

KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN
WETENSCHAPPEN

PROCEEDINGS OF THE
SECTION OF SCIENCES

VOLUME LII
(Nos. 1—5)

PUBLISHED BY
NORTH-HOLLAND PUBLISHING COMPANY
(N.V. Noord-Hollandsche Uitgevers Mij.)
AMSTERDAM, 1949

Proc. Kon. Ned. Akad. v. Wet., Vol. 52, Nos. 1-5, p. 1—593, A'dam, 1949

CONTENTS

Proceedings No. 1	1
" No. 2	97
" No. 3	193
" No. 4	311
" No. 5	463

KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN
WETENSCHAPPEN

PROCEEDINGS

VOLUME LII

No. 1

President: A. J. KLUYVER

Secretary: M. W. WOERDEMAN

1949

NORTH-HOLLAND PUBLISHING COMPANY

(N.V. Noord-Hollandsche Uitgevers Mij.)

AMSTERDAM

CONTENTS

Biochemistry

BUNGENBERG DE JONG, H. G., and H. J. VAN DEN BERG: Elastic-viscous oleate systems containing KCl. II. a) Period and logarithmic decrement as function of the radius of the sphere for a system containing 1.2% oleate, in 1.52 N KCl + + 0.08 N KOH at 15° and 23° C; b) Shear modulus and relaxation time as function of the temperature, p. 15.

Botany

FRETS, G. P.: De F₄-zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van Phaseolus vulgaris. I. (Communicated by Prof. J. BOEKE), p. 76.

Geology

HAMMEN, T. VAN DER: De Allerød-oscillatie in Nederland. Pollenanalytisch onderzoek van een laatglaciale meerafzetting in Drente. I. (Communicated by Prof. C. J. VAN DER KLAUW), p. 69.

Mathematics

BRUIJN, N. G. DE and P. ERDÖS: Sequences of points on a circle. (Communicated by Prof. W. VAN DER WOUDE), p. 46.
BRUINS, E. M.: On the symbolical method. II. (Communicated by Prof. L. E. J. BROUWER), p. 35.
MULLENDER, P.: Lattice points in non-convex regions. III. (Communicated by Prof. J. G. VAN DER CORPUT), p. 50.
WRONA, WŁODZIMIERZ: On multivectors in a V_n. II. (Communicated by Prof. J. A. SCHOUTEN), p. 61.

Zoology

BOSCHMA, H.: The Ampullae of Millepora, p. 3.
HOLTHUIS, L. B.: Note on the Species of Palaemonetes (Crustacea Decapoda) found in the United States of America. (Communicated by Prof. H. BOSCHMA), p. 87.
RAVEN, CHR. P. and SELMA DUDOK DE WIT: On the influence of lithium chloride on the eggs of Limnaea stagnalis at the 24-cell stage, p. 28.

Zoology. — The Ampullae of *Millepora*. By H. BOSCHMA.

(Communicated at the meeting of December 18, 1948.)

In 1884 QUELCH described the ampullae of *Millepora murrayi* Quelch as "receptacles covered by a very thin and porous layer, which is often broken away" (l.c., p. 539). In his report on the reef corals of the Challenger Expedition he described them as follows: "Ampullae developing as special cavities in the superficial meshwork of the coenenchyma; often crowded, about 0.75 mm. in diameter within, scarcely raised above the general surface, on which they are seen as small white spots or vesicles, which are about 0.5 mm. in diameter, the centre being generally pierced by a small pore" (QUELCH, 1886, p. 192).

HICKSON (1891 a) reported upon medusae found in the ampullae of *Millepora murrayi*, and in another paper (HICKSON, 1891 b) gave a detailed description of the male medusae present in the ampullae of this species. In a later paper HICKSON (1897) stated that he noticed several ampulla-bearing specimens other than those belonging to the species *Millepora murrayi*; of these he mentioned *M. schrammi* Duch. and Mich., *M. alcicornis* L., and *M. complanata* Lamk. In his next paper dealing with *Millepora* HICKSON (1898 a) remarked that ampullae may be found in plicate, ramosc, and digitate specimens, and that the presence or absence of ampullae cannot be used as a specific character. In the same year HICKSON (1898 b) found male medusae in a specimen of the facies "complanata" (undoubtedly a specimen of *M. platyphylla* Hempr. and Ehr.) of the same size as those found previously in *M. murrayi*. According to HICKSON (1899) a brief account of medusae of a West Indian form of *Millepora* was given by DUERDEN (1899). In the cited paper HICKSON published a detailed description of the structure of various stages of the female medusae of this West Indian *Millepora*. Here HICKSON moreover stated that the medusae of *Millepora* "complanata" mentioned in his 1898 b paper were identical in size and form with those of *M. murrayi*, so that "no specific distinction could be drawn between the two forms based on characters of the medusa before it is set free" (HICKSON, 1899, p. 4). Finally mention must be made of figure 68 in HICKSON (1924), representing a part of a colony of *Millepora* (probably *M. alcicornis*), the surface of which is profusely pitted with open ampullae.

During my previous investigations on *Millepora* (BOSCHMA, 1948 a, b) I found open ampullae in *M. alcicornis*, and ampullae still in possession of their covering in *M. murrayi*, *M. latifolia*, and *M. platyphylla*. I noted a difference in size of the ampullae in the various specimens, but concluded that as yet the available data were too scanty to allow of a conclusion concerning specific differences.

Since I wrote my previous papers dealing with *Millepora* a continued examination of numerous colonies of the genus resulted in ten specimens with closed ampullae, some of which possessed a few only among a multitude of open ampullae, whilst in others most of the ampullae were still in a closed condition. Although this material is still far from being sufficiently complete it seemed worth while to try whether the ampullae might furnish peculiarities to be regarded as characteristic of the species. HICKSON (1897, p. 3) wrote: "The most important problems that have still to be solved are these: Are the medusae of the different species of *Millepora* alike, or do they present specific differences? Are the medusae confined to the male sex, or do medusae occur bearing the ova?" Soon afterwards the second question could be answered in the affirmative (HICKSON, 1899), but the first question still remains open. Moreover the question whether the

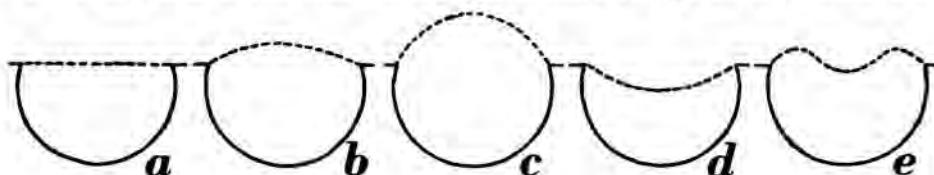


Fig. 1. Diagrams of sections of ampullae of various forms. Cavities of ampullae with full lines, coverings with broken lines.

ampullae of the different species of *Millepora* are alike or are presenting specific differences remains still unanswered. The new data contained in the present paper do show that the ampullae of different specimens of *Millepora* among each other may present striking differences, but it remains uncertain whether in all cases these data can be used for specific distinction.

For a comparison of the shape and size of the various ampullae it was necessary to obtain microphotographs of parts of the coralla, and I am strongly indebted to Professor S. T. BOK for placing his photographic apparatus at my disposal. As the surface of the corallum often is decidedly uneven in many cases it is difficult to obtain satisfactory results, but the photographs at least show the most important details of the ampullae.

In textfig. 1 diagrammatic sections of ampullae are represented showing the various forms of covering. Textfigs. 2 and 3 are more or less diagrammatic figures of some of the photographs of the plates, they may serve as an additional explanation of these photographs.

A short description of the ampullae of the various specimens follows here.

1. *Millepora alcicornis* L., Zoölogisch Museum Amsterdam (cf. BOSCHMA, 1948 b, p. 101, no. 10). Locality unknown. Textfig. 4 b of the present paper is an outline of a fragment of this colony.

Ampullae: Pl. I figs. 1—2, textfig. 2 a.

The colony shows a multitude of open ampullae, in certain parts only there are some ampullae which are still closed.

The ampullae have a diameter of 0.5—0.7 mm. They have a rather flat upper surface so that their shape approximately corresponds with the diagrammatic figure 1 a. The covering of the ampullae consists of trabeculae which have a strong tendency to a radial arrangement from the small central opening.

2. *Millepora alcicornis* L., Rijksmuseum van Natuurlijke Historie, Leiden (cf. BOSCHMA, 1948 b, p. 87, no. 3). Locality unknown. Textfig. 4 a of the present paper is an outline of a fragment of this colony.

Ampullae: Pl. I figs. 3—5, textfig. 2 c.

The colony shows numerous ampullae, the greater part of which are still closed.

The ampullae have a diameter of 0.3—0.45 mm. Their central part is slightly sunk beneath the surface of the corallum, so that their shape more or less corresponds with the diagrammatic figure 1 d. The covering of the ampullae is a calcareous plate irregularly pierced by small openings surrounding the somewhat larger central opening.

3. *Millepora platyphylla* Hempr. and Ehr., Island Edam, Bay of Batavia, May 25, 1921. Colony of a plate-like growth.

Ampullae: Pl. II figs. 1—3, textfig. 2 b.

In certain regions the colony shows a large number of ampullae which for the greater part are still closed.

The ampullae have a diameter of 0.4—0.55 mm. Their covering as a rule is decidedly concave so that the central part is pronouncedly below the surface of the corallum (Pl. II figs. 1 and 3). In some parts of the colony the ampullae are less strongly concave (Pl. II fig. 2). Generally the shape of the ampullae corresponds with the diagrammatic figure 1 d, the concavity even may be still more pronounced. The covering of the ampullae consists of a calcareous plate which is more or less irregularly pierced by holes, the central of which usually is slightly larger than the others.

4. *Millepora platyphylla* Hempr. and Ehr., Island Edam, Bay of Batavia, date unknown. Colony of a plate-like growth (cf. BOSCHMA, 1948 b, Pl. XV fig. 5).

Ampullae: Pl. II figs. 4—7, textfig. 2 d.

In a part of the colony there are numerous ampullae, usually in a closed condition.

The ampullae have a diameter of 0.4—0.5 mm. They are slightly or pronouncedly concave, so that their general shape corresponds with the diagrammatic figure 1 d. The covering of the ampullae consists of a mass of trabeculae which are not distinctly arranged in a radial manner (Pl. II fig. 6), often the trabeculae are rather broadened, so that the covering consists of a calcareous plate pierced by irregular holes (Pl. II figs. 4, 5, 7). As a rule the central opening is somewhat larger than the other holes.

5. *Millepora murrayi* Quelch, British Museum (Natural History), reg. no. 94. 6. 19. 1, Tongatabu, J. J. LISTER. Colony of the same shape and

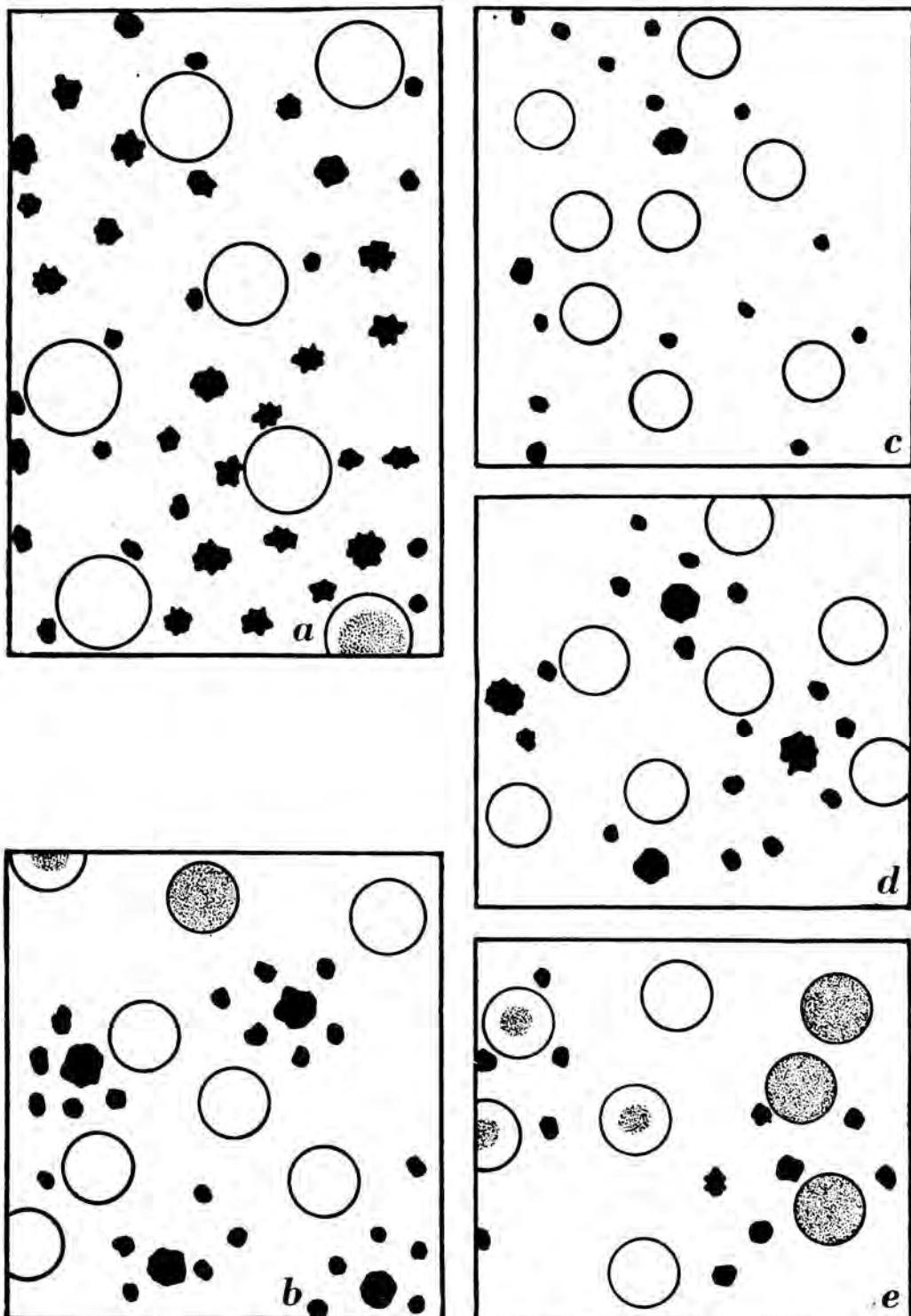
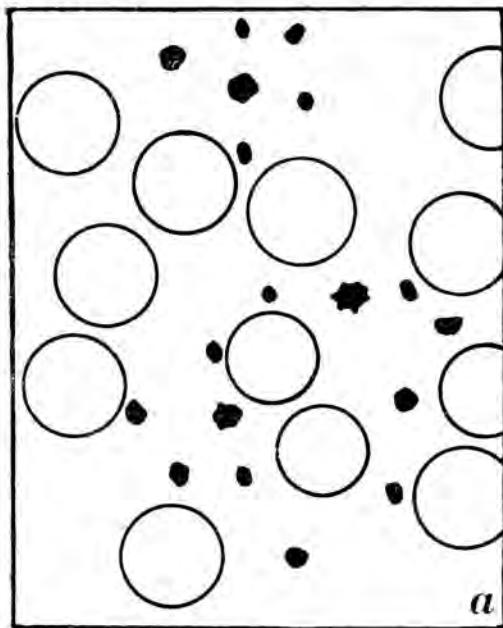
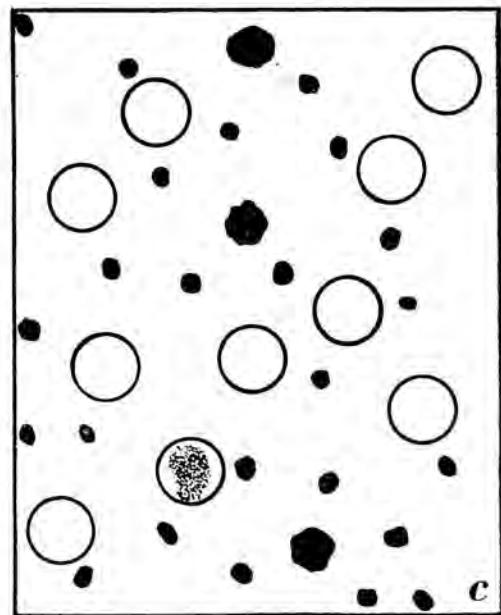


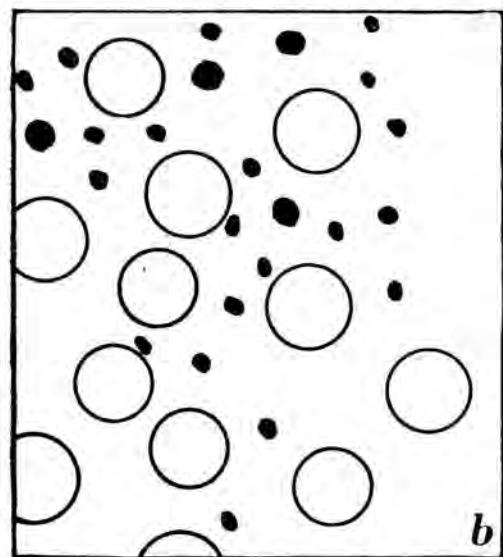
Fig. 2. Diagrams of some figures of the plates, indicating the place of the ampullae.
 a, *M. alcicornis* of Pl. I fig. 1; b, *M. platyphylla* of Pl. II fig. 2; c, *M. alcicornis* of Pl. I fig. 4; d, *M. platyphylla* of Pl. II fig. 7; e, *M. tenella* of Pl. V fig. 3.
 Open circles, closed ampullae; dotted parts, partly or wholly open ampullae; black,
 gastropores and dactylopoles. $\times 20$.



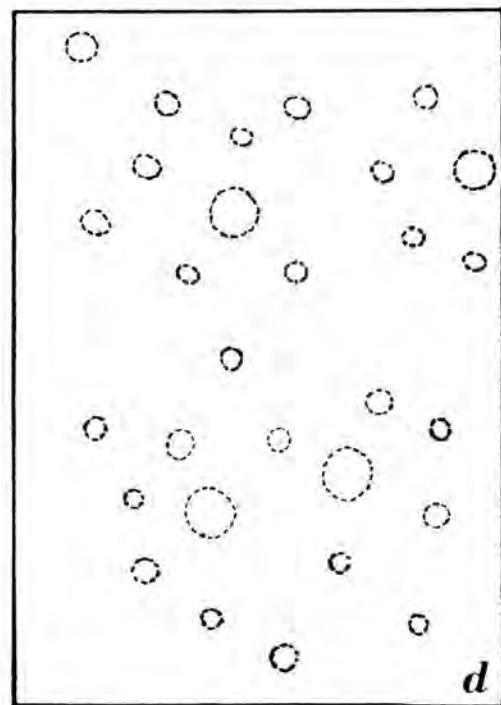
a



c



b



d

Fig. 3. Diagrams of some figures of the plates, indicating the place of the ampullae or (d) the partly closed pores. *a*, *M. murrayi* of Pl. III fig. 3; *b*, *M. murrayi* of Pl. III fig. 4; *c*, *M. tenella* of Pl. IV fig. 7; *d*, *M. platyphylla* of Pl. V fig. 3. For further explanation see fig. 2. $\times 20$.

manner of branching as the type specimen from Zamboanga, Philippine Islands (cf. QUELCH, 1886, Pl. VII figs. 5—5 a).

Ampullae: Pl. III figs. 1—3, textfig. 3 a.

The colony shows a great number of ampullae which for the greater part are still closed.

The ampullae have a diameter of 0.4—0.7 mm. As a rule their covering is flat so that in general appearance the ampullae correspond with the diagrammatic figure 1 a. In the greater part of the colony the ampullae are fairly large (Pl. III figs. 1, 3), in other parts they are decidedly smaller (Pl. III fig. 2). The covering of the ampullae consists of a mass of trabeculae with comparatively wide openings. The trabeculae are not arranged in a radial manner. As a rule the central opening is slightly larger than the other openings of the covering.

6. *Millepora murrayi* Quelch, Island Edam, Bay of Batavia, July 5—8, 1921. Colony very similar to that described in a previous paper (BOSCHMA, 1948 b, Pl. XI fig. 2).

Ampullae: Pl. III figs. 4—6, textfig. 3 b.

The colony shows a multitude of ampullae, in closed as well as in open condition.

The ampullae have a diameter of 0.4—0.6 mm. Their covering projects noticeably above the surface of the corallum, so that in general appearance the ampullae correspond with the diagrammatic figure 1 c. The trabeculae which form the covering of the ampullae are rather thick, and between themselves they leave but small openings. The central opening as a rule is slightly larger than the other openings of the covering.

7. *Millepora tenella* Ortm., Snellius Expedition, Ake Selaka, Kaoe Bay, Halmahera, May 28, 1930. Rather widely branched colony, textfig. 4 d of the present paper is an outline of the topmost part.

Ampullae: Pl. IV figs. 1—4.

The surface of the colony is largely covered with ampullae, nearly all of which are still in a closed condition.

The ampullae have a diameter of 0.5—0.6 mm. Their marginal part is raised above the surface of the corallum, whilst their central part shows a slight depression. In general appearance the shape of the ampullae corresponds with the diagrammatic figure 1 e. The covering of the ampullae consists of a mass of largely fused trabeculae forming a calcareous plate. The latter is pierced by numerous openings of comparatively large size, of which not always the central opening is larger than the others. There is no distinct radial arrangement of the trabeculae.

8. *Millepora tenella* Ortm., British Museum (Natural History), reg. no. 76. 5. 5. 110, Rodriguez Island, SLATER. Colony of plate-like growth with short marginal branches, textfig. 4 c of the present paper is an outline of a fragment of this colony.

Ampullae: Pl. IV figs. 5—7, textfig. 3 c.

The colony shows a great number of ampullae, many of which still are in a closed condition.

The ampullae have a diameter of 0.3—0.5 mm. They have a flat surface or are slightly convex, so that their shape corresponds with the diagrammatic figure 1 a or b. The covering of the ampullae consists of a mass of rather thick trabeculae, leaving between themselves some small openings. There is no distinct radial arrangement of the trabeculae.

9. *Millepora tenella* Ortm., Siboga Expedition, station unknown. Rather delicate, widely spreading colony, textfig. 4 e of the present paper is an outline of some top branches of this colony.

Ampullae: Pl. V figs. 3—4, textfig. 2 e.

Many branches of the colony show a multitude of ampullae which for the greater part are in open condition; on a few branches there are some ampullae which have remained closed.

The ampullae have a diameter of 0.5—0.6 mm. They are slightly convex so that their general appearance corresponds with the diagrammatic figure 1 b. The covering of the ampullae consists of an irregular mass of thin trabeculae. The central opening is larger than the other openings between the trabeculae. There is no indication of a radial arrangement of the trabeculae.

10. *Millepora latifolia* Boschma, Island Edam, Bay of Batavia, May 27, 1921. Colony figured in a previous paper (BOSCHMA, 1948 b, Pl. IV fig. 1).

Ampullae: Pl. V figs. 1—2.

The colony shows a very great number of ampullae, in closed as well as in open condition.

The ampullae have a diameter of 0.6—0.8 mm. As a rule they are slightly convex (corresponding with the diagrammatic figure 1 b), some ampullae have a flat surface, and some are slightly concave. The covering consists of a system of trabeculae which more or less distinctly show an arrangement radiating from the centre. This holds at least for the peripheral region of the ampullae, the central part of the covering is a calcareous plate with small openings irregularly surrounding the somewhat larger central opening.

The data given above show that the ampullae in many cases possess peculiarities which undoubtedly are characteristic of the species.

It is a curious fact that the ampullae of *Millepora platyphylla* are of comparatively small size, whilst the gastropores in this species are decidedly larger than those of the other species of the genus. In one of the specimens of *M. platyphylla* the ampullae as a rule are pronouncedly concave (Pl. II figs. 1 and 3), though in the same colony there are regions in which the covering of the ampullae is hardly below the surface of the corallum (Pl. II fig. 2). Here they are of a quite similar appearance as the ampullae of the second specimen of *M. platyphylla* (Pl. II figs. 4—7). In the two

specimens the covering of the ampullae consists of a similar mass of trabeculae in which a distinct radial arrangement cannot be observed. As far as definite conclusions may be drawn from the examination of the ampullae of the two colonies we may conclude that the ampullae of *M. platyphylla* differ from those of the other species of the genus by their strong tendency for a concave covering.

As compared with those of *M. platyphylla* the ampullae of *M. latifolia* (Pl. V figs. 1—2) have quite a different appearance. They are comparatively large (diameter 0.6—0.8 mm), they may be flat or slightly concave, but as a rule are more or less convex, and their covering consists of trabeculae which show a distinct tendency to a radial arrangement. In the two species the ampullae have such a pronouncedly different aspect that undoubtedly these differences point to specific distinction between *M. latifolia* and *M. platyphylla*.

Another specimen with ampullae of fairly large size is the colony of *M. murrayi* from Tongatabu. As a rule in this specimen the ampullae have a diameter of 0.5—0.7 mm (Pl. III figs. 1 and 3), in some parts of the colony they are somewhat smaller (0.4—0.5 mm, Pl. III fig. 2). The covering of the ampullae does not project above the surface of the corallum, it consists of a network of fine trabeculae in which a distinct radial arrangement cannot be observed. I could compare the ampullae of this specimen with those of the type specimen from the Philippine Islands, in the collection of the British Museum (Natural History). In the two specimens the ampullae are exactly alike in size and shape and structure, so that the ampullae of *M. murrayi* of Pl. III figs. 1—3 are of the typical shape. As far as concerns the two colonies taken as a whole the specimen from Tongatabu in every detail of shape and size of the branches and manner of branching corresponds with the type specimen from the Philippines.

In the specimen of *M. murrayi* from the Island Edam the ampullae have a diameter of 0.4—0.6 mm, and their covering consists of trabeculae which do not present a radial arrangement. They differ, however, from the other specimens examined by their pronouncedly convex shape of the covering (Pl. III figs. 4—6). This difference in shape of the ampullae might be an indication for a specific difference between the specimen from Tongatabu and that from the Island Edam. It must be admitted that it is not quite certain that the specimen from the Island Edam identified as *M. murrayi* really belongs to this species. The colony undoubtedly is a representative of the same species as the one figured in a previous paper (BOSCHMA, 1948 b, Pl. XI fig. 2), but this colony has not exactly the same manner of branching and shape and size of its component parts as typical specimens of *M. murrayi*. In the cited paper I came to the conclusion that there are ten distinct species of the genus *Millepora*. If this conclusion is right the specimen dealt with here undoubtedly belongs to *M. murrayi*. If on the other hand the two specimens of which the ampullae are shown on Pl. III

are specifically distinct the specimen from the Island Edam is a representative of an undescribed species which closely resembles *M. murrayi*.

The differences in the ampullae of the two specimens, however, not necessarily point to a specific distinction. It remains possible that one of the colonies is of the male sex, the other of the female. It is unknown whether there are differences in shape and size between the medusae of either sex in one species of *Millepora* (male medusae only are known of two forms of Indopacific, and female medusae of one form of West Indian *Millepora*), but if these occur they might easily develop in ampullae of different shapes.

The ampullae of the two specimens of *M. alcicornis* again show striking differences. In the one colony (Pl. I figs. 1 and 2) the ampullae have a diameter of 0.5—0.7 mm, they have a rather flat surface and their covering shows a pronouncedly radial arrangement of trabeculae. In the other colony (Pl. I figs. 3—5) the ampullae are much smaller (diameter 0.3—

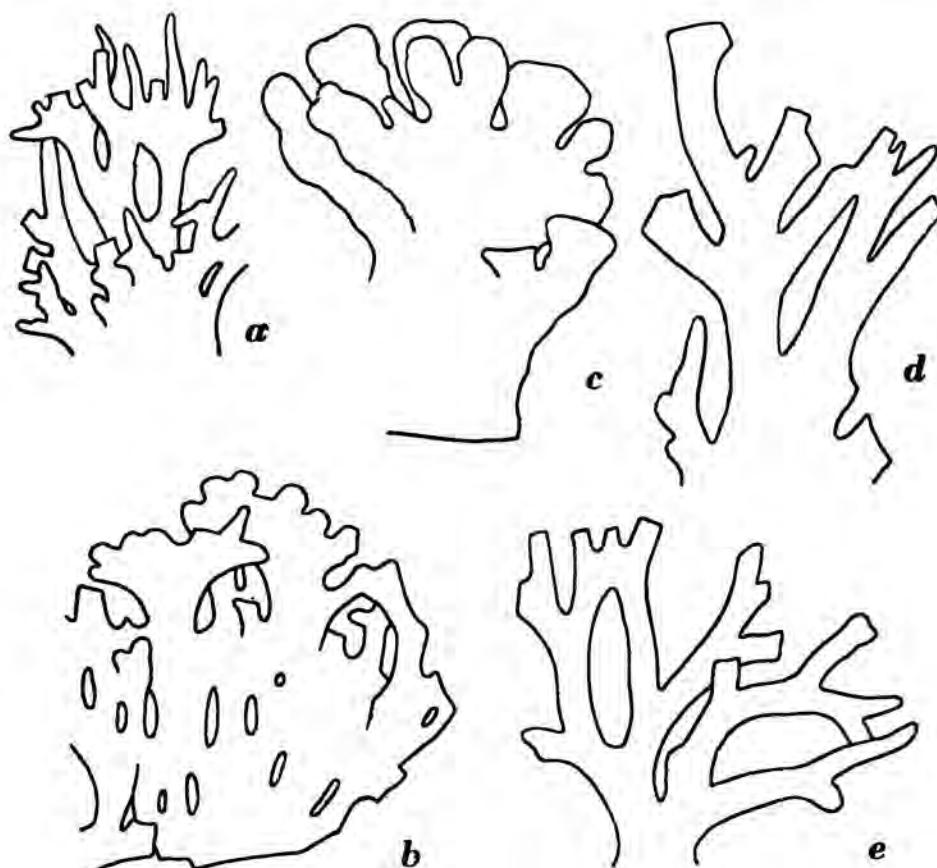


Fig. 4. Outlines of parts of colonies of *Millepora*. a, *M. alcicornis* from Leiden Museum; b, *M. alcicornis* from Amsterdam Museum; c, *M. tenella* from Rodriguez; d, *M. tenella* from Halmahera; e, *M. tenella* from Siboga Expedition. 5/6 natural size.

0.45 mm), they are slightly concave and their covering is composed of trabeculae which do not show a radial arrangement. These differences might point to a specific distinction of the two colonies, but it is highly improbable that we have to find a solution of the problem in this way. In their general appearance and in their manner of branching the two colonies are strongly similar. The fragment from which the photographs of Pl. I figs. 1 and 2 were taken is represented in outline in textfig. 4 b, a fragment of the colony from which the photographs of Pl. I figs. 3—5 were taken is shown in the same manner in textfig. 4 a. Both colonies are of the highly branched form of *Millepora alcicornis*, in both the branches have a strong tendency to unite into more or less plate-like forms. The differences in shape and in manner of branching of the two fragments are less striking than those often found in parts of the same colony.

Here the only possible explanation of the differences between the two sets of ampullae seems to be that one set represents the ampullae of the female colonies, the other set the ampullae of the male colonies.

When we compare the ampullae of the specimen of *M. tenella* from Rodriguez (Pl. IV figs. 5—7) to those of *M. tenella* from the Siboga Expedition (Pl. V figs. 3—4) we do not find striking differences. In the Rodriguez specimen the ampullae are rather small (diameter 0.3—0.5 mm), in the Siboga specimen they are slightly larger (diameter 0.5—0.6 mm), but in both specimens they are rather flat and do not show a pronouncedly radial arrangement of the trabeculae. The ampullae of the specimen of *M. tenella* from Halmahera have a diameter of 0.5—0.6 mm, they have no distinct radial arrangement of the trabeculae, but they differ in shape from those of the two other specimens as they do not possess a flat surface but have a convex marginal part and a concave central part. As a result they appear much more distinct than those of the two other specimens.

The difference in the shape of the ampullae in the three specimens of *M. tenella* remind of the differences found in the two specimens of *M. platyphylla*. In two specimens of *M. tenella* the ampullae have a more or less flat surface, in the third specimen they distinctly protrude over the surface of the corallum. These differences may be due to variation, possibly as a result of different circumstances of the localities in which the specimens were growing. On the other hand here again the differences in the shape of the ampullae may be the result of sexual differences of the colonies.

The three colonies of *M. tenella* in which closed ampullae were observed do not show important differences in shape and in manner of branching. The colony from Rodriguez (cf. textfig. 4 c) is of a more or less plate-like growth with short and blunt small branches in its marginal part, that from the Siboga Expedition (cf. textfig. 4 e) is of a slender growth form with spreading branches, whilst that from Halmahera (cf. textfig. 4 d) in its growth form is more or less intermediate between the two others, it is

spreadingly branched with slightly more robust branches than the Siboga specimen. Among each other the three specimens do not present more striking differences than those commonly found in different parts of one fairly large colony of *M. tenella*.

It is highly probable that ampullae develop in the colonies of *Millepora* in certain times of the year only. Unfortunately only a small part of the material dealt with here has the dates of collecting. One specimen of *M. platyphylla* was collected on May 25, the specimen of *M. murrayi* from the Island Edam on July 5—8, the specimen of *M. tenella* from Halmahera on May 28, the specimen of *M. latifolia* on May 27, the specimen of *M. murrayi* from Zamboanga (Philippine Islands) between October 24 and November 12 or between January 11 and February 5 (cf. MOSELEY, 1879). The time of development of the ampullae may be different in the various species. The dates given above do not point to a distinct season for all the species of the genus.

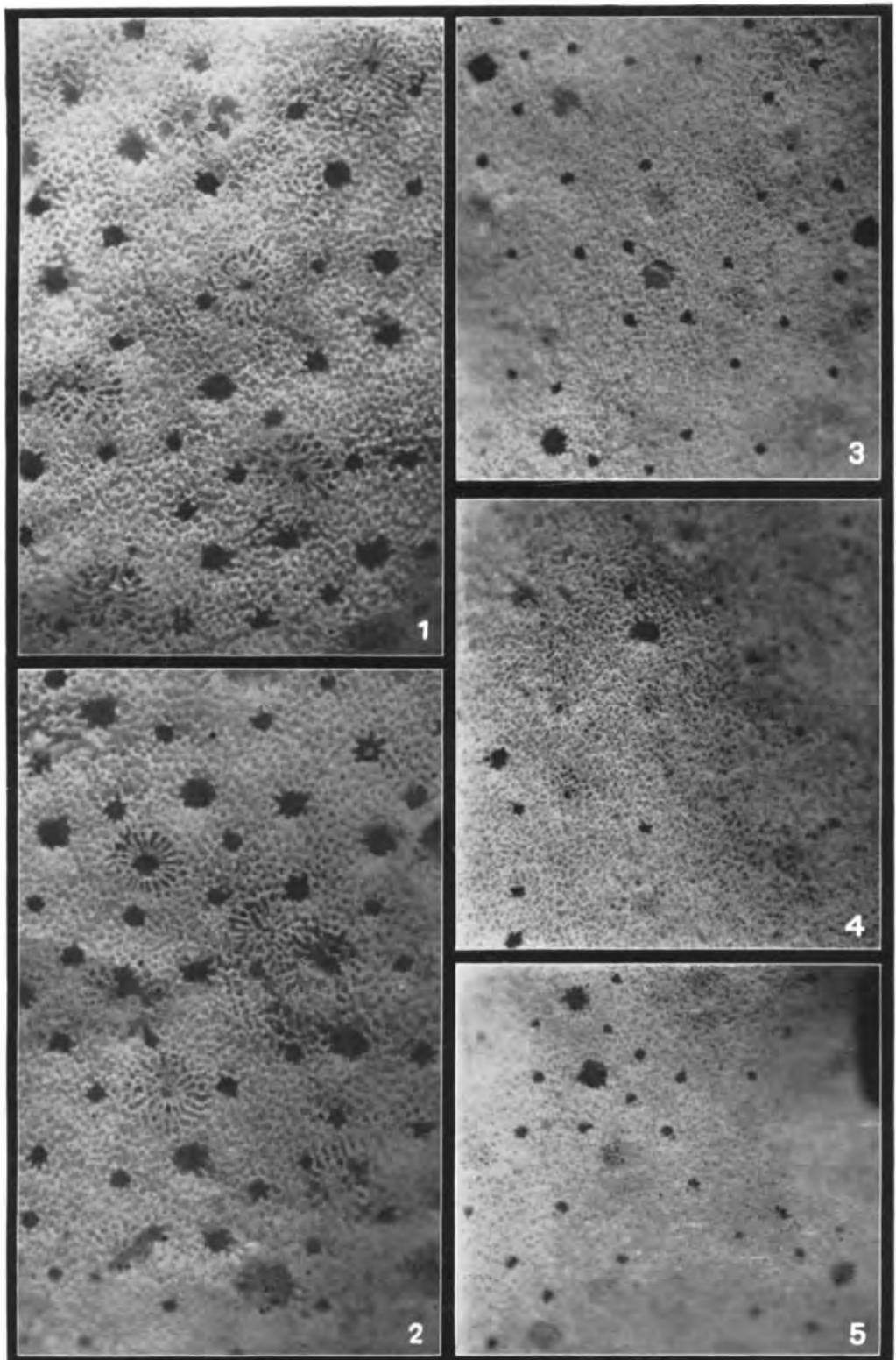
The peculiarities of one more colony of *Millepora* need to be described here, as it presents trabecular plates which easily might be mistaken for the coverings of ampullae. The colony is from Station 193 of the Siboga Expedition (Sanana Bay, Soela Besi, September 13—14, 1899), it is the topmost part of a plate-like growth of *Millepora platyphylla* (breadth of the fragment 14 cm, height 12 cm) which has broken off from the colony and apparently was collected in living condition after having been lying for some months on one of the flat sides of the colony on the reef. A large part of the new lower surface of the colony now was almost completely shut off from light, and on this surface it is to be observed that the size of the pores is gradually diminishing as their openings become covered by a thin plate consisting of trabeculae growing inwards from the margin of the pores. The figures (Pl. V fig. 5 and textfig. 3d) show the surface of a part of this colony that remained in darkness for some time, here the gastropores and the dactylo pores have become almost completely closed by the trabecular plates. Especially the almost closed gastropores show a marked resemblance to the ampullae of a number of specimens dealt with here.

Summarizing we may observe that the closed ampullae of the various specimens examined present striking differences which may be characteristic of the species. As long as it remains unknown whether or not the ampullae of a male colony of a certain species are different from those of a female colony of the same species no definite facts concerning the value of the ampullae as indicators of specific characters can be established.

LITERATURE.

- BOSCHMA, H., 1948a. Specific Characters in *Millepora*. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, vol. 51.
—, 1948b. The Species Problem in *Millepora*. Zool. Verh., no. 1.

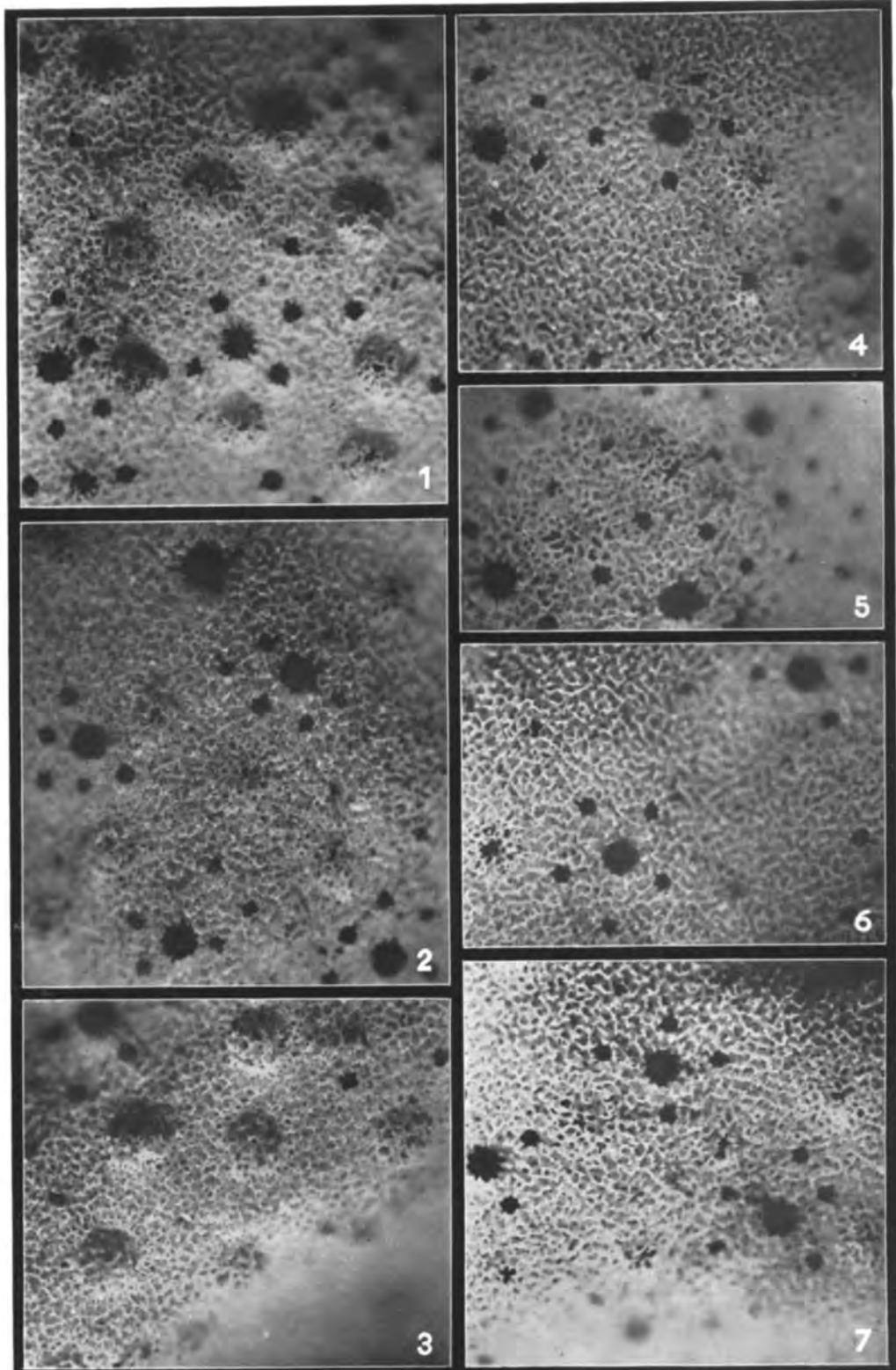
- DUERDEN, J. E., 1899. Zoophyte collecting in Bluefields Bay. *Journ. Inst. Jamaica* (cited after HICKSON, 1899).
- HICKSON, S. J., 1891a. The Medusae of Millepora and their Relations to the medusiform Gonophores of the Hydromedusae. *Proc. Cambridge Philos. Soc.*, vol. 7.
- _____, 1891b. The Medusae of *Millepora murrayi* and the Gonophores of *Allopora* and *Distichopora*. *Quart. Journ. Micr. Sci.*, n. s., vol. 32.
- _____, 1897. On the Ampullae in some Specimens of *Millepora* in the Manchester Museum. *Mem. and Proc. Manchester Lit. and Philos. Soc.*, vol. 41.
- _____, 1898a. On the Species of the Genus *Millepora*: a preliminary Communication. *Proc. Zool. Soc. London*.
- _____, 1898b. Notes on the Collection of Specimens of the Genus *Millepora* obtained by Mr. Stanley Gardiner at Funafuti and Rotuma. *Proc. Zool. Soc. London*.
- _____, 1899. The Medusae of *Millepora*. *Proc. Roy. Soc. London*, vol. 64.
- _____, 1924. An Introduction to the Study of recent Corals. *Publ. Univ. Manchester, biol. ser.*, no. 4.
- MOSELEY, H. N., 1879. Notes by a Naturalist on the "Challenger". London.
- QUELCH, J. J., 1884. The Milleporidae. *Nature*, vol. 30.
- _____, 1886. Report on the Reef-corals collected by H. M. S. Challenger during the Years 1873—76. *Rep. Challenger Exp., Zool.*, vol. 16.



Figs. 1 and 2. *Millepora alcicornis* L., Amsterdam Museum, ampullae.

Figs. 3—5. *Millepora alcicornis* L., Leiden Museum, ampullae.

All figures $\times 20$.



Figs. 1—3. *Millepora platyphylla* Hempr. & Ehr., Island Edam, May 25, 1921, ampullae.

Figs. 4—7. *Millepora platyphylla* Hempr. & Ehr., Island Edam, date unknown, ampullae.

All figures $\times 20$.

PLATE III

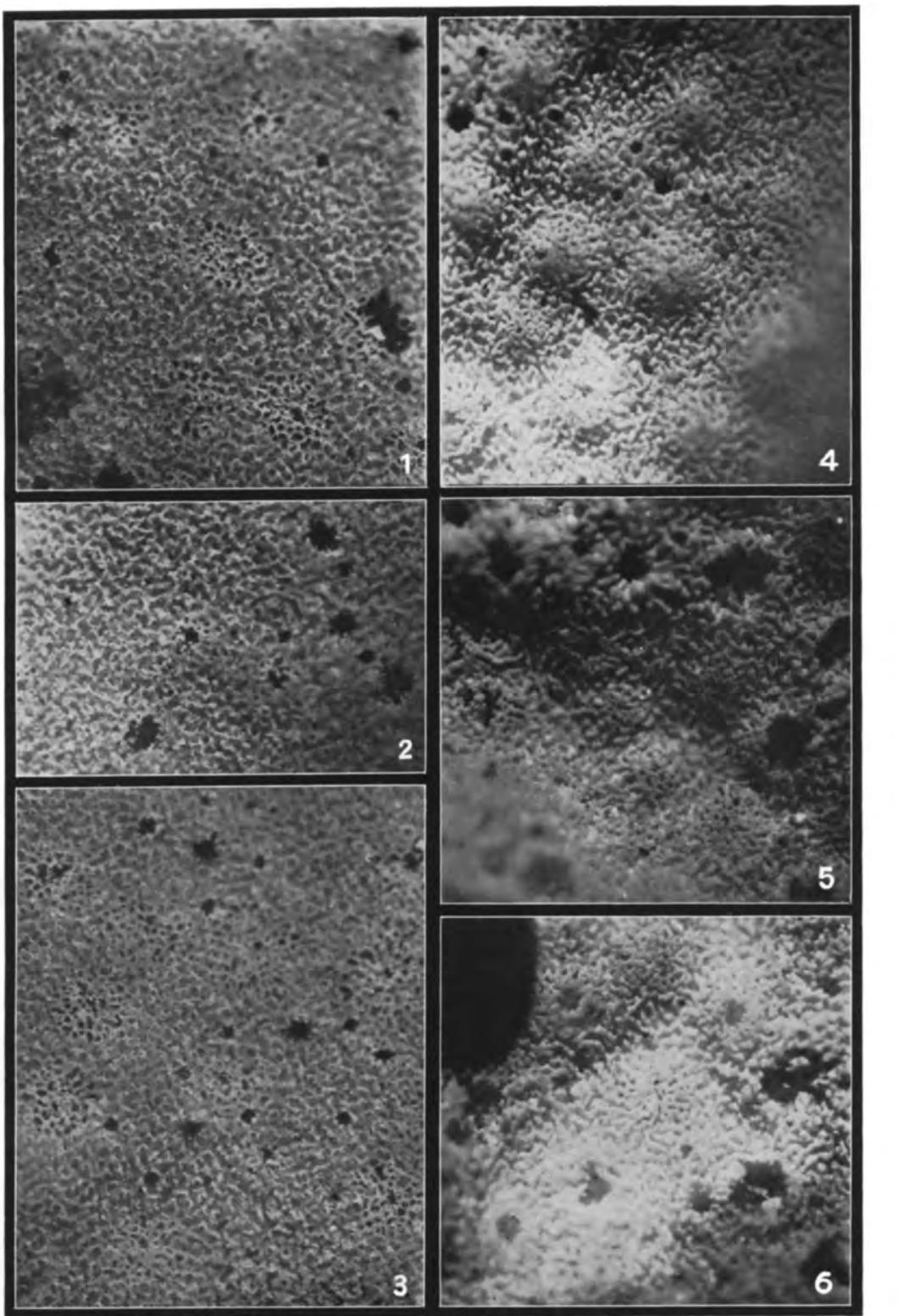
Figs. 1—3. *Millepora murrayi* Quelch, Tongatabu, ampullae.Figs. 4—6. *Millepora murrayi* Quelch, Island Edam, ampullae.All figures $\times 20$.

PLATE IV

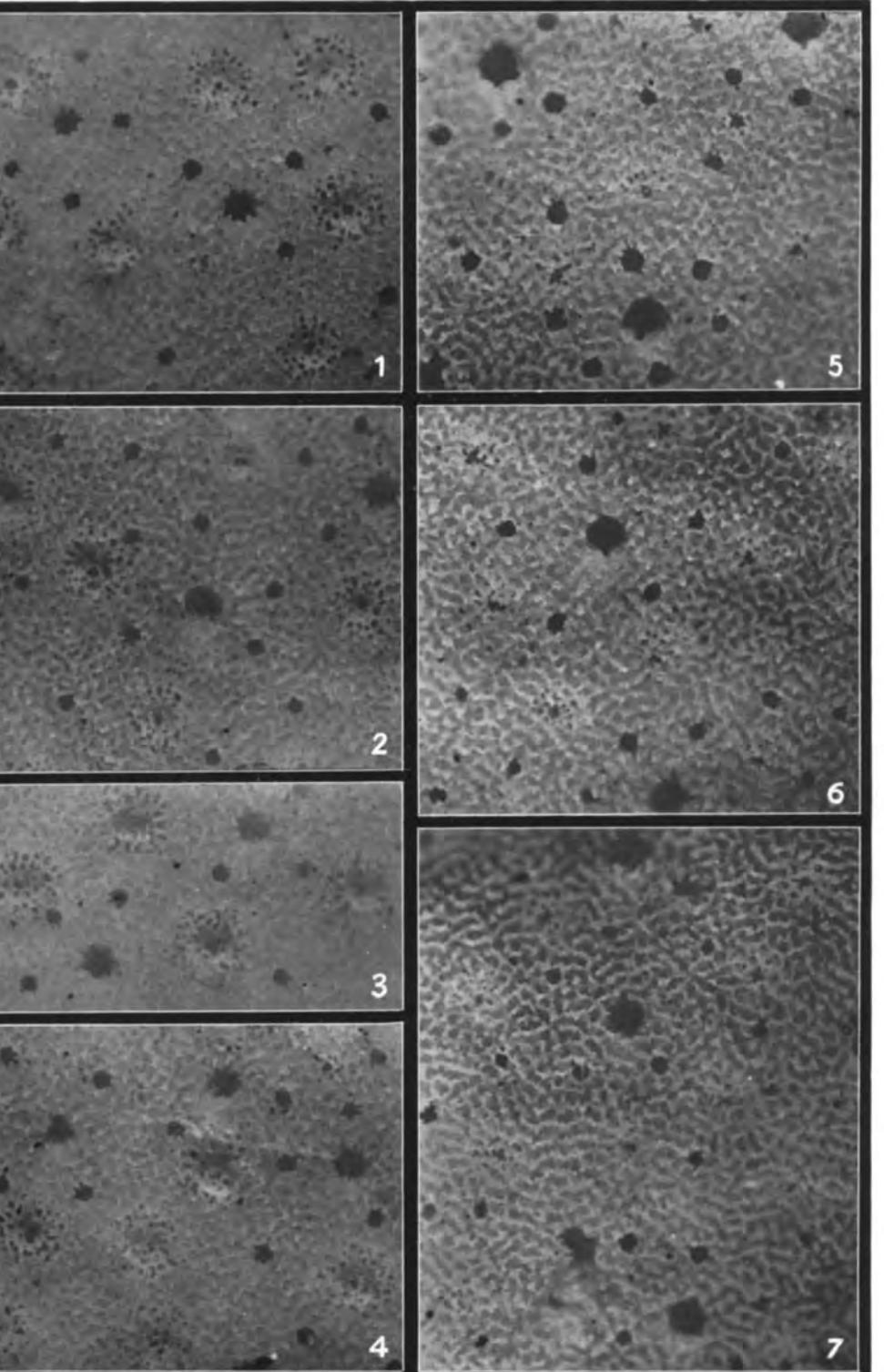
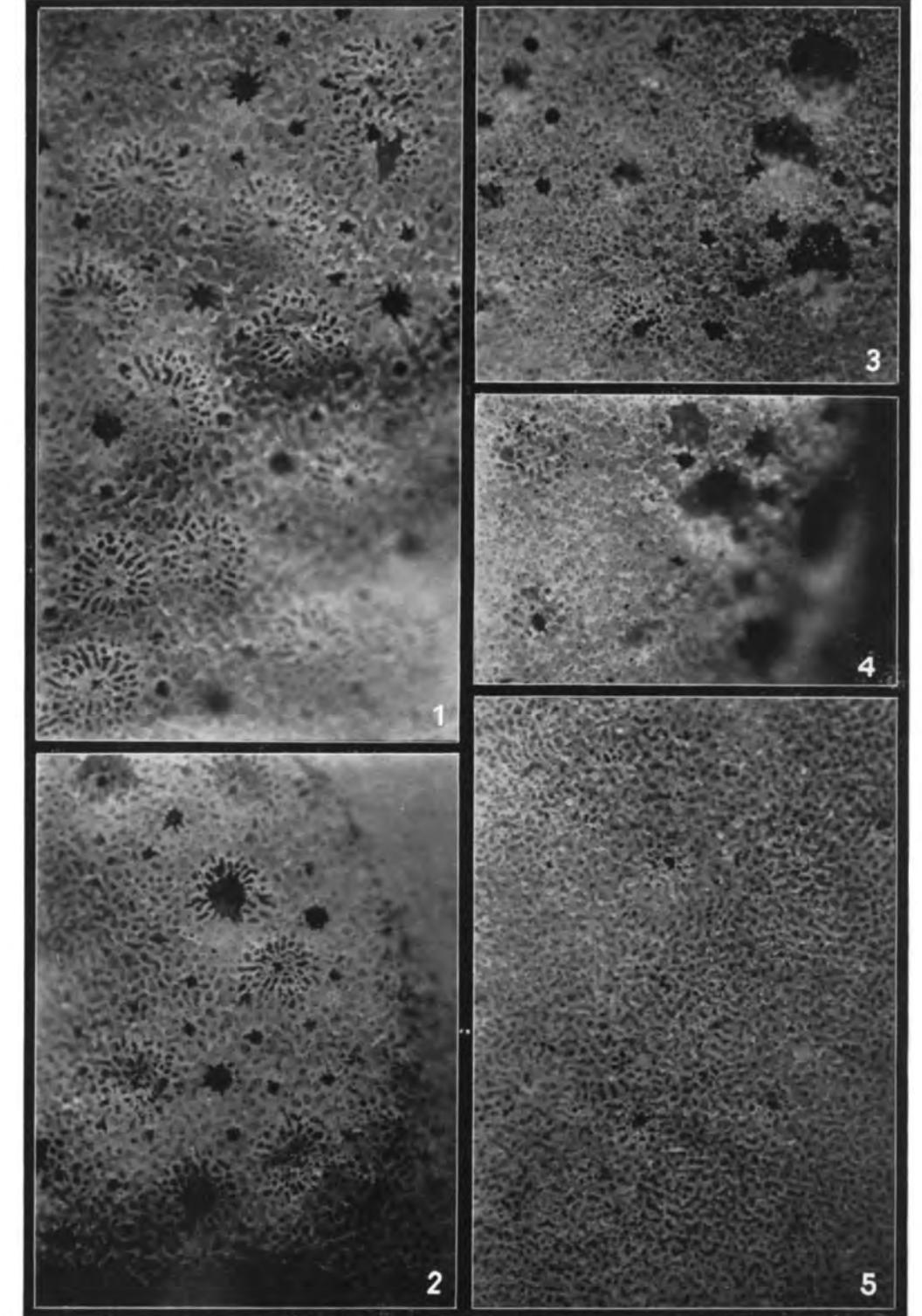
Figs. 1—4. *Millepora tenella* Ortm., Halmahera, ampullae.Figs. 5—7. *Millepora tenella* Ortm., Rodriguez, ampullae.All figures $\times 20$.

PLATE V

Figs. 1—2. *Millepora latifolia* Boschma, Island Edam, ampullae.Figs. 3—4. *Millepora tenella* Ortm., Siboga Expedition, station unknown, ampullae.Fig. 5. *Millepora platyphylla* Hempr. & Ehr., Siboga Expedition, Station 193, partially closed gastropores and dactylopoles.All figures $\times 20$.

Biochemistry. — *Elastic-viscous oleate systems containing KCl*¹⁾. II.

a) *Period and logarithmic decrement as function of the radius of the sphere for a system containing 1.2 % oleate, in 1.52 N KCl + + 0.08 N KOH at 15° and 23° C;* b) *Shear modulus and relaxation time as function of the temperature.* By H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG.

(Communicated at the meeting of December 18, 1948.)

1. Introduction.

The object of the investigations described in the following communication was to obtain insight into the character of the damped oscillations of KCl containing oleate systems²⁾. This was possible with the aid of formulae, developed by J. M. BURGERS³⁾ on the basis of older work by LAMB, giving the period T and the logarithmic decrement Δ for an elastic medium completely filling a spherical space, performing one of the three types of oscillations, already mentioned in Part I. BURGERS has considered three possible causes of damping:

(a) purely viscous damping; (b) damping through relaxation of elastic tensions, characterized by a single constant relaxation time; (c) damping through slipping along the wall of the vessel.

It is found in all cases that the period is directly proportional to the radius; hence the experimental confirmation of this proportionality (already obtained in Part I for the case of the rotational oscillation) does not allow to distinguish between the three possible causes of damping. Such a distinction, however, can be made when data are available concerning the dependence of the logarithmic decrement Δ on the radius R , as in the case of viscous damping $\Delta \sim R^{-1}$, in the case of relaxation damping $\Delta \sim R$, while in the case of damping through slipping Δ appears to be independent of R .

2. Experimental method.

As has been mentioned in preliminary investigations⁴⁾ on the elastic behaviour of KCl containing oleate systems, the elastic properties are a function of the KCl concentration, and it was found that in the case of the rotational oscillation the number of visible oscillations n has a maximum for a definite value of the KCl content.

¹⁾ Part I has appeared in these Proceedings 51, 1197 (1948).

²⁾ In the execution of the measurements we were assisted by D. VREUGDENHIL, to whom we express our thanks also here.

³⁾ J. M. BURGERS, these Proceedings 51, 1211 (1948).

⁴⁾ H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 50, 1227 (1947).

For the present work we have used a concentration, nearly corresponding to this maximum of n , so that the damping was as small as possible. In order to prevent the appearance of auto-sensibilisation as a consequence of hydrolysis, it is necessary to ensure a sufficiently high pH (above 11), for which purpose a small amount of KOH is added⁵⁾.

The solution used was prepared by mixing 4,5 litres of a solution of Na-oleate (90 gr Na-oleinicum pur. pulv. "Merck"⁶), dissolved in 4050 cm³ aqua destillata to which is added afterwards 450 cm³ KOH 2 N) with 3 litres of KCl 3,8 N. The mixture was thoroughly shaken for an extensive period in order to make it homogeneous, after which it was put into a thermostat at 15,0° C for two days, in order to obtain the desired temperature and to get rid of the air content. (In Part I we have already mentioned the disturbances that are caused by the presence of air bubbles).

The measurements were then started; the rotational, the meridional and the quadrantal oscillations were measured on three consecutive days; for details concerning the method applied we refer to Part I. We mention that the experiments with the very large reservoirs (the largest one has a capacity of 6 litres) had to be done outside the thermostat. The vessels were put on the turning table, described in Part I, and found themselves during 20 minutes in air of 18° C. The very large volume of these reservoirs and the low rate of heat exchange with the surroundings, combined with the relatively small values of the temperature coefficients of period and damping ratio for this system in the range of temperatures from 2° to 19° C (see Part I), make this permissible. It will appear that even in the less favourable case of measurements with a solution at a temperature of 23° C, the results concerning the functional relation between T and R or between A and R for a single type of oscillation still were satisfactory.

It is possible nevertheless that results obtained in this way are less satisfactory when it is desired to make comparisons between the three types of oscillation. There are indications that such comparisons can safely be made only when data are obtained with a vessel completely surrounded by thermostat water.

⁵⁾ H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 849 (1947).

H. G. BUNGENBERG DE JONG, H. L. Booij and G. G. P. SAUBERT, Protoplasma, **29**, 536 (1938).

⁶⁾ The concentration of Na-oleate in the final mixture is 1,2% = 0,04 N Na-oleate. Relatively to the large concentration of K-ions (1,60 N) the small concentration of the Na-ions is negligible, so that we may consider our system practically as a K-oleate system.

In principle it might be possible to dissolve oleic acid in KOH, in order to arrive at a system containing exclusively K-ions; actually, however, this procedure proved to be unsatisfactory, which probably must be ascribed to the circumstance that the liquid oleic acid is much more liable to chemical alterations than the solid Na-oleate. Nevertheless also the latter substance gradually changes, even when it is left in its original packing (paraffined bottles, not opened). Pure Na-oleate when shaken must behave as a fine dusty powder. After deterioration it takes the form, first of coarse grains, finally of a compact viscous mass. Compare the first paper mentioned in footnote⁵⁾.

3. Period and logarithmic decrement as functions of R . Measurements at $15^\circ C$.

Owing to the difficulties inherent in the measurement of the damping ratio, it was necessary to use vessels of large radius. For this purpose it was decided to use Pyrex and Jena "round bottom" vessels, of nominal capacities of 6 litres, 3 litres, 1.5 litres and 750 cm^3 . The actual volumes contained in these vessels till the beginning of the tube-shaped neck were: 6000, 3309, 1738 and 754 cm^3 respectively. Assuming an exact spherical shape, the radii become: 11.27; 9.24; 7.46; 5.65 cm.

TABLE I.
Measurements with a 1.2 % oleate system (containing 1.52 N KCl + 0.08 N KOH) at $15.0^\circ C$.

Type of oscillation	R (cm)	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	A $(= 2.303 \log \frac{b_1}{b_3})$	$10 \times \frac{T}{2}$ corr.	λ (sec) $(= \frac{T}{2A})$	G (dynes/cm ²)
Rotational	11.27	28.8 ± 0.1	12.31 ± 0.020	1.702 ± 0.029	0.532 ± 0.017	12.27	2.31	44.4
	9.24	32.2 ± 0.2	10.09 ± 0.029	1.553 ± 0.026	0.440 ± 0.017	10.07	2.29	44.3
	7.46	37.5 ± 0.3	8.28 ± 0.024	1.408 ± 0.014	0.342 ± 0.010	8.27	2.42	mean 42.8
	5.65	42.6 ± 0.2	6.41 ± 0.015	1.352 ± 0.008	0.301 ± 0.006	6.40	2.13	41.0
	2.33*	47.5 ± 0.5	2.44 ± 0.015	1.111 ± 0.003	0.105 ± 0.003	2.44	2.32	48.0
Meridional	11.27	29.5 ± 0.3	9.39 ± 0.019	1.440 ± 0.014	0.365 ± 0.009	9.37	2.57	46.2
	9.24	33.5 ± 0.2	7.67 ± 0.018	1.385 ± 0.023	0.326 ± 0.016	7.66	2.35	mean 46.5
	7.46	34.0 ± 0.3	6.26 ± 0.015	1.267 ± 0.006	0.237 ± 0.004	6.26	2.64	45.4
	5.65	37.3 ± 0.3	4.74 ± 0.020	1.207 ± 0.004	0.188 ± 0.003	4.74	2.52	45.4
	2.33*							
Quadrantal	11.27	35.1 ± 0.3	7.87 ± 0.024	1.369 ± 0.009	0.314 ± 0.007	7.86	2.50	44.6
	9.24	40.0 ± 0.3	6.33 ± 0.010	1.310 ± 0.012	0.270 ± 0.009	6.32	2.34	mean 46.4
	7.46	43.3 ± 0.3	5.13 ± 0.016	1.231 ± 0.006	0.208 ± 0.005	5.13	2.47	45.9
	5.65	47.2 ± 0.3	3.89 ± 0.013	1.193 ± 0.005	0.176 ± 0.005	3.89	2.21	45.8
	2.33*							

* In this case an approximately spherical reservoir of 52.7 cm^3 capacity has been used, immersed in the thermostat of $15^\circ C$. The damping ratio could be measured in the case of the rotational oscillation only, as for the other two forms the period of the oscillation was too short.

Table I for the temperature of $15^\circ C$ gives the experimental results concerning: type of oscillation, total number n of visible oscillations; $10 \times T/2$ (i.e. the time for 10 turning points, that is 5 complete oscillations); the damping ratio b_1/b_3 calculated from the position of 4 consecutive turning points, read off on the scale of the ocular micrometer⁷); and the corrected value of $10 \times T/2$, obtained with the aid of the formula

$$T_{\text{corr}} = \frac{T}{\sqrt{1 + (A/2\pi)^2}}.$$

⁷) The measurements of the period and of the damping ratio are performed with the aid of the telescope of a kathetometer, from a distance of approximately 1 m. The total

The last two columns of the table will be considered later. All data given are averages derived from 10 measurements of n , from 20 measurements of $10 \times T/2$, and from 20 measurements of b_1/b_3 . The mean error [calculated from $\sqrt{\sum \Delta^2/n(n-1)}$] has been stated in each case.

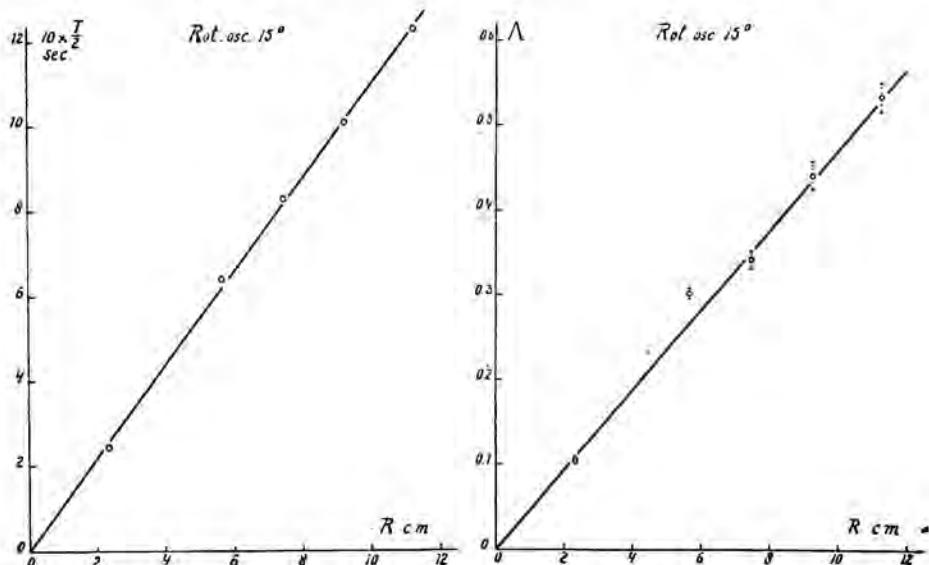


Fig. 1.

Fig. 1, 2 and 3 give the corrected values of $10 \times T/2$ and the values of A in function of the radius R .

It is evident that the period is directly proportional to R , for each of the three types of oscillation. The points are situated so closely to a straight line passing through the origin, that we did not consider it necessary to indicate the mean errors in the diagrams. In the case of the decrement, where the spreading is larger, the errors have been represented. When due regard is given to these errors it is found that also the A -points, on the whole, are sufficiently close to a straight line through the origin. In two cases, however, the distance of the points from the line is much larger than the mean error. We must suppose that here some irregularity has occurred, the nature of which is unknown. Nevertheless the general conclusion is allowed that for the system considered both T and A are pro-

length of the scale of the ocular micrometer (50 divisions) corresponds to a length of ca. 13 mm in the vessel filled with the oleate solution. The displacements of the air bubbles at first exceed the length of this scale, and measurements of the damping ratio can be carried out only when the amplitude has decreased. According to our notes the length b_1 (distance between the position of the bubbles at the first turning point and their position at the next turning point) ordinarily does not exceed 25 scale divisions, which represents an actual displacement of ca. 6 mm. The deviation from the equilibrium position thus is ca. 3 mm. For the points on which the telescope was focussed compare Part. I.

portional to R , which proves that the damping in this case must be ascribed to relaxation with a constant relaxation time⁸⁾.

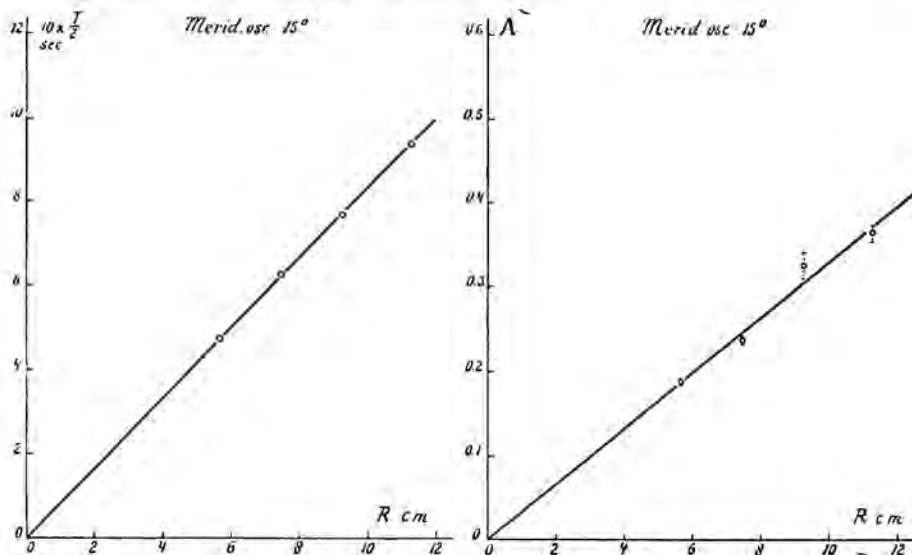


Fig. 2.

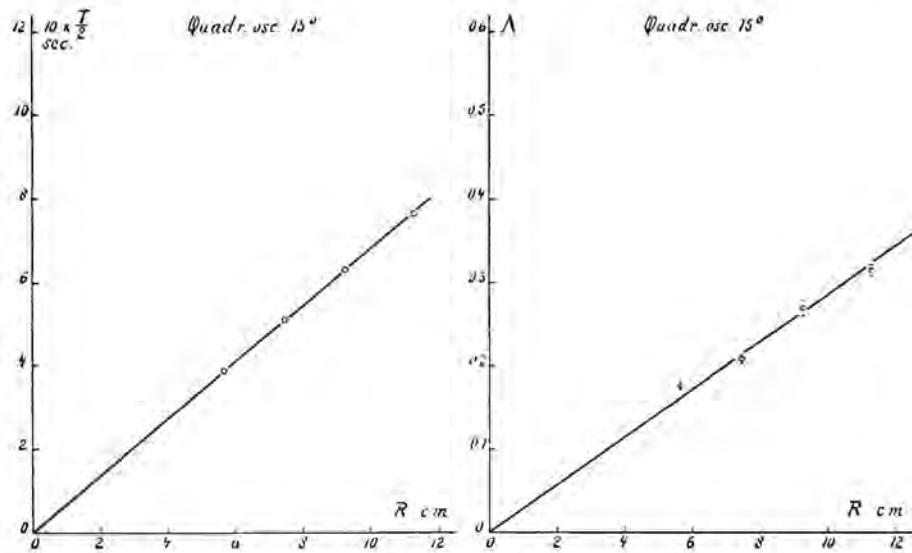


Fig. 3.

4. Period and logarithmic decrement as functions of R . Measurements at 23° C.

With the same oleate system similar measurements were performed at a

⁸⁾ The reader must be warned against the idea that this result should apply to all elastic oleate systems. In the next communication we shall encounter a case where the relation between A and R is different.

temperature of 23° (two days later). Whereas the measurements described in the preceding section referred to a point on that branch of the b_1/b_3 -temperature curve, which presents the smaller inclination, the higher temperature was chosen so as to give a point on the steep branch of this

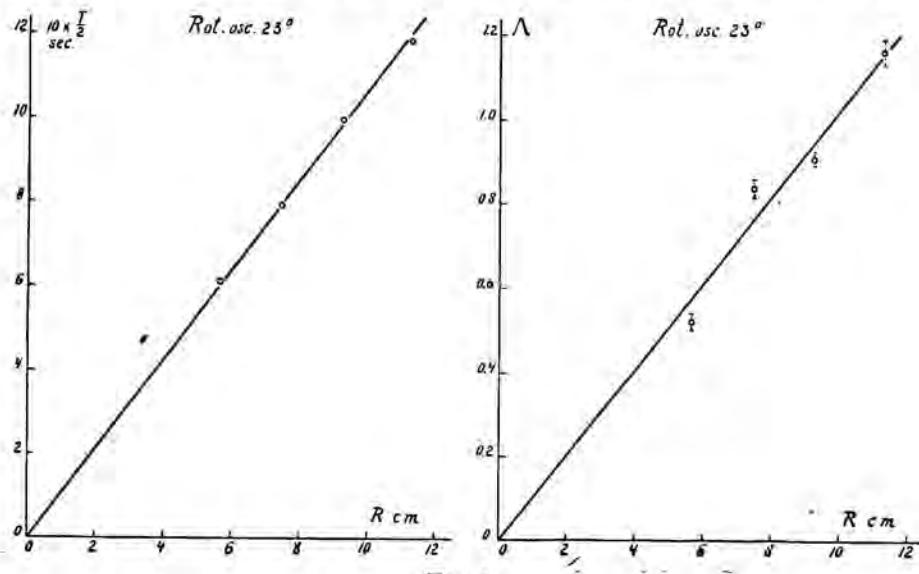


Fig. 4.

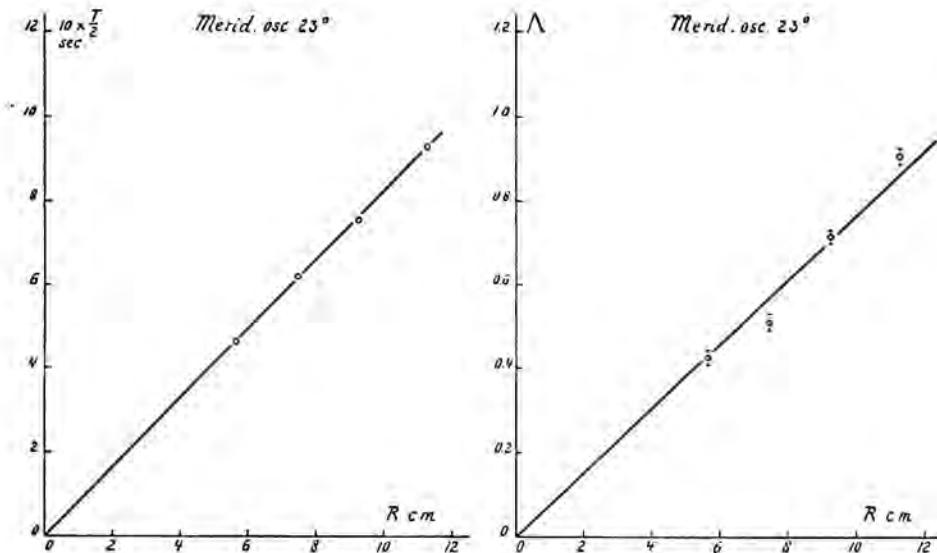


Fig. 5.

curve (compare Part I, fig. 4). The measurements have been executed in a single day, which, however, made it necessary to restrict to the rotational and the meridional types, as measurements with the quadrantal oscillation always are more difficult, in particular at the higher temperature.

The experimental results and the data derived from them are given in Table II.

TABLE II.
Measurements with the same system as that of Table I, at 23.0° C.

Type of oscillation	R (cm)	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	Λ (= $2.303 \log \frac{b_1}{b_3}$)	$10 \times \frac{T}{2}$ corr.	λ (sec) (= $\frac{T}{2\Lambda}$)	G (dynes/cm ²)
Rotational	11.27	7.1 ± 0.1	$12.04 \pm 0.058^*$	3.195 ± 0.098	1.162 ± 0.031	11.84	1.02	47.7
	9.24	9.3 ± 0.3	$10.08 \pm 0.042^*$	2.480 ± 0.042	0.909 ± 0.017	9.98	1.10	45.1
	7.46	12.7 ± 0.2	8.00 ± 0.027	2.311 ± 0.048	0.838 ± 0.021	7.93	0.95	46.6
	5.65	17.0 ± 0.1	6.13 ± 0.018	1.687 ± 0.032	0.523 ± 0.019	6.11	1.17	45.0
Meridional	11.27	7.1 ± 0.1	$9.38 \pm 0.060^*$	2.473 ± 0.055	0.906 ± 0.022	9.28	1.02	47.1
	9.24	10.1 ± 0.2	$7.60 \pm 0.038^*$	2.047 ± 0.036	0.717 ± 0.018	7.55	1.05	47.9
	7.46	12.9 ± 0.1	6.21 ± 0.017	1.669 ± 0.032	0.512 ± 0.019	6.19	1.21	46.4
	5.65	17.1 ± 0.1	4.65 ± 0.016	1.531 ± 0.023	0.426 ± 0.017	4.64	1.09	47.4

* As the number of observable oscillations was too small, we have measured $5 \times T/2$ instead of $10 \times T/2$. The results and the mean errors all have been multiplied by 2.

It will be seen that the mean errors have become larger at the higher temperature, in particular those for the damping.

In figures 4 and 5 the corrected values of $10 \times T/2$ and the values of Λ have been represented as functions of R .

The experimental points on the whole are again situated on straight lines through the origin, although in the case of Λ the distances of the points from the line sometimes exceed the mean errors. We believe that it is permitted to consider these deviations as the results of unsystematic errors, and conclude that the relations $T \sim R$, $\Lambda \sim R$ are also valid at 23° C. Hence also at this temperature damping must be ascribed to relaxation with a constant relaxation time. We shall come back to this point in the last section of this paper.

5. Coefficients occurring in BURGERS' formulae.

BURGERS has given the following expressions for the period and the logarithmic decrement of an elastic medium, confined in a spherical space and describing oscillations damped through relaxation:

$$\text{rotational oscillation} \dots T_0 = \frac{2\pi}{4.49} R \sqrt{\frac{\rho}{G}} \text{ and } \Lambda_0 = \frac{\pi}{4.49} R \frac{1}{\lambda} \sqrt{\frac{\rho}{G}}$$

$$\text{meridional oscillation} \dots T_1 = \frac{2\pi}{5.76} R \sqrt{\frac{\rho}{G}} \text{ and } \Lambda_1 = \frac{\pi}{5.76} R \frac{1}{\lambda} \sqrt{\frac{\rho}{G}}$$

$$\text{quadrantal oscillation} \dots T_2 = \frac{2\pi}{6.99} R \sqrt{\frac{\rho}{G}} \text{ and } \Lambda_2 = \frac{\pi}{6.99} R \frac{1}{\lambda} \sqrt{\frac{\rho}{G}}$$

in which:

- T = period for a complete oscillation in sec
- R = radius of the spherical vessel in cm
- ρ = density of the elastic medium in gr/cm³
- G = shear modulus in dyne/cm²
- λ = relaxation time in sec.

When these formulae are applied to our oleate system they allow the calculation of the characteristic quantities G and λ , so that it becomes possible to investigate the dependence of these quantities on various parameters (temperature, KCl concentration, oleate concentration etc.).

The formulae give a direct proportionality for T and A with R , which has already been confirmed experimentally.

The next point to be considered are the values of the ratios T_0/T_1 , A_0/A_1 , T_0/T_2 , A_0/A_2 , for which the equations give:

$$T_0/T_1 = A_0/A_1 = 5.76/4.49 = 1.283;$$

$$T_0/T_2 = A_0/A_2 = 6.99/4.49 = 1.557.$$

The values calculated from the experimental results (Table I) obtained at 15° C are given below:

R	T_0/T_1	A_0/A_1	T_0/T_2	A_0/A_2
11.27	1.309	1.458	1.561	1.694
9.24	1.315	1.350	1.593	1.630
7.46	1.321	1.324	1.443	1.603
5.65	1.350	1.601	1.645	1.704

The mean values of the ratios found experimentally are higher than the theoretical ones: 3 % and 3 % respectively for the period, and 14 % and 7 % respectively for the decrement. It must be kept in mind, however, that, as our primary purpose was to find the relation between T and R , and between A and R for a single type of oscillation, the three types have been investigated on consecutive days, so that perhaps the circumstances are not completely identical for the three types.

This supposition is supported by the following data, referring to the experiments (Table II) performed at 23° C:

R	T_0/T_1	A_0/A_1
11.27	1.276	1.283
9.24	1.322	1.268
7.46	1.281	1.372 (1.260)
5.65	1.317	1.228

The measurements from which these numbers have been deduced were performed on a single day, and it will be seen that the difference between the experimental values of the ratios and the theoretical ones is less (1 %

for T , 14 % for A). Amongst the experimental values of A there is one which apparently is in error (the value 1.637 for $R = 7.46$ cm); if this value is omitted, the mean becomes 1.260 (indicated between brackets), which deviates only 2 % from the theoretical value.

A further inspection of the data shows that the differences between the experimental and the theoretical values of the ratios generally increase with decrease of the radius of the vessel. It is possible that the deviation must partly be ascribed to the fact that the vessels, when put on the turning table, found themselves in air of a temperature differing from that inside the fluid. Whilst this circumstance was not of much importance in the investigations on the relation between T and R and that between A and R , it has apparently been more harmful in the present case, which requires a higher degree of precision.

We therefore decided to perform new determinations with a single vessel, executing all measurements in one consecutive series, in such a way that heat exchange with the surroundings was excluded. For this purpose we put on the turning table a wide glass cylinder, through which — also during the measurements — passed a constant flow of water of 15° C., taken from the thermostat. The vessels, placed on the cork-ring on the bottom of the cylinder (see Part I, fig. 2) found themselves surrounded with this thermostat water to the neck.

The results have been given in Table III.

TABLE III.
Determination of T and A ratios at 15° C.

Type of oscillation	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	A	$10 \times \frac{T}{2}$ corr.(sec)
Rotational	8.10 ± 0.024	1.400 ± 0.016	0.337 ± 0.011	8.09 ± 0.024
Meridional	6.31 ± 0.014	1.302 ± 0.008	0.264 ± 0.006	6.30 ± 0.014
Quadrantal	5.12 ± 0.021	1.233 ± 0.0043	0.209 ± 0.003	5.12 ± 0.021
$\frac{T_0}{T_1} = 1.284$			$\frac{A_0}{A_1} = 1.277$	
$\frac{T_0}{T_2} = 1.580$			$\frac{A_0}{A_2} = 1.612$	

It will be seen that in this case, where heat exchange with the surrounding air had been prevented, there is a satisfactory agreement with the theoretical values, in particular for T_0/T_1 and A_0/A_1 . The other two quotients, T_0/T_2 , A_0/A_2 still are slightly too high.

In this connection we think it useful to mention some measurements of the ratio T_0/T_2 , performed under conditions which prevented heat exchange with the surroundings, in an earlier part of this research⁹⁾. They refer to an oleate system of slightly different composition; the oscillations

⁹⁾ These measurements were made before we knew anything about the theoretical formulae.

were excited with the aid of the pendulum apparatus (see Part I, fig. 3). The vessels were wholly immersed in a thermostat of 15° C. The results of four series of measurements have been given in Table IV.

TABLE IV.
Determinations of T_0/T_2 at 15°.

Nominal volume of the spherical vessel in c.c.	$10 \times \frac{T_0}{2}$ (sec.)	$10 \times \frac{T_2}{2}$ (sec.)	T_0/T_2
1500	8.64	5.49	1.574
750	7.66	4.87	1.573
750	7.63 ± 0.04	4.90 ± 0.02	1.557
500	6.48 ± 0.02	4.18 ± 0.03	1.550

The results for the first two cases (nominal volumes 1500 and 750 cm³) are deduced each from ten determinations of $10 \times T/2$; those for the last two cases (nominal volumes 750 and 500 cm³) have been deduced from 50 and 60 determinations of $10 \times T/2$ respectively and the mean errors are very small. The values of the ratio T_0/T_2 obtained in these cases (1.550, 1.557) are even nearer to the theoretical value 1.557 than the value 1.580 given in Table III¹⁰.

Hence judging by the most reliable measurements we come to the conclusion that the experimental values of the ratios T_0/T_1 , T_0/T_2 , Λ_0/Λ_1 , Λ_0/Λ_2 are very near to the theoretical ones.

This proves that the theoretical formulae, deduced by BURGERS, are valid for the oleate system to which the measurements refer (1.2 % oleate, with 1.52 N KCl and 0.08 N KOH), and that the elastic behaviour of this system can be described by means of two constants; the shear modulus G and the relaxation time λ .

6. Shear modulus and relaxation time as functions of the temperature.

The values of G and λ , calculated with the aid of BURGERS' formulae, assuming the value $\varrho = 1.074$ for the density, have been inserted into the last columns of Tables I and II. The overall mean values of these quantities are found to be:

$$\begin{aligned} \text{at } 15^\circ \text{ C: } G &= 45.2 \text{ dynes/cm}^2; \quad \lambda = 2.40 \text{ sec.} \\ \text{at } 23^\circ \text{ C: } G &= 46.7 \text{ dynes/cm}^2; \quad \lambda = 1.08 \text{ sec.} \end{aligned}$$

¹⁰⁾ The value 1.580 has been obtained by using the turning table. It may be that this method of exciting the oscillations is the cause of the abnormal values of T_0/T_2 and in particular of Λ_0/Λ_2 in Table III. The exact measurement of quantities referring to the quadrantal oscillation is difficult, when it is markedly combined with the rotational oscillation, as the air bubbles describe complicated curves instead of straight lines. It is in particular the determination of b_1/b_3 which is influenced in this way. The application of the pendulum apparatus for exciting the oscillations has the advantage that with a sufficiently strong impulse the rotational oscillation appears to be nearly suppressed, so that the quadrantal oscillation is much better observable (see Part I, section 6).

Increase of temperature appears to have a small effect on the shear modulus and a much larger effect on the relaxation time.

In view of the remarkable form of the b_1/b_3 -curve, described in Part I, it is of interest to re-calculate the values for the series of experiments considered there, so as to obtain the corresponding values of G and λ . As the values of $10 \times T/2$ had been determined at temperatures not equal to those to which refer the values of b_1/b_3 , we have drawn smooth curves through the experimental points, from which we have read off the values of the period and of the damping ratio for a series of temperatures (2.5° ; 5° ; 7.5° , etc.). These values have been collected in Table V and the values calculated for G and λ have been added.

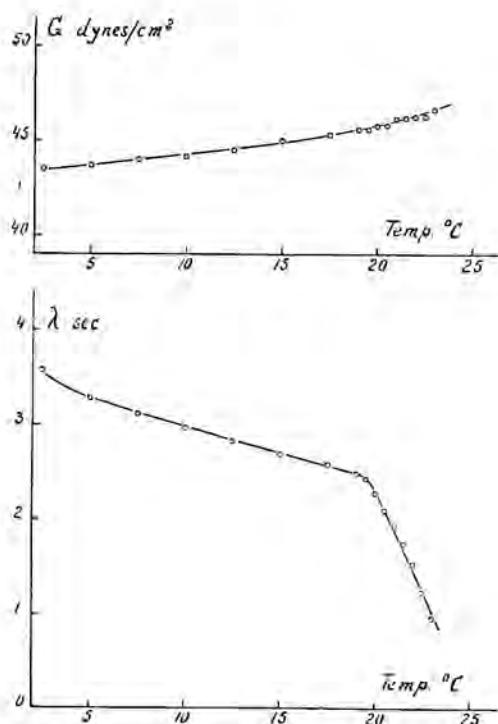


Fig. 6.

It is again found (compare fig. 6) that there is a slight increase of G with temperature, whereas there is a marked decrease of λ . It is remarkable, however, that the decrease of λ occurs in two stages: the curve exhibits two branches, joining with a relatively sharp bend. Hence there is a definite temperature (19.5°C) at which the behaviour of λ changes in such a way that its temperature coefficient suddenly increases. There is no indication of a similar change in the curve for G .

It seems premature to attempt an interpretation of the course of the curves for G and λ as functions of the temperature. A much more extended

TABLE V.

Shear modulus and relaxation time as functions of the temperature.

Temp. °C	$10 \times T/2$ (sec.)	b_1/b_3	Λ	$10 \times T/2$ corr. (sec.)	λ (sec.)	G (dynes/cm ²)
2.5	5.57	1.170	0.157	5.57	3.57	43.5
5	5.56	1.185	0.170	5.56	3.28	43.7
7.5	5.54	1.195	0.178	5.54	3.11	44.0
10	5.53	1.205	0.186	5.53	2.97	44.2
12.5	5.51	1.215	0.194	5.51	2.83	44.5
15	5.48	1.225	0.203	5.48	2.70	45.0
17.5	5.46	1.235	0.211	5.46	2.59	45.3
19	5.44	1.245	0.219	5.44	2.49	45.6
19.5	5.44	1.250	0.223	5.44	2.44	45.6
20	5.43	1.270	0.239	5.43	2.28	45.8
20.5	5.43	1.295	0.258	5.43	2.10	45.8
21	5.42	1.325	0.281	5.41	1.93	46.2
21.5	5.42	1.365	0.311	5.41	1.74	46.2
22	5.41	1.425	0.354	5.40	1.53	46.3
22.5	5.41	1.550	0.438	5.40	1.23	46.3
23	5.40	1.750	0.559	5.38	0.96	46.7

material will be necessary, allowing the discussion of the influence of a greater number of parameters.

In the next communication we shall consider the influence of the oleate concentration.

Summary of Part II.

- Measurements of the period and the logarithmic decrement of the rotational, the meridional and the quadrantal oscillations have been executed with an 1.2 % oleate system (containing 1.52 N KCl + 0.08 N KOH), at temperatures of 15° and 23° C. The dependence of these quantities on the radius of the vessel has been investigated.
- It has been found that both the period and the logarithmic decrement are proportional to the radius of the vessel, which proves that at both temperatures the damping of the elastic oscillations of this oleate system must be ascribed to relaxation of the elastic stresses, characterized by a single and constant relaxation time.
- Theoretical formulae developed by BURGERS make it possible to calculate the shear modulus and the relaxation time. The shear modulus is only slightly dependent on the temperature: $G = 45$ dyne/cm² at 15° and 47 dyne/cm² at 23° C. There is a larger change in the relaxation time: $\lambda = 2.4$ sec at 15° and 1.1 sec at 23° C.
- Using subscripts 0, 1 and 2 in order to distinguish between quantities referring to the rotational, the meridional and the quadrantal oscillations respectively, it has been found that the experimental values of the ratios T_0/T_1 , Λ_0/Λ_1 , T_0/T_2 , Λ_0/Λ_2 are very near to the theoretical values

$T_0/T_1 = A_0/A_1 = 1.28$ and $T_0/T_2 = A_0/A_2 = 1.56$, the differences being of the order 1 or 2 %.

5. Making use of results of experiments already mentioned in Part I, curves have been constructed giving the course of G and λ with the temperature. The shear modulus presents a slight gradual increase with temperature. The relaxation time decreases; the curve exhibits two straight branches, joining at ca. 19° C with a not very marked bend. Above 20° the temperature coefficient of λ is about 5 or 6 times as large as below 19° .

Zoology. — *On the influence of lithium chloride on the eggs of Limnaea stagnalis at the 24-cell stage.* By CHR. P. RAVEN and SELMA DUDOK DE WIT. (Zoological Laboratory, University of Utrecht.)

(Communicated at the meeting of December 18, 1948.)

When the eggs of *Limnaea stagnalis* are treated with weak solutions of LiCl, characteristic malformations are induced (RAVEN 1942). On the one hand, part of the eggs may exogastrulate, giving rise to pear- or dumb-bell-shaped vesicular embryos; on the other hand, embryos with various malformations of the head region are produced. It has been proved (RAVEN 1947) that these head malformations are due to a reduction of dorsomedian parts of the head, which are derived, in normal development, from cells adjacent to the original animal pole of the egg. It was concluded from these observations that the influence of lithium chloride can be explained by its action on a polar gradient field.

RAVEN, KLOEK, KUIPER and DE JONG (1947) showed that the action of LiCl on *Limnaea* eggs is phase-specific. There is a distinct maximum of sensibility for the production of exogastrulae shortly before and during second cleavage. With regard to the production of head malformations, a first period of sensibility exists immediately after laying; a second maximum is found at the 24-cell stage. RAVEN and RIJVEN (1948) demonstrated that the susceptibility for the induction of head malformations rises gradually from the 12-cell stage until 2—3 hours after the beginning of the 24-cell stage; then, a drop in susceptibility occurs. When the eggs are treated more than 8 hours after the beginning of the 24-cell stage, no head malformations are produced.

DE GROOT (1948) studied the immediate effects of the exposure of decapsulated *Limnaea* eggs at the uncleaved stage to stronger solutions of LiCl. Hypertonic LiCl solutions caused various disturbances of the nuclear and mitotic apparatus, leading to an arrest of development at different stages of maturation and fertilization. In isotonic LiCl solutions, the nuclear processes proceeded in a normal way, but the distribution of the subcortical protoplasm was very abnormal. The formation of the animal pole plasm was suppressed at all concentrations studied.

In view of these results it appeared interesting to study the immediate effects of a treatment of decapsulated eggs at the 24-cell stage with LiCl solutions of various concentrations, in order to compare the behaviour of the eggs at both stages at which head malformations can be easily induced by LiCl treatment.

Material and methods

Egg-masses were followed in their development till the 24-cell stage was reached. 2—3 Hours after the beginning of this stage, each egg-mass was

divided into 6 equal parts. The eggs contained in 5 of these parts were decapsulated and transferred to solutions of 0.4 %, 0.2 %, 0.1 %, 0.05 % LiCl and to distilled water, respectively. After an exposure of 1, 2 or 3 hours, the eggs were fixed. Those of the 6th part of the egg-mass, which had remained in the capsules meanwhile, now were decapsulated too and fixed immediately; they served as controls. Hence, the eggs were fixed 3½—5½ hours after the beginning of the 24-cell stage.

All eggs were fixed in BOUIN's fluid and cut in sections of 5 μ thickness, which were stained either with iron haematoxylin and eosin or with azan.

Results

Already at a superficial inspection of the sections it becomes clear that the treatment of the eggs with LiCl solutions or distilled water has produced peculiar changes in the size and position of the nuclei.

In normal eggs of this stage (fig. 1), the nuclei are situated at the boundary between ecto- and endoplasm in all cells. As the layer of ecto-

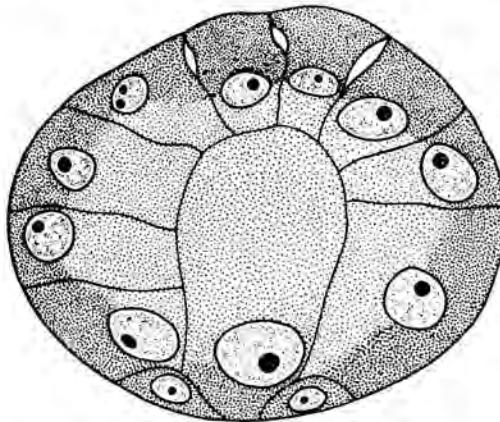


Fig. 1. *Limnaea stagnalis*. 24-cell stage. Normal egg.

plasm at the animal side of the egg is somewhat thicker the nuclei of the animal cells are lying at a greater distance from the surface than those of the vegetative cells. Moreover, there is a clear difference in size, the animal nuclei being, on an average, distinctly smaller than the vegetative ones (4 small cells surrounding the vegetative pole, probably the 3d micromeres, have, however, still smaller nuclei).

In the eggs treated with distilled water and with hypotonic solutions of LiCl (0.05 and 0.1 %), the size of the nuclei has considerably increased. Moreover, they have been displaced towards the surface. In many cells, both animal and vegetative, the nuclei are lying in immediate contact with the cell surface (fig. 2); sometimes, especially in vegetative cells, the surface is (in the sections) even slightly bulging at this place. This change of position is not due to a disappearance or reduction of the layer of

ectoplasm, but the nuclei have left their position at the boundary of the plasma layers and are now situated wholly or for the greater part in the ectoplasm.

In isotonic (0.2 %) or hypertonic (0.4 %) solutions of LiCl, the nuclei are smaller, but their position is the same as in hypertonic solutions.

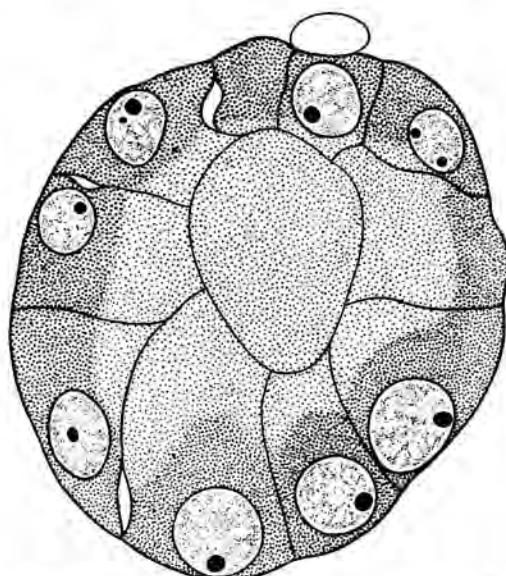


Fig. 2. *Limnaea stagnalis*. 24-cell stage. 0.05 % LiCl.

In order to get accurate data both on the swelling and the displacement of the nuclei in the different solutions, measurements have been made by means of an ocular micrometer, with oil immersion. The diameter of each nucleus was determined in 2 perpendicular directions; the product of these 2 measurements is indicated below as "nuclear value" (in μ^2). Moreover, the distance between the outer side of the nuclear membrane and the cell surface (in μ) is called "nuclear distance". The nuclei of the animal and vegetative half of the egg have been treated separately.

4 Egg-masses have been used for these measurements; they are indicated as series 1, 2, 3 and 4. In total, 832 nuclei have been measured.

Nuclear values.

The inspection of the nuclear values obtained in the controls shows that series 1 and 2 agree within the limits of error in the size of their nuclei; the same holds true as regards series 3 and 4, which show comparable nuclear values, but lower than those of series 1 and 2. Therefore, in the following table series 1 and 2, on the one hand, and 3 and 4, on the other, are taken together.

Table I shows the nuclear values (in μ^2) of controls and treated eggs.

TABLE I.

Treatment	Series (1 + 2)				Series (3 + 4)			
	Animal cells		Vegetative cells		Animal cells		Vegetative cells	
	n	M ± m	n	M ± m	n	M ± m	n	M ± m
Controls	54	120.5 ± 5.89	58	169.0 ± 8.98	72	92.9 ± 4.71	46	122.1 ± 8.67
Dist. water	15	119.4 ± 10.05	18	193.3 ± 15.37	17	132.3 ± 8.29	26	226.8 ± 10.43
0.05% LiCl	17	190.3 ± 18.72	34	246.6 ± 13.03	58	172.1 ± 7.23	68	216.8 ± 5.82
0.1% "	39	196.2 ± 9.21	28	253.6 ± 14.50	35	136.8 ± 8.23	37	177.8 ± 8.59
0.2% "	37	123.6 ± 7.80	24	171.3 ± 9.46	9	128.8 ± 10.89	23	178.4 ± 11.48
0.4% "	32	127.4 ± 6.40	35	178.2 ± 8.80	16	118.8 ± 11.81	33	177.0 ± 8.26

In order to judge the significance of the differences found, for each relevant comparison a *t* has been computed; for the sake of brevity, these values are omitted here, but they have been used to draw the following conclusions (probabilities below 0.01 have been considered as significant).

In the controls a significant difference in size between animal nuclei (A) and vegetative nuclei (V) exists, the nuclear values of V being about 1.3—1.4 times those of A; hence, the volumes of the vegetative nuclei are about 1.5—1.6 times as large.

In distilled water in series (3 + 4) a considerable swelling of the nuclei has taken place, especially at the vegetative side. In series (1 + 2) this swelling is either insignificant (V) or it is lacking altogether (A).

In 0.05 % LiCl the nuclei are considerably swollen; this swelling is much stronger than in distilled water, except in (3 + 4) V, where the nuclear values show no significant difference in both cases.

In 0.1 % LiCl the general swelling of the nuclei is still obvious. In series (1 + 2) the nuclear values agree with those in 0.05 % LiCl; in series (3 + 4) they are lower, in A they have reached again the value of distilled water, in V they are even significantly lower than this.

In 0.2 % and 0.4 % LiCl the swelling has decreased further. In series (1 + 2) the nuclear values of both A and V agree with those of the controls; in series (3 + 4), they are still higher than in the controls. It is very remarkable that the animal nuclei as well as the vegetative nuclei of both series agree in size in 0.2 % (isotonic) and 0.4 % (hypertonic) solutions (nuclear values: animal cells about $125 \mu^2$, vegetative cells about $175 \mu^2$), notwithstanding the initial size difference between both series in the controls and the different osmotic pressures of the solutions.

Though the behaviour of both series is not altogether identical, we may summarize the results as follows: the swelling of the nuclei is greatest in 0.05 % LiCl, somewhat less in 0.1 % and distilled water, whereas the size of the nuclei is still less in 0.2 % and 0.4 % LiCl, but may be still greater than in the controls in egg capsule fluid. Considering that the egg capsule fluid is considerably hypotonic to the eggs (RAVEN and KLOMP 1946), it is clear that the size of the nuclei is not only governed by osmotic forces.

Nuclear distances.

Table II gives the nuclear distances (in μ) of controls and treated eggs.

TABLE II.

	Treatment	Series 1		Series 2		Series 3		Series 4	
		n	$M \pm m$	n	$M \pm m$	n	$M \pm m$	n	$M \pm m$
a. Animal nuclei	Controls	22	6.50 ± 0.475	32	5.29 ± 0.464	36	4.55 ± 0.690	36	6.22 ± 0.666
	Dist. water	7	12.80 ± 3.032	8	5.78 ± 0.413	17	0.83 ± 0.242	—	—
	0.05% LiCl	4	5.95 ± 0.670	13	0.97 ± 0.401	31	0.32 ± 0.125	27	1.67 ± 0.246
	0.1%	26	6.83 ± 0.516	13	1.19 ± 0.310	8	1.93 ± 0.875	27	3.58 ± 0.657
	0.2%	6	14.00 ± 3.313	32	1.79 ± 0.182	9	0.00 ± 0.000	—	—
	0.4%	10	5.32 ± 0.952	22	2.80 ± 0.357	3	0.94 ± 0.469	13	1.40 ± 0.449
b. Vegetative nuclei	Controls	26	3.12 ± 0.808	32	2.10 ± 0.177	21	1.06 ± 0.396	25	5.43 ± 0.643
	Dist. water	8	1.92 ± 0.834	10	1.12 ± 0.280	26	0.38 ± 0.147	—	—
	0.05% LiCl	18	1.16 ± 0.325	16	0.87 ± 0.177	32	0.27 ± 0.099	37	1.29 ± 0.226
	0.1%	16	1.81 ± 0.725	12	0.70 ± 0.273	7	0.20 ± 0.198	30	2.06 ± 0.505
	0.2%	—	—	23	0.42 ± 0.140	23	0.49 ± 0.188	—	—
	0.4%	11	1.40 ± 0.462	24	1.75 ± 0.283	6	0.70 ± 0.314	27	2.65 ± 0.570

Also in this case the significance of differences has been tested by computation of *t* values.

In general, the nuclear distances show regular changes with the treatment; only the animal nuclei of series 1 exhibit large and quite irregular differences, which are, however, not significant, and may be largely due to the small numbers of nuclei measured in 3 of the 6 groups. Therefore, series 1 A is not further considered below.

In the controls, the nuclear distances of animal nuclei are greater than those of vegetative nuclei; except in series 4, these differences are significant.

In distilled water, in 4 out of 5 possible comparisons, the nuclear distances have diminished as compared with the controls; in 2 cases (series 3 A and 2 V), this decrease is significant.

In 0.05 % LiCl, in all 7 cases the nuclear distances have diminished as compared with the controls; in 5 cases (2—4 A, 2 V and 4 V) the decrease is significant. In all 5 possible comparisons with distilled water, the nuclear distances have further diminished in 0.05 % LiCl; this decrease is significant in 2 A.

In 0.1 % LiCl, in all 7 cases the nuclear distances are lower than those of the controls (significant difference in 2 A and V and 4 A and V), in 4 out of 5 cases lower than those in distilled water (significantly in 2 A). However, in 5 out of 7 possible comparisons they are higher than in 0.05 % LiCl; in 4 A this difference is significant.

In 0.2 % LiCl, in all 4 possible comparisons the nuclear distances have diminished with respect to the controls; the difference is significant in 3 cases (2 A and V, 3 A). In 3 out of 4 cases they are lower than in distilled

water (significantly in 2 A and 3 A). No significant differences exist in comparison with 0.1 % LiCl.

In 0.4 % LiCl, in all 7 cases the values are lower than those of the controls; the difference is significant in 4 cases (2—4 A, 4 V). In 2 A, the nuclear distance is also significantly lower than in distilled water. However, in all 4 possible comparisons they are higher than those in 0.2 % LiCl; this difference is significant in series 2 V.

Summarizing, it can be said that the nuclear distances are smallest in 0.05 % LiCl (on an average: about 1 μ), somewhat greater in 0.1 % and 0.2 % LiCl, still greater in 0.4 % LiCl and distilled water. Again, it may be concluded that the decrease of the nuclear distances is not directly related to the osmotic pressure of the solutions.

Discussion

The treatment of decapsulated *Limnaea* eggs at the 24-cell stage with distilled water or solutions of LiCl results in 1°. an increase in the size of the nuclei, 2°. a decrease of their distance from the cell surface. Both phenomena are most pronounced in 0.05 % LiCl, and diminish in intensity with increasing concentration of LiCl. However, they cannot be explained by osmotic forces only, since in eggs treated with distilled water the swelling of the nuclei is less and their distance from the surface is greater than in 0.05 % LiCl.

It may be asked if the decrease in the distance between nuclear membrane and cell surface is due to a real displacement of the nucleus, or is only a consequence of its swelling, the centre of the nucleus remaining at the same place. From the figures of table I the average increase of the nuclear radius at various concentrations can be computed; by comparing this with the decrease of the nuclear distance, an answer to this question may be obtained.

These comparisons show that e.g. in 0.05 % LiCl the radius of the animal nuclei have increased, on an average, with $1\frac{1}{2}$ — $1\frac{3}{4}$ μ as compared with the controls; at the same time, the nuclear distances have decreased with 4— $4\frac{1}{2}$ μ . Hence, it appears that at least the animal nuclei undergo a real displacement towards the surface. At the vegetative side, this is less evident; the decrease of the nuclear distances with, on an average, but 1—2 μ may be accounted for by a similar increase in nuclear radius; only in series 4 a real displacement of the vegetative nuclei may have taken place.

This shift in the position of the nuclei towards the outer part of the cells resembles the displacement of the nuclei in neural tube, notochord and somites of *Triton* embryos which LEHMANN and ANDRES (1948) have observed after treatment of the embryos with phenol solutions.

Our results do not prove that LiCl at the 24-cell stage of *Limnaea* influences the polar (animal-vegetative) gradient-field. Rather, the displacement of the nuclei might be explained by a disturbance of an exter-

interior gradient-field determining their position in normal development (cf. RAVEN 1946, p. 424). To be sure, in normal cleavage stage embryos the animal nuclei are further removed from the surface than the vegetative ones; this might be explained as being due to an interference between the extero-interior and animal-vegetative gradient-field, probably through the medium of the unequal distribution of ecto- and endoplasm over the cleavage cells. As table II shows, in Li-treated embryos this difference in "nuclear distances" between animal and vegetative cells has nearly disappeared. Moreover, as stated above, a distinct shift in the nuclei is only indicated in the animal cells, whereas in the vegetative cells the reduction of the nuclear distance may be entirely due to swelling. Hence, the disturbance of the extero-interior gradient is more pronounced at the animal side, and the Li-effect leads to a weakening of the manifestations of the animal-vegetative polarity. Whether this may be interpreted as an actual weakening of the polar gradient-field, is not certain.

Summary

1. Decapsulated eggs of *Limnaea stagnalis* at the 24-cell stage have been treated with distilled water and 0.05 %, 0.1 %, 0.2 % and 0.4 % LiCl solutions.
2. This treatment leads to 1. an increase in the size of the nuclei, 2. a decrease of their distance from the cell surface. Both phenomena are most pronounced in 0.05 % LiCl, and diminish in intensity with increasing concentrations of LiCl; in distilled water they are less pronounced, too.
3. At the animal side an actual shift of the nuclei towards the surface takes place; in the vegetative cells the decrease of the distance between nucleus and cell surface may be accounted for, in most cases, by the increase in nuclear radius.
4. The displacement of the nuclei may be explained by a disturbance of an extero-interior gradient-field, which is most pronounced at the animal side.

REFERENCES.

- GROOT, A. P. DE, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 588, 752 (1948).
- LEHMAN, F. E. and G. ANDRES, Rev. Suisse de Zool., **55**, 280 (1948).
- RAVEN, CHR. P., Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 856 (1942).
- _____, Arch. Néerl. Zool., **7**, 353 (1946).
- _____, Acta Anat. (Basel), **4**, 239 (1947).
- RAVEN, CHR. P., J. C. KLOEK, E. J. KUIPER and D. J. DE JONG, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 584 (1947).
- RAVEN, CHR. P. and H. KLOMP, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 101 (1946).
- RAVEN, CHR. P. and A. H. G. C. RIJVEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 427 (1948).

Mathematics. — *On the symbolical method.* II. By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 27, 1948.)

§ 5. Six points on a conic.

It is generally accepted that PASCAL discovered his theorem before the year 1644. Not before 1806 BRIANCHON found the dual theorem and afterwards lead the work of STEINER (1828), KIRKMAN (1849), CAYLEY-SALMON (1849) to the discovery of the VERONESE-properties of Hexagramma mysticum by VERONESE (1877 — Atti Linc., 3—I, 649—703 seq.). Not only the long period elapsed during this development but also the fact that prominent mathematicians as STEINER, HESSE, SCHRÖTER made statements and suggestions which were proved to be false, indicates that from a geometrical point of view these VERONESE-properties do not belong to the simplest ones. CREMONA's discovery of the relation between Hexagramma mysticum and the lines on a cubic surface with a conical point lead to a more-dimensional treatment of the problem.

In the following it will be shown, that, the general method being founded, and ternary geometry being reduced to binary geometry, the VERONESE-properties are indeed the simplest theorems one can write down, the PASCAL-theorem being proved.

I. PASCAL's theorem.

$$(lp)(km)(ni)P_{ik,mn} + (mi)(ln)(pk)P_{kl,np} + (nk)(mp)(il)P_{lm,pi} \equiv 0$$

i, k, l, m, n, p being six parameters of points on a conic.

Proof.

$$\begin{aligned} & (lp)(km)a_ia_n + (lp)(in)a_k a_m + (mi)(ln)a_k a_p + (mi)(kp)a_l a_n + \\ & \quad + (nk)(mp)a_l a_i + (nk)(li)a_m a_p \equiv \\ & - (il)(km)a_p a_n + (mp)(in)a_k a_l - (ml)(in)a_k a_p + (mi)(ln)a_k a_p + \\ & + (ki)(mp)a_l a_n + (nk)(mp)a_l a_i + (nk)(li)a_m a_p \equiv \\ & \quad [(mi)(ln) - (ml)(in) - (il)(mn)] a_p a_k \equiv 0. \end{aligned}$$

In virtue of the relation $P_{ik,mn} \equiv -P_{mn,ik}$ this relation is invariant under cyclical permutation and reversal of the order $iklmnp$. So we have 60 PASCAL-lines p_{iklmnp} .

$$\text{II. } p_{iklmnp} = (li)(pk)a_n a_m - (ln)(pm)a_l a_k = 0.$$

Proof: Breaking up the ternary brackets into binary cycles we have from

$$(ikl)(knlp)(mnx) + (nkl)(mnp)(ikx) = 0,$$

dividing by $(ik)(kl)(kn)(np)(mn)$ the equation stated in the theorem. From this it follows immediately:

$$p_{iklmnp} \equiv p_{pnmlki} \equiv -p_{kilmnp}.$$

Transforming a_m in (li) and a_k in (ln) we obtain

$$p_{iklmnp} \equiv (m i) (p k) a_n a_l + (p l) (k m) a_n a_l + (m p) (k n) a_l a_l,$$

giving the coordinates on the fundamental triangle i, l, n etc.

III. The three PASCAL-lines

$$p_{imknlp}, \quad p_{inkplm}, \quad p_{ipkmnl}$$

are concurrent in the STEINER-point

$$S_{ikl,mnp} \equiv \Delta_{ikl} - \Delta_{mnp} = 0.$$

$$\begin{aligned} \text{Proof: } S_{ikl,mnp} &\equiv \{ik\} + \{kl\} + \{li\} + \{nm\} + \{pn\} + \{mp\} \equiv \\ &\equiv P_{in,mk} + P_{kp,pl} + P_{lm,pi} \equiv \\ &\equiv P_{ip,nk} + P_{km,pl} + P_{ln,mi} \equiv \\ &\equiv P_{lm,pk} + P_{kn,ml} + P_{lp,ni}. \end{aligned}$$

The first three PASCAL-points are on p_{ipkmnl} , the second three on p_{imknlp} and the third three on p_{inkplm} .

As $S_{ikl,mnp} \equiv -S_{mnp,ikl}$ and because of the cyclical symmetry in ikl and mnp , there are 20 STEINER-points.

A triangle formed by $[ik] = 0$, $[lm] = 0$, $[np] = 0$ is called a VERONESE-triangle (ik, lm, np) if all i, k, l, m, n, p are different. There are 15 VERONESE-triangles.

IV. The 15 VERONESE-triangles are in 20 triples perspective with a STEINER-point as centre.

Proof: We can split up $S_{ikl,mnp}$ in three other ways:

$$S_{ikl,mnp} \equiv P_{in,mk} + P_{km,pl} + P_{lp,ni} \equiv \dots \quad (1)$$

$$\equiv P_{lm,pi} + P_{ip,nk} + P_{nk,lm} \equiv \dots \quad (2)$$

$$\equiv P_{kp,pl} + P_{ln,mi} + P_{lm,pk} \equiv \dots \quad (3)$$

which gives us the vertices of the VERONESE-triangles

$$(in, mk, pl), \quad (lm, ip, nk) \quad \text{and} \quad (kp, ln, mi).$$

Intersecting corresponding sides of:

$$(1) \text{ and } (2): \quad P_{mk,ip}, \quad P_{lp,kn}, \quad P_{nl,mi} \quad \text{on } p_{lpilmkn},$$

$$(1) \text{ and } (3): \quad P_{km,pl}, \quad P_{lp,im}, \quad P_{in,kp} \quad \text{on } p_{intpkm},$$

$$(2) \text{ and } (3): \quad P_{lp,ni}, \quad P_{nk,im}, \quad P_{ml,pk} \quad \text{on } p_{imlnkp}.$$

these three PASCAL-lines are the axis of perspectivity of the pairs of triangles and concurrent in $S_{ikl,mnp}$, the conjugate STEINER-point of $S_{ikl,mnp}$.

V. Two conjugate STEINER-points form the JACOBIAN (common harmonical pair) of Δ_{ikl} , Δ_{mnp} and the points of intersection of the line joining them with the conic.

$$\text{P r o o f : } S \equiv S_{ikl, mnp} \equiv \Delta_{ikl} - \Delta_{mnp} = 0$$

$$S_1 \equiv S_{ikl, mnp} \equiv \Delta_{ikl} + \Delta_{mnp} = 0$$

from which the first part is evident.

$(\Omega' S)(\Omega' S_1) \equiv (\Omega' \Delta_{ikl})^2 - (\Omega' \Delta_{mnp})^2 = 0$, as $(\Omega' \Delta_{ikl})^2$ is independent of the indices i, k, l , from which the second half follows.

VI. The linear relation between the three in $S_{iln, kmp}$ concurrent PASCAL-lines is

$$p_{iklmnp} + p_{ilmnpk} + p_{iplknm} \equiv 0.$$

$$\text{P r o o f : } p_{iklmnp} \equiv (li)(pk)a_n a_m - (ln)(pm)a_i a_k$$

$$p_{ilmnpk} \equiv (li)(km)a_n a_p - (ln)(kp)a_i a_m$$

$$p_{iplknm} \equiv (li)(mp)a_n a_k - (ln)(mk)a_i a_p$$

Adding these three forms we obtain according to the fundamental identity $(pk)a_m \equiv (mk)a_p - (mp)a_k$ etc. a sum $\equiv 0$.

$$\text{VII. } S_{ikl, mnp} \equiv S_{ikp, mn} + S_{int, mnp} + S_{mkl, imp}.$$

P r o o f : Inserting the brackets $\{ik\}$... etc. all terms cancel.

$$\text{VIII. } \langle iln \rangle \langle kmp \rangle S_{iln, kmp} + \langle kln \rangle \langle imp \rangle S_{kln, imp} + \\ + \langle imn \rangle \langle klp \rangle S_{imn, klp} + \langle ilp \rangle \langle kmn \rangle S_{ilp, kmn} \equiv 0.$$

P r o o f : Consider the points

$$S_{imn, klp} \text{ and } \lambda S_{iln, kmp} + \mu S_{kln, imp}.$$

Calculating the linear-factors for the line p_{iklmnp} and the second point the coefficient of λ vanishes, that of μ is, as the 12 terms cancel nearly all

$$2 \frac{\langle imn \rangle \langle kpl \rangle - \langle kmn \rangle \langle ipl \rangle}{(kl)(ip)(mn)},$$

whereas the linear-factor of the first point is

$$2 \frac{\langle ipl \rangle \langle nkm \rangle - \langle lnk \rangle \langle pm \rangle}{(kl)(ip)(mn)}.$$

Again, with p_{iklmnp} we find the coefficient of μ vanishing, that of λ being

$$2 \frac{\langle imn \rangle \langle kpl \rangle - \langle kmn \rangle \langle ipl \rangle}{(il)(kp)(mn)},$$

whereas the first point gives

$$2 \frac{\langle ilp \rangle \langle knm \rangle - \langle iln \rangle \langle m kp \rangle}{(il)(kp)(mn)}.$$

The three points $S_{imn, klp}$, $S_{iln, kmp}$, $S_{kln, lmp}$ are therefore collinear the relation being

$$\begin{aligned} & [\langle ilp \rangle \langle mkn \rangle - \langle iln \rangle \langle mkp \rangle] S_{iln, kmp} + \\ & [-\langle ilp \rangle \langle kmn \rangle - \langle lnk \rangle \langle pmi \rangle] S_{kln, lmp} \equiv \\ & [-\langle imn \rangle \langle klp \rangle - \langle kmn \rangle \langle ipl \rangle] S_{imn, klp} \end{aligned}$$

from which according to theorem VII we have VIII.

q.e.d.

R e m a r k : The four STEINER-points are collinear on a line of STEINER-PLÜCKER $j(ik, lm, np)$.

The process of forming the linear-factors from p and S will be given more in details in deducting the equations of the STEINER-PLÜCKER-lines.

$$\text{IX. } j(ik, lm, np) \equiv (kl)(mn)(pi)p_{klmnp} + (im)(lp)(nk)p_{ilmnk} = 0.$$

P r o o f :

$j(ik, lm, np) \equiv \lambda p_{iklmnp} + \mu p_{ilmnk}$, where $\lambda : \mu$ can be calculated from the condition that this line, through $S_{iln, kmp}$ contains the point $S_{kln, lmp}$ also.

Now

$$\begin{aligned} S_{kln, lmp} & \equiv \{kl\} + \{ln\} + \{nk\} + \{mi\} + \{pm\} + \{ip\} \\ p_{iklmnp} & \equiv (li)(pk)a_n a_m - (ln)(pm)a_i a_k. \end{aligned}$$

The linear-factor of this point and this line is given by the sum of twelve linear-factors. However eight cancel and we are left with

$$\begin{aligned} & \frac{(li)(pk)}{(kl)} [(nk)(ml) + (nl)(mk)] + \frac{(li)(pk)}{(ip)} [(nl)(mp) + (np)(mi)] + \\ & - (pm) [(il)(kn) + (in)(lk)] - (ln) [(ip)(km) + (im)(kp)] \equiv \\ & 2 \frac{(li)(pk)(nk)(ml)}{(kl)} + 2 \frac{(li)(pk)(np)(mi)}{(ip)} - 2(pm)(il)(kn) - 2(ln)(im)(kp) \equiv \\ & 2 \frac{(li)(nk)(mk)(pl)}{(kl)} + 2 \frac{(im)(kp)(ni)(lp)}{(ip)} \equiv \\ & \frac{2}{(ip)(kl)(mn)} [\langle imn \rangle \langle kpl \rangle - \langle kmn \rangle \langle ipl \rangle], \end{aligned}$$

which gives the coefficient of λ in the equation for $\lambda : \mu$ obtained by substituting $S_{kln, lmp}$ in $j(ik, lm, np) = 0$.

The coefficient of μ becomes:

$$\frac{2}{(nk)(mi)(lp)} [\langle klp \rangle \langle inm \rangle - \langle ipl \rangle \langle kmn \rangle],$$

which proves the equation for the STEINER-PLÜCKER-line given above.

From this we have

$$-(ik)(lm)(np) j(ik, lm, np) \equiv \langle iklnmp \rangle p_{iklmnp} + \langle imlpnk \rangle p_{ilmnk}$$

which changes sign under $k \leftrightarrow i$ and is invariant under the interchange of

$ik, lm; ik, np; lm, np$, thus showing that there are only 15 STEINER-PLÜCKER-lines forming a $(20_3, 15_4)$ configuration of three perspective triangles with the STEINER-points.

X. The three PASCAL-lines $p_{kmplin}, p_{mpknli}, p_{pkmln}$ are concurrent in the KIRKMAN-point

$$K_{iklmnp} \equiv \{ik\} + \{kl\} + \{lm\} + \{mn\} + \{np\} + \{pi\}.$$

$$\begin{aligned}\text{P r o o f: } K_{iklmnp} &\equiv P_{im,nk} + P_{kp,il} + P_{ln,pm} \equiv \\ &\equiv P_{il,mk} + P_{kn,pl} + P_{mp,in} \equiv \\ &\equiv P_{in,pk} + P_{km,tl} + P_{lp,im}.\end{aligned}$$

The first three points lie on p_{impknl} , the second three on p_{ilpmkn} , the third three on p_{intpkm} , which proves the theorem.

Evidently

$$K_{iklmnp} \equiv K_{kilmnp} \equiv -K_{pnmlki},$$

so there are only 60 KIRKMAN-points.

On the PASCAL-lines through K_{iklmnp} are six other KIRKMAN-points forming the triangles $K_{mpkln}, K_{klpinm}, K_{lnmipk}$ and $K_{lmpnki}, K_{pnikml}, K_{mikpnl}$. The sides of these triangles are PASCAL-lines; corresponding sides meet in three KIRKMAN-points on p_{iklmnp} . In this way a DESARGUES-configuration $(10_3, 10_3)$ is generated. There are in all six of these decades of KIRKMAN-points and PASCAL-lines.

XI. The linear relation between the PASCAL-lines concurrent in a KIRKMAN-point is

$$p_{kmplin} + p_{pkmln} + p_{mpknli} \equiv 0.$$

$$\begin{aligned}\text{P r o o f: } p_{mpknli} &\equiv (lm)(kp) a_n a_l - (ln)(ki) a_m a_p \\ &- p_{lnimkp} \equiv (ik)(pm) a_n a_l - (il)(pn) a_k a_m \\ & p_{impknl} \equiv (pi)(lm) a_n a_k - (pn)(lk) a_i a_m.\end{aligned}$$

Adding we obtain, carrying out the indicated identical transformations

$$a_m(ki) [(pn)a_l + (lp)a_n - (ln)a_p] \equiv 0.$$

q.e.d.

$$\text{XII. } \langle iklmnp \rangle K_{iklmnp} \equiv \langle klmnp \rangle a_i^2 + \langle ikmnp \rangle a_l^2 + \langle iklmp \rangle a_n^2 \equiv \\ \equiv \langle ilmnp \rangle a_k^2 + \langle iklnp \rangle a_m^2 + \langle iklmn \rangle a_p^2.$$

P r o o f: Dividing by $\langle iklmnp \rangle$ and carrying out the only possible identical transformations we have

$$\frac{(pk)a_i^2}{(pi)(ik)} + \frac{(km)a_l^2}{(kl)(lm)} + \frac{(mp)a_n^2}{(mn)(np)} \equiv \{pi\} + \{ik\} + \{kl\} + \{lm\} + \{mn\} + \{np\}$$

q.e.d.

$$\text{XIII. } K_{iklmnp} + K_{lmpnki} + K_{lpiknm} \equiv 0.$$

P r o o f: Inserting the symbols $\{ik\} \dots$ all terms cancel.

The corresponding KIRKMAN-points are therefore collinear on a CAYLEY-SALMON-line $c_{ltn, kmp}$.

Along the same line is evident:

$$K_{kmp\ln} + K_{pkminl} + K_{mpknll} \equiv 2 S_{kmp, ltn}$$

$$K_{kmp\ln} - K_{pkminl} + K_{mpknll} \equiv 2 (P_{ml, lp} + P_{pl, ml}) \equiv \frac{2(ip)(lm)}{(im)(lp)} \cdot P_{ml, lp}.$$

$$\text{XIV. } \langle iklnmp \rangle K_{iklnmp} + \langle mlpnk \rangle K_{mlpnk} + \langle plknm \rangle K_{plknm} \equiv \\ \equiv \langle inl \rangle \langle kmp \rangle S_{ltn, kmp}.$$

P r o o f :

$$- 2 \langle inl \rangle \langle kmp \rangle S_{ltn, kmp} \equiv \langle kmp \rangle \{ (li)^2 a_n^2 + (ln)^2 a_i^2 + (ni)^2 a_l^2 \} + \\ + \langle iln \rangle \{ (km)^2 a_p^2 + (mp)^2 a_k^2 + (pk)^2 a_m^2 \} \equiv \{ \langle kmp \rangle (ln)^2 + \langle kln \rangle (mp)^2 + \\ + \langle mln \rangle (pk)^2 + \langle pln \rangle (mk)^2 \} a_i^2 + \dots a_l^2 + \dots a_n^2.$$

The left-hand-side is

$$\{ \langle klmnp \rangle + \langle mlpnk \rangle + \langle plknm \rangle \} a_i^2 + \dots a_l^2 + \dots a_n^2,$$

Now we have

$$\begin{aligned} \langle kmp \rangle (ln)^2 + \langle kln \rangle (mp)^2 + \langle mln \rangle (pk)^2 + \langle pln \rangle (mk)^2 &\equiv \\ - \langle kmpln \rangle - \langle kmpnl \rangle - \langle klnmp \rangle - \underline{\langle klnpm \rangle} & \\ - \langle plnmk \rangle - \langle plnkm \rangle - \underline{\langle mlnpk \rangle} - \underline{\langle mlnkp \rangle} &\equiv \\ - 2 [\langle kmpln \rangle + \langle mpkln \rangle + \langle mkpln \rangle], & \end{aligned}$$

as follows inverting the order in the underlined cycles.

On the other hand is

$$\begin{aligned} \langle klmnp \rangle + \langle mlpnk \rangle + \langle plknm \rangle &\equiv \\ - (lm)^2 \langle knp \rangle - (lp)^2 \langle mnk \rangle - (lk)^2 \langle pnm \rangle - \langle kmlnp \rangle - \langle mpink \rangle + \\ - \langle pklnm \rangle &\equiv - [\langle kmpln \rangle + \langle mpkln \rangle + \langle mkpln \rangle] \end{aligned}$$

which proves the identity.

The line of CAYLEY-SALMON $c_{ltn, kmp}$ contains the point $S_{ltn, kmp}$.

Starting with the PASCAL-lines through a KIRKMAN-point or through the STEINER-point on a CAYLEY-SALMON-line we obtain different, equivalent equations.

$$\text{XV. } c_{ltn, kmp} \equiv \langle kmpln \rangle p_{kmpln} + \langle mpknl \rangle p_{mpknl} + \\ + \langle pkminl \rangle p_{pkminl}.$$

P r o o f : The equation is of the form

$$\lambda p_{kmpln} + \mu p_{pkminl} = 0,$$

where $\lambda : \mu$ can be obtained by substituting the coordinates of K_{mlpnk} .

Now with

$$\begin{aligned} p_{pkminl} &\equiv (mp)(lk) a_n a_l - (mn)(li) a_p a_k \\ \langle imlpnk \rangle K_{imnplk} &\equiv \langle mlpnk \rangle a_i^2 + \langle impnk \rangle a_i^2 + \langle imlpk \rangle a_n^2 \end{aligned}$$

we have:

$$\begin{aligned} (p'_{pkminl} K_{imnplk}) &\equiv \\ \frac{2(li)(ki)(pn)(nk)}{\langle imlpnk \rangle} [- (mp)(lk)(im)(mp)(nl) - (mn)(ml)(lp)(km)(pi) + & \\ - (mn)(im)(mp)(pl)(kl) + (mn)(im)(ml)(lp)(pk)] &\equiv \\ \frac{2(li)(ki)(pn)(nk)(ml)(mp)}{\langle imlpnk \rangle} [- (np)(lk)(im) + (mn)(lp)(ki)] &\equiv \\ 2 \frac{\langle liknmp \rangle - \langle limpnk \rangle}{(im)(lp)(nk)} = 2 \frac{\langle limpkn \rangle - \langle linkmp \rangle}{(im)(lp)(nk)}. & \end{aligned}$$

Again

$$(p'_{kmplin} K_{imnplk}) \equiv 2 \frac{\langle minlpk \rangle - \langle milnk \rangle}{(im)(lp)(nk)}.$$

So the CAYLEY-SALMON-line is

$$\begin{aligned} -[\langle limpkn \rangle - \langle linkmp \rangle] p_{kmplin} + [\langle minlpk \rangle - \langle milnk \rangle] p_{pkminl} &\equiv \\ \equiv \langle kmplin \rangle p_{kmplin} + \langle pkminl \rangle p_{pkminl} - \langle mpknli \rangle [p_{kmplin} + p_{pkminl}] & \end{aligned}$$

from which the theorem follows.

Evidently $c_{iln, kmp} \equiv c_{lni, mpk} \equiv +c_{kmp, lni}$.

Moreover; as is clear from the geometrical point of view

XVI. $c_{iln, kmp} - c_{iln, mpk} \equiv 0$.

P r o o f: Inserting the p_{kmplin}, \dots and summing up every two terms containing the same a -product the left-hand-side is

$$\begin{aligned} \langle kmp \rangle \langle lin \rangle [(nm)(pk) a_l a_l + (mp)(lk) a_n a_l + (ip)(km) a_n a_l] + \\ - \langle kmp \rangle \langle lin \rangle [(pi)(nl) a_k a_m + (kl)(in) a_m a_p + (mn)(li) a_p a_k] \end{aligned}$$

$$\begin{aligned} \text{Now: } (ip)(km) a_n a_l &\equiv (ip)(nm) a_k a_l - (ip)(nk) a_m a_l \equiv \\ (lp)(nm) a_k a_l - (li)(nm) a_k a_p - (ip)(lk) a_m a_n + (ip)(ln) a_m a_k &\equiv \\ - (pk)(nm) a_l a_i - (kl)(nm) a_p a_l - (np)(lk) a_m a_i + (in)(kl) a_m a_p + & \\ + (li)(mn) a_k a_p + (pi)(nl) a_m a_k &\equiv - (pk)(nm) a_l a_i - (kl)(pm) a_n a_l + \\ + (in)(kl) a_m a_p + (li)(mn) a_k a_p + (pi)(nl) a_m a_k & \end{aligned}$$

so all terms in $c_{iln, kmp} - c_{iln, mpk}$ cancel.

There are 20 Cayley-Salmon-lines.

XVII. $c_{iln, kmp} + c_{kln, imp} + c_{imn, klp} + c_{ilp, kmn} \equiv 0$.

P r o o f: Inserting the forms all terms cancel because of the identities

$$c_{iln, kmp} \equiv c_{lin, kmp} \quad , \quad p_{iklmnp} \equiv p_{pnmlki} \equiv -p_{klmipi}.$$

$$\text{XVIII. } c_{iln,kmp} \equiv \langle iknmlp \rangle p_{iknmlp} + \langle imnplk \rangle p_{imnplk} + \langle ipnklm \rangle p_{ipnklm}.$$

P r o o f: Substituting K_{iklmnp} in $\lambda p_{iknmlp} + \mu p_{imnplk} = 0$ we find for the coefficient of λ at first

$$\frac{2(ni)(pk)(mi)(lk)}{(ik)} + \frac{2(ni)(pk)(mi)(lp)}{(pi)} - \frac{2(nl)(pm)(kl)(im)}{(lm)} + \\ - \frac{2(nl)(pm)(kn)(im)}{(mn)},$$

as all other terms cancel. Multiplying by $\langle iklnmp \rangle$ this form contains a factor $-2[(ik)(lm)(np) + (ip)(lk)(mn)]$ the remaining factor being

$$\langle imnplk \rangle - \langle imlnp \rangle.$$

Again, the coefficient of μ is found to be

$$2(ni)(km) \left[\frac{(lk)(pi)}{(ik)} + \frac{(lm)(pn)}{(mn)} \right] - 2(nl)(kp) \left[\frac{(ik)(ml)}{(kl)} + \frac{(ip)(mn)}{(np)} \right],$$

which multiplied by $\langle iklnmp \rangle$ contains the same factor as the coefficient of λ the remaining factor being

$$\langle imlnp \rangle - \langle iknmlp \rangle.$$

Therefore the form $c_{iln,kmp}$ is, apart from a constant factor ϱ equal to the right-hand-side. Specialising $l \equiv m, p \equiv k$ we find

$$\varrho \langle ilnk lk \rangle p_{ilnk lk} \equiv \langle klklin \rangle p_{klklin}$$

which gives immediately $\varrho = 1$. q.e.d.

As in the case of the STEINER-points a second linear relation between the $c_{iln,kmp}$ could be derived in an analogous way. Leaving apart for the moment the question whether the equation Σ is irreducible we have directly:

XIX. The CAYLEY-SALMON-lines $c_{iln,mpk}, c_{ilk,nmp}, c_{ipn,mik}, c_{mpn,ilk}$ are concurrent in the point

$$\Sigma(im, nk, lp) \equiv S_{int,kmp} + K_{iklmnp}$$

P r o o f: Splitting up in { }-brackets we have

$$S_{int,kmp} + K_{iklmnp} \equiv S_{ikl,nmp} + K_{intmfp} \equiv S_{inp,mik} + K_{ikpmnl} \equiv \\ \equiv S_{mnl,lpk} + K_{lnpmkl} \equiv \{in\} + \{nl\} + \{li\} + \{mk\} + \{pm\} + \{kp\} + \\ + \{ik\} + \{kl\} + \{lm\} + \{mn\} + \{np\} + \{pi\}$$

q.e.d.

$\Sigma(im, nk, lp)$ is written in four ways as a sum of two points on each of the CAYLEY-SALMON-lines

Evidently the Σ, c form a $(15_4, 20_3)$ -configuration of two perspective tetragons c.q. three perspective tri-sides.

Writing

$$\Sigma(im, nk, lp) \equiv P_{im,kn} + P_{nk,pl} + P_{lp,mi} + P_{lm,nk} + P_{kn,pl} + P_{lp,im}$$

it is evident, that Σ is invariant under the interchange of two elements in a pair, changes sign if two pairs are commuted.

$$\text{XX. } \Sigma(im, lk, np) + \Sigma(ik, lp, nm) + \Sigma(ip, lm, nk) \equiv 3 S_{iln, kmp}$$

P r o o f: Splitting up in { }-brackets all terms cancel.

$$\text{XXI. } K_{iklmnp} \equiv P_{im, nk} + P_{kn, pl} + P_{lp, tm}$$

P r o o f: Evident; see XIX.

To each KIRKMAN-point corresponds one VERONESE-triangle. Every VERONESE-triangle corresponds to four KIRKMAN-points viz. (im, nk, lp) corresponds to $K_{iklmnp}, K_{mklinp}, K_{nlkimp}, K_{lkipmn}$, obtained from one of these by interchanging a pair of opposite indices. If we substitute the coordinates of K_{iklmnp} in $\lambda a_{im} + \mu a_{km} = 0$ we obtain as nearly all terms cancel:

$$\frac{(il)(ln)(km)}{(kl)(lm)} \lambda - \frac{(in)(mp)(pk)}{(np)(pi)} \mu = 0,$$

from which follows, because of the symmetry in l, p :

$$\text{XXII. } P_{in, km}, K_{iklmnp}, K_{lkipmn} \text{ are collinear on the VERONESE-line}$$

$$v(ik, mn) \equiv (in)(lk)(lm)(pk)(pm) a_i a_n + (km)(li)(ln)(pi)(pn) a_k a_m = 0.$$

Evidently: $v(ik, mn) \equiv v(ki, nm) \equiv -v(mn, ik) \equiv -v(nm, ki)$, so there are 90 VERONESE-lines, the sides of the 15 tetragons of KIRKMAN-points, corresponding to the 15 VERONESE-triangles.

XXIII. The PASCAL-point $P_{in, km}$ is incident with two VERONESE-lines, which are harmonical to the sides of the hexagon through $P_{in, km}$.

P r o o f: An interchange of k, m or i, n changes + in — in the equation $v(ik, mn) = 0$. q.e.d.

On the six sides of the K -tetragon are the PASCAL-points:

$$P_{in, km}, P_{il, mp}, P_{ln, kp}, P_{ik, mn}, P_{kt, np}, P_{lm, pi}.$$

The last three points are on p_{iklmnp} , whereas the other three are the vertices of a triangle with the sides $p_{ikpmnl}, p_{inlmkp}, p_{mklinp}$. So the K -tetragon and the corresponding p -tetragram form a DESARGUES ($10_3, 10_3$) configuration.

In all fifteen of these configurations can be formed from the sixty PASCAL-lines and KIRKMAN-points, and the 90 VERONESE-lines.

$$\text{XXIV. } v(ik, mn) + v(np, lk) + v(lm, ip) \equiv 0.$$

P r o o f: Evident.

These three lines are concurrent in a point, the equation of which can be found by intersecting $v(ik, mn) = 0$ and $v(np, lk) = 0$ which according to the fundamental formulae, dividing by $- <kmp><iln>$ is

$$-\langle imnplk \rangle P_{im, pl} + \langle imlknip \rangle P_{im, kn} + \langle iknmlp \rangle P_{nk, pl} = 0.$$

As the left-hand-side is invariant under cyclical permutation and reversal

of the order $iklmnp$ there are 60 of these points. We denote the left-hand-side by $\langle iklmnp \rangle Z_{iklmnp}$ and we obtain dividing by $\langle iklmnp \rangle$

$$\begin{aligned} & -\frac{(im)(pl)}{(lm)(pi)} [\{il\} + \{mp\}] + \frac{(im)(kn)}{(ik)(mn)} [\{in\} + \{mk\}] + \\ & + \frac{(kn)(lp)}{(kl)(np)} [\{nl\} + \{kp\}] \equiv \end{aligned}$$

$$\begin{aligned} & \{li\} + \{pm\} + \{ml\} + \{ip\} + \{in\} + \{mk\} + \{nm\} + \{ki\} + \{nl\} + \\ & + \{kp\} + \{lk\} + \{pn\} \equiv S_{int, kmp} - K_{iklmnp}. \text{ So we have:} \end{aligned}$$

$$\text{XXV. } Z_{iklmnp} \equiv S_{int, kmp} - K_{iklmnp}.$$

XXVI. The six VERONESE-lines through the vertices of a VERONESE-triangle are by three concurrent in four points Z .

Proof: The equations of the lines through the vertices of (ik, lm, np) are

$$\begin{aligned} & (nl)(nm)(pl)(pm)(ik) a_l a_k + \varepsilon_1 (ni)(nk)(pi)(pk)(lm) a_l a_m = 0 \\ & (in)(ip)(kn)(kp)(lm) a_l a_m + \varepsilon_2 (il)(im)(kl)(km)(np) a_n a_p = 0 \\ & \varepsilon_3 (nl)(nm)(pl)(pm)(ik) a_l a_k + (il)(im)(kl)(km)(np) a_n a_p = 0, \end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are ± 1 .

The determinant of the coefficients of $a_l a_k, a_l a_m, a_n a_p$ is

$$D(1 + \varepsilon_1 \varepsilon_2 \varepsilon_3),$$

where D is the discriminant $(ik)(il) \dots (ip)(kl) \dots (np)$.

The four points of intersection are $Z_{imnklp}, Z_{ilpkmn}, Z_{impkln}, Z_{ilnkmp}$ of which each can be obtained from the others by interchanging opposite indices.

XXVII. The three points $Z_{iklmnp}, Z_{implnk}, Z_{iplknm}$ are incident with the CAYLEY-SALMON-line $c_{iln, kmp}$.

Proof: We have to show that $v(ik, nm), v(np, lk), c_{iln, kmp}$ are concurrent.

Now $c_{iln, kmp}$ can be written:

$$\begin{aligned} & \langle kmplin \rangle [(mn)(ik) a_l a_p - (ml)(ip) a_n a_k] + \\ & \langle mpknli \rangle [(pi)(lm) a_n a_k - (pn)(lk) a_l a_m] + \\ & \langle pkminl \rangle [(kl)(np) a_l a_m - (ki)(nm) a_l a_p] = 0. \end{aligned}$$

The determinant of the three equations in $a_l a_m, a_k a_n, a_l a_p$ given by $v(ik, nm) = 0, v(np, lk) = 0, c_{iln, kmp} = 0$ is therefore apart from a factor

$(lk)(pn)(lm)(pi)(ik)(mn) \cdot (im)(ln)(pk) \cdot (kn)(li)(pm) \cdot (pl)(im)(nk)$ equal to

1	1	0
0	-1	1
$[(km)(in)(lp) + (mp)(kn)(li)] - [(pk)(nl)(im) + (km)(pl)(in)] - [(mp)(li)(nk) + (pk)(mi)(nl)]$		

which vanishes.

$$\text{XXVIII. } Z_{iklmnp} + Z_{imlpnk} + Z_{iplknm} \equiv 3 S_{int,kmp}.$$

P r o o f: The left-hand-side is

$$3 S_{int,kmp} - [K_{iklmnp} + K_{imlpnk} + K_{iplknm}]$$

and the form between brackets vanishes. q.e.d.

$$\text{XXIX. } Z_{kmplin} + Z_{pkminl} + Z_{mpknli} \equiv S_{iln,kmp}.$$

XXX. The STEINER-point and the KIRKMAN-points form harmonical groups with the points Z and the SALMON-points on a CAYLEY-SALMON-line.

P r o o f: We have the relations:

$$Z_{iklmnp} = S_{int,kmp} - K_{iklmnp}$$

$$\Sigma(im, nk, lp) = S_{int,kmp} + K_{iklmnp}.$$

q.e.d.

$$\text{XXXI. } Z_{kmplin}, Z_{pkminl}, Z_{mpknli} \text{ are collinear on } z_{iklmnp}.$$

P r o o f: We have the relations:

$$Z_{kmplin} = S_{kip,mtn} - K_{kmplin} : \quad \Sigma(im, lk, np) = S_{kip,mtn} + K_{kmplin},$$

$$Z_{pkminl} = S_{pnm,kll} - K_{pkminl} : \quad \Sigma(tp, lm, nk) = S_{pnm,kll} + K_{pkminl},$$

$$Z_{mpknli} = S_{mlk,pnl} - K_{mpknli} : \quad \Sigma(ik, lp, nm) = S_{mlk,pnl} + K_{mpknli}.$$

Now the STEINER-points are on the line of STEINER-PLÜCKER $j(im, lp, nk)$, the KIRKMAN-points are on p_{iklmnp} and the SALMON-points on $c_{iln,mkp}$. So the Z -points are collinear on a line through $S_{iln,kmp}$ the fourth harmonical of p_{iklmnp} to $j(im, lp, nk)$ and $c_{iln,mkp}$.

$$\text{XXXII. } z_{iklmnp} \equiv 2 \langle iklmnp \rangle p_{iklmnp} + \langle imlpnk \rangle p_{imlpnk} + \\ + \langle iplknm \rangle p_{iplknm}.$$

P r o o f:

$$c_{int,kmp} \equiv \langle iklmnp \rangle p_{iklmnp} + \langle imlpnk \rangle p_{imlpnk} + \langle iplknm \rangle p_{iplknm};$$

$$-(im)(lp)(nk)j(im, lp, nk) \equiv \langle imlpnk \rangle p_{imlpnk} + \langle iplknm \rangle p_{iplknm} \\ \equiv c_{int,kmp} - \langle iklmnp \rangle p_{iklmnp}.$$

$$\text{So } z_{iklmnp} \equiv c_{int,kmp} + \langle iklmnp \rangle p_{iklmnp}. \quad \text{q.e.d.}$$

We have $z_{iklmnp} + z_{imlpnk} + z_{iplknm} \equiv 4 c_{int,kmp}$, as is evident from the relation just deduced.

XXXIII. The linear relation between z_{kmplin} , z_{pkminl} , z_{mpknli} , which are concurrent in Z_{iklmnp} is

$$z_{kmplin} + z_{pkminl} + z_{mpknli} \equiv 0.$$

P r o o f: $z_{kmplin} = c_{kip,mtn} + \langle kmplin \rangle p_{kmplin}$ etc. gives by addition for the left-hand-side

$$c_{kip,mtn} + c_{pnm,kll} + c_{mlk,pnl} + c_{iln,mkp} \equiv 0.$$

(To be continued.)

Mathematics. — *Sequences of points on a circle.* By N. G. DE BRUIJN and P. ERDÖS. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of December 18, 1948.)

1. Introduction. We consider sequences $\{a\}$ of points a_1, a_2, a_3, \dots on a circle with radius $1/2\pi$, in other words numbers mod 1. The numbers a_1, a_2, \dots, a_n define n intervals with total length 1; denote by $M_n^1(a)$ and $m_n^1(a)$ the largest and the smallest length. Clearly

$$n M_n^1(a) \geq 1 \geq n m_n^1(a).$$

Analogously $M_n^r(a)$ and $m_n^r(a)$ denote the maximum and minimum length of the sum of r consecutive intervals, so that $n M_n^r(a) \geq r \geq n m_n^r(a)$. We put

$$\limsup_{n \rightarrow \infty} n M_n^r(a) = \Lambda_r(a)$$

$$\liminf_{n \rightarrow \infty} n m_n^r(a) = \lambda_r(a)$$

$$\limsup_{n \rightarrow \infty} M_n^r(a) / m_n^r(a) = \mu_r(a)$$

and

$$\Lambda_r = \text{g.l.b. } \Lambda_r(a), \quad \lambda_r = \text{l.u.b. } \lambda_r(a), \quad \mu_r = \text{g.l.b. } \mu_r(a).$$

We are able to determine

$$\Lambda_1 = 1/\log 2, \quad \lambda_1 = 1/\log 4, \quad \mu_1 = 2.$$

The problem of $\Lambda_r, \lambda_r, \mu_r$ is closely related to a problem concerning "just distributions" solved by Mrs VAN AARDENNE-EHRENFEST¹). All we can prove is that $\mu_r \geq 1 + 1/r$ (and analogous inequalities for Λ_r and λ_r); we conjecture that $r(\mu_r - 1)$ is unbounded. From this the theorem of Mrs VAN AARDENNE-EHRENFEST would follow.

2. A sequence which gives the best possible values of $\Lambda_1(a), \lambda_1(a), \mu_1(a)$. Take $a_k = \log(2k-1)$, reduced mod 1. We show that a_1, \dots, a_n occur in the following order

$${}^2\log n, {}^2\log(n+1), \dots, {}^2\log(2n-1). \quad \dots \quad (2.1)$$

Namely, no two of the a_k 's and no two of the numbers (2.1) are congruent mod 1, but each number in (2.1) is congruent to just one a_k .

It follows from (2.1) that the lengths of the intervals defined by a_1, \dots, a_n are

$${}^2\log \frac{n+1}{n}, {}^2\log \frac{n+2}{n+1}, \dots, {}^2\log \frac{2n-1}{2n-2}, {}^2\log \frac{2n}{2n-1}.$$

¹) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam 48, 266—271 (1945) = Indagationes Mathematicae, 7, 71—76 (1946).

and so

$$n M_n^1(a) = \frac{n \log \left(1 + \frac{1}{n}\right)}{\log 2}, \quad n m_n^1(a) = \frac{n \log \left(1 - \frac{1}{2n}\right)^{-1}}{\log 2}.$$

For $n \rightarrow \infty$, $n M_n^1(a)$ increases to the limit $1/\log 2$; $n m_n^1(a)$ decreases to the limit $1/\log 4$; $M_n^1(a)/m_n^1(a)$ increases to the limit 2. It follows that $\Lambda_1(a) = 1/\log 2$, $\lambda_1(a) = 1/\log 4$, $\mu_1(a) = 2$.

3. Lower bound for $\Lambda_r(a)$.

Let $\{a\}$ be a sequence, n a natural number, and suppose that ϱ is such that

$$k M_n^1(a) < \varrho. \quad (n \leq k < 2n) \dots \dots \quad (3.1)$$

Let the intervals determined by a_1, \dots, a_n be I_1, \dots, I_n , arranged in descending order of length. Denote the length of I_j by a_j ; so that

$$a_1 \geq a_2 \geq \dots \geq a_n; \quad a_1 + \dots + a_n = 1. \quad \dots \quad (3.2)$$

Now put in the points $a_{n+1}, a_{n+2}, \dots, a_{2n-1}$. Since any point "destroys" one I at most, there remains at least one interval of length $\geq a_p$ undisturbed after $a_{n+1}, \dots, a_{n+p-1}$ have been put in ($1 \leq p \leq n$). Hence

$$M_n^1(a) \geq a_1, \quad M_{n+1}^1(a) \geq a_2, \dots, \quad M_{2n-1}^1(a) \geq a_n;$$

consequently, by (3.1) and (3.2),

$$\varrho \left(\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1} \right) > 1.$$

It follows that for at least one k ($n \leq k < 2n$) we have

$$k M_k^1(a) \geq \left(\frac{1}{n} + \dots + \frac{1}{2n-1} \right)^{-1} = \sigma_n.$$

We have $\sigma_n < 1/\log 2$, $\sigma_n \rightarrow 1/\log 2$, and so $\Lambda_1(a) \geq 1/\log 2$. This holds for any $\{a\}$; the lower bound is attained for the sequence of section 2.

Similarly we can prove that for at least one k ($r n \leq k < (r+1)n$) we have

$$k M_k^r(a) \geq \left(\frac{1}{rn} + \frac{1}{rn+1} + \dots + \frac{1}{rn+n-1} \right)^{-1}$$

and so

$$\Lambda_r(a) \geq 1/\log \left(1 + \frac{1}{r}\right) > r.$$

4. Upper bound for $\lambda_r(a)$ ²⁾.

Let $\{a\}$ be a sequence, n a natural number, and suppose that ϱ is such that

$$k m_k^1(a) > \varrho \quad (n < k \leq 2n). \quad \dots \quad (4.1)$$

²⁾ The proof presented in this section was found by Mrs. VAN AARDENNE-EHRENFEST independently.

Let $a_{k_1}, a_{k_2}, \dots, a_{k_{2n}}$ be the cyclic order of the points a_1, \dots, a_{2n} on the circle (k_1, \dots, k_{2n} is a permutation of $1, \dots, 2n$); put $k_{2n+1} = k_1$. If $k_i^* = \max(k_i, k_{i+1}, n+1)$, then the interval $a_{k_i}, a_{k_{i+1}}$ is one of the intervals determined by $a_1, \dots, a_{k_i^*}$. It follows that its length is less than ϱ/k_i^* . Hence

$$1 > \varrho \sum_{i=1}^{2n} 1/k_i^*, \quad \dots, \quad (4.2)$$

We have $n < k_i^* \leq 2n$, and any k ($n+1 < k \leq 2n$) occurs ε_k times as a k^* ; $\varepsilon_k = 0, 1$ or 2 . It follows that

$$\sum_{i=1}^{2n} 1/k_i^* = \sum_{n+1}^{2n} \frac{2}{k} + \sum_{n+2}^{2n} (2 - \varepsilon_k) \left\{ \frac{1}{n+1} - \frac{1}{k} \right\} \geq \sum_{n+1}^{2n} \frac{2}{k}.$$

Finally, by (4.1) and (4.2) we infer that at least for one ($n < k \leq 2n$) we have

$$k m_k^1(a) \leq \left(\frac{2}{n+1} + \frac{2}{n+2} + \dots + \frac{2}{2n} \right)^{-1} = \tau_n.$$

We have $\tau_n > 1/\log 4$, $\tau_n \rightarrow 1/\log 4$, and so $\lambda_1(a) \leq 1/\log 4$. The example of section 2 again shows that $1/\log 4$ is best possible.

Similarly we can show that for at least one k ($r n < k \leq (r+1)n$) we have

$$k m_k^r(a) \leq r \left(\frac{r+1}{nr+1} + \dots + \frac{r+1}{nr+n-1} \right)^{-1}$$

and so

$$\lambda_r(a) \leq \frac{r}{r+1} \log \left(1 + \frac{1}{r} \right) < r. \quad \dots, \quad (4.3)$$

5. Lower bound for μ_r .

Let $\{a\}$ be a sequence. We first prove that, for $r \geq 1$, $n \geq 1$ we have

$$M_n^r(a) / m_{n+1}^r(a) \geq 1 + \frac{1}{r}. \quad \dots, \quad (5.1)$$

We first suppose that $r > 1$. Let I_1, I_2, \dots, I_n be the intervals of the n -th stage, i.e. the intervals determined by a_1, \dots, a_n . Let I_{k_0} be the one into which a_{n+1} falls, and let

$$I_{k_{-r+1}}, I_{k_{-r+2}}, \dots, I_{k_0}, I_{k_1}, \dots, I_{k_{r-1}} \quad \dots, \quad (5.2)$$

be consecutive on the circle³⁾.

Put $M = M_n^r(a)$, $m = m_{n+1}^r(a)$ and denote by M_1 the maximum length of the sum of r consecutive intervals from the set (5.2). Denote the length of I_{k_i} by β_i . Let γ_1 and γ_2 be the lengths of the parts into which I_{k_0} is divided by a_{n+1} .

³⁾ If $2r-1 > n$ the k_i are not all different.

Clearly at least one of the numbers $\beta_{-r+1}, \dots, \beta_{-1}, \beta_1, \dots, \beta_{r-1}$ (β_j say) is $\geq (M_1 - \beta_0)/(r-1)$; we may suppose that $j > 0$. Now we have

$$m \leq \beta_{j-r+1} + \dots + \beta_{-1} + \gamma_1 + \gamma_2 + \dots + \beta_{j-1}$$

and hence

$$m \leq M_1 - \beta_j \leq \frac{r-2}{r-1} M_1 + \frac{\beta_0}{r-1}. \quad \dots \quad (5.3)$$

On the other hand it follows from

$$m \leq \gamma_2 + \beta_1 + \dots + \beta_{r-1} \leq M_1 - \gamma_1$$

$$m \leq \beta_{-r+1} + \dots + \beta_{-1} + \gamma_1 \leq M_1 - \gamma_2$$

that

$$m \leq M_1 - \frac{1}{2} \beta_0. \quad \dots \quad (5.4)$$

Trivially we have $M_1 \leq M$. If $\beta_0 \leq 2 M_1 / (r+1)$ we infer $m \leq M_1 r / (1+r) \leq M r / (1+r)$ from (5.3); if $\beta_0 \geq 2 M_1 / (r+1)$ we deduce the same result from (5.4). This proves (5.1) for $r > 1$.

If $r = 1$, (5.1) immediately follows from

$$m \leq \min(\gamma_1, \gamma_2) \leq \frac{1}{2} \beta_0 \leq \frac{1}{2} M.$$

Now suppose that n is a natural number and that for $n r \leq k \leq n(r+1)$ we have

$$M_k^r(a)/m_k^r(a) < \left(1 + \frac{1}{r}\right) \left(1 + \frac{1}{k}\right)^2. \quad \dots \quad (5.5)$$

It follows, by (5.1) that

$$m_{k+1}^r/m_k^r < k^2/(k+1)^2 \quad (n r \leq k < n r + n)$$

and also

$$m_{rn+n}^r/m_{rn}^r < r^2/(1+r)^2. \quad \dots \quad (5.6)$$

Trivially we have $m_{rn}^r \leq 1/n$; on the other hand, by (5.5)

$$m_{rn+n}^r > \frac{r}{1+r} M_{rn+n}^r \geq \frac{r}{1+r} \cdot \frac{r}{rn+n-1} \geq \frac{r^2}{(r+1)^2} \cdot \frac{1}{n}.$$

This contradicts (5.6). Hence for at least one k ($n r \leq k \leq n r + n$) (5.5) is not true. It follows that

$$\mu_r \geq 1 + \frac{1}{r}. \quad \dots \quad (5.7)$$

6. The inequalities (3.3), (4.3) and (5.7) are probably not best possible if $r \geq 2$. We conjecture that the expressions

$$r(\lambda_r - 1), \quad r(1 - \lambda_r), \quad r(\mu_r - 1)$$

tend to infinity if $r \rightarrow \infty$.

We owe some useful remarks to Mrs. T. VAN AARDENNE-EHRENFEST and Mr. J. KOREVAAR with whom we first discussed the above problems.

Mathematics. — Lattice points in non-convex regions. III. By P. MULLENDER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 27, 1948.)

4. The Proof. Suppose $P_1 = (x'_1, \dots, x'_n)$ and $P_2 = (x''_1, \dots, x''_n)$ are points of \bar{K} . Then we have to prove that $P_1 + P_2$ is a point of \bar{R} , i.e.

$$F(x'_1 + x''_1, \dots, x'_n + x''_n) \equiv F(1, \dots, 1). \dots \quad (86)$$

We can divide the coordinates x_v into four groups, according as

- a. Both x'_v and x''_v are not greater than 1.
- b. Only x''_v is greater than 1, but not x'_v .
- c. Only x'_v is greater than 1, but not x''_v .
- d. Both x'_v and x''_v are greater than 1.

Of course it is possible that some of these groups do not contain any members at all. Without loss of generality we may suppose that the first p coordinates belong to group a, the next q to group b, the next r to group c and the last s to group d, where $p + q + r + s = n$.

Now it is convenient to choose our notation in accordance with the division of the coordinates into these four groups. For, in our proof, different coordinates of the same group have to be dealt with in the same way, whereas coordinates of different groups have to be dealt with in different ways. Hence we need not deal with all the coordinates, one by one, but we may take one representative for each group.

We denote the coordinates of group a by a_1, \dots, a_p , those of group b by b_1, \dots, b_q , etc. Then, writing a for a_1, \dots, a_p , etc., we have

$$F(x_1, \dots, x_n) \equiv F(a, b, c, d),$$

and in particular

$$F(1, \dots, 1) = F(1, 1, 1, 1).$$

Also ($\alpha = 1, \dots, p$ and $\beta = 1, \dots, q$ etc.)

$$\frac{\partial}{\partial x_\alpha} F(x_1, \dots, x_n) \equiv \frac{\partial}{\partial a_\alpha} F(a, b, c, d)$$

and

$$\frac{\partial}{\partial x_{p+\beta}} F(x_1, \dots, x_n) \equiv \frac{\partial}{\partial b_\beta} F(a, b, c, d), \text{ etc.}$$

We use, further, the following abbreviations:

$$\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1) = A, \quad \sum_{\beta=1}^q \frac{\partial}{\partial b_\beta} F(1, 1, 1, 1) = B \text{ etc.} \quad (87)$$

Now we have

$$P_1 = (a', b', c', d') , \quad P_2 = (a'', b'', c'', d''), \dots \quad (88)$$

with

$$a' \leq 1, b' \leq 1, a'' \leq 1, c'' \leq 1, \dots \quad (89)$$

and

$$c' > 1, d' > 1, b'' > 1, d'' > 1, \dots \quad (90)$$

That means P_1 is a point of $\bar{K}(1, \dots, p, p+1, \dots, p+q)$ and P_2 is a point of $\bar{K}(1, \dots, p, p+q+1, \dots, p+q+r)$.

By (73),

$$\vartheta' = \frac{A+B+C+D}{2(A+B)}, \quad \vartheta'' = \frac{A+B+C+D}{2(A+C)}, \quad (91)$$

and, by (74),

$$\left. \begin{aligned} x' &= \frac{\sum_{\alpha=1}^p a'_\alpha \frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1) + \sum_{\beta=1}^q b'_\beta \frac{\partial}{\partial b_\beta} F(1, 1, 1, 1)}{A+B}, \\ x'' &= \frac{\sum_{\alpha=1}^p a''_\alpha \frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1) + \sum_{\gamma=1}^r c''_\gamma \frac{\partial}{\partial c_\gamma} F(1, 1, 1, 1)}{A+C}. \end{aligned} \right\} \quad (92)$$

Finally, by (77),

$$\left. \begin{aligned} F(x' + \vartheta', x' + \vartheta', c', d') &\equiv F(1, 1, 1, 1), \\ F(x'' + \vartheta'', b'', x'' + \vartheta'', d'') &\equiv F(1, 1, 1, 1). \end{aligned} \right\} \quad (93)$$

We have to prove (86), or in our new notation,

$$F(a' + a'', b' + b'', c' + c'', d' + d'') \equiv F(1, 1, 1, 1). \quad (94)$$

We do this in four steps.

(i) First we assert that the group of coordinates a cannot be empty. For, if that were the case, we should have $A = 0$, and hence, by (91),

$$\vartheta' + \vartheta'' = \frac{B+C+D}{2B} + \frac{B+C+D}{2C} \geq \frac{(B+C)^2}{2BC} \geq 2.$$

This cannot be true, since $\vartheta' < 1$ and $\vartheta'' < 1$, because the regions to which P_1 and P_2 belong cannot be empty.

We now define y' and y'' by

$$\left. \begin{aligned} F(y' + \vartheta', y' + \vartheta', c', d') &\equiv F(1, 1, 1, 1) \\ \text{and} \quad F(y'' + \vartheta'', b'', y'' + \vartheta'', d'') &\equiv F(1, 1, 1, 1) \end{aligned} \right\} \quad (95)$$

respectively. Note that y' and y'' each denote one number only, whereas c', d', b'' and d'' each denote a set of numbers, namely c'_1, \dots, c'_r etc. Then, by (93) and (63),

$$y' \equiv x' \quad \text{and} \quad y'' \equiv x''.$$

or, by (92),

$$\left. \begin{aligned} y' &\equiv \frac{\sum_{\alpha=1}^p a'_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) + \sum_{\beta=1}^q b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1)}{A+B}, \\ y'' &\equiv \frac{\sum_{\alpha=1}^p a''_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) + \sum_{\gamma=1}^r c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1)}{A+C}. \end{aligned} \right\} . . . \quad (96)$$

This we may write in the form

$$\sum_{\alpha=1}^p a'_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) - y' A + \sum_{\beta=1}^q b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1) - y' B \equiv 0.$$

$$\sum_{\alpha=1}^p a''_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) - y'' A + \sum_{\gamma=1}^r c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1) - y'' C \equiv 0.$$

or, by (87),

$$\begin{aligned} \sum_{\alpha=1}^p \left\{ a'_\alpha - y' + \frac{\sum_{\beta=1}^q b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1) - y' B}{A} \right\} \frac{\partial}{\partial a_\alpha} F(1,1,1,1) &\equiv 0, \\ \sum_{\alpha=1}^p \left\{ a''_\alpha - y'' + \frac{\sum_{\gamma=1}^r c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1) - y'' C}{A} \right\} \frac{\partial}{\partial a_\alpha} F(1,1,1,1) &\equiv 0. \end{aligned}$$

Adding we obtain

$$\sum_{\alpha=1}^p (a'_\alpha + a''_\alpha - X) \frac{\partial}{\partial a_\alpha} F(1,1,1,1) \equiv 0, \quad (97)$$

where

$$X = y' + y'' + \frac{y' B + y'' C - \sum_{\beta=1}^q b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1) - \sum_{\gamma=1}^r c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1)}{A}. \quad (98)$$

However, for $a_1, a_2 = 1, \dots, p$,

$$\frac{\frac{\partial}{\partial a_{\alpha_1}} F(1,1,1,1)}{\frac{\partial}{\partial a_{\alpha_2}} F(1,1,1,1)} = \frac{\frac{\partial}{\partial a_{\alpha_1}} F(X, b' + b'', c' + c'', d' + d'')}{\frac{\partial}{\partial a_{\alpha_2}} F(X, b' + b'', c' + c'', d' + d'')}.$$

Hence (97) implies

$$\sum_{\alpha=1}^p (a'_\alpha + a''_\alpha - X) \frac{\partial}{\partial a_\alpha} F(X, b' + b'', c' + c'', d' + d'') \equiv 0.$$

From this it follows, by lemma 1, that

$$F(a' + a'', b' + b'', c' + c'', d' + d'') \equiv F(X, b' + b'', c' + c'', d' + d''). \quad (99)$$

(ii) We now prove that

$$F(X, b' + b'', c' + c'', d' + d'')$$

does not decrease, if one of the coordinates b' or c'' decreases, suppose b'_1 . We leave out this step, if both the groups b and c are empty.

By (98),

$$\frac{dX}{db'_1} = \frac{-\frac{\partial}{\partial b_1} F(1, 1, 1, 1)}{A},$$

since, by (95), y' and y'' depend only on the coordinates c' , d' , b'' and d'' . Hence

$$\begin{aligned} \frac{d}{db'_1} F(X, b' + b'', c' + c'', d' + d'') &= \\ &= \frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') + \\ &\quad + \sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(X, b' + b'', c' + c'', d' + d'') \cdot \frac{dX}{db'_1} = \\ &= \frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') - \\ &\quad - \frac{\frac{\partial}{\partial b_1} F(1, 1, 1, 1) \sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(X, b' + b'', c' + c'', d' + d'')}{A}. \end{aligned}$$

Now we have to prove that this expression is not positive.

Multiplication by

$$\begin{aligned} \frac{A}{\frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') \frac{\partial}{\partial b_1} F(1, 1, 1, 1)} &= \\ &= \frac{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1)}{\frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') \frac{\partial}{\partial b_1} F(1, 1, 1, 1)} \end{aligned}$$

does not alter the sign, since, by (63), none of the derivatives of is negative. Thus we obtain

$$\sum_{\alpha=1}^p \left\{ \frac{\frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1)}{\frac{\partial}{\partial b_1} F(1, 1, 1, 1)} - \frac{\frac{\partial}{\partial a_\alpha} F(X, b' + b'', c' + c'', d' + d'')}{\frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'')} \right\}.$$

According to our assumption that

$$\frac{\frac{\partial}{\partial x_k} F(x_1, \dots, x_n)}{\frac{\partial}{\partial x_l} F(x_1, \dots, x_n)}$$

increases, if x_k/x_l decreases, no term of the above sum will be positive, if

$$\frac{X}{b'_1 + b''_1} \leq 1. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (100)$$

By (98),

$$X \leq y' + y'' + \frac{y' B + y'' C}{A} = \frac{y' (A + B) + y'' (A + C)}{A}.$$

Further, by (63), (90) and (95),

$$y' + \vartheta' \leq 1, \quad y'' + \vartheta'' \leq 1.$$

Hence, by (91),

$$y' \leq \frac{A + B - C - D}{2(A + B)}, \quad y'' \leq \frac{A + C - B - D}{2(A + C)},$$

and so

$$X \leq \frac{A + B - C - D + A + C - B - D}{2A} = \frac{A - D}{A} \leq 1,$$

However, by (90), $b''_1 > 1$, and hence indeed (100) is true.

Thus we have proved that

$$F(X, b' + b'', c' + c'', d' + d'')$$

attains its largest value, when the coordinates b' and c'' are least, i.e. when $b'_1 = \dots = b'_q = c''_1 = \dots = c''_r = 0$. Hence

where $F(X, b' + b'', c' + c'', d' + d'') \leq F(X_1, b'', c', d' + d'')$, (101)

$$X_1 = \frac{y' (A + B) + y'' (A + C)}{A}. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (102)$$

(iii) We prove, further, that

$$F(X_1, b'', c', d' + d'')$$

does not decrease, if one of the coordinates b'' or c' decreases, suppose c'_1 . We also leave out this step, if there are no coordinates b and c .

By (102),

$$\frac{dX_1}{dc'_1} = \frac{A + B}{A} \cdot \frac{dy'}{dc'_1},$$

but, for $\alpha = 1, \dots, p$; $\beta = 1, \dots, q$,

$$\begin{aligned} \frac{\frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1)}{\frac{\partial}{\partial b_\beta} F(1, 1, 1, 1)} &= \frac{\frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}{\frac{\partial}{\partial b_\beta} F(y' + \vartheta', y' + \vartheta', c', d')} \\ &= \frac{\frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}{\frac{\partial}{\partial b_\beta} F(y' + \vartheta', y' + \vartheta', c', d')} \end{aligned}$$

and so

$$\frac{dX_1}{dc'_1} = \frac{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d') + \sum_{\beta=1}^q \frac{\partial}{\partial b_\beta} F(y' + \vartheta', y' + \vartheta', c', d')}{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')} \cdot \frac{dy'}{dc'_1}.$$

Further, by (95),

$$\frac{d}{dc_1} F(y' + \vartheta', y' + \vartheta', c', d') = 0,$$

or

$$\begin{aligned} \frac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d') + \left\{ \sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d') + \right. \\ \left. + \sum_{\beta=1}^q \frac{\partial}{\partial b_\beta} F(y' + \vartheta', y' + \vartheta', c', d') \right\} \frac{dy'}{dc_1} = 0, \end{aligned}$$

whence it follows that

$$\frac{dy'}{dc_1} = - \frac{\frac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')}{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d') + \sum_{\beta=1}^q \frac{\partial}{\partial b_\beta} F(y' + \vartheta', y' + \vartheta', c', d')}.$$

Substitution gives

$$\frac{dX_1}{dc_1} = - \frac{\frac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')}{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}.$$

Hence

$$\begin{aligned} \frac{d}{dc_1} F(X_1, b'', c', d' + d'') &= \\ &= \frac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'') + \sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(X_1, b'', c', d' + d'') \cdot \frac{dX_1}{dc_1} = \\ &= \frac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'') - \\ &\quad - \frac{\frac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d') \sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(X_1, b'', c', d' + d'')}{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')} \end{aligned}$$

Again we have to prove that this expression cannot be positive.

Multiplication by

$$\frac{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}{\frac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'') \frac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')}$$

does not alter the sign. Thus we obtain

$$\sum_{\alpha=1}^p \left\{ \frac{\frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}{\frac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')} - \frac{\frac{\partial}{\partial a_\alpha} F(X_1, b'', c', d' + d'')}{\frac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'')} \right\}.$$

and according to our assumption that

$$\frac{\frac{\partial}{\partial x_k} F(x_1, \dots, x_n)}{\frac{\partial}{\partial x_l} F(x_1, \dots, x_n)}$$

increases, if x_k/x_l decreases, no term of the above sum will be positive, if

$$\frac{X_1}{c'_1} \equiv \frac{y' + \vartheta'}{c'_1}$$

or

$$X_1 \equiv y' + \vartheta',$$

or, by (102),

$$y'(A+B) + y''(A+C) \leq (y' + \vartheta')A,$$

i.e.

$$y'B + y''(A+C) - \vartheta'A \leq 0.$$

As before we have

$$y' + \vartheta' \leq 1, \quad y'' + \vartheta'' \leq 1,$$

hence

$$y'B + y''(A+C) - \vartheta'A \leq B + A + C - (A+B)\vartheta' - (A+C)\vartheta'',$$

or, by (91),

$$y'B + y''(A+C) - \vartheta'A \equiv B + A + C - 2 \cdot \frac{1}{2}(A+B+C+D) = -D \equiv 0.$$

This proves our assertion. Hence

$$F(X_1, b'', c', d' + d'')$$

attains its largest value, when the coordinates b'' and c' are least, i.e., by (90), when $b'_1 = \dots = b'_q = c'_1 = \dots = c'_r = 1$. Consequently

$$F(X_1, b'', c', d' + d'') \equiv F(X_2, 1, 1, d' + d''), \dots \quad (103)$$

where

$$X_2 = \frac{y'(A+B) + y''(A+C)}{A}, \dots \quad (104)$$

with y' and y'' given by (95) under the assumption that

$$b'_1 = \dots = b'_q = c'_1 = \dots = c'_r = 1.$$

(iv) We finally prove that

$$F(X_2, 1, 1, d' + d'')$$

does not decrease, if one of the coordinates d' or d'' decreases, suppose d'_1 . We leave out this step, if there are no coordinates d .

By (104),

$$\frac{dX_2}{dd'_1} = \frac{A+B}{A} \cdot \frac{dy'}{dd'_1}.$$

From this it follows in the same way as before, that

$$\frac{dX_2}{dd'_1} = - \frac{\frac{\partial}{\partial d'_1} F(y' + \vartheta', y' + \vartheta', c', d')}{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}.$$

Hence

$$\begin{aligned} \frac{d}{dd'_1} F(X_2, 1, 1, d' + d'') &= \\ &= \frac{\partial}{\partial d'_1} F(X_2, 1, 1, d' + d'') + \sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(X_2, 1, 1, d' + d'') \frac{dX_2}{dd'_1} = \\ &= \frac{\partial}{\partial d'_1} F(X_2, 1, 1, d' + d'') - \\ &\quad - \frac{\frac{\partial}{\partial d'_1} F(y' + \vartheta', y' + \vartheta', c', d') \sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(X_2, 1, 1, d' + d'')}{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')} . \end{aligned}$$

Multiplication by

$$\frac{\sum_{\alpha=1}^p \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}{\frac{\partial}{\partial d'_1} F(X_2, 1, 1, d' + d'') \cdot \frac{\partial}{\partial d'_1} F(y' + \vartheta', y' + \vartheta', c', d')}$$

does not alter the sign. Thus we obtain

$$\sum_{\alpha=1}^p \left\{ \frac{\frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}{\frac{\partial}{\partial d'_1} F(y' + \vartheta', y' + \vartheta', c', d')} - \frac{\frac{\partial}{\partial a_\alpha} F(X_2, 1, 1, d' + d'')}{\frac{\partial}{\partial d'_1} F(X_2, 1, 1, d' + d'')} \right\} .$$

This is not positive, if

$$\frac{X_2}{d'_1 + d''} \leq \frac{y' + \vartheta'}{d'_1},$$

which is true, since X_2 is a special value of X_1 , and $X_1 \leq y' + \vartheta'$, which we have proved already.

This proves the assertion, and hence

$$F(X_2, 1, 1, d' + d'')$$

attains its largest value, when $d'_1 = \dots = d'_s = d''_1 = \dots = d''_s = 1$. So we have finally

$$F(X_2, 1, 1, d' + d'') \equiv F(X_3, 1, 1, 2), \quad \dots \quad (105)$$

where again

$$X_3 = \frac{y'(A+B) + y''(A+C)}{A}, \quad \dots \quad (106)$$

with y' and y'' given by (95), now, however, under the assumption, that not only $b_1'' = \dots = b_q'' = c_1' = \dots = c_r' = 1$, but also

$$d_1' = \dots = d_s' = d_1'' = \dots = d_s'' = 1.$$

Under this assumption we have, by (95),

$$y' + \vartheta' = 1 \quad \text{and} \quad y'' + \vartheta'' = 1.$$

so that, by (91),

$$y' = \frac{A+B-C-D}{2(A+B)} \quad \text{and} \quad y'' = \frac{A+C-B-D}{2(A+C)}.$$

Substituting these values in (106) we get

$$X_3 = \frac{A+B-C-D+A+C-B-D}{2A} = \frac{A-D}{A}.$$

By (99), (101), (103) and (105), we have now

$$F(a' + a'', b' + b'', c' + c'', d' + d'') \leq F(X_3, 1, 1, 2).$$

If some of the four groups of coordinates are empty, then the corresponding terms in this inequality have to be left out. E.g. if there are no coordinates b, c and d , i.e. if only group a is not empty, we have

$$F(a' + a'') \leq F(X_3),$$

where $X_3 = X = 1$.

Now, by lemma 1,

$$F(X_3, 1, 1, 2) - F(1, 1, 1, 1) \leq \left(\frac{A-D}{A} - 1 \right) A + (2-1) D = 0,$$

and hence also

$$F(a' + a'', b' + b'', c' + c'', d' + d'') \leq F(1, 1, 1, 1),$$

which proves the theorem.

5. The Application. If we define

$$F(x_1, \dots, x_n) \equiv (x_1 \dots x_n)^{\frac{1}{n}},$$

then $F(x_1, \dots, x_n)$ satisfies all the conditions mentioned in B, 1, as one can easily verify. Thus we get the solution of the problem of the product of n homogeneous linear forms we gave in A.

It is possible to obtain other solutions as well.

If we put

$$F(X, Y, Z) \equiv (X^p Y^q Z^r)^{\frac{1}{p+q+r}}, \dots, \dots, \dots \quad (107)$$

where

$$X = \frac{1}{p} \sum_{i=1}^p |x_i|, \quad Y = \frac{1}{q} \sum_{\mu=1}^q |x_{p+\mu}|, \quad Z = \frac{1}{r} \sum_{\nu=1}^r |x_{p+q+\nu}|, \quad (108)$$

with $p + q + r = n$, then the region R is three dimensional in the coor-

dinates X , Y and Z , but, by (108), if we take $p > \frac{1}{2}n$, it represents the n -dimensional region, considered by MORDELL, in order to apply his n -dimensional method to the product of the n homogeneous linear forms, i.e. the region we denoted in $A, 2$ by R'' .

Now, our method furnishes a construction of a three dimensional region K , for which theorem 6 holds with respect to the three dimensional region R . However, by (108), this region K also represents an n -dimensional region, and it is very easy to see that also for this region theorem 6 holds, but with respect to R'' .

In this case \bar{K} consists of four parts, namely $\bar{K}^*(1, 2, 3)$, $\bar{K}^*(1)$, $\bar{K}^*(1, 2)$ and $\bar{K}^*(1, 3)$.

By (107) and according to the definition in $B, 2$ the region $\bar{K}^*(1, 2, 3)$ is given by

$$x + \vartheta \leq 1 \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (109)$$

Since for this region $\vartheta = \frac{1}{2}$ and

$$x = \frac{pX + qY + rZ}{p+q+r},$$

by (68) and (69), we can write (109) in the form

$$pX + qY + rZ \leq \frac{1}{2}n.$$

Therefore, by

$$X = \frac{1}{p} \sum_{i=1}^p x_i, \quad Y = \frac{1}{q} \sum_{\mu=1}^q x_{p+\mu}, \quad Z = \frac{1}{r} \sum_{v=1}^r x_{p+q+v}, \quad (108a)$$

restricting ourselves again to the first hyperoctant, $\bar{K}^*(1, 2, 3)$ represents the n -dimensional region given by

$$x_1 + \dots + x_n \leq \frac{1}{2}n, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (110)$$

which region we denoted in $A, 2$ by \bar{K}_0 .

Further, $\bar{K}^*(1)$ is given by

$$\left. \begin{aligned} Y &> 1, \quad Z > 1, \\ \left(X + \frac{n}{2p} \right)^p Y^q Z^r &\leq 1, \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (111)$$

$\bar{K}^*(1, 2)$ by

$$\left. \begin{aligned} Z &> 1, \\ \left(\frac{pX + qY + \frac{1}{2}n}{p+q} \right)^{p+q} \cdot Z^r &\leq 1. \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (112)$$

and $\bar{K}^*(1, 3)$ by

$$\left. \begin{aligned} Y &> 1, \\ \left(\frac{pX + rZ + \frac{1}{2}n}{p+r} \right)^{p+r} \cdot Y^q &\leq 1. \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (113)$$

The inequalities (111) are the same as we used in $A, 2$ to define \bar{H}_2 .

Hence $\bar{K}^*(1)$ represents \bar{H}_2 , plus a set of points, which are also contained in $\bar{K}^*(1, 2, 3) = \bar{K}_0$.

Hence MORDELL's region \bar{K} , which we denoted in A, 2 by \bar{K}_2 , and which consisted of \bar{K}_0 and \bar{H}_2 , is wholly contained in the region \bar{K} we can construct by our method.

Moreover we may leave out the restriction $p > \frac{1}{2}n$. The only difference in the result is that we have to replace some the above mentioned four regions by their complementary ones, e.g. $\bar{K}^*(1)$ by $\bar{K}^*(2, 3)$.

To calculate V_K it is more convenient to regard \bar{K} as the sum of the non-overlapping regions $\bar{K}(1, 2, 3)$, $\bar{K}(1, 2)$, $\bar{K}(1, 3)$ and $\bar{K}(1)$. However, it does not seem worth while carrying out the calculation, as our first solution probably gives a much better result.

Generalising these results we can apply our method, to a region R defined by the function

$$F(X_1, \dots, X_N) \equiv (Y_1^{p_1} \dots X_N^{p_N})^{\frac{1}{p_1 + \dots + p_N}},$$

where e.g.

$$X_1 = \frac{1}{p_1} \sum_{\lambda_1=1}^{p_1} |x_{\lambda_1}|, \dots, X_N = \frac{1}{p_N} \sum_{\lambda_N=1}^{p_N} |x_{p_1+\dots+p_{N-1}+\lambda_N}|,$$

with $p_1 + \dots + p_N = n$.

Another application of our method can be given by putting

$$F(x_1, \dots, x_n) \equiv (x_1^\sigma + \dots + x_n^\sigma)^{\frac{1}{\sigma}} \quad (0 < \sigma < 1).$$

Then the region \bar{R} is defined by

$$x_1^\sigma + \dots + x_n^\sigma \leq n.$$

with $x_1 \geq 0, \dots, x_n \geq 0$.

In this case we have for the region $\bar{K}(k_1, \dots, k_p)$,

$$\vartheta = \frac{n}{2p} \quad , \quad x = \frac{1}{p} (x_{k_1}, \dots, x_{k_p}),$$

and hence $\bar{K}(k_1, \dots, k_p)$ is defined by the inequalities

$$x_{k_1} \leq 1, \dots, x_{k_p} \leq 1,$$

$$x_{l_1} > 1, \dots, x_{l_q} > 1,$$

$$p \left(\frac{x_{k_1} + \dots + x_{k_p} + \frac{1}{2}n}{p} \right)^\sigma + x_{l_1}^\sigma + \dots + x_{l_q}^\sigma \leq n.$$

The region is not empty, if $\vartheta < 1$, i.e. if $p > \frac{1}{2}n$.

Adding all the regions $\bar{K}(k_1, \dots, k_p)$, which are not empty, we obtain \bar{K} .

In conclusion I should like to thank Professor MORDELL for reading the manuscripts of this paper and the previous one under the same title, and for making several useful suggestions.

Mathematics. — *On multivectors in a V_n . II.* By WŁODZIMIERZ WRONA.
(Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of December 18, 1948.)

Consider now an arbitrary simple non-singular m -vector $F^{\lambda_1 \dots \lambda_m}$ and its dual $\circ F^{\lambda_{m+1} \dots \lambda_n}$ and denote by κ the scalar curvature of the m -direction which corresponds to $F^{\lambda_1 \dots \lambda_m}$ and by $\overset{(m)}{\circ \kappa}$ the scalar curvature of the $(n-m)$ -direction absolutely perpendicular to the former one which corresponds to the $(n-m)$ -direction of $\circ F^{\lambda_{m+1} \dots \lambda_n}$. By (2.21) the relation (2.62) takes now the form

$$\binom{m}{2} \kappa - \binom{n-m}{2} \overset{(m)}{\circ \kappa} = \frac{1}{2} (2m-n)(n-1) \kappa . \quad (2.67)$$

As by (2.46) equation (2.65) may be put in the form

$$W_{[\lambda_1 \lambda_2 [\mu_1 \mu_2} a_{\lambda_3 \mu_3} \dots a_{\lambda_m] \mu_m]} F^{\lambda_1 \dots \lambda_m} F^{\mu_1 \dots \mu_m} = 0,$$

where

$$W_{\lambda_1 \lambda_2 \mu_1 \mu_2} = \varrho^2 m! (m U_{[\lambda_1 [\mu_1} a_{\lambda_2] \mu_2]} + \mu a_{[\lambda_1 [\mu_1} a_{\lambda_2] \mu_2]}),$$

using the lemma I we can give to the last theorem the following equivalent form:

A necessary and sufficient condition that the space be EINSTEINian is that the expression

$$\binom{m}{2} \kappa - \binom{n-m}{2} \overset{(m)}{\circ \kappa}$$

be at every point independent of the choice of absolutely perpendicular non-singular m - and $(n-m)$ -directions.

It has then the value given by (2.67).

Similarly from (2.60) we have:

A necessary and sufficient condition that the space be conformal to a flat space is that the expression

$$\binom{n-m}{(n-m)} \overset{(n-m)}{\circ \omega} + m \omega$$

be independent of the choice of the non-singular m -vector at each point of the space. We have then

$$\binom{n-m}{(n-m)} \overset{(n-m)}{\circ \omega} + m \omega = 0. \quad (2.68)$$

from which in the case when the multivectors under consideration are simple we get by (2.21)

$$m \kappa + (n-m) \overset{(m)}{\circ \kappa} = n \kappa . \quad (2.69)$$

and the theorem:

A necessary and sufficient condition that the space be conformal to a flat space is that the expression

$$\frac{m}{(m)} \kappa + (n-m) \frac{\circ \kappa}{(n-m)}$$

be independent of the choice of absolutely perpendicular non-singular m - and $(n-m)$ -directions at each point of the space. It has then the value given by (2.69).

Consider the subspace V_m , $4 \leq m \leq n$, geodesic at a given point of V_n and tangent at this point to the given m -direction and denote by

$$'a_{\lambda}, 'K_{\lambda\mu\nu}, 'U_{\lambda\mu\nu}, 'U_{\lambda\mu}$$

and κ the fundamental tensor, the RIEMANN curvature affinor, the affinors defined by (2.1) and (2.2) and the scalar curvature of this V_m at the point under consideration. We get the following corollary from the last theorem:

If the expression

$$\frac{m_1}{(m_1)} \kappa + (m-m_1) \frac{\circ \kappa}{(m-m_1)}$$

where $2 \leq m_1 \leq m-2$ and κ and $\circ \kappa$ are the scalar curvatures of absolutely perpendicular non-singular m_1 - and $(m-m_1)$ -directions which are situated in the given m -direction, is independent of the particular choice of the m_1 -direction, we have

$$\frac{m_1}{(m_1)} \kappa + (m-m_1) \frac{\circ \kappa}{(m-m_1)} = m \kappa. \quad \quad (2.70)$$

It should be noted that the last condition is equivalent to the condition

$$'U_{\lambda\mu\nu} = -\frac{4}{m-2} 'U_{[\lambda} [\mu} 'a_{\lambda]\nu] \quad \quad (2.71)$$

Assume now that V_n is a C_n , i.e. the affinor $U_{\lambda\mu\nu}$ satisfies the relations (2.6), and denote by $\bar{K}_{\lambda\mu\nu}$, $\bar{U}_{\lambda\mu\nu}$ and $\bar{U}_{\lambda\mu}$ the V_m components of $K_{\lambda\mu\nu}$, $U_{\lambda\mu\nu}$ and $U_{\lambda\mu}$, V_m being defined in the preceding corollary. Taking the V_m -components of the both members of the equations (2.1) and (2.6) we obtain

$$\bar{U}_{\lambda\mu\nu} = \bar{K}_{\lambda\mu\nu} + \kappa 'a_{\lambda\mu\nu}^2 \quad \quad (2.72)$$

and

$$\bar{U}_{\lambda\mu\nu} = -\frac{4}{n-2} \bar{U}_{[\lambda} [\mu} 'a_{\lambda]\nu]. \quad \quad (2.73)$$

According to a well known theorem¹³⁾ we have from (2.72)

$$\bar{U}_{\lambda\mu\nu} = 'K_{\lambda\mu\nu} + \kappa 'a_{\lambda\mu\nu}^2 = 'U_{\lambda\mu\nu} + (\kappa - \circ \kappa) 'a_{\lambda\mu\nu}^2 \quad . . . \quad (2.74)$$

¹³⁾ See Ref. 10, V. II, p. 133, (14.1).

from which we get by (2.73)

$$'U_{\lambda\mu\nu} + (\kappa - \kappa) \frac{2}{(m)} 'a_{\lambda\mu\nu} = -\frac{4}{n-2} \bar{U}_{[\lambda[\mu} 'a_{\nu]\nu]} (2.75)$$

Multiplying (2.75) by ' $a^{\nu\nu}$ ' and summing for ν and ν we obtain

$$'U_{\lambda\mu} + (\kappa - \kappa)(1-m) 'a_{\lambda\mu} = \frac{m-2}{n-2} \bar{U}_{\lambda\mu} + \frac{1}{n-2} \bar{U}'a_{\lambda\mu}, (2.76)$$

where

$$\bar{U} = \bar{U}_{\nu\nu} 'a^{\nu\nu}.$$

From (2.76) we get further

$$(\kappa - \kappa)(1-m)m = \frac{m-2}{n-2} \bar{U} + \frac{m}{n-2} \bar{U}'a_{\lambda\mu}, (2.77)$$

from which

$$\bar{U} = -(\kappa - \kappa) \frac{m(n-2)}{2} (2.78)$$

From (2.76) and (2.78) we obtain

$$\bar{U}_{\lambda\mu} = \frac{n-2}{m-2} 'U_{\lambda\mu} - \frac{1}{2} (\kappa - \kappa)(n-2) 'a_{\lambda\mu}, (2.79)$$

and from (2.75)

$$'U_{\nu\lambda\mu\nu} = -\frac{4}{m-2} 'U_{[\lambda[\mu} 'a_{\nu]\nu]} (2.80)$$

Thus we get the lemma II: *The equation (2.80) is satisfied for every m-direction $2 \leq m \leq n$ at each point of the V_n conformal to a flat space.*

This implies that every total geodesic subspace immersed in a C_n must be a C_m itself.

From the last lemma it follows also that in a C_n the relation (2.70) is satisfied for all integers m and m_1 satisfying $4 \leq m \leq n$ and $2 \leq m_1 \leq m-2$.

Vice versa we shall prove, If for a number m , $4 \leq m \leq n$, at each point of a V_n the quantity

$$m_1 \kappa + (m-m_1) \frac{\kappa}{(m-m_1)}$$

is for every m -direction independent of the special choice of the non-singular m_1 -direction inside the m -direction then the V_n is a C_n .

Take namely an orthogonal ennumple i^1, i^2, \dots, i^n and denote by (h_1, h_2, \dots, h_n) an arbitrary permutation of the set $(1, 2, \dots, n)$. Consider further the m -direction defined by the vectors $i^{h_1} \dots i^{h_m}$, then the orthogonal components

of the RIEMANN curvature affinors of the V_n and the V_m , defined in the corollary, satisfy the relations

$$K_{hijk} = K_{h'ij'k'} , \quad h, i, j, k = 1, \dots, n (2.81)$$

From our assumption follows (2.71), from which we can easily see that all orthogonal components K_{hijk} with four unequal indices vanish. Thus by (2.81) we have

$$K_{hijk} = 0 \quad \text{for } h, i, j, k ; \quad h, i, j, k = 1, \dots, n$$

hence all orthogonal components of the RIEMANN curvature affinor of V_n with four unequal indices vanish and by a well known theorem¹⁴⁾ the V_n is conformal to a flat space¹⁵⁾.

We can now easily prove the following theorem of HAANTJES¹⁶⁾.

A necessary and sufficient condition that a V_n be a C_n is that at every point the scalar

$$m_1 \underset{(m_1)}{\times} + m_2 \underset{(m_2)}{\times} + \dots + m_p \underset{(m_p)}{\times} \dots \dots \dots \quad (2.82)$$

should be independent of the choice of the set of p mutually absolutely perpendicular non-singular m_i -directions, where the integers

$$m_i, (1 = 1, 2, \dots, p),$$

are arbitrary except for the conditions

$$2 \leq m_i \leq n-2 \text{ and } \sum_{i=1}^p m_i = n$$

and \times are scalar curvatures of corresponding m_i -directions.

The scalar (2.82) is then equal to $n \times$, \times being the scalar curvature of the space.

In fact, in the C_n from the last but one theorem follows

$$\begin{aligned} m_1 \underset{(m_1)}{\times} + m_2 \underset{(m_2)}{\times} + m_3 \underset{(m_3)}{\times} + \dots + m_p \underset{(m_p)}{\times} &= (m_1 + m_2) \underset{(m_1+m_2)}{\times} + m_3 \underset{(m_3)}{\times} + \dots + m_p \underset{(m_p)}{\times} = \\ &= \dots \dots \dots \dots = \\ &= n \times \end{aligned} \quad (2.83)$$

\times being scalar curvature of the corresponding $(m_1 + m_2)$ -direction.

Inversely if

$$m_1 \underset{(m_1)}{\times} + m_2 \underset{(m_2)}{\times} + m_3 \underset{(m_3)}{\times} + \dots + m_p \underset{(m_p)}{\times} = \tau, \quad \dots \dots \dots \quad (2.84)$$

where τ is independent of the choice of the assumed m_i -directions and if we do not change the m_3 -, m_4 -, m_p -directions, we get

$$m_1 \underset{(m_1)}{\times} + m_2 \underset{(m_2)}{\times} = \sigma, \quad \dots \dots \dots \quad (2.85)$$

where $\sigma = \tau - m_3 \underset{(m_3)}{\times} - \dots - m_p \underset{(m_p)}{\times}$ is independent of the choice of the m_1 - and m_2 -directions within the $(m_1 + m_2)$ -direction. Thus from the last theorem we obtain that the V_n must be a C_n .

¹⁴⁾ Ref. 10, V. II, p. 204.

¹⁵⁾ This idea of the proof of this theorem is due to Prof. J. HAANTJES and different from my original one.

¹⁶⁾ Ref. 5.

Section 3. Principal m -vectors.

In order to shorten our work we shall confine ourselves in this section to the RIEMANNian spaces of an even number of dimensions, $2m$, and we shall consider m -vectors in them. There is little difficulty in generalizing the most of the following results to spaces of an odd number of dimensions.

Consider the projective P_N associated with a point in V_{2m} and coordinates in it defined by

$$\begin{aligned} v^{\lambda_1 \dots \lambda_m}, & \quad \alpha = 1, 2, \dots, N+1, \\ \alpha & \quad \lambda_i = 1, 2, \dots, 2m. \end{aligned} \quad (3.1)$$

In P_N we have three $(N-1)$ -dimensional quadrics defined by the tensors

$$a_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 a_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} v^{\lambda_1 \dots \lambda_m}_{\alpha} v^{\mu_1 \dots \mu_m}_{\beta}, \quad (3.2)$$

$$U_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 U_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} v^{\lambda_1 \dots \lambda_m}_{\alpha} v^{\mu_1 \dots \mu_m}_{\beta}, \quad (3.3)$$

$${}^o U_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 {}^o U_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} v^{\lambda_1 \dots \lambda_m}_{\alpha} v^{\mu_1 \dots \mu_m}_{\beta}, \quad (3.4)$$

and we also have the affinor

$$I_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 I_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} v^{\lambda_1 \dots \lambda_m}_{\alpha} v^{\mu_1 \dots \mu_m}_{\beta}, \quad (3.5)$$

which defines a quadric or null polarity according to m being even or odd. We can easily prove that

$$a_{\alpha\beta} I^{\alpha\gamma} I^{\beta\delta} = a^{\gamma\delta} \quad \dots \quad (3.6)$$

$$U_{\alpha\beta} I^{\alpha\gamma} I^{\beta\delta} = \varrho^2 {}^o U^{\gamma\delta} \quad \dots \quad (3.7)$$

and

$$f^{\alpha} = \varrho^3 {}^o f_{\delta} I^{\delta\alpha} \quad \dots \quad (3.8)$$

where

$${}^o f_{\delta} = \frac{1}{m!} {}^o f_{\lambda_1 \dots \lambda_m} v^{\lambda_1 \dots \lambda_m}_{\delta} \quad \dots \quad (3.9)$$

The deviation of an m -vector is now given by

$$\omega = - \frac{U_{\alpha\beta} f^{\alpha} f^{\beta}}{a_{\alpha\beta} f^{\alpha} f^{\beta}} \quad \dots \quad (3.10)$$

At a point the finite maxima and minima of ω defined by (3.10) are given by the m -vectors satisfying

$$(U_{\alpha\beta} + \omega a_{\alpha\beta}) f^{\alpha} = 0 \quad \dots \quad (3.11)$$

at that point.

The m -vectors which satisfy equations (3.11) for some value of ω will be called the *principal m -vectors*.

Multiplying (3.11) by $\varrho I^{\beta\gamma}$ and summing for β we have from (3.8), (3.6) and (3.7)

$$({}^0U^{\alpha\beta} + \varrho^2 \omega a^{\alpha\beta}) {}^0f^\alpha = 0 \dots \dots \quad (3.12)$$

or equivalently

$$({}^0U_{\alpha\beta} + \varrho^2 \omega a_{\alpha\beta}) {}^0f^\alpha = 0 \dots \dots \quad (3.13)$$

Thus we have: If an m -vector f^α satisfies equation (3.11) for a characteristic root ω , then its dual ${}^0f^\alpha$ satisfies (3.13) for the characteristic root $\varrho^2\omega$.

Consider now an EINSTEINIAN V_{2m} . From (2.59), (3.3) and (3.4) we have in this case

$${}^0U_{\alpha\beta} = \varrho^2 U_{\alpha\beta} \dots \dots \quad (3.14)$$

and (3.13) becomes

$$(U_{\alpha\beta} + \omega a_{\alpha\beta}) {}^0f^\alpha = 0 \dots \dots \quad (3.15)$$

Hence: In an EINSTEINIAN V_{2m} the dual of a principal m -vector is also a principal m -vector, the characteristic roots are the same for both m -vectors.

The m -vectors

$$\bar{f}^\alpha = \frac{1}{2}(f^\alpha + {}^0f^\alpha) \dots \dots \quad (3.16)$$

and

$$\bar{\bar{f}}^\alpha = \frac{1}{2}(f^\alpha - {}^0f^\alpha) \dots \dots \quad (3.17)$$

satisfy the relations

$${}^0\bar{f}^\alpha = \bar{f}^\alpha \dots \dots \quad (3.18)$$

$${}^0\bar{\bar{f}}^\alpha = -\bar{f}^\alpha \dots \dots \quad (3.19)$$

and are therefore self-dual. If f^α is principal m -vector so are \bar{f}^α and $\bar{\bar{f}}^\alpha$. Thus: If at a point of an EINSTEINIAN V_{2m} the characteristic roots of (3.11) are distinct then there are $(N+1)$ linearly independent principal m -vectors at the point, they are all self-dual, half of them satisfying (3.18) and the other half (3.19).

Consider now a C_{2m} . From (2.60) we have

$${}^0U_{\alpha\beta} = -\varrho^2 U_{\alpha\beta} \dots \dots \quad (3.20)$$

and equation (3.13) becomes

$$(U_{\alpha\beta} - \omega a_{\alpha\beta}) {}^0f^\alpha = 0 \dots \dots \quad (3.21)$$

Hence: In a C_{2m} the dual of a principal m -vector is also a principal m -vector, and corresponding characteristic roots differ only in sign.

Thus in a C_{2m} we can arrange the characteristic roots of (3.11) in pairs in such a way that the roots in each pair differ only in sign.

By (2.60) and (2.46), we have in C_{2m}

$$\bar{U}_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} = -\frac{m!}{m-1} U_{[\lambda_1 \mu_1, \lambda_2 \mu_2 \dots \lambda_m \mu_m]} \dots \quad (3.22)$$

Consider now the RICCI principal directions in C_{2m} , the elementary divisors of the RICCI characteristic equation being assumed simple. Let k

be their mean curvatures. Take the orthogonal enneple of RICCI principal directions at a point of C_{2m} to be the basis of the coordinate-system in the P_N associated with this point. It is easily seen from (3.22) that the coordinate- m -vectors of this system are principal m -vectors and that their corresponding characteristic roots ω , ($\alpha = 1, 2, \dots, N+1$), are

$$\omega = \frac{1}{m(m-1)} \sum_{p=1}^{m-1} k - \frac{2m-1}{m-1} z, \quad \dots \quad (3.23)$$

where k are mean curvatures of the corresponding RICCI principal directions situated inside of the m -direction defined by the α -th principal m -vector.

