Physics. — *"On the application of* EINSTEIN'S *theory of gravitation to a stationary field of gravitation."* By H. A. KRAMERS. (Communicated by Prof. H. A. LORENTZ).

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§ 1. *Definition and invariant properties of a stationary field of gravitation.*

We will call a field of gravitation stationary when the expression for the line element can be put into such a form $ds^2 = g_{\mu\nu} dx_\mu dx_\nu$ [1]) (x_0 time-coordinate, x_1, x_2, x_3 space-coordinates) that the gravitation potentials $g_{\mu\nu}$ do not depend on the time x_0. A special case of the stationary field of gravitation, defined in this way, forms the so-called "static" field of gravitation, which appears when it is possible by a suitable transformation to make the quantities g_{01}, g_{02} and g_{03} equal to zero. It is simply seen, that when the line element of a stationary field of gravitation is brought in the above mentioned form, the most general transformation of coordinates, for which the $g^{\mu\nu}$'s remain independent of the time, and for which a point at rest remains at rest, is given by the formulae

$$\left.\begin{aligned} x_k &= \varphi_k (x'_1, x'_2, x'_3), \qquad (k = 1, 2, 3) \\ x_0 &= a x'_0 + \psi (x'_1, x'_2, x'_3). \end{aligned}\right\} \quad . \quad (1)$$

Here φ_k and ψ are arbitrary functions of x'_1, x'_2, x'_3, while a is a positive constant. The quantities $g_{\mu\nu}$ and their derivatives show, with regard to the transformation group expressed by (1), certain invariant and covariant properties, which we will now investigate. The line element may be written in the following form,

$$\left.\begin{aligned} ds^2 = g_{\mu\nu} dx_\mu dx_\nu = -\Sigma G_{kl} dx_k dx_l + \frac{1}{g_{00}} (g_{00} dx_0 + g_{01} dx_1 + g_{02} dx_2 + g_{03} dx_3)^2, \\ G_{kl} = - g_{kl} + \frac{g_{0k}\, g_{0l}}{g_{00}}, \end{aligned}\right\} . \quad (2)$$

[1]) Just as EINSTEIN we have omitted the signs of summation for summations which have to be extended over indices, which occur twice in a product.

where the summation has to be extended over $k,l = 1, 2, 3$. When in the following an index can assume one of the values 1, 2, 3, we will denote this index by a Latin letter. If on the contrary an index can assume one of the values 0, 1, 2, 3, it will always be denoted by a Greek letter. In case of summation over an index, occurring twice in a product, the sign of summation will be omitted in both cases. If now we perform the transformation (1), the expression $G_{kl}\, dx_k\, dx_l$ becomes again a quadratic form of the differentials of the space coordinates, and $\frac{1}{g_{00}} (g_{0\mu}\, dx_\mu)^2$ becomes again the square of a linear differential form. Since the separation in two parts of the expression of the line-element given by (2) is only possible in one way, we may conclude, that the expressions

$$G_{kl}\, dx_k\, dx_l \quad \text{and} \quad \frac{(g_{0\mu}\, dx_\mu)^2}{g_{00}}$$

are invariants with regard to the transformation (1). Consequently the quantities $g_{0\mu}/\sqrt{g_{00}}$ possess the character of a vector, and from this we conclude again, that the bilinear differentialform

$$g_{00}^{s-\frac{1}{2}} \left\{ \frac{\partial}{\partial x_\mu} \left(\frac{g_{0\nu}}{g_{00}^s} \right) - \frac{\partial}{\partial x_\nu} \left(\frac{g_{0\mu}}{g_{00}^s} \right) \right\} dx_\mu\, \delta x_\nu \quad . \quad . \quad . \quad . \quad (3)$$

is also invariant with regard to the transformation (1). The constant s may be chosen arbitrarily, because the quantity g_{00} appears only multiplied by a constant factor after the transformation. Choosing the special value $s = 1$, we see that all terms for which $\mu = 0$ or $\nu = 0$ become equal to zero, so that in this case we may omit the index 0 under the summation, and we obtain the result, that the expression

$$\sqrt{g_{00}} \left\{ \frac{\partial}{\partial x_k} \left(\frac{g_{0l}}{g_{00}} \right) - \frac{\partial}{\partial x_l} \left(\frac{g_{0k}}{g_{00}} \right) \right\} dx_k\, \delta x_l \quad . \quad . \quad . \quad . \quad (4)$$

is invariant. As the coefficients of this differential form are antisymmetrical with regard to the indices k and l, we may consider the expression (4) as a linear form of the differentials $dx_{kl} = dx_k \delta x_l - \delta x_k dx_l$. Now for a threedimensional extension, the expression $\Sigma \sqrt{G} Dx_m dx_{kl}$, remains invariant for an arbitrary transformation of coordinates, where G represents the determinant of the coefficients G_{kl} in the expression $d\varrho^2 = G_{kl}\, dx_k dx_l$ for the invariant line-element, and where under the summation the indices k,l,m assume the sets of values 1, 2, 3 and 2, 3, 1 and 3, 1, 2. Consequently the quantities $\sqrt{G} dx_{kl}$ are transformed as the components of a covariant vector (if we, in the usual way, call the transformation of the components dx_k of a small displacement contravariant). From the invariance of the expression (4) we may thus conclude, that the quantities [1])

$$R^m = \tfrac{1}{2}\sqrt{\frac{g_{00}}{G}}\left\{\frac{\partial}{\partial x_k}\left(\frac{g_{0l}}{g_{00}}\right) - \frac{\partial}{\partial x_l}\left(\frac{g_{0k}}{g_{00}}\right)\right\}, \quad . \quad . \quad . \quad . \quad (5)$$

where k,l,m again may assume the sets of values 1, 2, 3 and 2, 3, 1 and 3, 1, 2, with regard to the transformation (1) are the contravariant components of a vector in the three-dimensional extension with the invariant line-element $d\varrho^2 = G_{kl}\,dx_k dx_l$. The invariant absolute value of this vector is given by $R = \sqrt{G_{kl}\,R^k R^l}$.

If the components R^m are everywhere equal to zero, we have to do with a static field of gravitation. In fact, from (5) follows that in this case the quantities g_{0k}/g_{00} may be deduced from a potential φ in such a way that $g_{0k} = g_{00}\dfrac{\partial \varphi}{\partial x_k}$; but from this follows again, that the line element may be written in the form $ds^2 = -G_{kl}\,dx_k dx_l + g_{00}dx_0'^2$, where $x_0' = x_0 + \varphi$.

