations of motion in an external field follows automatically from the projective invariance in G_4.