But from (2.21) we get

$$\omega = z - z, \quad \dots \quad (3.24)$$

where z , ($\alpha = 1, 2, \dots, N+1$), is the scalar curvature of the m -direction defined by the α -th principal m -vector.

Hence

$$z = \frac{1}{m(m-1)} \sum_{p=1}^{m-1} k - \frac{m}{m-1} z, \quad \dots \quad (3.25)$$

Thus we have the theorem:

If in a C_{2m} the RICCI elementary divisors are assumed simple then the m -vectors defined by the m -directions composed of every set of m RICCI principal directions are the principal m -vectors and the scalar curvatures of these m -directions are given by (3.25).

Consider now a V_{2m} which is immersible in a flat $(2m+1)$ -dimensional space. In this case we have¹⁷⁾

$$K_{\lambda\mu\nu} = -2\varepsilon h_{[\lambda} h_{\mu} h_{\nu]}, \quad \dots \quad (3.26)$$

where $\varepsilon = \pm 1$ and $h_{\lambda\mu}$ is the second fundamental tensor. We have further

$$U_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m} = -m! \varepsilon h_{[\lambda_1 \mu_1} h_{\lambda_2 \mu_2} h_{\lambda_3 \mu_3} \dots h_{\lambda_m \mu_m]} + \varepsilon a_{\lambda_1 \dots \lambda_m \mu_1 \dots \mu_m}. \quad (3.27)$$

The principal directions of the tensor $h_{\lambda\mu}$ are the same as the RICCI principal directions. Let h be the principal normal curvatures of V_{2m} . Consider, in the P_N associated with a point of the V_{2m} under consideration, the basic coordinate-system having the orthogonal enneple of principal directions of $h_{\lambda\mu}$ as its basis. It is easy to show from (3.27) that the coordinate m -vectors of this system are the principal m -vectors and that the characteristic roots of (3.11) are

$$\omega = \frac{\varepsilon}{m(m-1)} \sum_{p \neq q}^{1 \dots m} h h - z, \quad \dots \quad (3.28)$$

¹⁷⁾ Ref. 10, V. II, p. 76.

From (3.24) and (3.28) we have

$$\kappa = \frac{\epsilon}{m(m-1)} \sum_{\alpha}^{1 \dots m} h_{\alpha p} h_{\alpha q} \quad , \quad , \quad , \quad , \quad (3.29)$$

Thus: In a V_{2m} which is immersible in a $(2m+1)$ -dimensional flat space, the m -vectors defined by all m -directions of the principal ennume of the tensor $h_{\lambda\mu}$ are the principal m -vectors; the scalar curvatures

$$\kappa_a (a = 1 \dots N+1),$$

of these m -directions, are given by (3.29), in which $h_{\alpha p}$ ($p = 1, \dots, m$) are corresponding principal normal curvatures.

REFERENCES.

1. CHURCHILL, R. V., "On the geometry of the Riemann tensor", Trans. American Math. Soc., **34**, 126—152 (1932).
2. EISENHART, L. P., "Riemannian geometry", Princeton (1926).
3. GIVENS, J. W., "Tensor coordinates of linear spaces", Annals of Math. Vol. 38, No. 2, 355—385 (1937).
4. HAANTJES, J. and W. WRONA, "Ueber konformeuklidische und Einsteinsche Räume gerader Dimension", Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLII, No. 7, 626—636 (1939).
5. HAANTJES, J., "Eine Charakterisierung der konf.-eukl. Räume", Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, No. 1, 91—94 (1940).
6. RUSE, H. S., "On the line geometry of the Riemann tensor", Proc. Roy. Soc. Edinburgh, Vol. LXII, 64—73 (1944).
7. ———, "The five dimensional geometry of the curvature tensor in a Riemannian V_4 ", Quart. J. of Math. Oxford, 7—15 (1946).
8. ———, "Multivectors and catalytic tensors", Phil. Mag. Ser. 7, Vol. XXXVIII, 408—421 (1947).
9. ———, "The self-polar Riemann complex for a V_4 ", Proc. London Math. Soc. (1948).
10. SCHOUTEN, J. A. and D. J. STRUIK, "Einführung in die neueren Methoden der Differentialgeometrie", Groningen (1935—1938).
11. SOMMERRVILLE, D. M. T., "Analytical Geometry of three Dimensions", Cambridge (1934).
12. STRUIK, D. J., "On sets of principal directions in a Riemannian manifold of four dimensions", Journ. of Math. and Ph., Vol. VII, No. 3, 193—197 (1928).
13. WRONA, W., "Eine Verallgemeinerung des Schurschen Satzes", Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIV, No. 8, 943—946 (1941).
14. ———, "Cond. néc. et suff. qui déterm. les espaces Einsteiniens conf.-enclid. et de courb. const.", Ann. de la Soc. Polonaise de Math. T. XX, Cracovie, 28—80 (1948).

Geologie. — *De Allerød-oscillatie in Nederland. Pollenanalytisch onderzoek van een laatglaciale meerafzetting in Drente.* (Met een diagram en een tabel). I. By T. VAN DER HAMMEN. (Rijksmuseum van Geologie en Mineralogie, Leiden.) (Communicated by Prof. C. J. VAN DER KLAUW.)

(Communicated at the meeting of December 18, 1948.)

Inleiding.

Sinds HARTZ en MILTHERS in 1901 voor de eerste maal in laatglaciale sedimenten op het Deense eiland Seeland een tijdelijke verbetering van het klimaat konden aantonen, die de Allerød-oscillatie werd genoemd, is deze voor de correlatie zo belangrijke klimaatsschommeling op vele plaatsen in Europa waargenomen. Zo werd het voorkomen van de Allerød oscillatie aangetoond onder meer bij Hamburg door SCHÜTRUMPF (o.a. 1936), in Oostpruisen door GROSZ (1937), bij Bremen door OVERBECK en SCHNEIDER (1938), in Ierland door JESSEN en FARRINGTON (1938), in het Untereichsfeld, ten Zuiden van de Harz, door STEINBERG (1944), in Frankrijk door G. en C. DUBOIS (1944), in het Bodenmeer gebied door INGE MÜLLER (1947) en in het Lake District in Engeland door WINIFRED PENNINGTON (1947). Voorts werd onze kennis van flora en klimaat in het Laatglaciaal belangrijk uitgebreid door een aantal publicaties van de hand van IVERSEN (o.a. 1946, 1947). In Nederland was het vooral FLORSCHÜTZ (o.a. 1939, 1941, 1944), die zich met de vegetatie-ontwikkeling in het Laatglaciaal heeft beziggehouden. Hij heeft enige diagrammen gepubliceerd die gedeeltelijk gebaseerd waren op de analyses van laatglaciaal veen. Het resultaat van zijn onderzoek stelde hem in staat, voor ons land een schema van de vegetatie-ontwikkeling te ontwerpen (zie de tabel). FLORSCHÜTZ (1939) was het ook, die voor de eerste maal de mogelijkheid opperde van het voorkomen van afzettingen uit de Allerödtijd in Nederland. Later wees WATERBOLK (1947) eveneens hierop. Tot nu toe heeft men echter de Allerød-oscillatie in Nederland nog niet met zekerheid aan kunnen tonen. De oorzaak daarvan moet gezocht worden in het feit, dat slechts een gering aantal laatglaciale meerafzettingen uit Nederland bekend is, in tegenstelling tot b.v. Denemarken, waar deze afzettingen zeer veelvuldig zijn aangetroffen. Meerafzettingen blijken n.l. zeer scherp veranderingen in het klimaat weer te geven, door een daarmede parallel gaande verandering in de sedimentatie: meer minerogeen tijdens slechtere, meer organogeen tijdens betere klimaatsomstandigheden. Bovendien zijn de locale invloeden op de polleninhoud van het sediment, zoals die vooral in veen optreden tengevolge van de vegetatie ter plaatse, aanzienlijk geringer in meerafzettingen. Dit is van groot belang, daar de polleninhoud van laatglaciale venen beïnvloed kan zijn door de lokale vegetatie van Cyperaceae. De pollen-

spectra geven dan geen zuiver algemeen beeld van de plantengroei ten tijde van de afzetting, maar representeren de locale begroeiing, gesuperponeerd op de regionale. Deze verwikkeling zal eerst ten volle te ontwarren zijn, wanneer met behulp van de analyse van meerafzettingen beter inzicht zal zijn verkregen in de regionale vegetatie-ontwikkeling in Nederland. Door de onderzoeken van enkele pollenanalytici, waaronder IVERSEN op de voorgrond treedt, is het door een vergelijkende studie van de morfologie van pollen en sporen mogelijk geworden, om naast de stuifmeelkorrels van bomen een groot aantal soorten kruidenpollen en sporen te determineren. Dit kruidenpollen blijkt nu juist in laatglaciale sedimenten van uitermate groot belang te zijn, niet alleen quantitatief maar ook qualitatief. In de eerste plaats zijn de zuivere botanische resultaten van het kruidenpollenonderzoek natuurlijk van veel gewicht voor de reconstructie der vegetatie. Daarnaast blijken die resultaten echter ook de stratigrafie te kunnen dienen, daar een bepaald soort kruidenpollen of een associatie van verschillende soorten alleen of hoofdzakelijk in bepaalde zones van het Laatglaciaal voor kan komen. IVERSEN heeft nu voor het weergeven van de uitkomsten der analyses van laatglaciale afzettingen een geheel nieuw diagram-type ontworpen, dat meer beantwoordt aan de bijzondere eisen, die aan een duidelijke afspiegeling van de vegetatie-veranderingen in die tijd mogen worden gesteld. Als basis van procentberekening neemt hij de som van al het pollen van bomen, anemophile kruiden en Ericaceae. Alle stuifmeelkorrels van andere planten en alle sporen blijven buiten deze „pollensom“. Voor het verkrijgen van één spectrum worden ongeveer 500 tot de „pollensom“ gerekende stuifmeelkorrels geteld. Dit is noodzakelijk, om de gewenste statistische zekerheid voor de percentages der verschillende soorten van het kruidenpollen te bereiken. Het zal duidelijk zijn, dat het op deze wijze samenstellen van een diagram zeer veel tijd in beslag neemt. De resultaten blijken er echter ten volle tegen op te wegen.

Vindplaats en aard van het onderzochte materiaal.

In April 1948 werd, met het doel een laatglaciale meerafzetting te vinden, geboord met de Dachnovsky-sonde aan de rand van de verlandingszone aan de Westoever van het Hijkemeer. Dit is gelegen bij Hijken ten Zuid-Westen van Hooghalen in de provincie Drente. Deze plaats werd uitgekozen naar aanwijzingen van Prof. FLORSCHÜTZ. Het meertje, dat afmetingen heeft van ongeveer 200 m × 150 m, is een van de vele kommen op het Drentse plateau, hier omringd door zandafzettingen, op de geologische kaart als fluvio-glaciaal aangegeven. In de naaste omgeving komt het keileem aan de oppervlakte voor. In het midden staat waarschijnlijk ongeveer 5 m water, reeds een opmerkelijke diepte voor een dergelijke kom. Bovendien werd nu op de boorplaats een sediment-pakket van 6,5 m aangetroffen, voor het onderliggende zand bereikt werd, waaronder dan nog de keileem te verwachten is. Op zijn minst moet de diepte in het midden oorspronkelijk dus 11 m bedragen hebben. Het is wel zeer waar-

schijnlijk, dat we hier met een kom te maken hebben, die zijn ontstaan aan een in de grondmoraine achtergebleven massa gletscherijs te danken heeft.

Aangetroffen werden (dieptecijfers globaal; zie verder het diagram)

0—	\pm 100 cm	veen van de verlandingszone;
\pm 100—	510	,, detritus-gyttja;
510—	580	,, klei-gyttja, met uiterst fijn „zand”;
580—	600	,, detritus-gyttja;
600—	630	,, klei-gyttja;
630—	?	,, zand.

Al dadelijk rees het vermoeden, dat het in de kleigyttja ingeschakelde laagje detritusgyttja zou corresponderen met de Allerød-oscillatie, daar deze afwisseling van meer minerogene en meer organogene afzettingen onder analoge omstandigheden elders karakteristiek is hiervoor. Het microbotanische onderzoek heeft dit vermoeden geheel bevestigd. De onder en boven de Allerödlaag aanwezige sedimenten bleken naar hun polleninhoud overeen te komen met resp. de Oudere en Jongere Dryas-lagen uit Denemarken. Hoewel overblijfselen van *Dryas octopetala* of dergelijke toendraplanten in deze lagen in Nederland nog niet aangetoond zijn, en het zelfs min of meer twijfelachtig is, of deze plant ten tijde van de afzetting der bovenste kleigyttja in Nederland voorkwam, menen we toch deze namen, als zuiver stratigrafische, over te moeten nemen. Bovendien hebben zij in de Europese literatuur reeds algemeen ingang gevonden.

Wijze van werken.

Bij de bereiding van het materiaal voor de analyses werd gebruik gemaakt van de verkorte methode ERDTMAN, zoals die bij de Deense Geologische Dienst toegepast wordt, steeds voorafgegaan door een behandeling met KOH en HF. Verder werden de berekeningen der procenten geheel uitgevoerd op de reeds in de inleiding vermelde methode van IVERSEN. Zodoende werd een grote mate van vergelijkbaarheid met de Deense diagrammen verkregen.

Secundair pollen.

Bij de analyse van het materiaal werden in een aantal monsters, afkomstig uit het laatglaciale deel van de afzettingen, enkele stuifmeelkorrels van warmteminnende bomen aangetroffen, die in dit milieu allerminst verwacht konden worden. IVERSEN (1936) heeft hetzelfde verschijnsel in Denemarken waargenomen, alleen in veel sterkere mate. Hij toonde aan, dat deze stuifmeelkorrels — waaronder zich typisch tertiaire en interglaciale typen bevinden — tezamen met het minerogene materiaal uit het keileem afkomstig moeten zijn. Steeds werden ze ook begeleid door een waarschijnlijk uit het Tertiair afkomstig marien micro-organisme, waaraan hij de naam *Hystrix* gaf, en dat eveneens in het keileem voor bleek te komen. Door een vernuftige correctie-methode wist hij, met behulp van het zeker

secundaire pollen, ook het ingespoelde pollen dat niet direct als zodanig herkenbaar was (b.v. van *Pinus* en *Betula*), te elimineren. Zo bleken bijvoorbeeld bijna alle in de Oudere Dryas-afzettingen voorkomende *Pinus* korrels secundair te zijn. Deze methode kon helaas op ons materiaal niet worden toegepast, maar gelukkig bleek de hoeveelheid zeker secundair pollen zeer gering te zijn, zodat de verontreiniging slechts een zeer geringe invloed op het diagram gehad kan hebben.

Wij hebben gemeend het weinige pollen van warmteminnende houtige gewassen, dat in ons laatglaciale materiaal gevonden werd, niet in het diagram te moeten aangeven, maar in een hieronder volgende tabel te moeten vermelden, daar wij het voor een groot deel als secundair beschouwen, en wel om de navolgende redenen:

1. Het voorkomen ervan in de min of meer minerogene afzettingen, terwijl het in de meer organogene afzettingen bijna geheel ontbreekt. Dit betekent dus een afname bij klimaatverbetering.

(Bij een verbetering van het klimaat treedt namelijk een meer volledige begroeiing op, waardoor de erosie minder sterk, en dus de lacustrine sedimentatie minder minerogeen wordt.)

2. Het eveneens voorkomen van stellig secundair pollen, bijvoorbeeld van *Tsuga*, *Rhus*, *Platycarya*, *Carya*, *Abies* en *Picea*.

3. Het samengaan met „*Hystrix*”.

4. De conservatietoestand, die vaak vrij slecht is, in tegenstelling tot het overige aanwezige pollen (uitgezonderd sommige *Pinus* korrels, die waarschijnlijk eveneens secundair zijn), terwijl bovendien de kleur afwijkt van het overige materiaal, wat doet denken aan een voorafgaande insluiting in veen of bruinkool.

Monster	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
<i>Corylus</i>	1	3	3	3	1	1	—	—	—	2	1	2	5	2	3	(1)	Aantal
<i>Quercus</i>	—	1	—	1	1	—	—	—	—	—	—	—	1	—	—	—	pollen-
<i>Alnus</i>	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	korrels
<i>Picea/Abies</i>	1	1	—	—	—	—	—	—	—	—	—	1	—	1	—	—	
Tertiaire typen	—	2	1	1	—	—	—	—	—	—	—	—	1	—	—	—	
<i>Hystrix</i>	—	x	x	?	—	—	—	—	—	—	x	—	x	—	—	—	

Helaas is de Allerød-laag niet geheel zuiver organogeen, want in het bovenste deel van de detritusgyttja komen kleine mineraal-partikeltjes voor. Daardoor is het niet uit te maken, of een enkele *Corylus*-korrel daarin primair voorkomt, een mogelijkheid, die misschien niet geheel uitgesloten mag worden.

We zullen na de analyse van niet geheel zuiver organogene en van minerogene sedimenten van laatglaciale ouderdom, de uiterste voorzichtigheid moeten betrachten bij het trekken van conclusies uit het voorkomen van een enkele korrel van meer-warmteminnende bomen. Voorts is er de

mogelijkheid van transport over grote afstand, dat in boomarme gebieden op kan treden, van het beboste achterland uit. Mogelijk zijn ook daardoor enkele *Corylus*-korrels in de sedimenten van het Hijkermeer terecht gekomen.

Over het voorkomen van *Corylus*-pollen in laatglaciale afzettingen zie men ook het laatste onderdeel van dit artikel.

Algemeen overzicht van de vegetatie-ontwikkeling.

Het diagram is verdeeld in drie afdelingen. A is het algemene diagram, tonend van links uitgezet de curven van de bomen en struiken, en van rechts uitgezet de percentages van de som der anemophile kruiden en van de Ericaceae. Deze beide groepen vormden tezamen de basis voor de procentberekening. *Corylus* is, zoals gebruikelijk, buiten de som gehouden. B geeft de curven van de anemophile kruiden en van *Empetrum* afzonderlijk, en C geeft 'hetzelfde voor de niet in de „pollensom“ begrepen planten, te beginnen met *Juniperus*. Deze laatste is, om resultaten te verkrijgen die vergelijkbaar zouden zijn met de gepubliceerde Deense diagrammen, buiten de „pollensom“ gehouden. Geheel rechts zijn de waarden opgegeven van de boompollenfrequentie (Tree Pollen Frequency, T.P.F.), het gemiddeld aantal boompollen-korrels per preparaat.

In A komt duidelijk de afwisseling van een open landschap met meer of minder dichte bossen tot uitdrukking. In het onderste deel toont dit diagram ons het beeld van een open landschap met slechts weinig bomen. Bovendien is waarschijnlijk hier een groot deel van het *Betula*-pollen afkomstig van *Betula nana*, naar een oppervlakkige grootte-beoordeling. Om hiervan meer zekerheid te krijgen zal een statistische meting van de korrels noodzakelijk zijn. De boompollenfrequentie is in de onderste monsters gering. We zouden het uit dit deel van het diagram sprekende vegetatie type, met enig voorbehoud, een „niet geheel boomloze toendra“ willen noemen. Wij hebben deze uitdrukking gekozen, in de veronderstelling, dat hier, evenals in Denemarken, toentertijd een landschap aanwezig was, dat alhoewel niet geheel boomloos, de meeste overeenkomst met een toendra zal hebben gehad. Met nadruk vestigen wij echter de aandacht op de omstandigheid, dat van echte toendraplanten tot dusver geen overblijfselen gevonden zijn.

Hierna vertoont het diagram een duidelijke stijging van de *Betula*-percentages, samengaand met een vermindering van de kruiden-procenten. We noemen het in die tijd aanwezige vegetatietype, in navolging van IVERSEN, een „parktoendra“, die we ons moeten voorstellen als een open landschap met verspreide boomgroepen.

Direct daarop volgt een nieuw, en ditmaal zeer hoog maximum van het kruidenpollen. De weinige *Betula*-pollenkorrels zijn ook hier bijna alle opvallend klein, terwijl de T.P.F. zeer gering is. Het lijkt aannemelijk, hier tot het bestaan hebben van een toendra te besluiten, daarbij dezelfde gedachtengang volgend, die tot de keuze van de uitdrukking „niet geheel

"boomloze toendra" heeft geleid, en met inachtneming van hetzelfde voorbehoud.

Hierna treedt een plotselinge en sterke teruggang op van de kruidenlijn, gepaard gaande met een scherpe stijging van de *Betula*-curve, en met een veel grotere T.P.F. Iets hoger in het diagram neemt het *Pinus*-percentage plotseling toe en vervolgens geleidelijk weer af, terwijl het kruidenprocent opnieuw groter wordt. Blijkbaar was het terrein om het meer in de tijd van het lage kruidenpercentage bebost, in tegenstelling met daarvoor en daarna. Ook de sedimentatie wijst op een klimaatsverbetering, want in deze bostijd was zij hoofdzakelijk organogeen, na tevoren minerogeen geweest te zijn. De bossen worden, gezien het hoge kruidenpercentage, weer lichter, maar ditmaal komt het niet verder dan tot de vorming van een „park-toendra". De *Empetrum*-heide moet in deze tijd een grote uitbreiding gehad hebben.

Ten slotte begint nu, zoals uit het bovenste deel van het diagram blijkt, de definitieve bebossing. Spoedig nemen *Pinus* en *Corylus* een belangrijke plaats in de bossen in, terwijl ook de componenten van het *Quercetum mixtum* verschijnen.

Parallelisatie en zone-indeling.

Vergelijkt men nu het diagram van het Hijkemeer met de Deense diagrammen van de hand van IVERSEN, dan blijkt met een oogopslag, dat in het onze een duidelijke Allerødzone aanwezig is. Ook overigens is een goede parallelisatie met de Deense diagrammen mogelijk, niet alleen wat het algemene verloop aangaat, maar ook wat betreft het voorkomen van bepaalde soorten kruidenpollen. We hebben dan ook niet geaarzeld, voor ons diagram de zone-indeling van IVERSEN te gebruiken (zie hiervoor het diagram en de overzichtstabel).

Het lijkt nuttig, om hier, waar voor de eerste maal de Allerød-oscillatie met zekerheid in Nederland herkend is, de waarnemingen op te sommen, die tot haar aanwezigheid deden besluiten. Deze zijn:

1. Een wijziging in de sedimentatie: een organogene afzetting, een laag detritus-gyttja, ingeschakeld in een pakket hoofdzakelijk minerogene sedimenten van laatglaciale ouderdom, wijzend op een tijdelijke klimaatsverbetering.
2. Het plotseling sterk afnemen van de kruidenpollenpercentages in de detritus-gyttja-laag en het toenemen van de boompollenprocenten, met stijging van de boompollenfrequentie en een daling van de *Salix*-percentages. Dit alles eveneens wijzend op een tijdelijke klimaatsverbetering.
3. De overeenstemming van de onder en boven deze laag liggende sedimenten — wat kruidenpolleninhoud betreft — met de sedimenten onder en boven de Allerød-laag in Denemarken, resp. overeenkomende met de Oudere *Dryas*-laag en de Jongere *Dryas*-laag. Zo komen b.v. voor *Hippophaë* en *Helianthemum* in de er onder liggende, *Empetrum* in de er boven liggende laag.

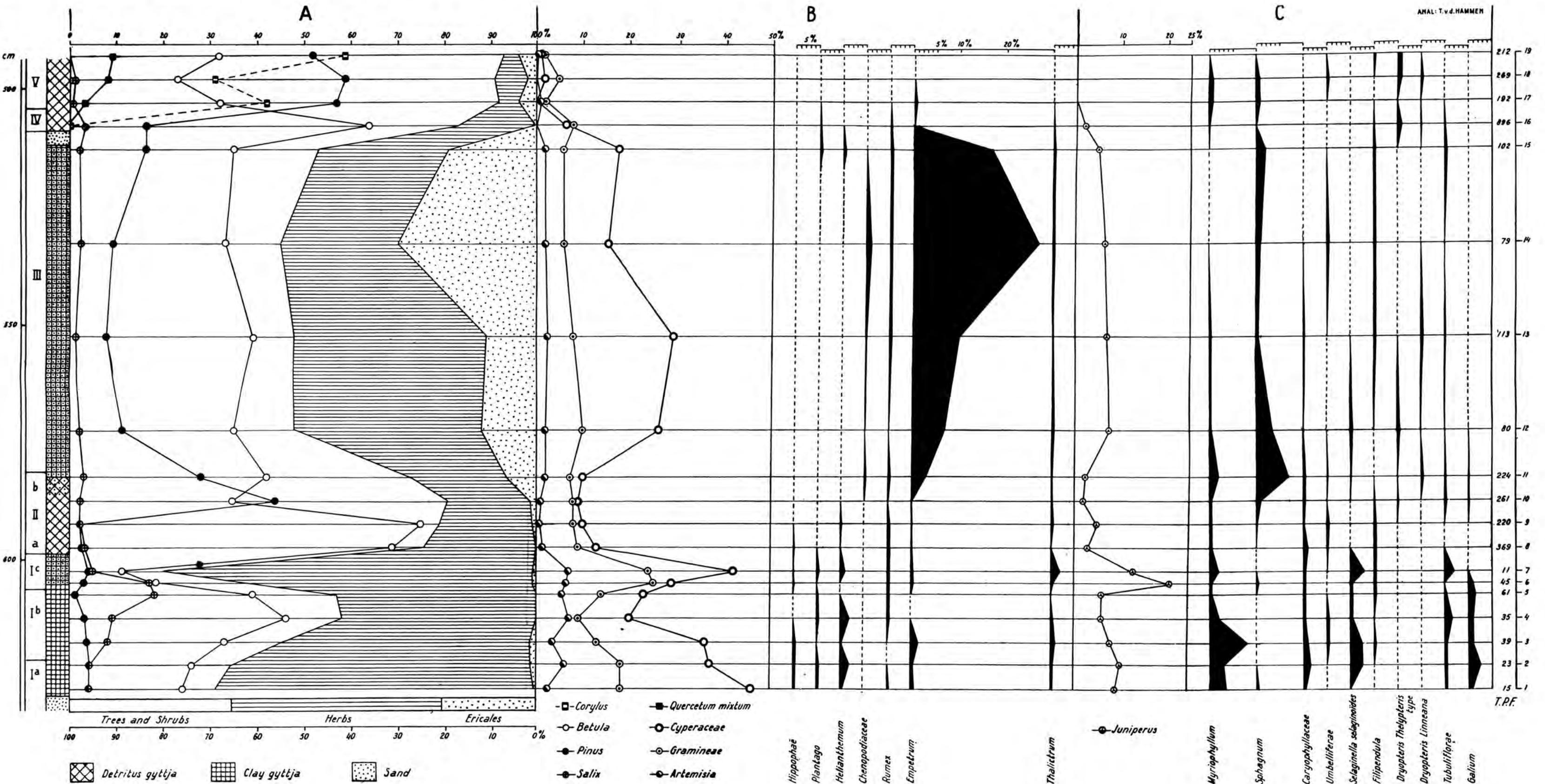


Diagram from the Hijkermeer (lake of Hijken), in the province of Drente, Holland. The sum of calculation of the pollen percentages consists of the pollen of all anemophile plants (with the exception of the water-plants).

The diagram consists of three parts:
A. The general part.
B. Pollen curves for the anemophile herbs (together with Hippophaë and Empetrum), which are included in the sum of calculation (striped and spotted part of A).
C. Curves for the pollen and spores of plants which are not included in the sum.

(T.P.F. = Tree Pollen Frequency = number of treepollen grains per slide 18 × 18 mm).

Hiermede zijn dus tevens de redenen van parallelisatie van de onder en boven onze Allerød-laag aanwezige laatglaciale sedimenten met IVERSENS zones I en III aangegeven.

Er blijkt nu in onze zone I nog een kleine oscillatie voor te komen, die wel is waar van veel geringere betekenis is dan de Allerød, doch niettemin een opvallend karakter heeft. IVERSEN vond een dergelijke klimaatsschommeling op geheel analoge plaats in het diagram in afzettingen uit het Bøllingmeer in Midden-Jutland (IVERSEN, 1947), die hij Bølling-oscillatie noemde. Zowel bij onze oscillatie als bij die van Bølling is de klimaatsterrugslag aan het eind veel scherper dan de klimaatsverbetering aan het begin, die geleidelijker geschiedde, bij beide uitkomend in het verloop van de Betula- en de kruidenlijn. Het laat dan ook geen twijfel, dat beide oscillaties identiek zijn. Daardoor zijn we in staat, hier ook IVERSENS onderverdeling van de Oudere Dryastijd te gebruiken, n.l.

- Ic Vroegere Dryas-tijd
- Ib Bølling-oscillatie
- Ia Vroegste Dryas-tijd.

We gebruiken deze benamingen, hoewel zij minder fraai zijn, in plaats van Oudere en Oudste Dryas-tijd, om verwarring met de naam Oudere Dryas-tijd voor de gehele zone I te voorkomen. De Allerød-tijd kan bij ons verdeeld worden in een oudere fase met zuivere Betula-bossen, en een jongere fase met Pinus-Betula-bossen. Merkwaardig is de plotseling steil opkomende Pinus-lijn. Een geheel analoog verloop heeft deze lijn op Bornholm (IVERSEN, schr. med.). Een indeling in tweeën van de Allerød-tijd op grond van een kleine klimaatsschommeling, zoals die in de Deense diagrammen vaak uitkomt, is in ons diagram niet aan te tonen. Volgens IVERSEN gaat het hier om een kleine oscillatie, die alleen in een zeker randgebied om de gletschers uit zal komen.

Op zone III volgt, met de definitieve klimaatsverbetering en bebossing, zone IV, het Praeboreaal, waarvan het begin samenvalt met een verandering in sedimentatie.

Slechts één monster (16) valt in deze zone, die hier dan ook slecht ontwikkeld is. In dit monster werd één *Corylus* korrel aangetroffen. De zone van de zuivere Pinus-Betula-bossen schijnt dus in ons diagram te ontbreken; het is evenwel onzeker, welke waarde aan de vondst van slechts één *Corylus*-korrel gehecht moet worden.

De monsters 17, 18 en 19 behoren tot het onderste deel van het Boreaal, zone V, met vrij hoge *Corylus*-percentages en opkomst van de componenten van het *Quercetum mixtum*.

Botany. — De F_4 -zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van *Phaseolus vulgaris*. I. By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of October 30, 1948.)

Het uitgangsmateriaal voor deze onderzoeken, die in 1932 werden begonnen, bestond uit enige bonen van twee zuivere lijnen. (Genetica, 1934). De bonen van de I-lijn zijn lang, breed en dun, die van de II-lijn zijn kort, iets minder breed en dik (Deze Proceed. 1948, Vol. 51, No. 2 en 3, Pl. 1). De bonen van de eerste bastaardgeneratie, F_1 , vertonen matroclinie, te verklaren uit de moederlijke zaadhuid van de F_1 -bonen (Genetica, 1947). In de F_2 -zaadgeneratie treedt splitsing op, die getemperd wordt door de uniforme zaadhuid van de F_1 -planten (Genetica, 1947, 2). De F_3 -zaadgeneratie wordt gevormd uit de bonenopbrengsten van F_2 -planten en deze kunnen zeer verschillen (Genetica, 1947, 2). Bij de kruisingen vertonen de bonen enige dominantie van de grote afmetingen over de kleine. Voor de verklaring van onze resultaten nemen we de werking van polymere factoren aan.

Deze mededeling over de F_4 -zaadgeneratie van 1936, die terug te voeren is tot kruisingen van 1933 sluit aan bij een publicatie in Genetica (1948) over de F_4 -zaadgeneratie van 1935, die tot kruisingen van 1932 teruggevoerd kan worden.

We groepeerden het materiaal volgens de formule van de uitgangsbonen van F_3 -1935 voor de F_4 -bonenopbrengsten van 1936. Volgens het vereenvoudigde tetrahybride schema onderscheiden we 8 klassen (1947; deze Proceedings, Vol. 50, No. 7).

C1 1a. De formule van de uitgangsbonen is $L_1 L_2 B Th$, d.w.z., alle 3 afmetingen zijn zeer groot. 14 gevallen.

In één geval is de formule van de gemiddelden van de bonenopbrengst (pl. 331, tab. 1 en 1a) ook $L_1 L_2 B Th$.

Van de bonenopbrengst van de pl. 84 (tab. 21, 1947, Genetica) d.i. de plant, waarvan de uitgangsboon voor pl. 331 genomen is, is de formule van de gemiddelden $L B Th$. De uitgangsboon voor pl. 84 is van pl. 82, F_2 -1934, en heeft de form. $L b Th$, cl 3; de breedte blijft iets beneden de grenswaarde ($b = 8.4$ mm). De gemiddelde dikte van de bonenopbrengst van pl. 331 is groter dan we bij bonenopbrengsten van de I-lijn van 1936 aantreffen; het gemiddelde gewicht is veel groter (tab. 4a). Van de individuele bonen van de bonenopbrengst van pl. 331 is de grootste lengte $l = 17.5$ mm, de grootste dikte $d = 7.3$ mm. Van de uitgangsboon van pl. 84 voor pl. 331 beantwoordt het genotype in hoge mate aan de form. $L_1 L_2 B Th$.

De formule van de gemiddelden van de bonenopbrengsten van de overige 13 gevallen is in 2 gevallen (pl. 274 en 404) $L_1 l_2 B Th$, cl 1b, in 6 gevallen (pl. 275, 277, 300, 306, 309 en 336) $L_1 l_2 B th$, cl 2b, in 2 gevallen (pl. 267 en 278, blz. II 76) $L B th$, cl 4, in één geval (pl. 293) $L b Th$, cl 3 (grensgeval van cl 1) en in 2 gevallen $l b th$, cl 8. Uit deze opsomming en uit het verdere onderzoek en de classificatie, blijkt, dat we met verwante gevallen te doen hebben. In het centrum staan de 6 gevallen met de form. $L B th$, cl 2, der gemiddelden. Van één van deze gevallen geven we een korte beschrijving (pl. 309, tab. 1 en 1a).

De uitgangsboon is 2 p 1 b van pl. 73, F₃-1935 (tab. 1). Van de gemiddelden van pl. 73 is de formule L₁ L₂ B th als van de I-lijn; de gemiddelde lengte is zeer groot ($l = 16.0$ mm, tab. 21, 1947, Genetica). De uitgangsboon voor pl. 73, F₃-1935, is 4 p 1 b van pl. 81, F₂-1934; de formule is L₁ l₂ B Th; de lengte is zeer groot, heeft de grenswaarde ($l = 15.5$ mm, tab. 21, 1947). De formule van de gemiddelden is eveneens L B Th.

De bonenopbrengst van 35 bonen van pl. 73 heeft een zeer grote gemiddelde lengte en breedte ($l_m = 15.9$, $b_m = 9.5$ mm) en een kleine gemiddelde dikte ($th_m = 6.3$ mm); de formule is L₁ L₂ B th, als van de I-lijn. Het gemiddelde gewicht is zeer groot ($w_m = 66$ cg). Vergelijken met de gemiddelden van bonenopbrengsten van de I-lijn, zijn de gemiddelde indices (60, 40 en 66 iets te hoog; ook het gemiddelde gewicht is iets te groot (tab. 2a). Onder de individuele bonen zijn er twee, zoals we ze onder de bonen van de I-lijn niet aantreffen. Met de afmetingen van de uitgangsboon van pl. 73 voor pl. 309 komen die van enkele bonen van de I-lijn van 1935 als hoge uitzondering overeen (tab. 3a). Het gewicht is echter groter ($w = 101$ cg); de 2 overige bonen van de peul hebben ook zeer grote afmetingen en een zeer groot gewicht ($w = 97$ en 92 cg).

De gemiddelden van de bonenopbrengst van 38 bonen van pl. 309 zijn groot (tab. 1), de formule is L B th als van de I-lijn. De gemiddelde breedte is zeer groot; daardoor is de gemiddelde L B-index hoog en de gemiddelde B Th-index laag, ofschoon de gemiddelde dikte groot is. De bonenopbrengst bevat volgens de aantekeningen, brede, minder brede en smalle peulen. Een enkele peul is zwart, door het weer aangetast; enkele bonen zijn donker en vlekkig. Sommige bonen vertonen de bizarheden van vorm, — een kuilje en een vlekje aan de uiteinden —, van de bonen van de I-lijn. De variabiliteit is groot ($l = 15.37 \pm 0.2$, $\sigma = 1.05 \pm 0.1$). Onder de bonenopbrengsten van de I-lijn van 1936 is er geen, van welke de gemiddelden geheel overeenkomen met die van pl. 309; de gemiddelde dikte is te groot, de gemiddelde L B-index te hoog, (vgl. tab. 1 en tab. 4a). Onder de individuele bonen van pl. 309 zijn er 2 met een zeer grote breedte ($b = 10.3$ mm) en een hoge L B-index ($L B = 72$), zoals we ze niet bij vergelijkbonen van de I-lijn van 1936 aantreffen (tab. 6). Ook de classificatie van de bonenopbrengst van pl. 309 (tab. 1a) verschilt van die van overeenkomstige bonenopbrengsten van de I-lijn (tab. 5a). Er zijn te veel bonen van cl 1a en 1b, form. L B Th. De formule van de uitgangsboon van pl. 73 voor pl. 309 is L₁ L₂ B Th in niet geheel homozygote vorm.

Van de gevallen met de form. L₁ L₂ B Th, cl 1a is de gemiddelde dikte niet zeer groot; de gemiddelde L Th- en B Th-indices zijn niet zeer hoog; we moeten daarbij rekening houden met „spurious correlation". De bonen hebben overeenkomst met die van cl 2. Een enkel geval voldoet goed aan de formule L₁ L₂ B Th; de uitgangsboon voor de bonenopbrengst heeft hier deze formule in overwegend homozygote vorm.

Cl 1b. De formule van de uitgangsbonen is L₁ l₂ B Th, d.w.z., alle 3 afmetingen zijn groot. 40 gevallen. De bonen met de form. L B Th vormen het grote aantal bonen na de kruisingen; ze geven aan het hele materiaal het intermediaire uiterlijk.

In 5 gevallen is de formule van de gemiddelden ook L B Th (pl. 298, 242, 243, 251 en 410). Volgens de classificatie zijn er in de bonenopbrengsten een overwegend aantal bonen in cl 1. Van één geval (pl. 298, tab. 1 en 1a) geven we een korte beschrijving.

De uitgangsboon voor pl. 298 is van pl. 70. De formule van de gemiddelden van pl. 70 is L B Th ($l_m = 14.5$ mm, tab. 21, 1947). De uitgangsboon voor pl. 70 is van pl. 81, F₂-1934; de formule is L B Th. De formule van de gemiddelden van pl. 81 is L B Th ($l_m = 14.2$ mm).

De uitgangsboon van pl. 70 voor pl. 298 heeft iets kleiner lengte dan de 1ste en de 3de, laatste, boon van de peul; de breedte en de dikte zijn iets groter. De formule van de gemiddelden van pl. 298 is L B Th. Bij de bonenopbrengst staat vermeld „mooie, gave peulen". Alle individuele bonen hebben een grote dikte; de grootste dikte is $th = 7.2$ mm. Volgens de classificatie (tab. 1a) zijn bijna alle bonen in cl 1. Van de 3 bonen in cl 2 heeft de dikte de grenswaarde ($th = 6.5, 6.5$ en 6.4 mm). We hebben hier met een mooi geval van een bonenopbrengst met de form. L B Th te doen. Ook de ascendentie wijst hierop.

In 15 gevallen is de formule van de gemiddelden der bonenopbrengsten L B th, als van cl 2. Vaak heeft hier de gemiddelde dikte de grenswaarde ($th_m = 6.5$ mm). Een geval (pl. 361) met een kleine gemiddelde dikte ($th_m = 6.1$ mm) kan beter tot cl 2 gerekend worden; dit geldt blijkens de classificatie voor nog 4 gevallen (pl. 258, 268, 1042 en 1044). Vier gevallen hebben zeer veel bonen in cl 1 en cl 2 en enkele in cl 4, 6 en 8. Van één van deze gevallen (pl. 194) volgt hieronder een korte beschrijving. In 6 gevallen zijn er ook vrij veel bonen in andere klassen. In cl 5 en in cl 7 wordt in al deze 15 gevallen slechts één boon aangetroffen (pl. 281, 1 p 4 b; form. 1 B Th, cl 5, de lengte heeft de grenswaarde, $l = 13.0$ mm, en pl. 294, 5 p 2 b. form. 1 b Th, cl 7; de dikte is even over de grenswaarde, $th = 6.6$ mm). In 21 bonenopbrengsten van de I-lijn van 1936, tezamen 555 bonen, zijn volgens de classificatie geen bonen in cl 5 en in cl 7.

Van pl. 194 (tab. 1 en 1a) is de uitgangsboon van pl. 47, F₃-1935. De formule van de gemiddelden van de bonenopbrengsten van pl. 47 is L B Th. De uitgangsboon voor pl. 47 is van pl. 66, F₂-1934; de formule is L B Th. Ook de formule van de gemiddelden van pl. 66 is L B Th.

De uitgangsboon van pl. 47 voor pl. 194 is van een peul met 8 bonen. Enkele van de overige bonen hebben een iets grotere lengte ($l = 14.2$ mm) en een iets grotere dikte ($th = 7.0$ — 7.4 mm). De formule van de gemiddelden van pl. 194 is L B th ($th_m = 6.5$ mm). Er zijn „brede en smalle peulen" volgens de aantekeningen. Van een „lange en smalle" peul met 7 bonen, is de breedte van de bonen 9.2—9.5 mm en de dikte 6.4—6.9 mm. Van een „brede" peul met 2 bonen is de breedte der bonen 10 en 10.4 mm, de dikte 6.6 en 7.2 mm; van een andere is $b = 10.5, 10.4$ en 10.1 mm. Van een smalle peul met 5 bonen is de breedte der bonen 8.2, 8.5—8.7 mm, en de dikte 5.3—5.6 mm. (Er zijn nog 2 peulen met 7 bonen in de gemeten bonenopbrengst. We hebben hier waarschijnlijk met erfelijkheid van het aantal bonen in de peul te doen. Dus als half-ras. De uitgangsboon van pl. 47 voor pl. 194 was van een peul met 8 bonen.) Volgens de classificatie zijn er zeer veel bonen in cl 1 en een minder groot aantal in cl 2. De opbrengst is samengesteld. De uitgangsboon van pl. 47 voor pl. 194 is niet homozygoot voor de form. L B Th.

In één geval (pl. 308) is de formule van de gemiddelden van de bonenopbrengst L b Th, cl 3; we bespreken het daar. Van 6 gevallen is de formule van de gemiddelden L b th, cl 4. Eén van deze gevallen (pl. 322) is een grensgeval van de gevallen met de formule der gemiddelden L B Th, cl 1 ($th_m = 6.5$ mm); ook volgens de classificatie. De 5 overige van deze 6 gevallen bespreken we bij cl 4. Een geval (pl. 185) met de formule 1 b Th, cl 7 der gemiddelden bespreken we daar. Er zijn 11 gevallen met de formule 1 b th, cl 8, van de gemiddelden der bonenopbrengsten. Eén van deze gevallen (pl. 213) bespreken we bij cl 7, form. 1 b Th, de overige bij cl 8, form. 1 b th.

Er zijn 6 gevallen, waar de uitgangsboon niet de form. L B Th heeft en waar de formule der gemiddelden L B Th, cl 1 is. In één van deze gevallen is de formule van de uitgangsboon van de bonenopbrengst (pl. 321) L B th, cl 2 (tab. 1 en 1a). Volgens de classificatie zijn er zeer veel bonen in cl 1 en enige in cl 2. Ze heeft overeenkomst met die van pl. 322 en 323, waarvan de uitgangsbonen van dezelfde plant (pl. 78, F₃-1935) zijn.

De formule van de gemiddelden van pl. 78 is L B th ($th_m = 6.5$ mm). De formule van de uitgangsboon voor pl. 78 is van pl. 81, F₂-1934; de formule is L₁ L₂ B th ($l = 16.1$ mm, $th = 6.3$ mm). De formule van de gemiddelden van pl. 81 is L B Th ($th_m = 7.0$ mm). De uitgangsboon van pl. 78 voor pl. 321 heeft een veel groter lengte dan de 3 overige bonen van de peul en een kleiner dikte ($l = 15.6$ contra 13.7—14.3 mm; $d = 6.0$ contra 6.4—7.0 mm). In de bonenopbrengst van pl. 321 zijn enkele bonen met een kleine dikte ($th = 6.2$ mm) en overeenkomende met bonen van de I-lijn. De uitgangsboon van pl. 78 voor pl. 321, met de form. L B th is volgens de classificatie van de bonenopbrengst van pl. 321 en op grond van de gegevens der ascendentie heterozygoot en stemt zeer overeen met de form. L B Th.

Eveneens in één geval is de formule van de uitgangsboon L b Th, cl 3, en die van de gemiddelden van de bonenopbrengst (pl. 228) L B Th, cl 1.

De lengte van de uitgangsboon is zeer groot ($l = 15.9$ mm); de formule is dus L₁ L₂ b Th, cl 3a. De breedte ($b = 8.5$ mm) heeft de grenswaarde voor de form. L₁ L₂ B Th, cl 1a. Volgens de classificatie van de bonenopbrengst van pl. 228 zijn er zeer veel bonen in cl 1 (één in cl 1a); ook enige in cl 3 en cl 4. We hebben hier bij de uitgangsboon dus wel met een erfelijk iets kleinere breedte te doen.

In twee gevallen is de formule van de uitgangsboon L b th, cl 4, terwijl die van de gemiddelden der bonenopbrengst (pl. 264 en 288) L B Th is.

Volgens de classificatie behoort de bonenopbrengst van pl. 264 tot cl 1, haar uitgangsboon van pl. 59 heeft de formule L B Th in heterozygote vorm. De classificatie van pl. 288 wijst zeer veel bonen aan in cl 1, meerdere in cl 4 en enige in andere klassen. De bonenopbrengst is samengesteld en de uitgangsboon van pl. 66 bevat nevens factoren in heterozygote vorm, ook enige b b- en th th-verbindingen.

Er is ook één geval met de form. I B Th, cl 5, van de uitgangsboon, terwijl de formule van de gemiddelden van de bonenopbrengst (pl. 364) L B Th is.

De gemiddelden zijn niet groot ($l_m = 13.2$ mm). De lengte van de uitgangsboon voor pl. 364 is weinig beneden de grenswaarde ($l = 12.8$ mm) voor de form. L B Th. Volgens de classificatie van pl. 364 zijn er veel bonen in cl 1 en enige over alle overige klassen verspreid.

Ten slotte zijn er 2 gevallen met de form. l b th van de uitgangsboon en L B Th van de gemiddelden der bonenopbrengsten (pl. 295 en 390).

De uitgangsboon voor pl. 295, F₄-1936 is van pl. 69, F₃-1935; ze heeft zeer kleine afmetingen; die van de 1e en 2e boon van de peul hebben veel grotere afmetingen; de 4e, laatste, boon van de peul heeft iets minder grote afmetingen dan de 1e en de 2e boon. De uitgangsboon is de enige zeer kleine boon van de hele bonenopbrengst van pl. 69. De formule van de gemiddelden van pl. 69 is L B Th. De uitgangsboon voor pl. 69 is van pl. 81 F₂-1934 en heeft de form. L B Th met zeer grote afmetingen. De formule van de gemiddelden van pl. 81 is eveneens L B Th. Volgens de classificatie van de

bonenopbrengst van pl. 295 zijn bijna alle bonen in cl 1, een enkele is in cl 3 en in cl 4. We nemen aan, dat de uitgangsboon van pl. 69 voor pl. 295 met het phaenotype $l\ b\ th$ een niet-erfelijke variant is te midden van de 3 overige bonen van de peul, die het phaenotype $L\ B\ Th$ en $L\ b\ Th$ hebben. Op grond van de samenstelling van de bonenopbrengst van pl. 295 nemen we verder aan, dat de uitgangsboon het genotype $L\ B\ Th$ heeft in een in hoge mate homozygote vorm.

Bij de groep gevallen met de form. $L\ B\ Th$, cl 1 van de uitgangsbonen, vinden we een enkel geval, waar we uit de classificatie van de bonenopbrengst en de ascendentie, tot een hoge mate van homozygotie voor de form. $L\ B\ Th$ mogen besluiten. In veel gevallen zijn er in de bonenopbrengsten, volgens de classificatie bonen in veel klassen, vaak in cl 1, 2, 4 en 8, zelden in cl 5 en 7. De eerste tonen hun verwantschap met bonenopbrengsten van cl 2, met de form. $L\ B\ th$ als van de I-lijn, de tweede met die van cl 7, form. $l\ b\ Th$ als van de II-lijn. In bonenopbrengsten van de I-lijn zijn er volgens de classificatie, behalve een groot aantal bonen in cl 2, ook enige bonen in cl 1, 4 en 8a (tab. 5a), in die van de II-lijn zijn er behalve veel bonen in cl 7, ook enige in cl 5 en 8b (tab. 5b).

Cl 2a. De formule van de uitgangsboon is $L_1\ L_2\ B\ th$ als van de bonen van de I-lijn met een zeer grote lengte (groter dan 15.5 mm). 6 gevallen. In 2 gevallen is de formule van de gemiddelden der bonenopbrengsten (pl. 291 en 338) $L_1\ L_2\ B\ th$, cl 2a.

De uitgangsboon voor pl. 291 is van pl. 68, F₃-1935. De formule van de gemiddelden van pl. 68 is $L\ B\ Th$ (tab. 21, 1947); de gemiddelde breedte is groot ($b_m = 9.7$ mm), de gemiddelde dikte is slechts even boven de grenswaarde ($th_m = 6.6$ mm). De uitgangsboon voor pl. 68 is van pl. 76, F₂-1934; de formule is $L\ B\ th$, cl 2 ($th = 6.4$ mm). De formule van de gemiddelden van pl. 76 is $L\ B\ Th$.

Van pl. 68, F₃-1935 zijn de uitgangsbonen genomen voor de 4 pl. 291—294, F₄-1936. Alle hebben een grote breedte (10.9, 10.3, 10.0, 9.6 mm). Die voor de pl. 292—294 hebben de form. $L\ B\ Th$; hun lengte is kleiner dan die voor pl. 291 (17.0 en 15.6, 14.3, 13.7 mm; fig. 1—4). Dergelijke bonen als de uitgangsboon van pl. 68 voor pl. 291 komen ook onder de bonen van de I-lijn van 1935 als hoge uitzondering voor (tab. 3a). Van de bonenopbrengst van pl. 291 is de gemiddelde breedte groot, de gemiddelde L B-index is hoog. De gemiddelde L B-indices van de pl. 291—294 komen zeer overeen ($L\ B_n = 69$ en $= 68$). Uit de characterogrammen (Fig. 1—4) zien we de kleine gemiddelde lengten en grote gemiddelde breedten en daardoor hoge gemiddelde L B-indices. Onder de bonenopbrengsten van de I-lijn van 1936 zijn er niet, waarvan de gemiddelden overeenkomen met die van de pl. 291—294 (fig. 1—4 en tab. 4): de gemiddelde L B-indices zijn veel te hoog (68 contra 62—64). De gemiddelde lengte van de bonenopbrengst van pl. 293 is kleiner dan bij bonenopbrengsten van de I-lijn van 1936 voor komt. De formule voor de uitgangsboon van pl. 68 voor pl. 293 bevat dus een kleiner aantal L-factoren dan die van de I-lijn. De gemiddelde dikte van de bonenopbrengsten van de pl. 291 en 292 ($th_m = 5.5$ mm) is kleiner dan bij bonenopbrengsten van de I-lijn van 1936 voorkomt (tab. 4). Als we hier met een genotypisch verschil te doen hebben, moeten we aannemen, dat de formule van de dikte van bonen van de I-lijn niet eenvoudig $th_1\ th_1 \dots th_3\ th_3$ is en die van bonen van de II-lijn $Th_1\ Th_1 \dots Th_3\ Th_3$, doch b.v., resp., $th_1\ th_1\ Th_2\ Th_2\ th_3\ th_3\ th_4\ th_4$ en $Th_1\ Th_1\ th_2\ Th_2\ Th_3\ Th_3\ Th_4\ Th_4$, zodat er door transgressieve variabiliteit een dikte met de formule $th_1\ th_2 \dots th_4\ th_4$ mogelijk is, die kleiner is dan de dikte van bonen van de I-lijn.

Van de kleine bonenopbrengst van 15 bonen van pl. 291 zijn er volgens de classificatie

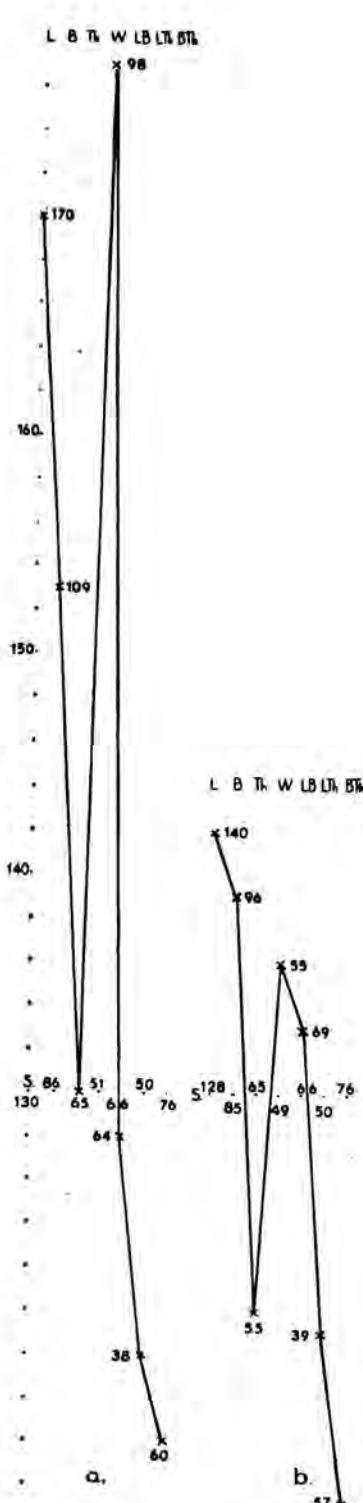


Fig. 1a. Characterogram of the initial bean 1 p. 1 b. (1. pod, 1. bean) of pl. 68 of F₃-1935 for pl. 291, F₄-1936.

Fig. 1b. Idem of the mean dimensions, weight and indices of pl. 291; n = 15. S = standardcharacterogram of 1935, S₁ = standardcharacterogram of 1936. Dimensions in 0.1 mm, weights in cg.

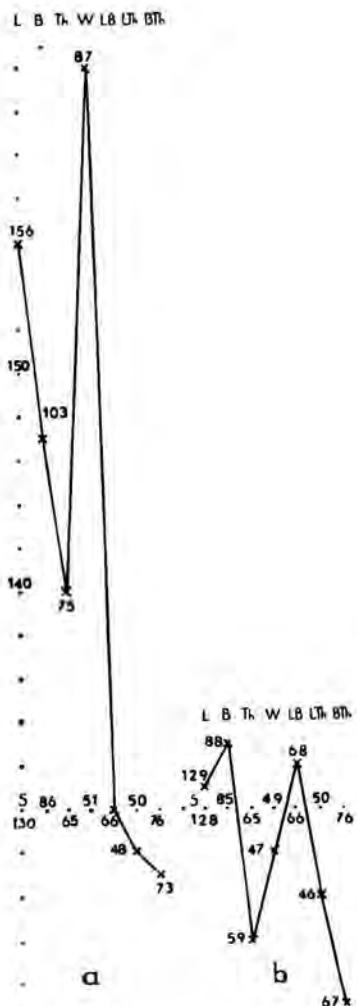


Fig. 2b. Idem of the averages of pl. 293; n = 23.

14 in cl 2 en 1 in cl 4. De bonen zijn zeer overeenkomstig; alle hebben een grote breedte ($b = 9.3-10.1$ mm en eenmaal, van een boon, die de laatste is in de rij van de peul, is $b = 8.5$ mm). De LB-index is zeer hoog, varieert van 65-73 (en is eenmaal = 78). Onder de individuele bonen zijn er niet, zoals ze bij bonen van de I-lijn voorkomen. We

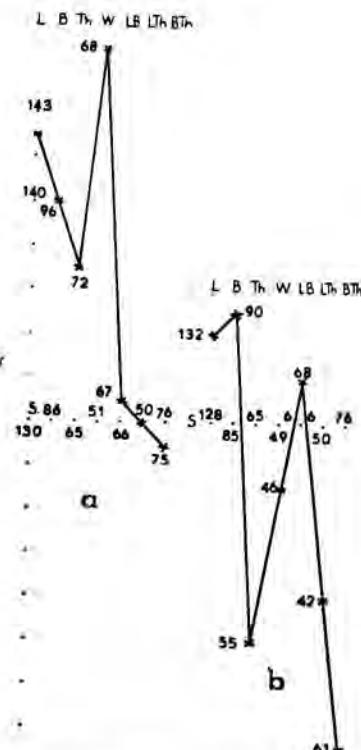


Fig. 3a. Characterogram of the initial bean 4 p. 4 b. of pl. 68 of F_3 -1935 for pl. 292, F_4 -1936.

Fig. 3b. Idem of the averages of pl. 292; $n = 25$.

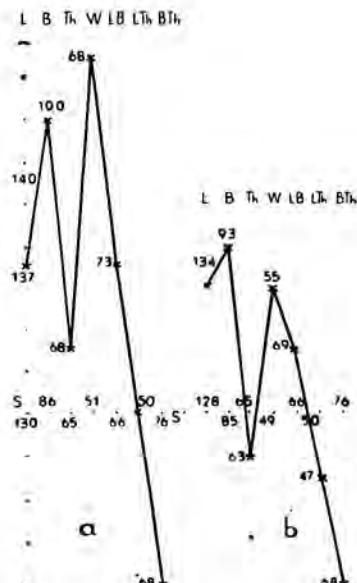


Fig. 4a. Characterogram of the initial bean 13 p. 2 b. of pl. 68 of F_3 -1935 for pl. 294, F_4 -1936.

Fig. 4b. Idem of the averages of pl. 294; $n = 14$.

hebben met een genotypisch verschil te doen. Het voornaamste kenmerk van de bonen is de grote breedte. Daardoor is de LB-index hoog en de B Th-index laag. De L Th-index komt overeen met die van bonen van de I-lijn. De dikte van de bonen is zeer klein ($th = 5.1-6.1$ en eenmaal = 4.8 mm). De lengte van de bonen is niet zeer groot, ($l = 13.6-14.6$ mm en eenmaal $l = 13.3$ en = 13.1 mm, daardoor is de L Th-index niet zeer laag, komt overeen met die van bonen van de I-lijn).

We hebben bij pl. 291, voor zover het kleine aantal bonen een beoordeling toelaat, met een bonenopbrengst te doen, waarvan het genotype van de uitgangsboon van pl. 68, in hoge mate homozygoot is voor een groot aantal B-factoren en een kleiner aantal L-factoren dan het genotype van de bonen van de I-lijn bevat. Ook het aantal th-factoren in homozygote vorm is misschien groter dan dit aantal van de I-lijn.

Eveneens in 2 gevallen is de formule van de gemiddelden van de bonen-

opbrengsten (pl. 318 en 346) L b th, cl 4. De bonenopbrengst van pl. 318 bestaat uit slechts 4 bonen. Pl. 346 vindt beter bij cl 4 bespreking. In één geval is de formule van de gemiddelden van de bonenopbrengst (pl. 260) 1B Th, cl 6 en in één (pl. 128) 1b th, cl 8.

Cl 2b. De formule van de uitgangsboon is $L_1 l_2 B th$ als van de bonen van de I-lijn, waarvan de lengte niet groter is dan 15.5 mm, 14 gevallen. In 3 gevallen (pl. 245, pl. 282 en pl. 337) is de formule van de gemiddelen ook L B th.

Van pl. 282 komt de uitgangsboon van pl. 64, F₃-1935, geheel overeen met bonen van de I-lijn, de breedte is zeer groot ($b = 10.0$ mm; tab. 3a). Bonenopbrengsten met de gemiddelden als van pl. 282 treffen we bij de I-lijn van 1936 niet aan; de gemiddelde LB-index is te hoog ($LB = 67$; tab. 4a). De classificatie komt overeen met die van bonenopbrengsten van de I-lijn van 1936. Van de individuele bonen van pl. 282 is de grootste dikte, $th = 7.1$ mm. Bij deze boon staat aangetekend „abnormaal“. Er zijn zeer veel bonen met een grote breedte en een niet grote lengte, waardoor de LB-index hoog is. De bonenopbrengst van pl. 282 komt in dit opzicht overeen met die van pl. 291 (blz. 80). Het genotype van de uitgangsboon van pl. 64 voor pl. 282 stemt overeen met dat van bonen van de I-lijn, maar bevat niet alle L-factoren.

Van pl. 337 (tab. 1 en 1a) is de uitgangsboon van pl. 87 (zie ook pl. 338, blz. 80 en pl. 336 blz. 76). De formule van de gemiddelden van pl. 87, F₃-1935, is L B Th ($th_m = 6.6$ mm). De uitgangsboon is van pl. 82, F₃-1934, de formule is LB Th ($l = 15.5$ mm). De formule van de gemiddelden van pl. 82 is L b Th ($b_m = 8.5$ mm). De uitgangsboon van pl. 87 (tab. 1) voor pl. 337 is van een peul met 7 bonen, die onderling veel verschillen. Drie bonen, waaronder de uitgangsboon hebben een kleine dikte ($th = 5.6$ en $= 5.9$ mm), drie andere hebben een grote dikte ($th = 7.1$ —7.2 mm). Vijf bonen, waaronder de uitgangsboon, hebben een grote breedte ($b = 10.0$ —10.4 mm). Bonenopbrengsten met gemiddelden als van pl. 337 treffen we niet bij de I-lijn van 1936 aan (tab. 4a). De gemiddelde LB-index is te hoog. Dezelfde te hoge LB-index hebben de pl. 336 en 338. De grootste dikte van de individuele bonen van pl. 337 is $th = 6.9$ mm. Onder de individuele bonen van pl. 337 zijn er geen, zoals ze, als hoge uitzondering, ook niet bij de I-lijn van 1936 voorkomen (tab. 6). Ook de classificatie (tab. 1a), treffen we zo bij bonenopbrengsten van de I-lijn van 1936 aan (tab. 1a en tab. 5a). Er zijn echter te veel bonen met een grote breedte en niet grote lengte. De uitgangsboon van pl. 87 voor pl. 337 komt zo onder de bonen van de I-lijn van 1935 als hoge uitzondering voor (tab. 1 en tab. 3a). De bonenopbrengst van pl. 337 lijkt op die van pl. 282 (blz. 83). De uitgangsboon van pl. 87 voor pl. 337 is in hoge mate homozygoot voor de formule van de bonen van de I-lijn, doch bezit een kleiner aantal L-factoren in homozygote vorm.

In 11 gevallen heeft de uitgangsboon de form. LB th, cl 2 en is de formule van de gemiddelden der bonenopbrengsten een andere. Eenmaal is ze L B Th, cl 1 (pl. 321, blz. 79), eenmaal L B th, cl 2 (pl. 374), eenmaal L b Th, cl 3 (pl. 121), eenmaal 1b Th, cl 7 en 8 maal is de formule der gemiddelden 1b th, cl 8. Deze gevallen vinden bij de resp. klassen bespreking.

Er zijn in het hele materiaal 33 gevallen, waar de formule van de gemiddelden van de bonenopbrengsten LB th, cl 2 is en waar de formule van de uitgangsbonen een andere dan LB th is. In 6 van deze gevallen is de formule van de uitgangsboon $L_1 l_2 B Th$ cl 1a, in 14 gevallen is ze $L_1 l_2 B Th$, cl 1b. Van 5 van deze gevallen komt de classificatie der bonenopbrengsten (pl. 258, 268, 361, 1042 en 1044) zeer met die van cl 2 overeen. Van pl. 361 (tab. 1 en 1a) volgt hier een korte beschrijving.

Pl. 361. De uitgangsboon is van pl. 95, F₃-1935. De formule van de gemiddelden van de bonenopbrengst van pl. 95 is L B Th; de gemiddelde lengte is zeer groot ($l_m = 15.5$ mm). Onder de bonenopbrengsten van de I-lijn van 1935 zijn er met nog grotere gemiddelde lengte ($l_m = 16.6$ mm). De formule van de uitgangsboon van pl. 82, F₂-1934 voor pl. 95 is L b th, cl. 4 ($l = 14.8$ mm); de formule van de gemiddelden van pl. 82 is L B Th.

De uitgangsboon van pl. 95 voor pl. 361 is één van de 3 weinig verschillende bonen van de peul. De formule van de gemiddelden van pl. 361 is L B th, cl 2; de gemiddelde dikte is klein ($th_m = 6.1$ mm, tab. 1). Bonenopbrengsten met dergelijke gemiddelden komen bij de I-lijn voor (tab. 4a). Onder de individuele bonen van pl. 361 is er een enkele met een zo grote dikte, als we onder de bonen van de I-lijn van 1936 niet aantreffen. Ook volgens de classificatie (tab. 1a) zijn er te veel bonen in cl 1. Ook de uitgangsboon van pl. 95 voor pl. 361 verschilt iets van overeenkomstige vergelijkbonen van de I-lijn van 1935 (tab. 1 en tab. 3a). De uitgangsboon voor pl. 361 van pl. 95 is, op grond van de samenstelling van de F₄-bonenopbrengst van 1936 en ook volgens zijn eigen phaenotype niet geheel homozygoot voor de form. L B th van de I-lijn.

Van de overige der genoemde 33 gevallen (blz. 83) is de formule van de uitgangsboon als volgt:

In 2 gevallen is ze L b Th, cl 3. In één er van (pl. 138) zijn er volgens de classificatie zéér veel bonen in cl 2; de bonenopbrengst heeft overeenkomst met bonenopbrengsten van de I-lijn. Er is geen enkele boon in cl 3. De grote dikte van de uitgangsboon van pl. 37, F₃-1935 voor pl. 138 is daarom opmerkelijk. Ook de bonenopbrengst (pl. 312) van het 2e geval stemt volgens de classificatie overeen met de I-lijn. Ook hier is geen enkele boon in cl 3. De uitgangsboon met de form. L b Th, cl 3 is waarschijnlijk een niet-erfelijke variant in deze classe. In één geval is de formule van de uitgangsboon L b th, cl 4 (pl. 374). Volgens de classificatie van pl. 374 zijn er zeer veel bonen in cl 2 en in cl 4, die dus deze bonenopbrengst met de form. L B th van de gemiddelden binnen het gebied van cl 4 brengt.

Er is ook één geval met de form. 1 B Th, cl 5 (pl. 1045) van de uitgangsboon. De lengte heeft de grenswaarde, $l = 13.0$ mm voor de form. L B Th, cl 1. De dikte is groot. Alle bonen van de bonenopbrengst van pl. 1045 hebben een kleine dikte en behoren volgens de classificatie tot cl 2, 4 en 8a. De grote dikte van de uitgangsboon kan hierdoor niet verklaard worden.

In 4 gevallen (pl. 165, pl. 201, pl. 373 en 398) is de formule van de uitgangsboon 1 b Th, cl 7. De samenstelling van de bonenopbrengst van pl. 165 (tab. 1 en 1a) geeft geen aanwijzing voor de formule, die de uitgangsboon heeft. Volgens de classificatie (tab. 1a) zijn er zeer veel bonen in cl 2, cl 1 en enige in cl 4. De gemiddelde dikte heeft de grenswaarde ($th_m = 6.5$ mm) voor de form. L B Th, cl 1. De uitgangsboon voor pl. 165 van pl. 43 F₂-1934 is de laatste boon van een peul met 6 bonen met verschillende dikten. Bonen als de uitgangsboon treffen we in het materiaal van de II-lijn van 1935 aan (tab. 7b). Ook van pl. 201 ($l = 12.8$ mm) behoort de uitgangsboon, volgens de classificatie van haar bonenopbrengst, ofschoon ze een grote dikte heeft, tot het erfgebied van de I-lijn. Van pl. 373 ($l = 12.9$ mm) is de uitgangsboon van een peul met 4 bonen, die alle een grote dikte hebben. Ze komen overeen met bonen van de II-lijn, maar de lengte is iets te groot (tab. 7b). Volgens de classificatie van de bonenopbrengst van pl. 373 zijn er zeer veel bonen in cl 2. Dit geldt ook voor de classificatie van pl. 378. We moeten aannemen, dat de bonen van de peul, waartoe ook de uitgangsboon van pl. 98 voor pl. 373 en voor pl. 378 behoren, niet-erfelijke variaties zijn en behoren tot de erfkring van de I-lijn. Er zijn onder de bonen van de I-lijn van 1935, er ook, die lijken op bonen van de II-lijn (tab. 7a).

In 6 gevallen ten slotte is de formule van de uitgangsboon 1 b th, cl 8, terwijl de formule van de gemiddelden der bonenopbrengsten L B th is (pl. 200, 329, 347, 354, 1046 en 1047). Meestal is de lengte van de uitgangsbonen niet zeer klein ($l = 12.8—12.1$

en 11.6 mm) en de gemiddelde lengte van de bonenopbrengsten niet zeer groot (14.6, 14.3, 13.9—13.4 mm). De uitgangsboon van pl. 48, F₃-1935 voor pl. 200 F₄-1936 is van een peul met 5 bonen, die alle nog al klein zijn, terwijl de bonenopbrengst van pl. 48 overigens peulen met grotere bonen bevat. Volgens de classificatie van de bonenopbrengst van pl. 200 zijn er veel bonen in cl 2, cl 1, cl 4 en cl 8. Ofschoon de uitgangsboon van pl. 48 voor pl. 200 een relatief grote dikte heeft ($th = 6.4$ mm), nemen we aan, dat ze behoort tot het erfgebied van de form. L B Th en L B th (z. boven). Overeenkomstige gegevens hebben we voor de uitgangsboon van pl. 82 voor pl. 329.

Iets uitvoeriger zijn we over een ander van deze gevallen (pl. 347, tab. 1 en 1a, cl 4). Van pl. 347, ook van pl. 346 en 345 (blz. II 77) zijn de uitgangsbonen van pl. 91. De uitgangsboon voor pl. 347 komt zeer overeen met de 4e en de 5e, laatste, boon van de peul en er zijn meer dergelijke bonen (tab. 9). De grootste lengte van de bonen van de bonenopbrengst van pl. 347 is $l = 16.3$ mm. De lengte gaat dus ver uit boven die van de uitgangsboon ($l = 12.8$ mm). De formule van de lengte van de uitgangsboon kan dus niet 11 zijn, daar de niet-erfelijke variabiliteit van de bonen van de II-lijn, dus haar variatie-breedte, voor materiaal van 1936, 7.3—13.1 mm is. Een boon met een lengte van 16.3 mm, zal in haar formule ook L-factoren in heterozygote en homozygote vorm bevatten. We nemen daarom aan, dat de lengte van de uitgangsboon in haar formule ook L L en L l-verbindingen bevat. De grootste breedte van de bonen van pl. 347 is $b = 9.1$ mm, die van bonen van de II-lijn is, $b = 9.5$ mm; de grootste dikte is $th = 7.1$ mm, die van bonen van de I-lijn, $th = 8.2$ mm; dan volgt $th = 7.2$ mm. De formule van de breedte en van de dikte van de uitgangsbonen van pl. 91 voor pl. 347 kan dus resp., b b en th th zijn. Volgens de classificatie (tab. 1a, cl 4) zijn er in de bonenopbrengst van pl. 347 zeer veel bonen in cl 1 (twee bonen zijn in cl 1a), ook in cl 4 en enkele in cl 2, 3 en 8a. De bonenopbrengst is samengesteld en de formule van de uitgangsboon van pl. 91 voor pl. 347 bevat heterozygoten voor de factoren van alle 3 afmetingen. De lengte heeft daarbij enige factoren als L L, de breedte en de dikte als b b en th th.

Van pl. 354 is de lengte van de uitgangsboon, die van pl. 93 is, zeer klein ($l = 11.6$ mm); de grootste lengte van de bonen, van pl. 354, is $l = 16.1$ mm. De 3 overige bonen van de peul van de uitgangsboon hebben grote afmetingen. Volgens de classificatie van pl. 354 zijn er zeer veel bonen in cl 2 en in cl 1. We nemen aan, dat de uitgangsboon van pl. 93 voor pl. 354 een niet-erfelijke variatie is uit het gebied met de form. L B Th en L B th.

Van de pl. 1046 en 1047 (tab. 1 en 1a) komen de uitgangsbonen, de gemiddelden van de bonenopbrengsten en de classificatie zeer overeen. Ze voldoen in hoge mate aan de eisen van homozygotie van de uitgangsbonen voor de form. L B th, cl 2. De formule is dus slechts l b th als phaenotype. Het kan ook zijn, dat de form. l b th hier enige factorenverbindingen als L l bevat en dat door ongeregelde dominantie de bonenopbrengst enige bonen met een grote lengte heeft; hetzelfde geldt voor de breedte. We vinden enige dominantie van de grote afmetingen over de kleine, doch ze kan ook omgekeerd zijn.

In de groep gevallen van cl 2, waar de uitgangsboon de formule L B th, cl 2, als van de I-lijn, heeft, zijn dus enkele gevallen, die geheel of bijna geheel overeenkomen met de I-lijn, waarvan dus de uitgangsboon van de bonenopbrengst de form. L B th van de I-lijn heeft in homozygote of bijna homozygote vorm. Twee gevallen komen overeen met het bij cl 2a (blz. 82) beschreven geval.

Cl 3. Er zijn 9 gevallen, waar de uitgangsboon van de bonenopbrengst de form. L b Th, cl 3 heeft. In één van deze gevallen (pl. 228) is de formule L₁ L₂ b Th, cl 3a; de formule van de gemiddelden van de bonenopbrengst van pl. 228 is L B Th, cl 1 (zie blz. 79).

Er is één geval (pl. 212, tab. 1 en 1a), waar de uitgangsboon de form. L b Th, cl 3 heeft en de formule van de gemiddelden van de bonenop-

brengst eveneens L b Th, cl 3 is. Van de uitgangsboon heeft de breedte de grenswaarde en de lengte is niet zeer groot (tab. 1); ze is dus niet een zeer kenmerkende boon van cl 3 (tab. 9). Dit geldt ook van de gemiddelden van de bonenopbrengst. Volgens de classificatie (tab. 1a) zijn er zeer veel bonen in cl 1 en enige in cl 3. Ook hier, evenals in het vorige geval (pl. 228) zijn er in de formule van de uitgangsboon enige factoren voor kleine breedte.

In 2 gevallen (pl. 138 en pl. 312) is de formule van de uitgangsboon L b Th, cl 3 en is de formule van de gemiddelden van de bonenopbrengst L B th, cl 2 (zie blz. 84). Eveneens in 2 gevallen (pl. 345 en 1037) is de formule van de uitgangsboon L b th, cl 3 en is de formule van de gemiddelden van de bonenopbrengst L b th, cl 4 (zie blz. II 77). Tenslotte is in 3 gevallen (pl. 207, 979 en 224) de formule van de uitgangsboon L b Th, cl 3 en die van de gemiddelden der bonenopbrengsten 1 b th, cl 8 (blz. III 79).

In 3 gevallen (pl. 308, 121 en 211), is de formule der gemiddelden van de bonenopbrengsten L b Th, cl 3, terwijl de uitgangsboon een andere formule heeft.

In één geval (pl. 308) is ze L B Th, cl 1. De gemiddelde breedte van de bonenopbrengst van pl. 308 is slechts iets kleiner dan de grenswaarde ($b_m = 8.4$ mm). Volgens de classificatie zijn er in de bonenopbrengst van 24 bonen van pl. 308, 5 bonen in cl 3, verder in alle klassen, behalve in cl 5 en 7. De bonenopbrengst is samengesteld. De uitgangsboon van pl. 172 voor pl. 308 behoort tot het erfgebied van cl 1 en cl 2 en heeft enige breedte-factoren als b b.

Van pl. 121 is de formule van de uitgangsboon L B th cl 2. Ook hier is de gemiddelde breedte weinig kleiner dan de grenswaarde ($b_m = 8.4$ mm). Volgens de classificatie van pl. 121 zijn er veel bonen in cl 1, meerdere (5 van 25) in cl 3 en enige in andere klassen, ook in cl 7, geen in cl 2. Volgens de gegevens van de bonenopbrengst en van de ascendentie bevat de formule van de uitgangsboon breedte-factoren als b b en dikte-factoren als Th Th.

Van één geval ten slotte (pl. 211) is de formule van de uitgangsboon, 1 b Th, cl 7. De gemiddelden van de bonenopbrengst komen zeer overeen met die van pl. 212 (blz. 85), tab. 1 en 1a. Ook de indices van de uitgangsbonen van pl. 49 voor pl. 211 en 212 komen zeer overeen. Volgens de classificatie zijn er in de bonenopbrengst van pl. 211, slechts 2 bonen in cl 3 (4 in die van pl. 212) en veel bonen in cl 1 en in cl 7. Ook van pl. 211 is de bonenopbrengst samengesteld. Naast breedte-factoren als b b zijn er hier vooral ook dikte-factoren als Th Th in de formule van de uitgangsboon.

We vinden in de groep gevallen van cl 3 enige aanwijzing voor de erfelijkheid van kleine breedte. We vinden ook, dat de lengte van de bonen van cl 3 niet zeer groot en de breedte niet zeer klein is. Hier komen ook de positieve correlatie van de lengte en de breedte door de niet-erfelijke variabiliteit en de erfelijkheid door polymere factoren tot uitdrukking.

**Zoology. — Note on the Species of *Palaemonetes* (*Crustacea Decapoda*)
found in the United States of America.** By L. B. HOLTHUIS. (Com-
municated by Prof. H. BOSCHMA.)

(Communicated at the meeting of December 18, 1948.)

During a year's stay at the U.S. National Museum, Washington, D.C., the American Palaemonid material of this institute as well as that of the Allan Hancock Foundation at Los Angeles, Calif. was studied for a revision of the American species of this family. As it probably will be some time before this revision is published, it was thought advisable to give a preliminary account of the species of the genus *Palaemonetes* occurring in the U.S.A. Up till now namely a large confusion existed in the conception of the taxonomic status of the various American species of this genus, while these species are very common in fresh, brackish, and salt waters of the U.S.A. and often are studied for their life history or mentioned in connection with faunistic or ecological studies.

Of all species dealt with here a short diagnosis is given, accompanied by some figures and an indication of the range of distribution. A key to the forms precedes the treatment of the separate species. More extensive descriptions and more detailed information as to the occurrence will be given in the forthcoming revision of the American Palaemonidae.

The first species of *Palaemonetes* described from the U.S.A. is *Palaemon vulgaris* of SAY (1818). SAY's description leaves no doubt whatsoever as to the identity of his material. In 1850 GIBBES described a species of prawn from freshwater of S. Carolina under the name *Hippolyte paludosa*, which species was recognized as late as 1878 by KINGSLEY to be a *Palaemonetes*, identical with *Palaemonetes exilipes*, a species described as new by STIMPSON (1871) also from fresh water of S. Carolina. In the same publication in which he described *P. exilipes*, STIMPSON gave a description of another new species of *Palaemonetes*, namely *P. carolinus*. An aberrant form, *Palaemonetes antrorum*, was found in an artesian well in Texas and described by BENEDICT (1896) as new. In 1902 RATHBUN described *Palaemonetes kadiakensis* from Alaska, and in 1921 *Palaemonetes hiltoni* was described by SCHMITT.

When KEMP (1925, Rec. Indian Mus., vol. 27, p. 315) gave his key to all species of the genus *Palaemonetes* he included the following species from the U.S.A.: *P. exilipes* (KEMP did not use the older name *paludosus*), from which he was not able to separate *P. kadiakensis*, *P. carolinus*, between which species and *P. hiltoni* he could not find good differences, *P. vulgaris*, and *P. antrorum*.

The study of the large material of the various species of the genus

represented in the collection of the U.S. National Museum resulted in the following changes in KEMP's arrangement:

1. *Palaemonetes antrorum* is found to be so strongly different from the other species, that a separate subgenus is created for it.
2. *Palaemonetes paludosus* (the older name of GIBBES has to be used instead of the name *exilipes* of STIMPSON) was found to be restricted to that part of the U.S.A. which lies east of the Alleghenies. West of this mountain range it is replaced by a species, which shows to be identical with *Palaemonetes kadiakensis* Rathbun; it even becomes very improbable that the type specimens of RATHBUN's species really originate from Alaska, they probably are incorrectly labelled.
3. It was found that *Palaemonetes carolinus* Stimpson is based on specimens of *Palaemonetes vulgaris*, so that the former name becomes a synonym of the latter and has to disappear.
4. The form named by KEMP *Palaemonetes carolinus* shows to consist of two good species. As no name is available for either of these species, they are given new names here.
5. *Palaemonetes hiltoni* is a good species, different from *P. vulgaris*.

At present thus 7 species of *Palaemonetes* are known from the U.S.A. These species may be distinguished as follows:

1. Eyes without pigment, cornea degenerated. First and second pairs of pereiopods not very different in shape and size. Exopod of uropod with outer margin ending in a single immovable tooth. Subterranean fresh waters of Texas. Subgenus *Alaocaris*. Only species: *antrorum*
- Eyes with dark pigment, cornea well developed, globular. Second pereiopods much stronger and longer than first pair. Exopod of uropod with outer margin ending in a tooth, which at its inner side is provided with a movable spine. Fresh, brackish or salt surface waters, only accidentally subterranean. Subgenus *Palaemonetes* s.s. 2.
2. Fused part of the two rami of upper antennular flagellum distinctly longer than free part. Carpus of second legs longer than chela. Fresh water 3.
- Fused part of the two rami of upper antennular flagellum shorter than free part. Carpus of second legs shorter than chela. Brackish or salt water 4.
3. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove. Posterior pair of dorsal spines of telson placed midway between anterior pair and posterior margin of telson. East of Alleghenies. *paludosus*
- Branchiostegal spine distinctly removed from anterior margin of carapace and situated some distance below branchiostegal groove. Posterior pair of dorsal spines of telson placed very close near posterior margin, and more close to that margin than to anterior pair of spines, often lying in one row with spines of posterior margin. Central U.S.A. west of Alleghenies, southern part of Central Canada, N.E. Mexico. *kadiakensis*
4. Anterior margin of basal segment of antennula strongly produced forwards and far overreaching anterolateral spine of basal segment. S. California, N.W. Mexico *hiltoni*
- Anterior margin of basal segment of antennula, though being convex, not overreaching anterolateral spine of basal segment. Atlantic coast of U.S.A. 5.

5. Carpus of second leg in adult female shorter than palm, in males carpus being only very slightly (1.1 times) longer or shorter than palm. Dactylus of second leg with 2, fixed finger with 1 tooth on cutting edge. Rostrum with first two teeth of dorsal margin behind orbit. Dorsal rostral teeth reaching up to apex. Lower margin of rostrum with 3 to 5 teeth *vulgaris*
- Carpus of second leg in adult female much longer than palm (1.3 to 1.5 times), in males carpus almost as long as whole chela. Dactylus of second leg with one or without teeth, fixed finger without teeth on cutting edge. Rostrum with only one dorsal tooth behind orbit. 6.
6. Dorsal teeth of rostrum reaching up to apex, which often is bifid. Lower margin of rostrum with 4 or 5, seldom 3 teeth. Dactylus of second leg with one distinct tooth on cutting edge, no teeth on cutting edge of fixed finger. *intermedius*
- Dorsal and ventral margins of rostrum with an unarmed stretch before tip; the latter thereby dagger shaped. Lower margin of rostrum with 2 to 4, generally 3 teeth. Dactylus as well as fixed finger of second leg without teeth on cutting edge. *pugio*

Alaocaris new subgenus

Diagnosis: The rostrum is compressed, serrate on the upper margin. The carapace bears an antennal and a branchiostegal spine, a branchiostegal groove is present. The telson bears two dorsal and two posterior pairs of spines, between the latter two feathered setae are present.

The eyes are strongly degenerated and have no pigment.

The mandible bears no palp, also the other mouthparts and the branchial formula are like in *Palaemonetes* s.s.

The first and second pereiopods are very similar in shape and size. The last three legs have a shape similar to those of *Palaemonetes* s.s.

The pleopods, except the first pair, each are provided with an appendix interna, while in the male moreover an appendix masculina is present in the second pair. The uropods differ from those of *Palaemonetes* s.s. by missing the movable spine at the inner side of the final tooth of the external margin of the exopod.

Type is the only species contained at present in the subgenus, *Palaemonetes antrorum* Benedict.

***Palaemonetes (Alaocaris) antrorum* Benedict (fig. 1a—e)**

Palaemonetes antrorum Benedict, 1896, Proc. U.S. Nat. Mus., vol. 18, p. 615.

Diagnosis: Rostrum (fig. 1a) reaching about to base of last segment of antennular peduncle, apex pointed. Rostral formula: $\frac{3}{0} \frac{8-12}{0}$. Branchiostegal spine removed from anterior margin of carapace.

Telson (fig. 1b) with anterior pair of dorsal spines placed slightly behind the middle, posterior pair situated close near posterior margin of telson.

Eyes bullet shaped, cornea degenerated, without pigment.

Antennular peduncle (fig. 1c) with rounded anterior margin of basal

segment reaching about as far forwards as anterolateral spine. Upper antennular flagellum with free part of shorter ramus much longer than fused part.

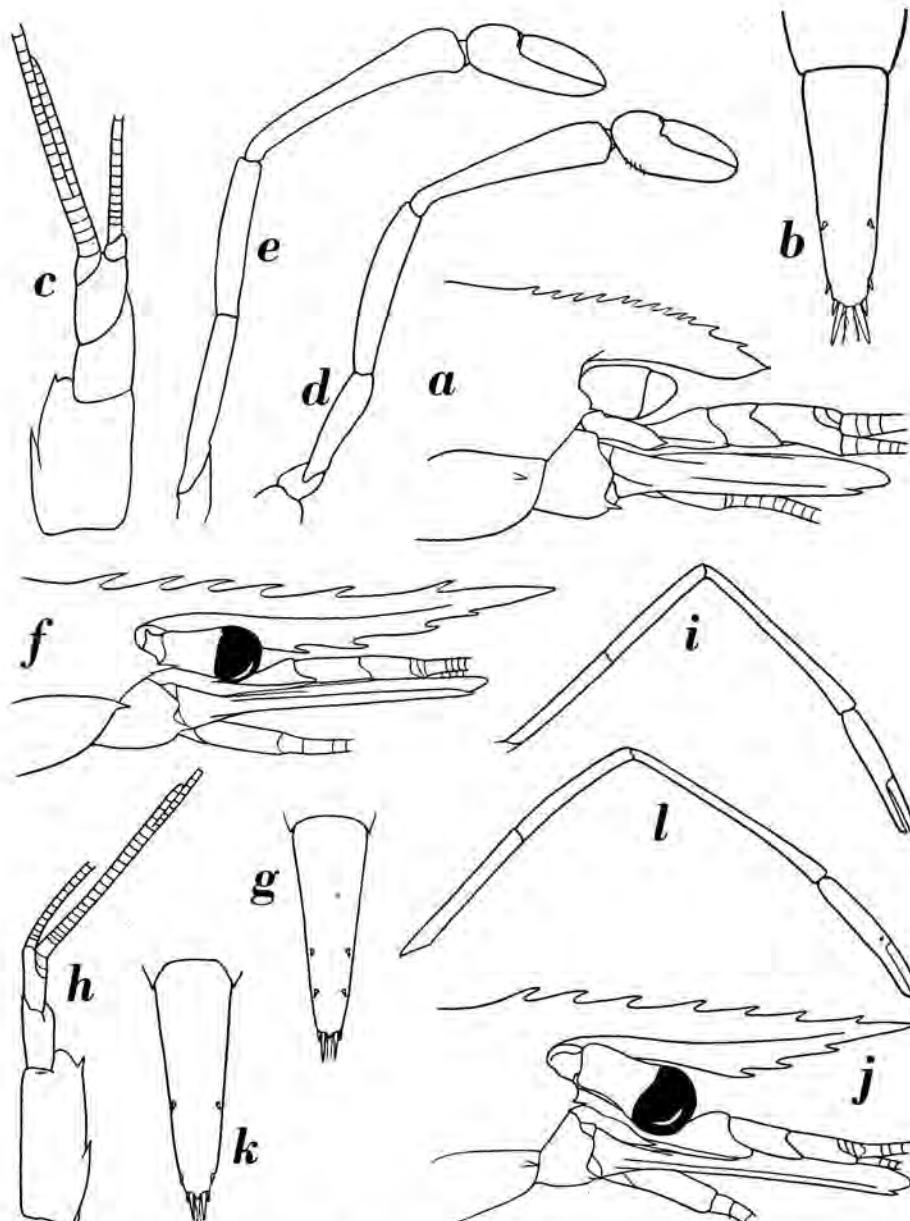


Fig. 1. a—e, *Palaemonetes (Alaocaris) antrorum* Benedict, paratype. a, anterior part of body in lateral view; b, telson in dorsal view; c, antennula; d, first pereiopod; e, second pereiopod. f—i, *Palaemonetes (Palaemonetes) paludosus* (Gibbes). f, anterior part of body in lateral view; g, telson in dorsal view; h, antennula; i, second pereiopod. j—l, *Palaemonetes (Palaemonetes) kadiakensis* Rathbun. j, anterior part of body in lateral view; k, telson in dorsal view; l, second pereiopod. a—e, $\times 15$; f—l, $\times 6$.

Fingers of first pereiopod (fig. 1d) about twice as long as palm. Second leg (fig. 1e) differing only slightly from first, being somewhat longer. Carpus half again as long as chela.

Uropods with outer margin of exopod ending in a tooth, which bears no movable spinule at its inner side.

Size: Up to 18 mm.

Distribution: The species is known only from subterranean waters near San Marcos, Texas, where it occurs in considerable numbers.

Palaemonetes antrorum differs from all other species by the absence of pigment in the eyes, and by the degenerate cornea. The blind prawns from Cuban caves, described in literature as *Palaemonetes eigenmanni* Hay, *P. calcis* Rathbun, *P. inermis* Chace and *P. gibarensis* Chace on examination proved to be no *Palaemonetes* at all as no branchiostegal spine is present. A new genus *Troglocubanus* is erected for them (type species: *Palaemonetes eigenmanni* Hay, 1903, Proc. U.S. Nat. Mus., vol. 26, p. 431, fig. 2). This genus is characterized by the compressed rostrum, the absence of hepatic and branchiostegal spines, by the unpigmented eyes, which have the cornea degenerated, and by the absence of a mandibular palp.

Subgenus *Palaemonetes* Heller, 1869 s.s.

Palaemonetes (Palaemonetes) paludosus (Gibbes) (fig. 1f—i)

Hippolyte paludosa Gibbes, 1850, Proc. Amer. Ass. Adv. Sci., vol. 3, p. 197.

Palaemonetes exilipes Stimpson, 1871, Ann. Lyc. nat. Hist. New York, vol. 10, p. 130.

Diagnosis: Rostrum (fig. 1f) reaching about to the end of the scaphocerite. Rostral formula: $\frac{1-2}{1-4} 5-9$, generally $\frac{1}{3-4} 6-8$. Branchiostegal spine situated on anterior margin of carapace just below branchiostegal groove; this spine sometimes slightly removed from anterior margin, but always reaching beyond the margin with larger part of its length.

Telson (fig. 1g) with anterior dorsal pair of spines placed slightly behind middle, posterior pair about midway between anterior pair and posterior margin of telson.

Eyes with cornea well developed, globular and well pigmented.

Basal segment of antennular peduncle (fig. 1h) with rounded anterior margin reaching slightly beyond anterolateral spine. Upper antennular flagellum with free part of shorter ramus about $\frac{1}{3}$ of length of fused part.

First legs with fingers about as long as palm. Second legs (fig. 1i) much stronger and longer than first. Fingers $\frac{3}{4}$ of length of palm, cutting edges without teeth. Carpus about 1.5 times as long as chela.

Exopod of uropod with outer margin ending in an immovable tooth, which bears at its inner side a movable spine.

Size: Up to 46 mm.

Distribution: Fresh waters east of the Alleghenies, from New Jersey to Florida. Occasionally in localities west of the Alleghenies (Mississippi, Louisiana, Oklahoma, Texas), possibly introduced there.

Palaemonetes (Palaemonetes) kadiakensis Rathbun (fig. 1j—l)

Palaemonetes kadiakensis Rathbun, 1902, Proc. U.S. Nat. Mus., vol. 24, p. 93.

Diagnosis: Rostrum (fig. 1j) straight, reaching about end of scaphocerite. Rostral formula: $\frac{1) 5-10}{0-4}$, generally $\frac{1) 7}{2}$. Branchiostegal spine placed some distance behind anterior margin of carapace, at most reaching with the tip beyond the margin. Branchiostegal groove touching anterior margin of carapace distinctly dorsally of branchiostegal spine.

Telson (fig. 1k) with anterior pair of dorsal spines behind middle of telson, posterior pair placed very close to posterior margin of telson, lying far closer to this margin than to anterior pair of spines.

Eyes as in *P. paludosus*. Antennula with anterior margin of basal segment sometimes much produced anteriorly. Upper flagellum with fused part more than thrice as long as free part of shorter ramus.

Pereiopods and uropods as in previous species.

Size: Up to 53 mm.

Distribution: Fresh waters of Central U.S.A. west of the Alleghenies from the Great Lakes to the Gulf coast, also in S. Ontario (Canada) and N.E. Mexico.

Palaemonetes (Palaemonetes) hiltoni Schmitt (fig. 2a—d)

Palaemonetes hiltoni Schmitt, 1921, Univ. Calif. Publ. Zool., vol. 23, p. 36, pl. 12 fig. 5.

Diagnosis: Rostrum (fig. 2a) reaching end of scaphocerite. Rostral formula $\frac{1) 8-11}{2-3}$. Dorsal teeth of rostrum generally regularly divided over upper margin. Branchiostegal spine on anterior margin of carapace, just below branchiostegal groove.

Telson with anterior pair of dorsal spines situated in its middle, posterior pair about halfway between anterior pair and posterior margin of telson.

Eyes like in *P. paludosus*.

Antennular peduncle (fig. 2b) with anterior margin of basal segment strongly produced anteriorly and distinctly overreaching the anterolateral spine. Free part of shorter ramus of upper antennular flagellum about 1.5 times as long as fused part.

First pereiopod with fingers as long as palm. Second leg (fig. 2c) much stronger than first. Dactylus (fig. 2d) in ovigerous females with two teeth, fixed finger with one tooth on cutting edge. Palm 1.3 to 1.4 times as long as fingers, carpus as long as palm and half the length of the fingers.

Uropods as in *P. paludosus*.

Size: Up to 24 mm long.

Distribution: S. California and N.W. Mexico (Sonora and Sinaloa States). Living probably in brackish water.

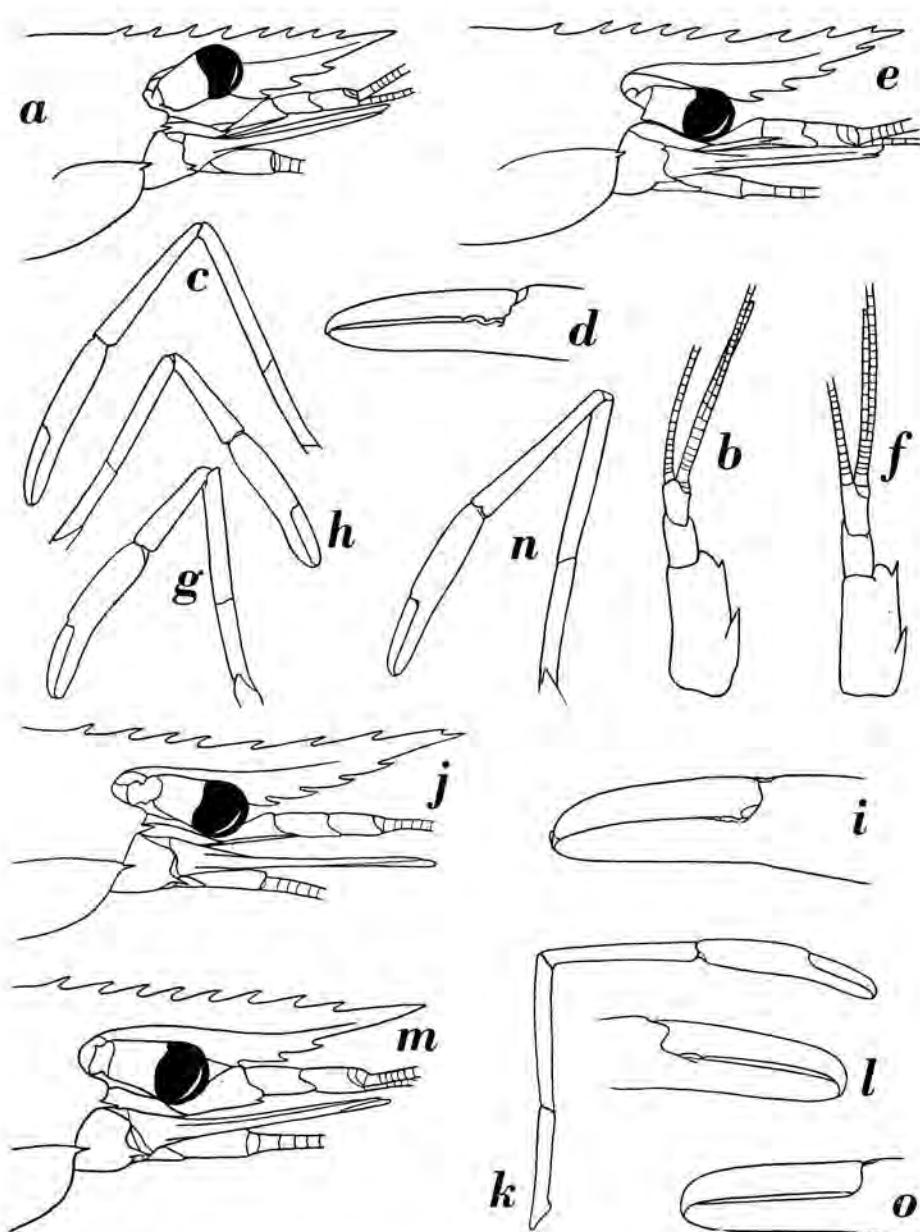


Fig. 2. a—d, *Palaemonetes (Palaemonetes) hiltoni* Schmitt. a, anterior part of body in lateral view; b, antennula; c, second pereiopod of female; d, fingers of second pereiopod. e—i, *Palaemonetes (Palaemonetes) vulgaris* (Say). e, anterior part of body in lateral view; f, antennula; g, second pereiopod of female; h, second pereiopod of male; i, fingers of second pereiopod of female. j—l, *Palaemonetes (Palaemonetes) intermedius* new species. j, anterior part of body in lateral view; k, second pereiopod of female; l, fingers of second pereiopod of female. m—o, *Palaemonetes (Palaemonetes) pugio* new species. m, anterior part of body in lateral view; n, second pereiopod of female; o, fingers of second pereiopod of female. a—c, e—h, j, k, m, n, $\times 6$; d, i, l, o, $\times 15$.

Palaemonetes (Palaemonetes) vulgaris (Say) (fig. 2e—i)

Palaemon vulgaris Say, 1818, Journ. Acad. nat. Sci. Phila., vol. 2, p. 248.

Palaemonetes carolinus Stimpson, 1871, Ann. Lyc. nat. Hist. New York, vol. 10, p. 129.

Diagnosis: Rostrum (fig. 2e) reaching about to end of scaphocerite.

Rostral formula: $\frac{2) \ 8-11}{3-5}$. Teeth placed regularly over upper margin of rostrum, tip never dagger shaped, often bifid. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove.

Pleura of fifth abdominal segment with the tip rectangular or slightly acute. Telson as in *P. paludosus*.

Eyes as in *P. paludosus*.

Antennular peduncle (fig. 2f) with the anterior margin of basal segment not reaching beyond anterolateral spine. Free part of shorter ramus of upper antennular flagellum 1.5 times as long as fused part.

First pereiopod with fingers as long as palm. Second legs (fig. 2g, h) much stronger and longer than first. Fingers 0.6 to 0.75 times as long as palm. Dactylus (fig. 2i) with two, fixed finger with one tooth on cutting edge. Carpus distinctly shorter than palm in female, as long as or slightly (1.1 times) longer than palm in males.

Uropods as in *P. paludosus*.

Size: Up to 42 mm in length.

Distribution: In salt or brackish water of the Atlantic coast of the U.S.A., from Massachusetts to Texas.

Palaemonetes (Palaemonetes) intermedius new species (fig. 2j—l)

Diagnosis: Rostrum (fig. 2j) reaching about end of scaphocerite, tip directed upwards. Rostral formula: $\frac{1) \ 7-10}{3-5}$, generally $\frac{1) \ 8-9}{4-5}$. The teeth are regularly divided over the upper margin of rostrum, tip often bifid. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove.

Pleura of fifth abdominal segment with apex rounded. Telson as in previous species.

Eyes and antennular peduncle as in *P. vulgaris*. Shorter ramus of upper antennular flagellum with free part 1.2 to 1.7 times as long as fused part.

First leg with fingers as long as palm. Second leg (fig. 2k) longer and stronger than first. Fingers (fig. 2l) 0.6 to 0.8 times as long as palm. Dactylus with one tooth on cutting edge, cutting edge of fixed finger unarmed. Carpus 1.2 to 1.5 times as long as palm.

Uropods as in *P. vulgaris*.

Size: Up to 37 mm in length.

Distribution: Brackish water of the Atlantic coast of the U.S.A. from Massachusetts to Texas.

Palaemonetes (Palaemonetes) pugio new species (fig. 2m—o)

Diagnosis: Rostrum (fig. 2m) reaching about to the end of scaphocerite, straight, with the top sometimes curved upwards. Rostral formula: $\frac{1) 7-10}{2-4}$, generally $\frac{1) 8-9}{3}$. Distal part of both upper and lower margin unarmed, tip thereby becoming dagger shaped. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove.

Fifth abdominal segment with the pleura ending in a small acute tooth, which sometimes is very small. Telson as in *P. vulgaris*.

Eyes and antennular peduncle as in *P. vulgaris*. Upper antennular flagellum with the free part of shorter ramus 1.1 times to twice as long as the fused part.

First leg with fingers as long as palm. Second legs (fig. 2n) stronger and longer than first. Fingers (fig. 2o) 0.6 to 0.8 times as long as palm, no teeth present on the cutting edges of dactylus and fixed finger. Carpus 1.3 to 1.5 times as long as palm in females, in males carpus being about as long as whole chela.

Uropods as in *P. vulgaris*.

Size: Up to 50 mm in length.

Distribution: Brackish to almost fresh water of the Atlantic coast of the U.S.A. from Massachusetts to Texas.

Palaemonetes vulgaris, *P. intermedius* and *P. pugio* have the same range of geographic distribution, but probably their ecological habitats are different. *P. pugio* at least prefers water of a much lower salinity than *P. vulgaris*.

KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN
WETENSCHAPPEN

PROCEEDINGS

VOLUME LII

No. 2

President: A. J. KLUYVER
Secretary: M. W. WOERDEMAN

1949

NORTH-HOLLAND PUBLISHING COMPANY
(N.V. Noord-Hollandsche Uitgevers Mij.)
AMSTERDAM

CONTENTS

Biochemistry

BUNGENBERG DE JONG, H. G. and H. J. VAN DEN BERG: Elastic-viscous oleate systems containing KCl. III. The elastic properties as a function of the concentration of the oleate at 15° C and constant KCl concentration, p. 99.

Biology

BEMMELEN, J. F. VAN: Dieren en planten, p. 125.

Botany

FRETS, G. P.: De F₄-zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van Phaseolus vulgaris. II. (Communicated by Prof. J. BOEKE), p. 185.

Chemistry

BOKHOVEN, C., J. C. SCHOONE and J. M. BIJVOET: On the Crystal structure of Strychnine Sulfate and Selenate. III. [001] projection, p. 120.

HAMAKER, H. C.: A simple technique for producing random sampling numbers. (Communicated by Prof. H. B. G. CASIMIR), p. 145.

Geology

HAMMEN, T. VAN DER: De Allerød-oscillatie in Nederland. Pollenanalytisch onderzoek van een laatglaciale meerafzetting in Drente. II. (Communicated by Prof. C. J. VAN DER KLAUW), p. 169.

Mathematics

BROUWER, L. E. J.: De non-aequivalentie van de constructieve en de negatieve orderrelatie in het continuum, p. 122.

BRUINS, E. M.: Some remarks on ancient calculation. (Communicated by Prof. L. E. J. BROUWER), p. 161.

MONNA, A. F.: Sur les espaces linéaires normés. VI. (Communicated by Prof. W. VAN DER WOUDE), p. 151.

Mechanics

BURGERS, J. M.: Note on the damping of the rotational oscillation of a spherical mass of an elastic fluid in consequence of slipping along the boundary, p. 113.

Medicine

HUIZINGA, J.: Somatometric relations between relatives of the first degree. (Preliminary note.) (Communicated by Prof. M. W. WOERDEMAN), p. 177.

Physics

GROENEWOLD, H. J.: Unitary Quantum Electron Dynamics. I. (Communicated by Prof. F. A. VENING MEINESZ), p. 133.

Physiology, Comparative

POSTMA, N., W. L. VETTER and Miss J. H. M. WITMER: The Strength of the Carbohydrases in the Cropjuice of *Helix pomatia* L. (Communicated by Prof. H. W. JULIUS), p. 164.

Biochemistry. — *Elastic-viscous oleate systems containing KCl. III¹⁾. The elastic properties as a function of the concentration of the oleate at 15° C and constant KCl concentration.* By H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG.

(Communicated at the meeting of January 29, 1949.)

1. Introduction.

The methods of measurement applied in the investigations described in this paper are the same as used before.

The concentration of KCl has been chosen slightly below the one used in Part II, owing to the circumstance that in the meantime we have determined more accurately the concentration necessary for obtaining the minimum damping with the oleate used in our investigations (Merck, Na-oleinicum medicinale pur. pulv.). The concentration used in Part II differs very little from this optimum concentration, so that the slight change in the composition of the electrolyte (1,43 N KCl + 0,12 N KOH at present, as compared with 1,52 N KCl + 0,08 N KOH in Part II) does not materially affect a comparison of the results. Moreover, experimenting with a system having an oleate concentration of 1,2 % we again found the proportionality between T and R , and that between A and R .

The extension of the investigations to systems of lower concentration, however, has shown that A presents a wholly different behaviour when the concentration decreases below a certain limit.

2. Period and logarithmic decrement as functions of the oleate concentration at 15° C (rotational oscillation).

As starting point we used a standard solution containing 36 gr Na-oleate per litre, with 0,36 N KOH; from this solution 8 mixtures have been prepared according to the scheme:

a cm³ of standard solution
(200—a) cm³ KOH 0,36 N
225 cm³ KCL 3,8 N
175 cm³ H₂O,

a being 200, 175, 150, 125, 100, 80, 60 and 40 cm³ respectively. The final concentration of KCL (1,43 N) and that of KOH (0,12 N) are the same in all mixtures; the oleate concentrations amount to 1,20; 1,05; 0,90; 0,75; 0,60; 0,48; 0,36; and 0,24 % respectively.

After heavy shaking for a sufficient time the mixtures were poured out into 8 round bottom vessels of nominally 500 cm³ capacities (the actual

¹⁾ Part I has appeared in these Proceedings 51, 1197 (1948); Part II in these Proceedings 52, 15 (1949).

capacities, measured up to the beginning of the neck, have been mentioned in Table I, together with the values of the radius R derived from them). The vessels were filled well into the neck, and were kept in a thermostat at 15°C during one night, in order to obtain the desired temperature and to get rid of air. The measurements took place in the usual manner on the following day. The results together with the values of $10 \times T/2$, corrected for the decrement, have been given in Table I²⁾³⁾.

TABLE I.

Elastic properties as a function of the oleate concentration at 15°C measured in completely filled spherical vessels with radius of approx. 5 cm.

Capacity in cm^3	R in cm	Oleate concentr. in gr. per 100 cm^3	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	A	$(10 \times \frac{T}{2})_{\text{corr.}}$	$\lambda = \frac{T}{2A}$	$G(\text{dynes})$	ν for $R = 5 \text{ cm}$ $\left(= \frac{R}{10 \times \frac{T}{2}} \right)$	$1/A$
539	5.05	1.35	53.5	5.12	1.197	0.180	5.12	2.85	51.0	0.986	5.56
500	4.92	1.20	47.7	5.81	1.252	0.225	5.81	2.58	37.6	0.845	4.44
501	4.93	1.05	40.3	6.61	1.360	0.307	6.60	2.15	29.3	0.747	3.26
500	4.92	0.90	31.2	7.84	1.590	0.464	7.82	1.69	20.8	0.629	2.16
498	4.92	0.75	24.0	9.65	1.860	0.621	9.60	1.55	13.8	0.513	1.61
534	5.03	0.60	18.8	12.82	2.151	0.766	12.73	1.66	8.2	0.395	1.31
535	5.04	0.48	13.3	16.78	2.791	1.027	16.56	1.61	4.86	0.304	0.97
528	5.01	0.36	8.0	25.70	4.764	1.561	24.94	1.60	2.12	0.201	0.64
527	5.01	0.24 ⁴⁾	2.5	49.3	(47.1)	(3.85)	42.0	(1.09)	0.75	0.119	(0.26)

²⁾ With a view to the investigation of the behaviour of $T/2A$ (see below), it was considered desirable to dispose of a system with an oleate concentration still higher than those obtainable according to the recipe given in the text. This system was prepared by taking 225 cm^3 of the standard solution, 225 cm^3 KCl and 150 cm^3 H_2O ; composition: 1.35 gr oleate per 100 cm^3 , 1.43 N KCl, 0.135 N KOH. Results referring to it have been given in the first line of Table I and in the first line of Table II. It was found that the values of ν , $1/A$ and G for this system are in good accordance with the lines drawn in the figures 1 and 2.

³⁾ Professor BURGERS has pointed out that the formulae used for the derivation of the corrected value of the period (compare Part II, these Proceedings, p. 17) is valid only in those cases where the damping is due either to viscous resistance or to relaxation of the elastic stresses. It cannot be applied when the damping is a consequence of slipping. As relaxation certainly is operative with the more concentrated systems and some influence may persist also with systems of smaller concentration, we have provisionally applied the formula in all cases. The influence of the correction usually is small.

A calculation based upon the formula derived for the case where damping is due exclusively to slipping, will be given by BURGERS in the article following this communication, for the system having an oleate concentration of 0.6%. For systems with still smaller concentrations, which show very large values of A , there appears to remain an unsolved problem.

⁴⁾ With this system the decrement appeared to be very large, so that it had to be determined from the ratio b_1/b_2 . The value of b_1/b_3 given in the Table, has been calculated from that of b_1/b_2 ; the results, and also the values of A , $T/2A$ and $1/A$ have been put between brackets, as their reliability will be less than that of the other results.

When the reciprocal value of $(10 \times T/2)_{\text{corr}}$ is represented as a function of the oleate concentration, a practically straight line is obtained. As the radii of the 8 vessels are slightly unequal, it is necessary to correct for the differences; making use of the proportionality between T and R found before (and checked also with the systems used now), we reduced all values to the same radius $R = 5,00$ cm with the aid of the formula:

$$T = \frac{5}{R} \times T_{\text{corr}} = \left(10 \times \frac{T}{2}\right)_{\text{corr}} \cdot \frac{1}{R}, \text{ or } \nu = \frac{1}{T} = \frac{R}{\left(10 \times \frac{T}{2}\right)_{\text{corr}}}.$$

The reduced frequencies ν have been mentioned in the 11th column of Table I and are represented in fig. 1a as a function of the oleate concentration. It will be seen that *the frequency of the rotational oscillation for constant radius is a linear function of the oleate concentration*.

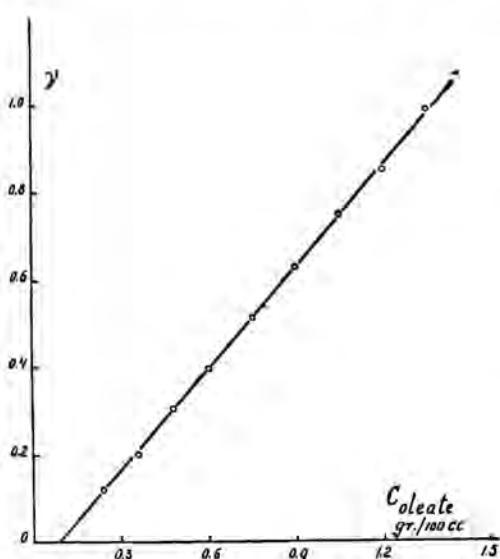


Fig. 1a.

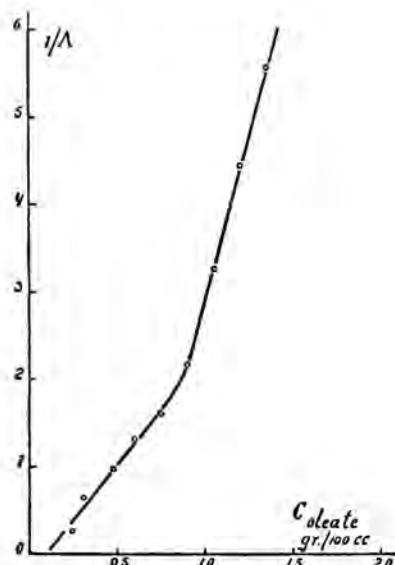


Fig. 1b.

If we may assume — what looks very probable — that the straight line can be extrapolated downward until it meets the horizontal axis, a point of intersection is obtained at an oleate concentration of 0.09 gr per 100 cm³ (approx. 3 millimol oleate per litre). Hence we can distinguish between two regions of oleate concentration: no elastic phenomena are found with a concentration in the region from zero to 3 millimol per litre; whereas elastic phenomena present themselves with concentrations exceeding 3 millimol per litre. — It will afterwards be seen that the second domain must still be subdivided.

We have further calculated the shear modulus G with the aid of BURGERS' formula⁵⁾:

$$T_0 = \frac{2\pi}{4.99} R \sqrt{\frac{\rho}{G}},$$

taking $\rho = 1.072$. The results obtained have been given in column 10 of Table I. Making use of the result obtained for the frequency ν , we may at once write down the formula:

$$G \sim (\text{oleate concentration} - 0.09)^2.$$

This relation is illustrated in fig. 2, which gives the values of G as a

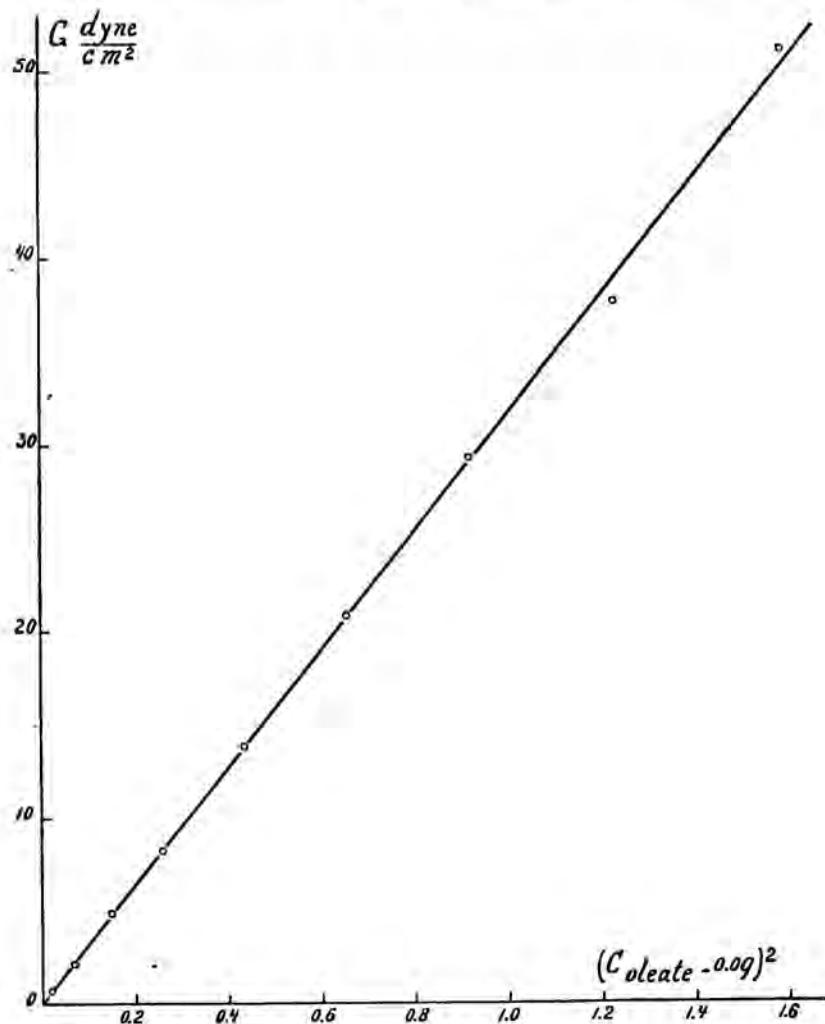


Fig. 2.

⁵⁾ J. M. BURGERS, Damped oscillations of a spherical mass of an elastic fluid, these Proceedings 51, 1211 (1948), eq. (17). — It should be noted that the quantity ν introduced by BURGERS has a different meaning from the one used here.

function of the square of (oleate concentration decreased by 0,09). The result undoubtedly will be of great importance for a future theory of the elastic oleate systems; however, at the present moment we will not bind ourselves to any assumptions concerning their internal structure.

The next point to be considered is the damping. When the value of $1/\Lambda$ (column 12 of Table I) is represented as a function of the oleate concentration (fig. 1b), a curve is obtained with two branches, the lower branch presenting a similar character as the curve for ν given in fig. 1a. The experimental points are situated closely to a straight line, cutting the horizontal axis at the oleate concentration 0,09, so that the following relation appears to be valid for this branch:

$$1/\Lambda \sim (\text{oleate concentration} - 0,09).$$

Provisionally assuming that the damping is caused by relaxation of the elastic tension — as had been proved in the case of the 1,2 % system considered in Part II — we have calculated a nominal relaxation time $\lambda = T/2\Lambda$ from our data. This quantity has been given in column 9 of Table I and is represented as a function of the oleate concentration in fig. 3 (full drawn curve, marked $R = 5$). At low concentrations (above 0,09) the curve exhibits a horizontal branch, as must be expected from what has been observed concerning the curves for $\nu (= 1/T)$ and $1/\Lambda$. This horizontal part is followed by a markedly ascending branch.

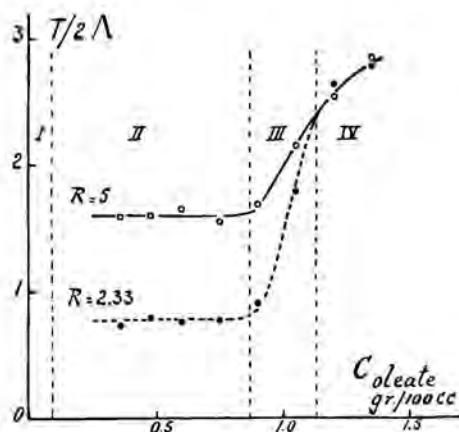


Fig. 3.

A few days afterwards measurements have been performed with the same series of oleate solutions in a vessel of radius $R = 2,33$ cm (the same vessel being used in all cases). The results have been collected in Table II. In the last column of this Table the frequencies have been reduced to a radius $R = 5,00$ cm. When represented as a function of the oleate concentration they give rise to a similar curve as that obtained from the data of Table I, although the values appear to be a few percent higher.

The corresponding differences in the values of G are somewhat larger, as is to be expected in view of the proportionality of G with ν^2 . This fact must be ascribed to a slight change within the system. What is much more interesting, however, are the values of $T/2A$, which have been represented by the broken curve (marked $R = 2.33$) in fig. 3.

TABLE II.

Elastic properties as function of the oleate concentration at 15° C measured in a completely filled spherical vessel of 2.33 cm radius.

Oleate concentr. in gr per 100 cm ³	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	A	$(10 \times \frac{T}{2})_{\text{corr.}}$	$\lambda = \frac{T}{2A}$	$G(\text{dynes})$	$\nu \text{ for } R = 5 \text{ cm}$ $\left(= \frac{R}{10 \times \frac{T}{2}} \right)$	$1/A$
1.35	59.3	2.28	1.085	0.082	2.28	2.78	54.8	1.022	12.2
1.20	50.7	2.57	1.102	0.097	2.57	2.64	43.1	0.907	10.3
1.05	43.4	3.00	1.182	0.167	3.00	1.79	31.7	0.767	5.98
0.90	34.2	3.64	1.488	0.398	3.63	0.91	21.6	0.642	2.52
0.75	24.8	4.49	1.790	0.582	4.47	0.77	14.3	0.521	1.72
0.60	20.8	5.86	2.182	0.781	5.82	0.76	8.4	0.400	1.28
0.48	14.2	7.68	2.608	0.959	7.59	0.79	5.0	0.307	1.04
0.36	9.0	11.48	4.581	1.522	11.16	0.73	2.29	0.209	0.66
0.30	7.0	15.52	6.751	1.910	14.88	0.78	1.29	0.157	0.52
0.24 ^{a)}	5.0	20.32	(8.010)	(2.081)	19.30	(0.93)	0.77	0.121	(0.48)

It will be seen that the two curves, deduced from the data of Tables I and II respectively, nearly coincide at the higher concentrations. Going downward they separate, and in the domain of the lower concentrations two distinct, approximately horizontal lines are found, giving the average values: 1.60 (derived from the experiments with $R = 5 \text{ cm}$) and 0.76 (derived from the experiments with $R = 2.33 \text{ cm}$). The ratio of these numbers is $1.60/0.76 = 2.11$, which is almost the same as the ratio of the two radii, *viz.* $5.00/2.33 = 2.15$.

This result is very remarkable and stands in sharp contrast with that of the experiments described in Part II, which showed a relaxation time independent of the radius. We must conclude that with the lower concentrations the damping is due to a phenomenon of a different nature. It is of importance to consider this matter in more detail.

3. Period and logarithmic decrement of the 0.6 % oleate system as functions of the radius for the rotational oscillation.

In order to have at our disposal a sufficient quantity of oleate solution, we mixed a number of the mixtures already used in the preceding experiments, in such a way that the final mixture obtained (2.35 litres in all) had an oleate concentration of 0.6 %.

^{a)} Compare footnote 4).

In a single day we investigated the elastic behaviour as a function of the radius, in the way as has been described in Part II. The vessels used were of smaller dimensions (the largest having a capacity of 1.5 litre); the radii, calculated from the measured capacities, varied from 7.46 till 2.33 cm. This made it possible to perform all experiments with the aid of the turning table, all vessels being completely immersed in thermostat water of 15° C.

TABLE III.

Period and logarithmic decrement as function of the radius at 15° C (0.6% Oleate in 1.43 N KCl + 0.12 N KOH; rotational oscillation).

R (cm)	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	Λ	$(10 \times \frac{T}{2})_{\text{corr.}}$	G (dynes)	α
7.46	19.7 ± 0.2	19.12 ± 0.03	2.156 ± 0.019	0.768 ± 0.009	18.98	8.11	5.37
9.36	20.0 ± 0.3	15.98 ± 0.02	2.140 ± 0.024	0.761 ± 0.011	15.86	8.44	5.53
5.01	19.8 ± 0.1	12.71 ± 0.02	2.141 ± 0.009	0.761 ± 0.004	12.62	8.27	5.44
4.15	20.5 ± 0.3	10.30 ± 0.02	2.196 ± 0.023	0.787 ± 0.010	10.22	8.65	5.41
3.00	20.4 ± 0.3	7.57 ± 0.02	2.173 ± 0.024	0.776 ± 0.011	7.51	8.37	5.40
2.33	21.5 ± 0.2	5.81 ± 0.01	2.160 ± 0.028	0.770 ± 0.013	5.76	8.58	5.51

The results of the measurements, together with the values of Λ and those of $(10 \times T/2)_{\text{corr.}}$ have been given in Table III. In fig. 4 these values have been represented as functions of the radius R . The same as with the system of 1.2 % oleate concentration, it is found that the period is directly proportional with R also for the 0.6 % system. The logarithmic decrement for the 0.6 % system on the contrary proves to be independent of the radius.

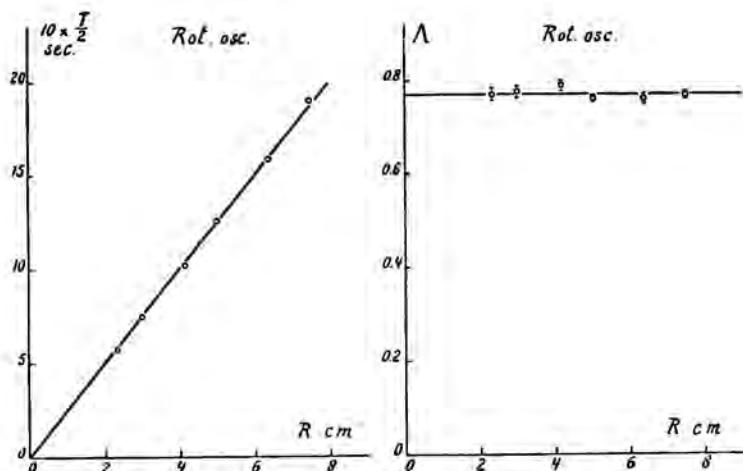


Fig. 4.

4. Comparison of periods and logarithmic decrements of the 0.6 % system as found for the three forms of oscillation.

In connection with the result just found BURGERS informed us that it would be of interest to have values of T and Λ for the three forms of oscillation, measured in one and the same vessel, so that R would be the same for all.

As the stock of Na-oleate (Merck Na-oleinicum medicinale) had been used up, we had to use a preparation from a different source (Na-oleate, neutral powder, made by BAKER⁷⁾). As we knew that there may always be found differences in the properties of various samples of Na-oleate (even when they are taken from two unopened bottles delivered by the same factory at the same time), and as we had no experience with the Baker preparation, it was considered desirable to perform measurements with two spheres, of radii $R = 5,62$ and $R = 2,99$ cm respectively. This would give an opportunity of checking whether the new material (having the same concentration of 0.6 % Na-oleate in 1.43 N KCl + 0.12 N KOH) likewise would show the proportionality between T and R , with Λ independent of R .

The method applied was the same as used before, but a slightly different way of exciting the rotational and the quadrantal oscillations was chosen. The principle is the same as with the turning table, but the vessels are clamped about the neck in the centre of a ball bearing, in such a way that they are hanging well until the neck in the thermostat water of 15° C. The measurements have been executed in one consecutive series in a single day. The results are given in Table IV.

The first point to be considered is whether the new preparation exhibited a similar behaviour as the old one. This does not primarily refer to the absolute values of T and Λ for the same radius, which indeed appear to

TABLE IV

Comparison of period and logarithmic decrement of the 0.6 % oleate system for the three forms of oscillation.

R (cm)	Type of oscillation	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	Λ	$(10 \times \frac{T}{2})_{\text{corr.}}$	$\frac{T}{2\Lambda}$	G	x
5.62	Rotational	24.9 ± 0.2	10.70 ± 0.01	1.696 ± 0.016	0.528 ± 0.009	10.66 ± 0.01	2.02	mean 2.01	14.6
	Meridional	24.7 ± 0.1	8.21 ± 0.02	1.498 ± 0.009	0.404 ± 0.006	8.19 ± 0.02	2.03		15.0
	Quadrantal	24.4 ± 0.3	6.81 ± 0.02	1.407 ± 0.012	0.342 ± 0.009	6.80 ± 0.02	1.99		14.8
2.99	Rotational	28.2 ± 0.1	5.63 ± 0.02	1.677 ± 0.017	0.517 ± 0.010	5.61 ± 0.02	1.09	mean 1.10	14.9
	Meridional	25.8 ± 0.1	4.39 ± 0.02	1.494 ± 0.022	0.402 ± 0.015	4.38 ± 0.02	1.09		14.9
	Quadrantal	27.4 ± 0.1	3.61 ± 0.02	1.382 ± 0.008	0.324 ± 0.006	3.60 ± 0.02	1.11		14.9

⁷⁾ As it was impossible to buy Na-oleate from MERCK, we have directed ourselves to the Rockefeller Foundation, which kindly put at our disposal a large quantity of the preparation mentioned in the text in order to make possible the continuation of our researches on oleate systems. We are glad to express our great thanks for this help.

be different⁸⁾, but rather to the functional relations between T and R , and between Λ and R . For this purpose we have represented the values of T_0 , T_1 , T_2 ; Λ_0 , Λ_1 , Λ_2 as functions of R in fig. 5. The diagram proves that indeed a state of the oleate system is present, fully comparable to that which was found in our former measurements.

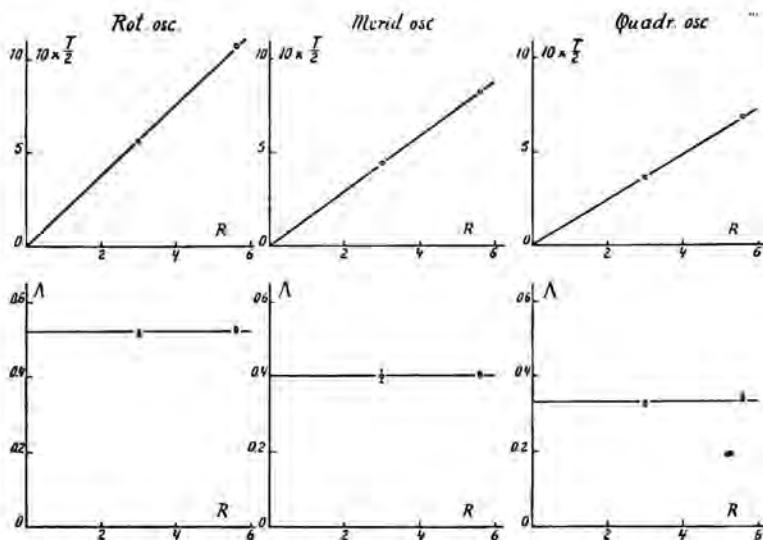


Fig. 5.

The quantity $T/2 \Lambda$ consequently cannot be interpreted as a relaxation time, as it appears to be different for the two radii. The ratio of the values obtained (average value for the three forms of oscillation) differs from the quotient of the radii by not more than 2 % ($2.01/1.10 = 1.84$; $5.62/2.99 = 1.88$). As T proved to be proportional to R , this means that for a given form of oscillation Λ is independent of R . Inspection of the data given in column 6 of Table IV shows that for the rotational oscillation, considered separately, the ratio differs from that of the radii by 2 %; for the meridional oscillation the difference is 1 %; while for the quadrant oscillation it is slightly larger, *viz.* 5 %. But we know from our former work (see Parts I and II) that with the latter type of oscillations the accurate measurement of the damping is very difficult. When now again we calculate the ratios of the periods and of the logarithmic decrements, obtained for the various types of oscillation, the

⁸⁾ When the data referring to the rotational oscillation, given in Table IV, are compared with those given in Table III for the same radius, it appears that the new preparation obtained from BAKER has both a smaller period and a smaller logarithmic decrement. The new preparation consequently has a larger shear modulus and a smaller damping. In conformity with the experience related in Parts I and II that n increases when the damping becomes smaller, it was found that the new preparation showed larger values for n .

following numbers are found:

$$\begin{array}{ll} R = 2,99 \text{ cm} & T_0/T_1 = 1,281 \\ & A_0/A_1 = 1,286 \end{array} \left. \begin{array}{c} \\ \end{array} \right\} \text{mean} \quad \begin{array}{ll} T_0/T_2 = 1,558 \\ A_0/A_2 = 1,596 \end{array} \left. \begin{array}{c} \\ \end{array} \right\} \text{mean}$$

$$\begin{array}{ll} R = 5,62 \text{ cm} & T_0/T_1 = 1,302 \\ & A_0/A_1 = 1,307 \end{array} \left. \begin{array}{c} 1,294 \\ \end{array} \right\} \quad \begin{array}{ll} T_0/T_2 = 1,568 \\ A_0/A_2 = 1,544 \end{array} \left. \begin{array}{c} 1,567 \\ \end{array} \right\}$$

If the fluctuations of the ratios are left out of account (having regard to the mean errors of the individual determinations these fluctuations on the whole are within the possible errors), we may conclude that there are to be found only two distinct ratios, one referring to T_0/T_1 and A_0/A_1 , the other one referring to T_0/T_2 and A_0/A_2 . The averages obtained for these two ratios are 1,294 and 1,567 respectively, which values differ by not more than 1 % from the values 1,283 and 1,557 predicted theoretically for the periods on the basis of BURGERS' formulae.

5. The nature of the damping in the 0.6 % oleate system.

After the measurements described in sections 3 and 4 had been finished, professor BURGERS informed us that all results obtained with the 0.6 % oleate system could be explained if it be assumed that the damping is a consequence of slipping of the elastic system along the wall of the vessel. In this case we have the following formulae⁹⁾:

$$T = \frac{2\pi R}{\beta} \sqrt{\frac{\rho}{G}}; \quad A = \frac{2\pi}{\beta} \frac{\sqrt{G\rho}}{\kappa}.$$

where κ is a coefficient characterising the friction experienced in slipping, while the factor β is a numerical quantity, having the well known values 4,49 for the rotational oscillation, 5,76 for the meridional oscillation and 6,99 for the quadrantal oscillation respectively. The formulae show that for every type of oscillation the period is proportional to the radius, whereas the logarithmic decrement is independent of it; moreover, when we pass from one type of oscillation to another type, both period and decrement change in the same ratio, so that there is full conformity with the experimental results for the 0,6 % oleate system. — The values of G and κ calculated with these formulae have been given in the final two columns of Tables III and IV. It is found that the preparation obtained from Baker gives higher values both for G and for κ , than the preparation obtained from Merck¹⁰⁾¹¹⁾.

⁹⁾ See J. M. BURGERS, paper quoted in footnote⁵⁾ above, eq. (18).

¹⁰⁾ When G has already been calculated, it is possible to obtain κ from the relation $\kappa = TG/4R$, which is an immediate consequence of the formulae for T and A given in the text.

¹¹⁾ As mentioned already in footnote³⁾ above, BURGERS has recalculated the values for the 0.6 % systems with the equations given in the article following this communication (Note on the damping of the rotational oscillations of a spherical mass of an elastic fluid in consequence of slipping along the boundary, these Proceedings p. 113).

6. The frictional coefficient operative in slipping as a function of the oleate concentration.

According to fig. 3 the nature of the damping appears to present the same character as that found with the 0.6 % oleate system, for a series of concentrations which extend upward to approximately 0.9 %. If we apply the formulae mentioned in the preceding section to all systems of Tables I and II with concentrations not exceeding 0.9 %, a series of values of the coefficient α is obtained, which shows a marked decrease of α with decreasing concentration, as will be evident from Table V¹²⁾.

TABLE V.
Frictional coefficient operative in slipping as function of the oleate concentration.

Oleate concen- tration in gr. per 100 cm ³	From the data of Table II		From the data of Table II	
	α	$\alpha/(C_{oleate} - 0.09)^2$	α	$\alpha/(C_{oleate} - 0.09)^2$
0.90	14.25	21.7	16.91	25.8
0.75	8.67	19.9	9.43	21.6
0.60	5.42	20.8	5.37	20.6
0.48	3.11	20.5	3.39	22.3
0.36	(1.35)	(18.5)	(1.44)	(19.8)
0.30	—	—	(0.86)	(19.5)
0.24	(0.33)	(14.5)	(0.61)	(27.1)

It is found that the quotient of α by the square of (oleate concentration — 0.09) presents values which may be considered as approximately constant (having regard to all errors inherent to the measurements). The numbers referring to the lower concentrations and put in brackets are not

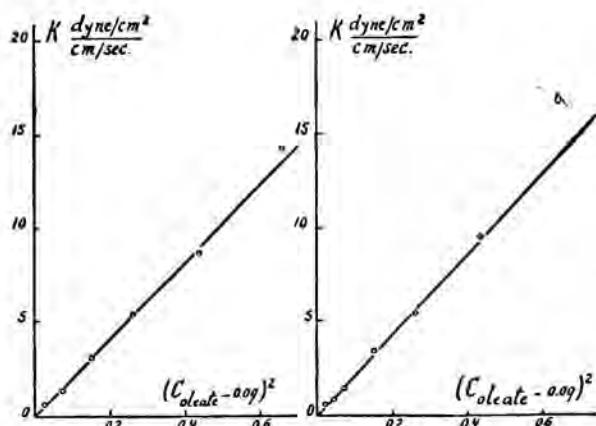


Fig. 6.

¹²⁾ The calculations on which are based the values of α given in Table V, are all made with the aid of the first order formulae mentioned in the preceding section, as obtained by BURGERS in his first article.

reliable, owing on one hand, to difficulties in the measurement of A and on the other hand, to the fact that the theory given by BURGERS cannot explain values of A exceeding approx. 1.06. — The fact that the value obtained with the concentration of 0.9 % seems to be somewhat in excess, may be an indication that we ought to limit our domain at a slightly lower value, approx. 0.87 %, as is proved also by the position of the bend in the curve for $1/A$, given in fig. 1 B.

For the rest of the domain one gets the impression that the relation:

$$\alpha \sim (\text{oleate concentration} - 0.09)^2$$

is rather satisfactory (compare fig. 6). The relation between α and concentration would then be the same as that between G and concentration, given in section 2.

7. Distinction of a number of domains of oleate concentration.

We return to the consideration of the dependence of the elastic properties on the oleate concentration.

In the first place it is of importance to delimit the domain in which the behaviour corresponds to that found with the 0.6 % system (T proportional with R and A independent of R). In fig. 3 this domain is characterized by the two distinct horizontal curves for $T/2A$. It has been found that this domain extends upwards to an oleate concentration of approx. 0.87 % (ca. 29 millimol per litre). Downward it extends at least to 0.3 % (ca. 10 millimol per litre). The values derived with a concentration of 0.24 % must be considered as unreliable, and we cannot check whether $T/2A$

Domain	Concentration in gr oleate per 100 cm ³	Dependence of T and A on the radius for constant oleate conc.	Ratios for T and A at constant radius and constant oleate concentration	Dependence of G , α and λ on oleate concentration
I	0 — 0.09	$\nu = 1/T = 0$		$G = 0$
II	0.09 — 0.87	$T \sim R$ A independent of R $T/2A \sim R$ $T/AR = \alpha/G$ independent of R	$\frac{T_0}{T_1} = \frac{A_0}{A_1} = 1.28$ $\frac{T_0}{T_2} = \frac{A_0}{A_2} = 1.56$	$G \sim (C_{\text{ol}} - 0.09)^2$ $\alpha \sim (C_{\text{ol}} - 0.09)^2$
III	0.87 — 1.1	$T \sim R$		$G \sim (C_{\text{ol}} - 0.09)^2$ with same factor as in II
IV	1.1 — 1.35 (possibly extending further)	$T \sim R$ $A \sim R$ $T/2A = \lambda$ independent of R	$\frac{T_0}{T_1} = \frac{A_0}{A_1} = 1.28$ $\frac{T_0}{T_2} = \frac{A_0}{A_2} = 1.56$	$G \sim (C_{\text{ol}} - 0.09)^2$ with same factor as in II λ increasing with oleate concentration

remains the same until the limiting concentration of 0,09 % (ca. 3 millimol per litre). Nevertheless as a provisional assumption we may suppose this to be the case.

At the highest concentrations investigated (1,2 % and 1,35 %) the quantity $T/2 \Lambda$ has the meaning of a relaxation time. This domain appears to extend downward until a concentration of approx. 1,1 %. Between 0,87 % and 1,1 % there is a transition region.

In this way we arrive at the four domains given in the survey.

It should be kept in mind that the limits between these domains, determined at a temperature of 15° C with a given concentration of KCl etc., may be functions of various parameters, including the purity of the oleate used.

Summary of Part III:

- 1) An investigation of the elastic properties as a function of the oleate concentration (with constant concentration of KCl and at a temperature of 15° C) has shown that:
 - a) the frequency of the rotational oscillation in completely filled spherical vessels of constant radius is a linear function of the oleate concentration, becoming zero at a concentration of 0,09 % (3 millimol oleate per litre);
 - b) the elastic shear modulus is proportional to (oleate concentration — 0,09)²;
 - c) the quantity $T/2 \Lambda$, which has the meaning of a relaxation time in the case of the 1,2 % oleate system, loses this meaning when the concentration decreases below 0,9 %.
- 2) In order to arrive at a more complete characterization of the elastic behaviour of oleate systems in the domain mentioned sub 1c), the following problems have been investigated with the 0,6 % oleate system:
 - a) Period and logarithmic decrement as functions of the radius of the spherical vessel; it appeared that the period is proportional to R , while the logarithmic decrement was independent of it;
 - b) period and logarithmic decrement as functions of the type of oscillation (rotational, meridional, quadrantal); it appeared that the quotients T_0/T_1 and Λ_0/Λ_1 had the value 1,28, the quotients T_0/T_2 and Λ_0/Λ_2 the value 1,56, the same as was the case with the oscillations damped in consequence of relaxation.

The results mentioned sub a) and b) are in conformity with a theoretical case, treated by J. M. BURGERS, referring to the oscillations of an elastic medium slipping along the walls of the spherical vessel.
- 3) It is possible to distinguish four domains of the oleate concentration:
 - (I) from 0 to 0,09 %, in which elastic phenomena are absent;
 - (II) from 0,09 % until 0,87 %, where Λ is independent of R ;

- (III) from 0.87 % until 1.1 %, in which the relation between A and R changes in character;
 - (IV) above 1.1 % where $A \sim R$.
- 4) The quantities G and either λ or κ are functions of the oleate concentration:
- a) in the domains II, III and IV a single formula
- $$G \sim (c_{o1} - 0.09)^2$$
- is obtained;
- b) the relaxation time λ , characteristic for the damping in domain IV, increases with the oleate concentration;
 - c) the frictional coefficient κ , operative in slipping in domain II, presents the same dependence on the concentration as G : $\kappa \sim (c_{o1} - 0.09)^2$.

Mechanics. — *Note on the damping of the rotational oscillation of a spherical mass of an elastic fluid in consequence of slipping along the boundary.* By J. M. BURGERS. (Mededeling No. 60 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of January 29, 1949.)

1. In a paper on the damped oscillations of a spherical mass of an elastic fluid¹⁾ there had been derived first order formulae for the period and the logarithmic decrement of three types of oscillations of an elastic fluid in a spherical vessel, as observed by BUNGENBERG DE JONG²⁾.

One of the cases considered referred to damping in consequence of slipping of the elastic system along the walls of the vessel. When effects of viscous friction and of relaxation of elastic stresses are absent, the result for this case is contained in the formula:

$$\nu = i \frac{\zeta}{R} \sqrt{\frac{G}{\rho}} \cong i \frac{\beta}{R} \sqrt{\frac{G}{\rho}} \left(1 - \frac{G}{\kappa \nu R} \right) \cong i \frac{\beta}{R} \sqrt{\frac{G}{\rho}} - \frac{G}{\kappa R}.$$

Here G is the elastic shear modulus, R is the radius of the spherical vessel and κ is the frictional coefficient operative in slipping, while β has been written for a numerical factor, having the value 4,49 for the (first) rotational oscillation, 5,76 for the meridional oscillation and 6,99 for the quadrantal oscillation³⁾. To this degree of approximation the period of the oscillation and the logarithmic decrement have the values:

$$T = \frac{2\pi R}{\beta} \sqrt{\frac{\rho}{G}} \quad ; \quad A = \frac{2\pi}{\beta} \frac{\sqrt{G\rho}}{\kappa}.$$

The logarithmic decrement proves to be independent of the radius, while in passing from one type of oscillation to another type, the decrement changes in the same ratio as the period.

These relations are found to be in good agreement with the experimental results obtained by BUNGENBERG DE JONG with systems having an oleate concentration of 0,6 %⁴⁾.

¹⁾ J. M. BURGERS, these Proceedings 51, 1211 (1948). — References to equations of this paper will be made by giving the numbers of the equations; the equations of the present communication will be indicated by means of letters.

²⁾ H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG, Elastic-viscous oleate systems containing KCl, these Proceedings 51, 1197 (1948); 52, 15 (1949).

³⁾ Compare eqs. (15) and (28) of the paper quoted in footnote¹⁾. The rotational oscillation considered in that paper, which is determined by the first root $\zeta = 4,493$ of the equation $\tan \zeta = \zeta$, is the first one of an infinite series of possible rotational oscillations; it is characterised by the absence of a nodal point in the function $\Phi(r)$, whereas the other solutions have 1, 2, nodal points respectively.

⁴⁾ H. G. BUNGENBERG DE JONG, these Proceedings 52, 99 (1949).

With oleate systems of smaller concentration the values of A observed by BUNGENBERG DE JONG became so large, that it appeared questionable whether first order formulae are sufficiently accurate. Moreover the equation:

$$T_{\text{corr}} = T_{\text{obs}} \{1 + (A_{\text{obs}}/2\pi)^2\}^{-1/2},$$

used by BUNGENBERG DE JONG in order to derive "corrected" values of the period of the oscillation, is valid only in the case of damping through viscosity or through relaxation, but does not apply when the damping is a consequence of slipping.

In the following lines we intend to deduce more accurate expressions for the latter case. We restrict ourselves to the rotational oscillation and for simplicity omit the subscript \circ ; hence in the following equations $\beta = 4.49$. We again assume that there is neither viscous resistance, nor relaxation of stresses, so that the quantity L occurring in the formulae of the preceding paper can be replaced by G .

2. We may start by calculating an expression for the root of eq. (13) in the form of a series proceeding according to powers of $\varepsilon = G/\kappa\nu R$. This gives:

$$\zeta = \beta(1 - \varepsilon - \varepsilon^2 \dots) \dots \dots \dots \quad (a)$$

As:

$$\nu R = i a R \sqrt{G/\rho} = i \zeta \sqrt{G/\rho} \dots \dots \dots \quad (b)$$

— compare eq. (8) — we may write:

$$\varepsilon = \frac{1}{i \zeta M}, \quad \text{where } M = \frac{\kappa}{\sqrt{G\rho}}.$$

Hence eq. (a) can be transformed into:

$$\zeta = \beta \left(1 + \frac{i}{\zeta M} + \frac{1}{\zeta^2 M^2} \dots\right) \dots \dots \dots \quad (c)$$

The root of this equation can be expressed as a series proceeding according to powers of $1/M$, the first few terms being:

$$\zeta = \beta \left(1 + \frac{i}{\beta M} + \frac{2}{\beta^2 M^2} \dots\right) \dots \dots \dots \quad (d)$$

The period and the logarithmic decrement are then given by the formulae:

$$T = \frac{2\pi R}{\beta} \sqrt{\frac{\rho}{G}} \left(1 - \frac{2}{\beta^2 M^2} \dots\right); \quad A = \frac{2\pi}{\beta M} \left(1 - \dots\right). \dots \quad (e)$$

The parameter M decreases with decreasing values of the frictional coefficient κ ; it will be seen that this entails a decrease of the period T together with an increase of the logarithmic decrement A . Such behaviour is contrary to what is found in the case of damping through viscosity or through relaxation, where T and A change in the same sense.

3. In order to make clear the meaning of this result and to obtain convenient formulae for the calculation of G and α from the measured values of T and A , a different procedure is more suitable. We observe that:

$$\nu = \frac{2\pi i}{T} - \frac{A}{T} = i \frac{2\pi + iA}{T} \dots \dots \dots \quad (f)$$

where the values to be inserted for T and A are those directly measured, without applying any correction to T . Referring to eq. (b) we then have:

$$\zeta = \frac{\nu R}{i \sqrt{G/\rho}} = \frac{(2\pi + iA)R}{T \sqrt{G/\rho}} \dots \dots \dots \quad (g)$$

Next eq. (13) is brought into the form:

$$\psi(\zeta) \equiv \frac{\zeta^2 \operatorname{tg} \zeta - 3 \operatorname{tg} \zeta + 3\zeta}{\zeta (\operatorname{tg} \zeta - \zeta)} = -\frac{\alpha \nu R}{G \zeta}$$

so that:

$$\psi(\zeta) = -iM \dots \dots \dots \quad (h)$$

After division by $-i$ this equation can be transformed into:

$$\frac{1}{i \cot \zeta - i/\zeta} - \frac{3i}{\zeta} = M \dots \dots \dots \quad (k)$$

Substituting a complex value $\zeta = \xi + i\eta$ into (k) it is possible, by numerical calculation, to find a series of solutions for which the left hand member has a real value. For any such solution the logarithmic decrement can be obtained from:

$$A = 2\pi\eta/\xi \dots \dots \dots \quad (l)$$

The calculations at the same time give the values to be assigned to M . Results are given in the accompanying table and in fig. 1.

$(\cos \xi)$	ξ	η	A	M	$1/M$
	4.493	0	0	∞	0
(0.15)	4.564	0.387	0.533	2.58	0.388
(0.10)	4.613	0.495	0.674	2.01	0.497
(0)	4.712	0.643	0.857	1.53	0.654
(-0.10)	4.812	0.741	0.968	1.36	0.735
(-0.30)	5.004	0.847	1.064	1.063	0.941
(-0.50)	5.176	0.872	1.059	0.927	1.078
(-0.75)	5.356	0.830	0.974	0.804	1.243
(-1.00)	5.498	0.737	0.842	0.694	1.44
(-1.40)	5.663	0.503	0.558	0.481	2.08
	5.763	0	0	∞	

It will be seen that the values of ζ are situated on a curve which starts from $\zeta = 4.49$ (with $M = \infty$) and ends at $\zeta = 5.76$ (with $M = 0$). The case $M = \infty$ represents an oscillation without slipping; this is the first rotational oscillation, which has no nodal point between $r = 0$ and $r = R$.

The case $M = 0$ leads to an oscillation for which τ becomes zero at the wall, so that the point $r = R$ is equivalent to a "free end". This solution

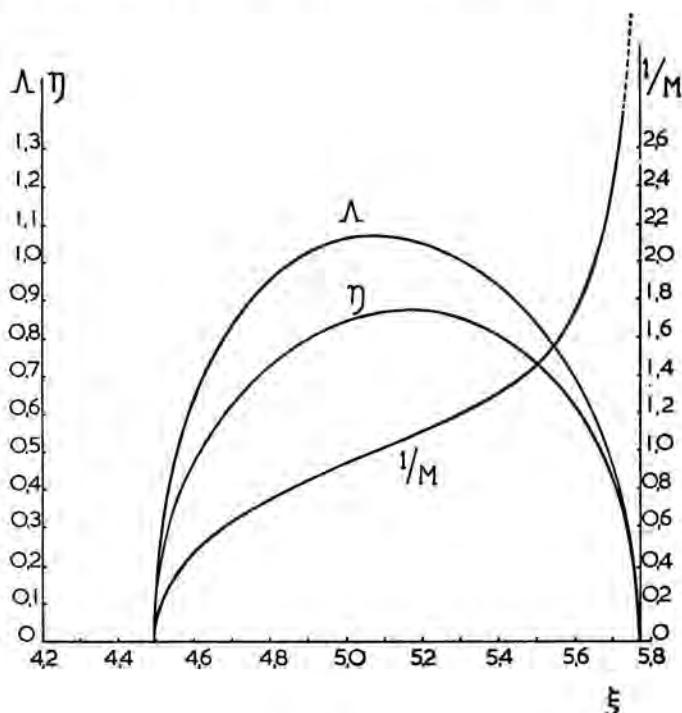


Fig. 1.

has one nodal point. The two limiting cases obtained in this way represent undamped oscillations. The other solutions determine a continuous series of damped oscillations, with a geometrical pattern intermediate between that of the limiting cases. It is found that there is a maximum value of the logarithmic decrement, which is approximately 1.06.

This result is a consequence of the circumstance that we have looked for a particular type of motions of the elastic system, so-called "normal oscillations", described by formula (3). The geometrical pattern of these motions is determined by the function $\Phi(r)$ and thus is independent of the time, which appears in the exponential factor only. The values of ν characteristic for such motions are restricted to a certain finite domain, as found above. It should be kept in mind that there are also other domains of values of ζ giving solutions of eq. (k). For instance, we can satisfy this equation by means of a purely imaginary value of ζ , which will lead to a damped motion of non-oscillatory character. We can also find a series of solutions starting from $\zeta = 7.72$ (second root of the equation $\operatorname{tg} \zeta = \zeta$, leading to the second rotational oscillation, having one nodal point between $r = 0$ and $r = R$) and ending with $\zeta = 9.09$ (second root of the equation $\zeta^2 \operatorname{tg} \zeta - 3 \operatorname{tg} \zeta + 3\zeta = 0$); etc.

4. Let us assume that the solutions represented by the left hand branch of the curve of fig. 1, starting from $\xi = 4,49$ (that is, the branch with the larger values of α) can be used to calculate the values of G and α from the results obtained by BUNGENBERG DE JONG with the 0,6 % oleate systems⁵⁾. In the series of experiments summarized in Table III of BUNGENBERG DE JONG's third paper, the mean value of A was found to be 0,770. Interpolation by means of the curves of fig. 1 then gives:

$$\xi = 4,658 : 1/M = 0,575.$$

From the observed data for the period we deduce that the mean value of $2\pi R/T$ is equal to 12,47; making use of the real terms of eq. (g) we find:

$$\sqrt{G/\xi} = 2,68,$$

so that, with $\varrho = 1,072$:

$$G = 7,70.$$

With the aid of the value of $1/M$ given above we obtain:

$$\alpha = 5,0.$$

These results are somewhat lower than those given by BUNGENBERG DE JONG, who applied the first order formulae after having corrected the period by means of the equation to be used in the case of relaxation.

In the case of the 0,6 % oleate system prepared with Na-oleate from a different source, to which refers Table IV of BUNGENBERG DE JONG's paper, the mean value of A for the rotational oscillations was 0,523; interpolation with the aid of the curves of fig. 1 now gives:

$$\xi = 4,560 : 1/M = 0,38.$$

From the observed data for the period we deduce $2\pi R/T = 16,60$; hence $\sqrt{G/\varrho} = 3,64$; $G = 14,2$; $\alpha = 10,2$. In this case the differences with the calculations made by BUNGENBERG DE JONG are even less.

5. The cases in which the experimental values of A , as found by BUNGENBERG DE JONG, exceed the limit 1,06 deduced from fig. 1, cannot be explained on the basis of the formulae deduced here. The reason for the appearance of these large values [1,561 and (3,85) in Table I; 1,522, 1,910 and (2,081) in Table II] may perhaps be sought either in the circumstance that the damping is partly due to other causes than slipping alone, or in the fact that the observed oscillations differ from the "normal oscillation" considered in the formulae.

With regard to the first possibility, if we assume that relaxation is

⁵⁾ In a letter to the author professor BUNGENBERG DE JONG stated that the maximum amplitude for the rotational oscillations of this system was still found at $r = \frac{1}{2}R$ approximately, so that the function $\Phi(r)$ cannot have a node in this neighbourhood. Hence we may expect that the course of this function will not differ very much from that described by (10). More serious is that BUNGENBERG DE JONG could not observe any motion in the immediate neighbourhood of the wall of the vessel.

operative along with slipping, the shear modulus in our present formulae must be changed back into L , with L given by (5c). Omitting the details of the calculation it is then found that eq. (h) should be replaced by:

$$\psi(\zeta)^2 + \frac{\alpha R}{\lambda G} \frac{\psi(\zeta)}{\zeta} + M^2 = 0 \quad \dots \quad (m)$$

The solutions of this equation will depend upon the parameter $\alpha R / \lambda G$, so that the value of ζ will no longer be independent of R . This may entail an influence of R on the magnitude of the logarithmic decrement. In the case of the 0.6 % oleate system no systematic influence of R has been detected, which induced us to suppose that in this case relaxation should be left out of consideration. With the systems of smaller concentration the influence of the radius has not been investigated, so that for these systems the question whether relaxation may be operative must be left open.

We might also suppose that viscous resistance was operative along with slipping. In that case we should use form. (5b) for L and it is found that eq. (h) must be replaced by:

$$\psi(\zeta)^2 + \frac{\alpha \eta}{\rho G R} \zeta \cdot \psi(\zeta) + M^2 = 0. \quad \dots \quad (n)$$

Here again there appears a term depending on R , so that similar remarks apply as before.

When account should be taken of both viscous resistance and relaxation, operating simultaneously with slipping, the resulting equation takes the form:

$$\psi(\zeta)^2 + \frac{\alpha}{G} \left\{ \frac{R}{\lambda \zeta} + \frac{\eta \zeta}{\rho R} \right\} \psi(\zeta) + M^2 = 0. \quad \dots \quad (o)$$

This brings no new point of view.

Turning to the other possibility, concerning the form of the oscillations, it is certain that the initial motion produced in the elastic system by the method applied to start the oscillatory phenomenon, will not exactly answer to the pattern of a single normal oscillation. This initial motion represents a combination of such oscillations (possibly an infinite number), in which, however, the first normal solution (the first rotational oscillation) can usually be expected to show the largest amplitude. The assumption is commonly made that all the other normal solutions are more heavily damped than the first one, so that after a few periods have elapsed, the first one will remain in a relatively pure form. But this assumption can be defended only in the case of damping through viscous resistance, which, although not observable in the first rotational oscillation, could perhaps be operative with the oscillations of higher index.

With both lines of thought we find ourselves before problems which cannot be easily worked out, as we have no sufficient data from which we might start. As both lead to rather complicated formulae with a number of unknown parameters, we are forced to leave the subject here and must

admit that a wholly satisfactory explanation of all results of BUNGENBERG DE JONG's observations has not been reached.

Résumé.

Dans cet article, qui fait suite à la communication parue dans Proceedings 51, 1211 (1948), il a été donné une expression plus exacte de la période et du décrément logarithmique de la première oscillation rotatoire d'un fluide élastique dans un réservoir sphérique, en supposant que l'amortissement soit causé par le glissement du fluide le long de la paroi. En faisant varier le coefficient de friction de l'infini à zéro, on obtient une série de formes d'oscillation, pour lesquelles le décrément logarithmique paraît d'abord croître de zéro vers un maximum, pour redescendre ensuite vers zéro.

Le résultat est appliqué à l'oscillation rotatoire présentée par une solution contenant 0,6 % d'oléate dans les recherches expérimentales de BUNGENBERG DE JONG, pour laquelle la valeur du décrément logarithmique est bien au-dessous du maximum. Des solutions à concentration plus petite montrent des décréments dépassant le maximum calculé; on doit supposer que dans ce cas l'amortissement est dû (au moins en partie) à des phénomènes d'un autre genre.

Resumo.

Ci-tiu artikolo estas daŭrigo de komunikajo aperita Proceedings 51, 1211 (1948). Oni donas pli ekzaktajn formulojn de la periodo kaj de la logaritma dekremento de la unua rotacia formo de osciloj de elasta fluidaĵo entenata en sfera vazo, supozante ke la amortizo estas kaŭzita per glito de la fluidaĵo laŭ la parojo de la vazo. Kiam koeficienteo de rezisto ŝanĝiĝas de infinito al nulo, oni trovas serion de formoj de osciloj, kun logaritmaj dekrementoj komence kreskantaj de nulo al maksimuma valoro kaj poste malkreskantaj denove al nulo.

Rezulto estas aplikata al rotacia formo de osciloj de solvajo de oleato de koncentriteco 0,6 %-a, observita en la esploroj de BUNGENBERG DE JONG; por ĉi-tiu kazo la logaritma dekremento estas malpli ol la kalkulita maksimuma valoro. Solvajoj de koncentriteco malpli ol 0,6 %-a donas logaritmajn dekrementojn superantajn ĉi-tiun maksimuman valoron; oni devas supozи ke en ĉi-tiuj kazoj amortizo estas (almenaŭ parte) kaŭzita per aliaj fenomenoj.

Chemistry.—On the Crystal structure of Strychnine Sulfate and Selenate.
 III. [001] projection ¹⁾). By C. BOKHOVEN, J. C. SCHOONE and J. M.
 BIJVOET.

(Communicated at the meeting of January 29, 1949.)

The expression for the electrodensity on (001) reads

$$\varrho_{x+y} = \frac{2}{a \cdot b} \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \left[|F_{hk0}| \cos 2\pi \frac{hx}{a} \cdot \cos \left(2\pi \frac{ky}{b} - \alpha_{hk0} \right) + \right. \\ \left. + |F_{\bar{h}k0}| \cdot \cos 2\pi \frac{hx}{a} \cdot \cos \left(2\pi \frac{ky}{b} - \alpha_{\bar{h}k0} \right) \right].$$

Now the unknown phase angles except their signs can be determined, as is seen from the vector diagram (fig. 1):

$$\cos \alpha_{\text{sulf.}} = \frac{|F_{\text{sel}}|^2 / |F_{\text{sulf.}}|^2 - |\Delta F|^2 / |\Delta F|}{2 |F_{\text{sulf.}}| / |\Delta F|}.$$

This value of $\alpha_{\text{sulf.}}$, calculated from directly observed data, will be denoted below by $\alpha_{\text{obs.}}$. The sign of α being unknown so far, a normal

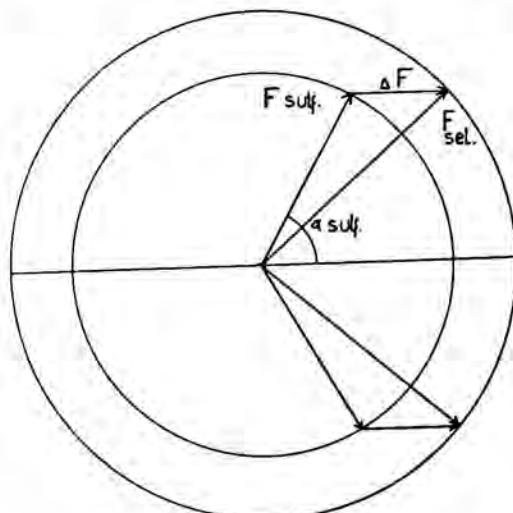


Fig. 1.

Fourier synthesis on (001) cannot be carried out at this stage, but it is possible to perform a "double" Fourier, in which each structure amplitude is included twice, namely with positive and with negative value of the

¹⁾ C. BOKHOVEN, J. C. SCHOONE and J. M. BIJVOET, These proceedings: L, 825 (1947), LI, 990 (1948).

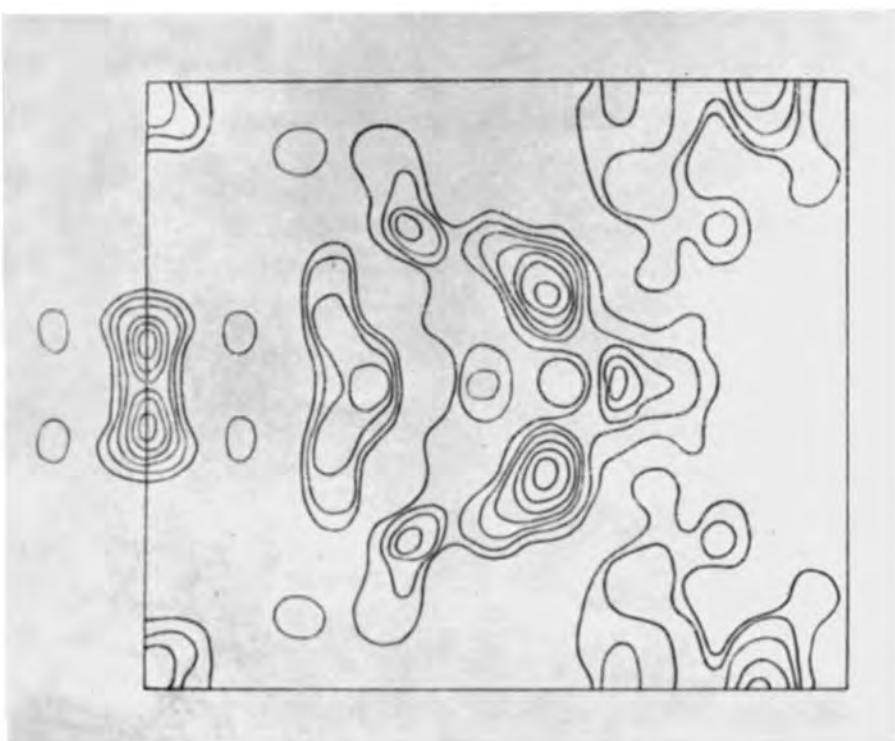


Fig. 2. [001] projection with its mirror image.

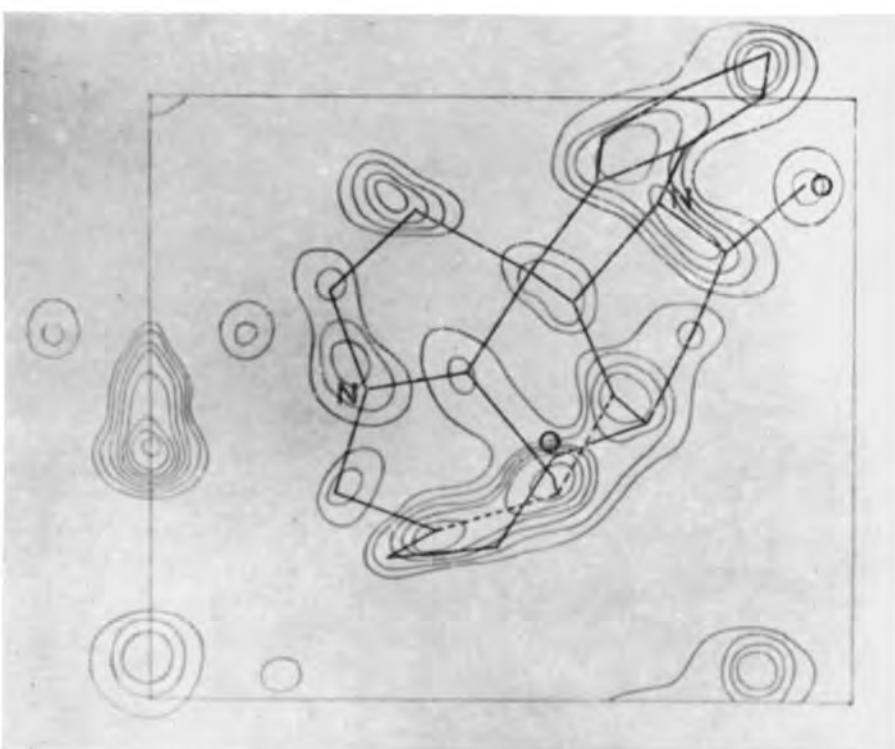


Fig. 3. [001] projection.

phase angle. The correct combination of F giving a maximum on x, y , the reversed set will give one on $-x, -y$.

In this way a set of symmetry centra is introduced and moreover, by operation of the twofold axis, a mirror plane through the S atom.

The "double" Fourier, calculated according to

$$\rho_{x+y} + \rho_{x-y}^* = \frac{8}{a \cdot b} \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} |F_{hk0}| \cdot \cos 2\pi \frac{hx}{a} \cdot \cos 2\pi \frac{ky}{b} \cdot \cos a_{hk0}$$

is shown in fig. 2.

It gives the strychnine molecule superimposed on its mirror image. The separation of the two superimposed images was accomplished on the basis of the [010] projection and the known interatomic distances and valency angles²⁾. In the projection the benzene ring clearly reveals itself, and it can be used as a starting point in this procedure, which was facilitated by the — non essential — use of those features of the structure which were beyond doubt from the chemical point of view.

It is easy to separate the image and mirror image of the sulfate group and the watermolecules, but for these separate groups it is impossible to make a choice between the two images on the lines indicated above.

To improve the preliminary y -coordinates we used an iterative process, minimizing the function:

$\sum_{hK} (\Sigma f_i A_i - A_{\text{obs}})^2$, where $\Sigma f_i A_i$ is the calculated real part of the structure amplitude

$$\left(A_i = \cos \frac{2\pi h x_i}{a} \cos \frac{2\pi K y_i}{b} \right) \text{ and } A_{\text{obs.}} = F_{\text{obs.}} \cos a_{\text{obs.}}$$

The advantage of this minimizing function is, that one can use it without knowing the sign of the y -coordinates of the oxygen atoms.

In order to obtain a decision in respect to these signs, the structure amplitudes of about 50 reflexions were calculated for their possible combinations and compared with the observed structure amplitudes. An unambiguous choice could be made, except for the position of the fifth water molecule, its most probable position being the vague maximum in the [010] projection. With these y -coordinates the "B" parts of the structure amplitudes were calculated, and so the signs of these parts were fixed. Now a Fourier synthesis on (001) could be performed with observed amplitudes and observed phases, signs included; the result is shown in fig. 3.

There is good agreement between calculated and observed amplitudes F_{hk0} , which proves the structure deduced. Further refining of the parameters is in progress, a full account will be published in the Acta Crystallographica.

*Van 't Hoff Laboratorium
der Rijksuniversiteit, Utrecht.*

²⁾ C. M. CARLISLE and D. CROWFOOT, Proc. Roy. Soc. 184, 74 (1945).

Mathematics. — De non-aequivalente van de constructieve en de negatieve orderrelatie in het continuum. By L. E. J. BROUWER.

(Communicated at the meeting of January 29, 1949.)

In een vroegere mededeeling¹⁾ is de onwaarschijnlijkheid toegelicht, dat voor het continuum de constructieve orderrelatie $\circ>$ en de negatieve orderrelatie $>$ ooit aequivalent zouden kunnen blijken. In het volgende zal worden aangetoond, dat het blijken dezer aequivalentie zelfs contradictoor is.

Hiertoe herinneren we aan de in een andere vroegere mededeeling²⁾ gegeven definitie van een *aanstuiving* als vereeniging γ van een positief convergente fundamentealreeks van van elkaar verwijderd liggende reële getallen $c_1(\gamma), c_2(\gamma), \dots$, de *telgetallen* der aanstuiving, en hun van iedere $c_r(\gamma)$ verwijderd onderstelde limiet $c(\gamma)$, de *kern* der aanstuiving. Een zoodanige aanstuiving zullen we in het bijzonder *links gevleugeld* resp. *rechts gevleugeld* noemen, als voor iedere r de relatie $c_r(\gamma) <^\circ c(\gamma)$ resp. voor iedere r de relatie $c_r(\gamma) \circ> c(\gamma)$ geldt. Verder zal, als de fundamentealreeks $c_1(\gamma), c_2(\gamma), \dots$ de vereeniging is van een fundamentealreeks van *linksche telgetallen* $l_1(\gamma), l_2(\gamma), \dots$, die alle $<^\circ c(\gamma)$ zijn, en een fundamentealreeks van *rechtsche telgetallen* $d_1(\gamma), d_2(\gamma), \dots$, die alle $\circ> c(\gamma)$ zijn, de betrokken aanstuiving *tweevleugelig* worden genoemd.

Zij a een wiskundige assertie. Dan kan het scheppende subject in verband met a en met een aanstuiving γ naar het volgende voorschrift een onbegrensd voortschrijdende sequentie $R(\gamma, a)$ van reële getallen $c_1(\gamma, a), c_2(\gamma, a), \dots$, creëeren: Zoolang bij de keuze der $c_n(\gamma, a)$ aan het scheppende subject *niet de juistheid*, resp. *nôch de juistheid nôch de absurditeit*, van a is gebleken, wordt iedere $c_n(\gamma, a)$ gelijk aan $c(\gamma)$ gekozen. Zoodra echter voor $r = 1$ vóór de keuze van $c_r(\gamma, a)$ of voor $r > 1$ tusschen de keuze van $c_{r-1}(\gamma, a)$ en die van $c_r(\gamma, a)$ aan het scheppende subject de *juistheid*, resp. *hetzij de juistheid hetzij de absurditeit*, van a is gebleken, wordt zowel $c_r(\gamma, a)$ als voor ieder natuurlijk getal r ook $c_{r+r}(\gamma, a)$ gelijk aan $c_r(\gamma)$ gekozen. Deze sequentie $Q(\gamma, a)$, resp. $R(\gamma, a)$, convergeert positief tot een reëel getal $C(\gamma, a)$, resp. $D(\gamma, a)$, dat een *conditioneel dempingsgetal van γ door a* , resp. een *direct dempingsgetal van γ door a* , zal worden genoemd³⁾.

¹⁾ Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 51, p. 963 (1948).

²⁾ Ibid. 51, p. 1239 (1948).

³⁾ Voor een nog niet toetsbaar gebleken a werd het begrip van *direct dempingsgetal van γ door a* onder den naam van *dempingsgetal van γ door a* reeds l.c.²⁾ ingevoerd.

Evenzoo kan het scheppende subject in verband met α en met een tweevleugelige aanstuiving γ naar het volgende voorschrift een onbegrensd voortschrijdende sequentie $S(\gamma, \alpha)$ van reële getallen $\omega_1(\gamma, \alpha), \omega_2(\gamma, \alpha), \dots$ creëeren: Zoolang bij de keuze der $\omega_n(\gamma, \alpha)$ aan het scheppende subject noch de juistheid noch de absurditeit van α is gebleken, wordt iedere $\omega_n(\gamma, \alpha)$ gelijk aan $c(\gamma)$ gekozen. Zoodra echter voor $r = 1$ vóór de keuze van $\omega_r(\gamma, \alpha)$ of voor $r > 1$ tusschen de keuze van $\omega_{r-1}(\gamma, \alpha)$ en die van $\omega_r(\gamma, \alpha)$ aan het scheppende subject de juistheid van α is gebleken, wordt zoowel $\omega_r(\gamma, \alpha)$ als voor ieder natuurlijk getal ν ook $\omega_{r+\nu}(\gamma, \alpha)$ gelijk aan $d_r(\gamma)$ gekozen. En zoodra voor $s = 1$ vóór de keuze van $\omega_s(\gamma, \alpha)$ of voor $s > 1$ tusschen de keuze van $\omega_{s-1}(\gamma, \alpha)$ en die van $\omega_s(\gamma, \alpha)$ aan het scheppende subject de absurditeit van α is gebleken, wordt zoowel $\omega_s(\gamma, \alpha)$ als voor ieder natuurlijk getal ν ook $\omega_{s+\nu}(\gamma, \alpha)$ gelijk aan $l_s(\gamma)$ gekozen. Deze sequentie $S(\gamma, \alpha)$ convergeert positief tot een reëel getal $E(\gamma, \alpha)$, dat een *tweezijdig dempingsgetal van γ door α* zal worden genoemd.

We beschouwen thans voor ieder natuurlijk getal ν de k_ν -intervallen $k'_\nu, k''_\nu, \dots, k^{(s_\nu)}_\nu$, d.w.z. de van links naar rechts geordende $\lambda^{(4\nu+1)}$ -intervallen, welker doorsnede met het *eenheidscontinuum* (d.w.z. de soort der reële getallen ≥ 0 en ≤ 1) een λ -interval bevat. Dan *valt het eenheidscontinuum samen met de „puntwaaier“ J* , d.w.z. met de soort der onbegrensd voortschrijdende sequenties $k_1^{(\mu_1)}, k_2^{(\mu_2)}, k_3^{(\mu_3)}, \dots$, die *telescopisch* zijn, d.w.z. waarbij ieder volgend interval in engeren zin binnen het er aan voorafgaande gelegen is. Zij f een willekeurig punt van J , verstaan we onder α_f de assertie, die f rationaal verklaart, en beschouwen we de soort ϱ der sequenties $R(\gamma, \alpha_f)$ en de soort δ der correspondeerende directe dempingsgetallen $D(\gamma, \alpha_f)$, waarbij γ steeds voorstelt de aanstuiving met kern 0 en telgetallen 2^{-n} ($n = 1, 2, \dots$), doch f vrij varieert binnen J , met dien verstande dat het n de element $k_n^{(\mu_n)}$ van f steeds na $c_n(\gamma, \alpha_f)$ doch vóór $c_{n+1}(\gamma, \alpha_f)$ wordt gekozen.

Zouden nu te eeniger tijd de orderrelaties $\circ >$ en $>$ voor het continuum aequivalent blijken, dan zou daarmee in het bijzonder voor elk element e van δ zijn gebleken, dat het $\circ > 0$ is, dus zou aan elk element e van δ een zoodanig natuurlijk getal $n(e)$ zijn toegewezen, dat $e \circ > 2^{-n(e)}$ is, dus zou tevens aan elk element p van ϱ een zoodanig natuurlijk getal $n(p)$ zijn toegewezen, dat bij de keuze van het $n(p)$ de element van p , dus ook bij de keuze van het $n(p)$ de element van het p bepalende element $f(p)$ van J , hetzij de rationaliteit hetzij de irrationaliteit van $f(p)$ is gebleken. Aldus ware aan een willekeurig element f van J een zoodanig natuurlijk getal $n(f)$ toegewezen, dat bij de keuze van het $n(f)$ de element van f hetzij de rationaliteit hetzij de irrationaliteit van f is gebleken. Wegens de waaierstructuur van J zou dan bij variatie van f binnen J voor $n(f)$ een natuurlijk getal m als maximum kunnen worden aangewezen⁴⁾. D.w.z. voor ieder

⁴⁾ Vgl. het Mathem. Annalen 97, p. 66, bewezen Theorem 2.

element f van J zou bij de keuze van $k_m^{(\mu_m)}$ hetzij de rationaliteit hetzij de irrationaliteit zijn gebleken, hetgeen klaarblijkelijk contradictoor is voor alle elementen f van J , die bij de keuze van $k_m^{(\mu_m)}$ hun volle voortzettingsvrijheid, inclusief vrijheid tot latere vrijheidsbeperking, hebben behouden.

Ook van sommige andere in de klassieke wiskunde aangenomen aequivalenties kan door toetsing aan passende soorten van dampingsgetallen van aanstuivingen, als boven gedefinieerd, worden aangetoond dat ze niet alleen onjuist, doch tevens contradictoor zijn.

Biology. — Dieren en planten. By J. F. VAN BEMMELEN.

(Communicated at the meeting of January 29, 1949.)

In „Het Leven Ontsluierd” schrijft KONINGSBERGER op bladzijde 4: „Wanneer men zich in speculaties begeeft omtrent de eerste levende wezens op aarde, dan zal men steeds moeten bedenken dat dit autotrophe organismen geweest moeten zijn, omdat er geen organische stoffen waren, waarvan heterotrophe hadden kunnen leven”. Op bladzijde 20 herhaalt hij dit nog eens, onder toevoeging, dat men het „met zekerheid kan zeggen om de eenvoudige reden, dat er vóór het leven geen organische stof op aarde geweest kan zijn, waar de heterotrophe van leven”.

Wanneer ik mij veroorloof deze „zekerheid” in twijfel te trekken, dan geschiedt dat om de „eenvoudige reden” dat ik niet vermag in te zien, waarom er „geen organische stof op aarde kan aanwezig geweest zijn, alvorens de eerste levende wezens ontstonden”. Integendeel: wilden er levende wezens ontstaan, dan moest er levenloze materie aanwezig zijn, waarin zich de eerste levensverschijnselen konden voordoen¹⁾.

Hoe dat in zijn werk is gegaan, weten wij niet en zullen wij vermoedelijk nimmer ervaren. Maar één ding meen ik met zekerheid te mogen aannemen: dat die eerste levende wezens van stoffelijken aard waren. Die stof zal dan vermoedelijk wel reeds protoplasma geweest zijn, of tenminste een organische stof, die gemakkelijk in protoplasma kon overgaan. Maar deze laatste gedachtengang is een zuiver chemische redenering, die zich niet in verband laat brengen met het onmiskenbare feit, dat protoplasma alleen bekend is in den vorm van afzonderlijke levende wezens en ook uitsluitend door zulke wezens kan voortgebracht worden.

Toch is, betrekkelijk nog niet lang geleden, de voorstelling gangbaar geweest van een ongeorganiseerde en vormloze massa ener levensstof, die nog heden ontstond op den bodem van diepe zeeën en waaruit zich gevormde levende kernen van omschreven gedaante konden losmaken, die zelfstandig gingen rondzwemmen. Zij sproot voort uit de waarneming der Challenger-expeditie, dat soms bij diepzeedreggingen een geleachtige massa opkwam, waaruit flagellate Infusoriën uitzwermden. Uitgaande van

¹⁾ De tegenovergestelde mogelijkheid, dat alle onbezielde stof oorspronkelijk uit bezielde (dus levende) stof ontstaan zou zijn, lijkt mij daarom minder denkbaar, omdat het „leven”, zoals wij het in zijn oneindige verscheidenheid waarnemen, toch een enkelvoudig natuurverschijnsel is, dat aan zeer beperkte voorwaarden der omgeving is gebonden, in de eerste plaats aan uiterst begrensde temperatuurgraden. Daar wij nu wel, „met zekerheid mogen aannemen”, dat de aarde ontstaan is als een gloeiend vloeibare bol, die zich aan zijn oppervlakte langzamerhand heeft afgekoeld, tot op een warmtegraad, waarop Leven mogelijk werd, moet op dat tijdstip een stof aanwezig zijn geweest, waaruit zich de eerste levende wezens konden ontwikkelen. Die stof kon dan tevens als voedsel voor die eerste bionten dienen.

de onderstelling, dat deze Infusoriën uit de vormloze geleï waren ontstaan, werd aan deze laatste de naam *Bathybius* gegeven. Lange tijd heeft die naam in de voorstellingen der Evolutieleer rondgespoekt, totdat afdoende werd aangetoond, dat men hierbij met een neerslag van zuiver anorganischen aard te doen had, waarin levende organismen bij de vorming van het substraat waren besloten geraakt.

Desniettegenstaande zou ik tot de voorstelling ener ongeorganiseerde levensstof willen terugkeren, omdat het mij nog het waarschijnlijkst voorkomt, dat er aanvankelijk slechts levenloze organische stof aanwezig was (laten wij zeggen eiwitten) en dat een klein gedeelte daarvan is gaan leven (men zou kunnen zeggen bezield is geworden). Die eerste levende wezens konden zich dan voeden met het niet-levend substraat, waaruit zij ontstaan waren, die dezelfde chemische (eiwitachtige) samenstelling had als zij. Die eerste levende protoplasten zullen wel een zeer eenvoudige bouw gehad hebben, misschien kropen zij bij wijze van Amoeben met behulp van pseudopodiën op de grens van lucht en water rond. Maar één grondeigenchap moeten zij m.i. bezeten hebben, namelijk de polariteit, daar deze aan alle levende wezens eigen is, hoezeer zij vaak door aanpassing aan bijzondere levensvoorraarden wordt verborgen. Hoe ontstonden nu daaruit de planten?

Door onbekende oorzaken verwierf een gedeelte der eerste levende Protoplanten het vermogen om koolzuur te assimileren en het daaruit bereide zetmeel onder opslorping van zouten uit den bodem tot eiwit te verwerken. Dit vermogen dankten zij aan de groene kleurstof, die zich in hen had gevormd, maar noodzaakte hen tevens om zich aan den bodem te hechten.

Deze noodzaak werd hun in zoverre noodlottig, dat zij in zintuigelijk en dus in geestelijk opzicht niet meer in staat waren zich te differentiëren tot het hogere niveau, dat vrijlevende organismen konden bereiken. Wel maakten zich enkele hunner weer van de bodem los en keerden tot het vrije leven terug, maar daar zij voor hun voeding en groei op de assimilatie aangewezen bleven, moesten zij in ondiep water of op vochtige plaatsen in het licht blijven en misten dus alle prikkel tot activiteit voor het vermeesteren van voedsel.

Ten gevolge van de al of niet volledige vasthechting maakte de oorspronkelijke bilateraal-symmetrische bouw plaats voor de spiraalsgewijze actinomorphie.

Maar in onderdelen, zoals bladen en bloemen, vruchten en zaden, handhaafde zich in vele gevallen de oorspronkelijke tweezijdigheid, zij het slechts voor een korte periode, zoals in de kiemplanten vooral die der Dicotylen.

In dit opzicht is het wel heel opmerkelijk dat bij *Salvinia*, een waterplant, de bladen in drie rijen gerangschikt zijn, en daarbij de middelste rij, die naar onderen in het water afhangt, een geheel andere bouw heeft dan de beide terweerszij.

Men kan dus zeggen, dat *Salvinia* volledig bilateraal-symmetrisch gebouwd is.

Daar de losdrijvende watervarens, evenals andere waterplanten, te beschouwen zijn als losgerakte landplanten, is het wel waarschijnlijk, dat deze bilateraliteit mag opgevat worden als een geval van atavisme.

De eigenschappen, waarin planten en dieren met elkaar overeenstemmen, zijn juist de hoofdkenmerken van het levende protoplasma: stofwisseling, ademhaling, uitscheiding, groei, voortplanting en dood. Van de oorspronkelijke bewegelijheid zijn bij planten slechts sporen overgebleven, zoals nutatie onder den invloed van het licht. Deze immobilisatie mag toegeschreven worden aan het wortelen in den bodem. Bij dieren heeft vasthechting dezelfde uitwerking gehad. Dat echter ook bij planten de gevoeligheid voor uitwendige prikkels nog behouden is gebleven, openbaart zich aan de reacties der planten op zwaartekracht, licht, warmte en vochtigheid en in enkele gevallen op schoksgewijze aanraking, zoals bij het kruidje-roer-mij-niet en de vleesetende planten, ook bij de meeldraden van *Berberis*, en vooral bij het ontluiken der bloemen.

Deze opvatting van den oorsprong der plantaardige natuur schijnt mij door de gegevens der Palaeontologie bevestigd²⁾, zoals die vermeld worden door Professor LAM. Als oudste landplanten beschouwt hij de Psilophytale, waarvan *Rhynia* als zeer oorspronkelijk en primitief gebouwd wordt geacht. Deze bestaat uit een stam en heeft dus klaarblijkelijk in den grond geworteld. Aan zijn top verdeelde deze stam zich vorksgewijs. De tanden der vork herhalen de bifurcatie, wanneer zij vegetatief van aard blijven, maar als zij aan hun top een sporangium voortbrengen, komt daaroor aan hun verdere groei een eind. Deze overgang tot de generatieve toestand kan aan beide takken van een bifurcatie plaats vinden of slechts aan een ervan; in dit laatste geval maakt de symmetrische bouw plaats voor een asymmetrische. Wanneer deze differentiatie zich in opvolgende bifurcaties herhaalt, ontstaat een sympodium, dat door strekking der opvolgende dragers van de vorktanden overgaat in een monopodium, en daardoor wederkeert tot de bilaterale symmetrie die ten gevolge der fixatie overgaat in de spiraalsgewijze vertakking. Blijven n.l. de opvolgende bifurcaties niet in hetzelfde vlak gericht staan, maar in vlakken, die een hoek met elkaar maken, dan verkrijgt het geheel een radiair-symmetrisch karakter. Dit heeft geleid tot de cormophytische bouw van het overgrote deel der hedendaagse landplanten.

De voorstelling van twee afzonderlijke rijken van levende wezens: Planten en Dieren, heeft zich vroegtijdig in het menselijk brein ontwikkeld door de onwillekeurige waarneming van het verregaande verschil tussen de hogere dieren en planten in 's mensen onmiddellijke omgeving. Naarmate ook de kennis der in het water levende wezens en vooral die der

²⁾ Ontleend aan H. J. LAM, Classification and the new Morphology, Acta Biotheoretica, Vol. VIII, 1948.

microscopische organismen tot onze voorstellingen doordrong, werd de grens tussen beide rijken hoe langer hoe onduidelijker. HAECKEL trachtte de moeilijkheden te ontgaan door een middenrijk te beramen: de Protisten, maar bereikte daarmee niet anders dan het bezwaar te verdubbelen.

Naar 't mij voorkomt, vervalt de moeilijkheid wanneer men de planten beschouwt als organismen, die in een zeer vroeg stadium der Evolutie van de levende wezens op een doodlopend spoor zijn geraakt door de vasthechting aan den bodem, waardoor alle mogelijkheid van geestelijke ontwikkeling werd afgesneden. Om het kort te zeggen: Planten zijn „verworden" Dieren.

In verschillende mate delen zij dit lot met tal van andere levende wezens, die zich aan bijzondere levensvoorraarden hebben vastgeketend, zoals parasieten, onderraads levende dieren (b.v. Termieten), vastzittende levenswijze (sponzen, koralen, zeelelies, paalwormen), terugkeer in zee (walvissen, zeekoeien) en tal van andere.

Om ons tot den invloed der vastzittende levenswijze te bepalen: degeneratie ten gevolge van immobilisatie kon (en kan) ieder type van organismen treffen, wat ook de hoogte zijner ontwikkelingstrap was op het tijdstip van intreden van den verwordenden invloed. Zij kan gepaard gaan met hoge specialisatie en differentiatie zowel in morphologische als in functionele zin, maar kan ook leiden tot algeheel verlies dier eigenschappen en vermogens. Om van beide uitwerkingen der fixatie enkele voorbeelden te kiezen uit de talloze, die de wereld der levende wezens ons biedt: De Sacculina, een parasiet op Kreeften en Krabben, is zelf een geheel gedegeneerd Crustacé, die alle bewegings- en waarnemingsorganen kwijtraakt, wanneer zij als vrijzwemmende Nauplius-larve zich aan haar slachtoffer vasthecht en met haar kopeind daarin doordringt. Dit vooreind groeit daarbij tot een vertakt stelsel van zuigdraden uit, dat doet denken aan het wortelstelsel der planten. Op geheel andere wijze werkt fixatie zonder parasitisme bij naverwanten der Sacculina: de Zeepokken en Eendenmossels onder de Cirripede Crustacé en, die weliswaar hun gezichtsvermogen en hun monddelen verliezen, wanneer hun vrijzwemmende larven zich op allerlei voorwerpen, zowel levende als levenloze, vasthechten met hun kopvoelers, maar daarna een zeer samengestelde behuizing van kalkplaten voortbrengen op de mantel, waarin zij zich hullen.

Hoe hardnekkig de erfelijke nawerking ener eenmaal aanvaarde fixatie is, blijkt bijzonder duidelijk in die gevallen, waarin sommige leden ener aan den bodem vastgehechte dier- of plantengroep zich weder uit dien toestand hebben vrijgemaakt.

De Stekelhuidigen leveren daarvan bijzonder duidelijke voorbeelden. Onder de Haarsterren, wier oudste vertegenwoordigers reeds met of zonder steel bevestigd waren, heeft de recente Antedon zich daarvan weten te bevrijden, maar doorloopt in zijn ontwikkeling nog steeds een stadium van gesteeldheid. Zeesterren en Zeeëgels zijn alle vrijlevend, en evenzo de Holothuriën, maar in de ontogenie der eerste treedt voorbijgaand een

voetvormig orgaan op, dat blijkbaar het rudiment van een steel is. Aan hun tweezijdig-symmetrische larven ontwikkelt zich de aanleg van het volwassen dier als een knop, die van den aanvang af meerstralig (meestal 5-stralig) is. De radiaire symmetrie hangt dus klaarblijkelijk met den overgang tot het vastgegroeide leven samen en de terugkeer tot een vrijlevend bestaan kan dien radiairen bouw niet meer uitwissen, evenmin als deze vermocht de sporen der oorspronkelijke bilateraliteit geheel te verbergen.

Ook de mens doorloopt in zijn ontogenie een vastzittend stadium en draagt de sporen daarvan gedurende het gehele extrauterine leven mee in den vorm van het navellitteken.

Als samenvatting van den gedachtengang, die tot het bovenstaande betoog heeft geleid, moge het volgende in het midden worden gebracht.

Leven op aarde werd slechts mogelijk toen deze aan haar oppervlakte voldoende was afgekoeld. Toen die toestand aanbrak, hadden zich waarschijnlijk wel allerlei chemische verbindingen gevormd uit de grondstoffen, die tevoren in ongebonden toestand aanwezig waren, of misschien zich uit een enkele oerstof hadden gedifferentieerd.

De Koolstof leverde enkele levensvatbare verbindingen, de Kiezelstof slechts levenloosblijvende. Hoe het leven in die meest samengestelde Koolstofverbindingen werd opgewekt, blijft even onverklaarbaar als alle andere natuurverschijnselen. Denkbaar is alleen, dat die eerste levende wezens zich konden voeden met de eiwitten, waaruit zij zelf ontstaan waren.

Men mag zich hen denken als vrij bewegelijk. Voor hun stofwisseling en groei waren zij behalve op het substraat waaruit zij zelf waren voortgekomen, aangewezen op het zouten bevattende water hunner omgeving.

Toen echter aan het proces der eiwitvorming buiten het leven om, door onbekende oorzaken een einde kwam, konden de daaruit gevormde levende wezens slechts op twee wijzen blijven bestaan, te weten of door andere op te eten, terwijl zij tevens zich door deling vermenigvuldigden, of door zelf eiwitten te gaan voortbrengen uit minder samengestelde z.g. anorganische verbindingen: Enkelen hunner verwierven dit laatstgenoemde vermogen, in den vorm van het koolzuur-assimilerende chlorophyl, maar werden daarbij gedwongen de benodigde zouten op te slurpen. Deze zouten vonden zij aanvankelijk voornamelijk in den bodem, zodat zij zich aan den grond moesten vasthechten, waardoor zij hun vrije bewegelijkheid kwijtraakten. Wel keerden in alle volgende tijdperken enkele tot de vrijzwemmende levenswijze terug, maar dat kon hen niet meer van den eenmaal onderganen invloed der fixatie bevrijden. Deze laatste overweging meen ik te mogen gronden op het feit, dat al dië groene organismen, die vrij in het water leven en daarbij geen kenmerken vertonen, welke op een oorspronkelijk verblijf op het droge wijzen, het niet verder gebracht hebben

dan de allereenvoudigste typen van organisatie (b.v. de Blauwwieren). De op het land gebleven planten differentieerden zich daarentegen even goed als de dieren in allerlei richtingen en graden, maar bleven daarbij toch steeds in de hogere levensfuncties bij deze laatste achter, ten gevolge van den remmenden invloed der fixatie.

De ontwikkelingsdrang die tot deze differentiatie drijft, is een der vele vormen van het natuurverschijnsel der onafgebroken veranderlijkheid van de gehele levende wereld.

Beschouwt men Leven als een vorm van energie, dan zou men kunnen zeggen, dat het daarvoor vatbare stoffelijke substraat met die energie „geladen” wordt, en bij den dood weder door haar verlaten, zonder dat de massa der stof verandert. Uit dat oogpunt vertoont het Levensverschijnsel overeenkomst met de energievormen in 't algemeen, zoals electriciteit, magnetisme, warmte, die om zich te manifesteren, materie behoeven waaraan zij zich kunnen meedelen, en die zij ook weer kunnen verlaten. Weliswaar leidt deze vergelijking niet tot dieper inzicht in het Levensverschijnsel, maar ook in dat opzicht verschilt dit mysterie niet van elk ander natuurverschijnsel. Al kan men de wetten waaraan zij gehoorzamen, nauwkeurig bepalen, hun eigenlijke wezen blijft ondoorgrondelijk.

Zusammenfassung.

Die ersten Lebenserscheinungen auf der Oberfläche der Erde können nur aufgetreten sein nachdem diese sich genügend abgekühlt hatte. Das Leben selbst ist eine Form der Energie, die gebunden ist an eine einzige Beschaffenheit der Materie: das Protoplasma. Um sich anfänglich manifestiren zu können, musz diese Substanz in zusammenhängender Masse da gewesen sein, woraus selbständige beseelte Organismen sich loslösen könnten. Diese fanden in der nicht beseelten Grundmasse ihre Nahrung. Es ist deshalb nicht notwendig anzunehmen, dasz diese ersten Organismen schon die Fähigkeit der Kohlensäure-Assimilation besaszen. Viel wahrscheinlicher ist, dasz dieses Vermögen sich erst nachträglich bei einem Teil der ersten Lebewesen entwickelte als eine Variation, wodurch sie von der Albumen-Ernährung unabhängig wurden, aber zugleicherzeit ihre freie Beweglichkeit verloren, weil sie genötigt waren mit ihrem aboralen Pole in den Boden einzudringen, zur Absorbtion der notwendigen Salze. Dadurch ging die ursprüngliche bilaterale Symmetrie verloren, die für die radiale Platz machte, obwohl Sie sich in Unterteilen wie Blätter, Blüten, Früchte, Samen, Keimpflanzen, in vielen Fällen deutlich handhabte.

Die ersten paläozoischen Pflanzen, wie Rhynia besaszen noch die bilaterale dichotomische Verästelung. Einige rezente Laubpflanzen wie Salvinia besitzen sie ebenfalls, wohl als Rückschlag auf älteste Anlagen. Alle im Wasser freilebende höhere Pflanzen müszen als Landpflanzen, die sich aus der Anheftung an den Boden freigelöst haben, betrachtet werden. Dadurch haben sie die Beweglichkeit zwar in beschränktem Maße wieder erlangt.

aber die Möglichkeit höherer geistiger Entwicklung ist für alle Pflanzen unwiderruflich verloren. In dieser Hinsicht können Pflanzen als degenerierte Tiere bezeichnet werden.

Summary.

Living animals could only appear on the surface of the earth, when this had sufficiently cooled down, and there had been formed a substance, in which the phenomena of life could manifest themselves. Though not yet living, this material already must have possessed a structure identical, or nearly so, to protoplasm. In accordance, the remaining, non-animated bulk of this substance could afford nutrition to the organisms, that had differentiated themselves from it. Consequently they did not need to possess from the first instance the faculty of assimilation. Afterwards, from unknown causes, this faculty arose in part of them, which thereby at the same time were obliged to fix themselves into the earth to absorb the necessary salts for the formation of albumens. Moreover their oral part grew out to the stem, which by bifurcation produced branches and leaves. In consequence of this fixation the original bilateral symmetry of the first green organisms changed into a radial configuration, passing through a spiral arrangement of their branches.

By this modification the contrast between dorsal and ventral surface of the originally zygomorphic organisms got effaced, but maintained itself in many details of its structure, e.g. leaves, flowers, fruits, seeds, prothallia, and even shows itself in some fullgrown plants as the palaeozoic Rhynia or the recent *Salvinia*; the latter is to be considered as a case of atavism.

Therefore we are justified to assert, that plants, by acquiring the faculty to produce chlorophyll, have lost the opportunity to reach a higher level of mental development, and so in a certain sense may be considered as degenerate animals.

Résumé.

Le phénomène de la vie, qui est une forme spéciale d'énergie, ne peut s'être manifesté qu'après un abaissement suffisant de température de la surface de la terre. Parmi les nombreux procédés chimiques à cet'époque une masse de matière inorganisée conforme au protoplasme peut s'être formée. Dans ces masses certaines parties se différencierent et devinrent des organismes vivants et indépendants, qui pouvaient se nourrir de la reste de la substance non-organisée. Par cette conception du commencement de la vie, nous ne sommes plus obligés de supposer que les premiers organismes possédaient déjà la faculté de l'assimilation de l'acide carbonique. Au contraire, cette faculté ne se développa que plus tard dans une partie des premiers organismes en guise d'une variation accidentelle. Grâce à elle ils devinrent indépendants de la nutrition avec la masse de l'albumen inorganisé, mais en même temps ils furent obligés de se fixer dans la terre,

pour l'absorption des sels nécessaires à la production de nouveau protoplasme. Cette fixation, tout en rendant les plantes indépendantes de la nutrition organique, en même temps leur fermait l'accès aux fonctions psychiques. Dans ce sens on peut prétendre que les plantes sont des animaux dégénérés.

En outre, cette immobilisation nécessitait un changement de leur structure originale selon le type de symétrie bilatérale, qui se transformait en un arrangement radial des parties de leur corps. Ce changement de symétrie était accompagné d'une différentiation spéciale des deux bouts de l'organisme bipolaire; l'extrémité orale devenant la couronne, l'aborale pénétrant dans la terre en forme de racine. La zygomorphie originale cependant se maintenait dans maintes parties, comme feuilles, fleurs, fruits, semences et surtout dans les germes des Angiospermes.

Dans ce regard il est remarquable que les plus anciennes plantes comme les *Rhynia* de l'ère paleozoologique montrent une ramification bifurquée, qui répond à la symétrie bilatérale, mais qui passe à la symétrie radiaire par torsion en forme de spirale.

Physics. — Unitary Quantum Electron Dynamics I. By H. J. GROENEWOLD.
(Koninklijk Nederlands Meteorologisch Instituut te De Bilt.)
(Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of January 29, 1949.)

1. Introduction.

1.1 *Source particles and carrier particles.* In a previous paper¹⁾ (here referred to by 1) we have distinguished source particles (e.g. electrons) and carrier particles (e.g. photons). The carrier particles are emitted and absorbed by source particles and in this way they lead to a (e.g. electromagnetic) interaction between source particles. Because the carrier particles have a finite velocity there will be a time lag in this interaction.

1.2 *Dualistic and unitary theories.* 1.21 *Dualistic theory.* In the usual dualistic theories the two kinds of particles are treated separately without conspicuous relations between the properties of carrier particles and those of source particles.