If the components R^m are not equal to zero, these quantities determine in every point what might be called the "rotatory" properties of the stationary field of gravitation. This may be illustrated by considering the motion of a masspoint, the velocity of which is small compared with the velocity of light. In general the "worldline" of a masspoint is determined by the equations

$$\frac{d^2 x_\lambda}{ds^2} + \left\{ {\mu\nu \atop \lambda} \right\} \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If now we assume, that by a suitable transformation of the kind (1), the line-element has been given the form $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ at a given point P of the worldline, it may be easily verified that the equations (6), looking apart from small terms of the same order of magnitude as the square of the velocity, in the point P assume the simple form

$$\frac{d^2 x_k}{dx_0^2} = 2\left(R^l \frac{dx_m}{dx_0} - R^m \frac{dx_l}{dx_0}\right) - \tfrac{1}{2}\frac{\partial g_{00}}{\partial x_k}, \quad . \quad . \quad . \quad . \quad (7)$$

where k,l,m just as before may assume the sets of values 1, 2, 3 and 2, 3, 1 and 3, 1, 2. From these equations we learn that the "force" which the field of gravitation in the point P exerts on a mass point of unit mass may be described as the sum of a Coriolis-force perpendicular to the velocity and proportional to it and of a force, which may be derived from the potential $\tfrac{1}{2} g_{00}$.

[1]) If we admit only such transformations, for which the functional determinant is positive, we may by the root-sign in this expression always understand the positive root.

If next we consider the motion of a mass-point in a conservative field of force, such as would take place according to Newtonian mechanics, we obtain equations of motion of the same form as (7), if the Cartesian coordinates describing the position of the mass-point refer to a system of coordinates which rotates uniformly in the space. In fact, if the equations of motion in the non-rotating system of coordinates possess the form

$$\frac{d^2 x_k}{dt^2} = -\frac{\partial \varphi}{\partial x_k}$$

they obtain in a system of coordinates x'_1, x'_2, x'_3, which rotates round an axis through the origin with an angular velocity R which, considered as a vector, possesses the components $-R^1$, $-R^2$ and $-R^3$, [1]) the form:

$$\frac{d^2 x'_k}{dt^2} = 2\left(R^l \frac{dx'_m}{dt} - R_m \frac{dx'_l}{dt}\right) - \frac{\partial(\varphi - \psi)}{\partial x'_k},$$

$$\psi = \tfrac{1}{2}(R)^2(x_1'^2 + x_2'^2 + x_3'^2) - \tfrac{1}{2}(R^1 x'_1 + R^2 x'_2 + R^3 x'_3)^2,$$

which coincides exactly with the form of the equations (7), if we put $t = x_0$ and $\varphi - \psi = \tfrac{1}{2} g_{00}$. The essential difference with the equations of motion in the non-rotating system of coordinates lies consequently in the appearance of the Coriolis-forces, and we are justified in denoting in the following the vector R^k as the "*rotation-vector*".

The character of the rotation-vector may also be examined in the following way. We will try by a transformation of coordinates of the form (1) to give the line-element of a stationary field of gravitation such a form, that in a given point P not only the relations $g_{\mu\nu} = \varepsilon_{\mu\nu}$ are valid, where the quantities $\varepsilon_{\mu\nu}$ are defined by

$$\varepsilon_{00} = -\varepsilon_{11} = -\varepsilon_{22} = -\varepsilon_{33} = 1, \qquad \varepsilon_{\mu\nu} = 0 \ (\mu \neq \nu), \quad . \quad (8)$$

but that at the same time the quantities $\dfrac{\partial g_{\mu\nu}}{\partial x_k} = g_{\mu\nu,k}$ as many of them as possible become equal to zero. If it was possible to make all the latter quantities equal to zero, we should obtain in this way a system of coordinates, which is "geodetic" in P. Now it is always possible in many ways by means of a transformation (1) to make the quantities $g_{\mu\nu}$ assume the values $\varepsilon_{\mu\nu}$ in the point P, but in general it will not be possible to make all quantities $g_{\mu\nu,k}$ equal to zero. In

[1]) Here and in the following we will assume the usual rule, that to a rotation in a plane corresponds a direction of the normal of this plane in such a way, that, for a rotation in the x_1, x_2-plane from the positive x_1-axis to the positive x_2-axis through an angle smaller than π, the corresponding normal points to the same half-space as the positive x_3-axis.

the first place this is easily seen to hold for the quantities $g_{00,k}$, because g_{00} by the transformation (1) is only multiplied by a constant factor; but neither the quantities $g_{0,kl}$ can all of them at the same time be reduced to zero, because this would mean, that the components of the rotation-vector would be equal to zero, and this is in general not the case. On the other hand it is obviously always possible to perform such a special transformation, that the system of coordinates in the three-dimensional extension with the line-element $d\sigma^2 = G_{kl}\, dx_k\, dx_l$ becomes geodetic in the point P. In this way we get $\frac{\partial G_{kl}}{\partial x_n} = 0$ and consequently also $g_{kl,n} = 0$, since $g_{kl} = -G_{kl} + \frac{g_{k0} g_{l0}}{g_{00}}$ and since the quantities g_{ko} are equal to zero in the point P.

Let us now imagine, that by means of (1) the line-element has been given such a form, that in a given point P the following relations are valid

$$\left.\begin{aligned} a)\quad & g_{\mu\nu} = \varepsilon_{\mu\nu} \\ b)\quad & g_{kl,\nu} = 0 \\ c)\quad & g_{0k,l} + g_{0l,k} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

As regards the third of these conditions it will be observed, that it is always possible by a suitable transformation of the time to effect, that the symmetrical quantities $A_{kl} = g_{0k,l} + g_{0l,k}$ become equal to zero in P. In fact, it is easily shown that, if the conditions (a) and (b) are already fulfilled, but not yet (c), the transformation $x'_0 = x_0 + \frac{1}{2} (A_{kl})_P (x_k - (x_k)_P)(x_l - (x_l)_P)$ leads us to the desired purpose. Thereby we have denoted the value of a quantity in the point P by adding the index P on the right below. Let us now perform a transformation of coordinates, which corresponds to a uniform rotation, around an axis through P, of the x_1, x_2, x_3-space (considered as a Euclidean space with the line-element $d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2$), the angular velocity of which considered as a vector has the components R^1, R^2, and R^3. After the performance of this "*rotation transformation*", which does not belong to the group of transformations (1), the relations (a) and (b) are still valid, but also all the quantities $g_{0k,l}$ have become equal to zero. This may be proved by a direct calculation, and the proof becomes especially simple, if we assume, that in P the quantities R^1 and R^2 are equal to zero, so that we have to do with a uniform rotation round the axis of the coordinate x_3 with an angular velocity $R^3 = \omega$. Since in the point P the quantities R_k reduce to $\frac{1}{2}(g_{0m,l} - g_{0l,m})$, we have in consequence of relation (c), that of all

quantities $g_{0k,l}$ only the two quantities $g_{01,2}$ and $g_{02,1}$ are different from zero in such a way that $\omega = g_{02,1} = -g_{01,2}$. The rotation-transformation can now be written in the form

$$x_1 - (x_1)_P = x'_1 \cos \omega x_0 - x'_2 \sin \omega x_0,$$
$$x_2 - (x_2)_P = x'_1 \sin \omega x_0 + x'_2 \cos \omega x_0.$$

If now by means of this formula the line-element (2) is transformed, and if we make use of the above mentioned relations (a), (b) and (c) it is easy to verify:

That for the transformed quantities $g'_{\mu\nu}$ in the point P the relations (a) and (b) are still valid.