1.22 *Unitary theories.* Unitary theories intend to give a complete description entirely either in terms of fields of carrier particles (f -theory) or in terms of source particles (p -theory). We shall only be concerned with the latter type and further omit the prefix p .

A unitary theory has to give the complete equations of motion of the interacting source particles. Because of the time lag in the interaction the description will be extremely complicated if not impossible. Not before the theory has been completely established in terms of source particles, fictitious carrier particles which are created and annihilated may be formally introduced in order to simplify the description.

1.23 *Balance.* 1.231 *Dualistic versus unitary theory.* The time lag in the interaction makes that in unitary theory the boundary conditions if they can be stated at all will be frightfully complicated. Further it makes that the energy-momentum which is lost by one source particle in a given region of time-space is gained by another source particle in a different region of time-space. In this way conservation of energy-momentum becomes rather intricate. The same holds for loss and gain of charge in case the carrier particles of the dualistic theory are supplied with some kind of charge.

In the latter case the source particles (e.g. nucleons) emit and absorb in the dualistic theory carrier particles (e.g. charged mesons), which in their turn act as source particles and in this capacity emit and absorb still other kinds of carrier particles (e.g. photons). The latter lead to a (e.g. electromagnetic) interaction between all charged particles. Strictly a somewhat similar situation always exists in general relativity theory in which

all particles, source particles as well as carrier particles, have gravitational interaction. In such cases one needs to be highly optimistic in order to have still expectations of a unitary theory.

Altogether if within certain limits a unitary theory might be possible at all it would be extremely complicated and intractable. On the other hand the dualistic theory gives a straightforward and relatively simple description which is continuous in time-space with proper differential equations of motion, clear boundary conditions and differential conservation relations. It is not astonishing that it is always the dualistic theory which is used in practice.

1.232 *Unitary versus dualistic theory.* In spite of the perhaps unsurmountable difficulties it might still be worth while to deal with some aspects of what might be a unitary theory if ultimately it were feasible. Even an imperfect unitary theory might be able to throw some light on the possibility of

U_2 another outlook on dualistic theory;

U_3 further developments in unitary theory, which are untranslatable into dualistic theory

(there is no U_1 ; it can be seen in 1.3 why not).

1.2321 *Other outlook.* As long as the unitary and dualistic descriptions are supposed to give the same observable behaviour of source particles they can be translated into each other. Yet each of them may show aspects which are not as easily brought to light in the other description.

In particular a unitary theory would completely derive all properties of carrier particles from the properties of interacting source particles and/or the way in which the carrier particles are introduced.

1.2322 *Untranslatable developments.* It is quite conceivable that in a further development a unitary theory would be modified in a way, which could no longer be translated into a dualistic description. In such a case the dualistic theory could be maintained as an extremely useful approximation, but the unitary theory would become of fundamental importance.

This possibility in particular refers to the divergence difficulties connected with pair processes and self interaction.

1.3 *Queries.* Corresponding to U_2 and perhaps to U_3 there are the two remaining problems (Q_3 slightly extended) of 1:

Q_2 how are the relations between the properties of carrier particles and those of source particles; can in particular the former be derived from the latter?

Q_3 how shall negative states and pair processes and self interaction be dealt with?

In this paper we shall be concerned with Q_2 . Q_3 will only incidentally be touched in connection with the negative states of carrier particles.

1.4 Conditions. We mention some of the most striking conditions which a unitary theory would have to satisfy.

1.41 Opaqueness condition. There is one condition which can directly be seen to be necessary in order that carrier particles can entirely be eliminated from dualistic theory. Every carrier particle which is absorbed by a source particle must have been emitted by another source particle and every carrier particle which is emitted by a source particle must be absorbed by another source particle. Because identical particles are indiscernible, no carrier particles at all can be allowed to enter into or to escape from the physical universe, not even such ones which never have been emitted and never will be absorbed. If the universe is not opaque for carrier particles, a unitary theory if possible at all cannot be equivalent to the dualistic theory.

1.42 Asymmetry. Processes of emission and absorption of carrier particles (radiation processes) are connected with a peculiar kind of asymmetry, which if suppressed at one place peeps up at another place. It appears in relation with retarded or advanced fields or in the properties of radiation damping. Emission of carrier particles occurs spontaneously or induced, absorption only induced. In unitary theory this asymmetry will have to appear in another form. If energy-momentum is transferred from one source particle in a certain region of time-space to another source particle in another region, the energy loss has always to occur earlier in time than the energy gain. This asymmetry, which appears to be rather fundamental, is obviously not an asymmetry in time; loss and gain (like emission and absorption) are interchanged under reversal of time.

This asymmetry condition will be considered in a later paper. That paper (here referred to by 3) will deal with some unitary aspects of what in dualistic theory are radiation processes.

1.43 Interference and diffraction. The periodic wave aspect of carrier particles (e.g. photons) is unrefutably established by interference and diffraction phenomena. In a unitary theory not only the carrier particles, but at the same time their periodic wave aspect will be lost. The periodicity conditions, which are responsible for the observable interference and diffraction effects, have then to be derived from the properties of the interacting carrier particles. Also this point will be considered in 3.

1.44 Vacuum effects. In dualistic theory there are effects of vacuum polarization by virtual creation and annihilation of the same pairs of source particles and of vacuum fluctuations of the carrier field by virtual emission and absorption of carrier particles by the same source particle (self interaction), which in spite of malignant divergences are of fundamental importance²⁾. It is not only a task of Q₃ to ask for such a modification of the theory which removes all divergences, but also to take care that the relevant effects are not removed at the same time.

1.5 Unification and quantization. One could try two different ways from dualistic classical theory to unitary quantum theory. The one is via unitary

classical theory, the other via dualistic quantum theory. We shall rove about a little along the latter way.

1.51 *Dualistic quantum theory.* Those principles of dualistic quantum theory which we shall need for this excursion will be reviewed in 2.

1.52 *Unitary classical theory.* Though we shall not directly be concerned with unification of classical theory, it will be profitable to keep in mind the problems and achievements in this process. The problems are narrowly related to those of the unification of quantum theory. To some extent they have been solved for electromagnetic interaction between electric charges.

As to the classical analogue of Q_2 the equations of motion have already been given by TETRODE³⁾ and FOKKER⁴⁾. A thorough physical interpretation and discussion has been given by WHEELER and FEYNMAN⁵⁾.

As to the classical analogue of Q_3 there is a recent attack by FEYNMAN⁶⁾. Also other suggestions could be thought of⁷⁾.

1.6 *Simplifications and limitations.* As our derivations will appear a bit complicated it will be advisable to omit all those complications which are not strictly essential for understanding the fundamental problem. Therefore we shall in the first place restrict the source particles to electrons and the carrier particles to photons. So we shall only consider electromagnetic interaction between electrons. The generalization to meson interaction between nucleons gives no fundamental new difficulties as long as nothing (like disintegration) happens to them but emission and absorption by nucleons.

Further we shall have to resort to very simple models and even then our considerations will remain rather poor. In spite of all this it is hoped that they can throw a bit of light on Q_2 .

2. Dualistic theory.

In 1 the interaction between particles of different kinds has hardly been mentioned. In the dualistic part of the present paper the interaction between source particles (electrons) and carrier particles (photons) is of primary importance.

2.1 *Photons.* Before dealing with this interaction, we first recollect some results of 1 for the case of photons (spin = 1, $m = 0$).

2.11 *Notation.* For photons we shall now write the time-space vectors, which occur as arguments in the wave functions, as $(cs, y^1, y^2, y^3) = (y^0, y^1, y^2, y^3)$ ($-g^{00} = g^{11} = g^{22} = g^{33} = 1$). The wave operators replace some of these sets by other sets, which we now write as $(ct, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$. Thus for photons the (\vec{x}, t) and (\vec{y}, s)

of 1 are now written as (\vec{y}, s) and (\vec{x}, t) respectively. The reason for this interchange will become clear later on in 2.2122, when the interaction with electrons is introduced.

2.12 *Equations of motion.* The equations of motion for free photons are

$$L \{ \vec{y}_k, \vec{s}_k \} (\vec{y}_1 \vec{s}_1, \vec{y}_2 \vec{s}_2, \dots |^{\alpha_1 \alpha_2 \dots} \Phi = 0 \quad (k=1, 2, \dots), \dots \quad (1)$$

where

$$\mathbf{L} \{ \vec{y}, s \} = \left(\frac{\hbar c}{i} \right)^2 \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y_\alpha} \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

In the absence of electrons the wave functions Φ are subjected to the supplementary conditions

$$\frac{\partial}{\partial y_k^\alpha} (\vec{y}_1 s_1, \vec{y}_2 s_2, \dots | \dots^a_k \dots \Phi) = 0 \quad (k=1, 2, \dots). \quad \dots \quad (3)$$

If the conditions (3) are satisfied together with their first order time derivatives at a given set of initial times, (1) makes that they are satisfied at all times.

2.13 Reference functions. A complete system of orthonormal solutions of the 1-particle equation

$$\mathbf{L} \{ \vec{y}, s \} (\vec{y} s) |^\alpha \varphi = 0, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

not subjected to supplementary conditions, is given by

$$(\vec{y} s) |^\alpha \varphi | \vec{\xi}_{\pm r} = {}^\alpha b | \vec{\xi}_{\pm r} e^{-\frac{i}{\hbar c} (\pm \xi_{\pm r} - \vec{\xi} \cdot \vec{y})} / \xi^{\pm} (2 \hbar^3 c^3)^{\frac{1}{2}} \quad \dots \quad (5)$$

with 4 4-vectors ${}^\alpha b | \vec{\xi}_{\pm r}$ ($r = 1, 2, 3, 4$) for which

$$\left. \begin{aligned} (\vec{\xi}_{\pm r} | b_\alpha^\dagger {}^\alpha b | \vec{\xi}_{\pm r}) &= \delta_{rs}, \\ \sum_r {}^\alpha b | \vec{\xi}_{\pm r} (\vec{\xi}_{\pm r} | b_\beta^\dagger) &= \delta_\beta^\alpha + 2 j^\alpha j_\beta, \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad (6)$$

where j^α is an arbitrarily chosen normalized time-like vector ($j^\alpha j_\alpha = -1$), which we take in positive time direction.

2.14 Positive and negative states. **2.141 Density states.** The indefinite density operator is $\left(\frac{\hbar}{i} \frac{\delta}{\delta s} - \frac{\hbar}{i} \frac{\partial}{\partial s} \right) g_{\alpha\beta}$. The positive and negative density states are distinguished by the eigenvalues +1 and -1 of the operator η , which is the product of the operator for which the exponentials (5) are eigenfunctions with eigenvalues ± 1 and the operator $(g_{\alpha\beta} + 2 j_\alpha j_\beta)$.

2.142 Energy states. The energy operator is $\left(\frac{\hbar}{i} \frac{\delta}{\delta s} - \frac{\hbar}{i} \frac{\partial}{\partial s} \right) / 2$. The positive and negative energy states are distinguished by the + and - sign in (5).

2.143 Charge conjugated states. The "charge" conjugated state of $(\vec{y} s) |^\alpha \varphi | \vec{\xi}_{\pm r}$ is $(\vec{y} s) |^\alpha \tilde{\varphi} | \vec{\xi}_{\pm r} = (\vec{y} s) |^\alpha \varphi | - \vec{\xi}_{\pm r}$.

2.15 Creation and annihilation operators. The creation and annihilation operators a and a^\dagger are defined by

$$\left. \begin{aligned} & (\vec{\xi}_{\pm r} | \mathbf{a}_\alpha^\dagger | \vec{y s}) (\vec{y_1 s_1}, \dots, \vec{y_n s_n})^{\alpha_1 \dots \alpha_n} \Phi = n^4 \int (dy_n) (\vec{\xi}_{\pm r} | \varphi_\beta^\dagger | \vec{y_n s_n}) \cdot \pm (\delta_\alpha^\beta + 2j^\beta j_\alpha) \\ & \quad \left(\frac{\hbar}{i} \frac{\partial}{\partial s_n} - \frac{\hbar}{i} \frac{\partial}{\partial s_n} \right) S_n (\vec{y_1 s_1}, \dots, \vec{y_n s_n})^{\alpha_1 \dots \alpha_n} \Phi, \\ & \{ \vec{y s} | \mathbf{a}^\alpha | \vec{\xi}_{\pm r}) (\vec{y_1 s_1}, \dots, \vec{y_{n-1} s_{n-1}})^{\alpha_1 \dots \alpha_{n-1}} \Phi \\ & \quad = n^4 S_n (\vec{y_n s_n})^\alpha \varphi | \vec{\xi}_{\pm r}) (\vec{y_1 s_1}, \dots, \vec{y_{n-1} s_{n-1}})^{\alpha_1 \dots \alpha_{n-1}} \Phi \end{aligned} \right\} \quad (7)$$

if acting to the right and by the HERMITIAN adjoint relation (7[†]) if acting to the left.

2.151 Wave operators. For photons we shall use dynamical wave operators (representation s_2). In the distinction made in 1 between DIRAC's 1942 theory (D -modification) and the current hole theories (H -revision), there remained an ambiguity just in the photon case (cf. 1 4.3224). In that case we had to do with positive and negative energy functions and with positive and negative vectors. Each of them can be treated either by D -modification or by H -revision. That gives (writing small types for the vector treatment) the 4 combinations Dd , Dh , Hd and Hh . We shall consider them all.

The dynamical wave operators are defined by

$$\left. \begin{aligned} & (\vec{x t} |^\alpha \varphi_\beta' | \vec{y s}) = \sum_{r \pm} \sum \int (d\xi) (\vec{x t} |^\gamma \varphi | \vec{\xi}_{\pm r}) (\vec{\xi}_{\pm r} | \mathbf{a}_\beta^\dagger | \vec{y s_0}) | \lambda_{\pm \gamma}^\alpha \\ & \quad + \sum_{r \pm} \sum \int (d\xi) (\vec{x t} |^\gamma \varphi | \vec{\xi}_{\mp r}) \{ \vec{y s_0} |^\beta \mathbf{a} | -\vec{\xi}_{\mp r} \} \mu_{\pm \gamma}^\alpha, \\ & \{ \vec{y s} |^\beta \varphi_\alpha^\dagger | \vec{x t}) = \sum_{r \pm} \sum \int (d\xi) \{ \vec{y s_0} |^\beta \mathbf{a} | \vec{\xi}_{\pm r}) (\vec{\xi}_{\pm r} | \varphi_\gamma^\dagger | \vec{x t}) \lambda_{\pm \alpha}^{+\gamma} \\ & \quad + \sum_{r \pm} \sum \int (d\xi) (-\vec{\xi}_{\mp r} | \mathbf{a}_\beta^\dagger | \vec{y s_0}) (\vec{\xi}_{\pm r} | \varphi_\gamma^\dagger | \vec{x t}) \mu_{\pm \alpha}^{+\gamma}. \end{aligned} \right\} \quad (8)$$

The λ 's and μ 's are for the 4 combinations given by

	Dd	Dh	Hd	Hh
$\lambda_{\pm \gamma}^\alpha$	δ_γ^α	$\delta_\gamma^\alpha + j^\alpha j_\gamma$	$\frac{1 \pm 1}{2} \delta_\gamma^\alpha$	$\frac{1 \pm 1}{2} \delta_\gamma^\alpha + j^\alpha j_\gamma$
$\mu_{\pm \gamma}^\alpha$	0	$\mp j^\alpha j_\gamma$	$\frac{1 \mp 1}{2} (\delta_\gamma^\alpha + 2j^\alpha j_\gamma)$	$\frac{1 \mp 1}{2} \delta_\gamma^\alpha + j^\alpha j_\gamma$
$\lambda_{\pm \gamma}^{+\alpha}$	$\pm (\delta_\gamma^\alpha + 2j^\alpha j_\gamma)$	$\pm (\delta_\gamma^\alpha + j^\alpha j_\gamma)$	$\frac{1 \pm 1}{2} (\delta_\gamma^\alpha + 2j^\alpha j_\gamma)$	$\frac{1 \pm 1}{2} \delta_\gamma^\alpha + j^\alpha j_\gamma$
$\mu_{\pm \gamma}^{+\alpha}$	0	$-j^\alpha j_\gamma$	$\frac{1 \pm 1}{2} \delta_\gamma^\alpha$	$\frac{1 \mp 1}{2} \delta_\gamma^\alpha + j^\alpha j_\gamma$

if acting to the right. If they act to the left λ and λ^+ have to be interchanged in (9) and also μ and μ^+ .

2.152 *Field operators.* From the mutually HERMITIAN adjoint wave operators φ and φ^\dagger we form for later use the self-adjoint field operators Φ_c and Φ_s

$$\left. \begin{aligned} \Phi_c(\vec{x}t) &= (2\pi)^{\frac{1}{2}} \hbar c (\{\vec{y}s_0|^\beta \varphi_\alpha^\dagger|\vec{x}t\} + (\vec{x}t)_\alpha \varphi_\beta^\dagger |\vec{y}s_0\}), \\ \Phi_s(\vec{x}t) &= (2\pi)^{\frac{1}{2}} \hbar c (\{\vec{y}s_0|^\beta \varphi_\alpha^\dagger|\vec{x}t\} - (\vec{x}t)_\alpha \varphi_\beta^\dagger |\vec{y}s_0\})/i. \end{aligned} \right\} \quad . . . \quad (10)$$

The numerical constant has been inserted for later purpose of normalization.

Though it is quite a relief that the cumbersome notation for the wave operators has to be abandoned, the role of the omitted variables and dashes should continually be kept in mind. We shall further write the argument $\vec{x}t$ in general as (x) .

2.16 *D-functions.* The asymmetrical and symmetrical invariant *D*-functions are (because of $m = 0$ degenerated into)

$$\left. \begin{aligned} D_a(x) &= -\frac{i}{\hbar c} \delta(x^\alpha x_\alpha) = -\frac{i}{\hbar c} (\delta(x-ct) - \delta(x+ct))/2x, \\ D_s(x) &= \frac{1}{\pi \hbar c} \frac{1}{x^\alpha x_\alpha} = \frac{1}{\pi \hbar c} \left(\frac{1}{x-ct} - \frac{1}{x+ct} \right)/2x. \end{aligned} \right\} \quad . . . \quad (11)$$

We shall also need their linear combinations

$$D_{\pm}(x) = D_s(x) \pm D_a(x). \quad . . . \quad (12)$$

2.17 *Commutation relations.* We check that for each kind of wave operators we do obtain the required commutation relations

$$\left. \begin{aligned} &[(\vec{x}t)_\alpha \varphi_\beta^\dagger |\vec{y}s_0\}, \{\vec{y}s_0|^\beta \varphi_\alpha^\dagger|\vec{x}'t'\}] = \\ &= \sum_r \sum_{\pm} (\lambda_{\pm\alpha\gamma} \lambda_{\pm\alpha'\gamma'}^\dagger - \mu_{\pm\alpha\gamma} \mu_{\pm\alpha'\gamma'}^\dagger) \int (d\xi) (\vec{x}t)_\gamma^\gamma \varphi_\gamma^\dagger |\xi \pm r\rangle (\xi \pm r| \varphi^{\dagger\gamma'} |\vec{x}'t') \\ &= g_{\alpha\alpha'} D_a(x-x'). \end{aligned} \right\} \quad (13)$$

The commutators which give zero have not been written down.

The field operators satisfy the commutation relations

$$\left. \begin{aligned} [\Phi_{ca}(x), \Phi_{ca'}(x')]^\perp &= [\Phi_{sa}(x), \Phi_{sa'}(x')]^\perp = 2\pi(\hbar c)^2 g_{\alpha\alpha'} 2D_a(x-x'), \\ [\Phi_{ca}(x), \Phi_{sa'}(x')]^\perp &= 0. \end{aligned} \right\} \quad (14)$$

2.18 *Empty-empty part.* 2.181 *Empty states.* With a view to unification we shall be particularly interested in states in which no photons are present

(empty states). The wave function of such a state contains no sets $(\vec{y}s_0)$. If it contains no variables of particles of another kind, it is a constant. The empty state projection operator E has the eigenvalue 1 in the empty states, 0 in all other states. As the empty-empty part of an operator

$Q \{ \vec{y}_1 s_{10}, \vec{y}_2 s_{20}, \dots \}$ we define the operator

$$E Q \{ \vec{y}_1 s_{10}, \vec{y}_2 s_{20}, \dots \} E. \quad . . . \quad (15)$$

It is that part of \mathbf{Q} , which gives an empty state function if it operates upon an empty state function and zero otherwise.

2.182 *Order in time.* Suppose we have a product of creation and annihilation operators, each containing a set (xt) . We shall call the product well ordered in positive/negative time direction if the values of the successive t 's do not increase/decrease if one goes from on factor to its neighbour in the direction to which the product operates.

2.183 *A product of exponentials.* We shall later need the empty-empty part of a product of exponentials in the field operators of the form

$$\prod_{k=1}^n e^{\frac{ie}{\hbar c} s_k^\alpha \Phi_\alpha(x_k)} \quad \dots \quad (16)$$

in which the n factors are well ordered in positive time direction. Because according to (14) the factors commute in world points with a space-like connection, the product (16) is independent of the choice of the time axis. Therefore we can order (16) with respect to the direction of the time-like vector j^α used in (6), even if the time axis has a different direction.

The product (16) can be expanded into a power series in the Φ 's. Each Φ is according to (10) a linear combination of a creation and an annihilation operator. Each creation operator adds a set (ys_0) , each annihilation operator takes away such a set. If no sets are left to be taken away (empty states) the annihilation operator yields zero. Only those terms of (16) can contribute to the empty-empty part, which are a product of a number of creation operators and an equal number of annihilation operators and in which no factor is preceded by more annihilation than creation operators. Each of these products begins with a creation operator, which is supposed to act on an empty state function and ends with an annihilation operator, which than produces an empty state function again. The creation and annihilation operators of such a product can be dovetailed together with the help of (8) and (7), observing that the product is well ordered in the time direction of j^α . The empty-empty part of (16) is finally obtained by summing all these terms. It is most easily seen by means of complete induction that the result is

$$\prod_{k,l=1}^n e^{\frac{ie}{\hbar c} s_k^\alpha s_l^{\alpha'} W_{\alpha\alpha'}(x_k, x_l)} \quad E. \quad \dots \quad (17)$$

The cross factors ($k \neq l$) are counted twice, the "self" factors ($k = l$) once. Whether Φ_c or Φ_s is used in (16), the function $W_{\alpha\alpha'}(x, x')$ is determined by

$$\begin{aligned} \frac{i}{\hbar c} W_{\alpha\alpha'}(x, x') = & \sum_r \sum_{\pm} \lambda_{\pm\alpha\gamma} \lambda_{\pm\alpha'\gamma'}^\dagger \int (d\vec{\xi}) (\vec{\xi}_{\pm r} | \varphi^{+r''} | x'') (x''' | r''' \varphi | \vec{\xi}_{\pm r}) \\ & + \sum_r \sum_{\pm} \mu_{\pm\alpha\gamma} \mu_{\pm\alpha'\gamma'}^\dagger \int (d\vec{\xi}) (x'' | r'' \varphi | \vec{\xi}_{\pm r}) (\vec{\xi}_{\pm r} | \varphi^{+r'''} | x'''). \end{aligned} \quad (18)$$

2 and 3 dashes stand for 0 and 1 dash or crosswise in that way for which with respect to the time direction of $j^\alpha x''$ is later than x''' . We regard to this order in time we write

$$\sigma(x) = -\frac{j_\alpha x^\alpha}{|j_\alpha x^\alpha|}; f^\sigma(x^\alpha) = f(x^\alpha + j^\alpha (j_\beta x^\beta + |j_\beta x^\beta|)), \dots \quad (19)$$

so that for even and odd functions $f(x)$

$$f_{\text{even}}^\sigma(x) = f_{\text{even}}(x), f_{\text{odd}}^\sigma(x) = \sigma(x) f_{\text{odd}}(x). \quad (20)$$

Then (18) reads

$$\frac{i}{hc} W_{\alpha\alpha'}(x, x') = \sum_{\tau} \sum_{\pm} (\lambda_{\pm\alpha\gamma} \lambda_{\pm\alpha'\gamma'}^\dagger + \mu_{\mp\alpha\gamma} \mu_{\mp\alpha'\gamma'}^\dagger) (g^{\gamma\gamma'} + 2j^\gamma j^{\gamma'}) D_\pm^\tau(x - x'). \quad (21)$$

The 4 combinations Dd , Dh , Hd and Hh give different results

	$\frac{i}{hc} W_{\alpha\alpha'}(x, x')$	
Dd	$g_{\alpha\alpha'} D_a^\sigma(x - x')$	}
Dh	$(g_{\alpha\alpha'} + 2j_\alpha j_{\alpha'}) D_a^\sigma(x - x')$	
Hd	$g_{\alpha\alpha'} D_+^\sigma(x - x')$	
Hh	$(g_{\alpha\alpha'} + j_\alpha j_{\alpha'}) D_+^\sigma(x - x') + j_\alpha j_{\alpha'} D_-^\sigma(x - x').$	

. . . (22)

The evaluation of the numerical coefficients in the expansion of (17) into a power series in the W 's is a matter of combinatorics. The corresponding coefficients in the dovetailed expansion of (16) are only partially a result of combinatorics, for the rest they are produced at the dovetails by the square roots which occur in (7). At this point the EINSTEIN-BOSE statistics of the photons play a decisive part.

The functions $D_s^\sigma(x)$ and $D_a^\sigma(x)$ are not only relativistic invariant but even independent of j^α , because $D_s(x)$ is even and $D_a(x)$, which is odd, vanishes outside the light cone.

2.2 Electrons and photons. Now turning to the electrons and their interaction with photons we first consider a single electron.

2.21 1 electron. **2.211 Notation.** For electrons we shall write the time-space vectors which occur as arguments in the wave functions as $(ct, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$. In this paper we shall not use creation and annihilation operators of electrons so that we need no other sets than (x) . Further we write the spin matrices $(1, \alpha^1, \alpha^2, \alpha^3) = (\alpha^0, \alpha^1, \alpha^2, \alpha^3)$. (α^α transforms as a 4-vector density, $\beta\alpha^\alpha$ as a 4-vector).

2.212 Equations of motion. **2.2121 No photons.** We put the electron with charge e and mass m in a given 4-potential field (A^0, A^1, A^2, A^3) . This field only serves to determine the unperturbed state of the electron and will not be quantized. The unperturbed equation of motion is

$$K^0 \{x\}(x|\psi^0 = 0 \dots \dots \dots \dots \quad (23)$$

with

$$\mathbf{K}^0\{\mathbf{x}\} = \alpha^\alpha \left(\frac{\hbar c}{i} \frac{\partial}{\partial x^\alpha} + e A_\alpha(\mathbf{x}) \right) + \beta mc^2. \quad (24)$$

The density operator is 1, the inner product of $\psi^\dagger | \mathbf{x} \rangle$ and $\langle \mathbf{x} | \psi'$ is

$$\int (\vec{dx}) \psi^\dagger | \mathbf{x} \rangle \langle \mathbf{x} | \psi'. \quad (25)$$

The wave function at a time t is determined by the wave function at a time t' according to the integral equation

$$\langle \mathbf{x} | \psi^0 = \int (\vec{dx}') \mathbf{N}(\mathbf{x}; \mathbf{x}') \langle \mathbf{x}' | \psi^0, \quad (26)$$

where the nucleus $\mathbf{N}(\mathbf{x}; \mathbf{x}')$ satisfies the differential equation

$$\mathbf{K}^0\{\mathbf{x}\} \mathbf{N}(\mathbf{x}; \mathbf{x}') = 0 \quad (27)$$

with the initial condition

$$\mathbf{N}(\mathbf{x}; \mathbf{x}')_{t=t'} = \delta(\vec{x} - \vec{x}'). \quad (28)$$

If $(\mathbf{x} | \psi^0 | \mu)$ is a complete orthonormal system of solutions of (23), $\mathbf{N}(\mathbf{x}; \mathbf{x}')$ is given by

$$\mathbf{N}(\mathbf{x}; \mathbf{x}') = \sum_\mu (\mathbf{x} | \psi | \mu) (\mu | \psi^\dagger | \mathbf{x}'). \quad (29)$$

2.2122 With photon interaction. In a photon field the equation of motion is

$$(\mathbf{K}^0\{\mathbf{x}\} + \mathbf{K}^1(\mathbf{x})) (\mathbf{x}; \vec{y}_{10}, \dots | \psi = 0 \quad (30)$$

with the interaction operator

$$\mathbf{K}^1(\mathbf{x}) = \alpha^\alpha e \Phi_\alpha(\mathbf{x}). \quad (31)$$

The wave functions are taken up to date in the electron coordinates (\mathbf{x}) , which stand for $(\vec{x} t)$, and at the beginning in the photon coordinates (\vec{y}_{k0}) , which stand for $(\vec{y}_k s_{k0})$, ($k = 1, \dots$). Speaking about the wave function at a certain time will therefore always refer to electron time. The 4-potential operator Φ_α may be either $\Phi_{c\alpha}$ or $\Phi_{s\alpha}$ of (10), the remaining one is redundant⁸⁾ and further on denoted by $\Phi_{rda\alpha}$. Φ_α in the equation of motion describes creation and annihilation of photons. It is formed from the wave operators of free photons. That means that apart from creation and annihilation the photons behave dynamically just as in the absence of charges.

In order to determine the wave function at a time t from the wave function at a time t' we shall prefer to use a not full-grown calculus introduced by FEYNMAN⁹⁾ without taking over his interpretation. This calculus which is mathematically in a rather undeveloped state provides a very apt tool for handling with the general formulation of our present problems.

We divide the time interval $t' t$ into a large number p of infinitesimal intervals $t' t'', t'' t''', \dots t^{(p)} t^{(p+1)}$ with $t^{(p+1)} = t$. Our expressions are

hoped to be valid in the limit $p \rightarrow \infty$, $\max |t^{(k+1)} - t^{(k)}| \rightarrow 0$. Then we write

$$\langle x; y_1, \dots | \psi = \lim_{p \rightarrow \infty} \prod_{m=1}^p \left(\int (d\vec{x}^{(m+1)}) \mathbf{N}(x^{(m+1)}; x^{(m)}) \right) \left. e^{\frac{i e}{\hbar c} \sum_{m=1}^p (x^{(m+1)\alpha} - x^{(m)\alpha}) \Phi_\alpha(x^{(m)})} \right\} \langle x'; y_1, \dots | \psi. \quad (32)$$

$x^{(m)\alpha}$ lies in the interval $x^{(m)\alpha} \dots x^{(m+1)\alpha}$. In order to get later on an easier notation we have symbolically written a sum in the exponent rather than writing the exponential under the product sign. That makes that whereas otherwise the product of exponentials had to be well ordered in time, this order has now to be observed in the sum of exponents. This comment has to be well observed for all following exponential operators, which without it would have to be understood in a different way. Also the \mathbf{N} 's have to be well ordered in time. They commute with the exponentials.

It is easily seen that (32) satisfies the equation (30) and the initial condition $\langle x | \psi = \langle x' | \psi$ for $t = t'$. This remains true if everywhere in (32) $(x^{(m+1)\alpha} - x^{(m)\alpha})$ is replaced by $\alpha^\alpha c(t^{(m+1)} - t^{(m)})$, provided the ordered \mathbf{N} 's are properly sandwiched between the ordered exponentials with which they no longer commute.

It would of course also be possible to treat the interaction with the field A_α in the same way as that with the field Φ_α . Doing so one could write

$$\mathbf{N}(x^{(m+1)}; x^{(m)}) = \mathbf{N}^0(x^{(m+1)}; x^{(m)}) e^{\frac{i e}{\hbar c} (x^{(m+1)\alpha} - x^{(m)\alpha}) A_\alpha(x^{(m)})}, \quad (33)$$

where \mathbf{N}^0 refers to free electrons. Dealing with \mathbf{N}^0 becomes urgent if one is interested in the self interaction of the electron. That belongs to Q_3 .

2.22 Many electrons. **2.221 Description.** In describing a system of many electrons we shall use many-times theory¹⁰⁾ ¹¹⁾.

We do not consider creation and annihilation of electrons, their number n is taken fixed. We shall not use electron creation and annihilation operators. Further in order to avoid unessential complications we shall not be concerned with electron exchange effects. Therefore we treat the electrons as if they were discernible and only account for the FERMI-DIRAC statistics by PAULI's exclusion principle.

2.222 Equations of motion. **2.2221 No photons.** The unperturbed field A_k will be taken different for different electrons. The unperturbed equations of motion

$$\mathbf{K}_k^0 \{x_k\} (x_1, \dots, x_n | \Psi^0 = 0 \quad (k = 1, \dots, n) \quad (34)$$

are independent of each other. There is no interaction between the electrons. The wave function at a set of times (t_1, \dots, t_n) is determined by the wave function at a set of times (t'_1, \dots, t'_n) according to

$$(x_1, \dots, x_n | \Psi^0 = \int \dots \int (d\vec{x}'_1) \dots (d\vec{x}'_n) \mathbf{N}_1(x_1; x'_1) \dots \mathbf{N}_n(x_n; x'_n) (x'_1, \dots, x'_n | \Psi^0, \quad (35)$$

The nuclei N_k and N_l for different electrons ($k \neq l$) commute with each other.

2.2222 With photon interaction. In a photon field the equations of motion are

$$(K_k^0 \{x_k\} + K^1(x_k)) (x_1, \dots, x_n; y_{10}, \dots) | \Psi = 0 \quad (k = 1, \dots, n), \quad (36)$$

where $K^1(x)$ is given by (31). The commutation relations (14) make that the n equations (36) are only compatible if all world points x_k and x_l ($k, l = 1, \dots, n$; $k \neq l$) lie outside each others light cone¹²⁾, i.e. if $c|t_k - t_l| < < |\vec{x}_k - \vec{x}_l|$. The wave function at a set of times (t_1, \dots, t_n) is determined by the wave function at a set of times (t'_1, \dots, t'_n) according to

$$(x_1, \dots, x_n; y_{10}, \dots) | \Psi = \lim \prod_{k=1}^n \prod_{m=1}^p \left(\int (d\vec{x}_k^{(m)}) N_k(x_k^{(m_k+1)}; x_k^{(m_k)}) \right) e^{\frac{i e}{\hbar c} \sum_{k=1}^n \sum_{m=1}^p (x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha}) \phi_\alpha(x_k^{(m_k)})} \quad (x'_1, \dots, x'_n; y_{10}, \dots) | \Psi. \quad (37)$$

For each k ($k = 1, \dots, n$) the product of the N_k 's and the (symbolical) sum of exponents each have to be well ordered in all electron times. Everywhere in (37) $(x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha})$ can again be replaced by $\alpha_k^\alpha c(t_k^{(m_k+1)} - t_k^{(m_k)})$, provided the ordered N 's are properly sandwiched between the ordered exponentials.

(37) can be maintained for electron world points which lie inside each others light cone.

2.23 Supplementary conditions. The wave functions are still subjected to supplementary conditions. They will be dealt with in 4.12.

(To be continued.)

Statistics. — A simple technique for producing random sampling numbers.

By H. C. HAMAKER. (Laboratorium voor Wetenschappelijk Onderzoek der N.V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands.) (Communicated by Prof. H. B. G. CASIMIR.)

(Communicated at the meeting of January 29, 1949.)

Summary.

This note describes a simple dice-throwing technique for producing random sampling numbers, together with the results of tests applied to prove the randomness of the procedure.

Having in war time no access to existing tables, I made some experiments to produce random sampling numbers for my own use which have led to the technique described below. The various tests as described by KENDALL and BABINGTON SMITH¹), when applied to series of from 10,000 to 40,000 throws, did not show any signs of bias, so that it may be concluded that the procedure is a truly random one. Besides, the technique is extremely simple and may easily be used by anyone wishing to construct his own tables; it might, for instance, be useful as an exercise in the training of students.

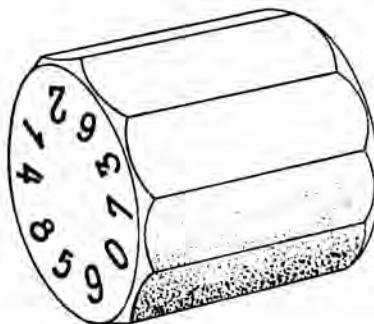


Fig. 1. The ten-sided die. Length 30 mm, diameter 30 mm.

The only tool needed is a ten-sided die as shown in fig. 1. My own dice were made of brass, but they should preferably be made of some lighter metal. The ten sides have been marked from 0 to 9 in random order by numbers engrafted in one of the top faces, as shown in the figure.

In a first series of experiments the die was thrown across the table and the uppermost number noted after it had come to rest, but bias was evident already after 1000 throws.

It was noted that the surface of the table was not perfectly horizontal, so that when rolling down-hill the die would find its final position more

hesitatingly than when rolling up-hill. As this might have some influence, separate series of throws were carried out in either direction with the results recorded in table I.

TABLE I.
Frequencies observed in 600 throws, up-hill and down-hill.

Digit	Down-hill		Up-hill	
	Die No 1	Die No 2	Die No 1	Die No 2
0	86	19	48	46
1	42	41	63	58
2	50	56	65	72
3	42	85	64	65
4	34	26	63	54
5	37	57	48	51
6	73	66	72	56
7	113	65	61	70
8	51	68	55	54
9	72	117	61	74
Total	600	600	600	600
χ^2	97.2	120.4	8.6	13.90
v	9	9	9	9
P	$\sim 10^{-15}$	$\sim 10^{-21}$	0.473	0.126

Two different dice were used, which both exhibit a pronounced bias in a set of 600 down-hill throws. By plotting the frequencies of the digits in the order in which they occur along the circumference of the dice, we obtain the two curves shown in fig. 2; die No 1 gives a curve with a single

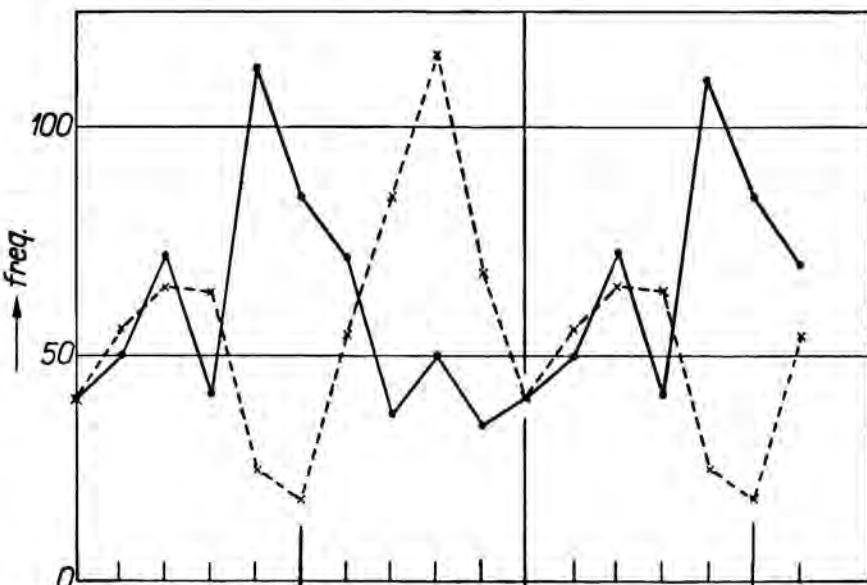


Fig. 2. The frequencies of single digits in 600 down-hill throws in consecutive order.

maximum, which might be due to eccentricity of the centre of gravity, but die No 2 shows a curve with two maxima, and this cannot be so explained.

A die rolling down hill loses energy by friction and gains energy by loss of height, the total being a loss since the die ultimately comes to rest. If the amount of energy lost in each step becomes very small, slight differences in the size of the faces of the die, or in the shape of the edges, may have a pronounced influence. I think it likely that the bias observed in the down-hill throws must be explained in this way.

In keeping with this interpretation the bias in the up-hill throws, though perhaps still present, is so much reduced that it is not yet apparent in 600 throws: the die now loses energy both by friction and by a gain in height, and its final position will mainly be determined by its initial energy.

It may be added that the slope of the surface was only $\frac{1}{2}^{\circ}$, a slope of 1° being already sufficient to keep the die rolling down-hill when once set in motion.

It was inferred from these observations that satisfactory results might probably be obtained, if a die in spinning motion is suddenly stopped by a strong frictional force, and this principle was brought in practice in the following simple way.

From the flat hand a die (as shown in fig. 1) was thrown spinning into the air and caught again in its downward fall; one of the ten side-faces was then selected by the thumb and the corresponding digit read off. To preclude personal bias the top face with the digits on it should not be visible while the choice is being made, a measure that can easily be effected.

This practice proved entirely satisfactory. By throwing with the left hand and recording the digits on a typewriter with the right, 1000 throws could be completed in 40 minutes. At a later stage two dice were thrown simultaneously, one with each hand, an assistant recording the scores; in that way 1000 throws took not more than half an hour, including the insertion of a new sheet in the typewriter and a few corrections for misprints.

As it does not seem likely that in this throwing technique one particular digit should have definite preference above the others, a correlation between successive throws must be considered as the main form of bias conceivable. And a bias of this kind is not very probable either, since both the throwing and the catching of the die are operations in which many random factors play a role, and in which it is not easy to achieve a great degree of regularity even after prolonged practice; it was estimated that a die made, on the average, about 5 revolutions per throw.

But the final decision as to the absence or presence of bias should rest with the scores obtained. To settle this question the frequencies of "overlapping" pairs have been recorded in a set of 10,000 singlehanded throws; that is, a series of scores

was recorded as 61, 18, 80, 03, 38, 80, 05, and so on. Thus each single throw is recorded twice; once as the second and once as the first digit in a pair. Hence the marginal totals of rows and of columns in the 10×10 frequency array (table II) are equal to each other and equal to the total

TABLE II.
Frequencies of 'overlapping' pairs in a series of 10,000 throws.

Second digit	0	1	2	3	4	5	6	7	8	9	Total
First digit	Frequencies										
0	112	103	98	99	105	98	91	98	96	101	1,001
1	75	100	91	95	93	108	107	103	100	105	977
2	107	87	105	93	106	100	103	91	105	96	993
3	87	105	96	82	88	103	106	100	115	84	966
4	100	102	103	86	111	102	104	73	103	112	996
5	112	101	105	105	102	100	98	95	100	99	1,017
6	104	108	85	117	102	94	99	97	104	99	1,009
7	102	83	98	95	86	112	93	103	102	109	983
8	103	97	104	101	96	95	115	121	94	105	1,031
9	99	91	108	93	107	105	93	102	112	117	1,027
Total	1,001	997	993	966	996	1,017	1,009	983	1,031	1,027	10,000

$$\chi^2 = 73.64 \quad \sqrt{2\chi^2} = 12.14 \quad P = 0.214$$

$$v = 90 \quad \sqrt{2v-1} = 13.30$$

$$\Delta = -1.16$$

TABLE III.
Frequencies of 'independent' pairs in 20,000 double-handed throws.

Second digit	0	1	2	3	4	5	6	7	8	9	Total
First digit	Frequencies										
0	101	102	99	91	97	114	102	98	91	92	987
1	101	85	103	93	122	100	110	115	100	93	1,022
2	108	90	111	91	104	102	110	102	102	92	1,012
3	97	101	103	108	95	94	97	89	103	97	984
4	103	91	106	120	109	96	110	95	91	105	1,026
5	83	101	109	97	114	91	98	94	79	92	958
6	96	127	98	98	100	95	107	100	101	102	1,024
7	99	107	89	110	103	98	98	87	76	121	988
8	101	113	105	107	107	105	83	94	100	100	1,015
9	103	88	85	94	108	107	101	92	115	91	984
Total	992	1,005	1,008	1,009	1,059	1,002	1,016	966	958	985	10,000

$$\chi^2 = 83.40 \quad \sqrt{2\chi^2} = 12.92 \quad P = 0.263$$

$$v = 99 \quad \sqrt{2v-1} = 14.04$$

$$\Delta = -1.12$$

frequencies with which the single digits were observed. It is easily deduced that the number of degrees of freedom for the entire array is 90.

As indicated at the bottom of the table the χ^2 -test applied to the entire array gives $P = 0.214$, and when applied to the marginal totals (see table IV A) $P = 0.893$; reasonable values which do not indicate bias.

A second series of 10,000 double-handed throws gave a sequence of 20,000 random digits, and the frequencies of these arranged in 10,000 'independent' pairs have been collected in table III.

The two sets of marginal totals are now independent and their sum is equal to the frequencies of the single digits; the number of degrees of freedom for the entire array is 99. The χ^2 -test yields $P = 0.263$ for the complete 10×10 -array and $P = 0.527$ for the single-digit frequencies (see table IV B).

Finally in another set of 10,000 single-handed throws the frequencies of the single digits were computed, and these were added to the sum of the corresponding frequencies in tables II and III. This gave us the single-digit frequencies for a total of 40,000 throws as shown in table IV C; $P = 0.696$, again a normal value.

TABLE IV.
Single-digit frequencies in various cases.

0	1	2	3	4	5	6	7	8	9	Total
<i>A. In 10,000 single-handed throws (table II)</i>										
1,001	977	993	966	996	1,017	1,009	983	1,031	1,027	10,000
$\chi^2 = 4.23$:			$\nu = 9$:				$P = 0.893$			
<i>B. In 20,000 double-handed throws (table III)</i>										
1,979	2,027	2,020	1,993	2,085	1,960	2,040	1,954	1,973	1,969	20,000
$\chi^2 = 7.93$:			$\nu = 9$:				$P = 0.527$			
<i>C. In a total of 40,000 throws (including tables II and III)</i>										
4,000	3,996	4,077	3,898	4,058	3,957	4,008	3,950	4,027	4,029	40,000
$\chi^2 = 6.42$:			$\nu = 9$:				$P = 0.696$			

Thus our data satisfy the frequency and the series tests¹⁾, which provides fairly conclusive evidence that the technique adopted is reliable.

In constructing their table of random sampling numbers KENDALL and BABINGTON SMITH¹⁾ used a disc with the numbers 0 to 9 inscribed on its circumference and rotating past a pointer; in momentaneous flashes at random intervals the number seen nearest to the pointer was recorded.

When using this method much will depend on the reaction velocity of the observer, and if he should fall short in this respect, there is a danger of personal bias being introduced into the records; in one case such a bias

could actually be demonstrated from the frequencies of the separate digits. Similarly there might be some danger of a person being disinclined to read a certain number after it has repeatedly been observed, or being inclined to read a specified number if it has not occurred for a long time.

Probably the numbers thrown as described above are less subject to bias of this kind; one should have to be a die-hard falsifier to introduce personal bias once the choice has been made, and the only bias possible is that which may be introduced in the throwing, the catching, and the choosing of one of the ten faces.

To test for the kind of bias just mentioned KENDALL and BABINGTON SMITH¹⁾ devised two other tests: the poker- and the gap-test. For the sake of completeness these have also been applied to the final series of 20,000 double-handed throws giving P 's of 0.672 and 0.968 respectively. I shall not record the data in detail here; a fuller account has been published in Dutch in the journal 'Statistica'²⁾.

I am indebted to Mr H. A. C. VAN DER LINDEN for his painstaking assistance in recording the throws and performing the tests.

Eindhoven, 15 December 1947.

LITERATURE.

- (1) M. G. KENDALL and B. BABINGTON SMITH, Jl. Roy. Stat. Soc. **101**, 147 (1938);
Jl. Roy. Stat. Soc. Suppl. **6**, 51 (1939).
- (2) H. C. HAMAKER, Statistica, **2**, 97—106 (1948).

Mathematics. — Sur les espaces linéaires normés VI. By A. F. MONNA.
(Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 29, 1949.)

Cet article fait partie d'une série d'articles I, II, III, IV et V, publiés sous le même titre¹⁾, ayant pour but l'étude systématique des espaces linéaires normés totalement-non-archimédiens. Ces articles sont supposés connus. Plus spécialement l'article est lié à l'article II, où est donné un développement en série des éléments d'un espace totalement-non-archimédien localement compact. On verra qu'un développement en série analogue est possible pour des espaces plus généraux. Valuation triviale du corps K , auquel appartiennent les coefficients, sera permis.

§ 1. Nous ne considérons dans tout ce qui suit que des espaces linéaires normés E totalement-non-archimédiens (voir I).

La valuation du corps K est alors non-archimédienne (voir I, p. 1046). Les éléments a de K tels que $|a| \leq 1$ constituent, comme il est connu, un anneau I . Les éléments a de K tels que $|a| < 1$ constituent dans I un idéal premier sans diviseurs \mathfrak{p} ; l'anneau résiduaire I/\mathfrak{p} est un corps. Si la valuation de K est triviale, les éléments de K appartiennent tous à des classes résiduaires mod \mathfrak{p} différentes et alors I/\mathfrak{p} est identique à K .

Pour chaque $C \in N_E$ l'ensemble des points $x \in E$ tels que $\|x\| \leq C$ est un groupe abélien additif; de même l'ensemble des $x \in E$ tels que $\|x\| < C$. On peut donc déterminer dans ce premier groupe le groupe quotient des classes résiduaires mod $\|x\| < C$. Par E_C nous désignons l'ensemble des points $x \in E$ tels que $\|x\| \leq C$. Cet ensemble E_C est un espace linéaire par rapport à l'anneau I . En effet, si $\|x\| \leq C, \|y\| \leq C$, on a $\|x+y\| \leq \max(\|x\|, \|y\|) \leq C$ et si de plus $a \in I$, donc $|a| \leq 1$, on a $\|ax\| = |a| \cdot \|x\| \leq C$.

Remarque. En se rapportant la définition des idéaux dans un anneau, on voit qu'il y a une analogie entre les idéaux et les sous-espaces E_C : on peut considérer ces espaces comme des idéaux de E par rapport à I . Supposons $a, \xi \in E_C (a \in I, \xi \in E)$, donc $\|a\xi\| \leq C$, et supposons ξ pas dans E_C , donc $\|\xi\| > C$; alors on a $|a| < 1$, de sorte que a appartient à l'idéal premier \mathfrak{p} de I . À cause de cette propriété on peut appeler, d'après l'analogie avec la définition ordinaire des idéaux primaires, E_C un idéal

¹⁾ Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 1045—1055, 1056—1062, 1134—1141, 1142—1152 (1946); 51, 197—210 (1948).

Voir aussi: I. S. COHEN, On non-Archimedean normed spaces; id. 51, 693—698.

primaire par rapport à I . Cette analogie devient encore plus claire si nous supposons que le nombre 1 appartient à N_E de sorte qu'il existe un élément e de E tel que $\|e\|=1$. On a alors: si $a\xi \in E_C$, ξ pas dans E_C , p. ex. $\|\xi\|=C'$ ($C' > C$), alors il existe un nombre entier $\lambda > 0$ tel que $a^\lambda e \in E_C$. En effet, on a $|a| \leq C \cdot C'^{-1}$ et il suffit de choisir λ tel que $(C \cdot C'^{-1})^\lambda \leq C$. Étant donné un $C < 1$, l'idéal \mathfrak{p} se compose des éléments a de I tels qu'il existe un $\lambda > 0$ tel que $a^\lambda e \in E_C$. Ici λ dépend de l'élément a . Si l'on suppose de plus que la valuation de K est discrète, on voit que l'ensemble des λ est borné: en effet, pour les a tels que $|a| < C$, on peut prendre $\lambda=1$; puisque $|a| \leq 1$ et N_K est discret, il n'y a qu'un nombre fini de valeurs $|a|$ entre C et 1, d'où il suit que l'ensemble des λ est borné. On peut donc, dans ce cas, considérer les E_C avec $C < 1$ comme des idéaux primaires par rapport à I , ayant tous le même idéal premier associé \mathfrak{p} .

Propriété A. *Par là nous entendons la propriété suivante: Si $a \in K$ parcourt un système complet S de représentants des classes résiduaires mod \mathfrak{p} dans I , alors pour chaque $C \in N_E$ et pour chaque $\xi \in E$ tel que $\|\xi\|=C$, ξ étant fixé mais arbitrairement choisi, $a\xi$ parcourt un système complet de représentants des classes résiduaires mod $\|x\| < C$ dans $\|x\| \leq C$.*

Remarques. 1. L'inverse de la propriété A est évident: si $b\xi$ (b fixé) parcourt un système complet de représentants, alors les éléments b figurants constituent un système complet mod $|a| < 1$. En effet, les éléments b appartiennent tous à des classes différentes: $|b_i - b_j| = 1$ si $i \neq j$. Si, inversement, a_1 et a_2 appartiennent à des classes mod $|a| < 1$ différentes, donc $|a_1 - a_2| = 1$, alors $\|a_1\xi - a_2\xi\| = C$, donc $a_1\xi$ et $a_2\xi$ appartiennent à des classes différentes mod $\|x\| < C$.

2. La propriété A est vérifiée dans chaque espace dont tous les éléments x puissent être représentés dans la forme $x = a\xi$ ($a \in K$; ξ un élément fixe). En effet, soit $x_0 = b\xi$ ($b \neq 0$), $\|x_0\| = C$. Alors si a' et a'' sont dans S ($a' \neq a''$) on a $\|(a' - a'')x_0\| = C$, de sorte que $a'x_0$ et $a''x_0$ appartiennent à des classes différentes. Si de plus $y = a_1\xi = a_1b^{-1}x_0$, $\|y\| = C$ est tel que $\|y - ax_0\| = C$ pour tout a de S , on a

$$\begin{aligned} \|a_1b^{-1}x_0 - ax_0\| &= C, \\ |a_1b^{-1} - a| &= 1. \quad , \quad |a_1b^{-1}| = 1 \end{aligned}$$

de sorte que S étant complet, $a_1b^{-1} = a_2$ appartient à S et $y = a_2x_0$.

3. Si l'espace E lui-même est un corps et si $K = E$, la propriété A est vérifiée puisque si $x \in E$, $\|x\| = C$, il existe un $a \in K$ tel que $x = a\xi$ et $|a| = 1$. Voir alors la remarque précédente.

4. La propriété A est une extension de l'hypothèse, faite dans II, p. 1058, de l'égalité des indices de K et de E , qui étaient là finis.

5. Dans le cas où $N_E = N^K$ on peut se borner à vérifier la propriété

A pour une seule valeur de C puisqu'il résulte alors que $a\xi$ parcourt un système complet pour tout autre valeur $C' \in N_E$.

Propriété B. *L'espace totalement-non-archimédien E satisfait à la propriété B si l'ensemble N_E n'a, éventuellement, que 0 comme point d'accumulation. Soit $\{C_i\}$ l'ensemble des nombres appartenant à N_E :*

$$0 \dots < C_n < C_{n+1} < \dots \dots \dots \quad (1)$$

Si la valuation de K est triviale, il se peut, que $C_n = 0$ pour $n < m$; dans ce cas l'espace est discret, c'est à dire n'a pas de points d'accumulations. Si l'espace satisfait à la propriété B, nous écrivons, pour abréger, E_n au lieu de E_{C_n} . On a alors $E_{n-1} \subset E_n$ et l'ensemble des x tels que $\|x\| < C_{n+1}$ est identique à E_n .

Théorème 1. *Supposons que E satisfait aux propriétés A et B et soit L un sous-espace linéaire par rapport à I tel que*

$$E_{n-1} \subset L \subset E_n. \dots \dots \dots \quad (2)$$

Alors on a $L = E_n$ ou bien $L = E_{n-1}$. Il n'y a donc pas d'espaces linéaires entre E_{n-1} et E_n au sens strict.

Démonstration. Supposons $\xi \in L$, ξ pas dans E_{n-1} . Donc $\|\xi\| > C_{n-1}$ et puisque $\xi \in E_n$ donc $\|\xi\| = C_n$. Si $a\xi$ parcourt le système S (pour la définition voir la propriété A), $a\xi$ parcourt un système de représentants mod $\|x\| < C_n$. Un x donné arbitraire dans E_n tel que $\|x\| = C_n$, appartient donc à une et une seule des classes résiduaires, supposons à la classe représentée par $a\xi$. On a donc

$$\|x - a\xi\| < C_n. \dots \dots \dots \quad (3)$$

Ecrivons alors

$$x = a\xi + (x - a\xi). \dots \dots \dots \quad (4)$$

$a\xi$ appartient à L à cause de la linéarité de L par rapport à I . En vertu de (2) et (3) on a $x - a\xi \in E_{n-1} \subset L$ et alors il suit de (4) encore à cause de la linéarité de L , que $x \in L$. On a donc $L = E_n$. S'il n'existe pas un ξ comme il a été défini ci-dessus, on a $L = E_{n-1}$.

Remarque. Dans la terminologie de la théorie des idéaux (voir ci-dessus) on peut exprimer le théorème 1 en disant que les E_n constituent une série de composition dans E .

Ils existent des espaces satisfaisants à la propriété A mais dans lesquels le théorème 1 n'est pas vrai. Par exemple les espaces de la forme $x = a\xi$ déjà considérés ci-dessus (voir la remarque 2 précédente), en supposant que la valuation de K est non-discrete. Puisque N_K est alors partout dense, il existe un espace linéaire entre chaque couple d'espaces linéaires du type E_C et différant des deux espaces du couple. Ces espaces ne satisfont pas à la propriété B. Ceci est clair: la validité de la propriété B est une condition nécessaire pour que le théorème 1 soit vrai.

Si N_E a un point d'accumulation $C \neq 0$ le théorème n'est pas vrai comme on voit aisément (voir p. ex. l'espace considéré dans V, p. 204).

Théorème 2. Supposons que E satisfait aux propriétés A et B et soit $x_0 \in E$ tel que $x_0 \in E_n$, x_0 pas dans E_{n-1} , donc $\|x_0\| = C_n$. Alors

$$E_n = (x_0, E_{n-1}),$$

où (x_0, E_{n-1}) désigne l'espace linéaire par rapport à I déterminé par x_0 et E_{n-1} (donc les points $\lambda x_0 + y$; $\lambda \in I$, $y \in E_{n-1}$).

Démonstration. On a

$$E_{n-1} \subset (x_0, E_{n-1}) \subset E_n$$

en vertu de la linéarité de E_{n-1} et E_n par rapport à I . Avec le théorème 1 il s'ensuit

$$E_n = (x_0, E_{n-1}).$$

Théorème 3. Supposons que E satisfait aux propriétés A et B. Soit E complet. Soient donnés pour tout n entier un $\xi_n \in E$ tel que $\|\xi_n\| = C_n$. Supposons que $a \in K$ parcourt le système complet S (voir la définition dans l'énoncé de la propriété A) et désignons ces éléments par des indices a_1, a_2, \dots . Alors chaque $x \in E$ peut s'écrire dans une façon unique dans la forme

$$\begin{aligned} x &= \sum_{i=0}^{\infty} a_{n-i} \xi_{n-i} \quad (a_{n-i} \in S) \\ \|x\| &= C_n. \end{aligned}$$

Démonstration. Posons $x_{-1} = \theta$ et supposons que pour $i < k$ on a déterminé les coefficients a_{n-i} dans S tels que

$$\begin{aligned} x_{k-1} &= \sum_{i=0}^{k-1} a_{n-i} \xi_{n-i}, \\ x - x_{k-1} &\subset E_{n-k}. \end{aligned}$$

Il suit du théorème 2 qu'il existe un $\eta_{n-k} \in E_{n-k}$ tel que

$$(x - x_{k-1}) - \eta_{n-k} \in E_{n-(k+1)}.$$

η_{n-k} appartient à une des classes résiduaires mod $\|x\| < C_{n-k}$. La propriété A étant supposée satisfaite, il existe un élément a_{n-k} de S tel que

$$\|\eta_{n-k} - a_{n-k} \xi_{n-k}\| < C_{n-k}.$$

Donc

$$\begin{aligned} \eta_{n-k} - a_{n-k} \xi_{n-k} &\in E_{n-(k+1)}, \\ (x - x_{k-1}) - a_{n-k} \xi_{n-k} - (\eta_{n-k} - a_{n-k} \xi_{n-k}) &\in E_{n-(k+1)}, \\ x - x_{k-1} - a_{n-k} \xi_{n-k} &\in E_{n-(k+1)}. \end{aligned}$$

En posant alors

$$x_k = x_{k-1} + a_{n-k} \xi_{n-k}$$

on voit que la condition, permettant le raisonnement par induction, est satisfaite. On a

$$\|x - x_k\| \leq C_{n-(k+1)} \rightarrow 0 \text{ si } k \rightarrow \infty$$

de sorte que

$$x = \sum_{i=0}^{\infty} a_{n-i} \xi_{n-i}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (5)$$

En remarquant que E est supposé complet on montre sans peine que chaque série de la forme (5) représentent un élément de E . Il reste à démontrer l'unicité du développement. Supposons

$$x = \sum_{i=0}^{\infty} a_{n-i} \xi_{n-i} = \sum_{i=0}^{\infty} a'_{n-i} \xi_{n-i} \quad (a_{n-i}, a'_{n-i} \in S)$$

et supposons de plus que pour $i < k$ on a démontré que $a_{n-i} = a'_{n-i}$. On peut supposer $C_{n-k} \neq 0$. Posons

$$\sum_{i=k}^{\infty} a_{n-i} \xi_{n-i} = a_{n-k} \xi_{n-k} + \eta$$

$$\sum_{i=k}^{\infty} a'_{n-i} \xi_{n-i} = a'_{n-k} \xi_{n-k} + \zeta.$$

On a

$$a_{n-k} \xi_{n-k} + \eta = a'_{n-k} \xi_{n-k} + \zeta = \chi,$$

$$\begin{aligned} \|a_{n-k} \xi_{n-k} - a'_{n-k} \xi_{n-k}\| &= \|a_{n-k} \xi_{n-k} - \chi - (a'_{n-k} \xi_{n-k} - \chi)\| \leq \\ &\leq \max(\|\eta\|, \|\zeta\|) \leq C_{n-(k+1)}. \end{aligned}$$

Si l'on avait $a_{n-k} \neq a'_{n-k}$, alors $|a_{n-k} - a'_{n-k}| = 1$ et le premier membre serait égal à C_{n-k} d'où une contradiction.

Théorème 4. *Supposons que l'espace totalement-non-archimédien E satisfait à la propriété B. Soient donnés pour tout n entier un $\xi_n \in E$ tel que $\|\xi_n\| = C_n$ et supposons que chaque $x \in E$ peut s'écrire dans la forme*

$$x = \sum_{i=0}^{\infty} a_{n-i} \xi_{n-i} \quad (a_{n-i} \in S).$$

Alors E satisfait à la propriété A.

Démonstration. Les éléments $a \xi_n$ et $b \xi_n$ ($a \neq b$, $a, b \in S$) appartiennent à des classes résiduaires différentes puisque

$$\|a \xi_n - b \xi_n\| = |a - b| \cdot \|\xi_n\| = C_n.$$

Soit donné un $x \in E$ tel que $\|x\| = C_n$. On a

$$x = \sum_{i=0}^{\infty} a_{n-i} \xi_{n-i},$$

$$x - a_n \xi_n = \sum_{i=1}^{\infty} a_{n-i} \xi_{n-i},$$

$$\|x - a_n \xi_n\| \leq C_{n-1} < C_n,$$

de sorte que x et $a_n \xi_n$ appartiennent à la même classe. Cette classe est donc représentée par $a_n \xi_n$.

En supposant que K est complet et $N_E = N_K$, de sorte qu'il existe un $\xi \in E$ tel que $\|\xi\| = 1$, il suit sans peine du théorème 3 que chaque espace qui satisfait aux conditions de ce théorème est de la forme $x = a\xi$, où a parcourt le corps K (comparer II p. 1060) ²⁾. Ceci est une inversion de la remarque 2 précédente, exprimant que chaque espace de la forme $a\xi$ satisfait à la propriété A. Bien entendu, la remarque 2 reste vraie même si la valuation de K est non-discrete; cependant le théorème 3 exige que la valuation est discrète.

Remarquons enfin que la série (5) n'a qu'un nombre fini de termes si $C_n = 0$ pour $n < m$.

Théorème 5. *Mêmes suppositions que pour le théorème 3. Supposons de plus que le corps I/\mathfrak{p} est fini. Alors E est localement compact.*

Démonstration. Soit $x^{(k)}$ une suite bornée: $\|x^{(k)}\| \leq C_k$ pour $k=1,2,\dots$. En vertu du théorème 3 on peut écrire

$$x^{(k)} = \sum_{i=0}^{\infty} a_{\lambda-i}^{(k)} \xi_{\lambda-i}$$

où éventuellement $a_{\lambda}^{(k)} = 0$. Puisque I/\mathfrak{p} est fini, le système S a un nombre fini d'éléments; posons ce nombre $= n$. Pour le premier coefficient $a_{\lambda}^{(k)} (k=1,2,\dots)$ il y a donc n possibilités. Il s'ensuit qu'il existe une suite infinie partielle avec la propriété que pour chaque élément de cette suite partielle le premier coefficient du développement est le même. On peut extraire de cette suite une nouvelle suite infinie avec la propriété que pour chaque élément le second coefficient est aussi le même. En continuant ainsi on arrive à une série de la forme (5), représentant donc un élément de E . On voit que ce point est un point d'accumulation de la suite $\{x^{(k)}\}$.

Théorème 6. *Mêmes suppositions que pour le théorème 3. Supposons que E est localement compact. Alors I/\mathfrak{p} est fini.*

Démonstration. Le développement (5) est valable dans E . En supposant que I/\mathfrak{p} est infini, il existe un système infini dénombrable $a_i (i=1,2,\dots)$ tel que $|a_i - a_j| = 1 (i \neq j)$, $|a_i| = 1$. La suite $\{x^{(k)}\}$ définie par

$$x^{(k)} = a_k \xi_n + \sum_{i=1}^{\infty} b_{n-i} \xi_{n-i} \quad (b_{n-i} \in S)$$

est bornée et on a

$$\|x^{(p)} - x^{(q)}\| = |a_p - a_q| \cdot \|\xi_n\| = C_n,$$

de sorte que cette suite n'a aucune suite partielle convergente; d'où une contradiction.

²⁾ On peut élargir la condition $N_E = N_K$ d'une façon inessentielle en la remplaçant par celle que $N_E = \{Ce^i\}$ si $N_K = \{e^i\}$, où $C \neq 0$ est une constante.

Les résultats obtenus dans V permettent de construire des séries analogues à (5) dans les espaces séparables et complets qui satisfont à la propriété B en supposant de plus que le corps K est complet. Selon V il existe une suite de vecteurs (éventuellement un nombre fini) y_1, y_2, \dots telle que chaque $x \in E$ s'écrit dans la forme

$$x = \sum_{n=1}^{\infty} a_n y_n \quad a_n \in K.$$

E est donc une somme directe d'un nombre fini ou d'une infinité dénombrable d'espaces de la forme $a y_n$. Chacun de ces espaces satisfait donc à la propriété A , de sorte qu'on peut appliquer le théorème 3. On arrive donc à un développement de la forme

$$x = \sum_{k=0}^{\infty} \lambda_{i_1-k}^{(1)} \xi_{i_1-k}^{(1)} + \sum_{k=0}^{\infty} \lambda_{i_2-k}^{(2)} \xi_{i_2-k}^{(2)} + \dots$$

Remarques. 1. Dans le théorème 5 on peut remplacer la supposition que la propriété A soit vraie dans l'espace par la condition suivante:

$$\dots \cong E_n/E_{n-1} \cong E_{n+1}/E_n \cong \dots \cong I/\mathfrak{p}$$

En effet puisqu'on y a supposé que I/\mathfrak{p} est fini, on en dérive la validité de la propriété A .

En supposant de plus dans ce théorème que $N_E = N_K$, on a

$$\dots \cong E_n/E_{n-1} \cong E_{n-1}/E_n \cong \dots$$

de sorte qu'on peut simplifier encore cette condition: il suffit que

$$E_n/E_{n-1} \cong I/\mathfrak{p}$$

pour une seule valeur de n .

2. On démontre tout analogue la compacité locale d'un espace qui est somme directe d'un nombre fini d'espaces $E^{(i)}$ dont chacun satisfait aux conditions du théorème 5 et qui ont tous le même corps K .

3. Dans I nous avons démontré (voir aussi l'étude de M. COHEN I.c.) que chaque espace totalement-non-archimédien localement compact possède une base finie, c'est à dire qu'il existe un système de vecteurs x_1, \dots, x_r tel que chaque $x \in E$ peut s'écrire dans la forme $x = a_1 x_1 + \dots + a_r x_r$ ($a_i \in K$). Valuation triviale de K n'était pas permise. En effet, comme nous avons déjà remarqué dans II, p. 1061, cette propriété n'est pas vraie si la valuation de K est triviale, ni même si N_E contient des nombres arbitrairement petits (dans ce dernier cas il faut que K est un corps fini; voir II, p. 1061).

4. Dans tout ce qui précède dans cet article valuation triviale de K est permise³⁾.

³⁾ Dans II, p. 1060, nous avons considéré comme exemple d'un espace, n'admettant pas le développement (5), un corps E , à valuation triviale, ayant une infinité d'éléments,

5. La théorie précédente admet comme cas particulier la théorie de M. VAN DANTZIG concernant les anneaux primitifs en tant qu'il s'agit des propriétés additives de ces anneaux; comparer la remarque p. 151⁴).

§ 2. Application. Dans la théorie des approximations diophantiennes on montre le théorème suivant:

Pour chaque système de nombres réels a_1, a_2, \dots, a_n et pour chaque $A > 0$ il existe au moins un système de fractions

$$\frac{p_1}{q}, \frac{p_2}{q}, \dots, \frac{p_n}{q}$$

où q, p_1, \dots, p_n sont des entiers rationnels, tel que

$$\left| a_i - \frac{p_i}{q} \right| < \frac{1}{q^{i+\frac{1}{n}}} \quad (i = 1, 2, \dots, n, q > A).$$

M. LOCK a donné un théorème analogue concernant les nombres p -adiques⁵). La théorie précédente permet d'étendre cette propriété à une classe d'espaces linéaires.

Soit pour $i = 1, \dots, n$, E_i un espace totalement-non-archimédien complet qui satisfait aux propriétés A et B . Le corps K soit le même pour tous ces espaces.

Supposons de plus que l'anneau résiduel I/\mathfrak{p} est fini et soit l le nombre des classes. Afin d'obtenir une simplification des formules on peut supposer sans nuire la généralité que les nombres $C_j^{(i)}$, constituant pour $-\infty > j > \infty$ l'ensemble N_{E_i} , sont numérotés de façon qu'on ait $C_0^{(i)} \leq 1$, $C_1^{(i)} > 1$.

Soit Γ un sous-ensemble fini de K ; soit $|p| \leq 1$ pour tout $p \in \Gamma$. Soit enfin t un nombre naturel > 0 fixé. Supposons que le nombre des éléments de Γ soit plus grand que l^{nt} .

La théorie précédente et le principe des tiroirs fournissent maintenant une propriété concernant l'approximation d'un système d'éléments a_i tel que

$$a_i \in E_i, \|a_i\| \leq 1 \quad (i = 1, \dots, n).$$

contenant un corps fini K (donc nécessairement à valuation triviale). Remarquons que cet espace ne satisfait pas à la propriété A .

L'hypothèse dans II, p. 1061, exprimant que la série (6) (II, p. 1058) représente la forme générale des espaces localement compacts, même si la valuation de K est triviale, est faux. En effet, si la valuation de K est triviale, la série (6) prend la forme $\lambda_1 \xi_1 + \dots + \lambda_r \xi_r$, et ceci n'est pas la forme générale des espaces localement compacts dans ce cas. Le théorème 4 montre en effet que l'espace $x = \lambda_1 \xi_1 + \dots + \lambda_r \xi_r + \dots$, $\|\xi_r\| \rightarrow 0$, où λ_i appartient au système fini S , est localement compact.

⁴⁾ D. VAN DANTZIG. Zur topologischen Algebra II. Compositio Mathematica 2, 201—223 (1935). Dans quelques démonstrations nous suivons cet article.

⁵⁾ D. J. LOCK. Metrisch-diophantische onderzoeken in $K(P)$ en $K^{(n)}(P)$. Diss. V.U. Amsterdam (1947).

Si $q \in \Gamma$, on a $\|qa_i\| \leq 1$ ($i = 1, \dots, n$) et l'application du théorème 3 donne

$$qa_i = \sum_{j=0}^{\infty} a_{-j}^{(i)} \xi_{-j}^{(i)},$$

où $a_{-j}^{(i)}$ appartient au système fini de représentants S . Soit $x_i \in E_i$ un élément ayant le développement

$$x_i = \sum_{j=0}^{nt-1} a_{-j}^{(i)} \xi_{-j}^{(i)}.$$

On a

$$qa_i - x_i = \sum_{j=nt}^{\infty} a_{-j}^{(i)} \xi_{-j}^{(i)}.$$

Considérons pour $i = 1, \dots, n$ les premiers t termes du développement de $qa_i - x_i$. On y trouve les nt coefficients

$$a_{-nt}^{(i)}, \dots, a_{-(nt+t-1)}^{(i)}.$$

$$i = 1, \dots, n.$$

Puisqu'il y a l valeurs possibles pour chaque a , il existe l^{nt} combinaisons différentes. Pour chaque $q \in \Gamma$ on trouve ainsi une suite de coefficients et puisque le nombre des q est plus grand que l^{nt} , il existe au moins deux q , supposons q' et q'' , ayant la même suite de coefficients. Pour tout $i = 1, \dots, n$ le développement de $(q' - q'')a_i - (x'_i - x''_i)$ commence donc par le terme

$$a_{-nt-t}^{(i)} \xi_{-nt-t}^{(i)}.$$

On a donc

$$\|(q' - q'')a_i - (x'_i - x''_i)\| \leq C_{-nt-t}^{(i)}$$

$$i = 1, \dots, n.$$

Ceci est le théorème d'approximation dans une forme très générale. Le théorème de LOCK concernant les nombres P -adiques y est contenu comme cas particulier.

En supposant que $N_{E_i} = N_K$, les E_i ont la forme $a\zeta_i$, où $a \in K$ et où ζ_i est un élément de E_i tel que $\|\zeta_i\| = 1$. L'inégalité devient alors

$$\|(q' - q'')a_i - (p'_i - p''_i)\zeta_i\| \leq C_{-nt-t}^{(i)}.$$

Dans ce cas (excluant le cas de valuation triviale de K) les éléments de K peuvent être représentés par une série. En effet, supposons que N_K consiste des éléments ϱ^j ($\varrho > 1$), et introduisons les éléments $\eta_j \in K$, $|\eta_j| = \varrho^j$. On a alors pour chaque $a \in K$

$$a = \sum_{k=0}^{\infty} a_{i-k} \eta_{i-k} \quad (a_{i-k} \in S).$$

Puisque $N_{E_l} = N_K$, on peut choisir pour les éléments $\xi_j^{(l)}$ les éléments $\xi_j^{(l)} = \eta_j \zeta_l$. On trouve

$$x_i = \sum_{j=0}^{nt-1} a_{-j}^{(l)} \eta_{-j} \zeta_l = \zeta_l \sum_{j=0}^{nt-1} a_{-j}^{(l)} \eta_{-j} = p_l \zeta_l.$$

Choisissons pour Γ le système des éléments de la forme

$$\sum_{j=0}^{nt} a_{-j} \eta_{-j}$$

où les a_{-j} parcoururent le système S indépendamment l'un de l'autre; le nombre de ces éléments est l^{nt+1} . On voit que dans ce cas p_l appartient à Γ .

On peut donc alors approximer le système a_1, \dots, a_n simultanément à l'aide de différences d'éléments de Γ .

Si E_l est le corps des nombres P -adiques on peut arranger par un choix convenable de Γ que $q' - q''$ aussi appartient à Γ : il suffit de prendre pour Γ le système des nombres $0, 1, \dots, P, \dots, P^{nt}$. En effet, alors, si $q', q'' \in \Gamma$, ou bien $q' - q''$ ou bien $q'' - q'$ appartient à Γ . La possibilité d'ordonner Γ est ici essentiel.

's-Gravenhage, décembre 1948.

Mathematics. — *Some remarks on ancient calculation.* By E. M. BRUINS.
(Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of December 18, 1948.)

When I wrote the paper "Square roots in Babylonian and Greek Mathematics" ¹⁾ it seemed excluded to me, that the indications I gave there could appear to be insufficiently clear. But this seems not to be the case. Consequently I should like to add the following elucidations.

a. $\frac{1093}{773} < \sqrt{2}$.

This approximation is not a result of tedious, time-devouring calculations. Totally six simple additions and four simple subtractions are at most necessary and as a multiplication by 10 or a doubling is not to be regarded as a multiplicative operation, the method given is *purely additive* starting from the evident relation

$$\frac{10}{8} < \frac{11}{8} < \sqrt{2} < \frac{10}{7}$$

and consists of a transition to tenfold denominators and *shifting these towards one another*. The calculation begins by:

$$110^2 = 12100 \quad 80^2 = 6400 \quad 79^2 = 6400 - 159 = 6241 \\ 78^2 = 6241 - 157 = 6084 \quad 77^2 = 6084 - 155 = 5929.$$

So from three subtractions we obtain

$$\frac{110}{78} < \sqrt{2} < \frac{110}{77}.$$

and as $12100 - 219 = 11881$ we have

$$\frac{110}{78} < \sqrt{2} < \frac{109}{77}.$$

Again, a transition to tenfold denominators and shifting 770 towards 780 ($\frac{109}{77}$ being the last obtained approximation) we obtain with six additions

$$\frac{1093}{773} < \sqrt{2} \quad \frac{1093}{773} = 1.41397\dots \text{deviation } 0.00024.$$

I wrote the calculation of the first step in *full*, but I think that the conclusion

$$\frac{110}{78} < \sqrt{2} < \frac{110}{77}$$

¹⁾ These Proceedings 51, 332 (1948).

is already clear from $80^2 = 6400 \quad 3 \times 160 > 3 \times 140 > 400$, so the calculation could be still shorter.

b. Because of the well known remarks of NEUGEBAUER (e.g. Q.S., B 2—295 (1932)), I did not think it necessary to remark once more that the Babylonian calculator, in order to obtain an upper and a lower bound for a square root, would write

$$d \sim a^2 + b^2$$

a being a first approximation of \sqrt{d} , the difference between $d - a^2$ and b^2 being neglected. Both from geometrical and numerical considerations I showed how a study of cuneiform texts would lead from

$$a < \sqrt{d} < a + b$$

to the lower bound

$$\sqrt{d} > a + \frac{b^2}{2a+b} = a + \frac{d-a^2}{2a+b}.$$

On the other hand I stated that the Greek calculator — with a corrected harmonical mean — would deduce from

$$a < \sqrt{d} < \beta$$

the lower bound

$$\sqrt{d} > \frac{a\beta + d}{a + \beta}.$$

Then we have for $\beta = a + b$

$$\sqrt{d} > \frac{a(a+b) + d}{2a+b} = a + \frac{d-a^2}{2a+b}.$$

The remark ²⁾, that the Babylonian and Greek formulae for deduction of a corrected lower bound from a given upper and lower bound are identical — the only difference lies in the method to obtain a first upper bound from a first lower bound, or vice versa — could not be misunderstood in my opinion.

c. Duplicatio, mediatio and multiplication by ten were the main tools for computation in all times. That the Greek calculator could not use our decimal system and how he had to make the calculations according to the ancient computation-methods seemed obvious to me. The meaning of the notation

$$571^2 + 153^2 = 349450 \quad a = 591$$

was to show that the value a was obtained *not* from 349450 but from $571^2 + 153^2$ correcting $591 = 571 + 20$.

Making use of the squares already calculated one gets

$$d^2 = 571^2 + 153^2 = 326041 + 23409 = 349450$$

$$a = 571 + 20 \quad a^2 = 326041 + 400 + 22840 = 349281.$$

²⁾ Loc. cit. pag. 336.

The correction is sufficient as $349450 - 349281 = 169$ and so addition of 591 to a^2 would surpass d^2 . $2 \times 591 + 1 = 1183$ so

$$d > 591 \frac{169}{1183} = 591\frac{1}{7} > 591\frac{1}{8}.$$

The more exact value $591\frac{1}{7}$ would only complicate the calculations, whereas $591\frac{1}{8}$ gives e.g. in the next step *)

$$(1162\frac{1}{8})^2 = 1162^2 + \frac{1}{4} \times 1162 + \frac{1}{64} \text{ and so on.}$$

$$1172 = 1162 + 10; \quad 2339 = 2334 + 5.$$

Nor non trivial multiplications are necessary for the computation of the lower bound for the next step³⁾.

In total only two second order corrections occur in *Dimensio circuli*

$$3013 = 2911 + 100 + 2 \quad 1838 = 1823 + 10 + 5.$$

The intermediate fractions may have been estimated by simple iterated mediatio or duplicatio in order to obtain useful values. This certainly has been done, with one exception where a great simplification can be obtained by the estimation *) $\frac{9}{11}$: the evident $163 : 1009 < \frac{1}{6}$ belongs moreover to the second fundamental series $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \dots$

d. Analysis of $\pi < 197888 : 62351$.

197888 has only two prime factors 2, 773; $773\sqrt{2} \sim 1093$; $1093 + 773 = 1866$ and

$$\frac{197888}{62351} = \frac{256 \times 773}{1866 \left[33 + \frac{773}{1866} \right]} \quad \text{whereas } 33 = 2 \times 16 + 1,$$

which shows that the hypothesis $t = \varrho_8 - R = \frac{1}{16}(R + \frac{1}{2}A_8)$ is sufficient to explain the correct upper bound for π given by HEROON.

$R + \frac{1}{2}A_8 = \varrho_4$, $R < \varrho_n < \varrho_4$. Because of $A_n : a_n = \varrho_n : R$, $t_n = \varrho_n - R$ measures the accuracy. From $\varrho_{2n}^2 - R_{2n}^2 = \varrho_n^2 - R_n^2$, $R_{2n} = R_n + \varrho_n$ follows $2t_{2n} \sim t_n$, $R_{2n} \sim 2R_n$, which shows that the accuracy increases by a factor ~ 4 at each step. The nominator 16 gives therefore a better upper bound of π than the for t exact nominator 17.2 ... (compare Euclides 1943, 1).

My conclusion was that the approximation was not meant to be a very accurate one, but that it is the result of a rough estimation, which according to the use of the hekkaidekagon must be older than that of *Dimensio circuli*.

³⁾ The calculation of a lower bound of a square root via a correction of a Heron upper bound requires controlling calculations. If a calculation schema is devised, then the systematical use of an upper bound for one and a lower bound *) for the other half makes controlling calculations superfluous.

*) This was already common knowledge in 1897 (see Heath, Works of Archimedes).

Comparative Physiology. — The Strength of the Carbohydrases in the Cropjuice of *Helix pomatia* L. By N. POSTMA, W. L. VETTER and Miss J. H. M. WITMER. (From the laboratory of Comparative Physiology, University of Utrecht.) (Communicated by Prof. H. W. JULIUS.)

(Communicated at the meeting of January 29, 1949.)

Introduction.

During digestion the breakdown of high-molecular nutritive substances takes place in steps, each step being under the influence of a different enzyme: starch is hydrolysed to maltose, and this maltose afterwards split to glucose, whereas proteins are broken down to peptones, the further cleavage of which leads to aminoacids. We can thus differentiate between enzymes starting the attack on the compound nutritive substances and those completing hydrolysis.

For mammals it was known that the completing enzymes only occur in the small intestine, and moreover show only a rather weak activity. Through this arrangement only small quantities of the final product of digestion are absorbed at a time and pass bye and bye in the bloodstream. This can be considered as a protection against "inundation" of the blood by these endproducts, which is of importance especially for the glucose.

About 20 years ago H. J. VONK (1927) studied this problem in fishes and the frog. The situation proved to be different from that present in mammals: the pancreatic secretion contains maltase as well as amylase. That study was followed by similar experiments on other animals by some associates of our laboratory. H. P. WOLVEKAMP (1928) worked on tortoises and found hardly any maltase in the pancreas, so a situation like in higher vertebrates.

For the lower vertebrates the problem presents moreover a different angle. Whereas in the higher vertebrates under consideration there exists a local differentiation between the secretion and the action of the forementioned enzyme groups, in the fishes and the frog the pancreatic juice contains both starting as well as completing enzymes. The rate at which endproducts of digestion arise, is solely governed by the ratio of activities of the two groups of enzymes. VONK found the maltase activity to be nearly one twenty-fifth of that measured for the amylolytic power; thus the glucose still arises in small quantities. A similar question presents itself in those invertebrates whose foregut serves as a reservoir for the digestive juices, both enzyme groups being present in the mixture. C. A. G. WIERSMA and R. v. D. VEEN (1928) experimented on the crayfish and found the maltase activity in the stomach juice to be equal to that of

amylase. There are, however, other factors that limit the ingestion by the midgut gland of the products of digestion.

The cropjuice of the edible snail was studied by the present author. While working on that enzyme mixture we had to take into consideration the results of KRIJGSMAN (1925), who found that the activity of the cropjuice may vary considerably, according to whether the juice was taken from the animal after a shorter or longer interval after feeding. KRIJGSMAN found that the cellulase in the cropjuice showed an optimum strength 2—3 hours after feeding had started. Histological research showed that the number of secreting cells in the digestive glands (salivary and midgut glands) showed a maximum at $\frac{1}{2}$ — $1\frac{1}{2}$ hours after feeding; this maximum precedes therefore the optimum activity of the cellulase (Graph I). The problem was now whether other enzymes, and especially the completing ones, also showed such an increased activity and to find out the time-relations of the optimum.

Results.

The experiments were made on hibernating snails, which were treated in the way indicated by KRIJGSMAN. In each series of experiments another group of snails was used. Cropjuice was collected from starved animals, while other animals were fed during $\frac{1}{2}$ hour with lettuce or slices of potatoes. The food was taken away and the cropjuice taken directly or after intervals of $\frac{1}{2}$, 1, $1\frac{1}{2}$ hours etc.

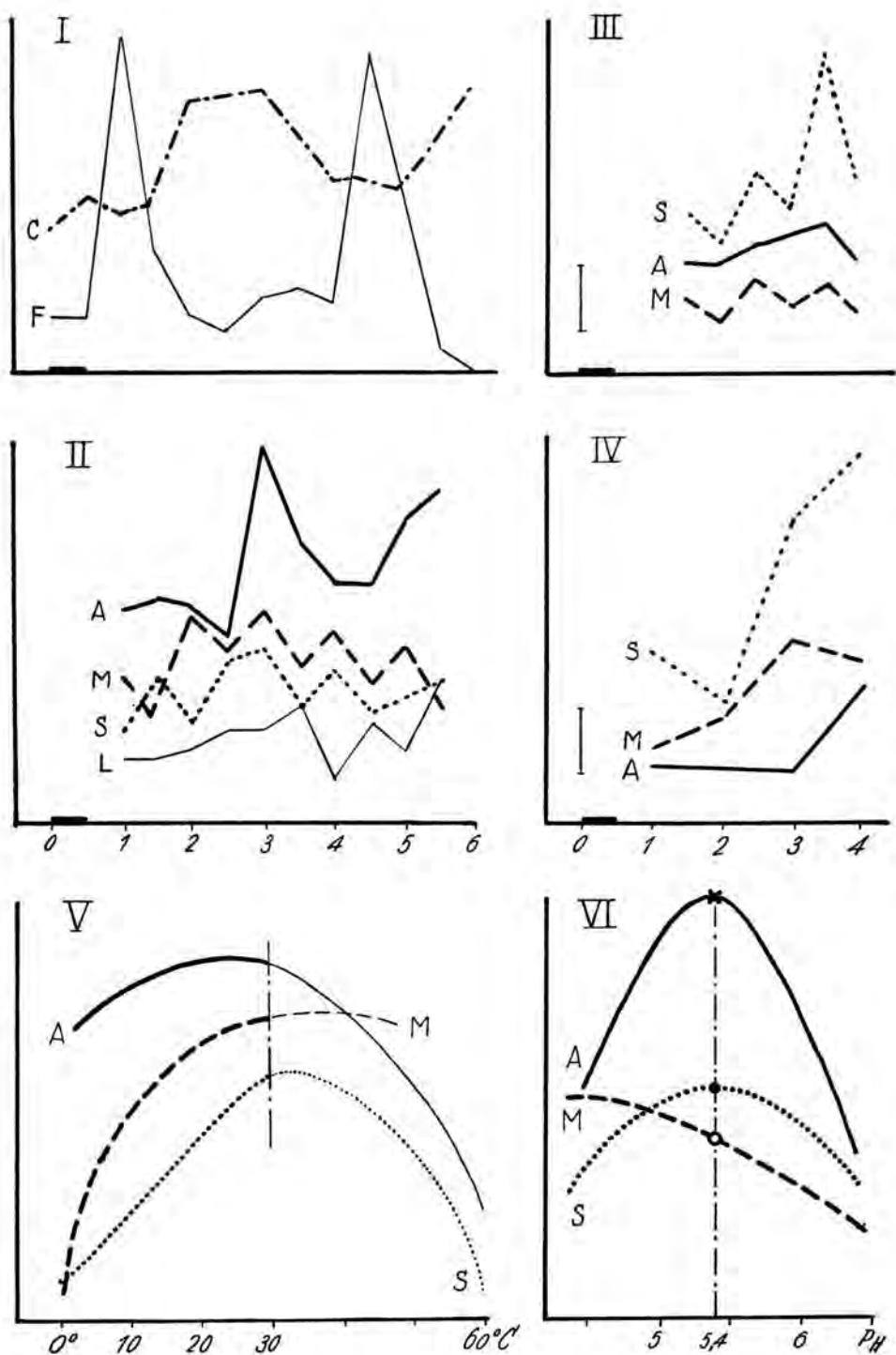
The cropjuice was analysed for the following enzymes: amylase, maltase, saccharase and lactase.

The 1st two series of experiments were executed in cooperation with Mr. VETTER, and the oxydometric method of sugar titration of MOHR as indicated by SCHOORL and REGENBOGEN (1917) was used. The cleavage of the substrates was effected in a thermostat at 25° C and in presence of a phosphate buffermixture (SÖRENSEN) warranting pH 5.4 which value was found for the cropjuice itself. That hydrogenion concentration agrees with the optimum pH found by us for amylase and saccharase; maltase showed a flat pH-curve with an optimum at 4.5 and a decrease with $\frac{1}{8}$ of the optimal activity for the pH realized in the cropjuice and in the experiments (Graph VI).

In the first series of experiments the results obtained seemed promising. Only amylase showed an increased activity, with an optimum $2\frac{1}{2}$ hours after feeding, i.e. a similar result to KRIJGSMAN's results for cellulase. The disaccharidases showed irregular fluctuations in the strength of their activities (Graph II).

In a second series of experiments the amylase did not show any optimum, and neither did the disaccharidases.

The problem after these contradictory results remained therefore unsolved.



Last year we re-started our experiments, this time in cooperation with Miss WITMER, while the method used was the iodometric sugar titration of LUUFF, according to SCHOORL (1929).

Like in 1928, the snails were wakened from their hibernating state, fed, and the cropjuice collected at different intervals after feeding. Three series of experiments were made, but the lactase-activity was not measured. The following results were obtained:

Series I: Only saccharase showed a clear optimum 3 hours after feeding, the increase in strength amounted to 80 % (Graph III).

Series II: Saccharase showed clearly an increase in strength, starting 1½ hour after feeding, and not yet having reached its optimum at 2 hours after feeding. Amylase and maltase showed a similar increase, but less pronounced. The increase in strength of amylase, maltase and saccharase are 15, 80 and 50 % respectively (Graph IV).

Series III: Only the saccharase showed a weak increase in strength (20 %) 5 hours after feeding, however, this increase does not exceed clearly the preceding fluctuations in strength.

The temperature optimum of amylase was found at 25° C, the optima of saccharase and maltase at 35°, the top of the latter being very flat. The activities at 15° (normal summer-temperature) showed a decrease with 4 % for amylase, with $\frac{1}{6}$ for maltase and with $\frac{1}{3}$ for saccharase, these values being related to those found at 25° and not to the optima (Graph V).

Summary.

The results of the experiments lead to the following conclusions:

1. A certain rhythm can be detected in the secretion of the carbohydrases, present in the cropjuice of *Helix pomatia*. The various groups of snails, however, show an arbitrary variability in the mutual ratio between the different carbohydrases.
2. Only in one of the five series of experiments the enzymes examined (amylase, maltase, saccharase) all shared simultaneously in the increase in strength of their activity.

Explanation of Figures:

Graphs relating time (in hours) after feeding to strength of enzyme activity. Ordinates: Strength of enzyme activity; Abscissae: Graphs I—IV. Time interval after beginning of feeding in hours. The small solid line represents the time of feeding ($\frac{1}{2}$ hour). Graph V. Temperature. Graph VI. pH. F = Secreting follicles. C = Cellulase. A = Amylase. M = Maltase. S = Saccharase. L = Lactase.

Graph I according to KRIJGSMA (1925). Graph II according to results of POSTMA and VETTER (1928, till now unpublished). Graphs III and IV according to results of POSTMA and WITMER (1947); the vertical lines over time 0 indicate the measure of variation in strength, which must be exceeded for an increase in strength to be considered real. Graphs V and VI according to results of POSTMA and VETTER.

3. The disaccharidases, as the completing enzymes, in principle do not behave differently from the starting enzymes, cellulase and amylase! One single enzyme may show a clear increase in strength at a certain interval after feeding as compared with its strength at earlier or later stages, whereas the other enzymes do not show any such increase.

4. The question arises therefore whether the agreement between the histological changes in the glands and the enzyme-strength as established by KRIJGSMAN for cellulase would remain generally valid if the total pattern of carbohydrases is taken into consideration.

5. The hydrogenion concentration of the cropjuice guarantees an optimal activity for amylase and saccharase while maltase has a low sensitivity for pH. The amylase is within natural limits (under 30° C) the least sensitive for temperature.

REFERENCES.

- KRIJGSMAN, B. J. 1925. Arbeitsrhythmus der Verdauungsdrüsen bei *Helix pomatia*. (I).
Zs. vergl. Physiol. 2, 264.
SCHOORL, N. and A. REGENBOGEN. 1917. Maat-analytische Suikerbepaling. Chem. Weekbl. 1917, 221.
SCHOORL, N. 1929. Suiker-titratie. Ibid. 1929, 130.
VONK, H. J. 1927. Die Verdauung bei den Fischen. Zs. vergl. Physiol. 5, 445.
WIERSMA, C. A. G. and R. V. D. VEEN. 1928. Die Kohlehydratverdauung bei *Astacus fluvialis*. Ibid. 7, 269.
WOLVEKAMP, H. P. 1928. Kohlehydratverdauung im Darme der Schildkröte. Ibid. 7, 454.

Geology. — *De Allerød-oscillatie in Nederland. Pollenanalytisch onderzoek van een laatglaciale meerafzetting in Drente.* (Met een diagram en een tabel). II. By T. VAN DER HAMMEN. (Rijksmuseum van Geologie en Mineralogie, Leiden.) (Communicated by Prof. C. J. VAN DER KLAUW.)

(Communicated at the meeting of December 18, 1948.)

Afzonderlijke besprekking van enkele pollenanalytisch aangetoonde planten.

Betula. De karakteristieke eigenschappen van de *Betula*-lijn zijn in het voorgaande al voldoende naar voren gebracht. Ook over *Betula nana*-pollen, dat in bepaalde delen van het diagram een sterkere vertegenwoordiging van bomen zou kunnen suggereren dan met de werkelijkheid overeenkomt, is reeds gesproken. IVERSEN heeft de pollenhoud van recente gyttja's op Groenland vergeleken met de in de omgeving aanwezige vegetatie (IVERSEN, 1947). Daarbij bleek het percentage *B. nana*-pollen ongeveer 2 à 3 maal zo hoog te zijn als het percentage van de totale vegetatie, dat de plant zelf innam, terwijl *Cyperaceae* en *Gramineae* juist het tegenovergestelde beeld vertoonden. Hoewel we natuurlijk voorlopig uiterst voorzichtig moeten zijn met het toepassen van deze resultaten op laatglaciale sedimenten, kunnen ze toch wel, zij het met het grootste voorbehoud, een aanwijzing geven voor de interpretatie van delen van een diagram, die bij vrij grote bosarmoede ook nog de invloed van *B. nana*-pollen zouden kunnen ondervinden.

Pinus. In zone I zijn zeer geringe *Pinus*-percentages gevonden, terwijl een aantal van de korrels bovendien corrosie-verschijnselen vertoonden, wat op het secundaire karakter van althans een deel ervan zou kunnen wijzen. Het een weinig omhooggaan der *Pinus*-lijn in dit diagram-onderdeel bij slechtere klimaatcondities kan hiermee verklaard worden, terwijl ook transport over grote afstand hier van invloed geweest kan zijn. In ieder geval lijkt het waarschijnlijk, dat hier in de Oudere Dryas-tijd geen of zeer weinig dennen voorkwamen.

Salix is van het meeste belang geweest in de Oudere Dryas-tijd. Merkwaardig is de *Salix*-top aan het eind van zone Ib en het begin van Ic. Het zal later moeten blijken, in hoeverre deze top in de *Salix*-lijn van regionaal stratigrafisch belang is.

Alnus. Bij de analyse der monsters van de postglaciale detritus-gyttja werd slechts éénmaal een pollenkorrel van *Alnus* gevonden, en wel in monster 18.

Quercus, Ulmus en Fraxinus. Eerst verschijnen *Quercus* en *Ulmus* (monster 17), dan pas *Fraxinus* (m. 19).

Juniperus. Pollen hiervan is voor het eerst in laatglaciale afzettingen

gevonden door IVERSEN (1946). Voordien meende men, dat dit pollen te teer was om bewaard te blijven. Een curve van *Juniperus* is tot nu toe, voor zover ons bekend, alleen nog maar door IVERSEN gepubliceerd van het bovenste deel der Jongere Dryas-laag. Die curve bereikt vrij hoge waarden, en daalt al direct na het begin van het Praeboreaal tot nul. Voor zone I en II staan ons dus geen gegevens voor vergelijking ter beschikking. We zullen daarom het verloop van de *Juniperus*-lijn iets nauwkeuriger beschouwen. De hoogste waarden worden in ons diagram in zone I bereikt, met een vrij sterk maximum in zone Ic, en een kleiner maximum in Ia. In zone II daalt de curve sterk, in zone III behaalt ze weer hogere waarden en ten slotte daalt ze in het Praeboreaal tot nul. Het schijnt dus dat de *Juniperus*-curve een functie is van de bosdichtheid, m.a.w. hoe opener het landschap was, hoe talrijker *Juniperus* voorkwam. Volgens HEGI omvat het tegenwoordige areaal van *J. communis* L.: Europa (in het Zuiden alleen in het gebergte), Voor-Azië en Centraal Azië. In het Hooggebergte en in het Noorden komt bijna alleen de subspecies *nana* Briq. (= *J. nana* Willd.) voor. Het lijkt dus wel waarschijnlijk, dat het in laatglaciale afzettingen voorkomende *Juniperus*-pollen hoofdzakelijk van deze subspecies afkomstig is, hoewel voor die veronderstelling nog geen bewijzen bestaan.

Populus-pollen werd aangetroffen voornamelijk in het laatglaciale deel der geanalyseerde monsters. Een curve werd niet aan het diagram toegevoegd, omdat bij de determinatie soms onzekerheid optrad.

Hippophaë. Over het voorkomen van de duindoorn in het laatglaciaal is reeds veel geschreven. Het tegenwoordige areaal van *H. rhamnoides* omvat een groot deel van Europa en Azië. Deze nu in West-Europa voornamelijk langs de kusten groeiende plant, schijnt in laatglaciale tijd een grote verspreiding in het binnenland gehad te hebben. FLORSCHÜTZ vond reeds in verscheidene laatglaciale afzettingen in Nederland pollen van *Hippophaë*. Evenals dat elders in West-Europa geconstateerd is, blijkt ook uit ons diagram, dat het voorkomen van dit pollen voornamelijk beperkt is tot de Oudere Dryas-laag.

Helianthemum. Pollenkorrels van *Helianthemum* werden door IVERSEN gevonden in laatglaciale sedimenten in Denemarken (IVERSEN, 1944, 1947). De meeste bleken de habitus te hebben van *H. oelandicum* (L.) Willd., terwijl enkele het minder slanke *H. nummularium*-type vertegenwoordigen. Het in ons materiaal aangetroffen pollen stemt geheel overeen met dat van *H. oelandicum* en is, evenals in Denemarken, voornamelijk beperkt tot de Oudere Dryas-laag.

Helianthemum oelandicum schijnt thans te ontbreken in het Scandinavische arctische gebied. Op Oeland heeft dit Zonneroosje een geïsoleerde groeiplaats, in een gezelschap op droge kalkgrond, terwijl het verder in de alpiene en subalpiene-vegetaties van Centraal- en Zuid-Europa aanwezig is. Een tweetal soorten van dit geslacht komt, zij het zeer zeldzaam, ook in Nederland voor, op droge gronden. IVERSEN rekende *Helianthemum*

bij de z.g. „steppe elementen” in de laatglaciale vegetatie (IVERSEN, 1944). (Zie hierover verder bij *Artemisia*.)

Artemisia. Een zeer opvallend pollentype in laatglaciale afzettingen is afkomstig van *Artemisia*. Gedurende het gehele Laatglaciaal is het tegenwoordigd. Het voorkomen van *Artemisia* en van enkele andere planten, als *Hippophaë* en *Helianthemum*, is door enkele auteurs (o.a. GAMS, IVERSEN) naar voren gebracht, daar het zou kunnen wijzen op een droog klimaat en een daarmee gepaard gaand zogenoemd „steppe-karakter” der vegetatie. Het tegenwoordige voorkomen van deze genera in West-Europa is voornamelijk beperkt tot droge gronden, terwijl ze voorts in gebieden met een meer continentaal klimaat een grotere verbreiding hebben. In het Scandinavische arctische gebied, dat door de geografische ligging een oceanische invloed ondervindt, ontbreken deze steppe-elementen blijkbaar. In het meer continentale Russische toendra-gebied komen echter enclaves van steppe voor.

Het lijkt zeer aannemelijk dat in laatglaciale tijd het klimaat in West-Europa veel continentaler geweest is dan nu. Immers, toen stond de zeespiegel aanmerkelijk lager, zodat een groot gedeelte van de tegenwoordige Noordzee droog lag. Zowel op grond van plantengeografische als van palaeogeografische overwegingen lijkt het dus waarschijnlijk, dat een continentaal klimaat zijn stempel gezet heeft op de laatglaciale vegetatie. We zullen hieronder zien, dat er toch nog een feit is, dat, voor de Jongere Dryas-tijd, op een relatief sterkere atlantische invloed in Nederland dan in andere delen van West-Europa zou kunnen wijzen.

Empetrum-pollen komt hoofdzakelijk in zone III voor, waar bijzonder hoge waarden worden bereikt. Ook elders is de uitbreiding van *Empetrum* typisch voor de Jongere Dryas-tijd. In Denemarken werden echter veel geringere procenten verkregen. In Noorwegen evenwel vond FAEGRI ook hoge *Empetrum*-percentages. Volgens IVERSEN (schr. med.) is dit blijkbaar een atlantisch kenmerk.

Plantago-pollen is hoofdzakelijk in zone I aangetroffen, doch steeds in geringe hoeveelheid. De korrels zijn bijna alle van het het *P. maritima* L.-type, en wijken, zoals ook IVERSEN in Denemarken vond (mond. med.), soms hiervan af door een iets meer uitstekende annulus om de poriën.

Rumex. Bijna alle gevonden pollenkorrels behoorden tot het *Rumex acetosa* L.-type, dat zowel bij deze soort als bij *R. acetosella* voorkomt.

Compositae. De meeste korrels waren van het *Tubuliflorae*-type. Slechts éénmaal werd een korrel aangetroffen, die tot het *Liguliflorae*-type behoorde (m. 5).

Sanguisorba minor Scopoli (syn. *Poterium sanguisorba* L.). In monster 4 werd een pollenkorrel aangetroffen, die door Dr. IVERSEN herkend werd als afkomstig te zijn van *Sanguisorba minor* Scopoli. Een tweede korrel, die naar alle waarschijnlijkheid ook hiertoe behoort, werd in monster 1 gevonden. Tezamen met Dr. IVERSEN hopen wij over deze nieuwe vondst in de naaste toekomst een documenterend artikel te publiceren. *S. minor*

zou gerekend kunnen worden bij de reeds eerder genoemde planten met een zgn. „steppe-karakter”. Hij komt recent voor (ook in Nederland) op droge gronden en bereikt in Europa zijn noordgrens op Gotland, in Denemarken, in Zuid-Zweden en in Engeland. Het areaal omvat verder o.a. Centraal- en Oost-Europa en Siberië. In het arctische gebied ontbreekt hij echter.

Lycopodium Selago L. Drie sporen hiervan werden gevonden in monster 14, op de plaats van het Empetrum-maximum. *L. Selago* wordt recent bij ons zeldzaam aangetroffen o.a. op vochtige heiden. In Dalarne schijnt hij in de Empetrumheide voor te komen.

Dryopteris Linneana Chr. Blijkbaar voorkomend in het subarctische bos. Evenals in Denemarken ontbreken sporen in zone I.

Dryopteris Thelypteris A. Gray-type. Hiertoe behoort een aantal verder niet te onderscheiden varensporen.

Equisetum-sporen werden in bijna alle monsters aangetroffen.

Potentilla. Twee korrels in monster 4.

Epilobium. In monster 2, 4 en 11 ieder een pollenkorrel.

Menyanthes, *Typha angustifolia L.*, *T. latifolia L.* en *Sparganium* werden verspreid aangetroffen in het gehele laatglaciale profiel.

Potamogeton. Wij geven hier een tabel van de berekende pollenpercentages.

Zone	I					II					III					IV			V	
Monster	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
Potam. %	40	20	16	14	4	6	5	0.2	1.5	—	0.2	4	2.5	1.5	1.5	0.5	—	—	—	

Toppen dus in zone I en een lagere in zone III.

Isoëtes en *Littorella*. Isoëtes-sporen werden aangetroffen in het bovenste deel der Allerød-laag en in het Post-glaciaal. Per preparaat kwamen de volgende aantallen voor

Allerød:	monster 10	130
	„	11 25
Postglaciaal:	„	16 92
	„	17 3
	„	18 6
	„	19 4

Littorella-pollen was alleen aanwezig in monster 16, en wel 30 korrels per preparaat.

In het geval van monster 16 doet de toenmalige vegetatie op de meer-bodem met veel Isoëtes en *Littorella* sterk denken aan een plantengeselschap uit de hedendaagse flora, voorkomend in voedselarm, helder water op zandgrond (zie hierover WESTHOFF, 1946). Dit is het Isoëteto-Lobelietum, behorend tot het *Littorellion uniflorae* (verbondskensoort o.a. *Littorella uniflora*). Kensoorten der associatie zijn o.a. *Isoëtes lacustris* L. en *I. echinospora* Dur. Het is natuurlijk zeer gevaarlijk om hier de naam

van deze recente associatie te gebruiken, daar we de verdere samenstelling van het toenmalige plantengeselschap niet kennen. Dat het klimaat in het Praeboreaal continentaler geweest is dan nu, behoeft niet in tegenspraak te zijn met het feit, dat het Littorellion in de huidige vegetatie beperkt is tot de atlantische provincie van Europa. Het is n.l. hierboven al gebleken, dat zich waarschijnlijk reeds in de Jongere Dryas-tijd bij ons zekere atlantische invloeden lieten gelden. In ieder geval is het waarschijnlijk, dat we hier met een associatie te doen hebben, die verwant is met het bovengenoemde recente plantengeselschap.

Vergelijking van de resultaten met die van reeds eerder in Nederland uitgevoerde onderzoeken.

Onze huidige kennis van het Laatglaciaal in Nederland hebben we voor het grootste deel te danken aan FLORSCHÜTZ, die een aantal publicaties hierover deed verschijnen. Pogen we nu de door hem geanalyseerde laatglaciale veenlagen (zie o.a. FLORSCHÜTZ, 1939) te plaatsen in ons schema, dan stuiten we op vooralsnog onoverkomelijke moeilijkheden. Deze vinden hun oorzaak ten dele in de invloed der „veen"-vegetatie, waarover reeds hierboven gesproken is, ten dele in het ontbreken van afzonderlijke curven voor verschillende soorten kruidenpollen. Het grote belang van die soorten als „gidsfossielen" is ook pas in zeer recente tijd duidelijk geworden, terwijl de genoemde onderzoeken alle van oudere datum zijn. In een nieuwere publicatie geeft FLORSCHÜTZ (1941) de analyse van een moeraskalkafzetting uit Midden-Limburg. Hoewel in de sedimentatie blijkbaar geen verschillen aanwezig zijn, lijkt het ons waarschijnlijk dat in dit diagram de Allerød-oscillatie tot uiting komt. Onderin komen hoge kruidenpollenpercentages voor, samen met Hippophaë (6 %!) en Selaginella. Na een Salixtop (!) treedt een sterke vermindering van het kruidenpollen op, gevolgd door een hernieuwde toename daarvan, weer gepaard gaand met een Salix-maximum. Hier kunnen dus de drie laatglaciale zones alle aanwezig zijn. Merkwaardig zijn voor het Laatglaciaal de vrij hoge Coryluswaarden. Het voorkomen van 200 % kruidenpollen, 6 % Hippophae, enz. in een laatglaciale meerafzetting tezamen met Corylus kan moeilijk anders verklaard worden dan óf door transport over grote afstand van het Corylus-pollen (en dan kan de Zuidelijke ligging wellicht een rol spelen), óf door de aanwezigheid van secundair pollen. Ook in andere diagrammen zijn misschien enkele aanduidingen te vinden van de aanwezigheid van de Allerød-oscillatie (b.v. in het diagram van Grollo, FLORSCHÜTZ, 1941), maar het is niet mogelijk hierover iets met zekerheid te zeggen.

FLORSCHÜTZ (1939) wees op de mogelijkheid, dat op den duur zou kunnen blijken, dat een der in Twenthe geconstateerde laatglaciale stuifzandperioden geparalleliseerd kon worden met de Allerød-tijd. WATERBOLK (1947) acht die verwachting ongegrond, daar zowel bij Hengelo als bij Usselo in het onder het bewuste stuifzand gelegen veen reeds Corylus gevonden is, hetgeen pleit voor een Praeboreale, dus veel jongere datering.

Deze auteur ziet daarbij vermoedelijk over het hoofd, dat FLORSCHÜTZ van meer dan één, door veenlagen gescheiden, stuifzand pakket spreekt.

Naar onze voorlopige mening, gebaseerd op een nog niet afgesloten onderzoek, behoren die veenlagen tot verschillende zones van het Laat-glaciaal.

Op de resultaten van dit onderzoek, dat waarschijnlijk tevens enkele nieuwe Allerød-profielen zal omvatten, hopen we later in een uitgebreidere publicatie terug te komen. Wij stellen ons voor, dan ook een uitvoeriger synthese te geven van de laatglaciale vegetatie-ontwikkeling in Nederland.

Tot slot mogen wij een bijzonder woord van dank richten tot Prof. Dr. F. FLORSCHÜTZ, die ons steeds, zowel bij het onderzoek als bij het samenstellen van dit artikel, met raad en daad ter zijde stond. Ook Dr. JOH. IVERSEN, palaeobotanicus bij de Deense Geologische Dienst, die het ons mogelijk maakte om onze kennis van kruidenpollen aanmerkelijk te vergroten en te verdiepen, zijn wij hiervoor uitermate erkentelijk.

Summary.

Proof of the Allerød oscillation in Holland.

(With a diagram and a table.)

The present article deals with the analysis of a late-glacial lake deposit from the lake of Hijken, in the province of Drente, Holland. The diagram, that is composed according to the method of IVERSEN, clearly shows that the Allerød oscillation is present here. It is the first time this interstadial has been recognised with certainty in Holland. The diagram shows also a smaller oscillation before the Allerød interstadial. It corresponds with the Bølling oscillation, which IVERSEN, found in Mid-Jutland. There is a striking resemblance between the diagram from Hijken and the diagrams from Denmark, published by IVERSEN, as well for the general part, as for the curves of the separate herbs. For instance is, just as in Denmark, *Helianthemum* and *Hippophaë* pollen mainly found in the Older Dryas deposits. We were thus able to use the zone division published by IVERSEN. It appears that during the late-glacial period the vegetation showed great conformity over large areas. A curve is given for *Juniperus*, showing tops at those places where the general part of the diagram proves an open country. The high *Empetrum* percentages found in the Younger Dryas deposits are curious. In some samples (from the late Allerød- and early Postglacial-sediments), spores of *Isoetes* were found. In one case also *Littorella* pollen was found, together with these spores. In the Older Dryas deposits a pollengrain was found, that Dr. IVERSEN recognised as belonging to *Sanguisorba minor* Scopoli. In a future joint publication Dr. IVERSEN and the present writer hope to present the data for this identification. Some pollengrains of thermophile trees were found in the late-glacial sediments. For the greater part this pollen is probably secondary, because also tertiary or interglacial pollen and "Hystrix" were found. It is not added in the diagram, but mentioned in a separate table. Perhaps some *Corylus* grains are primary, in which case they have probably been carried by the wind over a long distance. Finally

	Division DE GEER	Climatic periods Blytt- Sernander	Vegetation development for Holland FLORSCHÜTZ	Climatic periods (for Denmark etc.)	Zones of SCHÜT- RUMPF 1943	Zones of K. JESSEN 1935 IVERSEN 1942	Vegetation development		
							HOLSTEIN (SCHÜT- RUMPF)	Mid-Jutland (IVERSEN)	Drente (Holland)
Post-glacial	Fini-glacial	Preboreal	pine- and birchforest + therm. elem.	Preboreal	VI V	IV	pine- and birch-pine forest	birch-forest	birch-pine-forest
Late-glacial	Goti-glacial	sub-arcticum	Woods (Betula, Salix, Pinus)	Late Dryas-period	IV	III	park-tundra	tundra or park-tundra	"park-tundra"
				Allerød-oscillation	III	b a	birch-pine-forest	park-tundra	birch-pine-forest birch-forest
				Earlier Dryas-time	II	I c	park-tundra	tundra	"tundra"
				Bølling-oscillation		I b		park-tundra	"park-tundra"
	Dani-glacial		subarctic marshy park-landscape (B. nana, Arctostaph., Selaginella)	Earliest Dryas-time	I	I a	tundra	tundra	"tundra", not without trees tundra ?
		Arcticum	tundra, perhaps not without trees (Twenthean Dryasflora)						

a correlation with some of the diagrams published by Prof. FLORSCHÜTZ has been attempted.

LITERATUUR.

- BURCK, H. D. M., F. FLORSCHÜTZ en P. TESCH, 1948. De stratigrafische grens tussen het Plistocene en het Holoceen in Nederland. Geol. Mijnb., vol. 10 (5).
- DEGERBØL, MAGNUS and JOHS. IVERSEN, 1945. The Bison in Denmark. Danm. Geol. Und. (2), No. 73.
- DUBOIS, GEORGE et CAMILLE DUBOIS, 1944. L'oscillation chaude d'Allerød reconnue dans une deuxième tourbière du Cantal à Riom-ès Montagne. C. R. S. Soc. Geol. France No. 6.
- FIRBAS, F., 1944. Systematische und genetische Pflanzengeographie. Fortschr. Bot., Vol. 11.
- FLORSCHÜTZ, F., 1939. Spätkiaziale Torf- und Flugsandbildungen in den Niederlanden als Folge eines dauernden Frostbodens. Abh. Nat. Ver. Bremen, Vol. 31 (2).
- en I. M. VAN DER VLERK, 1939. Duizend eeuwen geschiedenis van den bodem van Rotterdam. De Maastunnel, vol. 2 (6).
- , 1941. Resultaten van microbotanisch onderzoek van het complex loodzandoerzand en van daaronder en daarboven gelegen afzettingen. In: Besprekingen over het heidepodsolprofiel.
- , 1941. Laatglaciale afzettingen in Midden- en Noord-Limburg. Tijdschr. K. N. Aardr. Gen., vol. 58 (6).
- , 1944. „Laagterrassen“ en „Veen op grotere diepte“ onder Velzen. Tijdschr. K. N. Aardr. Gen., vol. 61 (1).
- GODWIN, H., 1947. The late glacial period. Science Progress No. 138.
- GROSZ, H., 1937. Nachweis der Allerödschwankung im süd- und ostbaltischen Gebiet. Beih. Bot. Centr. bl., vol. 57, Abt. B.
- HEGÉ, GUSTAV. Illustrierte Flora von Mittel-Europa.
- IVERSEN, JOHS., 1936. Sekundäres Pollen als Fehlerquelle. Danm. Geol. Und. (4), vol. 2 (15).
- , 1944. Helianthemum som fossil Glacialplante i Danmark. Geol. Fören. Stockholm Förh., vol. 66.
- , 1946. Geologisk Datering af en senglacial Boplads ved Bromme. Aarbøger for Nordisk Oldkyndighed og Historie.
- , 1947. Plantevækst. Dyreliv og Klima i det senglaciale Danmark. Geol. Fören. Stockholm, Förh., vol. 69 (1).
- LITZELMANN, ERWIN, 1938. Pflanzenwanderungen im Klimawechsel der Nacheiszeit. Schriften des Deutschen Naturkundevereins, Neue Folge, vol. 7.
- MÜLLER, INGE, 1947. Ueber die spätkiaziale Vegetations- und Klimaentwicklung im Westlichen Bodenseegebiet. Planta, vol. 35 (1/2).
- OVERBECK, FRITZ und SIEGFRIED SCHNEIDER, 1938. Mooruntersuchungen bei Lüneburg und bei Bremen und die Reliktnatur von *Betula nana* L. in Nordwestdeutschland. Zeitschr. Bot. vol. 33.
- PENNINGTON, WINIFRED, 1947. Pollendiagrams from the bottom deposits of the north basin of Windermere. Phil. trans. Royal Soc. London, (B), vol. 233 (596).
- SCHÜTRUMPF, RUDOLF, 1936. Paläobotanisch-Pollenanalytische Untersuchungen der paläolithischen Rentierjägerfundstätte von Meiendorf bei Hamburg. Inaugural-Dissertation. Veröff. Arch. Reichsinst.
- , 1943. Die Pollenanalytische Untersuchung der Rentierjägerfundstätte Stellmoor in Holstein. In: A. RUST, Die alt- und mittelsteinzeitlichen Funde von Stellmoor.
- STEINBERG, KURT, 1944. Zur spät- und nacheiszeitlichen Vegetationsgeschichte des Untereichsfeldes. Hercynia, vol. 3 (7/8).
- WATERBOLK, H. Tj., 1947. De oudheidkundige verschijnselen in verband met de ontwikkeling van plantengroei en klimaat. Gedenkbl. A. E. van Giffen.
- WESTHOFF, V., J. W. DIJK, H. PASSCHIER en G. SISSINGH, 1946. Overzicht der plantengemeenschappen in Nederland. 2e dr.

Medicine. — Somatometric relations between relatives of the first degree.
(Preliminary note.) By J. HUIZINGA. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of January 29, 1949.)

With investigation into somatometric relations we enter the domain of genetics.

The factors which impede, if not prevent, the resolution of questions relating to heredity in man are many and sufficiently known. We can approach hereditary problems in man by cyto-genetic research, by study of twins, or by family investigation and statistical analyses of large populationgroups as HARRIS, for instance, did in connection with premature baldness (1946).

We chose for our research a particular type of family investigation. The choice was determined by difficulties which arise specifically with metrical data from age and sex differences. We did not try to solve these difficulties, but knowing they were there, rather sought to avoid them by:

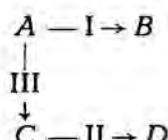
- a. entertaining only adults in our research (elimination of age-differences), and
- b. comparing solely sons with fathers and daughters with mothers (elimination of sex-differences).

It would be pointless to determine the degrees of resemblance or difference between relatives unless these degrees were also determined between non-relatives and the two results compared.

A certain resemblance between an adult son *A* and his father *B* can be accepted in view of the genetical relationship, provided such resemblance does not exist between this son *A* and a non-related adult man *C*.

Other investigators might, however, prefer to compare in this case two non-related adults *C* and *D* instead of again son *A* and *C*.

With regard to the scheme



it would seem most logical to compare the expression for the resemblance obtained by method I with that obtained by method III.

The difficulty is now to obtain an "expression of the resemblance". The protagonists of the biometrical school of PEARSON calculated for each of the many variables, correlation coefficients between fathers and sons, sons and daughters, daughters and mothers, etc.

As far as we can gather from their publications, they do not think it necessary, using this method, to take into account the age and sex differences of the correlated groups of relatives.

Their material is very often admirably extensive, so that their results should be thoroughly reliable statistically.

In the hands of the biologist-statistician the correlation coefficients can, moreover, allow of biologically reliable conclusions.

We endeavoured, using a simpler and clearer method, to gain an insight into somatometrical relationships.

Suppose the head length of son A (*S/A*) is 188 mm, that of his father (*F/A*) 193 mm, and that of a non-related male adult (*F/B*) 181 mm, then the absolute difference (i.e. the difference which, in contrast to the algebraic difference, takes no account of direction) is 5 mm between *S/A* and *F/A* and 7 mm between *S/A* and *F/B*.

(We always took as "a non-related male adult *F/B*" the father of the next-examined family. For this reason we chose the symbol *F* for this man too. As the examination of the families took place in an arbitrary order of succession, there can, in our opinion, be no objection to this method.)

If we determine such differences more than once, then we can also calculate an average absolute difference (*A.A.D.*) *S/A.F/A* and *S/A.F/B*.

As our material consisted of 65 families, we obtained *A.A.D.S/A.F/A* by adding 65 differences *S/A.F/A* and dividing by 65. Thus, in the head width there is an *A.A.D.* of 5.3 mm between *S/A* and *F/A* and of 6.3 mm between *S/A* and *F/B*.

In view of the genetic connection between *S/A* and *F/A* on the one hand and its total absence between *S/A* and *F/B* on the other hand, we could expect that the *A.A.D.* would be less for *S/A.F/A*, or at least not greater, than for *S/A.F/B*.

One could therefore calculate the ratio of the *A.A.D.* for *S/A.F/B* and for *S/A.F/A*, and one could expect that this ratio would be equal to or greater than unity.

In order to obtain easily manageable results, this expression was calculated as

$$\text{ratio} = \frac{100 \times \Sigma A.D. \text{ for } S/A.F/B}{\Sigma A.D. \text{ for } S/A.F/A}.$$

The same holds for the comparison of an adult daughter (*D/A*) with her mother (*M/A*) and with a non-related female adult (*M/B*).

Among 142 ratios calculated by us (from measured and calculated variables of head and hand for men and women), 126 were found to conform to our expectations, and thus only 16 were less than 100 (viz. 5 of 99, 4 of 98, 1 of 97, 3 of 96, 1 of 93, 1 of 92 and 1 of 91). Considering also the relatively small extent of the material from which the

averages were calculated (65 families), we can regard our expectations as substantially fulfilled.

"A variable with a ratio of 100" means that the difference between a son and his father, for this variable, is, on the average, as great as between this son and a non-related adult male.

Such variables have no value for indicating genetical connection.

The extent by which a ratio of a variable deviates from 100 is more or less an index of the importance of this variable for the indication of a genetically determined conformity.

In this manner, the ratio can be calculated for each measurable variable in the two sexes.

Thus, we calculated, after comparison of S/A with F/A and F/B , the ratio for the physiognomical face-height as 127 in males and 106 in females.

Though there is a sex-difference in the head length, for instance, which we have to take into consideration in comparing the head length of a man with that of a woman, this sex-difference in the ratio no longer exists.

This is connected with the factor of sex-relation by which the value of a female variable must be multiplied in order to obtain a figure which is directly comparable to the value of the same male variable (WEBER, 1935).

If we call this factor a , then each absolute difference between two females has been multiplied by a in order to obtain a result directly comparable to a male difference.

The $A.A.D.$ calculated for several female pairs also contains the factor a , which however is eliminated when we divide this difference by a second $A.A.D.$ containing the factor a , calculated from a group of other female pairs. The ratios of the female variables are thus without correction based on a sex-difference comparable with the ratios of the male variables.

An objection to the use of the *absolute* differences is that the real significance of a certain difference does not become manifest.

A difference of 4 mm in the mouth width is, of course, relatively more considerable than the same difference e.g. in the face height. This objection can be met by using the percentual difference, calculated from the absolute difference and the average of the two variables between which such difference exists.

A number of percentual differences $S/A . F/A$ can again be added and divided by this number, so that the average percentual difference ($A.P.D.$) is known.

From this difference for S/A and F/A and for S/A and F/B a ratio for the variable in question can be calculated as follows:

$$\text{ratio} = \frac{100 \times \sum P.D. \text{ between } S/A . F/B}{\sum P.D. \text{ for } S/A . F/A}.$$

Theoretically we preferred the $A.P.D.$ to the $A.A.D.$, and we were a priori inclined to attach greater value to ratios calculated by means

of the *A.P.D.*, but the 142 ratios, calculated in the two different ways, proved, to our surprise, to be almost identical.

As an example we give in order of magnitude the two ratios of 23 absolute measurements in women.

Nr.	Variable	Ratio <i>A.A.D.</i>	Ratio <i>A.P.D.</i>
1	face depth	145	146
2	head width	126	126
3	mandibular angle width	124	123
4	upper-lip height	124	124
5	frontal width	123	123
6	nose height	119	121
7	head length	116	116
8	chin height	114	114
9	orbit height	114	114
10	frontal height	113	111
11	frontal depth	111	109
12	ear width	111	111
13	nose width	110	110
14	nose length	108	107
15	head height	106	108
16	physiognomical face height	106	106
17	mouth width	103	102
18	interorbital width	101	103
19	ear length	101	101
20	face width	99	98
21	morphological face height	99	100
22	orbit width (right)	99	99
23	nose depth	96	95

Neither the author nor a mathematician has succeeded in discovering the mathematical grounds of this identity.

Forms representing the mathematical structure of the two ratios are not manageable.

We accepted this identity, though not understood, intentionally, because calculation of the percentual differences would entail considerably more work. For future researches this possibility of substitution is of great practical value.

In 1947 we described fully our investigation on *cephalometric* relations and in 1948 our investigation of *cheirometric* relations.

Now we will throw some light on a particular aspect of the two investigations and consider the results together with some data on *body-measurements*.

We divided the employed measurable variables of the head, the hand and the rest of the body into:

a. *height-measurements* (measurements taken perpendicular to the transversel plane) e.g. orbit height, middle finger length, body height.

b. *width-measurements* (measurements taken perpendicular to the sagittal plane) e.g. mouth width, hand width, shoulder width.

c. *depth-measurements* (measurements taken perpendicular to the frontal plane) e.g. head length, chest depth.

These terms are applied to the erect man, with the face turned forward and the arms straight down along the body with hands supinated.

Depth-measurements are not further considered.

The following tables give the ratios for the relevant variables of head and hand:

	A. Height-measurements:	Ratios (A.D.)		Predominant
		♂ ♂	♀ ♀	
I	<i>Neurocranium:</i>			
	1. head height	111	106	♂
II	<i>Face:</i>			
	2. physiogn. face h.	127	106	♂
	3. morphol. face h.	120	99	♂
	4. front h.	101	113	
	5. nose h.	114	119	
	6. upper lip h.	115	124	♀
	7. chin h.	106	114	
	8. orbit h.	109	114	
III	<i>Hand:</i>			
	9. hand length	121	120	♂
	10. metacarpal length	129	112	♂
	11. length I	98	103	
	12. " II	121	140	♀
	13. " III	122	126	♀
	14. " IV	126	129	
	15. " V	122	108	♂

B. Width-measurements:

I	<i>Neurocranium:</i>			
	1. head width	119	126	♀
	2. frontal width	100	123	♀
II	<i>Face:</i>			
	3. face width	108	99	
	4. mand. angle width	130	124	
	5. nose width	119	110	
	6. mouth width	129	103	
	7. interorbital width	117	101	
	8. orbit width	97	99	
III	<i>Hand:</i>			
	9. hand width	144	119	
	10. width I	133	101	
	11. " II	132	96	
	12. " III	144	99	
	13. " IV	169	96	
	14. " V	116	91	

We now observe that:

1. The ratios of the *height-measurements* calculated by comparison of Females show a fairly strong tendency to be greater than those calculated by comparison of Males.

Exceptions: Neurocranium and measurements which more or less concern a totality (physiognomic and morphological face height, hand length), and also metacarpal length and length V.

2. The ratios of the *width measurements* calculated by comparison of Males are greater than those calculated by comparison of Females.

Exceptions: Neurocranium (the ratios of the orbit width are both smaller than 100 and both are thus of no importance).

3. The neurocranium takes an exceptional position in both cases.

In our 65 families no body-measurements could be determined. In considering how far the above-stated regularity is valid also for body-measurements, we had to be satisfied with computing the ratios obtained by comparison of 9 sons with their fathers etc., and of 6 daughters with their mothers, etc.

These persons form a part — to be considered by us — of a material, collected by DE FROE in Nijmegen (Holland).

The *height-measurements* of the body determined by him were body height, upper arm length, arm length and crista height, for which we calculated the following ratios:

	Ratios (A.D.)		Predominant
	♂♂	♀♀	
1. Body height	64 66	47 33	
2. Upper arm length	28 25	13 7	
3. Arm length	40 36	30 14	♀
4. Crista height	57 43	38 27	

Thus, our expectation that ratios of height-measurements for women would be greater are fulfilled.

For the width-measurements of the body, of which only the chest width and shoulder width measurements were available, we find:

	Ratios (A.D.)		Predominant
	♂♂	♀♀	
1. Chest width	17 8	14 8	♂
2. Shoulder width	19 14	18 5	♀

in which thus the shoulder width deviates from the rule that we found to apply to the width-measurements.

It is not easy to explain this (possible) sex-difference existing in the significance of certain directions of growth in heredity problems. In connection with the shoulder width we must note that this is determined chiefly by the length of the clavicle.

The clavicle and the neurocranium constitute exceptions.

In the ossification of the skeleton these two bony structures occupy a special position, in that a large part ossifies primarily.

As the epiphyseal cartilages generally disappear earlier in women than in men, we can think that as exogenous and endogenous influences can act for a longer time in the male, the variability is somewhat greater, and thus a son has more chance to differ from his father as regard measurements taken perpendicular to the epiphyseal cartilage.

A better understanding of this startling problem will be obtained by study of the influence of the sex-hormones on growth-potencies in various directions.

At this stage of investigation any hypothesis is liable to be not only premature but even inconvenient.

The anthropologist will for this purpose have to turn his investigation to subjects with hormonal disturbances, such as eunuchs, girls with agenesis of the ovaries, hermaphrodites, etc., preferably together with their families.

Summary.

The author describes a fairly straightforward method of obtaining quantitative expression of the resemblance between related adults of the same sex (the ratio). There is a ratio for each measured or calculated variable, and the ratios proved, as theoretically expected, to be greater than or equal to 100. The extent by which a ratio of a variable deviates from 100 is more or less an index of the importance of this variable for the indication of a genetically determined conformity. Different groups of variables were found to be of different significance for the two sexes, namely:

1. Height-measurements (taken perpendicular to the transverse plane) tend to be of more significance in expressing the genetic connection between related females.
2. Width-measurements (taken perpendicular to be sagittal plane) are of more importance in expressing the genetic connection between related males.

Some exceptions are discussed.

The author believes that for a better understanding it is necessary to investigate the sex-differences in persons with hormonal disturbances, together with their families.

REFERENCES.

- HARRIS, H., The inheritance of premature baldness in men. *Annals of Eugenics*, 13/3, 172—181, 1946.
- HUIZINGA, J., Cephalometriche verwantschap tusschen verwanten van den eersten graad. *Dissertatie*, Amsterdam, 1947.
- _____. Cheirometric relations between relatives of the first degree. *Proc. Kon. Ned. Akad. v. Wetensch.*, Amsterdam, vol. II, 5, 575—588, 1948.
- WEBER, E., *Einführung in die Variations- und Erblichkeitsstatistik*, München, 1935.

Botany. — *De F_4 -zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van Phaseolus vulgaris. II.* By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of October 30, 1948.)

C1 4. De formule van de uitgangsboon is L b th, cl 4. 7 gevallen. De bonen hebben een grote lengte en een kleine breedte. Meestal is de lengte niet zeer groot en de breedte niet zeer klein (tab. 9). Daardoor zijn de bonen van cl 4 verwant aan die van cl 2. Ook de bonenopbrengsten van de I-lijn, de zuivere lijn, bevatten meestal enige bonen met de form. L b th, cl 4 (tab. 5a).

In één geval (pl. 1033) is de formule van de gemiddelden van de bonenopbrengst ook L b th (tab. 1 en 1a).

De grootste lengte van de bonen van pl. 1033 is $l = 14.6$ mm. Volgens de classificatie (tab. 1a) zijn er zeer veel bonen in cl 2, cl 4 en cl 8a. Van alle bonen is de dikte klein. Van de uitgangsboon van pl. 101 voor pl. 1033 is de dikte homozygoot voor th-factoren; de lengte en de breedte (ofschoon klein, $b = 8.4$ mm), zijn heterozygoot voor L, resp. B-factoren. Voor de uitgangsboon van pl. 101 en haar bonenopbrengst, pl. 1033, hebben we met een goed geval van de form. L b th, cl 4, te doen.

In 2 gevallen (pl. 264, tab. 1 en 1a, cl 3 en 288) is de formule van de gemiddelden van de bonenopbrengsten LB Th, cl 1, (blz. 188). In één geval (pl. 374) is ze L B th, cl 2 (blz. 192). Volgens de classificatie is het een goed geval van cl 4, komt overeen met pl. 1033 (zie boven). De uitgangsboon is van dezelfde plant, pl. 101. Eveneens in één geval, pl. 1035, is de formule van de gemiddelden van de bonenopbrengst 1B th, cl 6. De uitgangsboon stemt zeer overeen met die voor pl. 1033; de gemiddelde breedte van de bonenopbrengst is veel groter. Volgens de classificatie zijn er zeer veel bonen in cl 1 en in cl 6. Ten slotte zijn er nog 2 gevallen met de form. L b th, cl 4 van de uitgangsboon, waar de formule van de gemiddelden van de bonenopbrengst 1 b th, cl 8 is (pl. 976 en 1001). Bijna alle bonen van de bonenopbrengsten behoren hier tot cl 8 (zie artikel III in de volgende Proceedings).

In 16 gevallen is de formule van de gemiddelden L b th, cl 4 en is die van de uitgangsboon een andere. In 2 van deze gevallen (pl. 267 en 278) is de formule $L_1 L_2 B Th$, cl 1a.

Bij pl. 278 staat aangetekend „slecht, enkele peulen nog groen”; het is mogelijk, dat de bonen onvolgroeid zijn. Er zijn zeer veel bonen in cl 8, vele in cl 2 en meerdere in cl 4. De bonenopbrengst en haar uitgangsboon, behoort tot het gebied van cl 2, form. L B th. De bonenopbrengst van pl. 267 heeft volgens de classificatie veel bonen in cl 4 en cl 8 en meerdere in cl 2; alle bonen hebben een kleine dikte. Opmerkelijk daartegenover is de grote dikte, ($th = 7.4$ mm) van de uitgangsboon. Bonenopbrengsten van de I-lijn bevatten als hoge uitzondering ook wel bonen met een grote dikte. In 1935 was de grootste niet-erfelijke dikte-variant $th = 7.5$ mm. We nemen aan, dat ook pl. 267 behoort tot de erfkring van de I-lijn (cl 2 of cl 4).

In 7 van de bovengenoemde 16 gevallen is de formule van de uitgangsboon L B Th, cl 1b.

Volgens de classificatie van de bonenopbrengsten en de ascendentie van de uitgangsboon behoren ze tot cl 4 (pl. 1040, 1036 en 1019), cl 2 (pl. 1041, pl. 292 en pl. 353) en cl 1 (pl. 322). Van pl. 353 zijn 4 bonen in cl 5; van pl. 1019, 2 in cl 7 (zie art. III in de volgende Proceedings).

In 2 gevallen is de formule van de uitgangsboon L₁ L₂ B th, cl 2a (pl. 318, blz. 191 en pl. 346). Volgens de ascendentie van de uitgangsboon en de classificatie van de bonenopbrengst, behoort dit geval duidelijk tot cl 4, form. L b th. Eveneens in 2 gevallen is de formule van de gemiddelden L b Th, cl 3 (pl. 345, pl. 1037).

Van pl. 345 bestaat de bonenopbrengst uit slechts 6 bonen. Van pl. 1037 heeft de uitgangsboon van pl. 101 een grote dikte. Volgens de classificatie van pl. 1037 zijn er veel bonen in cl 4 en in cl 8, zoals bij bonenopbrengsten, waarvan de uitgangsboon de formule L b th, cl 4 heeft (zie ook art. III in de volgende Proceedings).

In 2 gevallen ten slotte is de formule van de uitgangsboon 1b th, cl 8 (pl. 247 en 1049).

De uitgangsboon voor pl. 247 is van pl. 55 en is de kleinste boon van de bonenopbrengst van pl. 55. Volgens de classificatie van pl. 247 zijn er bonen in bijna alle klassen, behalve in cl 5. De uitgangsboon is waarschijnlijk een modificatie van bonen met de form. L B Th en daar de afmetingen van de meeste bonen niet zeer groot zijn, zijn er ook 1l-, b b- en th th-factorenverbindingen in de formule. De uitgangsboon voor pl. 1049 heeft uiterst kleine afmetingen ($l = 8.6$ mm). Volgens de classificatie van de bonenopbrengst van slechts 10 bonen behoort het geval tot cl 4, form. L b th.

Uit de bespreking van de gevallen van cl 4 hebben we gezien, dat enkele gevallen goede vertegenwoordigers zijn van deze classe. De bonen van cl 4, form. L b th, zijn nauw verwant aan die van cl 2, form. L B th.

Cl. 5. Form. 1B Th. De formule van de uitgangsbonen is 1B Th. Deze bonen hebben dus een kleine lengte en een grote breedte en dikte (tab. 9). 9 gevallen. De lengte heeft meestal de grenswaarde (3 maal is $l = 13.0$, 2 ml = 12.8 en 1 ml = 12.7, 12.5, 12.4 en 11.8 mm). Dit geldt ook voor de breedte (5 maal is $b = 8.6$ mm, 1 ml is $b = 8.7$, 2 ml = 9.0 en 1 ml = 9.1 mm). De dikte daarentegen is meestal groot ($th = 7.3$, 2 ml = 7.2, 2 ml = 7.1 en 2 ml = 7.0 mm; slechts 2 ml is $th = 6.6$ mm). De grote dikte wijst op overeenkomst van de uitgangsboon met bonen van de II-lijn.

In geen der gevallen is de formule van de gemiddelden der bonenopbrengsten ook 1B Th, cl 5. Eenmaal is deze formule L B Th, cl 1 (pl. 364, blz. 189). Eveneens eenmaal is ze L B th, cl 2 (pl. 1045, zie art. III in de volgende Proceedings). In deze 2 gevallen heeft de lengte de grenswaarde ($l = 12.8$ en $= 13.0$ mm) van de formule L B Th, cl 1.

In 7 van de totaal 9 gevallen is de formule van de gemiddelden 1b th, cl 8. Twee van deze gevallen (pl. 314 en 124) beschrijven we hier, de 5 overige bij cl 8.

De uitgangsboon voor pl. 314 (tab. 1 en 1a) is de laatste van de peul met 3 bonen, die alle de formule 1B th, cl 6 hebben, de lengte is niet zeer klein ($l = 12.5$ mm), die van

de eerste is nog kleiner ($l = 12.1$ mm). Van de bonenopbrengst van pl. 314 hebben de gemiddelde breedte en dikte de grenswaarde ($b_m = 8.4$, $th_m = 6.4$ mm). De grootste lengte van de bonen is $l = 13.6$ mm; de grootste dikte is $th = 7.0$ mm. Volgens de classificatie (tab. 1a) zijn er bonen in alle klassen, behalve in cl 2; veel in cl 8. Er zijn veel bonen met hoge LB-indices, zoals van bonen met de form. 1B Th (tab. 9). De bonenopbrengst beantwoordt hier aan de uitgangsboon: de uitgangsboon heeft min of meer het genotype van bonen met de form. 1B Th. Tab. 10 geeft inderdaad goede voorbeelden van bonen met de form. 1B Th, volgens de indices, in de bonenopbrengst van pl. 314.

Van pl. 124 is de uitgangsboon van pl. 33, F₃-1935. Ze is van een peul met 6 overeenkomstige bonen, alle met de form. 1B Th. Pl. 33 bevat meerdere dergelijke bonen (tab. 9). De formule van de gemiddelden van de bonenopbrengst van slechts 5 bonen van pl. 124 is 1b th, waarbij de breedte en de dikte de grenswaarden hebben, dus de formule ook als 1B Th geschreven kan worden. Van de 5 bonen van pl. 124 hebben er 2 de form. 1B Th, cl 5; de overige 3 hebben de indices van bonen met de form. 1B Th; de afmetingen zijn klein (tab. 10). Ook pl. 124 met haar uitgangsboon is een geval van de form. 1B Th, cl 5.

We vinden dus enkele gevallen, waar de uitgangsbonen en haar bonenopbrengsten wijzen op erfelijkheid van bonen met de form. 1B Th, cl 5, die gekenmerkt zijn door een hoge LB-index en een niet zeer hoge B Th-index.

Cl 6. In één geval (pl. 203, tab. 1 en 1a) is de formule van de uitgangsboon 1B th, cl 6. De formule van de gemiddelden van de bonenopbrengst is 1B Th, cl 5 (pl. 203, tab. 1 en 1a).

De uitgangsboon voor pl. 203 is van pl. 49, F₃-1935. De formule van de gemiddelden van pl. 49 is 1b Th. De gemiddelde afmetingen zijn niet groot. Volgens de classificatie van de grote bonenopbrengst van 96 bonen van pl. 49, zijn er 21 bonen in cl. 1, 6 in cl. 2, 10 in cl 3, 15 in cl 5 en 1 in cl 6, dat zijn allen bonen met een grote breedte. Er zijn er 23 in cl 7, dus bonen met een kleine breedte. De uitgangsboon voor pl. 49 is van pl. 66, F₂-1934; de formule is 1B Th, cl 5; de formule van de gemiddelden van pl. 66 is LB Th met een nogal grote gemiddelde breedte ($b_m = 8.9$ mm). De uitgangsboon van pl. 49 voor pl. 203 is van een peul met 5 bonen, waarvan de 4 overige resp., tot cl 5, 6, 7 en 8a behoren. Van de bonenopbrengst van pl. 203 is de grootste lengte $l = 14.3$ mm, (dan volgt $l = 13.4$ mm). Bijna alle bonen hebben een grote dikte; de grootste dikte is $th = 7.4$ mm. Volgens de classificatie (tab. 1a) zijn er veel bonen in cl 1 en 5; er is slechts één boon in cl 6. Volgens de samenstelling van de bonenopbrengst van pl. 203 en de gegevens van de ascendentie, is in dit geval de uitgangsboon, die van cl 6 is, zeer verwant aan cl 5 (vgl. ook tab. 10). De uitgangsboon heeft veel B-factoren in homozygote vorm.

Cl 7. De formule van de uitgangsboon is 1b Th, als van de bonen van de II-lijn. 26 gevallen, daarvan 16 met uitgangsbonen van éénzelfde plant (pl. 51).

Onder de uitgangsbonen zijn er enkele, zoals ze ook als hoge uitzondering bij bonen van de II-lijn van 1935 worden aangetroffen. Meestal is de lengte te groot; tienmaal is ze groter dan 12.5 mm (tab. 7b).

In één geval (pl. 206) is de formule van de gemiddelden van de bonenopbrengst ook 1b Th.

De uitgangsboon is van pl. 49, F₃-1935, evenals van de pl. 208 en 211. De grootste lengte van de bonen van de bonenopbrengst van pl. 206 is $l = 13.8$ mm, (dan volgt $l = 13.4$ mm). De bonen hebben alle een grote dikte (van enkele heeft ze de grenswaarde,

$th = 6.5, 6.5$ en 6.4 mm, vgl. tab. 8). De uitgangsbonen van pl. 49 voor de pl. 206, 208 en 211 hebben alle een vrij grote lengte ($l = 13.0, 12.7$ en 12.7 mm, vgl. tab. 7b). Volgens de classificatie behoren pl. 206, 208 en 211 tot het erfgebied van de II-lijn, doch de uitgangsboon is heterozygoot voor de form. 1b Th; er zijn veel dikte-factoren in de homozygote vorm.

In 4 gevallen (pl. 165, 201, 373 en 398) is de formule van de gemiddelden L B th, cl 2 (zie art. III in de volgende Proceedings). In 2 van deze gevallen (pl. 165 en 398) heeft de gemiddelde dikte de grenswaarde ($th_m = 6.5$ mm). Van de overige 21 van de 26 gevallen is de formule van de gemiddelden der bonenopbrengsten 1b th, cl 8. Daarvan vinden pl. 311 en 1038 bij cl 8 bespreking.

Van de 16 gevallen (pl. 980—1011), waar de uitgangsbonen van pl. 51 genomen zijn, bespreken we hier pl. 984 (tab. 1 en 1a; fig. 5); de overige spreken we bij cl 8.

Van de bonenopbrengst van 60 bonen van pl. 51, F₃-1935, zijn volgens de classificatie 18 bonen in cl 7. De gemiddelde lengte van deze bonen met de form. 1b Th is $l_m = 12.3$ mm. Een dergelijk groot gemiddelde komt onder de gemiddelde lengten van de bonenopbrengsten van de II-lijn van 1935 niet voor (tab. 2b).

Zoals de uitgangsboon van pl. 51 voor pl. 984 is, komen er als hoge uitzondering ook bonen van de II-lijn van 1935 voor (vgl. tab. 1 en tab. 7b). Ook bonenopbrengsten met gemiddelden als van pl. 984 komen bij bonenopbrengsten van de II-lijn van 1936 voor (tab. 4b). Bij de bonenopbrengst van pl. 984 staat aangetekend „als 983” en bij die van 983 „matig, nog groene peulen”. Het kan dus zijn, dat niet alle bonen hun volle wasdom bereikt hebben. Pl. 984 heeft wit gebloeid. De grootste lengte van de bonenopbrengst van 22 bonen is $l = 12.1$ mm, (dan volgt $l = 11.6$ mm). De grootste dikte is $th = 7.0$ mm, (dan volgt $th = 6.7$ mm, de kleinste dikte is $th = 5.9$ mm; vgl. tab. 8). Onder de individuele bonen is er geen, zoals ze ook niet bij de II-lijn van 1936 voorkomen. De classificatie (tab. 1a) komt overeen met de classificatie van pl. 102 van de II-lijn, waarvan ook de gemiddelden met die van pl. 984 overeenkomen. Toch zijn er ook hier kleine verschillen. De uitgangsboon van pl. 51 voor pl. 984 komt niet geheel en al met vergelijkbonen van de II-lijn overeen. De lengte is iets te groot. Slechts van een heel enkele boon komen alle 3 indices overeen met de 3 indices van een vergelijkboon van de II-lijn van 1936. De vergelijk-bonenopbrengst van de II-lijn (pl. 102, tab. 4b en fig. 6), d.i. dus die, waarvan de gemiddelden het best met die van pl. 984 overeenkomen, verschilt nog iets van pl. 984. Uit de frekwentie-krommen van de bonenopbrengst van pl. 984 en van pl. 102 (II-lijn) blijkt, dat de frekwentie-krommen van pl. 984 van het gewicht en van de lengte iets meer naar rechts liggen (fig. 7), die van de dikte ligt iets meer naar links, de curven van de indices liggen iets meer naar links. Er zijn dus meer bonen met een grote lengte en een groot gewicht en minder met een grote dikte in de bonenopbrengst van pl. 984 dan in die van pl. 102 en het aantal bonen met hoge indices is groter bij pl. 102 dan bij pl. 984.

Er zijn 2 gevallen (pl. 185 en 400), waar de formule van de gemiddelden van de bonenopbrengsten 1b Th, cl 7 is en de uitgangsboon de form. L B Th, cl 1 heeft (blz. 187).

De uitgangsboon voor pl. 185 is van pl. 46 ($l = 13.5$ mm). De formule van de gemiddelden van pl. 46, F₃-1935 is 1b th ($l_m = 12.8$ mm). De uitgangsboon voor pl. 46 is van pl. 66, F₂-1934. De formule is 1b th ($th = 6.0$ mm). De formule van de gemiddelden van pl. 66 is L B Th ($l_m = 13.6$ mm).

De formule van de gemiddelden van pl. 185 is 1b Th ($l_m = 12.0$ mm, $th_m = 6.6$ mm). De grootste lengte is $l = 13.4$ mm, de grootste dikte, $th = 7.1$ mm. Volgens de classificatie zijn er zeér veel bonen in cl 7. Pl. 185 wijkt af van de bonenopbrengsten van de II-lijn

van 1936. Er is in ons materiaal van de II-lijn van 1936 slechts één bonenopbrengst met $l_m = 12.0$ mm, en van deze bonenopbrengst is de gemiddelde dikte veel groter $th_m = 7.3$ mm, tab. 4b).

Van pl. 400 (tab. 1 en 1a) is de uitgangsboon van pl. 105. De formule van de gemiddelden van pl. 105 is 1 b th ($th_m = 6.5$ mm). De formule van de uitgangsboon van pl. 83

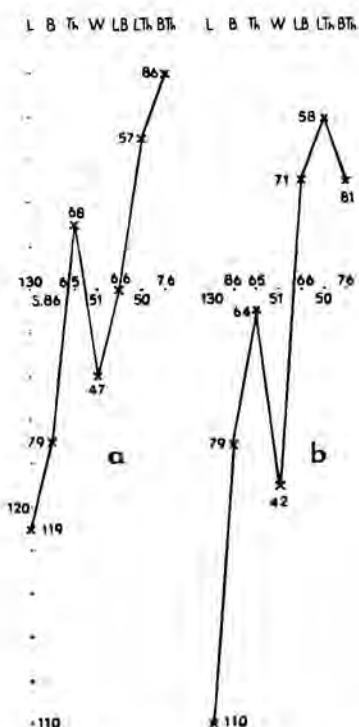


Fig. 5a. Characterogram of the initial bean 2 p. 3 b. of pl. 51 of F_3 -1935 for pl. 984, F_4 -1936.

Fig. 5b. Idem of the averages of pl. 984; $n = 22$.

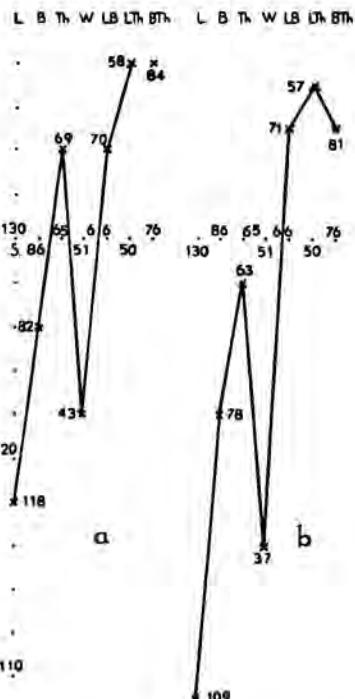


Fig. 6a. Characterogram of the initial bean 19 p. 3 b. of pl. 25 of the II-line of 1935.

Fig. 6b. Idem of the averages of pl. 102, II-line 1936; $n = 20$.

voor pl. 105 is LB Th, waarbij alle 3 afmetingen zeer groot zijn. De formule van de gemiddelden van pl. 83 is LB Th ($l_m = 14.2$ mm). Er is hier een groot verschil tussen het gewicht van de uitgangsboon ($W = 77$ cg) en het gemiddelde gewicht ($W_m = 62$ cg) van pl. 83 en het gemiddelde gewicht van de bonenopbrengst van pl. 105 ($W_m = 49$ cg). De grootste lengte van de bonen van pl. 400 is $l = 13.6$ mm, de grootste dikte is $th = 7.4$ mm. Volgens de classificatie zijn er vrij veel bonen in cl 7 met de form. 1 b Th. In de ascendentie is grote dikte. De bonenopbrengst van pl. 400 is samengesteld; de uitgangsboon van pl. 105 is niet homozygoot voor de form. 1 b Th, van cl 7, d.i., van bonen voor de II-lijn.

Van pl. 397 is de formule van de gemiddelden van de bonenopbrengst 1 b Th, cl 7 en die van de uitgangsboon LB th, cl 2; deze is van pl. 105 (evenals in het vorige geval, pl. 400). De bonenopbrengst bestaat uit slechts 7 bonen van het gebied van cl 1, form. LB Th.

Fig. 7, a—g. Frequency curves of the dimensions, the weight and the indices of pl. 984, F₄-1936, n = 22 and comparison frequency curves of pl. 102, II-line 1936, n = 19.

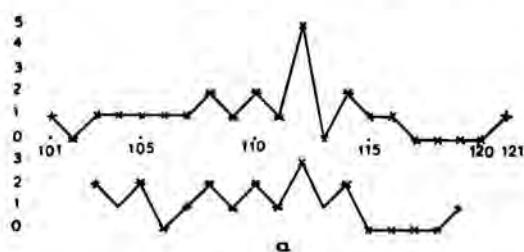


Fig. 7a. The length.

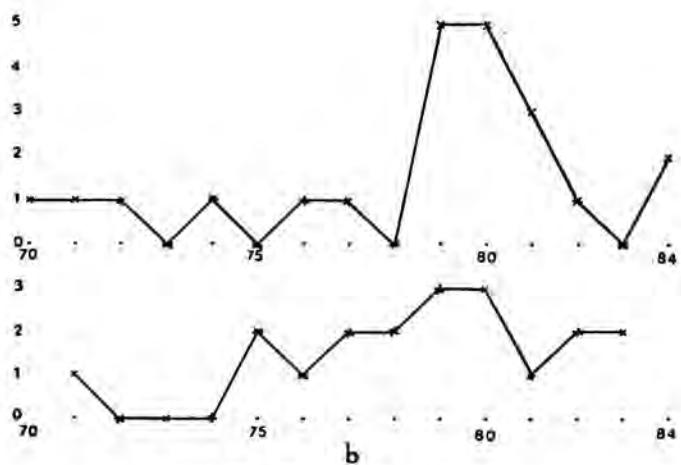


Fig. 7b. The breadth.

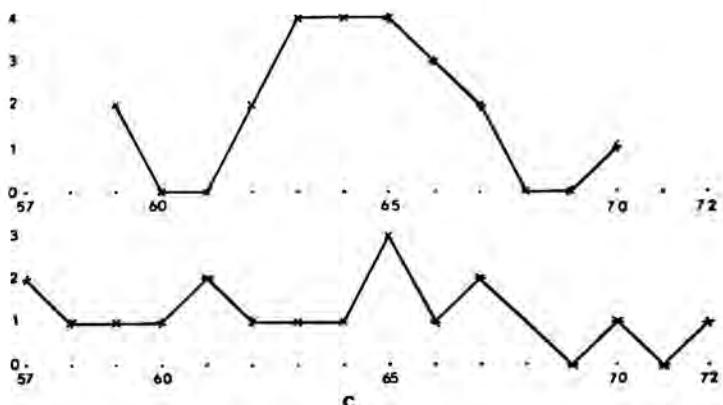


Fig. 7c. The thickness.

Er is één geval (pl. 123), waar de formule van de gemiddelden van de bonenopbrengst 1 b Th, cl 7 is en die van de uitgangsboon 1 b th, cl. 8. Deze is van pl. 32, F₃-1935.

Alle 4 bonen van de peul van de uitgangsboon hebben een grote dikte. De grootste lengte van de bonenopbrengst van pl. 123 is $l = 13.6$ mm, (dan volgen $l = 13.5, 13.2, 13.1$ mm). Er zijn dus enkele bonen met een grotere lengte dan we bij de bonen van de II-lijn van 1936 aantreffen. Volgens de classificatie zijn er veel bonen in cl 7, enige in cl 1, 3, 5 en 8b. Er zijn in deze bonenopbrengst geen bonen met een kleine dikte. Door de 3 bonen in cl 1 verschilt de classificatie van die van bonenopbrengsten van de II-lijn.

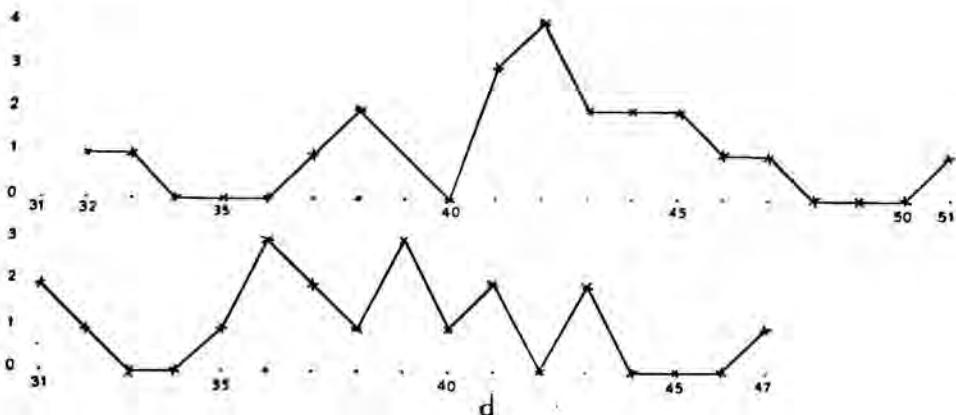


Fig. 7d. The weight.

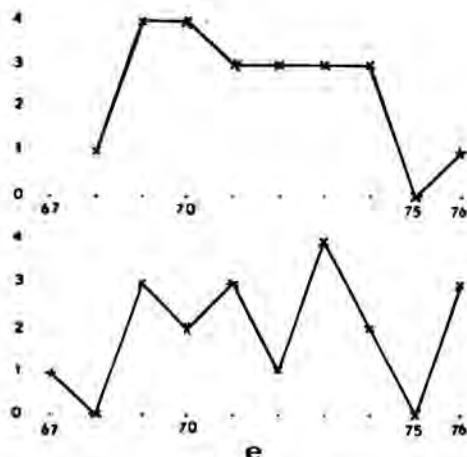


Fig. 7e. The L B-index.

De uitgangsboon van pl. 32 voor pl. 123 is in hoge mate homozygoot voor de formule 1b Th; de lengte bevat echter een enkele Ll- of LL-factorenverbinding.

In de groep gevallen van cl 7, form. 1b Th, vinden we enkele gevallen, die geheel of bijna geheel met bonenopbrengsten van de II-lijn overeenkomen.

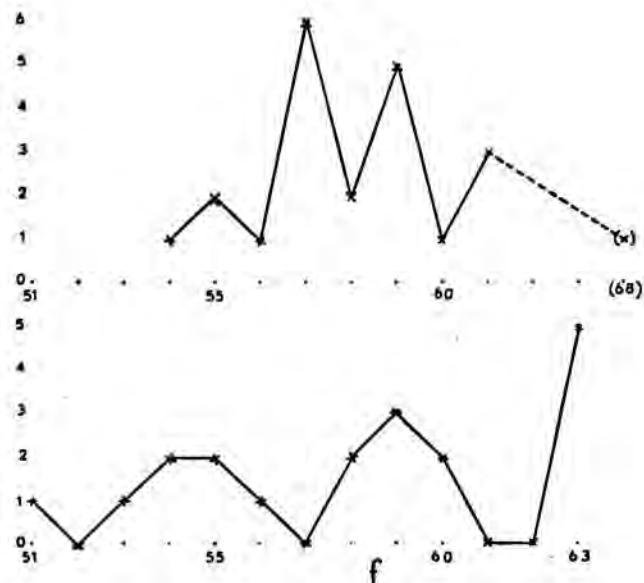


Fig. 7f. The L Th-index.

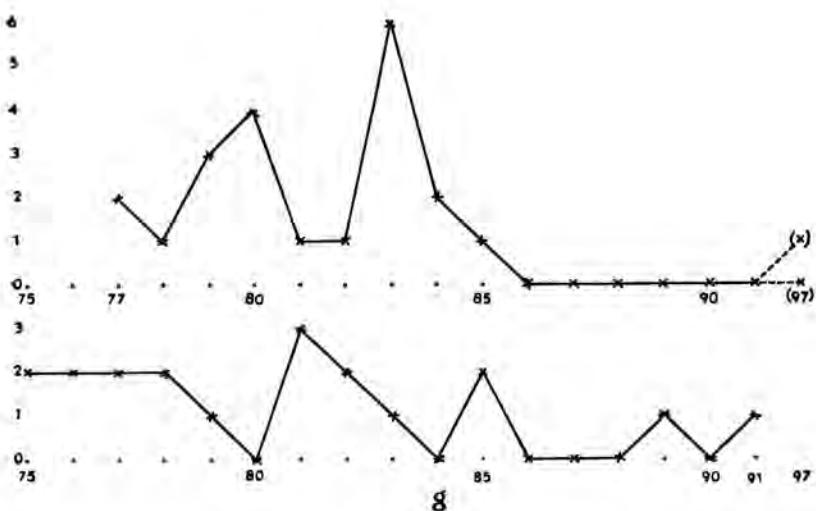


Fig. 7g. The B Th-index.

KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN
WETENSCHAPPEN

PROCEEDINGS

VOLUME LII

No. 3

Dep. President: J. M. BURGERS
Secretary: M. W. WOERDEMAN

1949

NORTH-HOLLAND PUBLISHING COMPANY
(N.V. Noord-Hollandsche Uitgevers Mij.)
AMSTERDAM

C O N T E N T S

Aerodynamics

BETCHOV, R.: Théorie non-linéaire de l'anémomètre à fil chaud. (Communicated by Prof. J. M. BURGERS), p. 195.

Botany

FRETS, G. P.: De F_4 -zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van Phaseolus vulgaris. III. (Communicated by Prof. J. BOEKE), p. 274.

Chemistry

SIXMA, F. L. J. and J. P. WIBAUT: Influence of temperature on the substitution type in the bromination and chlorination of aromatic compounds in the gas phase, p. 214.

Mathematics

ERDÖS, P. and J. F. KOKSMA: On the uniform distribution modulo 1 of lacunary sequences. (Communicated by Prof. J. G. VAN DER CORPUT), p. 264.

FUCHS, LADISLAS: Absolutes in partially ordered groups. (Communicated by Prof. J. G. VAN DER CORPUT), p. 251.

PRASAD, A. V.: A non-homogeneous inequality for integers in a special cubic field. (First communication.) (Communicated by Prof. J. G. VAN DER CORPUT), p. 240.

ROGERS, C. A.: The product of the minima and the determinant of a set. (Communicated by Prof. J. G. VAN DER CORPUT), p. 256.

Physics

GROENEWOLD, H. J.: Unitary Quantum Electron Dynamics II. (Communicated by Prof. F. A. VENING MEINESZ), p. 226.

Zoology

BRETSCHNEIDER, L. H.: An electron-microscopical study of bull sperm III. (Communicated by Prof. CHR. P. RAVEN), p. 301.

GRASVELD, MIEK S.: On the influence of various chlorides on maturation and cleavage of the egg of Limnaea stagnalis L. (Communicated by Prof. CHR. P. RAVEN), p. 284.

HOLTHUIS, L. B.: Zonophryxus dodecapus nov. spec., a remarkable species of the family Dajidae (Crustacea Isopoda) from the Canary Islands. (Communicated by Prof. C. J. VAN DER KLAUW), p. 208.

LEVER, J., J. MILTBURG and G. J. VAN OORDT: The effect of a short treatment with thiourea upon the fish thyroid gland. (Communicated by Prof. CHR. P. RAVEN), p. 296.

Aerodynamics. — *Théorie non-linéaire de l'anémomètre à fil chaud.* By R. BETCHOV. (Med. No. 61 uit het Laboratorium voor Aero- en Hydro-dynamica der Technische Hogeschool.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of February 26, 1949.)

Nous étudions ici les propriétés d'un anémomètre à fil chaud, en supposant que le transfert de chaleur du fil à l'air dépend d'une part de la différence de température et d'autre part du carré de cette différence. L'expérience confirme cette hypothèse et les conséquences peuvent être fort importantes. Cet effet de non-linéarité est plus grand que l'effet de conduction thermique.

I. *La loi de KING non-linéaire.*

La quantité de chaleur Q enlevée par seconde par un vent V à un fil de diamètre d et de longueur unité, est donnée par KING sous la forme:

$$Q = (a + b \sqrt{Vd}) T \quad \dots \dots \dots \quad (1)$$

où T désigne la différence de température entre le fil et l'air. Le calcul de KING, confirmé approximativement par l'expérience, donne:

$$a = \kappa' ; \quad b = \sqrt{2\pi\kappa'\delta'c'} \quad \dots \dots \dots \quad (2)$$

avec κ' = conductivité thermique de l'air, δ' = densité de l'air, c' = chaleur spécifique de l'air à volume constant.

Ces grandeurs peuvent varier avec T , et l'expérience montre que a augmente alors que b reste pratiquement constant. Intuitivement, on peut interpréter cet effet en disant que l'air en contact avec le fil est chauffé, ce qui augmente sa conductivité. En revanche, sa densité diminue, parce que la pression ne varie que très peu. Il faut admettre que les effets sur κ' et δ' se compensent, et que seul a varie.

Dans son mémoire original (Littérature, No. 1), KING signale que a augmente de 1,14 pour mille par degré; il décrit également un effet du diamètre sur ce terme a , effet dont nous ne discuterons pas ici. Il convient donc d'écrire:

$$Q = [a(1 + \gamma T) + b \sqrt{Vd}] T \quad \dots \dots \dots \quad (3)$$

où γ traduit la non-linéarité.

Il ne faut pas perdre de vue les hypothèses sur lesquelles repose le calcul de KING, qui admet que l'écoulement d'air se fait sans viscosité et que le flux de chaleur, tout près du fil, est constant. Il prend la chaleur spécifique

à volume constant, alors que la pression est certainement plus constante que la densité. C'est pourquoi nous considérons (3) comme une relation empirique, valable pour l'unité de longueur du fil, et nous souhaitons que le problème de KING fasse l'objet d'une étude plus approfondie.

Nous nous proposons ici d'étudier l'effet du terme γ sur les propriétés du fil chaud, et simplifions l'écriture en introduisant le nombre P tel que:

$$P = \frac{b}{a} \sqrt{Vd} \quad \dots \dots \dots \quad (4)$$

$$Q = a(1 + P + \gamma T) T \quad \dots \dots \dots \quad (5)$$

II. Equation générale du fil chaud.

Nous prendrons les symboles suivants:

S = résistance du fil, par unité de longueur, à la température de fonctionnement,

S_0 = résistance du fil, par unité de longueur, à la température de l'air ambiant,

I = intensité du courant électrique de chauffage du fil,

a = coefficient de variation de S selon la température,

m = masse du fil, par unité de longueur,

c = chaleur spécifique du fil, en joule/gramme et degré,

κ = coefficient de conductivité thermique du fil en watt/cm et degré,

σ = section du fil,

l = demi-longueur du fil,

t = temps,

x = coordonnée de position, variant de $+l$ à $-l$.

Nous posons:

$$A = \frac{a}{aS_0} (1 + P) = \frac{a + b \sqrt{Vd}}{aS_0} \quad \dots \dots \dots \quad (6)$$

L'équation du fil chaud doit exprimer l'équilibre entre la chaleur fournie par seconde, la chaleur de KING emportée par le vent, la chaleur nécessaire à éléver la température du fil et la chaleur transmise par conduction. On obtient:

$$SI^2 = A(S - S_0) + \frac{a\gamma}{a^2 S_0^2} (S - S_0)^2 + \frac{mc}{aS_0} \frac{\partial S}{\partial t} - \frac{\kappa\sigma}{aS_0} \frac{\partial^2 S}{\partial x^2} \quad \dots \quad (7)$$

En régime continu, et en introduisant les grandeurs:

$$\left. \begin{aligned} y &= x/l^* : \quad l^* = \sqrt{\frac{\kappa\sigma}{aS_0(A-I^2)}} : \quad Z = \frac{A-I^2}{S_0 I^2} (S - S_0) \\ G &= \frac{2}{3} \frac{a\gamma}{a^2 S_0} \frac{I^2/A^2}{(1-I^2/A)^2} = \frac{2}{3} \frac{\gamma}{a} \frac{1}{1+P} \frac{I^2/A}{(1-I^2/A)^2} \end{aligned} \right\} \quad (8)$$

on obtient l'équation (7) sous la forme:

$$Z + 3/2 G Z^2 - \frac{\partial^2 Z}{\partial y^2} = 1 \dots \dots \dots \quad (9)$$

III. Intégration exacte du cas statique.

En multipliant (9) par $\partial z / \partial y$ et intégrant, on obtient, avec une constante:

$$\partial z / \partial y = \sqrt{G Z^3 + Z^2 - 2 Z + cste} \dots \dots \dots \quad (10)$$

On voit que $\partial z / \partial y$ est nulle pour une valeur négative de Z et peut l'être pour deux valeurs positives de Z . Aux extrémités du fil, on a $S = S_0$ et $Z = 0$; au centre il faut que Z soit positif et $\partial z / \partial y$ nul. Le domaine qui nous intéresse est donc compris entre $Z = 0$ et la première racine positive donnant $\partial z / \partial y = 0$, que nous désignerons par $Z = B$. Posons:

$$Z(y) = B - X^2(y) \dots \dots \dots \dots \dots \quad (11)$$

En vertu de la relation:

$$G B^3 + B^2 - 2 B + cste = 0 \dots \dots \dots \dots \dots \quad (12)$$

on obtient:

$$4 (\partial x / \partial y)^2 = -G X^4 + E X^2 + D \dots \dots \dots \dots \quad (13)$$

avec:

$$E = 1 + 3 G B \quad ; \quad D = 2(1 - B) - 3 G B^2 \dots \dots \dots \quad (14)$$

Nous considérerons désormais B comme une nouvelle constante d'intégration, donnant la température au milieu du fil. Au centre du fil, on a $X = 0$ et:

$$4 (\partial x / \partial y)^2 = D \dots \dots \dots \dots \dots \dots \quad (15)$$

ce qui montre que D est positif, et en général petit. Aux extrémités du fil $Z = 0$ et $X = \pm \sqrt{B}$.

Les racines de l'équation (13) sont:

$$X^2 = \frac{-E \pm \sqrt{E^2 + 4 G D}}{-2 G} \dots \dots \dots \dots \dots \quad (16)$$

En introduisant le paramètre β , tel que:

$$\sin \text{hyp } \beta = sh \beta = \frac{2 \sqrt{G D}}{E} \dots \dots \dots \dots \dots \quad (17)$$

on peut mettre (13) sous la forme:

$$4 (\partial x / \partial y)^2 = G \left[\frac{E}{2 G} (ch \beta + 1) - X^2 \right] \left[\frac{E}{2 G} (ch \beta - 1) + X^2 \right]. \dots \dots \dots \quad (18)$$

Définissons l'angle φ , fonction de y , tel que:

$$X = \sqrt{\frac{D}{E ch \beta}} \frac{\sin \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \dots \dots \dots \dots \dots \quad (19)$$

avec:

$$k^2 = \frac{ch \beta + 1}{2 ch \beta} \quad \dots \quad (20)$$

L'équation (18) devient alors:

$$\frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{1}{2} \sqrt{E ch \beta} dy \quad \dots \quad (21)$$

et nous obtenons l'intégrale elliptique de première espèce:

$$U(k; \varphi) = \int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{1}{2} \sqrt{E ch \beta} y \quad \dots \quad (22)$$

La variable y varie de $-\xi$ à $+\xi$, avec:

$$\xi = l/l^* \quad \dots \quad (23)$$

et nous avons, pour $y = \xi$ et $X^2 = B$:

$$\frac{\xi}{2} \sqrt{E ch \beta} = \int_0^{\varphi_{\max}} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} \quad \dots \quad (24)$$

avec:

$$\sqrt{B} = \sqrt{\frac{D}{E ch \beta}} \frac{\sin \varphi_{\max}}{\sqrt{1-k^2 \sin^2 \varphi_{\max}}} \quad \dots \quad (25)$$

Cette dernière équation peut s'écrire:

$$\frac{1}{\sin^2 \varphi_{\max}} = k^2 + \frac{D}{EB ch \beta} \quad \dots \quad (26)$$

A partir de (20), on peut déduire:

$$\frac{1}{k^2} = 1 + Tgh^2(\beta/2) \quad \dots \quad (27)$$

Lorsque β va de 0 à $+\infty$, k^2 varie de 1 à 0,5.

Si l'on connaît donc G et B , on peut calculer successivement D , E , β , k et φ_{\max} . Une table de $U(k, \varphi)$ nous permet alors de calculer ξ . La figure 1 donne les résultats obtenus par ce procédé et permet, en partant d'un fil donné et en connaissant γ , de déterminer B à partir de G et de ξ .

La répartition de la température sur le fil est donnée par Z en fonction de y , soit par X fonction de φ et φ fonction de y . La relation $\varphi(y)$ est donnée par le quotient de (22) et (24), soit:

$$\frac{y}{\xi} = \frac{\int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}}{\int_0^{\varphi_{\max}} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}} \quad \dots \quad (28)$$

A partir de (19) et (25), on obtient une relation entre φ et X , soit:

$$\frac{X^2}{B} = \frac{\sin^2 \varphi}{\sin^2 \varphi_{\max}} \frac{1 - k^2 \sin^2 \varphi_{\max}}{1 - k^2 \sin^2 \varphi} \dots \quad (29)$$

La résistance totale R du fil est donnée par:

$$R = \int_{-l}^{+l} S dx = \frac{2S_0 I^2}{A - I^2} l \int_0^\xi Z dy + 2S_0 l \dots \quad (30)$$

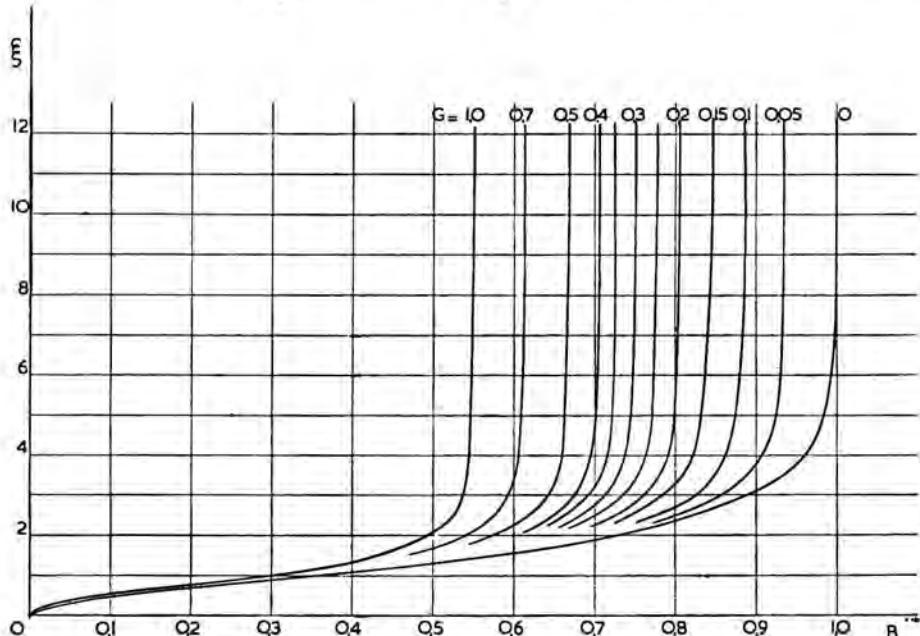


Fig. 1. Relation entre les grandeurs B , G et ξ .

Introduisons la résistance à froid R_0 et la fonction X :

$$R - R_0 = \frac{R_0 I^2}{A - I^2} \left(B - \frac{1}{\xi} \int_0^\xi X^2 dy \right) \dots \quad (31)$$

Remplaçons X selon (19) et dy selon (21), soit:

$$R - R_0 = \frac{R_0 I^2}{A - I^2} \left(B - \frac{2D}{\xi(E \operatorname{ch} \beta)^{3/2}} \int_0^{\varphi_{\max}} \frac{\sin^2 \varphi}{(1 - k^2 \sin^2 \varphi)^{3/2}} d\varphi \right) \dots \quad (32)$$

L'intégrale est égale à $2 \partial U / \partial k^2$ et nous obtenons:

$$R - R_0 = \frac{R_0 I^2}{A - I^2} \left(B - \frac{2D}{E \operatorname{ch} \beta} \frac{\partial U / \partial k^2}{U} \right) \dots \quad (33)$$

Les valeurs de $\partial U / \partial k^2$ peuvent être déduites d'une bonne table de $U(k, \varphi)$ avec une approximation suffisante.

Si le fil était parfait, l'expression entre () dans (33) serait à remplacer par l'unité; nous introduirons donc la grandeur M telle que:

$$M = 1 - B + \frac{2D}{E ch \beta} \frac{\partial U / \partial k^2}{U} \dots \dots \quad (34)$$

La formule (33) nous donne alors:

$$R = R_0 \frac{1 - MI^2/A}{1 - I^2/A} \dots \dots \quad (35)$$

et l'importante relation:

$$\frac{RI^2}{R - R_0} = A \left\{ 1 + \frac{M}{1-M} (1 - I^2/A) \right\} \dots \dots \quad (36)$$

Cette dernière équation permet un contrôle facile, parce que R , R_0 et I peuvent être mesurés avec précision, et que les courbes obtenues en fonction de R/R_0 par exemple, indiquent directement les effets de conduction et de non-linéarité. Nous avons calculé les valeurs de $M/1-M$, en fonction de G et ξ , et la figure 2 donne nos résultats.

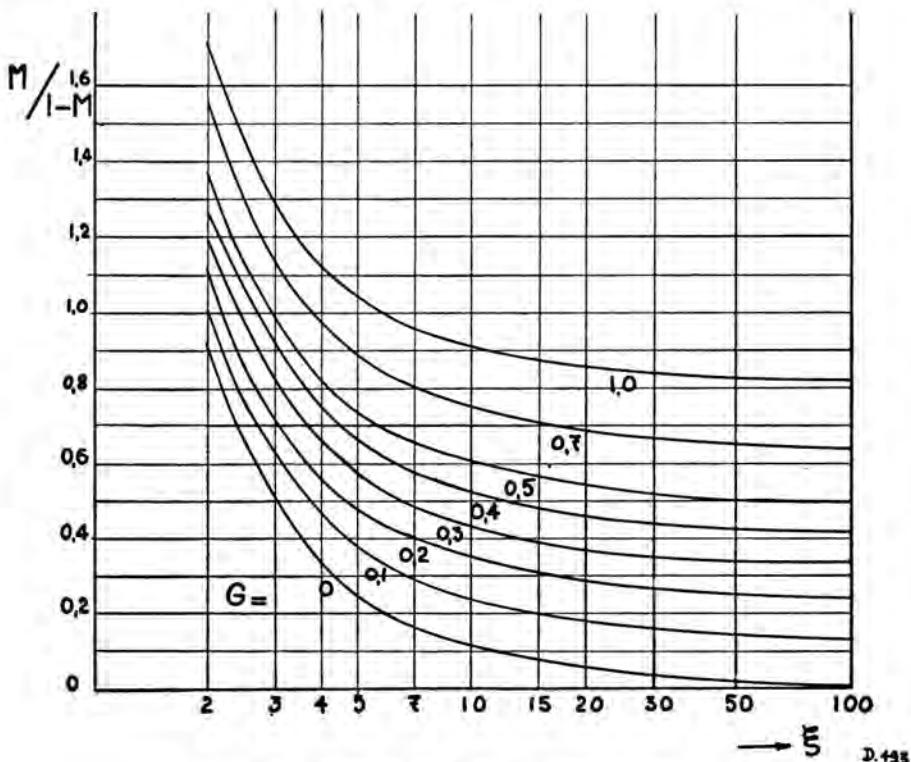


Fig. 2. Valeurs de $M/1-M$, selon G et ξ .

Une bonne approximation est donnée par:

$$\frac{M}{1-M} = \frac{1}{B_0} - 1 + \frac{1}{\xi - 1} \dots \dots \quad (37)$$

où B_0 correspond à B , dans le cas $\xi = \infty$, soit:

$$B_0 = \frac{+\sqrt{1+6G}-1}{3G} \quad (38)$$

IV. Quelques approximations utiles.

Nous avons remarqué, en effectuant les calculs nécessaires au tracé de la fig. 1, que l'on peut donner à k la valeur 1 sans introduire de grandes erreurs.

Cela implique $\beta = 0$ et (19) donne alors:

$$X = \sqrt{\frac{D}{E}} Tg \varphi \quad (39)$$

L'intégrale (22) devient:

$$U = \int_0^{\varphi} \frac{d\varphi}{\cos \varphi} = \frac{1}{2} L \frac{1 + \sin \varphi}{1 - \sin \varphi} = \frac{1}{2} \sqrt{E} y \quad (40)$$

d'où l'on déduit:

$$ch(2U) = \frac{1 + \sin^2 \varphi}{1 - \sin^2 \varphi} = ch(\sqrt{E} y) \quad (41)$$

$$X^2 = \frac{D}{2E} (ch \sqrt{E} y - 1) \quad (42)$$

Aux limites, on a:

$$B = \frac{D}{2E} (ch \sqrt{E} \xi - 1) \quad (43)$$

ce qui donne:

$$Z = \frac{B}{1 - 1/ch \sqrt{E} \xi} \left(1 - \frac{ch \sqrt{E} y}{ch \sqrt{E} \xi} \right) \quad (44)$$

Pour calculer M , il faut prendre:

$$\lim_{k \rightarrow 1} \partial U / \partial k^2 = \frac{1}{2} \int_0^{\varphi} \frac{\sin^2 \varphi}{\cos^3 \varphi} d\varphi = \frac{1}{4} \left(\frac{\sin \varphi}{\cos^2 \varphi} - U(\varphi) \right) . . . \quad (45)$$

ce qui donne:

$$M = 1 - \frac{B}{1 - 1/ch \sqrt{E} \xi} \left(1 - \frac{Th \sqrt{E} \xi}{\sqrt{E} \xi} \right) \quad (46)$$

On voit que la non-linéarité remplace ξ par $\sqrt{E} \xi$, et abaisse la température centrale.

Si l'on prend (44) comme solution de (9), on voit que l'équation est

satisfait à un terme en $(ch \sqrt{E} y / ch \sqrt{E} \xi)^2$ près, et que B est donné à peu près par:

$$B = \left(\frac{\sqrt{1+6G}-1}{3G} \right) (1 - 1/ch \sqrt{E} \xi) \dots \quad (47)$$

avec

$$E \approx \sqrt{1+6G} \dots \quad (48)$$

Prenons un exemple qui correspond à un cas extrême. Soit un fil de Pt avec 10 % d'Ir, diamètre de 7 microns, longueur $2l = 1,14$ mm. Exposé à un vent de 5 m/sec et chauffé avec 75 mA, il nous donne:

$$\begin{aligned} P &= 2,9 & A &= 1,2 \cdot 10^{-2} & I^2/A &= 0,47 \\ l^* &= 0,15 \text{ mm} & \xi &= 3,75. \end{aligned}$$

En prenant $\gamma = 1,2 \cdot 10^{-3}$, et à l'aide des fig. 1 et 2 nous avons déterminé:

$$G = 0,25 ; B = 0,76 ; E = 1,56 ; M = 0,4.$$

Les autres paramètres ont les valeurs calculées suivantes:

$$\begin{aligned} k &= 0,99756 & \varphi_{\max} &= 79,5 \text{ degrés} ; \\ sh \beta &= 0,14 & D &= 0,0477 & U &= 2,34. \end{aligned}$$

En fig. 3 nous donnons le profil des températures, calculé exactement, le profil calculé avec l'approximation $k = 1$, et le profil calculé avec $\gamma = 0$.

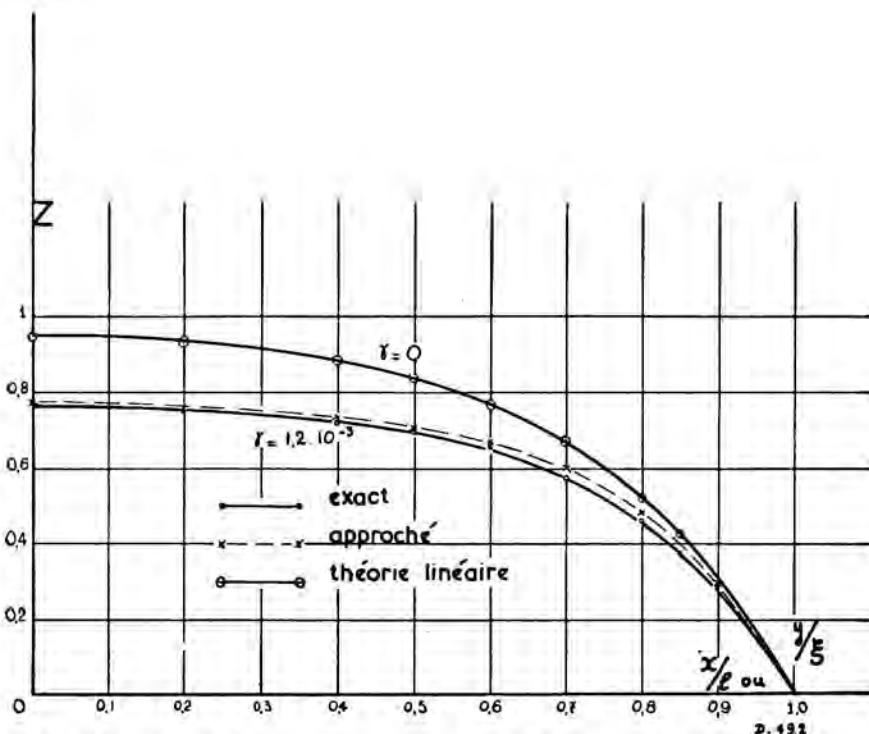


Fig. 3. Répartition de la température, avec un fil de 7 microns, selon la théorie linéaire et selon nos formules.

On peut remarquer que la non-linéarité donne une distribution plus régulière de la température, et que l'approximation est suffisante.

A partir de (35), on calcule $R/R_0 = 1,5$, alors que le calcul avec $\gamma = 0$ donnerait 1,9. La température moyenne donnant $R/R_0 = 1,5$ serait 380 degrés.

Quant au terme $RI^2/R - R_0$, il passe de la valeur 1,24 A, lorsque le courant est très faible, à la valeur 1,35 A lorsque I atteint 75 mA. Il varie donc de environ 10 % entre $R/R_0 = 1$ et $R/R_0 = 1,5$, ce qui est de l'ordre des variations observées.

Ainsi, cette grandeur est constante à 10 %, et la loi de KING est vérifiée, mais nous verrons plus loin que l'inertie thermique est très différente de la valeur attendue.

V. L'équation dynamique du fil chaud.

Pour calculer la variation de la résistance totale du fil, lorsque le courant ou le vent fluctuent, il faut remonter à l'équation (7). En y remplaçant I , S et V (contenu dans le terme A), par $I + i e^{j\omega t}$, $S + s e^{j\omega t}$ et $V + v e^{j\omega t}$ on obtient, après suppression des termes d'ordre zéro, deux et plus, ainsi que du facteur $e^{j\omega t}$:

$$2SIi + I^2s = \frac{aP}{2aS_0} \frac{v}{V} (S - S_0) + As + \frac{2a\gamma}{a^2 S_0^2} (S - S_0)s + \left. \begin{array}{l} \\ \\ \end{array} \right\} \dots \quad (49)$$

$$+ \frac{mc}{aS_0} j \omega s - \frac{\kappa\sigma}{aS_0} \frac{\partial^2 s}{\partial x^2}.$$

Introduisons:

$$z = s \frac{A - I^2}{S_0 I^2} \quad \dots \quad (50)$$

et, identique à la formule (9) de notre Med. 55:

$$\omega^* = \frac{aS_0}{mc} (A - I^2). \quad \dots \quad (51)$$

On obtient alors, après introduction de Z :

$$2 \frac{i}{I} + 2 \frac{i}{I} \frac{I^2/A}{1-I^2/A} Z - \frac{1}{2} \frac{P}{1+P} \frac{Z}{1-I^2/A} \frac{v}{V} = z(1+3GZ+j\omega/\omega^*) - \frac{\partial^2 z}{\partial y^2}. \quad (52)$$

La fonction $Z(y)$ figure deux fois dans cette équation différentielle, et il nous faut utiliser l'approximation $k = 1$ pour éviter de grandes complications. En prenant l'expression de Z selon (44), on obtient:

$$-\frac{\partial^2 z}{\partial y^2} + z \left(1 + \frac{3GB}{1-1/ch\sqrt{E}\xi} + j\omega/\omega^* - \frac{3GB}{ch\sqrt{E}\xi-1} ch\sqrt{E}y \right) = C_1 + C_2 \frac{ch\sqrt{E}y}{ch\sqrt{E}\xi} \quad (53)$$

avec:

$$C_1 = 2 \frac{i}{I} \left(1 + \frac{I^2/A}{1-I^2/A} \frac{B}{1-1/ch \sqrt{E} \xi} \right) - \frac{1}{2} \frac{v}{V} \left(\frac{P}{1+P} \frac{1}{1-I^2/A} \frac{B}{1-1/ch \sqrt{E} \xi} \right) \quad (54)$$

$$C_2 = -2 \frac{i}{I} \frac{1^2/A}{1-I^2/A} \frac{B}{1-1/ch \sqrt{E} \xi} + \frac{1}{2} \frac{v}{V} \left(\frac{P}{1+P} \frac{1}{1-I^2/A} \frac{B}{1-1/ch \sqrt{E} \xi} \right). \quad (55)$$

La solution sans second membre de (53) est une fonction de MATHIEU, prise pour une valeur purement imaginaire de l'argument. Mais si $\sqrt{E} \xi$ est assez grand, supérieur à 4 par exemple, on peut négliger le terme en $ch \sqrt{E} y$ du membre de gauche de (53). En effet, il n'est important que lorsque y est voisin de ξ ; mais nous prendrons comme condition aux limites $z(\xi) = 0$, et le produit $z ch \sqrt{E} y$ ne sera jamais important. Nous négligerons également le terme $1/ch \sqrt{E} \xi$ dans le dénominateur d'un des termes de gauche de (53), ce qui revient à prendre $Z = B$ dans le terme facteur de z .

Compte tenu de la définition de E , on a alors:

$$-\frac{\partial^2 z}{\partial y^2} + z(E + j\omega/\omega^*) = C_1 + C_2 \frac{ch \sqrt{E} y}{ch \sqrt{E} \xi} \quad . . . \quad (56)$$

dont la solution, nulle pour $y = \pm \xi$, est:

$$z = \frac{C_1}{Ep^2} \left(1 - \frac{ch p \sqrt{E} y}{ch p \sqrt{E} \xi} \right) + \frac{C_2}{j\omega/\omega^*} \left(\frac{ch \sqrt{E} y}{ch \sqrt{E} \xi} - \frac{ch p \sqrt{E} y}{ch p \sqrt{E} \xi} \right) \quad . . \quad (57)$$

avec

$$p = \sqrt{1 + j\omega/E \omega^*} \quad \quad (58)$$

On remarque que la non-linéarité remplace la fréquence propre ω^* par une nouvelle fréquence plus élevée d'un facteur E (Comparer avec (8) de notre article donné sous Littérature, No 5). Les amplitudes selon C_1 et C_2 sont aussi modifiées par la non-linéarité.

La variation r de la résistance totale R sera:

$$r = \int_{-l}^{+l} s dx = \frac{R_0 I^2}{A - I^2} \frac{1}{\xi} \int_0^\xi z dy \quad \quad (59)$$

L'intégration donne:

$$r = \frac{R_0 I^2}{A - I^2} \left\{ \frac{C_1}{Ep^2} \left(1 - \frac{Tgh p \sqrt{E} \xi}{p \sqrt{E} \xi} \right) + \frac{C_2}{j\omega/\omega^*} \frac{1}{\sqrt{E} \xi} \left(Tgh \sqrt{E} \xi - \frac{Tgh p \sqrt{E} \xi}{p} \right) \right\} \quad (60)$$

Nous avons supposé plus haut que $\sqrt{E} \xi$ était assez grand, aussi pouvons nous donner aux tangentes hyperboliques la valeur 1 (la présence d'un argument complexe est ici sans importance).

On obtient alors:

$$r = \frac{R_0 I^2}{A - I^2} \left\{ \frac{C_1}{E p^2} (1 - 1/p \sqrt{E} \xi) + \frac{C_2}{j \omega / \omega^*} \frac{1}{\sqrt{E} \xi} \left(\frac{p-1}{p} \right) \right\}. \quad . . . \quad (61)$$

VI. La réponse à une fluctuation du courant.

Prenons $v = 0$ dans (54) et (55), on peut alors transformer (61) en supprimant les termes en $1/ch \sqrt{E} \xi$:

$$rI = \frac{2i R_0 I^2 / A}{(1 - I^2 / A)^2} \left\{ \frac{(1 - I^2 / A(1-B))(1 - 1/p \sqrt{E} \xi)}{E p^2} - \frac{BI^2 / A}{\sqrt{E} \xi} \frac{\omega^* \left(\frac{p-1}{p} \right)}{j \omega} \right\}. \quad . . . \quad (62)$$

Cette formule donne la tension alternative produite aux bornes du fil chaud par le courant de modulation i , en plus de la tension normale Ri . Bien que cette formule paraisse compliquée, elle peut être adaptée aux besoins pratiques. Si la fréquence $\omega / 2\pi$ tend vers l'infini, (62) devient:

$$rI = \frac{2i R_0 I^2 / A}{(1 - I^2 / A)^2} \frac{(1 - I^2 / A(1-B) - I^2 / A B / \sqrt{E} \xi)}{j \omega / \omega^*}. \quad . . . \quad (63)$$

Compte tenu de (35) donnant R , (46) donnant M et (51) donnant ω^* , on trouve:

$$rI = \frac{-j}{\omega / 2\pi} i C R I^2 \quad \text{avec} \quad C = \frac{\alpha S_0}{\pi m c}. \quad . . . \quad (64) \quad (65)$$

La constante C correspond à celle de nos précédentes publications, et (64) montre que la tension rI prise en haute fréquence permet de mesurer C , sans être gêné par la conduction ni la non-linéarité. La méthode décrite sous Litt. 4 est donc plus indiquée que celle qui consiste à mesurer des déphasages, avec ω de l'ordre de ω^* .

On peut directement vérifier ce point en prenant ω très grand dans l'équation (49), ce qui rend négligeable l'effet des termes de conduction et de non-linéarité.

Lorsque ω tend vers zéro, la tension devient:

$$rI = \frac{2i R_0 I^2 / A}{(1 - I^2 / A)^2} \frac{1}{E} \left\{ 1 - \frac{I^2}{A} M - \frac{1}{\sqrt{E} \xi} (1 - I^2 / A(1 - \frac{1}{2}B)) \right\}. \quad . . . \quad (66)$$

La fonction complexe rI selon (62) doit donc passer de (64) à (66) lorsque ω varie de 0 à une grande valeur. Le tracé complexe de cette fonction donne pratiquement un demi-cercle, ce qui permet de poser approximativement:

$$rI = \frac{rI(\omega = 0)}{1 + j \omega / \omega^{**}} \quad . . . \quad (67)$$

où ω^{**} désigne la fréquence propre effective. En fig. 4 nous avons tracé le demi-cercle et indiqué quelques valeurs de ω / ω^{**} . A partir de (62), et dans le cas de l'exemple ci-dessus, nous avons calculé les tensions rI pour

quelques valeurs de $\omega/E\omega^*$. On voit que les deux fonctions se confondent, en basses fréquences, si l'on prend $\omega^{**} = 1,1 E \omega^* = 1,72 \omega^*$. En haute fréquence, (64) et (67) doivent être égales, ce qui permet de calculer une

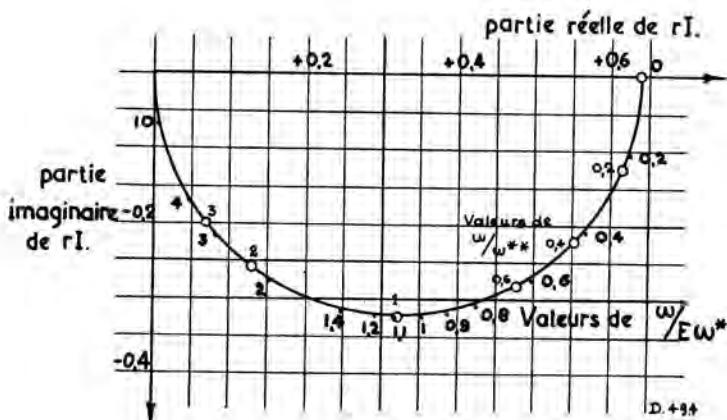


Fig. 4. Etude de la tension rI .

Le demi-cercle correspond à la formule (67); les points donnent rI , selon (62), pour quelques valeurs de $\omega/E\omega^*$.

valeur de ω^{**} satisfaisante, lorsque l'effet d'inertie thermique est important. On obtient:

$$\omega^{**} = \frac{E \omega^*}{1 - 1/\sqrt{E} \xi \left\{ \frac{1 - I^2/A (1 - 1/2 B)}{1 - I^2/A M} \right\}} \quad \dots \quad (68)$$

Dans le cas traité on trouve $\omega^{**} = 1,23 E \omega^*$, soit $\omega^{**} = 1,92 \omega^*$: la fréquence propre effective est presque le double de la valeur attendue. L'approximation (67) donne donc un tracé correcte de la fonction, mais les phases selon (68) ne seront exactes qu'à 10 %.

Le dénominateur de (68) dépend principalement de ξ , et il élève la fréquence propre. Au lieu de se compenser, comme dans le cas statique, les deux effets réduisent l'inertie thermique.

Intuitivement, on peut dire que la conduction raccourcit la partie chaude du fil, et réduit ainsi la chaleur nécessaire pour modifier la température centrale; la non-linéarité tient à la présence d'air chaud autour du fil, et l'inertie thermique de l'air est négligeable, ce qui améliore la réponse globale de l'anémomètre.

Lorsque le fil est poussiéreux, la quantité d'air immobile est plus forte, et l'expérience montre que le terme a en (3) est élevé, alors que b reste inchangé. Il faut donc prévoir une action dynamique de la poussière du fil, dans la mesure où E est modifié. L'effet dynamique peut être plus important que l'effet statique.

Le fil de l'exemple cité nous montre que, avec un terme $R I^2/R - R_0$

constant à 10 %, la fréquence propre peut être presque double de la valeur normalement prévue.

VII. La réponse à une fluctuation du vent.

En prenant $i = 0$ dans les formules (54) et (55), et en maintenant v , on peut transformer (61) on:

$$rI = -\frac{1}{2} \frac{v}{V} \frac{P}{1+P} \frac{R_0 I^3/A}{(1-I^2/A)^2} \left\{ \frac{B}{1-1/ch \sqrt{E} \xi} \right. \\ \left. \left\{ \frac{1-1/p \sqrt{E} \xi}{E p^2} - \frac{\omega^* p-1}{j \omega p} \frac{1}{\sqrt{E} \xi} \right\} \right\} \quad \dots \quad (69)$$

Avec ω tendant vers zéro, on a:

$$rI = -\frac{1}{2} \frac{v}{V} \frac{P}{1+P} \frac{R_0 I^3/A}{(1-I^2/A)^2} \frac{B}{1-1/ch \sqrt{E} \xi} \left\{ \frac{1-\frac{3}{2} \frac{1}{\sqrt{E} \xi}}{E} \right\} \quad \dots \quad (70)$$

Si ω tend vers l'infini, on a:

$$rI = -\frac{1}{2} \frac{v}{V} \frac{P}{1+P} \frac{R_0 I^3/A}{(1-I^2/A)^2} \frac{B}{1-1/ch \sqrt{E} \xi} \left\{ \frac{1-\frac{1}{\sqrt{E} \xi}}{j \omega / \omega^*} \right\} \quad \dots \quad (71)$$

et on peut remplacer approximativement (69) par le demi-cercle:

$$rI = \frac{rI(\omega=0)}{1+j\omega/\omega^{**}} \quad \dots \quad (72)$$

avec:

$$\omega^{**} = E \omega^* \frac{1-1/\sqrt{E} \xi}{1-\frac{3}{2} 1/\sqrt{E} \xi} \quad \dots \quad (73)$$

ce qui dans le cas de l'exemple traité donne $\omega^{**} = 1,81 \omega^*$.

Il n'y a donc pas, en principe, égalité entre la réponse dynamique à une variation i et à une variation v .

Cette difficulté est introduite par le terme $\sqrt{E} \xi$, soit par la conduction; et la non-linéarité tend à en diminuer l'importance (facteur \sqrt{E}).

Nous espérons publier prochainement quelques résultats d'expériences, et le calcul des écarts signalés par divers auteurs.

LITTERATURE.

1. L. V. KING, On the convection of heat from small cylinders Phil. Trans. of the Royal Society of London. Series A, Vol. 214, 373—432 (1914).
2. L. F. G. SIMMONS, Note on errors arising in measurements of turbulence, Aeron. Research Committee, Technical Report, R. & M. 1919 (4042), 1939.
3. R. BETCHOV et E. KUYPER, Un amplificateur pour l'étude de la turbulence, Med. 50, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 50, 1134—1141 (1947).
4. R. BETCHOV, L'inertie thermique des anémomètres à fils chauds, Med. 54, ibid. 51, 224—233 (1948).
5. ———, L'influence de la conduction thermique sur les anémomètres à fils chauds, Med. 55, ibid. 51, 721—730 (1948).

Zoology. — *Zonophryxus dodecapus nov. spec., a remarkable species of the family Dajidae (Crustacea Isopoda) from the Canary Islands.*
By L. B. HOLTHUIS. (Communicated by Prof. C. J. VAN DER KLAUW.)

(Communicated at the meeting of February 26, 1949.)

(31st Contribution to the Knowledge of the Fauna of the Canary Islands edited by Dr. D. L. UYTENBOOGAART, continued by Dr. C. O. VAN REGTEREN ALTENA.)¹⁾

During a trip to the Canary Islands in the spring of 1947, several Caridean Crustacea were collected by Dr. G. THORSON of Universitetets Zoologiske Museum, Copenhagen. Dr. THORSON was so kind as to entrust me with the study of this material, a report on which is now in the press.

During the examination of the Caridea, a number of Isopod parasites of *Parapandalus narval* (Fabr.) were found in a jar containing several specimens of this pandalid prawn. One of the parasites was still attached to the body of its host. Examination of these Isopods showed them to belong to a new species of the genus *Zonophryxus*, which proved to be highly interesting in many respects. The present paper deals with this new species.

Zonophryxus dodecapus nov. spec.

Material examined: Los Cristianos, Tenerife, Canary Islands; on specimens of *Parapandalus narval* (Fabr.); rocky bottom, depth 210 m; March 26, 1947; G. THORSON leg. — 5 ♀♀ and 2 ♂♂.

Situation: The female is situated on the middorsal region of the posterior half of the carapace of the Decapod prawn *Parapandalus narval* (Fabr.) (= *Parapandalus pristis* (Risso)). The longitudinal axis of the parasite is parallel to that of its host, but the head of the parasite is directed to the tail end of the host.

Size: The females are 9 to 12 mm long, 5 to 7 mm broad, and 2 to 3 mm high. The males are about 3 mm long.

♀. The body of the female is oval in outline, with the posterior end truncate. The dorsal surface (fig. 1b) is very convex, especially in the posterior and lateral parts, where the surface almost vertically (or even with a distinctly convex curve) dips down to the ventro-lateral border; anteriorly it slopes more gradually to this border. On the dorsal surface the various thoracic segments are rather distinctly defined. The second and third segments are least distinctly indicated, but the fourth to sixth are well marked. The first segment is completely fused with the cephalon, while the separation of the seventh thoracic segment from the abdomen is obscure,

¹⁾ Contribution 26 to 30 are published in *Tijdschrift voor Entomologie*, vol. 91.

if visible at all. The ventro-lateral border of the body (fig. 1a), which is one of the most characteristic features of the present genus, is oval, with the posterior end truncate. Small notches in this border are visible on the lines separating the various thoracic segments. The anterior half of the

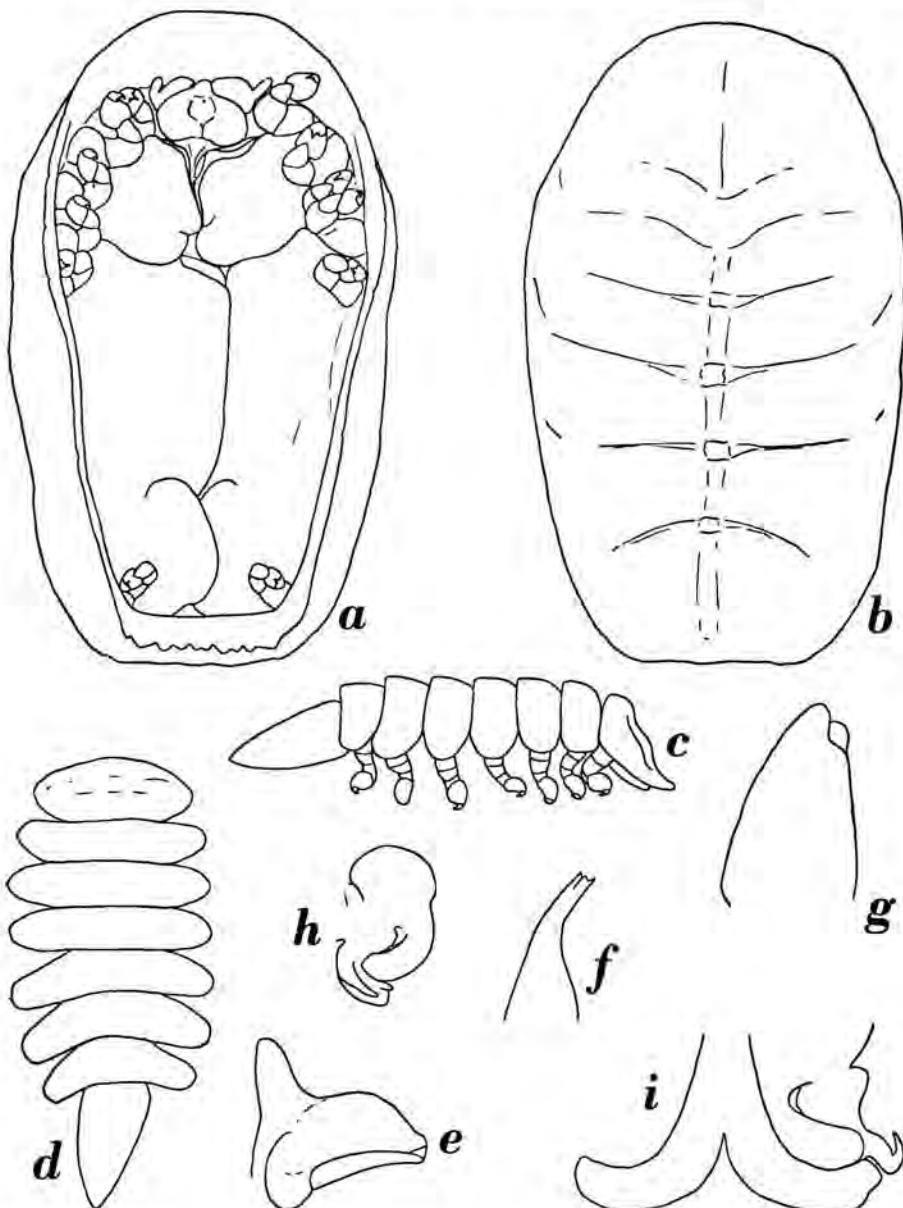


Fig. 1. *Zonophryxus dodecapus* nov. spec. a, body of female in ventral view; b, body of female in dorsal view; c, body of male in lateral view; d, body of male in dorsal view; e, internal and external antennae; f, mandible; g, left plate of lower lip; h, maxilliped; i, sternum in ventral view, with part of left maxilliped showing. a, b, $\times 7$; c-f, i, $\times 25$; g, $\times 50$; h, $\times 10$. (e-i of a female specimen.)

border shows four such notches at each side. The anterior of these notches is situated on the line separating the cephalon (with which the first thoracic segment is fused) from the second thoracic segment; the other three notches are placed on the lines which separate the second segment from the third, the third from the fourth, and the fourth from the fifth segment. In the posterior half of each lateral side of the border only one notch is visible: this notch is situated in about the middle of the posterior half of the border, on the line separating the fifth from the sixth thoracic segments. The truncated posterior part of the border bears 10 rather irregular teeth, which in one of the specimens are somewhat inconspicuous. As already remarked by RICHARDSON (1906) these teeth probably define the various abdominal segments. The ventral side of the body (fig. 1a) is strongly concave. This concavity is almost entirely covered by the oostegites of the pereiopods, so that it is not distinct at first, but may be seen when the oostegites are removed. The external antenna (fig. 1e) consists of a single papilliform joint. The internal antennae are formed by two broad lobes, which enclose the oral cone. Of the mouth parts the mandible (fig. 1f) is slender and ends in three small teeth. Behind and pressed against the oral cone are two oval plates, which touch each other in the middle and have a lobe at the inner side near the top. These two plates (one of which is figured here, fig. 1g) are considered by BARNARD to form the lower lip. Between the thoracic legs the ventral surface of the body bears the sternal process (fig. 1i), which is covered by the oostegites of the first pair of pereiopods. This process is anchor-shaped, the two flukes are curved and directed laterally, while the top is directed anteriorly; the posterior margin of this anchor is deeply incised in the middle. At each side of the sternal process and pressed close to the inner curve of the flukes lie the maxillipeds (fig. 1h). The maxillipeds consist of a thin anterior plate, which posteriorly ends in two broad curved processes, the inner of which curves outwards, the outer curves inwards, the processes partly covering each other. The outer process bears a sort of palp. BARNARD (1913) mentions the presence in one of his specimens of *Zonophryxus quinquedens* of a maxilla, which should be situated below the maxilliped. No such maxilla could be found by me in the specimens examined. There are six pairs of pereiopods. In this feature the present species differs from all Dajidae, except for the species *Paradajus tenuis* Nierstrasz & Brender à Brandis. The first five pereiopods are placed close together in the anterior half of the body, the sixth is situated in the posterolateral angle of the body. The pereiopods all are of the same shape, the sixth pair being slightly smaller than the others. Each leg ends in a strong unguis and is provided at the base with an oostegite (= marsupial plate = incubatory lamella). The oostegite of the first leg (figs. 2a, b) not only covers the maxillipeds, but also the oostegite of the second leg. This first oostegite is very curious in that it consists of various lamellae. When seen in situ only the ventral lamella is visible. This lamella is about as long as broad, is of an irregular

L. B. HOLTHUIS: *Zonophryxus dodecapus* nov. spec., a remarkable species of the family Dajidae (Crustacea Isopoda) from the Canary Islands.

PLATE I



Fig. a. *Parapandalus narval* (Fabr.) parasitized by *Zonophryxus dodecapus* nov. spec.
 $\times 0.8$.



Fig. b. *Zonophryxus dodecapus* nov. spec. shown on carapace of *Parapandalus narval* (Fabr.). $\times 3$.

outline and possesses a deep incision in the posterior part of the inner margin. A second lamella, which is about as broad as the first, is fused with its base to the first lamella along a line running transversely over the middle of the ventral surface of the first lamella. The second lamella

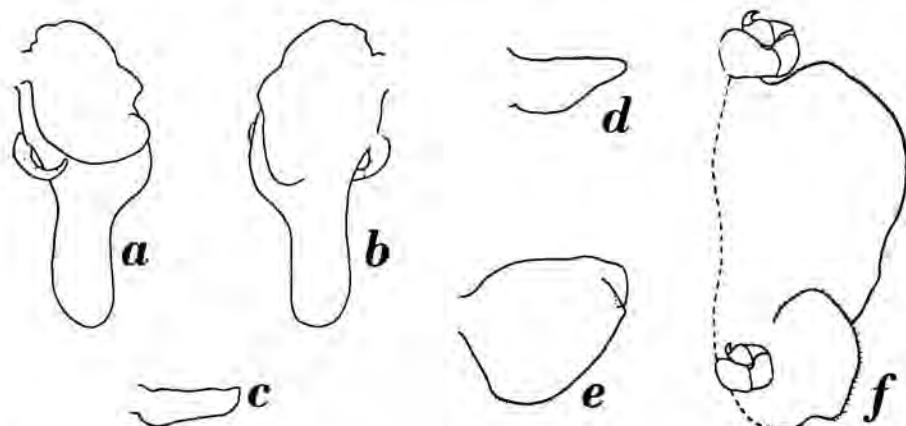


Fig. 2. *Zonophryxus dodecapus* nov. spec., female. a, oostegite of first pereiopod, ventral view; b, same, dorsal view; c, oostegite of second pereiopod; d, oostegite of third pereiopod; e, oostegite of fourth pereiopod; f, fifth and sixth pereiopods with their oostegites. a-f. $\times 10$.

posteriorly reaches far beyond the first, its tip is tongue-like narrowed. A third, smaller, lamella is fastened to the basal part of the underside of the second. Near the base of the second lamella a palp-like process is visible; this palp curves over the lateral margin of the second lamella, and in it very faint indications of a possible segmentation may be seen. The nature of this palp, which is not mentioned in any of the other descriptions of species of *Zonophryxus*, is not known to me. The oostegite of the second pereiopod (fig. 2c) is elongate in shape, it is directed towards the middle of the body and is entirely covered by the first oostegite. The oostegite of the third pair of legs (fig. 2d) is slightly larger than that of the second, it is more triangular, and it is covered by the oostegites of the fourth and fifth legs. The fourth oostegite (fig. 2e) is broadly laminar, being slightly broader than long, it is almost completely covered by the fifth oostegite. The latter (fig. 2f) is very large: the oostegites of the fifth legs together cover the larger part of the posterior half of the ventral side of the body. The fifth oostegite gradually narrows posteriorly, it has the outer margin entirely fused with the body wall, while the posterior margin partly is fused with the oostegite of the sixth leg. The sixth leg is situated in the postero-lateral angle of the body and bears a rounded oostegite, which is fused partly with the lateral and posterior parts of the body wall and partly with the fifth oostegite. No pleopods are present. The three raised pads in the extreme posterior end of the ventral surface of the body mentioned by

BARNARD (1913, p. 229) for *Zonophryxus quinquedens*, are visible in my specimens too.

♂. The male (figs. 1c, d) has the body flattened. The head is broad, being slightly narrower than the thoracic segments. The upper surface of the head is somewhat concave. The first thoracic segment is fused with the cephalon. The other six thoracic segments are free and distinct, the posterior segments are somewhat narrower than the anteriors. The abdomen is unsegmented, it is elongate triangular, with the greatest breadth slightly posterior of the base and regularly tapering towards the top. The seven pereiopods are uniform in shape.

Remarks. The genus is characterized by the curious ventro-lateral border, which surrounds the entire ventral surface of the female, and which invariably bears 9 to 11 teeth on the posterior margin. At present there are six species known to belong to this genus: *Zonophryxus retrodens* Richardson (1904, p. 678, figs. 30—33; 1906, p. 823, figs. 4, 5) from Oahu, Hawaiian Islands, host unknown (type of the genus); *Zonophryxus grimaldii* Koehler (1911, p. 16, figs. 13, 14), from S. of the Gulf of Cadiz, Spain, depth 1401 m, host *Heterocarpus grimaldii* A. Milne Edwards & Bouvier; *Zonophryxus trilobus* Richardson (1910, p. 41, figs.) from Caluya Island, Philippines, depth 562 m, host unknown; *Zonophryxus quinquedens* Barnard (1913, p. 228, pl. 22), from off Cape Point, S. Africa, depth 850—1250 m, host *Nematocarcinus* spec.; *Zonophryxus similis* Richardson (1914, p. 369, fig. 16), Pacific Ocean S. of Panama, depth 1000 m, host unknown; and the present form.

The present species differs from all 5 already known species by possessing six instead of five pereiopods. Curiously enough in the drawings of the species of *Zonophryxus* species, nearly always the oostegites of the sixth pereiopods are visible; they are, however, generally considered to be pleopods. In almost all other genera of Dajidae the female is known to possess 5 or less pairs of pereiopods only. The only exception is the genus *Paradajus* Nierstrasz & Brender à Brandis (1923, p. 108, pl. 8 fig. 30), in which the female is provided with six legs; the only species of this genus, *Paradajus tenuis* Nierstr. & Br. à Br., is a parasite found in the branchial cavity of a Brachyuran crab from the Malay Archipelago, it differs considerably in many respects from *Zonophryxus*. The character of the presence of six pereiopods at first sight seems to be of generic importance. I think, however, the erection of a new genus for the present species not justified, since in all other characters it is a typical *Zonophryxus*.

Richardson in her descriptions gives rather few details of the antennae, mouth parts and oostegites. Her *Z. retrodens* and *Z. similis* seem to have the external antennae divided into three distinct joints. The maxillipeds mentioned by Richardson (1914) for *Z. similis*, probably are those organs, which are considered by BARNARD (1913) to be the lower lip.

KOEHLER (1911) describes the antennae, the oral parts and the oostegites of his specimen quite accurate, though he did not dissect his material. His

description of the various details agrees very good with the characters found in my material. KOEHLER states the external antenna to consist of two joints. The drawing he gives of it shows the situation much like in *Zonophryxus dodecapus*. *Zonophryxus grimaldii* at once may be distinguished from the other species of this genus by having the ventro-lateral border posteriorly ending in a narrow point.

BARNARD (1913) gives many details of the oral parts and the antennae of his species. The antennae agree good with those of the present form, as also do the mandibles. The maxillipeds, however, are quite different, while BARNARD moreover reports the presence of maxillae in one of his specimens. No maxillae could be found in my material.

At present the host of three species of *Zonophryxus* is known. In all three cases it belongs to the Crustacea Decapoda Caridea: *Heterocarpus grimaldii* A. M. Edw. & Bouv. and *Parapandalus larval* (Fabr.) both belong to the family Pandalidae, while *Nematocarcinus* belongs to the family Nematocarcinidae. Up till now it was not known how the parasite was attached to its host. The present material of *Zonophryxus dodecapus* shows that the parasites are attached to the posterodorsal region of the carapace of their hosts, just like the species of *Holophrayxus*.

LITERATURE.

- BARNARD, K. H., 1913. Contributions to the Crustacean Fauna of South Africa. Ann. S. Afr. Mus., vol. 10, pp. 197—240, pls. 17—24.
- KOEHLER, R., 1911. Isopodes nouveaux de la famille des Dajidés provenant des campagnes de la "Princesse-Alice". Bull. Inst. océanogr. Monaco, n. 196, pp. 1—34, figs. 1—21.
- NIERSTRASZ, H. F. & G. A. BRENDER Å BRANDIS, 1923. Epicaridea. Die Isopoden der Siboga-Expedition. II. Isopoda genuina. I. Siboga Exped., mon. 32b, pp. 57—121, pls. 4—9.
- RICHARDSON, H., 1904. Contributions to the Natural History of the Isopoda. (Second Part.). Proc. U.S. Nat. Mus., vol. 27, pp. 657—681, figs. 1—39.
- , 1906. Isopods collected at the Hawaiian Islands by the U.S. Fish Commission Steamer Albatross. Bull. U.S. Fish Comm., vol. 23 pt. 3, pp. 819—826, figs. 1—8.
- , 1910. Marine Isopods collected in the Philippines by the U.S. Fisheries Steamer "Albatross" in 1907—8. Doc. U.S. Bur. Fish., n. 736, pp. 1—44, figs. (This paper has not been seen by me)
- , 1914. Isopoda. Reports on the scientific results of the Expedition to the Tropical Pacific, in charge of Alexander Agassiz, on the U.S. Fish Commission Steamer Albatross, from August, 1899, to March, 1900, Commander Jefferson F. Moser, U.S.N., Commanding. XVII. Reports on the scientific results of the Expedition to the Eastern Tropical Pacific in charge of Alexander Agassiz, by the U.S. Fish Commission Steamer Albatross from October 1904 to March, 1905, Lieut. Commander L. M. Garrett, U.S.N., Commanding. XXVIII. Bull. Mus. comp. Zool. Harvard, vol. 58, pp. 361—372, figs. 1—16.

Chemistry. — Influence of temperature on the substitution type in the bromination and chlorination of aromatic compounds in the gas phase. By F. L. J. SIXMA and J. P. WIBAUT.

(Communicated at the meeting of February 26, 1949.)

§ 1. The rules governing the introduction of substituents into the benzene nucleus during reactions in the liquid phase were determined by extensive investigations in the latter half of the nineteenth and the beginning of this century; in this respect the work of HOLLEMAN and his co-workers occupies a prominent place (1). Two cases can be distinguished:

- a. the new substituent preferably occupies the ortho and para positions (ortho-para substitution);
- b. the new substituent chiefly occupies the meta position (meta substitution).

To which of these two types the substitution reaction belongs was found to be in general independent of the nature of the new substituent and to be conditioned only by the substituent present already in the benzene nucleus. Investigations carried out by WIBAUT and co-workers, however, showed that when halogen atoms are introduced into aromatic compounds in the gas phase between 300° and 600°, the substitution type will in many cases be governed by the reaction temperature. Thus ortho and para dibromobenzene are the main products of the non-catalytic bromination of *monobromobenzene* (2,3) below 350° in the gas phase; in this case the reaction is therefore of the ortho-para type, which is also the case with the reaction in the liquid phase. Between 350° and 450°, however, a sudden change takes place: the quantity of *p*-dibromobenzene rapidly decreases, whereas the quantity of *m*-dibromobenzene rises; the relative quantity of *o*-dibromobenzene, on the contrary, remains almost unchanged. Upwards of 450° *m*-dibromobenzene is the main product of the reaction (fig. 1). According to HOLLEMAN's criterion the ortho-para substitution type therefore changes into the meta substitution type between 350° and 450°. A similar change was observed in the bromination of chlorobenzene (fig. 2) and of fluorobenzene¹⁾. In these experiments purified pumice and graphite were used as contact material.

If the bromination reaction is carried out by means of a ferric bromide (or ferric chloride) catalyst on a suitable carrier, the total reaction velocity is much greater. In the temperature range from 200—450° the substitution reaction develops according to the ortho-para type; the substitution type

¹⁾ Owing to experimental difficulties the quantities of ortho and meta fluorobenzene in the reaction product of the bromination of fluorobenzene could not be determined.

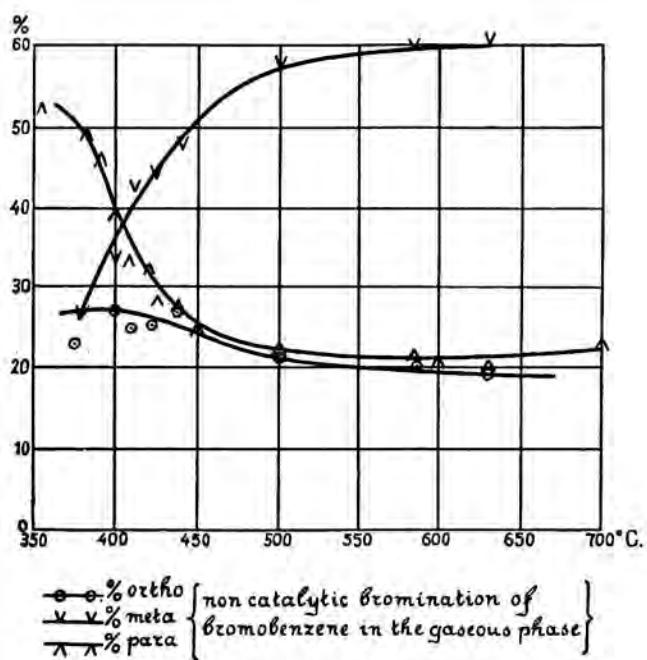


Fig. 1.

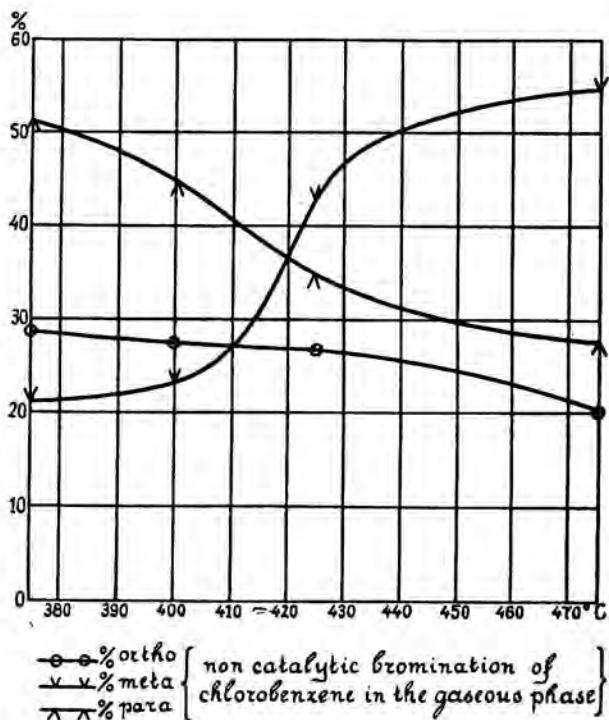


Fig. 2.

does not change. Experimental difficulties prevented the investigation of this heterogeneously catalytic reaction upwards of 450°.

The ratio of the isomeric dibromobenzenes formed in the catalytic bromination of bromobenzene is represented by the following formulae of SCHEFFER (4):

$$\ln \frac{2 \text{ conc.}_p}{\text{conc.}_m} = \frac{E_m - E_p}{RT} \quad \dots \dots \dots \quad (1)$$

and

$$\ln \frac{2 \text{ conc.}_p}{\text{conc.}_o} = \frac{E_o - E_p}{RT} \quad \dots \dots \dots \quad (2)$$

assuming that $E_m - E_p = 2890$ cal/mol and $E_o - E_p = 2036$ cal/mol; E_o , E_m and E_p in these formulae represent the energies of activation of the substitution in the ortho, meta and para positions²⁾. Fig. 3 shows satisfactory agreement between the values calculated according to formulae

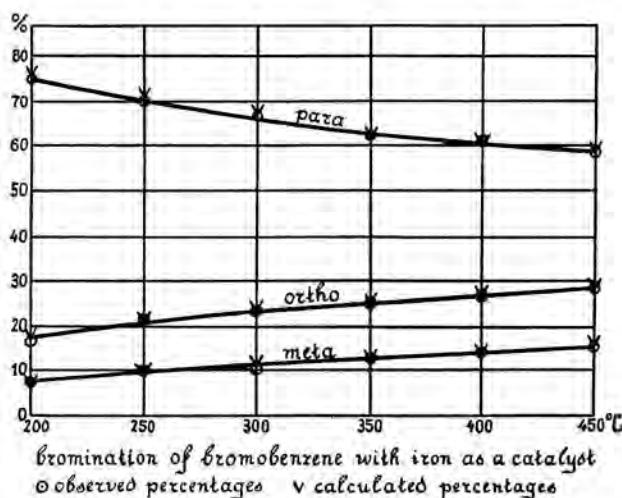


Fig. 3.

(1) and (2) and the experimental figures. This investigation carried out in 1937 by WIBAUT and VAN LOON (2) therefore showed that the formulae of SCHEFFER hold within a temperature range of 250°. Before that time these formulae had been tested for a narrow temperature range (-30° to ca + 30°) on a number of nitration reactions.

The results of the non-catalytic bromination of the halogenobenzenes (figs. 1 and 2) cannot be represented by formulae of the types (1) and (2). Neither can the more general formulae derived by SCHEFFER, which contain an energy term as well as an entropy term account for the change

²⁾ In our former publication the energy of activation was represented by ε ; for typographical reasons the symbol E is used in this paper.

in the para:meta ratio which takes place within a narrow temperature range. To gain a better insight into the influence of the temperature on substitution reactions in aromatic rings, the monobromination of naphthalene has been investigated extensively in our laboratory (6, 8, 9). Introduction of one substituent into the naphthalene molecule is theoretically simpler than into a mono-substituted benzene (C_6H_5A), because in the former case only two isomers are formed. Moreover, the naphthalene molecule consists only of carbon and hydrogen atoms, whereas the substituent in a monosubstituted benzene has in most cases a polar character, which enables it to influence the charge distribution in the molecule.

In our former publication (9) we subjected the results obtained in the bromination of naphthalene to a theoretical discussion, special attention being paid to the reversible conversion α -bromonaphthalene $\rightleftharpoons \beta$ -bromonaphthalene, which takes place under the influence of catalysts such as ferric chloride or ferric bromide. We shall now consider the *non-catalytic bromination* of naphthalene, which is not accompanied by conversion of the monobromonaphthalenes into each other — neither at elevated temperatures nor otherwise —, which has been proved experimentally.

The monobromination of gaseous naphthalene in the presence of a non specific contact material (pumice, glass wool) yields in the temperature range from 250° — 300° chiefly α -bromonaphthalene in addition to small quantities of β -bromonaphthalene. In this temperature range the $\alpha : \beta$ ratio satisfies SCHEFFER's formula and is therefore determined by the difference in energies of activation required for the substitution in the α - and β -positions. In the temperature range from 500 — 650° equal quantities of α - and β -bromonaphthalene are formed; the difference in energy of activation does no longer play a part and the $\alpha : \beta$ ratio is only determined by the probability of the collision of a bromine particle with the naphthalene molecule in the α - or β -position.

In the transition zone from 350 — 500° the relative quantity of β -isomer rapidly increases, which is reflected in the S-shape of the curve (see fig. 4). The question is how to account for the experimental result that from a certain temperature the difference in energy of activation no longer plays a part and that within a narrow temperature range a rather sudden change in the substitution type develops.

In our first paper reference was made to a theoretical consideration by one of us (S.), on the strength of which the existence of a transition temperature was made plausible (9). It was found, however, that the formulae derived lead to a less sudden change in the substitution type than had been observed experimentally.

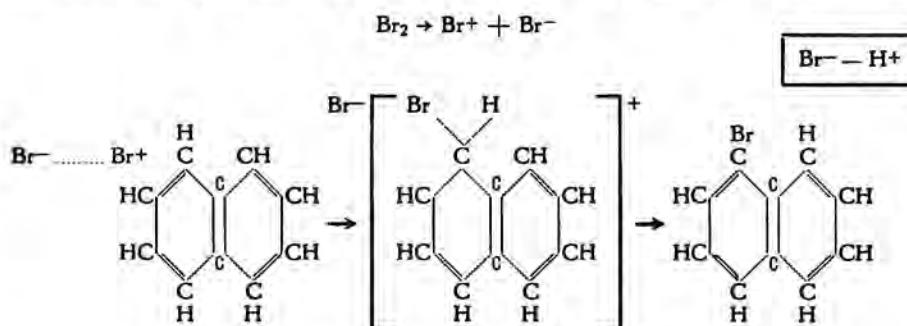
We shall now derive a formula by means of which the S-shaped curve found experimentally can be represented.

§ 2. In the bromination of gaseous naphthalene we must distinguish between a wall reaction and a real gas reaction. At a low temperature the

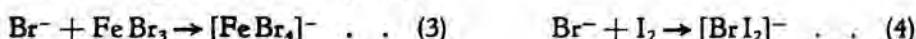
former will predominate (5). Our investigations into the isomerization of bromonaphthalene in the gas phase in the presence of a ferric bromide contact indicate that upwards of 400° the adsorption of the bromonaphthalene molecules to the catalyst surface is very slight. Obviously, we may assume that the adsorption of naphthalene on any solid surface at that temperature will also be negligible, so that we may put forward the hypothesis that the bromination of naphthalene vapour at about 300° and below will chiefly proceed as a wall reaction and from 400° upwards chiefly as a gas reaction.

We shall now explain how the change from wall reaction to gas reaction may lead to a pronounced change in the ratio of the isomeric substitution products. In our former publication we showed that the monobromination of liquid naphthalene, which takes place in the temperature range from $85-215^\circ$, can be described as an electrophilic substitution. By starting from this theory and using wave-mechanical calculations, one of us (S.) found an approximate value for the difference in energy of activation for substitution in the α - and β -positions ($E_\beta - E_\alpha$); this theoretical value was in fair agreement with the value calculated from the experimental data by means of SCHEFFER's formula (10).

We now assume that the bromination in the gas phase in the temperature range from $250-300^\circ$ also takes place according to an electrophilic substitution type. For this it is necessary that a bromine particle is strongly polarized in the transition state of the reaction, which can be represented schematically by a splitting into a positive and a negative bromine ion:



As stated formerly, catalysts such as ferric bromide, aluminium chloride or iodine can stabilize the transition state by the formation of complexes with the negative bromine particle:



This increases the reaction velocity.

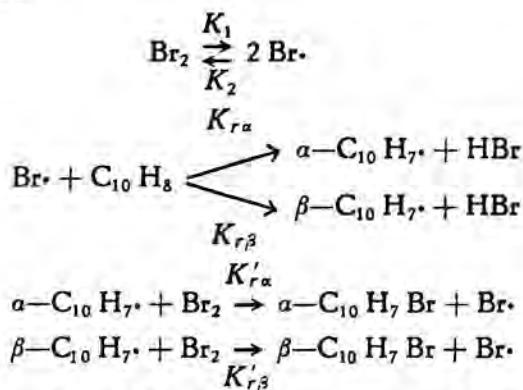
Such a stabilization of the transition state by a second bromine molecule is also conceivable:



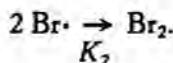
This is in agreement with the fact that some brominations in the liquid

phase proceed according to a second order reaction with respect to the bromine concentration (11).

In our experiments on the non-catalytic bromination in the absence of metal halides, the stabilization of the transition state may proceed according to equation (5). It should be borne in mind, however, that collisions between three molecules in a gas phase reaction are rare. In the temperature range from 250—300°, however, the reaction will chiefly proceed on the wall, so that in this range the electrophilic reaction is possible. According as the temperature rises, the adsorption of the reacting substances to the wall decreases; the normal rise in reaction velocity with the temperature is thus counteracted. The electrophilic substitution reaction on the wall is ultimately reduced to such an extent that the reaction in the gas phase predominates. For this reaction we assume a radical mechanism. As a result of the thermal dissociation of the bromine molecules bromine atoms are formed, which may introduce a chain reaction. This does not require adsorption to the wall. The chain reaction is represented by the following scheme:



By means of the values of bond energies mentioned by PAULING (12) the heat effect of these reactions can be computed. The reactions $K_{r\alpha}$ and $K_{r\beta}$ will show hardly any heat effect, while in the reactions $K'_{r\alpha}$ and $K'_{r\beta}$ about 8 kcal/mol of energy is liberated. It is therefore conceivable that the reactions should proceed according to this scheme. The chains can be broken by:



which reaction may proceed at the wall of the vessel.

At low temperatures the overall rate of this radical substitution will be low, because the dissociation of the bromine is then extremely slight. According as the temperature rises, the dissociation increases (at 800° 0.16 % of the bromine has been dissociated; at 900° 1.5 %); the rate of the atomic reaction rises, whereas the rate of the electrophilic reaction decreases. At a certain temperature the atomic substitution will therefore predominate.

§ 3. To obtain a quantitative formulation we assume that the dissociation equilibrium of the bromine is established so quickly that the concentration of the bromine atoms during the experiment may be considered constant. If the heat of dissociation of bromine is D cal/mol and C_D the corresponding entropy term, the concentration of the bromine atoms is:

$$[\text{Br}\cdot] = C_D^{1/2} \cdot e^{-D/2RT} \cdot [\text{Br}_2]^{1/2}.$$

The rate at which α - and β -monobromonaphthalene are formed according to the radical reaction is then:

$$\left(\frac{da}{dt} \right)_r = C_D^{1/2} \cdot C_{r\alpha} \cdot e^{(-E_{r\alpha} - 1/2 D)/RT} \cdot [\text{C}_{10}\text{H}_8] \cdot [\text{Br}_2]^{1/2} \dots \quad (6)$$

$$\left(\frac{d\beta}{dt} \right)_r = C_D^{1/2} \cdot C_{r\beta} \cdot e^{(-E_{r\beta} - 1/2 D)/RT} \cdot [\text{C}_{10}\text{H}_8] \cdot [\text{Br}_2]^{1/2} \dots \quad (7)$$

In these equations $E_{r\alpha}$ and $E_{r\beta}$ are the energies of activation for the reactions $K_{r\alpha}$ and $K_{r\beta}$; $C_{r\alpha}$ and $C_{r\beta}$ are the corresponding collision factors, whose dependence on the temperature is relatively small. Since each naphthalene molecule has as many α - as β -positions and both positions are sterically equally favoured, we put $C_{r\alpha} = C_{r\beta}$.

The electrophilic reaction, which develops exclusively on the wall, is based on adsorption. In the case of catalytically inactive adsorbents such as pumice or glass wool, we have adsorption by VAN DER WAALS forces; the number of adsorbed molecules is proportional to $e^{E_{\text{ads}}/RT}$, where E_{ads} is a function of the heat of adsorption of naphthalene, bromine and the reaction products. The rate at which α - and β -monobromonaphthalene are formed according to the electrophilic reaction is then:

$$\left(\frac{da}{dt} \right)_e = C_{ea} \cdot C_{\text{ads}} \cdot e^{(-E_{ea} + E_{\text{ads}})/RT} \cdot [\text{C}_{10}\text{H}_8] \cdot [\text{Br}_2]^n \dots \quad (8)$$

$$\left(\frac{d\beta}{dt} \right)_e = C_{e\beta} \cdot C_{\text{ads}} \cdot e^{(-E_{e\beta} + E_{\text{ads}})/RT} \cdot [\text{C}_{10}\text{H}_8] \cdot [\text{Br}_2]^n \dots \quad (9)$$

In these equations the bromine concentration occurs to the power n , n being 1 or 2 according as the reaction mainly proceeds with two or three collisions (cf. § 2). The collision factors C_{ea} and $C_{e\beta}$ are now equalized again; also E_{ads} has the same value in both equations, because of the simultaneous reactions between the adsorbed bromine and naphthalene molecules. E_{ea} and $E_{e\beta}$ are the energies of activation for the electrophilic substitution reactions in the α - and β -positions in the naphthalene nucleus. The ratio in which α - and β -monobromonaphthalene will occur in the reaction product is:

$$\frac{[a]}{[\beta]} = \frac{\left(\frac{da}{dt} \right)_e + \left(\frac{da}{dt} \right)_r}{\left(\frac{d\beta}{dt} \right)_e + \left(\frac{d\beta}{dt} \right)_r} \dots \quad (10)$$

After substitution of (6)–(9) in (10) and dividing numerator and denominator by $\left(\frac{d\beta}{dt}\right)_e$ we find:

$$\frac{[\alpha]}{[\beta]} = \frac{e^{(E_{e\beta}-E_{ea})/RT} + C \cdot e^{Q_\alpha/RT}}{1 + C \cdot e^{Q_\beta/RT}}, \quad \dots \quad (11)$$

where

$$C = \frac{C_D^b \cdot C_{ra} \cdot [Br_2]^{1-n}}{C_{e\beta} \cdot C_{ads}} = \frac{C_D^b \cdot C_{r\beta} \cdot [Br_2]^{1-n}}{C_{e\beta} \cdot C_{ads}}$$

and

$$\begin{aligned} Q_\alpha &= +E_{e\beta} - \frac{1}{2}D - E_{ads} - E_{ra}; \\ Q_\beta &= +E_{e\beta} - \frac{1}{2}D - E_{ads} - E_{r\beta}. \end{aligned}$$

As the bromination of naphthalene below 100° in the liquid phase proceeds fairly rapidly, E_{ea} and $E_{e\beta}$ will be of the order of 10 to 20 kcal/mol. The dissociation heat (D) of bromine is about 46 kcal/mol, so that Q_α and Q_β are negative magnitudes. From theoretical considerations as well as from the experiment we know that the difference between $E_{e\beta}$ and E_{ea} is a positive magnitude (about 4 kcal/mol) (9, 10). At elevated temperatures the terms $C \cdot e^{Q/RT}$ will therefore predominate in equation (11).

The experiment showed that at elevated temperatures α - and β -bromonaphthalene are formed in equal quantities (§ 1), so that $\frac{[\alpha]}{[\beta]} = 1$. This holds if $Q_\alpha = Q_\beta$. The absolute values of E_{ra} and $E_{r\beta}$ will be much lower than those of E_{ea} or $E_{e\beta}$. This renders it possible that even at moderate temperatures the average energy of the molecules is so great that any sterically favourable collision of a bromine atom with a naphthalene molecule may cause a reaction. The orienting effect is then eliminated and E_{ra} and $E_{r\beta}$ can be equalized, so that

$$\frac{[\alpha]}{[\beta]} = \frac{C \cdot e^{Q_\alpha/RT}}{C \cdot e^{Q_\beta/RT}} = 1.$$

On the other hand, $C \cdot e^{Q/RT}$ will be very small at a low temperature in the gas phase, so that in this case

$$\frac{[\alpha]}{[\beta]} = e^{(E_{e\beta}-E_{ea})/RT}, \quad \dots \quad (12)$$

The electrophilic reaction then predominates and SCHEFFER's relation is found again. For the bromination of naphthalene at a low temperature in the gas phase $E_{e\beta} - E_{ea} = 4215$ cal/mol.

§ 4. Unfortunately, there are no data available to calculate Q and C independently of the results of the bromination experiments. Therefore we can only calculate these magnitudes from the ratio in which the isomeric substitution products are formed at two different temperatures. The

difference in energy of activation for the electrophilic substitution ($E_{\alpha\beta} - E_{\alpha\alpha}$) in the non-catalytic bromination of naphthalene can be calculated from the ratio in which α - and β -isomers are formed at a temperature below the transition zone (6).

By means of the following values of the constants we find satisfactory agreement between the calculated and experimental values for the bromination and for the chlorination of naphthalene (see table I and II and fig. 4).

	$E_{\alpha\beta} - E_{\alpha\alpha}$	Q cal/mol	cal/mol	C
bromination of naphthalene	-3.8×10^4	4215	5×10^{12}	
chlorination of naphthalene	-2.8×10^4	(4215)	3×10^{12}	

For the chlorination of naphthalene ($E_{\alpha\beta} - E_{\alpha\alpha}$) cannot be calculated according to the above method, because the measurements cannot be extended to temperatures sufficiently below the transition zone (cf. fig. 4);

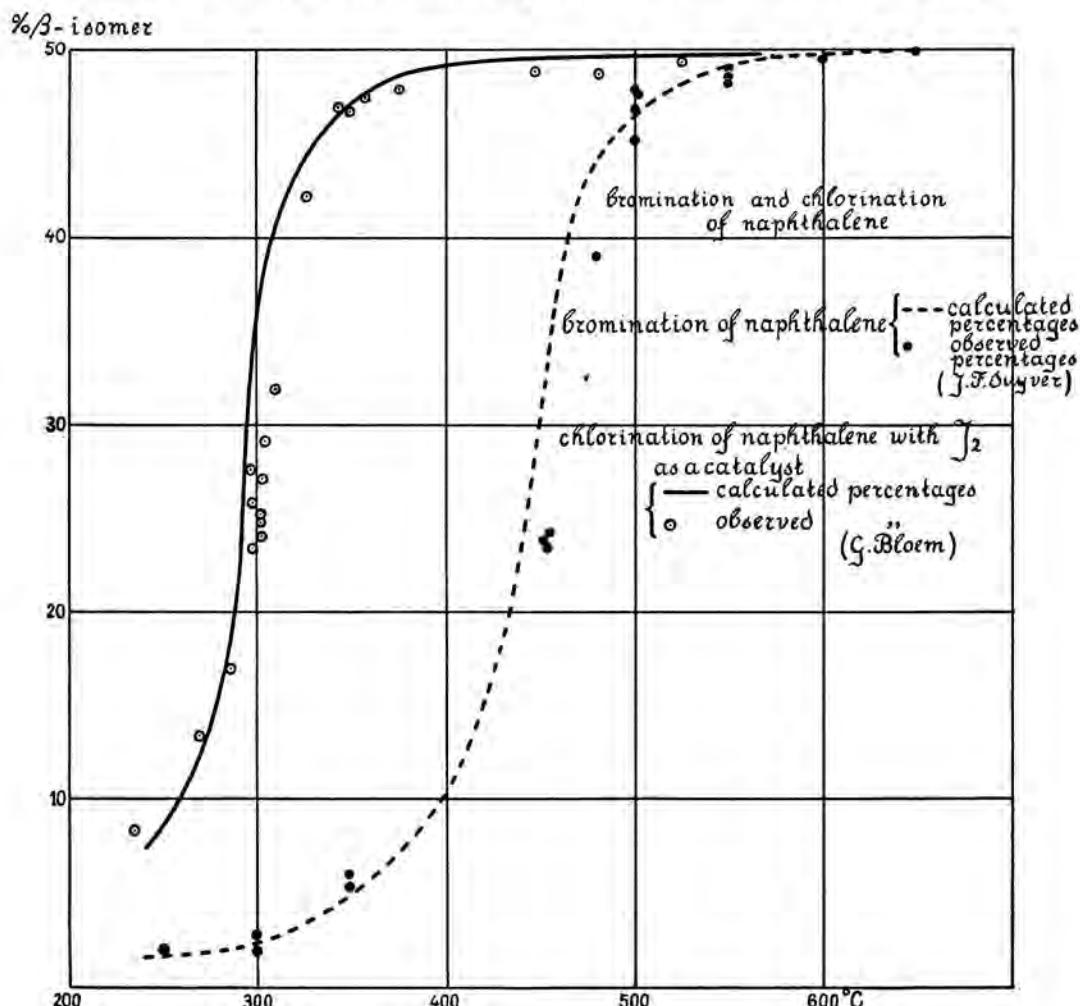


Fig. 4.

we assumed the same value as found for the bromination of naphthalene, i.e. 4215 cal/mol.

TABLE I.
Non-catalytic bromination of naphthalene in the gas phase (6).

Temperature °C	% β -monobromonaphthalene	
	experimental (J. F. SUYVER)	calculated
250	1.7	1.7
300	2.4	2.4
350	5.5	4.9
450	23.9	32.4
500	46.7	46.5
550	49.0	49.7
650	50.0	50.0

TABLE II.
Chlorination of naphthalene in the gas phase (under the influence of iodine) (7)

Temperature °C	% β -chloronaphthalene	
	experimental (G. BLOEM)	calculated
250	10.4	8.7
300	24—29	36.0
350	46.9	47.5
450	48.9	49.9
550	49.9	50.0

In the steepest part of the curve the deviation between the experimental and the calculated figures is fairly great. As the ratio of the isomers varies considerably with the temperature in this range, both the calculated and the experimental results are less accurate.

The ratio in which α - and β -bromonaphthalene are formed in the bromination of naphthalene in the gas phase with iron compounds as catalysts cannot be represented either by our own formula or by SCHEFFER's. As we showed in a previous publication, the two isomers are converted into each other under these conditions (9).

In the bromination of bromobenzene over iron compounds as catalyst such a conversion of the isomeric reaction products does not take place (2, 10), so that for this case formula (11) can be applied. These iron compounds, however, only catalyze the electrophilic reactions, the rate of the radical substitution reaction remaining unchanged. The result is that in the catalytic bromination of bromobenzene the electrophilic wall reaction will predominate up to 450° ; the sudden change in the substitution type is thus completely suppressed (fig. 3). In formula (11) the terms $C \cdot e^{Q/RT}$

are therefore smaller than $\exp(E_{e\text{meta}} - E_{e\text{para}})/RT$ so that this relation can be replaced by SCHEFFER's formula (form. 12; cf. § 1).

In the non-catalytic bromination of bromobenzene the substitution type changes. The ratio in which para and meta dibromobenzene are formed is then represented by formula (11). The ortho position is partly screened by steric influences, so that

$$C_{e\text{ortho}} \neq C_{e\text{para}} \quad \text{and}$$

$$C_{r\text{ortho}} \neq C_{r\text{para}}$$

As the magnitude of this effect is unknown we shall restrict the calculation to the ratio meta : para dibromobenzene. The difference in energy of activation for the electrophilic reaction ($E_{e\text{meta}} - E_{e\text{para}}$) is derived from the values found for the catalytic bromination. In this calculation we assume therefore that the catalyst reduces the energies of activation for the meta and para substitution reaction by the same amount, which supposition may hold only approximately.

In table III the experimental values for the para : meta isomer ratio in the non-catalytic bromination of bromobenzene are compared with the values derived from formula (11) with

$$Q = -5.65 \times 10^4 \text{ cal/mol}$$

$$E_{e\text{meta}} - E_{e\text{para}} = 2890 \text{ cal/mol}$$

$$C = 9.6 \times 10^{18}.$$

TABLE III.
non-catalytic bromination of bromobenzene in the gas phase (2).

Temperature °C	Ratio of the reaction velocities		$K_{\text{para}}^1)$ K_{meta}
	experimental (v. LOON)	calculated	
380	4.18		4.18
400	2.45		2.41
410	1.64		1.685
420	1.51		1.35
440	1.135		1.11

Experiments are in progress to test the hypothesis underlying this theory of the influence of the temperature on the substitution type, i.e. the development of a radical type of reaction at high temperatures.

January 1949.

*Laboratory of Organic Chemistry of the
University of Amsterdam.*

¹⁾ Calculation of percentages was impossible in this case, because the reaction mixture contains the ortho isomer as third component.

BIBLIOGRAPHY.

1. A. F. HOLLEMAN, Die direkte Einführung von Substituenten in den Benzolkern (1910); Rec. trav. chim. **42**, 355 (1923).
2. M. VAN LOON and J. P. WIBAUT, Rec. trav. chim. **56**, 815 (1937).
3. J. P. WIBAUT, L. M. F. VAN DE LANDE and G. WALLAGH, Rec. trav. chim. **52**, 794 (1933); **56**, 65 (1937).
4. F. E. C. SCHEFFER, Proc. Kon. Akad. v. Wetensch., Amsterdam, **15**, 1109, 1118 (1913).
F. E. C. SCHEFFER and W. F. BRANDSMA, Rec. trav. Chim. **45**, 522 (1926).
5. B. W. SPEEKMAN, Thesis, Amsterdam (1943).
6. J. F. SUYVER and J. P. WIBAUT, Rec. trav. chim. **64**, 65 (1945).
7. J. P. WIBAUT and G. BLOEM, Rec. trav. chim., in the press.
8. J. P. WIBAUT, F. L. J. SIXMA and J. F. SUYVER, Rec. trav. chim. in the press.
9. J. P. WIBAUT and F. L. J. SIXMA, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 776 (1948).
10. F. L. J. SIXMA, Thesis, Amsterdam (1948).
11. P. W. ROBERTSON, P. B. D. DE LA MARE and W. T. G. JOHNSTON, J. Chem. Soc., 277 (1943).
12. L. PAULING, The Nature of the Chemical Bond (sec. edition), page 53 (1945).

Physics. — Unitary Quantum Electron Dynamics II. By H. J. GROENEWOLD.

(Koninklijk Nederlands Meteorologisch Instituut te De Bilt.)

(Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of January 29, 1949.)

3. Unitary theory.

3.1 *Elimination of photons.* The equations of motion (36) show that an empty state cannot subsist in time. Photons are continually emitted and absorbed. That can still more clearly be seen from (37) where in each exponential corresponding to an infinitesimal time interval $t_k^{(m_k)} t_k^{(m_k+1)}$ the creation and annihilation operators describe (virtual or actual) emission and absorption of photons.

In spite of this we suppose for a moment that at a certain set of initial times (t'_1, \dots, t'_n) the wave function is an empty state function and that at a set of final times (t_1, \dots, t_n) it is again an empty state function. Or expressed less correctly no photons are supposed to be present at the initial and final electron times. In that case all photons emitted in the meantime are absorbed in the meantime and vice versa. These transient photons can be eliminated from the description indeed by dovetailing according to 2.183 the empty-empty part of the entire operator in (37) which transforms the initial wave function into the final one. Doing so we get

$$\begin{aligned} (x_1, \dots, x_n | \Psi = \lim & \prod_{k=1}^n \prod_{m=1}^p \left(\int dx_k^{(m_k)} \mathbf{N}_k(x_k^{(m_k+1)}, x_k^{(m_k)}) \right. \\ & \left. e^{\frac{i e^2}{\hbar c} \frac{1}{2} \sum_{k, l=1}^n (x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha})(x_l^{(m_l+1)\alpha'} - x_l^{(m_l)\alpha'}) W_{\alpha\alpha'}(x_k^{(m_k)}, x_l^{(m_l)})} \right) . \quad (38) \\ (x'_1, \dots, x'_n | \Psi. \end{aligned}$$

Now the underlying supposition is untenable because all photons emitted by the electrons during the time intervals $t_k' t_k$ ($k = 1, \dots, n$) cannot all be absorbed by other electrons during these time intervals and vice versa. That might suggest to solve the difficulty by shifting the initial and final times toward the infinite past and future. But that would be too much. For whereas it is doubtful whether the difficulties would not be merely shifted toward infinity, it is sure that all transitory adventures of the electrons would be shifted out of the picture. Now that not all electrons can simultaneously be saved from this deadly eternity, no much else seems to be left than trying to save them one at a time. So each electron will in its turn be taken separately with an arbitrarily variable finite initial and final time and the initial and final times of all other electrons are shifted toward the infinite past and future. That makes us sure that if the opaqueness con-

dition is fulfilled (which may require an infinite n) all photons emitted or absorbed by the chosen electron during the finite interval of time are absorbed or emitted by the other electrons during the infinite intervals of time. But we are not sure whether all other difficulties have been solved. Thus we cannot say that the photons have rigorously been eliminated from the description.

3.2 Equations of motion. Therefore we take another way. We forget everything about dualistic theory and try what happens in a unitary theory with equations of motion

$$\left. \begin{aligned} & \langle x_1, \dots, x_n | \Psi_{t_1=\dots=t_{r-1}=t_{r+1}=\dots=t_n \rightarrow +\infty} \\ &= \lim \prod_{k=1}^n \prod_{m=1}^p (\int d\vec{x}_k^{(m_k)}) \mathbf{N}_k(x_k^{(m_k+1)}; x_k^{(m_k)}) \\ & e^{\frac{i\theta^2}{\hbar c} \frac{1}{2} \sum_{k,l=1}^n \sum_{m=1}^p (x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha}) (x_l^{(m_l+1)\alpha'} - x_l^{(m_l)\alpha'}) W_{\alpha\alpha'}(x_k^{(m_k)}, x_l^{(m_l)})} \\ & \langle x'_1, \dots, x'_n | \Psi_{t'_1=\dots=t'_{r-1}=t'_{r+1}=\dots=t'_n \rightarrow -\infty} \quad (\epsilon = 1, \dots, n). \end{aligned} \right\} . \quad (39)$$

If a unitary theory of electron interaction cannot be like this it is hard to conceive what else it could be like.

3.3 Consistency and applicability. After having written down the equations (39) one is bound to state whether and under which conditions they form a consistent system. Further one is expected to give the conditions under which they are able to describe the processes, which are described by the dualistic theory. Finally after having established a complete self supporting unitary description one should be able to derive the dualistic theory by introducing in the formalism of the unitary theory wave operators of (from this point of view) fictitious photons. It is needless to say that we have written down (39) without being able to meet these obligations. We can do no more than making some comments on the general principles and illustrating the unitary description of some fundamental radiation processes in a very simple model. The comments are given in the present paper 2, the illustration will be given in a later paper 3.

3.31 Boundary conditions. One of the most difficult problems of consistency is that of boundary conditions. In 4-dimensional time-space an intricate network of mutual interaction is interwoven between the electrons. It looks like flogging a dead horse to disentangle this network in order to indicate in general initial conditions, starting from which the further course of the system can be derived. This is the main reason why in applications we have to resort to oversimplified models.

3.32 Supplementary conditions. It has already been announced in 2.23 that in dualistic theory one needs supplementary conditions. Therefore we have to be prepared that also in unitary theory one might need supplementary conditions. We shall see, however, in 4.33 that presumably one does not.

3.33 Opaqueness condition. From the dualistic point of view it has

already been remarked in 1.4 that the opaqueness condition (there stated in dualistic terms) is necessary for the equivalence between dualistic and unitary description if the latter is possible at all. From the unitary point of view we shall see in the simple model treated in 3 that the opaqueness condition (then stated in unitary terms) is even necessary for the consistency of the unitary description.

3.34 *Relativistic invariance.* The form of the dualistic and unitary integral equations of motion (37) and (39) is not relativistic invariant. Because, however, the dualistic differential equations of motion (36) (multiplied by β_k) are relativistic invariant, (37) describes invariant processes. The notion of empty states is also invariant. Therefore if (39) describes any processes at all, they may be expected to be relativistic invariant as well. An obvious invariant formulation²⁾ would be desirable.

3.35 *Independence of j^a .* Meanwhile the choice of the time-like vector j^a in (6) is a disguised form of fixing the direction of the time axis. Therefore in order to be genuinely relativistic invariant the theory has to be independent of the choice of j^a . Now in the dualistic description j^a only appears in the definition (22) of the interaction function $W_{\alpha\alpha'}$ and there even only in the cases Dh and Hh . The cases Dd and Hd , however, do lead to a description, which is independent of the choice of j^a .

3.36 *Asymmetry conditions.* In 3 we shall see that the asymmetry mentioned in 1.42 requires the occurrence of D_+ rather than D_a in the interaction function (22). That rules out the cases Dd and Dh for the purpose of unification.

3.4 *Self interaction.* The self terms ($k = l$) in the exponent in (39) ought to describe the self interaction of the electrons. They lead, however, to divergencies. These divergencies are provisionally removed by omitting the self terms. The relevant effects of self interaction are then supposed to be partially included in the unperturbed motion and therefore represented in the N 's. Meanwhile entirely similar divergencies arise from the cross terms ($k \neq l$) in case of pair creation and annihilation. It belongs to Q_3 to find a modification which removes all divergencies and which properly deals with self interaction and with pair creation and annihilation.

4. Radiative and COULOMB interaction.

Now we compare the separation of the photon states in the dualistic description and of the interaction function in the unitary description into a transverse and a non-transverse part. With the help of the time-like vector j^a this separation can be made in a relativistic invariant way. But then it depends on the choice of j^a .

4.1 *Dualistic theory.* 4.11 *Scalar, longitudinal and transverse states.* The photon states belonging to the 4 wave functions ($y |^a \varphi | \vec{\xi}_\pm r$) ($r = 1, 2, 3, 4$) of (5) can be distinguished in to 1 scalar state ($r = 4$),

1 longitudinal state ($r = 2$) and 2 transverse states ($r = 3, 4$) by choosing for the 1st and 2nd of the 4 4-vectors αb

$$\left. \begin{aligned} {}^{\alpha} b | \vec{\xi}_{\pm 1} \rangle &= j^{\alpha}, \\ {}^{\alpha} b | \vec{\xi}_{\pm 2} \rangle &= j^{\alpha} + \frac{\xi^{\alpha}}{j_{\gamma} \xi^{\gamma}}, \end{aligned} \right\} \quad \dots \quad (40)$$

where $\xi^0 = \pm |\vec{\xi}|$, and meanwhile observing (6). The scalar state vectors ($r = 1$) are the negative eigenvectors of $(g^{\alpha\beta} + 2j^{\alpha} j^{\beta})$ (cf. 2.141 and 2.151). The second relation of (6) can be split up into

$$\left. \begin{aligned} {}^{\alpha} b | \vec{\xi}_{\pm 1} \rangle (\vec{\xi}_{\pm 1} | b_{\beta}^{\dagger} &= j^{\alpha} j_{\beta}, \\ {}^{\alpha} b | \vec{\xi}_{\pm 2} \rangle (\vec{\xi}_{\pm 2} | b_{\beta}^{\dagger} &= (j^{\alpha} + \frac{\xi^{\alpha}}{j_{\gamma} \xi^{\gamma}})(j_{\beta} + \frac{\xi_{\beta}}{j_{\delta} \xi^{\delta}}), \\ \sum_{r=3,4} {}^{\alpha} b | \vec{\xi}_{\pm r} \rangle (\vec{\xi}_{\pm r} | b_{\beta}^{\dagger} &= \delta_{\beta}^{\alpha} - \frac{j^{\alpha} \xi_{\beta} + \xi^{\alpha} j_{\beta}}{j_{\gamma} \xi^{\gamma}} - \frac{\xi^{\alpha} \xi_{\beta}}{(j_{\gamma} \xi^{\gamma})^2}. \end{aligned} \right\} \quad \dots \quad (41)$$

The wave operators $\alpha \varphi$ and $\varphi_{\alpha}^{\dagger}$ can be split up into a scalar, longitudinal and transverse part by taking apart in (8) the terms with $r = 1$, $r = 2$ and $r = 3, 4$

$$\alpha \varphi = \alpha \varphi_{sc} + \alpha \varphi_{ln} + \alpha \varphi_{tr} \quad \dots \quad (42)$$

and similarly for $\varphi_{\alpha}^{\dagger}$. Observing that $-i\hbar c \partial / \partial x^{\alpha}$ operating under the integral sign in (8) gives a factor ξ_{α} we can symbolically write

$$\left. \begin{aligned} \alpha \varphi_{sc} &= -j^{\alpha} j_{\beta} \delta^{\beta} \varphi, \\ \alpha \varphi_{ln} &= (j^{\alpha} + \frac{\partial / \partial x_{\alpha}}{j_{\gamma} \partial / \partial x_{\gamma}})(j_{\beta} + \frac{\partial / \partial x_{\beta}}{j_{\delta} \partial / \partial x_{\delta}}) \delta^{\beta} \varphi, \\ \alpha \varphi_{tr} &= (\delta_{\beta}^{\alpha} - \frac{j^{\alpha} \partial / \partial x^{\beta} + j_{\beta} \partial / \partial x_{\alpha}}{j_{\gamma} \partial / \partial x_{\gamma}} - \frac{\partial^2 / \partial x_{\alpha} \partial x^{\beta}}{j_{\delta} \partial / \partial x_{\delta}}) \delta^{\beta} \varphi \end{aligned} \right\} \quad \dots \quad (43)$$

and similarly for $\varphi_{\alpha}^{\dagger}$. The separate parts satisfy the commutation relations

$$\left. \begin{aligned} [(x | {}_{\alpha} \varphi'_{\beta sc} | y_0), \{ y_0 | {}^{\beta} \varphi'_{\alpha sc}^{\dagger} | x' \}]^{-} &= \\ &= (j_{\alpha} j_{\alpha'}) (j^{\beta} j^{\beta'}) G_{\alpha \beta \beta'}(x - x'), \\ [(x | {}_{\alpha} \varphi'_{\beta ln} | y_0), \{ y_0 | {}^{\beta} \varphi'_{\alpha ln}^{\dagger} | x' \}]^{-} &= \\ &= (\delta_{\alpha}^{\beta} + j_{\alpha} j^{\beta}) (\delta_{\alpha'}^{\beta'} + j_{\alpha'} j^{\beta'}) G_{\alpha \beta \beta'}(x - x'), \\ [(x | {}_{\alpha} \varphi'_{\beta tr} | y_0), \{ y_0 | {}^{\beta} \varphi'_{\alpha tr}^{\dagger} | x' \}]^{-} &= \\ &= (g_{\alpha \alpha'} j^{\beta} j^{\beta'} - j_{\alpha} j^{\beta} \delta_{\alpha'}^{\beta'} - j_{\alpha'} j^{\beta'} \delta_{\alpha}^{\beta} - \delta_{\alpha}^{\beta} \delta_{\alpha'}^{\beta'}) G_{\alpha \beta \beta'}(x - x'). \end{aligned} \right\} \quad \dots \quad (44)$$

all other commutators give zero. $G_{\beta \beta'}(x)$ in (44) is together with $F(x)$

and $G_s(x)$, which will be used further on, defined by

$$\left. \begin{aligned} F_s(x) &= \frac{i}{\hbar c} \frac{1}{j_\gamma \partial/\partial x_\gamma} D_a(x), \\ G_s(x) &= \left(\frac{i}{\hbar c} \right)^2 \frac{1}{(j_\gamma \partial/\partial x_\gamma)^2} D_a(x), \\ G_{a\beta\beta'}(x) &= \left(\frac{\hbar c}{i} \right)^2 \partial^2/\partial x_\beta \partial x_{\beta'} G_s(x). \end{aligned} \right\} \quad \dots \quad (45)$$

For $F_s(x)$ we get outside the lightcone

$$F_s(x) = \left(\frac{i}{\hbar c} \right)^2 \frac{1}{4\pi} \frac{1}{(x_\alpha x^\alpha - (j_\beta x^\beta)^2)^{1/2}} \quad (\text{for } x > |t|) \quad \dots \quad (46)$$

and zero inside (for $x < |t|$). The $G_{\beta\beta'}$ satisfy the relations

$$\left. \begin{aligned} j^\beta j^{\beta'} G_{a\beta\beta'}(x) &= D_a(x), \\ g^{\beta\beta'} G_{a\beta\beta'}(x) &= 0. \end{aligned} \right\} \quad \dots \quad (47)$$

The function $W_{\alpha\alpha'}$ in (18) (which in the unitary description becomes the interaction function) splits up into

$$W_{\alpha\alpha'} = W_{\alpha\alpha'sc} + W_{\alpha\alpha'ln} + W_{\alpha\alpha'tr} \quad \dots \quad (48)$$

where

$$\left. \begin{array}{c|ccc} & \frac{i}{\hbar c} W_{\alpha\alpha'sc} & \frac{i}{\hbar c} W_{\alpha\alpha'ln} & \frac{i}{\hbar c} W_{\alpha\alpha'tr} \\ \hline Dd & -j_\alpha j^\beta j_{\alpha'} j^{\beta'} G_{a\beta\beta'}^\sigma & (j_\alpha j^\beta + \delta_\alpha^\beta) (j_{\alpha'} j^{\beta'} + \delta_{\alpha'}^{\beta'}) G_{a\beta\beta'}^\sigma & \\ Dh & + j_\alpha j^\beta j_{\alpha'} j^{\beta'} G_{a\beta\beta'}^\sigma & (g_{\alpha\alpha'} j^\beta j^{\beta'} - \delta_\alpha^\beta \delta_{\alpha'}^{\beta'} - \delta_\alpha^\beta j_{\alpha'} j^{\beta'} - j_\alpha j^\beta \delta_{\alpha'}^{\beta'}) G_{a\beta\beta'}^\sigma & \\ Hd & - j_\alpha j^\beta j_{\alpha'} j^{\beta'} G_{+\beta\beta'}^\sigma & (j_\alpha j^\beta + \delta_\alpha^\beta) (j_{\alpha'} j^{\beta'} + \delta_{\alpha'}^{\beta'}) G_{+\beta\beta'}^\sigma & \\ Hh & + j_\alpha j^\beta j_{\alpha'} j^{\beta'} G_{-\beta\beta'}^\sigma & (g_{\alpha\alpha'} j^\beta j^{\beta'} - \delta_\alpha^\beta \delta_{\alpha'}^{\beta'} - \delta_\alpha^\beta j_{\alpha'} j^{\beta'} - j_\alpha j^\beta \delta_{\alpha'}^{\beta'}) G_{-\beta\beta'}^\sigma & \end{array} \right\} \quad (49)$$

the arguments being the same as in (22).

4.12 Supplementary conditions. If the 4-vector wave function ${}^a\varphi$ is a solution of (4), the 4-scalar wave function

$$\frac{\partial/\partial x^\alpha}{j_\gamma \partial/\partial x_\gamma} {}^a\varphi \quad \dots \quad (50)$$

is also a solution. The states (50) are kept out of the Ψ 's by the supplementary conditions.

The two operators

$$\left. \begin{aligned} \mathbf{C}(x_1, \dots, x_n; x) &= \frac{\partial/\partial x^\alpha}{j_\gamma \partial/\partial x_\gamma} \Phi^\alpha(x) + e 2\pi (\hbar c)^2 \sum_{l=1}^n 2 F_s(x-x_l), \\ \mathbf{C}_{rd}(x_1, \dots, x_n; x) &= \frac{\partial/\partial x^\alpha}{j_\gamma \partial/\partial x_\gamma} \Phi_{rd}^\alpha(x) \end{aligned} \right\} \quad (51)$$

commute in all points with each other and with $\mathbf{K}_k^0 \{x_k\} + \mathbf{K}^1(x_k)$. Therefore they are for arbitrarily fixed x^α integrals of electron motion at least as long as all electron world points lie outside each others light cone. So we can write

$$\left. \begin{aligned} \mathbf{C}(x_1, \dots, x_n; x)(x_1, \dots, x_n; y_{10}, \dots) | \Psi &= \mathbf{C}(x)(x_1, \dots, x_n; y_{10}, \dots) | \Psi, \\ \mathbf{C}_{rd}(x_1, \dots, x_n; x)(x_1, \dots, x_n; y_{10}, \dots) | \Psi &= \mathbf{C}_{rd}(x)(x_1, \dots, x_n; y_{10}, \dots) | \Psi. \end{aligned} \right\} \quad (52)$$

Because $(\partial^2 / \partial x^\alpha \partial x_\alpha) \mathbf{C} = (\partial^2 / \partial x^\alpha \partial x_\alpha) \mathbf{C}_{rd} = 0$ the conditions

$$\left. \begin{aligned} \mathbf{C}(x_1, \dots, x_n; x)(x_1, \dots, x_n; y_{10}, \dots) | \Psi &= 0, \\ \mathbf{C}_{rd}(x_1, \dots, x_n; x)(x_1, \dots, x_n; y_{10}, \dots) | \Psi &= 0 \end{aligned} \right\} \quad . . . \quad (53)$$

are satisfied for every t if at an arbitrarily given t they are satisfied together with their first derivative in t . They are a generalization of the elementary conditions (3) for the charge free case.

4.13 Elimination of non-transverse wave operators. The non-transverse wave operators can be eliminated from the equations of motion in various ways^{11) 13)}. The procedure consists of two steps. One of these steps makes use of the supplementary conditions, the other step can be made by means of a gauge transformation. In behalf of later comparison with the unitary description we shall first make the latter step.

4.131 Transformed equations of motion. For our present purpose a suitable transformation is

$$(x_1, \dots, x_n; y_{10}, \dots) | \Psi = e^{-\frac{i e}{\hbar c} \sum_{l=1}^n j_\gamma \frac{\partial}{\partial x_\gamma} \Phi^\alpha(x_l)} (x_1, \dots, x_n; y_{10}, \dots) | \Psi^*, \quad (54)$$

By this transformation an operator \mathbf{Q} is transformed into

$$\mathbf{Q}^* = e^{\frac{i e}{\hbar c} \sum_{l=1}^n j_\gamma \frac{\partial}{\partial x_\gamma} \Phi^\alpha(x_l)} \mathbf{Q} e^{-\frac{i e}{\hbar c} \sum_{l=1}^n j_\gamma \frac{\partial}{\partial x_\gamma} \Phi^\alpha(x_l)} \quad . . . \quad (55)$$

If we apply this to the operators \mathbf{C} and \mathbf{C}_{rd} of (51) we get with the help of the commutation relations (14)

$$\left. \begin{aligned} \mathbf{C}^*(x_1, \dots, x_n; x) &= \frac{\partial / \partial x^\alpha}{j_\gamma \frac{\partial}{\partial x_\gamma}} \Phi^\alpha(x), \\ \mathbf{C}_{rd}^*(x_1, \dots, x_n; x) &= \frac{\partial / \partial x^\alpha}{j_\gamma \frac{\partial}{\partial x_\gamma}} \Phi_{rd}^\alpha(x). \end{aligned} \right\} \quad . . . \quad (51^*)$$

In the same way we obtain

$$\left(\frac{\hbar c}{i} \frac{\partial}{\partial x^\alpha} + e \Phi_\alpha(x) \right)^* = \frac{\hbar c}{i} \frac{\partial}{\partial x_\alpha} + e \Phi_\alpha^{(*)}(x), \quad . . . \quad (56^*)$$

where $\Phi_\alpha^{(*)}$ stands for

$$\left. \begin{aligned} \Phi_\alpha^{(*)}(x) &= \Phi_{atr}(x) + \left(j_\alpha + \frac{\partial / \partial x^\alpha}{j_\gamma \frac{\partial}{\partial x_\gamma}} \right) \frac{\partial / \partial x_\beta}{j_\delta \frac{\partial}{\partial x_\delta}} \Phi_\beta(x) - \\ &- e^2 2\pi (\hbar c)^2 j_\alpha \sum_{l=1}^n 2F_s(x - x_l), \end{aligned} \right\} \quad . \quad (57^*)$$

which is not the transformed Φ_α^* of Φ_α alone.

The principle of gauge invariance requires that the 4-gradient term in (57*) could also be removed. This can be done by the further transformation

$$(x_1, \dots, x_n; y_{10}, \dots | \Psi^*) = e^{-\frac{ie}{\hbar c} \sum_{l=1}^n \frac{\partial/\partial x^\alpha}{(j_\gamma \partial/\partial x_l)^2} \Phi^\alpha(x_l)} (x_1, \dots, x_n; y_{10}, \dots | \Psi^{**}). \quad (54^*)$$

The operators (51*) remain invariant under this transformation

$$\mathbf{C}^{**} = \mathbf{C}^*, \quad \mathbf{C}_{rd}^{**} = \mathbf{C}_{rd}^* \quad \dots \quad (51^{**})$$

and (56*) transforms into (56**) with two asterisks instead of one and Φ_α^{**} standing for

$$\Phi_\alpha^{**}(x) = \Phi_{atr}(x) + j_\alpha \frac{\partial/\partial x_\beta}{j_\gamma \partial/\partial x_\gamma} \Phi_\beta(x) - e^2 2\pi (\hbar c)^2 j_\alpha \sum_{l=1}^n 2F_s(x-x_l). \quad (57^{**})$$

The transformed equations of motion (36**) for Ψ^{**} can be obtained from the original equations (36) for Ψ in replacing Φ_α by Φ_α^{**} .

4.132 Effective equations of motion. Operating on transformed wave functions Ψ^* or Ψ^{**} which satisfy the transformed supplementary conditions (53*) or (53**), the operator $\Phi_\alpha^{(*)}$ (57*) or $\Phi_\alpha^{(**)}$ (57**) becomes effectively equal to

$$\Phi_{aef}^{(*)}(x) = \Phi_{atr}(x) - e^2 2\pi (\hbar c)^2 j_\alpha \sum_{l=1}^n 2F_s(x-x_l). \quad (57_{ef}^*)$$

The effective equations of motion (36_{ef}) for Ψ^* (or Ψ^{**} which is effectively the same) are obtained from the original equations (36) for Ψ in replacing Φ_α by $\Phi_{aef}^{(*)}$.

4.14 COULOMB interaction. By the two successive steps made in 4.131 and 4.132 the non-transverse wave operators have entirely been eliminated from the equations of motion. The effect of the non-transverse photon states is represented by the second term in the right hand member of (57_{ef}^{*}). According to (46) this term is equal to

$$e^2 j_\alpha \sum_{l=1}^n \frac{1}{((x_\beta - x_{l\beta})(x^\beta - x_l^\beta) - (j_\gamma(x^\gamma - x_l^\gamma)^2)^{\frac{1}{2}})} \quad \dots \quad (58)$$

where the sum is only extended over those points x_l^β , which lie outside the light cone of x^β ($|x-x_l| > c|t-t_l|$).

Practical applications of the dualistic theory are commonly in single-time description. If the effective equations of motion are reduced to single-time description with the time axis in the direction of j^α , the term (58) is reduced to

$$e^2 \delta_\alpha^0 \sum_{l=1}^n \frac{1}{|\vec{x} - \vec{x}_l|} \quad (t_l = t) \quad \dots \quad (59)$$

and represents simultaneous COULOMB interaction.

The transverse part of (57_{ef}^{*}) leads to the proper radiation processes.

4.2 Unitary theory. There are two reasons why it would not be correct

to infer transformed and effective unitary equations of motion from the transformed and effective dualistic equations of motion in the same way as the original unitary equations (39) have been inferred from the original dualistic equations (36). Neither the empty non-transverse states, nor the order in time of the non-transverse wave operators are invariant under the transformations. Therefore the unitary derivation has to start from the original unitary equations (39) straightforwardly. Meanwhile it is worth while to keep an eye on the dualistic derivation during the course of the unitary one.

4.21 Reduced equations of motion. In the equation of motion (39) we split up $W_{\alpha\alpha'}$ according to (49) into a transverse part $W_{\alpha\alpha'tr}$ and a non-transverse part $W_{\alpha\alpha'nt} = W_{\alpha\alpha'sc} + W_{\alpha\alpha'ln}$. $W_{\alpha\alpha'tr}$ will be left unchanged throughout the reduction. In $W_{\alpha\alpha'nt}$ we insert

$$\left. \begin{aligned} \delta_{\alpha}^{\beta} \delta_{\alpha'}^{\beta'} G_{\alpha\beta\beta'}^{\sigma}(x) &= \left(\frac{\hbar c}{i} \right)^2 \frac{\partial^2}{\partial x^{\alpha} \partial x^{\alpha'}} G_{\alpha}^{\sigma}(x) - i_{\alpha} j_{\alpha'} 2 \delta(j, x') \frac{\hbar c}{i} F_s^{\sigma}(x), \\ \delta_{\alpha}^{\beta} \delta_{\alpha'}^{\beta'} G_{s\beta\beta'}^{\sigma}(x) &= \left(\frac{\hbar c}{i} \right)^2 \frac{\partial^2}{\partial x^{\alpha} \partial x^{\alpha'}} G_s^{\sigma}(x) \end{aligned} \right\} \quad (60)$$

and

$$\left. \begin{aligned} j_{\alpha} j^{\beta} \delta_{\alpha'}^{\beta'} G_{\alpha\beta\beta'}^{\sigma}(x) &= j_{\alpha} \frac{\hbar c}{i} \frac{\partial}{\partial x^{\alpha'}} \sigma(x) F_s^{\sigma}(x) + j_{\alpha} j_{\alpha'} \frac{\hbar c}{i} 2 \delta(j, x') \frac{\hbar c}{i} F_s^{\sigma}(x), \\ j_{\alpha} j^{\beta} \delta_{\alpha'}^{\beta'} G_{s\beta\beta'}^{\sigma}(x) &= j_{\alpha} \frac{\hbar c}{i} \frac{\partial}{\partial x^{\alpha'}} \sigma(x) F_a^{\sigma}(x). \end{aligned} \right\} \quad (61)$$

In the exponent in (39) the sum

$$\left. \begin{aligned} \frac{ie^2}{hc2} \sum_{k,l=1}^n \sum_{m=1}^p (x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha}) (x_l^{(m_l+1)\alpha'} - x_l^{(m_l)\alpha'}) \\ \left(\frac{\hbar c}{i} \right)^2 \frac{\partial^2}{\partial x_{kl}^{\alpha} \partial x_{kl}^{\alpha'}} G^{\sigma}(x_k^{(m_k)} - x_l^{(m_l)}), \end{aligned} \right\} \quad \dots \quad (62)$$

where x_{kl} stands for $x_k - x_l$, might in the limit

$$p_k \rightarrow \infty, \max |t_k^{(m_k+1)} - t_k^{(m_k)}| \rightarrow 0$$

be treated as

$$-\frac{ie^2}{hc2} \sum_{k,l=1}^n \int_{(x')}^{(x)} dx_k'' dx_l'' \left(\frac{\hbar c}{i} \right)^2 \frac{\partial^2}{\partial x_k'' \partial x_l''} G^{\sigma}(x_k'' - x_l''). \quad \dots \quad (63)$$

It could already for finite initial and final times t_k and t_l be removed by the transformation

$$(x_1, \dots, x_n) |\Psi\rangle = e^{\frac{ie^2 \hbar c}{2} \sum_{k,l=1}^n G^{\sigma}(x_k - x_l)} (x_1, \dots, x_n) |\Psi^*\rangle \quad \dots \quad (64)$$

In (39), where for all electrons but one (the r th one) the initial and final times are shifted to $-\infty$ and $+\infty$ respectively, the contributions of (62) may be assumed to vanish even without this transformation. The same can

be assumed for the contributions of the terms

$$\left. \begin{aligned} & \frac{ie^2}{\hbar c 2} \sum_{k,l=1}^n \sum_{m=1}^p (x_k^{(m+1)\alpha} - x_k^{(m)\alpha}) (x_l^{(m_l+1)\alpha'} - x_l^{(m)\alpha'}) \\ & \left(j_\alpha \frac{\hbar c}{i} \frac{\partial}{\partial x_k^\alpha} + j_{\alpha'} \frac{\hbar c}{i} \frac{\partial}{\partial x_k^\alpha} \right) \sigma(x_k^{(m')\alpha} - x_l^{(m')\alpha'}) F^\sigma(x_k^{(m')\alpha} - x_l^{(m')\alpha'}) \end{aligned} \right\}. \quad (65)$$

except for those in which the differentiation is with respect to x_r . The latter ones might be replaced by

$$\left. \begin{aligned} & \frac{e^2}{2} \sum_{l=1}^n \sum_{m=1}^p (x_l^{(m_l+1)\alpha'} - x_l^{(m)\alpha'}) j_{\alpha'} (\sigma(x_r - x_l^{(m')\alpha}) F^\sigma(x_r - x_l^{(m')\alpha})) \\ & - \sigma(x_r - x_l^{(m')\alpha}) F^\sigma(x_r - x_l^{(m')\alpha}). \end{aligned} \right\}. \quad (66)$$

Once more because in (39) the initial and final times of all electrons except the r th one are shifted to $-\infty$ and $+\infty$ the contribution of the terms (66) can be removed by the transformation

$$(x_1, \dots, x_n | \Psi = e^{-\frac{e^2}{2} \sum_{k,l=1}^n \int_{x'_l}^{x_k} J_\alpha dx''_l^\alpha \sigma(x_k - x''_l) F^\sigma(x_k - x''_l)} (x_1, \dots, x_n | \Psi^*. \quad (67)$$

By that all 4-gradiant terms of $W_{\alpha\alpha'nt}$ have been removed. The remaining terms give the reduced non-transverse interaction function $W_{\alpha\alpha'nt}^*$

$$\frac{i}{\hbar c} W_{\alpha\alpha'nt}^*(x, x') = j_\alpha j_{\alpha'} (2\delta j_\gamma(x'' - x'^\gamma)) \frac{\hbar c}{i} F_s(x - x') + B(x - x') \quad (68)$$

with

$$\left. \begin{array}{c|c} & B(x) \\ \hline Dd & 0 \\ Dh & D_a^\sigma(x) \\ Hd & 0 \\ Hh & D_s^\sigma(x). \end{array} \right\} \quad (69)$$

The reduced equations of motion (39*) are obtained from the original ones (39) in replacing $W_{\alpha\alpha'}$ by $W_{\alpha\alpha'tr} + W_{\alpha\alpha'nt}^*$.

4.22 COULOMB interaction. After the reduction the non-transverse interaction is represented by (68). The part which the term with F_s contributes to the exponent of (39*) is according to (46)

$$\left. \begin{aligned} & \frac{ie^2}{\hbar c 2} \sum_{k,l=1}^n \sum_{m=1}^p j_\alpha (x_k^{(m_k+1)\alpha} - x_k^{(m)\alpha}) j_{\alpha'} (x_l^{(m_l+1)\alpha'} - x_l^{(m)\alpha'}) \delta(j_\gamma(x_k^{(m')\gamma} - x_l^{(m')\gamma})) \\ & \frac{1}{((x_k^{(m')\delta} - x_l^{(m')\delta})(x_{k\delta}^{(m')\delta} - x_{l\delta}^{(m')\delta}))^{\frac{1}{2}}} \end{aligned} \right\}. \quad (70)$$

If the time axis is chosen in the direction of j^α (70) reads

$$\frac{ie^2}{\hbar 2} \sum_{k,l=1}^n \sum_{m=1}^p (t_k^{(m_k+1)} - t_k^{(m)\alpha}) (t_l^{(m_l+1)} - t_l^{(m)\alpha'}) \delta(t_k^{(m')\alpha} - t_l^{(m')\alpha'}) \frac{1}{|x_k^{(m')\alpha} - x_l^{(m')\alpha}|}. \quad (71)$$

Even in many-times description this represents simultaneous COULOMB interaction.

In case Dh the term with D_a^r contributes according to (11) a part, which represents a similar COULOMB interaction along the past and future branches of the light cone. In case Hh the term with D_s^r leads according to (11) to a different type of interaction, which extends over all time-space with a singularity along the light cone.

The transverse interaction for Dd is the same as for Dh ; for Hd it is the same as for Hh . Though we now don't talk of radiation, we might call it radiative interaction.

4.3 Comparison of dualistic and unitary reduction. 4.31 *Transformed equations.* If from the transformed dualistic equations of motion (36**) we would thoughtlessly try to infer unitary equations in the same way as the equations (39) have been inferred from (36), the first two terms of Φ_α^{**} (57**) would just yield the interaction function $W_{\alpha\alpha' tr} + W_{\alpha\alpha' nt}^*$, which is the interaction function in the reduced unitary equations of motion (39*). The latter, however, does not contain a further part, which could be considered as a counterpart of the remaining terms with F_s in (57**). In fact it has been argued in 4.2 that this thoughtless procedure cannot be expected to be correct.

4.32 Effective equations. The objections against the application of the same procedure to the effective dualistic equations of motion (36_{ef}*) are less stringent because in the latter the non-transverse wave operators have entirely been removed and the transverse ones have been left unaffected. The first term of $\Phi_{\alpha ef}^{*}$ (57_{ef}*) would yield the radiative interaction function $W_{\alpha\alpha' tr}$. Now the remaining part $W_{\alpha\alpha' nt}^*$ of the interaction function in (39*) can in the cases Dd and Hd , in which it represents instantaneous COULOMB interaction, more or less be considered as the counterpart of the remaining terms with F_s in (57_{ef}*)¹⁾, which in those cases in which single-time description is used also represent instantaneous COULOMB interaction.

Therefore if anything could be expected of a unitary description, the cases Dd or Hd might be hoped to give the same results as the dualistic description, whereas in the cases Dh or Hh one might expect additional effects represented by D_a^r or D_s^r . Just the cases Dh and Hh have been rejected in 3.35 because they depend on j^a already before the distinction into transverse and non-transverse interaction has been made.

4.33 Supplementary conditions. It should be noticed that, whereas in the dualistic theory supplementary conditions were needed in order to reduce the total field to the transverse field and the COULOMB field, in the unitary theory the reduction of the total interaction to the transverse interaction and the COULOMB interaction has been made without supplementary conditions. (Also in classical theory the 4-divergence of the field potential vanishes without extra conditions if the field is derived from a unitary charge description).

5. Transitions.

5.1 *Transition representation.* Up till now we have used a time-space representation based on FEYNMAN's calculus. Everywhere in the exponents of (37) as well as (39) and (39*) ($x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha}$) can be replaced by $\alpha_k^\alpha c(t_k^{(m_k+1)} - t_k^{(m_k)})$, provided the N's and the exponentials are well ordered in time together instead of separately because they no longer commute. The latter restriction can easily be satisfied in (37) but in (39) and (39*) it requires that the exponentials shall be placed straddle-legged between the N's in a way, which cannot simply be symbolized in a formula. It is just a peculiar feature of FEYNMAN's calculus that thanks to the properties of N one can instead of the $\alpha^\alpha ct$ exponents use the x^α exponents, which do commute with the N's.

Now another way to get rid of the difficulties with the ordering of the N's and the exponentials is to get rid of the N's (29) by means of a unitary transformation. The transformation nucleus is supplied by a complete orthonormal system of solutions of the unperturbed electron equations (34) and in the dualistic theory also by complete systems of free particle wave functions (5). The transformed representation expresses in the unitary theory the interaction between the electrons in terms of transitions between unperturbed electron states and in the dualistic theory it expresses the interaction between electrons and photons in terms of transitions between unperturbed electron states together with creation and annihilation of free photons. Therefore this representation will be called the transition representation. If the interaction is small compared with the unperturbed electron motion, the transition representation provides a direct base for perturbation calculus.

We shall write down the transition representation only for the unitary description.

5.2 *Transformation.* In the transition representation the wave functions $(x_1, \dots, x_n | \Psi)$ are represented by their expansion coefficients with respect to the products $(x_1 | \psi_1^0 | \mu_1) \dots (x_n | \psi_n^0 | \mu_n)$ of unperturbed individual wave functions. These coefficients are given by

$$(\mu_1 t_1, \dots, \mu_n t_n | \Psi = \\ = \int \dots \int (\vec{dx}_1) \dots (\vec{dx}_n) (\mu_n | \psi_n^0 | x_n) \dots (\mu_1 | \psi_1^0 | x_1) (x_1, \dots, x_n | \Psi). \quad (72)$$

A 2-electron operator $\mathbf{Q} \{x_k, x_l\}$ is represented by $\mathbf{Q} \{\mu_k t_k, \mu_l t_l\}$, which is given by

$$\mathbf{Q} \{\mu_k t_k, \mu_l t_l\} (\mu_1 t_1, \dots, \mu_n t_n | \Psi = \sum_{\mu'_k \mu'_l} (\mu_k t_k, \mu_l t_l | \mathbf{Q} \{x_k, x_l\} | \mu'_k t_k, \mu'_l t_l) \\ (\mu_1 t_1, \dots, \mu'_k t_k, \dots, \mu'_l t_l, \dots, \mu_n t_n | \Psi, \dots. \quad (73)$$

with the matrix elements

$$(\mu_k t_k, \mu_l t_l | \mathbf{Q} \{x_k, x_l\} | \mu'_k t_k, \mu'_l t_l) = \\ = \int \int (\vec{dx}_k) (\vec{dx}_l) (\mu_l | \psi_l^\dagger | x_l) (\mu_k | \psi_k^\dagger | x_k) \mathbf{Q} \{x_k, x_l\} (x_k | \psi_k | \mu'_k) (x_l | \psi_l | \mu'_l). \quad (74)$$

The generalization to 1- and more-electron operators is obvious.

5.3 *Transition equations.* Because in the equations of motion (39) and (39*) the N 's are removed by the transformation, the limiting process can now formally be performed by replacing the sums in the exponents by integrals. The latter have still to be understood symbolically as the integrands are well ordered in time. Just as the sums of exponents actually stand for products of exponentials, the integrals stand for productals¹⁴⁾. The productants are well ordered in time.

The original equations of motion (39) are transformed into the transition equations

$$\left. \begin{aligned} & (\mu_1 t_1, \dots, \mu_n t_n | \Psi_{t_1=\dots=t_{r-1}=t_{r+1}=\dots=t_n \rightarrow +\infty} \\ & \frac{ie^2}{\hbar c^2} \int \int \int dt_k' dt_l' a_k^\alpha | \mu_k t_k' \rangle a_l^\alpha \langle \mu_l t_l' | W_{\alpha\alpha'} \langle \mu_k t_k' | \mu_l t_l' \\ & e t_k' t_l') \\ & (\mu_1 t_1, \dots, \mu_n t_n | \Psi_{t_1=\dots=t_{r-1}=t_{r+1}=\dots=t_n \rightarrow -\infty} \quad (r=1, \dots, n). \end{aligned} \right\} . \quad (75)$$

The reduced transition equations (75*) are obtained from the original ones (75) in replacing $W_{\alpha\alpha'}$ by $W_{\alpha\alpha' tr} + W_{\alpha\alpha' nt}$.

The transition representation makes no use of FEYNMAN's imperfect calculus. It is particularly suitable for applications such as in 3. The time-space representation, which has been made possible by FEYNMAN's calculus, is the more appropriate form for dealing with general topics such as the present derivations and problems of electron self interaction and pair processes, belonging to Q_3 .

5.4 *COULOMB interaction.* As the COULOMB interaction represented in (75*) by $W_{\alpha\alpha' nt}^*$ is in general not small compared with the unperturbed electron motion, it is often desirable to include it in the unperturbed motion. The unperturbed motions of the various electrons are then no longer independent of each other and the wave function of the entire system can no longer be written as a product of individual wave functions of separate electrons. The separation into individual functions can in many cases be restored by an approximation in which for each electron the instantaneous COULOMB fields due to the other electrons are replaced by an appropriate average field. For different electrons we get in general different average fields. The approximation will fail as soon as the correlations between the unperturbed motions of the various electrons begin to act a part. The average fields together with the fields due to other kinds of charges (e.g. protons) constitute the unperturbed fields A_k in (34).

If the radiative interaction represented by the remaining interaction function $W_{\alpha\alpha' tr}$ in (75*) is sufficiently small, it can be treated as a perturbation.

5.5 *Paired transitions.* Remembering (73) we see that the transition processes described by (75*) are composed of elementary paired transitions of ever 2 electrons during infinitesimal time intervals. These paired transitions and their compositions will be discussed in 3.

6. Conclusion.

6.1 *D-modification or H-revision.* Arguments against the cases *Dh* and *Hh* have been given in 3.35 and 4.32. A further argument can be found from a comparison between the photon case and the case of neutral carrier particles with spin 1 and $m \neq 0$ (neutral mesons) of which the photon case is a degenerate case ($m = 0$). For $m \neq 0$ there occur no negative vectors, only negative energy states. In *D-modification* and in *H-revision* the interaction function $W_{\alpha\alpha'}$ is given by

$$\left. \begin{array}{c} \frac{i}{\hbar c} W_{\alpha\alpha'}(\mathbf{x}, \mathbf{x}') \\ \hline D \quad g_{\alpha\alpha'} D_a^\sigma(\mathbf{x}-\mathbf{x}') \\ H \quad g_{\alpha\alpha'} D_+^\sigma(\mathbf{x}-\mathbf{x}'). \end{array} \right\} \dots \quad (76)$$

with D_a and D_+ given by (1.27). For $m \rightarrow 0$ the interaction functions for the cases *D* and *H* degenerate into the photon interaction functions respectively for the cases *Dd* and *Hd*. Therefore all evidence is against the cases *Dh* and *Hh*.

It has been mentioned already in 3.36 that it will be seen in 3 that the cases *Dd* and *Dh* are excluded by the asymmetry condition of 1.51.

All things considered, the only case which might leave some hope for unification is *Hd*. That means that for photons the negative energy states have to be treated according to hole theory (*H-revision*), the negative vectors according to the original theory (*D-modification*).

6.2 *Equations of motion.* If a unitary theory were feasible at all, it might be expected to be characterized by integral equations of motion of the type (39). This is an intractable type. The main difficulties lie in the consistency and boundary (initial) conditions.

6.3 *Introduction of photons.* If the unitary theory had been completely established, fictitious photon states with wave operators to match could be introduced as in 2.1. The exponential function in (39) could be written as the empty-empty part of the exponential operator in (37). Because no photons are supposed to be present in the initial and final states we could take the entire operator as well. Doing so we could then cut off at finite initial and final times not only for 1 but for all electrons at a time. If all this were allowed, we would have obtained a dualistic description. The correct form of the empty-empty part could only be obtained with BOSE-EINSTEIN statistics. The fundamental part of these statistics will become still more conspicuous in the treatment of transition processes in 3.

In the dualized description (52) would hold but (53) in general would not. Now in the unitary description the interaction function could be reduced to radiative interaction and instantaneous COULOMB interaction. In the dualized description a corresponding reduction would require the introduction of the supplementary conditions (53).

6.4 Applications. The solution of the consistency and boundary problems has to decide about the applicability of the unitary description. The results of the practical applications have to decide about the correctness of the unitary description as compared with the dualistic one.

6.5 Generalization. The present tentative treatment of the photon case could in principle be generalized to other kinds of carrier particles, provided they obey BOSE-EINSTEIN statistics. For neutral mesons no essentially new difficulties occur. For charged mesons, however, there is a further complication. During the time between their emission and absorption these particles cannot be treated as free because of the presence of other charges. That makes that in the unitary description the interaction function $W(x, x')$ will not only depend on the variables (x) and (x') of the interacting source particles (nucleons), but also on the electromagnetic field (or on all electric charges). That would considerably aggravate the intractability of the description.

Summary.

If a unitary charge theory with advanced and retarded interaction at a distance is possible at all, it will be almost intractable anyhow. In spite of this it has been asked how such a theory would have to look like. Even an imperfect answer might show some aspects of the dualistic charge-field theory, in particular about the properties of the field, which are less conspicuous in the common description. The present paper deals with the presumable type of the equations of motion of electrons. Further after the interaction has been split up into a transverse and a non-transverse part, the latter part has been reduced to simultaneous COULOMB interaction.

REFERENCES.

1. H. J. GROENEWOLD, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 977-1091 (1948).
2. J. SCHWINGER, Phys. Rev. (2) **73**, 416 (1948).
3. H. TETRODE, Z. Phys. **10**, 317 (1922).
4. A. D. FOKKER, Physica Ned. T. Natuurk. **9**, 33 (1929); **12**, 145 (1932); Z. Phys. **58**, 386 (1929); Arch. Teyler (3) **7**, 176 (1932).
5. J. A. WHEELER and R. P. FEYNMAN, Rev. Mod. Phys. **17**, 157 (1945).
6. R. P. FEYNMAN, Phys. Rev. (2) **74**, 939 (1948).
7. H. J. GROENEWOLD, Physica **13**, 79 (1947).
8. P. A. M. DIRAC, Proc. Roy. Soc. London, **A 180**, 1 (1942).
9. R. P. FEYNMAN, Rev. Mod. Phys. **20**, 367 (1948).
10. P. A. M. DIRAC, V. A. FOCK and B. PODOLSKI, Sow. Phys. **2**, 468 (1932).
11. V. A. FOCK, Sow. Phys. **6**, 425 (1934).
12. F. BLOCH, Sow. Phys. **5**, 301 (1933).
13. SONJA ASHAUER, Proc. Roy. Soc. London, **A 194**, 206 (1948).
14. H. REICHENBACH, Wahrscheinlichkeitslehre, Leiden (1935).

Mathematics. — *A non-homogeneous inequality for integers in a special cubic field.* (First communication.) By A. V. PRASAD. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 29, 1949.)

1. Let ξ, η, ζ be three real linear forms in integral variables u, v, w with determinant $\Delta \neq 0$. Let $M(\xi, \eta, \zeta)$ be the lower bound of the numbers λ such that, if a, b, c are any real numbers, then there exist integers u, v, w satisfying

$$|(\xi - a)(\eta - b)(\zeta - c)| \leq \lambda |\Delta|.$$

Then it follows by a well known theorem of REMAK¹⁾ that

$$M(\xi, \eta, \zeta) \leq \frac{1}{8}. \quad \dots \dots \dots \quad (1)$$

The value of $M(\xi, \eta, \zeta)$ will in fact be $\frac{1}{8}$, when ξ, η, ζ have certain special forms; in particular when $\xi = u, \eta = v, \zeta = w$. DAVENPORT²⁾ has proved that, if none of the forms ξ, η, ζ represents zero for integral values of u, v, w other than 0, 0, 0 then

$$M(\xi, \eta, \zeta) < \frac{1}{8}.$$

Further DAVENPORT³⁾ determines the exact value of $M(\xi, \eta, \zeta)$ in the two cases when

$$\xi = u + \theta_1 v + \theta_1^2 w, \eta = u + \phi_1 v + \phi_1^2 w, \zeta = u + \psi_1 v + \psi_1^2 w, \quad (2)$$

where θ_1, ϕ_1, ψ_1 are the roots of

$$t^3 + t^2 - 2t - 1 = 0,$$

and θ_2, ϕ_2, ψ_2 are the roots of

$$t^3 - 3t - 1 = 0.$$

In the first case the modulus of the determinant of the forms is 7 and $M(\xi, \eta, \zeta) = \frac{1}{14}$ and in the second case the modulus of the determinant is 9 and $M(\xi, \eta, \zeta) = \frac{1}{27}$. In each case the product of the three forms is the norm of the general integer of the cubic field $k(\theta_1)$; the fields $k(\theta_1)$ and $k(\theta_2)$ being the totally real cubic fields of least discriminants.

In this paper we are concerned with the product of one real linear form ξ and two conjugate complex linear forms $\eta, \bar{\eta}$ in integral variables u, v, w , with determinant $\Delta \neq 0$. Let $M(\xi, \eta, \bar{\eta})$ be the lower bound of the numbers

¹⁾ R. REMAK, Math. Zeitschrift, **17**, 1—34 (1923) and **18**, 173—200 (1923). For a simpler proof see H. DAVENPORT, Jour. London Math. Soc., **14**, 47—51 (1939).

²⁾ H. DAVENPORT, Proc. Camb. Philos. Soc., **43**, 137—152 (1947).

³⁾ Ibid.

λ such that, if a is real and b and \bar{b} are conjugate complex, then there are integers u, v, w satisfying

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})| < \lambda |\Delta|.$$

Now there is no theorem corresponding to that of REMAK giving an absolute bound for $M(\xi, \eta, \bar{\eta})$. There are forms $\xi, \eta, \bar{\eta}$ for which $M(\xi, \eta, \bar{\eta})$ is arbitrarily large. In particular, if we take $\varepsilon > 0$ and

$$\left. \begin{array}{l} \xi = u, \quad \eta = v + i\varepsilon w, \quad \bar{\eta} = v - i\varepsilon w \\ a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad \bar{b} = \frac{1}{2}. \end{array} \right\} \dots \quad (3)$$

then we have for all integers u, v, w

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})| = |(u - \frac{1}{2})(v - \frac{1}{2})^2 + \varepsilon^2 w^2| \geq \frac{1}{8};$$

since the determinant of the forms (3) is $-2i\varepsilon$, this proves that

$$M(\xi, \eta, \bar{\eta}) \geq \frac{1}{16\varepsilon}$$

for these forms. Hence $M(\xi, \eta, \bar{\eta})$ can take arbitrarily large values. But it is still of interest to find the exact value of $M(\xi, \eta, \bar{\eta})$ for particular forms $\xi, \eta, \bar{\eta}$. We consider the special forms

$$\xi = u + \theta v + \theta^2 w, \quad \eta = u + \phi v + \phi^2 w, \quad \bar{\eta} = u + \bar{\phi} v + \bar{\phi}^2 w, \quad (4)$$

where $\theta, \phi, \bar{\phi}$ are the real and complex roots of the equation

$$t^3 - t - 1 = 0. \dots \quad (5)$$

The determinant of these forms has modulus $\sqrt[3]{23}$; we prove in this paper that

$$M(\xi, \eta, \bar{\eta}) = \frac{1}{5\sqrt[3]{23}}, \dots \quad (6)$$

These forms are of special interest as $\xi \eta \bar{\eta}$ is the norm of the general integer of the field $k(\theta)$, which is the cubic field with least discriminant.

Our result (6) can be regarded from a different point of view. We use ξ to denote any integer of the field $k(\theta)$ and we use η and $\bar{\eta}$ to denote the conjugates of ξ in the fields $k(\phi)$ and $k(\bar{\phi})$. Then since $1, \theta, \theta^2$ are a basis of the field $k(\theta)$, it follows that $\xi, \eta, \bar{\eta}$ can be expressed in the form (4) for suitable rational integers u, v, w . Conversely it is clear that, if $\xi, \eta, \bar{\eta}$ are given by (4), where u, v, w are any rational integers, then $\xi, \eta, \bar{\eta}$ are conjugate integers of the fields $k(\theta), k(\phi), k(\bar{\phi})$. Consequently $M(\xi, \eta, \bar{\eta}) \cdot |\Delta|$ is identical with the lower bound $M(\theta)$ of the numbers λ such that for every real a and complex conjugate b, \bar{b} there is an integer ξ of $k(\theta)$ such that

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})| < \lambda.$$

With this notation our result (6) becomes

$$M(\theta) = \frac{1}{5}. \dots \quad (7)$$

This result is amply sufficient to prove the existence of EUCLID's algorithm in the field $k(\theta)$. For, if a and β are any integers of $k(\theta)$ with $\beta \neq 0$, there is an integer ξ of $k(\theta)$ satisfying ⁴⁾ $|N\left(\xi - \frac{a}{\beta}\right)| < 1$, and for this integer

$$|N(a - \xi\beta)| < |N(\beta)|.$$

We are able in fact to prove a result rather more precise than (7). We prove the following theorem.

Theorem. *Let a be any real number and let b and \bar{b} be any pair of conjugate complex numbers. Then there is an integer ξ of $k(\theta)$ with conjugates η and $\bar{\eta}$ in $k(\phi)$ and $k(\bar{\phi})$ for which*

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})| < \frac{1}{5 \cdot 0001}, \quad \dots \dots \quad (8)$$

unless a, b, \bar{b} are of the form

$$a = \frac{\sigma}{\theta^2 + 1} + \xi_1, \quad b = \frac{\tau}{\phi^2 + 1} + \eta_1, \quad \bar{b} = \frac{\bar{\tau}}{\bar{\phi}^2 + 1} + \bar{\eta}_1, \quad \dots \dots \quad (9)$$

where $\xi_1, \eta_1, \bar{\eta}_1$ are conjugate integers and $\sigma, \tau, \bar{\tau}$ are conjugate units of the fields $k(\theta), k(\phi), k(\bar{\phi})$. If a, b, \bar{b} are of the form (9) then there is no integer ξ of $k(\theta)$, with conjugates $\eta, \bar{\eta}$ satisfying

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})| < \frac{1}{5}.$$

but there are an infinite number of integers ξ of $k(\theta)$, with conjugates $\eta, \bar{\eta}$ satisfying

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})| = \frac{1}{5}. \quad \dots \dots \quad (10)$$

The proof is based on the ideas used by DAVENPORT in the case of three real forms, but a new technique is needed to handle the inequalities in the present case, and to prove that the result is "isolated".

I am grateful to Professor DAVENPORT for suggesting this problem to me and for his guidance. I am also grateful to Mr. C. A. ROGERS for some suggestions and for helping me to prepare the manuscript.

2. Outline of the proof of the theorem.

Our proof depends on a number of lemmas. We give here the proof of the main theorem assuming the truth of the lemmas, which will be proved later.

Let a be any real number and let b and \bar{b} be any pair of conjugate complex numbers.

Let $M = M(a, b, \bar{b})$ be the lower bound of

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})|$$

⁴⁾ Here $N(a)$ denotes as usual the norm of the number a of $k(\theta)$, i.e. the product of a by its conjugates in the fields $k(\phi)$ and $k(\bar{\phi})$.

for all integers ξ of $k(\theta)$. We suppose that

$$M \geq \frac{1}{5+\delta} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (11)$$

where we write, for convenience,

$$\delta = 0.0001, \cdot \quad (12)$$

and we deduce eventually that a, b, \bar{b} must then be of the form (9). This will prove the first assertion of the theorem. From the definition of M it is clear that corresponding to any positive number ε_1 , there is an integer ξ_1 of $k(\theta)$ satisfying

$$|(\xi_1 - a)(\eta_1 - b)(\bar{\eta}_1 - \bar{b})| = \frac{M}{1-\varepsilon} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (13)$$

where

$$0 \leq \varepsilon < \varepsilon_1, \cdot \quad (14)$$

We suppose that ε_1 is sufficiently small; in the course of our proof we suppose that ε_1 is less than a finite number of positive absolute constants.

We write

$$a = \frac{1}{a - \xi_1}, \quad \beta = \frac{1}{b - \eta_1}, \quad \bar{\beta} = \frac{1}{\bar{b} - \bar{\eta}_1} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (15)$$

Then, by (13) and (11),

$$|a\beta\bar{\beta}| = \frac{1-\varepsilon}{M} \leq (5+\delta)(1-\varepsilon), \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (16)$$

and by (15), (16) and the definition of M ,

$$\begin{aligned} & |(\alpha\xi - 1)(\beta\eta - 1)(\bar{\beta}\bar{\eta} - 1)| \\ &= |a\beta\bar{\beta}| \cdot \left| \left(\xi - \frac{1}{a} \right) \left(\eta - \frac{1}{\beta} \right) \left(\bar{\eta} - \frac{1}{\bar{\beta}} \right) \right| \\ &= |a\beta\bar{\beta}| \cdot |(\xi + \xi_1 - a)(\eta + \eta_1 - b)(\bar{\eta} + \bar{\eta}_1 - \bar{b})| \\ &\geq (1-\varepsilon), \end{aligned} \quad \left. \right\} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (17)$$

for all integers ξ of $k(\theta)$. Consequently we have

$$|(\alpha - \xi^{-1})(\beta - \eta^{-1})(\bar{\beta} - \bar{\eta}^{-1})| \geq (1-\varepsilon) |\xi\eta\bar{\eta}|^{-1}, \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (18)$$

for all integers $\xi \neq 0$ of $k(\theta)$.

Our object is to prove that, if α is real, and $\beta, \bar{\beta}$ are conjugate complex numbers, such that (16) is satisfied and (17) and (18) are satisfied for all integers $\xi \neq 0$ of $k(\theta)$ then $\alpha, \beta, \bar{\beta}$ must be of certain special forms. It follows from Lemma 1 that there is no real loss of generality in assuming that

$$\frac{\alpha}{\theta^2} \leq |\beta|^2 \leq \alpha, \cdot \quad (19)$$

Using this assumption together with (16) and (17) and (18) for various

integers $\xi \neq 0$ of $k(\theta)$, we prove by the Lemmas 2—20 that $a, \beta, \bar{\beta}$ are given by

$$a = \theta^2 + 1, \beta = \phi^2 + 1, \bar{\beta} = \bar{\phi}^2 + 1. \dots \quad (20)$$

It follows by Lemma 1 that in the general case, when (19) is not necessarily satisfied, we must have

$$a = \frac{(\theta^2 + 1)}{\sigma}, \beta = \frac{(\phi^2 + 1)}{\tau}, \bar{\beta} = \frac{(\bar{\phi}^2 + 1)}{\bar{\tau}}, \dots \quad (21)$$

for some conjugate units $\sigma, \tau, \bar{\tau}$ of the fields $k(\theta), k(\phi), k(\bar{\phi})$. It now follows by (15) that a, b, \bar{b} are of the form (9). This completes the proof of the first assertion of the theorem.

The proof of the last assertion of the theorem is given as Lemma 21.

3. Lemma 1. *Let a be any real number and let $\beta, \bar{\beta}$ be any conjugate complex numbers such that $|a\beta\bar{\beta}| \neq 0$ and*

(a) *for every integer ξ of $k(\theta)$*

$$|(a\xi - 1)(\beta\eta - 1)(\bar{\beta}\bar{\eta} - 1)| \geq 1 - \varepsilon. \dots \quad (22)$$

Then for some conjugate units $\sigma, \tau, \bar{\tau}$ of the fields $k(\theta), k(\phi), k(\bar{\phi})$ the numbers $a^ = a\sigma, \beta^* = \beta\tau$ and $\bar{\beta}^* = \bar{\beta}\bar{\tau}$ have the property (a) and satisfy*

$$|a^*\beta^*\bar{\beta}^*| = |a\beta\bar{\beta}| \dots \quad (23)$$

and

$$\frac{a^*}{\theta^2} \leq |\beta^*|^2 < a^*. \dots \quad (24)$$

Proof. We obviously have $a \neq 0$ and $\beta \neq 0$. We can choose a rational integer n so that

$$\frac{|\phi|^2}{\theta} \leq \frac{|\beta\phi^n|^2}{|\alpha\theta^n|} < 1. \dots \quad (25)$$

We take

$$\sigma = \pm\theta^n, \tau = \pm\phi^n, \bar{\tau} = \pm\bar{\phi}^n, \dots \quad (26)$$

the sign being chosen to ensure that

$$\pm a\theta^n > 0. \dots \quad (27)$$

Then as $\theta, \phi, \bar{\phi}$ are conjugate units of $k(\theta), k(\phi), k(\bar{\phi})$ it follows that $\sigma, \tau, \bar{\tau}$ are conjugate units of these fields. It follows from (25), (26) and (27) that the numbers $a^* = a\sigma, \beta^* = \beta\tau, \bar{\beta}^* = \bar{\beta}\bar{\tau}$ satisfy (23) and (24). Further, if ξ is any integer of $k(\theta)$, $\sigma\xi$ is also an integer of $k(\theta)$ and so using (22) we have

$$|(a^*\xi - 1)(\beta^*\eta - 1)(\bar{\beta}^*\bar{\eta} - 1)| = |(a\sigma\xi - 1)(\beta\tau\eta - 1)(\bar{\beta}\bar{\tau}\bar{\eta} - 1)| \geq 1 - \varepsilon,$$

for all integers ξ of $k(\theta)$. Thus $a^*, \beta^*, \bar{\beta}^*$ have the property (a). This proves the lemma.

4. Lemma 2. Let a be any real number and let b and \bar{b} be any pair of conjugate complex numbers. Then there is an integer ξ of $k(\theta)$ satisfying

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})| \leq \frac{1}{2} \dots \dots \dots \quad (28)$$

Proof. For any such a, b, \bar{b} we define u_0, v_0, w_0 by the equations

$$\left. \begin{aligned} a &= u_0 + v_0 \theta + w_0 \theta^2, \\ b &= u_0 + v_0 \phi + w_0 \phi^2, \\ \bar{b} &= u_0 + v_0 \bar{\phi} + w_0 \bar{\phi}^2, \end{aligned} \right\} \dots \dots \dots \quad (29)$$

these equations having a unique solution, since the determinant

$$(\phi - \bar{\phi})(\bar{\phi} - \theta)(\theta - \phi)$$

of the forms on the right hand side does not vanish. Writing the equations (29) in the form

$$\left. \begin{aligned} a &= u_0 + v_0 \theta + w_0 \theta^2, \\ b + \bar{b} &= 2u_0 + v_0(\phi + \bar{\phi}) + w_0(\phi^2 + \bar{\phi}^2), \\ i(b - \bar{b}) &= v_0 i(\phi - \bar{\phi}) + w_0 i(\phi^2 - \bar{\phi}^2), \end{aligned} \right\}$$

we see that u_0, v_0, w_0 are real, b, \bar{b} and $\phi, \bar{\phi}$ being conjugate. We take

$$\xi = u + v \theta + w \theta^2. \dots \dots \dots \quad (30)$$

where u, v, w are integers satisfying

$$|u - u_0| \leq \frac{1}{2}, \quad |v - v_0| \leq \frac{1}{2}, \quad |w - w_0| \leq \frac{1}{2}.$$

Then ξ is an integer of $k(\theta)$ and writing

$$u - u_0 = x, \quad v - v_0 = y, \quad w - w_0 = z,$$

we have

$$\left. \begin{aligned} &(\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &= \{(u - u_0) + (v - v_0)\theta + (w - w_0)\theta^2\} \\ &\times \{(u - u_0) + (v - v_0)\phi + (w - w_0)\phi^2\} \\ &\times \{(u - u_0) + (v - v_0)\bar{\phi} + (w - w_0)\bar{\phi}^2\} \\ &= (x + \theta y + \theta^2 z)(x + \phi y + \phi^2 z)(x + \bar{\phi} y + \bar{\phi}^2 z). \end{aligned} \right\} \quad (31)$$

where

$$|x| \leq \frac{1}{2}, \quad |y| \leq \frac{1}{2}, \quad |z| \leq \frac{1}{2}. \dots \dots \dots \quad (32)$$

Now multiplying out the expression on the right of (31) and using the fact that $\theta, \phi, \bar{\phi}$ are the roots of the equation (5) we obtain

$$(\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) = x^3 + y^3 + z^3 + 2x^2 z - y^2 x + z^2 x - z^2 y - 3xyz. \quad (33)$$

We use (32) and (33) to show that the integer ξ of $k(\theta)$ given by (30) satisfies (28). In proving this we may clearly suppose without loss of generality that $x \geq 0$. We consider four cases separately.

Case (i). When $x \geq 0, y \geq 0, z \geq 0$ we have

$$\begin{aligned} -\frac{5}{8} &\leq -y^2x - z^2y - 3xyz \\ &\leq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leq x^3 + y^3 + z^3 + 2x^2z + z^2x \leq \frac{6}{8}. \end{aligned}$$

Case (ii). When $x \geq 0, y \geq 0, z \leq 0$ we have

$$\begin{aligned} -\frac{5}{8} &\leq -|z|^3 - 2x^2|z| - y^2x - z^2y \\ &\leq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leq x^3 + y^3 + z^3 + 3xy|z| \leq \frac{6}{8}. \end{aligned}$$

Case (iii). When $x \geq 0, y \leq 0, z \geq 0$ we have

$$\begin{aligned} -\frac{5}{8} &\leq -|y|^3 - y^2x \\ &\leq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leq x^3 + z^3 + 2x^2z + z^2x + z^2|y| + 3x|y|z \leq \frac{6}{8}. \end{aligned}$$

Case (iv). When $x \geq 0, y \leq 0, z \leq 0$ we have

$$\begin{aligned} -\frac{5}{8} &\leq -|y|^3 - |z|^3 - 2x^2|z| - y^2x - 3x|yz| \\ &\leq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leq x^3 + z^3 + z^2|y| \leq \frac{6}{8}. \end{aligned}$$

These results prove that ξ satisfies (28), and so prove the lemma.

5. In Lemma 3—20 we suppose that $a, \beta, \bar{\beta}$ satisfy (16) and (19), and that (17) and (18) are satisfied for all integers $\xi \neq 0$ of $k(\theta)$, and we prove that $a, \beta, \bar{\beta}$ are necessarily given by (20).

We summarize here certain identities satisfied by $\theta, \phi, \bar{\phi}$ and some numerical results.

We have

$$\begin{aligned} \theta^3 &= \theta + 1, \quad \theta^4 = \theta^2 + \theta, \quad \theta^5 = \theta^2 + \theta + 1, \\ \theta^{-1} &= \theta^2 - 1, \quad \theta^{-2} = 1 + \theta - \theta^2, \quad \theta^{-3} = \theta^2 - \theta, \\ \theta^{-4} &= \theta - 1, \quad \theta^{-5} = 2 - \theta^2, \quad \theta^{-6} = 2\theta^2 - \theta - 2, \quad \theta^{-7} = 1 + 2\theta - 2\theta^2, \end{aligned}$$

and the same identities are satisfied by ϕ and $\bar{\phi}$. We have the symmetrical results

$$\theta\phi\bar{\phi} = 1, \quad \phi\bar{\phi} + \bar{\phi}\theta + \theta\phi = -1, \quad \theta + \phi + \bar{\phi} = 0.$$

The number ϕ is given by

$$\phi = -\frac{1}{2}\theta + i\sqrt{\frac{3}{4}\theta^2 - 1}.$$

Numerically,

$$\theta = 1 \cdot 3247 \ 1795 \dots, \quad \phi = -0 \cdot 6623 \ 5897 \dots + i0 \cdot 5622 \ 7951 \dots;$$

also

$$\theta^2 = 1 \cdot 7548 \ 7766 \dots, \quad \theta^4 = 1 \cdot 1509 \ 6392 \dots.$$

Using these results we obtain the following numerical values for the various powers of θ :

n	θ^n	n	θ^n
5	4.079595...	-4	0.868836...
4	3.079595...	-3	0.754877...
7	2.675666...	-2	0.655865...
3	2.324717...	-1	0.569840...
5	2.019800...	-6	0.430159...
2	1.754877...	-5	0.324717...
4	1.524702...	-4	0.245122...
1	1.324717...	-3	0.185037...
5	1.150963...	-2	0.139680...

Lemma 3.

$$\alpha \geq \{\frac{8}{9}(1-\varepsilon)\}^{\frac{1}{2}} \dots \dots \dots \quad (34)$$

Proof. By (17) we have

$$\left| \left(\xi - \frac{1}{\alpha} \right) \left(\eta - \frac{1}{\beta} \right) \left(\bar{\eta} - \frac{1}{\bar{\beta}} \right) \right| \geq \frac{1-\varepsilon}{|\alpha \beta \bar{\beta}|},$$

for all integers ξ of $k(\theta)$. It follows from Lemma 2 that

$$\frac{1-\varepsilon}{|\alpha \beta \bar{\beta}|} \leq \frac{8}{9} \dots \dots \dots \quad (35)$$

So, using (19),

$$\alpha \geq |\alpha| |\beta|^2 \geq \{\frac{8}{9}(1-\varepsilon)\}^{\frac{1}{2}},$$

which gives (34).

Lemma 4.

$$\alpha > 1 + \theta^{-7} \dots \dots \dots \quad (36)$$

Proof. Suppose, if possible, that $\alpha \leq 1 + \theta^{-7}$. It is easy to verify that

$$1 - \theta^{-7} = 2\theta^2 - 2\theta = 0.860 \dots < \{\frac{8}{9}(1-\varepsilon)\}^{\frac{1}{2}},$$

if ε is sufficiently small. So by (34) and our supposition, we have

$$1 - \theta^{-7} < \alpha \leq 1 + \theta^{-7}.$$

Hence, using (17), with $\xi = 1$,

$$|(\beta - 1)(\bar{\beta} - 1)| \geq \frac{1-\varepsilon}{|\alpha - 1|} \geq (1-\varepsilon) \theta^7,$$

so that

$$|\beta - 1| \geq \sqrt{(1-\varepsilon)} \theta^{\frac{7}{2}} > 2.6,$$

if ε is sufficiently small. Using this result and (19) we obtain

$$1.6 < |\beta| \leq \sqrt{\alpha} \leq \sqrt{1 + \theta^{-7}} < 1.1,$$

a contradiction. This proves the lemma.

Lemma 5.

$$\alpha > \theta + \theta^{-6} \dots \dots \dots \dots \quad (37)$$

Proof. Suppose, if possible, that $\alpha \leq \theta + \theta^{-6}$. Now

$$1 + \theta^{-7} = 2 + 2\theta - 2\theta^2 = \theta - \theta^{-6}.$$

It follows by Lemma 4 that

$$|\alpha - \theta| \leq \theta^{-6}.$$

Hence, using (18), with $\xi = \theta^{-1}$,

$$|(\beta - \phi)(\bar{\beta} - \bar{\phi})| \geq (1 - \varepsilon) \theta^6,$$

so that

$$|\beta - \phi| \geq \gamma (1 - \varepsilon) \theta^3 > 2 \cdot 3,$$

if ε is sufficiently small. Using this result and (19) we obtain

$$\begin{aligned} 1 \cdot 4 &< 2 \cdot 3 - \theta^{-1} = 2 \cdot 3 - |\phi| \\ &\leq |\beta| \leq \sqrt{\alpha} \\ &\leq \sqrt{\{\theta + \theta^{-6}\}} = \sqrt{2\theta^2 - 2} < 1 \cdot 3, \end{aligned}$$

a contradiction. This proves the lemma.

Lemma 6.

$$\alpha > \theta^2 \dots \dots \dots \dots \quad (38)$$

Proof. Suppose, if possible, that $\alpha \leq \theta^2$. By (18) with $\xi = \theta^{-1}$ and $\xi = \theta^{-2}$ we have

$$|(\alpha - \theta)(\beta - \phi)(\bar{\beta} - \bar{\phi})| \geq 1 - \varepsilon,$$

$$|(\alpha - \theta^2)(\beta - \phi^2)(\bar{\beta} - \bar{\phi}^2)| \geq 1 - \varepsilon.$$

Hence

$$|\beta - \phi|^2 + |\beta - \phi^2|^2 \geq \frac{1 - \varepsilon}{|\alpha - \theta|} + \frac{1 - \varepsilon}{|\alpha - \theta^2|} \dots \dots \quad (39)$$

But by Lemma 5 and our supposition

$$\theta < \alpha \leq \theta^2,$$

and so

$$\begin{aligned} \frac{1}{|\alpha - \theta|} + \frac{1}{|\alpha - \theta^2|} \\ = \frac{\theta^2 - \theta}{(\alpha - \theta)(\theta^2 - \alpha)} \geq \frac{\theta^2 - \theta}{|\frac{1}{2}(\alpha - \theta + \theta^2 - \alpha)|^2} \\ = \frac{4}{\theta^2 - \theta} = 4\theta^3. \end{aligned}$$

Thus, if ε is sufficiently small, we have by (39)

$$|\beta - \phi|^2 + |\beta - \phi^2|^2 > 9 \cdot 2 \dots \dots \dots \quad (40)$$

We now use the identity

$$|2\beta - \eta_1 - \eta_2|^2 = 2|\beta - \eta_1|^2 + 2|\beta - \eta_2|^2 - |\eta_2 - \eta_1|^2. \dots \quad (41)$$

with $\eta_1 = \phi$, $\eta_2 = \phi^2$. Using (41) and (40) we obtain

$$\left. \begin{aligned} |\beta| &\geq \frac{1}{2} \{ |2\beta - \phi - \phi^2| - |\phi + \phi^2| \} \\ &= \frac{1}{2} \{ 2|\beta - \phi|^2 + 2|\beta - \phi^2|^2 - |\phi^2 - \phi|^2 \}^{\frac{1}{2}} - \frac{1}{2} |\phi + \phi^2| \\ &> \frac{1}{2} \{ 18 \cdot 4 - |\phi|^{-6} \}^{\frac{1}{2}} - \frac{1}{2} |\phi|^4 \\ &= \frac{1}{2} \{ 18 \cdot 4 - \theta^3 \}^{\frac{1}{2}} - \frac{1}{2} \theta^{-2} \\ &> \frac{1}{2} \{ 18 \cdot 4 - 2 \cdot 4 \}^{\frac{1}{2}} - 0 \cdot 3 = 1 \cdot 7. \end{aligned} \right\}. \quad (42)$$

But, by (19) and our supposition concerning a ,

$$|\beta| \leq \sqrt{a} \leq \theta < 1 \cdot 4,$$

contrary to (42). This contradiction proves the lemma.

Lemma 7.

$$a > \theta^3 - \theta^{-6}. \quad \dots \quad (43)$$

Proof. Suppose, if possible, that $a \leq \theta^3 - \theta^{-6}$. By (18) with $\xi = \theta^{-2}$ $\xi = \theta^{-3}$ we have

$$\begin{aligned} |(\alpha - \theta^2)(\beta - \phi^2)(\bar{\beta} - \bar{\phi}^2)| &\geq 1 - \varepsilon, \\ |(\alpha - \theta^3)(\beta - \phi^3)(\bar{\beta} - \bar{\phi}^3)| &\geq 1 - \varepsilon. \end{aligned}$$

Hence

$$|\beta - \phi^2|^2 + |\beta - \phi^3|^2 \geq \frac{1 - \varepsilon}{|\alpha - \theta^2|} + \frac{1 - \varepsilon}{|\alpha - \theta^3|}. \quad (44)$$

But by Lemma 6 and our supposition

$$\theta^2 < a < \theta^3,$$

and so

$$\begin{aligned} \frac{1}{|\alpha - \theta^2|} + \frac{1}{|\alpha - \theta^3|} \\ = \frac{\theta^3 - \theta^2}{(a - \theta^2)(\theta^3 - a)} \\ \geq \frac{\theta^3 - \theta^2}{\{\frac{1}{2}(a - \theta^2 + \theta^3 - a)\}^2} = \frac{4}{\theta^3 - \theta^2} = 4\theta^2. \end{aligned}$$

Thus, if ε is sufficiently small, we have, by (44),

$$|\beta - \phi^2|^2 + |\beta - \phi^3|^2 > 7 \cdot 01. \quad \dots \quad (45)$$

Using the identity (41) with $\eta_1 = \phi^2$, $\eta_2 = \phi^3$, and (45), we obtain

$$\left. \begin{aligned} |\beta| &\geq \frac{1}{2} \{ |2\beta - \phi^2 - \phi^3| - |\phi^2 + \phi^3| \} \\ &= \frac{1}{2} \{ 2|\beta - \phi^2|^2 + 2|\beta - \phi^3|^2 - |\phi^3 - \phi^2|^2 \}^{\frac{1}{2}} - \frac{1}{2} |\phi^2 + \phi^3| \\ &> \frac{1}{2} \{ 14 \cdot 02 - |\phi|^{-4} \}^{\frac{1}{2}} - \frac{1}{2} |\phi|^5 \\ &= \frac{1}{2} \{ 14 \cdot 02 - \theta^2 \}^{\frac{1}{2}} - \frac{1}{2} \theta^{-\frac{5}{2}} \\ &> \frac{1}{2} \{ 14 \cdot 02 - 1 \cdot 76 \}^{\frac{1}{2}} - 0 \cdot 25 \\ &> \frac{1}{2} \{ 12 \cdot 25 \}^{\frac{1}{2}} - 0 \cdot 25 = 1 \cdot 5. \end{aligned} \right\} \quad (46)$$

But, by (19) and our supposition concerning α ,

$$|\beta| \leq \sqrt{\alpha} \leq \sqrt{\{\theta^3 - \theta^{-6}\}} < \sqrt{2 \cdot 33 - 0 \cdot 18} < 1 \cdot 5,$$

contrary to (46). This contradiction proves the lemma.

Lemma 8.

$$\alpha > \theta^3 + \theta^{-6}. \quad \dots \quad (47)$$

Proof. Suppose, if possible, that $\alpha \leq \theta^3 + \theta^{-6}$. Then, by Lemma 7,

$$|\alpha - \theta^3| \leq \theta^{-6}.$$

Hence, using (18), with $\xi = \theta^{-3}$,

$$|(\beta - \phi^3)(\bar{\beta} - \bar{\phi}^3)| \geq (1 - \varepsilon) \theta^6,$$

so that

$$|\beta - \phi^3| \geq \theta^3 \vee (1 - \varepsilon) > 2 \cdot 32,$$

if ε is sufficiently small. Using this result and (19) we obtain

$$\begin{aligned} 1 \cdot 66 &< 2 \cdot 32 - \theta^{-1} = 2 \cdot 32 - |\phi^3| \\ &< |\beta - \phi^3| - |\phi^3| \leq |\beta| \\ &\leq \sqrt{\alpha} \leq \sqrt{\{\theta^3 + \theta^{-6}\}} = \sqrt{2 \theta^2 - 1} \\ &< \sqrt{2 \cdot 51} < 1 \cdot 6, \end{aligned}$$

a contradiction. This proves the lemma.

(To be continued.)

Mathematics. — Absolutes in partially ordered groups. By LADISLAS FUCHS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 29, 1949.)

In a lattice-ordered group G , BIRKHOFF¹⁾ has defined the *absolute* of an element a as the element $a \cup -a$, that is, the join of a and $-a$, and has shown that many of the properties of the common absolute value are valid for the new absolute. We now generalize further this notion to every partially ordered group. The basic idea is the observation that a lattice-ordered group plays the same role in the theory of partially ordered groups as a principal ideal ring among the rings. Therefore, if we consider sets instead of elements, viz., instead of the sole element $a \cup -a$, the collection of the elements greater than or equal to $a \cup -a$, then the concept of absolute (considered now as a set) may be extended to every partially ordered group. Indeed, it may be defined in terms of the upper bounds of certain elements depending on a .

Let us recapitulate the definition of an ordered group²⁾. The group G , written additively, is called an *ordered group*, if a relation \equiv is defined in G which satisfies the following postulates:

$$\text{reflexive law: } a \equiv a; \dots \dots \dots \dots \dots \quad (1)$$

$$\text{antisymmetric law: } a \equiv b \text{ and } b \equiv a \text{ imply } a = b; \dots \dots \quad (2)$$

$$\text{transitive law: } a \equiv b \text{ and } b \equiv c \text{ imply } a \equiv c; \dots \dots \dots \quad (3)$$

$$\begin{aligned} \text{homogeneity law: } a \equiv b \text{ implies } c + a + d \equiv c + b + d \\ \text{for every } c, d \text{ in } G; \dots \dots \dots \dots \dots \quad (4) \end{aligned}$$

$$\begin{aligned} \text{the MOORE—SMITH property: for every } a, b \text{ there exists} \\ \text{a third element } c \text{ such that } c \equiv a \text{ and } c \equiv b. \dots \dots \dots \quad (5) \end{aligned}$$

The MOORE—SMITH property is known to be equivalent to the assertion that³⁾

$$\begin{aligned} \text{each element of } G \text{ is the difference of two positive} \\ \text{elements.} \dots \dots \dots \dots \dots \dots \dots \quad (6) \end{aligned}$$

Let $U(x_1, \dots, x_n) = U(x_i)$ and $L(x_1, \dots, x_n) = L(x_i)$ denote the set of all upper bounds and all lower bounds, respectively, for the finite

¹⁾ G. BIRKHOFF, Lattice-ordered groups, Ann. Math. **43**, 298—331 (1942). The notion of absolute is originally due to L. V. KANTOROVITSCH, Lineare halbgeordnete Räume, Mat. Sbornik, **44**, 121—168 (1937).

²⁾ C. J. EVERETT and S. ULAM, On ordered groups, Trans. Am. Math. Soc. **57**, 208—216 (1945).

³⁾ A. H. CLIFFORD, Partially ordered abelian groups, Ann. Math. **41**, 465—473 (1940). For the non-commutative case see BIRKHOFF, loc. cit.

number of elements x_1, \dots, x_n of G , that is, $u \in U(x_i)$ if and only if $u \geqq x_i$ for $i = 1, 2, \dots, n$, and dually, $v \in L(x_i)$ if and only if $v \leqq x_i$ for $i = 1, 2, \dots, n$. By (5) neither $U(x_i)$ nor $L(x_i)$ is empty.

There is an intrinsic connection between the sets $U(x_i)$ and $L(x_i)$, namely⁴⁾,

$$U(x_i) = -L(-x_i). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (7)$$

If $u \in U(x_i)$, that is, $u \geqq x_i$, then by making use of (4) one concludes $-u \leqq -x_i$, which shows that $-u \in L(-x_i)$, $u \in -L(-x_i)$; and vice-versa. (7) indicates that proving propositions for $L(x_i)$ is unnecessary if the corresponding one has been established for $U(x_i)$.

One can define addition between elements and sets U, L . By $a + U + b$ we shall understand the totality of the elements $a + u + b$ with $u \in U$. Similar definitions apply to $a + L + b$, $a - U + b$ and $a - L + b$. We now prove

$$a + U(x_i) + b = U(a + x_i + b). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (8)$$

Assume $u \in U(x_i)$, i.e. $u \geqq x_i$ for every i . By (4) we have $a + u + b \geqq a + x_i + b$ whence $a + u + b \in U(a + x_i + b)$. Conversely, if $y \in U(a + x_i + b)$, $y \geqq a + x_i + b$, then $u = -a + y - b \geqq x_i$, so that $u \in U(x_i)$ and $y = a + u + b \in a + U(x_i) + b$, by definition. Dually we get

$$a + L(x_i) + b = L(a + x_i + b). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (8a)$$

Another interesting and important law is

$$a - U(x_i) + b = L(a - x_i + b). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (9)$$

Let $u \in U(x_i)$, $u \geqq x_i$, then by making use of (4), $a - u + b \leqq a - x_i + b$, that is, $a - u + b \in L(a - x_i + b)$. On the other hand, if $v \in L(a - x_i + b)$, $v \leqq a - x_i + b$, then $x_i \leqq b - v + a = u$, $u \in U(x_i)$ and hence $v = a - u + b \in a - U(x_i) + b$. The dual of (9) is

$$a - L(x_i) + b = U(a - x_i + b). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (9a)$$

Putting $a = b = 0$ in (9) or in (9a) we get (7). Further applying (9) to two elements and substituting a for x_1 , b for x_2 , one is led directly to the result:

$$a - U(a, b) + b = L(a, b) \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (10)$$

and dually,

$$a - L(a, b) + b = U(a, b). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (10a)$$

By $U_1 + U_2$ is meant the totality of all elements of the form $u_1 + u_2$ where $u_1 \in U_1$, $u_2 \in U_2$. In similar ways are defined $L_1 + L_2$, $U - L$, $-L + U$, etc. It must be observed that an expression such as $U + L$ has no meaning, since the elements of the form $u + v$ ($u \in U$, $v \in L$) constitute the whole group. Indeed, if a is arbitrary in G by (6) we have $a = b - c$ with $b \geqq 0$, $c \geqq 0$. Choose an element d belonging to both U and $-L$; this is possible by (5). Then $b + d$ again belongs

⁴⁾ $-L$ consists of the elements of the form $-v$ with $v \in L$.

to U and $-d - b$ again belongs to L , consequently, $U + L$ contains $(b + d) + (-d - c) = b - c = a$.

It is obvious that the addition of the U 's is an associative operation $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$ and if G is commutative, it is commutative too.

Next we turn our attention to the proof of the relation:

$$U(x_i) + U(y_j) \subseteq U(x_i + y_j). \dots \quad (11)$$

To prove (11), suppose $t \in U(x_i)$, $u \in U(y_j)$, that is, $t \geq x_i$, $u \geq y_j$, then $t + u \geq x_i + u \geq x_i + y_j$, therefore, $t + u$ belongs to $U(x_i + y_j)$. In fact, (8) states that in case there is only one x_i or only one y_j (11) is true with the sign of equality instead of \subseteq . The same is the case even if the order is a lattice-order. However, in general, (11) is not true with $=$. For example, in the group G_2 of all two-dimensional vectors with integral components, $[m, n]$, let the order be defined by the axiom

$$[m, n] > [p, q] \text{ if and only if } m > p \text{ and } n > q.$$

Then $U_1 = U([0, 1], [1, 0])$ and $U_2 = U([1, 2], [2, 1])$ consist of vectors, both components of which are ≥ 2 and ≥ 3 , respectively, so that $U_1 + U_2$ contains the vectors with components ≥ 5 . Nevertheless $U([1, 3], [2, 2], [2, 2], [3, 1])$ consists of vectors whose components are ≥ 4 , i.e., it contains properly $U_1 + U_2$. If in G_2 it were allowed the components to be arbitrary rational numbers, or, even real numbers, then we could at once convince ourselves that instead of (11) the sharper

$$U(x_i) + U(y_j) = U(x_i + y_j) \dots \quad (12)$$

holds. Orders satisfying (12) are called *distributive*. The last example shows that lattice-order is not necessary for distributivity; but it suffices.

By the positive part of a is meant a set of elements of G such that $a^+ = U(a, 0)$, $a^- = L(a, 0)$ is called the negative part of a . From (7) we get

$$a^- = L(a, 0) = -U(-a, 0) = -(-a)^+. \dots \quad (13)$$

When we apply (10) and (10a) to $a = 0$, then to $b = 0$, we are led to the result:

$$a - a^+ = -a^+ + a = a^- \text{ and } a - a^- = -a^- + a = a^+. \dots \quad (14)$$

Hence it follows, immediately, that a commutes with the sets $-a^+$ and $-a^-$. It is also of importance that

$$a^+ \text{ and } -a^- = (-a)^+ \text{ are permutable: } a^+ + (-a)^+ = (-a)^+ + a^+. \quad (15)$$

In order to verify this assertion, use (10a) repeatedly, then we get

$$\begin{aligned} a^+ + (-a)^+ &= U(a, 0) + U(-a, 0) = (-L(0, a) + a) + (-a - L(-a, 0)) \\ &= -L(0, a) - L(-a, 0) = U(-a, 0) + U(a, 0) = (-a)^+ + a^+, \end{aligned}$$

as we wished to prove.

Let us now turn to the definition of the "absolute". The definition may be given in two different manners.

I. Absolute of the first type (1-absolute) is defined by

$$|a| = a^+ - a^- = a^+ + (-a)^+. \dots \quad (16)$$

(15) shows that an equivalent definition of $|a|$ would be $|a| = (-a)^+ + a^+$.

II. By absolute of the second type (2-absolute) we mean

$$\|a\| = U(a, -a). \dots \quad (17)$$

The connection between the two types of absolute is given by

$$|a| \subseteq \|a\| \text{ for every } a. \dots \quad (18)$$

Indeed, using (11) we obtain

$$|a| = U(a, 0) + U(-a, 0) \subseteq U(0, a, -a, 0) \subseteq U(a, -a) = \|a\|. \quad (18')$$

In general, the sign \subseteq can not be sharpened to the sign of equality; e.g. in the group G_2 considered above

$$\|[2, -1]\| \subset \|[2, -1]\|$$

for the first consists of vectors $[m, n]$ with $m \geq 4, n \geq 3$, while the second of those with $m \geq 3, n \geq 2$. But, whenever the order defined in G is distributive as well as *normal* in the sense that

$$na = a + a + \dots + a \geq 0 \text{ for some positive integer } n \text{ implies } a \geq 0, \quad (19)$$

then the 1-absolute and the 2-absolute are equal. In proving the equality, first we show that normality implies $U(a, -a, 0) = U(a, -a)$. It is clearly enough to prove that $U(a, -a) \subseteq U(a, -a, 0)$, or, in other words, $b \geq a, b \geq -a$ imply $b \geq 0$. From $b \geq a, b \geq -a$ it follows by (4) and (3) that $2b \geq b + a \geq -a + a = 0$, and hence by normality we conclude that $b \geq 0$, as stated. Now we have, by distributivity, the sign of equality instead of \subseteq in (18').

Since a lattice-ordered group is distributive and normal, therefore in every lattice-ordered group both absolutes coincide.

Next we exhibit the properties of the absolute.

$$|0| = G^+, \quad |a| \subseteq G^+ \text{ if } a \neq 0. \quad {}^5) \quad \dots \quad (20)$$

$|0| = G^+$ and $|a| \subseteq G^+$ are trivial consequences of the definition (16). Further it is evident that $|a| = U(-a, 0) + U(a, 0) = G^+$ if and only if $U(-a, 0) = G^+$ and $U(a, 0) = G^+$. These imply $-a \leq 0$ and $a \leq 0$, respectively, so that $a = 0$.

$$\|0\| = G^+ \text{ and if the order is normal, } \|a\| \subseteq G^+ \text{ when } a \neq 0. \quad (21)$$

As we have seen in the proof above, normality implies $U(a, -a, 0) = U(a, -a)$, so that $\|a\|$ contains only positive elements, $\|a\| \subseteq G^+$. Further $\|a\| = U(a, -a, 0) = G^+$ implies $a \leq 0, -a \leq 0$; therefore $a = 0$.

$$|-a| = |a| \text{ is a consequence of (15), (16).} \quad \dots \quad (22)$$

⁵⁾ G^+ denotes the set of all positive elements (together with 0), that is, $G^+ = U(0)$.

$\| -a \| = \| a \|$ is a consequence of the symmetrical definition of $\| a \|$. (23)

$$|a - b| = U(a, b) - L(a, b) \text{ for every pair } a, b \text{ in } G. \dots \quad (24)$$

This gives for $b = 0$ the definition (16). On using (8) we obtain

$$|a - b| = U(a - b, 0) - L(a - b, 0) = \{U(a, b) - b\} - \{L(a, b) - b\} = \\ U(a, b) - L(a, b).$$

The 2-absolute fails to have the corresponding property.

$$n |a| \leqq |na| \text{ for every positive integer } n. \dots \quad (25)$$

Obviously,

$$|n|a| = n \{U(a, 0) + U(-a, 0)\} = U(a, 0) + U(-a, 0) + U(a, 0) + U(-a, 0) + \dots$$

Since the sets $U(a, 0)$ and $U(-a, 0)$ are by (15) permutable, we get $n |a| = nU(a, 0) + nU(-a, 0)$. Hence, in view of (11) we infer that $n |a| \leqq U(na, \dots, a, 0) + U(-na, \dots, -a, 0) \leqq U(na, 0) + U(-na, 0) = |na|$. q.e.d. From the proof it is clear that distributivity does not suffice for proving the equality in (25).

$$\text{For every positive integer } n \text{ we have } n \|a\| \leqq \|na\|. \dots \quad (26)$$

$$\text{since } n \|a\| = nU(a, -a) \leqq U(na, -na) = \|na\|.$$

In case G is a commutative group, we can in addition prove the analogue of the triangular inequality: the sign of inequality must hold, of course, in the opposite sense:

$$|a + b| \geqq |a| + |b|. \dots \quad (27)$$

For, $|a| + |b| = U(a, 0) - L(a, 0) + U(b, 0) - L(b, 0) = U(a, 0) + U(b, 0) - L(a, 0) - L(b, 0) \leqq U(a + b, 0) - L(a + b, 0) = |a + b|$.

$$\|a\| + \|b\| = U(a, -a) + U(b, -b) \leqq U(a + b, -a - b) = \\ U(a + b, -b - a) = \|a + b\| \quad (28)$$

provided that G is commutative.

Mathematics. — *The product of the minima and the determinant of a set.*

By C. A. ROGERS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 26, 1949.)

1. We use X to denote the general point with coordinates (x_1, \dots, x_n) in n -dimensional space and we use a vector notation. If

$$A_1 = (a_{11}, \dots, a_{n1}), \dots, A_n = (a_{1n}, \dots, a_{nn})$$

are any n points, such that the determinant Δ of the matrix (a_{rs}) is not zero, the set A of all points X of the form

$$X = u_1 A_1 + \dots + u_n A_n,$$

where u_1, \dots, u_n take all integral values, is said to be the lattice with determinant $d(A) = |\Delta|$ generated by A_1, \dots, A_n .

A well known theorem of MINKOWSKI asserts that, if K is any n -dimensional convex body with the origin $O = (0, \dots, 0)$ as centre, and if A is any lattice with no point other than O in K , then the volume $V(K)$ of K satisfies the inequality

$$V(K) \leq 2^n d(A). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

The first minimum $\mu_1 = \mu_1(S, A)$ of any set S for a lattice A is defined to be the lower bound of the positive numbers μ such that the set μS of all points μX with X in S contains a point different from O of A . It is clear that, with this definition, MINKOWSKI's theorem can be restated in the following form. If K is any convex body with the origin O as centre and with volume $V(K)$, then for every lattice A

$$\mu_1^n V(K) \leq 2^n d(A), \quad (\mu_1 = \mu_1(K, A)). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

MINKOWSKI¹⁾ refined this inequality, by associating with the convex body K and the lattice A a succession of minima $\mu_1 = \mu_1(K, A), \dots, \mu_n = \mu_n(K, A)$ satisfying $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$, and by proving that

$$\mu_1 \dots \mu_n V(K) \leq 2^n d(A), \quad (\mu_i = \mu_i(K, A)). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

In this paper we obtain an analogous inequality for the product of the successive minima and the critical determinant of an arbitrary set.

There are various ways of defining the successive minima of an arbitrary set. The following is a natural generalization of MINKOWSKI's definition.

Definition 2) (1). *The k -th minimum $\mu_k = \mu_k(S, A)$ of a set S for a*

¹⁾ H. MINKOWSKI, *Geometrie der Zahlen*, (Berlin, 1910), Kapitel 5. For a simple proof see H. DAVENPORT, *Quart. J. of Math.*, **10**, 119–121 (1939).

²⁾ Our notation is consistent with that of V. JARNÍK, *Časopis pro pěstování matematiky a fysiky*, **73**, 9–15 (1948) (in English) and consequently differs from that of other papers on this subject.

lattice Λ is defined to be the lower bound of the positive numbers μ such that the set μS contains at least k linearly independent points of Λ . By convention $\mu_k = +\infty$ if there are no such numbers μ .

It is clear from this definition that

$$0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_n. \quad \dots \quad (4)$$

The critical determinant $\Delta(S)$ of a set S is defined to be the lower bound of the determinants of the lattices Λ with no point different from O in S . [Note that this definition of the critical determinant does not coincide with that given by MAHLER in *Revista, Tucuman, serie A*, 5 (1946), 113—124, and in the *Canadian Journal of Maths.*, 1 (1949). The definitions coincide when S is open, and they are equivalent to that of MAHLER, *Proc. Royal Soc. (A)*, 187 (1946), 151—187, when S is a star body³⁾.] With this definition it is easy to see that, for any⁴⁾ set S and any lattice Λ

$$\mu_1^n \Delta(S) \leq d(\Lambda), \quad (\mu_1 = \mu_1(S, \Lambda)). \quad \dots \quad (5)$$

Further MINKOWSKI's result (1) can be restated in the form that

$$V(K) \leq 2^n \Delta(K), \quad \dots \quad (6)$$

for any convex body K with O as centre.

When (6) is satisfied with equality, for example when K is an n -dimensional cube with O as centre, it follows by MINKOWSKI's result (3) that

$$\mu_1 \dots \mu_n \Delta(K) \leq d(\Lambda), \quad (\mu_i = \mu_i(K, \Lambda)) \quad \dots \quad (7)$$

This inequality (7) has also been proved by MINKOWSKI⁵⁾ when K is an n -dimensional ellipsoid with O as centre. Again (7) is true if K is any 2-dimensional convex domain symmetrical in O . It has been conjectured that (7) is valid for any convex body K symmetrical in O ; but the best general result⁶⁾ known is that, for any convex body K symmetrical in O ,

$$\mu_1^{n-1} \mu_n \Delta(K) \leq d(\Lambda), \quad (\mu_i = \mu_i(K, \Lambda)) \quad \dots \quad (8)$$

The first general inequality for the successive minima of non-convex sets was obtained by JARNIK⁷⁾. Using a different definition he established an inequality for the product of the successive minima of the difference

³⁾ For the definition of a star body see MAHLER, *loc. cit.* It is not very difficult to prove that the following geometrical definition is equivalent to MAHLER's definition. The set S is a star body, if it is closed and if, for every λ with $-1 < \lambda < 1$ and for every point X of S , the point λX is an inner point of S .

⁴⁾ If $\Delta(S) = +\infty$, we interpret (5) as implying that $\mu_1 = 0$; and, if $\mu_1 = +\infty$, we interpret (5) as implying that $\Delta(S) = 0$.

⁵⁾ H. MINKOWSKI, *loc. cit.* § 51.

⁶⁾ This can be proved by a simple modification of DAVENPORT's proof of MINKOWSKI's result (3), see DAVENPORT, *loc. cit.*

⁷⁾ V. JARNIK, *Věstnik Královské České Společnosti Nauk.* (Praha, 1941), in Czech with a German summary.

set of a set S and the inner JORDAN content of S . This result of JARNIK was improved first by JARNIK and KNICHAL⁸⁾ and then by myself⁹⁾. The first direct inequality for the product of the successive minima of a set and its critical determinant was obtained by CHABAUTY¹⁰⁾ who used JARNIK's method to prove that

$$\mu_1 \dots \mu_n \Delta(S) \leq 2^{n-1-(1/2)-\dots-(1/n)} d(\Lambda), \quad (\mu_i = \mu_i(S, \Lambda)) \quad . \quad (9)$$

for all¹¹⁾ star sets¹²⁾ S . Before the publication of CHABAUTY's result MAHLER obtained an inequality of this form but with the larger constant $n!$ on the right hand side¹³⁾. Dr. MAHLER was kind enough to let me see his manuscript. It then became plain to me that the inequality

$$\mu_1 \dots \mu_n \Delta(S) \leq 2^{n-1} d(\Lambda), \quad (\mu_i = \mu_i(S, \Lambda)), \quad . \quad . \quad . \quad (10)$$

for all star sets S was implicit in JARNIK's paper (*loc. cit.*, 1941); and that the inequality

$$\mu_1 \dots \mu_n \Delta(S) \leq 2^{(n-1)/2} d(\Lambda), \quad (\mu_i = \mu_i(S, \Lambda)), \quad . \quad . \quad . \quad (11)$$

for all sets S , was implicit in my paper (*loc. cit.*). The object of the present paper is to give a simple proof of this inequality (11). Dr. MAHLER, in a paper following this, shows by an ingenious n -dimensional example that the constant $2^{(n-1)/2}$ in (11) is the best possible, even for bounded star bodies.

In the next sections we give a proof of the inequality (11) in a slightly refined form. In the last section we state some results involving a second type of successive minima.

This paper owes its existence and its simplicity to many suggestions I have received at various times from Professors DAVENPORT and JARNIK and Drs. ESTERMANN and MAHLER. I am most grateful to these mathematicians.

2. Theorem 1. Let S be any set and let Λ be any lattice. Suppose that, for some positive number μ and some positive integers m_1, \dots, m_n ,

- (a) m_{k+1} is an integral multiple of m_k , if $1 \leq k < n$ and
- (b) $\mu m_k \leq \mu_k = \mu_k(S, \Lambda)$, if $1 \leq k \leq n$.

Then $\Delta(S) < +\infty$ and

$$\mu^n m_1 \dots m_n \Delta(S) \leq d(\Lambda). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

⁸⁾ V. JARNIK and V. KNICHAL, *Rozpravy II třídy České Akademie*, 53 (1943), Cislo 43, in Czech. For a translation into French see *Bulletin international de l'Académie tchèque des Sciences*, 47 (1946), N°. 18.

⁹⁾ C. A. ROGERS, *Proc. London Math. Soc.* (2), in the press.

¹⁰⁾ C. CHABAUTY, *Comptes rendus*, 227, 747—749 (1948).

¹¹⁾ If $\Delta(S) = +\infty$, we interpret (9) as implying that $\mu_1 = 0$; and, if $\mu_n = +\infty$, we interpret (9) as implying that $\Delta(S) = 0$.

¹²⁾ The set S is a star set if, for every point X of S and for all λ with $0 \leq \lambda \leq 1$, the point λX is in S .

¹³⁾ Both CHABAUTY and MAHLER obtained their results without knowledge of my *Proc. London Math. Soc.* paper; had either of them been able to study my paper they could not but have obtained the inequality (11).

Further, if S is a bounded star body and $\mu m_n < \mu_n$, then (12) is satisfied with strict inequality.

Proof¹⁴⁾. As $0 < \mu m_1 \leq \mu_1$ it is clear that $\Delta(S) < +\infty$. Let ν be any number satisfying $0 < \nu < \mu$. Let a_k be the smallest linear manifold (or the minimal subspace) containing O and the points of A in the set $\nu m_k S$; so that a_k is the set of all points X of the form

$$X = \xi_1 X_1 + \dots + \xi_r X_r$$

where ξ_1, \dots, ξ_r are real and X_1, \dots, X_r are points of A in $\nu m_k S$. Then, by the condition (b) and the definition of μ_k , the set a_k is of dimension $d(k)$, say, less than k . Now by the condition (a) it is clear that, if P is a point of A in $\nu m_k S$, then $(m_{k+1}/m_k)P$ is a point of A in $\nu m_{k+1} S$. Hence a_{k+1} contains a_k if $1 \leq k < n$. Consequently we can choose a set of lattice points P_1, \dots, P_n generating A and such that a_k is the subspace generated by $O, P_1, \dots, P_{d(k)}$, for $k = 1, \dots, n$.

Consider the lattice A' generated by the points

$$\frac{1}{\nu m_1} P_1, \dots, \frac{1}{\nu m_n} P_n.$$

Suppose, if possible, that X is a point other than O of A' in S . Then

$$X = \frac{u_1}{\nu m_1} P_1 + \dots + \frac{u_n}{\nu m_n} P_n,$$

for some integers u_1, \dots, u_n , not all zero. Let k be the integer with $1 \leq k \leq n$ such that

$$u_k \neq 0, u_{k+1} = 0, \dots, u_n = 0.$$

By the condition (a) it is clear that the point

$$\nu m_k X = \frac{u_1 m_k}{m_1} P_1 + \dots + \frac{u_k m_k}{m_k} P_k$$

is a point common to $\nu m_k S$ and A . But, since $u_k \neq 0$, it is clear that the point $\nu m_k X$ does not lie in the subspace generated by O, P_1, \dots, P_{k-1} and so does not lie in a_k . This is contrary to the definition of a_k . Consequently there is no point X different from O of A' in S . Hence

$$\Delta(S) \leq d(A') = \frac{1}{\nu^n m_1 \dots m_n} d(A).$$

As ν may be arbitrarily close to μ this proves (12).

Now suppose that S is a bounded star body and that $\mu m_n < \mu_n$. In this case we take $\nu = \mu$ and take a_k to be the linear subspace generated by O and the points of A which are inner points of $\mu m_k S$. The lattice A' is

¹⁴⁾ This proof is based on V. JARNÍK and V. KNÍČHAL, loc. cit., 1943, Důkaz věty 1, or see V. JARNÍK and V. KNÍČHAL, loc. cit., 1946, Démonstration du théorème 1.

constructed as before, and the same argument now leads to the conclusion that there is no point X other than O of Λ' in the interior of S .

Let β_n be the linear subspace generated by O and the points of Λ which are points of $\mu m_n S$. Then β_n contains a_n and since $\mu m_n < \mu_n$ the dimension of β_n is less than n . We may thus suppose that β_n is contained in the $(n-1)$ -dimensional space γ_n generated by O, P_1, \dots, P_{n-1} . It follows by the argument used above that, if X is any point of Λ' in S , then X is in γ_n .

We have now proved that there is no point other than O of Λ' in the interior of S and that there are at most $n-1$ linearly independent points of Λ' on the boundary of S . As S is a bounded star body, it follows by a result of MAHLER¹⁵⁾ that Λ' is not a critical lattice of S . Consequently by MAHLER's definition of a critical lattice

$$\Delta(S) < d(\Lambda') = \frac{1}{\mu^n m_1 \dots m_n} d(\Lambda).$$

This completes the proof of the theorem.

3. **Theorem 2.** Suppose that μ_1, \dots, μ_n are any numbers satisfying

$$0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \dots \dots \dots \quad (13)$$

Then there exist a positive number μ and positive integers m_1, \dots, m_n , such that

- (a) m_{k+1} is an integral multiple of m_k , if $1 \leq k < n$.
- (b) $\mu m_k \leq \mu_k$, if $1 \leq k \leq n$, and (c)

$$\mu_1 \dots \mu_n \leq 2^{(n-1)/2} \mu^n m_1 \dots m_n \dots \dots \dots \quad (14)$$

Further μ, m_1, \dots, m_n may be chosen to ensure that either (14) is satisfied with strict inequality or $\mu m_n < \mu_n$.

Proof¹⁶⁾. Write

$$\delta_k = \delta(k) = \log_2 \mu_k, \quad k = 1, \dots, n, \dots \dots \quad (15)$$

Let $r(x)$ be defined by the equation $r(x) = x - [x]$, where $[x]$ denotes as usual the integral part of x . Consider the sums

$$\sum_{k=1}^n r(\delta_k - \delta_h), \quad h = 1, \dots, n.$$

Since

$$r(\delta_k - \delta_h) + r(\delta_h - \delta_k) = 0$$

if $\delta_k - \delta_h$ is an integer, and

$$r(\delta_k - \delta_h) + r(\delta_h - \delta_k) = 1$$

otherwise, it is clear that

$$\sum_{h=1}^n \sum_{k=1}^n r(\delta_k - \delta_h) \leq \frac{1}{2} n(n-1).$$

¹⁵⁾ K. MAHLER, Proc. Royal Soc. A, 187 (1946), 151–187 (162), Th. 11.

¹⁶⁾ This proof is based on the proof of Theorem 2 of C. A. ROGERS, *loc. cit.*

Hence we can choose a number δ such that

$$\sum_{k=1}^n r(\delta_k - \delta) \leq \frac{1}{2}(n-1). \quad \dots \quad \dots \quad \dots \quad (16)$$

We can also, by subtracting a sufficiently large integer from δ , ensure that $\delta \leq \delta_1$. Then by (13) and (15)

$$\delta \leq \delta_1 \leq \delta_2 \leq \dots \leq \delta_n \quad \dots \quad \dots \quad \dots \quad (17)$$

Write

$$\left. \begin{aligned} q_k &= q(k) = [\delta_k - \delta], & \text{for } k = 1, \dots, n, \\ m_k &= 2^{q(k)}, & \text{for } k = 1, \dots, n. \end{aligned} \right\} \quad \dots \quad (18)$$

and $\mu = 2^\delta$. Then, by (17) and (18),

$$0 \leq q_1 \leq q_2 \leq \dots \leq q_n,$$

so that m_{k+1} is an integral multiple of the integer m_k , for $k = 1, \dots, n-1$. Also using (18), as

$$q_k = q(k) \leq \delta_k - \delta, \text{ for } k = 1, \dots, n,$$

we have

$$\mu m_k = 2^{q(k)+\delta} \leq 2^{\delta(k)} = \mu_k, \text{ for } k = 1, \dots, n,$$

by (15). Further, by (18), (15), the definition of $r(x)$, and (16), we have

$$\log_2(\mu_1 \dots \mu_n) - \log_2(\mu^n m_1 \dots m_n) = \sum_{k=1}^n \delta_k - n\delta - \sum_{k=1}^n [\delta_k - \delta] = \sum_{k=1}^n r(\delta_k - \delta) \leq \frac{1}{2}(n-1). \quad \left. \right\} \quad (19)$$

Thus (14) is satisfied and we have found a positive number μ and positive integers m_1, \dots, m_n satisfying the conditions (a), (b) and (c).

We have still to prove the last clause of the theorem. It is clear that, unless

$$\sum_{k=1}^n r(\delta_k - \delta_h) = \frac{1}{2}(n-1), \text{ for } h = 1, \dots, n, \quad \dots \quad (20)$$

we can choose δ to satisfy (16) with strict inequality and can deduce that (19) and (14) are satisfied with strict inequality. When (20) is satisfied the number δ satisfying (16) and (17) may be taken to be δ_1 . Then, if μm_n were equal to μ_n , the number $\delta_n - \delta_1 = \delta_n - \delta$ would be equal to the integer q_n . This would imply that

$$\sum_{h=1}^n \sum_{k=1}^n r(\delta_k - \delta_h) \leq \frac{1}{2} n(n-1) - 1,$$

contrary to (20). Hence when (20) is satisfied we can ensure that $\mu m_n < \mu_n$ by taking $\delta = \delta_1$. This completes the proof of the theorem.

4. **Theorem 3.** Let S be any set and let Λ be any lattice. Suppose that $\mu_1(S, \Lambda) > 0$ and that $\Delta(S) > 0$. Then $\mu_n(S, \Lambda) < +\infty$, $\Delta(S) < +\infty$ and¹⁷⁾

$$\mu_1 \dots \mu_n \Delta(S) \leq 2^{(n-1)/2} d(\Lambda), \quad (\mu_i = \mu_i(S, \Lambda)). \quad \dots \quad (21)$$

Further, if S is a bounded star body, (21) is satisfied with strict inequality.

Proof. Since $\mu_1 > 0$ it is clear that $\Delta(S) < +\infty$. Suppose, if possible, that $\mu_n = +\infty$. Write $\mu = \mu_1$ and

$$m_1 = m_2 = \dots = m_{n-1} = 1, \quad m_n = N,$$

where N is a large positive integer. Then by Theorem 1,

$$\mu_1^n N \Delta(S) \leq d(\Lambda).$$

As $\mu_1 > 0$ and as this is true for arbitrarily large values of N , it follows that $\Delta(S) = 0$. This is contrary to our hypothesis and consequently $\mu_n < +\infty$.

Now μ_1, \dots, μ_n are finite numbers satisfying the inequalities (13). Let μ, m_1, \dots, m_n be the corresponding numbers whose existence is asserted in Theorem 2. Then

$$\mu_1 \dots \mu_n \leq 2^{(n-1)/2} \mu^n m_1 \dots m_n, \quad \dots \quad (22)$$

and, since the conditions of Theorem 1 are satisfied,

$$\mu^n m_1 \dots m_n \Delta(S) \leq d(\Lambda). \quad \dots \quad (23)$$

Combining (22) and (23) we obtain (21).

It is clear from the last clause of Theorem 2 that we have strict inequality in (22) and (21) unless $\mu m_n < \mu_n$. But, when S is a bounded star body and $\mu m_n < \mu_n$, it follows from the last clause of Theorem 1 that we have strict inequality in (23) and (21). Hence (21) is satisfied with strict inequality when S is a bounded star body. This completes the proof of the theorem.

We now show how the main result of my previous paper can be deduced from Theorem 3. We use $\mathcal{D}S$ to denote the difference set of S , i.e. the set of all points of the form $X - Y$ where X and Y are in S . It has been essentially shown by BLICHFELDT¹⁸⁾ that, for any measurable set S with a finite LEBESGUE measure $V(S)$,

$$V(S) \leq \Delta(\mathcal{D}S); \quad \dots \quad (24)$$

and that, if S is measurable and has an infinite LEBESGUE measure, then $\Delta(\mathcal{D}S)$ is infinite. It now follows by Theorem 3 that, for any measurable set S with LEBESGUE measure $V(S)$, and for any lattice Λ ,

$$\mu_1 \dots \mu_n V(S) \leq 2^{(n-1)/2} d(\Lambda), \quad (\mu_i = \mu_i(\mathcal{D}S, \Lambda)); \quad \dots \quad (25)$$

¹⁷⁾ Note that the theorem implies that $\mu_1 = 0$ if $\Delta(S) = +\infty$, and that $\Delta(S) = 0$ if $\mu_n = +\infty$.

¹⁸⁾ H. F. BLICHFELDT, *Trans. Amer. Math. Soc.*, 15 (1914), 227–235, does not work with the LEBESGUE measure. For a general proof see FENCHEL, *Acta Arithmetica*, 2 230–241 (240–241) (1937).

with the interpretations that $\mu_1 = 0$, if $V(S) = +\infty$, and that $V(S) = 0$, if $\mu_n = +\infty$. This is the main result of my previous paper, in a different notation.

4. In this section we introduce a second set of successive minima.

Definition (2). *The k-th minimum $\nu_k = \nu_k(S, \Lambda)$ of a set S for a lattice Λ is defined to be the upper bound of the values of ν such that the set νS contains less than k linearly independent points of Λ .*

Clearly

$$0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_n \leq +\infty,$$

and comparing the Definitions (1) and (2)

$$\mu_1 \leq \nu_1, \dots, \mu_n \leq \nu_n.$$

Using this definition (but with another notation) I have proved (*loc cit.* Th. 3) that

$$\mu_1 \dots \mu_n \frac{\nu_k}{\mu_k} V(S) \leq 2^{n-1} d(\Lambda), \text{ for } k = 1, \dots, n, \quad \dots \quad (26)$$

and that

$$(\mu_1 \dots \mu_n)^{1-(1/n)} (\nu_1 \dots \nu_n)^{1/n} V(S) \leq 2^{n-1} d(\Lambda), \quad \dots \quad (27)$$

where

$$\mu_i = \mu_i(DS, \Lambda), \quad \nu_i = \nu_i(DS, \Lambda),$$

for every measurable set S with a finite positive LEBESGUE measure. JARNIK¹⁹⁾ has shown that, in a certain sense, these results are of the best possible nature.

Just as these results can be obtained by a simple modification of the method of proving (25), the following results can be obtained by a simple modification of the method used above for proving (21).

Theorem 4. *Let S be any set and let Λ be any lattice. Suppose that $\mu_1(S, \Lambda) > 0$ and that $\Delta(S) > 0$. Then $\nu_n(S, \Lambda) < +\infty$, $\Delta(S) < +\infty$, and*

$$\mu_1 \dots \mu_n \frac{\nu_k}{\mu_k} \Delta(S) \leq 2^{n-1} d(\Lambda), \text{ for } k = 1, \dots, n, \quad \dots \quad (28)$$

and

$$(\mu_1 \dots \mu_n)^{1-(1/n)} (\nu_1 \dots \nu_n)^{1/n} \Delta(S) \leq 2^{n-1} d(\Lambda), \quad \dots \quad (29)$$

where

$$\mu_i = \mu_i(S, \Lambda), \quad \nu_i = \nu_i(S, \Lambda).$$

Note that the inequalities (26) and (27) are consequences of the inequality (24) and the inequalities (28) and (29) with S replaced by DS . Further it follows from (24) and JARNIK's work that, in a certain sense, (28) and (29) are of the best possible nature.

University College, London.

¹⁹⁾ V. JARNIK, *loc. cit.*, (1948).

Mathematics. — *On the uniform distribution modulo 1 of lacunary sequences.* By P. ERDÖS and J. F. KOKSMA. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 26, 1949.)

§ 1. As is well known one calls the sequence of real numbers u_1, u_2, \dots uniformly distributed modulo 1, if the number N' of those among the numbers

$$u_1 - [u_1], u_2 - [u_2], \dots, u_N - [u_N]$$

which fall into an arbitrarily given part $\alpha \leq u < \beta$ of the unit interval $0 \leq u < 1$ satisfies the condition

$$\frac{N'}{N} \rightarrow \beta - \alpha, \text{ if } N \rightarrow \infty.$$

The difference $\left| \frac{N'}{N} - (\beta - \alpha) \right|$ is always ≤ 1 and for fixed $N \geq 1$ one calls its upper bound, (if (α, β) is supposed to run through all couples with $0 \leq \alpha < \beta \leq 1$), the discrepancy $D(N)$ of the sequence. If

$$ND(N) = o(N), \dots \dots \dots \quad (1)$$

it is trivial that the sequence is uniformly distributed modulo 1 and as was proved by WEYL¹⁾, inversely (1) is a consequence of the distribution modulo 1, defined above.

One gets an interesting special case when putting

$$u_n = \theta \lambda_n \quad (n = 1, 2, \dots) \dots \dots \quad (2)$$

where

$$\lambda_1 < \lambda_2 < \dots \dots \dots \quad (3)$$

denotes an increasing sequence of integers. FATOU¹⁾ already proved that such a sequence is everywhere dense modulo 1 in the unit interval for almost all values of θ , provided that the sequence (3) is lacunary, i.e. that for some positive constant δ

$$\lambda_{n+1} \geq (1 + \delta) \lambda_n \quad (n = 1, 2, \dots) \dots \dots \quad (4)$$

HARDY-LITTLEWOOD¹⁾ and WEYL¹⁾ proved that for each sequence of integers (3) the sequence of numbers (2) is uniformly distributed modulo 1 for almost all θ . Hence for such sequences (1) holds. FOWLER¹⁾, KOKSMA¹⁾ and DREWES²⁾ deduced improvements of (1). In the special case

$$\lambda_n = 2^n,$$

¹⁾ References in "Diophantische Approximationen", Erg. d. Math. IV, 4 (1936) by J. F. KOKSMA (Kap. VIII and IX).

²⁾ A. DREWES, Diophantische Benaderingsproblemen, Thesis Free University, Amsterdam (1945).

the problem is equivalent to the question how the digits 0, 1 are distributed in the dyadic expansion of θ . Here KHINTCHINE¹⁾ proved very sharp results.

Generally speaking, the problem is somewhat easier to handle for lacunary sequences (4) than in the general case (3). In this paper we consider the case of lacunary sequences of numbers

$$u_n = f(n, \theta),$$

which form a generalisation of the sequences defined by (2). The method used in this paper leads to great difficulties, if one tries to apply it in the general case. In a following paper we treat the *general* case with an other method, which in the specialised cases which are considered in the present paper would give a slightly less sharp result than we deduce here.

§ 2. In this paper we prove a general theorem in which as a special case is contained the following

Theorem 1. Let δ denote an arbitrary positive constant and $\omega(n)$ a positive increasing function of $n = 1, 2, \dots$ with $\omega(n) \rightarrow \infty$, if $n \rightarrow \infty$. Then for any sequence of positive numbers $\lambda_1, \lambda_2, \dots$, which satisfy (4), the discrepancy $D(N)$ of the sequence (2) satisfies the inequality

$$ND(N) = o(N^{\frac{1}{2}} \log^{\frac{1}{2}} N (\log \log N)^{\frac{1}{2}} \omega(N)) \dots \quad (5)$$

for almost all θ .

This estimate is sharper than all known results. The exponent $\frac{1}{2}$ in the factor $N^{\frac{1}{2}}$ cannot be improved, as KHINTCHINE proved that in the special case $\lambda_n = 2^n$, we have

$$ND(N) = \Omega(N^{\frac{1}{2}} \sqrt{\log \log N}).$$

Another application of our theorem is the following

Theorem 2. For almost all values of $\theta \geq 1$ the discrepancy of the sequence

$$\theta, \theta^2, \theta^3, \dots$$

satisfies the inequality (5), if $\omega(n)$ denotes a positive increasing function such that $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$.

That the sequence θ, θ^2, \dots for almost all θ is uniformly distributed (modulo 1) had already been proved by KOKSMA¹⁾, whereas the sharpest estimate for the discrepancy of this sequence known till now was given by DREWES²⁾.

§ 3. The theorems quoted above are contained in the following Theorem 3, which itself is a special case of the main Theorem 5.

Theorem 3. Let $a < b$, $\delta > 0$ be given real numbers. Let $f(1, \theta)$, $f(2, \theta), \dots$ denote a sequence of real functions which are defined on $a \leq \theta \leq b$, such that

$$f'_\theta(n+1, \theta) \geq (1+\delta) f'_\theta(n, \theta) > 0; \quad f''_\theta(n+1, \theta) \geq (1+\delta) f''_\theta(n, \theta) \geq 0 \quad (n = 1, 2, \dots)$$

for all values of θ on $a \leq \theta \leq b$. Let $\omega(n)$ denote an increasing function of $n = 1, 2, \dots$, such that $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Then for almost all θ on $a \leq \theta \leq b$ the discrepancy in the uniform distribution of the sequence

$$f(1, \theta), f(2, \theta), \dots$$

satisfies the relation (5).

Remark. It is clear that the sequences of the theorems 1 and 2 satisfy the conditions of Theorem 3. In the first case we put without loss of generality $a = 0$, $b = 1$ and in the second case we put $a = 1 + \delta$, $b > a$ and after application of Theorem 3, we let

$$\delta \rightarrow 0, \quad b \rightarrow \infty.$$

The reader will find the deduction of Theorem 3 from Theorem 4 in § 8.

§ 4. For the proof of our main theorem we deduce a lemma (Lemma 2), which has some interest in itself. For the special case, considered in Theorem 3, it runs as follows:

Theorem 4. Suppose that the conditions of Theorem 3 are satisfied. Let K denote a positive constant. Then for almost all θ the following statement is true: If N and k are integer such that $1 \leq k \leq N^K$, then

$$\left| \sum_{n=1}^N e^{2\pi i k f(n, \theta)} \right| \leq C(\theta) N^{\frac{1}{2}} \log^{\frac{1}{2}} N (\log \log N)^{\frac{1}{2}} \omega(N),$$

where $C(\theta)$ does not depend on N or k .

The reader finds its deduction in § 9.

§ 5. Before we state the main theorem, we make some

Preliminary Remarks. Let N and r denote positive integers. Out of the N integers $n = 1, 2, \dots, N$, we can form N^r different r -tuples; such an r -tuple we shall denote by (n_1, \dots, n_r) . There are $\binom{N}{r}$ different r -tuples among them for which $n_1 \leq n_2 \leq \dots \leq n_r$. Such a special r -tuple we shall denote also by $\{n_1, \dots, n_r\}$. The elements n_1, \dots, n_r of the r -tuple $\{n_1, \dots, n_r\}$ have a number of different permutations, which we shall denote by $A\{n_1, \dots, n_r\}$. Then we obviously have

$$A\{n_1, \dots, n_r\} \equiv r! \dots \dots \dots \dots \quad (6)$$

and

$$\sum_{\{n_1, \dots, n_r\}} A\{n_1, \dots, n_r\} = N^r \dots \dots \dots \dots \quad (7)$$

For later purposes we put

$$A_N^r = \sum_{\{n_1, \dots, n_r\}} A^2 \{n_1, \dots, n_r\}, \dots, \dots, \dots \quad (8)$$

Definition. If $\{n_1, \dots, n_r\}$ and $\{m_1, \dots, m_r\}$ are two different r -tuples of the above kind, we say that the first is greater than the second:

$$\{n_1, \dots, n_r\} > \{m_1, \dots, m_r\},$$

if and only if for some τ ($1 \leq \tau \leq r$)

$$n_\tau > m_\tau, \quad n_\varrho = m_\varrho \quad (\varrho = \tau + 1, \tau + 2, \dots, r).$$

Condition A. Let $g(x, \theta)$ for $x = 1, 2, \dots, N$ denote a function of θ on the segment $a \leq \theta \leq b$ such that for each couple of r -tuples $\{n_1, \dots, n_r\} > > \{m_1, \dots, m_r\}$ the function

$$\phi(\theta) = \phi(n_1, \dots, n_r; m_1, \dots, m_r; \theta) = \sum_{\varrho=1}^r g(n_\varrho, \theta) - \sum_{\varrho=1}^r g(m_\varrho, \theta) \quad (9)$$

has a derivative for $a \leq \theta \leq b$, which is continuous, $\neq 0$, and either non-decreasing or non-increasing in the segment $a \leq \theta \leq b$. We then put

$$\begin{aligned} \Psi &= \Psi(n_1, \dots, n_r; m_1, \dots, m_r) = \\ &\text{Min } \{\phi'_0(n_1, \dots, n_r; m_1, \dots, m_r; a), \phi'_0(n_1, \dots, n_r; m_1, \dots, m_r; b)\}, \end{aligned} \quad (10)$$

$$\begin{aligned} B_N &= N^{-r} \sum_{\{n_1, \dots, n_r\} > > \{m_1, \dots, m_r\}} A \{n_1, \dots, n_r\} A \{m_1, \dots, m_r\} \\ &\quad \Psi^{-1}(n_1, \dots, n_r; m_1, \dots, m_r). \end{aligned} \quad (11)$$

Now we state our main theorem:

Theorem 5. I. Let a and b denote real constants with $a < b$. Let $f(n, \theta)$ for $n = 1, 2, \dots$ denote a real function of θ on the segment $a \leq \theta \leq b$. Let N_0 be a positive integer. Let $r = r(N)$ and $s = s(N)$ be positive integers which are defined for each integer $N \geq N_0$, such that

$$S(N) \equiv N.$$

Let for each integer $N \geq N_0$ and each integer $\sigma = 1, \dots, s(N)$ the N_σ functions

$$g_\sigma(x, \theta) = f(\sigma + (x-1)s, \theta) \quad \left(x = 1, 2, \dots, N_\sigma = \left[\frac{N-\sigma}{s} \right] + 1 \right)$$

be considered and let the condition A be satisfied with g_σ instead of g and with N_σ instead of N .

II. Putting

$$B_N^* = \max_{1 \leq \sigma \leq s} B_{N_\sigma}, \dots, \dots, \dots, \dots \quad (12)$$

we assume that a non-decreasing sequence $\psi(1), \psi(2), \dots$ of positive numbers exists such that the series

$$\sum_{n=N_0}^{\infty} s \cdot \{(b-a)r/N! + B_N^* \log N\} \left\{ \psi \left(\left[\frac{N-s}{s} \right] + 1 \right) \right\}^{-2r} \quad (13)$$

converges. Then almost all numbers θ of $a \leq \theta \leq b$, have the property that the discrepancy $D(N)$ of the sequence $f(1, \theta), f(2, \theta), \dots$ satisfies the inequality

$$ND(N) \leq K_1 s^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \cdot \psi \left(\left[\frac{N-1}{s} \right] + 1 \right) \log N \text{ for } N \geq N_0. \quad (14)$$

where K_1 denotes a numerical constant, whereas N_0 denotes an index depending on θ .

§ 6. We now prove

Lemma 1. Let N and r denote positive integers and let $g(x, \theta)$ satisfy the condition A. Then for each fixed integer $h \neq 0$ we have

$$\int_a^b \left| \sum_{x=1}^N e^{2\pi i h g(x, \theta)} \right|^{2r} d\theta = (b-a) A_N^r + \frac{2\vartheta}{\pi h} B_N N^r (|\vartheta| \leq 1). \quad (15)$$

Proof.

$$\begin{aligned} \left| \sum_{x=1}^N e^{2\pi i h g(x, \theta)} \right|^{2r} &= \left\{ \sum_{x=1}^N e^{2\pi i h g(x, \theta)} \right\}^r \left\{ \sum_{x=1}^N e^{-2\pi i h g(x, \theta)} \right\}^r \\ &= \sum_{(n_1, \dots, n_r)} e^{2\pi i h(g(n_1, \theta) + \dots + g(n_r, \theta))} \cdot \sum_{(n_1, \dots, n_r)} e^{-2\pi i h(g(n_1, \theta) + \dots + g(n_r, \theta))}, \end{aligned}$$

where both sums are to be expanded over all N^r r -tuples of integers

$$n_\vartheta = 1, 2, \dots, N.$$

Applying the preliminary remark of § 5, we write

$$\begin{aligned} \left| \sum_{x=1}^N e^{2\pi i h g(x, \theta)} \right|^{2r} &= \sum_{(n_1, \dots, n_r)} A\{n_1, \dots, n_r\} e^{2\pi i h(g(n_1, \theta) + \dots + g(n_r, \theta))} \\ &\quad \cdot \sum_{(n_1, \dots, n_r)} A\{n_1, \dots, n_r\} e^{-2\pi i h(g(n_1, \theta) + \dots + g(n_r, \theta))} \\ &= \sum_{(n_1, \dots, n_r)} \sum_{(m_1, \dots, m_r)} A\{n_1, \dots, n_r\} A\{m_1, \dots, m_r\} e^{2\pi i h \left\{ \sum_{\vartheta=1}^r g(n_\vartheta, \theta) - \sum_{\vartheta=1}^r g(m_\vartheta, \theta) \right\}} \\ &= \sum_{(n_1, \dots, n_r)} A^2\{n_1, \dots, n_r\} + 2 \sum_{(n_1, \dots, n_r) > (m_1, \dots, m_r)} \sum_{A\{n_1, \dots, n_r\} A\{m_1, \dots, m_r\}} \\ &\quad A\{n_1, \dots, n_r\} A\{m_1, \dots, m_r\} \cos 2\pi h \phi(\theta) \end{aligned}$$

by (9). Hence by (8)

$$\begin{aligned} \int_a^b \left| \sum_{x=1}^N e^{2\pi i h g(x, \theta)} \right|^{2r} d\theta &= (b-a) A_N^r + \\ &\quad + 2 \sum_{(n_1, \dots, n_r) > (m_1, \dots, m_r)} \sum_{A\{n_1, \dots, n_r\} A\{m_1, \dots, m_r\}} \int_a^b \cos 2\pi h \phi(\theta) d\theta. \end{aligned} \quad (16)$$

Now choosing the new variable of integration u by the substitution

$$u = \phi(\theta),$$

we find

$$du = \phi'(\theta) d\theta$$

and

$$\int_a^b \cos 2\pi h \phi(\theta) \cdot d\theta = \int_{\phi(a)}^{\phi(b)} \cos 2\pi h u \frac{du}{\phi'(\theta)}$$

and therefore using BONNET's theorem and (10)

$$\left| \int_a^b \cos 2\pi h \phi(\theta) \cdot d\theta \right| \leq \frac{1}{\pi h} \Psi.$$

Hence by (16) and (11)

$$\int_a^b \left| \sum_{x=1}^N e^{2\pi i h g(x, \theta)} \right|^{2r} d\theta = (b-a) A_N^r + \frac{2\vartheta N^r}{\pi h} B_N \cdot (|\vartheta| \leq 1).$$

Q.e.d.

Lemma 2. Let the conditions I of the main theorem 5 be satisfied. II. Defining B_N^* by (12), we suppose that a non-decreasing sequence $\psi(1), \psi(2), \dots$ and a non-decreasing sequence of integers $A(1), A(2), \dots$ ($A(N) \geq 3$, if $N \geq N_0$) exist, such that the series

$$\sum_{n=N_0}^{\infty} s \{ (b-a) r! A(N) + 2 B_N^* \log A(N) \} \left\{ \psi \left(\left[\frac{N-s}{s} \right] + 1 \right) \right\}^{-2r}. \quad (13a)$$

converges. Then almost all numbers θ of $a \leq \theta \leq b$ have the property that for all integers $h = 1, 2, \dots, A(N)$

$$\left| \sum_{n=1}^N e^{2\pi i h f(n, \theta)} \right| \leq 2s^{\frac{1}{2}} N^{\frac{1}{2}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right), \text{ if } N \geq N_0^*(\theta). \quad (17)$$

Proof. Let N be a fixed integer $\geq N_0$. Let (h, σ) denote a couple of integers which satisfy the inequalities

$$1 \leq h \leq A(N), \quad 1 \leq \sigma \leq s.$$

Then the Lemma 1 with

$$g(x, \theta) = g_\sigma(x, \theta) = f(\sigma + (x-1)s, \theta), \quad N = N_\sigma$$

learns

$$\int_a^b \left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x, \theta)} \right|^{2r} d\theta \leq (b-a) r! N_\sigma^r + \frac{1}{h} B_{N_\sigma} N_\sigma^r \quad . . . \quad (18)$$

because of (6) and (7).

Now let $S(N, h, \sigma)$ denote the set of all numbers θ on $a \leq \theta \leq b$ for which

$$\left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x, \theta)} \right| \equiv N_\sigma^{\frac{1}{2}} \psi(N_\sigma) \dots \dots \quad (19)$$

Then by (18) we obviously find for its measure $mS(N, h, \sigma)$ the inequality

$$mS(N, h, \sigma) N_\sigma^r \{ \psi(N_\sigma) \}^{2r} \equiv (b-a) r! N_\sigma^r + \frac{1}{h} B_{N_\sigma} N_\sigma^r,$$

hence

$$mS(N, h, \sigma) \equiv \left\{ (b-a) r! + \frac{1}{h} B_{N_\sigma} \right\} \{ \psi(N_\sigma) \}^{-2r}$$

and therefore by

$$N_\sigma \geq \left[\frac{N-s}{s} \right] + 1 \text{ and (12):}$$

$$mS(N, h, \sigma) \equiv \left\{ (b-a) r! + \frac{1}{h} B_N^* \right\} \left\{ \psi \left(\left[\frac{N-s}{s} \right] + 1 \right) \right\}^{-2r}.$$

Now let $S(N)$ denote the set

$$S(N) = \sum_{(h, \sigma)} S(N, h, \sigma),$$

where the summation is to be extended over all couples (h, σ) which satisfy $1 \leq h \leq A(N)$, $1 \leq \sigma \leq s$. Then we have

$$\begin{aligned} mS(N) &\equiv \left\{ (b-a) r/s A(N) + s B_N^* \sum_{h=1}^{A(N)} \frac{1}{h} \right\} \left\{ \psi \left(\left[\frac{N-s}{s} \right] + 1 \right) \right\}^{-2r} \\ &< s \left\{ (b-a) r! A(N) + 2 B_N^* \log A(N) \right\} \left\{ \psi \left(\left[\frac{N-s}{s} \right] + 1 \right) \right\}^{-2r}. \end{aligned} \quad (20)$$

Each θ of $a \leq \theta \leq b$, which does not belong to $S(N)$ ($N \geq N_0$) has the property that the inequality

$$\left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x, \theta)} \right| \equiv N_\sigma^{\frac{1}{2}} \psi(N_\sigma) < \left(\frac{N-1}{s} + 1 \right)^{\frac{1}{2}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right)$$

is valid for all couples (h, σ) which satisfy the inequalities

$$1 \leq h \leq A(N), \quad 1 \leq \sigma \leq s.$$

Therefore we have for such a θ

$$\left| \sum_{n=1}^N e^{2\pi i h f(n, \theta)} \right| \leq \left| \sum_{\sigma=1}^s \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x, \theta)} \right| \leq 2s^{\frac{1}{2}} N^{\frac{1}{2}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right),$$

for all integers $h = 1, 2, \dots, A(N)$.

Now as $mS(N)$ satisfies (20) and as the series (13a) converges, almost all numbers θ of $a \leq \theta \leq b$ belong to at most a finite number of the sets

$S(N)$ ($N = N_0, N_0 + 1, \dots$). Therefore for almost all θ of $a \leq \theta \leq b$ an index N_0^* can be found, such that

$$\left| \sum_{n=1}^N e^{2\pi i h f(n, \theta)} \right| \leq 2s^{\frac{1}{s}} N^{\frac{1}{s}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right),$$

whatever the value of the integer $h = 1, 2, \dots, A(N)$ may be. Q.e.d.

§ 7. In order to prove the main theorem, we quote the following theorem, which is an improvement proved by ERDÖS-TURÁN³⁾ of the one dimensional case of a theorem of VAN DER CORPUT-KOKSMA¹⁾.

Lemma 3. *If u_1, u_2, \dots is a real sequence and if $D(N)$ denotes its discrepancy, then for each integer $m \geq 1$, we have*

$$ND(N) \leq K \left\{ \frac{N}{m+1} + \sum_{h=1}^m \frac{1}{h} \left| \sum_{n=1}^N e^{2\pi i h u_n} \right| \right\}, \quad \dots \quad (21)$$

where K denotes a numerical constant.

Proof of the Theorem 5. Put

$$u_n = f(n, \theta), \quad m = [\sqrt{N}], \quad A(N) = [\sqrt{N}].$$

Then (using Lemma 2) for almost all θ we have by (17) and (21), if $N \geq N_0^*(\theta)$

$$\begin{aligned} ND(N) &\leq K \sqrt{N} + K \sum_{h=1}^{[\sqrt{N}]} \frac{2}{h} s^{\frac{1}{s}} \cdot N^{\frac{1}{s}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right) \\ &\leq K_1 s^{\frac{1}{s}} N^{\frac{1}{s}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right) \log N. \end{aligned}$$

Q.e.d.

§ 8. **Proof of Theorem 3.** Be $\omega(N)$ the function of Theorem 3 and let N_0 be a sufficiently large integer. We shall prove that the functions $f(n, \theta)$ of Theorem 3 satisfy the conditions of Theorem 5, if we put for $N \geq N_0$

$$\left. \begin{aligned} s(N) &= \left[\frac{2}{\log(1+\delta)} \log \log N \right]; \\ r(N) &= \left[\frac{\log N}{\log \sqrt{\omega([\sqrt{N}])}} \right] + 1; \quad \psi(N) = \sqrt{\log N^2} \cdot \sqrt{\omega(N)} \end{aligned} \right\}. \quad (22)$$

where δ denotes the constant of Theorem 3. Now for $N \geq N_0$ we consider the $s = s(N)$ sequences

$$g_x(x, \theta) = f(\sigma + (x-1)s, \theta) \left(1 \leq \sigma \leq s; x = 1, 2, \dots, N_s = \left[\frac{N-\sigma}{s} \right] + 1 \right).$$

³⁾ P. ERDÖS and P. TURÁN, On a problem in the theory of uniform distribution. Proc Kon. Ned. Akad. v. Wetensch., Amsterdam, 51, 1146–1154, 1262–1269 (1948); Ind. Math. 10, 370–378, 406–413 (1948).

By hypothesis we have (the prime meaning differentiation by θ):

$$\frac{f'(n+1, \theta)}{f'(n, \theta)} \geq 1 + \delta,$$

hence for $N \geq N_0$

$$\frac{g'_\sigma(x+1, \theta)}{g'_\sigma(x, \theta)} \geq (1 + \delta)^s > (1 + \delta)^{\frac{s}{s} \frac{\log \log N}{\log(1+\delta)}} = (\log N)^{\frac{s}{s}} > r + 1.$$

Therefore if we take two r -tuples

$$\{n_1, \dots, n_r\} > \{m_1, \dots, m_r\},$$

we have, using the notation of the Preliminary Remarks:

$$\begin{aligned} \phi'(\theta) &= g'_\sigma(n_1, \theta) + \dots + g'_\sigma(n_r, \theta) - g'_\sigma(m_1, \theta) - \dots - g'_\sigma(m_r, \theta) \\ &\geq g'_\sigma(n_r, \theta) - rg'_\sigma(n_r - 1, \theta) > g'_\sigma(n_r - 1, \theta) \equiv f'(1, a) = c_0 \end{aligned} \quad (23)$$

and we conclude that if we range the r -tuples $\{n_1, \dots, n_r\}$ in order of increasing magnitude, the corresponding sums

$$g'_\sigma(n_1, \theta) + \dots + g'_\sigma(n_r, \theta)$$

with each step will increase by at least the amount c_0 , whatever the value of θ ($a \leq \theta \leq b$) may be. Hence, if the r -tuple $\{n_1, \dots, n_r\}$ is fixed, we have by (10) and (9)

$$\sum_{\{n_1, \dots, n_r\} > \{m_1, \dots, m_r\}} \{\Psi(n_1, \dots, n_r; m_1, \dots, m_r)\}^{-1} \leq \frac{1}{c_0} \sum_{k=1}^{N_r^r} \frac{1}{k} \leq \frac{1 + r \log N}{c_0},$$

as there are at most $N_r^r \leq N^r$ r -tuples $\{m_1, \dots, m_r\}$.

Therefore we find by (11) and (6)

$$B_{N_\sigma} \leq N_\sigma^{-r} r! \frac{1 + r \log N}{c_0} \sum_{\{n_1, \dots, n_r\}} A\{n_1, \dots, n_r\} = \frac{r!}{c_0} (1 + r \log N)$$

by (7). Hence we find by (12) a fortiori

$$B_N^* \leq \frac{2r}{c_0} r! \log N < r^r \log N \text{ for } N \geq N_0.$$

Thus we find that the general term of our series (13) is at most

$$\begin{aligned} \{(b-a)s r^r N^{\frac{1}{r}} + s r^r \log^2 N\} \{\psi([\sqrt[N]{N}] + 1)\}^{-2r} &\leq \\ &\leq c_1 s (\log N)^r N^{\frac{1}{r}} \cdot \sqrt[r]{\log N}^{-2r} \{ \sqrt[r]{\omega([\sqrt[N]{N}])} \}^{-2r} \end{aligned}$$

by (22) and therefore by (22)

$$\leq c_1 s N^{\frac{1}{r}} \{ \sqrt[r]{\omega([\sqrt[N]{N}])} \}^{-\frac{2 \log N}{\log \sqrt[r]{\omega([\sqrt[N]{N}])}}} < N^{-\frac{5}{4}}, \text{ if } N \geq N_0.$$

Hence, our series (13) converges.

By hypothesis we further have $f''_{(n+1, \theta)} \geq (1 + \delta) f''_{(n, \theta)} \geq 0$.

Repeating the proof of (23) with f'' instead of f' and g''_σ instead of g'_σ , we find $\phi''(\theta) \geq 0$. Hence $\phi'(\theta)$ is non-decreasing.

From our result we conclude that for almost all θ the inequality (14) holds, i.e. because of (22):

$$ND(N) \leq K_2 N^{\frac{1}{2}} (\log N)^{\frac{1}{2}} (\log \log N)^{\frac{1}{2}} \sqrt{\omega(N)}, \text{ if } N \geq N_0^*(\theta).$$

Hence (5) follows immediately.

§ 9. Proof of Theorem 4. We shall use Lemma 2 and we put

$$\left. \begin{aligned} A(N) &= [N^K], s(N) = \left[\frac{2}{\log(1+\delta)} \log \log N \right], r(N) = \\ &= \left[\frac{(K+2) \log N}{\log \omega([\sqrt{N}])} \right] + 1, \psi(N) = \sqrt{\log N^2 - \omega(N)}, \end{aligned} \right\}. \quad (24)$$

where K , δ and $\omega(N)$ are defined in Theorem 4. Then in exactly the same way as in § 8 it follows that

$$B_N^* \leq r \log N \text{ for } N \geq N_0$$

and thus the general term of the series (13a) is at most

$$\begin{aligned} &\{(b-a)s r^r N^K + s r^r \cdot 2K \log^2 N\} \{\psi([\sqrt{N}]) + 1\}^{-2r} \leq \\ &\leq c_2 s (\log N)^r N^K (\sqrt{\log N})^{-2r} \{\omega([\sqrt{N}])\}^{-2r} \end{aligned}$$

by (24) and therefore by (24)

$$< c_2 s \cdot N^K \{\omega([\sqrt{N}])\}^{\frac{-(2+K) \log N}{\log \omega([\sqrt{N}])}} < N^{-\frac{1}{2}}, \text{ if } N \geq N_0.$$

Hence, the series (13a) converges. From our result we conclude that (17) holds for almost all θ on $a \leq \theta \leq b$, i.e.

$$\left| \sum_{n=1}^N e^{2\pi i h f(n, \theta)} \right| \leq K_3 N^{\frac{1}{2}} \log^{\frac{1}{2}} N (\log \log N)^{\frac{1}{2}} \omega(N), \text{ if } N \geq N_0^*(\theta).$$

From this Theorem 4 follows immediately.

Botany. — *De F₄-zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van Phaseolus vulgaris. III.* By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of October 30, 1948.)

Cl. 8, Form. 1b th. Alle 3 afmetingen zijn klein. 30 gevallen. Bij de bonen van cl 8, met de form. 1b th, kunnen we 3 vormen onderscheiden. Van bonen van cl 8a komen de indices overeen met die van bonen van cl 2, met de form. L B th als van de I-lijn, — de dikte van deze bonen is relatief klein —. De formule is 1b th I. Van bonen van cl 8b komen de indices overeen met die van bonen van cl 7, met de form. 1b Th als van de II-lijn, de dikte van deze bonen is relatief groot. De formule is 1b th II. In onderclasse 8c brengen we de bonen met de form. 1b th van de uitgangsbonen, waarbij alle 3 afmetingen gelijkmatig klein zijn. Deze laatste uitgangsbonen zijn, genotypisch, uitsplitsingen met de 3 afmetingen als geëxtraheerde recessieven (ll bb thth). Het zijn bonen met de form. 1b th in engere zin. De L B-index van bonen van cl 8c is $L B = 100 \times b/l$, komt overeen met die van bonen van de II-lijn en is dus $L B = 70$. De L Th-index ($= 100 \times th/l$) en de B Th-index ($= 100 \times th/b$) zijn intermediair ten opzichte van deze indices van de I- en de II-lijn. Deze verdeling in 3 onderklassen hebben we in tab. 1 en 1a nog niet doorgevoerd.

In 17 gevallen is de formule van de gemiddelden der bonenopbrengsten ook 1b th.

Van pl. 270 (tab. 1 en 1a) is de uitgangsboon van pl. 62. De formule van de gemiddelden van pl. 62 is L B Th ($l_m = 13.8$ mm). De uitgangsboon voor pl. 62 is van pl. 70, F₂-1934, de formule is L B Th. De formule van de gemiddelden van pl. 70 is eveneens L B Th. De 5 bonen van de peul van de uitgangsboon van pl. 62 voor pl. 270 zijn nog al verschillend. Drie bonen hebben grotere afmetingen, vooral ook een grotere breedte. Bij de bonenopbrengst van pl. 270 staat aangetekend „slecht, onrijp?”. Deze bonen kunnen onvolgroeid en daardoor klein zijn. Het gewicht van de bonen is nog al verschillend (van de verschillende peulen: 31—38, 26—31, 21—30, 24—28 cg, en 40, 48—52 cg). Alleen deze laatste peul met 4 bonen heeft 2 bonen met een grote lengte ($l = 13.7$ en $= 13.6$ mm). Alle bonen op 2 na behoren tot cl 8a (2 tot cl 4, tab. 1a). Alleen verder kweken zou kunnen leren, of we in dit geval met erfelijkheid van de kleine afmetingen te doen hebben.

Pl. 341 (tab. 1 en 1a) en pl. 342 (zie ook pl. 340, blz. 275) zijn van uitgangsbonen van pl. 89. De uitgangsboon voor pl. 341 is de 2de van de 4 bonen van de peul, die alle kleine afmetingen hebben. Alle bonen van pl. 89 zijn klein; er zijn slechts 4 bonen met een iets grotere lengte ($l = 13.8—12.9$ mm). De gemiddelde lengte van pl. 341 is $l_m = 12.0$ mm. Een dergelijke gemiddelde lengte treffen we ook bij bonenopbrengsten van de II-lijn van 1936 aan (tab. 4b). Bij de bonenopbrengst van pl. 341 staat aangetekend „heel goed, van de bonen echter de meeste gevlekt (en

rimpelig)". De grootste lengte van de bonen is $l = 14.4$ mm, dan volgt $l = 12.9$ mm. Deze ene boon met een grote lengte is opmerkelijk, want deze lengte ligt buiten de variatiebreedte van de lengte van de bonen van de II-lijn, dus met de formule $l_1 l_2$. We moeten aannemen, dat de formule van de lengte van de uitgangsboon ook één of meer L-factoren bevat (zie ben.). Er is in de bonenopbrengst van 27 bonen maar één boon met een grote lengte. Het is moeilijk, om het verschijnen van deze ene boon met een grote lengte door uitsplitsing te verklaren. Wellicht zijn er onder de 6 bonen met een niet zeer kleine lengte ($l = 12.5-12.9$ mm) enkele bonen met een LL- of Ll-verbinding in de formule van de lengte. De variatiebreedte van bonen van de I-lijn is zeer groot. Volgens de classificatie van de bonenopbrengst van pl. 341 (tab. 1a) zijn er 41 bonen in cl 8. Hiervan behoren er 13 tot cl 8c. De formule van de uitgangsboon van pl. 89 voor pl. 341 is 1b th in bijna homozygote vorm. Ze heeft slechts enkele lengte- en breedtefactoren als Ll en Bb. Van de bonenopbrengst van pl. 342 is de grootste lengte der bonen $l = 13.3$ mm (dan volgt $l = 12.9$ mm). Ook hier treffen we dus een boon aan met een grotere lengte dan bij de II-lijn voorkomt. Bovendien zijn er te veel bonen met een nog al grote lengte. Volgens de classificatie zijn er 19 bonen in cl 8, waarvan 3 in cl 8a, 12 in cl 8c en 4 in cl 8b. Bovendien zijn er 4 bonen in cl 7 en één in cl 1. De uitgangsboon van pl. 89 voor pl. 342 verschilt in haar genotype iets van die voor pl. 341. Ze is 1b th en bevat ook enkele Th Th-verbindingen. Van de slechte bonenopbrengst van pl. 340 behoren alle 14 bonen tot cl 8; 8 tot cl 8c en 6 tot cl 8a. Er is overeenkomst met pl. 341.

Van pl. 343 is de uitgangsboon van pl. 90; haar lengte is nog al groot ($l = 12.8$ mm). Er is in de ascendentie een vrij grote lengte en een kleine breedte. De bonenopbrengst van pl. 343 bevat 2 bonen met een grote lengte ($l = 15.1$ en $= 14.7$ mm). Van alle bonen is de breedte klein ($b = 8.5-7.1$ mm); ook de dikte ($th = 6.2-5.1$ mm). Volgens de classificatie zijn er 10 bonen in cl 4, form. Lb th en 16 bonen in cl 8. De uitgangsboon van pl. 90 voor pl. 343 heeft de form. 1b th met enkele lengtefactoren als LL en Ll en de overige als 11.