That, although the $g_{\mu\nu}$'s will contain the time x_0, their first derivatives with respect to the time will be equal to zero in P, and that equally all derivatives of g_{01} g_{02} and g_{03} have become equal to zero.

We thus see in the first place that, after the rotation-transformation, the equations for the world-line of a mass point, the velocity of which is small compared with that of light, assume in the point P the simple form

$$\frac{d^2 x_k}{dx_0^2} = -\frac{1}{2} \frac{\partial g_{00}}{\partial x_k},$$

so that the term corresponding to a Coriolis-force appears no more, which was naturally to be expected from the above considerations. Let us further consider the special case, that in the point P the mass point can remain in equilibrium; that is, that in this point the quantities $g_{00\,k}$ in the original system of coordinates are equal to zero. In this case we find, that in the new system of coordinates, to which the rotation-transformation has given rise, all quantities $g_{\mu\nu,\rho}$ without exception disappear in the point P, so that this system of coordinates is geodetic in that point.

§ 2. *On the field of gravitation, which is produced by stationarily moving masses.*

Let us consider a space-time-extension, for which the line-element at large distances from the zero-point of the coordinates approaches to $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ [1]) and in which there exist masses, which perform stationary motions; that is, the components $T_{\mu\nu}$ of the energy-tensor of matter do not depend on the time. The field of gravitation, to which these masses give rise will then be stationary

[1]) Here and in the following we shall always assume that the centimeter has been chosen as unit of length. The unit of time is then determined by the condition that the velocity of light is equal to 1.

in the sense described in § 1, and is determined by the equations of EINSTEIN

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R = -\, 8\,\pi\,\varkappa\, T_{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which $R_{\mu\nu}$ is a tensor of the second order, which depends only on the g's and their derivatives:

$$R_{\mu\nu} = \frac{\partial}{\partial x_\nu}\begin{Bmatrix}\mu\varrho\\ \varrho\end{Bmatrix} - \frac{\partial}{\partial x_\rho}\begin{Bmatrix}\mu\nu\\ \varrho\end{Bmatrix} + \begin{Bmatrix}\mu\varrho\\ \sigma\end{Bmatrix}\begin{Bmatrix}\nu\sigma\\ \varrho\end{Bmatrix} - \begin{Bmatrix}\mu\nu\\ \varrho\end{Bmatrix}\begin{Bmatrix}\varrho\sigma\\ \sigma\end{Bmatrix},$$

while $R = g^{\mu\nu} R_{\mu\nu}$. $\varkappa$ is the gravitation constant of EINSTEIN, which, if we choose the gram as unit of mass, is equal to $7{,}4\,.\,10^{-29}$. If we assume that the $g_{\mu\nu}$'s of the line-element which corresponds to the field of gravitation, only differ little from $\varepsilon_{\mu\nu}$, there exists a simple method indicated by EINSTEIN to obtain in first approximation a solution of the equations (10). This solution is obtained by writing

$$g_{\mu\nu} = \varepsilon_{\mu\nu} + \gamma_{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where the functions $\gamma_{\mu\nu}$ everywhere possess a very small value, and introducing the quantities $\gamma'_{\mu\nu}$ defined by

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2}\,\varepsilon_{\mu\nu}\,(\varepsilon_{\alpha\beta}\,\gamma_{\alpha\beta}),$$

which give

$$\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \tfrac{1}{2}\,\varepsilon_{\mu\nu}\,(\varepsilon_{\alpha\beta}\,\gamma'_{\alpha\beta}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the values of the $\gamma'_{\mu\nu}$ in the point $\overline{x}_1,\ \overline{x}_2,\ \overline{x}_3$ and at the moment $\overline{x}_0$ may be calculated as retardated potentials by means of the formulae

$$\gamma'_{\mu\nu} = -\,4\varkappa\int\frac{T_{\mu\nu}\,[x_0 = \overline{x}_0 - r]}{r}\,dS \quad . \quad . \quad . \quad . \quad (13)$$

Here dS represents the space-element $dx_1\,dx_2\,dx_3$ and r represents the distance from that space-element to the point

$$\overline{x}_1,\ \overline{x}_2,\ \overline{x}_3\ \ (r^2 = (x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2 + (x_3 - \overline{x}_3)^2).$$

The addition $[x_0 = \overline{x}_0 - r]$ means that everywhere the value of $T_{\mu\nu}$ at the moment $\overline{x}_0 - r$ has to be used. If we apply the formulae (13) to our case where the $T_{\mu\nu}$'s do not depend on the time, we may clearly omit the latter condition, and we obtain the usual formulae for static potentials

$$\gamma'_{\mu\nu} = -\,4\varkappa\int\frac{T_{\mu\nu}}{r}\,dS\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

We will now calculate the components of the rotation-vector in the point $\overline{x}_1, \overline{x}_2, \overline{x}_3$. Neglecting small terms which relative to the main terms are of the same order of magnitude as the $\gamma'_{\mu\nu}$'s, we obtain for the x_1-component of the rotation-vector

$$R^1 = \tfrac{1}{2}\sqrt{\frac{g_{00}}{G}}\left\{\frac{\partial}{\partial x_2}\left(\frac{g_{30}}{g_{00}}\right) - \frac{\partial}{\partial x_3}\left(\frac{g_{20}}{g_{00}}\right)\right\} = \tfrac{1}{2}\left(\frac{\partial\gamma'_{30}}{\partial x_2} - \frac{\partial\gamma'_{20}}{\partial x_3}\right)$$

$$= -\,2\varkappa\int\left\{T_{03}\frac{\partial}{\partial \overline{x}_2}\left(\frac{1}{r}\right) - T_{02}\frac{\partial}{\partial \overline{x}_3}\left(\frac{1}{r}\right)\right\}dS =$$

$$= 2\varkappa\int\frac{(x_2 - \overline{x}_2)\,T_{03} - (x_3 - \overline{x}_3)\,T_{02}}{r^3}\,dS,$$

and analogous expressions for the components R^2 and R^3. With neglect of small terms of the same order of magnitude as the square of the velocities of the masses we have further the formulae

$$T_{00} = m, \qquad T_{01} = -\,m\,v_1, \qquad T_{02} = -\,m\,v_2, \qquad T_{03} = -\,m\,v_3,$$

where m is the density of mass of matter, and v_1, v_2, v_3 denote the components of the velocity of matter. If we substitute these values in the expression found above, we get

$$R^1 = 2\varkappa\int m\,\frac{(x_3 - \overline{x}_3)\,v_2 - (x_2 - \overline{x}_2)\,v_3}{r^3}\,dS\,. \quad . \quad . \quad . \quad (15)$$

and correspondingly R^2 and R^3. This formula teaches, that the contribution of every mass particle to the rotation-vector in point P is equal to the moment of momentum of the mass particle with respect to the point P, divided by the cube of the distance from P, and multiplied by twice the gravitation constant of EINSTEIN.

Formula (15) can be applied to a problem which has been treated by H. THIRRING [1]) in order to illustrate the influence of rotating masses on the field of gravitation. A homogeneous spherical shell with mass M and radius a rotates with constant angular velocity ω in a space, in which no other matter is present, and for which the quantities $g_{\mu\nu}$ approach to $\varepsilon_{\mu\nu}$ at infinite distance from the centre. It is asked to determine the influence of the spherical shell on the motion of a mass point, which is lying just at the centre O. The field of gravitation produced by the shell is stationary. From symmetry we may further conclude that in O a mass point can remain in equilibrium; that means that in this point the quantities $\dfrac{\partial g_{00}}{\partial x_k}$ disappear. Approximately, that means omitting small terms proportional to ω^2, g_{00} will even be constant in the space within the shell, and the quantities $\dfrac{\partial^2 g_{00}}{\partial x_k\,\partial x_l}$, which determine the force exerted on a mass point at rest just outside O, will in general be proportional to ω^2, but cannot be determined if the constitution of the shell has not

[1]) H. THIRRING, Phys. Zeitschr. XIX, p. 33 (1918).

been defined in nearer details. The rotation-vector in O, however, may directly be calculated approximately by means of (15). Its direction is of course parallel to the axis of rotation. We introduce a system of Cartesian coordinates x, y, z, the origin of which coincides with O, and the z-axis of which coincides with the axis of rotation. Let the mass of unit surface of the shell be denoted by m. The contribution to the value of R^z which is due to a ring of the shell, the angular distance of which to the z-axis is equal to ϑ, will then be equal to

$$2\varkappa \times 2\pi a^2 \sin\vartheta\, d\vartheta \times m \frac{x \,.\, x\omega + y \,.\, y\omega}{a^3} = \frac{\varkappa M \omega}{a} \sin^3\vartheta\, d\vartheta$$

and for R^z itself we thus get

$$R^z = \frac{\varkappa M \omega}{a} \int_0^\pi \sin^3\vartheta\, d\vartheta = \frac{4\varkappa M\omega}{3a} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From this we learn that if in O we introduce a system of coordinates, which rotates uniformly in the same sense as the shell with an angular velocity equal to $\frac{4M\varkappa}{3a}$ times that of the shell, the Coriolis-forces will disappear from the equations of motion for a mass point in O. This result is in agreement with the results obtained by THIRRING in his above mentioned paper.

Another application of formula (15) may be obtained in connection with the following problem. Let us imagine a uniformly rotating sphere, such as e.g. the earth, and let us suppose that the FOUCAULT's pendulum experiments are performed at the northpole. Then it will be found, that the plane in which the pendulum moves, will not remain at rest with respect to the fixed stars, but will rotate slowly in the same sense as the earth. The angular velocity of this slow rotation is given by the absolute value of the rotation vector at the pole, which by means of (15) may be found by simple integration. We find

$$R = \frac{4\varkappa M \omega}{5a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where M denotes the mass of the earth, which is supposed to be homogeneous, while a and ω represent the radius and the angular velocity of the earth. The factor $\frac{4\varkappa M}{5a}$ is of course so small (circa 5.10^{-10}), that it will be impossible to detect this rotation of the plane of the pendulum. Also at lower latitudes a similar influence on the result of FOUCAULT's experiments must be expected, but we will not enter here into this problem.

§ 3. *Influence of a stationary field of gravitation on the motion of a rigid body round its centre of gravitation.*

In the former § we have given an example of the appearance of the rotation-vector; the present § forms a direct continuation of § 1 and gives the necessary preparation for the treatment of the problem, which will be discussed in § 4, and which deals with the influence of the sun's field of gravitation on the precession of the axis of the earth.

If in the following we speak of a rigid body, we mean only a body, which is practically rigid, and which can move in the way well known from classical mechanics, characterised by 6 degrees of freedom, without changes of form or the appearance of enormously high stresses. Thus we will assume that the linear dimensions of the body are so small, that the "geometry" inside the body, which is determined by the quantities $g_{\mu\nu}$ and their derivatives deviates very little from the Euclidean geometry, and also that the relative velocities, which the different parts of the body possess relative to each other, are very small compared with the velocity of light. For such a rigid body it is possible directly to determine the values of the components of the energy-tensor of matter to an approximation, which may be exactly defined. In fact, if we introduce such a system of coordinates that in every point within the body the line element only differs very little from $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ — as a consequence of the above mentioned assumptions this will always be possible — and if we denote by v a small quantity of the same order of magnitude as the velocity $\frac{dx_k}{dx_0}$ of the different parts of the body, we have — neglecting small terms, which relative to the main terms are of the order v^2 and of the same order as the small deviations of the $g_{\mu\nu}$'s from the $\varepsilon_{\mu\nu}$'s (see (8)) —:

$$T_{oo} = m, \qquad T_{ok} = -m \frac{dx_k}{dx_o} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

while the quantities T_{kl} $(k,l = 1, 2, 3)$ which are connected with the stresses existing in the body, and which can only be determined, if the constitution of the body is known more closely, will be small compared with the quantities T_{ok} and may be considered as being of the order of magnitude v^2. The quantity m in the formulae (18) represents the mass per unit of volume. Further it may be remarked, that apart from the sign, we need not distinguish between covariant, contravariant and mixed components of the energy-tensor [1]). Theoretically there will be a difference, but this difference will be of the same order of magnitude as the small deviations of the $g_{\mu\nu}$'s from the $\varepsilon_{\mu\nu}$'s, and small terms of this order have already been neglected in the establishment of the formulae (18).