De uitgangsboon voor pl. 351 is van pl. 92; ze heeft zeer kleine afmetingen. De grootste lengte van de bonen van pl. 351 is $l = 14.1$ mm, (2 bonen, dan volgt $l = 13.1$ mm). Bij de uitgangsboon, die een zeer kleine lengte heeft ($l = 10.7$ mm) moeten we dus aannemen, dat ook L-factoren in de formule aanwezig zijn. In ons materiaal van de I-lijn van 1936 zijn ook bonen met een zeer kleine lengte ($l = 10.2$ en 11.2 mm). Volgens de classificatie van pl. 351 zijn er zeer veel bonen in cl 8 en een enkele in cl 1, 2 en 6. Onder de bonen van cl 8 zijn er verschillende, die behoren tot de onderclasse 8c. Tot overeenkomstige resultaten voert het onderzoek van 2 andere gevallen (pl. 365 en 368). Ook hier bevatten de bonenopbrengsten goede voorbeelden van bonen van cl 8c. Van een ander geval tenslotte nog (pl. 402) is de uitgangsboon iets meer heterozygot voor de form. 1b th. Volgens de classificatie zijn hier ook bonen in cl 7, 5 en 4.

Van de overige gevallen, waarin de uitgangsboon de formule 1b th, cl 8 heeft en ook de formule van de gemiddelden der bonenopbrengsten de formule 1b th cl 8 is, zijn er 8, waar de uitgangsbonen van dezelfde plant (pl. 51, F₃-1935, fig. 5) genomen zijn.

Van pl. 51, F₃-1935 zijn in 1936 de F₄-bonenopbrengsten van 35 uitgangsbonen onder-

zocht (pl. 977—1011 en 224). Voor pl. 51 is de uitgangsboon van pl. 66, F₂-1934 (tab. 21, 1947) genomen. De formule is L B th, cl 2 (th = 6.5 mm); de formule van de gemiddelden van pl. 51 is 1 b th ($l_m = 12.9$ mm). Volgens de classificatie zijn er in de bonenopbrengsten van 60 bonen van pl. 51, 18 bonen in cl 7. Van de 35 uitgangsbonen, die van pl. 51 genomen zijn, behoren er 17 tot cl 7, 8 tot cl 8 en 10 tot de overige klassen (de grootste lengte is $l = 14.2$ en $= 13.8$ mm). Er zijn daarbij relatief zeer veel bonen van cl 7. De bonenopbrengsten van de pl. 977—1011 en 224 behoren alle tot cl 8. We nemen aan, dat aan de grootte-verschillen van de bonen van de bonenopbrengsten van F₄-1936 en F₃-1935 ook milieu-invloeden aandeel hebben. De bonen van 1936 zijn gemiddeld iets kleiner dan die van 1935.

Het blijkt, dat bonenopbrengsten met de form. 1 b th der gemiddelden in haar samenstelling overeenkomst hebben met bonenopbrengsten met de form. L B Th. In beide gevallen is er meestal een grote mate van heterozygotie.

De gemiddelde lengten van de 35 bonenopbrengsten variëren van 11.0—13.0 mm, die van 25 bonenopbrengsten van de II-lijn van 1936 variëren van 10.1—12.0 mm. De formule van de II-lijn is ll bb Th Th. Van de bonenopbrengsten met een groter gemiddelde lengte dan 12.0 mm, moet de formule van de lengte van de uitgangsboon dus enkele L l- of L L-verbindingen bevatten. Deze bonenopbrengsten bevatten meestal enkele bonen met een lengte, die groter is dan 13.0 mm, (bonen met een grotere lengte komen onder de bonen van de II-lijn van 1936 niet voor, zie tab. 8). Ook bij gevallen, waar de uitgangsbonen een andere formule hebben dan 1 b th (d.w.z., met L in de formule, terwijl de formule van de gemiddelden 1 b th is, nemen we aan, dat de uitgangsboon L l- of L L-verbindingen bevat. Van deze verschillende mogelijkheden vinden we voorbeelden bij de 35 gevallen. Volgens de classificatie bevatten de bonenopbrengsten in deze gevallen bijna alleen bonen in cl 8, resp., in cl 8a, cl 8b, cl 8a en 8c, en cl 8a en 8b.

Van de 17 van de 35 bonenopbrengsten met uitgangsbonen van pl. 51, waar deze de form. 1 b Th hebben, bevatten, volgens de classificatie der bonenopbrengsten, er enige zeer veel bonen in cl 8b en ook daarbij soms enige in cl 7. De uitgangsbonen hebben in deze gevallen de form. 1 b Th in meer of minder homozygote vorm. In andere gevallen zijn de indices intermediair met bonen in cl 8 en de meeste andere klassen.

Van de 8 van de 35 gevallen, waar de formule van de uitgangsboon 1 b th is en ook die van de gemiddelden van de bonenopbrengsten (z. boven), kunnen we ook, volgens de classificatie, de bonenopbrengsten onderscheiden naar gelang de uitgangsbonen de formules 1 b th I, 1 b th II of 1 b th hebben. Goede voorbeelden van deze bonenopbrengsten, dus met overwegend de eigenschappen van de bonen van cl. 8a, 8b en 8c zijn de pl. 978, 981 en 1000 (tab. 1 en 1a).

Van de uitgangsboon van pl. 51 voor pl. 978 hebben de afmetingen de grenswaarden, de indices zijn intermediair. De bonenopbrengst van pl. 978 is „goed”. Er zijn te veel bonen met een niet zeer kleine lengte (5 bonen met $l = 12.7$ —12.3 mm). De formule van de lengte van de uitgangsboon bevat dus een enkele L l of L L-verbinding. Er zijn 10 bonen met de L B-index 59—63, 11 met de L Th-index 46—50 en 13 met de B Th-index 75—78. De bonen van pl. 978 zijn overwegend intermediair, hebben slechts enkele bonen in cl 8b, de overige in cl 8a. De uitgangsboon voor pl. 1000 komt zeer met die voor pl. 978 overeen, ook de gemiddelden van de bonenopbrengst. De gemiddelde L B-index is iets hoger. Er zijn, volgens de classificatie, 24 bonen in cl 8a, waarvan 16 in cl 8c. De samenstelling van de bonenopbrengst is iets eenvormiger dan die van pl. 978, beantwoordt iets meer aan die, waarvan de formule van de uitgangsboon 1 b th in engere zin is. Pl. 981 verschilt duidelijk van pl. 978 en 1000, komt overeen met bonenopbrengsten van de II-lijn. Van de uitgangsboon van pl. 51 naderen de indices zeer tot die van bonen van de II-lijn (tab. 7). Gemiddelden als van pl. 1000 treffen we onder de bonenopbrengsten van de II-lijn van 1936 niet aan (tab. 4b). Ook van de individuele bonen van pl. 981 naderen de indices tot die van bonen van de II-lijn van 1936 (tab. 7), maar komen er niet geheel mee overeen.

Het onderzoek van de bonenopbrengst van pl. 51 F₃-1935 voerde tot het resultaat, dat haar uitgangsboon van pl. 66, F₂-1934 de formule 1b Th heeft met veel heterozygotie (1948). Uit het onderzoek van de 35 bonenopbrengsten van de pl. 977-1011 en 224, F₄-1936, waarvan de uitgangsbonen van pl. 51 genomen zijn, blijkt, dat deze uitgangsbonen genotypische verschillen bezitten, overwegend binnen het gebied van bonen met de formule 1b th en dat geen der uitgangsbonen de formule 1b th, cl 8a, 8b en c of de form. 1b Th, cl 7 in geheel homozygote vorm hebben.

Van de gevallen, waar de uitgangsboon de formule 1b th, cl 8 heeft en ook de formule van de gemiddelden 1b th is, zijn er nu nog 4 te vermelden (pl. 365, 368, 1048 en 402). Alleen van pl. 1048 (tab. 1 en 1a) volgt hier een korte beschrijving.

De uitgangsboon is van pl. 101, F₃-1935; ze is de voorlaatste boon uit de peul en heeft veel kleinere afmetingen dan de, 4 voorafgaande bonen van de peul. Bonen met deze afmetingen en indices komen onder de bonen van de II-lijn en de I-lijn van 1935 niet voor. Dit is opmerkelijk. Indien we met een erfelijke variatie te doen hebben, is ze een voorbeeld van transgressieve variabiliteit (Proc. 52, blz. 80). Volgens de classificatie (tab. 1a) zijn bijna alle bonen in cl 8a, form. 1b th I. De gemiddelde indices naderen zeer tot die van bonenopbrengsten van de I-lijn. Van de individuele bonen komen enkele geheel met bonen van de I-lijn overeen. De uitgangsboon voor pl. 1049 (tab. 1 en 1a) is de laatste boon uit dezelfde peul, waarvan de uitgangsboon voor pl. 1048 genomen is. Ze is zeer klein. Waarschijnlijk hebben we bij deze laatste boon van de peul en eveneens bij de voorlaatste voor pl. 1048, met modificaties te doen van bonen met de form. 1b th.

Van pl. 101, F₃-1935 zijn in 1936 de bonenopbrengsten van 19 uitgangsbonen onderzocht (pl. 1019—1049 en 390). De uitgangsbonen behoren tot verschillende klassen en de meeste bonenopbrengsten vonden reeds bespreking (cl 1b, 2b, 4, 5). Volgens de classificatie zijn er in de bonenopbrengst van 83 bonen van pl. 101, 36 bonen in cl 1 en de overige bonen zijn over alle klassen verspreid (behalve cl 6). Bij de uitzaai van F₄-1936, staat bij de uitzaaibonen van pl. 101 aangetekend „fraise F₃-generatie, zelf”.

Voor pl. 101 is de uitgangsboon van pl. 83, F₂-1934 genomen. Van 12 planten (pl. 97—109) zijn de uitgangsbonen van pl. 83 (tab. 21, 1948). De formule van de gemiddelden van pl. 83 is L B Th. De gemiddelde afmetingen zijn nog al groot ($l_m = 14.2$ mm). De formule van de gemiddelden van pl. 101 is ook L B Th ($l_m = 13.5$ mm). De uitgangsbonen van pl. 101 voor de pl. 1019—1049 en 390 zijn nog al verschillend (de grootste lengten zijn $l = 15.1$ en $= 15.0$ mm, dan volgen 14.7—12.1 mm, en de kleinste lengten zijn $l = 10.9$ en $= 8.6$ mm).

Van de 19 uitgangsbonen van de bonenopbrengst van 83 bonen van pl. 101, behoren er 9 tot cl 1, 1 tot cl 3, 2 tot cl 4, 1 tot cl 5, 1 tot cl 7 en 5 tot cl 8. Ook de gemiddelden van de bonenopbrengsten verschillen zeer; $l_m = 14.2$ —12.0 mm. Volgens de classificatie zijn er in geen van deze 19 bonenopbrengsten bonen in cl 7 en cl 5. Hieruit blijkt, dat al de uitgangsbonen van deze 19 bonenopbrengsten het genotype van de bonen van het gebied van de I-lijn hebben, d.i., van cl 2 en cl 4, ook van cl 1 en cl 8. Deze 19 bonenopbrengsten van F₄-1936 geven dus een aanvulling van onze gegevens van de bonenopbrengst van pl. 101, F₃-1935 en haar classificatie.

Van 8 gevallen is de formule van de uitgangsboon 1b th, cl. 8 en hebben de gemiddelden van de bonenopbrengsten een andere formule. We bespraken ze reeds (blz. 4, 9, 12).

In 24 gevallen is de formule van de gemiddelden 1b th, cl 8 en heeft de uitgangsboon een andere formule. Er zijn uitgangsbonen uit alle klassen, behalve uit cl 6.

Van de pl. 279 en 280 hebben de uitgangsbonen de form. $L_1 L_2 B Th$, cl 1a. Ook van de pl. 275, 277 en 278 alle van dezelfde pl. 63, F_3 -1935, hebben de uitgangsbonen grote afmetingen en de bonenopbrengsten hebben kleine of vrij kleine gemiddelde afmetingen. De ascendentie geeft geen duidelijke aanwijzing voor de verschillen. Waarschijnlijk hebben we hier vooral te doen met modificatie. In 1935 bevatten de bonenopbrengsten vaak grote bonen, in 1936 waren de bonen vaak kleiner. Meermalen is aangetekend: „oogst slecht, enkele peulen nog groen, onrijp, waarschijnlijk te vroeg geoogst”. Er is een grote niet-erfelijke variabiliteit. Toch is er verschil met de niet-erfelijke variabiliteit van de zuivere lijnen. We vinden voor de bonenopbrengsten van de I-lijn van 1936 na uitgangsbonen van 1935 (zie fig. 1, 1947, l.c.) b.v., van de uitgangsboon van 1935, $l = 21.6$ mm, en van de gemiddelden van de bonenopbrengst van 1936, $l_m = 14.5$ mm; evenzo $l = 19.6$ mm, $l_m = 13.4$ mm; ook $l = 15.5$ mm, $l_m = 12.9$ mm; ook $l = 11.2$ mm, $l_m = 13.8$ mm. De uitgangsbonen van planten van 1935 met de allergrootste lengten leveren bonenopbrengsten in 1936 met dezelfde gemiddelden als de uitgangsbonen van 1935 met de kleinste lengten. (Er is geen selectie.) Hier, bij ons F_4 -materiaal van 1936, zien we, dat uitgangsbonen F_3 -1935 met een zeer grote lengte, bonenopbrengsten van F_4 -1936 hebben met zeer kleine gemiddelden. De erfelijke samenstelling van de uitgangsbonen heeft hier ook een aandeel in het resultaat, nemen we aan.

Onzeker in hunne betekenis zijn ten dele ook de overige 22 gevallen. De uitgangsbonen hebben de formule van alle klassen, behalve van cl 6, form. 1B th. In meerdere gevallen zijn de afmetingen van de uitgangsboon niet zeer groot. Volgens de classificatie van de bonenopbrengsten zijn er meestal zeer veel bonen in cl 8, en ook in cl 4, ook een enkele maal in cl 7. Er is enige overeenstemming tussen de samenstelling van de bonenopbrengst en de formule van de uitgangsboon.

We vinden onder het grote aantal gevallen van cl 8 met de formule 1b th er enkele, waar de uitgangsboon voor de bonenopbrengsten in hoge mate de formule 1b th in homozygote vorm heeft. We kunnen daarbij de formules 1b th I, 1b th II en 1b th onderscheiden. Het is moeilijk, om de erfelijkheid van deze bonen vast te stellen. De invloed van het milieu is hier groot.

Summary.

The results are given of the F_4 -seedgeneration of 1936, from initial beans of F_3 -1935, that go back to crosses of 1934.

The material is grouped into 8 classes according to the slightly modified tetrahybrid scheme (1947; Proceed. Roy. Ac. of Sc., Amsterdam, Vol. 50, N. 7).

In a single case of cl 1a, according to the classification, the formula of the initial bean is $L_1 L_2 B Th$ in the homozygous form to a high degree. In many cases the beanyield shows agreement with that of cl 2 with the form. L B th as of the I-line. Among the cases of cl 1b too there is a single one with the form. L B Th of the initial bean in an almost homozygous form. In many cases there are beans in many classes according to the classification, namely in cl 2, 4 and 8.

The beanyields of cl 1, form. L B Th show in many cases a relation to those of cl 2, form. L B th as of the I-line. According to the classification there are, besides a great number of beans in cl 2, also some beans in cl 1, 4 and 8 (tab. 5a).

TABLE 1. Some examples of the dimensions, weights and indices of F_3 -startingbeans of 1935 and the mean dimensions, weights and indices of the F_4 -beanyields of 1936.

F_1 -pl. of 1934 F_2 -seed gener.	F_2 -pl. of 1935 F_3 -s. gen.	Flow. col. of F_2 -pl.	Initial beans of F_3 - 1935	L	B	Th	W	LB	L Th	B Th	F_3 -pl. of 1936 F_4 -seed gener.	Flow. col. of F_3 -pl.	n of beans	L	B	Th	W	LB	L Th	B Th
Cl 1 a. The formula of the startingbeans is $L_1 L_2 B Th$.																				
82	84	w	8 p 2 b	167	94	74	80	56	44	79	331	w	25	158	93	67	74	59	43	72
81	73	—	2 p 1 b	183	112	73	101	61	40	65	309	—	38	154	100	65	66	66	42	64
Cl 1 b. The formula of the startingbeans is $L_1 l_2 B Th$.																				
81	70	—	1 p 2 b	136	98	68	65	72	50	69	298	—	24	147	96	68	71	65	47	71
66	47	v	17 p 5 b	135	90	69	61	67	51	77	194	w	50	143	95	65	62	67	46	69
81	78*)	—	17 p 1 b	152	95	60	62	63	39	63	321	w	22	145	93	67	66	64	46	72
65	43*)	v	5 p 6 b	126	85	68	43	68	54	80	165	—	38	142	88	65	58	63	46	73
Cl 2 b. The formula of the startingbeans is $L_1 l_2 B th$.																				
82	87	—	6 p 2 b	141	106	59	60	75	42	56	337	w	24	146	93	63	60	64	43	68
82	95*)	w	11 p 2 b	152	98	72	74	65	47	74	361	w	25	151	94	61	61	62	40	65
83	101*)	w	7 p 4 b	121	81	63	44	67	52	78	1047	—	28	139	87	60	54	62	44	69
Cl 3. The formula of the startingbeans is $L b Th$.																				
66	49	—	17 p 3 b	136	85	70	58	63	52	82	212	v	25	133	85	67	55	64	50	79
70	59*)	v	7 p 3 b	134	85	61	50	63	46	72	264	w	26	145	87	70	65	60	48	80
Cl 4. The formula of the startingbeans is $L b th$.																				
83	101	w	4 p 3 b	131	84	64	50	64	49	76	1033	w	31	132	83	51	41	63	39	61
82	91*)	w	5 p 3 b	128	81	60	45	63	47	74	347	w	24	146	86	65	59	59	44	76
Cl 5. The formula of the startingbeans is $1 B Th$.																				
81	374	—	10 p 3 b	125	90	71	53	72	57	79	314	v	24	124	84	65	51	68	53	77
Cl 6. The formula of the startingbeans is $1 B th$.																				
66	49	—	4 p 1 b	128	94	63	53	73	49	67	203	w	28	128	87	68	54	68	53	78
Cl 7. The formula of the startingbeans is $1 b Th$.																				
66	51	—	2 p 3 b	119	79	68	47	66	57	86	984	w	22	110	79	64	42	71	58	81
83	105*)	—	16 p 1 b	138	89	70	62	65	51	71	400	w	27	126	84	67	49	67	54	80
Cl 8. The formula of the startingbeans is $1 b th$.																				
70	62	w	7 p 4 b	130	80	60	45	62	46	75	270	—	27	111	70	54	31	63	49	77
82	89	v	5 p 2 b	112	82	51	32	73	46	62	341	v	27	120	79	59	41	66	49	75
66	51	—	1 p 2 b	129	84	65	50	65	50	77	978	w	21	118	74	59	39	63	50	79
66	51	—	1 p 5 b	119	78	65	43	66	55	83	981	w	25	113	73	60	38	65	53	82
66	51	—	7 p 3 b	127	85	65	50	67	51	77	1000	v	25	118	77	58	40	65	49	75
83	101	w	7 p 5 b	109	75	59	34	69	54	79	1048	w	22	122	75	54	37	62	44	72
83	101	w	7 p 6 b	86	65	48	17	76	56	74	1049	—	10	137	84	57	47	62	42	68

TABLE 1a. Classification according to the simplified tetrahybrid scheme of the beanyields of 1936 of tab. 1.

1 a L ₁ L ₂ B Th	1 b L ₁ L ₂ B Th	2 a L ₁ L ₂ B th	2 b L ₁ L ₂ B th	3 L b Th	4 L b th	5 L B Th	6 L B th	7 L b Th	8 a L b th I	8 b L b th II
12	5	1	4	0	3	0	0	0	0	0
9	8	6	14	0	1	0	0	0	0	0
1	20	0	3	0	12	0	1	0	2	0
4	28	0	12	0	4	0	0	1	0	1
1	15	0	4	0	0	0	0	1	0	1
13	0	19	1	4	0	0	0	0	0	1
1	0	0	0	15	0	5	0	0	0	7
2	0	1	1	1	1	10	1	1	0	3
2	10	0	2	1	8	0	0	0	0	1
2	10	0	2	1	8	0	0	0	0	1
5	0	0	1	0	8	0	7	4	6	9
5	0	0	1	0	8	0	7	4	6	9
1	2	2	2	2	2	2	2	2	2	2
1	2	2	2	2	2	2	2	2	2	2
1	18	3	6	19	24	1	24	20	24	1

*) The initial bean has in these cases another formula than that of the group in which it is placed.

w = white, v = violet, p = pod, b = bean.

Cl 2a, with the form. $L_1 L_2 B th$ contains one case of which the initial bean of the beanyield is to a high degree homozygous for the form. $L B th$, as of the I-line, but the length contains a smaller number of L-factors as $L L$ and perhaps also a smaller number of th-factors as $th th$.

Cl 2b with the form. $L_1 l_2 B th$ contains some cases that agree with beanyields of the I-line, so where the initial bean is to a high degree homozygous for the form. $L B th$. Two cases correspond to the above mentioned case of cl 2a. There are also indistinct cases. The initial bean f.i. has, sometimes a great thickness, whereas according to the classification of the beanyield, the genotype answers the form. $L B th$. There are also beanyields with the form. $L B th$ of the mean dimensions where the formula of the initial bean is $l b th$. We assume here that the genotype of the initial bean is $L B th$ or $L B Th$. The phaenotype of the initial bean is $l b th$.

The beans of cl 3 have a small breadth. Mostly all three dimensions are not very great, owing to positive correlation and polymere heredity-factors. We find some heredity of small breadth.

The beans of cl 4 have the form. $L b th$. The length is mostly not very great. The beans are related to those of cl 2, form. $L B th$, the formula of the I-line. There are some good examples of this class in the material.

The beans of cl 5, form. $l B Th$ are characterized by a small length and a high $L B$ -index. They are related to the beans of the II-line by their great thickness. According to the classification, beans of the II-line have also beans in cl 5 and 8b (tab. 5b), besides many beans in cl 7. There is a single case with heredity of this beanform.

The beans of cl 6, form. $l B th$ are little characteristic in our material; they closely agree with beans of cl 5. We describe such a case.

In the group of cases of cl 7, form. $l b Th$, as of the II-line we find some cases that wholly or almost wholly agree with beanyields of the II-line. The case too (pl. 984) that most closely approaches the II-line and that was extensively investigated is not yet identical with cases of the II-line.

The beans of cl 8 with the form. $l b th$ may be subdivided in cl 8a, form. $l b th$ I; cl 8b, form. $l b th$ II, and cl 8c, form. $l b th$, in the strict sense. The group of cl 8 contains a great number of cases. The beans with the form. $l b th$ can be compared with those with the form. $L B Th$. The dominance of the great dimensions over the small ones is only slight. There is much heterozygousness among these beans. Moreover the milieu-influence is great. It is difficult to establish the heredity here. We find some cases where the initial bean for the beanyield is more or less homozygous for the formula $l b th$. It is possible to distinguish here the formulas $l b th$ I, $l b th$ II and $l b th$.

So far we found no cases in our investigation, that wholly agree with beanyields of the I-line and of the II-line, i.e., of the two parent forms.

TABLE 2a. The averages of comparison-beanyields of the I-line of 1935.

Pl	n	L	B	Th	W	LB	LTh	BTh
2	15	166	100	62	71	60	37	63
15	36	165	90	61	63	54	37	68
4	24	163	98	61	66	61	38	62
16	45	160	95	62	61	59	39	65
1	37	158	96	62	61	61	39	65
10	49	149	91	57	52	61	39	64
7	60	143	88	57	48	62	40	64
6	68	142	87	54	45	62	39	64
8	65	140	86	55	45	61	39	64

TABLE 2b. Of the II-line of 1935.

Pl	n	L	B	Th	W	LB	LTh	BTh
22	48	121	90	74	54	75	61	82
26	50	115	84	70	44	74	61	83
27	52	114	83	70	43	74	62	84
23	49	113	84	70	43	75	63	84
28	66	111	84	72	43	76	65	85
29	56	109	82	68	40	75	63	84
24	50	106	79	67	37	74	64	85
30	30	105	76	64	33	73	61	85

TABLE 3a. Comparison-beans of the I-line of 1935 with which initial beans of the F₃-generation of 1935 for the F₄-generation of 1936 agree.

Pl	Bean	L	B	Th	W	LB	LTh	BTh
9	1p 1b	191	113	71	96	59	37	63
2	7p 1b	183	115	67	95	63	37	58
4	5p 2b	184	98	70	86	56	38	71
16	9p 2b	182	110	72	89	60	40	65
1	1p 3b	178	107	71	86	60	40	66
7	13p 2b	168	104	70	86	62	42	67
8	3p 2b	162	104	72	80	64	44	69
4	1p 3b	152	107	60	65	70	40	56
1	4p 3b	148	111	65	78	75	44	69
20	3p 2b	148	105	61	64	71	41	58
4	1p 4b	147	104	52	55	71	35	50

TABLE 3b. Comparison-beans of the I-line of 1935 for individual beans of F₃-beanyields of 1935, that agree with beanyields of the I-line of 1935.

Pl	Bean	L	B	Th	W	LB	LTh	BTh
16	3p 2b	168	98	73	77	58	44	75
6	8p 1b	165	105	69	80	64	42	60
1	1p 2b	161	103	68	82	64	42	66
16	2p 3b	160	110	72	84	69	45	66
16	2p 2b	158	107	74	84	68	47	69
3	9p 1b	158	93	67	61	59	42	72
13	2p 2b	153	93	67	64	61	44	72
14	12p 2b	152	91	67	65	60	44	74
5	4p 3b	151	98	67	66	65	44	68
7	11p 2b	151	92	67	62	62	44	73

TABLE 4a. The averages of comparison-beanyields of the I-line of 1936.

Pl	n	L	B	Th	W	LB	LTh	BTh
81	24	155	95	61	63	61	40	64
28	25	151	91	62	62	60	42	69
89	24	149	93	63	61	62	42	68
42	27	149	90	59	57	61	39	66
70	25	145	86	62	56	59	43	73
40	25	143	89	57	52	63	40	65
39	25	137	85	61	51	62	45	72
68	25	130	77	57	41	60	44	74

TABLE 4b. Of the II-line of 1936.

Pl	n	L	B	Th	W	LB	LTh	BTh
113	36	120	87	73	51	73	61	84
114	21	118	85	69	47	72	58	81
108	20	116	87	70	48	75	60	80
115	22	114	86	72	47	75	63	84
101	46	112	84	71	46	75	63	85
116	20	112	81	66	40	72	59	81
102	20	109	78	63	37	72	58	81
103	22	108	81	68	42	75	63	84
96	20	106	79	65	40	74	61	82
119	20	103	77	63	33	75	61	82
97	23	101	75	61	33	74	60	81

TABLE 5a. Classification of beanyields of the I-line of 1936. Some examples.

Plant	Numb. of beans	1 a L ₁ L ₂ BTh	1 b L ₁ BTh	2 a L ₁ L ₂ BTh	2 b L ₁ BTh	3 LBTh	4 LbTh	5 IBTh	6 IBth	7 lbTh	8 a lbth I	8 b lbth II
81	24	4	0	10	9	0	1					
28	25	2	0	6	14	0	2					
42	27	2	0	2	19	0	4					
39	25	1	0	11	0	0	10					
80	27			8	0	0	9				3	
78	25			9	0	6		1			10	
											9	

TABLE 5b. Idem of the II-line of 1936.

Plant	Numb. of beans	1 a L ₁ L ₂ BTh	1 b L ₁ BTh	2 a L ₁ L ₂ BTh	2 b L ₁ BTh	3 LBTh	4 LbTh	5 IBTh	6 IBth	7 lbTh	8 a lbth I	8 b lbth II
113	36							26		10		
117	65							38		25	1	1
104	21							6		15		
101	45							13		27		5
106	21							1		18		2
105	20		1	0	0	0	0	0	0	13		6
97	23							1	4	2	16	

TABLE 6. Some comparison-beans of the I-line of 1936 for individual beans of beanyields of F₄-1936, that agree with beanyields of the I-line of 1936.

Pl	Bean	L	B	Th	W	LB	LTh	BTh
93	2p 2b	180	104	73	80	58	41	70
93	2p 1b	167	102	70	75	60	42	69
43	3p 1b	166	99	74	86	60	45	75
43	3p 2b	163	97	71	82	60	44	73
59	4p 1b	160	95	74	71	59	46	78
43	1p 1b	159	89	69	66	56	43	78
42	2p 4b	153	101	69	70	66	45	68
29	7p 1b	151	87	67	65	58	44	77
89	4p 3b	150	96	69	73	64	46	72
89	4p 2b	145	94	70	72	65	48	75
39	5p 4b	142	88	68	62	62	48	77
37	5p 2b	141	95	67	67	67	48	71
92	3p 1b	136	89	65	57	65	48	73

TABLE 7a. Comparison-beans of the I-line of 1935 with a small length for initial beans of F₃-1935 for F₄-1936.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
5	202	125	82	43	27	66	34	53
6	242	124	84	57	39	68	46	68
	276	125	78	57	35	62	46	73
8'	15129	126	80	50	34	64	40	63
5	209	127	84	45	30	66	35	54
8'	15161	128	80	58	40	63	45	73
9	429	129	87	56	42	68	43	65
11	555	132	81	62	42	61	47	77
9	458	135	94	66	55	70	49	70

TABLE 7b. Comparison-beans of the II-line of 1935 for initial-beans of F₃-1935 for F₄-beanyields of 1936.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
22	4p 2b	130	92	72	59	71	55	78
22	3p 1b	128	93	77	61	73	60	82
22	2p 1b	127	95	77	61	75	61	85
25	19p 2b	127	86	67	48	68	53	78
23	11p 3b	126	85	66	45	68	52	78
23	11p 4b	122	82	64	41	67	53	78
25	13p 3b	120	82	63	41	68	53	77
24	11p 3b	119	83	63	40	70	53	76
25	19p 3b	118	82	69	43	70	58	84
25	10p 4b	115	76	62	36	66	54	82

TABLE 8. Exceptional beans of the II-line of 1936 to be compared to F₄-beans of 1936.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
113	1p 2b	130	91	73	62	70	56	80
113	1p 1b	129	91	72	57	71	56	79
114	4p 1b	125	89	69	51	71	55	78
108	3p 1b	123	90	72	55	73	59	80
116	3p 3b	121	87	66	45	72	55	76
108	1p 4b	120	87	68	48	73	58	78
108	1p 1b	117	85	67	46	73	57	79
114	5p 3b	115	83	66	44	72	57	80
105	1p 1b	114	85	67	45	75	59	79
117	11p 5b	107	80	62	37	75	58	78
97	2p 2b	121	87	59	44	72	49	68
119	1p 6b	97	74	58	28	76	60	58

TABLE 9. Examples of beans of F₃-1935 of various classes.

P1	Bean	L	B	Th	W	LB	L Th	B Th
cl 3, form. L b Th								
91	1p 4b	149	84	68	60	56	46	81
91	1p 2b	143	83	67	56	58	47	81
91	1p 3b	142	82	69	56	58	49	84
91	1p 1b	140	76	65	50	54	46	86
49	17p 3b	136	85	70	58	63	52	82
74	2p 1b	135	81	68	53	60	50	84
cl 4, form. L b th								
99	1p 1b	143	80	57	41	56	40	71
51	7p 4b	133	82	63	51	62	47	77
101	4p 3b	131	84	64	50	64	49	76
cl 5, form. 1B Th								
33	16p 1b	130	92	68	55	71	52	74
74	12p 3b	125	90	71	53	72	57	79
33	6p 6b	122	88	69	49	72	57	78
33	6p 2b	121	92	69	50	76	57	75
33	6p 1b	118	91	71	49	77	60	78
cl 7, form. 1b Th								
51	2p 1b	128	84	68	54	66	53	81
51	2p 2b	126	80	69	52	64	55	86
51	2p 4b	116	80	68	45	69	59	85
49	6p 1b	127	80	69	52	63	54	86
98	8p 3b	126	85	71	55	68	56	84

TABLE 10. F₄-1936. Beans of cl 5 form. 1B Th, and of cl 8, form. 1b th with the indices of cl 5.

P1	Bean	L	B	Th	W	LB	L Th	B Th
314	3p 2b	125	87	67	55	70	54	77
	3p 1b	124	87	70	54	70	57	81
	3p 3b	121	86	66	51	71	55	75
	3p 5b ¹⁾	122	87	65	52	71	53	75
	4p 5b	123	87	65	53	71	53	75
	3p 4b ²⁾	118	85	66	50	72	56	78
	2p 5b	117	84	65	48	72	56	77
	2p 3b	116	82	65	48	71	56	79
	2p 4b	115	85	63	47	74	55	74
	2p 2b	113	81	62	44	72	55	77
124	1p 1b	119	88	67	50	74	56	76
	1p 3b	118	87	66	49	74	56	76
	1p 2b ¹⁾	114	86	63	43	75	55	73
	1p 4b	113	83	66	44	73	58	80
	1p 5b ³⁾	103	75	62	34	73	60	83

¹⁾ Form. cl 6. ²⁾ Form. cl 7. ³⁾ The last bean of the row in the pod.

Zoology. — *On the influence of various chlorides on maturation and cleavage of the egg of Limnaea stagnalis L.* By MIEK S. GRASVELD, (Zoological Laboratory, University of Utrecht). (Communicated by Prof. CHR. P. RAVEN).

(Communicated at the meeting of February 26, 1949.)

1. Introduction.

RAVEN and KLOMP (1946) described an abnormal cleavage of the eggs of *Limnaea stagnalis* in Ca-free media: the blastomeres remain spherical in shape and lose connection at an early stage; no cleavage cavity is formed; when cleavage advances, a loose aggregate of spherical cells is formed, surrounded by the vitelline membrane as by a loose envelope. Addition of CaCl_2 (minimum concentration 0.005 %) leads to normal cleavage; in 0.01—0.08 % CaCl_2 this effect is clearly observable. The primary action of the lack of Ca-ions seems to be a change in the properties of the vitelline membrane, which loses its contact with the egg surface.

OLGA HUDIG (1946) observed that the vitelline membrane of *Limnaea stagnalis* surrounds the egg from the very beginning till it is left by the embryo; initially it is very thin, at a later moment it becomes more distinctly visible. The membrane changes its consistency between the formation of the first and second polar body. It undergoes an alteration immediately after the egg is put into distilled water, probably through swelling; in consequence of this, the first polar body is situated inside the membrane and the "protoplasmic strand" which in normal eggs is seen between the first polar body and the egg and, later, between the first and second polar body is not visible.

This can be explained by supposing that the first polar body is always inside the membrane, which is pulled tightly around it in normal media. This view is supported by the observation that the first polar body always comes to lie inside the membrane, when eggs are transferred to distilled water. On the other hand, the first polar body of normal eggs detaches itself easily from the egg surface, when the eggs are stirred; this speaks strongly against the enclosure of the first polar body by the membrane; the connecting strand might be a spindle remnant.

The vitelline membrane does not swell in a mixture of distilled water and CaCl_2 . In solutions of 0.2—0.0125 % CaCl_2 the first polar body is mostly situated outside the membrane, in 0.00625 % and less concentrated solutions inside the membrane. In 0.04—0.005 % CaCl_2 cleavage is normal; in the latter concentration the blastomeres do not join quite so closely as they do in the higher concentrations and the cleavage cavity is formed late.

The vitelline membrane must be considered to be a chorion; it does not take an active part in the cleavage and does not form the cell walls. It is concluded that not only the membrane, but also a protoplasmic cortex layer, lying immediately beneath it, is influenced by Ca-ions.

In KCl cleavage may be normal, perhaps because the K-ions liberate the Ca-ions, present in the cortex.

RAVEN and MIGHORST (1946) studied the influence of highly concentrated CaCl_2 -solutions on maturation in the egg of *Limnaea stagnalis*. No polar bodies are formed in 3 % and 1.5 % CaCl_2 . One small, spindle-shaped polar body is sometimes formed in 1.1 % CaCl_2 ; a larger percentage of the eggs forms a first polar body and sometimes a second polar body in 0.75 %. In 0.5 % CaCl_2 depolarization of the second maturation spindle occurs in some egg-masses. In 0.4 % nearly all eggs develop to the 2-cell stage. In 0.2 % a morula stage may be attained.

In higher concentrations small clear spots appear in the vitelline membrane; after some hours small vesicles have been formed on the surface of the egg in these places; still later the membrane is lifted locally from the egg surface, forming large blebs, in which the polar bodies are sometimes lying. This forms an argument in favour of the view that the polar bodies lie inside the membrane. The eggs can accumulate Ca-ions from a concentrated CaCl_2 -solution.

DE GROOT (1948) treated the eggs of *Limnaea stagnalis* with LiCl-solutions, varying from 0.05—4 %. Eggs, transferred to a 4 % solution immediately stop their development and cytolysis occurs. In 2 % a partly extruded polar body is formed; in 1.5 and 1 % its formation is prevented, when treatment begins more than 30—35 minutes before; if it begins at a later moment the first polar body is deformed; in 0.6 % and 0.5 % a first polar body is always extruded, mostly deformed; in 0.4 % a second polar body may be formed during intensive amoeboid movements of the egg; in 0.35—0.05 % both polar bodies are nearly always formed; 0.35 % is the highest concentration, in which a cleavage furrow is observed, appearing with considerable delay; in 0.05 % there is no delay and sometimes a second cleavage occurs.

The observed effects in concentrations above 0.2 % need not to be due to a specific influence of LiCl, but may be caused primarily by hypertonicity of the medium, if the extrusion of polar bodies has to be understood as a process of osmotic activity.

In 0.1 % and 0.05 % the blastomeres flatten themselves against each other and a cleavage cavity is formed. Hence, not only CaCl_2 but also LiCl is able to cause flattening.

It is likely that the vitelline membrane is affected by LiCl.

In the following investigation the influence of CaCl_2 , MgCl_2 , LiCl, NaCl, and KCl on maturation and cleavage of the egg of *Limnaea stagnalis* has been compared. The position of the polar bodies, the behaviour of the vitelline membrane and the type of cleavage were especially observed.

2. Material and methods.

Eggs were obtained in the usual way by stimulation of the snails with *Hydrocharis* (cf. RAVEN and BRETSCHNEIDER 1942).

Each egg-mass was divided into 6 parts; the eggs were decapsulated immediately after oviposition and washed in distilled water in order to remove the capsule fluid; 10 eggs of each egg-mass were transferred to equimolar solutions of CaCl_2 , MgCl_2 , LiCl , NaCl and KCl , respectively.

Stock solutions of 0.1 M. were made from the crystallized chlorides; they were diluted with distilled water. With CaCl_2 and MgCl_2 , which contain crystal water, the percentages given are computed to net weights.

The eggs were put in a small drop of the solution on a hollow slide, on which the cover glass was fixed on two sides with paraffine; in this way, the preparations can be placed vertically on a horizontal microscope in order to study the polar bodies at the animal pole, which is, as a rule, turned upwards. To get a clear view of the vitelline membrane, it is necessary to close the diaphragm of the microscope a great deal. In the long run, the eggs beneath the cover glass do not get sufficient oxygen and, after a day, abnormalities ensue and development stops. The control eggs developed inside the egg capsules. All eggs were kept at a temperature of 25° C in a water bath.

Table 1 gives a general survey of the experiments.

TABLE I. General survey of the experiments.

Solutions	CaCl_2			MgCl_2			LiCl			NaCl			KCl		
Concentration (Mol.)	Conc. (%)	Relative 1) osmotic pressure	Number 2) of eggmasses	Conc. (%)	Relative osmotic pressure	Number of eggmasses	Conc. (%)	Relative osmotic pressure	Number of eggmasses	Conc. (%)	Relative osmotic pressure	Number of eggmasses	Conc. (%)	Relative osmotic pressure	Number of eggmasses
0.1	1.1	3	10	0.95	3	8	0.42	2	11	0.58	2	9	0.75	2	8
0.05	0.55	$1\frac{1}{2}$	12	0.475	$1\frac{1}{2}$	14	0.21	1	15	0.29	1	17	0.375	1	17
0.025	0.275	$\frac{3}{4}$	9	0.2375	$\frac{3}{4}$	9	0.105	$\frac{1}{2}$	9	0.145	$\frac{1}{2}$	9	0.1875	$\frac{1}{2}$	9
0.0125	0.1375	$\frac{3}{8}$	5	0.1188	$\frac{3}{8}$	5	0.0525	$\frac{1}{4}$	5	0.0725	$\frac{1}{4}$	5	0.0938	$\frac{1}{4}$	5
0.00625	0.0688	$\frac{3}{16}$	9	0.0594	$\frac{3}{16}$	9	0.0262	$\frac{1}{8}$	6	0.0363	$\frac{1}{8}$	6	0.0469	$\frac{1}{8}$	6
0.00312	0.0344	$\frac{3}{32}$	8												
0.00156	0.0172	$\frac{3}{64}$	6	0.0148	$\frac{3}{64}$	1	0.0066	$\frac{1}{32}$	1	0.0091	$\frac{1}{32}$	1	0.0117	$\frac{1}{32}$	1
0.00078	0.0086	$\frac{3}{128}$	2												
0.00039	0.0043	$\frac{3}{256}$	4												
0.00019	0.0022	$\frac{3}{512}$	1												

1) Relative osmotic pressure = $\frac{\text{osm. pr. of solution}}{\text{internal osm. pr. of eggs}}$

2) On an average, 10 eggs pro egg-mass were treated with each solution.

3. Results of experiments.

A. Normal development and influence of distilled water.

The development of the eggs in the capsules has been described by O. HÜDIG (1946). Some additional observations have been made:

1. The "protoplasmic strand" between first and second polar body does not always disappear, but it becomes smaller in all cases.

2. At the 24-cell stage and later the second polar body is no longer flattened, but spherical in shape and swollen; the first polar body is no longer spherical, but shows an irregular outline.

The development in distilled water has also been observed by O. HÜDIG; she describes that the membrane takes part in the movements of the egg surface during cleavage. This cannot be the case at the animal pole, however, for both polar bodies are distinctly visible here inside the membrane. Moreover, the bridge at the vegetative pole is formed already before the cleavage groove is complete. The first polar body has a less distinct outline than the second polar body in distilled water.

B. The development in the various chlorides.

a. The stage of development, reached in the different media, is shown in table 2. The results are arranged according to the relative osmotic

TABLE 2. The stage of development reached in the different media.

Rel. osm. press.	CaCl ₂	MgCl ₂	LiCl	NaCl	KCl
3	30 % 1st p.b.	15 % 1st p.b.			
2			50 % 1st p.b. 15 % 2nd p.b.	50 % 1st p.b. sometimes 2nd p.b.	35 % 1st p.b. sometimes 2nd p.b.
1 1/2	100 % 1st p.b. 50 % 2nd p.b.	100 % 1st p.b. 50 % 2nd p.b.			
1			4-8-c. st. (delay)	morula-st.	12-24-c. st.
3/4	16-24-c. st.	24-c. st.	4-c. st.	16-24-c. st.	16-24-c. st.
1/2					
3/8	16-c. st.	morula-st.	2-4-c. st.	8-c. st.	4-8-c. st.
1/4					
3/16	16-c. st.	8-16-c. st.	2-4-c. st.	4-8-c. st.	4-c. st.
1/8					
3/32	16-c. st.				
3/64	8-c. st.	4-16-c. st.	4-c. st.	4-c. st.	4-c. st.
1/32					
3/128	12-c. st.				
3/256	8-c. st.				
3/512	4-c. st.				

Abbrev.: p.b. = polar body; st. = stage.

pressures of the salt solutions as compared with the internal osmotic pressure of freshly-laid eggs, which equals that of a 0.09—0.10 M. solution of a non-electrolyte according to RAVEN and KLOMP (1946). Hence, the eggs of *Limnaea stagnalis* are about isotonic with 0.03 M. CaCl₂ and MgCl₂, 0.045—0.05 M. LiCl, NaCl and KCl.

In judging the results, it has to be kept in mind that in all solutions after some time development stops in consequence of the lack of oxygen. As a rule, this occurs at the 16—24 cell stage. Hence, the data of table 2 have only a relative value.

In solutions of CaCl₂ and MgCl₂ which are highly hypertonic with regard to the eggs, the first polar body is rarely, the second polar body

never formed. With decreasing hypertonicity, the number of first polar bodies rises and also second polar bodies appear. Finally, in solutions which are only slightly hypertonic, the first polar body is formed in 100 %, the second polar body in 50 % of cases in CaCl_2 and MgCl_2 .

In iso- and hypotonic solutions of all chlorides both polar bodies are always formed. Development stops after extrusion of the polar bodies in hypertonic solutions; in isotonic and hypotonic solutions cleavage occurs in all chlorides.

The highest stage of development is reached in 0.025—0.00312 M. CaCl_2 and MgCl_2 , and 0.05—0.025 M. LiCl , NaCl and KCl . In less concentrated solutions development stops sooner. In 0.00019 M. CaCl_2 and 0.00156 M. KCl and NaCl development stops at the same moment as in distilled water.

One may conclude that hypertonicity of the solutions causes interruption of maturation, while for further development an optimum exists at a moderate degree of hypotonicity. This agrees with the results of RAVEN and KLOMP (1946), HUDIG (1946), RAVEN and MIGHORST (1946) and DE GROOT (1948).

In none of the LiCl solutions development proceeded beyond the 4—8 cell stage. Hence, LiCl seems to be the most noxious of these chlorides.

b. Further effects of hypertonic solutions.

1. Rotation of the eggs.

When the eggs are in the capsule fluid, they have a tendency to turn their animal pole upwards. This does not occur in hypertonic solutions, viz. in 0.1 M. CaCl_2 and MgCl_2 , 0.05 M. CaCl_2 and in 0.1 M. LiCl , NaCl and KCl .

2. Shape of the first polar body.

Deformed first polar bodies were observed in hypertonic solutions, viz. in 0.1 M. and 0.05 M. CaCl_2 and MgCl_2 and in 0.1 M. LiCl , NaCl and KCl . As a rule, they are round at first, then they become pear-shaped (after $\frac{1}{4}$ — $\frac{1}{2}$ hour) and still later pointed, with a conical protuberance at the top (fig. 1). The deformation in MgCl_2 is greater than in CaCl_2 , in

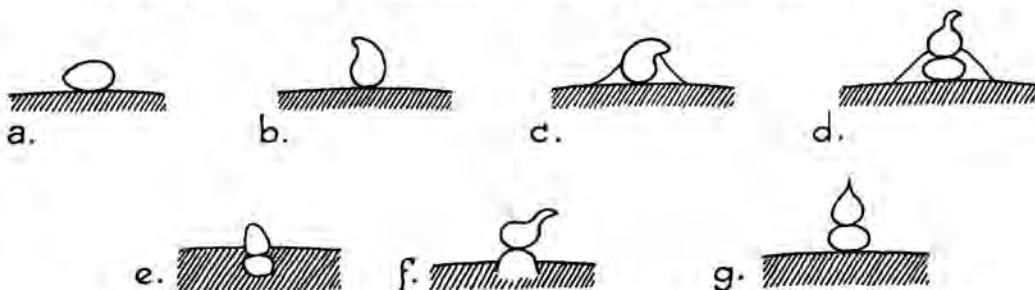


Fig. 1. Deformation of first polar body in a) 0.1 M. CaCl_2 ; b) 0.05 M. CaCl_2 ; c) 0.1 M. MgCl_2 ; d) 0.05 M. MgCl_2 ; e) 0.1 M. LiCl ; f) 0.1 M. NaCl ; g) 0.1 M. KCl .

NaCl greater than in KCl, in KCl greater than in LiCl. The polar bodies are normally shaped in all other solutions. Evidently, water is withdrawn from the polar bodies in hypertonic solutions.

Deformed polar bodies have been observed in hypertonic solutions of LiCl by DE GROOT (1948).

3. Vesicles and blebs at the egg surface.

Lipoid-looking vesicles and blebs at the egg surface appear in hypertonic solutions. Such vesicles were described as clear spots in the vitelline membrane, which forms great blebs at a later moment, by RAVEN and MIGHORST (1946).

Probably, the vesicles change into blebs; partly the vesicles remain, however; may-be they are vacuoles, containing a lipid substance; they are situated, however, inside and not in the membrane (fig. 2).

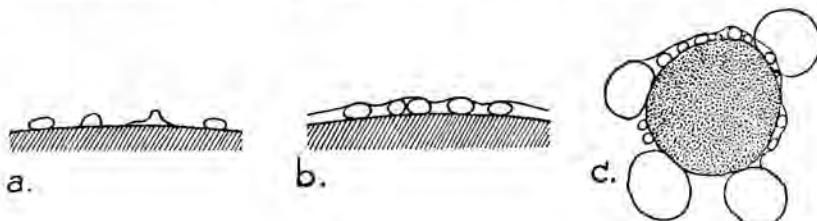


Fig. 2. Formation of "lipoid" vesicles and blebs on the egg-surface in 0.1 M. CaCl_2 after
a) 6 hours, b) 9 hours, c) 2—3 days.

4. Amoeboid movements in hypertonic LiCl-solutions.

The eggs show considerable amoeboid movements in 0.1 M. LiCl 3—7 hours after the first cleavage in the control eggs. They show the same movements in 0.05 M. LiCl after the extrusion of the first polar body and sometimes after the first cleavage.

DE GROOT also describes a strong amoeboid activity in hypertonic LiCl-solutions. As no increase of amoeboid activity of the eggs has been observed in the other chlorides, this seems to be a specific Li-effect.

c. Position of polar bodies.

In the following description the term of O. HUDIG "polar bodies outside the membrane" is used; this means a position of the first polar bodies as in control eggs.

Attention has been paid to:

1. The "protoplasmic strand" (O. HUDIG).
2. The vitelline membrane.
3. The flattening of the polar bodies.

Table 3 gives a survey of the results. They can be summarized in this way:

TABLE 3. The position of the 1st polar bodies.

Concentr. (Mol.)	CaCl_2	MgCl_2	LiCl	NaCl	KCl
0,1	vitelline membrane not visible				
0,05	± normal	inside	inside	inside	inside
0,025	± normal	normal	inside	inside	inside
0,0125	inside	normal	inside	inside	inside
0,00625	normal	normal	inside	inside	inside
0,00312	normal				
0,00156	normal	inside	inside	inside	inside
0,00078	inside				
0,00039	inside				
0,00019	inside				

1. In LiCl , NaCl and KCl solutions of all concentrations studied both polar bodies are situated inside the membrane.
2. In concentrated solutions of CaCl_2 and MgCl_2 , both polar bodies are inside the membrane or the latter is passing directly from the egg surface toward the first polar body.
3. In weaker solutions of these salts, the position of the polar bodies resembles that in normal eggs (first polar body outside, second polar body inside membrane); most so in 0.0625—0.00156 M. CaCl_2 and 0.025—0.00625 M. CaCl_2 . In these cases, a distinct "protoplasmic strand" is present between first polar body and egg, later between both polar bodies.
4. In still weaker solutions, it is obvious that this "protoplasmic strand" actually is the vitelline membrane, which passes now to the outside of the first polar body, so that both polar bodies are inside the membrane.

d. The type of cleavage.

Attention has been paid to:

1. Flattening of the blastomeres.
2. Formation of membrane bridges between the blastomeres.
3. Formation of a cleavage cavity.

Table 4 gives a survey of the types of cleavage obtained in the various solutions. From the observations, the following conclusions can be drawn:

1. Though no absolute correlation between the 3 phenomena exists, in general there is a direct relation between the degree of flattening and the development of the cleavage cavity, whereas an inverse relation exists between both these phenomena and the formation of membrane bridges at cleavage. The only exception to this rule is formed by cleavage in 0.05 M. LiCl , where a high degree of flattening is not attended with the formation of a cleavage cavity.
2. The cleavage type is most normal in CaCl_2 , MgCl_2 and LiCl at a moderate degree of hypotonicity (0.0125—0.00312 M.), somewhat less in NaCl at this same concentration, very abnormal in KCl . At both higher

TABLE 4. Cleavage in the different chlorides.

Conc. (Mol.)	CaCl ₂			MgCl ₂			LiCl			NaCl			KCl		
	F	C	B	F	C	B	F	C	B	F	C	B	F	C	B
0,05							+++D	-	+	+	-	+	-	-	++
0,025	+	-	+	+	-	-	+++D	++D	+	+	+	++	+	-	++
0,0125	++	+++	-	+++	+++	-	+++D	+++D	+	++	+++	++	+	-	++
0,00625	+++	+++	-	+++	+++	-	+++D	+++D	+	++	+++	++	+	-	++
0,00312	+++	+++	-	+++	+++	-	+++D	+++D	+	++	+++	++	+	-	++
0,00156	++	++	-	+	+D	+	-	-	+++	-	-	+++	-	-	++
0,00078	+	++D	-												
0,00039	+	+	+												
0,00019	±	±	++												

Abbrev.: F = Flattening of the blastomeres.

C = Formation of cleavage cavity.

B = Formation of membrane bridges.

The number of plus marks denotes the strength of the phenomenon

(+ = slight, ++ = moderate, +++ = strong).

D = delayed.

and lower concentrations the flattening of the blastomeres and the formation of a cleavage cavity are suppressed.

3. The degree of flattening decreases in the order LiCl—NaCl—KCl.
4. The incidence of membrane bridges increases in the order CaCl₂ and MgCl₂—LiCl—NaCl—KCl.
5. The flattening of the blastomeres can be preceded by formation of a bridge at the vegetative pole, but the latter disappears during the flattening. If flattening does not occur, the bridge is permanent.
6. The polar bodies are mostly inside the membrane, when bridges are formed and no flattening and cleavage cavity are observed then.
7. Both the flattening of the blastomeres and the formation of a cleavage cavity are delayed in LiCl. The second cleavage shows a delay of $\frac{1}{2}$ —2 hours in LiCl solutions.

e. Abnormal cleavage.

In some cases the eggs cleave into three blastomeres: the first cleavage is a little delayed, three blastomeres with three nuclei are formed. Secondarily, two of these cells may fuse, so that a 2-cell stage is formed, one of the cells containing two nuclei. The second cleavage of 3-cell stages, which is somewhat delayed, leads to a six-cell stage, which does not flatten, and forms membrane bridges; then development stops.

Though this phenomenon only occurs in cultures which are cut off from the air, it attracts attention that it occurs more frequently in some solutions (table 5), chiefly in CaCl₂ and MgCl₂, seldom in KCl and NaCl, never in LiCl.

It is evident that the egg is most susceptible to lack of oxygen in CaCl₂ and MgCl₂.

TABLE 5. Abnormal cleavages in various chlorides.

Conc. (Mol.)	CaCl ₂		MgCl ₂		LiCl		NaCl		KCl	
	Number of eggs	Number of abn. cleav.								
0.05	—	—	—	—	150	0	170	9	170	seldom
0.025	90	3	90	4	90	0	90	0	90	1
0.0125	50	8	50	4	50	0	50	0	50	2
0.00625	90	16	90	18	60	0	60	0	60	0
0.00312	80	2	—	—	—	—	—	—	—	—
0.00156	60	2	10	0	10	0	10	0	10	0
0.00078	20	1	—	—	—	—	—	—	—	—
0.00039	40	6	—	—	—	—	—	—	—	—
0.00019	10	1	—	—	—	—	—	—	—	—
Total	440	39	240	26	360	0	380	9	380	>3
%	9		11		0		2		± I	

4. Discussion.

Considering the results of my experiments, it is clear that the salt solutions studied have influenced 3 groups of factors:

1. the consistency of the vitelline membrane, which determines the position of the polar bodies with respect to the latter;
2. the "adhesiveness" of the vitelline membrane to the egg surface, determining its behaviour during cleavage;
3. the properties of the egg cortex, responsible for the flattening of the blastomeres and, indirectly, for the formation of the cleavage cavity.

1. It is evident from my observations that the protoplasmic strand mentioned by O. HUDIG is not a spindle remnant, but the same as the membrane, which undergoes an alteration between the formation of the first and second polar body, as supposed by RAVEN (1945). Normally the membrane is almost liquid at the formation of the first polar body; it is situated closely around it and is visible as a "protoplasmic strand" between the first polar body and the egg. This strand shrinks at a latter moment, probably by hardening; in consequence of this the first polar body remains attached to the egg surface. I never observed (as O. HUDIG did) that it can be detached from the egg surface by stirring the cover glass. The membrane gets another consistency before the formation of the second polar body; in consequence of this it is distinctly visible as a pellicle, tightened over the second polar body, the latter being flattened by it.

In some concentrations of CaCl₂ and MgCl₂ this process occurs likewise. In distilled water, LiCl, KCl and NaCl the change in the consistency of the membrane takes place at an earlier stage than in the control eggs; the same occurs in the less concentrated solutions of CaCl₂ and MgCl₂; in consequence of this there is no "protoplasmic strand" and the membrane

is visible as a pellicle. The normal course of the process requires a minimum concentration of CaCl_2 or MgCl_2 ; it never occurs in LiCl , NaCl and KCl .

Considering the protoplasmic strand as membrane, one can conclude that the first polar body is always inside the membrane. This agrees with the fact that in some cases polar bodies, which appear to lie outside the membrane at first cleavage are found inside at the 24-cell stage.

The swelling of the second polar body in the control eggs also points to the fact that the membrane, tightened over it, has another consistency than near the first polar body; the latter, surrounded by the hardened membrane, is irregular in shape.

Although the membrane is not immediately visible, it encloses the egg from the very beginning. This may be concluded from:

a. Wrinkling of the membrane in distilled water and low salt concentrations (0.00039 M. CaCl_2 , 0.00156 M. LiCl).

b. Lifting of the membrane by the lipoid-looking vesicles in hypertonic solutions.

c. The existence of the protoplasmic strand or the pellicle after the formation of the first polar body.

2. The vitelline membrane does not take an active part in cleavage, for cleavage continues when membrane bridges are formed; the walls of the blastomeres cannot be formed by the membrane, for it does not follow the cleavage furrows.

The behaviour of the membrane at cleavage parallels the position of the polar bodies. The accelerated change of the membrane consistency by lack of 2-valent kations, which causes a position of the polar bodies inside the membrane, is attended with formation of bridges between the blastomeres. However, there is no strict correspondence between both phenomena. Whereas the changes in consistency of the membrane, as evidenced by the position of the polar bodies, are alike in LiCl , NaCl and KCl , the incidence of membrane bridges in these solutions increases in the order mentioned. This points to a difference between the properties of the membrane which have been called "consistency" and "adhesiveness", respectively.

3. The changes in cleavage type, brought about by a diminished flattening of the blastomeres, which leads to a suppression of the cleavage cavity, may be explained by an influence of the salts on the properties of the egg cortex. Normal cleavage may occur in CaCl_2 , MgCl_2 and LiCl of optimal (moderately hypotonic) concentrations; in NaCl cleavage type is less normal; in KCl it is very abnormal.

It might be supposed that among the properties of the egg cortex involved its viscosity and "tension at the surface" play a prominent part. In table 6, our results have, therefore, been compared with those of DE VRIES (1947) on the influence of CaCl_2 and LiCl on these properties of the egg, as determined with the centrifuge method. The table shows, however, that no correspondence between both groups of observations exists. As a matter of fact, the centrifuge method yields only "gross"

TABLE 6. Comparison with the results of DE VRIES (1947).

CaCl ₂					LiCl				
Mol.	Viscosity	Tension	Flattening	Cleavage cavity	Mol.	Viscosity	Tension	Flattening	Cleavage cavity
0,12	-	+			0,25	-	+		
0,04	-	0			0,15	-	+		
0,025			+	-	0,1				
0,0125			++	+++	0,05				
0,009	-	0			0,025	-	-	+++	-
0,008	0	0			0,02	0	0	+++	++
0,00625			+++	+++	0,015	0	0		
0,005	0	0			0,0125	0	0	+++	+++
0,0045	+	0			0,0075	0	0	+++	+++
0,00312			+++	+++	0,00625				
0,003	0	0			0,005	0	0		
0,002	0	0			0,0025	+	0		
0,00156			++	++	0,002	0	0		
0,00078			+	++	0,00156			-	
0,0007	0	0			0,0015	0	0		
0,00039			+	+					
0,00019			±	±					

Decrease of viscosity (tension): -

Increase " " " : +

Flattening and cleavage cavity cf. table 4.

results, whereas in cleavage local changes in both properties may play a predominant part. This might explain the lack of correspondence between both experiments.

4. The question arises if either the various chlorides affect specifically maturation and cleavage of the egg of *Limnaea stagnalis*, or the influence only depends on the number of ions present.

Hypertonicity of the medium stops development at an early stage, prevents normal rotation of the eggs and suppresses formation of polar bodies; it causes formation of vesicles and blebs at the egg surface. If polar bodies are extruded, they are deformed.

These phenomena are not characteristic for the presence of special ions. In hypotonic solutions, however, characteristic differences between the ions become evident.

In CaCl₂ and MgCl₂ solutions of optimal (moderately hypotonic) concentrations, development is most normal. The consistency and adhesiveness of the vitelline membrane resemble those of normal eggs in egg capsule fluid, cleavage type is normal with good flattening of the blastomeres and well-developed cleavage cavity. The eggs are, however, rather susceptible to lack of oxygen.

In LiCl, hypertonic solutions cause a great amoeboid activity. In all

concentrations, development is scanty and does not proceed beyond the 8-cell stage. The consistency of the vitelline membrane is abnormal, but its adhesiveness only slightly affected. Cleavage type may be normal in optimal concentrations.

NaCl allows of a good development, but causes an abnormal consistency of the membrane. The adhesiveness of the latter is more affected than in LiCl and cleavage type is not entirely normal.

In KCl-solutions, a good development is possible, but both the consistency and the adhesiveness of the vitelline membrane are heavily affected and the cleavage type is very abnormal.

5. Summary.

1. Hypertonic solutions of CaCl_2 , MgCl_2 , LiCl , NaCl , and KCl interrupt maturation.
2. Deformation of the first polar body occurs in hypertonic solutions of all chlorides used; normal polar bodies are formed in iso- and hypotonic solutions.
3. Hypertonic solutions of all chlorides used cause formation of vesicles and blebs at the egg surface, inside the membrane.
4. The first polar body is always inside the membrane, which undergoes a change in normal development between the formation of the first and second polar body; it is almost liquid during the formation of the first polar body, it is tightened as a pellicle over the second polar body.
5. The change in consistency of the membrane occurs earlier by lack of bivalent kations.
6. The "adhesiveness" of the vitelline membrane to the egg surface is diminished by lack of bivalent kations; this is most evident in KCl, somewhat less in NaCl, still less in LiCl.
7. The cleavage type may be normal in CaCl_2 , MgCl_2 and LiCl ; less normal it is in NaCl; very abnormal in KCl.
8. Hypertonic LiCl solutions cause a great amoeboid activity of the eggs.
9. The eggs are most susceptible to lack of oxygen in CaCl_2 .

REFERENCES.

- GROOT, A. P. DE, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 588 (1948).
 HUDIG, O., Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 554 (1946).
 RAVEN, CHR. P., Arch. néerl. zool. **7**, 91- (1945).
 _____, Arch. néerl. zool. **7**, 353 (1946).
 RAVEN, CHR. P. and L. H. BRETSCHNEIDER, Arch. néerl. zool. **6**, 255 (1942).
 RAVEN, CHR. P. and H. KLOMP, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 101 (1946).
 RAVEN, CHR. P. and J. C. A. MIGHORST, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 1003 (1946).
 VRIES, G. A. DE, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 1335 (1947).

Zoology. — *The effect of a short treatment with thiourea upon the fish thyroid gland.* By J. LEVER, J. MILTENBURG and G. J. VAN OORDT. (Zoological Laboratory, Dept. of Endocrinology, University of Utrecht, and Station Biologique de Roscoff, France.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of February 26, 1949.)

The effect of thiourea upon the fish thyroid has been shown for the first time by GOLDSMITH, NIGRELLI, GORDON, CHARIPPER and GORDON (1944), who found that immersion of fish of a hybrid strain of *Xiphophorus helleri* and *Platypoecilus maculatus* in 0.033 — 0.67 % solutions of this anti-thyroid drug resulted in hyperplasia of the thyroid gland, which was first observed after 49 days of treatment; the number of follicles increased and cell proliferations took place. This was mentioned also by CHARIPPER and GORDON (1947).

Similar results were reported by NIGRELLI, GOLDSMITH and CHARIPPER (1946) after having treated guppies (*Lebistes reticulatus*) with a 0.03 % solution of thiourea for about 4 months.

In view of these results it became of interest to ascertain the effect of a short treatment with thiourea upon the thyroid glands not only of a fresh-water, but also of a marine fish, viz. *Lebistes reticulatus* and *Callionymus lyra*, respectively.

Immediately after parturition adult females of *Lebistes reticulatus* were placed separately in glass-bowls, containing 200 cc of tapwater, in which 0.01, 0.1, or 0.6 % thiourea had been dissolved. The last mentioned concentration was highly toxic, however. All glass-bowls were kept in a room with a constant temperature of 24° C.

The specimens of *Callionymus lyra*, having a length of about 9 cm, were kept at approximately 16° C. separately in glass-bowls, containing 750 cc of seawater, in which thiourea was dissolved in concentrations of 0.01 or 0.1 %, the latter being also highly toxic. These solutions were changed every 2 days.

Untreated controls were kept in both cases under similar conditions.

The *Lebistes*-heads and the thyroid glands of *Callionymus* were fixed in Bouin's solution and embedded in paraffin; the sections (3 μ) were stained with haematoxylin-eosin.

For histological examinations of structure and activity of the thyroid glands LEVER's mathematical method (1948) was applied, which is based on the relations between the cell number, the outer and inner diameter and the height of the epithelium cells of circular or almost circular follicles. A short survey of this method, illustrated by figure 1, will follow here.

Fig. 1 A shows a graph in which the diameter of the whole follicle (outer diameter, upper line) and the diameter of the follicular cavity (inner diameter, bottom line) of an inactive thyroid of an untreated guppy is plotted against the number of epithelium cells.

In Fig. 1 B similar regression lines are constructed for a thyroid gland of a guppy, treated with thiourea. In both graphs the following details are visible:

1. In the inactive thyroid the vertical distance between the almost parallel regression lines is very small. This distance demonstrates the

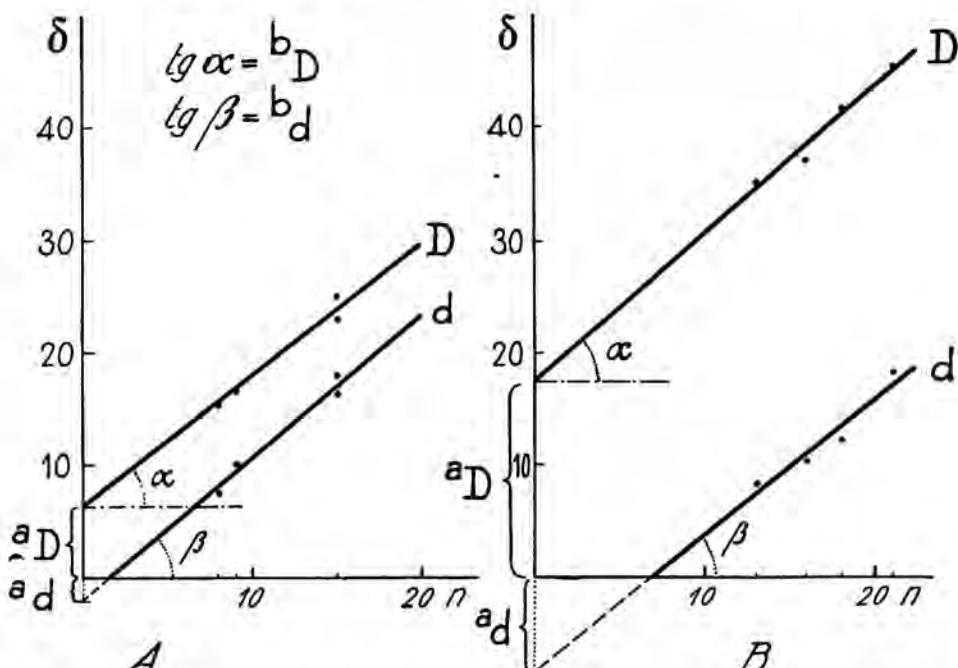


Fig. 1. *Lebistes reticulatus*. A. Graph, showing structure of thyroid follicles of an untreated control. B. The same of a specimen, treated with 0.1 % thiourea during 21 days.
For coefficients cf. text.

difference between the outer and inner diameter and equals twice the height of the epithelium cells. In the active thyroid this distance is large and demonstrates the increase in height of the epithelium cells. By studying frequency diagrams of the outer diameter in hundreds of follicles of inactive and active thyroids of cockerels, treated with several antithyroid drugs during approximately 14 days, LEVER has found that the epithelium-height increases in both directions.

2. The inclination of the regression lines demonstrates the relation between the diameter of the follicles and the cell number.

3. The distance between the bottom line and the abscis demonstrates the colloid content in the follicles. As in the graph of the activated thyroid

the bottom line has moved to the right, the colloid content of the thyroid follicle decreases by activation.

The regression lines delineated in fig. 1 A and 1 B may be described by the formulae $D = a_D + b_D n$ and $d = a_d + b_d n$, in which D is the outer, d the inner diameter, n the number of cells of the follicle section, a_D and a_d the parts of the ordinate cut off by the regression lines and b_D and b_d the tangents of the angles of inclination. In inactive glands b_D and b_d are practically equal, as the regression lines run almost parallelly. Therefore in this case we can derive from these 2 lines one with the formula

$$Y = A + Bn \text{ in which } A = a_D - a_d \text{ and } B = \frac{b_D + b_d}{2}.$$

A being related to the epithelium height and B to the number of cells dependent on the diameter of the follicle; if many cell divisions take place, the value of B decreases.

This method is easy to apply as sections stained by the ordinary laboratory routine techniques can be used; they must all have the same thickness; 3 μ -sections are the best, as these contain only one cell layer.

The results of our calculations are given in tables 1 and 2, from which the following conclusions may be drawn.

TABLE 1. Effect of thiourea on thyroid activity in *Lebistes reticulatus* (for coefficients cf. text).

	Duration of treatment in days	a_D	a_d	b_D	b_d	A	B
Untreated controls	7	8.7	4.1	0.8	0.8	4.6	0.8
	14	2.3	-6.1	1.5	1.4	8.4	1.5
	14	7.0	-2.7	1.4	1.4	9.7	1.4
	21	3.1	0.1	1.1	0.9		
	28	6.2	-2.0	1.2	1.3	8.4	1.3
	36	5.8	-0.7	1.2	1.1	6.5	1.2
0.01 %	8	9.8	-2.6	1.2	1.1	12.4	1.2
	14	11.1	-3.7	1.1	1.1	14.8	1.1
	14	4.5	-3.2	1.4	1.2		
	21	8.5	-3.9	1.4	1.1		
	28	5.1	-6.6	1.3	0.9		
	35	6.9	-0.3	1.3	1.3	7.2	1.3
0.1 %	7	4.6	-0.4	1.2	1.2	5.0	1.2
	12	2.7	-5.4	1.6	1.2		
	21	17.4	-8.4	1.3	1.2	25.8	1.3
	35	5.8	-1.5	1.9	1.0		
0.6 %	5	12.5	-0.8	1.0	1.0	13.3	1.0
	6	4.3	2.0	1.3	1.2	6.3	1.3
	21	8.1	-8.5	1.3	1.2	16.6	1.3

TABLE 2. Effect of thiourea on thyroid activity in *Callionymus lyra*
(for coefficients cf. text).

	Duration of treatment in days	a_D	a_d	b_D	b_d	A	B
Untreated controls	0	7.9	-3.0	1.6	1.5	10.9	1.5
	0	16.2	3.6	1.3	1.3	12.6	1.3
	6	-4.7	-10.3	1.5	1.4	5.6	1.5
	6	4.7	-10.2	1.2	1.2	14.9	1.2
0.01 %	1	16.9	0.5	1.2	1.1	16.4	1.2
	5	13.3	3.0	1.2	1.0		
	6	21.6	1.9	1.1	1.1	19.7	1.1
	7	12.5	2.1	1.2	1.1	10.4	1.2
	7	9.2	-9.7	1.4	1.4	18.9	1.4
	7	16.5	-3.8	1.2	1.0		
	9	13.4	-15.3	1.0	1.0	28.7	1.0
	11	4.4	-15.8	1.5	1.6	20.2	1.6
0.1 %	2	2.2	-13.9	1.7	1.6	16.1	1.7
	3	20.9	-0.2	1.0	1.0	21.1	1.0
	7	12.3	-6.8	1.3	1.3	19.1	1.3

1. A, i.e. the height of the follicle epithelium increases under the influence of thiourea in the concentrations used. This was especially distinct in *Callionymus*-specimens, treated with a 0.01 % solution of thiourea for only 7 days (fig. 2a and b).

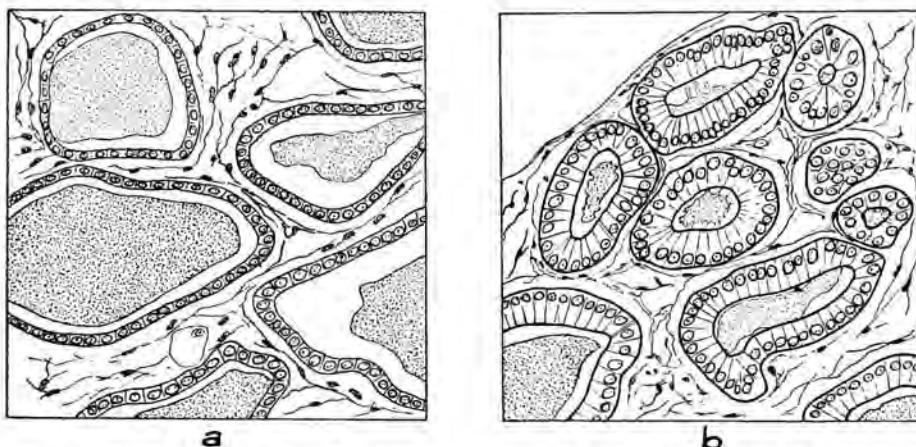


Fig. 2. *Callionymus lyra*. a. Section of a normal thyroid. b. Section of a thyroid of a specimen treated with 0.01 % thiourea during 7 days. ($\times 720$)

2. B does not show a distinct change which points to the fact that cell divisions have not taken place.

3. a_D has the tendency to increase, a_d to decrease, which demonstrates

that also in these fishes the thickening of the epithelium has taken place in both directions, centrifugally as well as centripetally.

In addition it must be mentioned that the follicles generally decrease in size after activation, this being caused by colloid resorption and by the formation of new follicles by budding.

Summary.

From these data it may be concluded that as early as in the first week of treating *Lebistes reticulatus*, a freshwater fish, and *Callionymus lyra*, a marine fish, with thiourea-solutions of 0.01, 0.1, 0.6 and 0.01, 0.1 % respectively, an increase in the activity of the thyroid gland is observed.

Acknowledgement.

We wish to express our thanks to Prof. P. DRACH, Sous-directeur de la Station Biologique de Roscoff (Finistère, France) for providing us with *Callionymus*-specimens during our stay in his laboratory in September 1947.

REFERENCES.

- CHARIPPER, H. A. and A. S. GORDON, The biology of antithyroid agents. Vitamins and hormones, V, 273 (1947).
- GOLDSMITH, E. D., R. F. NIGRELLI, A. S. GORDON, H. A. CHARIPPER and M. GORDON, Effect of thiourea upon fish development. Endocrinology, 35, 132 (1944).
- LEVER, J., A mathematical method for the determination of the state of activity of the thyroid gland. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 51, 1302 (1948).
- NIGRELLI, R. F., E. D. GOLDSMITH and H. A. CHARIPPER, Effects of mammalian thyroid powder on growth and maturation of thiourea-treated fishes. Anat. Rec. 94, 79 (1946).

Zoology. — *An electron-microscopical study of bull sperm III.* By L. H. BRETSCHNEIDER. (From the Zoological Laboratory at Utrecht and the Netherlands Institute for Electron-microscopy at Delft.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of February 26, 1949.)

A. Introduction.

Following our first, more or less preliminary communication (1) on this subject, our investigation of bull spermatozoa was continued as one of the activities of the Unit for the study of Artificial Insemination, under the auspices of the National Council for Agricultural Research.

Investigations were again made with the aid of the electron microscope of the Institute for Electron Microscopy at Delft, the instrument having meanwhile undergone some improvements.

In a further communication (2), the measurement was described, by means of electron-microscopic photographs, of the different head-axes, and their ratios to one another expressed in indices, an "ideal" type being construed from the average values found.

The present, third communication purports to state various new and supplementary facts arrived at, partly by the use of electron-microscopic techniques specially adapted to this end. In addition to this we were able to clarify and rectify certain points in our first communication, which were based on incorrectly interpreted details. Owing to the, as yet, limited experience in the use of electron-microscopic methods, and the completely different manner of "reading" the images obtained by the electron microscope as compared to that with which we are familiar in the light-microscope, the investigator is often faced with a difficulty in coming to a decision, to be solved only by further research with the aid of different techniques.

B. Techniques.

As a full and detailed description of the techniques employed by us is yet to follow, we will confine ourselves here to enumerating the techniques used in treating the material dealt with in the present communication.

(1) To free the spermatozoa from the disturbing colloids of the spermoplasm, the ejaculated semen was rinsed 2—3 times with a phosphate buffer solution "400" according to ROMIJN (3). This solution contains, for 100 cc' aq. dest., 0.4 g KH_2PO_4 (SÖRENSEN) and 1.954 g $\text{Na}_2\text{HPO}_4 \cdot 12$ aqu.

After this one of the following techniques was applied:

(2) Preservation of the spermatozoa in $\frac{1}{4}$ — $\frac{1}{2}$ % mercurochrome (the disodium salt of 2.7 dibrom-4-hydroxymercurifluorescein) in aq. dest. during from 3 hours to several days.

- (3) Treatment with 1 % AuCl_4 during 2 days.
- (4) Treatment with 1/10— $\frac{1}{2}$ % chloramine-T (Sodium salt of toluene-sulfochloramide) in aq. dest., from 12 hours to 2 days, in darkness.
- (5) Treatment with 0.005 % chromic acid in aq. dest., for 3 days at 40° C., in which a certain bacterial flora developed.
- (6) Digestion with 0.3 % trypsin in 0.3 % Na_2CO_3 , during from 2 to 48 hours at 38° C.; followed by rinsing in aq. dest. and fixing in OsO_4 2 %, 4 drops to 10 cc $\frac{1}{2}$ % chromic acid for 24 hours.
- (7) Bacterial maceration by developing the bacteria present in the semen, at room temperature, for 1—2 weeks, followed by rinsing in water and fixing in OsO_4 -chromic acid as sub (6).
- (8) Reagents for thymonucleic acid:
 - a) fixation in 2 parts concentrated corrosive sublimate (mercuric chloride) + 1 part absolute alcohol; 1 hour;
 - b) rinsing with alcoholic iodine solution, 1 hour;
 - c) lixiviation in 1 % "demedon" in alcohol 90 % for 24 hours;
 - d) gradually reduced alcoholic sequence down to aq. dest.;
 - e) hydrolysis in 1 n HCl at 55° C. for 12 minutes;
 - f) thorough rinsing in aq. dest.;
 - g) impregnation in 10 % AgNO_3 solution, brought to a pH of 8—9 by a few drops of 8 % NH_4OH , after which the resultant precipitate is filtered away; 24 hours at 40° C.;
 - h) thorough rinsing in aq. dest. with a trace of NH_4OH to remove any remaining Ag_2O precipitate;
 - i) digestion in 1 % pepsin in 0.1 n HCl, at 40° C. for 24 hours.

Following all these reactions, and before commencing the electron-microscopic examination, repeated rinsing in aq. dest., in which the objects then come on the container of the preparation.

Some preparations were shadowcast with gold according to WILLIAMS and WYCKOFF (4). The electron accelerating voltage varied between 80, 90 and 100 kV.

C. Examination of the sperm head.

- (a) *The head-cap.* Given efficient fixation, it is possible to demonstrate the fragile head-cap electron-microscopically.

In the living state the substance of which this cap consists is rich in water, so that, after the necessary dehydration, next to nothing remains of it. As will be seen from fig. 1 and 2 in our first communication (1) nothing much can be seen of it after treatment with proteolytic agents. For this reason we had to find a fixative, which we found in mercurochrome (vide above, technique 2) (2). The sublimate component of the mercurochrome causes the substance of the cap to coagulate, so that it is preserved in its original shape, also in a vacuum (fig. 1, 2). The thickness of this cap, i.e. the distance between the outer edge and the head-profile, measures, in these preparations, from 250—370 m μ . The structure of the

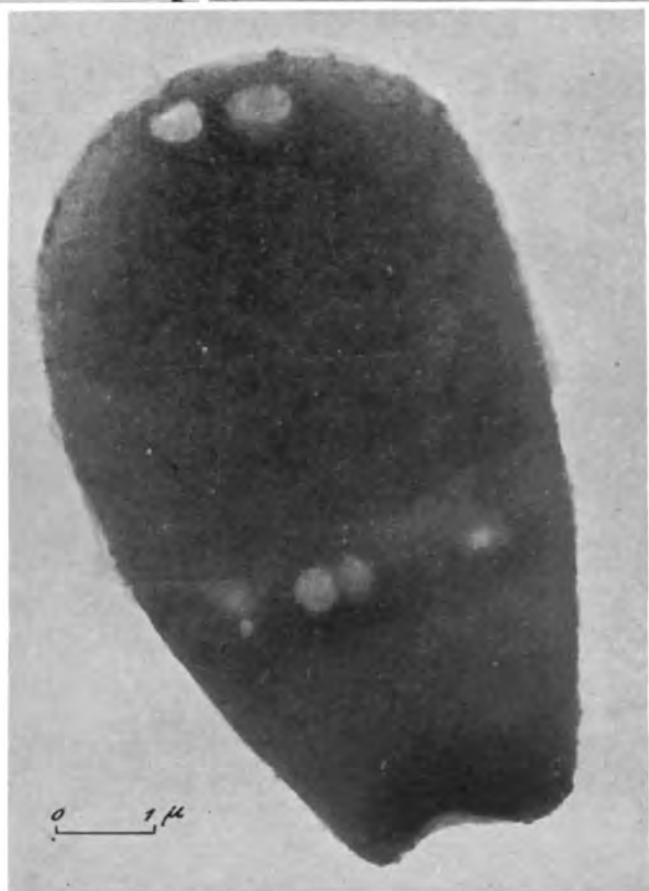
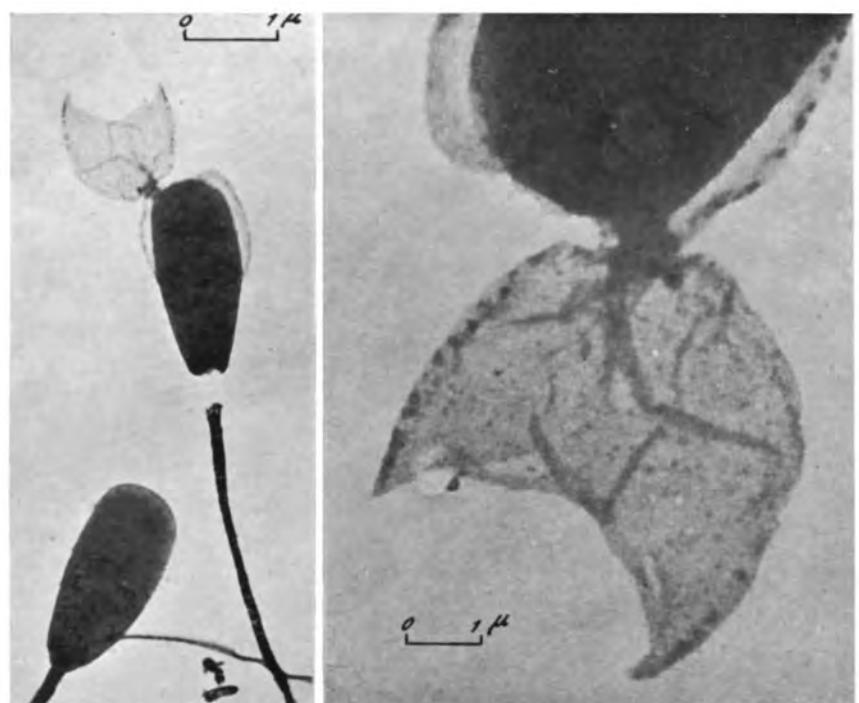


Fig. 1. Sperm head with head cap after fixation in mercurochrome; orig. magn. 2500 X.

Fig. 2. Sperm head with head cap after fixation in mercurochrome; orig. magn. 10000 X.

Fig. 3. Sperm head with vacuoles; orig. magn. 12500 X.

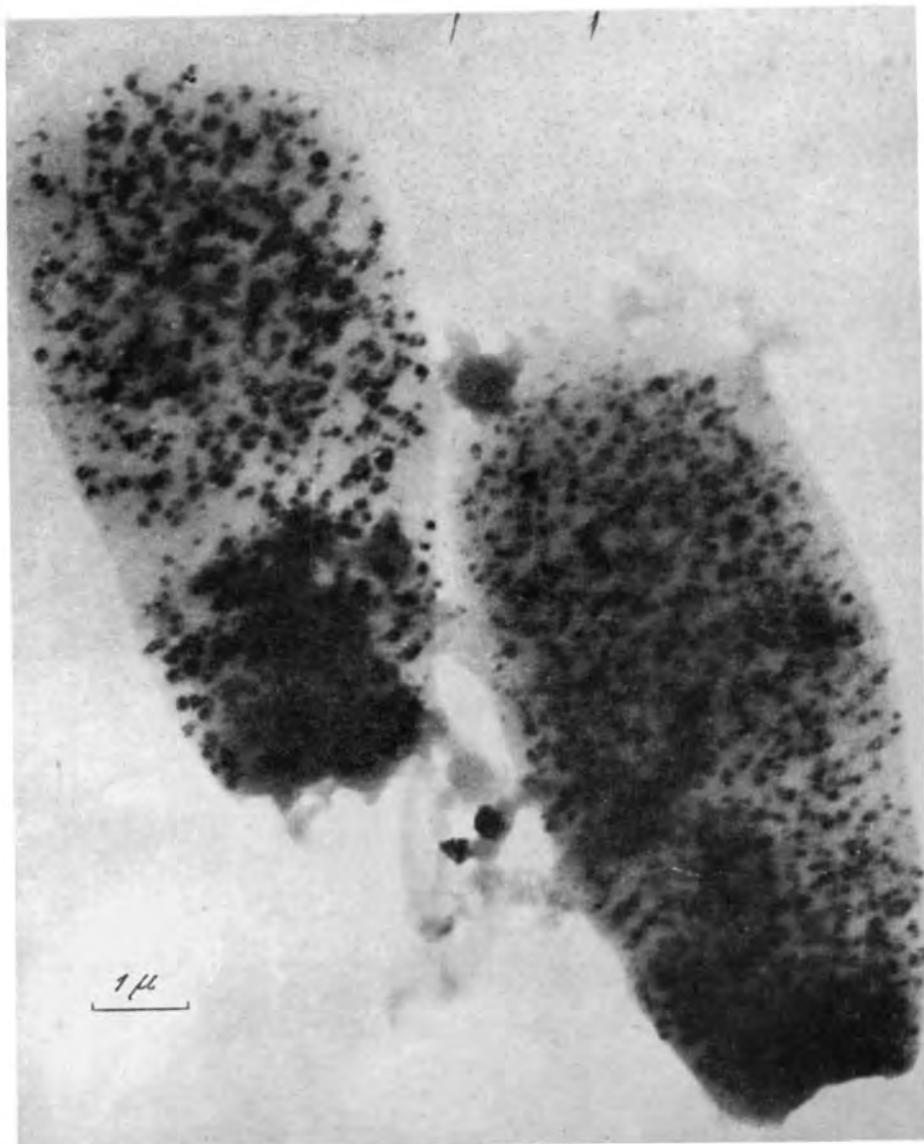


Fig. 4. Sperm head after hydrolysis and AgNO_3 impregnation; orig. magn. 30000 X.



Fig. 5. Middle piece with spiral body, after bacterial maceration and impregnation with chromic acid. Orig. magn. 8000 X.

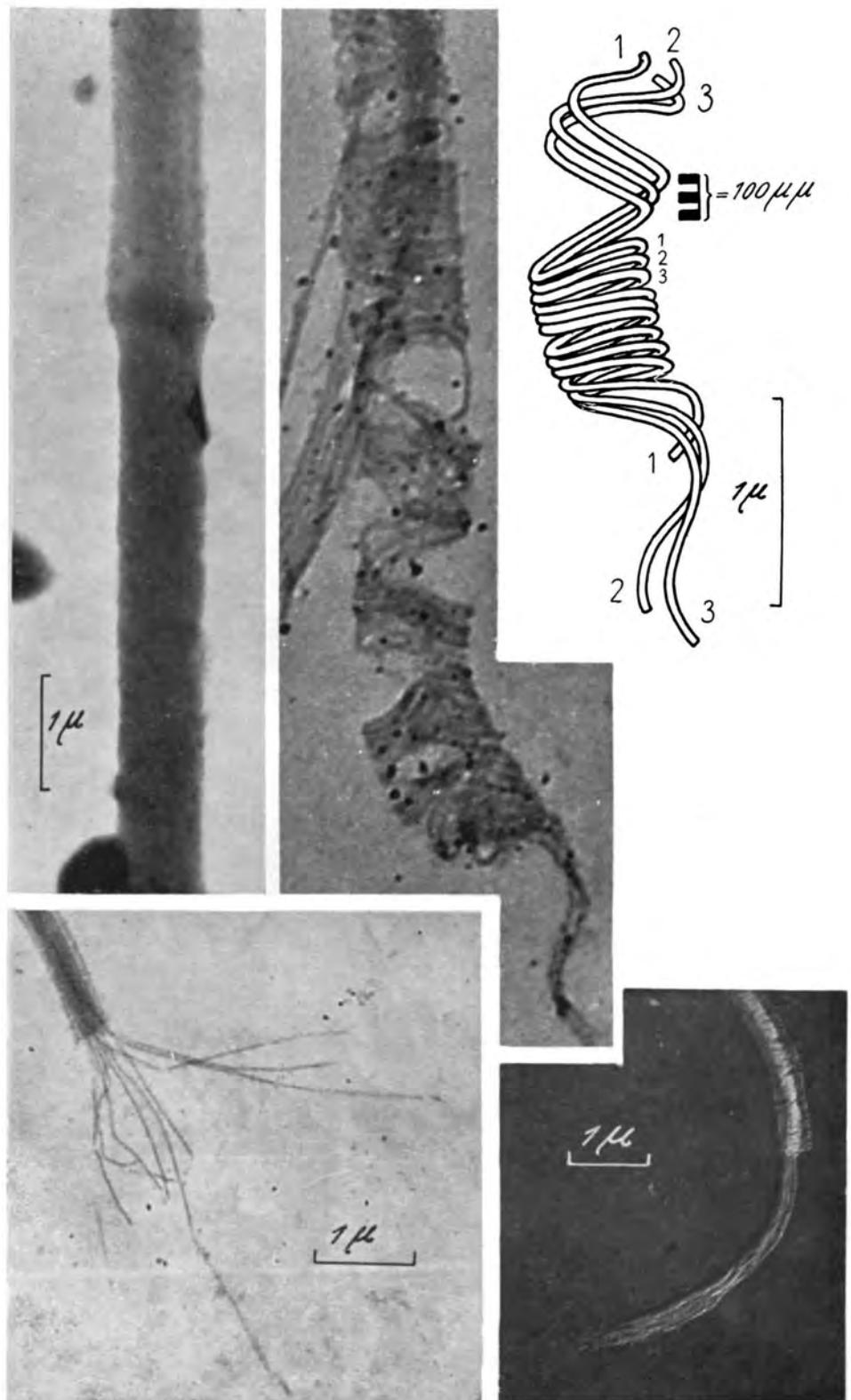


Fig. 6. JENSEN's ring after AuCl_4 impregnation. Orig. magn. 17000 X.

Fig. 7. Subfibrils and cortical spiral of tail, after maceration and OsO_4 impregnation. Orig. magn. 36000 X.

Fig. 8. Terminal piece of tail after chloramine; orig. magn. 15000 X.

Fig. 9. Terminal piece of tail after shadow-casting; orig. magn. 12000 X.

outer part is slightly denser than the inner zone, which usually contains vacuoles and small granules. The vacuoles measure $\pm 80 \text{ m}\mu$, the granules $\pm 50 \text{ m}\mu$.

(b) *The distribution of the head-plasma.* The occurrence of certain zones and vacuoles in the sperm-head is probably a result of a corresponding distribution of a so-called head-plasma. The head of a normal bull sperm does not show any vacuoles within its membrane. Whenever such vacuoles are visible, we must certainly have to do with artifacts. Vacuoles may occur through autolysis of dead spermatozoa in the ejaculated semen, or by the treatment with substances which dissolve the content of the head; such vacuoles probably originate from the products of disintegration of the contents of the head. Especially after proteolysis by bacterial enzymes (fig. 3, technique 7), or after chloramine (fig. 1 in our first communication), numerous vacuoles appear. These vacuoles measure from 300—500 $\text{m}\mu$, are generally sharply outlined, and may break through the membrane of the head when the latter is treated with a proteolytic. Owing to lack of adequate material for comparison, we took such a burst-open vacuole to be a "porus" (Communication 1, p. 94), which erroneous interpretation we hereby rectify.

What also strikes one in fig. 3 is the "preference" of these vacuoles for a certain position in the head. They accumulate especially below the equator, in a straight line, and also at the apex of the head. We examined statistically the number and position of the vacuoles in 50 other heads in which vacuoles were found, and found the following distribution curve (fig. A). If one divides the head into 7 zones, then it will be seen that

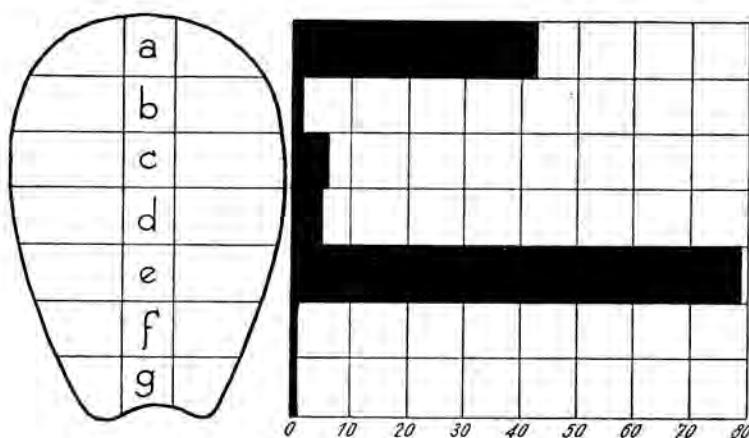


Fig. A. Frequency of the occurrence of vacuoles in the head.

most of the vacuoles occur in zone a (at the apex) and zone e (below the equator), while they only occur sporadically in the remaining zones. In our first communication (1) we already pointed to different zones, especially the lighter one in the equator which, in this case, appeared after the treat-

ment with chloramine-T. We now see that a similar zonal distribution also takes place after the action of proteolytic enzymes, from which one might conclude that the products of the disintegration collect on the spot and there form vacuoles. In correspondence with this assumption we see that in this equatorial zone, which has a width of 400—600 m μ , the chromatinic material is practically lacking (fig. 4). We may conclude from these facts that there is an apical and, especially, an equatorial accumulation of nuclear plasma whose lysis or dissolution gives rise to vacuoles.

(c) *The chromosomes.* Apparently, the chromosomes or their chromatinic derivatives can be demonstrated electron-microscopically in the head of the bull sperm by means of impregnation with silver.

When the staining method to thymonucleic acid according to FEULGEN is applied light-microscopically, the whole of the head is coloured an even red with parafuchsin, only the base being slightly darker. It was concluded from this that the chromatin filled the whole of the head as a compact mass. It was to be expected that the so much greater magnification possible with the electron-microscopic method would enable us to get to know more about the distribution of the chromatin. To this end, however, dyes are required with a greater electronic dispersion than that of the organic ones. For this reason we applied a variant of FEULGEN's reaction, i.e. replacing the parafuchsin by an ammoniacal silver nitrate solution (technique 8). In this, the Ag₂O is reduced by the lytic products of the thymonucleic acid, the Ag being precipitated in the nucleus. Fig. 4 shows the result of the electron-microscopic examination of sperm heads treated by technique 8. We then find the places, where the thymonucleic acid was, to be spread all over the head, in the form of smaller or larger granules. The densest concentration occurs in the base of the head; the smallest in the almost granule-free zone of equator, which corresponds to the nuclear plasma zone already mentioned above.

Although in the majority of cases the chromatin is seen to be very widely dispersed, we still find groups of granules, particularly in the front half of the head, which lie together in shorter or longer rows. One has the impression that parts of chromosomes are still present here. Closer analysis generally shows that in these parts pairs of granules — which might be chromomeres and chromioles — are linked up into a chain; they resemble parts which have not dispersed as widely as the rest. Since the size of these granules varies between 35 m μ and 260 m μ and their spatial distribution is very dense, it is understandable that they give only an evenly-coloured picture at the enlargements of the light-microscope. Not in all spermatozoa is the chromatin so finely distributed; sometimes the chromomeres still lie together in chromosome-formation. In view of the large number of chromosomes — according to KRALLINGER (6) 30 in the haploid form, — the chromosomes individually can not be so very long. Comparison of the configuration of the chromatin in the sperm head with that of the nucleus in the metasperm stage (vide 17) reveals a considerable similarity:

in all cases the chromatin formations lie together very compactly in the base of the head; a few series of granules pass the equatorial plasma zone, after which a looser spread of the granules or parts of chromosomes is seen in the front half of the head.

D. Examination of the sperm tail.

(a) JENSEN's "*spiral body*". By this is meant the spiral ("broad helix", see our first communication) in the middle piece, originating from the mitochondria. Owing to its being rich in lipoids it can be convincingly demonstrated electron-microscopically by appropriate techniques.

JENSEN's spiral body consists of two equally thick spirals running parallel with a pitch of 15—25 degrees around the fibrillary axis. In our first communication we were able to identify the spiral body only fragmentarily and after chloramin treatment and shadow-casting. Starting from the assumption that this structure might in some way be connected with the generation of energy, we endeavoured to demonstrate it by different electron-microscopic methods. When the middle piece is examined without preliminary treatment, the spiral windings are observable only as slightly darker strips (fig. 5). Application of technique No. 5 causes the spiral body to be more distinctly outlined, bacterial enzymes, coupled with the chromatization of the lipoids rendering the spiral-like structure plainly visible. Here too it is seen for the first time that the structure consists of two spirals, as, owing to the greater pitch of 25 degrees the distance of the windings increases in different places. The real thickness of each spiral band, too, is more easily measured here, i.e. about 170 m μ .

(b) JENSEN's *ring*. The electron-microscopic image shows a ring-formed structure at the transition between the middle piece and the tail. This structure, called after its discoverer CARL OLUF JENSEN (1887), is impregnated with Au ions (fig. 6). The diameter of the ring, which is 150 m μ thick, is usually slightly wider than the diameter of the middle piece, with the result that the tail here begins with its own, broader base, and no direct transition exists from the middle piece to the tail.

A number of pathologically changed sperms, which are described in more detail further on, confirm the independent position of this caudal structure.

(c) *The cortical helix (tail-spirals)*. These can be demonstrated electron-microscopically and without shadow-casting, after impregnation with OsO₄.

After treatment with bacterial enzymes, whose action is less radical than that of either trypsin or pepsin, the tail plasma is dissolved, after which the remaining axial fibrils and the cortical spirals become visible by impregnation with OsO₄ and chromic acid, which gives a sharper contrast (fig. 7).

Already in 1947 we gave it as our opinion, from the manner in which these spiral fibrils, also in other images, generally shift from their original position in groups of 3 — vide the accompanying sketch (fig 7), that there

are three spiral fibrils running around the tail. As the coarsening action of the gold particles after shadow-casting is eliminated here (vide fig. 10 of 1), a more exact measurement of the thickness of the fibrils is possible. Each fibril is about $30 \text{ m}\mu$ thick, and makes about 150 spiral windings around the tail.

(d) *The terminal piece* consists of the 9 sub-fibrils which, bound together to form a rod-like axis, protrude from the body of the tail. Usually this piece is slightly curved or sharply bent at the end (fig. 9). Either the quick dehydration or proteolysis causes this terminal piece to fall apart to form an artifact resembling a tiny brush (fig. 8). Whereas the intact terminal piece has equally long sub-fibrils, the latter, after unravelling, break off artificially at different lengths. The subfibrils can be demonstrated also without shadow-casting and preliminary treatment, as fig. 8 shows. No definite structure can be observed to finish off the tail spirals; the spiral fibrils here finish abruptly.

E. Discussion and Summary.

However praiseworthy may be the endeavours made, in applying electron-microscopy, to examine the objects without any preliminary treatment after necessary dehydration only, it appears that this ideal is difficult of realization in the case of biological objects.

The desirability of the development of a special electron-microscopic preparation technique has been understood from the very beginning at the Netherlands Institute for Electron-Microscopy. With many objects, the inner structure is obscured by their thickness, as they give uniformly black images, unless measures are taken to dissolve and remove part of their substance. When the preparation transmits the electron rays, a fruitful analysis is often prevented by the lack of contrasting atoms having a higher atom number than those which compose the organic substance.

The structures described above, such as the head-cap, the chromosomes, the double spiral body and JENSEN's ring were rendered visible only after certain technical expedients had been applied. This indeed makes the further development and elaboration of a special electron-microscopic preparatory technique desirable. The electron-microscopic investigation of the spermatozoon forms the immediate sequel to light-microscopic examination at the point where the latter has long since reached its extreme limit of applicability, culminating in the work of RETZIUS (7). Already in light-optical observation in the range of 1/10 micron, subjective guessing often takes the place of objective observation.

Electron-microscopic examination not only eliminates the necessity for such guessing but also reveals structural details that would be invisible light-microscopically. As such we may mention, in the present investigation, the following: the granules and vacuoles in the head-cap, of 50—80 $\text{m}\mu$; the distance between the two spiral bodies, of 10—30 $\text{m}\mu$; the "chromioles", 35—100 $\text{m}\mu$, and the cortical fibrils, 30 $\text{m}\mu$. In this way the continued

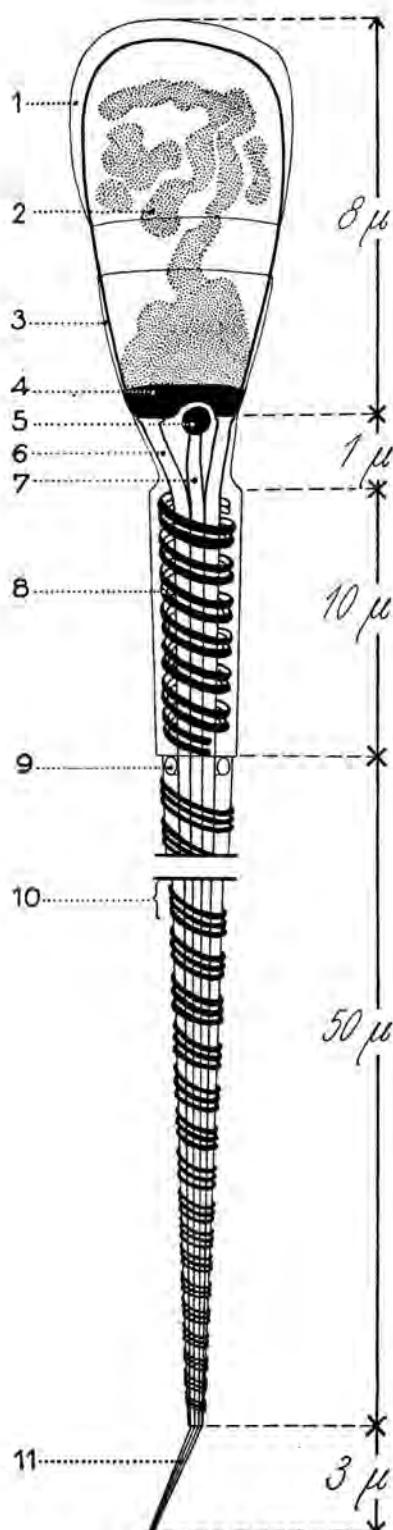


Fig. B. Diagram of the structure of the sperm.

1. head cap;
2. chromosomes;
3. head tunica, (external layer);
4. ring-shaped membrane;
5. centrosome;
6. articular strands;
7. axial filament;
8. double broad helix
(JENSEN's spiral body);
9. Jensen's ring;
10. cortical helix;
11. terminal piece.

development of the investigation and the new details discovered causes us to modify our conception of the structure of the sperm, enabling us to add the following supplementary structures to our original scheme (fig. B).

1. *Chromatin*, partly still to some extent in the form of individual chromosomes, or dispersed in the form of smaller or larger fragments, down to chromiole formation. It is interesting to note in this connexion that on account of the duplicity of the chromatin parts many of the fragments appear to be composed of 2 chromatids. From the similarity in the configuration of the chromatin in the spermatid nuclei and the sperms we may conclude that chromosomes, or parts thereof persist in this interphase. That chromosomes are retained in the sperm-head is known also of the lower animals. Thus, we see in the case of Nematoda such as *Ancyracanthus*, according to MULSOW (8), the chromosomes already *in vivo*. SOKOLOW (9) found, in different arachnids, by light-optical methods, the chromosomes in various degrees of condensation or dispersal. POLLISTER and MIRSKY (10) distinctly observed the presence of chromosomes in the dilating sperm-head of the trout, after treatment with 1/M NaCl. Not until now has this been possible in the case of mammals.

2. *The head-plasma*, particularly as a separate zone in the centre of the head. It is known that the sperm contains relatively little water. The head is considerably reduced in size in respect of the original spermatid nucleus; and it seems feasible that nothing more remains of the original nuclear fluid, after this condensation of substances composing the sperm, than a more or less condensed gelatinous colloid — the so-called head plasma, in which the chromatin is embedded. In addition the substance of the nucleoli, after their disintegration is incorporated with this head-plasma. Through the dissolution of the head-plasma, vacuoles are formed which burst open outwardly. In our first communication (1) we erroneously interpreted these artifacts as "pores". WILSON (11) already put forward the theory that the light-optically visible homogeneous and solid nuclei of most sperms largely consist of substances from the nuclear framework which, after insemination cause the nucleus to swell again owing to the incorporation of water.

3. *The double spiral body and the middle piece*. The extent to which the principle of the double spiral body is carried through in the spermatozoa of the vertebrates can be ascertained only after the electron-microscopic examination of other spermatozoa.

Light-optically only a single spiral body was known to us up to the present — from RETZIUS' investigations (7); but it is possible nevertheless that the distance between both spirals is too small for the resolving capacity of the light-microscope. Of invertebrates, double spirals in the tail of the sperm are already known; thus, WILSON (12) found, in the scorpion *Centrurus*, two strands originating from the chondriome which, during spermio-cytogenesis transform themselves into two intertwined spiral bodies.

BÖHMING (13) found a similar double spiral in the Turbellaria *Plagiotoma sulphureum* and *Plag. maculatum*.

4. JENSEN's ring. By its stronger colour it drew JENSEN's attention as early as 1887 (14), since when it has been regarded as a derivative of the hindmost centrosome. It is a disc pierced by the fibrillary axis, and forms the border between the middle piece and the tail.

5. *The normal terminal piece.* Our more detailed and extensive examination proved that the brush-like terminal piece described, in the case of the bull, by BAYLOR, NALBANDOV and CLARK (15), and ourselves (1), is in effect based on an artifact, since in the majority of sperms, or after preliminary fixation, it is found to consist of a fixed axis composed of 9 subfibrils. In the majority of sperms we have so far examined, the terminal piece was also shown to consist of a solid bundle of fibrils.

A renewed examination of the cortical tail spirals finally confirmed the interpretation already put forward in 1947, to the effect that there are, in fact, three fibrils present which run side by side. Such a threefold spiral was also found by KOLZOFF (16) in the spermatozoa of *Planorbis* by causing the sperms to dilate; our own findings in the case of the bull, therefore, constitute a sequel to KOLZOFF's discovery and point to the existence of a more widely spread principle.

I wish to express my gratitude to the National Council for Agricultural Research for their support and assistance given me in this investigation, and Miss W. VAN ITERSON and Dr. A. L. HOUWINK for their excellent cooperation with the electron-microscopical documentation.

LITERATURE.

1. BRETSCHNEIDER, L. H. and W. VAN ITERSON, Proc. Kon. Ned. Akad. v. Wetensch., **50** (1947).
2. BRETSCHNEIDER, L. H., Tijdschr. v. Diergeneeskunde, **73** (1948).
3. ROMIJN, C., Tijdschr. v. Diergeneeskunde, **72** (1947).
4. WILLIAMS, R. C. and R. W. G. WIJCKOFF, J. Appl. Phys., **17** (1946).
5. BRETSCHNEIDER, L. H., Tijdschr. v. Diergeneeskunde, **72**, 19—20 (1947).
6. KRALLINGER, Arch. Tierernaehr., B. 5 (1931).
7. RETZIUS, G., Biolog. Untersuchungen, Neue Folge (1909—1912).
8. MULSOW, K., Arch. f. Zellforsch. 9 (1912).
9. SOKOLOW, I., Zeitschr. f. Zellforschung, **21** (1934).
10. POLLISTER, A. W. and A. E. MIRSKY, J. gen. Physiol., **30** (1946).
11. WILSON, E. B., The cell in Development. New York (1937).
12. ———, Proc. Nat. Acad. Sc. U.S.A., **2** (1916).
13. BÖHMIG, L., Zeitschr. f. Wiss. Zool., **51** (1890).
14. JENSEN, O. S., Arch. f. mikr. Anat., **30** (1887).
15. BAYLOR, M. R. B., G. NALBANDOV and L. CLARK, Proc. Soc. Exper. Biol. Med., **54** (1943).
16. KOLZOFF, N. K., Biol. Centr.bl. **26** (1906).
17. MEVES, F., Arch. mikr. Anatomie, **54** (1899).

KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN
WETENSCHAPPEN

PROCEEDINGS

VOLUME LII

No. 4

President: A. J. KLUYVER
Secretary: M. W. WOERDEMAN

1949

NORTH-HOLLAND PUBLISHING COMPANY
(N.V. Noord-Hollandsche Uitgevers Mij.)
AMSTERDAM

CONTENTS

Anatomy

HUIZINGA, J.: The digital formula in relation to age, sex and constitutional type. I.
(Communicated by Prof. M. W. WOERDEMAN), p. 403.

Biochemistry

BUNGENBERG DE JONG, H. G. and H. J. VAN DEN BERG: Elastic-viscous oleate systems containing KCl. IV. The flow properties as a function of the shearing stress at 15° and constant KCl concentration, p. 363.

BUNGENBERG DE JONG, H. G., H. J. VAN DEN BERG and L. J. DE HEER: Elastic viscous oleate systems containing KCl. V. Viscous and elastic behaviour compared, p. 377.

BERG, H. J. VAN DEN and L. J. DE HEER: On stearate systems containing methyl-hexylcarbinol with viscous and elastic properties comparable to elastic viscous oleate systems containing KCl. (Communicated by Prof. H. G. BUNGENBERG DE JONG), p. 457.

Botany

FRETS, G. P.: De hypothese voor de erfelijkheidsformules van de twee zuivere lijnen I en II van Phaseolus vulgaris op grond van kruisingsproeven. I. (Communicated by Prof. J. BOEKE), p. 423.

Chemistry

BIJVOET, J. M.: Phase determination in direct Fourier-synthesis of crystal structures, p. 313.

Crystallography

DIJKSTRA, D. W.: Transformation of gnomograms and its application to the microchemical identification of crystals. I. (Communicated by Prof. J. M. BIJVOET), p. 440.

Geology

WAARD, D. DE: Tectonics of the Mt. Aigoual pluton in the southeastern Cevennes, France. Part I. (Communicated by Prof. H. A. BROUWER), p. 389.

Mathematics

BROUWER, L. E. J.: Contradictoriteit der elementaire meetkunde, p. 315.

CORPUT, J. G. VAN DER and H. MOOIJ: Approximate division of an angle into equal parts, p. 317.

DRONKERS, J. J.: Een iteratieproces voor de oplossing van een randwaardeprobleem bij een lineaire partiële differentiaalvergelijking van de tweede orde. I. (Communicated by Prof. W. VAN DER WOUDE), p. 329.

PRASAD, A. V.: A non-homogeneous inequality for integers in a special cubic field. (Second communication.) (Communicated by Prof. J. G. VAN DER CORPUT), p. 338.

RUBINOWICZ, A.: SOMMERFELD's Polynomial Method in the Quantum Theory. (Communicated by Prof. H. A. KRAMERS), p. 351.

Physics

CLAY, J.: Photons in extensive Cosmic-Ray-Showers, p. 450.

Physiology

WASSINK, E. C., J. E. TJA and J. F. G. M. WINTERMANS: Phosphate-exchanges in purple sulphur bacteria in connection with photosynthesis. (Communicated by Prof. A. J. KLUYVER), p. 412.

Statistics

HAMAKER, H. C.: Random sampling frequencies; an implement for rapidly constructing large-size artificial samples. (Communicated by Prof. H. B. G. CASIMIR), p. 432.

Chemistry. — Phase determination in direct Fourier-synthesis of crystal structures. By J. M. BIJVOET.

(Communicated at the meeting of March 26, 1949.)

A direct Fourier synthesis of the electrondensity in crystals is based on the fact that the amplitude of a density wave is proportional to that of the X -ray beam reflected by the corresponding netplane. The phases of the reflected waves being lost in ordinary X -ray methods¹⁾, the phase relations between the different density components remain unknown at this stage.

The method of isomorphous substitution $A_1R \rightarrow A_2R$ introduces, along familiar lines, known reference waves, viz. the density components of the A -configuration (for atomic scattering power $A_2 - A_1$; we suppose this configuration to be centrosymmetrical and its determination accomplished).

The reference waves being known, their superposition effect on the density waves of the A_1R structure reveals their mutual phase differences, i.e. the unknown phases φ_{hkl} of the density waves of the A_1R structure relative to the centre of symmetry of the A -configuration:

$$F_{hklA_2R}^2 = F_{hklA_1R}^2 + F_{hklA_2-A_1}^2 + 2 |F_{hkl}|_{A_1R} F_{hklA_2-A_1} \cos \varphi_{hklA_1R}.$$

Here $|F_{A_1R}|$ is derived from the diffraction-intensities of A_1R ; $|F_{A_1R}|$ idem for crystal A_2R ; $F_{A_2-A_1}$ is calculated for the A -configuration of atomic scattering power $A_2 - A_1$.

Now this procedure gives us φ_{hkl} except for its sign, thus performing the phase (sign) determination only for the case of a centrosymmetrical A_1R structure (projection). Otherwise a synthesis with both φ_{hkl} and $-\varphi_{hkl}$ can be resorted to, resulting in a duplicated structure-model²⁾.

Now we wish to call attention to the fact that in this non-symmetrical case there is, in principle, a general way of determining the sign of φ_{hkl} . We can use the abnormal scattering of an atom for a wavelength just beyond its K -absorption limit. This effect is made use of already in X -ray analysis to discriminate between atoms of scattering power nearly equal under normal conditions³⁾. Even the abnormal phase shift has been used, as long ago as 1930, by COSTER in a manner similar to that proposed here⁴⁾. Let the abnormal phase shift introduced at atom A be δ . Then the

¹⁾ C.f. those methods in which the scattered beam is made to interfere with the direct beam, i.a. H. OTT, Ann. Physik **31**, 264 (1938), D. GABOR, Nature **161**, 777 (1948).

²⁾ C.f. C. BOKHOVEN, J. C. SCHOONE, J. M. BIJVOET, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **52**, 120 (1949).

³⁾ See R. W. JAMES, The Cryst. State II (1948) Chapter IV (1).

⁴⁾ D. COSTER, K. S. KNOL and J. A. PRINS, Z. f. Physik, **63**, 345 (1930).

phase differences $\pm \varphi_{hkl}$ between the density waves concerned result in the phase differences $\varphi_{hkl} + \delta$ resp. $-\varphi_{hkl} + \delta$ for the scattered radiation, and so will become distinguishable. At present we are testing the applicability of his method in an actual analysis.

It will be clear that by the above method it also becomes possible to attribute the *d* or *l* structure to an optically active compound on actual grounds and not merely by basic convention as in organic chemistry.

*Van 't Hoff Laboratorium der
Rijksuniversiteit, Utrecht.*

Mathematics. — Contradictoriteit der elementaire meetkunde. By L. E. J. BROUWER.

(Communicated at the meeting of March 26, 1949.)

In een vroegere mededeeling¹⁾ is de contradictoriteit der aequivalentie der relaties „ > 0 ” en „ $\circ > 0$ ” aangetoond²⁾. Hieruit volgt tevens de contradictoriteit der aequivalentie der relaties „ ≥ 0 ” en „ $\delta f = 0 \text{ of } \circ > 0$ ”. Immers laatstgenoemde aequivalentie zou eerstgenoemde impliceeren.

Beschouwen we de *snijpuntsstelling der Euclidische planimetrie*, luidende dat voor elke twee lijnen a en l van het Euclidische vlak, die noch kunnen samenvallen noch evenwijdig kunnen zijn, een gemeenschappelijk punt kan worden aangewezen. Nemen we een rechthoekig coördinatenstelsel aan, en kiezen we voor a de X -as en voor l uitsluitend lijnen door het punt $P(0, 1)$ met richtingscoëfficiënt ≤ 0 en ≥ -1 , dan leert de snijpuntsstelling in het bijzonder, dat voor elke lijn l door P met richtingscoëfficiënt < 0 en ≥ -1 een snijpunt S met de X -as kan worden aangewezen, derhalve een natuurlijk getal $n(l)$ kan worden aangegeven zoodanig dat $x_s < 2^{n(l)}$, derhalve voor de richtingscoëfficiënt $\varrho(l)$ van l de relatie $\varrho(l) < -2^{-n(l)}$, dus de relatie $\varrho(l) < 0$ geldt. Daar derhalve uit de snijpuntsstelling der planimetrie de contradictoor gebleken aequivalentie der relaties „ < 0 ” en „ $\circ < 0$ ” volgt, is de *snijpuntsstelling der Euclidische planimetrie eveneens contradictoor gebleken*.

De hieruit voortvloeiende *contradictoriteit der Euclidische planimetrie* is evenwel door deze uitsluitend in de verte plaatsvindende manifestatie nauwelijks afdoende gedemonstreerd, terwijl de *snijpuntsstelling der projectieve planimetrie*, luidende dat voor elke twee lijnen van het projectieve vlak, die niet kunnen samenvallen, een gemeenschappelijk punt kan worden aangewezen, hierdoor nog niet wordt aangetast.

Beschouwen we wederom¹⁾ den met het eenheidscontinuum samenvallenden puntwaaier J . Zij f een willekeurig punt van J , verstaan we onder a_f de assertie, die f rationaal verklaart, en beschouwen we de soort σ der sequenties $S(\gamma, a_f)$ en de soort η der corresponderende tweezijdige dempingsetallen $E(\gamma, a_f)$, waarbij γ steeds voorstelt de tweevleugelige aanschuiving met kern 0 en telgetallen $(-1)^n 2^{-n}$ ($n = 1, 2, \dots$), doch f vrij varieert binnen J , met dien verstande dat het n de element $k_n(\mu_n)$ van f steeds na $c_n(\gamma, a_f)$ doch vóór $c_{n+1}(\gamma, a_f)$ wordt geschapen.

¹⁾ Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 52, p. 122 (1949).

²⁾ Als resultaat der mededeeling is de non-aequivalentie van $>$ en $\circ >$ geformuleerd. De bewijsgang levert echter het verder strekkende resultaat der non-aequivalentie van „ > 0 ” en „ $\circ > 0$ ”.

Zouden nu te eeniger tijd de relaties „ $\neq 0$ ” en „ $\delta f < 0$ of > 0 ” voor het continuum aequivalent blijken, dan zou daarmee in het bijzonder voor elk element e van η kunnen worden aangetoond:

hetzij $e < 0$, in welk geval de onmogelijkheid van $e > 0$, dus de absurditeit der rationaliteit van f zou zijn gebleken,

hetzij $e > 0$, in welk geval de onmogelijkheid van $e < 0$, dus de niet-contradictoriteit der rationaliteit van f zou zijn gebleken.

Derhalve zou het eenheidscontinuum zijn gesplitst in twee puntkernsoorten, in elk van welke een element zou kunnen worden aangewezen, hetgeen onmogelijk is³⁾. Zoodat van de aequivalentie der relaties „ $\neq 0$ ” en „ $\delta f < 0$ of > 0 ” voor het continuum de absurditeit is gebleken.

Hieruit volgt tevens de contradictoriteit der aequivalentie van „ $a = a'$ en „ $\delta f a \leq 0$ of $a \geq 0$ ” voor het continuum. Immers laatstgenoemde aequivalentie zou eerstgenoemde impliceeren.

Beschouwen we een Euclidisch vlak met een rechthoekig coördinatenstelsel. Zij a de X -as, m de lijn $x = 1$, a een willekeurig reëel getal $\neq 0$, β de absolute waarde van a , P het punt $(-1, 2\beta)$, Q het punt $(1, a)$, l de verbindingslijn der punten P en Q , γ een zoowel van $\frac{1}{2}$ als van 3 verwijderd liggend reëel getal. Dan snijdt de verbindingslijn van P met het punt $(\gamma, 0)$ de lijn m in het punt $\left(1, 2\beta \frac{\gamma-1}{\gamma+1}\right)$, dat met Q slechts kan samenvallen, als *hetzij* $2\beta \frac{\gamma-1}{\gamma+1} = \beta$, *hetzij* $2\beta \frac{\gamma-1}{\gamma+1} = -\beta$, d.w.z. als *hetzij* $\beta(\gamma-3) = 0$, *hetzij* $\beta(3\gamma-1) = 0$, welke vergelijkingen beide valsch zijn, zoodat het onmogelijk is, dat de verbindingslijn l van P en Q de X -as a snijdt in het punt $(\gamma, 0)$.

Mocht derhalve voor de lijnen l en a een snijpunt kunnen worden aangewezen, dan is dit of het punt $(\frac{1}{2}, 0)$, of het punt $(3, 0)$. In het eerste geval is a noodzakelijk < 0 , in het laatste geval > 0 .

Gold dus de snijpuntsstelling der projectieve planimetrie, kon derhalve voor iedere $a \neq 0$ een gemeenschappelijk punt van a en l worden aangewezen, dan zouden de relaties „ $\neq 0$ ” en „ $\delta f < 0$ of > 0 ” voor het continuum aequivalent zijn. Zoodat wegens de contradictoriteit dezer aequivalentie eveneens de snijpuntsstelling der projectieve planimetrie contradictor is gebleken, waarmee zoowel van de Euclidische als van de projectieve planimetrie de contradictoriteit afdoende is gedemonstreerd.

Verwacht mag worden dat, met behulp van uit vrije variatie van f voortvloeiende soorten van dempingsgetallen door a_f , ook van andere reeds onjuist gebleken theorieën der klassieke wiskunde de contradictoriteit zal kunnen worden vastgesteld.

³⁾ Vgl. Mathem. Annalen 97, p. 66, noot ¹⁰).

Mathematics. — *Approximate division of an angle into equal parts.* By J. G. VAN DER CORPUT and H. MOOIJ.

(Communicated at the meeting of March 26, 1949.)

The editors¹⁾ of *Mathematica A* published an approximate construction of the trisection, due to M. MARTENS, a former head-master. Although this construction is very accurate for acute angles (with an error smaller than $21' 24''$), it is surpassed in this respect by the very simple construction given by S. C. VAN VEEN²⁾, of which the error for acute angles is less than $2' 36''$.

These articles led H. MOOIJ to deal in his thesis³⁾ with the problem of dividing a given angle approximately into a number of equal parts and even of the approximate construction of $r\alpha$, where α is a given angle and r denotes a number between zero and one, which can be constructed with a pair of compasses and a ruler.

MOOIJ's construction is based on the following theorem.

First theorem.

Describe a circle, the centre of which coincides with the vertex O of the given angle $AOB = 2\alpha \leq 180^\circ$.

Suppose that the circle intersects the legs of the angle at A and B .

Let C be the middle of the segment AB . Produce AB to D in such a way, that

$$CD = \frac{r^2 + 3}{4r} AC.$$

Describe a circle with D as centre and $\frac{3-3r^2}{4r} AC$ as radius, which intersects the smaller arc AB at E . Then $\angle COE$ is approximately equal to $r\alpha$ and the difference is smaller than

$$\frac{4}{3}r(1-r^2)(\operatorname{tg}\frac{1}{2}\alpha + \frac{1}{4}\sin\alpha - \frac{3}{4}r\alpha).$$

Thus the trisection ($r = \frac{1}{3}$) gives the following construction, which is identical to VAN VEEN's.

Describe a circle with the unit as radius, the centre O of which is the vertex of the given angle $AOB = 2\alpha \leq 180^\circ$. Assume that the circle intersects the legs of the given angle 2α at A and B . Let C be the middle of AB . Produce AB to D such that

$$CD = \frac{7}{3} AC.$$

¹⁾ Trisectie, *Mathematica A* 6 (1937—38), p. 1—4.

²⁾ S. C. VAN VEEN, Benaderde trisectie, *Mathematica A* 7 (1938—39), p. 229—237.

³⁾ H. MOOIJ, Over de didactiek van de meetkunde benevens benaderingsconstructies ter verdeling van een hoek in gelijke delen, Thesis Amsterdam 1948.

Describe a circle with D as centre and AB as radius, which intersects the smaller arc AB at E . Then $\angle COE$ is approximately equal to $\frac{1}{3} \alpha$ and the difference is smaller than

$$\frac{3}{8} \frac{r^2}{1} (\operatorname{tg} \frac{1}{2} \alpha + \frac{1}{4} \sin \alpha - \frac{1}{4} \alpha).$$

In this article the proof of the results found by Mooij will again be given. Further we shall give a second approximate construction of $r\alpha$, which is a little more complicated, but gives a more accurate approximation.

The new construction is based on the following theorem.

Second theorem.

Describe a circle with the unit as radius, the centre O of which coincides with the vertex of the given angle $AOB = 2\alpha \leq 180^\circ$.

Assume the circle intersects the legs of the given angle at A and B ; let C be the middle of AB . Produce AB to D such that

$$CD = \frac{1}{4r} (3 + r^2) AC + \frac{1}{240r} (1 - r^2)(9 - r^2) AC^3. \quad \dots \quad (1)$$

Describe a circle with D as centre and the radius

$$\frac{3}{4r} (1 - r^2) AC + \frac{1}{240r} (1 - r^2)(9 - r^2) AC^3. \quad \dots \quad (2)$$

If this circle intersects the smaller arc AB of circle O at E , then $\angle COE$ is approximately equal to $r\alpha$ and the difference is smaller than

$$\frac{4}{3} r (\operatorname{tg} \frac{1}{2} \alpha + \frac{1}{4} \sin \alpha + \frac{3}{80} \sin^3 \alpha - \frac{3}{80} \alpha \sin^2 \alpha - \frac{1}{4} \alpha).$$

As for the trisection ($r = \frac{1}{3}$) we get the following approximate construction.

Describe a circle with the unit as radius, the centre O of which is the vertex of the given angle $OAB = 2\alpha \leq 180^\circ$.

Assume that the circle intersects the legs of the given angle 2α at A and B and that C is the centre of AB . Produce AB to D , such that

$$CD = \frac{1}{3} AC + \frac{3}{8} AC^3.$$

Describe a circle with D as centre and with

$$2 AC + \frac{3}{8} AC^3$$

as radius. This circle intersects the smaller arc AB at E . Then $\angle COE$ is approximately equal to α and the difference is smaller than

$$\frac{4}{3} (\operatorname{tg} \frac{1}{2} \alpha + \frac{1}{4} \sin \alpha + \frac{3}{80} \sin^3 \alpha - \frac{3}{80} \alpha \sin^2 \alpha - \frac{1}{4} \alpha).$$

This construction becomes somewhat simpler, if the factor $(9 - r^2)$ in the second term of (1) en (2) is replaced by 9.

If $r = \frac{1}{3}$, we get

$$CD = \frac{1}{3} AC + \frac{1}{16} AC^3 \quad \dots \quad (3)$$

and the radius of the circle with D as centre becomes

$$2AC + \frac{1}{16}AC^3. \dots \dots \dots \quad (4)$$

This approximation is extraordinarily accurate. In order to have a survey of the accuracy of the approximation of the trisection, we give the following tables, referring to the three constructions I, II and III; here I denotes VAN VEEN's construction (which also occurs in MOOIJ's thesis); II is the construction according to the formulae (1) en (2) and finally III is the simpler construction in which the formulae (3) and (4) are applied.

Difference from $\frac{1}{3}\alpha$.

2α	I	II	III
180°	$1^\circ 35' 17''$	$1^\circ 4' 9.5''$	$1^\circ 3' 7.1''$
120°	$11' 19''$	$4' 21.2''$	$4' 16.9''$
90°	$2' 38''$	$37.3''$	$35.7''$
60°	$18''$	$2.29''$	$2.25''$

For the proof of the above assertions we need the following lemma

Lemma.

Assume the real numbers A, B, C and D satisfy the inequalities

$$A^2 + B^2 \geq C^2; \quad A^2 + B^2 > D^2; \quad C \neq D$$

and the real number t satisfies

$$A \sin t + B \cos t = D.$$

Then the equation

$$A \sin x + B \cos x = C \quad \dots \dots \dots \quad (5)$$

has at least one real solution x for which the inequality

$$|x - t| < \frac{2|C - D|}{\sqrt{A^2 + B^2 - C^2} + \sqrt{A^2 + B^2 - D^2}}$$

holds and for which $x - t$ has the same sign as

$$\frac{C - D}{A \cos t - B \sin t}.$$

Moreover

$$A \cos t - B \sin t = \pm \sqrt{A^2 + B^2 - D^2} \neq 0.$$

For the proof we substitute $x = t + u$ in (5) and we introduce z determined by

$$\operatorname{tg} \frac{1}{2}u = z \quad (-\pi < u < \pi).$$

Equation (5) becomes

$$A \sin(t + u) + B \cos(t + u) = C,$$

whence

$$A \left(\frac{1-z^2}{1+z^2} \sin t + \frac{2z}{1+z^2} \cos t \right) + B \left(\frac{1-z^2}{1+z^2} \cos t - \frac{2z}{1+z^2} \sin t \right) = C.$$

consequently

$$z^2(A \sin t + B \cos t + C) - 2z(A \cos t - B \sin t) - (A \sin t + B \cos t - C) = 0.$$

hence

$$(D+C)z^2 + 2(B \sin t - A \cos t)z + C - D = 0. \quad . . \quad (6)$$

The discriminant of this quadratic equation is

$$\Delta = (B \sin t - A \cos t)^2 - (C^2 - D^2)$$

and therefore

$$\Delta = A^2 + B^2 - C^2 \geq 0.$$

The roots of (6) are

$$\begin{aligned} \operatorname{tg} \frac{1}{2}u &= \frac{A \cos t - B \sin t \pm \sqrt{A^2 + B^2 - C^2}}{C + D} \\ &= \frac{A^2 + B^2 - D^2 - A^2 - B^2 + C^2}{(C + D)(A \cos t - B \sin t \mp \sqrt{A^2 + B^2 - C^2})}. \end{aligned}$$

The sign of $\pm \sqrt{A^2 + B^2 - C^2}$ can be taken equal to that of $A \cos t - B \sin t$ and then the sign of $\operatorname{tg} \frac{1}{2}u$ is the same as that of

$$\frac{C - D}{A \cos t - B \sin t}.$$

According to our convention u lies between $-\pi$ and π , so that $\frac{1}{2}u$ has the same sign as $\operatorname{tg} \frac{1}{2}u$, thus

$$\begin{aligned} |\frac{1}{2}u| < |\operatorname{tg} \frac{1}{2}u| &= \frac{|C - D|}{\sqrt{A^2 + B^2 - C^2} + |A \cos t - B \sin t|} \\ &= \frac{|C - D|}{\sqrt{A^2 + B^2 - C^2} + \sqrt{A^2 + B^2 - D^2}} \end{aligned}$$

where $u = x - t$. This establishes the proof.

For the proof of the first theorem we put

$$AO = 1; \angle COE = x; CD = p \sin \alpha; DE = q \sin \alpha \text{ and } \angle BDE = y.$$

By projecting OED on OC and on AD , we get the relations

$$\cos x - q \sin \alpha \sin y = \cos \alpha$$

and

$$\sin x + q \sin \alpha \cos y = p \sin \alpha.$$

By eliminating y we obtain

$$p \sin \alpha \sin x + \cos \alpha \cos x = 1 + \frac{1}{2}(p^2 - q^2 - 1) \sin^2 \alpha.$$

thus

$$A \sin x + B \cos x = C,$$

where

$$A = p \sin \alpha; \quad B = \cos \alpha \text{ and } C = 1 + \frac{1}{2}(p^2 - q^2 - 1) \sin^2 \alpha. \quad (7)$$

We have to determine A and C in such a way, that x is approximately equal to $r\alpha$; therefore we put

$$A \sin r\alpha + B \cos r\alpha = D. \quad \dots \quad \dots \quad \dots \quad \dots \quad (8)$$

Then, according to the above lemma,

$$|r\alpha - x| < \frac{2|C - D|}{\sqrt{A^2 + B^2 - C^2} + \sqrt{A^2 + B^2 - D^2}}. \quad \dots \quad \dots \quad (9)$$

Expanding $\cos r\alpha$ and $\sin r\alpha$ in powers of $\sin \alpha$, we get

$$\left. \begin{aligned} \cos r\alpha &= 1 - \frac{r^2}{2!} \sin^2 \alpha - \frac{r^2(2^2 - r^2)}{4!} \sin^4 \alpha - \frac{r^2(2^2 - r^2)(4^2 - r^2)}{6!} \sin^6 \alpha - \dots \\ \sin r\alpha &= r \sin \alpha + \frac{r(1^2 - r^2)}{3!} \sin^3 \alpha + \frac{r(1^2 - r^2)(3^2 - r^2)}{5!} \sin^5 \alpha + \dots \end{aligned} \right\} \quad (10)$$

From the relations (7) and (10) it follows that (8) becomes

$$D = p \sin \alpha \left\{ r \sin \alpha + \frac{r(1^2 - r^2)}{3!} \sin^3 \alpha + \dots \right\} + \cos \alpha \cos r\alpha.$$

Differentiation of (10) gives

$$r \cos r\alpha = r \cos \alpha \left\{ 1 + \frac{1^2 - r^2}{2!} \sin^2 \alpha + \frac{(1^2 - r^2)(3^2 - r^2)}{4!} \sin^4 \alpha + \dots \right\}.$$

By substituting this result in the above value of D , we get

$$\left. \begin{aligned} D &= p \sin \alpha \left\{ r \sin \alpha + \frac{r(1^2 - r^2)}{3!} \sin^3 \alpha + \frac{r(1^2 - r^2)(3^2 - r^2)}{5!} \sin^5 \alpha + \dots \right\} + \\ &+ (1 - \sin^2 \alpha) \left\{ 1 + \frac{1^2 - r^2}{2!} \sin^2 \alpha + \frac{(1^2 - r^2)(3^2 - r^2)}{4!} \sin^4 \alpha + \dots \right\}. \end{aligned} \right\} \quad (11)$$

Putting

$$a_h = \{(2h-1)^2 - \varrho\} a_{h-1}; \quad a_0 = 1; \quad \psi(h) = -(2h-1+\varrho) + 2prh$$

and replacing $\sin \alpha$ by s and r^2 by ϱ , we find

$$D = \sum_{h=0}^{\infty} \frac{s^{2h} a_{h-2} \{(2h-3)^2 - \varrho\}}{(2h)!} \psi(h). \quad \dots \quad \dots \quad (12)$$

In order to get a good approximation of $r\alpha$, we choose p and q in such a way, that the expansion of $C - D$ begins with $\sin^6 \alpha$. This gives

$$\frac{1}{2}(p^2 - q^2 - 1) = pr - 1 + \frac{1}{2}(1 - r^2) \text{ and}$$

$$\frac{pr(1^2 - r^2)}{3!} + \frac{(1^2 - r^2)(3^2 - r^2)}{4!} - \frac{1^2 - r^2}{2!} = 0.$$

hence

$$p = \frac{r^2 + 3}{4r} \text{ and } q = \frac{3 - 3r^2}{4r}. \quad \dots \dots \dots \quad (13)$$

In what follows an upperbound is given for the error, provided that r lies between 0 and 1. Substituting in (12) the value of p given by (13), we get

$$\psi(h) = -2h + 1 - \varrho + \frac{r^2 + 3}{2r} hr = \frac{1}{2}(h-2)(\varrho-1)$$

hence

$$D = \sum_{h=0}^{\infty} \frac{s^{2h} a_{h-2} \{(2h-3)^2 - \varrho\}}{2 \cdot (2h)!} (h-2)(\varrho-1).$$

In $C-D$ the terms with s^0 ; s^2 and s^4 disappear, hence

$$\begin{aligned} C-D &= - \sum_{h=3}^{\infty} \frac{s^{2h} a_{h-2} \{(2h-3)^2 - \varrho\}}{2 \cdot (2h)!} (h-2)(\varrho-1) \\ &= (1-\varrho)^2 (3^2 - \varrho) U s^6, \end{aligned}$$

where

$$U = \frac{1}{2 \cdot 6!} + \frac{2(5^2 - \varrho) 5^2}{2 \cdot 8!} + \frac{3 \cdot (5^2 - \varrho) (7^2 - \varrho) 5^4}{2 \cdot 10!} + \dots$$

By virtue of $0 \leq \varrho \leq 1$ the inequality $U \leq U_0$ is evident, consequently

$$C-D \leq s^6 (1-\varrho)^2 (3^2 - \varrho) U_0$$

where U_0 denotes the value of U at the point $\varrho = 0$. Hence

$$C-D \leq (1-\varrho)^2 (1 - \frac{1}{2}\varrho) (C-D)_0. \quad \dots \dots \dots \quad (14)$$

where $(C-D)_0$ denotes the value which $C-D$ takes for $\varrho = 0$.

The denominator of the right hand side of (9) contains the terms $\sqrt{A^2 + B^2 - C^2}$ and $\sqrt{A^2 + B^2 - D^2}$. Here

$$\begin{aligned} A^2 + B^2 - C^2 &= p^2 s^2 + \cos^2 \alpha - \{1 + \frac{1}{2}(p^2 - q^2 - 1)s^2\}^2 \\ &= q^2 s^2 - \frac{1}{4}(p^2 - q^2 - 1)^2 s^4. \end{aligned}$$

Substitution of the values of p and q , found in (13), gives

$$A^2 + B^2 - C^2 = \frac{9(1-\varrho)^2}{16\varrho} s^2 - \frac{(1-\varrho)^2}{16} s^4,$$

hence

$$\sqrt{A^2 + B^2 - C^2} = \frac{3(1-\varrho)}{4r} s \sqrt{1 - \frac{\varrho}{9}s^2}. \quad \dots \dots \quad (15)$$

By putting $t = r\alpha$ in the last formula of our lemma and substituting (7), we obtain

$$\pm \sqrt{A^2 + B^2 - D^2} = p \sin \alpha \cos r\alpha - \cos \alpha \sin r\alpha. \quad \dots \dots \quad (16)$$

With the aid of the series (10), by substituting $p = \frac{r^2+3}{4r}$, and, putting $c_0 = 1$ and $c_h = (4h^2 - \varrho) c_{h-1}$ ($h \geq 1$), we get

$$\begin{aligned}\pm \sqrt{A^2 + B^2 - D^2} &= \frac{r(1-\varrho)}{4} \left\{ \frac{3}{\varrho} s - \frac{s^3}{3!} + \frac{2^2 - \varrho}{5!} s^5 + \frac{3(4^2 - \varrho)(2^2 - \varrho)}{7!} s^7 + \dots \right\} \\ &= \frac{r(1-\varrho)}{4} \sum_{h=1}^{\infty} \frac{(2h-5) c_{h-2}}{(2h-1)!} s^{2h-1}.\end{aligned}$$

The right hand side is ≥ 0 , consequently

$$\sqrt{A^2 + B^2 - D^2} \geq \frac{3(1-\varrho)}{4r} s \left(1 - \frac{\varrho}{18} s^2 \right)$$

and a fortiori

$$\sqrt{A^2 + B^2 - D^2} > \frac{3(1-\varrho)}{4r} s \sqrt{1 - \frac{\varrho}{9} s^2} \quad \dots \quad (17)$$

The relations (9), (14), (15) en (17) give

$$|ra - x| < \frac{2(1-\varrho)^2(1 - \frac{1}{9}\varrho)(C-D)_0}{\frac{3(1-\varrho)s}{4r} \sqrt{1 - \frac{1}{9}\varrho s^2} + \frac{3(1-\varrho)s}{4r} \sqrt{1 - \frac{1}{9}\varrho s^2}}.$$

From $C_0 = 1 + \frac{1}{4}s^2$ and $D_0 = \frac{3}{4}\alpha \sin \alpha + \cos \alpha$ it follows that

$$\begin{aligned}(C-D)_0 &= 1 + \frac{1}{4} \sin^2 \alpha - \frac{3}{4} \alpha \sin \alpha - \cos \alpha \\ &= \sin \alpha \left(\operatorname{tg} \frac{\alpha}{2} + \frac{1}{4} \sin \alpha - \frac{3}{4} \alpha \right).\end{aligned}$$

Hence

$$|ra - x| < \frac{2(1-\varrho)^2(1 - \frac{1}{9}\varrho)s \left(\operatorname{tg} \frac{\alpha}{2} + \frac{1}{4} \sin \alpha - \frac{3}{4} \alpha \right)}{\frac{2.3(1-\varrho)s}{4r} \sqrt{1 - \frac{1}{9}\varrho s^2}}.$$

Consequently

$$|ra - x| < \frac{1}{2} r(1-\varrho) (\operatorname{tg} \frac{1}{2} \alpha + \frac{1}{4} \sin \alpha - \frac{3}{4} \alpha).$$

This establishes the proof of the first theorem. For the proof of the second theorem we put

$$p = \frac{P}{r} + \frac{Q}{r} s^2 \quad \dots \quad (18)$$

Then

$$C = 1 + \frac{1}{2}(p^2 - q^2 - 1)s^2 = 1 + Ks^2 + Ls^4, \quad \dots \quad (19)$$

where P , Q , K and L are properly chosen functions of r . Formula (8) gives

$$D = \left(\frac{P}{r} + \frac{Q}{r} s^2 \right) s \sin ra + \cos \alpha \cos ra,$$

hence

$$D = (P s^2 + Q s^4) \sum_{h=0}^{\infty} \frac{a_h s^{2h}}{(2h+1)!} + (1-s^2) \sum_{h=0}^{\infty} \frac{a_h}{(2h)!} s^{2h},$$

where

$a_h = (1^2 - \varrho)(3^2 - \varrho) \dots ((2h-1)^2 - \varrho)$ and $a_0 = 1$, so $a_h = \{(2h-1)^2 - \varrho\} a_{h-1}$, consequently

$$\begin{aligned} D &= \sum_0^{\infty} s^{2h} \left\{ \frac{a_h}{(2h)!} - \frac{a_{h-1}}{(2h-2)!} + \frac{Pa_{h-1}}{(2h-1)!} + \frac{Qa_{h-2}}{(2h-3)!} \right\} \\ &= \sum_0^{\infty} s^{2h} a_{h-2} \left\{ \frac{(2h-1)^2 - \varrho}{(2h)!} ((2h-3)^2 - \varrho) - \frac{(2h-3)^2 - \varrho}{(2h-2)!} + \right. \\ &\quad \left. + \frac{(2h-3)^2 - \varrho}{(2h-1)!} P + \frac{Q}{(2h-3)!} \right\}. \end{aligned}$$

This gives

$$D = \sum_0^{\infty} \frac{s^{2h} a_{h-2}}{(2h)!} \varphi(h)$$

where

$$\begin{aligned} \varphi(h) &= -\{(2h-1) + \varrho\} \{(2h-3)^2 - \varrho\} + 2h \{(2h-3)^2 - \varrho\} P + \\ &\quad + 2h(2h-1)(2h-2) Q. \end{aligned}$$

The functions P, Q, K and L are chosen such that the expansion of $C-D$ begins with s^8 . By virtue of

$$\begin{aligned} D &= 1 + \frac{s^2}{2!} (-1 - \varrho + 2P) + \frac{s^4}{4!} \{-(3 + \varrho)(1 - \varrho) + 4(1 - \varrho)P + 4 \cdot 3 \cdot 2 \cdot Q\} + \\ &\quad + \frac{s^6(1 - \varrho)}{6!} \{-(5 + \varrho)(9 - \varrho) + 6(9 - \varrho)P + 6 \cdot 5 \cdot 4 \cdot Q\} + \dots \end{aligned}$$

we get in connection with (19)

$$2K = -1 - \varrho + 2P; \quad \dots \quad (20)$$

$$24L = -(3 + \varrho)(1 - \varrho) + 4(1 - \varrho)P + 4 \cdot 3 \cdot 2 \cdot Q; \quad \dots \quad (21)$$

$$-(5 + \varrho)(9 - \varrho) + 6(9 - \varrho)P + 6 \cdot 5 \cdot 4 \cdot Q = 0. \quad \dots \quad (22)$$

Substitution in (19) of the values of p and $2K$, found respectively in (18) and (20) furnishes

$$q^2 \varrho = (P - \varrho)^2 + 2s^2(PQ - L\varrho) + s^4 Q^2. \quad \dots \quad (23)$$

To simplify we put the right hand side of this equation equal to a perfect square, by choosing

$$PQ - L\varrho = (P - \varrho)Q, \text{ hence } L = Q.$$

From (21) follows

$$4(1 - \varrho)P = (3 + \varrho)(1 - \varrho), \text{ hence } P = \frac{3 + \varrho}{4}. \quad \dots \quad (24)$$

Formula (22) gives

$$120 Q = (5 + \varrho)(9 - \varrho) - 6(9 - \varrho) \cdot \frac{3 + \varrho}{4},$$

hence

$$Q = L = \frac{(1-\varrho)(9-\varrho)}{240}.$$

By (20) and (24) we obtain

$$K = \frac{1-\varrho}{4}.$$

Now (23) becomes

$$\varrho q^2 = (P - \varrho + Q s^2)^2 \text{ and } r q = \pm (P - \varrho + Q s^2).$$

The left hand side of the last relation and also both $P - \varrho$ and $Q s^2$ are positive, so that the plus sign holds good. This gives

$$r q = \frac{1-\varrho}{240} \{ 180 + (9-\varrho) s^2 \}$$

and by (18)

$$r p = \frac{3+\varrho}{4} + \frac{(1-\varrho)(9-\varrho)}{240} s^2. \quad \dots, \quad (25)$$

In order to obtain an upper bound of the error, we deduce from our lemma that this error is

$$|r a - x| < \frac{2 |C - D|}{\sqrt{A^2 + B^2 - C^2} + \sqrt{A^2 + B^2 - D^2}},$$

here

$$C - D = - \sum_{h=1}^{\infty} \frac{s^{2h} a_{h-2}}{(2h)!} \varphi(h)$$

and

$$\begin{aligned} \varphi(h) = & -\{2h-1+\varrho\} \{(2h-3)^2-\varrho\} + \\ & + 2h \{(2h-3)^2-\varrho\} P + 2h(2h-1)(2h-2) Q. \end{aligned}$$

Consequently

$$\begin{aligned} \varphi(h) = & -\{(2h-1)+\varrho\} \{(2h-3)^2-\varrho\} + \\ & + 2h \{(2h-3)^2-\varrho\} \frac{3+\varrho}{4} + 2h(2h-1)(2h-2) \frac{(1-\varrho)(9-\varrho)}{240}, \end{aligned}$$

which gives after reduction

$$\varphi(h) = \frac{(1-\varrho)(h-3)}{60} \{-102h^2 + 267h - 180 - \varrho(2h^2 + 3h - 20)\},$$

hence

$$C - D = \frac{1-\varrho}{60} \sum_{h=1}^{\infty} (h-3) \frac{s^{2h} a_{h-2}}{(2h)!} \{102h^2 - 267h + 180 + \varrho(2h^2 + 3h - 20)\}.$$

In this relation we have for $h \geq 4$

$$a_{h-2} = (1-\varrho)(1 - \frac{1}{6}\varrho) b_{h-2}(\varrho),$$

where $b_{h-2}(\varrho)$ is a polynomial in ϱ , which has in the interval $0 \leq \varrho \leq 1$ its maximum value at the point $\varrho = 0$, therefore

$$0 \leq a_{h-2} \leq (1-\varrho)(1 - \frac{1}{6}\varrho) b_{h-2}(0) = (1-\varrho)(1 - \frac{1}{6}\varrho) a_{h-2}(0).$$

Further

$$102h^2 - 267h + 180 + \varrho(2h^2 + 3h - 20) \leq (1 + \frac{1}{3}\varrho)(102h^2 - 267h + 180);$$

for this relation is equivalent to the inequality

$$62h^2 + 93h - 620 \leq 102h^2 - 267h + 180,$$

which is evident in virtue of $h^2 - 9h + 20 \geq 0$. Hence

$$C - D \leq$$

$$\leq \frac{(1-\varrho)^2}{60} (1 - \frac{1}{6}\varrho) \sum_{h=4}^{\infty} (h-3) \frac{s^{2h}}{(2h)!} a_{h-2}(0) (1 + \frac{1}{3}\varrho) (102h^2 - 267h + 180).$$

This furnishes

$$C - D \leq (1-\varrho)(1 - \frac{1}{6}\varrho)(1 + \frac{1}{3}\varrho)(C - D)_0. \quad \dots \quad (26)$$

From (7), (8) and (25) we deduce

$$D = \frac{3+\varrho}{4r} \sin \alpha \sin r\alpha + \frac{(1-\varrho)(9-\varrho)}{240r} \sin^3 \alpha \sin r\alpha + \cos \alpha \cos r\alpha,$$

hence

$$D_0 = \frac{3}{4} \alpha \sin \alpha + \frac{3}{80} \alpha \sin^3 \alpha + \cos \alpha.$$

Further

$$C_0 = 1 + \frac{1}{4} s^2 + \frac{3}{80} s^4,$$

hence

$$(C - D)_0 = 1 + \frac{1}{4} s^2 + \frac{3}{80} s^4 - \frac{3}{4} \alpha s - \frac{3}{80} \alpha s^3 - \cos \alpha,$$

consequently

$$(C - D)_0 = \sin \alpha (\operatorname{tg} \frac{1}{2} \alpha + \frac{1}{4} \sin \alpha + \frac{3}{80} \sin^3 \alpha - \frac{3}{80} \alpha \sin^2 \alpha - \frac{3}{4} \alpha). \quad (27)$$

Further

$$\begin{aligned} A^2 + B^2 - C^2 &= p^2 \sin^2 \alpha + \cos^2 \alpha - (1 + K \sin^2 \alpha + L \sin^4 \alpha)^2 \\ &= \left(\frac{P}{r} + \frac{Qs^2}{r} \right)^2 s^2 + 1 - s^2 - (1 + K^2 s^4 + L^2 s^8 + 2Ks^2 + 2Ls^4 + 2KLs^6), \end{aligned}$$

where $Q = L$, hence

$$\begin{aligned} A^2 + B^2 - C^2 &= \frac{P^2}{\varrho} s^2 + \frac{2PQ}{\varrho} s^4 + \frac{Q^2}{\varrho} s^6 + \\ &\quad + 1 - s^2 - 1 - K^2 s^4 - Q^2 s^8 - 2Ks^2 - 2Qs^4 - 2KQs^6 \\ &= s^2 \left\{ \frac{P^2}{\varrho} - 1 - 2K + s^2 \left(\frac{2PQ}{\varrho} - K^2 - 2Q \right) + s^4 \left(\frac{Q^2}{\varrho} - 2QK \right) - Q^2 s^6 \right\}. \end{aligned}$$

Substitution of the values of P , Q and K gives

$$\sqrt{A^2 + B^2 - C^2} = \frac{1-\varrho}{4} s \sqrt{\left\{ \frac{9}{\varrho} + \frac{9-11\varrho}{10\varrho} s^2 + \right.} \\ \left. + \frac{(9-\varrho)(9-121\varrho)}{3600\varrho} s^4 - \frac{(9-\varrho)^2}{3600} s^6 \right\}}.$$

In order to deduce from this relation the inequality

$$\sqrt{A^2 + B^2 - C^2} \geq \frac{1-\varrho}{4} \frac{3s}{\sqrt{\varrho}} (1 - \frac{1}{5}\varrho) (1 + \frac{1}{31}\varrho), \dots \quad (28)$$

we remark that this inequality is equivalent to

$$\frac{9}{\varrho} + \frac{9-11\varrho}{10\varrho} s^2 + \frac{(9-\varrho)(9-121\varrho)}{3600\varrho} s^4 - \frac{(9-\varrho)^2}{3600} s^6 \\ \geq \frac{9}{\varrho} (1 - \frac{1}{5}\varrho + \frac{1}{31}\varrho^2) \left(1 + \frac{2}{31}\varrho + \frac{1}{31^2}\varrho^2 \right)$$

and therefore equivalent to

$$\left(\frac{81}{3600} s^4 + \frac{9}{10} s^2 \right) \frac{1}{\varrho} + \left(\frac{44}{31} - \frac{11}{10} s^2 - \frac{1098}{3600} s^4 - \frac{81}{3600} s^6 \right) \\ + \left(\frac{121}{3600} s^4 + \frac{18}{3600} s^6 + \frac{74}{9.961} \right) \varrho - \left(\frac{1}{3600} s^6 + \frac{44}{9.961} \right) \varrho^2 - \frac{1}{9.961} \varrho^3 \geq 0.$$

To prove this inequality it is sufficient to consider the most unfavourable case, viz. $s = 1$; in this case the inequality becomes

$$\frac{369}{400} \frac{1}{\varrho} - \frac{101}{12400} + \frac{163179}{961.3600} \varrho - \frac{18561}{961.3600} \varrho^2 - \frac{1}{9.961} \varrho^3 \geq 0,$$

which is true for every value of ϱ between 0 and 1.

By substituting in (16) the value of p found in (25) we get

$$\sqrt{A^2 + B^2 - D^2} = \frac{3(1-\varrho)}{4r} s + \frac{(9-11\varrho)(1-\varrho)}{240r} s^3 - \frac{r(1-\varrho)}{4 \cdot 4!} s^5, \dots \\ + \sum_{n=4}^{\infty} \frac{r(2^2-\varrho) \dots ((2n-6)^2-\varrho)(1-\varrho)(2n-7) \{ 102n^2 - 369n + 339 + (n-3)(2n+7)\varrho \} s^{2n-1}}{120(2n-1)!}.$$

This gives

$$\sqrt{A^2 + B^2 - D^2} \geq \frac{3(1-\varrho)}{4r} s + \frac{(9-11\varrho)(1-\varrho)}{240r} s^3 - \left. \begin{aligned} & - \frac{r(1-\varrho)}{4 \cdot 4!} s^5 \geq \frac{1-\varrho}{4} \frac{3s}{\sqrt{\varrho}} (1 - \frac{1}{5}\varrho) (1 + \frac{1}{31}\varrho). \end{aligned} \right\}. \quad (29)$$

In fact this relation becomes after reduction

$$\frac{1}{20} s^2 + \frac{2}{779} \varrho - \frac{1}{180} \varrho s^2 + \frac{1}{779} \varrho^2 - \frac{1}{2} \varrho s^4 \geq 0,$$

that is

$$\frac{1}{2} s^2 + \varrho \left(\frac{1}{27} - \frac{1}{80} s^2 - \frac{1}{72} s^4 \right) + \frac{1}{72} \varrho^2 \geq 0$$

and this inequality holds because for $0 \leq s \leq 1$ the factor of ϱ is at least equal to

$$\frac{1}{27} - \frac{1}{80} - \frac{1}{72} > 0.$$

Now by virtue of (26), (27), (28) and (29) the relation (9) becomes

$$|\tau a - x| < \frac{2(1-\varrho)(1-\frac{1}{9}\varrho)(1+\frac{1}{3}\varrho)\sin a (\operatorname{tg} \frac{1}{2}a + \frac{1}{4}\sin a + \frac{3}{80}\sin^3 a - \frac{3}{80}\alpha \sin^2 a - \frac{3}{4}a)}{2\frac{1-\varrho}{4}\frac{3s}{\sqrt{\varrho}}(1-\frac{1}{9}\varrho)(1+\frac{1}{3}\varrho)}$$

which gives after reduction

$$|\tau a - x| < \frac{1}{3} \tau (\operatorname{tg} \frac{1}{2}a + \frac{1}{4}\sin a + \frac{3}{80}\sin^3 a - \frac{3}{80}\alpha \sin^2 a - \frac{3}{4}a).$$

This becomes in the special case $\tau = \frac{1}{3}$

$$|\tau a - x| < \frac{1}{9} (\operatorname{tg} \frac{1}{2}a + \frac{1}{4}\sin a + \frac{3}{80}\sin^3 a - \frac{3}{80}\alpha \sin^2 a - \frac{3}{4}a).$$

This establishes the proof of the second theorem.

Mathematics. — *Een iteratieproces voor de oplossing van een randwaardeprobleem bij een lineaire partiële differentiaalvergelijking van de tweede orde. I.* By J. J. DRONKERS. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of February 26, 1949.)

§ 1.

In dit artikel wordt een iteratieproces behandeld, ter verkrijging van een oplossing $z(x, y)$ van een lineaire partiële differentiaalvergelijking van de tweede orde:

$$\frac{\partial^2 z}{\partial x^2} + a \frac{\partial^2 z}{\partial y^2} + b \frac{\partial^2 z}{\partial x \partial y} + c \frac{\partial z}{\partial x} + d \frac{\partial z}{\partial y} + e z + f = 0 \quad \dots \quad (1)$$

waarbij de randwaarden gegeven zijn voor $x = 0$:

$$\left(\frac{\partial z}{\partial x} \right)_{x=0} \text{ en } z(0, y). \quad \dots \quad (2)$$

Hierbij zijn a, b, c, d, e en f functies van x en y , waarbij in het vervolg die waarden van x en y beschouwd worden, die gelegen zijn in de intervallen $0 \leq x \leq A$ en $0 \leq y \leq B$.

Nu wordt het volgende verondersteld:

Van de functies b en c bestaan haar afgeleiden naar x van de eerste orde. Deze zijn tevens continu in x , hetgeen ook het geval is met de overige functies a, d, e en f .

In het bijzonder zijn alle functies a t.m. f , de genoemde afgeleide functies van b en c naar x van de eerste orde, benevens de randwaarden $\left(\frac{\partial z}{\partial x} \right)_{x=0}$ en $z(0, y)$ analytisch in y en begrensd.

Voor de afgeleide naar y van de k^e orde van ieder der genoemde functies geldt dan analoog als voor de functie a :

$$\left| \frac{\partial^k a}{\partial y^k} \right| < M_1 \varrho^k k! \quad (k = 1, 2, \dots, n, \dots)$$

waarbij M_1 en ϱ onafhankelijk zijn van x, y en k , terwijl M_1 zo gekozen kan worden, dat ze ook voor alle andere functies toegepast kan worden.

Voor alle functies geldt dus, dat de afgeleiden naar y van de k^e orde na deling door $k!$ en ϱ^k begrensd zijn.

De iteratiemethode, die in dit artikel wordt behandeld, is verschillend van die, welke PICARD¹⁾ toepast om een integraal te verkrijgen van de hyperbolische partiële differentiaalvergelijking van de tweede orde, indien

¹⁾ E. PICARD: Mémoire sur la théorie des équations aux dérivées partielles et la

deze — eventueel na een coördinatentransformatie — van de vorm is:

$$\frac{\partial^2 z}{\partial x \partial y} = p \frac{\partial z}{\partial x} + q \frac{\partial z}{\partial y} + rz + s.$$

Een differentiaalvergelijking van deze vorm wordt zelfs in het vervolg uitgezonderd.

Ook is van PICARD een iteratiemethode afkomstig om een oplossing te verkrijgen van de elliptische partiële differentiaalvergelijking van de tweede orde.

In dit artikel wordt de partiële differentiaalvergelijking van de tweede orde (1), die van het hyperbolische, elliptische of parabolische type kan zijn, vervangen door 2 lineaire partiële differentiaalvergelijkingen van de eerste orde. Daarna wordt een iteratiemethode aangegeven om een oplossing te verkrijgen van deze twee vergelijkingen met randwaarden, die afhankelijk zijn van de randwaarden, welke in (2) zijn aangegeven. Dan wordt tevens de integraal $z(x, y)$ verkregen, die aan (1) en de gestelde randwaarden voldoet.

Alle iteratiemethoden betreffende partiële differentiaalvergelijkingen vertonen veel overeenkomst. De in dit artikel weergegeven oplossing onderscheidt zich door de voorwaarden, die aan de verschillende functies van (1) en (2) zijn opgelegd in verband met het convergentiebewijs van de iteratiemethode. Dit bewijs is van een andere structuur, dan gewoonlijk bij overeenkomstige iteratiemethoden voorkomt.

Om de convergentie van de iteratiemethoden van PICARD te bewijzen moet o.a. verondersteld worden, dat het gebied waarvoor de oplossing geldt, zo klein is, dat hiervoor de te bepalen functie $|z(x, y)|$, benevens

$$\left| \frac{\partial z}{\partial x} \right| \text{ en } \left| \frac{\partial z}{\partial y} \right| \text{ kleiner zijn dan een constante } c.$$

Een dergelijke veronderstelling wordt in dit artikel niet gemaakt, zoals uit de hierboven aangegeven voorwaarden blijkt. Er worden alleen voorwaarden opgelegd aan de bekende functies $a, b, \text{ enz.}$ van (1) en aan de randwaarden.

Betreffende de variabele x wordt alleen continuïteit verondersteld, terwijl de functies in y analytisch aangenomen worden.

Bij vele in de literatuur der toegepaste wiskunde voorkomende verhandelingen betreffende de convergentie van iteratiemethoden, is het o.a. nodig te veronderstellen, dat alle afgeleiden naar x of y van de bekende functies bestaan en tevens begrensd zijn. In verband hiermede moet vaak voor x of y of beiden een bepaalde lengte-eenheid gekozen worden (bv. $\sin 2x$ voldoet niet aan deze voorwaarden, $\sin x$ wel).

méthode des approximations successives. Journal de mathématiques pures et appliquées. Paris 1890.

E. GOURSAT: Cours d'analyse mathématique, troisième édition, tome III, pages 133 et 229 etc. Paris 1923.

COURANT und HILBERT: Mathematische Physik II, pag. 317 e.v.

Doordat nu echter aangenomen wordt, dat de bekende functies analytisch in y moeten zijn, worden de mogelijkheden aanzienlijk uitgebreid, zo kan nu bijv. ook de functie $\frac{1}{y+h}$ beschouwd worden. Eveneens is het niet meer nodig om een bepaalde lengte-eenheid te kiezen.

Betreffende de gemaakte veronderstellingen worden nog enkele opmerkingen gemaakt:

1e. In (1) is de coëfficiënt van $\frac{\partial^2 z}{\partial x^2}$ gelijk aan één genomen. Indien deze term ontbreekt en dit niet het geval is met $\frac{\partial^2 z}{\partial y^2}$, wordt de coëfficiënt hiervan gelijk aan één gesteld. In de randgegevens (2) moeten dan x en y verwisseld worden, terwijl ook, hiermede corresponderend, hetgeen van de functies a t.m. f gegeven is, gewijzigd zal moeten worden.

Wij zullen echter wel veronderstellen, dat zeker één der coëfficiënten van $\frac{\partial^2 z}{\partial x^2}$ en $\frac{\partial^2 z}{\partial y^2}$ gelijk aan één gemaakt kan worden. Zoals reeds gezegd, beschouwde PICARD speciaal de gevallen, waarbij of de beide coëfficiënten nul zijn, of aan elkaar gelijk en dus gelijk aan één gemaakt kunnen worden.

2e. In het voorgaande is als randwaarde $z(0, y)$ aangenomen. Laat een dergelijke kromme algemeen voorgesteld worden door:

$$x = \varphi_1(u); \quad y = \varphi_2(u); \quad z = \varphi_3(u) \quad \dots \quad (3)$$

terwijl verder $\frac{\partial z}{\partial x} = \varphi_4(u)$ en $\frac{\partial z}{\partial y} = \varphi_5(u)$ bekende functies zijn, mits het verband bestaat:

$$\varphi_4(u) \varphi_1^{(1)}(u) + \varphi_5(u) \varphi_2^{(1)}(u) = \varphi_3^{(1)}(u).$$

Wordt in plaats van x , de coördinaat x_1 ingevoerd, zodat $x = \varphi_1(u) + x_1$, dan is de kromme (3) weer in het vlak $x_1 = 0$ gelegen.

3e. Zoals bekend volgen de karakteristieke krommen van de partiële differentiaalvergelijking (1), die in het xy vlak gelegen zijn, uit de vergelijking:

$$dy^2 - b dx dy + adx^2 = 0.$$

Wordt nu een kromme volgens (3) gegeven, dan mogen de functies φ_1 en φ_2 voor geen enkele waarde van u voldoen aan de betrekking:

$$\varphi_2^{(1)2} - b \varphi_1^{(1)} \varphi_2^{(1)} + a \varphi_1^{(1)2} = 0$$

dat wil zeggen de raaklijn in een punt P van de projectie van de kromme (3) op het xy vlak mag niet samenvallen met de raaklijn in P aan een karakteristieke kromme, die door het punt P gaat.

Daar de coëfficiënt van $\frac{\partial^2 z}{\partial x^2}$ in (1) ongelijk aan nul is, zal de projectie van de kromme $z(0, y)$ op het xy vlak nooit samenvallen met een

karakteristieke kromme of een deel daarvan. Wel kan in één of meer punten P de raaklijn aan de projectie samenvallen met de raaklijn aan de karakteristieke kromme in P .

Zoals bekend, worden in verband met het al of niet reëel zijn van de karakteristieke krommen, de partiële differentiaalvergelijkingen van de tweede orde onderscheiden in hyperbolische, parabolische en elliptische differentiaalvergelijkingen. Alleen in het laatst genoemde geval zijn de karakteristieke krommen imaginair. Dan zal dus een gesloten kromme in het xy vlak in geen enkel punt aangeraakt kunnen worden door een karakteristieke kromme, hetgeen wel het geval kan zijn bij de hyperbolische en parabolische differentiaalvergelijkingen. Deze raakpunten zullen dan uitgezonderd moeten worden.

In het vervolg wordt niet verondersteld dat de differentiaalvergelijking noodzakelijk elliptisch moet zijn. Voor dit geval is de in dit artikel behandelde iteratiemethode uit praktisch oogpunt wel het meest aangewezen.

4e. In vele praktische vraagstukken betreffende de elliptische partiële differentiaalvergelijkingen is de kromme $x = 0$, waarop $z(0, y)$ gegeven is, gesloten.

Laten wij weer aannemen, dat binnen het gebied G door $x = 0$ omsloten, de functies a, b, c, d, e en f voldoen aan de eisen in het voorgaande genoemd.

Als de integraal $z(x, y)$ op G bepaald is door haar waarden voor $x = 0$ en de eis dat zij en haar partiële afgeleiden tot de tweede orde binnen G doorlopend moeten zijn, kan de functie $\left(\frac{\partial z}{\partial x}\right)_{x=0}$ niet willekeurig gegeven zijn, maar moet ze zo bepaald worden, dat een dergelijke integraal $z(x, y)$ kan worden verkregen.

§ 2.

In plaats van (1) beschouwen wij de twee partiële differentiaalvergelijkingen van de eerste orde:

$$\begin{aligned} (a) \quad & \frac{\partial z}{\partial x} = a \frac{\partial u}{\partial y} + \beta_1 \frac{\partial z}{\partial y} + \gamma_1 z + \delta \\ (b) \quad & \frac{\partial u}{\partial x} = \beta_2 \frac{\partial z}{\partial y} + \gamma_2 z. \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \dots \quad (4)$$

Ook nu zijn $a, \beta_1, \gamma_1, \delta, \beta_2$ en γ_2 functies van x en y .

Elimineren wij u uit (4), dan wordt gevonden:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} = & \beta_1 \frac{\partial^2 z}{\partial y \partial x} + a \beta_2 \frac{\partial^2 z}{\partial y^2} + \left(\gamma_1 + \frac{1}{a} \frac{\partial a}{\partial x} \right) \frac{\partial z}{\partial x} + \left(\frac{\partial \beta_1}{\partial x} + a \gamma_2 - \frac{\beta_1}{a} \frac{\partial a}{\partial x} + \right. \\ & \left. + a \frac{\partial \beta_2}{\partial y} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial \gamma_1}{\partial x} + a \frac{\partial \gamma_2}{\partial y} - \frac{\gamma_1}{a} \frac{\partial a}{\partial x} \right) z + \left(\frac{\partial \delta}{\partial x} - \frac{\delta}{a} \frac{\partial a}{\partial x} \right). \end{aligned} \quad (5)$$

Nu zullen (1) en (5) identiek zijn, indien de zes functies $a, \beta_1, \gamma_1, \delta, \beta_2$ en γ_2 voldoen aan de zes vergelijkingen:

$$\left. \begin{aligned} \beta_1 &= -b; \quad a\beta_2 = -a; \quad \gamma_1 + \frac{1}{a} \frac{\delta a}{\delta x} = -c; \\ \frac{\delta \beta_1}{\delta x} + a\gamma_2 - \frac{\beta_1}{a} \frac{\delta a}{\delta x} + a \frac{\delta \beta_2}{\delta y} &= -d; \quad \frac{\delta \gamma_1}{\delta x} - \frac{\gamma_1}{a} \frac{\delta a}{\delta x} + a \frac{\delta \gamma_2}{\delta y} = -e; \\ \frac{\delta \delta}{\delta x} - \frac{\delta}{a} \frac{\delta a}{\delta x} &= -f. \end{aligned} \right\} \quad (6)$$

Wij bepalen hiervan een particulier stelsel oplossingen.

De functie β_1 is onmiddellijk bekend, terwijl volgens de tweede en derde betrekking γ_1 en β_2 in a zijn uit te drukken. Elimineren wij daarna γ_1, β_2 en γ_2 uit de vierde en vijfde vergelijking, dan wordt een ingewikkelde partiële differentiaalvergelijking gevonden voor a , namelijk:

$$\left. \begin{aligned} \frac{\delta^2 a}{\delta x^2} + a \frac{\delta^2 a}{\delta y^2} + b \frac{\delta^2 a}{\delta x \delta y} - 2 \left(\frac{\delta a}{\delta x} \right)^2 - \frac{2a}{a} \left(\frac{\delta a}{\delta y} \right)^2 - \frac{2b}{a} \frac{\delta a}{\delta x} \frac{\delta a}{\delta y} - \\ - \frac{\delta a}{\delta x} \left(c - \frac{\delta b}{\delta y} \right) - \frac{\delta a}{\delta y} \left(d - \frac{\delta b}{\delta x} - \frac{2\delta a}{\delta y} \right) + a \left(\frac{\delta d}{\delta y} + \frac{\delta c}{\delta x} - e - \frac{\delta^2 b}{\delta x \delta y} - \frac{\delta^2 a}{\delta y^2} \right) = 0. \end{aligned} \right\} \quad (7)$$

Volgens de gegevens (zie § 1) bestaan alle coëfficiënten voor $0 \leq x \leq A$; $0 \leq y \leq B$ en zijn in ieder geval in x continu. Ten opzichte van y zijn alle coëfficiënten analytisch.

Deze partiële differentiaalvergelijking is ingewikkelder dan de partiële differentiaalvergelijking (1). Nu is het echter alleen nodig om een partiële oplossing voor a te bepalen, zodanig dat $a(x, y)$ en $\frac{\delta a}{\delta x}$ in y analytisch zijn en ten opzichte van x continu. Dan behoeft niet aan bepaalde randwaarden voldaan te zijn. Daarna kunnen de overige functies β_1, γ_1 enz. direct bepaald worden en ook hun partiële afgeleiden naar y van de k^{de} orde.

Zodra a bepaald is, is δ volgens de laatste betrekking van (6) onmiddellijk te bepalen als een oplossing van een gewone lineaire differentiaalvergelijking.

Wij kunnen uit (6), in verband met hetgeen in § 1 is verondersteld, $a, \beta_1, \gamma_1, \delta, \beta_2$ en γ_2 steeds zo bepalen, dat haar partiële afgeleiden naar y aan de voorwaarden voldoen, dat de afgeleiden naar y van de k^{e} orde na deling door $k!$ en een factor ϱ^k begrensd zijn en dus ook analytische functies in y zijn.

In het bijzondere geval dat:

$$\frac{\delta d}{\delta y} + \frac{\delta c}{\delta x} - e - \frac{\delta^2 b}{\delta x \delta y} - \frac{\delta^2 a}{\delta y^2} = 0 \quad \dots \quad (8)$$

zal $a = \text{constant} = k$ een particuliere oplossing zijn.

Dan is volgens (6):

$$\beta_1 = -b; \quad \beta_2 = -\frac{a}{k}; \quad \gamma_1 = -c; \quad \gamma_2 = -\frac{d}{k} + \frac{1}{k} \frac{\delta b}{\delta x} + \frac{1}{k} \frac{\delta a}{\delta y}; \quad \frac{\delta \delta}{\delta x} = -f.$$

Ook zal er vereenvoudiging optreden, als de functies a, b, c, d, e en f alleen van x of y afhankelijk zijn.

Het stelsel (6) kan dan vereenvoudigd worden door bijv. te veronderstellen, dat bij afhankelijkheid van x ook de functies $a, \beta_1, \gamma_1, \delta, \beta_2$ en γ_2 slechts van x afhankelijk zijn, zodat

$$\frac{\delta \beta_2}{\delta y} = \frac{\delta \gamma_2}{\delta y} = 0.$$

Dan zal a een particuliere oplossing moeten zijn van de gewone differentiaalvergelijking:

$$a \frac{d^2 a}{dx^2} - 2 \left(\frac{da}{dx} \right)^2 - c a \frac{da}{dx} + \left(\frac{dc}{dx} - e \right) a^2 = 0.$$

Ten slotte nog een opmerking over het geval, dat in (1) de functie a gelijk is aan nul. Dan kan het mogelijk zijn om (1) door één partiële lineaire differentiaalvergelijking van de eerste orde, nl. van de vorm (4a) te vervangen, waarbij dan $a = 0$. Volgens de tweede betrekking van (6) is nl. $a\beta_2 = 0$, dus of $\beta_2 = 0$ of $a = 0$.

Als $\beta_2 = 0$ en $a \neq 0$, is (1) weder door twee lineaire partiële differentiaalvergelijkingen van de eerste orde te vervangen. Zodra echter $a = 0$ kan zijn, kan (1) worden vereenvoudigd tot één lineaire partiële diff.verg. van de eerste orde, waarin een willekeurige functie van y voorkomt, die met behulp van de randwaarden (2) nader kan worden bepaald.

Dan moet verder nog deze lineaire partiële differentiaalvergelijking van de eerste orde worden opgelost.

In plaats van de partiële differentiaalvergelijking (1) en de gegeven randwaarden te beschouwen, trachten wij deze te vervangen door het stelsel vergelijkingen (4) met daarbij behorende randwaarden, op de wijze die in het voorgaande beschreven is.

Deze randwaarden worden dan als volgt bepaald:

Daar $z(0, y)$ en $\left(\frac{\partial z}{\partial x} \right)_{x=0}$ gegeven zijn, volgt uit (4a):

$$u^{(1)}(0, y) = \frac{1}{a(0, y)} \left(\frac{\partial z}{\partial x} \right)_{x=0} - \frac{\beta_1(0, y)}{a(0, y)} z^{(1)}(0, y) - \frac{\gamma_1(0, y)}{a(0, y)} z(0, y) - \delta(0, y). \quad (9)$$

Alle termen van het rechterlid zijn bekend, zodat dit ook het geval is met $u^{(1)}(0, y)$ en dus $u(0, y)$, waarbij aan de integratieconstante een willekeurige getallenwaarde gegeven kan worden. Tevens zijn alle afgeleiden naar y van $u(0, y)$ te berekenen.

Wij bepalen nu eerst de functies $z(x, y)$ en $u(x, y)$ zodanig, dat zij aan (4) voldoen, terwijl deze functies voor $x = 0$ overgaan in de bekende functies $z(0, y)$ en $u(0, y)$.

De functie $z(x, y)$ zal dan tevens aan (1) en de gestelde randwaarden voldoen.

§ 3.

In deze paragraaf wordt aangegeven hoe de functies $z(x, y)$ en $u(x, y)$, die aan (4) en gegeven randwaarden moeten voldoen, volgens een iteratiemethode kunnen worden bepaald.

De randwaarden $z(0, y)$ en $u(0, y)$ geven wij dan kortweg aan met z_0 en u_0 en beschouwen deze als gegeven functies.

Verder wordt onder $z_0^{(k)}$ de afgeleide naar y van de k^e orde verstaan enz.

Van al deze k^e afgeleiden naar y ($k = 1, \dots, n, \dots$) wordt weer verondersteld, dat zij na deling door $k!$ en ϱ^k begrensd zijn en dan kleiner zijn dan een bepaald getal M . Het verband met een bepaalde lineaire partiële differentiaalvergelijking van de 2^e orde, dat in de vorige paragraaf is aangenomen, wordt echter voorlopig buiten beschouwing gelaten.

Als eerste benadering van $z(x, y)$ en $u(x, y)$, \bar{z}_1 en \bar{u}_1 genaamd, wordt gesteld:

$$\left. \begin{aligned} \bar{z}_1 &= z_0 + z_1 = z_0 + u_0^{(1)} \int_0^x a dx + z_0^{(1)} \int_0^x \beta_1 dx + z_0 \int_0^x \gamma_1 dx + \int_0^x \delta dx \\ \bar{u}_1 &= u_0 + u_1 = u_0 + z_0^{(1)} \int_0^x \beta_2 dx + z_0 \int_0^x \gamma_2 dx. \end{aligned} \right\} \quad (10)$$

Zij geldt voor $0 \leq x \leq A$; $0 \leq y \leq B$.

Voor $x = 0$ is $\bar{z}_1 = z_0$, zodat \bar{z}_1 aan de gegeven randwaarden voldoet. In het vervolg zullen wij in de formules de integratiegrenzen 0 en x niet meer aangeven.

Laat in het algemeen:

$$\bar{z}_{n-1} = z_0 + \dots + z_{n-1} \quad \text{en} \quad \bar{u}_{n-1} = u_0 + u_1 + \dots + u_{n-1}$$

berekend zijn. Dan worden z_n en u_n als volgt gevonden:

Voor $\frac{\delta \bar{z}_n}{\delta x}$ en $\frac{\delta \bar{u}_n}{\delta x}$ wordt gesteld:

$$\begin{aligned} \frac{\delta \bar{z}_n}{\delta x} &= a \sum_{k=0}^{n-1} u_k^{(1)} + \beta_1 \sum_{k=0}^{n-1} z_k^{(1)} + \gamma_1 \sum_{k=0}^{n-1} z_k + \delta. \\ \frac{\delta \bar{u}_n}{\delta x} &= \beta_2 \sum_{k=0}^{n-1} z_k^{(1)} + \gamma_2 \sum_{k=0}^{n-1} z_k. \end{aligned}$$

Hieruit volgt:

$$\bar{z}_n = z_0 + \int a \left(\sum_{k=0}^{n-1} u_k^{(1)} \right) dx + \int \beta_1 \left(\sum_{k=0}^{n-1} z_k^{(1)} \right) dx + \int \gamma_1 \left(\sum_{k=0}^{n-1} z_k \right) dx + \int \delta dx,$$

$$\bar{u}_n = u_0 + \int \beta_2 \left(\sum_{k=0}^{n-1} z_k^{(1)} \right) dx + \int \gamma_2 \left(\sum_{k=0}^{n-1} z_k \right) dx.$$

Dus is:

$$\left. \begin{array}{l} z_1 = \int (\alpha u_0^{(1)} + \beta_1 z_0^{(1)} + \gamma_1 z_0 + \delta) dx, \\ z_2 = \int (\alpha u_1^{(1)} + \beta_1 z_1^{(1)} + \gamma_1 z_1) dx, \\ \vdots \\ z_n = \int (\alpha u_{n-1}^{(1)} + \beta_1 z_{n-1}^{(1)} + \gamma_1 z_{n-1}) dx \end{array} \right\} \dots \quad (11)$$

en:

$$\left. \begin{array}{l} u_1 = \int (\beta_2 z_0^{(1)} + \gamma_2 z_0) dx \\ u_2 = \int (\beta_2 z_1^{(1)} + \gamma_2 z_1) dx \\ \vdots \\ u_n = \int (\beta_2 z_{n-1}^{(1)} + \gamma_2 z_{n-1}) dx \end{array} \right\} \dots \quad (12)$$

In het bijzondere geval, dat z_{n-1} en u_{n-1} identiek gelijk aan nul zijn, is dat eveneens het geval met z_n en u_n enz.

In § 6 wordt aangetoond, dat als de functies a , b , enz. voldoen aan de voorwaarden in § 1 genoemd, voor de variabelen x en y intervallen bestaan, waarvoor de genoemde reeksen gelijkmataig convergeren en z en u oplossingen zijn van de differentiaalvergelijkingen (4), die tevens aan de gestelde randwaarden voldoen. Worden daarna in de formule voor z , de functies u_0 , u_1 enz. uitgedrukt in $\left(\frac{\partial z}{\partial x}\right)_{x=0}$ en $z(0, y)$ en haar afgeleiden naar y , dan wordt de functie gevonden, die aan (1) en de gegeven randwaarden voldoet.

Elimineren wij $u_{n-1}^{(1)}$ met behulp van (12) uit de uitdrukking voor z_n (zie (11)), dan wordt gevonden:

$$\left. \begin{array}{l} z_n = \int a \int \beta_2^{(1)} z_{n-2}^{(1)} dx^2 + \int a \int \beta_2 z_{n-2}^{(2)} dx^2 + \int a \int \gamma_2 z_{n-2}^{(1)} dx^2 + \\ + \int a \int \gamma_2^{(1)} z_{n-2} dx^2 + \int \beta_1 z_{n-1}^{(1)} dx + \int \gamma_1 z_{n-1} dx. \end{array} \right\} \quad (13)$$

In verband met de gegevens is het nl. geoorloofd om de differentiatie naar y en de integratie naar x te verwisselen.

Door tweemaal achtereenvolgens naar x te differentiëren, blijkt:

$$\left. \begin{array}{l} \frac{\partial^2 z_n}{\partial x^2} = \beta_1 \frac{\partial^2 z_{n-1}}{\partial y \partial x} + \alpha \beta_2 \frac{\partial^2 z_{n-2}}{\partial y^2} + \frac{1}{a} \frac{\partial a}{\partial x} \frac{\partial z_n}{\partial x} + \gamma_1 \frac{\partial z_{n-1}}{\partial x} + \\ + \left(-\frac{\beta_1}{a} \frac{\partial a}{\partial x} + \frac{\partial \beta_1}{\partial x} \right) \frac{\partial z_{n-1}}{\partial y} + \left(\alpha \frac{\partial \beta_2}{\partial y} + \alpha \gamma_2 \right) \frac{\partial z_{n-2}}{\partial y} + \\ + \left(\frac{\partial \gamma_1}{\partial x} - \frac{\gamma_1}{a} \frac{\partial a}{\partial x} \right) z_{n-1} + a \frac{\partial \gamma_2}{\partial y} z_{n-2}. \end{array} \right\} + \quad (14)$$

Volgens het gegeven is ook:

$$\frac{\partial^2 z_{n-1}}{\partial y \partial x} = \frac{\partial^2 z_{n-1}}{\partial x \partial y}.$$

Wordt ten slotte toch weer aangenomen, dat er tussen de vergelijkingen

(1) en (4) het verband bestaat, dat in § 2 is verondersteld, dan kan voor (14) geschreven worden:

$$\left. \begin{aligned} \frac{\delta^2 z_n}{\delta x^2} = & -b \frac{\delta^2 z_{n-1}}{\delta x \delta y} - a \frac{\delta^2 z_{n-2}}{\delta y^2} - (c + \gamma_1) \frac{\delta z_n}{\delta x} + \gamma_1 \frac{\delta z_{n-1}}{\delta x} + \\ & + \left(a \frac{\delta \beta_2}{\delta y} + a \gamma_2 \right) \frac{\delta z_{n-2}}{\delta y} - \left(d + a \gamma_2 + a \frac{\delta \beta_2}{\delta y} \right) \frac{\delta z_{n-1}}{\delta y} - \\ & - \left(e + a \frac{\delta \gamma_2}{\delta y} \right) z_{n-1} + a \frac{\delta \gamma_2}{\delta y} z_{n-2}. \end{aligned} \right\} . \quad (15)$$

Deze formule geldt voor $n \geq 2$.

Voor $n = 1$ luidt zij als volgt:

$$\frac{\delta^2 z_1}{\delta x^2} = -(c + \gamma_1) \frac{\delta z_1}{\delta x} - \left(d + a \gamma_2 + a \frac{\delta \beta_2}{\delta y} \right) \frac{\delta z_0}{\delta y} - \left(e + a \frac{\delta \gamma_2}{\delta y} \right) z_0 - f. \quad (15a)$$

Mathematics. — *A non-homogeneous inequality for integers in a special cubic field.* By A. V. PRASAD. (Second communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 26, 1949).

Lemma 9.

$$\alpha < \theta^4 - \theta^{-5}. \quad \dots \quad (48)$$

Proof. Suppose, if possible, that $\alpha \geq \theta^4 - \theta^{-5}$. Now by (16), (19) and (12)

$$\text{so } \alpha \leq \sqrt{\{\alpha|\beta|^2\theta^2}} \leq \theta \sqrt{(5+\delta)(1-\varepsilon)} < 3 < \theta^4,$$

$$|\alpha - \theta^4| < \theta^{-5}.$$

Hence, using (18) with $\xi = \theta^{-4}$,

$$|(\beta - \phi^4)(\bar{\beta} - \bar{\phi}^4)| \geq (1-\varepsilon)\theta^5,$$

so that

$$|\beta - \phi^4| \geq \theta^4 \sqrt{(1-\varepsilon)} > 2 \cdot 019,$$

if ε is sufficiently small. Thus

$$|\beta| \geq |\beta - \phi^4| - |\phi^4| > 2 \cdot 019 - \theta^{-2} > 2 \cdot 019 - 0 \cdot 570 = 1 \cdot 449. \quad (49)$$

But, by (16) and (12) and our supposition about α ,

$$\begin{aligned} |\beta| &= \sqrt{\left\{\frac{\alpha|\beta|^2}{\alpha}\right\}} \leq \sqrt{\left\{\frac{(5+\delta)(1-\varepsilon)}{\alpha}\right\}} \leq \sqrt{\left\{\frac{5+\delta}{\theta^4 - \theta^{-5}}\right\}} \\ &< \sqrt{\left\{\frac{5 \cdot 001}{2\theta^2 + \theta - 2}\right\}} < \sqrt{\left\{\frac{5 \cdot 001}{2 \cdot 834}\right\}} < 1 \cdot 4, \end{aligned}$$

contrary to (49). This contradiction proves the lemma.

6. The object of the next eight lemmas is to prove that α must be close to $\theta^2 + 1$ and that β must be close to $\phi^2 + 1$.

For this purpose it is convenient to work in polar coordinates. We write

$$\phi = r e^{i\psi} \quad \dots \quad (50)$$

$$\beta = d e^{i\chi} \quad \dots \quad (51)$$

where $r > 0$, $d > 0$ and ψ and χ are real.

We suppose that ψ is chosen so that $0 \leq \psi < 2\pi$, and that χ is chosen so that

$$3\psi - 2\pi \leq \chi < 3\psi \quad \dots \quad (52)$$

Then, as $\theta \phi \bar{\phi} = 1$ and $\theta + \phi + \bar{\phi} = 0$, we have

$$r = |\phi| = \theta^{-1},$$

and

$$\cos \psi = \frac{\phi + \bar{\phi}}{2|\phi|} = -\frac{1}{2}\theta^4 = -0.7623512\dots,$$

so that

$$\psi = 139^\circ 40' \cdot 31 \dots \dots \dots \dots \quad (53)$$

We note that

$$\begin{aligned} 2\psi &= 279^\circ 20' \cdot 6 \dots \\ 3\psi &= 419^\circ 0' \cdot 9 \dots \\ 4\psi &= 558^\circ 41' \cdot 2 \dots \\ 5\psi &= 698^\circ 21' \cdot 6 \dots \end{aligned} \quad (54)$$

We also write

$$\phi^2 + 1 = R e^{i\omega} \dots \dots \dots \dots \quad (55)$$

where $R > 0$ and $0 \leq \omega < 2\pi$. Taking $x = 1, y = 0, z = 1$ in (31) and (33) we have

$$(1 + \theta^2)(1 + \phi^2)(1 + \bar{\phi}^2) = 5 \dots \dots \dots \dots \quad (56)$$

Hence

$$R = \sqrt{(1 + \phi^2)(1 + \bar{\phi}^2)} = \sqrt{\left\{ \frac{5}{\theta^2 + 1} \right\}} = 1.3472054 \dots$$

and

$$\begin{aligned} \cos \omega &= \frac{2 + \phi^2 + \bar{\phi}^2}{2R} = \frac{2 + (\theta^2 + \phi^2 + \bar{\phi}^2) - \theta^2}{2R} \\ &= \frac{4 - \theta^2}{2R} = 0.833251 \dots \end{aligned}$$

Thus

$$\omega = 326^\circ 26' \cdot 0 \dots \dots \dots \dots \dots \dots \quad (57)$$

Lemma 10. For $n = 2, 3, 4$ or 5 ,

$$\begin{aligned} \cos(\chi - n\psi) &\leq \frac{1}{2} \left(\frac{a\theta^n}{5 + \delta} \right)^{\frac{1}{2}} \left\{ \frac{5 + \delta}{a} + \frac{1}{\theta^n} - \frac{1 - \varepsilon}{|a - \theta^n|} \right\} \\ &\quad - \frac{1}{2} \theta^{n/2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\}. \end{aligned} \quad (58)$$

Proof. For any rational integer n , we have

$$\begin{aligned} (\beta - \phi^n)(\bar{\beta} - \bar{\phi}^n) &= (d e^{i\chi} - r^n e^{in\psi})(d e^{-i\chi} - r^n e^{-in\psi}) \\ &= d^2 + r^{2n} - 2r^n d \cos(\chi - n\psi), \end{aligned}$$

so that

$$\cos(\chi - n\psi) = \{d^2 + r^{2n} - (\beta - \phi^n)(\bar{\beta} - \bar{\phi}^n)\}/2r^n d.$$

Using (18) with $\xi = \theta^{-n}$ we obtain, for all rational integers n ,

$$\begin{aligned} \cos(\chi - n\psi) &\leq \frac{1}{2r^n d} \left\{ d^2 + r^{2n} - \frac{1 - \varepsilon}{|a - \theta^n|} \right\} \\ &= \frac{\theta^{n/2}}{2d} \left\{ d^2 + \frac{1}{\theta^n} - \frac{1 - \varepsilon}{|a - \theta^n|} \right\}. \end{aligned} \quad (59)$$

Since, by Lemma 9,

$$\alpha < \theta^4 - \theta^{-5} = 2\theta^2 + \theta - 2 < 2\theta^2$$

we have

$$|\alpha - \theta^n| < \theta^n$$

for $n = 2, 3, 4$ or 5 . Thus if ϵ is sufficiently small

$$\frac{1}{\theta^n} - \frac{1-\epsilon}{|\alpha - \theta^n|} < 0, \quad \text{for } n = 2, 3, 4, \text{ or } 5. \dots \quad (60)$$

But by (51) and (16)

$$d^2 = \beta \bar{\beta} \leqslant \frac{5+\delta}{\alpha}.$$

Now (58) follows from this result together with (59) and (60).

Lemma 11. *If $\alpha \leq \theta^2 + 1$, then*

$$322^\circ < \chi < 333^\circ. \dots \quad (61)$$

Proof. By Lemma 8,

$$\alpha > \theta^3 + \theta^{-6} = 2\theta^2 - 1. \dots \quad (62)$$

Thus

$$d^2 \leqslant \frac{5+\delta}{\alpha} < \frac{5 \cdot 0001}{2\theta^2 - 1},$$

and by Lemma 10 and (60)

$$\cos(\chi - n\psi) < \theta^{n/2} \frac{1}{2} \left(\frac{2\theta^2 - 1}{5 \cdot 0001} \right)^{\frac{1}{2}} \left\{ \frac{5 \cdot 0001}{2\theta^2 - 1} + \frac{1}{\theta^n} - \frac{1-\epsilon}{|\alpha - \theta^n|} \right\},$$

for $n = 2, 3$ or 4 . Now numerically

$$\frac{1}{2} \left(\frac{2\theta^2 - 1}{5 \cdot 0001} \right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{2 \cdot 50975 \dots}{5 \cdot 0001} \right)^{\frac{1}{2}} < \frac{1}{2} (0 \cdot 50194)^{\frac{1}{2}} < 0 \cdot 35424,$$

and

$$\frac{5 \cdot 0001}{2\theta^2 - 1} = \frac{5 \cdot 0001}{2 \cdot 50975 \dots} < 1 \cdot 99263.$$

Also we have

$$\begin{aligned} |\alpha - \theta^2| &\leqslant 1, \\ |\alpha - \theta^3| &\leqslant \theta^2 + 1 - \theta^3 = \theta^{-3}, \\ |\alpha - \theta^4| &\leqslant \theta^4 - 2\theta^2 + 1 = \theta^{-2}. \end{aligned}$$

Thus, if ϵ is sufficiently small,

$$\begin{aligned} \cos(\chi - 2\psi) &< (1 \cdot 32472)(0 \cdot 35424) \{1 \cdot 99263 + 0 \cdot 56985 - 1\} \\ &< 0 \cdot 73323 < \cos 42^\circ 50', \end{aligned}$$

$$\begin{aligned} \cos(\chi - 3\psi) &< (1 \cdot 52471)(0 \cdot 35424) \{1 \cdot 99263 + 0 \cdot 43016 - 2 \cdot 32471\} \\ &< 0 \cdot 05298 < \cos 86^\circ 57', \end{aligned}$$

$$\begin{aligned} \cos(\chi - 4\psi) &< (1 \cdot 75488)(0 \cdot 35424) \{1 \cdot 99263 + 0 \cdot 32472 - 1 \cdot 75487\} \\ &< 0 \cdot 34967 < \cos 69^\circ 31'. \end{aligned}$$

Using the numerical values for 2ψ , 3ψ and 4ψ we see that it is not possible for χ to satisfy any of the inequalities

$$\begin{aligned} |\chi - 279^\circ 20' \cdot 6| &\leq 42^\circ 49' \cdot 9, \\ |\chi - 59^\circ 0' \cdot 9| &\leq 86^\circ 56' \cdot 9, \\ |\chi - 419^\circ 0' \cdot 9| &\leq 86^\circ 56' \cdot 9, \\ |\chi - 198^\circ 41' \cdot 2| &\leq 69^\circ 30' \cdot 9. \end{aligned}$$

Hence, by (52),

$$322^\circ < \chi < 333^\circ.$$

Lemma 12. If $a \geq \theta^2 + 1$, then

$$326^\circ < \chi < 327^\circ. \quad \dots \quad (63)$$

Proof. If $a \geq \theta^2 + 1$ we have

$$d^2 \leq \frac{5 + \delta}{a} < \frac{5 \cdot 0001}{\theta^2 + 1}.$$

Thus by Lemma 10 and (60),

$$\cos(\chi - n\psi) < \theta^{n/2} \frac{1}{2} \left(\frac{\theta^2 + 1}{5 \cdot 0001} \right)^{\frac{1}{2}} \left\{ \frac{5 \cdot 0001}{\theta^2 + 1} + \frac{1}{\theta^n} - \frac{1 - \varepsilon}{|\alpha - \theta^n|} \right\},$$

for $n = 3, 4$ or 5 . Now, numerically

$$\frac{1}{2} \left\{ \frac{\theta^2 + 1}{5 \cdot 0001} \right\}^{\frac{1}{2}} = \frac{1}{2} \left\{ \frac{2 \cdot 75487 \dots}{5 \cdot 0001} \right\}^{\frac{1}{2}} < \frac{1}{2} \{0 \cdot 55097\}^{\frac{1}{2}} < 0 \cdot 37114,$$

$$\frac{1}{2} \left\{ \frac{\theta^2 + 1}{5 \cdot 0001} \right\}^{\frac{1}{2}} = \frac{1}{2} \left\{ \frac{2 \cdot 75487 \dots}{5 \cdot 0001} \right\}^{\frac{1}{2}} > \frac{1}{2} \{0 \cdot 55096\}^{\frac{1}{2}} > 0 \cdot 37113,$$

and

$$\frac{5 \cdot 0001}{\theta^2 + 1} = \frac{5 \cdot 0001}{2 \cdot 75487 \dots} < 1 \cdot 81534.$$

Also, since $\theta^2 + 1 \leq a < \theta^4 - \theta^{-5}$,

$$|\alpha - \theta^3| < \theta^4 - \theta^{-5} - \theta^3 = 2\theta^2 - 3 < 0 \cdot 50976 < (1 \cdot 96170)^{-1},$$

$$|\alpha - \theta^4| \leq \theta^4 - \theta^2 - 1 = \theta^{-4},$$

$$|\alpha - \theta^5| \leq \theta^5 - \theta^2 - 1 = \theta.$$

Thus, if ε is sufficiently small,

$$\begin{aligned} \cos(\chi - 3\psi) &< (1 \cdot 52471)(0 \cdot 37114) \{1 \cdot 81534 + 0 \cdot 43016 - 1 \cdot 96170\} \\ &< 0 \cdot 16060 < \cos 80^\circ 45'. \end{aligned}$$

$$\begin{aligned} \cos(\chi - 4\psi) &< (1 \cdot 75487)(0 \cdot 37113) \{1 \cdot 81534 + 0 \cdot 32472 - 3 \cdot 07959\} \\ &< -0 \cdot 61190 < \cos 127^\circ 43'. \end{aligned}$$

$$\begin{aligned} \cos(\chi - 5\psi) &< (2 \cdot 01981)(0 \cdot 37114) \{1 \cdot 81534 + 0 \cdot 24513 - 0 \cdot 75487\} \\ &< 0 \cdot 97872 < \cos 11^\circ 50'. \end{aligned}$$

Using the numerical values for 3ψ , 4ψ and 5ψ , we see that it is not possible χ to satisfy any of the inequalities

$$\begin{aligned} |\chi - 59^\circ 0' \cdot 9| &\leq 80^\circ 44' \cdot 9, \\ |\chi - 419^\circ 0' \cdot 9| &\leq 80^\circ 44' \cdot 9, \\ |\chi - 198^\circ 41' \cdot 2| &\leq 127^\circ 42' \cdot 9, \\ |\chi - 338^\circ 21' \cdot 6| &\leq 11^\circ 49' \cdot 9. \end{aligned}$$

Hence, by (52)

$$326^\circ < \chi < 327^\circ.$$

Lemma 13. For $n = 2, 3, 4$ or 5 ,

$$\left. \begin{aligned} \cos(\chi - n\psi) &< \frac{1}{2} \left(\frac{a\theta^n}{5} \right)^{\frac{1}{2}} \left\{ \frac{5}{a} + \frac{1}{\theta^n} - \frac{1}{|a-\theta^n|} \right\} \\ &+ 1 \cdot 3\delta - \frac{1}{2} \left\{ \left(\frac{5+\delta}{a} \right)^{\frac{1}{2}} - d \right\}. \end{aligned} \right\} \quad . . . \quad (64)$$

Proof. It is clear from (12) that

$$\begin{aligned} \left(\frac{5+\delta}{a} \right)^{\frac{1}{2}} &< \left(\frac{5}{a} \right)^{\frac{1}{2}} (1 + \frac{1}{6}\delta), \\ \left(\frac{a}{5+\delta} \right)^{\frac{1}{2}} &> \left(\frac{a}{5} \right)^{\frac{1}{2}} (1 - \frac{1}{6}\delta). \end{aligned}$$

Using these inequalities together with (60) in (58) we obtain

$$\left. \begin{aligned} \cos(\chi - n\psi) &< \frac{1}{2} \left(\frac{a\theta^n}{5} \right)^{\frac{1}{2}} \left\{ \frac{5}{a} + \frac{1}{\theta^n} - \frac{1}{|a-\theta^n|} \right\} \\ &+ \frac{1}{10} \left(\frac{a\theta^n}{5} \right)^{\frac{1}{2}} \left\{ \frac{5}{a} - \frac{1}{\theta^n} + \frac{1-\varepsilon}{|a-\theta^n|} \right\} \delta \\ &+ \frac{1}{2} \left(\frac{a\theta^n}{5} \right)^{\frac{1}{2}} \frac{\varepsilon}{|a-\theta^n|} - \frac{1}{2}\theta^{n/2} \left\{ \left(\frac{5+\delta}{a} \right)^{\frac{1}{2}} - d \right\}, \end{aligned} \right\} \quad . . . \quad (65)$$

for $n = 2, 3, 4$ or 5 . Now since

$$\theta^3 + \theta^{-6} < a < \theta^4 - \theta^{-5},$$

we have

$$\left. \begin{aligned} \left(\frac{a\theta^n}{5} \right)^{\frac{1}{2}} \left\{ \frac{5}{a} - \frac{1}{\theta^n} + \frac{1-\varepsilon}{|a-\theta^n|} \right\} \\ < \left(\frac{\theta^4 \cdot \theta^5}{5} \right)^{\frac{1}{2}} \left\{ \frac{5}{\theta^3 + \theta^{-6}} - \frac{1}{\theta^5} + \theta^6 \right\} \\ < \left(\frac{3 \cdot 1 \times 4 \cdot 1}{5} \right)^{\frac{1}{2}} \left\{ \frac{5}{2 \cdot 5} - 0 \cdot 2 + 5 \cdot 5 \right\} \\ = (2 \cdot 542)^{\frac{1}{2}} (7 \cdot 3) < (1 \cdot 6) (7 \cdot 3) < 12. \end{aligned} \right\} \quad . . . \quad (66)$$

Now provided ε is sufficiently small (65) and (66) prove (64),

Lemma 14. If $\alpha \leq \theta^2 + 1$ and $n = 2$ or 3 , or if $\alpha \geq \theta^2 + 1$ and $n = 4$ or 5 , then

$$\cos(\chi - n\psi) < \cos(\omega - n\psi) + 1 \cdot 3\delta \\ - \frac{1}{6}|\alpha - \theta^2 - 1| - \frac{1}{2}\left\{\left(\frac{5+\delta}{\alpha}\right)^{\frac{1}{n}} - d\right\}. \quad \dots \quad (67)$$

Proof. We suppose that $n = 2, 3, 4$ or 5 and write

$$f_n(x) = \frac{1}{2} \left(\frac{x\theta^n}{5} \right)^{\frac{1}{n}} \left\{ \frac{5}{x} + \frac{1}{\theta^n} - \frac{1}{|x - \theta^n|} \right\}, \dots \quad (68)$$

for x satisfying

$$\theta^3 + \theta^{-6} < x < \theta^4 - \theta^{-5}. \quad \dots \quad (69)$$

Then

$$\begin{aligned} \frac{d}{dx} f_n(x) &= \frac{1}{2} \theta^{n/2} \left[-\frac{1}{2x} \left(\frac{5}{x} \right)^{\frac{1}{n}} + \frac{1}{2x} \left(\frac{x}{5} \right)^{\frac{1}{n}} \left\{ \frac{1}{\theta^n} - \frac{1}{|x - \theta^n|} \right\} \right. \\ &\quad \left. + \left(\frac{x}{5} \right)^{\frac{1}{n}} \frac{1}{(x - \theta^n)|x - \theta^n|} \right] \\ &= \frac{1}{4} \theta^{n/2} \left(\frac{x}{5} \right)^{\frac{1}{n}} \frac{1}{x(x - \theta^n)} \times \left[\frac{2x}{|x - \theta^n|} - \left(\frac{5}{x} - \frac{1}{\theta^n} + \frac{1}{|x - \theta^n|} \right) (x - \theta^n) \right] \end{aligned} \quad (70)$$

We consider two cases. First suppose that $n = 2$ or 3 and

$$2\theta^2 - 1 = \theta^3 + \theta^{-6} < x \leq \theta^2 + 1. \quad \dots \quad (71)$$

Then

$$\theta^{-6} < x - \theta^n \leq 1,$$

and by (70)

$$\begin{aligned} \frac{d}{dx} f_n(x) &> \frac{1}{4} \theta^{\left(\frac{1}{n}\right)\frac{1}{2}} \frac{1}{\sqrt{(\theta^2+1)}} \times \left[2(2\theta^2 - 1) - \left| \frac{5}{2\theta^2 - 1} - \frac{1}{\theta^3} \right| - 1 \right] \\ &> \frac{1}{4} \sqrt{\left\{ \frac{\theta^2}{5(\theta^2+1)} \right\}} \left[2(2 \cdot 5) - \frac{5}{2 \cdot 5} - 1 \right] \\ &> \frac{1}{4} \sqrt{\left\{ \frac{1}{5(1+0.6)} \right\}} \times [2] = \frac{1}{2\sqrt{8}} > \frac{1}{6}. \end{aligned}$$

Consequently

$$f_n(x) \leq f_n(\theta^2 + 1) - \frac{1}{6}|x - \theta^2 - 1| \quad \dots \quad (72)$$

if x satisfies (71). Thus, if $n = 2$ or 3 and $\alpha \leq \theta^2 + 1$, it follows from (64), (68) and (72) that

$$\begin{aligned} \cos(\chi - n\psi) &< \frac{1}{2} \theta^{n/2} \left(\frac{\theta^2 + 1}{5} \right)^{\frac{1}{n}} \left\{ \frac{5}{\theta^2 + 1} + \frac{1}{\theta^n} - \frac{1}{|\theta^2 + 1 - \theta^n|} \right\} \\ &\quad + 1 \cdot 3\delta - \frac{1}{6}|\alpha - \theta^2 - 1| \\ &\quad - \frac{1}{2} \left\{ \left(\frac{5+\delta}{\alpha} \right)^{\frac{1}{n}} - d \right\}. \end{aligned} \quad \dots \quad (73)$$

Now suppose that $n = 4$ or 5 and

$$\theta^2 + 1 \leq x < \theta^4 - \theta^{-5} = 2\theta^2 + \theta - 2. \dots \quad (74)$$

Then

$$\theta^{-5} < \theta^n - x \leq \theta^5 - \theta^2 - 1 = \theta,$$

and, by (70)

$$\begin{aligned} \frac{d}{dx} f_n(x) &< -\frac{1}{4} \theta^{2(\frac{1}{5})} \frac{1}{\sqrt{\{\theta^4 - \theta^{-5}\}}} \frac{1}{\theta} \\ &\times \left[\frac{2(\theta^2 + 1)}{\theta} + \left(\frac{5}{\theta^4 - \theta^{-5}} - \frac{1}{\theta^4} \right) (\theta^{-5}) + 1 \right] \\ &< -\frac{1}{4} \left(\frac{1}{5} \right)^{\frac{1}{5}} \frac{1}{\theta} [2(\theta + \theta^{-1}) + 1] \\ &< -\frac{2(1 \cdot 3 + 0 \cdot 7) + 1}{4(2 \cdot 3)(1 \cdot 4)} = -\frac{5}{12 \cdot 88} < -\frac{1}{8}. \end{aligned}$$

Consequently

$$f_n(x) \leq f_n(\theta^2 + 1) - \frac{1}{8} |x - \theta^2 - 1| \dots \quad (75)$$

if x satisfies (74). Thus, if $n = 4$ or 5 and $\alpha \geq \theta^2 + 1$, it follows from (64), (68) and (75) that (73) is satisfied.