In the former we have fixed the properties of a rigid body with an approximation, sufficient for our purpose. Let us now imagine such a body at a certain time to be placed in a stationary field of gravitation in such a way, that its centre of gravitation is at rest and coincides with a point P of the field, where all the derivatives $\frac{\partial g_{00}}{\partial x_k}$ are equal to zero, and we propose to discuss the influence, which the stationary field of gravitation will have on the motions, which will be executed by the body. We will begin by proving, that the centre of gravity will remain at rest in P. For this purpose we will use the equations of energy and impulse of matter:

$$\frac{\partial \mathfrak{T}_\varkappa^\lambda}{\partial x_\lambda} - \tfrac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_\varkappa}\mathfrak{T}^{\mu\nu} = 0, \quad \mathfrak{T}^{\mu\nu} = T^{\mu\nu}\sqrt{-g}, \quad . \quad . \quad . \quad (19)$$

where g represents the determinant of the quantities $g_{\mu\nu}$. We will assume, that by means of a suitable transformation of the form (1) the coordinates in P are made to fulfill the conditions (9). Then the $g_{\mu\nu}$'s may in the neighbourhood of P be represented by

$$\begin{aligned} g_{oo} &= \varepsilon_{oo} + \tfrac{1}{2}(g_{oo,mn})_P\, x_m x_n, \qquad \left(g_{\mu\nu,mn} = \frac{\partial^2 g_{\mu\nu}}{\partial x_m \partial x_n}\right) \\ g_{ok} &= (g_{ok,m})_P\, x_m + \tfrac{1}{2}(g_{ok,mn})_P\, x_m x_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20) \\ g_{kl} &= \varepsilon_{kl} + \tfrac{1}{2}(g_{kl,mn})_P\, x_m x_n. \end{aligned}$$

Here we have assumed for the sake of simplicity, that the coordinates x_1, x_2 and x_3 are equal to zero in the point P and have neglected small terms of the order of magnitude x^3, x^4 etc., that is, terms which would contain products of three or more x'_k's.

Let us now consider a closed surface in the x_1-x_2-x_3-space, which encloses the body under consideration, and in the inside of which the relations (20) hold, and let us integrate both sides of (19) over the space inside this surface. Then we get, denoting the space element $dx_1\, dx_2\, dx_3$ by dS:

[1]) $T^{0k} = T^k_0 = - T^0_k = - T_{0k} = m\frac{dx_k}{dx_0}, \qquad T^{00} = T^0_0 = T_{00} = m.$

$$\frac{d}{dx_o}\left(\int T^o_i\, dS\right) = \tfrac{1}{2}(g_{oo,ik})_P \int x_k\, T^{oo}\, dS + (g_{ok,i})_P \int T^{ok}\, dS + $$
$$+ (g_{ok,li})_P \int x_l\, T^{ok}\, dS + \ldots . \quad . \quad . \quad . \quad . \quad (21)$$

Here we have omitted terms, which would be of the order v^2 (terms with T^{kl}) and of the order x^2. The left side of (21) represents, apart from the sign, the variation with the time of the total momentum of the body in the direction of the x_i-axis. The integral in the first term on the right side (order of magnitude x) represents the total moment of the body with respect to a plane through P perpendicular to the x_k-axis and is equal to zero, because we have assumed that P coincides with the centre of gravity. The integral in the second term on the right side (order of magnitude v) is equal to the total momentum of the body in the direction of the x_k-axis and is also equal to zero, because we have assumed that the centre of gravity was at rest at the moment under consideration; finally the third term is a small term of the order of magnitude xv and may be neglected, since we already have omitted terms of the order of magnitude x^2 and v^2. From this we see that in first approximation the momentum of the body remains zero in the course of time, and that consequently also the centre of gravity remains at rest. (Here it may be of interest to mention, that it is impossible to fix the centre of gravity of a body in an invariant way; if we try to keep to the classical definition, there always exists a small uncertainty in the position of the centre of gravity, the order of magnitude of which may be easily indicated). If we assume that the equilibrium of the body in P is stable, equation (21) allows us also to calculate the small oscillations, which the centre of gravity can perform in the neighbourhood of P, but we shall not enter further into this point.

We will now proceed to consider the possible motion, characterised by three degrees of freedom, of the rigid body round its centre of gravity. This may be done most easily by calculating the rate of variation with the time of the moments of momentum of a body round the axes of coordinates. For this purpose it will be of advantage to introduce the system of coordinates, which was discussed at the end of the first §, and which appears by the "rotation-transformation" mentioned there (see p. 1056). This new system of coordinates rotates uniformly round the point P with respect to a system of coordinates, which is at rest in the stationary field of gravitation, with an angular velocity, the components of which coincide with the components R_k of the "rotation-vector", and we shall investigate the moment of momentum of the body with respect to these coordinates. In the point P this system is geodetical at any moment, but the quantities $g_{\mu\nu}$ will, in contrast to what was the case before, depend on the time x_0 and will be periodical with respect to the time in P with a period $T = \frac{2\pi}{R}$, where R represents the absolute value of the rotation-vector. In the neighbourhood of P we have now, instead of (20), the following formulae

$$g_{\mu\nu} = \varepsilon_{\mu\nu} + \tfrac{1}{2}\,(g_{\mu\nu,mn})_P\, x_m\, x_n, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

where the quantities $(g_{\mu\nu,mn})_P$ are periodical functions of the time. In order now to determine the variation with the time of the moment of momentum round the x_1-axis we make again use of the impulse equations (19), and get from these:

$$x_2 \frac{\partial T_3^{\lambda}}{\partial x_\lambda} - x_3 \frac{\partial T_2^{\lambda}}{\partial x_\lambda} = + \tfrac{1}{2}\,(x_2\, g_{\mu\nu,3} - x_3\, g_{\mu\nu,2})\, T^{\mu\nu}.$$

Integrating again over a closed surface in the x_1—x_2—x_3-space, which encloses the body, we find with neglect of terms of the order xv^2, xv^2, xv^3, and higher orders:

$$-\frac{d}{dx_0}\left(\int (x_2\, T_3^{\,0} - x_3\, T_2^{\,0})\, dS\right) = -\tfrac{1}{2}\int (x_2\,(g_{00,3k})_P -$$
$$- x_3\,(g_{00,2k})_P)\, x_k\, T^{00}\, dS. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

The left side represents the variation with the time of the moment of momentum of the body round the x_1-axis; the right side may directly be interpreted as the x_1-component of the couple, which a field of acceleration with potential $+\frac{1}{2}\,g_{00}$ exerts on the body, and is obviously closely connected with the integrals $\int x_k x_l\, T_{00}\, dS$, which determine the ellipsoid of inertia of the body. In case of a homogeneous spherical body they are as is well known equal to zero. By means of (23) and of the two analogous equations, which refer to the moment round the x_2-axis and the x_3-axis, the motion of the body round its centre of gravity in the stationary field of gravitation may thus be determined completely. It may be described as a Poinsot-motion, which is more or less disturbed by the influence of a field of acceleration with potential $+\frac{1}{2}\,g_{00}$ (right side of (23)), and on which is superposed a uniform rotation, the components of the angular velocity of which are given by $R_1\, R_2$ and R_3. The latter rotation is quite independent of the properties of the body, in contrast to the influence of the field of acceleration, which is intimately connected with these properties, and which e.g. disappears, if we have to do with a homogeneous spherical body.

Until now we have neglected the influence on the field of gravitation due to the body itself, but in the applications to special cases such a neglect might not be justifiable. When e.g. in the next § we will discuss the praecession of the axis of the earth, we have to do with a body, the "own" field of gravitation of which is much stronger, e.g. at its surface, than the field of gravitation arising from the sun (which appears as is well known in the forces, which cause the tides). We might imagine that in such a case other forces might influence the motion round the centre of gravity, which are much stronger than the forces just considered, or which disturb these forces essentially. A closer consideration shows, however, that if the mass of the body is so small, that at large distances it can only cause small changes in the original stationary field of gravitation, the own field of gravitation will only cause a small change in the motion of the body, which may be considered superposed on the influences of the stationary field of gravitation considered above, and which will be proportional to the mass of the body.

In order to show this let us first imagine the body placed in a space, in which no other matter is present, and the line-element of which approaches to $ds^2 = dx_0^{\,2} - dx_1^{\,2} - dx_2^{\,2} - dx_3^{\,2}$ at infinite distance from the origin. Then it is easily seen, that in first approximation the own field of gravitation will have no influence at all on the Poinsot-motion of the body, because the "forces" determined by the $g_{\mu\nu}$'s, which the different parts of the rigid body exert on each other in first approximation will fulfill the principle of action and reaction, just as is the case in Newton's theory of gravitation. This may easily be proved by applying Einstein's approximative solution of the field equations, described on page 1058, on the impulse energy equations (19), but for the sake of brevity we will not enter into this proof.

Let us again imagine the body placed in a stationary field of gravitation with its centre of gravity at the point P. Let us suppose, that the original values of the $g_{\mu\nu}$'s only undergo small changes $\Delta_{\mu\nu}$ on account of the presence of the body, and let the new values of the $g_{\mu\nu}$'s be denoted by $g'_{\mu\nu}$, so that

$$g'_{\mu\nu} = g_{\mu\nu} + \Delta_{\mu\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

Then we obtain, by applying the field-equations (10), for the $\Delta_{\mu\nu}$'s a set of 10 partial linear inhomogeneous differential-equations of the second order, of which we will assume that there exists a regular solution. (If necessary boundary conditions must be given. If the stationary field of gravitation is such that the line-element everywhere differs very little from $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ and becomes equal to this expression at infinity, the $\Delta_{\mu\nu}$'s might simply with a high degree of approximation be calculated by means of the formulae (13) of EINSTEIN). Inside the body and in its neighbourhood this solution will be the same as in the absence of the stationary field of gravitation, and just as before there will be no direct influence of the own field of gravitation on the motion of the body. Further it is easily seen, that the values of the $\Delta_{\mu\nu}$'s at large distance from the body to a high degree of approximation will depend only on the total mass of the body, because at such large distances the influence of the body can be considered as that of a singular point (or as a singular line in the space-time-extension) characterised by the integrals of the quantities $T_{\mu\nu}$ over the volume of the body. But, always neglecting small quantities of the order v^2 and higher orders, the integrals involving the T_{kl}'s may be neglected, while those of the T_{k0}'s disappear, because the centre of gravitation is at rest; so that we only have to do with the integral of T_{00} extended over the volume, that is with the total mass M of the body. Thus we find that the body will exert small forces proportional to M on the bodies, which give rise to the stationary field of gravitation. The motion of these bodies will therefore undergo a small perturbation, and as a consequence of this the $g_{\mu\nu}$'s of the stationary field of gravitation itself will again undergo a modification. Instead of (24) we must therefore write

$$g'_{\mu\nu} = g_{\mu\nu} + \Delta_{\mu\nu} + \Delta'_{\mu\nu}, \quad \ldots\ldots\ldots \quad (25)$$

where the $\Delta'_{\mu\nu}$'s represent the modifications just mentioned. The $\Delta'_{\mu\nu}$'s are terms, which will be small compared with the $\Delta_{\mu\nu}$'s, and which will be proportional to M; in contrast to the terms $\Delta_{\mu\nu}$ they will, however, in general have an influence on the motion of a body, which clearly will be proportional to M.

In order to discuss this influence we will confine ourselves, for simplicity, to the case that the $\Delta'_{\mu\nu}$'s are independent of the time. In this case the quantities $\frac{\partial \Delta'_{00}}{\partial x_k}$ will in general not be equal to zero in the point P, so that there must be found a point P' in the neighbourhood of P, where the centre of gravity of the body may remain at rest. Further in order to determine the components of the rotation-vector we will have to introduce in the formulae (5) instead of the quantities $g_{\mu\nu}$ the values of the quantities $g_{\mu\nu} + \Delta'_{\mu\nu}$ and of their derivatives in the point P'. In this way a small modification proportional to M will be found in the values of the rotation-

vector at the point, where the body is situated, and in consequence of this a corresponding modification in the perturbing influence, which the field of gravitation exerts on the motion of the body. It will also be clear that the influence which according to equation (23) is exerted on the motion of the body, will undergo a modification proportional to M.

The results of this § may be briefly stated as follows. If a rigid body is situated in a stationary field of gravitation with its centre of gravity at rest, the POINSOT-motion, which the body in the absence of the field of gravitation would perform in the way well known from classical mechanics, will be disturbed by a superposed uniform rotation, which is independent of the properties of the body, and at the same time by the influence of a conservative field of acceleration, an influence, which is closely connected with the properties of the body (e.q. with the ellipsoid of inertia). The "own" field of gravitation of the body will — in first approximation — have the effect that all quantities, which characterise the position and the just mentioned perturbations in the motion of the body, will undergo small modifications proportional to the mass M of the body.

In practical applications all these influences may of course be of quite different orders of magnitude, and it may happen that some of them practically may be neglected, while others are so large, that the approximation perhaps has to be carried on further than indicated in this §.

§ 4. *Influence of the field of gravitation of the sun on the rotation of the earth.*

The line element of the field of gravitation, to which the sun, which is supposed to be at rest, gives rise, can be written in the following form, which for the first time was given by SCHWARZSCHILD:

$$ds^2 = \left(1 - \frac{a}{r}\right) dT^2 - \frac{1}{1 - \frac{a}{r}} dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2), \; . \; (26)$$

where T represents the time, the unit of time being chosen such that the velocity of light at large distance from the sun approaches to unity. r, ϑ, φ are polar coordinates, which determine the position of a point in space, and a is a constant, which is connected with the mass M_Z of the sun by the formulae

$$\alpha = 2\varkappa M_Z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

where $\varkappa$ again denotes the constant of gravitation (see p. 1058).