We now show that (67) follows from (73) in the required cases by proving that

$$\cos(\omega - n\psi) = \frac{1}{2} \theta^{n/2} \left(\frac{\theta^2 + 1}{5} \right)^{\frac{1}{5}} \left\{ \frac{5}{\theta^2 + 1} + \frac{1}{\theta^n} - \frac{1}{|\theta^2 + 1 - \theta^n|} \right\}. \quad (76)$$

for $n = 2, 3, 4$ or 5 . Since

$$\phi = r e^{i\psi} \text{ and } \phi^2 + 1 = R e^{i\omega}$$

where

$$r = \frac{1}{\sqrt{\theta}} \text{ and } R = \sqrt{\left\{ \frac{1}{\theta^2 + 1} \right\}},$$

we have, as in the proof of Lemma 10,

$$\begin{aligned} \cos(\omega - n\psi) &= \{R^2 + r^{2n} - |\phi^2 + 1 - \phi^n|^2\}/2r^n R \\ &= \frac{1}{2} \theta^{n/2} \left(\frac{\theta^2 + 1}{5} \right)^{\frac{1}{5}} \left\{ \frac{5}{\theta^2 + 1} + \frac{1}{\theta^n} - |\phi^2 + 1 - \phi^n|^2 \right\}. \end{aligned} \quad (77)$$

Now

$$\theta^2 + 1 - \theta^2 = 1,$$

$$\theta^2 + 1 - \theta^3 = \theta^{-3},$$

$$\theta^2 + 1 - \theta^4 = -\theta^{-4},$$

$$\theta^2 + 1 - \theta^5 = -\theta,$$

and ϕ and $\bar{\phi}$ satisfy similar identities. Thus

$$|\theta^2 + 1 - \theta^n| \cdot |\phi^2 + 1 - \phi^n|^2 = 1,$$

for $n = 2, 3, 4$ and 5 . Consequently when n has one of these values the

expression on the right hand side of (77) reduces to that on the right hand side of (76). This proves (76) and completes the proof of the lemma.

Lemma 15. If $a \leq \theta^2 + 1$, then

$$|\sin \frac{1}{2}(\chi - \omega)| < \delta, \dots \dots \dots \quad (78)$$

$$|a - \theta^2 - 1| < 7 \cdot 8 \delta, \dots \dots \dots \quad (79)$$

and

$$\left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d < 2 \cdot 6 \delta, \dots \dots \dots \quad (80)$$

Proof. If $a \leq \theta^2 + 1$, it follows by Lemma 14, that

$$\cos(\chi - n\psi) < \cos(\omega - n\psi) + 1 \cdot 3 \delta - \frac{1}{6} |a - \theta^2 - 1| - \frac{1}{2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\},$$

for $n = 2$ and 3. Thus

$$\begin{aligned} & -2 \sin \{ \frac{1}{2}(\chi + \omega) - n\psi \} \sin \frac{1}{2}(\chi - \omega) \\ & < 1 \cdot 3 \delta - \frac{1}{6} |a - \theta^2 - 1| - \frac{1}{2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\}. \end{aligned} \quad (81)$$

for $n = 2$ and 3.

By Lemma 11 and the numerical values of ω and ψ we have

$$324^\circ < \frac{1}{2}(\chi + \omega) < 330^\circ,$$

so that

$$44^\circ < \frac{1}{2}(\chi + \omega) - 2\psi < 51^\circ,$$

$$-96^\circ < \frac{1}{2}(\chi + \omega) - 3\psi < -89^\circ.$$

Thus

$$2 \sin \{ \frac{1}{2}(\chi + \omega) - 2\psi \} > 2 \sin 44^\circ > 1 \cdot 3,$$

$$2 \sin \{ \frac{1}{2}(\chi + \omega) - 3\psi \} < -2 \sin 96^\circ < -1 \cdot 9.$$

These results together with (81) imply that

$$1 \cdot 3 |\sin \frac{1}{2}(\chi - \omega)| < 1 \cdot 3 \delta - \frac{1}{6} |a - \theta^2 - 1| - \frac{1}{2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\},$$

so that

$$1 \cdot 3 |\sin \frac{1}{2}(\chi - \omega)| + \frac{1}{6} |a - \theta^2 - 1| + \frac{1}{2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\} < 1 \cdot 3 \delta.$$

As the three terms on the left are non-negative, this inequality implies the inequalities (78), (79) and (80).

Lemma 16. If $a \geq \theta^2 + 1$ then

$$|\sin \frac{1}{2}(\chi - \omega)| < 3 \cdot 7 \delta, \dots \dots \dots \quad (82)$$

$$|a - \theta^2 - 1| < 7 \cdot 8 \delta, \dots \dots \dots \quad (83)$$

and

$$\left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d < 2 \cdot 6 \delta, \dots \dots \dots \quad (84)$$

Proof. If $a \geq \theta^2 + 1$, it follows from Lemma 14 that

$$-2 \sin \left\{ \frac{1}{2}(\chi + \omega) - n\psi \right\} \sin \frac{1}{2}(\chi - \omega) \\ < 1 \cdot 3\delta - \frac{1}{6}|a - \theta^2 - 1| - \frac{1}{2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\}. \quad (85)$$

for $n = 4$ and 5 .

By Lemma 12 and the numerical values of ω and ψ we have

$$326^\circ < \frac{1}{2}(\chi + \omega) < 327^\circ,$$

so that

$$127^\circ < \frac{1}{2}(\chi + \omega) - 4\psi + 2\pi < 129^\circ,$$

$$-13^\circ < \frac{1}{2}(\chi + \omega) - 5\psi + 2\pi < -11^\circ.$$

Thus

$$2 \sin \left\{ \frac{1}{2}(\chi + \omega) - 4\psi \right\} > 2 \sin 129^\circ > 1.5,$$

$$2 \sin \left\{ \frac{1}{2}(\chi + \omega) - 5\psi \right\} < -2 \sin 11^\circ < -0.38.$$

These inequalities together with (85) imply that

$$0.38 |\sin \frac{1}{2}(\chi - \omega)| < 1 \cdot 3\delta - \frac{1}{6}|a - \theta^2 - 1| - \frac{1}{2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\},$$

so that

$$0.38 |\sin \frac{1}{2}(\chi - \omega)| + \frac{1}{6}|a - \theta^2 - 1| + \frac{1}{2} \left\{ \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d \right\} < 1 \cdot 3\delta.$$

As the three terms on the left are non-negative, this inequality implies the inequalities (82), (83) and (84).

Lemma 17.

$$|\theta^2 + 1 - a| < 7.8\delta, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (86)$$

and

$$|\phi^2 + 1 - \beta| < 13\delta, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (87)$$

Proof. The inequality (86) follows immediately from Lemmas 15 and 16. By those lemmas we also have

$$|\sin \frac{1}{2}(\chi - \omega)| < 3.7\delta, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (88)$$

and

$$\left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - d < 2.6\delta. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (89)$$

Thus

$$|d - R| = \left| d - \left(\frac{5}{\theta^2 + 1} \right)^{\frac{1}{2}} \right| \\ \leqslant \left| d - \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} \right| + \left| \left(\frac{5 + \delta}{a} \right)^{\frac{1}{2}} - \left(\frac{5}{\theta^2 + 1} \right)^{\frac{1}{2}} \right| \\ < 2.6\delta + \left| \left(\frac{5 + \delta}{\theta^2 + 1 - 7.8\delta} \right)^{\frac{1}{2}} - \left(\frac{5}{\theta^2 + 1} \right)^{\frac{1}{2}} \right| \\ < 2.6\delta + \left(\frac{5}{\theta^2 + 1} \right)^{\frac{1}{2}} \left\{ (1 + \frac{1}{6}\delta) \left(1 + \frac{7.8\delta}{\theta^2 + 1} \right) - 1 \right\} \\ < 2.6\delta + 1.35 \left(\frac{5}{\theta^2 + 1} + \frac{8}{\theta^2 + 1} \right) \delta \\ < 7\delta.$$

Now, as in Lemma 10,

$$\begin{aligned} |\beta - \phi^2 - 1|^2 &= |de^{i\chi} - Re^{i\omega}|^2 \\ &= \{d^2 + R^2 - 2dR \cos(\omega - \chi)\} \\ &= (d-R)^2 + 4dR \sin^2 \frac{1}{2}(\omega - \chi) \\ &< 49\delta^2 + 4dR(3 \cdot 7)^2\delta^2 \\ &< 49\delta^2 + 4R(R+7\delta)(3 \cdot 7)^2\delta^2 \\ &< 169\delta^2. \end{aligned}$$

Consequently

$$|\beta - \phi^2 - 1| < 13\delta.$$

7. In this section we prove that $\alpha = \theta^2 + 1$ and $\beta = \phi^2 + 1$. We write

$$\alpha = \frac{\theta^2 + 1}{1 - \alpha'} , \quad \beta = \frac{\phi^2 + 1}{1 - \beta'} \quad (90)$$

and if $|\beta'| > 0$ we write

$$\beta' = \varrho e^{i\lambda} \quad (91)$$

where $\varrho > 0$ and λ is real.

Lemma 18. *For every rational integer n , we have*

$$|(a'(-\theta^2)^n - 1)(\beta'(-\phi^2)^n - 1)(\bar{\beta}'(-\bar{\phi}^2)^n - 1)| > 1 - \frac{1}{5}\delta. \quad (92)$$

Proof. It is clear that the number

$$\xi_n = \frac{1 - (-\theta^2)^n}{\theta^2 + 1} = \frac{1 - \{1 - (\theta^2 + 1)\}^n}{\theta^2 + 1} = (-\theta^2)^n \frac{\{1 - (\theta^2 + 1)\}^{-n} - 1}{\theta^2 + 1} \quad (93)$$

is an integer of $k(\theta)$ for all rational integers n . So, by (17) with $\xi = \xi_n$,

$$\left| \left(\alpha \frac{1 - (-\theta^2)^n}{\theta^2 + 1} - 1 \right) \left(\beta \frac{1 - (-\phi^2)^n}{\phi^2 + 1} - 1 \right) \left(\bar{\beta} \frac{1 - (-\bar{\phi}^2)^n}{\bar{\phi}^2 + 1} - 1 \right) \right| \geq 1 - \varepsilon,$$

for all rational integers n . Thus, by (90)

$$\left| \left(\frac{1 - (-\theta^2)^n}{1 - \alpha'} - 1 \right) \left(\frac{1 - (-\phi^2)^n}{1 - \beta'} - 1 \right) \left(\frac{1 - (-\bar{\phi}^2)^n}{1 - \bar{\beta}'} - 1 \right) \right| \geq 1 - \varepsilon,$$

so that, using (16) and (56),

$$\begin{aligned} &|(a'(-\theta^2)^n)(\beta'(-\phi^2)^n)(\bar{\beta}'(-\bar{\phi}^2)^n)| \\ &\geq (1 - \varepsilon) |(1 - \alpha')(1 - \beta')(1 - \bar{\beta}')| \\ &= (1 - \varepsilon) \frac{|(\theta^2 + 1)(\phi^2 + 1)(\bar{\phi}^2 + 1)|}{|\alpha \beta \bar{\beta}|} \\ &\geq \frac{5}{5 + \delta} > 1 - \frac{1}{5}\delta. \end{aligned} \quad \quad (94)$$

for all rational integers n . Replacing n by $-n$ and using the identity $\theta \phi \bar{\phi} = 1$ we obtain (92) from (94).

Lemma 19.

$$\alpha = \theta^2 + 1. \quad \dots \quad (95)$$

Proof. We have, by (90)

$$\alpha' = 1 - \frac{\theta^2 + 1}{\alpha}, \quad \beta' = 1 - \frac{\phi^2 + 1}{\beta}.$$

So, by Lemma 17,

$$|\alpha'| < \frac{7 \cdot 8 \delta}{\theta^2 + 1 - 7 \cdot 8 \delta} < \frac{7 \cdot 8 \delta}{2 \cdot 6} = 3 \delta. \quad \dots \quad (96)$$

and

$$|\beta'| < \frac{13 \delta}{|\phi^2 + 1| - 13 \delta} = \frac{13 \delta}{R - 13 \delta} < \frac{13 \delta}{1 \cdot 3} = 10 \delta. \quad \dots \quad (97)$$

Suppose, if possible, that $|\alpha'| > 0$. Then since

$$3 \delta < \theta^{-2},$$

it is clear that there will be two consecutive non-negative rational integral values of n satisfying

$$\theta^{-4} \leq |\alpha'(-\theta^2)^n| < 1.$$

So we can choose a non-negative rational integer n satisfying

$$\theta^{-4} \leq \alpha'(-\theta^2)^n < 1.$$

For this integer we have

$$|\alpha'(-\theta^2)^n - 1| \leq 1 - \theta^{-4},$$

and

$$|(\beta'(-\phi^2)^n - 1)(\bar{\beta}'(-\bar{\phi}^2)^n - 1)| \leq (1 + |\beta'|)^2 < (1 + 10 \delta)^2.$$

Thus

$$\left. \begin{aligned} & |(\alpha'(-\theta^2)^n - 1)(\beta'(-\phi^2)^n - 1)(\bar{\beta}'(-\bar{\phi}^2)^n - 1)| \\ & < (1 - \theta^{-4})(1 + 10 \delta)^2 \\ & < 1 - \theta^{-4} + 21 \delta. \end{aligned} \right\} \quad \dots \quad (98)$$

Since

$$22 \delta < \theta^{-4},$$

the inequalities (92) and (98) are incompatible for this rational integer n . This contradiction proves that $\alpha' = 0$. Now (95) follows from (90).

Lemma 20.

$$\beta = \phi^2 + 1. \quad \dots \quad (99)$$

Proof. Suppose, if possible, that $\beta' \neq 0$. Write

$$\mu = 2\psi - \pi = 99^\circ 20' \cdot 6 \dots \quad \dots \quad (100)$$

so that

$$-\phi^2 = \theta^{-1} e^{i\mu}.$$

Then it follows from Lemmas 18 and 19 and (91) that for every rational integer n ,

$$\left. \begin{aligned} 1 - \frac{1}{5} \delta &< |(\beta'(-\phi^2)^{-n}-1)(\bar{\beta}'(-\bar{\phi}^2)^{-n}-1)| \\ &= |(\varrho \theta^n e^{i(\lambda-n\mu)}-1)(\varrho \theta^n e^{-i(\lambda-n\mu)}-1)| \\ &= 1 + \varrho^2 \theta^{2n} - 2\varrho \theta^n \cos(\lambda-n\mu). \end{aligned} \right\} \quad \dots \quad (101)$$

Now, since $\varrho > 0$, there will be three consecutive rational integers n satisfying

$$\theta^{-3} \cos 86^\circ < \varrho \theta^n \leq \cos 86^\circ.$$

Further, it follows, from (100), that for at least one of these three consecutive integers

$$\cos(\lambda-n\mu) > \cos 86^\circ.$$

For this integer n ,

$$\begin{aligned} 2\varrho \theta^n \cos(\lambda-n\mu) - \varrho^2 \theta^{2n} \\ = \varrho \theta^n \{2 \cos(\lambda-n\mu) - \varrho \theta^n\} \\ > \theta^{-3} \cos 86^\circ \{2 \cos 86^\circ - \cos 86^\circ\} \\ = \theta^{-3} \cos^2 86^\circ > 0 \cdot 001 > \frac{1}{5} \delta, \end{aligned}$$

contrary to (101). This contradiction proves that $\beta' = 0$. Now (99) follows from (90).

8. Lemma 21. *If a, b, \bar{b} are of the form (9), where $\xi_1, \eta_1, \bar{\eta}_1$ are conjugate integers and $\sigma, \tau, \bar{\tau}$ are conjugate units of the fields $k(\theta), k(\phi), k(\bar{\phi})$ then there is no integer ξ of $k(\theta)$ with conjugates $\eta, \bar{\eta}$ satisfying*

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})| < \frac{1}{5},$$

but there are an infinite number of integers ξ of $k(\theta)$, with conjugates $\eta, \bar{\eta}$ satisfying

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})| = \frac{1}{5}, \dots \dots \dots \quad (102)$$

Proof. It is clearly sufficient to prove this lemma in the special case when

$$a = \frac{1}{\theta^2+1}, \quad b = \frac{1}{\phi^2+1}, \quad \bar{b} = \frac{1}{\bar{\phi}^2+1},$$

Then

$$\begin{aligned} |(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})| &= \left| \left(\xi - \frac{1}{\theta^2+1} \right) \left(\eta - \frac{1}{\phi^2+1} \right) \left(\bar{\eta} - \frac{1}{\bar{\phi}^2+1} \right) \right| \\ &= \left| \frac{N(\xi(\theta^2+1)-1)}{N(\theta^2+1)} \right| \\ &= \frac{1}{5} |N(\xi(\theta^2+1)-1)|. \end{aligned}$$

Now for any integer ξ of $k(\theta)$,

$$N(\xi(\theta^2 + 1)) = N(\theta^2 + 1)N(\xi) = 5N(\xi),$$

so that either

$$N(\xi(\theta^2 + 1)) = 0, \quad \text{or} \quad |N(\xi(\theta^2 + 1))| \geq 5.$$

So there is no integer ξ of $k(\theta)$ satisfying

$$\xi(\theta^2 + 1) = 1.$$

Consequently

$$|(\xi - a)(\eta - b)(\bar{\eta} - \bar{b})| = \frac{1}{5} |N(\xi(\theta^2 + 1) - 1)| \geq \frac{1}{5}$$

for every integer ξ of $k(\theta)$. This proves the first part of the lemma.

But, when $\xi = \xi_n$, where n is any rational integer and ξ_n is the corresponding integer of $k(\theta)$ given by (93),

$$\begin{aligned} & |(\xi_n - a)(\eta_n - b)(\bar{\eta}_n - \bar{b})| \\ &= \left| \left(\frac{-(-\theta^2)^n}{\theta^2 + 1} \right) \left(\frac{-(-\phi^2)^n}{\phi^2 + 1} \right) \left(\frac{-(-\bar{\phi}^2)^n}{\bar{\phi}^2 + 1} \right) \right| \\ &= \frac{1}{|N(\theta^2 + 1)|} = \frac{1}{5}. \end{aligned}$$

This completes the proof of the lemma and the theorem.

University College, London.

Mathematics. — SOMMERFELD's *Polynomial Method in the Quantum Theory*. By A. RUBINOWICZ. (Communicated by Prof. H. A. KRAMERS.)

(Communicated at the meeting of February 26, 1949.)

§ 1. *Scope of the paper.* A. SOMMERFELD¹⁾ has devised a method which enables us to treat many of the most simple but also most important eigenvalue problems in a very easy manner. His polynomial method starts with a splitting of the eigenfunction f of the originally given differential equation into two factors E and P

$$f = EP. \dots \dots \dots \quad (1)$$

The factor E takes care of the convergence of the normalization integral and of the fulfilment of the boundary conditions. The factor P however is assumed to be a polynomial, so that it does not disturb these properties of E . We suppose moreover that P is a solution of a differential equation of the second order with a recurrence formula containing only two terms, that is one of the form

$$(A_2 + B_2 x^h) x^2 \frac{d^2 P}{dx^2} + 2(A_1 + B_1 x^h) x \frac{dP}{dx} + (A_0 + B_0 x^h) P = 0. \quad (2)$$

A_i and B_i ²⁾ are constants and h is a positive integer.

In the present paper we will indicate the eigenvalue problems which can be treated by SOMMERFELD's polynomial method. In § 2 we suppose that the polynomial P in (1) is a solution of (2). Then a relation connecting the coefficients of the differential equation (2) and the original one determines the general form of the potential V which appears in the original differential equation. But the function V obtained in such a way contains the eigenvalue parameter λ and the coefficients A_i and B_i , which depend generally on λ . If V has a real physical meaning it can however not depend on λ . That means that there are some relations between λ and the coefficients A_i and B_i . Further relations follow if we require that E guarantees the fulfilment of boundary conditions. All these equations determine not only uniquely the potential function V but settle also the eigenvalues.

From this point of view we treat in the following sections some special problems completely. In § 3 we deal with the radial functions of the spherical symmetric case. In § 4 we discuss the differential equation of associated Legendre functions. In § 5 we start from the same differential equation as in § 4, but apply the polynomial method after a linear transformation of the independent variable. In this way we obtain the differential equation of JACOBI polynomials.

§ 2. *The connection between the original and the polynomial differential equations.* Going over from the "original" differential equation

$$f'' + 2af' + bf = 0 \dots \dots \dots \quad (1)$$

with the aid of (1.1) ³⁾ to the differential equation of the polynomials

$$P'' + 2aP' + \beta P = 0 \dots \dots \dots \quad (2)$$

we get between the coefficients of both the differential equations the relations

$$a = \frac{E'}{E} + a, \dots \dots \dots \dots \quad (3a)$$

$$\beta = \frac{E''}{E} + 2a \frac{E'}{E} + b. \dots \dots \dots \quad (3b)$$

Eliminating E we obtain a relation between the coefficients of both the differential equations (1) and (2)

$$a^2 + a' - b = a^2 + a' - \beta. \dots \dots \dots \quad (4)$$

We denote this expression in the following considerations by S .

We assume that the coefficients of (1) are real numbers and that (1) is the differential equation of an eigenvalue problem. (1) is then selfadjoint and has the form

$$\frac{d}{dx} \left(p \frac{df}{dx} \right) - qf + \lambda \varrho f = 0 \dots \dots \dots \quad (5)$$

λ denotes the eigenvalue parameter and $\varrho(x)$ the density function. It appears in the integral

$$\int_{x_1}^{x_2} f_2^* f_1 \varrho dx \dots \dots \dots \quad (6)$$

which decides in case of discrete eigenvalues about normalization and orthogonality of eigenfunctions. x_1 and x_2 are the boundaries of the fundamental interval.

Comparing (1) and (5) we obtain

$$a = \frac{1}{2} \frac{p'}{p} \dots \dots \dots \dots \quad (7a)$$

$$b = -\frac{q - \lambda \varrho}{p} \dots \dots \dots \dots \quad (7b)$$

According to (4) and (7) we can represent S in the form

$$S = \frac{1}{p} \left(\frac{1}{2} p'' - \frac{1}{4} \frac{p'^2}{p} + q - \lambda \varrho \right). \dots \dots \dots \quad (8)$$

so that q becomes

$$q = pS + \lambda \varrho + \frac{1}{4} \frac{p'^2}{p} - \frac{1}{2} p''. \dots \dots \dots \quad (9)$$

The form of the expression S which appears in (9) is known because we suppose that the differential equation for P is given by (1.2) and therefore

$$\alpha = \frac{1}{x} \frac{A_1 + B_1 x^h}{A_2 + B_2 x^h} \quad \beta = \frac{1}{x^2} \frac{A_0 + B_0 x^h}{A_2 + B_2 x^h} \quad \dots \quad (10)$$

so that according to (4)

$$S = \alpha^2 + \alpha' - \beta = \left. \begin{aligned} & \frac{(A_1 + B_1 x^h)[A_1 - A_2 + (B_1 - (h+1)B_2)x^h]}{x^2(A_2 + B_2 x^h)^2} \\ & - \frac{A_0 + (B_0 - h B_1)x^h}{x^2(A_2 + B_2 x^h)} \end{aligned} \right\} \quad (11)$$

Our final result is: If the differential equation (5) with given p and q is solvable by $f = EP, P$ being a solution of (1.2), q must be of the form (9) where S has the form (11).

We apply this proposition in cases where we can split the SCHROEDINGER equation

$$\Delta \psi + \kappa(\mathcal{E} - V)\psi = 0. \quad \kappa = \frac{2m}{\hbar^2}. \quad \dots \quad (12)$$

into a number of differential equations of the form (5). Both p and q are then completely determined by the coordinates used for the separation of the variables and q contains generally an expression given by the potential function V . By (9) and (11) are settled the forms of the q 's of all these differential equations therefore also the form of the potential function V .

A more exact determination of V we obtain by the demand that the coefficients of V can not depend on the eigenvalue parameter λ , if V has a real physical meaning. This takes place if the expressions (9) for q do not depend on λ^4). But this can be fulfilled only if the coefficients B_i which appear in S are functions of λ . To find out this dependence we can expand the expression

$$pS + \lambda \varrho \quad \dots \quad (13)$$

contained in (9) in a power series in powers of $x - x_0$ (x_0 arbitrary) eventually after multiplication with a function of x . On this occasion we also find, that only for distinct values of h the expression (13) can be made independent of λ .

Further conclusions as to the A_i and B_i and so as to the potential V we can draw from the boundary conditions for the eigenfunctions. As a rule the fundamental interval in the quantum theory is bounded by two singular points of the equation (5). In such points the solution has a tendency towards becoming infinite. From the mathematical point of view it is the task of the boundary conditions to suppress this tendency.

If we confine ourselves to the discrete eigenvalue spectrum we must claim that the integral (6) is convergent for any two eigenfunctions f_1 and f_2 , since otherwise we can not speak of their normalization or orthogonality.

Also in case of singular boundary points x_1 and x_2 we prove the orthogonality of two eigenfunctions f_1 and f_2 with the eigenvalues λ_1 and λ_2 by the aid of the well known relation

$$(\lambda_2 - \lambda_1) \int_{x_1}^{x_2} f_2^* f_1 \varrho dx = \lim_{x \rightarrow x_2} \Phi(x) - \lim_{x \rightarrow x_1} \Phi(x) \quad \dots \quad (14)$$

where

$$\Phi(x) = p(x) \left[f_2^*(x) \frac{df_1(x)}{dx} - f_1(x) \frac{df_2^*(x)}{dx} \right], \quad \dots \quad (15)$$

To apply (14) in this case in the usual manner we have to claim not only the convergence of the integral (6) appearing in (14) but also the existence and equality of both the limits

$$\lim_{x \rightarrow x_1} \Phi(x) = \lim_{x \rightarrow x_2} \Phi(x). \quad \dots \quad (16)$$

Physical arguments can stipulate to claim more, e.g. that (16) is zero or the eigenfunctions are periodic. But also mathematical motives can do it if e.g. the eigenfunctions have to form a complete set of orthogonal functions.

Both factors E and P which form f can facilitate or make more difficult the fulfilment of the boundary conditions. P is a product of x^σ with a "genuine" polynomial, beginning with a constant and consisting of integer powers of x^h . The exponent σ is given here by the determining fundamental equation

$$\sigma(\sigma-1)A_2 + 2\sigma A_1 + A_0 = 0. \quad \dots \quad (17)$$

But also the form of E is determined. According to (3a) we get

$$E = \exp(\int a dx).$$

From (7a) we find

$$\exp - \int a dx = \frac{\text{const}}{p^{1/h}}$$

whereas $\exp \int a dx$ may be evaluated by the use of (10). Carrying out this calculation we have to distinguish three cases, according to the disappearance or non-disappearance of the constants A_2 and B_2 . We get without an insignificant constant

(I) in case of $A_2 \neq 0, B_2 \neq 0$

$$E = p^{-1/h} x^{A_1/A_2} (A_2 + B_2 x^h)^{\frac{1}{h} \left(\frac{B_1}{B_2} - \frac{A_1}{A_2} \right)} \quad \dots \quad (18a)$$

(II) in case of $A_2 = 0, B_2 \neq 0$

$$E = p^{-1/h} x^{B_1/B_2} e^{-\frac{1}{h} \frac{A_1}{B_2} x^h}, \quad \dots \quad (18b)$$

(III) in case of $A_2 \neq 0, B_2 = 0$

$$E = p^{-1/h} x^{A_1/A_2} e^{\frac{1}{h} \frac{B_1}{A_2} x^h}, \quad \dots \quad (18c)$$

The role played by the singular and zero points of ϱ, p, P and E in the fulfilment of the boundary conditions we shall discuss when considering the different special cases.

To these relations we have to add SOMMERFELD's condition of breaking off the power series

$$(\sigma + n)(\sigma + n - 1)B_2 + 2(\sigma + n)B_1 + B_0 = 0 \dots \quad (19)$$

which makes of P a polynomial and determines the eigenvalues of λ in their dependence of an integer n , divisible by h .

Finally we indicate a very useful property of the polynomial equation (2). It does not change its form, if we multiply the polynomial P with a given power of x . Putting $E = x^\nu$ we get from (3)

$$a = a - \frac{\nu}{x}, \quad b = \beta - 2a \frac{\nu}{x} + \frac{\nu(\nu+1)}{x^2}.$$

We obtain therefore for $f = x^\nu P$ the differential equation (1), in which the coefficients a, b are of the form (10) of the coefficients a, β .

In both the cases (I) and (III) in which $A_2 \neq 0$ we shall use the abbreviations

$$a_i = \frac{A_i}{A_2}, \quad b_i = \frac{B_i}{A_2}.$$

§ 3. *Spheric symmetric field of force.* Splitting off from a Schrödinger eigenfunction a spherical harmonic we obtain for the radial function $R(x)$ the differential equation

$$\frac{d}{dx} x^2 \frac{dR}{dx} + \kappa \left[\mathcal{E}x^2 - Vx^2 - \frac{1}{\kappa} l(l+1) \right] R = 0 \dots \quad (1)$$

i.e. the differential equation (2.5) with

$$p = \varrho = x^2, \quad \lambda = \kappa \mathcal{E}, \quad q = \kappa V x^2 + l(l+1) \dots \quad (2)$$

V is here the potential function and l the azimuthal quantum number.

From (2) and (2.11) we obtain V in the form

$$V = \frac{S}{\kappa} - \frac{l(l+1)}{\kappa x^2} + \mathcal{E}. \dots \quad (3)$$

We assume that the fundamental interval is given by $0 < x < +\infty$ and use for our considerations the factor E first.

To make the normalizing integral (cp. (2.6))

$$\int_0^\infty x^2 R^* R dx \dots \quad (4)$$

convergent at its upper limit we have to assume case III i.e.

$$A_2 \neq 0, \quad B_2 = 0. \dots \quad (5)$$

According to (2.18c) and (2) E becomes

$$E = x^{\frac{A_1}{A_2} - 1} e^{\frac{1}{h} \frac{B_1}{A_2} x^h} \dots \quad (6)$$

and the convergence for $x \rightarrow +\infty$ requires $B_1/A_2 < 0$. Finally we assume

$$A_1 = A_2, \dots, \dots, \dots, \dots, \dots \quad (7)$$

to unite the x -power from E with the polynomial P .

For further considerations we use V . According to (2.11), (5) and (7) the expression S is given here by

$$S = \frac{1}{x^2} [-a_0 + ((h+1)b_1 - b_0)x^h + b_1^2 x^{2h}].$$

To free V , Eq. (3), from \mathcal{E} we have here only both the possibilities $h = 1$ or $= 2$. For $h = 1$ the potential becomes

$$V = \frac{c_{-2}}{x^2} + \frac{c_{-1}}{x} + c_0, \dots, \dots, \dots, \dots \quad (8)$$

where the constants

$$c_{-2} = -\frac{1}{\kappa}(a_0 + l(l+1)), \dots, \dots, \dots, \dots \quad (9a)$$

$$c_{-1} = \frac{1}{\kappa}(2b_1 - b_0), \dots, \dots, \dots, \dots \quad (9b), \quad c_0 = \mathcal{E} + \frac{b_1^2}{\kappa}, \dots, \dots, \dots, \dots \quad (9c)$$

are independent of \mathcal{E} . Therefore we obtain for V a Coulomb potential superposed by a potential inversely proportional to the square of the distance. The coefficients c_i in (8) are arbitrary because their dependence on A_l , B_l does not imply any connection between them.

To the potential (8) belongs the RYDBERG formula. From (2.19) and (5) follows

$$b_0 = -2b_1(n+\sigma), \dots, \dots, \dots, \dots, \dots, \dots \quad (10)$$

and hence, in accordance with (9b), $c_{-1} = \frac{b_1}{2}(n+\sigma+1)$ so that we obtain from (9c) in fact the RYDBERG formula

$$\mathcal{E} = -\frac{\kappa}{4} \frac{(c_{-1})^2}{(n+\sigma+1)^2} + c_0, \dots, \dots, \dots, \dots, \dots, \dots \quad (11)$$

Supposing further, that the Coulomb potential has the right constant $c_{-1} = -e^2 Z$ we obtain in (11) the RYDBERG constant. Eq. (9a) not yet used determines σ and hence the RYDBERG correction. From (2.17), (7) and (9a) we obtain

$$\sigma(\sigma+1) = -a_0 = l(l+1) + \kappa c_{-2}, \dots, \dots, \dots, \dots, \dots, \dots \quad (12)$$

In a pure Coulomb field, there is $c_{-2} = 0$ and therefore $\sigma = l$ (for $\sigma = -l-1$ the normalizing integral (4) is not convergent) so that we obtain the BALMER formula.

In case $h = 2$ we have

$$V = \frac{c_{-2}}{x^2} + c_2 x^2 + c_0$$

where

$$c_{-2} = -\frac{1}{x}(a_0 + l(l+1)) \dots \dots \dots \quad (13a)$$

$$c_2 = \frac{1}{x} b_1^2 \dots \quad (13b), \quad c_0 = \mathcal{E} + \frac{1}{x}(3b_1 - b_0) \dots \quad (13c)$$

The potential V is consequently given by a superposition of an elastic potential and of a potential inversely proportional to the square of the distance.

The dependence of \mathcal{E} upon the quantum numbers we obtain from (10), (13b) and (13c)

$$\mathcal{E} = -2 \frac{b_1}{x}(n + \sigma + \frac{3}{2}) + c_0 = 2 \sqrt{\frac{c_2}{x}(n + \sigma + \frac{3}{2}) + c_0}. \quad (14)$$

The positive sign of the square root is determined by (6).

If to the pure elastic field of force corresponds a frequency ω (in 2π sec), we have to put $c_2 = \frac{m}{2}\omega^2$. Like in case $h = 1$ the constant σ is given by (12).

In case of a pure elastic potential there is $c_{-2} = 0$ and therefore $\sigma = l$. For the eigenvalues of the spatial harmonic oscillator we obtain then

$$\mathcal{E} = (n + l + \frac{3}{2})\hbar\omega + c_0. \quad \dots \dots \dots \quad (15)$$

The general case (14) we can conceive now as (15) with a RYDBERG correction σ . The constant c_0 is in all the formulae of Legendre functions arbitrary and we can put $c_0 = 0$, if V is normalized as usual.

§ 4. The differential equation for associated Legendre functions. To have an example of an eigenvalue problem in a finite fundamental interval we generalize the equation for associated Legendre functions

$$\frac{d}{dx} \left((1-x^2) \frac{dK}{dx} \right) + \left(\lambda - \frac{m^2}{1-x^2} \right) K = 0$$

putting $V(x)$ for $m^2/(1-x^2)$

$$\frac{d}{dx} \left((1-x^2) \frac{dK}{dx} \right) + (\lambda - V(x)) K = 0. \quad \dots \dots \dots \quad (1)$$

We obtain hence the differential equation (2.5) with

$$p = 1-x^2, \quad q = 1, \quad q = V. \quad \dots \dots \dots \quad (2)$$

From (2.9) and (2) we get therefore

$$V = \frac{1}{1-x^2} + S(1-x^2) + \lambda. \quad \dots \dots \dots \quad (3)$$

For the fundamental interval we choose $-1 < x < +1$ and use first for our considerations the factor E , given by one of the Eqs. (2.18).

The zero points of p , which according to (2) are situated in $x = \pm 1$, endanger the convergence of the normalizing integral given according to (2.6) and (2) by

$$\int_{-1}^{+1} K^* K dx. \dots \dots \dots \quad (4)$$

because $p^{-\frac{1}{h}}$ appears in the factors E in all the cases (2.18). This danger can be eliminated only in case I, Eq. (2.18a), where

$$A_2 \neq 0, B_2 \neq 0 \dots \dots \dots \quad (5)$$

and where according to (2.18a) and (2)

$$E = x^{\frac{A_1}{A_2}} \frac{(A_2 + B_2 x^h)^{\frac{1}{h}} \left(\frac{B_1}{B_2} - \frac{A_1}{A_2} \right)}{(1-x^2)^{\frac{1}{h}}}.$$

To avoid E disturbing the convergence of the normalization integral (4) in the endpoints $x = \pm 1$ of the fundamental interval and to secure the finiteness of the expressions (2.15) the binomial $A_2 + B_2 x^h$ must be divisible by $1-x^2$. Hence follows that

$$A_2 = -B_2 \dots \dots \dots \quad (6)$$

and

$$h = \text{even integer.} \dots \dots \dots \quad (7)$$

Both our demands restrict also the variability of the exponent of $A_2 + B_2 x^h$; the demand that the expressions (2.15) are finite goes further and claims in accordance with (6) that

$$\frac{A_1 + B_1}{A_2} + 1 \equiv 0. \dots \dots \dots \quad (8)$$

To transpose $x^{\frac{A_1}{A_2}}$ from E to P we must put

$$A_1 = 0. \dots \dots \dots \quad (9)$$

Now let us use V . In accordance with (5), (6) and (9) we obtain from (2.11)

$$S = x^{h-2} b_1 \frac{-1 + (b_1 + h + 1)x^h}{(1-x^h)^2} - \frac{a_0 + (b_0 - hb_1)x^h}{x^2(1-x^h)}. \dots \quad (10)$$

Developing S in a power series we get from the expression $S p$ appearing in Eq. (3)

$$Sp = S(1-x^2) = -\frac{a_0}{x^2} + a_0 - M(x^{h-2}-x^h) - (M+N)(x^{2h-2}-x^{2h}) + \left. \begin{array}{l} \\ + (M+2N)(x^{3h-2}-x^{3h}) + \dots \end{array} \right\} \quad (11)$$

where $M = a_0 + b_0 - (h-1)b_1$, $N = -b_1(b_1 + h)$.

If V is independent of λ the coefficients of a development of V in a power series can also not depend on λ . Hence in accordance with (3) the coefficients of a power series development of

$$S(1-x^2) + \lambda \dots \dots \dots \quad (12)$$

must be independent of λ .

First we assume that $h \neq 2$, i.e. according to (7): $h = 4, 6, 8, \dots$. In accordance with (11) and (12) then the values of $a_0, a_0 + \lambda, M, N$ must be constant. But from the simultaneous constancy of a_0 and $a_0 + \lambda$ it follows that also the eigenvalue parameter λ is invariable. In such a case however our problem is not an eigenvalue problem.

For $h = 2$ we have to demand, that only the three quantities $a_0, a_0 - M + \lambda$ and N are constant. From this and the Eqs. (2.17) and (2.19) we could determine the dependence of the eigenvalues λ upon the quantum numbers. But we may come to this conclusion in an easier way.

For $h = 2$ we obtain from (3) and (10) for V the expression

$$V = \frac{A}{1-x^2} + \frac{B}{x^2} + C, \dots, \dots \quad (13)$$

where the constants are given by

$$A = (b_1 + 1)^2 \quad (14a), \quad B = -a_0 \quad (14b), \quad C = \lambda - (b_1^2 + b_1 + b_0). \quad (14c)$$

If we put $b_1 + 1 = -m$ and remark that from (2.18) it follows $b_0 = (\sigma + n)(\sigma + n - 1) - 2b_1(\sigma + n)$ we finally obtain from (14c)

$$\lambda = (\sigma + m + n)(\sigma + m + n + 1) + C. \dots \quad (15)$$

In accordance with (8) and (9), m is here a positive constant.

From (2.17) and (9) we obtain for σ the relation

$$\sigma(\sigma - 1) = -a_0 = B. \dots \dots \dots \quad (16)$$

By SOMMERFELD's polynomial method we can therefore solve a slightly more general differential equation than the one for the associated Legendre functions. This last differential equation we obtain by putting $B = 0$. In this case is $\sigma = 0$ or $= 1$. Remembering that n is an even integer ($h = 2!$) and hence $n + \sigma$ an arbitrary positive integer we see that (15) represents the well known eigenvalues of the differential equation for the associated Legendre functions.

To demonstrate by an example the simplifications caused by the transposition of the x -power from E to P , we indicate the results arrived at without the supposition (8). Instead of (11) we obtain an expression in which is substituted

$$a_0 - a_1(a_1 - 1) \text{ for } a_0, \quad M - a_1(2a_1 + 2b_1 + h - 1) \text{ for } M, \\ N - a_1(a_1 + 2b_1 + h) \text{ for } N.$$

But this does not alter the conclusion that $h = 2$.

We obtain V in the same way from (13) but have to substitute

$$A' = (a_1 + b_1 + 1)^2, \quad B' = B + a_1(a_1 - 1), \quad C' = C$$

for A, B, C .

Putting $a_1 + b_1 + 1 = -m$ so that we have again $A = m^2$ we obtain for the eigenvalues the expression

$$\lambda = (\sigma + a_1 + m + n)(\sigma + a_1 + m + n + 1) + C' \quad . . \quad (15')$$

where $\sigma + a_1$ is given by the equation

$$(\sigma + a_1)(\sigma + a_1 - 1) = B. \dots \dots \dots \quad (16')$$

The equations (15') and (16') we obtain from (15) and (16) if we write $\sigma + a_1$ instead of σ . But this changes only the notation.

§ 5. The differential equation of JACOBI polynomials. If SOMMERFELD's polynomial method is not applicable to a certain differential equation, we can try to give the latter a new form by a transformation of the independent variable and then to apply this method. We expect to succeed in this way from the fact, that in SOMMERFELD's polynomial equation (1.2) the zero point plays a distinguished role which is after a transformation taken over by another point of the fundamental interval. That means: If we replace in a "given" differential equation the independent variable by a new one and regard such an obtained equation as the "original" differential equation (2.1) or (2.5) we can generally solve the "given" differential equation by the polynomial method for other potentials V , as by direct application of this method to the "given" differential equation.

To verify this statement we use the differential equation of the associated Legendre functions (4.1) i.e.

$$\frac{d}{dx'} \left((1-x'^2) \frac{dK}{dx'} \right) + (\lambda - V(x')) K = 0 \dots \dots \quad (1)$$

where we have denoted the independent variable by x' . Substituting here by

$$x' = x - 1. \dots \dots \dots \quad (2)$$

the new independent variable x , we obtain the differential equation

$$\frac{d}{dx} x(2-x) \frac{dK}{dx} + (\lambda - V(x)) K = 0$$

which we will consider as the "original" equation of SOMMERFELD's polynomial method. It has the form (2.5) with

$$p = x(2-x), \quad q = 1, \quad \varrho = V \dots \dots \dots \quad (3)$$

so that according to (2.9) and (3) the potential V has the form

$$V = \frac{1}{x(2-x)} + Sx(2-x) + \lambda. \dots \dots \dots \quad (4)$$

In the variable x' the fundamental interval is bounded by ± 1 , in x it is therefore given by $0 < x < 2$.

For further conclusions we use first the factor E . We have to choose it in the form (2.18a) to guarantee finiteness of the normalizing integral

$$\int_0^2 K^* K dx. \dots \dots \dots \dots \quad (5)$$

Otherwise $p = x(2-x)$ endangers the convergence of (5) at its upper limit. Therefore we have to put $A_2 \neq 0$, $B_2 \neq 0$ and obtain

$$E = x^{\frac{A_1}{A_2}-1} \frac{(A_2 + B_2 x^h)^{\frac{1}{h}} \left(\frac{B_1}{B_2} - \frac{A_1}{A_2}\right)}{(2-x)^{\frac{1}{h}}}.$$

To guarantee the convergence of (5) for $x = 2$ we must suppose that the expression $A_2 + B_2 x^h$ is divisible by $2-x$, i.e. that

$$B_2 = -\frac{1}{2^h} A_2. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (6)$$

Further we have to assume according to the higher demands of (2.15) that

$$2 \left[\frac{1}{h} \left(\frac{B_1}{B_2} - \frac{A_1}{A_2} \right) - \frac{1}{2} \right] + 1 = \frac{2}{h} \left(\frac{B_1}{B_2} - \frac{A_1}{A_2} \right) > \frac{1}{2}. \quad \dots \quad (7)$$

Finally the removal of the x -power from E to P gives

$$A_1 = \frac{1}{2} A_2. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (8)$$

To use V for further considerations we remark that according to (6) and (8) the expression S is given by

$$S = -\frac{(1+my^h)(1+ny^h)}{16y^2(1-y^h)^2} - \frac{a_0}{4} \frac{1+py^h}{y^2(1-y^h)}$$

where $y = x/2$ and

$$m = 2^{h+1} b_1, \quad n = -2^{h+1} b_1 - 2(h+1), \quad p = \frac{2^h(b_0 - h b_1)}{a_0}.$$

Developing S in a power series in y we obtain for the expression $Sx(2-x)$ appearing in Eq. (4) for V

$$\begin{aligned} Sx(2-x) = 4Sy(1-y) = -(&a_0 + \frac{1}{4}) \left(\frac{1}{y} - 1 \right) + M(y^{h-1} - y^h) + \\ &+ (M+N)(y^{2h-1} - y^{2h}) + (M+2N)(y^{3h-1} - y^{3h}) + \dots \end{aligned}$$

where

$$M = \frac{h}{2} - a_0 - 2^h(b_0 - h b_1), \quad N = \frac{1}{4} [2h - 1 + 2^{h+1} b_1 (2^{h+1} b_1 + 2h + 2)].$$

To fix the value of h we can now use the demand, that the coefficients of $Sx(2-x) + \lambda$ (cp. (2.13) and (4)) are independent of λ . For $h = 2, 3, 4 \dots$ we have to claim that $a_0 + 1/4$, $a_0 + 1/4 + \lambda$, M , N are constant, so that λ would be constant.

For $h = 1$ the expressions

$$a_0 + \frac{1}{4}, \quad a_0 + \frac{1}{4} + M + \lambda, \quad N$$

only have to be constant. But this means that

$$a_0 = \text{const}, \quad b_1 = \text{const} \quad (9a), \quad \lambda = -M + \text{const} = 2b_0 + \text{const} \quad (9b)$$

λ can now depend on quantum numbers because b_0 is not constant now.

To consider the case $\hbar = 1$ in detail we remark that according to (2.11), (4), (6) and (8) the potential V has the form

$$V = \frac{A}{x-2} + \frac{B}{x} + C$$

where

$$A = -8b_1(b_1+1) - 2 \quad (10a), \quad B = -2a_0 \quad (10b), \quad C = \lambda - 2(b_0 + b_1 + 2b_1^2). \quad (10c)$$

The relations (9a) follow also from (10a) and (10b) and the relation (9b) follows from (10c).

Reintroducing by (2) again the variable x' , we obtain V in the form

$$V = \frac{A}{x'-1} + \frac{B}{x'+1} + C = \frac{x'(A+B) + A-B}{x'^2-1} + C.$$

But (1) represents with this V the differential equation of JACOBI polynomials. It is therefore situated at the limit of the applicability of SOMMERFELD's polynomial method.

In the quantum theory of a spinning symmetrical top we have to do with this equation with

$$A = -\frac{1}{2}(\tau - \tau')^2, \quad B = \frac{1}{2}(\tau + \tau')^2 \dots \quad (11)$$

where τ and τ' are positive or negative integers.

Using that we have according to (2.19) and (6)

$$(\sigma + n)(\sigma + n-1) - 4(\sigma + n)b_1 - 2b_0 = 0,$$

we obtain from (10) the eigenvalues

$$\lambda = (\sigma + n - 2b_1)(\sigma + n - 2b_1 - 1) + C.$$

According to (2.17), (8) and (10) we have to calculate σ from

$$\sigma^2 = -a_0 = -\frac{1}{2}B. \quad \dots \quad (12)$$

Supposing especially the case (11) we get from (10) according to (7):

$b_1 = -\frac{1}{2} + \frac{|\tau - \tau'|}{2}$ and from (12) in accordance with the fact that $\sigma > 0$

(otherwise we would obtain for $x' = -1$ an inadmissible singularity):

$\sigma = \frac{|\tau - \tau'|}{2}$. Hence we get the well known result

$$\lambda = (n + \tau^*)(n + \tau^* + 1) + C$$

where $\tau^* = \frac{|\tau + \tau'|}{2} - \frac{|\tau - \tau'|}{2}$ is the larger of both the integers $|\tau|$ and $|\tau'|$.

REFERENCES.

1. A comprehensive treatment was given by A. SOMMERFELD in Atombau und Spektrallinien, Vol. II, Braunschweig 1939, cp. p. 716.
2. SOMMERFELD denotes our coefficients $2A_1$ and $2B_1$ by A_1 and B_1 .
3. (a, b) means equation b of section a.
4. Compare however (3.1) where q depends on l .

Biochemistry. — Elastic-viscous oleate systems containing KCl. IV. The flow properties as a function of the shearing stress at 15° and constant KCl concentration. By H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG.

(Communicated at the meeting of February 26, 1949.)

1. Introduction.

An orientating investigation¹⁾ on the viscous behaviour of KCl containing oleate sols at 25° C. was performed with a viscometer which worked with constant hydrostatic head. The maximum shearing stress (at the wall of the capillary) was therefore nearly constant (of the order of 40 dynes/cm²)²⁾. As PHILIPPOFF³⁾ has shown that under certain conditions Na oleate sols may show "structural viscosity", it seemed safe to adopt for a further investigation of the viscous behaviour of the KCl containing oleate systems methods which allow of a wide variation of the shearing stress. In Part I, II and III of this series⁴⁾ the elastic properties of these systems have been studied at 15° C (at which temperature the damping of elastic oscillations is in general much smaller than at 25° C) and at a favourable KCl concentration (at or very near to the minimum damping at the given temperature). We chose therefore the same conditions for the present work and performed all experiments with "Naoleinicum, medicinale pur.pulv. Merck", which is present in the KCl (+ KOH) medium in a relatively low concentration.

The first impression these systems make on the observer is that they are gels. The definition of gels often includes the presence of a yield value. Our first aim will therefore be to investigate if a yield value can be detected, which involves the study of the flow phenomena at low shearing stresses.

2. Measurements at very low shearing stresses.

The viscometer used (see fig. 1) is in principle that of MICHAUD⁵⁾, whose characteristic feature is that slight level differences are obtained by immersion of a glass rod (in our case 0.55 cm diameter) into one of the

¹⁾ H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 849, 1011 (1947).

²⁾ Dimensions of the capillary: $R = 0.0425$ cm, $l = 24$ cm; constant hydrostatical head $h = 43.1$ cm.

³⁾ W. PHILIPPOFF, Viskosität der Kolloide, Dresden (1942). Cf. p. 116, 122, 279—280 and fig. 61 (p. 128) and fig. 162 (p. 280).

⁴⁾ H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Part. I: **51**, 1197 (1948); Part II: **52**, 15 (1949); Part III: **52**, 99 (1949).

⁵⁾ F. MICHAUD, Ann. de Phys. **19**, 63 (1923).

wide tubes (we used 3.6 cm diameter) and that the movement in the capillary of the system investigated, is observed in the microscope.

For the viscometer used a level difference of 0.0239 cm is obtained by screwing the glass rod downwards or upwards over a distance of one centimeter. With the aid of the formula $P = R \cdot p / 2l$ we obtain from the dimensions of the capillary ($R = 0.0425$ cm; $l = 24$ cm) and from the

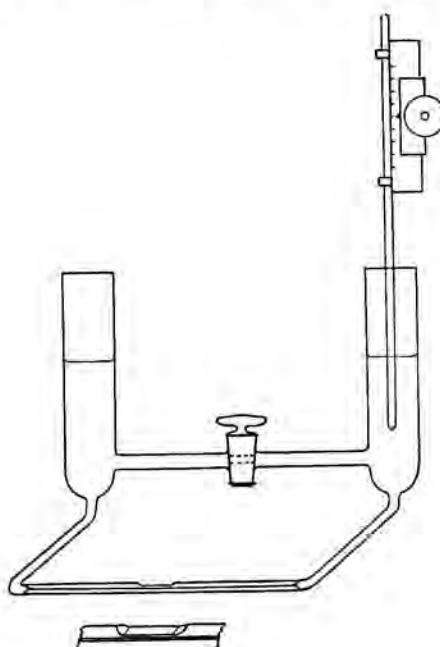


Fig. 1.

density of the soap system ($\rho = 1.066$) a value of 0.0169 dynes/cm² for P , the shearing stress at the wall of the capillary for one centimeter immersion of the rod.

Before starting a series of measurements the stopcock in the horizontal tube connecting the wide vertical tubes is closed and a certain quantity of the oleate system is brought into the left hand wide tube. With the aid of suction the capillary and the horizontal tubes connected with it are next completely filled with the oleate system. The apparatus is then put into its definite position, the capillary plus attached horizontal tubes ⁶⁾ being immersed in a tray (standing on the stage of the microscope and provided with a glass window at the bottom) through which flows water of constant temperature ⁷⁾. Now the glass rod is moved into such a position that after

⁶⁾ Only these parts of the apparatus need be at the desired temperature, because the very high viscosity of the oleate systems practically excludes all convection currents.

⁷⁾ Running tapwater was used, which practically did not change in temperature during the measurements (constant within 0.2° C.).

completely filling the apparatus it will be immersed some 5 cm into the oleate system. This original position of the glass rod will be called position "zero".

The levelling of the oleate system and the drainage of the walls of the wide tubes above the levels (which during the manipulations have become wetted by the oleate system) is then obtained by opening the stopcock (wide bore) for a sufficiently long time (we took 2—3 hours).

One is not sure, however, that after closing the stopcock exact levelling has really been accomplished (e.g. on account of very high viscosity of the system or on account of any presence of a yield value).

But the position of the glass rod which corresponds to real levelling follows from the measurements themselves⁸⁾.

The latter consist in measuring the velocity of displacement (v) of suspended particles in the axis of the capillary at various positions (p) of the glass rod. These positions are varied into both directions (reckoned from position "zero"), so that a number of values p are obtained in which the system in the capillary flows to the left and another number in which it flows to the right. Providing these p and v values with + or — signs and putting them in a graph, a symmetrical figure must result, the symmetrical point of which lies on the abscis axis (position of the glass rod) and represents that position of the rod at which real levelling exists. Various forms of such symmetrical figures can be imagined, from which a few are given under a , b , c and d in fig. 2. The vertical, dotted line represents the

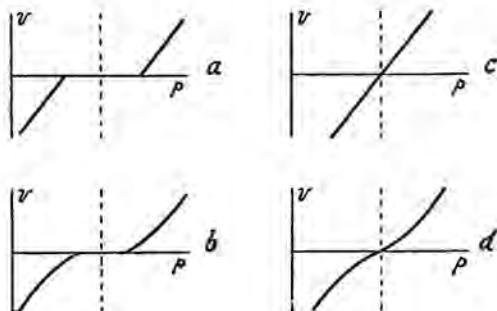


Fig. 2.

position of the rod at real levelling (which will in general differ slightly from position "zero").

In a and b a yield value is present, in c and d it is absent (or smaller than the experimental error). In c the velocity of flow is proportional to

⁸⁾ This position must remain constant during a series of measurements. This will only reasonably be the case if the stopcock after filling the apparatus has been left open for a sufficiently long time to allow the drainage of the walls. Otherwise a gradual shift of this position will occur during a series of measurements.

the shearing stress, in *a* it is a linear function of the latter and in *b* and *d* this function is more complicated.

Two oleate systems were investigated with the technique described above. They were made by mixing 50 and 125 cc respectively of a stock solution (10 gr. Na oleate + 500 cc H₂O + 50 cc KOH 2 N) with 75 cc H₂O + 75 cc KCl 3.8 N and with 75 cc KCl 3.8 N respectively. Practically the systems differ only in the final concentration of the oleate (0.45 and 1.14 gr per 100 cc). The electrolyte concentration (1.43 N KCl + + 0.05 N KOH or 1.43 N KCl + 0.11 N KOH) is such that the damping of the elastic oscillations is near to its minimum value. The mixtures had now to be provided with particles, that would indicate the velocity of flow.

We will discuss this problem together with our experiments on the 0.45 % oleate system. In preparing the mixture it was thoroughly shaken and had to be left to itself for a considerable time to become approximately free from air, though some small bubbles still happened to be present. It was now tried if such entrapped air bubbles could be used as "indicators" of the velocity of flow in the axis of the capillary. An ideal indicator should have the following properties:

- a. it shall not alter the properties of the oleate system;
- b. the dimension of the particles (bubbles or drops) of the indicator shall be small in comparison with that of the capillary;
- c. its particles, etc. shall neither rise nor sink during the measurement;
- d. a sufficient number of particles should always be present in the microscopic image, so that one can select one in the axis of the capillary for each single measurement.

For entrapped air bubbles only a. is fulfilled. Very small bubbles are absent or relatively rare (contrary to b. and d.), so that one must select larger ones, which, however, are all rising upwards (contrary to c.). Soon after the beginning of a series of measurements no air bubbles are present any longer in the axis of the capillary and all lie now pressed at the upper part of the glasswall of the capillary.

The use of air bubbles is therefore not possible for indicating the velocity of flow in the axis of the capillary. Nevertheless it was decided to start a series of measurements on such an air bubble. There were no indications of the presence of a three-phase contact (air — soapsystem — glass), so that it was supposed that the air bubble could freely move along the glass boundary. If this assumption is correct, we would get information on the mean velocity of flow of the oleate system in a zone⁹⁾ close to the wall as function of the shearing stress and the results could be compared

⁹⁾ This zone of course depends on the relative diameter of bubble and radius of the capillary. It is further necessary that the bubble shall not alter its diameter during the whole series of experiments. In our case this requisite was nearly fulfilled.

with measurements in the axis of the capillary which were to be performed with a more appropriate indicator later on.

The results are given in fig. 3 in which v is expressed in scale-divisions of the eye piece micrometer per 100 seconds. The order in which the

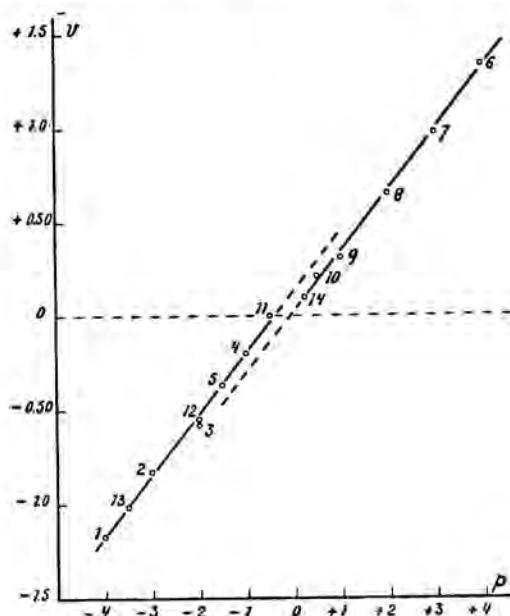


Fig. 3.

successive measurements have been taken is given by the numbers next to the experimental points. The figure obtained is of the type represented in fig. 2a, which would mean that there is a simple type of functional linear relation between rate of flow and shearing stress in the investigated range considered (0 — 0.07 dynes/cm 2) as well as a yield value. The latter expressed in dynes/cm 2 can be calculated from half the horizontal distance of the two parallel straight lines through the experimental points, bearing in mind that every cm change in the position of the glass-rod corresponds with 0.0169 dynes/cm 2 . The horizontal distance being approximately 0.35 cm change in position of the rod, we obtain a yield value in the order of only 0.003 dynes/cm 2 .

But it may be seriously doubted if a yield value of the oleate system itself is really indicated by fig. 3. It is very well conceivable, that the assumption of wholly free movability of the air bubble along the glasswall (as we did) is not true and that a certain difference in pressure between the two sides of the air drop must be exceeded before the air bubble itself "yields" from its original position (pressed against the glasswall).

The measurements of the velocity of flow in the axis of the capillary were performed by using paraffine oil as an indicator. We added two drops

of paraffine oil, which slowly dripped from a 5.5 mm diameter glass rod held vertically into 200 cc of the 0.45 % oleate system. This was shaken vigourously in order to emulsify the paraffine oil. The oleate system being full of entrapped air was then left standing till the next day, to become free from air bubbles.

The system was turbid in consequence of the paraffine oil having emulsified into very small drops, which were extremely useful for indicating the rate of flow in the axis of the capillary. Indeed, paraffine oil is the nearest approach to the ideal indicator, fulfilling the requirements mentioned sub b., c. and d. on page 366.

As to exigence a. (according to preliminary experiments) the influence on the period of the rotational oscillation was found to be very slight or negligible. The velocity of flow was measured with a stopwatch by checking the time necessary for a very small drop (diameter ± 0.2 scale divisions) to move over a number of scale divisions of the micrometer, and by calculating from this the number of scale divisions per 100 sec. See table I (upper part) and fig. 4, from which it appears that:

- a. There is no indication of the presence of a yield value;
- b. the velocity of flow in the axis of the capillary is proportional to the shearing stress at the glasswall¹⁰⁾.

Comparing fig. 4 with fig. 3 (in which the same arbitrary units for the velocity of flow have been used) we perceive from the far smaller inclination of the straight lines in the latter figure, that the mean velocity in a zone next to the wall (in which zone the air bubble moved) is much smaller than in the axis of the capillary. Unfortunately we did not measure the dimension of the air bubble and of the capillary (the diameter of the bubble may have been in the order of $1/5$ of the capillary diameter), so that no quantitative calculations can be made to examine whether this difference in rate of flow is compatible with laminar flow. Except for any future indications to the contrary it seems probable that in the range of small maximum shearing stresses (0 — approximately 0.07 dynes/cm²) the oleate system, although it shows marked elastic properties, behaves as a Newtonian liquid.

We therefore conclude that our oleate systems, although they make the impression of being gels, can be described better by the term: "elastic fluids".

¹⁰⁾ Fig. 4 deviates slightly from the simple scheme of fig. 1c, in as much the experimental points do not lie exactly on one straight line. Instead of it the points lie on two straight lines, with a slightly different inclination, meeting each other at the same point of the horizontal level at $v = 0$. In connection with the order of the measurements (alternately a point with a positive v and a point with a negative v , and further choosing these points in such a way that their absolute values become increasingly smaller) the shape of fig. 3 can be explained by a slight gradual shifting to the left as has been discussed in note 8. The conclusions a. and b. in the text are of course not endangered by this circumstance.

TABLE I.

Velocity of flow (v , in scale divisions per 100 sec) in the axis of the capillary.

System investigated	Immersion depth (p) of the glassrod in cm, relative to the original position									
	+ 4	+ 3	+ 2	1 +	+ 0.5	- 0.5	- 1	- 2	- 3	- 4
0.45% oleate system at 15.4° (see fig. 4)	6.85 7.06	5.36 5.45	3.69 3.61	1.94 1.90	1.04 1.09	-0.65 -0.69	-1.55 -1.52	-3.32 -3.27	-5.22 -5.18	-6.96 -6.96
mean:	6.96	5.41	3.62	1.90	1.05	-0.65	-1.52	-3.29	-5.20	-6.96
1.14% oleate system at 14.4° (see fig. 5)	0.357 0.350 0.351	0.248 0.241 0.242	0.164 0.158 0.161					-0.202 -0.204 -0.195	-0.276 -0.259 -0.277	-0.351 -0.348 -0.349
mean:	0.351	0.244	0.161					-0.200	-0.271	-0.349

The results obtained with the 1.14 % oleate system on the whole are the same. Because of the much higher viscosity of this more concentrated oleate system the drainage factor becomes of great importance now. (We waited 3 hours for levelling with open stopcock). The influence of this factor becomes plainly visible during the measurements, for after each displacement of the glassrod it will now take a long time until the new position of the levels is exactly reached in consequence of the slow drainage of the walls of the tubes and of the glassrod after the latter has been moved up, and of slowly renewed positions of the menisci against these

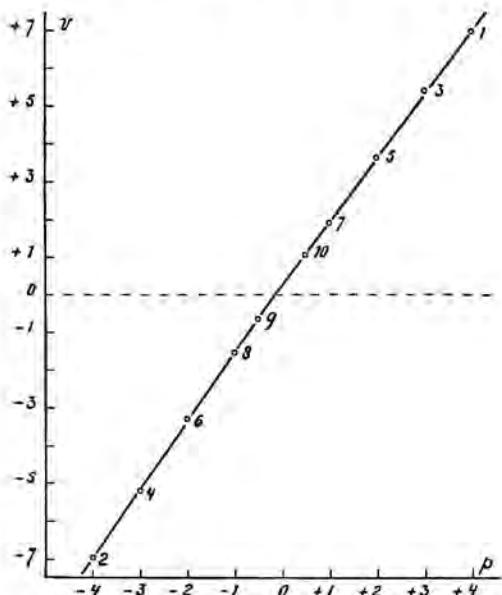


Fig. 4.

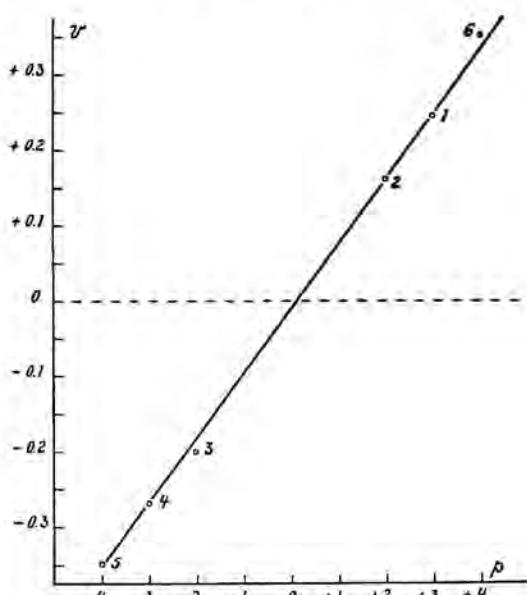


Fig. 5.

glass-surfaces after the glassrod has been lowered). The experimental results show this influence of the drainage factor (see table II and fig. 5). The apparently irregular positions of the experimental points around the straight line drawn can be accounted for from the above if we take the order of the separate measurements into consideration (indicated by the figure next to each experimental point).

We perceive, for instance, that the greatest deviation of a point from the straight line occurs if the glassrod has been displaced over a great distance just before the measurement. See point 3 and point 6, which both show too high a velocity (as can be expected from the above). Taking these disturbing influences of slow drainage and of slowly renewed wetting into consideration, we feel justified in concluding that also in the case of the 1.14 % oleate system no yield value is present and that the velocity of flow is proportional to the maximum shearing stress at the wall. In the domain of very low shearing stresses ($0-0.07 \text{ dynes/cm}^2$) this oleate system, since it shows very marked elastic properties and at the same time behaves as a Newtonian liquid, should not be described as a gel but as an elastic fluid.

3. Measurements at higher shearing stresses.

An investigation of the flow behaviour in a large range of much higher shearing stresses (order of $1-1000 \text{ dynes/cm}^2$) showed that the simple flow behaviour we met at very low shearing stresses (section 2) is lost.

In principle we followed the method described by PHILIPPOFF¹¹⁾ and calculated from the experimental data the quantities V ($= 4Q/\pi R^3$), the mean velocity of flow expressed in cc/sec., and P ($= R.p/2L$), the shearing stress at the wall of the capillary expressed in dyne/cm².

A survey of the flow behaviour is then obtained by drawing a curve — the flow rate curve — through the experimental points in a diagram in which $\log V$ is represented as a function of $\log P$.

But for the differences mentioned below, the same device was adopted for observing the velocity of displacement of a drop of petroleum in a calibrated tube placed horizontally. Differences: a. only one capillary was used, and as a consequence a number of tubes of different diameters for the drops of petroleum; b. the viscometer was calibrated with a Newtonian liquid of known viscosity, from which measurement the radius R of the capillary (its length being known = 11.8 cm) was calculated (which value $R = 0.0542 \text{ cm}$ was needed for the calculation of the quantities V and P mentioned above).

The use of only one capillary, though not in every respect a happy choice¹²⁾, made it possible to exclude, that the peculiar shape of the flow

¹¹⁾ W. PHILIPPOFF, Kolloid Z., 75, 155 (1936).

¹²⁾ It is of course not to be expected that with one capillary it will be possible to measure the rate of flow accurately in the large tract of P -values used.

At very high P -values and low viscosity the rate of flow of the oleate system is so great

rate curve, obtained with oleate systems containing KCl, was somehow connected with the successive use of capillaries of different diameters. The curves obtained in b. made it further possible to control that in our viscometer the Newtonian calibrating liquids showed, indeed, a quite normal behaviour.

Fig. 6 gives in a $\log V$ vs $\log P$ diagram the flow rate curves of the two calibration liquids used (60 % sucrose at 15° and a mineral oil purchased

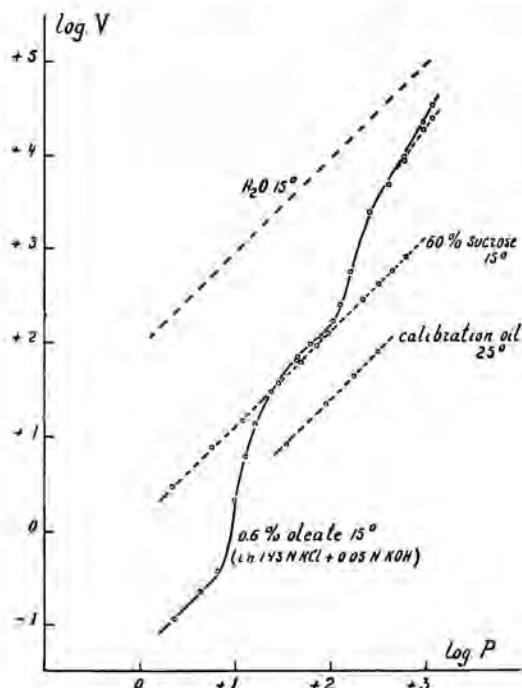


Fig. 6.

from and calibrated by the laboratory of the "Bataafsche Petroleum Maatschappij", Amsterdam¹³⁾ and the flow rate curve of an approximately 0.6 % oleate system (containing 1.43 N KCl + 0.05 N KOH) at 15° C.

that the correction for the kinetic energy (HAGENBACH) had to be made (see note 14). At very low P -values the rate of flow of the oleate system is extremely small and a capillary tube had to be used for the drop of petroleum. The displacement of this latter drop (read off on the scale divisions of an eye piece micrometer) was observed in the telescope of a cathetometer. Certain difficulties, which seriously interfered with the accurate measurements of the rate of flow of very high viscous oleate systems, will be discussed in note 16.

¹³⁾ The mineral oil had a kinematic viscosity of 4,424 Stokes at 25°. According to F. J. BATES and associates, Polarimetry, Saccharimetry and the Sugars, circular of the National Bureau of Standards, C 440, Washington 1942, see table p. 671, the 60 % sucrose solution must have a viscosity of 80.1 centipoises at 15°. The sucrose used was recrystallized and from the flow data obtained with the two calibration liquids, followed a viscosity of 79.6 ± 0.7 centipoises of the 60 % sucrose solution which agrees well with the value given above.

The curves (dotted ones) of the two calibration liquids, as is to be expected, are straight lines with a slope of practically 45° .

The curve for the oleate system, although presenting a similar course at low values of $\log P$, considerably deviates with further increase of $\log P$ ¹⁴⁾. The shape of this curve, as we will see in the next communication of this series, is characteristic for the oleate systems containing KCl, which show marked elastic phenomena, although its position in the $\log V$ vs $\log P$ diagram still depends on a number of variables.

The typical flow behaviour is perhaps more conveniently represented by fig. 7, in which the abscissae are once more $\log P$ values, the ordinates, however, giving the values of $\log P - \log V$. The latter quantity has the meaning of the logarithm of a viscosity η (expressed in poises) at the given value of $\log P$. These curves will therefore be called viscosity-shear stress curves.

We see from fig. 7 curve A that our 0.6 % oleate system behaves as a Newtonian liquid ($\log P - \log V$ independent of $\log P$) at low values of

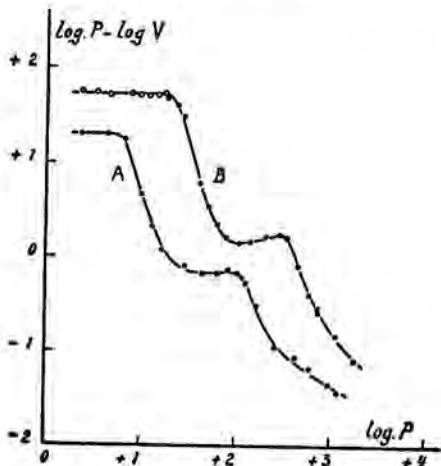


Fig. 7.

the shearing stress only. Obviously we have here the continuation of the simple flow behaviour we met in section 2, at much lower $\log P$ values.

The viscosity of the 0.6 % oleate system in fig. 7 in this range is of the order of 20 poises ($\log P - \log V = \pm 1.30$). However, with increasing $\log P$ values the viscosity decreases extremely strongly. At the highest P values investigated, its viscosity (of the order of 4 centipoises only) is still on its way downward.

It seems probable that at still higher $\log P$ values the viscosity-shear

¹⁴⁾ Application of the correction of HAGENBACH had a visible influence on the course of the flow rate curve in fig. 6 at the three highest P -values used. These corrected values lie on the drawn curve, those that have not been corrected on the dotted curve.

stress curve will reach asymptotically an end value, which, according to PHILIPPOFF, may be denoted by η_∞ . If we denote the highest horizontal level by η_0 , the peculiarity of our oleate systems containing KCl consists in the occurrence of an additional level¹⁵⁾ situated between η_0 and η_∞ . This level may be denoted by η_i ($i = \text{intermediate}$).

If it is permitted to interpret the viscosity-shear stress curve of the 0.6 % oleate system from the point of view of its structure, we obtain the following picture: At very low shearing stresses (0—0.07 dynes/cm², see section 2) there is a structure in the oleate fluid which allows a steady flow proportional to the shearing stress. This is still possible at slightly higher shearing stresses (the η_0 level in fig. 7) but above a certain value (in fig. 7 in the order of 8 dynes/cm²) this structure breaks down in two steps. In the first step the "viscosity" decreases from approximately 20 poises to approximately 0.7 poises, in the second step, beginning in fig. 7 at approximately 100 dynes/cm², it decreases to values which are only a few times larger than that of the corresponding KCl solution (order of 0.01 poise) without the oleate. It is further important to remark, that the two steep curve branches and the intermediate level of the viscosity curve are well defined. One may switch over to other P values situated far apart and into both directions without influencing the position of the flow rate curve. This means that if we spoke above of a structure in the oleate fluid, this structure is not one which breaks down irreversibly. On the contrary, the structure postulated is one which in accordance with the prevailing shearing stresses, practically immediately breaks down to or rebuilds itself to certain equilibrium states.

In this connection it seems of importance, that soaps belong to "Association Colloids" and it suggests that the very high viscosity at the η_0 level of rather diluted oleate systems is connected with large scale associations of the oleate molecules. Such associations in principle represent equilibria.

It might be supposed that these equilibria are influenced by sufficiently large shearing stresses and that the above mentioned "breaking down or rebuilding the structure" is connected with it.

Fig. 7, curve *B*, gives the results of measurements with an approximately 1.2 % oleate system at 15° C. The black dots represent measurements with the same viscometer used for the 0.6 % oleate system. At the low log P values the rate of flow of this far more viscous system becomes so small, that as a result of disturbances connected with the small radius of the capillary¹⁶⁾, it became impossible to obtain reliable measurements.

¹⁵⁾ A similar type of flow rate curve was found by PHILIPPOFF in 13,84 % Na-oleate solutions, containing NaOH and 1,62—7 % *m*-Kresol at 20°. See his book (quoted in note 2) on p. 279—280, fig. 162.

¹⁶⁾ The entrapped air between the meniscus of the oleate system and the drop of petroleum in the capillary makes a sensitive air thermometer. Slight fluctuations in the temperature of the thermostat reflect themselves in an irregular displacement of the

That here too a η_0 level (analogous to that of the 0.6 % oleate system) is actually present¹⁷⁾, follows from measurements (white dots) with a viscometer provided with a capillary of wider bore ($R = 0.1062$ cm).

Curve B shows the same general characteristics as curve A so that we need not discuss its form again. Increase of the oleate concentration has obviously two consequences, a. the values of the η levels become higher (for curve B $\eta_0 = \pm 52$ poises, $\eta_1 = \pm 1.5$ poises, η_∞ , not reached, lies below 0.08 poises), b. the values of the (max.) shearing stress at which "the structure breaks down" are greater (first step: order of 20 dynes/cm²; second step: order of 300 dynes/cm²)¹⁸⁾.

Comparing the curves A and B more closely we perceive, however, a difference in the position of the intermediate η "level". In B the experimental points do not lie on a horizontal line, but on one that is slightly sloping upwards to the right.

We are here dealing with "shear rate thickening", the viscosity increasing ($\pm 1.3 \rightarrow 1.7$ poises) with increasing shearing stress. This might indicate, that the "residual structure" left after the first step downwards (from the η_0 niveau) is modified at further increase of the shearing stress in such a way that it offers more resistance to flow, before definitely breaking up (fall of the curve towards the η_∞ level).

This shear rate thickening at the intermediate "level" is obviously facilitated by an adequate concentration of the oleate, which point will be of importance for a theory on the internal state of our oleate systems.

In the next communication, in which another viscometer will be used, we will meet with more examples of this shear rate thickening and once

petroleum drop (standstills with time and occasional temporary changes in the direction of displacement occurred). This irregular movement of the drop does not reflect a property of the oleate system; indeed in section 2, where we used still lower shearing stresses, but observed the flow of the oleate system itself in the capillary, we always found completely steady flow. It is still possible to investigate highly viscous oleate systems with the petroleum drop method, but one must for that end follow the displacement of the drop for a long time. Yet such a determination may be wrong, if apart from the relatively fast fluctuations in temperature, there happens to occur a slow and slight displacement (e.g. a few 0.01°) of the mean temperature of the thermostat during the single measurement. Therefore one ought to repeat such measurements many times, which would take many hours for the location of a single point on the η_0 level. It is easier to use a viscometer having a capillary of wider bore, which reduces the influence of the said disturbances on the measurements considerably (the quantity of fluid flowing through the capillary per second being proportional to the fourth power of the radius). The viscometer used to measure the η_0 level of the 1.2% system is, however, not suited to measure the whole flow curve with it.

¹⁷⁾ From the results of section 2 the occurrence of an η_0 level at low values of the shearing stresses was already certain (proportionality between rate of flow and shearing stress).

¹⁸⁾ The approximately 0.6 and 1.2% oleate systems used for these experiments are not wholly comparable in age and composition. It is therefore not allowed to compare them quantitatively, e.g. as regards the functional relation between η_0 and oleate concentration.

more we will find it clearly present at higher oleate concentrations (1.2 and 1.8 %) and absent at lower ones (0.6 and 0.3 %). This absence might of course be only apparent, as the method used is not accurate enough to detect a very slight shear rate thickening.

4. Certain particularities of the oleate systems containing KCl explained by the shape of their flow curves.

If we have a relatively large vessel (e.g. 1L) partially filled with an 1.2 % oleate system and try to pour out its contents, we have not the least difficulty in doing so. Certainly the fluid does not impress us as being extremely viscous. This is caused by sufficient shearing stresses being set up, so that the system does no longer behave as a fluid of ± 50 poises viscosity, but as one of materially lower viscosity. When the fluid is poured out, our working point is no longer on the η_0 level, but much lower.

In Part I (see section 12) we described the appearance of a dimple in the surface of the oleate system soon after excitation of the rotational oscillation. It is followed by a slight elevation of the surface, which may last some seconds (see fig. 6g and h in Part I). Now we can understand why this elevation may last several seconds. During the vigorous motion attending the formation of the dimple and its disappearance, the oleate fluid does not behave as a very high viscous system (too large shearing stresses).

A certain quantity of the oleate system is then brought above the horizontal level. As the large shearing stresses have meanwhile disappeared, a certain quantity of oleate system with the viscosity corresponding to the η_0 level, lies a fraction of a millimeter higher than the surrounding surface. So the shearing stresses which try to level out this elevation are very small, and therefore even its edge may remain visible for a couple of seconds, before it rounds off.

5. Summary.

1. The viscous behaviour of a few oleate systems containing KCl which showed marked elastic properties, has been investigated at 15° , with the techniques given by MICHAUD in the range of very small shearing stresses and by PHILIPPOFF in the range of larger shearing stresses.

2. In the domain of very small shearing stresses (shearing stresses at the wall of the capillary varying from 0—0.07 dynes/cm²) the oleate system showed a steady rate of flow proportional to the shearing stress.

3. A yield value could not be demonstrated. Our elastic oleate systems, which at first sight may make the impression of gels, can for this reason be better characterised as elastic fluids.

4. In the domain of larger shearing stresses (order 1—1000 dynes/cm²) flow rate curves were obtained, which show a very strong decrease in the viscosity (more than 500 times) in a two step process.

5. After the first step downwards (from the η_0 level) the viscosity retains over a certain range of shearing stresses, nearly the same value (the 0.6 % oleate system) or it increases slightly ("shear rate thickening; 1.2 % oleate system) before the second step downwards (towards the η_∞ level) sets in.

6. The sequence in which the shearing stress is varied does not change the position of the flow rate curve. If a structure is postulated in the original oleate system, which is broken down in a two step process, it must be one which, in accordance with the prevailing shearing stresses, allows a rapid and completely reversible rebuilding. In this connection some suggestions are made regarding the role of large scale associations of the oleate molecules.

7. A few particularities of the elastic oleate systems containing KCl described previously find an explanation from the shape of the flow rate curves.

*Department of Medical Chemistry,
University of Leiden.*

Biochemistry. — Elastic viscous oleate systems containing KCl. V¹⁾.

Viscous and elastic behaviour compared. By H. G. BUNGENBERG
DE JONG, H. J. VAN DEN BERG and L. J. DE HEER.

(Communicated at the meeting of March 26, 1949.)

Introduction.

According to MAXWELL's well known formula $\eta = G \times \lambda$, the product of the shear modulus G (in dynes/cm²) and the relaxation time λ (in sec.) must have the meaning of a viscosity coefficient (in poises).

In this communication measurements of the elastic and of the viscous behaviour have been performed side by side at 15° to compare the product $G \times \lambda$ with the viscosity levels η_0 , η_i and η_∞ , which according to the results obtained in Part IV of this series, occur in the viscosity shearing stress diagram of markedly elastic fluids containing KCl.

2) Methods used.

Previous to measuring the flow behaviour each oleate system was investigated as to its elastic properties, for which purpose we used the rotational oscillation in completely filled spherical vessels of different radii. For particulars see Parts I, II and III of this series. The method used for obtaining a survey of the flow behaviour was in principle the same as in Part IV, in which only one capillary and a number of tubes of different diameters were used to observe the rate of displacement of a drop of petroleum.

In connection with this choice it was no longer necessary to keep the original shape of PHILIPPOFF's viscometer. The capillary and the other tubes, all of Jena glass, were therefore sealed together, and instead of the original spherical reservoirs of relatively small capacity, we used relatively wide cylindrical reservoirs (see fig. 1).

The difference in level of the oleate system (or of the calibration liquids) could be easily read off by means of a cathetometer at the beginning and the end of each separate measurement. This mean hydrostatic pressure added to or subtracted from the air pressure applied (read off on a H₂O or Hg-manometer) gave the pressure over the capillary. From this corrected pressure we calculated the value P , the (maximum) shearing stress at the wall of the capillary, for which purpose
$$\left(P = \frac{R \cdot p}{2L} \right)$$
 the length of the capillary ($L = 11.8$ cm) and its radius must be known. This

¹⁾ Part I has appeared in these Proceeding 51, 1197 (1948), Parts II, III and IV in these Proceedings 52, 15, 99, 363 (1949).

radius ($R = 0.0716$ cm) was calculated from the calibration of the viscometer with the same Newtonian liquids of known viscosity as were used in Part IV.

The other quantity necessary for the plotting of flow rate curves, V , the mean rate of flow, was calculated from the rate of displacement of the drop of petroleum by using the formula $V = 4Q/\pi R^3$.

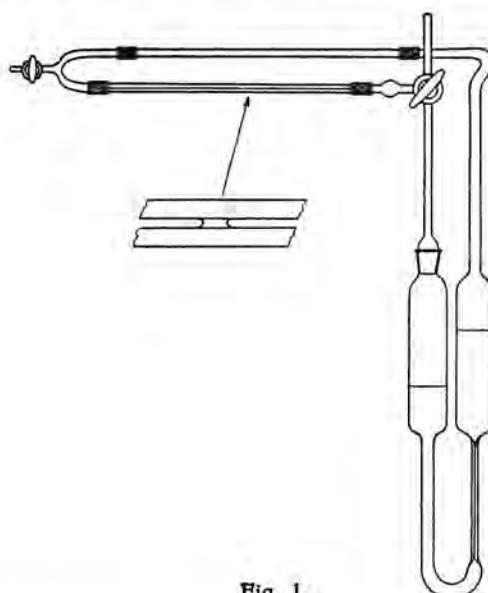


Fig. 1.

This time a wider radius of the capillary was chosen than in Part IV, as this diminishes the irregularities in the displacement of the drop of petroleum (cf. Part IV section 3, note 16), which do occur in the range of low shearing stresses with the very high viscous 1.2 and 1.8 % oleate systems. These irregularities could be further diminished by using in this range exclusively level differences between the oleate system in the cylindrical tubes and by closing the apparatus by a bent connecting tube as depicted in fig. 1. The drop of petroleum now separates two volumes of entrapped air (viz. between its own menisci and the menisci of the oleate system in the left and right reservoirs). Slight fluctuations in the temperature of the thermostat now effect slight variations in pressure on either side of the drop of petroleum which compensate one another partially. Although this device gave an improvement, nevertheless irregularities were still present, so that the measurements at low shearing stresses and very high viscosities showed a relatively bad reproducibility and the mean of a number of separate measurements may still embody a considerable error.

To check the conclusion in section 3, based on the results obtained with this viscometer, we will use in section 4 a technique by which the quantity of fluid flowing through the capillary is measured directly.

3) Comparison of the values $G \cdot \lambda$ with η_0 , η_i and η_∞ .

As in all the preceding parts of this series, we will also restrict ourselves in this section to a KCl (+ KOH) concentration, which at 15° C coincides or lies very near to the minimum damping of the elastic oscillations for the preparation of Na-oleate used (Na-oleinic acid med. pur. pulver. Merck).

TABLE I.

Measurements of the rotational oscillation with 1.8, 1.2, 0.6 and 0.3% oleate systems (containing 1.43 N KCl + 0.18 N KOH at 15° C).

Na oleate g/100 cc	R (cm)	n	$10 \times \frac{T}{2}$ (sec.)	A	$10 \times \frac{T}{2}$ corr.	λ (sec.)	G (dynes/cm ²)	$G \times \lambda$ (poises)
1.8	7.46	50.0	5.41	0.193	5.41	2.80	99.8	263
	4.92	57.8	3.57	0.137	3.57	2.61	99.7	
	4.12	65.3	2.97	0.124	2.97	2.40	101.0	
	7.46	—	5.40	0.201	5.40	2.69	100.2	
1.2	7.46	35.3	8.44	0.348	8.43	2.42	41.1	97
	4.92	39.5	5.60	0.246	5.60	2.27	40.6	
	4.12	41.6	4.55	0.196	4.55	2.33	43.0	
0.6	7.46	18.8	18.09	0.755	17.96	—	9.06	—
	5.01	19.0	12.35	0.757	12.26	—	8.77	
	4.12	19.1	10.03	0.763	9.96	—	8.98	
0.3	5.65	7.7	35.48	1.105	34.94	—	1.37	—
	5.03	8.0	31.53	1.105	31.05	—	1.38	
	4.12	8.4	24.48	1.112	24.11	—	1.53	
	3.18	9.0	20.75	1.105	20.44	—	1.27	
	1.70	9.4	10.03	1.109	9.88	—	1.55	

In a first series of measurements we investigated four oleate systems of different concentrations (1.8, 1.2, 0.6 and 0.3 g per 100 cc 1.43 N KCl + 0.18 N KOH). The results of the elastic measurements (rotational oscillation at 15° C) have been collected in Table I and represented in

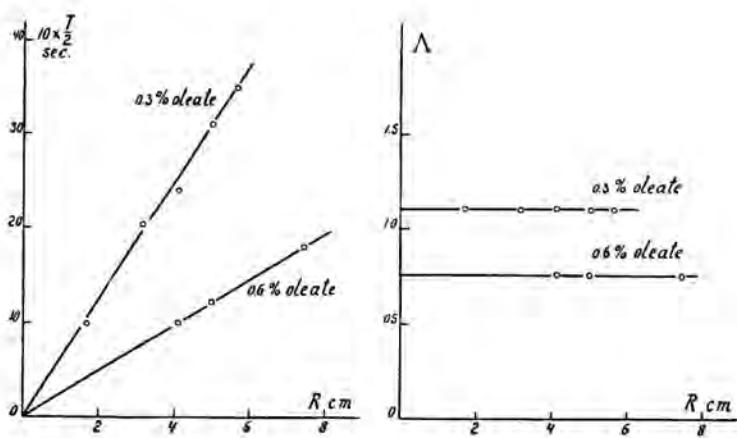


Fig. 2.

fig. 2 and 3. In accordance with the results obtained in Parts II and III of this series, we find that for oleate systems containing 0.3 and 0.6 g per 100 cc (see fig. 2 B) A is independent of the radius of the sphere and

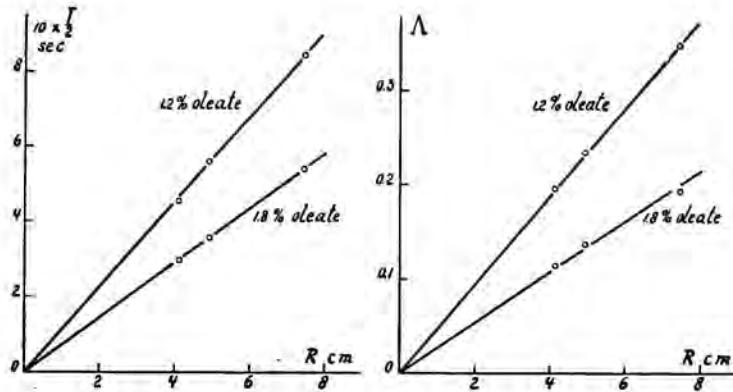


Fig. 3.

TABLE II.
Flow measurements at 15° C of the oleate systems of Table I.

1.8 g. oleate per 100 cc.			1.2 g. oleate per 100 cc.			0.6 g. oleate per 100 cc.			0.3 g. oleate per 100 cc.		
log V	log P	log P/V	log V	log P	log P/V	log V	log P	log P/V	log V	log P	log P/V
0.14-2	0.15	2.01	0.56-2	0.29	1.74	0.97-2	0.15	1.18	0.99-2	0.50-1	0.52
0.53-2	0.50	1.97	0.75-2	0.56	1.82	0.10-1	0.36	1.26	0.33-1	0.81-1	0.49
0.70-2	0.76	2.05	0.59-1	1.18	1.59	0.25-1	0.51	1.26	0.55-1	0.00	0.45
0.98-2	1.09	2.11	1.75	1.73	0.98-1	0.72-1	0.82	1.10	0.81-1	0.17	0.35
0.41-1	1.37	1.96	2.03	1.88	0.86-1	0.99	1.05	0.06	0.35	0.38	0.03
0.76-1	1.57	1.81	2.35	2.26	0.91-1	1.53	1.21	0.67-1	0.69	0.47	0.79-1
0.38	1.78	1.40	2.52	2.53	0.01	1.86	1.51	0.66-1	0.83	0.53	0.70-1
1.99	2.05	0.07	2.71	2.66	0.95-1	2.14	1.76	0.62-1	1.24	0.68	0.45-1
2.27	2.29	0.03	3.05	2.79	0.74-1	2.40	1.92	0.52-1	1.60	0.84	0.24-1
2.43	2.51	0.09	3.76	3.07	0.31-1*	2.53	2.06	0.53-1	1.88	1.00	0.12-1
2.51	2.65	0.15	4.24	3.31	0.07-1*	2.89	2.23	0.33-1	2.17	1.30	0.14-1
2.68	2.86	0.18				3.46	2.59	0.13-1*	2.62	1.67	0.05-1*
2.69	2.88	0.20				4.00	2.89	0.89-2*	3.15	1.96	0.81-2*
3.61	3.19	0.58-1*				4.66	3.36	0.70-2*	3.50	2.23	0.73-2*
4.18	3.43	0.25-1*							4.13	2.73	0.60-2*
$\log \eta_0 = 2.02$ (mean of first five values of $\log P/V$)			$\log \eta_0 = 1.78$ (mean of first two values of $\log P/V$)			$\log \eta_0 = 1.23$ (mean of first three values of $\log P/V$)			$\log \eta_0 = 0.49$ (mean of first three values of $\log P/V$)		
$\log \eta_i = 0.03 \rightarrow 0.20$			$\log \eta_i = 0.86-1 \rightarrow 0.01$			$\log \eta_i = 0.65-1$ (mean)			$\log \eta_i = 0.13-1$ (mean)		
$\log \eta_\infty < 0.25-1$			$\log \eta_\infty < 0.07-1$			$\log \eta_\infty < 0.70-2$			$\log \eta_\infty < 0.60-2$		
$\eta_0 = 105$ poises			$\eta_0 = 60$ poises			$\eta_0 = 17$ poises			$\eta_0 = 3.1$ poises		
$\eta_i = 1.1 \rightarrow 1.6$ poises			$\eta_i = 0.7 \rightarrow 1.0$ poises			$\eta_i = 0.45$ poises			$\eta_i = 0.11$ poises		
$\eta_\infty < 0.18$ poises			$\eta_\infty < 0.12$ poises			$\eta_\infty < 0.05$ poises			$\eta_\infty < 0.04$ poises		

* After application of the correction of HAGENBACH.

that at higher oleate concentrations (1.2 and 1.8 g per 100 cc) A is proportional to this radius (see fig. 3 B).

In these two latter cases only the relaxation time λ and consequently a value $\eta = G \cdot \lambda$ can be calculated. Though for the purpose of comparing $G \cdot \lambda$ with η_0 , η_1 and η_∞ it would suffice to measure the viscous behaviour of the 1.2 and 1.8 % oleate systems only, we have investigated the 0.3 and 0.6 % systems as well, in order to gain some insight into the flow behaviour as a function of the oleate concentration. The results have been given in Table II and represented in fig. 4 A and 7 A.

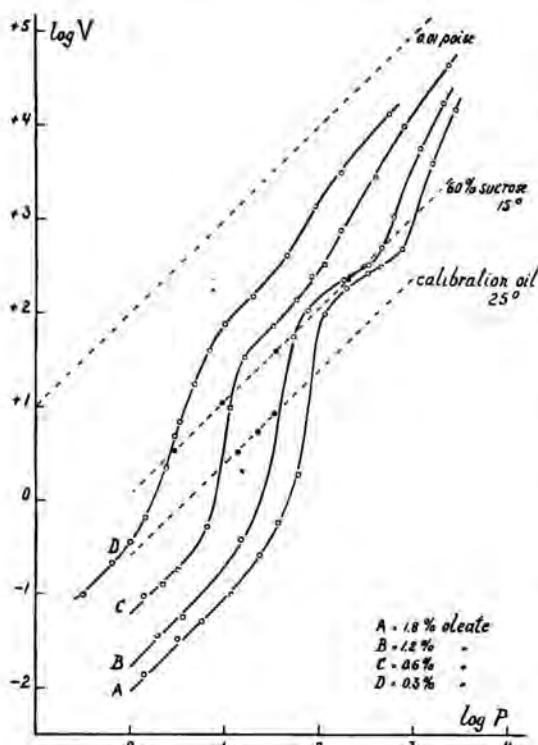


Fig. 4 A.

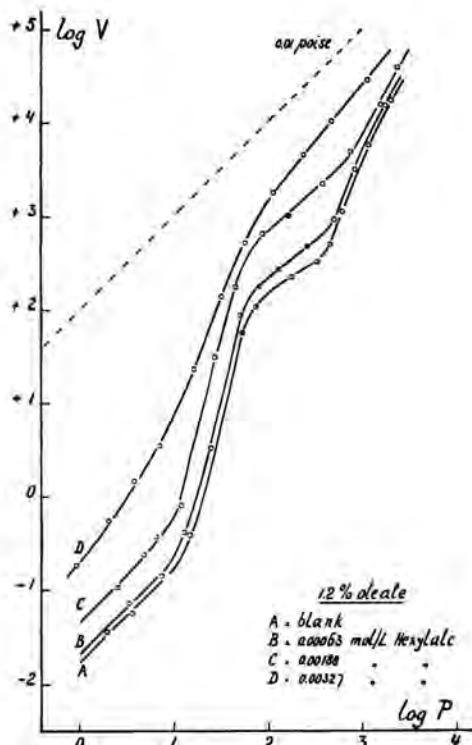


Fig. 4 B.

A second series of measurements was started from the 1.2 % oleate system (which served as a blank) to which small amounts of n.hexyl-alcohol were added. The results of the elastic measurements (rotational oscillation at 15° C) have been collected in Table III and represented in fig. 5 and 6.

It was found that the addition of hexylalcohol in the two concentrations which still permitted elastic measurements, does not change the proportionality between A and the radius of the spherical vessel. Therefore λ values and consequently $\eta = G \cdot \lambda$ values could be calculated. For the third concentration of hexylalcohol elastic measurements were no longer possible. It was nevertheless interesting to include also this latter oleate

TABLE III.

Measurements of the rotational oscillation with 1.2% oleate systems (1.43 N KCl + 0.18 N KOH) containing n. Hexylalcohol at 15° C. (For blank see Table I.)

n. Hexylalcohol mol/L	R (cm)	<i>n</i>	$10 \times \frac{T}{2}$ (sec.)	Λ	$10 \times \frac{T}{2}$ corr.	λ (sec.)	G (dynes/cm ²)	$G \times \lambda$ (poises)
0.00063	7.46	33.6	8.12	0.459	8.09	1.76	mean	71.2
	5.04	34.7	5.45	0.337	5.44	1.61		
	4.12	38.1	4.44	0.288	4.44	1.54		
	3.18	39.4	3.70	0.224	3.70	1.65		
0.00188	7.46	10.2	7.68	1.528	7.46	0.49	mean	25.9
	4.92	13.7	4.95	1.026	4.89	0.48		
	4.12	17.2	4.22	0.804	4.19	0.52		
							52.5	5.21

system in our investigations of the viscous behaviour. The results of these measurements have been collected in Table IV and represented in fig. 4 B and 7 B.

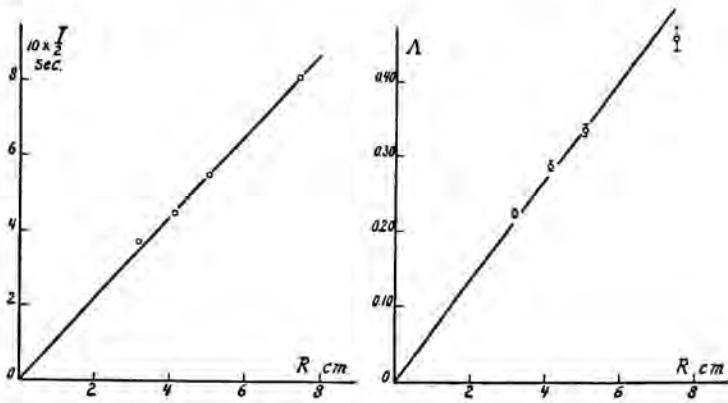


Fig. 5.

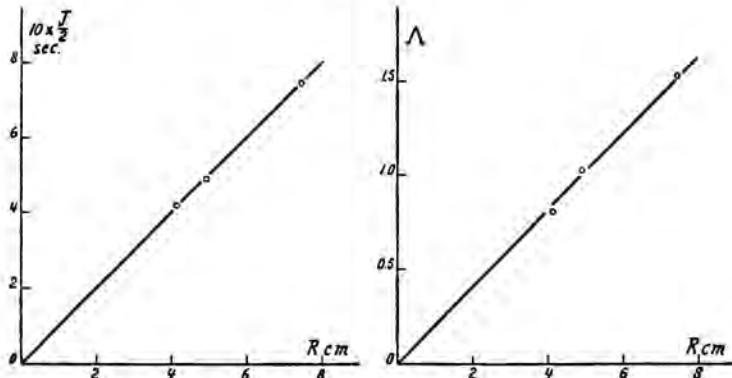


Fig. 6.

In discussing the results of the above two series of measurements we will first say a few words on the viscous behaviour. The flow rate curves of fig. 4 A and 4 B (log V as a function of log P) and the viscosity-

TABLE IV.

Flow measurements at 15° C of 1.2% oleate systems containing various concentrations of *n* Hexylalcohol..

0.00063 mol/L hexylalcohol			0.00188 mol/L hexylalcohol			0.00327 mol/L hexylalcohol		
log V	log P	log P/V	log V	log P	log P/V	log V	log P	log P/V
0.86-2	0.53	1.67	0.03-1	0.41	1.38	0.26-1	0.97-1	0.71
0.15-1	0.87	1.72	0.38-1	0.68	1.31	0.74-1	0.32	0.58
0.62-1	1.11	1.50	0.57-1	0.81	1.24	0.17	0.59	0.42
0.52	1.40	0.88	0.91-1	1.09	1.19	0.55	0.86	0.31
1.94	1.72	0.78-1	1.47	1.44	0.97-1	1.36	1.22	0.86-1
2.25	1.92	0.67-1	2.24	1.66	0.42-1	2.14	1.51	0.37-1
2.43	2.12	0.69-1	2.82	1.95	0.14-1	2.72	1.76	0.03-1
2.68	2.42	0.74-1	3.01	2.22	0.21-1	3.25	2.07	0.82-2*
2.97	2.71	0.75-1	3.35	2.58	0.22-1	3.65	2.39	0.74-2*
3.49	2.93	0.45-1*	3.69	2.89	0.20-1*	4.01	2.68	0.67-2*
4.19	3.26	0.07-1*	4.19	3.21	0.01-1*	4.46	3.06	0.61-2*
		4.60	3.38	0.78-2*				
$\log \eta_0 = 1.70$ (mean of first two values of $\log P/V$)			$\log \eta_0 = 1.35$ (mean of first two values of $\log P/V$)			$\log \eta_0 > 0.71$		
$\log \eta_i = 0.66-1 \rightarrow 0.75-1$			$\log \eta_i = 0.14-1 \rightarrow 0.22-1$			η_i level degenerated		
$\log \eta_\infty < 0.07-1$			$\log \eta_\infty < 0.78-2$			$\log \eta_\infty < 0.61-2$		
$\eta_0 = 50$ poises			$\eta_0 = 22$ poises			$\eta_0 > 5$ poises		
$\eta_i = 0.46 \rightarrow 0.56$ poises			$\eta_i = 0.14 \rightarrow 0.17$ poises			η_i = level degenerated		
$\eta_\infty < 0.12$ poises			$\eta_\infty < 0.06$ poises			$\eta_\infty < 0.04$ poises		

* After application of the correction of HAGENBACH.

shear stress curves of fig. 7 A and 7 B ($\log V - \log P$ as a function of $\log P$) show with one exception (the lowest curve in fig. 7 B) the same characteristics viz. η_0 level, a tendency to reach an η_∞ level and the presence of an intermediate "level", η_i , the experimental points on it lying either on a practically horizontal line or on lines that are sloping upwards to the right: shear rate thickening (as the corresponding curves in Part IV of this series, cf. fig. 7 in that publication). These characteristics were already discussed in Part IV, so that we need not discuss them again.

It is remarkable that the curve deviating from this typical curve form, (at its lowest in fig. 7 B) belongs to an oleate system, the elastic properties of which could not be measured on account of the exceedingly great damping of the oscillations (at most only two turning points were present). The viscosity-shear stress curve characteristic of the markedly elastic oleate systems (marked on account of smaller damping) has now degenerated to one in which an η_i level is no longer present and it already strongly resembles the curve form found in many cases of non Newtonian viscosity, in which there are only two levels (η_0 and η_∞).

We might further draw attention to a similarity on the one hand and a difference on the other between the influences of lowering the oleate concentration and of adding n. hexylalcohol at a constant oleate concentration. Comparing fig. 7 A with fig. 7 B we see in both cases that the viscosity-shear stress curve as a whole is displaced downwards. In lowering

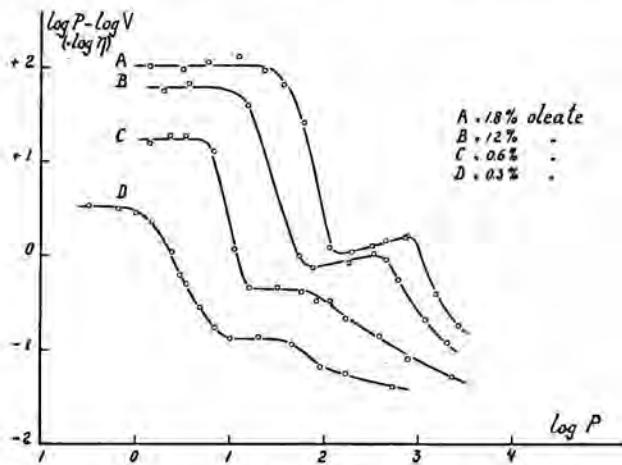


Fig. 7 A.

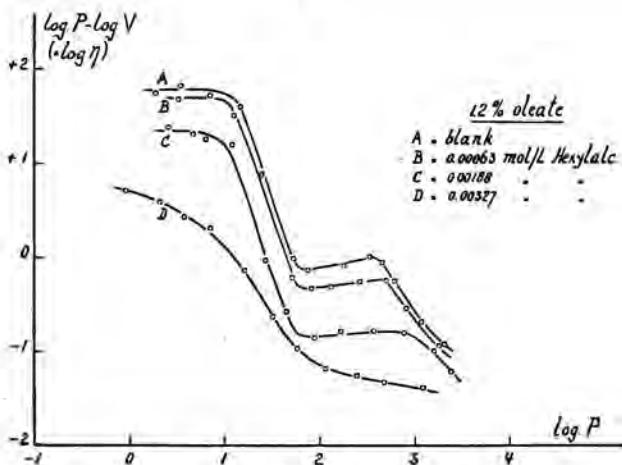


Fig. 7 B.

the oleate concentration, see fig. 7 A, the viscosity-shear stress curve is at the same time displaced to the left in a highly marked way. When adding hexylalcohol at a constant oleate concentration this displacement is absent, however.

We have now come to the comparison of the values $G \cdot \lambda$ with η_0 , η_i and η_∞ , and refer to Table V, in which we have collected the numerical values of these quantities for the four oleate systems in which this comparison is possible.

We perceive from this Table that the product $G \cdot \lambda$ does not coincide with any of the three η levels observed. The product is extremely different from η_∞ , still very different from η_t , but it is of the same order of magnitude as η_0 .

TABLE V.
Comparisson of $G \times \lambda$ with η_0 , η_t and η_∞

Composition		$G \times \lambda$ (poises)	η_0 (poises)	η_t (poises)	η_∞ (poises)
Oleate (g. per 100cc)	Hexylalcohol (mol/L)				
1.8	—	263	105	1.1 → 1.6	< 0.18
1.2	—	97	60	0.7 → 1.0	< 0.12
1.2	0.00063	71	50	0.46 → 0.56	< 0.12
1.2	0.00188	26	22	0.14 → 0.17	< 0.06

4) *The discrepancy between the values found for η_0 and for $G \cdot \lambda$.*

The same order of magnitude of the values found for η_0 and $G \cdot \lambda$ gives rise to the question if these values are really equal and if only experimental errors cause the discrepancy formed.

The difficulties we met with in measuring high η_0 values at low shearing stresses, and the discussion of these difficulties²⁾ made us much more suspicious regarding the reliability of the average η_0 values than regarding that of the average G or λ values.

We decided therefore to compare a new $G \cdot \lambda$ with η_0 , the η_0 value no longer being obtained with the aid of the petroleum drop method, but by directly measuring the quantity of the oleate fluid flown through the capillary of the viscometer. The latter was of the type as depicted in fig. 1 and the measurements consisted in reading at stated intervals the position of the levels of the oleate fluid in the equally wide reservoirs, (which were in direct communication with the air) with the aid of a cathetometer. A larger radius of the capillary³⁾ ($R = 0.1062$ cm, $L = 12.1$ cm) was taken and a smaller diameter of the cylindrical reservoirs (= 1.773 cm) to reduce the time which the measurements still required (the series of seven given in the table below took two whole days).

From the effective hydrostatic height⁴⁾ $\left(\Delta h / \ln \frac{h_1}{h_2} \right)$, the density of the oleate fluid, the change in position of the levels during the time elapsed, the dimensions of the capillary and the diameter of the reservoirs the viscosity was calculated with the aid of POISEUILLE's formula.

This method gave satisfactory results only if, after filling the apparatus and bringing about an initial level difference one does not start with the

²⁾ See section 2 and in Part IV section 3, note 16.

³⁾ The radius of the capillary was calculated from flow measurements with the same calibration liquids as were used in Part IV.

⁴⁾ See E. HATSCHKEK, Die Viskosität der Flüssigkeiten, Dresden 1929.

measurements immediately, but checks the flow of the oleate fluid for several hours by means of a counter air pressure (to give the walls of the cylindrical reservoirs time to drain)⁵⁾. The counter pressure is then removed and some hours later one begins with the measurements.

The apparatus functions well if after the intermission between two level readings, the level has risen in one reservoir quite as much as it has fallen in the other.

As we had no more Na oleinicum "Merck" in stock and it could not be purchased either, we used Na oleate, neutral powder, "Baker"⁶⁾ for the renewed comparison of $G \cdot \lambda$ and η_0 at 15° C. The composition of the system investigated was 1.2 g oleate per 100 cc (1.25 N KCl + 0.05 N KOH)⁷⁾. Table VI contains the η_0 values obtained by the above method

TABLE VI.

Measurements of the viscosity of 1.2% oleate system (oleate "Baker" in 1.25 N KCl + 0.05 N KOH) at 15° C.

Mean P (dynes/cm ²)	32.10	27.97	23.98	23.26	17.82	14.56
η (poises)	68.2	76.0	74.9	77.9	76.6	74.7
η_0 mean = 76.0 poises						

at a number of (mean) shearing stresses at the wall of the capillary⁸⁾. One perceives that compared with the large fluctuations of the log η_0 values in the Tables I and II the fluctuation of η_0 is small now, so that we may trust its mean value much better. Table VII gives the results

TABLE VII.

Measurements of the rotational oscillation of 1.2% oleate system (Oleate "Baker" in 1.25 N KCl + 0.05 N KOH) at 15° C.

R (cm)	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	A	λ (sec)	G (dynes/cm ²)	$G \times \lambda$ (poises)
2.98	60.3	3.13	1.118	0.112	2.79	46.6	
4.99	63.5	5.21	1.198	0.181	2.88	47.1	
7.16	68.0	7.40	1.291	0.255	2.90	48.1	46.6
4.99*	61.2 ± 0.2	5.35 ± 0.03	1.202 ± 0.0034	0.184 ± 0.003	2.91	44.7	134

* Measurement several days later during the flow experiments. From this single measurement would follow $G \cdot \lambda = 130$ poises. One is however not sure to conclude from a single measurement to a real decrease of $G \cdot \lambda$ with time.

⁵⁾ Cf. in Part IV section 2, notes 8 and 10 from which follows the necessity of taking a long time for this drainage.

⁶⁾ We are glad to express our profound gratitude to the Rockefeller Foundation, which kindly put at our disposal a large quantity of KCl and of the preparation mentioned in the text in order to enable us to continue our researches on oleate systems.

⁷⁾ The KCl (+ KOH) concentration at which the damping of the elastic oscillations is a minimum lies lower than for the oleate preparation of MERCK. It was recently ascertained that it lay even a little lower than the concentration mentioned in the text, which lies still close to this minimum.

⁸⁾ In the execution of these measurements we were assisted by W. W. H. WEIJZEN and W. A. LOEVEN, to whom we also wish to express our thanks here.

obtained in the investigation on the rotation oscillation (see also fig. 8). When we compare now the mean η_0 value ($= 76$ poises) with the mean $G \cdot \lambda$ value ($= 134$ poises) we still come to the same result as in the preceding section: η_0 is not equal to, but smaller than $G \cdot \lambda$, though both are of the same order of magnitude.

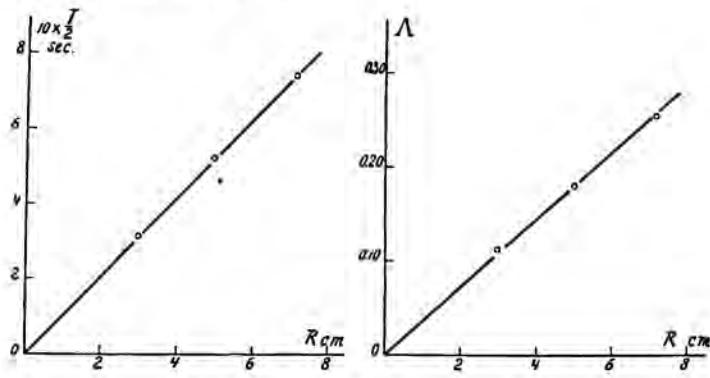


Fig. 8.

We could leave it at this general conclusion, but it is interesting that there appears to be a system in the disagreement between the two values. See the following survey. (Table VIII), giving the five cases in which η_0 and $G \cdot \lambda$ could be compared, arranged in the order of decreasing η_0 .

TABLE VIII.

Concentration Oleate	n. Hexylalc. Mol/L	η_0 (mean) (poises)	$G \cdot \lambda$ (mean) (poises)	$\eta_0/G \cdot \lambda$
1.8 % (MERCK)	—	105	263	0.40
1.2 % (BAKER)	—	76	134	0.57
1.2 % (MERCK)	—	60	97	0.62
1.2 % (MERCK)	0.0006	50	71	0.70
1.2 % (MERCK)	0.0018	22	25.9	0.85

We see from column 5 that the disagreement between η_0 and $G \cdot \lambda$ decreases with a decreasing value of η_0 . Cf. fig. 9, which gives $\eta_0/G \cdot \lambda$ as a function of η_0 and which shows that the experimental points fairly coincide with a curve drawn through $\eta_0/G \cdot \lambda = 1$ at $\eta_0 = 0$. This might mean that MAXWELL's relation $L = G \cdot \lambda$ in principle holds good, but that certain unknown systematic errors⁹⁾ in the experimental methods we used for determining G or λ or η , distort this equality and the more so as the absolute values of these quantities are higher. But it might also mean that the equation representing the connection between η , G and λ for our

⁹⁾ The fear of any occurrence of a systematic error in the mean η_0 values obtained in section 3 has been considerably diminished by the fact that the experimental point obtained in this section with a quite different viscosimetric technique lies on the same curve as the other experimental points (see fig. 9).

systems is really of a more complicated nature and approximates to $\eta = G \cdot \lambda$ at low values of these quantities.

Be that as it may, the same order of magnitude found for $G \cdot \lambda$ and for η_0 seems to indicate that the mechanism of viscous flow in the range of small shearing stresses is mainly due to relaxation of elastic stresses.

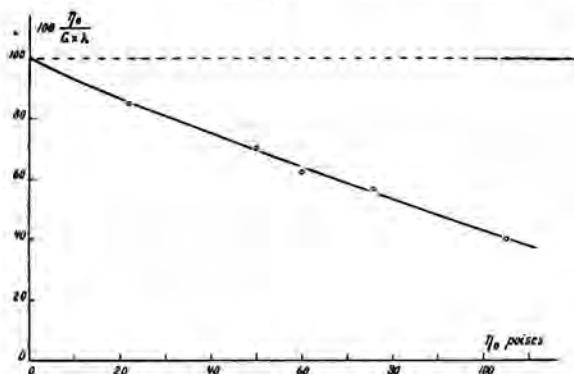


Fig. 9.

Summary.

- 1) The viscous and the elastic behaviour of a number of oleate systems containing KCl have been investigated at 15° C side by side to compare the viscosity coefficient calculated from elastic measurements ($\eta = G \cdot \lambda$, MAXWELL) with the viscosity coefficients η_0 , η_i and η_∞ manifesting themselves as levels in the viscosity-shearing stress diagram of the oleate systems.
- 2) It has been found, that both η_∞ and η_i are far smaller and of a quite different order than $G \cdot \lambda$, that η_0 , however, the viscosity coefficient at low shearing stresses, is of the same order of magnitude as $G \cdot \lambda$.
- 3) The percentual difference between η_0 and $G \cdot \lambda$ (η_0 has always turned out to be smaller than $G \cdot \lambda$) decreases in a marked way with decreasing absolute value of these quantities.
- 4) This suggests that either still unknown systematic errors, which increase percentually with increasing values of $G \cdot \lambda$ or η_0 , bring about this difference; or that a more complicated relation between $G \cdot \lambda$ and η is operative for the oleate fluid, which at small values of these quantities may practically be simplified to $\eta = G \cdot \lambda$.

Department of Medical Chemistry,
University of Leyden.

Geology. — Tectonics of the Mt. Aigoual pluton in the southeastern Cevennes, France. Part I. By D. DE WAARD. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of March 26, 1949.)

1. Introduction.

In the summer of 1947 detailed tectonic field-work was carried out in the southeastern Cevennes of France. The purpose of the investigation was to tackle structural problems in the little-known granite massif near the Mt. Aigoual. This massif is shown in the southwesterly corner of sheet Alais of the French geological map, 1 : 80,000, as a red dot of granite stretching away to the north in slates. Mapping has been done in and around this granite dot (fig. 1) in a purely structural sense. The well-exposed country in this part of the Cevennes facilitated detailed study, though mapping proved to be hampered sometimes by glacial deposits, hillside waste and afforestation. Excellent exposures are provided by a large number of small rivers.

During the survey, the area of the red dot of granite on the geological map appeared to comprise a small batholith surrounded by a complicated pattern of granitic, porphyritic and dark coloured dikes in the slaty country rock. The large scale map at the back shows the geological units of the mapped area.

The author's thanks are due to Professor W. NIEUWENKAMP for helpful discussions and criticism in field and laboratory, to Mr. R. C. HEIM for valuable data and good companionship during the field-work and to Miss D. E. WISDEN, M.Sc., Southampton for assistance with the English.

The writer acknowledges the receipt of a Z.W.O. grant (Dutch Organization of Pure Sciences) which made possible some comparative studies in adjacent granite massifs of the Cevennes.

2. Synopsis of literature.

The Central Plateau is situated in the middle of France as a large isle of mainly metamorphic and crystalline rocks with a local cover of young volcanic material on its peneplaned surface, surrounded by Mesozoic and Tertiary sediments (fig. 1).

The age of the metamorphic rocks has long been uncertain. The "Archean age" theory has been rejected since the discovery of Cambrian and Ordovician fossils. Precambrian rocks are still mentioned however. According to most authors, folding and metamorphism of the Palaeozoic geosynclinal strata has taken place in the Sudetic phase of the variscan orogeny. The tectonic units with metamorphic imbricate and nappe struc-

tures have been analysed especially by the studies of DEMAY (e.g. 1931b, 1934), summarized by VON GAERTNER (1937).

Also from this part of France granitization phenomena are extensively

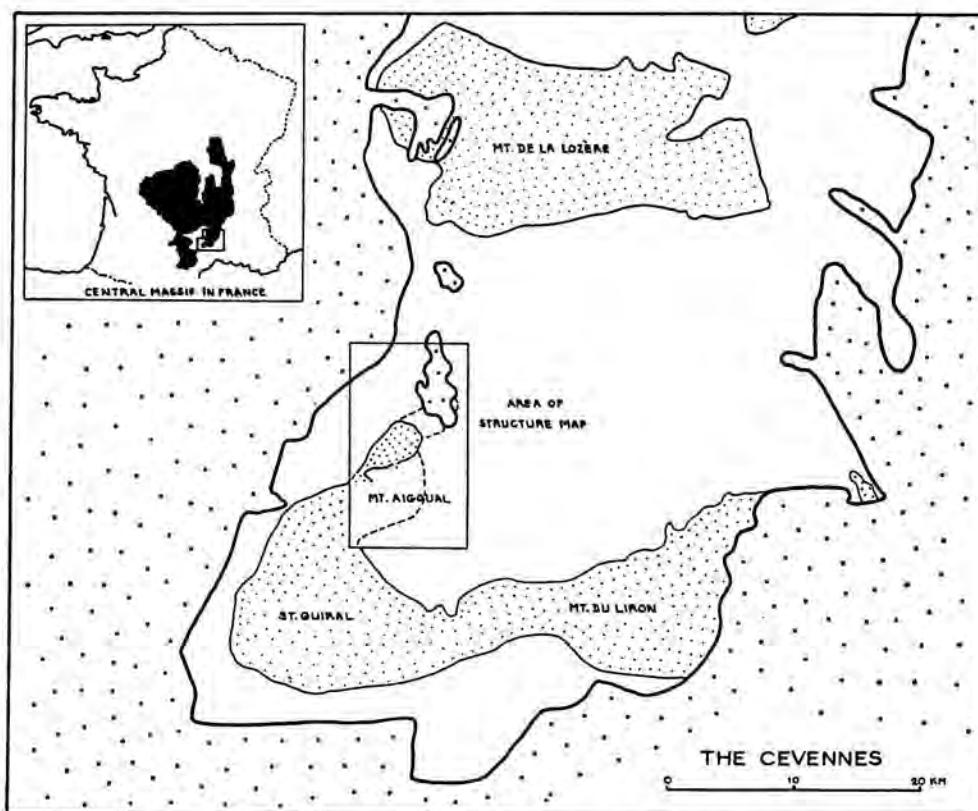


Fig. 1. Map of the southeastern part of the Central Massif showing the geographical position of the mapped area between slates (white) and adjoining granite massifs (dotted). The surrounding mesozoic strata are coarsely dotted.

described by French authors (e.g. DEMAY, 1935, 1942, RAGUIN, 1930, 1946, ROQUES, 1941). According to these publications the granite plutons in the Central Massif may be distinguished in 'pretectonic', 'syntectonic anatectic', 'synkinetic intrusive' and 'posttectonic intrusive' with respect to the Middle Carbonian orogeny. There seems to be no apparent relation between anatectic and intrusive granites. No rule could be laid down for the mode of occurrence, or for the texture of the posttectonic intrusive granites. Sometimes they are porphyritic, but syntectonic plutons too may have a similar texture (ROQUES, 1941, RAGUIN, 1946).

The Mt. Aigoual granite pluton, subject of this paper, is briefly mentioned in several French publications (e.g. BERGERON, 1889, 1904, DEMAY, 1931a). According to all authors it is a posttectonic intrusion, cutting the schistosity of the slates. A narrow metamorphic aureole is further evidence of its intrusive character. The texture of the granite is

described as homogeneous and porphyritic, with large crystals of orthoclase. Because of similarity in these properties, connections are postulated with the adjacent St. Guiral pluton. The Mt. Aigoual pluton is mentioned by DEMAY (1931a) as an apophysis of the much larger St. Guiral massif.

The first mapping of this area was achieved by FABRE and CAYEUX (1901) in sheet Alais of the geological map of France and for the second edition by THIÉRY (1923). Of the numerous granitic dikes north and south of the granite body only a few have been mapped separately in this sheet. Those nearer to the intrusive body are lumped together in one large granite outcrop extending southward as far as the St. Guiral massif.

The morphological study of BAULIG (1928) of the Central Plateau, as well as his maps, show many faults and fault-blocks. Two of these faults, continuing in one, are drawn through the Mt. Aigoual granite. In the mapped area, however, little evidence for these faults could be found.

According to VON GAERTNER (1937) and based on DEMAY (1931a), the Mt. Aigoual region forms part of the Orthocévennes complex, a pennine nappe-like tectonic unit of the Central Massif. The granite of the Mt. Aigoual, together with the southern adjacent massif, is mentioned and drawn in section by VON GAERTNER as a late orogenic (posttectonic) intrusive sheet, flatlying in the cleavage plane of the slates.

3. Geology.

In the centre of the mapped area a small elliptical pluton of granite crops out surrounded by slates. The granite body, 5 km long and 2.5 km wide is very homogeneous; no endomorphism could be observed.

The granite is a light coloured, black and white rock of coarse grained texture with 10—15 cm long idiomorphic phenocrysts of orthoclase. Main components are quartz, orthoclase, plagioclase and biotite. Thanks to the granite-porphyritic texture, planar and linear flow structures could be measured nearly everywhere in the pluton. Joint systems occur usually in a variety of directions; only in few cases have slickensides been formed. Aplite, lamprophyre and quartz porphyry dikes have been observed inside the granite massif.

Nearly everywhere round the granite body results of contact metamorphism are visible. The contact plane is always sharp; its strike and dip vary locally.

The country rock farther away from the contact is in general slaty. According to DEMAY (1931a) a Cambrian age of the slates would be probable. The slates are not subdivided on sheet Alais of the geological map of France; on sheet Séverac, some kilometers west of the mapped area, a probable Potsdamian age is indicated.

Normally the slate is veined with white quartz. Though differing in quantity, usually numerous thin veins and lenses as well as small dikes to 40 cm wide occur. Intensely folded veins have been observed. They do not penetrate in joint systems or faults and very seldom in cleavage planes.

In several parts the slaty rock is a well-developed phyllite. Locally quartzite layers parallel to the schistosity of the slates are observed; elsewhere microscopical folds in bedded slates cross the cleavage planes. Spotted slates too are found in some places.

Usually the slates possess one or two systems of ribs on their cleavage planes. In the south of the mapped area they have minor folds with wave lengths between 10 and 100 cm. In this region they often pass into folded gneiss. Directions of ribs and fold axes as well as cleavage planes and joint systems in the country rock have been measured for structural purposes.

Dikes cut through the country rock round the pluton. Granitic dikes in large numbers present themselves as a fringe-like extension of the pluton. This edging has a limited extension at the northern and eastern side. Dikes are found much farther towards the south, but they are almost absent on the western pluton border.

Lamprophyre dikes occur in groups, mostly outside the edging of granitic dikes. They are often intruded along the cleavage planes of the slates. Their occurrence within the granite and in porphyry dikes prove them to be younger. More basic dike rocks have been observed.

Quartz porphyry dikes of light coloured material with phenocrysts of quartz and also of both quartz and orthoclase are observed between granite dikes and inside the granite pluton. They cross granitic dikes and are crossed by lamprophyre dikes.

Faults and faultzones, with and without drag, occur frequently in the mapped region. Faults cutting dikes have not been found. In one case drag is observed in the country rock at the contact with a granitic dike. Quartz veins always are disturbed by faulting; quartz does not occur within the fault fissures, only crushed in fault breccia.

The geological history in the mapped region according to the observed phenomena may be summarized in the following succession.

Sedimentation of strata probably in Cambrian time. Folding and metamorphism accompanied by exudation of quartz (according to most authors in the Middle Carbonian, Sudetic phase of the variscan orogeny). Development of schistosity and joint systems. Both segregation of quartz and jointing will be products of the same cause, the latter somewhat younger than the former. Intrusion of granite accompanied by contact metamorphism and intrusion of quartz porphyry, lamprophyre, aplite and pegmatite.

4. Petrology and microtectonics.

The microscopic compositions and microtectonic features of the mapped rocks will be mentioned here in brief. The petrology of the Mt. Aigoual area is discussed in a separate paper by HEIM (1949).

The granite of the pluton is a coarse-grained, light-coloured rock with up to 15 cm large twinned automorphic phenocrysts of white K-feldspar.

Main components are white K-feldspar, plagioclase, dark grey grains of quartz and many small black biotite crystals dispersed through the rock and in a dark rim around the large feldspars. Thin sections show Carlsbad-twinned orthoclase, much albite- and pericline-twinned oligoclase, brown biotite, quartz and occasionally hornblende. According to the mineralogical composition the rock may be classified in the granodioritic subdivision of the granites *sensu lato*. Quartz shows in general slight undulatory extinction.

In the field little difference could be observed between the granite in the pluton and in the dikes. The latter may show however some more variation in size of the smaller minerals between the large feldspars. Quartz too may thus have an automorphic habitus, giving the rock frequently a granite-porphyritic texture. In that case the quartz phenocrysts are corroded and rounded. Evidence of stronger dynamo metamorphism in the granite dikes is shown by wide-spread and strongly undulatory extinction of quartz and by bent and frayed biotite crystals. Sometimes recrystallised nonundulatory, suture-grained parts within quartz crystals do occur. The microscopic texture as a rule is lacking in orientation.

The quartz porphyry dikes contain a light-coloured, dense rock with many automorphic, dark-coloured quartz grains and irregularly distributed feldspar phenocrysts. The dikes are sharply jointed in small regular blocks. The rock is composed of quartz, orthoclase, biotite and albite phenocrysts in a dense, partly spherulitic or micrographic groundmass of mainly quartz and orthoclase. The quartz phenocrysts are largely corroded and rounded and sometimes show undulatory extinction.

Dikes of a dark quartz porphyry have been mapped within the granite body, near the contact of the pluton, cutting through granitic dikes and forming bordering zones of the latter. It is a dark rock of biotite and amphibole with quartz and plagioclase phenocrysts and a few K-feldspar phenocrysts. Under the microscope this rock proved to be a tonalite porphyry. Large rounded, corroded and undulatory quartz phenocrysts, zonal, Carlsbad-, albite- and pericline-twinned, corroded andesine, brown biotite, hornblende and occasionally orthoclase phenocrysts are enclosed in a fine-grained groundmass, often with fluidal texture, of andesine, biotite, hornblende and some quartz.

As mentioned above different types of basic rocks have been observed. They show difference in darkness, in quantity of feldspar and in size and orientation of biotite. Usually they are dark coloured, fine-grained, glittering rocks, largely composed of biotite and feldspar. In dike outcrops they often show rounded blocks caused by weathering along joints or they may be weathered all through into yellow-brown, spotted, sandy material. In some dikes the rock has an oriented texture, due to relatively large biotite crystals, being arranged parallel to the dike contacts.

Most dikes have kersantite-like composition; one consists of greenish-grey porphyrite.

The aplite dikes normally contain quartz, orthoclase and some biotite; no exceptional minerals have been found in the aplites, nor in the few pegmatite or runite dikes and veins.

The country rock is usually microfolded slate with quartz veins, phyllitic with sericite and recrystallised quartz, blastophyllitic with garnet and kyanite, and schistose to almost gneissic with sericite, muscovite, biotite, chlorite, quartz and some albite. In the narrow contact zone a fine grained gneiss is locally developed with orthoclase and plagioclase. Andalusite, sillimanite and cordierite also have been observed near the contact.

5. Granite contacts.

Many exposures allowed a detailed observation of the pluton borders. The western contact forms a nearly straight line in the plane of the structure map. Slates border the granite along a sharply defined contact plane which dips, at an angle of 70° to 55° , under the slate. This contact has all the aspects of a set of joints, a pre- or syn-intrusive joint system without doubt, because of the existing contact metamorphism in the slate. A narrow zone of slightly metamorphosed slates and phyllites borders the contact with penetrating veins of aplite and pegmatite. No offshoots such as granitic dikes are to be found on this side of the pluton.

Joint-like contact planes have been observed nearly everywhere around the pluton. The northern border is complicated by the branching off of many dikes. These offshoots, splitting apart euhedral blocks of slate, could be observed in detail. It has not been feasible however to reproduce these complications in the structure map; the curved lines merely represent a general outline of the granite body in these places. The northern contact planes show great variety of orientation; steep discordant and low angled often concordant contacts have been measured. Blocks of slates, surrounded by granite dikes could be observed.

Contact metamorphism in the north and northeast of the pluton may locally be stronger than near the western border, though more than one meter of the country rock has usually not been changed megascopically. The slates are transformed in these places in a fine-grained gneiss often veined or brecciated with aplite and occasionally with pegmatite. The slates and metamorphosed country rock near the contact show variations in strike and dip. Dips proved to be usually steeper and strikes are systematically bent to northeasterly direction.

Parts of the eastern and southeastern contacts are hidden by hillside waste. Where visible the same joint-like contact planes with shallow metamorphism, acid veins and offshoots of granitic dikes occur. Here too, strikes and dips of the slates change near the contact. The contact plane usually dips under the slates with varying angles.

On the map there are two gaps in the southern border. In the eastern one the contact loses itself under debris. In the western gap there has

been no occasion for mapping further south; it seems quite possible however that across this track a bottleneck may join up the Mt. Aigoual pluton with the St. Guiral massif. The southern contact too has little contact metamorphism. Gneiss and acid veins are observed; mostly there is hardly any change beyond a hardening of the country rock.

Though the contact may be clean and sharp as in the western border, complicated contacts have been found. In a river exposure of the northern contact the following details have been observed. The granite near the contact, being quite normal with dark inclusions and at most a little less large orthoclase crystals is bordered by a small tonalite porphyry dike of 20 cm followed by a quartz porphyry dike of 30 cm thickness with inclusions. Then, along the contact there is a kind of conglomerate zone, 25 cm thick with rounded pieces of granite and metamorphosed slate in a fine-grained crystalline matrix. On the outer side of the contact and parallel to it a second tonalite porphyry dike cuts the slightly metamorphosed and hardened slate. In the south an exposure is found with the succession of normal granite, a 3 m zone of somewhat darker granite, a 50 cm zone of granite, normal in colour but more crowded with large orthoclase crystals than usual, a 75 cm dike of tonalite porphyry in contact with hardened slate of which a slice of 5 m is cut off by a quartz porphyry dike running parallel with the contact.

Still less of metamorphism has been found near the contacts of the granite or granite porphyry dikes, though local differences occur.

The contacts of the granite massif, usually cross-cutting and occasionally concordant with the slaty cleavage and the slight but never missing contact metamorphism points undoubtly to an intrusive character of the granite. As a whole the pluton seems to be bordered by joint systems. They must have been developed before or during the intrusion. During the intrusion movement along these joint systems has taken place. The intrusion has been accompanied by an uplift of the country rock. This is proved by strike and dip alterations near the contact. Post-intrusive movements along contact planes cannot have been of much importance because of the uninterrupted zone of contact metamorphism.

The contact zone described above with rounded granite and gneissic pebbles in a fine-grained crystalline mass, points to intrusive movements along this plane by which parts of both sides of the contact were crushed, rounded and embedded in a fluidal mass, causing an intrusion bordering mylonite.

6. Outline of tectonic phenomena.

The main point of the investigations has been to collect tectonic data in order to work out the movements and origin of the granite pluton. A summary is given in the structure map at the back. The detailed mapping in a relatively small area, with heights between 800 and 1400 meters, made it desirable to have all data transposed in one horizontal plane of 1000

meter. Topographic distortion is eliminated in this way. The structure map shows as it were a hypothetical peneplain at 1000 m in this area. Most measurements have been made near this 1000 m plane.

A fairly close network of tectonic data has been collected in the granite massif. Special attention was given to fluidal phenomena and fracture systems in order to be able to reconstruct the intrusive movements. Slicken-sides and directions of dikes were likewise of much importance with regards to these movements. Marginal observations on contact planes and inclusions complete the network of structural data.

In the country rock many important tectonic features could be measured. Strike and dip of cleavage planes of the slates are disturbed near the contact as mentioned in the preceding paragraph. Joint systems, ribs and minor folds in the slates give details of pre-intrusive tectonics. Dike measurements will possibly reveal relations between pre-intrusive tectonics and intrusive movements. Faults with drag phenomena may be connected with block movements during the intrusion.

These tectonic data, plotted in equiareal diagrams, will be discussed systematically in the following paragraphs.

7. Flow structures.

Flow structures in the Mt. Aigoual granite are mainly marked by oriented orthoclase crystals and, in its marginal parts, also by small inclusions.

The orthoclase crystals are mostly tabular after (010) and elongated in the direction of the *c*-axis. Their orientation may be linear with parallelism of the *c*-axes or planar with parallelism of the (010) planes. Usually they are twins after the Carlsbad law, showing interrupted (001) cleavage on their narrow sides.

Primary flow phenomena in crystalline rocks may be divided into linear and planar structures. Ordinarily one of the two is predominant or exclusive in a massif. Only when there are suitable minerals can both these flow structures be seen. Feldspar is one of these, because of its elongated tabular shape. In the Mt. Aigoual granite planar structures are favoured for its feldspars are distinctly more tabular than columnar in shape.

In spite of the possibility of recognizing flow structures by measuring the directions of the phenocrysts, their orientation at a first glance seems to be chaotic in many places. Only a careful investigation of several surfaces of different orientation of the granite reveals a predominant arrangement. A judicious choice of the order in which the surfaces are investigated, e.g. horizontal planes first, followed by surfaces at right angles to it assists in deciding between linear and planar orientation. In the latter case a surface parallel to this plane of flow was selected in order to reveal any tendency to a linear arrangement within this plane. As a rule only after a prolonged scrutiny and with many measurements of generally existing, local directions on many different planes of the granite, could

the predominant orientation be detected. Thus the field observations do not consist of measurements of single feldspar crystals which could only give reliable information if a large number (e.g. 100) were treated at each locality, but of measuring predominant directions on fairly large surfaces of granite outcrops.

Parallelism of minerals is caused by differential motion in the intrusive mass. Consolidation preserved a record of the ultimate movements. Linear parallelism is originated by preponderant one-dimensional elongation of the mass. Stretching in one direction results in a parallel orientation of the longest axes, the *c*-axis of the feldspars, in that direction. Their (010) planes have arbitrary positions rotated about the *c*-axes (fig. 2b). If differential motion in the mass effects equal stretching in all directions within (shortening normal to) a plane, planar parallelism will develop.

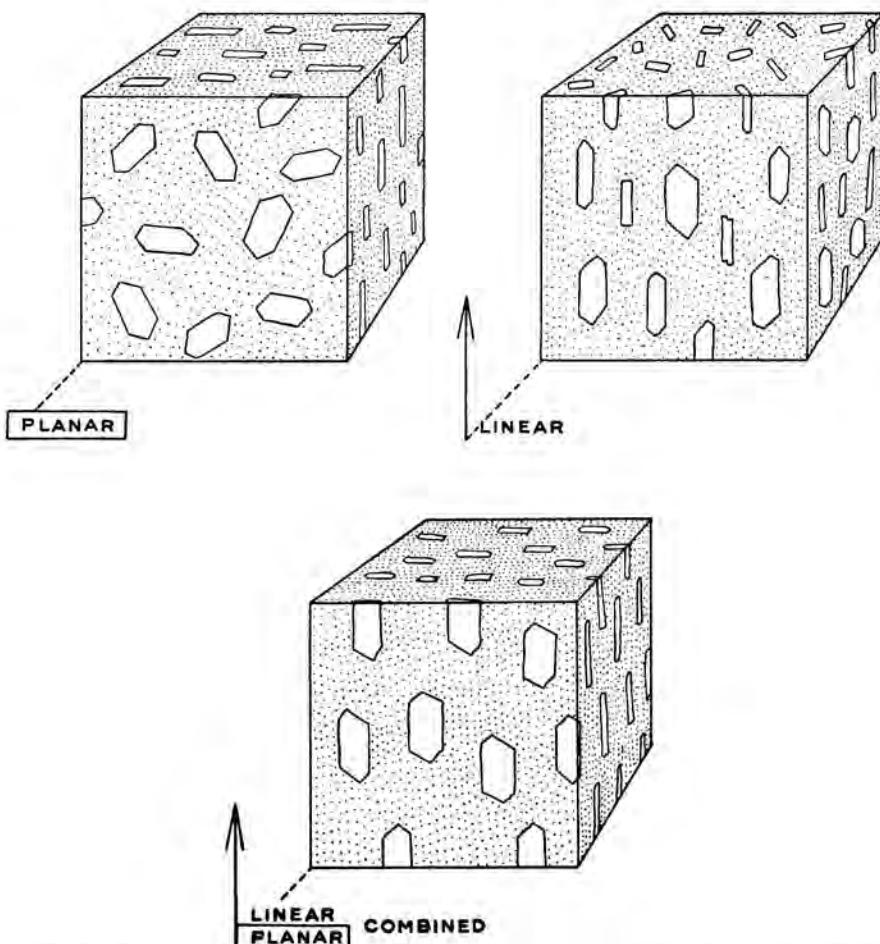


Fig. 2. Ideal arrangement of feldspar phenocrysts in primary oriented granite. a. (010) planes of feldspars in planar arrangement. b. *c*-axes of feldspar crystals in linear orientation. c. combination of linear flow structure by parallelism of *c*-axes and planar flow structure by platy arrangement of (010) planes.

Feldspars will tend to have (010) parallel to this plane, while their elongated c-axes show haphazard orientation (fig. 2a).

In the intermediate case, when planar structures are developed together with somewhat stronger stretching in a certain direction within this plane, both (010) planes are oriented in planar parallelism and c-axes in linear parallelism in the same plane (fig. 2c). Only locally are linear and planar orientation found associated; usually one of the two is predominant and the other weak or absent.

Because differential motion is strongest near the contacts of intrusions flow structures will be developed very well in their marginal parts. Conversely strongly oriented structures in the central part of the intrusion indicate that considerable differential movements also in these parts of the mass have existed. Usually these structures are planar. If linear parallelism exists too, strong one-dimensional stretching has been predominant. In parts of the intrusion where no planar structures are developed a weak linear orientation may be formed by distinct expansion of the pluton in a certain direction.

The movements of flow are very well determined if both planar and linear flow structures could be measured. Planar parallelism shows the plane along which flow has taken place and linear parallelism the direction of flow in that plane.

8. *Intrusive flow.*

The general aspect of the registered and interpolated flow data in the structure map of the pluton suggests comparison with a pan of boiling porridge. In at least three sections of this relatively small pluton upward culmination of flow is recorded.

This reconstruction is based on about 65 flow determinations. Construction of flow planes and lines is done by interpolation of the measured data in the least complicated way. These lines record in a nice manner the last movements of the intrusive mass before ultimate consolidation. The given interpolation is a reconstruction in the simplest way; in reality the flow structures may be much more complicated in detail.

West of the centre of the pluton a half concentric doming is made visible by flow phenomena. It has varying but mostly rather steep flow structures, planes as well as lines. The planar structure has conformable contacts with the westerly border, dips to the south against another, incompletely-known doming and continues to the E.N.E. To the north, structures are more complicated but presumably still conformable with the pluton contacts, continuing into dike offshoots. Conformable structures are also well-developed in granite dikes. Linear parallelism in north and west cross flow plane strikes nearly perpendicularly, indicating probably a strong upward flow. Near the top of the doming the lineation grows parallel with planar structures.

In the north-east of the pluton the continuation or easterly flank of the

doming is found. As far as observed it has identical conformable structures. Flow lines here may show stronger convergence in upward flow, also from northerly directions.

Flow structures in the south-east are different. They seem to be conformable to parts of the southern contact but disconformable to the easterly wall rock, representing a mass with parallel oriented planar structures, complicated by vertical and changing dip directions. They suggest a lacking or hidden part of the mass still S.E. of the southern pluton border.

These structures might have been caused by motion of a mass ascending from south-easterly direction; a northward motion which produces at the same time a drawing-out of the northern doming or elongation of the northern part of the pluton in S.W.—N.E. direction. Such an elongation may have originated the flow lines in that direction, which are bent in the central part of the pluton.

In the south-western part of the pluton a fragmentarily-known small doming is visible. The same parallel and steep flow lines as in the south-eastern section have been found in the south-western appendix or bottleneck.

All together the flow structures in the pluton suggest an upward moving mass, converging from all directions, which was strongly affected by the ascending mass from the south, causing the elongation and bending out of the northern doming.

9. Fracturing of granite.

In the pluton joint systems are found in amazing quantity. Normally the Mt. Aigoual granite is fractured by about six different systems of joints in each locality. Up to twelve systems have been measured in some places. By fracturing the granite falls apart in different sized and irregular blocks, frequently parallelepiped and column shaped. In fresh rock, joint planes are hardly visible; in a few cases faint marks of dislocation are observed. The explanation of the existence of fracturing may be contraction during cooling of the granite, or continued motion within the pluton after consolidation of the upper shell, or a combination of both causes. In either case, however, the resulting fracture systems cannot be quite arbitrary, there certainly will be a relation with the fabric of the rock and the shape of the massif.

The connection of flow structures and some joint systems is advocated by HANS CLOOS, BALK and others. They make a distinction between e.g. cross joints perpendicular to flow lines and longitudinal joints parallel to them. Many joint systems could not be placed systematically by these authors.

In the case in question it proved always to be possible to find some of the six or twelve joint systems in regular association with flow structures. But there is nothing specific which marks the thus found "cross" or "longitudinal" joints; apart from their relations to flow structures nothing

may distinguish them from the other fractures. Most joints however cannot be explained with reference to these directions of use; they seem to be useless from tectonic point of view. In this case it would be a rather arbitrary application, to select and use "cross" joints, etc., according to recipe.

In the method here applied all measured joint data are used in a statistical diagram. This is done in order to bring out clearly the possible regularities of jointing in the pluton as a whole. Though it has proved, so far, not to be possible to arrange all existing joint systems according to structural or other principles, this of course in no way excludes the possibility of a systematic arrangement. There may be suggested here a systematic coincidence between the fracturing of the rock and its mechanical properties in different directions. Any alteration in equilibrium of the mass, e.g. during cooling, contraction, renewed motion etc., will thus cause systematic joint systems. They cannot all be explained locally, but they will relate as a whole to the fabric of rock and massif.

Measurements of joint systems have been made at regular intervals throughout the pluton outcrop. An average diagram of fracturing of the pluton may reveal special regularities in the structure of the massif. Projections of poles of all joint planes are plotted in an equal area net, thus giving a petrofabric analysis of the whole pluton as far as its joints are concerned. Fig. 3 shows the statistical pole diagram of 80 measured joint planes.

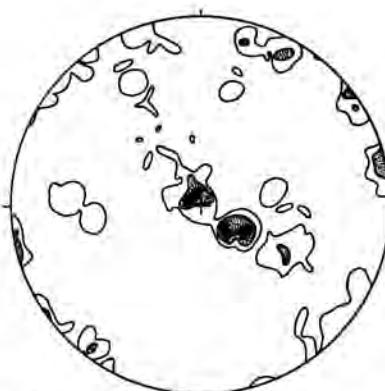


Fig. 3. Pole diagram of 80 joints throughout the pluton. Contours 5—4—2½%
(Southern hemisphere as in fig. 5, 6, 7, 9 and 10.)

The first striking regularity is the evident symmetry of the diagram. The N 120° E. central line of the diagram shows the same density distribution at both sides. The direction of this symmetry plane harmonizes with shape and directions of flow structures of the pluton except for the south-western corner. Probably this symmetry in the diagram is connected with the predominant motion in the direction of the line of symmetry and with the mentioned probability of supply of granite mass from a south-easterly direction.

By further analysis of the diagram two girdles may be observed. They resemble *B*-tectonites or a cross girdle diagram in petrofabrics, one NW—SE and the other horizontal, crossing each other at right angles. This indicates an orientation of nearly all measured joints either parallel to a horizontal NE—SW axis or in a more or less vertical arrangement.

Because joints largely will be related to, or oriented as a result of the fabric of the rock, in this case the only visible variable viz. flow structures, this diagram may show an average orientation of joints associated with the average orientation of flow. A glance at the structure map shows a predominance of flow lines in NE—SW direction which may be associated here with domination of cross joints in the densest parts of the horizontal girdle viz. in the NE and SW of the diagram.

In the same way a relation does not seem accidental between the densest part of the vertical girdle in the middle and south-east of the middle of the diagram with the ascent of mass from a south-easterly direction. This densest part of the diagram indicates the nearly everywhere observed "bedding planes" or primary flat-lying joints throughout the pluton.

10. *Dikes and faults in the pluton.*

Besides numerous ordinary fractures described in the preceding paragraph, dikes and faults have been observed in the pluton. In this paragraph joints filled with aplite, lamprophyre and porphyry and faults with well-developed slickensides or with visible dislocation will be discussed. More than fractures they may be expected to furnish information about motion in the rigid or semirigid upper part of the pluton. These data are plotted in the structure map.

The discussion of these phenomena is hampered however, by their limited number. The pluton body proved to be relatively poor in aplite dikes. Three of them may be called of cross joint origin, two are along primary flat-lying joints, two others show affinity to longitudinal joints and the few remaining cannot be labelled according to flow structures and shape of the massif. It will be clear that use of this nomenclature has no practical value here for tectonic purposes because of the small quantity of data. The few measurements of lamprophyre and porphyry dikes cannot be used either.

Faults usually with striae could be observed clearly in the north-western sector of the pluton. They are plotted in tectonogram fig. 4. The diagram is composed of equiangular projections of: a fault zone striking N 37 E, dipping steeply W; the sharp bisectrix N 50 E, 7 S of a system of diagonal joints or faults, crossing each other nearly perpendicular (84°); horizontal striae in a fault zone striking N 45 E; and three sets of striae on different fault planes in one locality, striking in a north-western direction and dipping S. The last three fault systems with striae are likely to represent the result of the same direction of motion recorded on different planes in this locality; combined they may give the direction of the original

motion, viz. S (1.2.3.) in the diagram, striking N 38 E and dipping 25 SW.

These plotted data show remarkable conformity; they have uniform directions between N 35 E and N 50 E, with gentle dips, mostly to the

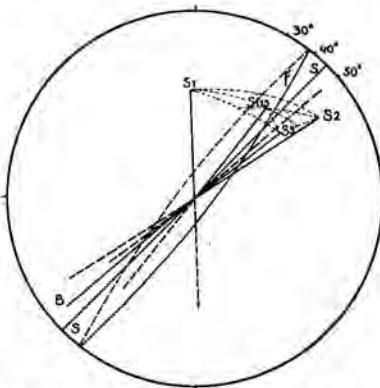
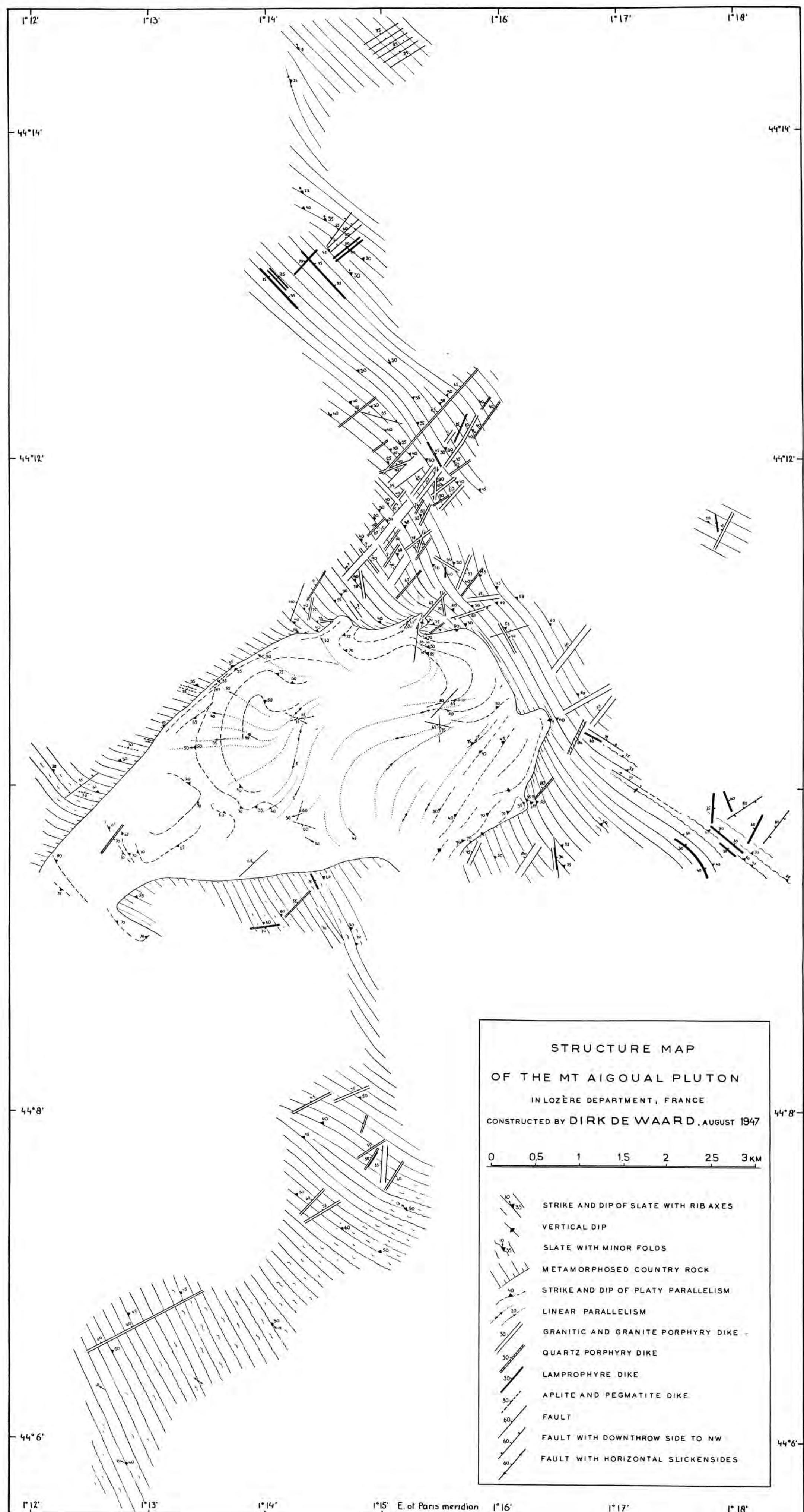


Fig. 4. Tectonogram of faulting and slickensiding in the north-eastern sector of the pluton. Equiangular projections of striae (S), the sharp bissectrix of diagonal joints (B) and a fault zone without striae (F). (Projections in the northern hemisphere.)

SW. Thus, motion is recorded in the rigid upper shell of the granite in SW—NE direction, slightly upward to the north-east.

In an earlier phase of the consolidation, motion in the same direction has taken place in the area north-west of the centre of the pluton. Two aplite dikes are several times broken and irregularly dislocated by sets of parallel steep faults, striking about N 40 E. Bent parts of the aplite dikes and fixing of the fault planes indicate a semirigid condition of the granite in which SW—NE motion has occurred.

The SW—NE motion thus found in the northern part of the massif may be interpreted as a result of differential expansion of the pluton in that direction during the last phases of the intrusion. Similar indications have been found by means of flow structures in earlier intrusive phases.



Anatomy. — *The digital formula in relation to age, sex and constitutional type.* I. By J. HUIZINGA. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of March 26, 1949.)

I. *Introduction.*

If we ask various persons to lay the pronated hand on a flat surface in such a way that the longitudinal axis of the hand is a prolongation of that of the forearm we shall find that:

1. The third finger ends the most distally.
2. The first finger ends the least distally.
3. The fifth finger follows the first in this respect.
4. Sometimes the second finger ends more distally than the fourth, sometimes the converse is true and sometimes these fingers are equal in length.

We may represent a given case as follows:

$$\text{III} > \text{IV} > \text{II} > \text{V} > \text{I}$$

This WOOD JONES (1944) terms the 'digital formula'.

If we confine ourselves to man, we find that the interindividual difference in digital formula consists in the varying relation between the fingers II and IV.

Although it is far from our intention to recommend memoirs such as those of CASANOVA as a source of scientific information, we feel justified in quoting CASANOVA's description of his conversation with the painter RAFAEL MENGS by way of introduction to our problem, the more so as it is our experience that the same dispute about the digital formula can be provoked in any group of people at the present day.

CASANOVA writes as follows (1871):

"Je me souviens qu'un jour je pris la liberté de lui faire observer, en voyant un de ses tableaux, que la main d'une certaine figure me paraissait manquée. En effet, le quatrième doigt était plus court que le second.

— Voilà une plaisante observation, me dit-il, voyez ma main! et il l'étendit.

— Voyez la mienne, répondis je, je suis convaincu qu'elle ne diffère pas de celle des autres enfants d'Adam.

— De qui donc me faites vous descendre? répliqua-t-il.

— Ma foi! lui dis-je après avoir examiné sa dextre, je ne sais à quelle espèce vous rattachez, mais vous n'appartenez pas à la mienne.

— Alors votre espèce n'est pas l'humaine, car la forme manuelle de l'homme et de la femme est bien celle que voilà.

— Je parie 100 pistoles que vous vous trompez, lui dis-je. Furieux de mon défi, il jette palette et pinceaux, sonne ses gens, et leur fait à tous exhiber leurs mains; sa colère fut grandi quand il reconnut que chez tous le doigt annulaire était plus long que l'index. Cependant il voulut bien sentir le ridicule de sa conduite et termina la scène par cette plaisanterie:

— Je suis charmé du moins d'être unique en mon genre sur un certain point."

In 1875 ECKER, was the first anatomist-anthropologist to bring up the problem of the individually-differing prominence of the fingers. Since then numerous publications have appeared; these may be classified as follows on the basis of certain general principles:

1. Prominence and sexual dimorphism
2. Prominence and age
3. Prominence in the light of typology
4. Prominence in the light of racial differences
5. Prominence in the light of problems of evolution
6. Prominence differences between the two hands.
7. Discussion of the causation of differences in prominence.

No single author, however, has dealt with all these aspects at once; this is partly due to the lack of due insight into anthropologo-phenomenological problems. In some cases the contradictory nature of the statements made can be ascribed to the fact that the groups studied were non-comparable. For instance, data furnished by the study of a group of females aged 4 to 71 years, in the absence of any previous investigation of differences according to age, may well lead to premature conclusions about the phenomenon in women.

Although the population of the Netherlands can certainly not be regarded as racially homogenous (in addition to definite Nordic, Alpine and Mediterranean characteristics we also find Baltic and Dinaric features), analysis of our material from this point of view is vitiated by so many uncertainties of race-diagnosis that its results are not worth reporting.

The amount of anthropoid material available was so small that we do not feel justified in including it in our study.

To give some idea of the scope of the problem of prominence, a discussion of the points 1 to 7 mentioned above will precede the description of our own observations.

WOLOTZKOV (1924) devised a useful nomenclature and we shall follow him in speaking of hands of the *radial type* (Rd.) when the *index finger extends more distally* than the ring finger, and of hands of the *ulnar type* (Uln.) when the converse holds. The results of our own investigations

lead us to use the term transitional type (T.) for hands in which the index and ring fingers extend equally far distally. This nomenclature will also be used in discussion of the work of other authors.

II. Survey of the literature.

1. Prominence and sexual dimorphism.

ECKER, who was the first anthropologist to publish an article on this 'oscillating character in the hand of men' (1875), found 24 examples of the ulnar type and one of the T. type in 25 outlines of the hands of American negroes aged 19 to 65 years. In his group of negresses (age 4 to 71 years) he found 15 Uln., 6 Rd. and one T.

With a total absence of criticism as to the age-composition of his groups (note the women) he concludes that there is an unmistakable sex difference: Rd. occurs more in women than in men. From a group of Europeans (composition unknown) he drew the same conclusions, although with some reservations.

The data reported by MANTEGAZZA (1877) (ages not stated) make it possible to calculate 75 % Uln. for 258 Italian women and 92 % Uln. for 336 Italian men. In these groups of individuals examined by him, only Rd. or Uln. is found. Thus the Uln. type predominates in both sexes, as found by ECKER (1875).

PFITZNER (1893) found just the contrary for skeleton hands of adult Alsatians: 70 % of 175 male hands were of the Rd. type and 79 % of 90 female hands, while RUGGLES (1930) and BAKER (1888) found in white Americans that more Uln. types on the whole occurred among men and more Rd. types among women.

WEISSENBERG (1895) is more inclined to agree with MANTEGAZZA; he found more Rd. in women than in men but the percentage of Rd. was always below 50 %, so that Uln. predominated in both sexes.

The next publication on sex differences in the digital formula did not appear until 1924 (WOLOTZKOI): in adult Russians Rd. was found in 62 % of 190 men and 77 % of 159 women. For adult Russian Jews the figures are 59 % Rd. in 29 men and 62 % Rd. in 58 women. These figures agree with those of PFITZNER (1893).

WOLOTZKOI then draws the inaccurate conclusion 'that the radial form is a special property of the female hand'.

RUGGLES (1930) concluded from a study of 402 male and 218 female 'white adults' that 'In white adults the ring finger in males is generally longer than the index finger and in females the reverse is found', a conclusion which he (*wrongly*) believes to be identical with those of PFITZNER, ECKER and MANTEGAZZA and contradictory to those of SCHULTZ (1924) and WOOD JONES, neither of whom, however, (the latter at any rate not in his book published in 1944) makes any mention of sexual dimorphism in a comparative sense. As we have not been able to obtain

a copy of the publication of BAKER (1888), we are doubtful as to what RUGGLES describes as BAKER's results (see above).

WECHSLER (1939) found more Uln. types in men than in women.

A number of the older anatomists (e.g. GEGENBAUER (1885), KOLLMANN (1886) believed Rd. to occur more in women than in men, although they did not give quantitative expression to this. The more frequent occurrence of Rd. in women together with the greater beauty of form (in the opinion of many) of the female hand, led various investigators to study the way in which artists depict the hands of their models. ECKER (1875) makes the following pronouncement: ... 'wherever a great artist has endeavoured, whether instinctively or consciously, to depict a hand of perfect beauty he certainly never makes the index finger appreciably shorter than the ring finger as this formation definitely gives the stamp of a lower type'.

WEISSENBERG (1895) did not confirm this in his study of Egyptian and Assyrian art. However, we shall confine ourselves to the mere outline of this aspect of the problem. Summing-up we may remark that the literature fails to provide us with unequivocal information on the sex differences in the digital formula. We shall return to this question in connection with our own investigations.

2. *Prominence and age.*

Much less has been written about the connection between relative length of fingers and age than about the difference between the sexes in this respect. We have already seen how ECKER (1875) put females aged 4 to 71 years in a single group labelled 'women' and then came to the conclusion that there was an unmistakable difference between the sexes.

WEISSENBERG (1895) classified his 574 male Jews according to age as well as sex. For the right hand he gives:

	5-10 yr.		11-20 yr.		21-30 yr.		31 yr. and older	
Rd.	30	45.5%	57	18.9%	29	23.6%	21	25.0%
Uln.	34	51.5%	222	73.8%	86	69.9%	59	70.2%
T.	2	3.0%	22	7.3%	8	6.5%	4	4.8%

From this it follows that boys from 5 to 10 years of age show the Rd type more frequently (45 %) than older boys (25 % for age about 20 yr.). Although the proportion of Rd types is higher between the ages of 5 and 10, WEISSENBERG's data show it to remain still below 50 %. He also remarks 'both types of hand may be found even in new-born infants'.

WOLOTZKOI (1924) arranged his Russian and Jewish men and women in age-groups as proposed by STRATZ (1903). In the periods from 1 to 4 and 8 to 10 years growth in breadth is regarded as predominating over that in height (first and second filling-out periods; turgor primus et turgor secundus), while those of 5 to 7 and 11 to 14 years correspond to relatively

greater increase in height (first and second periods of extension; proceritas prima et secunda). Then follows the maturation period from 15 to 20 years.

In order to facilitate comparison with our own findings we give WOLOTZKOI's figures in full:

Russians.

age-groups	Males				Females			
	number	% Rd.	% Uln.	% T	number	% Rd.	% Uln.	% T
1—5	13	77	21	2	18	60	28	12
5—7	59	81	15	4	67	76	21	3
8—10	53	64	30	6	53	77	21	2
11—14	78	50	42	8	184	63	29	8
15—20	61	53	44	3	52	67	33	—
21—older	190	62	34	4	159	77	21	2

Jews (Moscow).

age-groups	Males				Females			
	number	% Rd.	% Uln.	% T	number	% Rd.	% Uln.	% T
5—7	9	99	10	—	9	100	—	—
8—10	10	70	30	—	13	46	38	16
11—14	10	20	80	—	25	64	28	8
15—20	13	54	38	8	20	80	20	—
21—older	29	59	27	14	58	62	31	7

WOLOTZKOI concludes that the hands of children show a predominance of the Rd. type, but that with increasing age the number of Uln. forms increases. He also remarks that after the 20th year the converse phenomenon appears and the number of Uln. decreases in favour of Rd. forms. (This is not the case with the Jewesses, J. H.).

Although both WEISSENBERG and WOLOTZKOI examined Russian Jews, there is an enormous difference between their percentages for the different hand-types. WEISSENBERG finds 70 % of Uln. forms in adult males, whereas WOLOTZKOI gives 27 %. To what extent the cause of these discrepancies is to be sought in the number of subjects examined, we need not discuss here. It seems justifiable to conclude from both these investigations that the *Rd. type* is more frequent in *youth* than at a more advanced age, although the actual percentage reported by these two authors differ widely.

We shall discuss this conclusion further in connection with our own work.

As far as we know the only publication in which mention is made of the hands during intrauterine life is that of MIERZECKI (1946). Without specifying the number of subjects examined or their sex, he gives the following percentages for negroes (N) and whites (W)

	3th. month		4th. month		9th. month		10th. month	
	W	N	W	N	W	N	W	N
Uln.	33.3	57.1	14.7	50.0	33.5	63.2	31.8	54.6
T.	64.7	42.9	69.0	50.0	53.4	36.8	54.6	45.4
Rd.	2.0	—	16.3	—	13.1	—	13.6	—

In whites, thus, it appears that the index finger is longer than the ring finger in the third month in only 2 % of cases, whereas at birth the occurrence of this relation is 14 %. According to MIERZECKI, in negroes the hand during intrauterine life is invariably non-Rd. 'In the development of the hand of the negro, as in that of apes, no tendency whatever is seen for the length relation between the fingers to change in favour of the index finger.'

The high percentage of T. in the above table is remarkable. The number of cases in which the investigator is unable to make a decision is largely dependent on the method of examination used. What looks like a T form on simple inspection may be shown by accurate measurements to be Rd. or Uln., or vice versa. We may also agree to use the terms Rd. or Uln. only in cases where the difference in prominence exceeds a given number of millimetres. MIERZECKI is silent on this point.

In addition to this, the uncritical fashion in which he makes use of ECKER's data makes it impossible for us to have very great faith in his percentages.

3. Prominence in the light of typology.

In 1875 (long before the time of Kretschmer) ECKER remarked in his publication that, where the Rd type occurs in European men, this 'is found more frequently in tall, thin individuals than in those of short, stocky build'. We understand him to mean that this does not hold for women.

The Viennese investigator ROMICH (1932) describes the distribution of the two hand forms among the constitutional types distinguished by him.

The *progressive* constitutional type (P. T.) 'includes the sum of all progressive characteristics that are necessary for the static functions and that find expression in the transformations of the entire locomotor apparatus which are brought about by and adapted to these functions'. This part of his definition will, we believe, suffice to show the direction in which the characteristics are oriented.

The *conservative* constitutional type (C. T.) 'adapts itself, with respect to the locomotor apparatus, for the dynamic function and shows a pronounced accentuation of the rudimentary formation'.

Among 300 adults of both sexes (proportions not stated, J. H.) consisting of 150 P. T. and 150 C. T. individuals, 40 % showed Uln.,

51 % Rd. and 9 % T. forms. Classifying these according to C. T. and P. T. types, however, ROMICH found:

	P.T.	C.T.
Uln.	60%	20%
T.	8%	10%
Rd.	32%	70%

The percentage of Rd. forms for the whole group (51 %) appears thus to result from the occurrence of Rd. in 70 % of the conservative and 32 % of the progressive groups (these groups being numerically equal). ROMICH further states that in his progressive type (i.e. that in which the static function is to the fore) a narrow, gracile hand with long fingers is usually found, while the hand of the conservative type is short and broad with short fingers.

This would mean that the progressive hand corresponds to that of the leptosome type, in which, therefore, the Uln. form must predominate (60 %). If we are really justified in using the term leptosome here, it seems that this conclusion conflicts with the findings of ECKER already mentioned.

We are, further, inclined to believe that ROMICH's typology is no great acquisition.

4. *Prominence and racial differences.*

We have already mentioned the distribution of the two hand forms found by ECKER (1875) in American negroes. Although ECKER makes no definite statement, SCHAAFFHAUSEN (1884) after comparing the 'so-called savages with civilized human beings' considers himself justified in remarking that it is 'as ECKER was the first to show, a characteristic of culture versus savagery that the index finger increases in length relative to the fourth or ring finger'. Although we ourselves have not investigated racial differences, we do not wish to cast doubt on the possibility that differences may exist, even to a very large extent. But we see in the fact that judgements on this problem, pronounced at the end of the 19th century, were frequently based on the examination of 2, 3, 4 or 5 individuals of little-known races — and often without any attention to age or sex — a reason for not attaching undue value to such statements.

VIRCHOW (as stated by WECHSLER in 1939) asserted in 1898 that cultured peoples have the Rd. type while primitive peoples show the less elegantly proportioned hand of the Uln. type. VIRCHOW also remarked (in our opinion humorously): 'the tendency to a longer index finger happened to be greatest in the chief of the negro tribe studied: in him there was no difference between these two fingers'.

We have already mentioned the results of PFITZNER (1893) with Alsatians and those of WOLOTZKOI (1924) with Russians and Jews, as

well as those of WEISSENBERG (1895) with Jews. According to SCHULTZ (1924) the rule that primates (see below) have the Uln. type holds for negroes.

RUGGLES (1930) states in connection with his American whites: 'There is no indication of any relationship between the finger type and (either handedness or) eye color'.

Comparison of the results of racial investigations gives a confused picture.

5. Prominence and problems of evolution.

Here again we must mention ECKER who was the first to include anthropoids in his investigations. He remarks that in all the apes examined — but least in the Gorilla — the Uln. type occurs. The number of apes examined was very small.

SCHAFFHAUSEN (1884), the student of 'savages' already mentioned, states in connection with the occurrence of the Rd. type 'this is seen in none of the anthropoid animals; in these the ring finger is invariably the longer and the index finger the shorter'.

HARTMANN (1883) whom WEISSENBERG calls one of the greatest experts on the anthropoids, states, however, that Rd. prominence does occur in apes.

SCHULTZ (1924) remarks: 'Among all primates, except in a large percentage of white men and perhaps of some other human races, the fourth finger surpasses the second in length'. WOOD JONES (1944) also describes the Uln. form as typical of all 'monkeys and apes'. The Rd. type is 'definitely non-simian and constitutes a characteristic human specialisation'. He also remarks that, although the Rd. type is found only in man 'this formula is found only in a certain number of cases, that it may be present in one hand and not in the other and that it depends upon the greater development of the index finger'.

It is precisely this 'elongated index' that we may regard as a 'distinctly human specialisation'. In this connection WOOD JONES points to the differentiation of a separate deep index flexor muscle from the *musculus flexor digitorum profundus vel perforans*. 'The factor underlying the differentiation of this portion is undoubtedly the human specialisation of the index.'

From the literature it seems justifiable to conclude that, while the Rd. type may perhaps occur sometimes in anthropoids, the Uln. type is the rule in apes and anthropoids.

6. Difference in prominence between the two hands.

Little attention seems on the whole to have been paid to a possible difference in prominence between the right and the left hand. According to WEISSENBERG (1895) the Rd. type is commoner in the left hand. We quote the figures for his 574 Jewish boys and men.

RUGGLES (1930) comes to exactly the opposite conclusion, stating that the Rd. type predominates in the right hand in both men and women.

Rd. type (WEISSENBERG 1895).

	5–10 yr.		11–20 yr.		21–30 yr.		31 yr. and older	
	n = 30	45.50%	n = 57	18.90%	n = 29	23.60%	n = 21	25.00%
Right	n = 30	45.50%	n = 57	18.90%	n = 29	23.60%	n = 21	25.00%
Left	n = 29	43.90%	n = 75	24.90%	n = 39	31.70%	n = 27	32.10%

We have already seen that WOOD JONES (1944) says of the Rd. type 'that it may be present in one hand and not in the other'.

According to RUGGLES (1930) there is no relation between hand type and 'handedness'.

Rd. type (RUGGLES 1930).

	Men		Women	
	n = 56	28%	n = 57	52%
Right	n = 56	28%	n = 57	52%
Left	n = 40	20%	n = 51	47%

As is the case with practically all the aspects of the prominence problem which we have discussed here, a study of the literature once again presents us with contradictory opinions.



WOLOTZKOV (1924) determines the 'mean' prominence of the two hands of an individual by placing the hands, each with its longitudinal axis along the prolongation of that of the forearm, in the same plane, with the middle fingers tip to tip. If the distance between the tips of the index fingers is smaller than that between the tips of the ring fingers we have a (mean) Rd. type. As this method doubles the difference in prominence between II and IV and thus shows it more clearly, we also used it for our investigations (see photo).

Physiology. — *Phosphate-exchanges in purple sulphur bacteria in connection with photosynthesis.* By E. C. WASSINK, J. E. TJA and J. F. G. M. WINTERMANS. (From the Laboratory of Plantphysiological Research, Agricultural University, Wageningen.) (21st Communication on Photosynthesis *). (Communicated by Prof. A. J. KLUYVER.)

(Communicated at the meeting of March 26, 1949.)

Introduction.

Since the time VOGLER and his collaborators made the important discovery that in cultures of the chemo-autotrophic sulphur bacterium *Thiobacillus thiooxidans* the shift from the energy-producing to the energy-consuming phase of metabolism is accompanied by a phosphate exchange (1), suggestions as to something analogous in photo-autotrophic metabolism have not been lacking (2—7). Already VOGLER himself put the question: "..... is it possible to irradiate photosynthetic organisms in the absence of CO₂ and to store at least a portion of the radiant energy within the cell in a form which can later be used for CO₂ fixation in the dark?" (2). A certain uptake of CO₂ in the dark by photosynthetic organisms had already been demonstrated by sensitive methods (8), so that there appeared to be a fair chance that VOGLER's question would be answered in the affirmative. Nevertheless, the attempts undertaken so far to furnish direct proof herefor cannot be said to have shown very definite results (5, 9).

These attempts all made use of green cells. Now, in our opinion, there are some good reasons to give preference to purple sulphur bacteria for these studies. In the first place they are more closely related to the organisms VOGLER used. Some strains of purple bacteria even have been shown to be capable of an anaerobic, photosynthetic mode of life as well as of a heterotrophic, oxydative one. In the second place, in purple bacteria, both carbon dioxide and the hydrogen donor can be supplied at will. In the third place, previous studies had given some general idea of the kinetics of the metabolism of at least one strain (*Chromatium*, strain D).

We, therefore, decided to look for possible connections of photosynthesis and phosphate exchange in suspensions of *Chromatium*, strain D.

The method of phosphate determination was worked out chiefly by the second author, the results reported below were collected by the last mentioned author.

Methods.

The bacteria were grown in the medium, described earlier as "combination 23".

*) 20th Comm.: Ann. Rev. Biochem. 17, 559—578 (1948).

containing 0.24% sodium-l-malate and 0.16% sodium thiosulphate in an inorganic stock-solution (10). In the reported experiments, cultures from a small amount of dense inoculum were used after 1 day of incubation at about 27° in a light cabinet. They were centrifuged, suspended in the medium used in the experiments (see below), and centrifuged again. After this, the bacteria were resuspended in the same medium, and the experiments were started.

The general trend of an experiment was very simple. A suspension of bacteria in a glass cylinder of about 20 cm height and about 50 ml. contents was ventilated with a suitable gas stream, mostly consisting of oxygen-free nitrogen with either hydrogen or carbon dioxide added, and exposed either to light or to darkness. At the moment of changing conditions, or at intervals during an exposition, inorganic phosphate-P in the suspension medium was determined.

To this purpose, it was deemed advisable to separate the cells from the suspension medium as quickly as possible, and under sensibly the same conditions as during the experiment. In view of the experiments in light, centrifuging appeared unsuitable in this respect. At first, an experimental set-up was made in which the bottom of a glass cylinder was replaced by a cylindrical funnel with glass filter plate. Suspension liquid could be removed by suction during the fully undisturbed experiment.

Unfortunately, however, bacteria entered into the pores of the filter plate — of about 1 μ width — soon preventing rapid filtering. After a few trials the following simple procedure was found satisfactory. Cylindrical glass funnels were made, about 5 cm in diameter, and 6 cm in height, with a nearly flat bottom and a tube of about 1 cm width. A stiffly rolled strip of ordinary filter paper, about 2 cm broad and 30 cm long, was pressed into the upper end of the tube. About 5 ml. of suspension were now removed from the glass cylinder in which the bacteria were exposed to the experimental conditions, introduced into the funnel — which was either illuminated or in darkness — and ventilated with the same gas stream as used in the experiment. Then suction was quickly applied, upon which the suspension medium rapidly went through clear.

In order to have a chance upon measurable relative changes in PO₄-P, solutions of low phosphate contents had to be used; in general about 10 μg P/ml was taken. Initially, dilute phosphate buffers of pH ~ 8.0, according to CLARK and LUBBS, and to SÖRENSEN were used. Since the bacteria soon became inactive in these dilute media, 1% NaCl was added. A remaining drawback was the weak buffering capacity of the medium. In case of ventilation with gas containing CO₂, the buffering capacity could be increased by addition of sodium bicarbonate, which, however, had the disadvantage of preventing subsequent removal of CO₂ — e.g., when applying N₂ + H₂ afterwards — and, moreover, turned out to have an influence upon the phosphate determination.

In search for other suitable buffering systems it has to be observed that systems containing organic acids, as e.g., acetate, citrate, etc. are less advisable since these substances are likely to be used as sources of hydrogen and/or of CO₂ by the bacteria. Finally, a borate mixture was found suitable and was stood very well by the bacteria; PALITZCH' mixtures were used. Ten ml. solution containing 3 ml. M/20 borax, and 7 ml. M/5 boric acid + M/20 NaCl, pH ~ 8.0 were added to 40 ml. of a phosphate mixture containing about 10 μg P/ml. When ventilated in dark with N₂ + 1% CO₂, a decrease of pH was observed which, after about 3 hours, remained at about 7.15. B-concentrations used did not interfere with P-determination.

Phosphate-P was determined colorimetrically, according to the phospho-molybdic acid method first devised by OSMOND (1887). BELL and DOISY (1920) first used this method in connection with an organic reductant. Several modifications, differing in the sort of reductant, have since been described. We used that of LOWRY and LOPEZ (11), with ascorbic acid as reductant. Advantages of this method are the operation at moderate pH and the use of a rather dilute solution of molybdate.

Two ml. of suspension medium sucked through the filter (cf. above) were introduced into a measuring flask of 25 ml., and some acetate buffer, pH 4.0, was added. Subsequently,

2 ml. of a solution containing 1% of ammonium molybdate in 0.05 N. sulfuric acid, and 2 ml. of a 1% solution of ascorbic acid were added, and filled up to 25 ml. with acetate buffer. A stopwatch is started after addition of the reductant, and the blue color, gradually developing, is measured after 5 and 10 minutes with a "lumetron, model 400 A" colorimeter, using the red filter, and a solution without molybdic acid as reference. In general, the readings after 5 and 10 minutes turned out to be the same, so that each moment between was suitable for the determination. About 10 readings were taken from each sample, yielding one measurement. The phosphate content according to the reading was computed from a calibration curve obtained by submitting solutions of known phosphor contents to the same procedure.

The accuracy obtained may be indicated by the following figures: 20 measurements of blank P + B-solution (before introducing bacteria into it) yielded $P = 8.9 \pm 0.05$; $\sigma = 0.23 \mu\text{g P/ml}$.

The following experimental details still have to be mentioned. In the earlier experiments glass cylinders were used to take up the bacteria in the experiments (cf. above). As a rule two of these were used in each experiment, submitted to different conditions (e.g. light versus darkness, $N_2 + H_2$ versus $N_2 + CO_2$). The suspensions contained about 5 cmm bacteria/ml. The cylinders stood in a thermostatic water bath with glass sides, at about 29°, illuminated from 2 opposite sides with a 100 Watt incandescent lamp, yielding on either side an intensity of 8—9000 lux. In later experiments flat glass boxes of about $20 \times 10 \times 0.6$ cm were used instead of the cylinders.

During the experiments a flow-meter-controlled gas stream from a bomb was passed through the suspensions. Either $N_2 + 30\%$ H_2 , $N_2 + 30\%$ $H_2 + 5\%$ CO_2 , or $N_2 + 1\%$ CO_2 , obtained by mixing pure nitrogen with nitrogen containing 5% of carbon dioxide, were applied. The gasses were freed from oxygen by passing them over electrically heated copper gauze. With the aid of flow meters the gas streams in both cylinders or glass boxes were adjusted to equal velocity. At the top the vessels were closed except inlet and outlet of gas, the first ending in the suspension near the bottom. Before entering into the vessel with the bacteria, the gas passed a washing bottle with water at room temperature, to control evaporation of the suspension.

Unfortunately, so far, we were not in a position to follow simultaneously the gas exchange of the cells used, owing to the lack of suitable apparatus herefor. This is planned for a subsequent part of this investigation.

Experimental results and their preliminary discussion.

The first experiments tended to "translate" VOGLER's crucial experiments in terms of photosynthesis as closely as possible. A suspension of *Chromatium*, Strain D, in a borate mixture with about 10 $\mu\text{g PO}_4\text{-P/ml}$, pH ~ 8.0, under $N_2 + 30\%$ H_2 was submitted to a short dark period for "adaptation", and then illuminated. During the illumination period, a decrease of phosphate in the external medium is observed. After some hours, the light is turned off, and the gas phase replaced by $N_2 + 5\%$ CO_2 . Then, an increase in external phosphate is observed. The procedure may be repeated with the same result (fig. 1, curve a). It is obvious that these observations are concordant with a concept, analogous to the one proposed by VOGLER for the chemo-autotrophic sulphur bacteria. In the light, hydrogen may be taken up, and fixed in some form, involving phosphate uptake. If this form is an "energy rich phosphate", able to aid in the reduction of carbon dioxide in the dark, release of phosphate in a subsequent dark period may be expected.

It may be remarked already now that, in a M/15 phosphate buffer of pH ~ 8.0, WASSINK and KUIPER observed a long-lasting uptake of gas from $N_2 + H_2$, in the absence of CO_2 , even in cells submitted to special "starvation" treatments (in an investigation on the relation between redox potential and photosynthesis [12–14]). So far studies

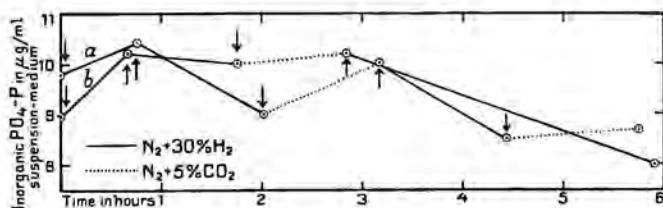


Fig. 1. Phosphate changes in suspension medium with *Chromatium*, strain D, in relation to light and darkness, and under various gas phases. Exp.s of 13, 14. 10. 48.

↓ : shift to darkness. ↑ : shift to light.

as to a "dark pick-up" of CO_2 have not yet been made. In the course of the mentioned investigation some incidental indications were obtained that cells, allowed to take up hydrogen in the light for a long time, already quickly thereafter produce considerable amounts of CO_2 in the dark.

Not always, however, the cells showed clear responses as to phosphate in the various phases of the above mentioned type of experiment (cf., e.g., fig. 1, curve b). This forced us to a more systematic study of the various phenomena involved.

The next observation made was a release of PO_4 in the dark (increase in the external medium) when freshly harvested, and washed cells were suspended as described. This PO_4 -liberation starts at once, and shows a tendency to decrease with time. In $N_2 + 30\%$ H_2 , and in $N_2 + 1\%$ CO_2 it was studied in greater detail and only little difference between these two gas phases was found (fig. 2, 3, 4, 5, 6, 8, 10; Table I). This release

TABLE I.
Release of PO_4 -P (increase of P in suspension medium) by *Chromatium*, strain D,
in darkness under various gas phases.
Suspension medium: borate buffer, pH ~ 8.0, with ~ 10 μg P/ml.

Ventilated with	Time	Total PO_4 -P released	Number of observations
<i>a</i> { $N_2 + 30\% H_2$	35'	0.53 ± 0.05	15
	120'	1.1 ± 0.13	12
	180'	1.8 ± 0.14	5
	285'	2.15 ± 0.25	2
<i>b</i> { $N_2 + 1\% CO_2$	120'	1.08 ± 0.18	12
	270'	2.19 ± 0.55	5

of PO_4 probably may be compared with that accompanying "endogenous respiration" in VOGLER's experiments. Notwithstanding the fact that, owing to the absence of oxygen, the *Thiorhodaceae* can not show a respiration-proper, some form of energy liberating metabolism is likely to continue in darkness, which may well be accompanied by a release of PO_4 .

Since neither hydrogen nor carbon dioxide are likely to be active in the dark in these bacteria — unless, eventually, after special treatments — it is perhaps not very surprising that the rate of release is fairly independent of the gas phases applied.

Next, the behaviour of the cells after admitting light, in $N_2 + 30\% H_2$, was studied in greater detail. The phosphate release is now replaced by phosphate uptake (fig. 1, 2, 3, 4, 7b, 8b); after a short dark period, when the tendency to release phosphate apparently still was strong, the uptake at first was sometimes small or even slightly negative still (cf. fig. 4, 7b) especially in dense suspensions (low average light intensity). Some typical experiments are represented in fig. 2 and 3; in each case, after a short dark period, the two halves of the same suspension were submitted oppositely to light and darkness, whereas, some 2 hours later, the conditions were reversed (fig. 2) or continued (fig. 3) without changing the gas phase.

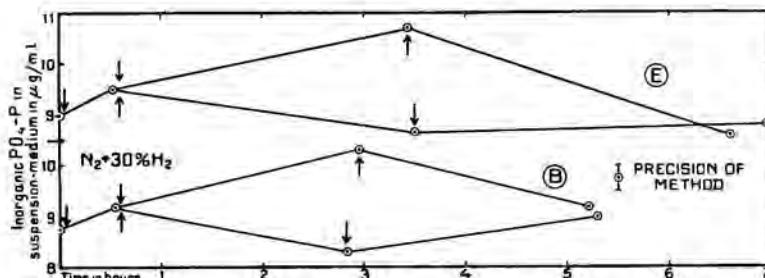


Fig. 2. Phosphate changes in suspension medium with *Chromatium*, strain D, in relation to light and darkness. Flushed with $N_2 + 30\% H_2$. E: Exp. of 17.11.48; B: Exp. of 10.11.48.
↓ : shift to darkness. ↑ : shift to light.

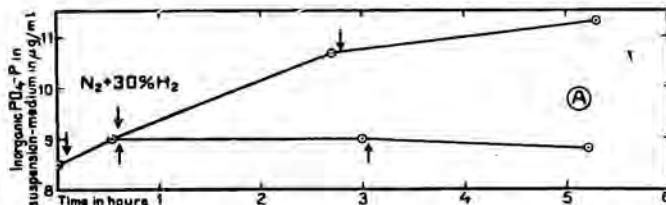


Fig. 3. Phosphate changes in suspension medium with *Chromatium*, strain D, in relation to light and darkness. Flushed with $N_2 + 30\% H_2$. Exp. of 9.11.48.
↓ : shift to darkness. ↑ : shift to light.

A number of experiments of the type—fig. 2 is summarized in fig. 4, showing the different behaviour in light and darkness very clearly. It may be concluded too, provisionally, that the rate of phosphate uptake in the light is higher after a longer dark period (fig. 4, Table II, a). The reaction upon renewed darkness will be discussed below.

Also in $N_2 + 1\% CO_2$, illumination as a rule causes uptake of phosphate, replacing the release occurring in the dark (fig. 5, a few experiments are summarized in fig. 6). Here, too, after a prolonged dark period the rate of uptake appeared increased (cf. Table II, b).

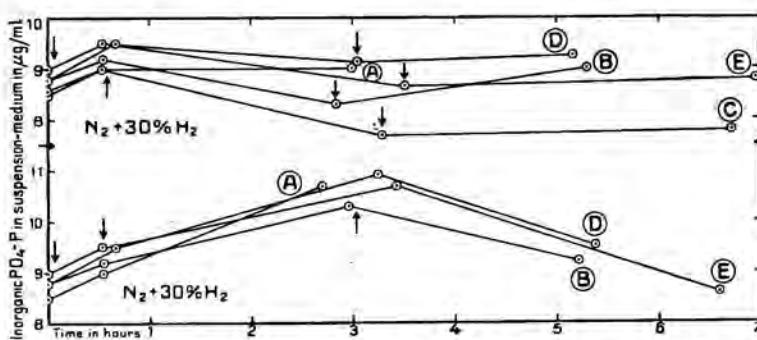


Fig. 4. Phosphate changes in suspension medium with *Chromatium*, strain D, in relation to light and darkness. Flushed with $N_2 + 30\% H_2$. Aliquots of the same culture indicated by equal characters. Exp.s of various dates.
 ↓ : shift to darkness. ↑ : shift to light.

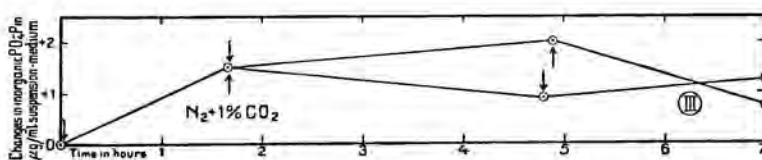


Fig. 5. Phosphate changes in suspension medium with *Chromatium*, strain D, in relation to light and darkness. Flushed with $N_2 + 1\% CO_2$. Exp. of 2.12.48.
 ↓ : shift to darkness. ↑ : shift to light.

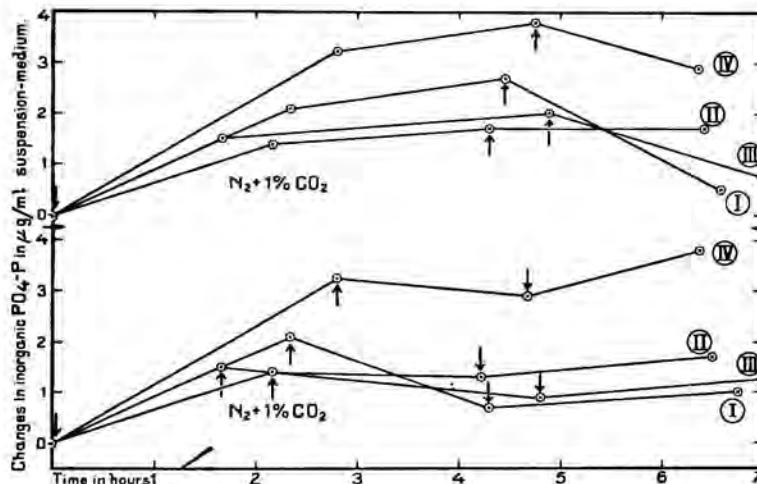


Fig. 6. Phosphate changes in suspension medium with *Chromatium*, strain D, in relation to light and darkness. Flushed with $N_2 + 1\% CO_2$. Aliquots of the same suspension indicated by equal numbers. Exp.s of various dates.
 ↓ : shift to darkness. ↑ : shift to light.

TABLE II.

Uptake of $\text{PO}_4\text{-P}$ (decrease of P in suspension medium) by *Chromatium*, strain D, in illumination under various gas phases.

Suspension medium: borate buffer, pH ~ 8.0 , with $\sim 10 \mu\text{g P/ml}$.

Condition during dark period	Duration of dark period	Condition during subsequent illumination	Time of exposure to light	Total $\text{PO}_4\text{-P}$ taken up in light	Number of observations
<i>a</i>	$\text{N}_2 + 30\% \text{H}_2$	30'	$\text{N}_2 + 30\% \text{H}_2$	75'	$0.29 \pm 0.1 \mu\text{g/ml.}$
				135'	$0.49 \pm 0.15 \text{ "}$
				195'	$0.80 \pm 0.32 \text{ "}$
				235'	$1.38 \pm 0.70 \text{ "}$
	same	2 hours	same	120'	0.80 ± 0.26
				240'	1.11 ± 0.16
	same	3 hours	same	120'	1.20 ± 0.10
	same	4.5 hours	same	120'	2.40 ± 0.2
	$\text{N}_2 + 10\% \text{CO}_2$	2 hours	$\text{N}_2 + 10\% \text{CO}_2$	120'	0.31 ± 0.16
				240'	0.20 ± 0.17
<i>b</i>	same	4.5 hours	same	120'	1.0 ± 0.4
					5
<i>c</i>	$\text{N}_2 + 30\% \text{H}_2$	30'	$\text{N}_2 + 30\% \text{H}_2 + 5\% \text{CO}_2$	150'	0.0 ± 0.14
					5

However, another feature is very obvious, *viz.*, that the uptake in light under $\text{N}_2 + 1\% \text{CO}_2$ is much smaller than under $\text{N}_2 + 30\% \text{H}_2$ (fig. 8). In some cases with CO_2 it even hardly reaches significantly positive values (cf. Table II, *b*). The same can be said, comparing $\text{N}_2 + \text{H}_2 + \text{CO}_2$ with $\text{N}_2 + \text{H}_2$, also here the presence of CO_2 reduces the phosphate uptake in the light practically to zero, and the difference in the general trend of the curves is very obvious, notwithstanding some incidental exceptions (fig. 7, Table II, *c*).

It should be emphasized that most of the curves of *a* and *b* in fig. 7 and 8 have been obtained one by one in the same experiment so that the variable "activity" of the cells to take up and give off phosphate cannot have interfered with these results.

In the type of experiment as shown in fig. 2 and 5 (cf. also fig. 4 and 6) darkness was again given after light in one of the two aliquots of suspension. The number of these observations is still too small to draw definite conclusions. It is obvious, however, that phosphate uptake stops or is converted into release. It would seem that in $\text{N}_2 + \text{CO}_2$ the release is somewhat more definitely pronounced than in $\text{N}_2 + \text{H}_2$ (cf. fig. 4 and 6). The reaction upon darkness after light without changing the gas phase throws some doubt upon the meaning of the release of phosphate after changing from H_2 to CO_2 . More and quantitative studies, taking into account the gas exchange, are required here. In general, however, it may be admitted that under all conditions the cells are provided with organic

substances, which may be converted under release of phosphate and probably of CO_2 , as soon as conditions for active rebuilding of $\sim\text{ph}$ are absent, especially when the source of energy; light, is lacking.

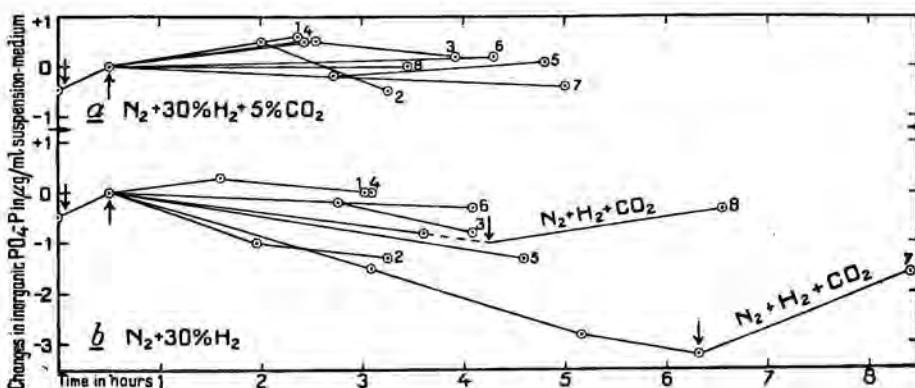


Fig. 7. Phosphate changes in suspension medium with *Chromatium*, strain D. Influences of various gas mixtures during illumination and eventually subsequent darkness. Aliquots of suspensions were separated after $\pm \frac{1}{2}$ hour adaptation in dark with $\text{N}_2 + 30\% \text{H}_2$.

(Numbers indicate parallels with the same culture.) Exp.s of various dates.

↓ : shift to darkness. ↑ : shift to light.

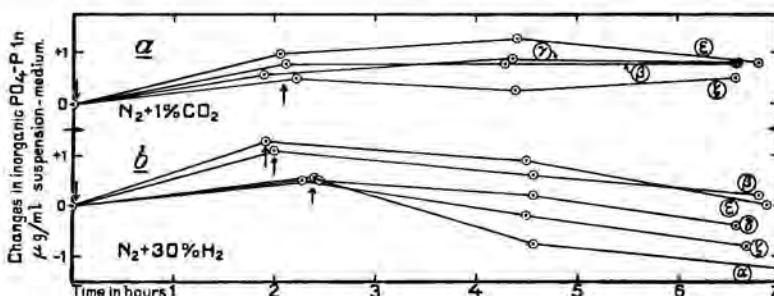


Fig. 8. Phosphate changes in suspension medium with *Chromatium*, strain D. Influence of various gas mixtures in dark and in light in parallel experiments (indicated by characters.) Exp.s of various dates.

↓ : shift to darkness. ↑ : shift to light.

General Discussion.

In the foregoing, uptake and release of phosphate have been observed. It is obvious, *a priori*, that neither of the two can last infinitely with the same speed, even under as such favorable conditions. It is obvious too, therefore, that both uptake and release will show an asymptotic course in relation to time. Furtheron, it may be admitted that the rate of reversion upon a change of conditions also depends upon the nature and duration of the condition before the change. Finally, if indeed there is a relation between phosphate metabolism and photosynthesis of the type VOGLER found in *Thiobacillus thiooxidans*, we may expect that in the light, in the

absence of carbon dioxide, cells will accumulate phosphate to a higher degree than in its presence. In darkness, we may expect phosphate bond energy to be released as part of the energy reserve of the cell, used up in darkness.

From these general statements, and founded upon the results of VOGLER and UMBREIT, we would expect purple sulphur bacteria to influence the phosphate content of a limited amount of medium in which they are suspended, in a way schematically represented in fig. 9.

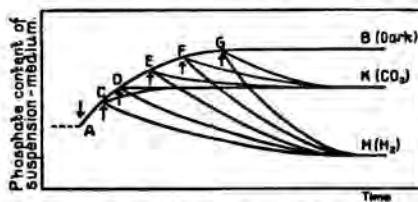


Fig. 9. Schematical representation of expected PO_4 -shifts under different conditions.
See text.

↓ : shift to darkness. ↑ : shift to light.

Cells brought into darkness from a well-fed condition will start releasing phosphate from a level A, which phosphate accumulates in the medium. Since neither hydrogen nor carbon dioxide are known to act as energy sources in the dark for *Chromatium*, there is no reason to expect that the release of phosphate will depend either on the presence of hydrogen or of carbon dioxide. The release will show an asymptotic course (AB). Upon illumination, e.g. in $\text{N}_2 + \text{H}_2$, an uptake of phosphate will occur which may be expected to start at higher rate, the more phosphate has been released before, so, e.g., the longer the dark period has lasted. The curves starting at C, D, E, etc., will be expected to strive towards virtually the same level, indicated by H. The probability that this level is, e.g., dependent on light intensity, will not be discussed here. Under $\text{N}_2 + \text{CO}_2$ we may expect a behaviour fundamentally analogous to that under $\text{N}_2 + \text{H}_2$, only pointing to a significantly higher ultimate phosphate content in the external medium (level K).

In fig. 10 we have collected from our total experimental material, the

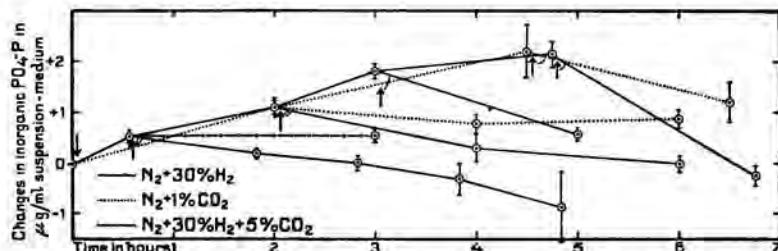


Fig. 10. Summarized representation of actual PO_4 -shifts from table I and II.
Cf. fig. 9 and text.

↓ : shift to darkness. ↑ : shift to light.

data pertaining to the points outlined above. Notwithstanding the fact that on various points the material still is somewhat scanty (expressed by the large mean errors of some points and the lack of data on shorter intervals) it is clear that, in general, fig. 10 shows a very distinct analogy with the scheme of fig. 9. This would seem to indicate that, in general, the situation answers to the outline given.

It should, however, not be overlooked that the experiments made so far, do not prove that, in photosynthesis of *Chromatium*, in the absence of CO₂, part of the energy of the light actually is accumulated as phosphate bond energy. To prove this would require recovery of these energy rich phosphates from the cells which determinations have not yet been attempted. The findings merely are very well in accordance with the supposition that indeed light energy is accumulated in this form.

It may be useful still to emphasize a rather fundamental difference between VOGLER's case and that of the photosynthetic bacteria. In *Thiobacillus*, the oxidation of the hydrogen donor is achieved by oxygen, and is, or at least may be conceived to be fundamentally independent of carbon dioxide. This easily leaves room for a storage of energy from the oxidation process in some form which may, or may not, be used for the reduction of carbon dioxide. In *Chromatium*, on the other hand, carbon dioxide itself is the ultimate oxidant of the hydrogen donor; for the transfer of hydrogen light is essential. This, however, would seem to leave much less room for an independent storage of (light) energy at the expense of the hydrogen donor, and requires a much more specialized mechanism. The observation, alluded to above, that excessive hydrogen uptake may occur without supply of carbon dioxide (13, 14), indicates that indeed under certain conditions — e.g. pH ~ 8.0 — hydrogen may be assimilated more or less independent of carbon dioxide. It is interesting in this respect that so far no indications were found for an appreciable CO₂-uptake without hydrogen during prolonged illuminations. The results reported in the present paper suggest that the mentioned uptake of hydrogen — which was not measured in the present study — is connected with phosphate uptake. It is perhaps most plausible to assume, that phosphate-shifts are connected with the transfer of energy from the pigment-protein system (cf. [15, 16, 12]).

In view of an estimation of the quantitative importance of the phosphate shifts observed, it may be of interest to compare their magnitude with the presumable phosphate content of the cells. If we suppose a "normal" phosphor content in *Chromatium*-cells, this may be fixed at ~ 5 % of the dry weight, as P₂O₅; the dry weight may be 15 % of the fresh weight. This would mean that the bacteria contain 0.75 % of their fresh weight as P₂O₅, or 0.33 % P. The suspension density as a rule was about 5 cmm/ml., or about 5 mg/ml. Thus, the bacteria in 1 ml. contain about 17 µg P. On exposure to N₂ + H₂ in the light, an uptake of 1.5 µg P/ml is not rare. It thus appears that the bacteria may undergo shifts in phosphate amounting to about 10 %. In order to achieve that these shifts will definitely surpass

the limits of the method of determination, rather dense bacterial suspensions are advisable. In order to obtain sufficient illumination inside the suspension, in our more recent series we used glass boxes instead of cylinders to suspend the bacteria.

Summary.

In suspensions of *Chromatium*, strain D, in borate buffer, pH ~ 8, containing some phosphate, shifts in phosphate content of the medium were found correlated with shifts from light to darkness, and from a gas phase consisting of $N_2 + H_2$ to one containing $N_2 + CO_2$.

The principal observations were as follows. Bacteria taken from the culture medium ("comb. 23", see [10]) release phosphate in darkness, at about equal rates under $N_2 + H_2$ as under $N_2 + CO_2$. Shift to light results in a marked uptake of phosphate under $N_2 + H_2$ and in a much smaller one under $N_2 + CO_2$. Shift from light under $N_2 + H_2$ to darkness under $N_2 + CO_2$ results in a marked release of phosphate. Change from light to darkness without change of gas phase results in a certain release, too.

As a whole the data collected so far show a marked similarity with VOGLER's findings with *Thiobacillus thiooxidans*, if $N_2 + H_2$ in light is parallelized with sulphur oxidation. They thus form a preliminary support for the hypothesis that *Chromatium* in the light is capable of building up energy rich phosphate bonds which are broken down in part when CO_2 becomes available and/or light is withdrawn. The phosphate exchanges observed so far can amount to 10 % of the phosphate content of the bacteria.

Correlations with gas exchange were not made so far. The investigation is being continued.

REFERENCES.

1. K. G. VOGLER and W. W. UMBREIT: J. Gen. Physiol. **26**, 157—167 (1943).
2. ———: J. Gen. Physiol. **26**, 103—117 (1943).
3. S. RUBEN: J. Amer. Chem. Soc. **65**, 279—282 (1943).
4. C. B. VAN NIEL: Physiol. Reviews **23**, 338—354 (1943).
5. R. L. EMERSON, J. F. STAUFFER and W. W. UMBREIT: Amer. J. Bot. **31**, 107—120 (1944).
6. H. GAFFRON: Currents in biochem. research 325—348, (Interscience Publishers, N. Y., N. Y., 1946).
7. B. KOK: Enzymol. **13**, 1—56 (1948).
8. E. D. MCALISTER and J. MYERS: Smithsonian Inst. Publ. Misc. Coll. **99** (6) (1940).
9. S. ARONOFF and M. CALVIN: Plant Physiol. **23**, 351—358 (1948).
10. J. G. EYMERS and E. C. WASSINK: Enzymol. **2**, 258—304 (1938).
11. O. H. LOWRY and J. A. LOPEZ: J. Biol. Chem. **162**, 421—428 (1946).
12. E. C. WASSINK: Antonie v. Leeuwenhoek, J. Microbiol. Serol. **12**, 281—293 (1947).
13. ———: Abstracts Comm. 4th Intern. Congr. Microbiol. 173—174 (Copenhagen, 1947) and 4th Intern. Congr. Microbiol. Rep. Proc. 455—456 (Copenhagen, 1949).
14. E. C. WASSINK and F. J. KUIPER: Enzymologia (in preparation).
15. E. C. WASSINK, E. KATZ and R. DORRESTEIN: Enzymol. **10**, 285—354 (1942).
16. R. DORRESTEIN, E. C. WASSINK and E. KATZ: Enzymol. **10**, 355—372 (1942).

Botany. — *De hypothese voor de erfelijkheidsformules van de twee zuivere lijnen I en II van Phaseolus vulgaris op grond van kruisingsproeven. I. By G. P. FRETS. (Communicated by Prof. J. BOEKE.)*

(Communicated at the meeting of January 29, 1949.)

In een vroegere mededeling (Proceed. 50, p. 798, 1947) stelden we voor de bonen van de I- en II-lijn de formules $L_1 L_1 \dots L_6 L_6 B_1 B_1 \dots B_3 B_3 b_4 b_4 \dots b_6 b_6 th_1 th_1 \dots th_6 th_6$ en $l_1 l_1 \dots l_6 l_6 b_1 b_1 \dots b_6 b_6 \dots Th_1 Th_1 \dots Th_3 Th_3 th_4 th_4 \dots th_6 th_6$ op, of eenvoudiger geschreven $L_1 L_2 B_1 b_2 th_1 th_2$ en $l_1 l_2 b_1 b_2 Th_1 Th_2$. De bonen van de I-lijn zijn lang, breed en dun, die van de II-lijn zijn kort, iets minder breed en dik. Het verschil van de lengten is veel groter dan dat van de breedten en de dikten. Bij de formules worden voor de erfelijkheid van de verschillen der afmetingen 6 paar genen aangenomen. De grootte-toename door één gen is voor alle 3 afmetingen relatief even groot. Steeds zijn één gen voor de lengte, één voor de breedte en één voor de dikte tegelijk werkzaam. Om het zoveel grotere verschil van de lengten dan van de breedten en de dikten te verklaren, wordt aangenomen, dat bij de bonen van de I-lijn er 6 genen voor de lengte in de homozygote, dominante vorm (als LL) aanwezig zijn, terwijl er slechts 3 genen voor de breedte in de homozygote, dominante vorm (als BB) aanwezig zijn. Bij bonen van de II-lijn zijn er slechts 3 genen voor de dikte in de homozygote, dominante vorm (als Th Th) aanwezig. Van de 6 genen voor de afmetingen komen dus de genen 4—6 bij de breedte en bij de dikte alleen in de homozygote, recessieve vorm (als bb en th th) voor. Daarom schrijven we de formules ook $L_1 L_2 B_1 b_2 th_1 th_1$ voor de bonen van de I-lijn en $l_1 l_2 b_1 b_2 Th_1 th_2$ voor de bonen van de II-lijn. Volgens deze schrijfwijze hebben we te doen met een kruising volgens het tetrahybride schema. Na de kruising ontstaan er in de splitsingsgeneraties verschillende nieuwe vormen als resultaat van genencombinaties op grond van het kruisingsschema zo b.v., voor de lengte en de breedte de vormen $L_1 L_2 B_1 b_2$ en $L_1 l_2 B_1 b_2$. Er zullen dus, — we wezen hier reeds op in onze eerste mededeling (l.c.) —, erfelijke variaties met een zeer grote lengte en een grote breedte en ook die met een niet zeer grote lengte en een grote breedte voorkomen, m.a.w. onder de bonen met de grootste breedten ($B_1 b_2$) zullen er zijn met zeer grote lengte ($L_1 L_2$), doch ook met een niet zeer grote lengte ($L_1 l_2$). De grens tussen bonen met een zeer grote en die met een niet zeer grote lengte, legden we bij 15.5 mm. Van bonen met een grote breedte is de breedte groter dan 8.5 mm. Bonen van de 2e groep, dus met de form. $L_1 l_2 B_1 b_2$ hebben een hoge LB-index.

In onze eerste mededeling voegden we hieraan de opmerking toe, dat

individueel voortkweken van verschillende varianten de beslissing over de juistheid der hypothese zou kunnen brengen (I.c., p. 804). Met het oog op dit gezichtspunt bespreken we hier de varianten van de lengte en de breedte $L_1 L_2 B_1 b_2$ en $L_1 l_2 B_1 b_2$ van F_3 -1935. We bedenken daarbij, 1e dat er duidelijk erfelijkheid is bij de op elkaar volgende generaties, — er is een vrij grote positieve correlatie, I.c., p. 800 —, doch er is ook een grote niet-erfelijke variabiliteit, blijkende uit de hoge standaard-deviatie en variatie-coëfficiënt (tab. 1 en 2). De grote niet-erfelijke variabiliteit zal het resultaat van ons onderzoek minder evident maken. Ook is het volstrekte bewijs der juistheid onzer hypothese niet te leveren. 2e De grond voor onze aanname, dat steeds een gen voor de lengte, één voor de breedte en één voor de dikte tegelijk werkzaam zijn, vinden we in de positieve correlatie tussen de afmetingen voor een willekeurig materiaal en voor de negatieve correlatie in groepen bonen met een zelfde gewicht, als uiting van „compensational growth". Er is bij de groei der bonen een regulerende factor

TABLE I. The variation of beanyields of the F_3 - and the F_4 -seedgenerations and of the I- and the II-line of 1935. (Only the variation of the length is given in this table.)
The length

	Pl.	n	$M \pm m$	$\sigma \pm m$	V	Gr. var.	Sm. var.	Var. r.
I-line	11	39	16.59 ± 0.24	1.54 ± 0.18	9.3	18.5	12.0	6.5
	3	38	15.19 ± 0.26	1.61 ± 0.18	10.6	18.3 ¹⁾	11.1 ²⁾	7.2
	9	58	14.33 ± 0.22	1.59 ± 0.15	11.1	19.1	11.1	8.0
	7	60	14.3 ± 0.14	1.08 ± 0.1	7.6	17.5	11.3	6.2
II-line	22	48	12.08 ± 0.09	0.64 ± 0.07	5.3	13.0	9.8	3.2
	25	87	11.02 ± 0.09	0.87 ± 0.07	7.9	12.9	8.6	4.3
	30	30	10.49 ± 0.13	0.73 ± 0.1	7	11.9	8.6	3.3
F_3 -seedgen- eration	73	35	15.9 ± 0.24	1.39 ± 0.17	8.8	18.3	12.0	6.3
	38	52	14.58 ± 0.17	1.21 ± 0.12	8.2	17.2	11.9	5.3
	44	90	13.66 ± 0.09	0.84 ± 0.06	6.2	15.4	10.9	4.5
	61	122	12.72 ± 0.06	0.65 ± 0.04	5.1	15.0	10.4	4.6
	97	58	12.01 ± 0.12	0.94 ± 0.09	7.8	14.3	10.2	4.1
F_4 -seedgene- ration	119	51	16.32 ± 0.17	1.18 ± 0.11	7.2	19.2	13.2	6.0
	133	51	15.16 ± 0.11	0.78 ± 0.08	5.2	16.5	13.5	3.0
	150	46	14.21 ± 0.16	1.1 ± 0.11	7.7	16.3	11.9	4.4
	178	45	13.26 ± 0.11	0.72 ± 0.08	5.4	14.5	11.8	2.7
	187	42	12.19 ± 0.1	0.63 ± 0.07	5.2	13.5	10.1	3.4
	198	41	11.04 ± 0.1	0.67 ± 0.07	6.1	12.8	9.5	3.3

¹⁾ Then follows 17.6. ²⁾ Then follows 12.

The variation of the beanyields of the pure lines I and II, — that is non-hereditary variation —, is great; it is greater of the I-line than of the II-line. The variation of the beanyields of the F_3 - and the F_4 -generation, that is hereditary and non-hereditary variation is not greater than that of the I-line and of the II-line (cf. tab. 2, p. 425).

TABLE 2. The mean dimensions and weights and the variation of the beans of the I- and the II-line in 1935—1937. (Only the lengths and the weights are mentioned in this table) ¹⁾.

The length									
Year	The lines	n	M ± m	$\sigma \pm m$	V	Gr. var.	Sm. var.	Var. r.	
1935	I-line	774	15.2 ± 0.07	1.78 ± 0.05	11.7	21.6 ²⁾	10.1	11.5	
1935	II-line	802	11.1 ± 0.03	0.84 ± 0.02	7.5	13.7	8.6 ²⁾	5.1	
1936	I-line	1617	14.2 ± 0.02	0.99 ± 0.02	7	18.1	10.2	7.9	
1936	II-line	605	11.16 ± 0.03	0.81 ± 0.02	7.3	13.3	7.3	6	
1937	I-line	2383	15.2 ± 0.02	1.13 ± 0.02	7.4	19.0	9.7	9.3	
1937	II-line	1292	11.0 ± 0.02	0.80 ± 0.02	7.3	14.9 ³⁾	6.4	8.5	
The weight									
1935	I-line	895	52.5 ± 0.5	16.4 ± 0.4	31	127 ⁴⁾	10	117	
1935	II-line	799	42.4 ± 0.3	7.9 ± 0.1	19	63	15	48	
1936	I-line	1610	50.7 ± 0.3	10.4 ± 0.2	21	89	21	68	
1936	II-line	605	45.1 ± 0.4	8.6 ± 0.2	19	66	10	56	
1937	I-line	1303	63.8 ⁵⁾ ± 0.4	13.1 ± 0.3	21	99	22	77	
1937	II-line	1298	43.8 ± 0.2	6.5 ± 0.1	15	64	15	49	

¹⁾ The F_3 - and following generations are not included in the table. The averages are not comparable here: at the basis of the material is the selection of the initial beans.

²⁾ Then follows 19.6 mm.

³⁾ Then follows 13.2 mm.

⁴⁾ Then follows 105 cg.

⁵⁾ The mean weight of 1937 is much greater than that of 1935. The mean thickness is also very great, whereas that of 1935 is small.

tussen de afmetingen (1947,l.c.). 3e Onder de uitsplitsingen van de F_3 - en F_4 -zaadgeneratie zullen ook bonen als de oudervormen zijn. Dergelijke uitzonderlijke uitsplitsingen zullen echter phaenotypisch als regel verschillen van de uiterste varianten van de I- en de II-lijn (1947).

We groepeerden van het materiaal van F_3 -1935 de bonen met de grote breedten volgens breedte-klassen en tekenden de bij de bonen van iedere breedte-klasse behorende lengte aan. Hetzelfde deden we voor vergelijk-materiaal van de I- en de II-lijn. We gaan dan na, of in de groepen van bonen met de grootste breedten meer bonen met een niet zeer grote of kleine lengte voorkomen dan in de overeenkomstige groepen van bonen van de I-lijn. We vinden het volgende: In tab. 3 zijn van het F_3 -materiaal van 1935 de bonen met de grootste breedten opgenomen. Ter vergelijking zijn bonen van de I-lijn en van de II-lijn er naast geplaatst. We zien (tab. 3), dat de grootste breedte van bonen van de II-lijn van 1935 is, $b = 9.8$ mm. De bonen van de II-lijn komen welhaast voor ons onderzoek niet in aanmerking. Er zijn in het I-materiaal 5 bonen met een breedte van $b = 13.0$ —11.5 mm, dit zijn grotere breedten dan bij de F_3 -bonen van 1935 voorkomen. Al deze 5 bonen hebben een zeer grote lengte; ook der-

TABLE 3. The F_3 -seedgeneration of 1935. The number of beans with a length that is not very great (15.5 mm and smaller) and with a very great length (15.6 mm and greater) for beans with a great breadth (8.6 mm and greater) and comparison beans of the I- and the II-line. sm. l. = number of beans with a length of 15.5 mm and smaller. gr. l. = number of beans with a length of 15.6 mm and greater. gr. v. = greatest variation; sm. v. = smallest variation.

1935

F_3 -seedgeneration							I-line ¹⁾					
Br.	n	Sm. l.	Gr. l.	m_1	Gr. v.	Sm. v.	n	Sm. l.	Gr. l.	m_1	Gr. v.	Sm. v.
130							1	0	1		216	
121							1	0	1		179	
117							1	0	1		193	
115							2	0	2	184.5	186	183
113	1	1	0			152	2	0	2	179	191	167
112	1	0	1		183		5	0	5	180.8	196	161
111	1	0	1		164		5	1	4	161.6	173	148
110	4	1	3	162.3	172	144	6	0	6	173.8	183	162
109	3	1	2	159.3	170	141	6	0	6	168.8	182	158
108	4	0	4	167.8	172	160	9	0	9	170.9	183	156
107	7	4	3	158	176	146	15	2	13	164.7	180	146
106	2	1	1	149.5	158	141	9	1	8	174.7	185	155
105	12	3	9	162.3	183	146	11	1	10	166.3	177	148
104	10	6	4	154	166	141	18	1	17	168.8	186	147
103	34	19	15	154.6	176	137	20	2	18	168.8	196	153
102	31	20	11	151.8	166	136	19	0	19	165.4	181	157
101	32	22	10	151.8	178	136	12	1	11	171.2	183	155
100	47	38	9	149.1	174	130	17	1	16	167.1	184	148
99	61	49	12	148.6	182	132	16	0	16	166.9	176	160
98	77	63	14	149.3	175	136	26	5	21	165.2	192	142
97	97	80	17	147.1	169	127	39	4	34	163.6	187	151
96	134	115	19	146.5	167	132	20	6	14	160.5	181	140

1) We find for the material of the II-line that there are 2 beans with the great breadth (i.e. the greatest breadth that occurs) $b = 9.8$ mm, the length is resp. 12.0 and 11.8 mm. There is one bean with the breadth, $b = 9.7$ mm; the length is $l = 13.0$ mm. There are 3 beans with the breadth, $b = 9.6$ mm; the length is resp. 12.6, 12.4 and 12.2 mm.

gelijke zeer grote lengten ontbreken onder de F_3 -bonen. Onder de 57 bonen van de I-lijn van 1935 met een breedte van 11.3—10.6 zijn er twee met een lengte van resp. 14.8—14.6 mm, d.w.z., dat er onder de bonen met de grote breedten 11.3—10.6 mm, van de I-lijn, d.i. van bonen met de form. $L_1 L_2 B_1 b_2$ van de lengte en breedte, er 2 zijn met de niet zeer grote lengte 14.8 en 14.6 mm, als niet-erfelijke varianten hier. Bovendien is er één boon met de lengte $l = 15.5$ mm, d.i. de grenswaarde. Er zijn 23 bonen in het F_3 -materiaal van 1935 met de grote breedten 11.3—10.6 mm. Deze aantallen bonen van de F_3 -generatie en van de I-lijn zijn niet direct vergelijkbaar: het totale aantal bonen van de I-lijn is veel kleiner dan van de F_3 -generatie. De I-lijn echter is een zuivere lijn en bevat slechts niet-

erfelijke variaties, terwijl de F_3 -generatie een zeer heterogene erfelijke samenstelling heeft.

Onder deze 23 F_3 -bonen met de breedten 10.6—11.3 mm, zijn 8 bonen met een niet zeer grote lengte (d.w.z. $l = 15.5$ mm, en kleiner). Er zijn daarbij met een lengte van 14.1—14.4 mm, d.i. met een zo kleine lengte, als ze onder bonen van de I-lijn met deze breedten niet voorkomen. Onder de 11, resp. 18 bonen van de I-lijn met een breedte van 10.5 en 10.4 mm,

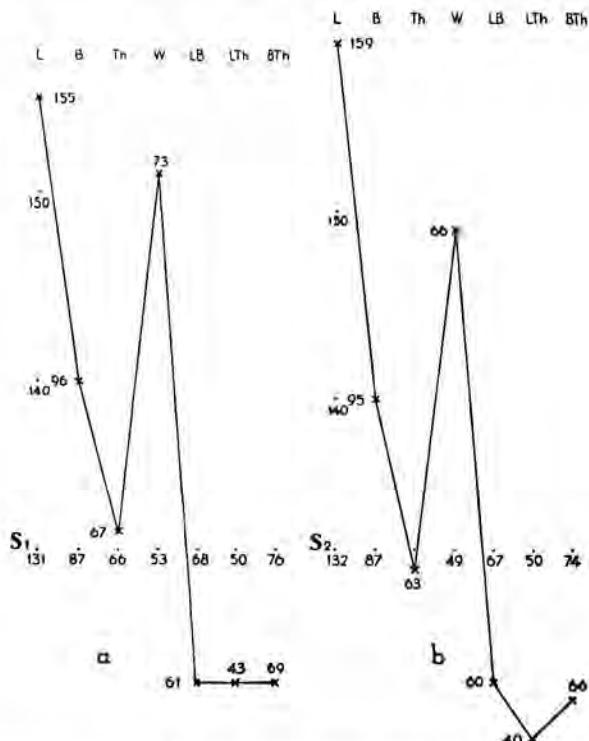


Fig. 1. a. Characterogram of 4 p 1 b, initial bean of pl. 81, F_2 -1934 for pl. 73, F_3 -1935.
b. Idem of the averages of the beanyield of pl. 73, F_3 -1935. S_1 = Standardcharacterogram of 1934. S_2 = Standardcharacterogram of 1935.

vinden we in ieder van deze groepen slechts één boon met een niet zeer grote lengte, $l = 14.7$ en $= 14.8$ mm. Onder de 12, resp. 10 bonen van de F_3 -generatie van 1935 vinden we 3, resp. 6 bonen met een niet zeer grote lengte. De kleinste lengte van de 1ste groep is 14.6, die van de 2de is 14.1 mm.

In de volgende breedte-klassen $b = 10.3$ mm e.v. (tab. 3) blijven we in de groepen van de I-lijn slechts enkele bonen met een niet zeer grote lengte aantreffen. De kleinste lengte, $l = 14.8$ mm, is van een boon met de breedte $b = 10.0$ mm. Van de bonen van de breedte-klassen $b = 10.3$ mm, e.v. van de F_3 -generatie wordt de gemiddelde lengte regelmatig kleiner; het aantal bonen in ieder van deze klassen is vrij groot. Van de bonen

met een breedte $b = 10.3$, $b = 10.2$ en $b = 10.1$ mm, is de kleinste lengte resp. $l = 13.7$, $= 13.6$ en $= 13.6$ mm.

We vinden dus onder de bonen met een grote breedte, $b = 11.3—10.1$ mm, van de F_3 -generatie meerdere bonen met een niet zeer grote lengte, $l = 15.5—13.6$ mm.

Bij bonen met een dergelijke grote breedte $b = 11.3—10.1$ mm van de I-lijn van 1935, vinden we slechts een zeer enkele boon met een niet zeer grote lengte. De boon met de kleinste lengte is $l = 14.6$ mm.

Bij het F_3 -materiaal zijn bonen met een zo kleine lengte, als ze onder de bonen van de I-lijn niet voorkomen. Deze verschillende bevinding laat zich verklaren uit de verschillende erfelijke samenstelling van deze groepen van bonen van de F_3 -generatie en van de I-lijn, iedere groep met dezelfde grote breedte. De formule voor de lengte en de breedte is van alle bonen van de I-lijn dezelfde, nl. $L_1 L_2 B_1 b_2$. Onder deze bonen van de F_3 -generatie zijn er met de formule $L_1 L_2 B_1 b_2$ als van de I-lijn (of tot dit gebied behorend), maar ook met de formule $L_1 l_2 B_1 b_2$.

De geleidelijke afname van de gemiddelde lengte der bonen van de op elkaar volgende breedte-klassen (tab. 3) wijst in de eerste plaats op de positieve correlatie van de lengte en de breedte, — we treffen de geleidelijke afname ook aan voor de I-lijn —, en ook op de erfelijkheid door polymere factoren. L_1 en L_2 , B_1 en b_2 staan ieder voor een groep polymere factoren ($L_1—L_3$ en $L_4—L_6$; $B_1—B_3$ en $b_4—b_6$).

Het aantal bonen van ons F_3 -materiaal is voor de breedten 10.0—9.6 mm, vrij groot, het aantal bonen met een niet zeer grote lengte neemt, vergeleken bij dat met een zeer grote lengte, zeer toe. Onder de bonen van de I-lijn met dezelfde breedten van 10.0—9.6 mm, zijn naar verhouding veel meer bonen met een zeer grote lengte. Dit verschil van bonen van de F_3 -generatie en van die van de I-lijn, berust hierop, dat in de formule van de lengte en de breedte van deze bonen met geleidelijk minder grote breedte, de homozygotie der bonen afneemt. Het best voor ons onderzoek lenen zich de groepen der grootste breedten en hier vinden we een goede aanwijzing voor het voorkomen, in het F_3 -materiaal, van bonen met voor de lengte en de breedte de formule $L_1 l_2 B_1 b_2$ naast bonen met de formule $L_1 L_2 B_1 b_2$, d.w.z. onder de bonen met de grootste breedten zijn er met de formule $L_1 L_2 B_1 b_2$ en $L_1 l_2 B_1 b_2$.

In onze eerste mededeling (1947) groepeerden we de bonen volgens afnemende lengte en als aanwijzing voor de bevestiging van onze hypothese, vonden we daar, dat in de groepen met de grootste lengten, de gemiddelde breedte dezelfde was; eerst in de latere groepen, daalde met de lengte, de gemiddelde breedte der bonen regelmatig. Ook hier de aanwijzing, dat bonen met de grootste lengten (formule $L_1 L_2$), dezelfde grote gemiddelde breedte ($B_1 b_2$) hebben als bonen met minder grote lengten (formule $L_1 l_2$). In tab. 4 zijn enige voorbeelden van F_3 -bonen van 1935 met de grootste breedten en één met zeer grote lengte bijeengebracht en

TABLE 4. F_3 -1935. Some examples of beans with great breadths and not very great lengths and comparisonbeans of the I and the II-line.

Pl.	Bean	L	B	Th	W	LB	LTh	BTh
36	1p 1b	148	107	66	70	72	45	62
36	1p 3b	146	107	65	66	73	45	61
87	6p 2b	141	106	59	60	75	42	56
68	9p 1b	146	105	77	83	72	53	73
87	10p 2b	146	105	73	74	72	50	70
87	6p 6b	141	104	71	74	74	50	67
87	6p 4b	137	103	56	59	75	41	54
58	5p 2b	139	102	65	67	73	47	64
82	3p 3b	136	102	69	73	75	51	68
65	9p 3b	137	101	76	73	74	56	75
103	1p 2b	140	101	65	65	72	46	64
Comparisonbeans of the I-line (very rare)								
1	4p 3b	148	111	65	78	75	44	69
4	8p 2b	146	107	66	70	73	45	62
20	5p 2b	148	105	61	64	71	41	58
4	1p 4b	147	104	52	55	71	35	50
10	16p 2b	153	103	67	72	67	44	65
Comparisonbeans of the II-line (very rare)								
28 ¹	5p 4b	118	98	75	58	83	64	77
28 ¹	12p 2b	122	98	74	59	80	61	76
22	8p 2b	130	97	67	55	75	52	69

enkele zeer zeldzame vergelijkbonen van de I- en de II-lijn. Deze F_3 -bonen hebben een hoge LB-index, evenals de bonen van cl. 5 met de formule 1B Th van ons schema. De dikte is verschillend. Van bonen met een kleine dikte is de B Th-index laag, ze komen overeen met de bonen van cl. 6 met de formule 1 B th. Deze F_3 -bonen met een grote dikte hebben een matig hoge B Th-index. Ze komen overeen met bonen van cl. 1a met de formule $L_1 l_2 B Th$. Bonen met de formule $L_1 l_2 B_1 b_2$ voor de lengte en de breedte hebben een grote absolute breedte en een hoge LB-index.

De hypothese (1947, l.c., blz. 800), waarbij aangenomen wordt, dat het grootte-verschil der lengten op de werking van een even groot aantal polymere factoren in de dominante homozygote vorm berust als dat der breedten en dikten, doch dat de werking van een factor voor het grootte-verschil van een lengte-factor beduidend groter is dan die van een factor voor de breedte en voor de dikte kan het voorkomen van bonen met een grote breedte en een niet zeer grote lengte ook verklaren, maar de verwachting van dergelijke bonen (dus met de formule voor de lengte en de breedte $L_1 l_2 B_1 b_2$ naast die met de formule $L_1 L_2 B_1 b_2$) is bij de door ons aangenomen hypothese veel groter.

De verschillende erfelijke samenstelling van F_2 -bonen brengt mee, dat de F_3 -bonenopbrengsten onderling zeer zullen verschillen. In tab. 5 zijn

enige bonenopbrengsten van de F_3 -zaadgeneratie met de grootste gemiddelde breedten en enkele vergelijk-bonenopbrengsten van de I- en de II-lijn bijeengebracht. We vinden, dat van de bonenopbrengsten met de grootste gemiddelde breedten, de gemiddelde lengte van de bonenopbrengsten van de I-lijn steeds zeer groot is ($l_m = 16.6 - 15.8$ mm, eenmaal

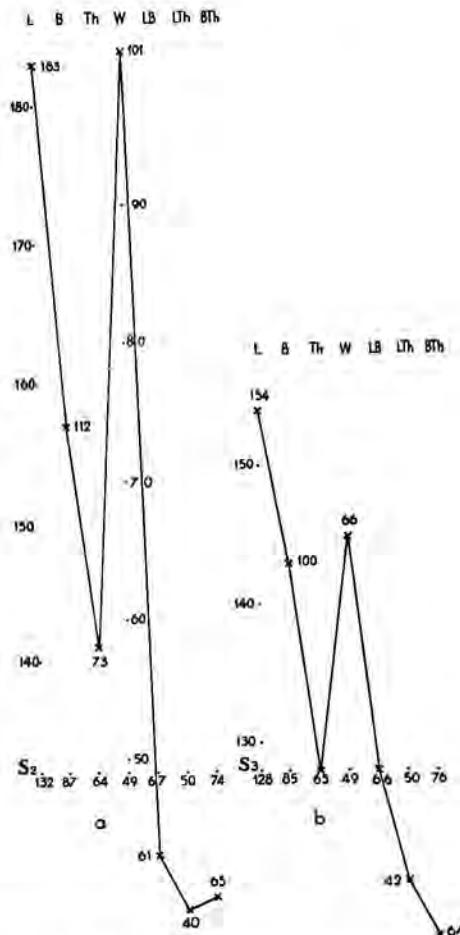


Fig. 2. a. Characterogram of 2 p 1 b, initial bean of pl. 73, F_3 -1935 for pl. 309, F_4 -1936.
b. Idem of the averages of the beanyield of pl. 309, F_4 -1936. S_2 = Standardcharacterogram of 1936.

is $l_m = 151.9$ en $= 149.3$ mm). Onder bonenopbrengsten van de F_3 -zaadgeneratie zijn er enkele, die, — ofschoon de gemiddelde breedte zeer groot is —, een niet zeer grote gemiddelde lengte hebben. De grootste gemiddelde breedte van bonenopbrengsten van de II-lijn is 9.04 mm, d.i. kleiner dan de gemiddelde zeer grote breedten, die we hier beschouwen; de gemiddelde lengte is 12.08 mm.

Bij de F_3 -bonenopbrengsten van 1935 met een grote gemiddelde breedte

zijn er enkele, met een kleine gemiddelde lengte, zoals ze bij de bonenopbrengsten van de I-lijn met eenzelfde grote gemiddelde breedte niet voorkomen. Deze bevinding wijst er weer op, dat, onder de F_3 -bonenopbrengsten met een grote gemiddelde breedte, er met een andere erfelijke samenstelling van de lengte en de breedte zijn dan bij de bonenopbrengsten van de I-lijn. Van alle bonenopbrengsten van de I-lijn is de formule voor de lengte en de breedte $L_1 L_2 B_1 b_2$, onder die van de F_3 -zaadgeneratie zijn er met de formule $L_1 L_2 B_1 b_2$ en ook andere met de formule $L_1 l_2 B_1 b_2$.

Statistics. — *Random sampling frequencies: an implement for rapidly constructing large-size artificial samples.* By H. C. HAMAKER. (Laboratorium voor Wetenschappelijk Onderzoek der N.V. Philips' Gloeilampenfabrieken Eindhoven-Nederland.) (Communicated by Prof. H. B. G. CASIMIR.)

(Communicated at the meeting of March 26, 1949.)

Artificial sampling is, no doubt, the chief purpose to which tables of random sampling numbers have been constructed and applied. To fix our argument, let us suppose that we wish to sample the Poisson distribution with parameter $m = 2$, which yields the cumulative frequencies listed below.

TABLE I.
Cumulative frequencies of the Poisson distribution with $m = 2$.

0	.1353	5	.9834
1	.4060	6	.9955
2	.6767	7	.9989
3	.8571	8	.9998
4	.9474	9	1.0000

Then referring to a table of random sampling numbers arranged in sets of 4 digits, we score a 0 for each number lying between 0000 and 1353, we score a 1 for numbers between 1353 and 4060, a 2 for those between 4060 and 6767, and so on; this is a common procedure which does not need further explanation.

If it is our purpose to investigate certain properties of small samples, the taking of individual scores will be unavoidable, even though this may involve some tedious labour; but if we intend to study the properties of large samples, the work involved will soon become prohibitive, and the question arises whether it is possible to devise a more rapid and convenient method for the constructing of random samples of a large size. Such a method will be described below.

We begin by observing that as we score a 0 for all numbers from 0000 to 1353, we may score a 0 for all numbers beginning with 00 to 12 without any further knowledge of the two digits that follow; likewise all numbers beginning with a pair between 14 and 39 will score a 1 regardless of the two final digits. It is only for numbers beginning with 13, 40, etc., that we require to know the last two digits as well.

Let us then for a moment neglect the last two digits. We take 1000 pairs of two-digit random sampling numbers and we score these in a 10×10 array, thereby obtaining the frequencies recorded in table II.

The classes and the frequencies in this table will be designated as the array-classes and the array-frequencies, to distinguish them from the sample-classes and the sample-frequencies relating to the sample that we are going to construct from them.

TABLE II.
1000 two-digit random sampling numbers scored in a 10×10 array.

First digit	Second digit										Totals
	0	1	2	3	4	5	6	7	8	9	
	Array-frequencies										
0	13	14	9	7	5	11	5	11	10	10	95
1	4	5	9	7	10	14	9	11	9	9	87
2	12	9	7	11	8	9	4	5	15	10	90
3	9	9	16	11	9	9	7	8	10	8	96
4	8	8	5	11	10	8	10	17	16	15	108
5	16	14	14	9	8	17	5	9	13	7	112
6	10	7	7	13	10	11	10	6	12	14	100
7	7	6	5	11	12	11	11	7	10	14	94
8	4	15	9	15	15	9	11	14	7	13	112
9	8	8	14	10	7	14	11	15	6	13	106
Totals	91	95	95	105	94	113	83	103	108	113	1000

With the aid of the array of table II we may now rapidly construct a random sample of 1000 scores of the population specified by table I in the following manner:

The frequencies in the array-classes 00 to 12 are added giving a score of 113 entirely belonging to sample-class 0. In the array-class 13 we have a score of 7 and to allot these to sample classes 0 and 1 we have to tag two more digits behind them, and then to divide the resultant set of 4-digit numbers into those below and those above 1353 (see table I); from a table of random sampling numbers we select a random set of 7 two-digit figures, for instance

12 17 77 82 46 33 97;

four of these are less than 53, and hence we decide that of the seven scores in array-class 13, 4 have to go into sample-class 0, and the other 3 into sample-class 1. The total score in sample class 0 will then be $113 + 4 = 117$.

Next we add the frequencies in the array-classes 14—39, giving a total of 248; the score of 8 in array-class 40 is again subdivided by a set of 8 two-digit random sampling numbers, 3 going into sample-class 1, and 5 into sample-class 2. The total in sample class 1 is now also complete, viz: 3 (from array-class 13) + 248 (from array-classes 14—39) + 3 (from array-class 40) = 254.

Thus we proceed until we have reached the array-class 99, which has to be subdivided into four sets, those below 55, those between 55 and 89,

those between 89 and 98, and those above 98, going into the sample-classes 6, 7, 8 and 9 respectively (compare table I). The sample will then be complete. The entire procedure is illustrated in table III.

TABLE III.

Illustrating the construction of a random sample of 1000 from the distribution of table I by means of the array-frequencies of table II.

Sample class	Array classes	Array frequencies	Sample frequencies	Theory
0	00–12	113	117	135
	13	7 ↗ 4 3 ↘		
1	14–39	248	254	271
	40	8 ↗ 3 5 ↘		
2	41–66	280	291	271
	67	6 ↗ 6 0 ↘		
3	68–84	178	185	180
	85	9 ↗ 7 2 ↘		
4	86–93	85	92	90
	94	7 ↗ 5 2 ↘		
5	95–97	40	43	36
	98	6 ↗ 1 5 ↘		
6			9	
7	99	13 ↗ 7 2 ↘	7	
8			2	17

$$\chi^2 = 6.28; v = 6; P = 0.395$$

So far this method cannot claim to effect much saving of labour. In constructing our sample, however, we started in the left-hand top corner of the array of table II, and we proceeded from left to right and from the top to the bottom of this array; and a considerable saving of labour can be achieved by observing that there is no reason for using the array-frequencies precisely in this order. All the array-frequencies are equivalent to one another, and in constructing a sample we may start at any point in the array and proceed in any direction; or, more general still, we may rearrange the frequencies of table II at random in any of their 100! permutations before we start making up our sample. The simplest method to achieve this is to write the array frequencies on 100 chips of card board, mix these thoroughly and lay them out in random order while constructing the sample.

If we use invariably the same set of array-frequencies (for instance those of table II) we may expect our samples to be on the average slightly biased in one way or another, though the variability obtained by rearranging is so large that serious bias is hardly to be expected. Nevertheless it seems inadvisable to use one set only, and I have therefore constructed ten independent arrays (as that in table II), from which one may be chosen at random every time we wish to construct a sample; thereby the possibility of systematic bias should be ruled out.

TABLE IV.
Ten sets of random sampling frequencies for sample size 1000.

Set Nr	1	2	3	4	5	6	7	8	9	10
Array-freq. f	Frequencies of the array-frequencies = n									
2	—	1	1	—	—	—	—	—	—	—
3	1	—	1	1	—	1	—	1	—	1
4	2	1	4	1	3	1	1	3	5	3
5	3	7	4	4	7	2	5	3	5	8
6	4	6	5	8	3	7	9	9	6	3
7	8	9	6	3	11	9	10	11	10	8
8	11	11	8	17	9	12	7	6	8	6
9	16	14	15	11	15	9	12	12	14	14
10	17	9	17	13	12	19	14	13	10	13
11	8	10	5	10	12	9	9	8	9	20
12	9	7	15	12	3	13	11	10	3	2
13	11	9	6	6	5	8	10	8	15	5
14	5	7	3	6	9	6	5	5	5	7
15	1	5	7	6	6	1	2	6	5	4
16	2	—	—	1	3	1	3	4	2	3
17	1	2	1	1	2	—	2	1	2	2
18	—	—	—	—	—	1	—	—	1	1
19	1	2	—	—	—	1	—	—	—	—
20	—	—	1	—	—	—	—	—	—	—
21	—	—	1	—	—	—	—	—	—	—
Σn	100	100	100	100	100	100	100	100	100	100
Σnf	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
s_f^2	8.42	11.26	11.50	8.62	10.50	8.20	9.14	10.54	11.32	10.70

If we use the array-frequencies not in the form of table II, but as a lottery with 100 lots drawn without replacement, the actual array will be quite immaterial; it will only be important to know the frequencies of the array-frequencies. These I shall call the "*random sampling frequencies*"; for the ten arrays actually constructed they have been recorded in table IV above; set Nr 5 corresponds to table II.

Also in table V an independent set of random sampling frequencies have been provided for the construction of samples of size 500, and in table VI a set for sample size 200.

Samples of other sizes may be obtained by combination; for instance, if we require a sample of 1500 we may construct two samples, one of

1000 and one of 500 scores, and then add their frequencies; or alternatively by laying out one set of 100 random sampling frequencies for sample size 1000 and one set for sample size 500 in a 10×10 array, and by adding the array-frequencies we obtain a set of random sampling frequencies for sample size 1500, from which by permutation any number of samples of this size may be constructed. If we desire a sample of exactly 1537 scores we have only to add 37 more made up in the usual way. Thus the sets of frequencies in tables IV, V, and VI should be amply sufficient for a great variety of purposes.

TABLE V.
Ten sets of random sampling frequencies for sample size 500.

Set Nr	1	2	3	4	5	6	7	8	9	10
Array freq. f	Frequencies of the array-frequencies = n									
0	1	2	—	—	1	—	—	—	—	2
1	3	3	2	3	6	—	3	3	1	2
2	9	9	5	13	7	10	10	5	11	8
3	13	13	15	20	10	19	21	19	12	13
4	23	17	25	11	21	17	15	15	18	15
5	15	13	19	16	16	18	16	23	17	20
6	8	19	13	12	16	13	10	12	20	18
7	12	12	10	6	10	11	7	12	11	12
8	8	6	3	5	5	4	10	7	7	5
9	5	3	6	10	5	5	2	1	1	2
10	1	1	—	2	1	1	3	1	—	1
11	2	—	2	1	—	1	1	1	2	1
12	—	2	—	1	1	1	—	—	—	1
13	—	—	—	—	1	—	2	1	—	—
Σn	100	100	100	100	100	100	100	100	100	100
Σnf	500	500	500	500	500	500	500	500	500	500
s_f^2	5.34	5.40	4.10	6.32	5.64	4.66	6.36	4.48	3.96	4.74

TABLE VI.
Ten sets of random sampling frequencies for sample size 200.

Set Nr	1	2	3	4	5	6	7	8	9	10
Array-freq. f	Frequencies of array-frequencies = n									
0	18	12	16	14	10	14	16	12	14	14
1	18	30	21	26	27	24	27	24	28	27
2	30	25	32	31	31	33	25	32	24	30
3	18	18	18	12	24	13	16	21	21	12
4	12	10	7	12	3	11	9	7	7	11
5	4	4	3	3	3	3	5	3	3	4
6	—	1	3	1	2	2	1	1	3	1
7	—	—	—	1	—	—	—	—	—	1
8	—	—	—	—	—	—	1	—	—	—
Σf	100	100	100	100	100	100	100	100	100	100
Σnf	200	200	200	200	200	200	200	200	200	200
s_f^2	1.92	1.88	2.06	2.10	1.62	1.92	1.76	1.64	2.08	2.16

It may be added that these random sampling frequencies were all derived from tables of random sampling numbers constructed as described in a previous paper^{1).}

To conclude this paper let us briefly discuss a practical application. In tables VII and VIII we have represented two sets of ten samples each of 1000 units taken from the population specified in table I; the first set (table VII) was constructed by using the random sampling frequencies of set Nr 1 in table IV in ten different permutations, while the second set (table VIII) was obtained by using the ten sets of table IV each once. For each sample we have computed the first, second and third moments about the origin, and χ^2 , and from these the average moments and the standard deviations have been calculated.

TABLE VII.

Ten random samples of the Poisson distribution with $m = 2$ constructed from one set of random sampling frequencies in ten different permutations.

Sample class	10 samples constructed from set Nr 1, table IV										Theory
	Frequencies										
0	127	146	147	148	155	146	145	128	138	129	135
1	282	267	257	301	247	272	270	265	264	277	271
2	285	271	259	236	269	265	253	271	276	266	271
3	157	178	178	170	178	190	183	177	176	179	181
4	90	88	98	91	89	89	97	110	89	100	90
5	42	35	40	31	48	29	36	30	38	30	36
6	14	10	17	18	11	8	11	13	17	13	
7	1	4	3	4	3	1	3	5	2	6	
8	2	1	1	1	—	—	2	1	—	—	16
	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	
m'_1	2.000	1.966	2.032	1.946	2.002	1.928	2.996	2.049	2.006	2.016	
m'_2	6.01	5.86	6.29	5.91	6.09	5.53	6.05	6.23	6.04	6.06	
m'_3	22.2	21.3	23.6	22.3	22.3	18.9	22.4	23.0	22.0	22.2	
χ^2	5.85	1.21	4.22	13.03	9.53	6.26	2.52	6.30	0.98	3.09	

$$\bar{m}'_1 = 1.994; \quad s_{m'_1} = 0.036$$

$$\bar{m}'_2 = 6.007; \quad s_{m'_2} = 0.200$$

$$\bar{m}'_3 = 22.02; \quad s_{m'_3} = 1.19$$

In computing χ^2 the frequencies of the last three rows were pooled, which reduces the number of classes to seven and the number of degrees of freedom to six; the 95 % and 5 % points of the χ^2 -distribution are then 1.64 and 12.59 respectively. On comparing with tables VII and VIII we see that out of a total of 20 values of χ^2 two (0.98 and 1.21) lie below the 95 % point and one (13.03) lies above the 5 % point: quite an acceptable result. It will be noted that these three somewhat outlying

¹⁾ H. C. HAMAKER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 52, 145—150 (1949).

values all occur in table VII, and if we restrict our attention to this table, we have an event with a chance 1/10 that has occurred 3 times in a series of 10 trials; the binomial probability of three or more such events is easily computed to be 0.07, which is still above the customary significance limit.

TABLE VIII.

Ten random samples of the Poisson distribution with $m = 2$ constructed from ten different sets of random sampling frequencies.

Set Nr	1	2	3	4	5	6	7	8	9	10	Theory
Sample class	Frequencies										
0	123	131	138	124	143	132	131	141	146	121	135
1	276	249	253	288	265	285	257	267	268	295	271
2	267	279	278	273	267	278	296	263	247	264	271
3	199	187	173	176	194	162	174	178	187	185	181
4	77	88	98	90	84	91	90	91	98	86	90
5	41	45	38	34	37	34	32	38	42	35	36
6	12	19	12	11	10	14	14	16	7	10	
7	4	1	7	3	—	3	5	4	4	4	
8	1	1	1	1	—	1	1	2	1	—	
9	—	—	2	—	—	—	—	—	—	—	
	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	
m'_1	2.028	2.074	2.057	1.987	1.962	1.974	2.018	2.021	2.003	1.985	
m'_2	6.08	6.38	6.44	5.86	5.71	5.88	6.06	6.23	6.07	5.82	
m'_3	22.3	23.7	25.1	21.1	19.8	21.5	22.3	23.6	22.2	20.8	
χ^2	5.71	5.86	4.45	2.36	4.59	3.23	4.61	2.57	6.19	4.43	

$$\bar{m}'_1 = 2.011; \quad s_{m'_1} = 0.024$$

$$\bar{m}'_2 = 6.053; \quad s_{m'_2} = 0.230$$

$$\bar{m}'_3 = 22.24; \quad s_{m'_3} = 1.47$$

Moreover, the ten samples of table VII were derived from the set of random sampling frequencies Nr 1 in table IV which possesses a standard deviation somewhat below the theoretical value; if anything, we should therefore expect the variability between the samples of table VII to be less than normal, though it is extremely unlikely that such a tendency should already be apparent in so small a series of samples. These arguments render it highly probable, however, that the comparatively large variations in χ^2 in table VII are accidental, and are not in any way connected with the particular method by which the samples were obtained.

The average moments in tables VII and VIII lie close to their theoretical values, which are 2.00, 6.00, and 22.00 respectively. The standard deviation of the moments computed from each of these tables separately are in satisfactory agreement, and these standard deviations offer perhaps the best illustration of the usefulness of the methods described in this paper;

by constructing a set of large-size samples we may rapidly estimate the standard error of any statistic computed from them, while avoiding involved mathematical arguments which are all too often incomprehensible to the practical statistician; and the method also applies to cases not amenable to mathematical treatment.

I should like to express my indebtedness to Mr. H. A. C. v. D. LINDEN for his careful assistance in carrying out this investigation.

Eindhoven, 19 December 1947.

Crystallography. — *Transformation of gnomograms and its application to the microchemical identification of crystals.* I. By D. W. DIJKSTRA. (Memorandum of the Crystallographic Institute of the Rijks-Universiteit at Groningen.) (Communicated by Prof. J. M. BIJVOET.)

(Communicated at the meeting of February 26, 1949.)

Summary. The method of construction, dealt with in this article, allows: 1^o the transition to another plane of projection, crystallographic axes of reference and parametral plane remaining the same (rotation, Umwälzung), 2^o the transition to other axes and parametral plane (transformation in a narrower sense).

Here it is not the crystal, that will be rotated (as usual) into another position with respect to the plane of drawing, which does not alter its position, but the crystal remaining stationary the projecting lines are made to cut out another gnomogram in a new plane, which may have any position with respect to the original plane of drawing. This plane is revolved to coincide with the picture plane, and then the old gnomogram and the new one will be found to be related by the principle of central collineation. The form of the collineation is a very simple one, so the constructions can be effected with only few lines and practically only with a ruler (§ 6, 8, 9 and 10).

The most usual types of gnomogram are considered to be particular cases of one general gnomogram (§ 1, 2 and 3).

As an application the identification of a crystal of silverdichromate, measured by microscope, will be described (§ 11b).

§ 1. In the most usual gnomogram the zone lines are equidistant parallel lines because of the particular position of the plane of projection, the latter being perpendicular to one of the axes of reference.

In the general form of the gnomogram, however, the plane of projection may have any position.

In practice this may occur inter alia in the under mentioned cases:

a. In measuring efflorescent or deliquescent crystals time may be insufficient to adjust the crystal on the two-circle goniometer. Measurements are taken with the crystal in an arbitrary position and this will result in a general gnomogram.

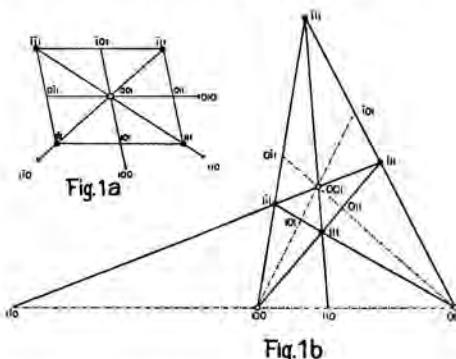
b. With some crystals the planes are arranged in a small number of zones, intersecting in a face actually occurring on the crystal. In measuring these "face adjustment"¹⁾ will be profitable. When plotting the measure-

¹⁾ M. H. HEY, Min. Mag. 23, 560 (1932).

ments one will find a gnomogram with the plane of projection coinciding with a face of the crystal, and the latter being in general not perpendicular to a possible edge, the form of the gnomogram will be the general one.

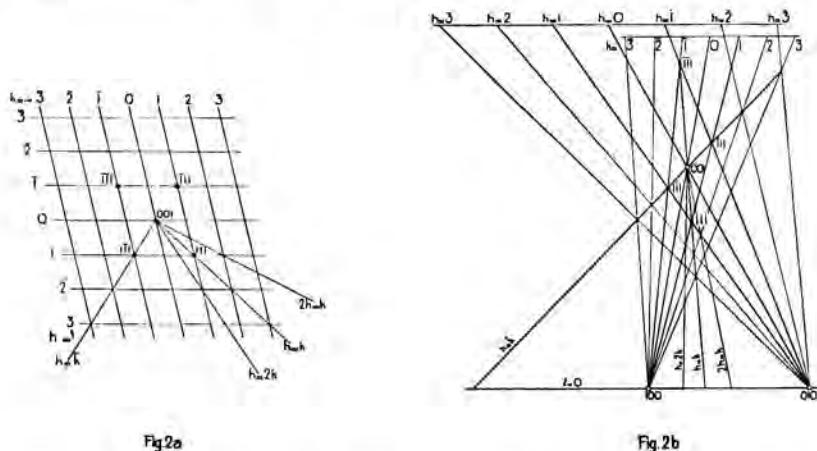
c. When reducing Laue photographs one often meets with gnomograms with arbitrary planes of projection ²⁾.

§ 2. When passing from the particular to the general gnomogram the normal pattern of fig. 1a changes into that of fig. 1b.



This figure 1b is a complete quadrangle with as its angular points the "octahedral faces": (111), (111), (111) and (111) (for the sake of clearness its sides are represented by full lines, its diagonals by dotted ones). The "cube faces" (100), (010) and (001) are the diagonal points; the four zone lines through each "cube point" form a harmonious pencil of lines, being two sides and two diagonals of the complete quadrangle.

In order to construct in this general gnomogram the zone lines corresponding to the equidistant parallel zone lines of the ordinary gnomogram represented in fig. 2a, the rays (100)-(001) and (100)-(111) in



²⁾ See e.g. R. W. G. WYCKOFF, "The structure of Crystals" (1924) p. 139, fig. 108.

fig. 2b (the points (100) , (010) , (001) and (111) being given) are intersected with a line parallel to the ray $(100)-(010)$, and then on this line pieces are laid off equal to the piece cut off by the rays just mentioned (fig. 2b). The connection of the division points, found in this way, with the point (100) results in the pencil through (100) , and then a corresponding construction gives the one through (010) .

§ 3. a. The first particular case of this general gnomogram mentioned here, is the ordinary gnomogram; the line $(100)-(010)$ in this case is the line at infinity of the plane, and that converts the pencils into systems of equidistant parallel lines.

b. Another particular case is that of the trigonal notation according to MILLER. Here the "dodecahedron zone" $(\bar{1}\bar{1}0)-(\bar{0}\bar{1}1)-(\bar{1}01)$ ³⁾ is the line at infinity (fig. 3).

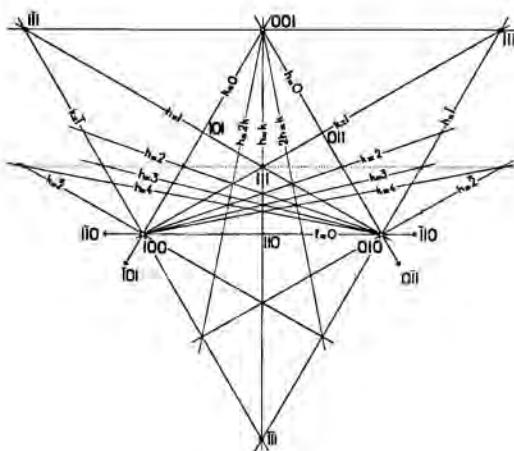


Fig. 3

c. In the regular system the "triangular gnomograms" are identical with those according to MILLER's notation; in the other systems their form is the general one⁴⁾.

§ 4. In the usual methods⁵⁾ of transforming gnomograms the crystal will be made to rotate the plane of drawing remaining stationary (Umwälzung), which results in the gnomogram thus constructed having the same gnomon circle as the original one. The new place of the pole is found by construction or by means of a "gnomonic net".

³⁾ This is the zone $[111]$.

⁴⁾ R. L. PARKER, Schw. Min. Petr. Mitt. XVII, 475 (1939).

P. TERPSTRA, "Kristallometrie", p. 103 seqq.

⁵⁾ See e.g. V. GOLDSCHMIDT, "Über Projection und graphische Krystallberechnung" (1887) p. 68.

H. E. BOEKE, "Die gnomonische Projection etc." (1913) p. 26.

F. E. WRIGHT, Am. Min. 17, 422 (1932).

Let us take WRIGHT's method by way of example. (In § 7 WRIGHT's construction will be shown to become formally identical with this method of ours, by only a slight alteration.)

In fig. 4 the crystal is to be rotated, the result being the coincidence of the pole P' with the centre of the gnomon circle. First thing to do is to construct the angle of rotation $P'C_nO$; this is bisected by the line $C_nR_{p'}$, and in $R_{p'}$ the line i_1 is drawn perpendicular to $P'O$. At the same distance from O , but this time on the other side, the line $i_2 \parallel i_1$ is drawn. These lines i_1 and i_2 are WRIGHT's "isometric lines".

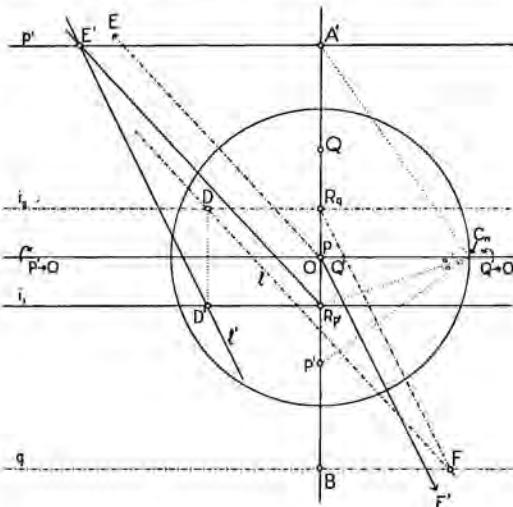


Fig. 4

The rotation makes a point D' on i_1 coincide with a point D on i_2 .

Next the trace p' of the plane P' is constructed; now the point $R_{p'}$ will be the angle point of the line p' ($\angle A'R_{p'}C_n = 90^\circ - \frac{1}{2}\angle OC_nP'$, $\angle A'C_nR_{p'}$ is so too; so $A'C_n = A'R_{p'}$). By rotation p' goes to infinity, making O the new angle point. This means that after the rotation a point E' on p' has come at the point at infinity of the line drawn through $O \parallel R_{p'}E'$. Thus the point at infinity of the line $R_{p'}E'$ will represent the point E corresponding to the point E' .

The zone line l conjugate to a given zone line l' through D' and E' is obtained by connecting D and E , this being the line through D parallel to $R_{p'}E'$.

§ 5. In our method the crystal itself is not rotated; the projecting rays from the centre O , however, are made to intersect not only the original plane of drawing but another one too, which may have any position with respect to the former, provided that its distance to the centre of projection — as that of the former plane — is equal to the radius of the gnomon-circle. Each pair of corresponding points lying on the same ray from the

centre of projection, the figures in both planes are in perspective with one another.

In fig. 5 τ' represents the original plane of drawing in which lies the gnomogram to be transformed, and τ the new plane of drawing. Both planes are tangent to the sphere, having the centre of projection O for centre and the gnomon for radius, and their intersection is the line d . The plane through O perpendicular to d is the plane of drawing in our figure. O' is the centre of the gnomon circle and P' is the pole of the plane τ in the original gnomogram. The plane τ_p , through O parallel to τ , intersects τ' by the vanishing trace v' ; the plane τ'_p , through O parallel to τ' , intersects τ in the vanishing axis w .

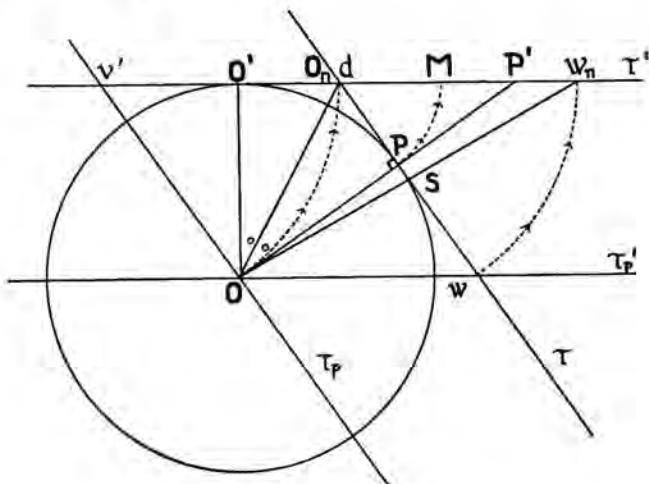


Fig. 5

The distances of the parallel sides in the parallelogram $v'Owd$ being equal $v'Owd$ is a rhomb, which makes $v'O = v'd$ and $v'd = dw$.

Now τ_p is revolved round v' into τ' , and τ in the same direction round d , likewise into τ' . This revolution makes O (afterwards called O_n) come on d and the vanishing axis w at w_n , $v'd$ being equal to dw_n . The centre P of the gnomon circle in the new plane τ comes at M , while $O'd = dP = dM$. Now the figure in the plane τ' will look like fig. 9 without the point O drawn there.

The revolution has made the old gnomogram in τ' and the new one in τ , formerly being in perspective to one another, now to be related by the principle of central collineation, the line d being the principal axis of the collineation, the lines v' and w_n , on either side of and at an equal distance from d , being the secondary axes and the point O_n on the line d the centre of collineation.

The central collineation has the following properties, as will be proved in § 6:

- a. Each line l' of the original gnomogram intersects its conjugate l of the new one in a point on the principal axis d .
- b. Each line through O_n coincides with its conjugate.
- c. A set of parallel lines l' changes into a pencil l through Q , the point of intersection of the secondary axis w_n and the line through O_n parallel to l' (fig. 6a).

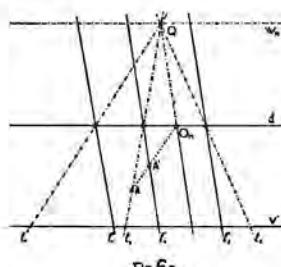


Fig. 6a

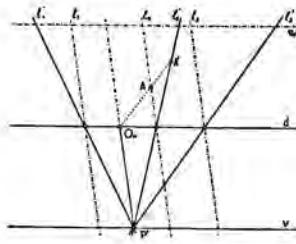


Fig. 6b

- d. A pencil l' through the point P' on the secondary axis v' changes into a set of parallel lines l , parallel to the line connecting P' and O_n (fig. 6b).
- e. Conjugate points A' and A lie on one straight line through O_n .

§ 6. In order to prove these properties fig. 5 has been copied stereometrically in fig. 7. The letters are the same as in figg. 5 and 6, in figg. 5 and 7 the index n being added after revolution.

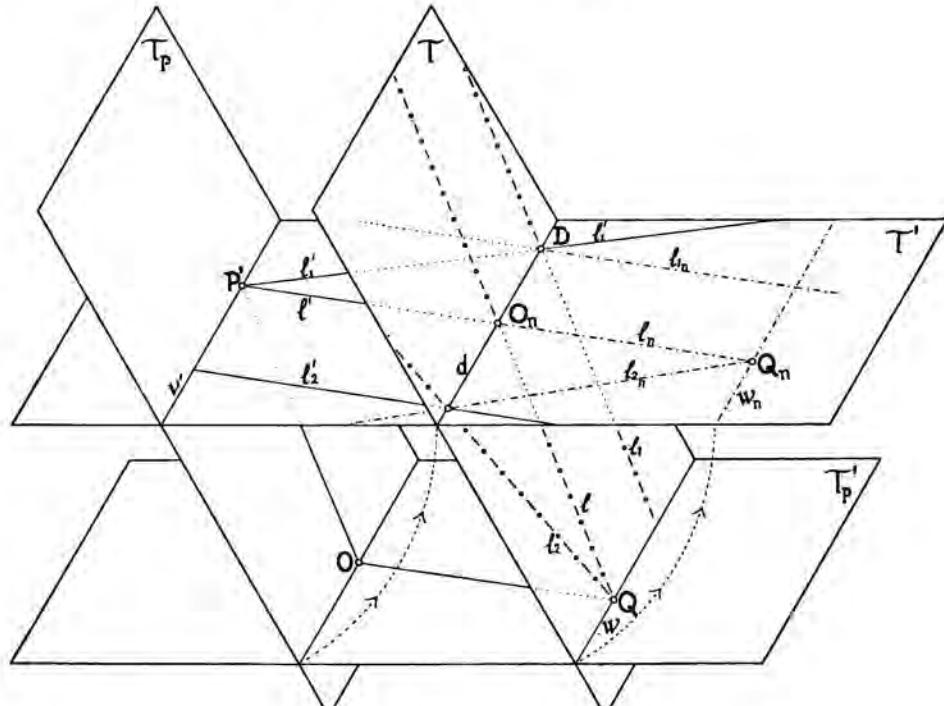


Fig. 7

a. The rays from O , projecting the line l_1 , describe a plane through O and l_1 ; the intersection of this plane and τ' is the line l'_1 conjugate to l_1 . The intersection D of this plane and the line d is a point common to l_1 and l'_1 and this point D remains stationary during the revolution, so the point of intersection of l'_1 and l_{1n} too will lie on the principal axis d .

b. The rays from O projecting the line l through O_n will lie in a plane through O and l . The parallelogram $O_nP'QO$ formed by the intersections of this plane with the planes τ , τ' , τ_p and τ'_p will be a rhomb.

During the revolution $P'O_n$ remains stationary, the parallelogram hinges in its corners, OQ remaining parallel to $P'O_n$. When finally O has come at O_n , Q coincides with Q_n on $P'O_n$ produced, so O_nQ ($= l$) becomes O_nQ_n ($= l_n$), l_n being l' produced.

c. The intersections of any set of planes through OQ with τ will form a pencil of lines $l, l_2 \dots$, having Q for its vertex, and those with τ' a set of parallel lines $l', l'_2 \dots$. After the revolution the pencil will lie in the plane τ' with Q_n for its vertex, O_nQ_n being parallel to the set $l', l'_2 \dots$

d. The intersections of any set of planes through OP' with τ' will form a pencil of lines $l', l'_1 \dots$, having P' for its vertex, and those with τ a set of parallel lines $l, l_1 \dots$, forming, after revolution into τ' , a set of lines parallel to $P'O_n$.

e. The line O_nA' is transformed into a conjugate line O_nA , coinciding — as shown in section (b) — with O_nA' ; which is another way of saying that O_n, A' and A will lie on one straight line.

§ 7. In this passage will be shown, that only a slight alteration in WRIGHT's construction will bring about formally the same results as those obtained in § 5.

In fig. 4 all the zone lines l' , meeting at the point E' on p' , are shown to change into a set of lines l , parallel to one another because they are all parallel to $R_p'E'$.

In order to deduce from the unaccented zone lines the corresponding accented ones the rotated crystal is to be turned back through the same angle, which results in Q coming at O (fig. 8). q , the trace of the plane Q , being located, the construction is an analogous one.

The zone line l meets i_2 at D and q at F ; so l' is drawn through D' and E' , that is through $D' \parallel OF^* \parallel R_qF$.

This way of constructing results in a set of accented lines, parallel to one another, all of them being parallel to R_qF , corresponding to a set of unaccented ones, meeting at the point F on q . In order to complete the conformity with the method based on the principle of central collineation, another modification is made in the drawing represented in fig. 4. The full lines in fig. 8 are the same as those in fig. 4 and the letters too are placed in the same way. In fig. 4 two gnomograms are drawn with the same gnomon circle, one accented, the other one unaccented.

The unaccented gnomogram is, as a whole, shifted downwards across a

distance $a = OR_q = OR_p$; the accented one is translated upwards across this distance a . After translation lines and points all have got an asterisk and the lines have been changed as follows: ——— into - - - - - and - - - - - into - - - - - . Now the gnomon circles will be separated and i_1 and i_2 will coincide with one another.

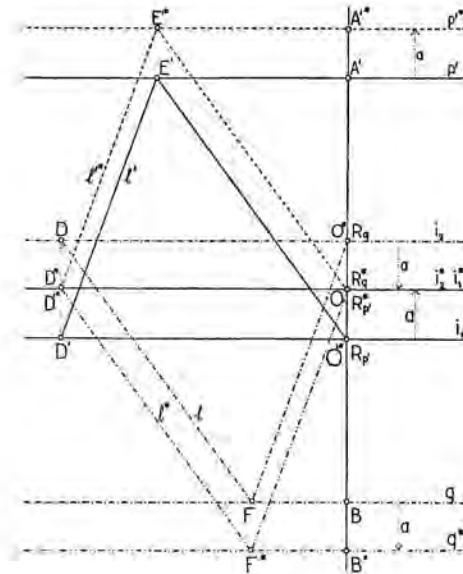


Fig. 8

The way of constructing conjugate lines is now identical with that described above in § 5; i_1^* , i_2^* taking the place of the principal axis of the collineation (fig. 7), p'^* that of the vanishing trace v' and q^* that of the vanishing axis w_n . R_p^* (R_q^*) is the centre of the collineation O_n .

§ 8 The transformation dealt with in § 5 can be performed easily, when the lines d , v' and w_n and the centre O_n have been located.

In most cases these lines and the point O_n will have to be constructed. Three of them will be described here:

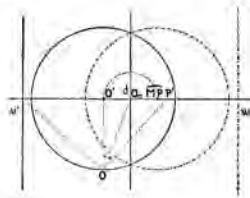


Fig. 9

a. A given gnomogram is to be transformed into a gnomogram on a plane indicated by its pole P' (fig. 9). With regard to an arbitrary plane

one has to solve this problem, when wishing to reproduce a face in its true form by means of orthogonal parallel projection onto the plane of drawing, e.g. for the construction of a model of some crystal or other in pasteboard.

The zone line to the pole P' (cf. fig. 5) is constructed; this is the secondary axis v' . The line bisecting $\angle O'OP'$ intersects the original plane of projection at O_n ; the line through $O_n \parallel v'$ will be the principal axis d ; the other secondary axis w_n is the line parallel to d and at an equal distance from d as v' . The centre M of the new gnomon circle is found by making $O'O_n = O_nM$, the radius not being changed. In the gnomogram now constructed the pole of face P will coincide with M .

b. A transformation of a given gnomogram is required, resulting in a certain line v' in the original one changing into the line at infinity of the new one, this being the case when a gnomogram in the general form is to be transformed into one with equidistant parallel zone lines.

First P' , the pole to the zone line v' , is located and then the construction is completed as above (problem (a)).

If the zone line v' should lie beside the drawing paper and so the pole close to the centre of the gnomon circle, to begin with, a transformation should be performed promoting an arbitrary but farther removed point to centre of the gnomon circle, which then should be followed by another transformation converting the line v' of the second gnomogram into line at infinity.

c. For the sake of completeness the case will be dealt with, that the line at infinity is to be transformed into a given line w_n (figg. 5 and 10). In fig. 10 only the line w_n and the point O' are given.

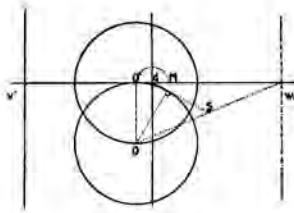


Fig 10

Now (fig. 5) O_nw_nwO will be a parallelogram, Ow and O_nw_n being parallel and equal to one another. The point of intersection S of the diagonals bisects the line Ow_n . So from S (fig. 10) the tangent to the circle is drawn and the point d results. The point M and the line v' are constructed as above (problem (a)).

§ 9. In order to change to other axes of reference and another parametral plane the complete quadrangle through the given poles is constructed, as indicated in the first part of this article, and this is transformed into an ordinary gnomogram as in § 8b.

§ 10. The construction is performed as follows. The line l conjugate to the given line l' is to be constructed.

a. The line l' (fig. 11) can be regarded as one out of the pencil through B (cf. fig. 6b), which will change into a set of parallel lines. The ray BO_n out of this pencil coincides with its conjugate and the line l conjugate to l' can be drawn through $A \parallel BO_n$.

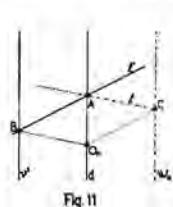


Fig. 11



Fig. 12

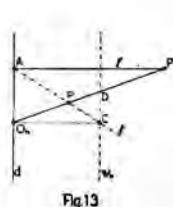


Fig. 13

b. The line l' (fig. 12) can be regarded as one of a set of parallel lines (cf. fig. 6a), which will change into a pencil with its vertex on w_n . The ray O_nC out of this set coincides with its conjugate, which makes C the required vertex. The line l conjugate to l' is drawn through A and C .

This construction of the point P conjugate to P' (fig. 13) turns out to be quite the same as that of the image formed by a negative lens, O_n performing the function of the optic centre and C that of the focus. Accordingly the relation between the positions of conjugate points complies with the well known formula for lenses, giving here $\frac{1}{O_n P} - \frac{1}{O_n P'} = \frac{1}{O_n D}$.

Physics. — Photons in extensive Cosmic-Ray-Showers. By J. CLAY.
(Natuurkundig Laboratorium Amsterdam.)

(Communicated at the meeting of March 26, 1949.)

Summary.

The results are given of 6 and 8-fold coincidence measurements of extensive showers with layers of lead above and between the counters, by which the absorption of the electrons and the production of electrons in the showers could be estimated. It is clear that there are a great deal of energetic photons in these showers and the amount increases with the density of the shower particles.

By measuring 4- and 6-fold coincidences for surfaces of relative sizes from 1 to 6 the value of the coefficient S in the frequency-density relation $I = I_0 \cdot N^{-S}$ could be found and we come to the conclusion that S is not constant but varies from $S = 3,0$ for very small densities to $S = 1,5$ for high densities, which is in agreement with the frequency-density relation for bursts.

Some time ago in the extensive atmospheric showers we found an indication of a great number of photons. We were carrying out an experiment for measuring on one hand the absorption of the electrons in these showers and on the other hand the percentage of the particles therein which penetrate at least 10 cm Pb. To this purpose we were using 4 trains of two counter-sets each, which were placed in a quadrangle, each train arranged so, that the lower box could be protected by 10 cm Pb. We were anxious to measure afresh the number of penetrating particles in relation to the total number of particles, as we had found in two separate sets of measurements this relation to be 1 to 10. At the Pasadena congress of 1948 COCCONI protested that he had found a relation which was totally different from the one found by us. Therefore we repeated our experiments with two different counter arrangements in two different places, in our laboratory and in a wooden shed with a thin wooden roof where we had already made several measurements before. In both places we again found the relation 1 to 10. Vide table.

We have already stressed that when we observe two particles there are two possibilities. It may be two particles belonging to one extensive shower and it may be two coherent mesons produced in one and the same process. The variation with distance found while measuring 4-fold coincidences caused by two particles in two vertical counter-trains at not too large a distance from each other appears in either case. We thought however that we would be able to distinguish between the two cases when measuring 2 or 3 particles. And indeed when we suppose the variation of frequency of the coincidences with distance to be given by $N = N_0 e^{-ax}$, we find in the case regarding 2 particles at distances of 120 cm and 60 cm $a = 0,47$

and $a = 0.37$ respectively and in the case of 3 particles at the distances mentioned $a = 0.62$ and $a = 0.64 \text{ cm}^{-1}$ resp. In the frequency-distance curve for the coincidences we find a very distinct difference between the variation for slightly larger distances (up till 5 m) for two penetrating particles of extensive showers and the variation of the frequency of coincidence of two coherent mesons which here are measured together¹⁾. Above 5 m distance the influence of the coherent mesons is negligible. JÁNOSSI²⁾ is of opinion that also the showers are to be divided in two groups, viz. extensive showers and locally more limited ones.

Two years ago already we came to the conclusion that the extensive showers were very rich in photons. We therefore tried to find the proportion of the photons to the electrons in the following way. We used four trains of two sets of counters each, these sets separated by a space of 10 cm designed to be filled with lead screens of different thickness. In case there is no lead between the counters, the particles, mesons and electrons, give coincidences and we can measure at the same time 4 and 6-fold (2 and 3 particles) and 6 and 8-fold (3 and 4 particles). When we apply lead between the counters, the absorption of the soft part of the radiation, the electrons, can be measured. From the absorption curve the energy distribution may be derived. With 10 cm of lead the electrons are practically all absorbed, and we find that the incident radiation contained a number of mesons equalling about 1/10 of the number of electrons. This ratio between the two kinds of particles is totally different from their ratio in the radiation of the atmosphere outside the showers.

When next we put a sheet of lead above the upper set of counters we find an increase. This is on account of the photons in the shower which produce in the top layer of lead Compton electrons or electron-pairs. These secondaries are able to operate the counters under the layer. The absorption of the primary electrons is known from measurements with lead between and above the counters. To determine the number of photons we proceed in the following way.

We know that when the active area of m counters, each of an area f , is called F and N is the density per m^2 , the probability of an n -fold coincidence is $(1 - e^{-mFN})^n$ or $(1 - e^{-FN})^n$. If there are P photons which have an opportunity of producing in the top layer an electron able to operate the counters under the layer, with an efficiency λ , then the probability of an n -fold coincidence in which n_1 counter-sets are discharged by either primary electrons or photon-produced electrons and $n-n_1$ by primary electrons only, is

$$\{1 - e^{-F(N+P\lambda)}\}^{n_1} (1 - e^{-FN})^{n-n_1}; \quad 0 \leq n_1 \leq n.$$

For each value n_1 this expression has a special weight g_{n_1} and the total probability for an n -fold coincidence with lead on top of the counter-sets is given by

$$\sum_{n_1=0}^{n_1=n} g_{n_1} \{1 - e^{-F(N+P\lambda)}\}^{n_1} (1 - e^{-FN})^{(n-n_1)}.$$

The experimental results however gave us an indication that this formula might be simplified in the following way. The $\frac{1}{n}$ th power of the ratio a of n -fold coincidences with and without a layer of lead on top is nearly a constant, as appears from the table hereunder:

N	a	$\sqrt[n]{a}$
2	1.20	1.10
3	1.39	1.12
4	1.80	1.16

Consequently we may conclude that in the case with lead above the counter-trains, the electron density N increases by λP , giving $(1 - e^{-F(N+\lambda P)})^n$ for the probability of an n -fold coincidence.

The ratio of n -fold coincidences with and without lead may therefore be expressed as

$$\frac{(1 - e^{-F(N+\lambda P)})^n}{(1 - e^{-FN})^n} = a$$

in agreement with the above-mentioned experimental results.

If F is known, we can compute N from the measurements without lead, so that e^{-FN} is known too. Substituting then $e^{-F\lambda P} = x$ and $e^{-FN} = y$ enables us to calculate λP in the following way:

$$a = \frac{(1 - e^{-FN - F\lambda P})^n}{(1 - e^{-FN})^n}$$

$$\sqrt[n]{a} = \frac{1 - yx}{1 - y}$$

$$\sqrt[n]{a} - y \sqrt[n]{a} = 1 - yx$$

$$x = \frac{y \sqrt[n]{a} - \sqrt[n]{a} + 1}{y}$$

From earlier experiments we found for counters $0.03 < \lambda < 0.06$. If we now introduce $\lambda = 0.06$ the value may turn out to be too small in this case. But we are experimenting at the time to find a more appropriate value.

The measurements treated in the present paper were made in two series. One of them was made in our laboratory under a roof with a concrete ceiling of 20 cm. Four vertical trains of two sets of counters each were used, so that it was possible to measure 2, 3 and 4 particles mainly in a vertical direction.

Next we experimented with the counter-sets placed side by side and 6 and 8 coincident particles were measured. These measurements were repeated with 1 cm Pb above the counter trains. With this arrangement we measured the particles and photons from every direction.

Each set contained 3 counters of 30 cm length and 3.5 cm diametre so that the active area of a set was 315 cm^2 . N was found 35 p m^2 . Using this value, we find $\lambda P = 16$, giving $P = 320$ for $\lambda = 0.06$.

In a second series of measurements 4 and 6-fold coincidences were counted with sets of counters of 70 cm length and 3 cm diametre. 1, 2, 3, 4, 5 and 6 of these counters in turn were used in parallel. As we will see, the minimum density of the showers which we are able to measure depends on the active surface and is smaller when this area is larger, as we had already observed in 1941. In several cases the photons were measured too, with lead (1, 3, 5 and 6 counters in parallel). In all these cases the frequency of coincidence was measured moreover with a cover of 10 cm Pb to the counters.

In one case we have determined the photon-density of the showers using very small surfaces and in doing so we have found a mean density of 150 times the density of the total of the particles of the shower. The density of the photons in this case was 4000 per m^2 for $\lambda = 0.06$.

For the first mentioned experiments the results are given in the graph 1, which shows that the increase by the lead layer is considerable.

Likewise a very interesting result is that, when 1 cm Pb is placed between the upper and lower sets of counters, there is still an increase in the number of coincidences in comparison to that in counters without lead. How is this possible? This is a consequence of the coherence between electrons and

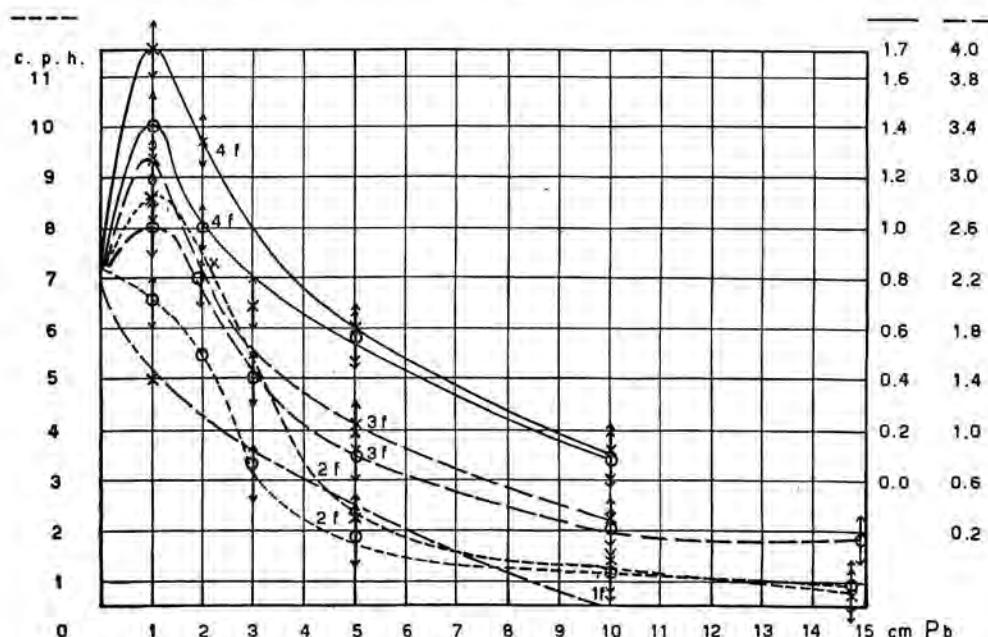


Fig. 1. 3 and 4-fold coincidences open and with different layers of Pb on top (drawn curves) and with Pb between the counter-sets (dotted curves). For comparison the absorption curve is given for the common soft electron component (1f).

photons. Due to the absorption of the electrons by the lead sheet between

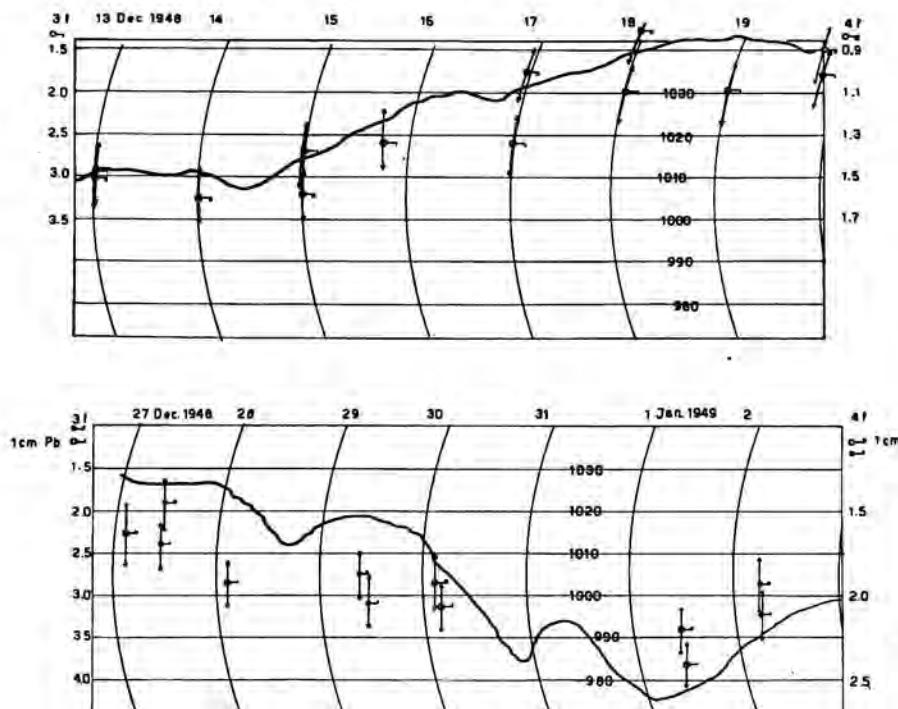


Fig. 2 and 3. The variation of 6 (3p) and 8 (4p) fold coincidences with barometric variations.

the counters a great number of the electrons cannot operate the counters under that sheet and this will decrease the number of coincidences. But there are photons in coherence with the electrons. And the presence of the lead sheet causes these photons to produce secondary electrons. Therefore a coincidence may occur caused by a primary electron traversing the upper set and by a secondary electron produced by a coherent photon traversing the lower set of counters. It may be possible to calculate the effect from the respective numbers of electrons and photons but it is difficult to estimate the energy spectrum of the electrons which is certainly different from that of the electron of the ordinary soft electron component in the radiation which is given also in the graph for comparison.

It is evident that one cannot measure the actual absorption of the primary electrons directly by putting lead over the counters nor by putting the lead between the counters.

Another interesting phenomenon is discernible in this series of observations, but this complicates the issues considerably. It became manifest that barometric fluctuations caused great variations in the frequency of the observed showers. This influence is given in the graphs 2 and 3 and the coefficient of the variation in frequency is 13 % for 1 cm Hg. Conse-

quently we can compare the results of different situations only for identical barometric pressure. This big coefficient is further quite in agreement with the increase of the frequency of showers with increasing altitude observed by HILBERRY in the lower part of the atmosphere and it indicates that the production of the showers may lie not very high in the atmosphere.

We also made a prolonged series of measurements in another building with sets arranged in a hexagon, observing 4 and 6-fold coincidences as mentioned above. Every set contained six counters and measurements were first made with one counter functioning in each box, then two, three ... up till six. From counting 4 and 6-fold coincidences in this manner we could compute the frequency of showers and their mean density. By putting 0.5 cm Pb over the counters the photon density could be estimated and by placing 10 cm Pb over these boxes the penetrating part was determined.

We know that with the increase of the active surface the ratio of the showers of low density to those of high density increases. It is even a well-known fact that with very small surfaces only the densest showers are found. There must be a relation between the frequency and the density of the showers and in analogy with what was found in the case of the meson and the electron spectrum, the most plausible assumption was to suppose that the frequency would be proportional to a power of the density. $I(N)dN = I_0 N^{-s} dN$. We may test by experiments whether the value of S is of the same order as the exponent for the energy for the mesons. This has also been done by COCCONI, LOVERDO and TONGIORGI³⁾.

If N is the density and f the active surface of the counters, the chance that a counter system of m counters will be hit is $(1 - e^{-mfN})^m$ and the total probability of the coincidence for an area $F = mf$ will be

$$I(F) = I_0 \int_0^F N^{-s} (1 - e^{-FN})^m dN.$$

TABLE.

Number of counters	Absorber	Surface in cm^2	$4f$	$6f$	$\frac{6f}{4f}$	N	P	M
1	1 - 10 cm Pb	64				140	250 (4000)	
	0	230	2,0	1,6	0,8			
	1 cm Pb	230	3,8	2,1				
2	10 cm Pb	230				60	26 (410)	
	0	460	4,6	3,1	0,68			
3	0	690	9,7	6,3	0,68	19	5 (80)	
	0,5 cm Pb	690	12,1	7,9				
4	10 cm Pb	690	0,6	0,4		16	5	
	0	920	13,3	7,9	0,60			
	0	1150	19,1	9,7	0,51			
5	0	1400	26,0	14,1	0,55	9	1,5 (24)	
	0,5 cm Pb	1400	30,1	17,0				
	10 cm Pb	1400	3,6	1,7				4

After the substitution of $FN = x$ this becomes

$$I(F) = I_0 F^{S-1} \int_0^{\infty} x^{-S} (1 - e^{-x})^n dx.$$

What we next want to find is the value of S . There are two ways: by varying F and by measuring for different values of n . The first has been done by COCCONI, LOVERDO and TONGIORGI and gives mathematically the simplest result, as the integrals for two values of F are the same and

$$\frac{I(F_1)}{I(F_2)} = \left(\frac{F_1}{F_2} \right)^{S-1}$$

gives directly the value of S which is constant for different N 's, but it is more correct to derive the relation from different areas which have different observed mean densities. Therefore it is preferable to find S for different n 's and use the same surface per set. We have made measurements in both ways, as we have observations of the relation of 4 and 6-fold coincidences with sets of 1, 2, 3, 4, 5 and 6 counters in every set.

For the relation of sixfold and fourfold coincidences the values which we found were:

S	Theoretical	Experimental	N (density)
1.5	0.89	0.84	100
2.0	0.78	0.68	19
2.5	0.63	0.60	16
3.0	0.58	0.55	9

The integrals $I(F)$ were calculated for a series of S and n values by Mr H. F. JONGEN to whom we are much indebted.

With these results it is obvious that we may not take the value of S constant for all densities, but that with the increase of density the value S decreases, and therefore our assumption about S in $I(F)$ is not perfectly correct, but this was a first attempt to solve the problem.

These different values of S are in agreement with our results for the track density of bursts¹), which is for the low density of the order of 3.0 and for high densities 1.6. Also the value of proportion of two coincidences of different numbers n is different.

	6/4 f	5/4 f	4/3 f	3/2 f
theoretical	0.78	0.77	0.72	0.33
experimental	0.8	0.6	0.36	0.11

This agreement is not too good, but the tendency in the calculations and the observations is the same.

Finally I wish to thank Mr G. KLEIN for his help with part of the experiments.

Amsterdam, Febr. 1949.

REFERENCES.

1. J. CLAY, Phys. Soc. Cambridge Rep. 47 (1947).
2. L. JÁNOSSY & ROCHESTER, Phys. Roy. Soc. A 181, 399 (1943).
3. G. COCCONI, A. LOVERDO, V. TONGIORGI, Phys. Rev. 70, 841 (1946).

Biochemistry. — *On stearate systems containing methyl-hexylcarbinol with viscous and elastic properties comparable to elastic viscous oleate systems containing KCl.* By H. J. VAN DEN BERG and L. J. DE HEER.
(Communicated by Prof. H. G. BUNGENBERG DE JONG.)

(Communicated at the meeting of March 26, 1949.)

1. Introduction.

The results obtained in this laboratory¹⁾ with elastic viscous oleate systems containing KCl opened up the question if any systems with analogous properties also exist with other soaps. A priori one would be inclined to answer in the affirmative, as at sufficiently high temperature palmitates and stearates also form elastic systems at KCl concentrations which are too low for coacervation.

Difficulties to be foreseen in the measuring technique at such high temperatures withheld us for some time to start with such an investigation.

The elastic systems obtained at high temperatures cannot be investigated after having cooled down to room temperature as they are no longer stable then (the soaps separate in an insoluble form). Prof. BUNGENBERG DE JONG informed us that the temperature at which the elastic stearate system of the composition 1.2 % K-stearate in 0.2 N KOH becomes unstable, can considerably be lowered by adding the proper amount of a suitable organic substance, e.g. methyl-hexylcarbinol. It was therefore decided to perform some measurements on the above stearate systems for which we used "K-stearat, doppelt gereinigt, RIEDEL DE HAEN", as this preparation makes it possible to get stable and clear systems at even 15°.

2. Preparation of the elastic system.

When the system (1.2 gr K-stearate p. 100 cc in 0.2 N KOH) has cooled down to room temperature, it is a thick semiliquid somewhat pasty system, of a white silky appearance. If at 15° one adds increasing amounts of methyl-hexylcarbinol and homogenizes it by vigorous shaking, it is found that gradually the white silky appearance becomes less and the system changes into a clear markedly elastic system, which on further addition of the alcohol grows turbid and finally loses its elastic properties completely. It was found in preliminary experiments (rotational oscillation in a spherical vessel with a radius of 3.58 cm) that as a function of the added amount of methyl-hexylcarbinol, the values

¹⁾ Cf. H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG, these Proceedings part I, 51, 1197 (1948), parts II, III, IV, V, 52, 15, 99, 363, 377 (1949).

$\nu (= 1/T)$, $1/\Lambda$ and n increase to a certain maximum and next decrease. The maxima, which nearly coincide, are situated in the region of the completely clear systems. For the experiments in the following section we made a large quantity of elastic system corresponding to the position of the above maxima, for which an addition of 0.42 cc methyl-hexyl-carbinol was needed per 100 cc. 1.2 % K-stearate in 0.2 N KOH.

3. Period and logarithmic decrement as function of the radius of the sphere at 15.7°.

The methods used and the calculations of the elastic constants were the same as described by H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG in their publication of their investigations on elastic viscous oleate systems (see note 1, Part I, II and III).

The results obtained with the rotational oscillation at 15.7° are collected in Table I and depicted in fig. 1.

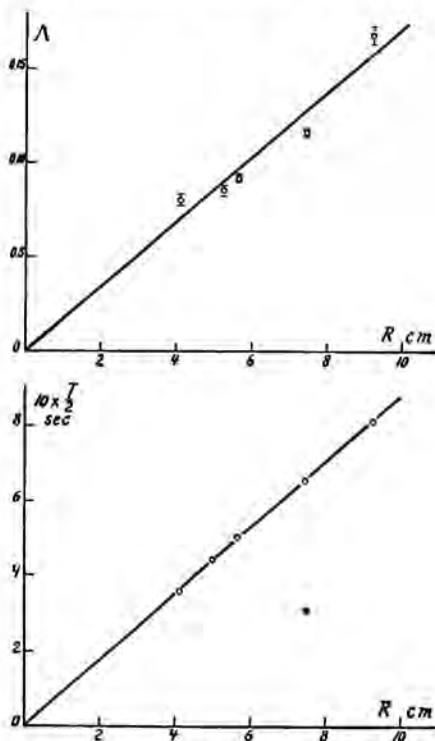


Fig. 1.

We see from fig. 1 lower that the period is proportional to the radius R of the sphere. Compared with oleate systems this is quite the same result.

In oleate systems we meet two different cases as regards the dependence of Λ on R . In 1.2 % systems it was found to be proportional to R , in 0.6 % systems, however, it was found to be independent of R . We

TABLE I.

Measurement with a 1.2 % stearate system (containing 0.2 N KOH and 0.42 cc methyl-hexylcarbinol per 100 cc) at 15.7° C (rotational oscillation).

R (cm)	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_3	Λ	$\lambda = (T/2\Lambda)$ (sec)	G (dynes/cm ²)	
9.24	96.5	8.10	1.183	0.168 ± 0.005	4.82	64.2	
7.46	93.8	6.54	1.124	0.116 ± 0.002	5.64	64.2	
5.65	86.2	5.02	1.096	0.092 ± 0.002	5.47	mean	
5.01	83.5	4.40	1.089	0.085 ± 0.003	5.16	5.12	
4.12	74.2	3.60	1.083	0.080 ± 0.003	4.51	63.9	
						64.6	

see from fig. 1 upper that in our 1.2 % stearate system too, Λ is proportional to the radius of the sphere. This means that here too the damping of the elastic oscillations must be ascribed to relaxation of the elastic stresses with a constant time of relaxation λ^2). These values of λ (calculated from $T/2\Lambda$) are given in column 6 of Table I and fluctuate considerably around the mean value 5.12.

This is most probably connected with the relatively low damping (far lower values of Λ as were ever found in oleate systems). The lower the damping ratio b_1/b_3 the more experimental errors exert their influence on Λ (which is $\ln b_1/b_3$). Such great fluctuations are not found in the calculated G values (Table I, column 7), which brings out that the b_1/b_3 measurements are far more difficult than the $10 \times T/2$ measurements.

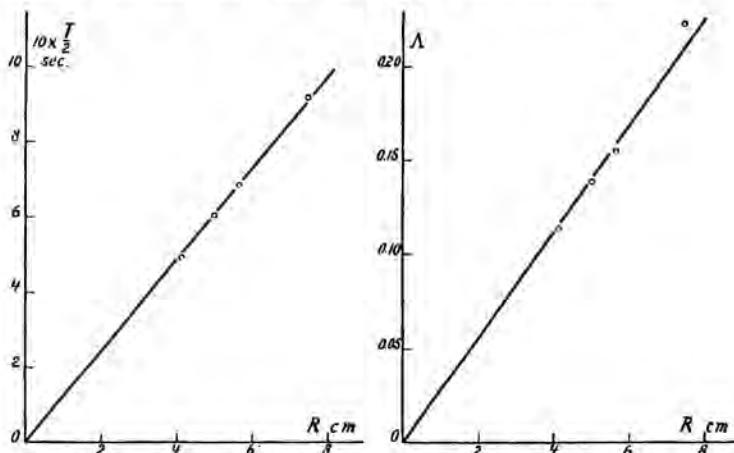


Fig. 2.

We therefore decided also to perform elastic measurements on a lower concentrated system, which we obtained by diluting three volumes of the above 1.2 % stearate system with one volume 0.2 N KOH. As it is highly probable that the methyl-hexylcarbinol is practically wholly bound to the stearate, an extra addition of it was deemed to be unnecessary.

²⁾ J. M. BURGERS, Proc. Kon. Ned. Akad. v. Wetensch., Amstendam, 51, 1211 1948.

TABLE II.

Measurements with a 0.9% stearate system (containing 0.2 N KOH and 0.32 cc methyl-hexylcarbinol per 100 cc) at 15.7° C (rotational oscillation).

R (cm)	n	$10 \times \frac{T}{2}$ (sec)	b_1/b_2	Λ	$(\lambda = \frac{T}{2\Lambda})$ (sec)	G (dynes/cm ²)	
7.46	54.7	9.18	1.250	0.2232	4.11	32.6	$\eta = G \times \lambda =$
5.65	63.3	6.86	1.168	0.1553	4.42	32.4	$= 4.30 \times 3.33$
5.01	61.3	6.06	1.149	0.1389	4.36	33.7	$= 143$ poises
4.12	59.4	4.92	1.121	0.1138	4.31	34.6	

The results obtained with this 0.9% stearate system are collected in Table II and depicted in fig. 2, from which we once more see that $T \sim R$ and $\Lambda \sim R$, the experimental Λ points now deviating much less from the drawn straight line than in fig. 1 upper. Therefore the calculated λ values (column 6) also show far smaller fluctuations percentually.

That is why we feel justified in assuming that the irregularities in the Λ and λ values met with in the 1.2% stearate system, were caused by the difficulties in the measurement of the damping ratio.

In Table I we have also calculated the value $G \times \lambda$, which we will need in the next section.

4. Viscous behaviour of the 1.2% stearate system at 15.7°.

For this investigation we used the same method as described by H. G. BUNGENBERG DE JONG, H. J. VAN DEN BERG and L. J. DE HEER (see note 1, Part. V). It may suffice to give here only the (logarithmic) viscosity-shearing stress diagram obtained. See fig 3, in which we notice the same characteristic shape as we met in the markedly elastic oleate system, though due to the very high viscosity at low shearing stresses it was not possible to make sufficiently accurate measurements to locate the η_0 level. The bending off at the left upper end of the curve, however, suggests that it must lie approximately as indicated by the dotted line. ($\log \eta_0 = 2.08$, that is $\eta_0 = 120$ poises). At the other end of the curve we have not yet reached the η_∞ level, which we may expect to lie at an η value in the order of 10 centipoises or lower.

The total fall in viscosity from η_0 to η_∞ by increasing the shearing stress is therefore very great, viz. at least to a thousandfold lower value.

The most characteristic feature of the viscosity curve in fig. 3 is, however, the occurrence of an intermediate level. The total fall in viscosity by increasing the shearing stress takes place in two separate steps, the first from ± 120 poises down to ± 4 poises, the second from ± 4 poises down to 0.1 poise. (or lower).

Such viscosity curves with three levels are characteristic of the markedly elastic oleate systems.

Finally we may discuss a further point of resemblance between stearate and oleate systems. It was found for the latter that the viscosity coefficient

calculated from the elastic measurements (using MAXWELL's formula $\eta = G \times \lambda$) is of the same order as the experimentally determined η_0 level, though η_0 was found to be always smaller than $G \times \lambda$ ³⁾.

We have drawn in figure 3 the level corresponding with the value $G \times \lambda$, which according to table I was found to be 327 poises ($\log \eta$ is therefore 2.51). Assuming the η_0 level to lie at approximately 120 poises ($\log \eta = 2.08$), as drawn in figure 3, η_0 would be approximately 2.7 times smaller than $G \times \lambda$.

For oleate systems the disagreement between η_0 and $G \times \lambda$ has been found to increase with the absolute value of η_0 and from this one would have expected for an η_0 value of 120 poises a disagreement which comes near to the above mentioned value of 2.7 times smaller. (i.e. $\eta_0/G \times \lambda = 0.37$).

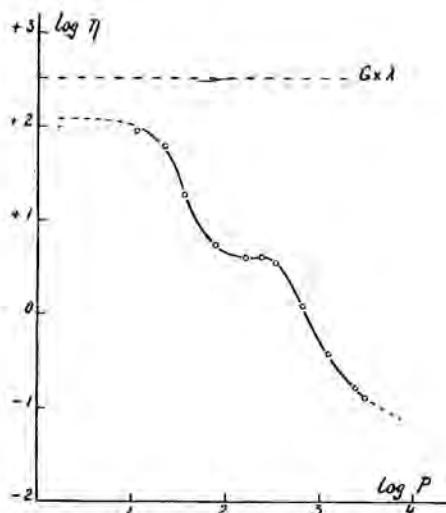


Fig. 3⁴⁾

5. Final remarks.

The investigations described above show that the characteristics of the elastic and viscous properties of the soap system used are the same as those of oleate systems containing KCl. They do not claim to give accurate figures for a pure stearate, however. In spite of the indication "doppelt gereinigt", the preparation of K-stearate used, appeared in further experiments to be anything but pure. BUNGENBERG DE JONG and DE HEER, in comparing a number of K-stearates and stearic acids of different origin, have found very great differences as to the damping under

³⁾ Cf. paper quoted in note 1, part V.

⁴⁾ P = shearing stress at the wall of the capillary in dynes/cm²; V = mean rate of flow in cm³/sec. In figure 3 the ordinate does not indicate $\log V$, but $\log P - \log V$, which stands for the meaning of the logarithm of a viscosity coefficient η .

comparable conditions. Really pure stearate systems (using a preparation of Stearic acid "Kahlbaum" with m. p. 69°) cannot be transformed into clear elastic systems with methyl-hexylcarbinol at 15°, nor at 20°, but it does succeed at 40°. At 40° however, RIEDEL DE HAEN's preparation, which we used in this investigation, was no longer measurable because of the now very great damping (n , the number of turning-points being only 3) though it had a small damping at 15° (n having now values in the order of 60).

Summary.

1. It has appeared to be possible to make elastic-viscous systems, with analogous properties, with other soaps than oleates. One must slightly alter the methods of preparation though, in order to obtain systems which are measurable at a conveniently low temperature.
2. Analogous to the results found for elastic measurements of oleate systems of more than 1.1 % Na-oleate, a direct proportionality was found between A and R , and T and R .
3. The viscous behaviour was the same as has been found for oleate systems viz. a breakdown of viscosity with increasing shearing stress in a two-step process, which seems to be characteristic for these elastic soap systems.
4. There exists a disagreement between the viscosity η_0 , measured at low shearing stress (at the wall of the capillary) and the viscosity predicted by MAXWELL's formula $\eta = G \times \lambda$, which disagreement is found to be of the same order of magnitude as could be expected from the behaviour of the oleate systems.

*Department of Medical Chemistry,
University of Leiden.*

KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN
WETENSCHAPPEN

PROCEEDINGS

VOLUME LII

No. 5

President: A. J. KLUYVER
Secretary: M. W. WOERDEMAN

1949

NORTH-HOLLAND PUBLISHING COMPANY
(N.V. Noord-Hollandsche Uitgevers Mij.)
AMSTERDAM

C O N T E N T S

Anatomy

HUIZINGA, J.: The digital formula in relation to age, sex and constitutional type. II. (Communicated by Prof. M. W. WOERDEMAN), p. 587.

Astronomy

CLAAS, W. J.: The Partition function for the elements in the Solar Atmosphere. (Communicated by Prof. M. G. J. MINNAERT), p. 518.

GATHIER, P. J.: Magnitude effects in G-type stars. (Communicated by Prof. M. G. J. MINNAERT), p. 569.

Biochemistry

BUNGENBERG DE JONG, H. G., H. J. VAN DEN BERG and D. VREUGDENHIL: Elastic viscous oleate systems containing KCl. VI. a) The elastic properties as a function of the KCl concentration. b) Influence of some alcohols and fatty acid anions on the elastic behaviour, p. 465.

Botany

FRETS, G. P.: De hypothese voor de erfelijkheidsformules van de twee zuivere lijnen I en II van Phaseolus vulgaris op grond van kruisingsproeven. II. (Communicated by Prof. J. BOEKER), p. 577.

Crystallography

DIJKSTRA, D. W.: Transformation of gnomograms and its application to the micro-chemical identification of crystals. II. (Communicated by Prof. J. M. BIJVOET), p. 563.

Geology

WAARD, D. DE: Tectonics of the Mt. Aigoual pluton in the southeastern Cevennes, France. Part II. (Communicated by Prof. H. A. BROUWER), p. 539.

Mathematics

DRONKERS, J. J.: Een iteratieproces voor de oplossing van een randwaardeprobleem bij een lineaire partiële differentiaalvergelijking van de tweede orde. II. (Communicated by Prof. W. VAN DER WOUDE), p. 479.

HLAVATÝ, V.: Affine embedding theory I: Affine normal spaces. (Communicated by Prof. J. A. SCHOUTEN), p. 505.

POPKEN, J.: A property of a DIRICHLET series, representing a function satisfying an algebraic difference-differential equation. (Communicated by Prof. J. G. VAN DER CORPUT), p. 499.

ZAANEN, A. C.: Note on a certain class of Banach spaces. (Communicated by Prof. W. VAN DER WOUDE), p. 488.

Petrology

EGELER, C. G.: On metamorphic rocks from the island of Kabaëna in the East-Indian Archipelago. (Communicated by Prof. H. A. BROUWER), p. 551.

Zoology

BRETSCHNEIDER, L. H.: An electron-microscopical study of sperm IV. (The sperm-tail of bull, horse and dog.) (Communicated by Prof. CHR. P. RAVEN), p. 526.

OORDT, G. J. VAN, F. CREUTZBERG and N. SPRONK: Spermiation in Rana and Salamandra. Preliminary note. (Communicated by Prof. CHR. P. RAVEN), p. 535.

Biochemistry. — Elastic viscous oleate systems containing KCl. VI¹⁾.

- a) *The elastic properties as a function of the KCl concentration.*
- b) *Influence of some alcohols and fatty acid anions on the elastic behaviour.* By H. G. BUNGENBERG DE JONG, H. J. VAN DEN BERG and D. VREUGDENHIL.

(Communicated at the meeting of April 23, 1949.)

1) *Introduction, and the measuring technique with half filled spheres.*

Having kept in the preceding Parts of this Series the KCl concentration constant (viz. at or very near to that concentration at which the damping of the elastic oscillation is a minimum), we will in the present communication investigate the dependence of the elastic properties on the KCl concentration (section 2).

The knowledge of this dependence is, of course, of importance for a future theory of the elastic viscous oleate systems, it is so too for studying the influence of organic substances on the elastic behaviour. A broader survey of the influence of an organic substance can indeed be obtained if its action is investigated not only at the KCl concentration of minimum damping (of the blank), but also at lower and higher KCl concentrations (section 2).

In the next sections, 3 and 4, in which the influence of a number of terms of the *n.* primary alcohols and of the fatty acid anions is studied, we were compelled (see section 3) to restrict ourselves to the KCl concentration of minimum damping. The points of view obtained in section 2 will, however, guide us in interpreting the results and will heed us from making erroneous generalizations.

For the methods (rotational oscillation in spherical vessels of known capacity at 15°) we refer to Parts I, II and III of this series, with the only exception that instead of using completely filled spherical vessels we used exactly half filled vessels. We know from the experiments in Part I (cf. ibid. section 2) that the period depends on the degree of filling of the vessel and that at 50 % degree of filling (by volume) the period is practically the same as in the completely filled sphere. According to not published previous work this practical equality also holds for the damping ratio b_1/b_3 , which is used for calculating Λ , the logarithmic decrement.

Half filled vessels of nominally 500 cc capacity (provided with ground in glass-stoppers) highly serve the purpose of investigating the influence of added organic substances. We first measure the elastic properties of the blank, add a small known quantity of the organic substance into the vessel (e.g. drops from a suitable micropipette, which delivers drops of known weight), shake vigorously (e.g. 5 minutes or more) and put the vessel in the thermostat (15°) till the next morning (to get rid of the

¹⁾ Part I has appeared in these Proceedings 51, 1197 (1948).
Part II—V in these Proceedings 52, 15, 99, 363, 377 (1949).

enclosed air) and measure the elastic properties anew. We continue in this way with alternately adding more substance and measuring the next morning until a sufficient number of experimental points has been obtained to characterize the action of the added substance.

This method has the advantage of economizing oleate, KCl and organic substance²⁾, its disadvantage is that it takes many days. This disadvantage is compensated however by running a number of half filled vessels simultaneously.

The reliability of the method depends of course on the assumption that during the whole process no changes take place in the oleate system when no organic substance is added. To control this, we take a half filled sphere and treat it in the same way as the other spheres (measuring, opening the glass stopper for a short time, closing it again, vigorously shaking of the vessel, placing it in the thermostat and so on repeating this for several days). With 500 cc stoppered vessels and oleate systems containing the usual small quantity of KOH we found no changes which obviously surpassed the experimental errors.

The method described above is very convenient for added substances which exert their full influence in small concentrations already. The quantity of substance added is very small then (e.g. n. hexyl-, n. amyl-, n. butyl alcohol) and no corrections are necessary.

If the substance exerts its full influence at concentrations which correspond to additions of one to several percent, of the oleate system (e.g. methanol, ethanol), corrections are necessitated by the increase in the degree of filling and its consequences, viz. decrease of the oleate concentration and of the KCl concentration and change in the density.

The addition of one percent by volume changes the degree of filling from 50% to 50.5%. From Table I in Part I we calculate for this increase an increase of 0.35% of the period. Later, more accurate measurements gave the somewhat smaller value of 0.25%. The addition decreases also the oleate concentration by one percent, which gives an increase in the period of approximately 1% (see Part III, where we found that $\nu (= 1/T)$, is roughly speaking, proportional to the oleate concentration).

The two changes considered result therefore already in an increase of 1.25% in the period, and as the shear modulus G is proportional to ν^2 , the addition of one percent to the volume will lead, if not corrected, to a value of G which is 2.5% too low. The two other changes above mentioned will, in general, have only a slight influence when compared to the two considered here.

To come to a practical limit for the volume of an added substance, below which no corrections are needed, it is necessary to know the reproducibility of G , calculated from measurements on a number of exactly half filled vessels, using an identical oleate system.

The next survey gives the results of the measurements on the eight vessels half filled with the blank oleate system which were used for the experiments in section 3.

R (cm)	5.00	5.01	5.01	5.01	5.03	5.04	5.04	5.05
n	48.0	48.2	47.9	48.0	48.1	47.9	48.2	47.8
A	0.222	0.222	0.222	0.223	0.219	0.224	0.219	0.222
λ (sec)	2.57	2.56	2.57	2.57	2.60	2.54	2.60	2.60
G (dynes/cm ²)	40.4	40.7	40.4	40.3	41.2	41.1	41.0	40.8

²⁾ It would of course also be economical to use spherical vessels of much smaller capacity than 500 cc, for instance 100 cc vessels, but experience taught us that it is not advisable to use them as the experimental errors are greater then. This is partly due to the more difficult measurement of the period and especially of the damping ratio (in 1.2% oleate systems T and A are proportional to the radius, see Part II, and are therefore much smaller in 100 cc vessels than in 500 cc vessels). This may also be due to the fact that smaller vessels do not approximate the ideal spherical form so well.

We see from this example that in practice differences up to 2% in the G value may occur. Therefore we may content ourselves not to object to the introduction of an error in the order of 1%; and corrections may be neglected, if the volume of the added substance does not exceed 0.5% of that of the oleate system.

For methanol and ethanol larger quantities must be added to characterize their influence on the oleate system and therefore corrections are necessary.

By measuring the influence of known quantities of distilled water on a vessel half filled with the blank system we determined in section 3 experimentally the correction to be applied in order to account for three of the four changes mentioned above (viz. the increase in the degree of filling, decrease in the oleate concentration and decrease in the KCl concentration. The correction for the fourth change (decrease of the density) was applied separately.

In the ethanol series in section 2, the necessity of applying corrections for the changes in the degree of filling and in the oleate concentration was made redundant by adding a volume b of ethanol and a volume $b/2$ of the oleate stocksol and, after mixing the contents of the vessel, by removing a volume of $1.5 b$ with a pipette. After the addition the oleate concentration is not changed (the stock solution having a threefold larger oleate concentration) and after the removal the original degree of filling is restored. Only a slight change in the KCl concentration remains which can be calculated. For this reason the corresponding points on the curves A, B and C of fig. 4 do not longer lie at precisely the same abscis-values (as in fig. 2).

2) *The elastic behaviour as a function of the KCl concentration and the influence of n. hexylalcohol and aethanol thereon.*

Starting from a stock solution (36 g. Na oleinicum medicinale pur. pulv. Merck + 820 cc H₂O + 180 cc KOH 2 N) we made 11 mixtures according to the formula 100 cc stock solution + a cc KCl 3.8 N + (200- a) cc H₂O.

In these mixtures the olcate concentration (1.2 g. p. 100 cc) and the KOH concentration (0.12 N) are constant and the KCl concentration increases by degrees from 0.57 N to 1.90 N. The highest concentration used closely approaches the coacervation limit (slightly above 2 N KCl).

Eleven "500 cc" spherical vessels of known real capacity, the radius of which (R) had been calculated, were each exactly half filled with one of the above mixtures and put in the thermostat of 15°. The next morning the elastic behaviour was measured, and afterwards the same small quantity of n. Hexylalcohol was added to each vessel.

The mixtures were thoroughly shaken, put in the thermostat and once more measured the following morning. Two further additions of hexylalcohol were measured in the same way. Table I gives the results of these measurements ³⁾.

³⁾ From experiments in Part II of this series it appeared that for the 1.2% oleate system at or near to the KCl concentration of minimum damping $A \sim R$, which enables us to calculate λ , the relaxation time. This proportionality characteristic in the 1.2% oleate system is not lost if by various causes A is increased very considerably (e.g. with increase of temperature, see Part II, or the addition of hexylalcohol, see Part V). We have therefore assumed that this proportionality is still present in the 1.2% oleate system if A is increased considerably by changing the KCl concentration to other values than that of minimum damping, or by adding the various substances used in section 3 and 4, others than hexylalcohol. As this assumption is not entirely safe of course, we give in

TABLE I.

The elastic behaviour as a function of the KCl concentration and the influence of n hexylalcohol thereon.

KCl concentration <i>mol/l</i>	G (dynes/cm ²)				λ (sec.)			
	Hexylalcohol concentr. mol/l				Hexylalcohol concentr. mol/l			
	blank	0.0016	0.0042	0.0094	blank	0.0016	0.0042	0.0094
0.57	—	—	13.2	28.7	—	—	0.32	0.61
0.76	17.4	21.9	30.4	44.2	0.32	0.47	0.74	0.64
0.95	30.4	32.2	38.1	80.5	0.90	1.14	1.02	0.15
1.14	35.0	37.5	46.9	— *	1.58	2.00	0.88	— *
1.235	37.7	40.9	51.0	—	2.12	1.93	0.69	—
1.33	38.9	43.1	56.7	—	2.52	1.40	0.41	—
1.425	40.2	48.7	67.8	—	2.68	0.90	0.23	—
1.52	42.2	52.5	—	—	2.47	0.76	—	—
1.615	45.5	54.9	— *	—	1.32	0.63	— *	—
1.71	51.5	62.4	—	—	0.90	0.29	—	—
1.90	63.1	— *	—	—	0.30	— *	—	—
KCl concentration <i>mol/l</i>	1/A				n			
	Hexylalcohol concentr. mol/l				Hexylalcohol concentr. mol/l			
	blank	0.0016	0.0042	0.0094	blank	0.0016	0.0042	0.0094
0.57	—	—	0.33	0.93	0	0	4.0	13.1
0.76	0.38	0.63	1.17	1.21	5.1	10.7	17.8	17.0
0.95	1.40	1.84	1.78	0.37	18.6	29.3	31.4	4.0
1.14	2.65	3.45	1.70	— *	39.8	41.4	27.2	— *
1.235	3.60	3.40	1.36	—	45.1	40.5	21.0	—
1.33	4.33	2.53	0.86	—	48.1	38.5	14.9	—
1.425	4.69	1.74	0.52	—	49.4	31.8	7.0	—
1.52	4.42	1.51	—	—	48.5	24.7	1.0	—
1.615	2.46	1.29	— *	—	38.3	18.9	— *	—
1.71	1.75	0.62	—	—	29.5	9.5	—	—
1.90	0.66	— *	—	—	12.2	— *	—	—

* First mixture in each column which shows coacervation.

We will first discuss the blank series (see fig. 1). The curves for λ (time of relaxation in sec.), $1/A$ (the reciprocal value of the logarithmic decrement A), n (the number of oscillations visible through the telescope of the kathetometer) show a maximum at approximately 1.43 N KCl, and at this concentration the curve for G (shear modulus in dynes/cm²) shows an inflexion point (compare the vertical, dotted line B at 1.43 N KCl). On either side of this concentration the λ , $1/A$ and n curves descend and, if extrapolated further downwards, reach the abscis axis at approximately

the following tables and figures besides the calculated values of λ the values for $1/A$ as well to characterize the damping in every case. We also give n , i.e. the maximum number of observable oscillations, which, as will be discussed in section 5, gives an approximate measure of $1/A$.

the two same KCl concentrations (0.63 and 2.05 N) indicated by the vertical, dotted lines A and C.

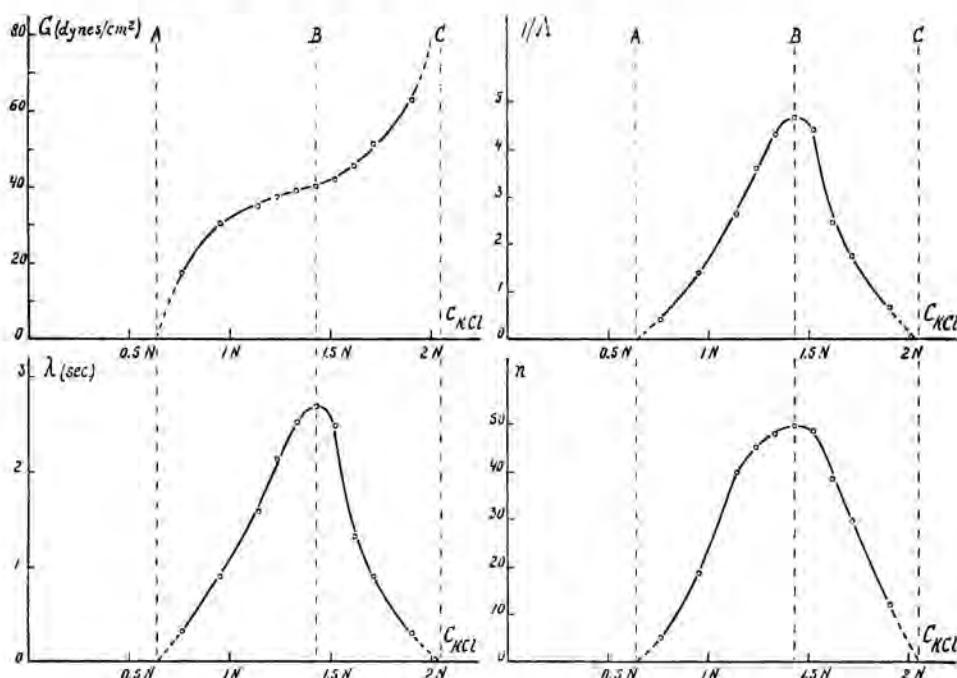


Fig. 1.

The latter of these KCl concentrations lies very near to the coacervation limit (at 2.00 N KCl not yet coacervation, at 2.09 N KCl coacervation). In the case of n we know from previous experiments that it indeed reaches the value zero just before the separation into two liquid phases (coacervate and equilibrium liquid) sets in (i.e. at slightly further increase of the KCl concentration). We may therefore say that the vertical, dotted line C indicates practically the coacervation limit.

If we now direct our attention to the G curve, we notice that it has quite another character than the λ , $1/A$ or n curves. By increasing the KCl concentration G always increases, though transitorily at a slower rate when nearing the KCl concentration of minimum damping (inflection point of the G curve).

It seems further probable that the G curve begins with the value zero at the KCl concentration, indicated by the dotted vertical line A and that the G curve still continues sloping upwards if the KCl concentration approaches the coacervation limit C. Any future theory of the elastic viscous oleate systems must be able to explain the characteristic shapes of the curves in fig. 1 as discussed above. For the present we do not yet wish to speculate on this matter, as we are convinced, that more facts should be known as described hitherto in this series of communications.

We now come to the influence of added hexylalcohol. Fig. 2 shows that n hexylalcohol displaces the G , λ , $1/\Lambda$ and n curves. Let us consider this displacement more closely:

While retaining their character (curve with a maximum) the λ , $1/\Lambda$ and n curves are displaced into the direction of smaller KCl concentrations. The height of the maximum decreases in case of displacement to the left.

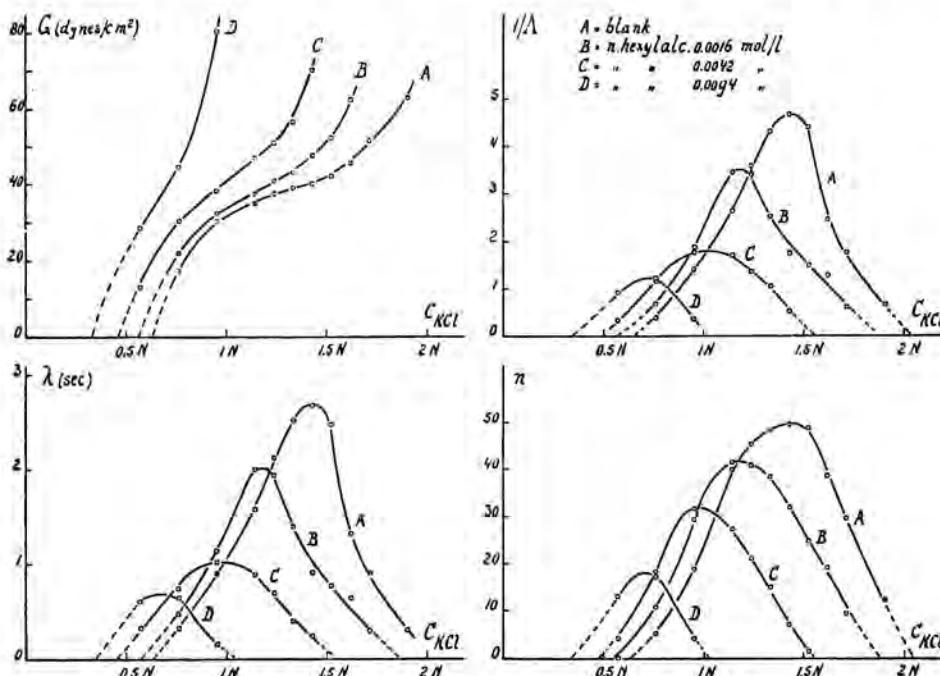


Fig. 2.

and the distance of the footpoints of these curves on the abscis axis grows smaller ⁴⁾.

When we consider the displacement of a characteristic point of these curves, viz. the maximum point, we see that it is displaced downwards to the left, which may be looked upon as to consist of a horizontal and vertical component (see fig. 3 scheme A, lower graph.)

The G curve (see fig. 2) while retaining its character of having an inflection point is also displaced towards smaller KCl concentrations; besides it becomes steeper, in consequence of which the inflection point becomes less marked.

When we consider the displacement of a characteristic point of this curve, viz. the inflection point, (in practice determined by reading G at the KCl concentration of the maxima of the λ , $1/\Lambda$ or n curves) we obtain

⁴⁾ The coacervation limit, which still practically coincides with the right footpoint of these curves, is therefore shifted to smaller KCl concentrations. See asterisks in Table I.

only a displacement in a horizontal direction⁵⁾ (see fig. 3 scheme A, upper graph.).

The above induces one to attribute a twofold action to the added hexylalcohol. In its presence smaller KCl concentrations are needed to set up the typical elastic viscous system (horizontal displacement of the inflection point on the G curve and horizontal component of the displacement of the maximum point on the λ , $1/\lambda$ and n curves), i.e. hexylalcohol facilitates the large scale associations of the soap molecules which we assume to be present in the elastic viscous system. The existence of a vertical component downwards in the displacement of the maximum point of the λ , $1/\lambda$ and n curves, however, indicates that hexylalcohol at the same time influences these elastic structures in such a way that the damping is generally increased (without altering the shear modulus).

If for the two above actions of hexylalcohol different points of attack on the elastic structure are supposed, it may be conceived that there also exist substances, which, when added to the oleate system, share with hexylalcohol the general increasing influence on the damping, but which counteract KCl in setting up the elastic viscous system (compare fig. 3 scheme C). In between the two extreme schemes A and C of fig. 3 a

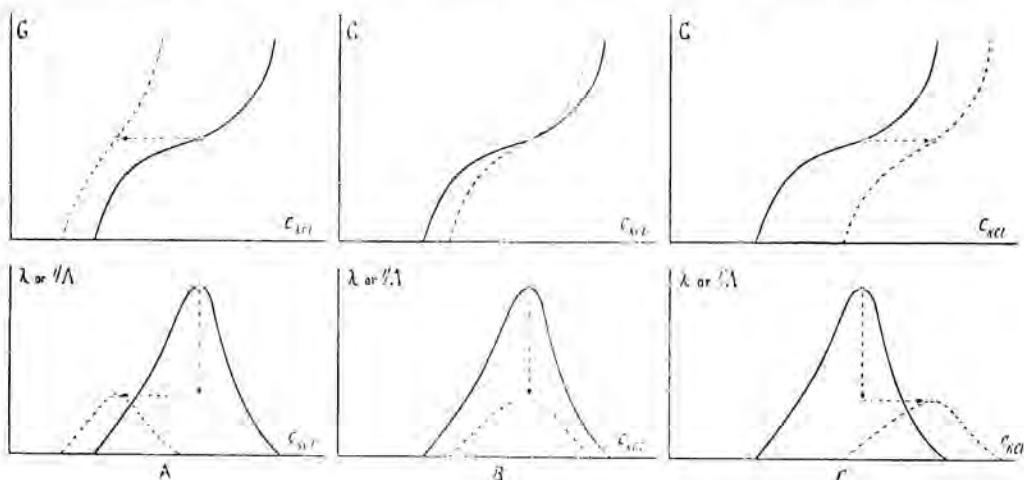


Fig. 3.

whole series of intermediate schemes may be conceived and half way it the scheme, indicated by B in fig. 3, represents a substance which neither helps nor counteracts KCl in setting up the elastic system. In the discussion of further experiments in this and following sections the schemes of fig. 3 will play a helpful rôle.

⁵⁾ At the KCl concentrations corresponding with the maxima of the λ , $1/\lambda$ and n curves A, B, C and D, we read of the following values of G on the corresponding G curves A, B, C and D: 40.5, 39, 40 and 38 dynes/cm², which cannot with certainty be considered as really different.

We also investigated the influence of ethanol on G , λ , $1/\Lambda$ and n in an analogous way as described above for the influence of n hexylalcohol.

We shall not give elaborate tables of the results but represent them graphically in fig. 4. When we compare this figure with fig. 3 we see that

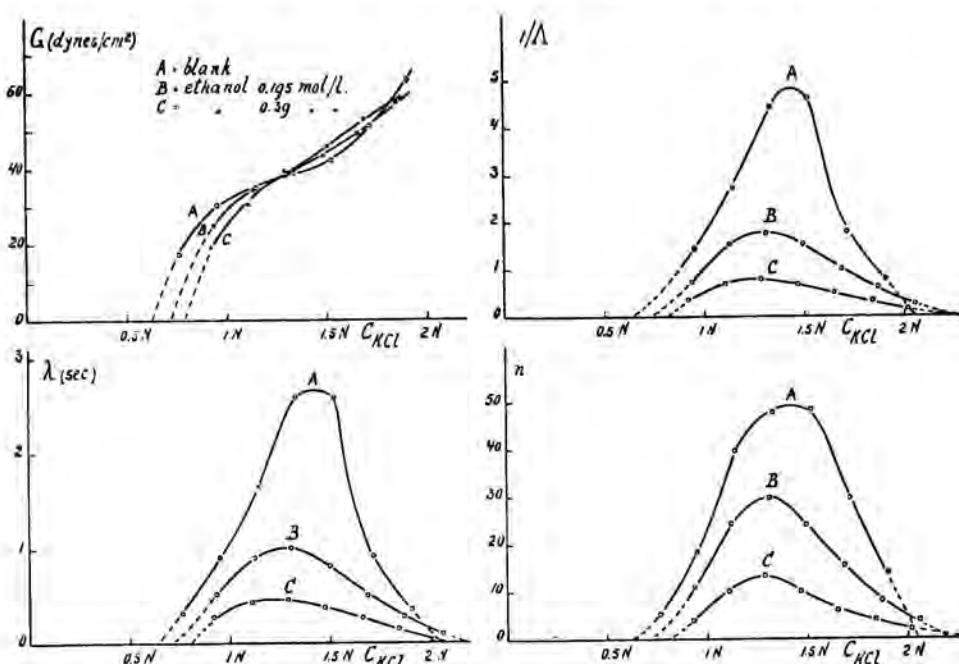


Fig. 4.

it greatly resembles scheme *B*. The following points show however, that it is not identical with it. In the first place the maximum points on the λ , $1/\Lambda$ and n curves are not displaced downwards in the vertical direction only, but there is still a slight horizontal component of the displacement to the left.

Secondly, the G curves do not intersect at the KCl concentration corresponding to the minimum damping of the blank (1.43 N) but at lower concentrations (± 1.25 N).

Ethanol therefore corresponds to a case intermediate between the schemes *A* and *B*, which stands much closer to *B* than to *A* on account of the slight horizontal displacement to the left relative to the vertical displacement downwards.

We must further remark that ethanol shows another complication, which cannot be predicted from the simple schemes of fig. 3. If we direct our attention to the course of the λ , $1/\Lambda$ and n curves at KCl concentrations above approximately 1.7 N, we perceive that those in the presence of ethanol cut the blank curve. Here the coacervation limit is shifted not to smaller KCl concentrations but to higher. A similar phenomenon is present in the course of the G curves above 1.7 N KCl. Those corresponding to systems to which ethanol has been added, tend now to cut the G curve of the blank. Both facts seem to

indicate that the action of ethanol still depends on the absolute value of the KCl concentrations. With concentrations up to ± 1.7 N ethanol is a substance which slightly helps KCl in building up the elastic system, with concentrations higher than ± 1.7 N it counteracts KCl slightly.

3) *Influence of the first six terms of the n. primary alcohols on the elastic behaviour of the 1.2 % oleate system at the KCl concentration of minimum damping.*

As our stock of Na oleinicum medicinale pur. pulv. Merck was nearly exhausted and could not possibly be replenished, we decided to use for the experiments in this and the following section the large volume of oleate system, which had already served in previous experiments (on the dependence of T and Λ on R , see Part II of this series). It had the composition 1.2 g. oleate per 100 cc 1.52 N KCl + 0.08 N KOH, which electrolyte composition lies near to the minimum damping of the elastic oscillations. Eight spherical vessels of nominally 500 cc capacity (with radii varying from 5.01—5.04 cm) were exactly half filled with this system, put in the thermostat (15°) and measured the next morning.

Six of these vessels were used to determine the influence of the six alcohols, the seventh to control that the blank does not alter its properties with time and the eighth to ascertain part of the corrections to be applied to the experiments with methanol, ethanol and the highest concentrations of n propanol (cf. small print in section 1). We shall not give elaborate tables of the results, but represent them graphically in fig. 5. In this figure G , λ , $1/\Lambda$ and n are given as functions of the logarithm of the alcohol concentration (the latter in moles/l).

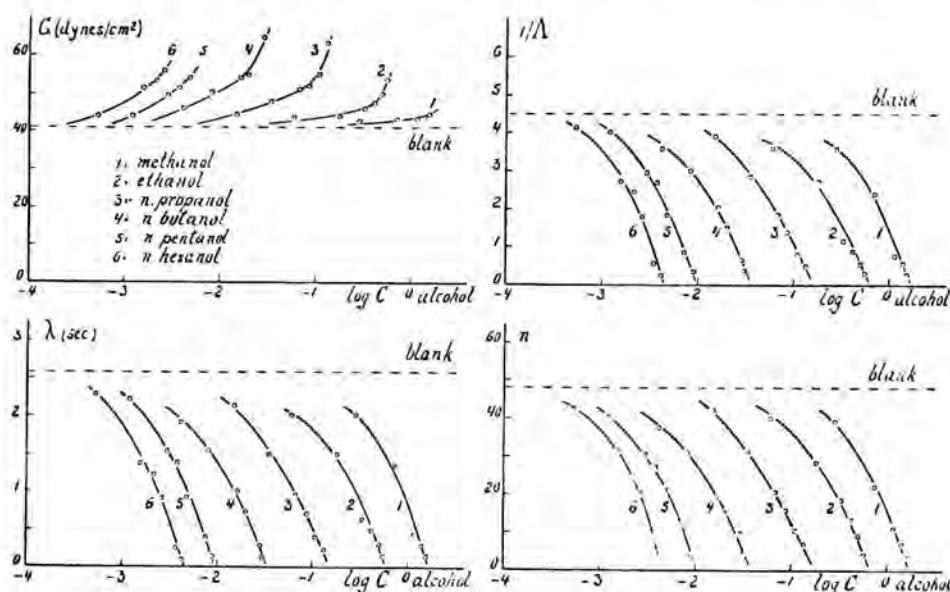


Fig. 5.

We see that all alcohols decrease λ , $1/A$ and n , and increase G . In the case of methanol the increase of G is so slight however, that one may doubt if it surpasses the experimental errors (taking into account that here the greatest correction had to be applied, which of course diminishes the reliability of the actual augmentation of the G curve above the blank level).

In connection with what was said in section 2, we come to the conclusion that the shift to the left in horizontal direction of the inflection point on the G curve in fig. 1 diminishes in the order $6 > 5 > 4 > 3 > 2 > 1$, by which it is perhaps not excluded that $1 = \text{methanol}$ shows already a very slight shift to the right.

As we have investigated here (fig. 5) the action of the alcohols at the KCl concentration of minimum damping, λ , $1/A$ and n can only decrease of course. Here too we have the same order of the curves as we had for G .

For a future theory of the action of n primary alcohols it is important that their action increases very considerably every time the alcohol is lengthened with one carbon atom. Below we give for instance the logarithms of the alcohol concentration corresponding to a decrease of λ , $1/A$ and n to 50 % of its original value.

	n hexyl alcohol	n amyl alcohol	n butyl alcohol	n propyl alcohol	ethanol	methanol
λ	0.27-3	0.58-3	0.06-2	0.67-2	0.32-1	0.84-1
$1/A$	0.32-3	0.63-3	0.15-2	0.74-2	0.34-1	0.85-1
n	0.35-3	0.64-3	0.12-2	0.73-2	0.37-1	0.84-1
mean logarithmic difference	0.31	0.49	0.60	0.63	0.50	

The differences in the logarithms correspond (at least when going from methanol to amylalcohol) to ratios of approximately 3 or 4 (the logarithmic differences of two succeeding G curves is of the same order or even greater). This reminds us of the ratios occurring in TRAUBE's rule and suggests that the alcohol molecules exert their influence in the adsorbed state, and that in the latter state the carbon chain of the alcohol lies flat against certain either external or internal surfaces of the elastic soap structures.

4) *Influence of the fatty acid anions C_8-C_{14} on the elastic behaviour of the 1.2 % oleate system at the KCl concentration of minimum damping.*

Preliminary experiments showed that the viscous and elastic properties of the oleate systems containing KCl are not only sensitive to alcohols, but to a great many classes of organic non electrolytes as well, hydrocarbons included.

This sensitivity to organic non electrolytes of all kinds is also found in

the two phase (coacervate/equilibrium liquid) oleate systems, which are formed from the one phase elastic viscous oleate systems at a somewhat higher KCl concentration. The characteristic influence of organic non electrolytes manifests itself here in the change of the partial solubility of the two phases (at constant KCl concentration) and can be conveniently studied by measuring the shift of the coacervation limit (i.e. the KCl concentration which is just needed for the separation into two phases) ⁶⁾.

So far as our experience reaches and speaking very generally it appears that as regards the connection between constitution and action of organic non electrolytes analogous rules hold for both the one phase elastic viscous oleate systems and the two phase (coacervate/equilibrium liquid) oleate systems.

It was recently shown in our laboratory by H. L. Booij that oleate coacervates are also sensitive to organic anions and that in the case of the fatty acid anions, the action of the same bound quantity depended in an unexpected way on the number of carbon atoms of the fatty acid anion ⁷⁾. From C₇ onward the coacervation limit is shifted to higher KCl concentrations, but this shift reaches a maximum value for undecylic acid (C₁₁) and diminishes thereafter very considerably (at C₁₄—C₁₆), to increase one more to approximately the level as reached with C₁₁ if the hydrocarbon chain of the fatty acid anion is sufficiently lengthened (C₂₀—C₂₂).

To see if now again a parallel could be drawn between elastic oleate systems and oleate coacervates, we decided also to include in the present investigation the influence of a number of fatty acid anions (from C₈—C₁₄) on the elastic properties of the 1.2 % oleate system at the KCl concentration of minimum damping. The method we followed was the same as in section 3 (half filled vessels of nominally 500 cc capacity 15°) and because a sufficient amount of KOH is present in the oleate system, we added known amounts of the fatty acids themselves (the terms with m.p. below room temperature as drops of known weight, the solid terms weighed amounts of the crystals). To accelerate the dissolution of the added acids, the vessels were agitated for 5 min. while immersed in a waterbath of 52°, the contents were thoroughly shaken then and the vessels replaced in the thermostat of 15° and the measurements made the next morning.

In order to check if this warming up of the oleate system has in itself no influence on the elastic behaviour after cooling down to 15° (though at the temperature of ± 45° as was reached during the heat treatment the oleate system is no longer elastic) a spherical vessel half filled with a blank system was subjected to the same treatment for three consecutive

⁶⁾ H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, these Proceedings 50, 1011 (1947).

⁷⁾ H. L. BOOIJ and H. G. BUNGENBERG DE JONG, Biochimica Acta, in press.

days. It was found that an influence exceeding the experimental errors could not be detected⁸⁾.

The results of these measurements are represented in fig. 6. In connection with the discussion in section 2 (c.f. fig. 3) it is easy to see that the results are compatible with the assumption that the lower fatty acid anions (C_8-C_{11}) shift the G , the λ , the $1/\Lambda$ and n curves of the blank in the reverse directions as hexylalcohol did in fig. 2. This assumption⁹⁾ explains why in fig. 6 the G curves for C_8 , C_9 , C_{10} and C_{11} bend downwards (in contrast to the G curves for the higher terms of the alcohols in fig. 5, which bend upwards). With the alcohols (fig. 5) any contrast in the course of the λ , $1/\Lambda$ and n curves, which here (fig. 6) too only¹⁰⁾ bend downwards, is, however, not to be expected, since here too we started with a blank oleate system at the KCl concentration of minimum damping (see scheme C of fig. 3).

In the four diagrams of fig. 6 we notice that the action of the fatty acid anions increases in the order $C_8 < C_9 < C_{10}$.

We see, however, that C_{11} is slightly less or very slightly more active than C_{10} (c.f. 100 G/G_0 diagram and the remaining ones, respectively). The next term C_{12} is decidedly less active than C_{11} .

All this means, that for a same concentration of the fatty acid anions, the above mentioned shifts of various curves (G , λ etc. in fig. 3, scheme C) towards higher KCl concentrations do not increase indefinitely with the length of the carbon chain of the fatty acid anion, but that these shifts reach a maximum value at about C_{10} or C_{11} . A further lengthening diminishes this shift rapidly (C_{12}) and the characteristic property of the lower fatty acid anions is already lost at C_{14} . Obviously this anion (myristate-) resembles the oleate anion that much already, that it no longer counteracts the elastic structures built up by the KCl in the oleate system, but even positively contributes to the elastic properties of the oleate system. This positive contribution reveals from the strong increase of G , λ , $1/\Lambda$ and n above the blank values, just as a further increase of the oleate concentration of the blank will also bring about.

⁸⁾ In the survey below four numbers are placed each time behind the symbols $10 \times \frac{T}{2}$, b_1/b_3 and n . They give the values found before the heat treatment, and after the first, the second and third heat treatment:

$$\begin{aligned} 10 \times \frac{T}{2} &: 5.60; 5.59; 5.59 \text{ and } 5.59 \text{ sec.} \\ b_1/b_3 &: 1.248; 1.250; 1.247 \text{ and } 1.250 \\ n &: 46.5; 46.6; 46.6 \text{ and } 46.8 \end{aligned}$$

⁹⁾ For the undecylate ion subsequent work has confirmed this assumption.

¹⁰⁾ We neglect here that the curves mentioned, first run in fig. 6 in the opposite direction and after reaching a maximum located some 6—10 percent above the blank value take the course discussed in the text. This curve form is to be expected if the KCl concentration in the blank oleate system (here 1.52 N) does not exactly correspond with the KCl concentration of the minimum damping (according to table I = 1.43 N) but is slightly higher.

When we summarize the result of the above measurements it appears that there is really a parallel with the action of the fatty acid anions on

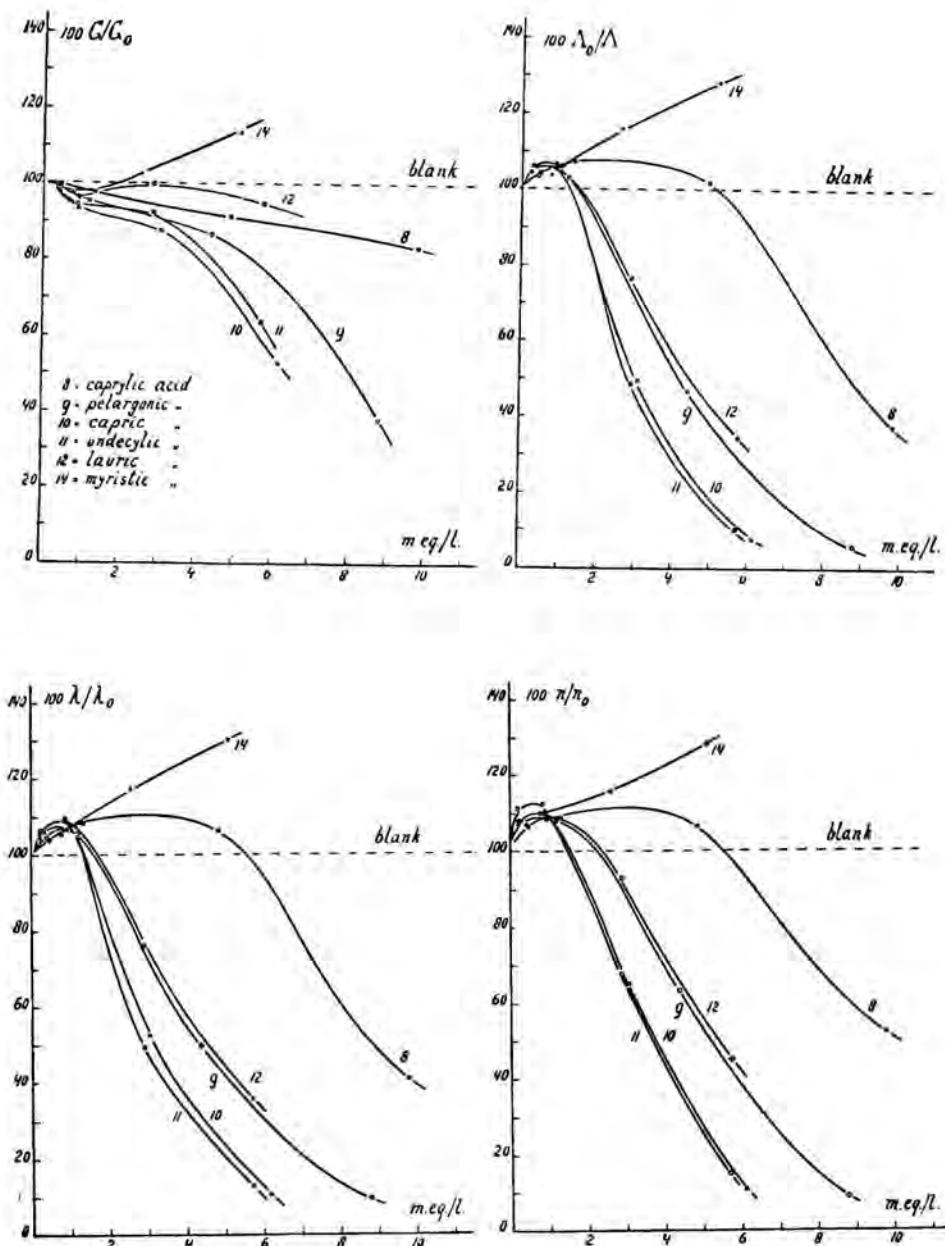


Fig. 6.

oleate coacervates, where the undecylate ion also takes an extreme position.

5) Correlation between n and Λ .

In the preceding Parts of this series, we have also inserted in many Tables referring to elastic measurements the values for n , i.e. the maximum number of observable oscillations (cf. already Part I, section 7, 9 and 10). An inspection of the above mentioned Tables will reveal that n decreases whenever Λ increases and reversely, but in none is the correlation between n and Λ a very simple one. In these Tables we considered the influence of the type of oscillation, the radius of the vessel, the temperature and the concentration of the oleate.

In the present communication, these four data are held constant while the KCl concentration is varied or when held constant small amounts of organic substances are added. Now the correlation between n and Λ is far much simpler and can be expressed very roughly speaking by $n \cdot \Lambda = \text{constant}$.

To show this, we have inserted in the figures of this communication graphs in which the ordinates are n and $1/\Lambda$ or $100 n/n_0$ and $100 \Lambda_0/\Lambda$.

It is true $n \cdot \Lambda$ is not really constant so that the graphs for n/n_0 give a deformed picture of the graphs for Λ_0/Λ (see e.g. fig. 1 and 2, and 4). Nevertheless the former give qualitatively the same details as the latter. Compare fig. 5 and especially fig. 6.

The above may be of practical importance, as it is not always easy to find an observer who is able to perform the difficult measurement of the decrement. In such a case the determination of the period and of the maximum number of observable oscillations can help to continue investigations of the type given in section 3 and 4, viz. those on the connection between the action of an organic substance and its structure.

Summary.

- 1) If the KCl concentration is increased the values of G , λ , $1/\Lambda$ and n are very low (presumably zero) at the first appearance of elastic phenomena. The last three increase to a maximum and then decrease to zero just before the coacervation limit is reached. G increases in this whole tract of KCl concentrations. The G curve shows an inflexion point at the KCl concentration corresponding to the maxima of the λ , $1/\Lambda$ and n curves.
- 2) The influence on the elastic behaviour of a number of n primary alcohols and of fatty acid anions has been studied and it appeared that the former (C_1-C_6) help the KCl in setting up the typical elastic viscous systems, the latter (C_8-C_{11}) counteract KCl in doing so.
- 3) The action of the alcohols increases considerably with increasing length of the carbon chain, each following term of the homologous series requiring for the same effect a 3-4 fold lower concentration than the preceding one.
- 4) The action of the fatty acid anions increases with the length of the carbon chain but this holds only up to C_{11} (undecylate), the action diminishes rapidly by further lengthening the carbon chain (C_{12}) and it has wholly disappeared in C_{14} .
- 5) For a certain type of work and under conditions described, the maximum number n of observable oscillations, may in emergency cases be used as an approximate measure of $1/\Lambda$.

Mathematics. — *Een iteratieproces voor de oplossing van een randwaardeprobleem bij een lineaire partiële differentiaalvergelijking van de tweede orde. II.* By J. J. DRONKERS. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of February 26, 1949.)

§ 4.

Bij de bepaling van z_1 (zie (11)) moet één keer naar x geïntegreerd worden. De integrand bestaat uit drie termen, ieder met twee factoren, bijv. a en $u_0^{(1)}$ en één term met één factor nl. δ . Verder bestaat u_1 uit twee termen met twee factoren, terwijl ook één keer naar x geïntegreerd moet worden.

De integrand van de functie z_2 bestaat vóór de eliminatie van u_1 en z_1 ook weer uit drie termen met twee factoren; zij bevat echter geen term met één factor.

Worden u_1 en z_1 geëlimineerd, dan zal, als de integraties buiten beschouwing gelaten worden, z_2 onder meer uit termen met drie factoren bestaan, waaronder δ niet voorkomt. Eén dier termen heeft bijv. de vorm:

$$z_0^{(2)} \int a \int \beta_2 dx^2.$$

Zodra echter in een term δ of één van haar partiële afgeleiden naar y voorkomt, bevat zij twee factoren.

Bij de berekening van z_n moet na de eliminaties, in iedere term n keer naar x geïntegreerd worden, terwijl zij $n+1$ of n factoren bevat. Iedere term is dan een n voudige integraal van de vorm:

$$z_0^{(k)} \int a^{(l_1)} \int \gamma_1^{(l_2)} \dots \int \gamma_n^{(l_n)} dx^n. \quad \dots \quad (16)$$

waarbij dan $z_0^{(k)}$ ook door $u_0^{(k)}$ vervangen kan zijn.

Zoals gemakkelijk uit (11) en (12) blijkt, geldt dan dat

$$k + l_1 + l_2 + \dots + l_n \leq n.$$

Dan kunnen een aantal grootheden l_i of k , gelijk aan nul zijn.

Evenzo bestaat u_n uit overeenkomstige termen met $n+1$ of n factoren.

Nu wordt door $A_{n,k} z_0^{(k)}$ de som van alle termen van z_n aangeduid, die dezelfde factor $z_0^{(k)}$ bevatten. Evenzo is $u_0^{(k)} B_{n,k}$ de som van alle termen met dezelfde factor $u_0^{(k)}$, enz.

Bij de achtereenvolgende eliminaties van z_i en u_i wordt gevonden, dat de hoogste afgeleiden naar y van z_0 en u_0 , die in z_n voorkomen, $z_0^{(n)}$ en $u_0^{(n)}$ zijn. Tevens komen ook alle andere afgeleiden $z_0^{(k)}$ en $u_0^{(k)}$ voor ($k = 1, 2, \dots, n-1$).

....., n). Dan kunnen z_n en u_n in de volgende vorm geschreven worden:

$$\left. \begin{array}{l} (a) \quad z_n = \sum_{k=1}^n z_0^{(k)} A_{n,k} + z_0 A_{n,0} + V_n + \sum_{k=1}^n u_0^{(k)} B_{n,k} \text{ en} \\ (b) \quad u_n = \sum_{k=1}^n z_0^{(k)} C_{n,k} + z_0 C_{n,0} + W_n + \sum_{k=1}^n u_0^{(k)} D_{n,k}. \end{array} \right\} . \quad (17)$$

Eveneens kan voor z_{n+1} geschreven worden:

$$z_{n+1} = \sum_{k=1}^{n+1} z_0^{(k)} A_{n+1,k} + z_0 A_{n+1,0} + V_{n+1} + \sum_{k=1}^{n+1} u_0^{(k)} B_{n+1,k},$$

terwijl ook:

$$z_{n+1} = \int (a u_n^{(1)} + \beta_1 z_n^{(1)} + \gamma_1 z_n) dx.$$

Door (17) in dit rechterlid te substitueren wordt het verband gevonden, dat bestaat tussen de coëfficiënten $A_{n+1,k}$ enz. van z_{n+1} en $A_{n,k}$ enz. van z_n en u_n .

Dan blijkt voor $k = 1, 2, \dots, n+1$.

$$\left. \begin{array}{l} A_{n+1,k} = \int [a(C_{n,k-1} + C_{n,k}^{(1)}) + \beta_1(A_{n,k-1} + A_{n,k}^{(1)}) + \gamma_1 A_{n,k}] dx, \\ B_{n+1,k} = \int [a(D_{n,k-1} + D_{n,k}^{(1)}) + \beta_1(B_{n,k-1} + B_{n,k}^{(1)}) + \gamma_1 B_{n,k}] dx, \\ A_{n+1,0} = \int (a C_{n,0}^{(1)} + \beta_1 A_{n,0}^{(1)} + \gamma_1 A_{n,0}) dx \text{ en} \\ V_{n+1} = \int (a W_n^{(1)} + \beta_1 V_n^{(1)} + \gamma_1 V_n) dx. \end{array} \right\} \quad (18a)$$

Hierbij moet worden opgemerkt, dat

$$C_{n,n+1} = A_{n,n+1} = D_{n,n+1} = B_{n,n+1} = 0.$$

Evenzo wordt voor de coëfficiënten $C_{n+1,k}$ enz. van u_{n+1} gevonden:

$$\left. \begin{array}{l} C_{n+1,k} = \int [\beta_2(A_{n,k-1} + A_{n,k}^{(1)}) + \gamma_2 A_{n,k}] dx; \\ C_{n+1,0} = \int (\beta_2 A_{n,0}^{(1)} + \gamma_2 A_{n,0}) dx; \quad W_{n+1} = \int (\beta_2 V_n^{(1)} + \gamma_2 V_n) dx; \\ D_{n+1,k} = \int [\beta_2(B_{n,k-1} + B_{n,k}^{(1)}) + \gamma_2 B_{n,k}] dx. \end{array} \right\} \quad (18b)$$

In de formule (17a) kunnen daarna de functies $u_0^{(k)}$ worden vervangen, immers volgens (9) kan $u_0^{(1)}$ lineair uitgedrukt worden in:

$$\left(\frac{\delta z}{\delta x} \right)_{x=0}, \quad z^{(1)}(0, y) \text{ en } z(0, y).$$

Achtereenvolgens kunnen dan ook de afgeleiden $u_0^{(k)}$ lineair worden uitgedrukt in de afgeleiden naar y van de zojuist genoemde functies.

Na substitutie in (17a) wordt dan gevonden:

$$z_n = \sum_{k=1}^n z_0^{(k)} \bar{A}_{n,k} + z_0 \bar{A}_{n,0} + \bar{V}_n + \sum_{k=1}^{n-1} \left(\frac{\delta z}{\delta x} \right)_{x=0}^{(k)} \bar{B}_{n,k} + \left(\frac{\delta z}{\delta x} \right)_{x=0} B_{n,0}.$$

Ten slotte is dan voor de integraal $z(x, y)$ te schrijven:

$$\left. \begin{aligned} z(x, y) = & \sum_{m=1}^{\infty} z^{(m)}(0, y) A_m(x, y) + z(0, y) A_0(x, y) + V_0(x, y) + \\ & + \sum_{m=1}^{\infty} \left(\frac{\partial z}{\partial x} \right)^{(m)}_{x=0} B_m(x, y) + \left(\frac{\partial z}{\partial x} \right)_{x=0} B_0(x, y). \end{aligned} \right\} \quad (19)$$

Daar de randwaarden volgens (2) bekend zijn, zijn van het rechterlid alle functies gegeven.

§ 5.

De functies z_n en u_n bestaan uit termen van de vorm (16). Om een overzicht te krijgen, wordt in deze paragraaf het aantal termen (16) tussen grenzen gesloten.

In het vervolg geven wij het aantal termen van z_n en u_n , die uit $n+1$ factoren bestaan, resp. door (z_n) en (u_n) aan. Zo bestaat z_1 uit drie termen met twee factoren en 1 term met 1 factor, terwijl u_1 uit twee termen met twee factoren bestaat.

Volgens (11) is het aantal termen van z_2 en u_2 met 3 factoren resp. gelijk aan:

$$\begin{aligned} (z_2) &= (u_1^{(1)}) + (z_1^{(1)}) + (z_1) = 2(u_1) + 3(z_1) \quad \text{en} \\ (u_2) &= (z_1^{(1)}) + (z_1) = 3(z_1). \end{aligned}$$

In het algemeen is:

$$\left. \begin{aligned} (z_n) &= n(u_{n-1}) + (n+1)(z_{n-1}) \quad \text{en} \\ (u_n) &= (n+1)(z_{n-1}). \end{aligned} \right\} \quad \dots \quad (20)$$

Met behulp van de betrekkingen (20) kan het aantal termen van z_n en u_n met $n+1$ factoren achtereenvolgens worden berekend. In het vervolg worden echter (z_n) en (u_n) benaderd, door deze tussen twee grenzen in te sluiten.

Voor $n \geq 3$ geldt nl.:

$$2 \frac{1}{6} n^2 (z_{n-2}) < (z_n) < 3 n^2 (z_{n-2}). \quad \dots \quad (21)$$

Bewijs: Uit (20) wordt eerst de volgende betrekking afgeleid voor $n \geq 4$:

$$(z_n) = (3n^2 + n - 1)(z_{n-2}) - (n-2)^2(n^2 - 1)(z_{n-4}). \quad \dots \quad (22)$$

Uit (20) volgt nl.:

$$\begin{aligned} (z_n) &= n^2(z_{n-2}) + (n+1)(z_{n-1}) \quad \text{dus:} \\ (z_{n-2}) &= (n-2)^2(z_{n-4}) + (n-1)(z_{n-3}) \quad \text{en} \\ (z_{n-1}) &= (n-1)^2(z_{n-3}) + n(z_{n-2}). \end{aligned}$$

Door (z_{n-1}) en (z_{n-3}) te elimineren wordt (22) afgeleid.
Het tweede deel van (21) wordt eerst aangetoond:

Immers uit $(z_n) < (3n^2 + n - 1)(z_{n-2})$ is af te leiden:

$$(z_{n-4}) > \frac{(z_{n-2})}{3n^2 - 11n + 9} > \frac{(z_{n-2})}{3(n-2)(n-1)} \text{ voor } n \geq 4.$$

Dus is volgens (22):

$$(z_n) < (2\frac{8}{3}n^2 + 1\frac{1}{3}n)(z_{n-2}) \leq 3n^2(z_{n-2}) \text{ als } n \geq 4.$$

Voor de laatste ongelijkheid kan vanzelfsprekend nog een scherpere benadering worden aangegeven.

De ongelijkheid $(z_n) < 3n^2(z_{n-2})$ geldt ook nog voor $n = 3$.

Wij zullen verder bewijzen dat $(z_n) > 2\frac{8}{5}n^2(z_{n-2})$.

Daar $(z_0) = 1$; $(z_1) = 3$; $(z_2) = 13$; $(z_3) = 79$; $(z_4) = 603$, is aan deze ongelijkheid voldaan voor $n = 3, 4$ enz.

Laat ze gelden tot $n-2$ toe, zodat:

$$(z_{n-2}) > 2\frac{8}{5}(n-2)^2(z_{n-4}).$$

Dan is volgens (22):

$$(z_n) > (2\frac{8}{3}n^2 + n - 1\frac{8}{3})(z_{n-2}) > 2\frac{8}{5}n^2(z_{n-2}); n \geq 3.$$

Het voorgaande is nog uit te breiden. Er bestaat een getal n_l , zodat voor $n > n_l$:

$$(z_n) < 3(n-l)(n-l-1)(z_{n-2}) \dots \dots \dots \quad (23)$$

Beweis: Uit (21) volgt:

$$(z_{n-4}) > \frac{(z_{n-2})}{3(n-2)^2}.$$

Dus volgens (22):

$$(z_n) < (z_{n-2}) [3n^2 - (\frac{1}{3}n^2 - n + \frac{8}{3})].$$

Wij bepalen nu het gehele getal n_l , zodat:

$$\frac{1}{3}n_l^2 - n_l \geq (6l+3)n_l - 3l^2 - 3l \text{ of } n_l \geq 6 + 9l + 3\sqrt{4 + 11l + 8l^2} \quad (23a)$$

Kiezen wij n groter dan dit getal n_l , dan zal in ieder geval (23) gelden. Zo wordt gevonden voor $l = 0$; $n_l \geq 12$.

Uit (23) volgt verder, dat voor $n > n_l$:

$$(z_n) < \frac{(z_{n_l+1})}{(n_l+1-l)!} 3^{\frac{n-n_l}{2}} (n-l)! \dots \dots \dots \quad (24)$$

Dan kan n of n_l zowel even als oneven zijn.

Bij de afleiding van deze ongelijkheid moet dan voor het geval n en n_l beiden even of oneven zijn, rekening gehouden worden met het feit, dat

$$\frac{(z_{n_l+1})}{(n_l+1-l)!} > \frac{(z_{n_l})}{(n_l-l)!}$$

Deze ongelijkheid is direct met behulp van (20) aan te tonen.

Voor (u_n) kan een overeenkomstige schatting worden gemaakt, terwijl dit eveneens kan geschieden voor het aantal termen van (z_n) en (u_n) met n factoren. Dit aantal termen is vanzelfsprekend kleiner dan het aantal met $n + 1$ factoren.

§ 6.

In deze paragraaf wordt aangetoond, dat de reeksen:

$$z = \sum_{k=0}^{\infty} z_k \text{ en } u = \sum_{k=0}^{\infty} u_k, \dots, \quad (25)$$

die in § 3 zijn opgesteld [zie (11) en (12)], oplossingen zijn van de partiële differentiaalvergelijkingen (4a) en (4b) en de gestelde randwaarden.

De functie $z(x, y)$, die wordt verkregen door de daarin voorkomende functies $u_0, u_1, u_2, \dots, u_n$ uit te drukken in de randwaarden $z(0, y)$ en $\left(\frac{\partial z}{\partial x}\right)_{x=0}$ en haar afgeleiden naar y , zal dan aan (1) en de randwaarden (2) voldoen.

Wij bewijzen nu de volgende stelling:

Laat de functies a, b, c, d, e en f van de partiële differentiaalvergelijking (1), benevens de randwaarden $z(0, y)$ en $\left(\frac{\partial z}{\partial x}\right)_{x=0}$, voldoen aan de voorwaarden in § 1 genoemd.

Dan geldt:

1e. De reeksen (25) convergeren gelijkmatig voor de intervallen:

$$0 \leq y \leq B; \quad 0 \leq x < \frac{1}{4M\varrho} \leq A$$

(zie voor de bepaling van M het hiernavolgende bewijs).

2e. Voor dit interval zullen de functies z en u voldoen aan de partiële differentiaalvergelijkingen (4a) en (4b), terwijl voor $x = 0$, $z(x, y)$ in de gegeven functie $z(0, y)$ en $u(x, y)$ in $u(0, y)$ zal overgaan.

3e. Worden de functies u_0, u_1, u_2 enz., die in de functie $z(x, y)$ als som van de reeks (25) voorkomen, in de gegeven randwaarden van $z(x, y)$ en haar afgeleiden uitgedrukt, dan wordt de oplossing verkregen, die aan (1) en de gestelde randwaarden voldoet.

Bewijs:

1e. Als de coëfficiënten a, b, c, d, e en f aan de gestelde eisen voldoen, zullen de partiële differentiaalvergelijkingen (6) waarin de genoemde functies a, b enz. als bekenden voorkomen, oplossingen voor $a, \beta_1, \gamma_1, \beta_2, \gamma_2$ en δ geven, waarvoor analoge voorwaarden gelden als voor a, b enz. Zo bestaat er een getallenwaarde M , die de absolute waarden van deze functies a enz. en haar partiële afgeleiden naar y van de k^e orde, na deling door $k!$ en ϱ^k voor de genoemde intervallen overschrijdt.

Deze partiële afgeleiden van de functies a, β_1 enz. kunnen uit (6) worden berekend, waarna een grootste waarde kan worden aangegeven (zie hiervoor § 2).

Wij zullen nu bewijzen, dat de reeks $z = \sum_{k=0}^{\infty} z_k$ gelijkmatig convergeert voor $0 \leq x < \frac{1}{4M\varrho}$.

Volgens de formule voor z_1 is (zie § 3):

$|z_1| < 3\varrho M^2 x + Mx < 4\varrho Mx$, als $M < 1$; of $< 4\varrho M^2 x$, als $M \geq 1$, terwijl

$$|u_1| < 2\varrho M^2 x.$$

Hierbij is verondersteld, dat $\varrho \geq 1$. Dit zullen we in het vervolg ook steeds aannemen. Indien $\varrho < 1$ kunnen analoge schattingen worden uitgevoerd. Het is duidelijk, dat als de convergentie is aangetoond voor $\varrho \geq 1$, de convergentie ook zal gelden voor $\varrho < 1$ als $0 \leq x \leq \frac{1}{4M}$. Het blijkt echter, dat ook voor $\varrho < 1$, gelijkmatige convergentie voor $0 \leq x \leq \frac{1}{4M\varrho}$ zal optreden.

Nu bestaat z_2 uit de som van een aantal termen van de vorm:

$$z_0^{(k)} \int a^{(l_1)} \int \beta_1^{(l_2)} dx^2, \text{ waarbij } k + l_1 + l_2 \leq 2.$$

Ook kan $z_0^{(k)}$ door $u_0^{(k)}$ vervangen zijn, terwijl een dergelijke factor ook kan ontbreken. Door het aantal termen van z_2 te bepalen, blijkt dat in ieder geval:

$$|z_2| < 4^2 \varrho^2 M^3 x^2 \text{ als } M \geq 1; \text{ of } < 4^2 \varrho^2 M^2 x^2, \text{ als } M < 1.$$

Bij nauwkeuriger schatting blijkt zelfs $|z_2| < 9\varrho^2 M^3 x^2$ als $M \geq 1$ enz. Zowel voor $M \geq 1$, als $M < 1$ is dus zeker:

$$|z_2| < 4\varrho M x \cdot \text{majorant van } |z_1|.$$

In het algemeen bestaat z_n uit een som van n voudige integralen van de vorm (16) (zie § 3):

$$\sum z_0^{(k)} \int a^{(l_1)} \int a^{(l_2)} \dots \int \gamma_2^{(l_n)} dx^n.$$

Verder geldt:

$$k + l_1 + \dots + l_n \leq n.$$

Volgens het gegeven dat $|\alpha^{(l_i)}| < M l_i! \varrho^{l_i}$ enz., is dus

$$|z_n| < M^{n+1} \varrho^n \frac{x^n}{n!} \sum k! l_1! l_2! \dots l_n!$$

Hierin is M groter dan 1 verondersteld; indien $M < 1$, moet M^{n+1} vervangen worden door M^n .

Wij beschouwen nu:

$$z_{n+1} = \sum z_0^{(k')} \int a^{(l_1)} \int \dots \int \gamma_2^{(l_{n+1})} dx^{n+1}$$

$$k' + l'_1 + l'_2 + \dots + l'_{n+1} \leq n + 1.$$

Anderzijds is echter:

$$z_{n+1} = \int_0^x (a u_n^{(1)} + \beta_1 z_n^{(1)} + \gamma_1 z_n) dx.$$

Hierin worden u_n en z_n gesubstitueerd, die respectievelijk bestaan uit een som van termen van de vorm (16). Door de differentiaties uit te voeren en daarna de termen te schatten, wordt gevonden:

$$\begin{aligned} |z_{n+1}| &\leq \frac{x^{n+1}}{(n+1)!} \varrho^{n+1} M^{n+2} \{ \sum 2 [(k+1)! l_1! \dots l_n!] + \\ &+ k!(l_1+1)! l_2! \dots l_n! + \dots + k! l_1! \dots (l_n+1)!] + \sum k! l_1! \dots l_n! \} < \\ &< \frac{x^{n+1}}{(n+1)!} \varrho^{n+1} M^{n+2} \sum k! l_1! \dots l_n! [2(k+1+l_1+1+\dots+l_n+1)+1] < \\ &< \frac{x^{n+1}}{(n+1)!} \varrho^{n+1} M^{n+2} (4n+3) \sum k! l_1! \dots l_n! < 4 \frac{x^{n+1}}{n!} \varrho^{n+1} M^{n+2} \sum k! l_1! \dots l_n!, \\ &(k+l_1+l_2+\dots+l_n \leq n). \end{aligned}$$

Uiteindelijk wordt dus gevonden, dat:

$$|z_{n+1}| < 4\varrho M x \cdot \text{majorant } |z_n|.$$

Dit geldt zowel voor $M \geq 1$ als $M < 1$. Dus:

$$|z_0| + |z_1| + |z_2| + \dots + |z_n| < |z_0| + 4\varrho M^2 x + 4^2 \varrho^2 M^3 x^2 + 4^3 \varrho^3 M^4 x^3 + \dots + 4^n \varrho^n M^{n+1} x^n + \dots.$$

Hierbij is $M \geq 1$; als $M < 1$ moeten de exponenten van M met één verminderd worden. In het vervolg wordt $M \geq 1$ aangenomen.

De z reeks (zie (25)) zal dus gelijkmatig convergeren voor $x < \frac{1}{4M\varrho}$.

Wordt de reeks bij z_n afgebroken, dan is:

$$|R_{n+1}| < \sum_n^\infty 4^{n+1} \varrho^{n+1} M^{n+2} x^{n+1} = \frac{4^{n+1} \varrho^{n+1} M^{n+2} x^{n+1}}{1 - 4M\varrho x}, \quad \dots \quad (26)$$

Op dezelfde wijze kunnen wij bewijzen, dat de u reeks in ieder geval gelijkmatig convergeert voor $x < \frac{1}{4M\varrho}$. Het aantal termen van u_n is o.a. kleiner dan dat van z_n .

2e. Wij bewijzen nu dat de functies z en u bepaald volgens (25), voldoen aan de partiële differentiaalvergelijkingen (4) en de bijbehorende randwaarden. Dit laatste is volgens (11) en (12) evident.

Wij substitueren in (4a) en (4b):

$$z = z_0 + z_1 + z_2 + \dots + z_n + R_{n+1},$$

$$u = u_0 + u_1 + u_2 + \dots + u_n + R_{n+1}^+,$$

en vinden dan voor het verschil tussen de linker- en rechterleden, in verband met (11) en (12):

$$\left. \begin{aligned} \frac{\partial R_{n+1}}{\partial x} - \alpha R_n^{(1)} - \beta_1 R_n^{(1)} - \gamma_1 R_n &= \bar{R}(x, y) \\ \frac{\partial R_{n+1}^+}{\partial x} - \beta_2 R_n^{(1)} - \gamma_2 R_n &= \bar{R}_2(x, y). \end{aligned} \right\} \quad \dots \quad (27)$$

Worden de termen van

$$\frac{\partial R_{n+1}}{\partial x} = \sum_{q=1}^{\infty} \frac{\partial z_{n+q}}{\partial x}$$

op overeenkomstige wijze geschat als in het voorgaande voor z_n enz. is geschiedt, dan wordt gevonden:

$$\left| \frac{\partial R_{n+1}}{\partial x} \right| < 4\varrho^{n+1} M^{n+2} \frac{x^n}{(n-1)!} \left[\sum_{p=0}^{\infty} (4\varrho Mx)^p \left(1 + \frac{p+1}{n} \right) \right] \Sigma k! l_1! \dots l_n!$$

Op analoge wijze wordt gevonden:

$$\left| \beta_1 \frac{\partial R_n}{\partial y} \right| < 2\varrho^{n+1} M^{n+2} \frac{x^n}{(n-1)!} \left[\sum_{p=0}^{\infty} (4\varrho Mx)^p \left(1 + \frac{p+1}{n} \right) \right] \Sigma k! l_1! \dots l_n!$$

Dan is $k + l_1 + \dots + l_n \leq n$.

Dergelijke benaderingen kunnen ook worden opgesteld voor de overige termen in (27).

Er is dus een waarde N te bepalen, zodanig dat voor $n > N$ en $x < \frac{1}{4M\varrho}$ de verschillende termen van de linkerleden van (27) kleiner zijn dan een willekeurig klein gekozen getal ε_1 , terwijl dan $\bar{R}_1(x, y)$ en $\bar{R}_2(x, y)$ kleiner zijn dan $\bar{\varepsilon}$.

Hiermede is aangetoond, dat de functies $z(x, y)$ bepaald volgens (25), voldoen aan de partiële differentiaalvergelijkingen (4a) en (4b).

3e. Ten slotte is nog te bewijzen, dat de functie $z(x, y)$ nadat de functies u_0, u_1, u_2 enz. zijn uitgedrukt in $z(0, y)$, $\left(\frac{\partial z}{\partial x}\right)_{x=0}$ en haar afgeleiden naar y , eveneens aan (1) voldoet. De algemene term van de z reeks, die verkregen wordt door $u_{n-1}^{(1)}$ te elimineren, is door (13) aangegeven. Ook nu substitueren wij in (1):

$$z = z_0 + z_1 + z_2 + \dots + z_n + R_{n+1}.$$

Bij de herleiding wordt dan gebruik gemaakt van (15) en (15a), terwijl

$$\frac{\partial z_0}{\partial x} = \frac{\partial^2 z_0}{\partial x^2} = \frac{\partial^2 z_0}{\partial x \partial y} = 0.$$

De formule (15) wordt dan toegepast voor $n = 2, 3$, enz.

Na de herleiding wordt gevonden:

$$\left. \begin{aligned} & \frac{\partial^2 R_{n+1}}{\partial x^2} + b \frac{\partial^2 R_n}{\partial x \partial y} + a \frac{\partial^2 R_{n-1}}{\partial y^2} + c \frac{\partial R_{n+1}}{\partial x} + d \frac{\partial R_n}{\partial y} + \\ & + e R_n - \gamma_1 \frac{\partial z_n}{\partial x} - \left(a \frac{\partial \beta_2}{\partial y} + a \gamma_2 \right) \frac{\partial z_{n-1}}{\partial y} - a \frac{\partial \gamma_2}{\partial y} z_{n-1} = \bar{R}(x, y). \end{aligned} \right\} (28)$$

Ook nu is weer gemakkelijk aan te tonen, dat een getal N te bepalen is, zodat voor $n > N$, $R(x, y) < \varepsilon$. Ieder der termen van het linkerlid van (28) is nl. willekeurig klein te maken.

Zo is bijvoorbeeld:

$$\left| \frac{\partial^2 R_{n+1}}{\partial x^2} \right| < \varrho^n M^{n+1} \frac{x^{n-2}}{(n-2)!} \left[\sum_{p=1}^{\infty} (4\varrho Mx)^p \left(1 + \frac{p}{n} \right) \left(1 + \frac{p}{n-1} \right) \right] \cdot \sum k! l_1! l_2! \dots l_n!$$

waarbij weer $k + l_1 + \dots + l_n \leq n$; terwijl voor

$$\left| a \frac{\partial^2 R_{n-1}}{\partial y^2} \right| \quad \text{en} \quad \left| b \frac{\partial^2 R_n}{\partial x \partial y} \right|$$

analoge uitdrukkingen gelden.

De functie $z(x, y)$ voldoet dus aan (1) en de gestelde randwaarden, hetgeen aangetoond moet worden.

Mathematics. — *Note on a certain class of Banach spaces.* By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of April 23, 1949.)

1. *Introduction.* The abstract theory of Banach space is usually illustrated by examples from the functions spaces $L_p(\Delta)$, $1 \leq p \leq \infty$, where, for $1 \leq p < \infty$, $L_p(\Delta)$ is the class of all LEBESGUE-measurable, complex-valued functions $f(x)$ such that $|f(x)|^p$ is summable over the interval Δ (one- or more-dimensional, finite or infinite), and where $L_\infty(\Delta)$ is the class of all measurable, complex-valued functions $f(x)$ such that $|f(x)|$ is bounded almost everywhere in Δ . ORLICZ [1] (cf. also [2]), in 1932, introduced the class of functions spaces $L_\varphi(\Delta)$, which contains the spaces $L_p(\Delta)$ as special cases for $1 < p < \infty$, but not for $p = 1$ and $p = \infty$. Shortly, the definition of the space $L_\varphi(\Delta)$ runs as follows: Let $\varphi(\bar{u})$, $\bar{u} \geq 0$, $\psi(\bar{v})$, $\bar{v} \geq 0$, be two continuous functions, vanishing at the origin, strictly increasing, tending to infinity, and inverse to each other. Then, for u and $v \geq 0$, we have YOUNG's inequality [3]

$$uv \leq \Phi(u) + \Psi(v), \text{ where } \Phi(u) = \int_0^u \varphi(\bar{u}) d\bar{u}, \Psi(v) = \int_0^v \psi(\bar{v}) d\bar{v}.$$

By $L_\varphi^*(\Delta)$ we mean the class of all complex, measurable functions $f(x)$ in Δ , for which $\Phi|f(x)|$ is summable over Δ , and the class $L_\varphi^*(\Delta)$ is introduced similarly. The space $L_\varphi(\Delta)$ is now defined to consist of all complex, measurable functions $f(x)$ in Δ , such that $f(x)g(x)$ is summable over Δ for every $g(x) \in L_\varphi^*(\Delta)$, and the norm $\|f\|_\varphi$ is defined by

$$\|f\|_\varphi = \text{l.u.b. } \left| \int_{\Delta} f(x) g(x) dx \right| \text{ for all } g(x) \text{ with } \int_{\Delta} |\psi(g)| dx \leq 1.$$

The space $L_\varphi(\Delta)$ is introduced similarly. (The notations used here differ slightly from those in [1] and [2]). It is proved then that $L_\varphi(\Delta)$ and $L_\varphi^*(\Delta)$, thus defined, are Banach spaces, that is, normed complete vectorspaces. Obviously, by YOUNG's inequality, L_φ^* is contained in L_φ . The case that $\Phi(2u) \leq C\Phi(u)$ for all $u \geq 0$ and a fixed positive constant C deserves special attention, because in this case L_φ and L_φ^* (but not necessarily L_φ and L_φ^*) are identical. Recently the present author [4] has proved, under this additional condition, that L_φ is separable and that every bounded linear functional $F(f)$ in L_φ is of the form

$$F(f) = \int_{\Delta} f(x) g(x) dx,$$

where $g(x) \in L_\varphi$ and $\frac{1}{2}\|g\|_\varphi \leq \|F\| \leq \|g\|_\varphi$. It is easily verified that

for $\varphi(\bar{u}) = \bar{u}^{p-1}$ ($1 < p < \infty$), hence $\Phi(u) = u^p/p$, the space $L_\Phi(\Delta)$ is identical with the space $L_p(\Delta)$, and that

$$\|f\|_\Phi = q^{1/q} \|f\|_p, \text{ where } 1/p + 1/q = 1, \quad \|f\|_p = \left(\int_{\Delta} |f|^p dx \right)^{1/p}.$$

It will be felt as a defect, both of aesthetical and practical nature, that the spaces $L_p(\Delta)$, $p=1$ and $p=\infty$, are excluded in this way from the class of spaces $L_\Phi(\Delta)$, so that, to obtain completeness, these "boundary cases" would need a separate treatment. It is our purpose to show here that with some care, altering some of the mentioned definitions and revising or supplementing some proofs, these exceptional cases may be included into the general theory of spaces $L_\Phi(\Delta)$. The principal change is that we no longer suppose the function $\varphi(\bar{u})$ to be continuous, strictly increasing and tending to infinity, but merely to be non-decreasing.

2. YOUNG's inequality. Let $v = \varphi(u)$, $u \geq 0$, be non-decreasing, $\varphi(0) = 0$. Then, for every $u > 0$, the number $\varphi(u-)$ exists; the same is true, for every $u \geq 0$, of the number $\varphi(u+)$, and $\varphi(u-) \leq \varphi(u) \leq \varphi(u+)$, if we define $\varphi(0-) = \varphi(0) = 0$. At those values of u where $\varphi(u-) = \varphi(u+)$, the function $\varphi(u)$ is continuous; at those values of u where $\varphi(u-) < \varphi(u+)$, we say that $\varphi(u)$ has a jump from $\varphi(u-)$ to $\varphi(u+)$. It is well-known that for any $u_0 \geq 0$ the set of values $u \leq u_0$ at which $v = \varphi(u)$ has a jump is at most enumerable infinite.

We shall suppose now that $v = \varphi(u)$ is continuous from the left, hence $\varphi(u) = \varphi(u-)$ for every $u \geq 0$. (This is no loss of generality since we are ultimately interested in $\int_0^u \varphi(\bar{u}) d\bar{u}$, and not in $\varphi(u)$ itself). By $u = \psi(v)$ the inverse function is defined, with this understanding that if $\varphi(u)$ makes a jump at $u = a$, then $\psi(v) = a$ for $\varphi(a-) < v \leq \varphi(a+)$, while, if $\varphi(u) = c$ for $a < u \leq b$ (but $\varphi(u) < c$ for $u < a$), then $\psi(c) = a$. Furthermore $\psi(0) = 0$ and, if $\lim_{u \rightarrow \infty} \varphi(u) = l$ is finite, then $\psi(v) = +\infty$ for $v > l$. Thus defined, $u = \psi(v)$ is evidently a non-decreasing function of v for $v \geq 0$, and continuous from the left for those v for which $\psi(v) < \infty$. We observe that $v = \varphi(u)$ implies $u \geq \psi(v) = \psi(v-)$, and $u = \psi(v)$ implies $v \geq \varphi(u) = \varphi(u-)$ for finite u .

Lemma 1. If $v < \varphi(u)$ for a point (u, v) in the uv -plane, then $u > \psi(v)$. If $v > \varphi(u)$, then $u \leq \psi(v)$.

Proof. From $v < \varphi(u)$ follows $v + p = \varphi(u)$ for a positive p , hence $u \geq \psi(v + p) \geq \psi(v)$. But $u = \psi(v)$ is impossible, for then $v \geq \varphi(u)$ in contradiction with the hypothesis. Hence $u > \psi(v)$.

Let now $v > \varphi(u) = \varphi(u-)$. If $v \leq \varphi(u+)$, then $u = \psi(v)$ by definition. If $v > \varphi(u+)$, then $v - p = \varphi(u+)$ for a positive p , hence

$$u = \psi(v - p) \leq \psi(v).$$

This completes the proof.

On account of this lemma we may now divide the quadrant $u \geq 0, v \geq 0$ into four sets E_1, E_2, E_2' and E_3 , where

E_1 consists of all points (u, v) such that $v < \varphi(u)$, hence $u > \psi(v)$.

E_2 consists of all points (u, v) such that $v = \varphi(u)$,

E_2' consists of all points (u, v) such that $v > \varphi(u)$ and $u = \psi(v)$,

E_3 consists of all points (u, v) such that $v > \varphi(u)$ and $u < \psi(v)$.

The set $E_2 = E_2 + E_2'$ consists therefore of all points (u, v) for which one at least of the relations $v = \varphi(u)$ and $u = \psi(v)$ is satisfied.

Lemma 2. *The sets E_1 and E_3 are open (relative to the quadrant $u \geq 0, v \geq 0$), so that the complementary set E_2 is closed.*

Proof. Suppose that $(u_0, v_0) \in E_1$, hence $v_0 < \varphi(u_0)$ or $v_0 = \varphi(u_0) - p$, where p is positive. Since $\varphi(u_0) = \varphi(u_0 -)$, there exists a positive number δ such that $\varphi(u) > \varphi(u_0) - p$ if only $u \geq u_0 - \delta$. In the rectangle $|u - u_0| \leq \delta, |v - v_0| \leq p$ (insofar as this rectangle is contained in the quadrant $u \geq 0, v \geq 0$), we have now $v \leq v_0 + p = \varphi(u_0) - p < \varphi(u)$, which shows that this rectangle belongs to E_1 . If $(u_0, v_0) \in E_3$, the proof is similar.

Lemma 3. *Let, for arbitrary $u_0 > 0, v_0 > 0$, the closed interval $[0, u_0; 0, v_0]$ be denoted by Δ . Then the measure of the productset $\Delta \cdot E_2$ satisfies $m(\Delta \cdot E_2) = 0$.*

Proof. Since $\Delta \cdot E_2$ is bounded and closed, it is measurable. If $c(u, v)$ is its characteristic function, we have therefore by FUBINI's theorem

$$m(\Delta \cdot E_2) \int_{\Delta} c(u, v) du dv = \int_0^{u_0} du \int_0^{v_0} c(u, v) dv = 0,$$

because the inner integral can differ from zero only for those values of u at which $\varphi(u)$ has a jump, so that it vanishes almost everywhere in $[0, u_0]$.

Definition. *If the non-decreasing functions $v = \varphi(u)$ and $u = \psi(v)$, inverse to each other, satisfy the above conditions, then*

$$\Phi(u) = \int_0^u \varphi(\bar{u}) d\bar{u}, \quad \Psi(v) = \int_0^v \psi(\bar{v}) d\bar{v},$$

defined for $u \geq 0, v \geq 0$, are called complementary in the sense of YOUNG.

Theorem (YOUNG's inequality). *If $\Phi(u)$ and $\Psi(v)$ are complementary in the sense of YOUNG, then*

$$uv \leq \Phi(u) + \Psi(v)$$

for arbitrary $u, v \geq 0$, where equality holds if and only if one at least of the relations $v = \varphi(u)$ and $u = \psi(v)$ is satisfied.

Proof. The geometrical meaning of the theorem is evident, which does not mean that no proof is necessary.

Let $u_0 \geq 0$, $v_0 \geq 0$ be given, and denote the interval $[0, u_0; 0, v_0]$ by Δ . Then, using the same notations as before, we have by Lemma 3

$$\begin{aligned} u_0 v_0 &= m(\Delta) = m(\Delta \cdot E_1) + m(\Delta \cdot E_2) + m(\Delta \cdot E_3) = \\ &= m(\Delta \cdot E_1) + m(\Delta \cdot E_3) = \int_{\Delta} d(u, v) du dv + \int_{\Delta} e(u, v) du dv, \end{aligned}$$

where $d(u, v)$ and $e(u, v)$ are the characteristic functions of the measurable sets $\Delta \cdot E_1$ and $\Delta \cdot E_3$. Now, by FUBINI's theorem,

$$\int_{\Delta} d(u, v) du dv = \int_0^{u_0} du \int_0^{v_0} d(u, v) dv = \int_0^{u_0} du \int_0^{m(u)} dv,$$

where $m(u) = \min[v_0, \varphi(u)]$; hence

$$\int_{\Delta} d(u, v) du dv \leq \int_0^{u_0} du \int_0^{\varphi(u)} dv = \int_0^{u_0} \varphi(u) du = \Phi(u_0),$$

with equality if and only if $v_0 \geq \varphi(u_0)$. In the same way

$$\int_{\Delta} e(u, v) du dv \leq \Psi(v_0),$$

with equality if and only if $u_0 \geq \psi(v_0)$. Hence

$$u_0 v_0 \leq \Phi(u_0) + \Psi(v_0).$$

with equality if and only if $v_0 \geq \varphi(u_0)$, $u_0 \geq \psi(v_0)$ hold simultaneously. But, since $v_0 > \varphi(u_0)$, $u_0 > \psi(v_0)$ are incompatible by Lemma 1, this means that one at least of the relations $v = \varphi(u)$ and $u = \psi(v)$ must be satisfied in the case of equality.

Remark. If $v = \varphi(u) = u^{p-1}$ ($1 < p < \infty$), $1/p + 1/q = 1$, YOUNG's inequality takes the form $uv \leq u^p/p + v^q/q$, with equality if and only if $v = u^{p-1}$.

3. *The space L_{Φ} .* Let Δ be a finite or infinite interval in the m -dimensional Euclidean space R_m ($m \geq 1$), and let $\Phi(u)$, $\Psi(v)$ be complementary in the sense of YOUNG, such that $\Phi(u)$ does not vanish identically (equivalent with assuming that $\varphi(u)$ does not vanish identically). It is easy to see that if $f(x)$ is measurable in Δ , then $\varphi|f(x)|$ is also measurable in Δ . Indeed, with every number $a \geq 0$ corresponds a number $b \geq 0$ such that $\varphi(u) \leq a$ holds if and only if $u \leq b$. In the same way it is seen that $\psi|f(x)|$, $\Phi|f(x)|$ and $\Psi|f(x)|$ are measurable in Δ .

Definition. By $L_{\Phi}^*(\Delta)$ we shall mean the class of all complex, measurable functions $f(x)$ on Δ , for which $\Phi|f(x)|$ is summable over Δ . The class $L_{\Psi}^*(\Delta)$ is defined similarly.

Remark. If $\Phi(u) = c u^p$ ($c > 0$, $p \geq 1$), the class $L_{\Phi}^*(\Delta)$ is identical with $L_p(\Delta)$. For $p > 1$ the complementary class $L_{\Psi}^*(\Delta)$ is identical with

$L_q(\Delta)$, where $1/p + 1/q = 1$, and for $p = 1$ the class $L_\Psi^*(\Delta)$ consists of all measurable functions for which $|f(x)| \leq c$ holds almost everywhere in Δ (since in this case $\Psi(v) = 0$ in $0 \leq v \leq c$, $\Psi(v) = \infty$ for $v > c$).

The classes $L_\Phi^*(\Delta)$ and $L_\Psi^*(\Delta)$ are, as a rule, not linear, that is, the summability of $\Phi|f_1|$ and $\Phi|f_2|$ does not necessarily imply the summability of $\Phi|f_1 + f_2|$. For this reason, linear classes $L_\Phi(\Delta)$ and $L_\Psi(\Delta)$ are defined, containing $L_\Phi^*(\Delta)$ and $L_\Psi^*(\Delta)$ as subclasses.

Definition. By $L_\Phi(\Delta)$ we shall mean the class of all complex, measurable functions $f(x)$ on Δ , such that $f(x)g(x)$ is summable over Δ for every $g(x) \in L_\Psi^*(\Delta)$, and such that

$$\|f\|_\Phi = \text{l.u.b. } \int_{\Delta} |f(x)g(x)| dx \text{ for all } g(x) \text{ with } \int_{\Delta} \Psi|g| dx \leq 1$$

is finite.

The complementary class $L_\Psi(\Delta)$ is defined similarly.

Remark. It will be seen in what follows (Theorem 5) that $\|f\|_\Phi < \infty$ is automatically satisfied if only $f(x)g(x)$ is summable for every $g \in L_\Psi^*$.

Theorem 1. The class L_Φ is linear and contains L_Φ^* as a subclass. For $f_1, f_2 \in L_\Phi$ the inequality $\|f_1 + f_2\|_\Phi \leq \|f_1\|_\Phi + \|f_2\|_\Phi$ holds. For $f \in L_\Phi^*$ we have $\|f\|_\Phi \leq \int_{\Delta} \Phi|f| dx + 1$. Furthermore $\|f\|_\Phi = 0$ if and only if $f(x) = 0$ almost everywhere in Δ . A similar theorem holds for L_Ψ .

Proof. The only statement which is not immediately evident is that $\|f\|_\Phi = 0$ implies $f(x) = 0$ almost everywhere in Δ . We observe first that we might have defined $\|f\|_\Phi$ as well by

$$\|f\|_\Phi = \text{l.u.b. } \int_{\Delta} |fg| dx \text{ for all } g(x) \text{ with } \int_{\Delta} \Psi|g| dx \leq 1.$$

If $\|f\|_\Phi = 0$, we have therefore $f(x)g(x) = 0$ almost everywhere in Δ for every $g(x)$ with $\int_{\Delta} \Psi|g| dx \leq 1$. Since we have supposed that $\varphi(u)$

does not vanish identically, there exist positive u_0, v_0 such that $v_0 = \varphi(u_0)$, hence $u_0 \geq \psi(v_0)$, so that $\psi(v) \leq u_0$ for $v \leq v_0$. It follows that $\Psi(v) \leq u_0 v$ for $v \leq v_0$ or $\lim_{v \rightarrow 0} \Psi(v) = 0$. If therefore Δ_1 is an arbitrary finite

subinterval of Δ with measure $m(\Delta_1)$, there exists a positive number p such that $\Psi(p) \leq 1/m(\Delta_1)$. Taking now $g(x) = p$ for $x \in \Delta_1$ and $g(x) = 0$ elsewhere, we have $\int_{\Delta} \Psi|g| dx = \int_{\Delta_1} \Psi(p) dx \leq 1$, so that, on account of $f(x)g(x) = 0$, also $f(x) = 0$ almost everywhere in Δ_1 . Hence $f(x) = 0$ almost everywhere in Δ .

Theorem 2. If $f(x) \in L_\Phi$, and $\|f\|_\Phi \neq 0$, then

$$\int_{\Delta} \Phi[|f|/\|f\|_\Phi] dx \leq 1.$$

A similar theorem holds for any $g(x) \in L_\Psi$.

Proof. As in [2], we observe that $\Psi(qv) \geq q\Psi(v)$ or $\Psi(v) \leq \Psi(qv)/q$ for $q \geq 1$ and every $v \geq 0$, which is easily proved. Supposing now that $f \in L_\Phi$ and $g \in L_\Psi$, we have $\int_{\Delta} |fg| dx \leq \|f\|_\Phi$ if $\varrho(g) = \int_{\Delta} \Psi|g| dx \leq 1$. If $\varrho(g) > 1$, then $\int_{\Delta} |\Psi|g|\varrho(g)| dx \leq \int_{\Delta} |\Psi|g|/\varrho(g)| dx = 1$, hence $\int_{\Delta} |fg| dx \leq \|f\|_\Phi \cdot \varrho(g)$. We conclude therefore that

$$\int_{\Delta} |fg| / \|f\|_\Phi dx \leq \varrho'(g) = \max[\varrho(g), 1].$$

In the case that $f(x)$ is bounded on Δ and Δ is finite, both functions $\Phi[|f|/\|f\|_\Phi]$ and $\Psi[\varphi\{|f|/\|f\|_\Phi\}]$ are bounded on Δ (observe in particular that this is also true in the case that $\lim_{u \rightarrow \infty} \varphi(u)$ is finite), and therefore summable over Δ ; hence

$$\varrho'(g) \geq \int_{\Delta} |fg| / \|f\|_\Phi dx = \int_{\Delta} \Phi[|f|/\|f\|_\Phi] dx + e(g),$$

where $g = \varphi\{|f|/\|f\|_\Phi\}$, so that YOUNG's inequality becomes an equality. The result follows now.