Let us now introduce a new system of coordinates, which rotates round the axis of the original system of coordinates with an angular velocity ω. The line element in the new coordinates may then be calculated by means of the transformation

$$\varphi = \psi + \omega T$$

This gives

$$\left.\begin{aligned} ds^2 = & \left(1 - \frac{\alpha}{r} - r^2 \sin^2 \vartheta\, \omega^2\right) dT^2 \\ & - 2r^2 \sin^2 \vartheta\, \omega\, d\psi\, dT - \\ & - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2 \left(d\vartheta^2 + \sin^2 \vartheta\, d\psi^2\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

The field of gravitation corresponding to this line element is stationary. We will first try to find a point P, where a mass-point can remain in equilibrium. In such a point the first derivatives of $g_{TT} = 1 - \frac{\alpha}{r} r^2 \sin^2 \vartheta \omega^2$ are equal to zero. This gives the following conditions, which must be fulfilled by the coordinates of P:

$$\frac{\partial g_{TT}}{\partial r} = \frac{\alpha}{r^2} - 2r \sin^2 \vartheta\, \omega^2 = 0, \qquad \frac{\partial g_{TT}}{\partial \vartheta} = -\, 2r^2 \sin \vartheta \cos \vartheta\, \omega^2 = 0.$$

From this we see, that a mass-point can remain at rest at every point P, which lies in the equatorial plane $\vartheta = \frac{\pi}{2}$, and for which the distance A from the sun fulfills the relation

$$2\, A^3\, \omega^2 = \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

This relation gives us therefore the connection between the angular velocity and the orbital radius of a planet, which moves in a circle round the sun.

In order to discuss the rotation of such a planet round its axis we shall begin by calculating the rotation-vector in P. In order to find, by means of (5), its contravariant components we want to know the value of the determinant G of the quantities

$$G_{kl} = -\, g_{kl} + \frac{g_{ko} g_{lo}}{g_{oo}}.$$

We find

$$G_{rr} = \frac{1}{1 - \frac{\alpha}{r}}, \quad G_{\vartheta\vartheta} = r^2, \quad G_{\psi\psi} = r^2 \sin^2 \vartheta + \frac{(r^2 \sin^2 \vartheta\, \omega)^2}{1 - \frac{\alpha}{r} - r^2 \sin^2 \vartheta\, \omega^2} =$$

$$= r^2 \sin^2 \vartheta \frac{1 - \frac{\alpha}{r}}{1 - \frac{\alpha}{r} - r^2 \sin^2 \vartheta\, \omega^2},$$

$$G_{r\vartheta} = G_{\vartheta\psi} = G_{\psi r} = 0, \quad G = \frac{1}{1 - \frac{\alpha}{r}} \,.\, r^2 \,.\, r^2 \sin^2 \vartheta \frac{1 - {}^{\alpha}/_{r}}{1 - {}^{\alpha}/_{r} - r^2 \sin^2 \vartheta\, \omega^2} =$$

$$= \frac{r^4 \sin^2 \vartheta}{g_{TT}}.$$

Since the derivatives of g_{TT} are equal to zero in the point P, the expressions (5) reduces in that point to

$$(R^m)_P = \tfrac{1}{2} \frac{1}{\sqrt{g_{oo} G}} \left\{ \frac{\partial g_{ol}}{\partial x_k} - \frac{\partial g_{ok}}{\partial x_l} \right\}.$$

This gives

$$(R^r)_P = -\, \tfrac{1}{2} \frac{1}{\sqrt{g_{oo} G}} \frac{\partial}{\partial \vartheta} (r^2 \sin^2 \vartheta \omega) = 0,$$

$$(R^\vartheta)_P = \tfrac{1}{2} \frac{1}{\sqrt{g_{oo} G}} \frac{\partial}{\partial r} (r^2 \sin^2 \vartheta\, \omega) = \left(\frac{r \sin^2 \vartheta\, \omega}{\sqrt{r^4 \sin^2 \vartheta}} \right)_P = \frac{\omega}{A}, \quad (R^\psi)_P = 0.$$

Consequently the rotation-vector in P is perpendicular to the equatorial plane, and for its absolute value R we find

$$R = \sqrt{G_{kl} R^k R^l} = R^\vartheta \sqrt{G_{\vartheta\vartheta}} = \frac{\omega}{A} \,.\, A = \omega, \quad . \quad . \quad . \qquad (30)$$

According to § 1 p. 1057 this result means, that for an observer placed at the earth the sun rotates with an angular velocity ω with respect to a system of coordinates, in which no Coriolis-forces are present, i. e. in which the Galileian law of inertia holds. On the other hand, from the point of view of the same observer, the sun rotates in the same sense with respect to the fixed stars with an angular velocity equal to the product of ω and the ratio of the time-unit of an observer on the earth and an observer, which is at rest at infinite distance from the sun, since, according to the formulae in the beginning of this §, ω represents the angular velocity of the earth round the sun, if the last mentioned time unit is used [1]).

[1]) Originally the writer had simply put the angular velocity of the sun with respect to the fixed stars equal to ω, and as a consequence of this obtained he result that EINSTEIN's theory of gravitation did not claim a non-Newtonian

From formulae (28) the ratio of the two time units in question is found to be equal to the value of $\sqrt{1-\frac{a}{r}-r^2\,\omega^2 \sin^2\vartheta}$ at the point where the earth is situated. Since at this point $r=A$, $\vartheta=\frac{\pi}{2}$, the ratio in question is equal to $1-\frac{3a}{4A}$.

It is therefore seen, that a system of coordinates in which the law of inertia holds, at the point where the earth is situated will rotate with an angular velocity $\frac{3a}{4A}\,\omega=\frac{3}{2}\,v^2\,\omega$ (where v is the velocity of the earth in its circular orbit) with respect to the fixed stars in the same sense as that, in which the earth rotates round the sun.

From this result, and from the results in the former §, we may therefore conclude that according to the gravitation theory of EINSTEIN there will be a contribution to the precession of the axis of the earth in progressive sense, which is independent of the constitution of the body of the earth, and which amounts to an angle equal to one and a half times the ratio of the velocity of the earth to the velocity of light, i.e. to 0,019 arc seconds anually. The existence of a non-Newtonian precession of this kind has for the first time been suggested in a paper by Professor SCHOUTEN.[1]) In this paper attention was drawn to the circumstance, that the field of gravitation of the sun is such, that a small body, which was made to move geodetically along a circle round the sun with radius A, would no longer have the same position as before at its return to the same point, but that it would be turned by a small angle equal to $\frac{\pi a}{A}$, in the same sense as that, in which the body had moved along the circle, and it was pointed out, that this result suggested a possible precession of the axis of the earth with respect to the fixed stars.

Now it remains to investigate the influence, which what we have called the "own" field of gravitation of the earth, may exert on its motion. According to what has been said in the former § (p. 1067), we may expect, that this influence will cause perturbations as well in the orbit of the earth as in its motion round the centre of gravity, and these perturbations will be small quantities proportional to the mass M_A of the earth. From classical mechanics we know already in first approximation the influence on the orbit: the sun is not at rest, but describes round the centre of gravity O of sun and earth an orbit similar to that of the earth, in such a way that its distance to O is always equal to the distance of the earth to O multiplied by $\frac{M_A}{M_Z}$. Assuming that the orbit of the earth is again a circle, we shall still have, that the product of the square of the angular velocity ω and the cube of the distance earth-sun will have a constant value, but this value will no longer be equal to $\frac{a}{2}$, but to $\frac{a}{2}\left(1+\frac{M_A}{M_Z}\right)=\frac{(M_A+M_Z)a}{2\,M_Z}$. If we again introduce a system of coordinates rotating with angular velocity ω, with respect to which the centre of gravity of the earth is at rest, the field of gravitation will again be stationary in these rotating coordinates, (if we look apart from the motion of the sun and of the earth round their respective centres of gravity [1])) but the distance from the earth to O corresponding to this angular velocity is no longer the same as when the mass of the earth was neglected, but smaller in the proportion $\left(1-\frac{2M_A}{3M_Z}\right)$. Thus we have to do with a displacement of the point, where the centre of gravity of the earth may remain at rest, which is a consequence of the own field of gravitation of the earth, and which is proportional to the mass of the earth. According to the considerations on p. 1066 such a displacement was to be expected.

We must further consider the possibility that the absolute value of the rotation-vector at the centre of the earth is no longer exactly equal to ω, but may, e. g. be written in the form $\omega\left(1+k\frac{M_A}{M_Z}\right)$, where k is a numerical factor of the same order of magnitude as unity. Here it must be remembered, that when speaking of the rotation-vector at the centre of the earth, we mean the quantity, which according to the scheme indicated in the former § (p. 1066)

precession of the kind described. I am indebted to Dr. FOKKER for the remark, that in doing so I had overlooked the difference in the time unit, in which the two angular velocities in question were expressed. Compare A. D. FOKKER, These Proc. Vol. XXIII. N°. 5, p. 729 (1920).

[1]) These Proc. Vol. XXI, p. 533 (1918).

[1]) From some interesting considerations by EINSTEIN, Berliner Berichte 1916 p. 695, it follows, that the field of gravitation in question may only be regarded as stationary to a certain degree of approximation, because we must expect that, analogous to what according to the classical theory of electrons would take place in a system of moving electrified particles, a system as that considered here will radiate energy in space in the form of socalled gravitational waves.

may be calculated by means of formula (5), if we so to say neglect the field of gravitation, which is directly due to the earth, or, more exactly, if we replace the $g_{\mu\nu}$'s of the original stationary field of gravitation by the quantities $g_{\mu\nu} + \Delta'_{\mu\nu}$, where the $\Delta'_{\mu\nu}$'s represent the small terms proportional to M_A, which are discussed on p. 1066. We shall now prove, that the just mentioned factor k is equal to zero, at any rate with neglect of small quantities of the order $\frac{\alpha}{A}$. This may most easily be proved by using the results of classical mechanics, which — with neglect of small quantities of the relative order of magnitude $\frac{\alpha}{A}$ — will coincide with the results of EINSTEIN's theory of gravitation. In fact, it is well known, that according to classical mechanics, the precession of the axis of rotation of the earth will entirely be due to the inhomogeinity of the Newtonian field of gravitation due to the sun at the place of the earth. In the mathematical expression for this precession there will therefore, whether we take the mass of the earth into account or not, not occur terms independent of the constitution of the earth and of the kind discussed on p. 1070, terms, which only appear, when the modifications in the phenomena of gravitation required by EINSTEIN's theory are taken into account. Hence we have the result, mentioned above that, with neglect of small quantities of the order $\frac{\alpha}{A}$, that is of the same order as the square of the velocity of the earth, the mentioned factor k will be zero.

Consequently we find, that the influence of the earth's own field of gravitation on the non-Newtonian contribution to the precession of the rotation-axis of the earth, considered in the present §, will at most be a small quantity of the order $\omega \times \frac{\alpha}{A} \times \frac{M_A}{M_Z}$. It is furthermore clear that, if the mass of the earth is not neglected, also the contribution to the value of the precession, which is due to the inhomogeinity of the sun's field of gravitation at the place of the earth will be altered by small quantities, which relative to the main term are of the order $\frac{M_A}{M_Z}$, and that this alteration will be the same as that calculated by means of Newtonian mechanics.

In connection with this result it might be of interest to draw attention to the fact that we have made use of the circumstance, that for the system under consideration, which consisted of bodies moving under mutual gravitational influence, the results obtained on Newton's theory will differ from those obtained on EINSTEIN's theory only by small quantities of the same order as the square of the velocities of the bodies. This circumstance is remarkable on account of the fact, that in EINSTEIN's theory all gravitational influence is propagated in space with the velocity of light, as it is e.g. indicated by the formulae (13), which have the form of retarded potentials. From this we might at first hand infer, that there might be discrepancies between the results of EINSTEIN's and of NEWTON's theories of the same order of magnitude as the first power of the velocity of the earth. A closer consideration, into which for the sake of brevity we will not enter here, shows however, that for the system under consideration small terms of this order will just compensate each other, a circumstance which is completely analogous to similar well-known phenomena with which we meet in the theory of electrons, whose interaction may be calculated by means of retarded potentials.

Conclusions of this paragraph.

It has been found that, in confirmation of an idea for the first time put forward by SCHOUTEN, the gravitation theory of EINSTEIN leads to the result, that theoretically there will exist a contribution to the value of the precession of the rotation axis of the earth, which did not appear on NEWTON's theory, and which is independent of the constitution of the body of the earth, and which amounts to a progressive precession of 0,019 arc seconds annually. In the calculations, the influence of the mass of the earth was neglected; if it is taken into account, there may arise a modification in the value of the precession, which relative to the main term in the expression for this precession is of the same order of magnitude as the ratio of the mass of the earth to that of the sun. In the considerations no regard has been paid to the contribution to the precession arising from the influence of the moon.