Supposing now Δ to be still finite but $f(x)$ no longer to be bounded, we put $f_n(x) = f(x)$ if $|f(x)| \leq n$ and $f_n(x) = 0$ elsewhere ($n = 1, 2, \dots$). Then the non-decreasing sequence of bounded functions $|f_n(x)|$ converges to $|f(x)|$. It is easily seen that $\|f_n\|_\Phi \leq \|f\|_\Phi$ for every n , hence $\int_{\Delta} \Phi[|f_n|/\|f\|_\Phi] dx \leq 1$ for every n , so that

$$\int_{\Delta} \Phi[|f|/\|f\|_\Phi] dx = \lim_{n \rightarrow \infty} \int_{\Delta} \Phi[|f_n|/\|f\|_\Phi] dx \leq 1.$$

The extension to infinite Δ is similar.

Let us suppose now that $g \in L_\Psi$. If $\Psi(v) < \infty$ for all $v < \infty$ (which is equivalent with $\lim_{u \rightarrow \infty} \varphi(u) = \infty$), the proof of $\int_{\Delta} \Psi[|g|/\|g\|_\Psi] dx \leq 1$ offers no new aspects. If however $\lim_{u \rightarrow \infty} \varphi(u) = l < \infty$, we observe first that $|g(x)|/\|g\|_\Psi \leq l$ almost everywhere in Δ . Indeed, if $|g(x)| > l\|g\|_\Psi$ on a set $E \subset \Delta$ of positive measure $m(E)$, we may put $f(x) = [l \cdot m(E)]^{-1}$ in E and $f(x) = 0$ elsewhere. Then, on account of $\Phi(u) \leq l u$,

$$\int_{\Delta} \Phi|f| dx = \Phi[l \cdot m(E)]^{-1} \cdot m(E) \leq 1,$$

but

$$\int_{\Delta} |fg| dx = \int_E |g| dx / [l \cdot m(E)] > \|g\|_\Psi,$$

which is a contradiction. Supposing now Δ to be finite, and choosing the arbitrary but fixed number δ such that $0 < \delta < 1$, we see that both $\Psi[\delta|g|/\|g\|_\Psi]$ and $\Phi|f|$, where $f = \varphi\{\delta|g|/\|g\|_\Psi\}$, are bounded and therefore summable over Δ . Hence, if $\varrho(f) = \int_{\Delta} \Phi|f| dx$ and $\varrho'(f) = \max[\varrho(f), 1]$, $\delta\varrho'(f) \geq \int_{\Delta} |\delta g f| / \|g\|_\Psi dx = \int_{\Delta} \Psi[\delta|g|/\|g\|_\Psi] dx + \varrho(f)$,

so that, since $\varrho'(f)=1$ on account of $\varrho(f) \leq \delta \varrho'(f) < \varrho'(f)$, we have $\int_{\Delta} \Psi[\delta |g| / \|g\|_{\psi}] dx \leq \delta < 1$. Making δ tend to 1, we obtain the desired result. The extension to an infinite interval is now evident.

Corollary. If there exists a constant $C > 0$ such that $\Phi(2u) \leq C\Phi(u)$ for all $u \geq 0$, the classes L_{Φ} and L_{Ψ} are identical.

Theorem 3. If $L_{\Phi}(\Delta)$ and $L_{\Psi}(\Delta)$ are complementary, $f \in L_{\Phi}$, $g \in L_{\Psi}$, then fg is summable over Δ and

$$\left| \int_{\Delta} fg dx \right| \leq \int_{\Delta} |fg| dx \leq \|f\|_{\Phi} \cdot \|g\|_{\Psi}.$$

Proof. If $\|g\|_{\Psi} = 0$, then $\int_{\Delta} fg dx = 0$; we may suppose therefore that $\|g\|_{\Psi} \neq 0$. If $f \in L_{\Phi}$, then fg is summable over Δ for every $g \in L_{\Psi}^*$ by definition, so that $fg/\|g\|_{\Psi}$ is summable for every $g \in L_{\Psi}$. But then fg is summable as well. Furthermore

$$\left| \int_{\Delta} fg dx \right| \leq \int_{\Delta} |fg| dx = \int_{\Delta} |fg| / \|g\|_{\Psi} dx \cdot \|g\|_{\Psi} \leq \|f\|_{\Phi} \cdot \|g\|_{\Psi},$$

since $\int_{\Delta} \Psi[|g| / \|g\|_{\Psi}] dx \leq 1$.

Remark. If $\Phi(u) = c u^p$ ($c > 0$, $1 \leq p < \infty$), then $\Phi(2u) = 2^p \Phi(u)$ for every $u \geq 0$, hence $L_{\Phi}(\Delta) \equiv L_{\Phi}^*(\Delta) \equiv L_p(\Delta)$. For $1 < p < \infty$ the complementary class $L_{\Psi}(\Delta)$ is identical with $L_q(\Delta)$, where $1/p + 1/q = 1$. For $p = 1$, $L_{\Psi}(\Delta) \equiv L_{\infty}(\Delta)$, the class of all measurable functions bounded almost everywhere in Δ . Indeed, if $g \in L_{\Psi}(\Delta)$, we have already seen in the proof of Theorem 2 that $|g(x)| / \|g\|_{\Psi} \leq c$ almost everywhere in Δ , hence $g \in L_{\infty}(\Delta)$. Conversely, if $g \in L_{\infty}(\Delta)$, we have $|pg(x)| \leq c$ almost everywhere in Δ for some suitable positive p , hence $pg(x) \in L_{\Psi}(\Delta)$ or $g \in L_{\Psi}(\Delta)$.

Considering the particular case that $\Phi(u) = u^p/p$ ($1 < p < \infty$), so that $\Psi(v) = v^q/q$, where $1/p + 1/q = 1$, we have

$$\|f\|_{\Phi} = \text{l.u.b. } \left| \int_{\Delta} fg dx \right| \text{ for all } g(x) \text{ with } \int_{\Delta} |g|^q dx \leq q,$$

$$\|f\|_{\Phi} = \left(\int_{\Delta} |f|^p dx \right)^{1/p} = \max_{\Delta} \left| \int_{\Delta} fg dx \right| \text{ for all } g(x) \text{ with } \int_{\Delta} |g|^q dx \leq 1,$$

hence $\|f\|_{\Phi} = q^{1/q} \|f\|_p$. For $p = 1$ we find in a similar way

$$\|f\|_{\Phi} = \|f\|_1 = \int_{\Delta} |f| dx \text{ and } \|f\|_{\Psi} = \|f\|_{\infty},$$

where $\|f\|_{\infty}$ is the number uniquely determined by $|f(x)| \leq \|f\|_{\infty}$ almost everywhere in Δ and $|f(x)| > \|f\|_{\infty} - \varepsilon$ for every $\varepsilon > 0$ on a set of positive measure.

Theorem 4. The function spaces $L_{\Phi}(\Delta)$ and $L_{\Psi}(\Delta)$ are complete; in other words, pronounced for L_{Φ} , if $f_n(x) \in L_{\Phi}$ ($n = 1, 2, \dots$) and $\lim \|f_n - f_m\|_{\Phi} = 0$ as $m, n \rightarrow \infty$, there exists a function $f(x) \in L_{\Phi}$ such

that $\lim \|f_n - f\|_\varphi = 0$. This function $f(x)$ is uniquely determined apart from sets of measure zero.

Proof. As in [2].

Theorem 5. If the complex, measurable function $f(x)$, defined on the finite or infinite interval Δ , has the property that $f(x)g(x)$ is summable over Δ for every $g(x) \in L_\psi^*(\Delta)$, then $f(x) \in L_\Phi(\Delta)$. In the same way $g(x) \in L_\psi(\Delta)$ if only $f(x)g(x)$ is summable for every $f(x) \in L_\Phi^*(\Delta)$.

Proof. We have to show that $\|f\|_\varphi = \text{l.u.b. } |\int fg dx|$ for all $g(x)$ with $\int \Psi |g| dx \leq 1$, is finite. We observe first that $f(x)g(x)$ is also summable over Δ for every $g(x) \in L_\psi$. Putting now $f_n(x) = f(x)$ ($n = 1, 2, \dots$) whenever $|f(x)| \leq n$ and $x \in \Delta \cdot \Delta_n$, where Δ_n is the interval $[-n, n; \dots; -n, n]$, while $f_n(x) = 0$ elsewhere, we have $f_n(x) \in L_\Phi(\Delta)$ and $\lim f_n(x) = f(x)$. The functional $T_n(g) = \int f_n(x) g(x) dx$ is therefore a bounded linear functional in $L_\psi(\Delta)$ as may be seen from $|T_n(g)| \leq \|f_n\|_\varphi \cdot \|g\|_\psi$. Furthermore the sequence $T_n(g)$ ($n = 1, 2, \dots$) is bounded for every $g \in L_\psi$, as follows from $\lim \int f_n g dx = \int fg dx$, since $\lim f_n g = fg$, $|f_n g| \leq |fg|$ and $|fg|$ is summable. Hence, by the BANACH-STEINHAUS theorem, $|T_n(g)| \leq M \|g\|_\psi$ for a fixed M . Taking now $g(x)$ such that $\int \Psi |g| dx \leq 1$, so that $\|g\|_\psi \leq 2$, we have therefore $|\int f_n(x) g(x) dx| = |T_n(g)| \leq 2M$, hence also $|\int f(x) g(x) dx| \leq 2M$. This shows that $\|f\|_\varphi \leq 2M$.

The proof that $g \in L_\psi$ if only fg is summable for every $f \in L_\Phi^*$, is similar. The crucial part is that the functions $g_n(x)$, equal to $g(x)$ for $|g(x)| \leq n$ and $x \in \Delta \cdot \Delta_n$, and zero elsewhere, all belong to $L_\psi(\Delta)$, even if $\lim_{u \rightarrow \infty} \varphi(u) = l < \infty$.

4. Concluding remarks. If $\Phi(2u) \leq C\Phi(u)$ for every $u \geq 0$ and a fixed $C > 0$, the space $L_\Phi(\Delta)$ is separable and every bounded linear functional $F(f)$ in $L_\Phi(\Delta)$ is of the form $F(f) = \int f(x) g(x) dx$, where $g(x) \in L_\psi(\Delta)$ and $\frac{1}{2} \|g\|_\psi \leq \|F\| \leq \|g\|_\psi$. For the proofs we refer to [4], as well as for the application of these facts to integral equations in L_Φ . (We permit ourselves however one remark. In the cited paper the interval Δ is the linear interval $[a, b]$, and, if $F(f)$ is a bounded linear functional in $L_\Phi(\Delta)$, it is proved first that $F(\xi_x) = G(x)$ is an absolutely continuous function, where ξ_x is the characteristic function of $[a, x]$. The derivative $G'(x) = g(x)$ exists therefore almost everywhere in $[a, b]$ and $F(\xi_x) = G(x) = \int_a^x g(u) du = \int_a^x \xi_x(u) g(u) du$. The analogous fact in the case of a moredimensional, possibly infinite interval Δ is

proved as follows: Letting $f(x)$ run through the characteristic functions of all measurable sets X contained in a finite interval $\Delta_1 \subset \Delta$, it is proved that $F(f)$ is an additive, absolutely continuous function of a set X on Δ_1 , so that, by the RADON-NIKODYM theorem (cf. [5], Ch. I, § 14), there exists a summable function $g(x)$ in Δ_1 such that

$$F(f) = \int g(x) dx = \int_{\Delta_1} f(x) g(x) dx.$$

Since Δ may be considered as a sum of finite intervals, $g(x)$ may be extended to the whole of Δ and $F(f) = \int_{\Delta} f(x) g(x) dx$ holds for the characteristic function of any finite measurable set contained in Δ .

We finally mention some other applications (variations upon examples from [6], Ch. 12).

1. If $f_n(x)$ ($n = 1, 2, \dots$) is measurable on the finite interval Δ , and $|f_n(x)| \leq C$, $f(x) = \lim f_n(x)$ for $x \in \Delta$, then $\lim \int_{\Delta} |\Phi| |f - f_n| dx = 0$.

The same holds with Φ replaced by Ψ , provided $\Psi(v) < \infty$ for $v \leq 2C$.

2. If $f_n(x) \in L_{\Phi}(\Delta)$, $f(x) \in L_{\Phi}(\Delta)$, $\lim \|f - f_n\|_{\Phi} = 0$, and $g(x) = \lim f_n(x)$ almost everywhere in Δ , then $f(x) = g(x)$ almost everywhere in Δ . A similar statement is true if Φ is replaced by Ψ .

(Observe that $\lim \|f - f_n\|_{\Phi} = 0$ implies the existence of a subsequence, tending almost everywhere in Δ to $f(x)$).

3. If $f_n(x)$, $f(x) \in L_{\Phi}(\Delta)$; $g_n(x)$, $g(x) \in L_{\Psi}(\Delta)$; $\lim \|f - f_n\|_{\Phi} = 0$, $\lim \|g - g_n\|_{\Psi} = 0$, then $\int_{\Delta} fg dx = \lim \int_{\Delta} f_n g_n dx$.

4. If $\Phi(2u) \leq C \Phi(u)$ for $u \geq 0$, $f(x) \in L_{\Phi}(\Delta)$ and $\varepsilon > 0$ is given, there is a continuous function $g(x)$ such that $\|f - g\|_{\Phi} < \varepsilon$.

(There is a stepfunction $h(x)$ such that $\|f - h\|_{\Phi} < \varepsilon/2$ (cf. [4]), and it is obvious that there is a continuous $g(x)$ such that $\|h - g\|_{\Phi} < \varepsilon/2$).

5. If $\Phi(2u) \leq C \Phi(u)$ for $u \geq 0$, $f(x) \in L_{\Phi}(\Delta_1)$ and Δ is a finite interval contained in the interior of Δ_1 , then

$$\lim_{h \rightarrow 0} (\|f(x+h) - f(x)\|_{\Phi} \text{ over } \Delta) = 0.$$

(For continuous $f(x)$ the statement is easy to prove. For non-continuous $f(x)$ we make use of the previous example).

6. If $f(x) \in L_{\Phi}(R_m)$, where R_m is the whole m -dimensional Euclidean space, and $\Delta_i = [-i, i; \dots; -i, i]$ ($i = 1, 2, \dots$), then $\|f\|_{\Phi}$ over $R_m = \lim_{i \rightarrow \infty} (\|f\|_{\Phi} \text{ over } \Delta_i)$.

(Obviously $\|f\|$ over $\Delta_i \leq \|f\|$ over R_m . Furthermore, if $\varepsilon > 0$ is given, there exists a function $g(x)$ such that $\int_{R_m} \Psi |g| dx \leq 1$ and

$\|f\| - \varepsilon < \int_{R_m} |fg| dx$. Hence, for i sufficiently large, $\|f\| - \varepsilon < \int_{\Delta_i} |fg| dx \leq \|f\|$ (over Δ_i).

7. It is evident, if E is a measurable set, what $f(x) \in L_\Phi(E)$ means. If E_1 and E_2 are measurable sets having no common points, and $f \in L_\Phi(E_1 + E_2)$, then

$$\|f\| \text{ (over } E_1 + E_2) \leq \|f\| \text{ (over } E_1) + \|f\| \text{ (over } E_2).$$

(Put $f = f_1 + f_2$, where $f_1 = f$ on E_1 and $f_1 = 0$ on E_2).

8. If $\Phi(2u) \leq C\Phi(u)$ for $u \geq 0$, $f(x) \in L_\Phi(R_m)$ and

$$\Delta_i = [-i, i; \dots; -i, i] \quad (i = 1, 2, \dots),$$

then

$$\lim_{i \rightarrow \infty} (\|f(x+h)\|_\Phi \text{ over } R_m - \Delta_i) = 0,$$

uniformly for all points h having distance ≤ 1 from the origin.

(Denoting $\|f(x)\|$ (over R_m) = $\|f(x+h)\|$ (over R_m) simply by $\|f\|$, we have $\int_{R_m} \Phi[|f(x+h)|/\|f\|] dx \leq 1$; hence, if $\varepsilon > 0$ is given,

$$\int_{R_m - \Delta_i} \Phi[|f(x+h)|/\|f\|] dx < \varepsilon \text{ for } i \geq i_0(\varepsilon),$$

uniformly for all h having distance ≤ 1 from the origin. The result follows now from the fact that $\lim_{R_m} \int \Phi|f_i| dx = 0$ implies $\lim_{R_m} \|f_i\|_\Phi = 0$.

9. If $\Phi(2u) \leq C\Phi(u)$ for $u \geq 0$ and $f(x) \in L_\Phi(R_m)$, then

$$\lim_{h \rightarrow 0} \|f(x+h) - f(x)\|_\Phi = 0.$$

($\|f(x+h) - f(x)\| \leq \|f(x+h) - f(x)\| \text{ (over } \Delta_i) + \|f(x+h)\| \text{ (over } R_m - \Delta_i) + \|f(x)\| \text{ (over } R_m - \Delta_i)$).

10. If $\Phi(u)$ and $\Psi(v)$ are complementary, $\Phi(2u) \leq C\Phi(u)$ for $u \geq 0$, $f \in L_\Phi(R_m)$, $g \in L_\Psi(R_m)$, then

$$F(t) = \int_{R_m} f(x+t) g(x) dx$$

is a continuous function of t .

$$(|F(t+h) - F(t)| \leq \|f(x+t+h) - f(x+t)\|_\Phi \cdot \|g\|_\Psi).$$

11. If, with the notations of the previous example, also $\Psi(2v) \leq C\Psi(v)$ for $v \geq 0$, the function $F(t)$ satisfies $\lim F(t) = 0$ as t tends to infinity. (In linear space R_1 we may write

$$|F(t)| \leq \int_{-\infty}^{-t/2} |f(x+t)g(x)| dx + \int_{-t/2}^{\infty} |f(x)g(x-t)| dx \leq A\|f\|_\Phi + B\|g\|_\Psi,$$

where $\lim A = \lim B = 0$ as $t \rightarrow +\infty$. For $t \rightarrow -\infty$ the proof is similar. In a space of more dimensions t tends to infinity if and only if one at least of its coordinates tends to $\pm\infty$. If e.g. the first coordinate t_1 tends to $+\infty$, we split $\int_{-\infty}^{\infty} dx_1$ up into $\int_{-\infty}^{-t_1/2} + \int_{-t_1/2}^{\infty}$.

12. If $\Phi(2u) \leq C\Phi(u)$ for $u \geq 0$ and $f(x) \in L_\Phi(-\infty, \infty)$, then

$$F(t) = \int_{-\infty}^{\infty} \frac{f(x)}{1+(x-t)^2} dx$$

is a continuous function of t .

(In the proof of Theorem 1 we have seen that there exist positive numbers u_0, v_0 such that $\Psi(v) \leq u_0 v$ for $v \leq v_0$, hence

$$\int_{-\infty}^{\infty} \Psi[v_0/(1+x^2)] dx \leq u_0 v_0 \int_{-\infty}^{\infty} dx/(1+x^2),$$

which shows that $1/(1+x^2) \in L_\Psi(-\infty, \infty)$. Hence

$$\begin{aligned} |F(t+h) - F(t)| &\leq \int_{-\infty}^{\infty} |f(x+h) - f(x)| / \{1+(x-t)^2\} dx \leq \\ &\leq \|f(x+h) - f(x)\|_q \cdot \|1/(1+x^2)\|_p. \end{aligned}$$

REFERENCES.

1. W. ORLICZ, Über eine gewisse Klasse von Räumen vom Typus B, Bull. d. l'Acad. Polonaise, classe A (1932), pp. 207—220.
2. A. ZYGMUND, Trigonometrical Series, 4, 541, Warsaw (1935).
3. W. H. YOUNG, On classes of summable functions and their Fourier series, Proc. of the Royal Soc. (A), vol. 87 (1912), pp. 225—229.
4. A. C. ZAANEN, On a certain class of Banach spaces, Annals of Math., vol. 47 (1946), pp. 654—666.
5. S. SAKS, Theory of the integral, Warsaw (1937).
6. E. C. TITCHMARSH, The theory of functions, Oxford (1932).

University of Indonesia, Bandoeng (Java).

Mathematics. — *A property of a DIRICHLET series, representing a function satisfying an algebraic difference-differential equation.* By J. POPKEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of April 23, 1949.)

By definition a function $f(s)$ satisfies an algebraic difference-differential equation if there exists a polynomial $F(x_0, x_1, \dots, x_t) \not\equiv 0$, a set of t real numbers u_1, u_2, \dots, u_t and a set of t non-negative integers n_1, n_2, \dots, n_t , such that the t systems $(u_1, n_1), (u_2, n_2), \dots, (u_t, n_t)$ are different and

$$F(s, f^{(n_1)}(s+u_1), \dots, f^{(n_t)}(s+u_t)) = 0.$$

In this paper we will deduce the following

Theorem: *Let the convergent DIRICHLET series $\sum_{h=1}^{\infty} a_h e^{-\lambda_h s}$, with non-vanishing coefficients a_h , represent a function $f(s)$ satisfying an algebraic difference-differential equation. Then there exists a positive number c , such that*

$$\lambda_h > h^c \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

is valid for all but a finite number of values of h .

This theorem can be used to derive the transcendental-transcendency of certain functions. Let us take for example

$$\zeta(s) = \sum_{h=1}^{\infty} e^{-s \log h},$$

then it is clear, that the exponents $\lambda_h = \log h$ in this series do not have the property (1), hence $\zeta(s)$ cannot satisfy an algebraic difference-differential equation; it follows in particular that $\zeta(s)$ is transcendental-transcendent¹.

In this paper we confine ourselves to DIRICHLET series $\sum_{h=1}^{\infty} a_h e^{-\lambda_h s}$, where the "exponents" λ_h are real numbers with

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad \lambda_h \rightarrow +\infty \text{ for } h \rightarrow \infty.$$

An "empty" sum, i.e. a formal sum with no terms at all, will be equal to zero by definition.

¹⁾ However this proof for HILBERT's theorem on the transcendental-transcendency of the ζ -function is not essentially new, but closely related to OSTROWSKI's proof in the paper cited below.

A well-known theorem of OSTROWSKI²⁾ states: Let the convergent DIRICHLET series $\sum_{h=1}^{\infty} a_h e^{-\lambda_h s}$ with non-vanishing coefficients a_h , represent a function, satisfying an algebraic difference-differential equation. Then there exists for the set of all exponents λ_h a finite linear integral basis.

In the same paper of OSTROWSKI we find another result, connected with the foregoing theorem, which will serve us as a basis for the proof of our own theorem:

Lemma 1 3): *Let the convergent DIRICHLET series $\sum_{h=1}^{\infty} a_h e^{-\lambda_h s}$, with all coefficients a_h different from zero, represent a function, satisfying an algebraic difference-differential equation. Then there exists a real number Λ and a positive integer N , such that for sufficiently large values of h every number $\lambda_h + \Lambda$ is a sum of N exponents at most, taken from the sequence $\lambda_1, \lambda_2, \dots, \lambda_{h-1}$.*

Now for sufficiently large values of h the number $\lambda_h + \Lambda$ is positive, hence the sum mentioned in the lemma is not empty. Putting $\lambda_0 = -\Lambda$ we can give this lemma the following form: *There exists a positive integer r , such that*

$$\lambda_h = \lambda_{h_1} + \lambda_{h_2} + \dots + \lambda_{h_n} + \lambda_0 \text{ for } h \geq r, \dots \quad (2)$$

where $n = n(h)$ denotes a positive integer $\leq N$ and $h_1 \geq h_2 \geq \dots \geq h_n$ represent positive integers $< h$.

Evidently

$$\lambda_h > \lambda_{h_1} \geq \lambda_{h_2} \geq \dots \geq \lambda_{h_n} \geq \lambda_0.$$

We can take r so large, that

$$\lambda_h \neq 0 \text{ for } h \geq r. \dots \quad (3)$$

In order to prove our theorem we first deduce the following

Lemma 2: *Let the conditions of lemma 1 be fulfilled, so that (2) holds. Then there exists a positive integer $H \geq r$, such that*

$$\lambda_h^2 \geq \lambda_{h_1}^2 + \lambda_{h_2}^2 + \dots + \lambda_{h_n}^2 + \lambda_0^2 \text{ for } h \geq H.$$

P r o o f. We put

$$|\lambda_0| + N |\lambda_1| = c_1,$$

then it follows $c_1 \geq 0$ and

$$\lambda_0^2 + N \lambda_1^2 \leq c_1^2.$$

Let h take the values $r, r+1, r+2, \dots$, so that

$$\lambda_h = \lambda_{h_1} + \lambda_{h_2} + \dots + \lambda_{h_n} + \lambda_0.$$

²⁾ A. OSTROWSKI, Über Dirichletsche Reihen und algebraische Differentialgleichungen. Mathem. Zeitschrift 8, 241—298 (1921), Satz 6, p. 260.

³⁾ A. OSTROWSKI, loc. cit. p. 261—262; compare the proof of "Satz 7". OSTROWSKI's notations for h , N and Λ are respectively i , n and Λ_1 .

Now we distinguish the following two cases:

$$1^0. \quad n = 1 \text{ or } n \geq 2, \lambda_{h_1} \leq 3c_1.$$

$$2^0. \quad n \geq 2, \lambda_{h_1} > 3c_1.$$

In the first case the expression

$$\lambda_h - \lambda_{h_1} = \lambda_{h_2} + \lambda_{h_3} + \dots + \lambda_{h_n} + \lambda_0$$

can take only a finite number of different values; for this is clear for $n = 1$, and, if $n \geq 2$, then the positive integers n, h_2, h_3, \dots, h_n satisfy

$$n \leq N, h_n \leq h_{n-1} \leq \dots \leq h_2,$$

where h_2 is bounded on account of $\lambda_{h_1} \leq 3c_1$, $\lambda_h \rightarrow +\infty$ for $h \rightarrow \infty$. Hence the numbers $\lambda_h - \lambda_{h_1}$, being positive, have a positive minimum. It follows the existence of a positive integer $H \geq r$, such that

$$\lambda_h^2 - \lambda_{h_1}^2 = (\lambda_h + \lambda_{h_1})(\lambda_h - \lambda_{h_1}) > 9Nc_1^2 \text{ if } h \geq H. \quad \dots \quad (4)$$

Now

$$\lambda_{h_1}^2 + \lambda_{h_2}^2 + \dots + \lambda_{h_n}^2 \leq (N-1)(\lambda_1^2 + \lambda_{h_1}^2).$$

on account of

$$\lambda_1 \leq \lambda_{h_1} \leq \lambda_{h_{n-1}} \leq \dots \leq \lambda_{h_2} \leq \lambda_{h_1}; \text{ also } \lambda_1 \leq \lambda_{h_1} \leq 3c_1, \lambda_1^2 \leq c_1^2, \text{ hence } \lambda_{h_1}^2 \leq 9c_1^2,$$

so that

$$\begin{aligned} \lambda_{h_1}^2 + \lambda_{h_2}^2 + \dots + \lambda_{h_n}^2 + \lambda_0^2 &\leq \lambda_{h_1}^2 + \lambda_0^2 + (N-1)\lambda_1^2 + (N-1)\lambda_{h_1}^2 \\ &\leq \lambda_{h_1}^2 + c_1^2 + 9(N-1)c_1^2 \\ &\leq \lambda_{h_1}^2 + 9Nc_1^2. \end{aligned}$$

From (4) it follows for $h \geq H$

$$\lambda_{h_1}^2 + \lambda_{h_2}^2 + \dots + \lambda_{h_n}^2 + \lambda_0^2 < \lambda_{h_1}^2 + \lambda_h^2 - \lambda_{h_1}^2 = \lambda_h^2,$$

and the assertion of our lemma is true in the considered case.

If, on the other hand, $n \geq 2$, $\lambda_{h_1} > 3c_1$, then we denote the positive exponents in the right-hand member of $\lambda_h = \lambda_{h_1} + \lambda_{h_2} + \dots + \lambda_{h_n} + \lambda_0$ by $\lambda_{h_1}, \lambda_{h_2}, \dots, \lambda_{h_m}$ ($m \geq 2$) and we write

$$\lambda_h = \lambda_{h_1} + \lambda_{h_2} + \dots + \lambda_{h_m} + \delta_h,$$

where

$$|\delta_h| \leq (n-m)|\lambda_1| + |\lambda_0| \leq N|\lambda_1| + |\lambda_0| = c_1.$$

Now

$$\begin{aligned} \lambda_h^2 &\geq \left(\sum_{\mu=1}^m \lambda_{h_\mu} \right)^2 + 2\delta_h \sum_{\mu=1}^m \lambda_{h_\mu} \\ &\geq \sum_{\mu=1}^m \lambda_{h_\mu}^2 + 2\lambda_{h_1} \sum_{\mu=2}^m \lambda_{h_\mu} - 2c_1 \sum_{\mu=1}^m \lambda_{h_\mu} \\ &= \sum_{\mu=1}^m \lambda_{h_\mu}^2 + (2\lambda_{h_1} - 2c_1) \sum_{\mu=3}^m \lambda_{h_\mu} + \lambda_{h_1}(\lambda_{h_1} - 2c_1) + \lambda_{h_1}(\lambda_{h_1} - 2c_1). \end{aligned}$$

We have $\lambda_{h_1} - 2c_1 > c_1$, hence $\lambda_{h_1} - 2c_1 > 0$, $2\lambda_{h_1} - 2c_1 > 0$; it follows

$$\lambda_h^2 \geq \sum_{\mu=1}^m \lambda_{h_\mu}^2 + \lambda_{h_1} c_1 \geq \sum_{\mu=1}^m \lambda_{h_\mu}^2 + 3c_1^2.$$

The exponents $\lambda_{h_{m+1}}, \lambda_{h_{m+2}}, \dots, \lambda_{h_n}$ are negative or zero, hence

$$\lambda_0^2 + \sum_{r=m+1}^n \lambda_{h_r}^2 \geq \lambda_0^2 + N \lambda_1^2 \geq c_1^2.$$

It follows

$$\lambda_h^2 \geq \sum_{\mu=1}^m \lambda_{h_\mu}^2 + \sum_{r=m+1}^n \lambda_{h_r}^2 + \lambda_0^2.$$

This proves the lemma.

Proof of the theorem: On account of lemma 1 we have

$$\lambda_h = \lambda_{h_1} + \lambda_{h_2} + \dots + \lambda_{h_n} + \lambda_0 \text{ for } h \geq r,$$

with $1 \leq n \leq N$, $h_n \leq h_{n-1} \leq \dots \leq h_1 < h$.

Now we consider r arbitrary numbers x_0, x_1, \dots, x_{r-1} and the infinite sequence $x_0, x_1, \dots, x_{r-1}, x_r, x_{r+1}, \dots$, defined by the recurrent relations

$$x_h = x_{h_1} + x_{h_2} + \dots + x_{h_n} + x_0 \text{ for } h = r, r+1, r+2, \dots \quad (5)$$

where n, h_1, h_2, \dots, h_n are the same integers as in the foregoing formula. It follows that the sequence x_0, x_1, x_2, \dots has an integral linear basis consisting of the numbers x_0, x_1, \dots, x_{r-1} ; we even have

$$x_h = p_{h0} x_0 + p_{h1} x_1 + \dots + p_{h,r-1} x_{r-1} \quad (h = 0, 1, 2, \dots) \quad (6)$$

where $p_{h0}, p_{h1}, \dots, p_{h,r-1}$ denote integers ≥ 0 .

In particular

$$\lambda_h = p_{h0} \lambda_0 + p_{h1} \lambda_1 + \dots + p_{h,r-1} \lambda_{r-1} \quad (h = 0, 1, 2, \dots).$$

Let

$$X_\varrho = \begin{cases} 1 & \text{if } \lambda_\varrho \neq 0 \\ 0 & \text{if } \lambda_\varrho = 0 \end{cases} \text{ for } \varrho = 0, 1, \dots, r-1 \quad \dots \quad (7)$$

(so that at most two of the numbers X_0, X_1, \dots, X_{r-1} are zero, the others being equal to unity). Hence

$$X_0 \lambda_0 = \lambda_0, X_1 \lambda_1 = \lambda_1, \dots, X_{r-1} \lambda_{r-1} = \lambda_{r-1},$$

so that

$$\lambda_h = p_{h0} X_0 \lambda_0 + p_{h1} X_1 \lambda_1 + \dots + p_{h,r-1} X_{r-1} \lambda_{r-1} \quad (h = 0, 1, 2, \dots).$$

Now all exponents $\lambda_1, \lambda_2, \lambda_3, \dots$ are different; it follows that all systems

$$(p_{h0} X_0, p_{h1} X_1, \dots, p_{h,r-1} X_{r-1}) \text{ for } h = 1, 2, 3, \dots,$$

are different.

Let P denote an arbitrary positive integer. The total number of different systems

$$(p_0 X_0, p_1 X_1, \dots, p_{r-1} X_{r-1}).$$

where p_0, p_1, \dots, p_{r-1} are integers with $0 \leq p_\varrho \leq P-1$ ($\varrho = 0, 1, \dots, r-1$) is $P^{r-\tau}$, τ denoting the number of zeros in the sequence X_0, X_1, \dots, X_{r-1} . Hence there exists in the sequence $1, 2, \dots, P^r + 1$ at least one number k , such that the system

$$(p_{k0} X_0, p_{k1} X_1, \dots, p_{k,r-1} X_{r-1})$$

contains at least one integer $p_{k_0} X_0 \geq P$, and therefore

$$p_{k0} X_0 + p_{k1} X_1 + \dots + p_{k,r-1} X_{r-1} \geq P.$$

We introduced already the numbers X_0, X_1, \dots, X_{r-1} in (7); let X_r, X_{r+1}, \dots satisfy the recurrent relations (5) with X_h in stead of x_h . It follows from (6)

$$X_h = p_{h0} X_0 + p_{h1} X_1 + \dots + p_{h,r-1} X_{r-1} \quad (h=0,1,\dots).$$

Hence the sequence $1, 2, \dots, p^r + 1$ contains at least one number k , such that

Let H be the integer of lemma 2. By (3) we know $\lambda_h \neq 0$ for $h \geq r$; it follows from (7) that $\lambda_h = 0$ implies $X_h = 0$ ($h = 0, 1, 2, \dots$). Hence there exists a positive number c_2 , such that

$$X_h \equiv c_2 \lambda_h^2 \text{ for } h = 0, 1, \dots, H-1.$$

We shall show by induction that this inequality also holds for $h \geq H$.

Let $h \geq H$ and let

$$X_l \equiv c_3 \lambda_l^2 \text{ for } l = 0, 1, \dots, h-1.$$

We have $h \geq H \geq r$, hence

$$X_h = X_{h_1} + X_{h_2} + \dots + X_{h_n} + X_0,$$

where $1 \leq h_v \leq h-1$ ($v = 1, 2, \dots, n$), so that

$$X_h \equiv c_2 \lambda_{h_1}^2 + c_2 \lambda_{h_2}^2 + \dots + c_2 \lambda_{h_n}^2 + c_2 \lambda_0^2.$$

Applying lemma 2 we find, on account of $h \geq H$,

$$\lambda_{h_1}^2 + \lambda_{h_2}^2 + \dots + \lambda_{h_n}^2 + \lambda_0^2 \leq \lambda_h^2.$$

hence

$$X_h \equiv c_2 \lambda_h^2$$

holds for any number $h = 0, 1, 2, \dots$

Taking $h = k$, where k is the integer in formula (8), we deduce $c_2 \lambda_k^2 \geq P$, so that every sequence $1, 2, \dots, P^r + 1$ contains at least one number k , such that $c_2 \lambda_k^2 \geq P$.

Now it is easy to prove the assertion of our theorem. Let h be an arbitrary integer $\geq 4^r$, hence $h^r \geq 4$; put $P = \left[h^{\frac{1}{r}} \right] - 1$, then

$$P > h^{\frac{1}{r}} - 2 \geq \frac{1}{2} h^{\frac{1}{r}} \text{ and } P + 1 \leq h^{\frac{1}{r}},$$

hence $P^r + 1 \leq h$. It follows, that every sequence $1, 2, \dots, h$ contains at least one number k , such that $c_2 \lambda_k^2 \geq P > \frac{1}{2} h^{\frac{1}{r}}$. For sufficiently large h clearly $\lambda_k^2 > \lambda_1^2$, hence λ_k is positive and it follows $\lambda_k > c_3 h^{\frac{1}{2r}}$, where $c_3 = \frac{1}{\sqrt{2c_2}}$. Now $h \geq k$, hence $\lambda_h \geq \lambda_k > c_3 h^{\frac{1}{2r}}$. If we take for c a positive number $< \frac{1}{2r}$, then

$$\lambda_h > h^c$$

for all but a finite number of values of h .

Mathematics. — *Remark on my paper „On LAMBERT's proof for the irrationality of π “.* By J. POPKEN.

In these Proceedings¹⁾ I have given an elementary proof for the irrationality of π . However Dr M. VAN VLAARDINGEN kindly informed me, that the method I used nearly is the same as that applied by HERMITE in the fourth edition of "Cours de la faculté des Sciences" (1891), p. 74—75²⁾.

¹⁾ Vol. XLIII (1940) p. 712—714.

²⁾ See also: A. PRINGSHEIM, Vorlesungen über Zahlen- und Functionenlehre II, 1, p. 471—474; p. 613.

Mathematics. — *Affine embedding theory I: Affine normal spaces.* By V. HLAVATÝ. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of February 26, 1949.)

Synopsis. In a sequence of papers we shall establish an embedding theory of a X_m in a A_n ($1 \leq m < n, n \geq 3$). In this first paper some fundamental transformation laws are found which enable us a construction of a connection (which stamps our X_m to an A_m) as well as of higher connections which lead to a well defined set of affine normal spaces of the A_m .

1. Introduction.

- a) Let A_n be an affine space of n dimensions ($n \geq 3$) referred to a coördinate system ξ^r ¹⁾. $\Gamma_{\alpha\beta}^r = \Gamma_{\beta\alpha}^r$ its connection coefficients and ∇_μ by the corresponding covariant derivative operator. Any scalar or tensor defined by means of the usual transformation formula (with respect to a system of transformations $\xi^r \leftrightarrow \xi^{r'}$) will be referred to as a ξ -scalar or ξ -tensor.
 b) Let X_m be defined in A_n ($1 \leq m < n$) by parametric equations

$$\xi^r = \xi^r(\eta^1, \dots, \eta^m), \quad \dots \quad (1, 1)$$

where the rank of the JACOBIAN matrix $\left(\frac{\partial \xi^r}{\partial \eta^a} \right)$ is m . Any scalar or tensor defined by means of the usual transformation formula (with respect to a system of transformations $\eta^a \leftrightarrow \eta^{a'}$) will be referred to as a η -scalar or η -tensor. A $(\xi\eta)$ -scalar (or $(\xi\eta)$ -tensor) is a ξ -scalar (ξ -tensor) as well as η -scalar (η -tensor). Thus for instance

$$T_a^r \equiv \frac{\partial \xi^r}{\partial \eta^a} \quad \dots \quad (1, 2 a)$$

is a $(\xi\eta)$ -tensor because of its transformation law

$$T_{a'}^{r'} = \frac{\partial \xi^{r'}}{\partial \xi^r} \frac{\partial \eta^a}{\partial \eta^{a'}} T_a^r. \quad \dots \quad (1, 2 b)$$

Consequently this $(\xi\eta)$ -tensor is independent of a choice of coördinate system ξ^r as well as of a choice of a parameter system η^a . On the other hand $T_1^r, T_2^r, \dots, T_m^r$ may be thought of as a set of m linearly independent ξ -vectors which depend on the choice of parameter systems.

¹⁾ It is understood that small greek (latin) indices run from 1 to n (from 1 to m unless otherwise stated).

c) Introducing the operator $\nabla_a \equiv T_a^\mu \nabla_\mu$ we define an infinite set of ξ -vectors

$$T_{a_x \dots a_1}^r = \nabla_{a_x} T_{a_{x-1} \dots a_1}^r = \partial_{a_x} T_{a_{x-1} \dots a_1}^r + \Gamma_{\lambda\mu}^\nu T_{a_x}^\mu T_{a_{x-1} \dots a_1}^\lambda = \left. \begin{array}{l} \\ = T_{a_x \dots a_1}^r \end{array} \right\} (1, 3)$$

$(x = 2, 3, \dots).$

each of them being dependent on the choice of a parameter system η^a . Because n is a finite integer, a least integer N may be found for which any ξ -vector $T_{a_M \dots a_1}^r$ (for $M > N$) is a linear combination of the ξ -vectors $T_{a_x \dots a_1}^r, x = 1, 2, \dots, N$. Throughout this paper we shall assume $N > 1$.

The linear space $E_{m_y}^y(P)$ spanned by the ξ -vectors

$$T_{a_1}^r(P), T_{a_1 a_1}^r(P), \dots, T_{a_y \dots a_1}^r(P), y = 1, 2, \dots, N. \quad (1, 4)$$

will be referred to as the y -th osculating space of our X_m at its generator point P . Any of these spaces is obviously $(\xi \eta)$ -invariant and moreover we have

$$E_{m_1}^1 \subset E_{m_2}^2 \subset E_{m_3}^3 \subset \dots \subset E_{m_N}^N.$$

The ξ -vectors T_a^r being linearly independent we have

$$m_1 = m, \dots, \dots, \dots, \dots, \quad (1, 5a)$$

and because of (1, 3) we obtain

$$m_x \equiv m + \sum_1^{x-1} \frac{m^x (m+1)}{2}, \quad x = 2, 3, \dots, N. \quad (1, 5b)$$

If the equality sign holds our space X_m will be called a *maximal* X_m , the remaining case being termed a *restricted* X_m .

2. Fundamental transformation formulae.

Our starting point will be the analytic expression of the statement (contained in 1c)) on the integer N , namely

$$F_{a_{N+1} \dots a_1}^r \equiv T_{a_{N+1} \dots a_1}^r - \sum_1^N \Phi_{a_{N+1} \dots a_x \dots a_1}^{b_x \dots b_1} T_{b_x \dots b_1}^r = 0. \quad (2, 1)$$

where the Φ 's are some functions of the η 's. If we are dealing with a maximal X_m , all ξ -vectors (1, 4) are linearly independent and consequently (2, 1) admits *only one solution for the Φ 's*. This solution is obviously

of the ξ -order $N+1$ ²⁾. On the other hand if our X_m is a restricted X_m , the Φ 's are not uniquely determined by (2, 1) unless this equation be a given one with an unique given set of Φ 's. We shall confine ourselves to this case if dealing with a restricted X_m .³⁾. Hence the Φ 's are of the ξ -order $N+1, (0)$ ⁴⁾. Later on we shall see that the η -order of $\Phi_{a_{N-1} \dots a_x \dots a_1}^{b_x \dots b_1}$ is $N+2-x$ ($N+2-x$).

Let us now find the transformation law for the Φ 's. If we introduce the abbreviations

$$\left. \begin{aligned} P_b^{a'} &= \frac{\partial \eta^{a'}}{\partial \eta^b}, & P_{b_x \dots b_1}^{a'_x \dots a'_1} &= P_{b_x}^{a'_x} P_{b_{x-1}}^{a'_{x-1}} \dots P_{b_1}^{a'_1} \\ P_{b'}^a &= \frac{\partial \eta^a}{\partial \eta^{b'}}, & P_{b'_x \dots b'_1}^{a_x \dots a_1} &= P_{b'_x}^{a_x} P_{b'_{x-1}}^{a_{x-1}} \dots P_{b'_1}^{a_1} \\ P_{b'_x \dots b'_1}^{a_x \dots a_1} &= \frac{\partial^x \eta^a}{\partial y^{b'_x} \dots \partial y^{b'_1}}, & x &= 1, 2, 3, \dots \end{aligned} \right\} \quad (2, 2)$$

we may write the transformation law for the $T_{a_x \dots a_1}^r$ in the following way

$$\left. \begin{aligned} T_{a_x \dots a_1}^r &= P_{a_x \dots a_1}^{a'_x \dots a'_1} T_{a_x \dots a_1}^r \\ &+ \sum_2^{x-1} \left[\left\{ P_{a_x \dots a_r}^{a'_x} P_{a_{r-1} \dots a_1}^{a'_{r-1} \dots a'_1} \right\} + P_{a_x \dots a_{r-1} \dots a_1}^{a'_x \dots a'_{r-1} \dots a'_1} \right] T_{a_r \dots a_1}^r \\ &+ P_{a_x \dots a_1}^{a'_x \dots a'_1} T_a^r, \quad x = 2, 3, \dots \end{aligned} \right\} \quad (2, 3)$$

Here $\left\{ P_{a_x \dots a_r}^{a'_x} P_{a_{r-1} \dots a_1}^{a'_{r-1} \dots a'_1} \right\}$ symmetric in $a_2 a_1$, stands for a sum of products of the type as introduced in the braces {} with a conveniently chosen permutation of the upper as well as lower indices,

$$P_{a_x}^{a_x} P_{a_{x-1} \dots a_1}^{a_{x-1} \dots a_1} = 0 = P_{a_x \dots a_1}^{a_x \dots a_1}$$

²⁾ An object is said to be of the ξ -order (η -order) r if its construction in any coördinate system (parameter system) requires the derivatives of the ξ ($\eta^{a'}$) with respect to the η^a up to the r -th order only.

³⁾ In other words: If dealing with a maximal case we start with (1, 1) and build up the equations (2, 1) which admit only one solution for the Φ 's. If our X_m is a restricted one we do not start with (1, 1) and we consider (2, 1) as an à priori given set of partial differential equations with uniquely given coefficients Φ 's.

⁴⁾ The first number refers always to the maximal case, the number in parenthesis refers to the restricted case.

and $P_{a'_x \dots a'_r \dots a'_1}^{a_r \dots a_1}$ (symmetric in $a_2 a_1$) represents for $r = x-2, x-3, \dots, 2$ a sum of products of at least two of the expressions

$$P_{a'_1 a'_1}^{a_1}, P_{a'_1 a'_2 a'_1}^{a_2}, \dots, P_{a'_{x-r} \dots a'_1}^{a_r} \dots \quad (2, 4)$$

and eventually of some $P_{a'_1}^{a_1}$.⁵⁾

The equation (2, 1) has to be $(\xi \eta)$ -invariant. Hence

$$F'_{a'_{N+1} \dots a'_1} = \varrho P_{a'_{N+1} \dots a'_1}^{a_{N+1} \dots a_1} F'_{a'_{N+1} \dots a_1} \dots \quad (2, 5)$$

Comparing this equation with (2, 3) for $x = N + 1$ we get an once

$$\varrho = 1 \dots \quad (2, 6)$$

5) A straightforward computation leads to

$$a) \quad \{P_{c' b'}^{b'} P_{a'}^{a'}\} = P_{c' b'}^{(b)} P_{a'}^{(a)} + P_{b' a'}^{(a)} P_{c'}^{(b)} + P_{a' c'}^{(b)} P_{b'}^{(a)},$$

$$P_{d' c' b' a'}^{b' a'} = P_{d' c'}^{(b)} P_{b'}^{(a)} + P_{d' b'}^{(b)} P_{a'}^{(a)} + P_{d' a'}^{(a)} P_{c'}^{(b)}$$

and

$$b) \quad \left\{ P_{a'_{x+1} \dots a'_1}^{a_x} P_{a'_1}^{a_1} \right\} = \left\{ \left(\partial_{a'_{x+1}} P_{a'_{x+1} \dots a'_1}^{(a_x)} \right) P_{a'_1}^{(a_1)} \right\} + P_{a'_{x+1}}^{(a_1)} P_{a'_{x+1} \dots a'_1}^{a_x}$$

$$\left\{ P_{a'_{x+1} \dots a'_s}^{a_s} P_{a'_{s-1} \dots a'_1}^{a_{s-1} \dots a_1} \right\} = \left\{ \left(\partial_{a'_{x+1}} P_{a'_{x+1} \dots a'_s}^{(a_s)} \right) P_{a'_{s-1} \dots a'_1}^{a_{s-1} \dots a_1} \right\} +$$

$$+ P_{a'_{x+1}}^{(a_s)} \left\{ P_{a'_{x+1} \dots a'_{s-1}}^{a_{s-1}} P_{a'_{s-2} \dots a'_1}^{a_{s-2} \dots a_1} \right\}, \quad s = 3, \dots, x-1$$

$$\left\{ P_{a'_{x+1} \dots a'_x}^{a_x} P_{a'_{x-1} \dots a'_1}^{a_{x-1} \dots a_1} \right\} = \partial_{a'_{x+1}} P_{a'_{x+1} \dots a'_x}^{(a_x \dots a_1)} + P_{a'_{x+1}}^{(a_x)} \left\{ P_{a'_{x+1} \dots a'_{x-1}}^{a_{x-1}} P_{a'_{x-2} \dots a'_1}^{a_{x-2} \dots a_1} \right\}.$$

$$c) \quad P_{a'_{x+1} \dots a'_s \dots a'_1}^{a_s \dots a_1} = \left\{ P_{a'_{x+1} \dots a'_s}^{a_s} \partial_{a'_{x+1}} P_{a'_{s-1} \dots a'_1}^{a_{s-1} \dots a_1} \right\} + \partial_{a'_{x+1}} P_{a'_{x+1} \dots a'_s \dots a'_1}^{a_s \dots a_1}$$

$$+ P_{a'_{x+1} \dots a'_{s-1} \dots a'_1}^{a_{s-1} \dots a_1} P_{a'_{x+1}}^{(a_s)}, \quad s = 3, \dots, x-1$$

$$P_{a'_{x+1} \dots a'_x a'_1}^{a_x a_1} = \left\{ P_{a'_{x+1} \dots a'_x}^{a_x} \partial_{a'_{x+1}} P_{a'_1}^{(a_1)} \right\} + \partial_{a'_{x+1}} P_{a'_{x+1} \dots a'_x a'_1}^{a_x a_1}, \quad x = 3, 4, \dots$$

These formulas enable us to compute all the symbols $\{\}$ as well as all the p 's.

Theorem (2, 1). If our X_m is a maximal one then

$$\left. \begin{aligned}
 (a) \quad \Phi_{a_{N+1} \dots a_1}^{b_N \dots b_1} &= P_{a_{N+1} \dots a_1}^{a'_N \dots a'_1} \left[P_{b'_N \dots b'_1}^{b_N \dots b_1} \Phi_{a'_{N+1} \dots a'_1}^{b'_N \dots b'_1} - \right. \\
 &\quad \left. - \left\{ P_{a'_{N+1} \dots a'_1}^{b_N} P_{a'_{N-1} \dots a'_1}^{b_{N-1} \dots b_1} \right\} \right], \\
 b) \quad \Phi_{a_{N+1} \dots a_r \dots a_1}^{b_r \dots b_1} &= P_{a_{N+1} \dots a_1}^{a'_N \dots a'_1} \left[P_{b'_r \dots b'_1}^{b_r \dots b_1} \Phi_{a'_{N+1} \dots a'_r \dots a'_1}^{b'_r \dots b'_1} + \right. \\
 &\quad + \sum_{r+1}^N \Phi_{a'_{N+1} \dots a'_q \dots a'_1}^{b'_q \dots b'_1} \left(\left\{ P_{b'_q \dots b'_r}^{b_r} P_{b'_{r-1} \dots b'_1}^{b_{r-1} \dots b_1} \right\} + p_{b'_q \dots b'_r \dots b'_1}^{b_r \dots b_1} \right) \\
 &\quad \left. - \left\{ P_{a'_{N+1} \dots a'_r \dots a'_1}^{b_r} P_{a'_{r-1} \dots a'_1}^{b_{r-1} \dots b_1} \right\} - p_{a'_{N+1} \dots a'_r \dots a'_1}^{b_r \dots b_1} \right], \quad r = 2, \dots, N-1 \\
 c) \quad \Phi_{a_{N+1} \dots a_1}^{b_1} &= P_{a_{N+1} \dots a_1}^{a'_N \dots a'_1} \left[P_{b'_1}^{b_1} \Phi_{a'_{N+1} \dots a'_1}^{b'_1} + \right. \\
 &\quad \left. + \sum_2^N \Phi_{a'_{N+1} \dots a'_q \dots a'_1}^{b'_q \dots b'_1} P_{b'_q \dots b'_1}^{b_1} - P_{a'_{N+1} \dots a'_1}^{b_1} \right].
 \end{aligned} \right\} (2, 7)$$

In the case of a restricted X_m these equations represent one of the possible conclusions taken from (2, 5). According to (2, 7) the $\Phi_{a_{N+1} \dots a_x \dots a_1}^{b_x \dots b_1}$ are of the η -order $N+2-x$ ($N+2-x$), $x=1, \dots, N$.

Proof. Substituting from (2, 6) and (2, 3) in (2, 5), we get an equation of the form

$$\sum_1^N T_{b_u \dots b_1}^u X_{a_{N+1} \dots a_u \dots a_1}^{b_u \dots b_1} = 0. \quad \dots \quad (2, 8)$$

If our X_m is a maximal one, then all the ξ -vectors $T_{b_u \dots b_1}^u$ are linearly independent and consequently (2, 8) leads to

$$X_{a_{N+1} \dots a_u \dots a_1}^{b_u \dots b_1} = 0, \quad u = 1, \dots, N. \quad \dots \quad (2, 9)$$

In the case of a restricted X_m the equations (2, 9) represent one of the possible conclusions taken from (2, 9). The equations (2, 9) are equivalent to (2, 7). — The last part of our theorem is obvious.

3. The connection Γ_{cb}^a of X_m .

Let us introduce the functions

$$\Pi_{a_{N+1} \dots a_1}^{b_N \dots b_1} = \Phi_{(a_{N+1} \dots a_1)}^{b_N \dots b_1}, \quad \Pi_{a'_{N+1} \dots a'_1}^{b'_N \dots b'_1} = \Phi_{(a'_{N+1} \dots a'_1)}^{b'_N \dots b'_1} \quad (3, 1a)$$

and

$$\left. \begin{aligned} A_{cb}^a &= A_{bc}^a = \prod_{c' a' N \dots a_1 b}^{a_N \dots a_2 a}, \quad A_c = A_{ca}^a \\ A_{c'b'}^{a'} &= A_{b'c'}^{a'} = \prod_{c' a' N \dots a'_2 b'}^{a'_N \dots a'_1 a'}, \quad A_{c'} = A_{c'a'}^{a'} \end{aligned} \right\} , \quad (3, 1 b)$$

which we shall need later on. According to the definition of the symbols in braces we have obviously

$$\left. \begin{aligned} \left\{ P_{b'e'}^e \delta_{(a_{N-1} \dots a_1 c)}^{a_{N-1} \dots a_2 a} \right\} P_{b'e}^{b'e'} &= r_1 P_{b'e'}^a P_{c'b}^{e'b'} \\ + r_2 \left(P_{b'e'}^e P_{b'e}^{b'e'} \delta_c^a + P_{b'e'}^e P_{c'e}^{b'e'} \delta_b^a \right) \end{aligned} \right\} , \quad (3, 2 a)$$

where r_1, r_2 are suitably chosen integers different from zero. Consequently

$$\left\{ P_{b'e'}^e \delta_{(a_{N-1} \dots a_1)}^{a_{N-1} \dots a_2 a} \right\} P_{b'e}^{b'e'} = [r_1 + r_2(m+1)] P_{b'e'}^e P_{b'e}^{b'e'} . \quad (3, 2 b)$$

Theorem (3, 1). The functions

$$\Gamma_{cb}^a = \Gamma_{bc}^a = \frac{1}{r_1} \left[A_{cb}^a - \frac{r_2}{r_1 + r_2(m+1)} (\delta_c^a A_b + \delta_b^a A_c) \right] . \quad (3, 3)$$

are coefficients of a connection in X_m which stamps our X_m to an A_m .

This connection is of the $\xi - \eta - \frac{1}{2}$ order $\left\{ \begin{array}{ll} N+1 & (0) \\ 2 & (2) \end{array} \right.$

Proof. From the first of (2, 7) and from (3, 1) we have

$$\left. \begin{aligned} a) \quad A_{cb}^a &= P_{c'b}^{c'b'a} A_{c'b'}^{a'} - \left\{ P_{b'e'}^e \delta_{(a_{N-1} \dots a_1 c)}^{a_{N-1} \dots a_2 a} \right\} P_{b'e}^{b'e'} \\ b) \quad A_b &= P_{b'e'}^e A_{b'e'} - \left\{ P_{b'e'}^e \delta_{(a_{N-1} \dots a_1)}^{a_{N-1} \dots a_2 a} \right\} P_{b'e}^{b'e'} \end{aligned} \right\} . \quad (3, 4)$$

Consequently it follows from (3, 2)

$$\left. \begin{aligned} \left\{ P_{b'e'}^e \delta_{(a_{N-1} \dots a_1 c)}^{a_{N-1} \dots a_2 a} \right\} P_{b'e}^{b'e'} &= r_1 P_{b'e'}^a P_{c'b}^{e'b'} + \\ + \frac{r_2}{r_1 + r_2(m+1)} \left[\delta_c^a (P_{b'e'}^e A_{b'e'} - A_b) + \delta_b^a (P_{c'b}^{c'b'a} A_{c'b'}^{a'} - A_c) \right] . \end{aligned} \right\} \quad (3, 5)$$

Substituting from (3, 5) in (3, 4a) we get

$$\Gamma_{cb}^a = P_{c'b}^{c'b'a} \Gamma_{c'b'}^{a'} - P_{c'b}^{c'b'} P_{c'b'}^a . \quad (3, 6 a)$$

or

$$P_{c'b'}^a = P_{a'}^a \Gamma_{c'b'}^{a'} - P_{c'b'}^{c'b} \Gamma_{c'b}^a . \quad (3, 6 b)$$

6) δ_c^a is the Kronecker delta and $\delta_{a_r \dots a_1}^{b_r \dots b_1} = \delta_{a_r}^{b_r} \dots \delta_{a_1}^{b_1}$.

where the Γ_{cb}^a are defined by (3, 3) and the $\Gamma_{c'b'}^{a'}$ are defined by a similar equation to (3, 3). Hence Γ_{cb}^a are the coefficients of a connection which is obviously of the $\frac{\xi - \eta}{\eta}$ order mentioned in the theorem.

4. Higher degree connections.

The r -th covariant derivative (with respect to the connection (3, 3)) of an η -tensor contains the $(r-1)$ -derivatives of Γ_{cb}^a which is of the ξ -order $N+r(r-1)$. In this section we shall replace these derivatives by another set of functions which are of the same ξ -order $N+1(0)$.

Lemma (4, 1). $P_{a'_N \dots a'_1}^{b_{N-1} \dots b_1}$ may be put in the form

$$P_{a'_N \dots a'_1}^{b_{N-1} \dots b_1} = P_{b'_{N-1} \dots b'_1}^{b_{N-1} \dots b_1} \left. \begin{aligned} & \omega_{a'_N \dots a'_1}^{b'_{N-1} \dots b'_1} + \xi_{a'_N \dots a'_1}^{b_{N-1} \dots b_1} \\ & - P_{a'_N \dots a'_1}^{a_{N+1} \dots a_1} \gamma_{a'_N \dots a'_1}^{b_{N-1} \dots b_1} \end{aligned} \right\} \quad (4, 1)$$

where $\gamma_{a'_N \dots a'_1}^{b_{N-1} \dots b_1} \left(\omega_{a'_N \dots a'_1}^{b'_{N-1} \dots b'_1} \right)$ is a sum of products of the $\Gamma_{cb}^a \left(\Gamma_{c'b'}^{a'} \right)$ only and does not contain $P_{b'}^a$, P_b^a and $\xi_{a'_N \dots a'_1}^{b_{N-1} \dots b_1}$ is a sum of products of at least one of the Γ_{cb}^a and some $\Gamma_{c'b'}^{a'} \left(\text{and of some } P_{b'}^a, P_b^a \right)$.

Proof. $P_{a'_N \dots a'_1}^{b_{N-1} \dots b_1}$ is a sum of products of at least two $P_{c'b'}^a$ and eventually of some $P_{b'}^a$. If the expressions (3, 6b) the substituted in these products we have at once the equation (4, 1), were γ is a function of the Γ_{cb}^a 's and does not contain the $\Gamma_{c'b'}^{a'}$'s. Let us suppose that it contains at least one of the $P_{b'}^{a'}$'s, say $P_d^{a'}$. Then in γ one accentuated index (namely c') has to occur which makes the index c' of the $P_d^{a'}$ a dummy index. Because γ does not contain the $\Gamma_{c'b'}^{a'}$'s the required index may appear only in some of the $P_{b'}^a$, say in P_c^e . Then $P_c^e P_d^{a'} = \delta_d^e$. Hence γ does not contain the $P_{b'}^a$'s and similarly it does not contain the $P_{b'}^{a'}$'s. The remaining statements of the lemma may be proved by a similar device.

Theorem (4, 1). The equation (2, 7b) for $r=N-1$ admits only one solution

$$P_{d'c'b'}^a = P_{a'}^a \Omega_{d'c'b'}^{a'} + P_{a'}^a \Xi_{d'c'b'}^{a'} - P_{d'c'b'}^{d\ c\ b} \Gamma_{dcb}^a. \quad \dots \quad (4, 2)$$

where Ω , Ξ and Γ are symmetric in the lower indices and

- a) $\Gamma_{dcb}^a (\Omega_{d'c'b'})$ is a function of the $\Phi_{a_{N+1}\dots a_r\dots a_1}^{b_r\dots b_1}$ and $(\Phi_{a'_{N+1}\dots a'_r\dots a'_1}^{b'_r\dots b'_1})$ only [$r=N-1, N$] and does not contain $P_{b'}^a$, $P_{b'}^{a'}$.
- b) $\Xi_{d'c'b'}^{a'}$ represents a sum of products of at least one Γ_{cb}^a and some $\Phi_{a'_{N+1}\dots a'_N\dots a'_1}^{b'_N\dots b'_1}$ (and eventually of some $P_b^{a'}$, P_b^a).

The functions Γ_{dcb}^a are of the $\left. \begin{matrix} \xi - \\ y - \end{matrix} \right\} \text{order } \frac{N+1}{3} (0)$. No derivatives $\partial_e \Gamma_{fg}^h$, $\partial_{e'} \Gamma_{f'g'}^{h'}$ occur in (4, 2).

Proof. Let us introduce the abbreviations

$$\left. \begin{aligned} \Pi_{a_{N+1}\dots a_{N-1}\dots a_1}^{b_{N-1}\dots b_1} &= \Phi_{(a_{N+1}\dots a_{N-1}\dots a_1)}^{b_{N+1}\dots b_1}, \quad A_{deb}^a = \Pi_{dca_{N-1}\dots a_2a}^{a_{N-1}\dots a_2a} \\ \Pi_{a'_{N+1}\dots a'_{N-1}\dots a'_1}^{b_{N-1}\dots b_1} &= \Phi_{(a'_{N+1}\dots a'_{N-1}\dots a'_1)}^{b'_{N-1}\dots b'_1}, \quad A_{d'c'a'}^a = \Pi_{d'c'a'_{N-1}\dots a'_2a'}^{a'_{N-1}\dots a'_2a'} \end{aligned} \right\}. \quad (4, 3)$$

From (2, 7b) for $r=N-1$, from (3, 6b) and from (4, 1) we obtain

$$\begin{aligned} A_{deb}^a - \gamma_{(dca_{N-1}\dots a_2b)}^{a_{N-1}\dots a_2a} &= \\ = P_{d'c'b'}^{d'c'b'a'} \left[A_{d'c'b'}^{a'} + \Pi_{d'c'a'_{N-1}\dots a'_2b'}^{d'_N\dots d'_1} \left\{ \Gamma_{d'_N d'_{N-1}}^{a'_{N-1}} \delta_{d'_{N-2}\dots d'_1}^{a'_N\dots a'_2a'} \right\} - \omega_{(d'c'a'_{N-1}\dots a'_2b')}^{a'_{N-1}\dots a'_2a'} \right] \\ - P_{d'c'b'}^{d'c'b'} \left[\Pi_{d'c'a'_{N-1}\dots a'_2b'}^{d'_N\dots d'_1} P_e^{a'_{N-1}} P_{d'_N d'_{N-1} a'}^{f\dots g} \left\{ \Gamma_{fg}^e \delta_{d'_{N-2}\dots d'_1}^{a'_{N-2}\dots a'_2a'} \right\} + P_{d'_{N-1}\dots d_2}^{a'_{N-1}\dots a'_2} \xi_{(d'c'a'_{N-1}\dots a'_2b')}^{d_{N-1}\dots d_2a} \right] \\ - \left\{ P_{d'c'h'}^h \delta_{(d'_{N-2}\dots d_2b)}^{d_{N-2}\dots d_2a} \right\} P_{h'dc}^{h'd'c'}. \end{aligned} \quad (4, 4)$$

Applying on (4, 4) the same device as before we obtain the statement of our theorem.

Note. If we put

$$\gamma_{dcb}^a = \gamma_{(dca_{N-1}\dots a_2b)}^{a_{N-1}\dots a_2a}, \quad \gamma_{dc}^a = \gamma_{dca}^a, \quad A_{dc}^a = A_{dca}^a$$

then we have

$$\Gamma_{abc}^a = c_1 (A_{abc}^a - \gamma_{abc}^a) - c_2 \delta_{(a}^a (A_{cb)}^a - \gamma_{cb)}^a) \dots \quad (4,5)$$

where c_1, c_2 are suitably chosen integers different from zero.

The theorem (4,1) may be easily generalized:

Theorem (4,2). The equations (2,7b) for $r=3, 4, \dots, N-1$ admit only one solution

$$P_{b'_q \dots b'_1}^a = P_{a'}^a \Omega_{b'_q \dots b'_1}^{a'} + P_{a'}^a \Xi_{b'_q \dots b'_1}^{a'} - P_{b'_q \dots b'_1}^{b_q \dots b_1} \Gamma_{b'_q \dots b'_1}^a \quad (4,6)$$

$q = 3, 4, \dots, N.$

where Ω , Ξ and Γ are symmetric in the lower indices and

a) $\Gamma_{b'_q \dots b'_1}^a (\Omega_{b'_q \dots b'_1}^{a'})$ is a function of the $\Phi_{a'_{N+1} \dots a_s \dots a_1}^{b_s \dots b_1} (\Phi_{a'_{N+1} \dots a_s' \dots a_1}^{b'_s \dots b'_1})$

only $[s=N+2-q, \dots, N]$ and does not contain $P_{b'}^a$, $P_{b'}^{a'}$.

b) $\Xi_{b'_q \dots b'_1}^{a'}$ represents a sum of products of at least one $\Gamma_{b_u \dots b_1}^a$,

$(u=2, \dots, q-1)$, and some $\Phi_{a'_{N+1} \dots a_p \dots a_1}^{b'_p \dots b'_1}$, $[p=N+3-q, \dots, N]$ and eventually of some $P_{b'}^{a'}$, $P_{b'}^a$. The functions $\Gamma_{b'_q \dots b'_1}^a$ are of the $\xi - \eta -$

order $\begin{cases} N+1 & (0) \\ q & (0) \end{cases}$.

No derivatives of $\Gamma_{b_u \dots b_1}^a$, $\Omega_{b_u \dots b_1}^{a'}$, $\Xi_{b_u \dots b_1}^{a'}$, $(u=2, 3, \dots, q-1)$ occur in (4,6).

The proof is similar to the previous one⁷⁾.

Note. Let (T) be an η -tensor and L be the transformation law for its r -th derivatives (with respect to the η 's). If we start with (3,6b) we

7) Let us suppose we have already proved (4,6) for $q=3, 4, \dots, u < N$. Then the following lemma (whose proof is similar to that of the lemma (4,1)) has to be used:

$$P_{a'_x \dots a'_r \dots a'_1}^{b_r \dots b_1} = P_{b'_r \dots b'_1}^{b_r \dots b_1} \omega_{a'_x \dots a'_r \dots a'_1}^{a_x \dots a_r \dots a_1} + \xi_{a'_x \dots a'_r \dots a'_1}^{b_r \dots b_1} - P_{a'_x \dots a'_1}^{a_x \dots a_1} \gamma_{a_x \dots a_r \dots a_1}^{b_r \dots b_1}$$

$x \leq N+1, x-r=2, \dots, u$

where γ is a sum of products of at least two of the $\Gamma_{a_s \dots a_1}^b$, $s=2, \dots, x-r$ only and does not contain $P_{a'}^a$, $P_{a'}^{a'}$, ω is a function of the $\Phi_{a' \dots}^{b' \dots}$ and does not contain $P_{a''}^a$, $P_{a''}^{a'}$ and ξ is a sum of products of at least one of the $\Gamma_{a_s \dots b_1}^b$, $s=2, \dots, x-r$ and some $\Phi_{a' \dots}^{b' \dots}$ (and some $P_{a''}^a$, $P_{a''}^{a'}$).

could deduce by derivation an equation of the type (4, 6) (which we shall designate by (4, 6)'), where the $\Gamma_{b_q \dots b_i}^a$ is now a function of the $\Gamma_{a_b}^c$ and its first $q-2$ derivatives and consequently is of the ξ -order $N-1+q$ ($q-2$). Eliminating the $P_{b_r \dots}^a$ from L and (4, 6)', we obtain the transformation law for the r -th covariant derivative (with respect to Γ_{cb}^a), say $(T)_{(r)}$, of (T) . The tensor $(T)_{(r)}$ contains the Γ_{cb}^a and the $\Gamma_{b_s \dots b_i}^a$, ($s=3, 4, \dots, r+1$) involved in (4, 6)', which consequently are of the ξ -order $N-1+s$ ($s-2$). If we replace these $\Gamma_{b_s \dots b_i}^a$ in $(T)_{(r)}$ by the $\Gamma_{b_s \dots b_i}^a$ involved in (4, 6), we obtain another tensor (cf. the next section) which is of the ξ -order $N+1$ (0) (as far as the $\Gamma_{b_s \dots b_i}^a$ are concerned), provided $r \leq N-1$. For that reason the $\Gamma_{b_q \dots b_i}^a$ involved in (4, 6) may be thought of as coefficients of a „higher degree connection”.

In the next lemma (we shall need later on) we denote by $\eta^{a'}$ a given parameter system (so that the $\Phi_{a'}^{b' \dots}$ are known functions of the $\eta^{a'}$), by $P_{b'}^a$ a given system of constants of the rank m and by P a generator point $\eta^{a'} = 0$:

Lemma (4, 2). The parameter system y^a defined by

$$y^a = P_{a'}^a \left[y^{a'} + \sum_2^N \frac{1}{q!} (\Omega_{b'_q \dots b'_1}^{a'})_P \eta^{a'_q} \dots \eta^{a'_1} \right] \quad \left. \begin{array}{l} \\ \\ \left(\Omega_{c' b'}^{a'} = \Gamma_{c' b'}^{a'} \right) \end{array} \right\} \quad (4, 7)$$

(the so called privileged parameter system at P) satisfies the following conditions at P

$$\left. \begin{array}{lll} a) \quad P_{b'_r \dots b'_1}^{a'} = \Omega_{b'_r \dots b'_1}^{a'}, & b) \quad \Xi_{b'_r \dots b'_1}^{a'} = 0, & c) \quad \Gamma_{b_r \dots b_1}^{a'} = 0 \\ & & \left(r=2, \dots, N, \quad \Omega_{c' b'}^{a'} = \Gamma_{c' b'}^{a'} \right) \end{array} \right\} \quad (4, 8)$$

Proof. Put

$$\eta^a = P_{a'}^a \left[\eta^{a'} + \sum_2^N \frac{1}{q!} X_{b_q \dots b_1}^{a'} \eta^{b'_q} \dots \eta^{b'_1} \right] \quad \dots \quad (4, 9)$$

where the constants X are to be found. If $X_{b' c'}^{a'} = (\Gamma_{b' c'}^{a'})_P$, then we get by virtue of (3, 6b) the equations (4, 8a, c) for $r=2$ (while (4, 8b) for $r=2$ is identically satisfied). Consequently, owing to the property of $\Xi_{b' c' d'}^{a'}$ mentioned in the theorem (4, 1) we have (4, 8b) for $r=3$. Hence

if $X_{c' b' d}^{a'} = (\Omega_{c' b' d'})_P$, we have (4, 8a, c) for $r=3$ and owing to the property of $\Xi_{b'_1 \dots b'_r}^{a'}$ mentioned in the theorem (4, 2) for $q=4$ we have (4, 8b) for $q=4$. Proceeding in this way we finally obtain (4, 8a, b, c) for $r=2, \dots, N$.

5. Affine normal spaces.

Theorem (5, 1). If we eliminate from (2, 3) (for $x=2, \dots, N$), (3, 6)b and (4, 2), (4, 6) the $P_{b'_1 \dots b'_r}^a$, we obtain the usual transformation law for a $(\xi \eta)$ -tensor (which we denote by $\Delta_{a_x} \dots \Delta_{a_r} T_{a_1}^r$ and call the $(x-1)$ st covariant pseudoderivative of T_a^r)⁸⁾ with x covariant indices

$$\left. \begin{aligned} a) \quad & \Delta_{a_x} T_{a_1}^r = T_{a_1 a_1}^r - \Gamma_{a_1 a_1}^b T_b^r \text{ for } x=2, \\ b) \quad & \Delta_{a_x} \dots \Delta_{a_r} T_{a_1}^r = \\ & = T_{a_x \dots a_1}^r - \sum_2^{x-1} T_{b_r \dots b_1}^r \left[\left\{ \Gamma_{a_x \dots a_r}^{b_r} \delta_{a_{r-1} \dots a_1}^{b_{r-1} \dots b_1} \right\} + \gamma_{a_x \dots a_r}^{b_r \dots b_1} \right] - \\ & \quad - T_b^r \Gamma_{a_x \dots a_1}^b \quad (x=3, \dots, N). \end{aligned} \right\} (5, 1)$$

Proof. The equation (5, 1a) is obvious. If $N \geq x > 2$, then the elimination mentioned in the theorem leads to

$$\left. \begin{aligned} P_{a'_x \dots a'_1}^{a_x \dots a_1} \Delta_{a_x} \dots \Delta_{a_r} T_{a_1}^r & = - \sum_1^{x-1} T_{b_r \dots b_1}^r * \Xi_{a'_x \dots a'_r \dots a'_1}^{b_r \dots b_1} \\ & + T_{a'_x \dots a'_1}^r - \sum_2^{x-1} \left(\left\{ \Omega_{a'_x \dots a'_n}^{b'_r} \delta_{a'_{p-1} \dots a'_1}^{b'_{r-1} \dots b'_1} \right\} + \omega_{a'_x \dots a'_r \dots a'_1}^{b'_r \dots b'_1} \right) P_{b'_r \dots b'_1}^b T_{b_r \dots b_1}^r - \\ & \quad - P_{b'_r}^b \Omega_{a'_x \dots a'_1}^{b'_r} T_b^r \end{aligned} \right\} (5, 2)$$

where $*\Xi_{a'_x \dots a'_r \dots a'_1}^{b_r \dots b_1}$ is a (linear) function of the $\Xi_{a'_x \dots a'_r}^{b'_r}$ and the $\xi_{a'_x \dots a'_r \dots a'_1}^{b_r \dots b_1}$ and contains some $P_{b'_r}^{a'_r}$, $P_{b'_r}^a$. In the right hand side we have to express $T_{b_q \dots b_1}^r$ by $T_{a'_s \dots a'_1}^r$ ($q=1, \dots, r$, $s=1, 2, \dots, q$) using a similar formula to (2, 3) (with interchanged rôle of the parameters)⁹⁾. In doing so we have finally the equation

⁸⁾ The same device (the generalized Christoffel elimination) may be applied to any $(\xi \eta)$ -tensor.

⁹⁾ And the device

$$P_{a_1 a_1}^{b'} = - P_{a_1 a_1 b}^{a'_1 a'_1 b'} P_{a'_1 a'_1}^b = \Gamma_{a_1 a_1}^b P_b^{b'} - \Gamma_{a'_1 a'_1}^b P_{a_1 a_1}^{a'_1 a'_1}$$

$$P_{a_1 a_2 a_1}^{b'} = \dots$$

and so on.

$$\Delta_{a'_x} \dots \Delta_{a'_1} T^v_{a_1} = P^{a'_x \dots a_1}_{a'_x \dots a'_1} \Delta_{a_x} \dots \Delta_{a_1} T^v_{a_1} + X^v_{a'_x \dots a'_1} . . . (5, 3)$$

where $X^v_{a'_x \dots a'_1}$ represents a function of the arguments $\Phi_{a_x \dots a_s \dots a_1}^{b_s \dots b_1}$, $\Phi_{a'_x \dots a'_s \dots a'_1}^{b'_s \dots b'_1}$, $T^v_{a_r \dots a_1}$ and $T^v_{a'_r \dots a'_1}$ ($r = 1, \dots, x$, $s = N+2-x, \dots, N$) as well as of the $P^a_{a'}$, $P^{a'}_a$ and does not contain the derivatives of the $P^a_{a'}$, $P^{a'}_a$. Let us now consider a transformation $\eta^a \leftrightarrow \eta^{a'}$ with constant $P^a_{a'}$ (and $P^{a'}_a$). Then any of the arguments already mentioned (with the exception of the $P^a_{a'}$ and $P^{a'}_a$) is a tensor with respect to this special transformation and consequently $\Delta_{a_x} \dots \Delta_{a_1} T^v_{a_1}$ is a $(\xi\eta)$ -tensor, which means that

$$X^v_{a'_x \dots a'_1} = 0 \text{ for constant } P^a_{a'}, P^{a'}_a (5, 4)$$

Because the lefthand member does not contain any derivatives of the $P^a_{a'}$, $P^{a'}_a$, the statement (5, 4) is equivalent to

$$X^v_{a'_x \dots a'_1} = 0 \text{ for any transformation at a generator point } P. \quad (5, 5)$$

Hence (5, 3) simplifies to an ordinary tensor law transformation at any generator point, which proves our statement.

Theorem (5, 2). The $(\xi\eta)$ -tensor

$$H^v_{a_x \dots a_1} \equiv \Delta_{a_x} \dots \Delta_{a_1} T^v_{a_1}; \quad (x=2, \dots, N) (5, 6)$$

has the following properties:

a) It is symmetric in the indices $a_1 a_2$

$$H^v_{a_x \dots a_2 a_1} = H^v_{a_x \dots a_1 a_2} (5, 7)$$

b) It is of the $\frac{\xi}{\eta} - \left\{ \begin{array}{c} N+1 \\ x \\ (x) \end{array} \right\}$ order

c) In the maximal case the ξ -vectors T^v_a , $H^v_{a_x \dots a_1}$ ($x=2, \dots, N$) are linearly independent.

d) The linear space $N_{n_{x-1}}^{x-1}$ spanned by $H^v_{a_x \dots a_1}$ is in \dot{E}_{m_x} but not in \dot{E}_{m_y} , $y < x$ and its dimension is

$$n_{x+1} = m_x - m_{x-1}; \quad (m_1 = m) (5, 8)$$

Moreover $N_{n_{x-1}}^{x-1}$ is of the $\frac{\xi}{\eta} - \left\{ \begin{array}{c} N+1 \\ x \\ (x) \end{array} \right\}$ order.

Proof. We have in a privileged parameter system because of (4, 8c) and (5, 1)

$$H_{a_x \dots a_1}^x = T_{a_x \dots a_1}^x \text{ at } P; \quad (x=2, \dots, N) \quad \dots \quad (5, 9)$$

The statement *b*) of the theorem follows from (5, 1), the remaining statements follow at once from (5, 9).

Definition (5, 1). The space $N_{n_{x-1}}^{x-1}$ as defined in the theorem (5, 2) will be termed the $(x-1)$ st affine normal space of A_m , $(x=2, \dots, N)$.

In the next paper we shall establish the FRENET formulas for the affine normal spaces.

*Indiana University
Department of Mathematics*

Bloomington (Ind.) USA

Astronomy. — The Partition function for the elements in the Solar Atmosphere. By W. J. CLAAS. (Communicated by Prof. M. G. J. MINNAERT.)

(Communicated at the meeting of March 26, 1949.)

1. The influence of perturbations.

The distribution of the atoms of an element over all the stages of excitation is of considerable importance for the interpretation of the solar spectrum. For conditions of local thermodynamic equilibrium, this reparation is described by BOLTZMANN's law $g_n e^{-E_n/kT}$, except for very high values of the quantum number n . Especially important is the partition function $a = \sum_n g_n \cdot e^{-E_n/kT}$ for a neutral atom and for the corresponding ion, which comes in when the amount of ionization is computed.

We shall consider particularly the influence of perturbing particles in the atmosphere, by which the formation of high electron orbits is made impossible. The nature of these perturbations is an interesting problem. Already long ago, physicists have pointed out two possibilities: UREY¹⁾ and FERMI²⁾ compared the perturbations to these which occur in a VAN DER WAALS gas, while PLANCK³⁾ considered the electrical perturbations due to the Coulomb-field of perturbing ions. This last effect has been recently treated by UNSÖLD⁴⁾; he calculates the maximum potential energy which a bound electron can have in the neighbourhood of a perturbing ion, and puts this equal to the energy of the highest excited level which actually occurs. The probability that no perturbing ion is found within a distance r of the atom will be $e^{-\frac{4\pi}{3}N_1 r^3}$, where N_1 is the number of perturbing ions per cm³. For the case when this probability is e^{-1} , UNSÖLD finds that the highest quantum number which occurs is:

$$\log n^* = 1.62 + \frac{1}{2} \log Z - \frac{1}{6} \log P_e - \frac{1}{6} \log \vartheta, \dots \quad (1)$$

Z being the effective nuclear charge of the perturbed particle, P_e the electron pressure in dynes/cm²; $\vartheta = \frac{5040}{T}$.

2. Perturbations by neutral atoms.

Since the ionisation in the exterior layers of a star is relatively small, we may ask whether the perturbations by ions will be more considerable than these by neutral atoms; practically among these last ones we shall have to consider only the hydrogen atoms, because of their overwhelming abundance. These perturbations have been studied by FÜCHTBAUER⁵⁾ and his collaborators in beautiful experiments; they are ascribed by FERMI⁶⁾ to two different effects:

1. the perturbing atoms penetrate between the excited electron and the atomic nucleus; they are polarized in the field of the perturbed atom, which produces a change in the energy of the excited electron;

2. the excited electron in its orbit encounters several sharp potential holes, due to the presence of perturbing atoms; by this the total energy of the atom is modified.

In FÜCHTBAUER's experiments, the influence of the potential holes proved more important than that of the polarization. However, these experiments were made at pressures of at least 1 atmosphere and at temperatures between 350° K and 750° K. Such temperatures and pressures are very different from these, prevailing in a stellar atmosphere. A rough estimate shows, that in the solar atmosphere the polarization effect is the more important one; for this is proportional to the number of disturbing particles N , while the first effect is only proportional to $N^{1/2}$. Thus the low pressure and the high temperature both tend to diminish the perturbations, but this influence is more important for the potential holes. In the solar atmosphere, at an optical depth $\tau_0 = 0.75$, where $\log T = 3.79$, $\log P_g = 4.9$, the perturbation by polarization is about 5 times stronger than the perturbation by potential holes.

3. A comparison between the perturbing effect of atoms and that of ions.

The polarization effect is due to an interaction between the disturbed atoms or ions and the perturbing H -atoms. This problem has been treated by WEISSKOPF⁷⁾. The interaction energy amounts to $E = \frac{e^2 a \bar{R}^2}{r^6}$, where a = polarisability of the hydrogen atom, \bar{R}^2 = the quadratic mean value of the radius of the excited orbit. The mean value of R^2 over all atoms with the same principal quantum number n and different auxiliary quantum numbers l , is found to be $\bar{R}^2 = \frac{n^2}{2} a^2 \cdot 4n^2$ (a = radius of the first BOHR orbit).

Ionization will take place if the interaction energy is equal to the energy which is needed for the ionization of the excited atom:

$$e^2 \cdot \frac{a}{r^6} \cdot 2a^2 n^4 = \frac{RhcZ^2}{n^2} \dots \dots \dots \quad (2)$$

By substituting the physical constants, we find

$$r = 8.56 \times 10^{-9} \times n \dots \dots \dots \quad (3)$$

The probability that there will be no perturbing H -atom within a distance r is: $e^{-\frac{4\pi}{3} N_H \cdot r^3}$.

So the correction factor which has to be applied to BOLTZMANN's distribution law, due to the perturbing influence of the hydrogen atoms, is found to be:

$$W_1 = e^{-\frac{4\pi}{3} \cdot \frac{P_g}{kT} \cdot r^3} = e^{-\frac{4\pi}{3} \cdot \frac{P_g}{kT} \cdot (8.56 \times 10^{-9})^3 \cdot n^3} \dots \dots \quad (4)$$

The correction factor due to the influence of the ions is similarly:

$$W_2 = e^{-\frac{4\pi}{3} \cdot \frac{27 e^6 \cdot P_e}{(Rhc)^2 Z^4 \cdot kT} \cdot n^6} = e^{-\frac{4\pi}{3} \cdot 3.18 \cdot 10^{-23} \cdot \frac{P_e}{Z^4 kT} \cdot n^6}. . . . (5)$$

The perturbations by ions are more important than these by H -atoms, if $W_2 < W_1$ vid. lic. if

$$\frac{4\pi}{3} \cdot 3.18 \cdot 10^{-23} \cdot \frac{P_e}{Z^4 kT} \cdot n^6 > \frac{4\pi}{3} \cdot \frac{P_g}{kT} \cdot (8.56 \times 10^{-9})^3 \cdot n^3.$$

Or, putting $Z = 1$, if the perturbed particles are neutral:

$$\log n > \frac{1}{3} (\log P_g - \log P_e) + 0.431 - 1. (6)$$

Taking again $\tau_0 = 0.75$, $\log P_g = 4.9$ and $\log P_e = 1.5$, substitution in (6) yields: $\log n > 0.46$, $n > 3$. For disturbed ions $W_2 < W_1$, if $n > 9$.

Consequently the influence of the ions is more important than that of the neutral H -atoms; for the correction factor becomes appreciable only for levels of high excitation ($n > 10$).

4. Calculation of the partition function.

When applying the laws of BOLTZMANN and SAHA, the partition function u is needed. We separate u into two parts:

$$u = \sum_{n_0} g_n e^{-\frac{E_n}{kT}} + u_1 = u_0 + u_1. (7)$$

The number n_0 is selected according to the following conditions:

1. Levels for which $n < n_0$ are populated according to the exponential distribution law;
2. levels for which $n = n_0$ may be considered as hydrogen-like.

In view of the computation of u_1 we choose the same $n_0 = 7$ for all elements. It is true that for several elements lower levels could already be treated as hydrogen-like. However, a uniform choice of n_0 has the advantage of an easier computation of the u_1 term in (7). Let r be the multiplicity of the levels for which $n \geq 7$; then

$$u_1 = \int_7^\infty r n^2 \cdot e^{-\frac{E_n}{kT} - \frac{4\pi}{3} \cdot \frac{27 e^6}{(Rhc)^2 Z^4} \cdot \frac{P_e}{kT} \cdot n^6} \cdot dn. (8)$$

We substitute $E_n = E_i - \frac{13.53 Z^2}{n^2}$ with E_i = ionization potential of the atom ($Z = 1$) or of the ion ($Z = 2$). We introduce $\vartheta = \frac{5040}{T}$, and take exponentials of 10:

$$u_1 = r \cdot 10^{-E_i \vartheta} \int_7^\infty n^2 \cdot 10^{\frac{13.53 Z^2 \vartheta}{n^2} - \gamma \cdot \frac{P_e \vartheta}{Z^4} \cdot n^6} dn. (9)$$

where

$$\gamma = \frac{4\pi}{3} \cdot \frac{e^6}{(Rhc)^3} \cdot \frac{27}{k} \cdot \frac{0.4343}{5040}; \log \gamma = 0.922 - 11.$$

Now put

$$v(\vartheta) = \int_1^\infty n^2 \cdot 10^{\frac{C_1}{n^2} - C_2 \cdot n^6} \cdot dn \quad \dots \dots \dots \quad (10)$$

where the parameters $C_1 = 13.53 Z^2 \vartheta$ and $C_2 = \gamma \cdot \frac{P_e \cdot \vartheta}{Z^4}$ are easily computed. Then finally:

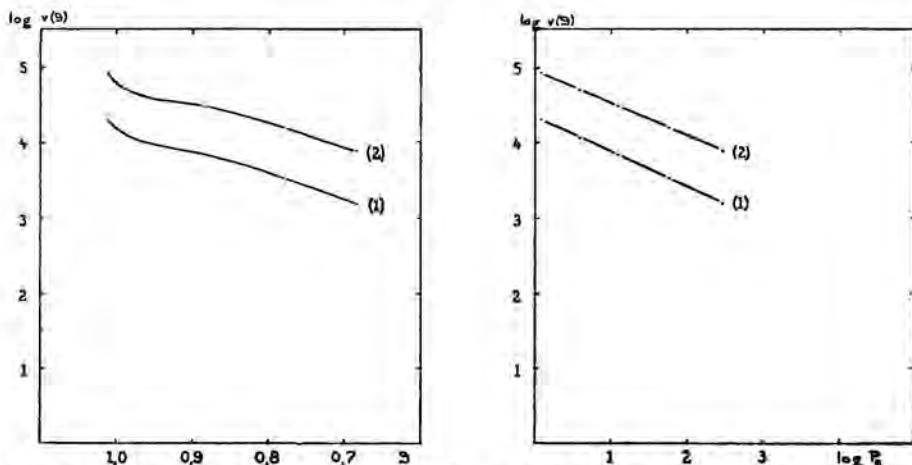
$$u_1 = r \cdot 10^{-E_i \vartheta} v(\vartheta). \quad \dots \dots \dots \quad (11)$$

The function $v(\vartheta)$ is computed by subdividing the integration interval and by developing the integrand within each segment in a Taylor series of the first or second degree. If, for $n > n_1$, $\frac{C_1}{n^2} \ll C_2 n^6$,

$$\begin{aligned} \int_{n_1}^\infty n^2 \cdot 10^{\frac{C_1}{n^2} - C_2 n^6} \cdot dn &= \int_{n_1}^\infty n^2 \cdot 10^{-C_2 n^6} \cdot dn = \\ &= \frac{1}{6} \sqrt{\frac{\pi}{2.3026 \times C_2}} \cdot \{1 - \Phi(n_1^3 \cdot \sqrt{2.3026 \cdot C_2})\}. \end{aligned}$$

The integral $v(\vartheta)$ converges and so does the partition function u . We computed this function $v(\vartheta)$ for neutral and for singly ionized atoms, for five values of ϑ . The relation between $\log v(\vartheta)$ and ϑ or $\log P_e$ is represented in fig. 1 and 2, for neutral atoms by the curves (1) and for ions by the curves (2). Practically $\log v$ is a linear function of $\log P_e$; in both cases $v \sim P_e^{-0.44}$.

The contribution $u_1 = r \cdot 10^{-E_i \vartheta} v(\vartheta)$ to the partition function u proves negligible for the ions of all elements, because of their high ionization potentials. For neutral atoms u_1 must be taken into account only for elements with low ionization potentials. If $E_i > 8$ Volt, u_1 can be entirely neglected.



The auxiliary functions $v(\vartheta)$ and $v(P_e)$ in the solar atmosphere: (1) if the perturbed particles are atoms; (2) if they are ions.

For the easily ionized alcali metals, the contribution of u_1 is important; for these atoms, an appreciable influence of the electron pressure is found, next to the temperature influence. As we pass from the outer layers into the deeper parts of the solar atmosphere, the increasing temperature produces a stronger excitation of the high levels, while the increasing electron pressure produces an opposite effect. For most elements the influence of the temperature is preponderant; however, for the alcali metals and for Ba the pressure has the greater influence and u_1 decreases at first and increases only from a certain depth on (table 1).

TABLE I. *Value of u_1 for easily ionized metals.*
Contribution of the higher levels ($n \geq 7$) to the partition function, at different levels of the solar atmosphere.

Element \ \vartheta	1.022	0.980	0.939	0.817	0.781	0.684
Li	0.13	0.13	0.16	0.36	0.44	0.68
Na	0.23	0.22	0.28	0.58	0.68	1.00
K	1.54	1.36	1.57	2.62	2.85	3.48
Rb	2.22	1.92	2.22	3.55	3.80	4.53
Cs	4.42	3.68	4.11	6.15	6.46	7.11
Ba	0.40	0.38	0.49	1.03	1.21	1.82

We computed the contribution $u_0 = \sum g_n \cdot 10^{-E_n \cdot \vartheta}$ by adding the contributions of the individual levels. For elements with complicated electron systems so as *Ti, Fe, V, Co, Ni, ...* terms with excitation potentials above $2V$ were combined into groups, and for each group a mean excitation potential was taken. Since u_0 depends only on ϑ , our computations for u_0 may be used for all stellar atmospheres; however, u_1 applies only to the solar atmosphere, since the concomitant influence of P_e which comes in is different from star to star. Table 2 gives u_0 and u for 4 values of ϑ .

5. Instead of introducing a correction factor to the BOLTZMANN distribution function, UNSÖLD⁴⁾ proposes to limit the integration at the quantum number n^* , following from equation (1). The integral $v = \int_7^{n^*} n^2 \cdot 10^{n \cdot \frac{C_1}{\vartheta}} dn$ is then to be computed by developing the exponential in a series and by integrating term by term. The result is practically the same as by our method, which gives bothways a very satisfactory confirmation. In UNSÖLD's method the influence of P_e is embodied in the choice of n^* . From a theoretical point of view our computation is slightly more satisfactory, since it does not cut off the integration in a rather artificial way at n^* .

It is a pleasure to thank Professor MINNAERT for stimulating discussions and criticism in the course of this investigation. I am grateful to Mr. B. B. VAN DEN HOORN for his kind help in translating FERMI's paper from "Il Nuovo Cimento".

TABLE 2. *The Partition Function.*

The contribution u_0 of the lower levels applies to any stellar atmosphere; the contribution of the higher levels and consequently the total partition function u applies only to the sun.

Element \ \theta	u_0				u			
	0.980	0.939	0.781	0.684	0.980	0.939	0.781	0.684
H	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
He	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
He+	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
Li	2.11	2.12	2.27	2.46	2.24	2.28	2.71	3.14
Li+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Be	1.02	1.03	1.07	1.13	1.02	1.03	1.07	1.13
Be+	2.00	2.00	2.01	2.01	2.00	2.00	2.01	2.01
B	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.01
B+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
C	9.29	9.33	9.53	9.70	9.29	9.33	9.53	9.70
C+	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00
N	4.05	4.06	4.15	4.26	4.05	4.06	4.15	4.26
N+	8.87	8.90	9.01	9.12	8.87	8.90	9.01	9.12
O	8.71	8.74	8.89	8.99	8.71	8.74	8.89	8.99
O+	4.01	4.01	4.03	4.05	4.01	4.01	4.03	4.05
F	5.66	5.68	5.73	5.77	5.66	5.68	5.73	5.77
F+	8.37	8.39	8.52	8.61	8.37	8.39	8.52	8.61
Ne	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Ne+	5.36	5.39	5.49	5.55	5.36	5.39	5.49	5.55
Na	2.07	2.08	2.21	2.40	2.29	2.36	2.89	3.40
Na+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Mg	1.02	1.03	1.07	1.14	1.02	1.03	1.09	1.18
Mg+	2.00	2.00	2.00	2.01	2.00	2.00	2.00	2.01
Al	5.91	5.91	5.97	6.07	6.01	6.04	6.42	6.88
Al+	1.00	1.00	1.00	1.01	1.00	1.00	1.00	1.01
Si	9.61	9.71	10.06	10.35	9.61	9.71	10.07	10.37
Si+	5.77	5.79	5.81	5.84	5.77	5.79	5.81	5.84
P	4.46	4.53	4.90	5.26	4.46	4.53	4.90	5.26
P+	8.48	8.50	8.58	8.64	8.48	8.50	8.58	8.64
S	8.23	8.25	8.38	8.46	8.23	8.25	8.38	8.46
S+	4.17	4.20	4.40	4.61	4.17	4.20	4.40	4.61
Cl	5.30	5.32	5.44	5.50	5.30	5.32	5.44	5.50
Cl+	7.68	7.74	7.93	8.06	7.68	7.74	7.93	8.06
A	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
A+	4.90	4.93	5.11	5.20	4.90	4.93	5.11	5.20
K	2.22	2.25	2.62	3.07	3.58	3.82	5.47	6.55
K+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Ca	1.20	1.24	1.58	2.00	1.25	1.31	1.82	2.44
Ca+	2.23	2.27	2.50	2.74	2.23	2.27	2.50	2.74
Sc	12.21	12.67	15.50	18.94	12.23	12.70	15.62	19.19
Sc+	23.27	23.82	26.26	28.28	23.27	23.82	26.26	28.28
Ti	30.55	31.97	39.93	48.09	30.57	32.00	40.08	48.41
Ti+	56.49	58.18	64.30	69.41	56.49	58.18	64.30	69.41
V	49.45	51.23	61.15	71.31	49.48	51.28	61.37	71.78
V+	43.73	44.93	50.43	55.15	43.73	44.93	50.43	55.15
Cr	10.69	11.17	13.63	16.43	10.73	11.23	13.91	17.02
Cr+	7.34	7.59	9.11	10.79	7.34	7.59	9.11	10.79

TABLE 2. (Continued.)

Element	θ	u_0				u			
		0.980	0.939	0.781	0.684	0.980	0.939	0.781	0.684
Mn	6.48	6.59	7.44	8.55	6.49	6.61	7.54	8.80	
Mn ⁺	7.77	7.89	8.57	9.23	7.77	7.89	8.57	9.23	
Fe	26.93	27.86	32.59	36.60	26.93	27.87	32.63	36.76	
Fe ⁺	41.54	42.45	46.70	49.37	41.54	42.45	46.70	49.37	
Co	31.94	32.97	38.14	42.80	31.94	32.98	38.17	42.89	
Co ⁺	28.13	29.06	32.95	35.90	28.13	29.06	32.95	35.90	
Ni	28.95	29.42	31.67	33.52	28.95	29.43	31.70	33.61	
Ni ⁺	10.30	10.68	12.46	14.08	10.30	10.68	12.46	14.08	
Cu	2.33	2.38	2.67	2.96	2.33	2.38	2.69	3.01	
Cu ⁺	1.03	1.05	1.10	1.21	1.03	1.05	1.10	1.21	
Zn	1.00	1.00	1.01	1.02	1.00	1.00	1.01	1.02	
Zn ⁺	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	
Ga	5.35	5.37	5.49	5.59	5.38	5.41	5.64	5.85	
Ga ⁺	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Ge	8.21	8.34	8.87	9.27	8.21	8.34	8.88	9.29	
Ge ⁺	4.68	4.74	4.92	5.05	4.68	4.74	4.92	5.05	
As	4.54	4.60	5.02	5.40	4.54	4.60	5.02	5.40	
Se	5.91	6.00	6.43	6.70	5.91	6.00	6.43	6.70	
Br	4.07	4.12	4.32	4.47	4.07	4.12	4.32	4.47	
Br ⁺	28.62	28.66	28.96	29.26	28.62	28.66	28.96	29.26	
Kr	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Kr ⁺	3.68	3.72	3.91	4.06	3.68	3.72	3.91	4.06	
Rb	2.28	2.33	2.75	3.28	4.20	4.55	6.55	7.81	
Rb ⁺	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Sr	1.28	1.34	1.74	2.25	1.41	1.51	2.25	3.09	
Sr ⁺	2.18	2.21	2.42	2.65	2.18	2.21	2.42	2.65	
Y	12.24	12.79	16.12	19.77	12.27	12.83	16.29	20.11	
Y ⁺	14.49	14.89	17.02	18.85	14.49	14.89	17.02	18.85	
Zr	36.08	38.15	48.95	59.68	36.10	38.18	49.07	59.94	
Zr ⁺	47.72	49.54	57.75	64.52	47.72	49.54	57.75	64.52	
Cb	25.68	25.85	26.82	27.63					
Cb ⁺	36.12	36.86	39.64	41.68	36.12	36.86	39.64	41.68	
Mo	8.82	9.12	10.93	12.85	8.83	9.13	11.00	13.03	
Mo ⁺	7.56	7.77	8.91	9.98	7.56	7.77	8.91	9.98	
Ru	28.05	29.32	35.02	39.71	28.05	29.33	35.05	39.79	
Ru ⁺	19.74	20.50	23.75	26.40	19.74	20.50	23.75	26.40	
Rh	24.39	25.32	29.70	33.44	24.39	25.32	29.72	33.49	
Rh ⁺	19.27	20.03	23.47	25.54	19.27	20.03	23.47	25.54	
Pd	2.87	3.07	4.01	4.85	2.87	3.07	4.01	4.86	
Pd ⁺	6.86	6.95	7.33	7.62	6.86	6.95	7.33	7.62	
Ag	2.00	2.00	2.01	2.02	2.00	2.00	2.02	2.04	
Ag ⁺	1.00	1.00	1.00	1.01	1.00	1.00	1.00	1.01	
Cd	1.00	1.00	1.01	1.02	1.00	1.00	1.01	1.02	
Cd ⁺	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	
In	4.62	4.66	4.85	4.97	4.67	4.73	5.07	5.33	
In ⁺	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Sn	6.49	6.63	7.22	7.69	6.49	6.64	7.25	7.75	
Sn ⁺	3.92	3.97	4.17	4.31	3.92	3.97	4.17	4.31	
Sb	4.83	4.93	5.44	5.89	4.83	4.93	5.45	5.91	

TABLE 2. (*Continued.*)

Element	ϑ	u_0				u			
		0.980	0.939	0.781	0.684	0.980	0.939	0.781	0.684
Te	4.87	5.00	5.59	6.05	4.87	5.00	5.59	6.05	
X	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Cs	2.50	2.57	3.17	3.87	6.18	6.68	9.63	10.98	
Cs+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Ba	2.49	2.66	3.87	5.06	2.87	3.15	5.08	6.88	
Ba+	4.38	4.48	5.16	5.69	4.38	4.48	5.16	5.69	
La	20.74	21.47	25.15	28.56	20.99	21.80	26.08	30.08	
La+	30.93	31.90	36.71	40.82	30.93	31.90	36.71	40.82	
Eu+	13.44	13.55	13.99	14.42	13.44	13.55	13.99	14.42	
Yb	1.05	1.07	1.17	1.28	1.09	1.12	1.36	1.63	
Yb+	2.00	2.00	2.01	2.02	2.00	2.00	2.01	2.02	
Lu	8.01	8.11	8.61	9.06					
Lu+	1.46	1.54	2.00	2.45	1.46	1.54	2.00	2.45	
Hf+	13.71	14.40	17.75	20.73	13.71	14.40	17.75	20.73	
Re	6.08	6.10	6.26	6.48	6.08	6.11	6.30	6.58	
Pt	16.27	16.55	17.95	19.14	16.27	16.55	17.95	19.15	
Au	2.40	2.46	2.70	2.92	2.40	2.46	2.70	2.92	
Hg	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Hg+	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	
Tl	3.34	3.37	3.53	3.66	3.37	3.41	3.65	3.88	
Tl+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
Pb	3.50	3.57	3.85	4.09	3.50	3.57	3.87	4.15	
Pb+	3.06	3.07	3.13	3.19	3.06	3.07	3.13	3.19	
Bi	4.28	4.32	4.57	4.82	4.29	4.33	4.61	4.93	
Bi+	3.10	3.11	3.22	3.34	3.10	3.11	3.22	3.34	
Ra+	2.01	2.01	2.03	2.06	2.01	2.01	2.03	2.06	

LITERATURE.

1. UREY, Ap. J. **49**, 1 (1924).
2. FERMI, Zs. f. Physik **26**, 54 (1924).
3. PLANCK, Annalen der Physik **75**, 673 (1924).
4. UNSÖLD, Zs. f. Ap. **24**, 355 (1948).
5. FÜCHTBAUER and collaborators, Zs. f. Physik **87**, 89 (1934); **89**, 63 (1934); **90**, 403 (1934).
6. FERMI, Il Nuovo Cimento, Nuova Serie, **11**, 157 (1934).
7. WEISSKOPF, Phys. Zs. **34**, 1 (1933); see also UNSÖLD, Physik der Sternatmosphären, p. 282 (1938).

Zoology. — *An electron-microscopical study of sperm IV.* (The sperm-tail of bull, horse and dog.) By L. H. BRETSCHNEIDER. (From the Zoological Laboratory, Utrecht, and the Department of Electron Microscopy, Delft.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of March 26, 1949.)

I. Introductory.

It was to be expected that a better insight into the structure of the tail of the spermatozoon could be obtained with the aid of electron-microscopic investigation than was previously possible by light-microscopy. To obtain a more or less satisfactory explanation of the complicated motorial automatism of the tail, it was necessary first to determine its morphological foundation. Our hitherto published communications (I, II, III) on this side of the question were confined to a description of the general structure of the tail; meanwhile, however, we have gained further knowledge of additional details, thanks to our being able to examine also the sperms of other animals besides the bull. Although the structural principle of the tail, so far as mammals are concerned, shows considerable mutual correspondence, our investigation nevertheless showed that a given structure can be analyzed far more easily in one species than in another, so that the facts obtained from the various species may be taken as mutually supplementary. One may pose the problem in the following way: "what is the structure of the mammalian sperm, as shown bij electron-microscopic examination?"

II. Technique.

As in our previous work, the investigation was carried out with the aid of the electron microscope of the Institute for Electron-Microscopy at Delft, at an emission voltage of between 80, 90, and 100 kv. For the technique we refer to our Communication No. III (L. H. BRETSCHNEIDER, 1949).

In addition to the techniques described therein, we also used in the present case a 5 % phenol solution, 10—20 drops of which, to 10 cm³ aq. dest., was added to the suspension of spermatozoa in aq. dest. or in buffer solution.

This substance has a markedly corrosive action on proteins.

III. The Neck region.

The neck region, the articulation between the middle piece and the base of the head consists essentially of a sort of wreath formed by a number of points of attachment of sub-fibrils. The fasciculation of these

sub-fibrils into at least 2 articular bundles must be regarded as a secondary phenomenon. The neck region is covered on the outside by a thick membrane originating in the tail-collar which, only in the neck region, is strengthened by spiral bands and runs along covering the middle piece as a smooth membrane.

The variations in size and arrangement of the different parts of the neck, to be observed repeatedly in our electron-microscopic pictures, are caused by artificial changes occurring during the necessary drying process in the vacuum. The neck, as articulation point, is a most vulnerable part even in the living sperm; it is possible, under certain conditions, for the head to break off at this place. Thus, for example, the tail and the head may during the drying process, attach themselves to the substrate over a relatively large area, so that, during dehydration and its attendant shrinking effect, there is a force pulling at the neck region on both ends. The result of this is that the dimensions of the neck region, which is thus stretched, become greater than they naturally are. Added to this is the fact that, during dehydration, the originally cylindric neck region gets flattened, and becomes much broader. Only after fixation — when, may-be, the proteins have become more resistant against stretching —, say, with OsO_4 , mercurochrome, Carnoy's mixture do the shape and size of the neck most conform to the natural state. But in this condition, by the compactness of the neck region, it becomes most difficult to get anything like an accurate picture of the various structural details. Thus, after previous fixation, the neck of the bull sperm had a length of $400 \text{ m}\mu$ and a width of $600 \text{ m}\mu$, whereas after maximum stretching and without previous fixation the length was $800 \text{ m}\mu$ and the width 1μ . In fact, the appearance of the neck region changes all according to the extent of these artificial modifications. This fact has led to differences of opinion, both in former light-microscopic research, but again in electron-microscopic investigations. The greater this flattening and deformation of the neck, the more details become visible, because it causes the enveloping membrane to tear open, whereby the fibrillar contents of the neck are spread out on a flat surface, whereas in the natural position, these structures cover one another and are closed in by the membrane. In the case of only slight flattening, we see two lateral bundles; when the flattening is more marked the number of fibrillar structures increases through spreading. Moreover, by proteolysis with bacterial enzymes, trypsin, chloramine-T or phenol (see technique in Part III), which act as solvents on all structures surrounding the fibrillar axis, we laid it bare as far as possible and so were better able to analyze it.

We also obtained further insight by comparing the spermatozoon of the bull with those of other species (horse and dog). It appears from the various data obtained that 8 out of the nine subfibrils of which the axis consists are each individually attached to the base of the head by a thickened insertion, as shown in fig. 1 of the horse's sperm. These 8

insertion points are arranged in the form of a wreath around the central fibril No. 9, which, in its turn, is attached to the centrosome (*vide* the diagram, L. H. BRETSCHNEIDER, 1949). For the sake of better comparison, we have numbered these insertion points from left to right. In the projection — i.e. as we view the spermatozoon, as a rule, from one of its flat sides — we observe three fasciae, covering one another, whereas, in the case of maximum broadening of the neck region, the fibrils arrange themselves side by side, so that a greater number are laid bare. Owing to lack of room near the narrow base of the head, the sub-fibrils join up already at their insertion points in sets of two, three or four, thus forming a common insertion basis with a form of its own. This fusion may be clearly seen in fig. 3 and 5. The common basis of such an articular fascia, seen from above, shows a horse-shoe-like shape; viewed laterally, it is claviform, and fits with its slightly dome-shaped top into a correspondingly pan-shaped hollow of the base of the head. Between the respective surfaces of the two, there is a probably proteinic binding substance, which, when strongly stretched, pulls out into threads. The united sub-fibrillae too appear to be connected by a binding substance, because, after proteolytic treatment, their common insertion base falls apart into clean, separate sub-fibrils.

The centre of the base of the head has a crater-shaped, sunken part and contains the centrosome, whence the sub-fibril No. 9 begins. The entire neck region is covered by a membrane, which, anteriad, closes over the distal half of the head, and posteriad over the middle piece; this membrane originates in the tail-collar. In this membrane, only in the part covering the neck region proper, there are, at regular distances, band-shaped thickenings, by way of strengthening structures. Around the wreath of fibrillae-insertions and against the base of the head, the 100—140 m μ wide "neck-ring" closes like a band. Just behind this neck-ring, a narrower band commences, of about 40—60 m μ , which runs in the form of a sinistrogyrate spiral as far as the beginning of the middle-piece, where it ends. We found, in the bull sperm, 10—12 windings (fig. 5); in the dog's sperm, 15—17 windings (fig. 6). The distance between the band-shaped windings is only \pm 20 m μ , generally even less. Especially shadow-casts show up these bands to be elevations that should be regarded as local thickenings of the neck-membrane. The blacker colouring in the pictures is caused by the greater thickness of the structures that cover each other. From the last winding of this band, the membrane runs like a smooth enclosure along the rest of the middle piece as far as JENSEN's ring, at which point the cortical spirals of the tail membrane begin. Our original assumption that the neck region itself contains no spirals, and that the cortical spirals run uninterruptedly from the middle piece to the end of the tail, was shown on closer examination to be erroneous. We believed at first that the deeper black shown by the spiral bodies on the photographs covered and concealed

the course followed by the cortical spirals, so that they only became visible again at the tail. But both after shadow-casting and in one particular case, in which, owing to a secondary breach in the middle piece, the membrane at this spot was laid bare (fig. 2), it was shown that this membrane contains no spirals whatever. In the figure mentioned, we see the membrane, within the small curvature of the breach, torn away from the deeper layer, and slightly curled up on both sides, but otherwise perfectly smooth and homogeneous. It is evident, from the greater material now at our disposal, that, so long as the neck-membrane is not stretched and remains in its natural state, the spiral bands in the neck region close around it quite smoothly (*vide* 1947, fig. 4). Bij stretching, either longitudinally or laterally or both, the membrane bursts, and, with that, the spiral band also breaks to pieces. When this happens, it appears that the fibrillar structures underneath and the neck-membrane are so firmly joined together that the spiral fragments *ad locum* still adhere to the sub-fibrils (Fig. 1 and 7). This phenomenon at first led us to assume that the sub-fibrils showed regular cross-striations (L. H. BRETSCHNEIDER, 1947). Another point to be noted is that, in the case of extreme stretching of the sub-fibrils in the neck region, those parts of the fibrils to which a spiral fragment adheres are not stretched so much as the parts in between, a phenomenon which causes the sub-fibrils to show, at regular distances, thickenings and narrowings (Fig. 1). (*Vide* also L. H. BRETSCHNEIDER and W. VAN ITERSON, 1947).

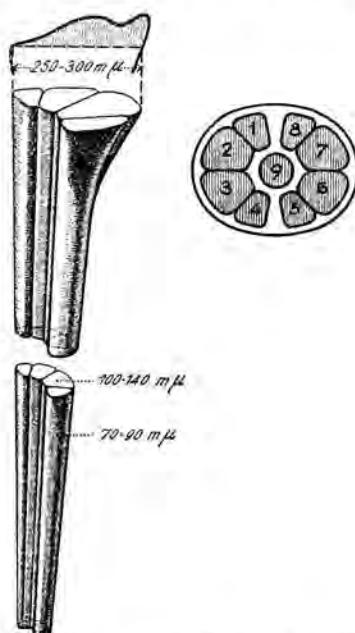
IV. *The axial filament.*

The axial filament, which forms the fibrillar axis of the whole of the tail, consists — in the spermatozoa of the three species named — of 9 sub-fibrils, which, in the middle piece, are of unequal thickness. We can distinguish 2—3 classes of thickness; not until they get to the tail proper do the fibrils become of equal thickness. The widest of the sub-fibrils show an ellipsoid cross-section; the thinner ones are round, while their thickness decreases from the anterior towards the posterior end. In the middle piece the fibrils group themselves into two articular fasciae and a central fibril; in the tail proper they gradually form more and more a common compact, rod-shaped axis.

This composition of the axial filament, i.e. of 9 sub-fibrils, is not clearly seen until the different components surrounding the axis are removed. To this end we used trypsin (technique 6), chloramine-T (technique 4), and phenol. After careful corrosion nothing but the axial filament remains, which divides up into the single sub-fibrils. In pathologically modified sperms, in which the axial filament is characterized bij unequal length of the sub-fibrils, the latter, thanks to their being more loosely interconnected, can be analyzed more easily (fig. 8).

After treatment with trypsin or chloramine-T, the head usually breaks off owing to the binding substance being dissolved, so that the tail is laid

bare for inspection up to the point of its insertion. By this treatment JENSEN's spiral bodies, the tail plasma and the cortical membrane are also dissolved, so that nothing remains but the axial filament. It appears from a reconstruction of three successive photographs of such an axial filament (fig. 3) that 4 of the sub-fibrils are thicker, whereas 5 are thinner. From other pictures we may conclude that each articular fascia at the beginning of the filament is composed of two thick and two thinner fibrils. Each time two thicker fibrils lie together; they are adjoined by two thinner ones, so that we get the arrangement shown in fig. A. If we



A. Articular strand and arrangement of the sub-fibrils in the middle piece.

project these on to the base of the head, and number the fibrils from left to right, they show a symmetrical picture in respect of the two transverse axes of the head. The size of the band-shaped sub-fibrils Nos. 2, 3, 6 and 7, immediately behind the neck, is $\pm 100-140 \text{ m}\mu$; at the beginning of the tail, 80—100 $\text{m}\mu$, and at the end of the tail, like the other sub-fibrils, 30—40 $\text{m}\mu$. The fibrils Nos. 1, 4, 5, 8 and 9 measure, behind the neck, 70—90 $\text{m}\mu$; at the beginning of the tail, 50—70 $\text{m}\mu$, and at the end again 30—40 $\text{m}\mu$. In some cases the central fibril (9) was already at the beginning of the tail, only 40—50 $\text{m}\mu$ thick, and again, at the end, 30—50 $\text{m}\mu$. From the arrangement of the fibrils in the dog's sperm (fig. 7) it follows that, in this case, the pairs of fibrils Nos. 1 and 8, and 4 and 5 each possess a common insertion, and are placed crosswise in respect of the two pairs 2 and 3, and 6 and 7. The independent position of fibril No. 9 is evident, among other things, from fig 8, taken

from a bull sperm, in which one of the articular fasciae (right) possesses longer sub-fibrils than the other (left). The difference in thickness, too, is plainly visible here. Whereas, in some cases, the tail has a somewhat twisted course, it may be seen from the fact that it gets thinner at the points where it is twisted that it is not cylindrical in shape, but slightly flattened in the same direction as the flattening of the head. In the middle piece the four fibrils that are united into one articular fascia are shifted slightly towards one side, so that there is a small space between the two articular fasciae. Viewed, therefore, from the flattened sides, a light zone is visible between the fasciae (fig. 4). On looking from the narrow edge, one sees the entire width of an articular fascia. Only after the four thicker fibrils, referred to above, become thinner does the space between also disappear, after which the 8 sub-fibrils run like a regularly formed cylinder around the central fibril No. 9.

The axial filament is enveloped in a plasma covering. In the middle piece the two spiral bodies of JENSEN are embedded in this plasma covering, while, in the tail, the plasma fills the space between the filament and the cortical membrane in a homogeneous layer.

V. Discussion.

On comparing our electron-microscopic findings in regard to the mammalian spermatozoon with the light-microscopic observations of previous authors, we shall be able either to rectify or to formulate more concretely certain statements. Thus, it has been customary — for understandable reasons, such as the extreme smallness of the structures to be investigated, whose sizes are in the neighbourhood of $0.1\text{ }\mu$ — to regard the thickened insertion base of the articular fasciae, light-microscopically, as granula. In the same way, the fragments of the spiral bands in the neck region, which fall asunder after dehydration, present the appearance, when seen light-microscopically, of more markedly stainable granula. Since the separate bands, as such, are invisible by light-microscopic means ($40-60\text{ m}\mu$), they can only be observed by this means, if a fairly large group of them are joined together. As early as 1886, BALLOWITZ described, in the two diverging articular fasciae, the insertion with the aid of a granulum which he took to be a divided centrosome. Later, MEVES (1899), in his classic investigation on the *Cavia* sperm, described three proximal and three distal granula in the neck region, which he considered to be derivatives of the proximal and distal centrosome, respectively. RETZIUS (1910) gives a number of illustrations showing, on the articular fasciae, and even on the sub-fibrils, three granula (centrosome derivatives), one behind the other; this shows that this eminent investigator already succeeded in distinguishing certain separate, adjoining bands in the neck region. Thanks to the theory of the identity of centrosomes and basal granula of cilia and flagella — formulated independently by both LENHOSSEK and HENNEGUY (1898), this

problem was also transferred to the granula in the neck of the sperm, which were regarded as belonging to the same class. Many an explanation relative to the movement of the tail was also connected up with the same theory. In spite of the fact that this theory has been abandoned long since, and that no indication has ever been found of the existence of a division of the centrosome into basal granula during spermiogenesis, the term "centrosome derivatives" to denote a series of neck region granula has maintained itself up to the present. No doubt we may conceive the thickened insertion bases of the sub-fibrils — correspondingly to the structure of similar motorial organella as vibrissae, cilia and flagella — as "basal granula". On comparing these organella with the sperm tail we find, indeed, an essential correspondence and only a difference in degree. According to KOLTZOFF (1906), ERHARD (1909), KLEIN (1926–32), the vibrissae of cells and the cilia of the Ciliata possess a central fibril by way of a fixed axis. BROWN (1945) found, for the flagella of Flagellates — electron-microscopically — two axial fibrils, while, already since BALLOWITZ (1886–1912), the structure of the axial filament in spermatozoa has been known to consist of several fibrils. Thus, we found, up to the present, 9 sub-fibrils in the bull, the horse, the dog *Myotis* and *Gallus*. HARVEY & ANDERSON (1943) found, in *Arbacia*, 9–10 sub-fibrils. BALLOWITZ found 9 sub-fibrils in *Fringilla*, *Copris*, *Chrysomela*, *Colothus*, and *Hydrophilus*. We ourselves found 18 sub-fibrils, i.e. a multiple of 9 — in *Cavia*. In cilia, flagella and the sperm tail this fibrillary axis is enveloped by a plasmatic enclosure forming a more or less firm membrane, which finishes before the end of the fibrillary axis, the latter finishing as a free terminal piece. Both in the case of the flagella of Flagellates and in that of the sperm tail, spiral fibrils run in the cortical membrane around the organellum (BROWN, 1945); while KLEIN (1929) also observed in the cilia of the Ciliates — with the aid of silver nitrate — rings placed at regular distances (although explaining these differently). In every case there is a basal granulum at the basis. All according to the size of the motorial organellum, we therefore find more or fewer fibrils in this axis, the plasmatic envelope being accordingly thicker or thinner, respectively, and the basal granulum either larger or smaller. We may say, therefore, that the differences between cilia, flagella and sperm tails are merely gradual ones.

Notwithstanding the fact that BALLOWITZ (1886–1912) and RETZIUS (1909–1912) already demonstrated the insertion of the articular fasciae immediately at the base of the head, we still continue to find — even in the most recent textbooks — in the schemata of the mammalian sperm, the insertion of the tail being placed only behind the "neck": MAXIMOV (1945); ROMEIS (1938); METZ (in Cowdry, 1934); BRANCA and VERNE (1942); DE GROOT (1944), amongst others. The neck is described as an envelope which is supposed to contain merely a clear "filling" or "connecting substance". The composition of the axial filament of

individual sub-fibrils, too, which was discovered as early as 1879 by JENSEN, and by BALLOWITZ in 1886, still continues to be known under the indifferent name of "involucrum". If there is anything at all that "envelops", then it is the plasma envelope surrounding the axial filament, not the sub-fibrils, which, on the contrary, constitute the central, fixed axis.

Following the above rectification of certain older results, we would briefly revert to our own observations, which too need some correction. As we intend to explain in more detail in subsequent communications, the beginning of the spiral body of JENSEN is situated, in different spermatozoa — also in those of the same species — at varying levels. In cases where it is not placed immediately at the beginning of the middle piece, we see that the 10—17 spiral bands of the neck region continue their course some way into the middle piece. This fact led us to assume, at first (1947), that these spiral bands run along the entire tail, including the middle piece, without interruption. We have rectified this erroneous supposition in the diagram in fig. B (Part. III). Further, the cross striation of the sub-fibrils, which we believed to have observed, turned out to have been suggested simply by an artificial interruption of the spiral bands in the neck region. It was probably a similar erroneous interpretation that must have led HARVEY & ANDERSON (1943) to affirm the presence of cross striations in the sperm tail of *Arbacia*. It was suggested electron-optically by the over-crossing of longitudinal fibrils and spiral fibrils.

VI. Summary.

With the aid of a renewed electron-microscopic investigation of three different mammalian sperms (bull, horse and dog), further structural details were described, supplementary to those previously published. By the application of proteolysis, or of phenol as a solvent, the axial filament was freed of all surrounding structures, thereby rendering it accessible to separate analysis. Comparative examination of the spermatozoa of different animal species facilitated the interpretation of structural details which could hardly be done justice to in the electron-optical investigation of a single species. A description is given of the insertion of the sub-fibrils with basal granula at the base of the head; their varying arrangement into articular fasciae; their thickness and their shape. A number of spiral bands giving firmness to the structure were convincingly observed in the membrane of the "neck", which bands, in contrast to our previous assumption, come to an end at the beginning of the middle piece. The middle piece itself is surrounded by a smooth membrane; the cortical spiral fibrils begin only at JENSEN's ring. The structures supposed by us to be "cross striations", at the beginning of the sub-fibrils, proved to be caused by artificial fragmentation of the band structures in the membrane of the neck region.

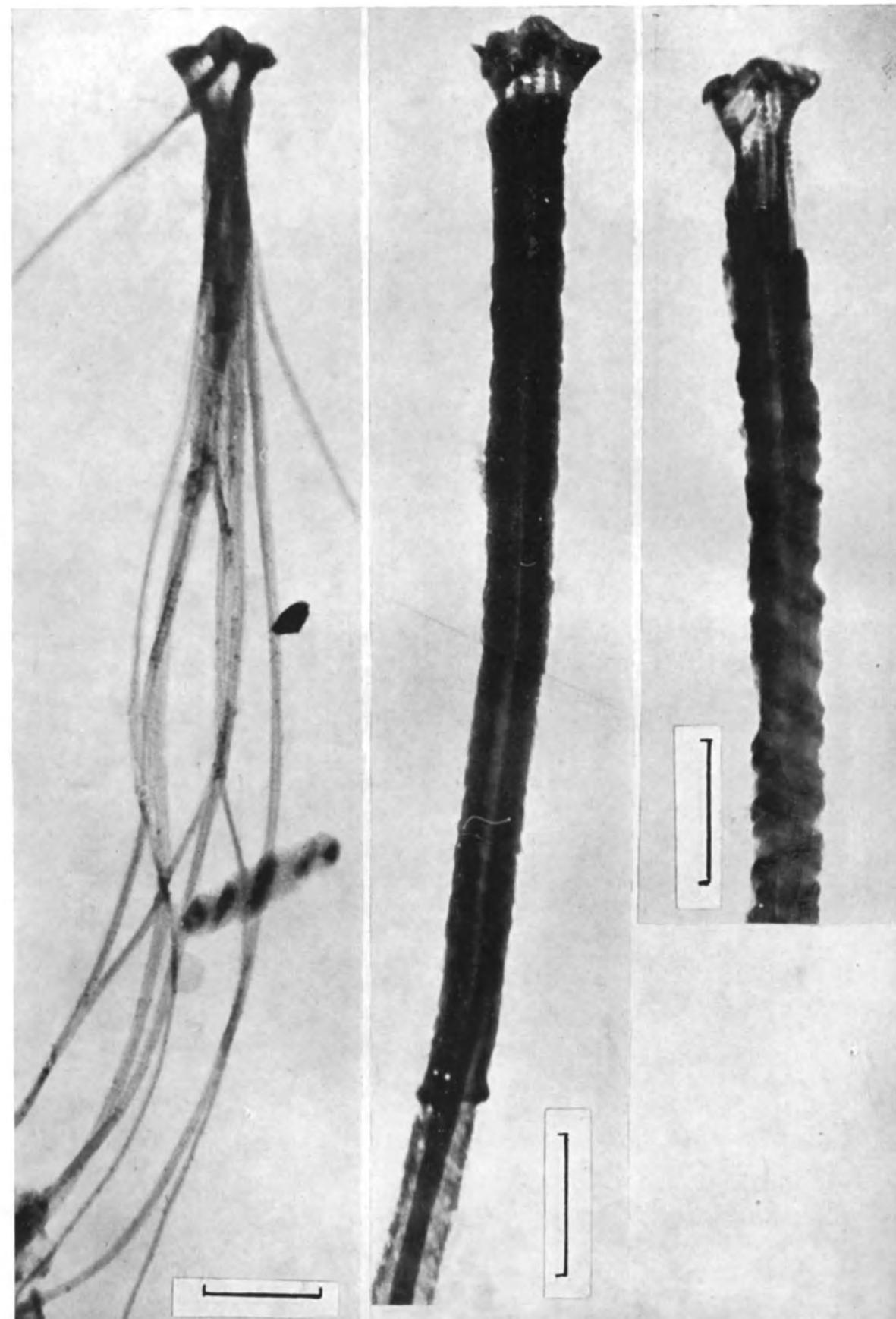
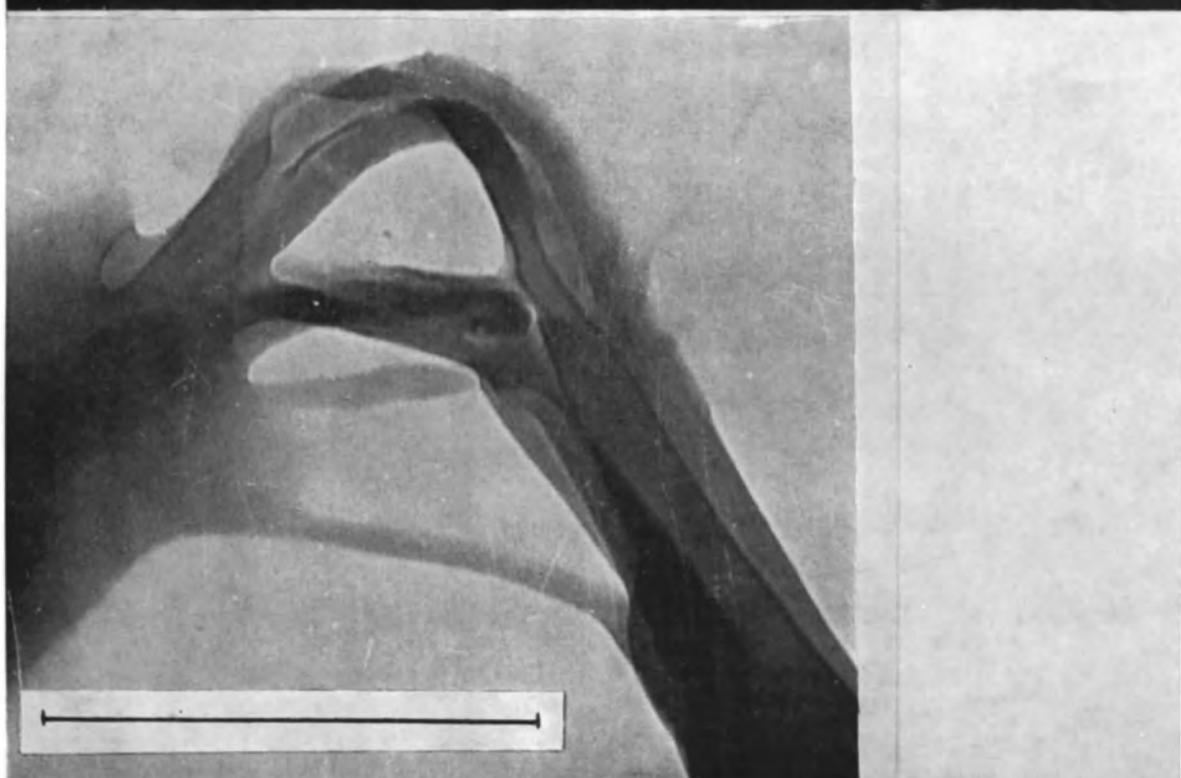
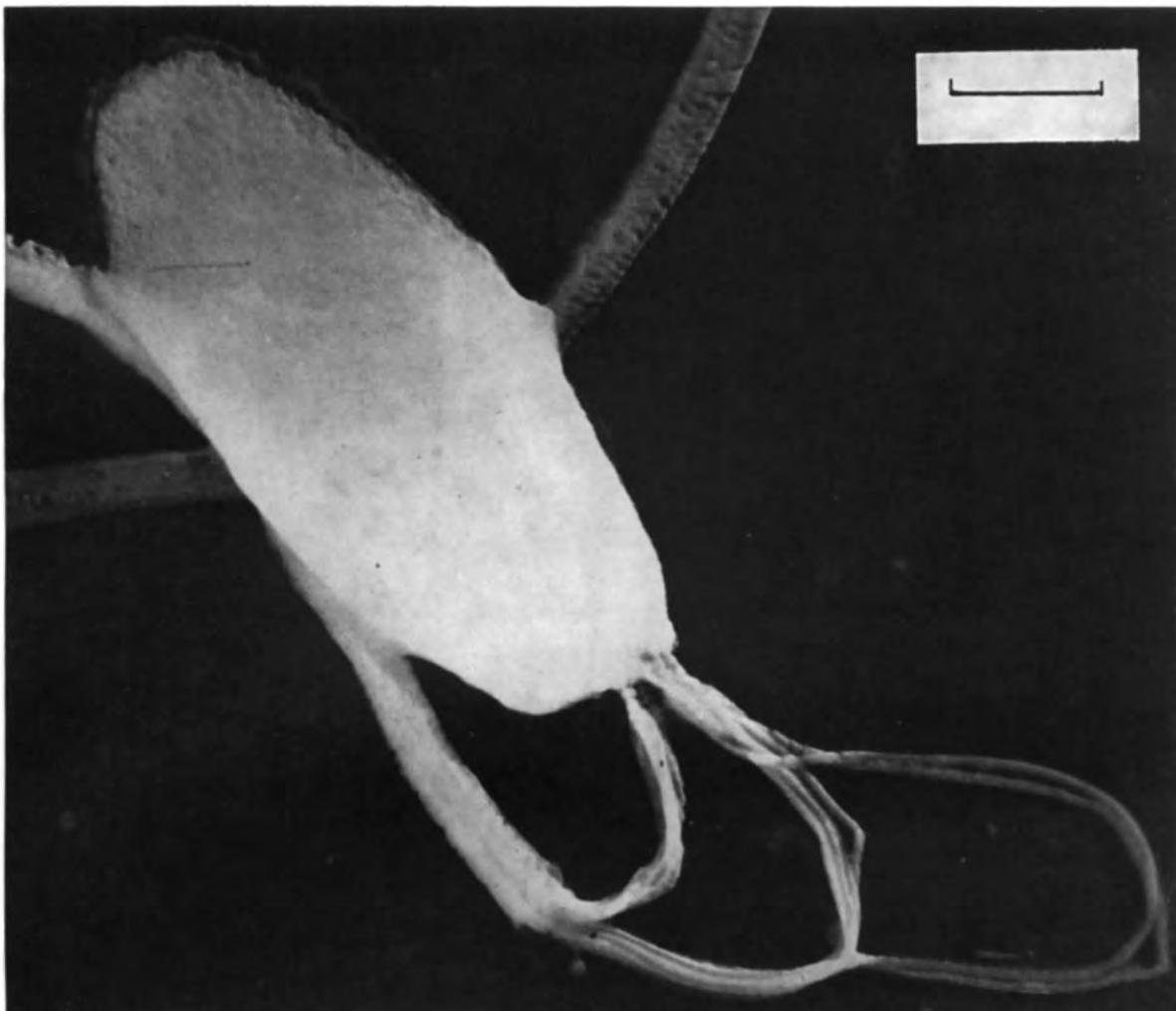
The writer desires to express his appreciation and gratitude to Miss W. VAN ITERSON and Dr A. L. HOUWINK for their assistance in the electron-microscopic documentation, and to the Landbouw-organisatie T.N.O. (Agricultural Organization for Applied Scientific Research), for the financial aid granted him for this research work.

LITERATURE.

- BALLOWITZ, E., Handwörterbuch d. Naturwissenschaften, 9 (1913).
 BRANCA, A. and J. VERNE, Precis d'Histologie. Masson, Paris (1942).
 BRETSCHNEIDER, L. H. and W. v. ITERSON, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 50 (1947).
 BRETSCHNEIDER, L. H., Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 52 (1949).
 BROWN, H. P., Ohio Journ. of Sci. 85 (1945).
 EIMER TH., Verhandl. physic. med. Gesellsch. Wuerzburg, 6 (1874).
 GROOT, A. DE, Leerboek bijzondere weefselleer, Oosthoek, Utrecht (1947).
 HARVEY E. B. and TH. F. ANDERSON, Biol. Bull. 85 (1943).
 HENNEGUY, Journ. d'Anat. et de la Phys. 27 (1891).
 JENSEN, O. S., Struktur der Samenfaeden, Bergen (1879).
 KLEIN, B. M., Arch. f. Protistenk., 65 (1929).
 KOLTZOFF, N. K., Biol. Zentralbl. 26 (1906).
 LENHOSSEK, M. v., Arch. f. mikr. Anat. 51 (1898).
 ——, Verhandl. Anat. Ges., Kiel (1898).
 MAXIMOW A. A. and W. BLOM, Textbook of Histology, Saunders, N. York (1930).
 METZ, CH. W. in E. V. COWDRY, Special Cytology, Hoeber, N. York (1934).
 MEVES, F., Arch. f. mikr. Anat. 54 (1899).
 RETZIUS, G., Biologische Untersuchungen, Stockholm (1881—1912).

EXPLANATION OF FIGURES.

- Fig. 1. Horse sperm with sub-fibrils after chloramine. Shadow cast.
 Orig. magn. 20.000 X.
- Fig. 2. Bull sperm. Fragment of middle piece with subfibrils and membrane.
 Orig. magn. 65.000 X.
- Fig. 3. Bull sperm. Axial filament after digestion with trypsin and impregnation with OsO₄. Orig. magn. 20.000 X.
- Fig. 4 and 5. Bull sperm. Articular strands and neck with spiral bands.
 Orig. magn. 25.000 X.
- Fig. 6 and 7. Dog sperm. Spiral band of the neck and sub-fibrils after phenol.
 Orig. magn. fig. 6: 23.000 X, fig. 7: 20.000 X.
- Fig. 8. Abnormal bull sperm with loose sub-fibrils. Orig. magn. 20.000 X.



— Indicates 1 μ in all pictures.

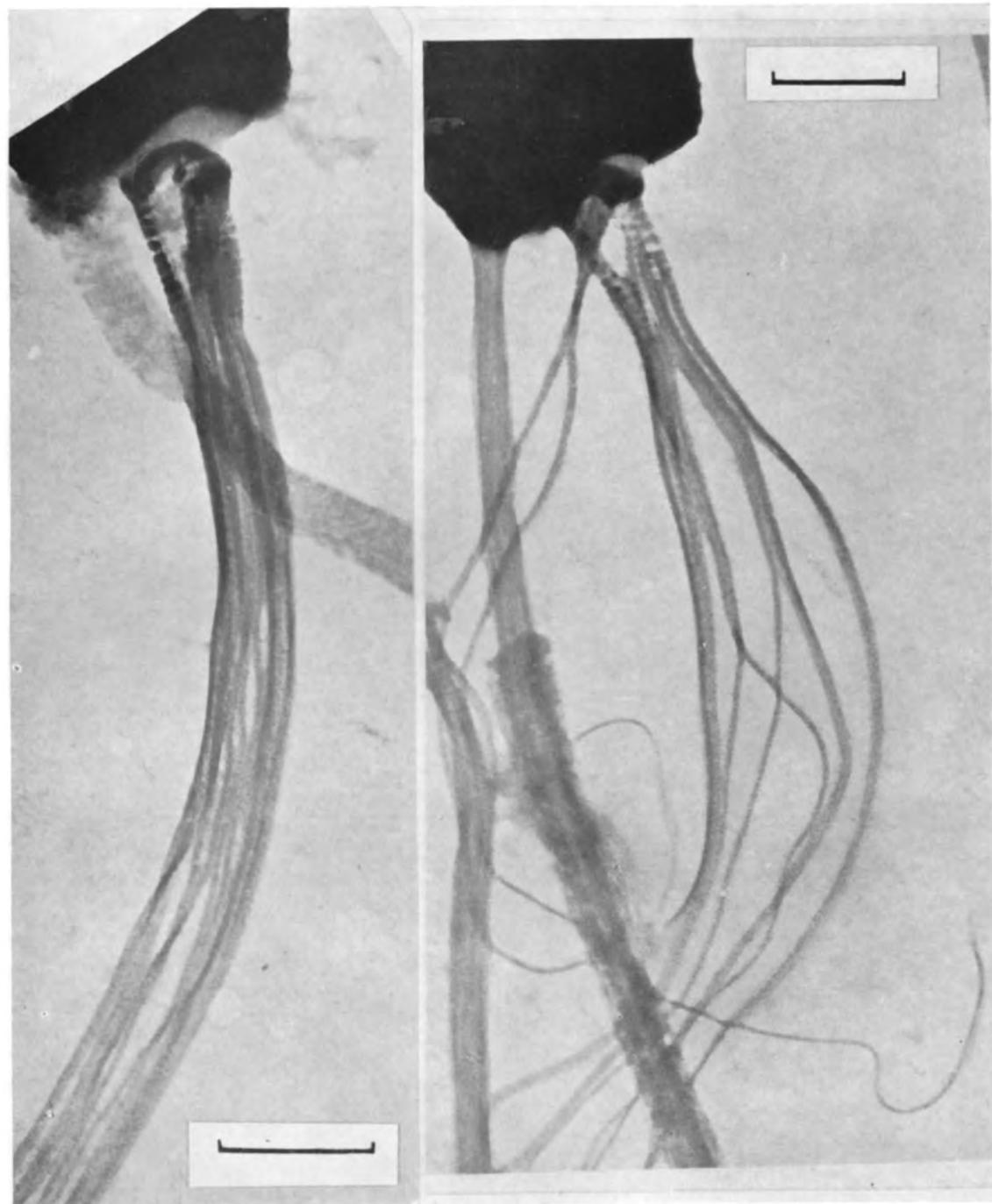


Fig. 6 and 7.

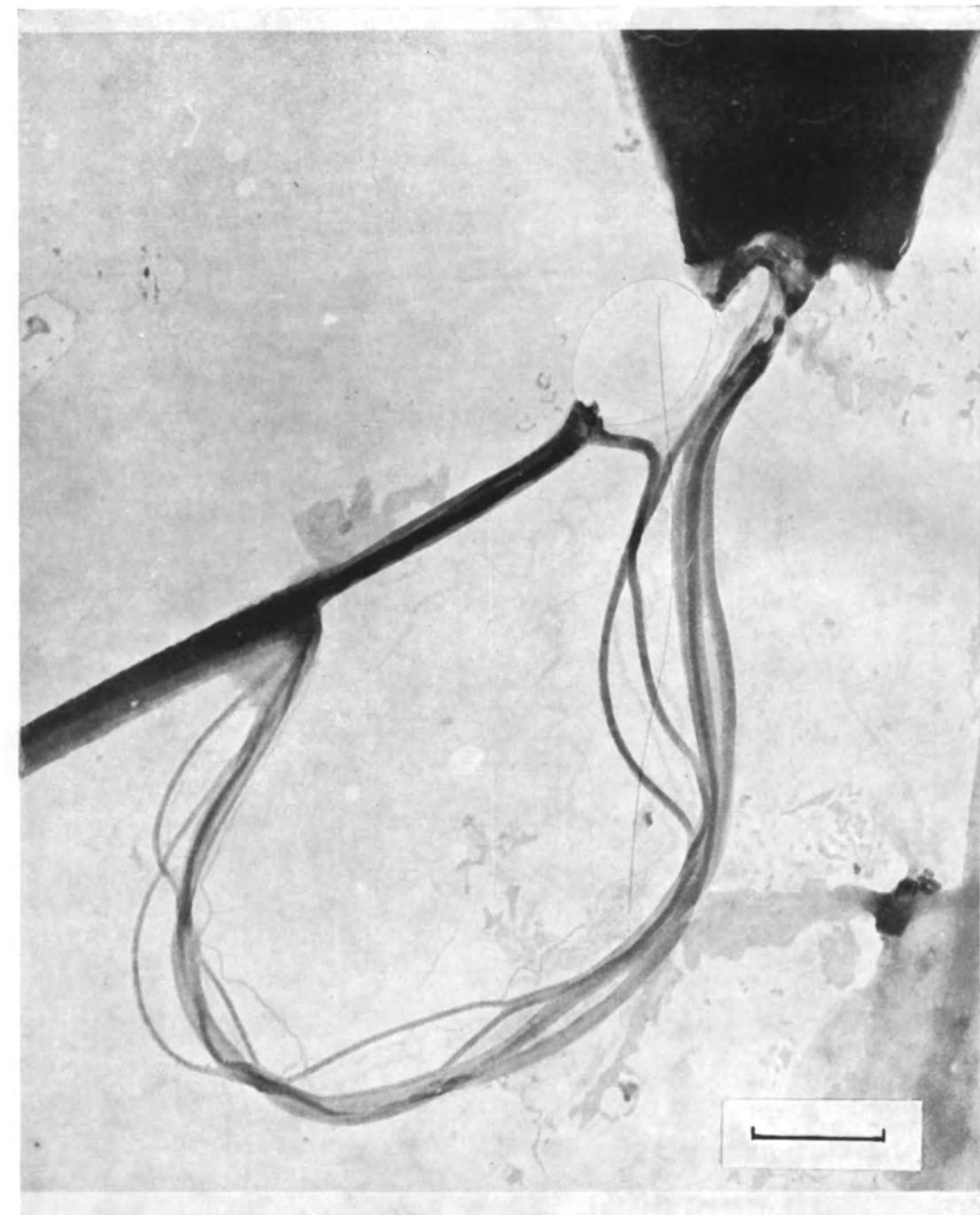


Fig. 8.

Zoology. — Spermiation in *Rana* and *Salamandra*. Preliminary note¹⁾.

By G. J. VAN OORDT, F. CREUTZBERG and N. SPRONK. (Zoological Laboratory, Dept. of Endocrinology, University of Utrecht.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of April 23, 1949.)

In 1929 HOUSSAY and LASCANO GONZALEZ showed that after transplantation of the anterior part of the mammalian pituitary in male specimens of *Bufo arenarum* sperms are released from the testes in a remarkably short time. This observation has been confirmed later by RUGH (1935) in *Rana* and by many others; from it GALLI-MAININI (1947) derived his pregnancy-test, for which many species of *Bufo*, as well as of *Rana*, may be used.

In a recent communication (VAN OORDT and KLOMP, 1946) we have introduced the term spermiation for the liberation of sperms from the testis, a term which is analogous to ovulation, the process of discharging the ova from the ovary.

The subject of the present investigation is a study of spermiation in representatives of Anurans and Urodeles. As experimental animals hibernating *Rana esculenta* and *temporaria* and *Salamandra salamandra* were used.

We have found on the one hand that spermiation can be provoked easily by gonadotrophins²⁾ in these species; on the other hand important differences were observed in the way in which this process takes place in Anurans and Urodeles respectively.

Rana spec. In intact *Rana*-specimens spermiation is accomplished easily after administration of gestyl, a pregnant mare serum gonadotrophin, or pregnyl, a pregnancy urine preparation. In a specimen of *Rana esculenta*, which has been injected with a dosis of 2–3 I.U. of pregnyl pro gram body-weight, sperms are already found in the cloacal urine after about 1 hour.

As we were also interested in the resumption of spermatogenesis in the testis tubules after spermiation, we have tried, by repeated injections, to get testis tubules totally free from sperm. However, in *Rana* it is extremely difficult to accomplish this phenomenon. After 9 doses of pregnyl had been administered in 22 days to a specimen of *Rana esculenta*, most of the testis-tubules were not yet empty (fig. 1).

In this species it was difficult to ascertain, whether a resumption of spermatogenesis took place after administration of such a large quantity

¹⁾ 26th communication of the "Werkgemeenschap voor Endocrinologie", part of the "National Council for Agricultural Research T.N.O."

²⁾ These preparations were kindly supplied by the Direction of Organon N.V., Oss.

of pregnyl, as control specimens may also show many spermatogenetic cysts in the tubules of the winter-testis. In *Rana temporaria* however, in which the testis-tubules contain only very few spermatogonia and numerous compact sperm-bundles in winter, the formation of spermatogenetic cysts and therefore the resumption of spermatogenesis was very distinct in several cases after 3 or more gestyl-injections (fig. 2). Of course this spermatogenesis may be provoked by the pituitary, as the experimental frogs were not hypophysectomized, but it seems more likely that the resumption is (directly?) caused by the injected gonadotrophins.

From the above follows that in *Rana* spec. there is a distinct, direct (?) influence of pregnyl on the testis-tubules, which results in the liberation of sperms from these tubules.

Salamandra salamandra. It is a well-known fact that in Urodeles the testis structure is very different from that of the Anurans. In the newts (*Triturus* spec.) as well as in the Spotted Salamander (*Salamandra salamandra*) the winter-testis possesses numerous tubules, filled with bundles of sperm, loosely attached to Sertoli-cells, and with very few spermatogonia and tubules, totally filled with germ cells in the early spermatogenetic stage (especially spermatogonia) which show very few or no cell-divisions. These tubules form two areas, which may be called the sperm- and the spermatogenetic part, with a transition-zone between them (fig. 3). The testis of *Salamandra* belongs to the multiple type; in every lobe a sperm- and a spermatogenetic part are present, whereas in the narrow connecting tissue only male primary sex cells are to be found.

Spermiation is very easily caused in the sperm-part. Already after 2 doses of 4—5 I.U. of gestyl pro gram body-weight (the second dose given 48 h. after the first) practically all sperms are released from the testis-tubules and 5 h. after the second injection the testis is totally devoid of sperms (fig. 4).

Moreover, it is interesting that in experimental as well as in control salamanders, killed by means of chloroform, sperm and mucus were present in the cloaca, even before they died, but that in the cloaca of control salamanders which had been decapitated, no sperm or mucus secreted by the cloacal glands could be found. Between the experimental animals and the controls, however, there was a conspicuous quantitative difference, which became especially distinct on microscopical investigation of the testes: control specimens which had been killed by chloroform showed little spermiation, whereas in gonadotrophin-treated animals spermiation was mostly total and the efferent ducts of the testis were packed with sperm.

After spermiation the following processes take place in the "empty" testis-tubules (fig. 4):

1. a distinct increase in size of the Sertoli cells, which pass over into large inflated cells, filling almost the whole diameter of the tubules, and
2. the formation of large cells, very similar to male primary sex-cells.

From the above it follows that spermiation can be caused by gonadotrophins in *Rana* as well as in *Salamandra*. The way in which spermiation takes place is very different, however: gradually in frogs, very rapidly in the salamander. Though we have given the salamanders a relatively larger quantity of gonadotrophic hormones than the frogs, we think that the differences in the way in which spermiation is accomplished may be explained with the help of the biology of the investigated animals:

Male frogs of most species generally possess large seminal vesicles in which sperm is stored before and during the breeding season. Therefore it is not necessary that sperms are liberated from the testis tubules at once, and we may assume that under a rather prolonged but weak influence of the anterior lobe of the pituitary, spermiation takes place gradually in spring. In the salamander, however, the formation of spermatophores, consisting of a large quantity of sperm and of mucus, secreted by the cloacal wall, is a process which must take place rapidly; hence, in the reproductive period the sperms are suddenly released from the testis in one large mass, presumably under the influence of a large amount of hypophyseal gonadotrophin. Therefore total spermiation can only be provoked experimentally by many repeated injections of gonadotrophins in *Rana* and by already 2 consecutive injections of these hormones in *Salamandra*.

Summary.

Spermiation, i.e. the process of liberation of sperms from the testis, can be provoked easily in *Rana* and *Salamandra* by means of gestyl as well as pregnyl. Consequently both gonadotrophins have a distinct (direct?) influence on the testis-tubules.

In *Rana* it is almost impossible to get total spermiation: after more than 9 dosages of pregnyl administered in 22 days, most of the testis-tubules of a specimen of *Rana esculenta* still possessed sperms. Resumption of spermatogenesis after gonadotrophin-injections was especially distinct in *Rana temporaria*.

On the other hand spermiation is easily accomplished in *Salamandra salamandra*; all sperms are released from the sperm-part of the Salamander-testis after 2 dosages, the second being given 48 h. after the first.

The differences in the way in which spermiation is accomplished in these Amphibians are explained with the help of the biology of the investigated animals.

REFERENCES.

- GALLI-MAININI, C., J. Clin. Endocrin. 7 (1947).
- HOUSSAY, B. A. & J. M. LASCANO-GONZALEZ, Rev. Soc. Argent. Biol. 5 (1929).
- OORDT, G. J. VAN and H. KLOMP, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49 (1946).
- RUGH, R. R., Proc. Soc. exp. Biol. a. Med. 36 (1937).

DESCRIPTION OF PLATE.

- Fig. 1. Spermiation of *Rana esculenta* after 9 doses of 2—3 I.U. of pregnyl pro gram body-weight, administered in 22 days. Large quantities of sperm are still present in the testis-tubules.
- Fig. 2. *Rana temporaria*. Distinct resumption of spermatogenesis after 3 injections of 2—3 I.U. of gestyl pro gram body-weight.
- Fig. 3. *Salamandra salamandra*. Section through testis of a control specimen. At the upper side of the figure the spermatogenetic, at the bottom-side the sperm-part.
- Fig. 4. Spermiation of *Salamandra salamandra* after 2 injections of 4—5 I.U. of gestyl pro gram body-weight. In the tubules of the sperm-part only enlarged Sertoli-cells and a few male primary sex cells are present; no sperms.

G. J. VAN OORDT, F. CREUTZERG and N. SPRONK: *Spermiation in Rana and Salamandra.*

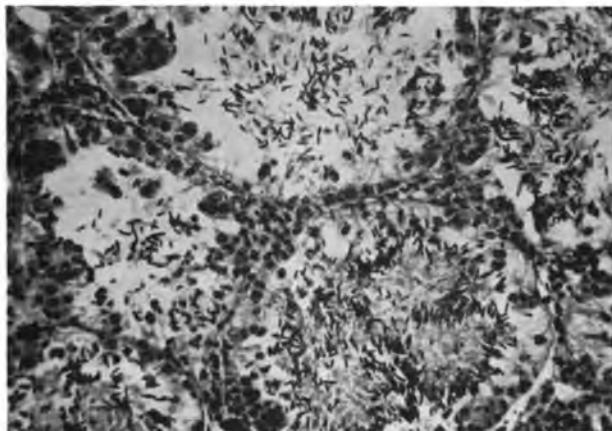


Fig. 1.

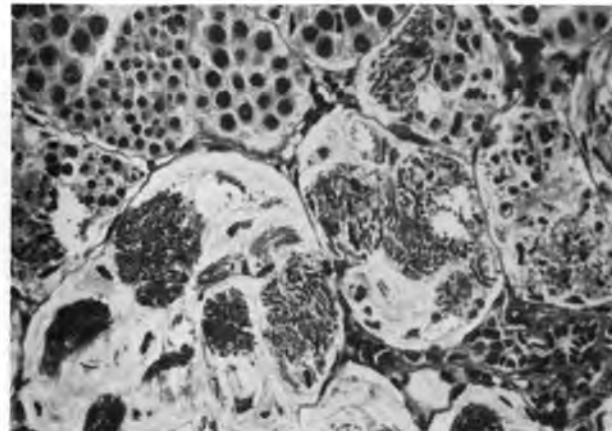


Fig. 3.



Fig. 2.

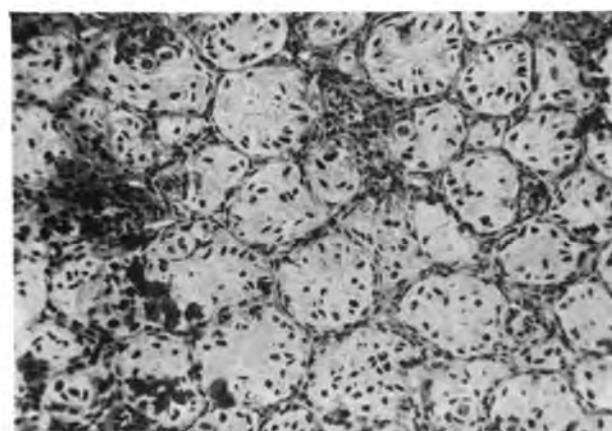


Fig. 4.

Geology. — Tectonics of the Mt. Aigoual pluton in the southeastern Cevennes, France. Part II. By D. DE WAARD. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of March 26, 1949.)

11. *Tectonics of the slaty country rock.*

In the slates cleavage planes have been measured in radial strips as far as 10 km from the pluton and round about the contact. The poles of the cleavage planes of the measurements not too close to the contact in the whole area are plotted in tectonogram fig. 5. The local density with close

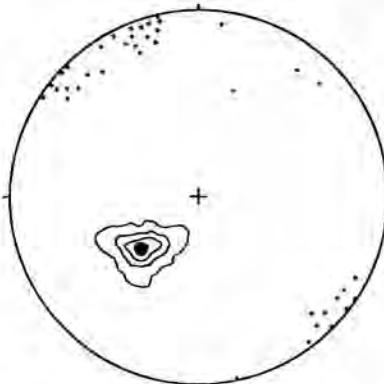


Fig. 5. Pole diagram (55 poles) of the slaty cleavage except near the granite contact. Contours 45—27—18—9 %. Crosses indicate the axes of ribs and minor folds.

contour intervals points to a very regular strike and dip of the slaty cleavage with an average of about N 138° E, 34° NE.

Generally, cleavage planes in slates have been developed without being the direct consequence of stratification. Locally however cleavage planes may be found parallel to the original bedding. The original bedding in the studied area is usually obliterated. In some localities hard beds of quartzite have been found parallel to the slaty cleavage, indicating a local parallelism of primary and secondary structures. Elsewhere microscopic examination of slates brings out that ordinary flat cleavage planes intersect small folds in the original stratification.

Slaty cleavage, being a better developed kind of fracture cleavage, is usually ascribed to shearing action of tectonic movements. The rock material gives way to the stress by differential movement along shearing planes. The direction of motion of the tectonic unit of which the Mt. Aigoual region is a small part, must thus be found within the plane of cleavage. In the same way ribs or minor folds and most of the joint systems

in the slates have been formed during the orogeny. They give additional clues about the direction of orogenic motion.

Minor folds are usually well developed in the southern part of the area. In the northern area an analogous phenomenon is to be seen in small straight ribs of about 2 mm wave length in the cleavage planes. Connection between ribs and small folds have been found as transitions and combinations of both forms in the field and after plotting the axes of folds and ribs in the diagram, fig. 5, they clearly exhibit uniformity of directions.

The average trend of these folds (N 140 E) is about the strike of the slates, most of them pitching a few degrees north-west. In some cases a second system perpendicular to the first is developed. Their axes are plotted in the north-east section of the diagram.

As folds usually originate by compression of material, the compression or motion of the strata in the Mt. Aigoual area must have taken place in NW—SE direction. Small folds and ribs however may be interpreted as drag folds. In that case the pitch of the drag folds is about parallel to the pitch of the major structures, being here nearly horizontal. Together with the relationship of cleavage planes to tectonic motion the direction in which the motion occurred is found, viz. perpendicular to the minor folds and within the cleavage planes, resulting in a direction of about NE—SW, dipping NE. Whereas French geologists consider the southern part of the Central Massif to be originated by southward motion, the foregoing conclusions with respect to the Mt. Aigoual area point to a tectonic motion directed about 30° upward to the S.W. with a folding axis slightly dipping to the N.W.

Most joints in slates are to be considered too as caused by tectonic motion. Everywhere in the Mt. Aigoual region slates are fractured in at

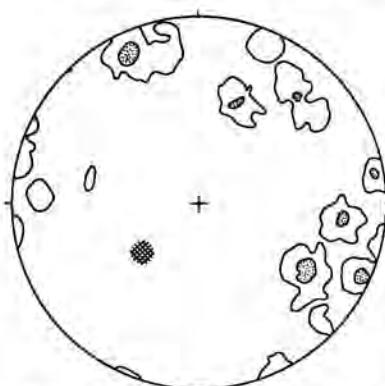


Fig. 6. Pole diagram (72 poles) of joint systems in slates. Contours 5½—3 %. Hatched area indicates the average of schistosity of fig. 5.

least two systems of joints. A combination of all measured joints in slates not too close to the contact is shown in fig. 6. The poles of the joint planes appear to form densities in a broad girdle whose axis is the average

cleavage of the slate. This indicates a preponderant jointing perpendicular to the cleavage plane.

In this girdle dominant densities of poles are broadly concentrated in a NW—SE direction, which indicates a preponderance of joints perpendicular to both cleavage and minor folds and they may thus be considered as tension joints. The smaller concentration of poles in the NE corner of the diagram — being joints in the direction of the minor folds — may be seen as the less developed set of the two shearing systems of which the slaty cleavage is by far the most important.

In fig. 7 the plane of the diagram is made the plane of the average slaty cleavage by rotation of this average to the centre of the diagram. The concentration of perpendicular joints is now clearly shown as a peripheral girdle. Addition of the rotated axes of ribs and minor folds of

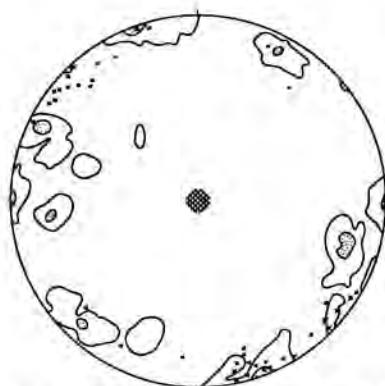


Fig. 7. Same diagram as fig. 6 after rotation of the average of schistosity to the centre of the diagram. Crosses are the rotated axes of ribs and minor folds of fig. 5.

fig. 5 in the same tectonogram gives the relation between the directions of the ribs and folds and the NW—SE densities of joints which are considered as to be caused by tension.

The above mentioned features of the slates are combined in block diagram fig. 8. It shows in outline the relation of joints and folds to each

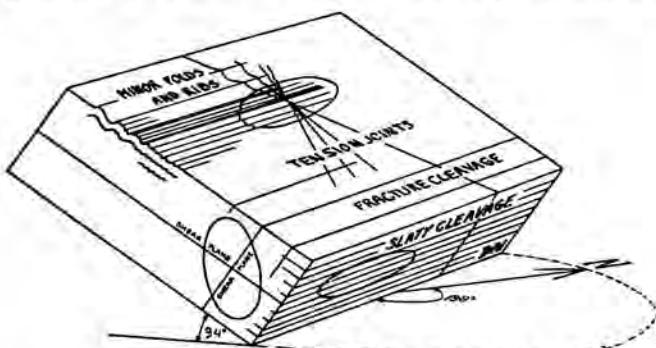


Fig. 8. Block diagram of the tectonic features in the slates in relation with the strain ellipsoid.

other and to the strain ellipsoid of which sections are drawn on the sides of the diagram.

12. Tectonics of the country rock near the contact.

Besides a normal small variability in strike and dip of the slaty cleavage, big divergences have been measured near the contacts of the granite. The structure map (in: DE WAARD, 1949) shows these variations especially in the country rock near the northern and eastern boundary of the pluton. The other contacts too, nearly always have divergences in the strikes of the slates. As a whole these divergences are found in a zone up to 500 meters round about the pluton. Strikes in this zone are systematically bent to the ENE. As much as 70° divergence has been observed. Bending of the strike in this zone is accompanied by alteration in dip. Usually dips near the contact are much stronger. The normal dip of the slates of about 30° tot 35° increases to 40° and 45° and sometimes over 60° towards the granite contact.

These divergences are plotted in tectonogram fig. 9. The poles of the

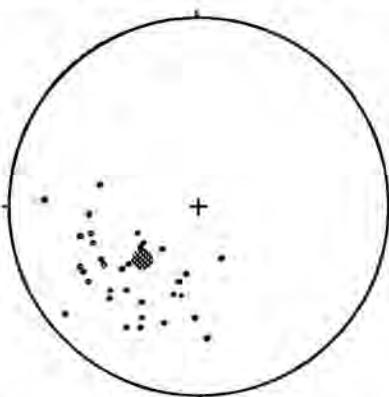


Fig. 9. Pole diagram of the slaty cleavage in the neighbourhood of the granite contact. Dots indicate data in the northern and northwestern, circles in the southern and southeastern contacts. The average of the normal schistosity is shown by the hatched area.

slaty cleavage of the country rock are shown together with the average of the normal schistosity (hatched spot). They occupy a much larger area in the diagram than those of the slates farther away from the contact. Variations of about 60° at both sides of the average occur and most dips are considerable steeper. In the diagram a separation has been made between the country rock NW and SE of a plane perpendicular to the average strike of the slates through the centre of the pluton. North and west of this plane strikes are mostly turned anti clockwise (dots) and in the southeastern part in clockwise direction (circles) in the same measure and both with increase of dip.

Clearly these phenomena have been caused by the intrusive movement during the *mise en place* of the granite. The upward movement of the

granite mass pushed upward the slates close to its contacts. Lifting up strata with a certain strike and dip can mean alteration in dip as well as in strike.

This is illustrated in the theoretical block diagram fig. 10. In the case

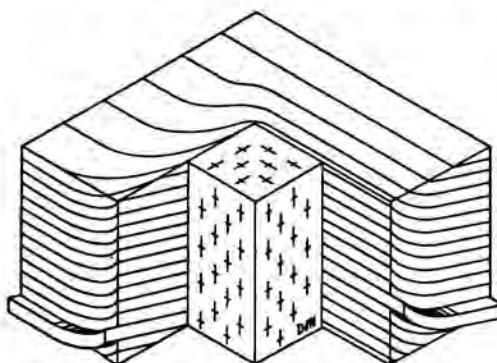


Fig. 10. Block diagram showing the theoretical alteration in strike and dip of the slates by upward bending at the contacts of the pluton, parallel and at right angles to the strike of the slaty cleavage.

in question the uplift of the slates in the country rock is shown in the increase in dip as well as in bending of its strike to the NE. These phenomena may thus be seen as large drag features. They are one more evidence of the intrusive character of the pluton.

13. *Faults and dikes in the country rock and the contact planes of the pluton.*

A glance at the structure map shows striking regularities in the directions of faults, dikes and contact planes of the pluton. This is indicative of a similar age and origin of these different features.

As mentioned in paragraph 3 faults have not been found cutting granite dikes. They are younger however than quartz veins and the development of schistosity and joint systems in the slates. In one locality drag features are observed in the adjoining country rock of a granite dike, but this drag is caused before the consolidation of the dike. Thus faults must have been active during the intrusive motion of the granite. This age determination of course does not imply any denial of the possibility of faults having been formed along existing fractures in the slate.

All observed faults, dikes and contact planes in the mapped area are compiled in tectonogram fig. 11. Besides some scattered pole axes two areas of increased density are shown. Most of the pole axes are concentrated in the SE corner of the diagram. A second and smaller concentration is visible in the middle of the SW quadrant. Compared with the diagram of the joint systems in the slates of fig. 6, the largest concentration coincides with the area of joints in the SE corner of this diagram which is interpreted as the area of tension joints in the slates. The smaller

concentration in the SW quadrant is the area of the slaty cleavage in fig. 5. The great majority of faults, dikes and contact planes thus may be considered as occurring along existing joint systems in the slates.

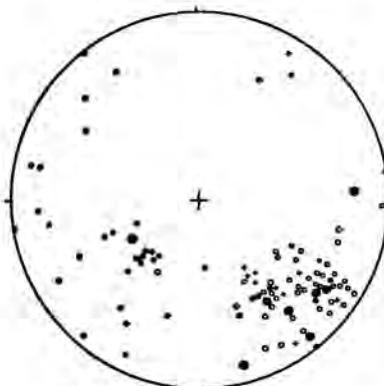


Fig. 11. Pole diagram of faults (crosses), granitic dikes (circles) and lamprophyre dikes (dots) in the country rock and the contact planes (double circles) of the pluton.

Most of the contact planes of the pluton occupy the area of tension joints in the diagram. Only in one case the cleavage plane of the slate has been used. More than 80 % of all granite dikes penetrated tension joints. Few of them intruded along the slaty cleavage. Most lamprophyre dikes preferred the cleavage planes of the slates. About 30 % penetrated tension joints. There may be a relation between the usually far away intrusions of lamprophyre dikes and their preference of penetration in the slaty cleavage. Faults occur in the joint system areas as well as the secondary fracture cleavage areas in the NE and SW margins of the diagram.

As appears from the structure map faults are arranged more or less parallel to the nearest granite contact. If drag features have been observed uplift proved to be at the side of the contact and downthrow at the other. The drag features indicate in this way an uplift of the country rock round about the pluton.

The faults nearest to the granite are shown in outline in fig. 12. The dip of all faults is rather steep; less than 50° has not been found. In the north faults are dipping N. All except one have downthrow to northern directions. The exception near the NE contact is presumably connected with the peninsula of slates in the pluton and has in this way only local significance. East and west of the pluton faults dip away from the contact except two in the south-east of the map which are very steep.

As mentioned above, movements along these faults must have taken place somewhat before or during the intrusion of the granite. It is obvious that these movements are in causal relation with the intrusive motion of the granite. The intrusive forces must have arched the country rock which caused tension in the slates and penetration of the granite mass along

D. DE WAARD: *Tectonics of the Mt. Aigoual pluton in the southeastern Cévennes, France.*



Photo 1. Scenery in the northern part of the Mt. Aigoual region. The reticular pattern in the vegetation of this hill is caused by erosion differences on its surface as a result of dikes and faults in the slates.



Photo 2. Ordinary type of the Mt. Aigoual granite with oriented phenocrysts of orthoclase.



Photo 3. Joint systems in the granite.



Photo 4. Example of joint systems in the slates.

open joints. This was accompanied by movements of blocks of rock along joints of the slates which effected uplift of the country rock in the neighbourhood of the intrusion.

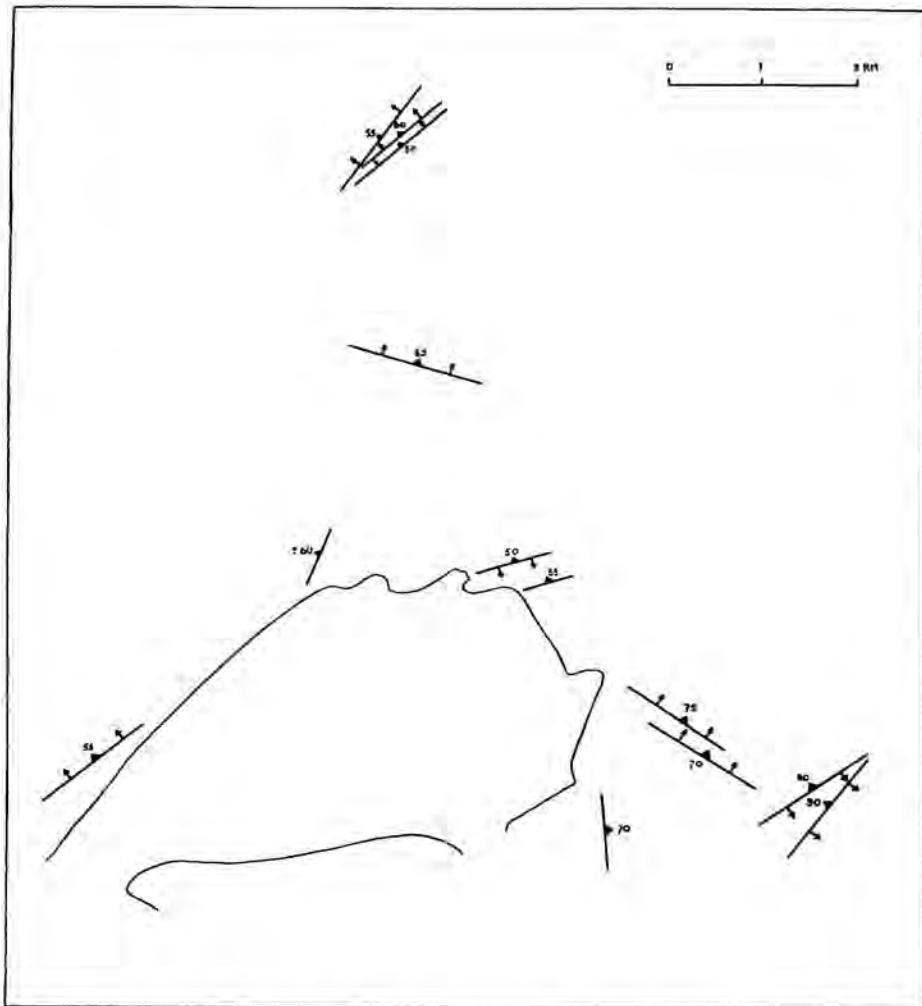


Fig. 12. Outline map of all faults near the pluton. Observed drag features are shown by arrows; the direction of the arrows indicates the downthrow side of the fault.

14. *The mise en place of the granite.*

In the preceding paragraphs all observed structural phenomena of the Mt. Aigoual pluton and its country rock have been discussed successively. They are subordinate parts in a complicated occurrence, each giving its own piece of evidence. Taken together they tell a rather complete story of an important geological event, the story of the emplacement of the granite.

The history of this area starts with Palaeozoic — probably Cambrian —

sedimentation of mainly argillaceous material, exudation of quartz veins and folding probably in the sudetic phase of the variscan orogeny. The latter created slates, locally phyllite and quarzite, and developed ribs, minor folds, slaty cleavage, tension joints and secondary fracture cleavage, indicating the stress effects on the rock in this part of the orogeny.

In this rock the granite mass occupied room. The substance cannot have been rich in easily volatilized constituents and the temperature has not been very high; the contact metamorphic zone is poorly developed, no assimilation features have been found at the contacts, acid dikes and veins are scarce. Nor can it have been very liquid during its *mise en place*; oriented crystals compose flow structures which indicate the presence already of a great part of the crystals. The occupied room must have been relatively high in the earth's crust, which is shown by the straight and sharp contacts of the pluton and the faulting features.

Thus the substance — magma *sensu lato* or better "mush" — has been a viscous flowing, partly crystallized mass of relatively low temperature and poor in easily volatilized constituents in a relatively shallow depth in the earth's crust. The mush consolidated slowly within the pluton into porphyritic granite and somewhat quicker in most dikes into granite porphyry having the same composition of quartz and feldspar phenocrysts in a matrix with different coarseness of grain. Somewhere inside the pluton body differentiation processes must have taken place and acid and basic dikes were formed. The source of the lamprophyre dikes has probably been deeper in the pluton which coincides perhaps with their relatively great distance from the plutonic outcrop.

The mush flowed upward as appears from the dome-shaped flow structures. It forced its way upward according to the upward bended slates near the contact and it arched even the intruded area which became evident by the uplifted blocks of country rock round the contact. The structural habitus of the pluton may be summarized according to the nomenclatures of HANS CLOOS (1928) and BALK (1937) as follows. A small, oval, mainly periclinal, autonomous pluton of porphyritic granite with flow facies in at least two domes of flow lines and layers, is discordantly and parallel to joint systems, mainly conformably, disharmoniously and posttectonic intruded as a nuclear pluton in the variscan orogen.

As mentioned in paragraph 8 the flow structures of the pluton seem to be incomplete. The structures in the SE of the outcrop are part of a dome of flow layers of which the other part must be hidden SE of the pluton border. Inconformable structures in the south and the east of the outcrop give the impression of continuation of internal structures under the adjoining country rock. The drawn-out flow structures and the enormous dike swarms north and east of the pluton outcrop also seem to point to a hidden extension of the pluton body.

All mentioned phenomena indicate a larger pluton than the exposed

outcrop not far below the present surface. Most of the adjoining country rock must thus be part of the roof of the pluton. Upthrown blocks of slates, faulted or separated along pre-existing fractures form part of this roof. The granite must have forced its way upward, arched the roof as a whole and made room by further lifting up parts of the roof and by penetrating its fractures.

This is illustrated by the series of sections in block diagram fig. 13. Faults and dikes border blocks of walls and roof. Their bottom is the cleavage plane of the slates. The NW contact is steep; continuation of the pluton is drawn to the north and east. North of the outcrop the granite body must be very near. The numerous dikes border floating blocks of slates.

15. Comparative observations in adjacent massifs of the Cévennes.

By a rapid survey in 1948 comparative studies have been done in surrounding massifs (fig. 1). South of the Mt. Aigoual pluton is the much larger St. Guiral pluton which is possibly connected with the Mt. Aigoual pluton by the bottle-neck in the SW of the latter. From the St. Guiral pluton a massif — about 30 km long — stretches away to the east, dilating in the Mt. du Liron massif. North of the Mt. Aigoual is the Mt. Lozère massif, the largest massif of the Cévennes region. The mentioned massifs form together an interrupted horseshoe including a large area of slates open to the NE.

Preponderant resemblances in structure and petrology have been observed in the visited areas. The massifs consist mainly of identical grey-coloured porphyritic granite with large phenocrysts of orthoclase. In the Mt. Lozère massif and to a less extent in the Mt. du Liron massif the phenocrysts may be missing locally, the type of rock in all other aspects remaining the same. In the Mt. du Liron massif masses of aplitic granite have been found. It is the same type of granite but without phenocrysts and with less biotite often passing into normal aplite. As near the Mt. Aigoual pluton, dikes of granite porphyry, quartz porphyry, tonalite porphyry, aplite, pegmatite and lamprophyre occur. Flow structures, sharp and straight contacts with narrow and feebly developed contact zones are also typical in these massifs. Their posttectonic and intrusive character are equally evident.

This wide-spread homogeneity, petrological as well as structural, in the massifs of the Cévennes points to a same source of the material which intruded after the orogenic movements into different places in the mainly slaty country rock. Of less importance — but mentioned because of petrological resemblances — is at least one small outcrop of augen gneiss in the neighbourhood of one of the granite contacts. There is no relationship however between the posttectonic granite and the much older augen gneiss. The latter has been mapped in detail and it turned out to be a small massif of orthogneiss, intruded as well but much earlier. There may be some relation in the source of both rocks.

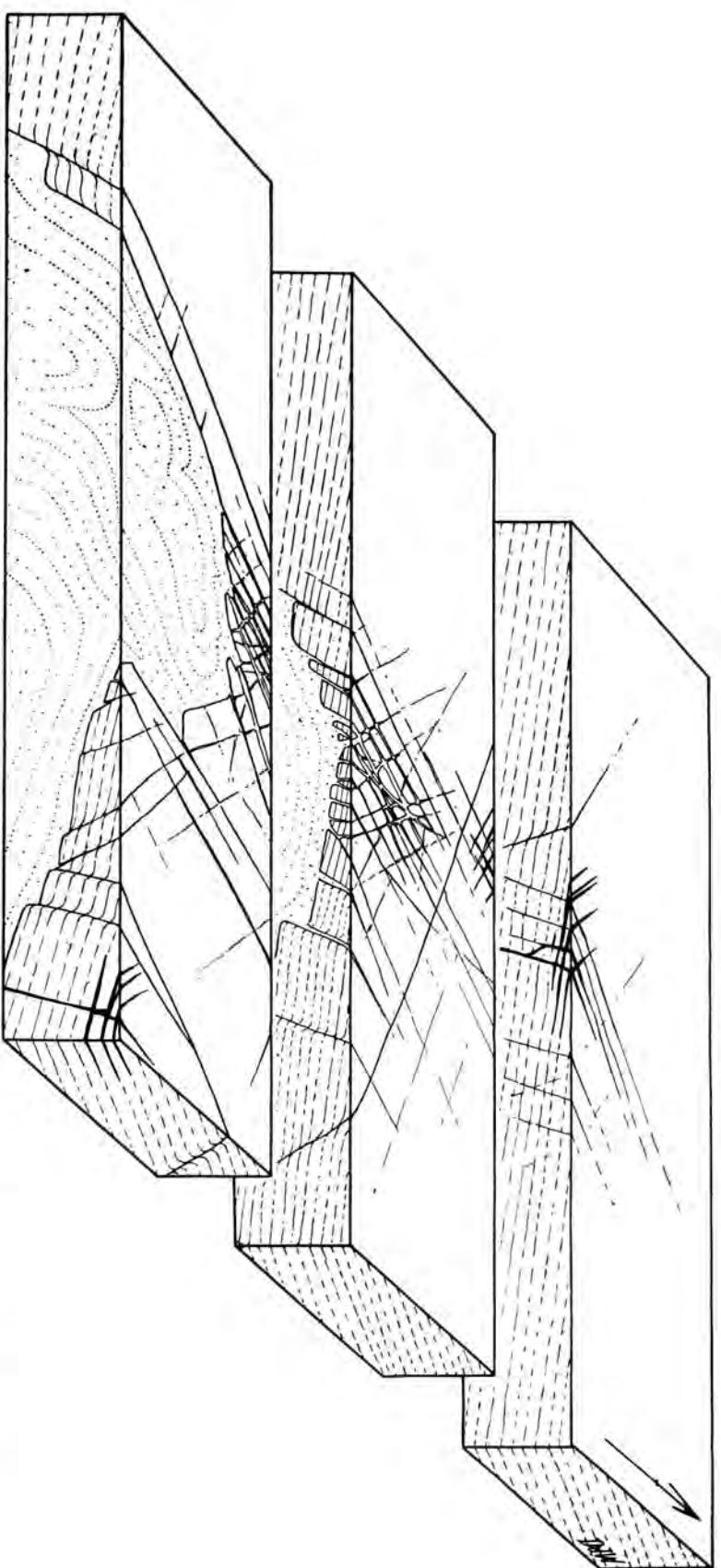


Fig. 13. Block diagram of the northern part of the mapped area showing the mean structural phenomena and their continuation into the depth.

16. *Concluding remarks in relation with granite problems.*

For a long time the mechanism of intrusion has been a matter of discussion. Stoping, enlargement of a magmatic chamber by the breaking off of blocks from the walls and roof is opposed by organized upward flow of the magma through the crust and doming of its roof. In the case in question there is nothing found supporting stoping. On the contrary all features point to a doming mechanism of intrusion. Domes of flow structures, slates dragged upward near the contacts, dragged faults indicating raised blocks of rock near the pluton and long-distance faults showing arching of the roof give evidence of a forcing upward of the magma and enlargement of the magmatic chamber by doming of the roof.

During recent years the discussions concerning the origin of granite have increased considerably. The purely magmatic origin of granite is opposed by migmatization, refusion and metamorphic diffusion. The structure of the Mt. Aigoual pluton and as far as observed in all granite massifs of the Cévennes proves evidently the existence of magma or mush and magma flow. The granite has been a mobile substance during its *mise en place* and there are no indications for metamorphic diffusion in the present magmatic deposit. No indications have been found, supposing a special conception about the origin of the mobile granite mass in the depth before its intrusion.

REFERENCES.

- BALK, R. (1937) — Structural behavior of igneous rocks — Geol. Soc. America, Memoir 5, 1937.
- BAULIG, H. (1928) — Le Plateau Central de la France et sa bordure méditerranéenne; étude morphologique — Paris, 1928.
- BERGERON, J. (1889) — Étude géologique du massif ancien situé au sud du Plateau Central — Ann. Sc. Géol., XXII, 1889.
- (1904) — Note sur les nappes de recouvrement du versant méridional de la Montagne Noir et des Cévennes aux environs du Vigan — Bull. Soc. Géol. France, IV série, 4, 1904.
- CLOOS, H. (1928) — Zur Terminologie der Plutone — Fennia, 50, 2, 1928.
- DEMAY, A. (1931a) — Contribution à l'étude de la tectonique hercynienne antestéphanienne dans les Cévennes méridionales et dans le Rouergue — Bull. Soc. Géol. France, V série, 1, 1931.
- (1931b) — Les nappes cévenoles — Mém. Carte Géol. France, 1931.
- (1934) — Contribution à la synthèse de la chaîne hercynienne d'Europe. Étude du plan axial de l'évolution et de l'orogénèse hercyniennes — Bull. Soc. Géol. France, V série, 4, 1934.
- (1935) — Sur le jeu alternant ou simultané des phénomènes magmatiques et dynamiques dans les Cévennes septentrionales — Comptes Rendus Ac. Sc., Paris, 200, 1935, pp. 2197—2199.
- (1942) — Microtectonique et tectonique profonde. Cristallisations et injections magmatiques syntectoniques — Mém. Carte Géol. France, 1942.
- FABRE, G. et L. CAYEUX (1901) — Alais, avec notice explicative — Carte Géol. France au 1 : 80.000, 209, 1901.

- GAERTNER, H. R. VON (1937) — Der Bau des Französischen Zentralplateaus — Geol. Rundschau, 28, 1937.
- HEIM, R. C. (1949) — Petrology of the Mt. Aigoual area in the southeastern Cévennes, France — (in print).
- RAGUIN, E. (1930) — Problèmes tectoniques dans les terrains cristallins du Centre de la France — Bull. Soc. Géol. France, IV série, 30, 1930.
- (1946) — Géologie du granite — Paris, 1946.
- ROQUES, M. (1941) — Les schistes cristallins de la partie Sud-Ouest du Massif Central Français — Mém. Carte Géol. France, 1941.
- THIÉRY, P. (1923) — Alais, 2-ème éd. avec notice explicative — Carte Géol. France au 1 : 80.000, 209, 1923.
- WAARD, D. DE (1949) — Tectonics of the Mt. Aigoual pluton in the southeastern Cévennes, France. Part I. — Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 52, 389, 1949.

Petrology. — *On metamorphic rocks from the island of Kabaëna in the East-Indian Archipelago.* By C. G. EGELER. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of April 23, 1949.)

In this paper a description will be given of a number of metamorphic rocks from Kabaëna, a small island situated directly South of the south-eastern peninsula of Celebes and West of the islands of Moena and Boeton. These rocks form part of the collections of the Geological Institute of the University of Amsterdam. They were partly collected by Prof. H. A. BROUWER and co-workers when visiting Kabaëna during the Celebes expedition of 1929 and partly by geologists of the Mining Department at Bandoeng, which presented the Geological Institute with a type-collection¹⁾.

I herewith wish to thank Prof. BROUWER for his courtesy in placing this material at my disposal and especially for discussing with me the results of my microscopical investigations in connection with the metamorphism on Celebes.

I. Thermally metamorphosed rocks Garnetiferous biotite-hornfelses

2691 Schistose garnet-biotite-hornfels with bands of a garnet-epidote-amphibole-rock.
in the Oe. Lakambola.

The only rock of this type examined is a somewhat folded variety, showing violet-brown and greenish grey bands in irregular alternation.

Under the microscope the brown bands appear to consist of a schistose garnet-biotite-hornfels, the structure of which is truly hornfelsic, notwithstanding the parallel arrangement of the mica flakes. The biotite is developed in small reddish brown crystals of a uniform size. For the rest the matrix is mainly formed by granular quartz and plagioclase (andesine). Garnet is represented by reddish, six-sided or more or less rounded crystals, generally of less than 0.5 millimetre in diameter, intensely cracked and often containing abundant inclusions. Both the garnet and the biotite are somewhat chloritized. Some parts of the rock are rich in colourless mica, which is considered an alteration-product of the felspar. Minerals of the epidote-group occur in a considerable amount, sometimes in separate zonary crystals partly recognized as clinozoisite, and also in turbid sausuritic aggregates. Titanite and rutile are accessory constituents, together with some pyrite.

The greenish bands contain much garnet, while further a pale coloured

¹⁾ The rocks collected by Prof. BROUWER c.s. may be distinguished from those from the Mining Department by the series-number 18.

amphibole is found in a considerable quantity, partly formed as a transition-product of a monoclinic pyroxene which still appears to be fairly abundant locally. The epidote-group is again represented by various members, especially zoisite. The interstitial mass consists mainly of quartz. Some muscovite occurs, presumably formed out of felspar.

This rock is of some interest as it closely resembles some of the schistose biotite-hornfelses which are so abundantly represented in western Celebes, where they are considered to have originated out of low-grade crystalline schists by the thermal influence of granodioritic intrusions (Lit. 4). On Kabaëna, however, no acid intrusive rocks are known to occur and the micaceous hornfelses of the type mentioned above are attributed to the thermal metamorphism caused by basic magmas²⁾.

Lime-silicate-hornfelses

- 18—34 *Grossularite-hornfels rich in pumpellyite.* In contact with an albite-diabase in the upper course of the Oe. Lakambola, closely above the point where the road from Tangkeno crosses the river.
- 18—36 *Banded lime-silicate-rock with bands of partly pumpellyitized grossularite-hornfels and of crystalline limestone.* Same locality.
- 18—37a *Grossularite-hornfels.* In the Oe. Lakambola, somewhat more upstream.
- 18—37b *Banded lime-silicate-rock with bands of partly pumpellyitized grossularite-hornfels and of grossularite-bearing crystalline limestone.* Same locality.

The sample of the *grossularite-hornfels* 37a is a massive grey rock of a characteristic hornfels type. Under the microscope a colourless lime-garnet appears to be by far the most important constituent, occurring in a finely granular "groundmass" consisting of albite together with minute (some hundredths of a millimetre in length) pale green prisms determined as monoclinic pyroxene. Garnet is developed in six-sided or more or less rounded crystals, strikingly equigranular and averaging 0.2 millimetre in diameter. Optical anomalies are common, the birefringent areas showing a characteristic sector arrangement. The garnet contains abundant inclusions, especially of ore, pyroxene and minute needles considered as rutile. Sometimes a number of grossularite grains are cemented by a single calcite crystal. One or two grains of titanite are found. Ore is very abundant; it is chiefly represented by pyrrhotite, scattered through the rock in irregular grains.

Megascopically the rock 37b also has the appearance of a normal *lime-silicate-hornfels*, in which light grey bands alternate with darker grey ones.

Under the microscope this rock shows several features of interest.

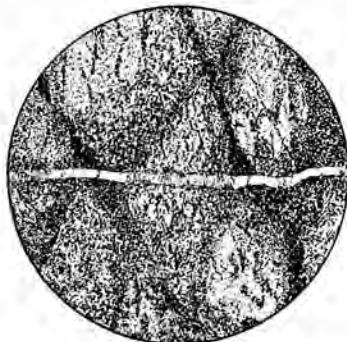
The lighter coloured bands appear to consist of crystalline limestone, the calcite developed in fairly large, closely interlocking crystals. Small, rounded crystals of grossularite are noticed in some parts of these bands. Swarms of small, colourless prisms, presumably consisting of zoisite, occur locally. The limestone band examined is intersected by veins of pumpel-

²⁾ A rock from the North of Kabaëna was described as a diorite by WUNDERLIN, but an approximate analysis yielded a SiO₂-content of only 48% (Lit. 12).

lyite; irregular patches of pumpellyite are also noticed within the carbonate mass. In the veins the pumpellyite occurs in fibrously developed crystals, averaging several tenths of a millimetre in length. The mineral is very pale coloured and shows a faint pleochroism from pale green (n_β) to almost colourless (n_a and n_γ). The optic axial plane has a transverse position, so that n_β is parallel to the longitudinal direction of the crystals and the elongation varying. The refringence is fairly strong; with the aid of the immersion method in sodium light a value of ± 1.68 was found for n_β . The birefringence is moderate; with the aid of the universal rotation stage and the Berek-compensator a birefringence of 0.018 was found. A measurement in a single crystal of the optic axial angle gave a value of 32° for $2V_\gamma$; it should be noted, however, that the mineral shows a distinct zonary structure. The dispersion $\rho < \nu$ appears to rather weak for pumpellyite.

The pumpellyite vein continues from the limestone band into one of the darker grey bands, and here the mineral is associated with varying quantities of albite, some calcite, deep green chlorite-like matter and pyrrhotite.

The dark grey band shows a rather complicated structure. It partly consists of a very finely granular aggregate of albite, minute crystals of monoclinic pyroxene, some grossularite especially concentrated near to the contact with the limestone band, and much pyrrhotite. In this fine-grained lime-silicate-rock rounded or lenticular patches occur, consisting of a felty aggregate of very fine-crystalline pumpellyite. Evidently the process of pumpellyitization began at certain points causing the formation of these concentrations which apparently have grown at the expense of the other minerals. In other places this process appears to have become so intensive that entire areas now consist of closely packed lenticular pumpellyite patches, often of more than a millimetre in length and showing a distinct tendency towards a parallel arrangement (see figure). The matrix between



Lenticular pumpellyite aggregates in a partly pumpellyitized lime-silicate-hornfels (18—37b) found near a diabase dike in the upper course of the Oe. Lakambola. An albite-vein is seen intersecting the rock; where this vein cuts the very fine-crystalline pumpellyite patches it is filled up with pumpellyite crystals of a considerably larger size, which appear to be more or less parallel-ranged conforming with the general orientation of the patches. Approx. $\times 30$.

these pumpellyite concentrations is formed by a greyish turbid aggregate, presumably mainly consisting of very fine-crystalline pyroxene together with much ore (pyrrhotite), some interstitial albite and occasionally some titanite. A few grossularite crystals are also present. It should be noted that the actual determination of the very fine-crystalline pumpellyite concentrations was made possible by the fact that the band in question is

intersected by narrow veins of albite and pumpellyite. These sometimes cut the pumpellyite "eyes", in which case the two parts generally appear connected in the vein by larger pumpellyite crystals.

Another of the dark grey bands examined contains much well-developed grossularite, again embedded in a finely granular matrix of albite, pyroxene, pyrrhotite and some calcite. It is a grossularite-hornfels resembling the rock 18—37a.

The lime-silicate-rocks 18—34 and 18—36 are closely related to the type described above, so no separate description need be given.

The lime-silicate-hornfelses described above are formed by the contact-metamorphism induced by albite-diabases on a series of dark coloured limestones and calcareous slates. It is an interesting feature of these grossularite-rich rocks that the felspar is albite. It seems likely that this phenomenon should be attributed to albitization caused by the soda-rich diabases responsible for the metamorphism.

Another noteworthy fact is the abundant occurrence of pumpellyite as a hydrothermal mineral both in the diabases and in the contact-rocks. This again points to the fact that this mineral — which was described for the first time in 1925 by PALACHE and Miss VASSAR — has a considerable distribution in the East-Indian archipelago in rocks of various type. In the last few years occurrences have been recorded from several localities: in eastern Celebes, for instance, pumpellyite appears to be an important constituent of various igneous as well as metamorphic rocks (Lit. 8), in the western part of Celebes it has been found in albite-diabases (Lit. 3, p. 26), whereas lately the mineral has also been found in eastern Borneo, in spilites and albite-diabases of the Danau-formation (Lit. 9).

The peculiar structure arising from the pumpellyitization of some of these lime-silicate-hornfelses (see the description of the rock 18—37b and the figure) is of special interest.

II. Regionally metamorphosed rocks

Micaceous schists

18—6 *Muscovite-schist*. In the Oe. Emeloro, downstream of Menoero.

18—39 *Piedmontite-bearing garnet-muscovite-quartz-schist with porphyroblastic plagioclase*. In the upper course of the Oe. Lakambola, upstream of the point where the road from Tangkono crosses the river.

The *muscovite-schist* 18—6 is a phyllitic rock showing no special features of interest.

The *piedmontite-bearing garnet-muscovite-quartz-schist* 18—39 is a considerably higher grade type, especially characterized by the occurrence of plagioclase porphyroblasts. The chief constituents are quartz and muscovite. In some highly micaceous bands a distinct folding is observed. The garnets vary much in size (up to 1.5 millimetres in diameter); irregular shapes predominate; some of the crystals are shattered and occasionally the mineral seems to be partly recrystallized. Trains of ore particles which

emphasize the foliation pass mostly undisturbed through the larger garnet specimens; sometimes also a certain degree of rotation is observed. The piedmontite is developed in small, elongated prisms, parallel-ranged and showing the characteristic intensive pleochroism from deep yellow (n_a) to violet (n_β) and carmine red (n_γ); the mineral is locally associated with epidote³⁾. The lenticular felspar porphyroblasts, attaining sizes up to 2.5 millimetres, consist of sodic andesine. Inclusions are abundant, especially of parallel-ranged piedmontite prisms and of ore, but also of quartz, muscovite, tourmaline and apatite. Some chlorite is associated with the muscovite; some brown mica-like matter is locally concentrated.

Gneissic rocks

- 18—41 *Amphibole-garnet-zoisite-gneiss*. In the upper course of the Oe. Lakambola, upstream of the point where the road from Tangkono crosses the river.
- 2691 *Biotite- and amphibole-bearing garnet-plagioclase-gneiss*. In the Oe. Lakambola.
- 2629 *Gneissic garnet-epidote-augite-quartz-rock*. Along the beach at Dongkala.

The rather fine-grained *amphibole-garnet-zoisite-gneiss* 18—41 shows some relationship to some of the amphibolitic rocks described below. Garnet is very abundant, occurring in porphyroblastic crystals. Often the garnets are shattered and dragged out and locally recrystallization appears to have taken place. Changes to biotite and chlorite are common. Amphibole is present in bluish green porphyroblasts, which are often much chloritized. Clinozoisite and zoisite are abundant, often accompanied by sericitic mica which is considered as an alteration-product of felspar. Sometimes the sodic plagioclase is still preserved as turbid crystals partly changed to albite. In general, however, the interstitial mass consists of quartz. Some much chloritized biotite is also present. A few, apparently relict crystals of monoclinic pyroxene are observed, partly changed to chlorite and amphibole. Carbonate occurs in a considerable quantity in between the other minerals; titanite is the main accessory constituent.

The *biotite- and amphibole-bearing garnet-plagioclase-gneiss* 2691 is a somewhat divergent type, with plagioclase, quartz and garnet as characteristic minerals, together with smaller quantities of amphibole, biotite and chlorite. Retrograde metamorphism has again caused shattering and chloritization of the garnet, chloritization of the biotite and sericitization and sausuritization of the felspar (oligoclase).

The *gneissic garnet-epidote-augite-quartz-rock* 2629 is a completely divergent type. The sample shows an irregular alternation of deep green, lighter green and greyish bands, all three speckled with red garnets. Under the microscope the deep green bands appear to consist mainly of pyroxene and quartz, with varying amounts of epidote, plagioclase, garnet and amphibole. The pale green augite is developed on irregular poiciloblastic crystals of varying size (up to 2.5 mm), which appear partly changed to

³⁾ An occurrence of piedmontite on the neighbouring island of Boeton was stated by the author in a metamorphic radiolarite (18—27), found as a cobble in the Oe. Moekito.

fibrous bluish green amphibole. The amphibole also occurs separately, sometimes formed at the expense of the epidote. The larger amphibole crystals sometimes show deeper coloured cores. Quartz forms an irregular mosaic. The plagioclase (oligoclase-andesine to andesine), which occurs in varying amounts in the quartzitic matrix, is generally much altered. Clusters of garnet locally occur; the crystals, which attain sizes up to several millimetres in diameter, are generally irregularly shaped, though more or less idioblastically developed specimens are also observed. Magnetite is very abundant.

In the lighter green bands epidote is the main rock-forming mineral; certain discontinuous streaks consist almost exclusively of yellow, equigranular epidote. Generally, however, pyroxene is fairly abundant too. Some layers are very rich in quartz, sometimes in association with much altered plagioclase. Garnet is again abundant locally. A considerable amount of titanite occurs, while ore is much less abundant than in the deep green bands.

The greyish bands are very rich in quartz, with varying amounts of felspar, augite, epidote and ore.

Amphibolites and amphibole-schists

- 18—39× *Epidote-amphibole-schist*. In the upper course of the Oe. Lakambola, upstream of the point where the road from Tangkeno crosses the river.
- 18—46× *Epidote-amphibole-schist*. Cobble in the Oe. Lakambola.
- 2680 *Epidote-amphibole-schist*. In the Oe. Rantinoli.
- 300 *Epidote-amphibolite*. On Poelou Damalawa Besar, NE of Dongkala.
- 18—40× *Quartz-rich plagioclase-amphibole-schist*. In the upper course of the Oe. Lakambola, upstream of the point where the road from Tangkeno crosses the river.
- 246 *Quartz-rich plagioclase-amphibole-schist*. Along the road from Dongkala to Sikele.
- 626 *Quartz-rich plagioclase-amphibole-schist*. In the Oe. Kala-Ero.
- 1523 *Quartz-rich plagioclase-amphibolite*. In the Oe. Lakambola.
- 298 *Amphibole-schist*. On Poelou Damalawa Besar, NE of Dongkala.
- 299 *Amphibole-schist*. Same locality.
- 2636 *Amphibole-schist*. Along the path N of the Oe. Lambale.
- 18—46 *Garnetiferous plagioclase-amphibolite*. Cobble in the Oe. Lakambola.
- 18—41× *Garnet-epidote-amphibolite*. In the upper course of the Oe. Lakambola, upstream of the point where the road from Tangkeno crosses the river.
- 18—42 *Garnet-epidote-amphibolite*. Same locality.
- 2647 *Garnet-amphibolite*. In the Oe. Lambo.
- 2654 *Quartz-rich garnet-amphibole-schist*. In the Oe. Lapondoowe.

The collection investigated contains a considerable number of amphibolitic rocks. They are distinguished as amphibolites or amphibole-schists merely in accordance with respectively a massive or a more schistose structure.

The *epidote-amphibole-schists* 18—39×, 18—46× and 2680 are fairly low-grade rocks containing bluish green amphibole, epidote and plagioclase as characteristic constituents. The felspar, which has a composition near oligoclase, is much sausuritized. Some quartz is always present.

The *epidote-amphibolite* 300 is a divergent variety, consisting of epidote

and amphibole, almost to the exclusion of other minerals. Originally some felspar has been present but now this mineral is entirely changed to sericite and zoisite. A few relict crystals of monoclinic pyroxene are present.

The quartz-rich plagioclase-amphibole-schists 18—40 \times , 246 and 626 differ from the rocks above mentioned in so far that here epidote does not occur as an important rock-forming constituent. Amphibole and lime-bearing plagioclase are the chief minerals, together with varying amounts of quartz. The plagioclase, which ranges in the various rocks from oligoclase-andesine to calcic andesine, is often much altered. Some epidote may occur; a small quantity of colourless mica and chlorite is also sometimes present. Titanite is generally the main accessory mineral.

A more highly amphiboliferous type is represented by the amphibole-schists 298, 299 and 2636. These mainly consist of stout crystals of bluish green amphibole, mostly attaining a length of several millimetres. Some felspar occurs, though always much broken down and changed to zoisite, sericite or prehnite; often turbid patches are the only indication of the original presence of plagioclase. Rutile is always a characteristic accessory constituent.

The garnetiferous plagioclase-amphibolite 18—46 is a coarsely granular rock. The pale greenish amphibole is developed in large, broad crystals, often with abundant inclusions. Garnet is represented by irregularly shaped poeciloblastic individuals, with inclusions i.a. of plagioclase, amphibole, rutile, titanite and ore. The plagioclase is an intermediate andesine, well-twinned and somewhat sausuritized. Some quartz occurs. Rutile, often associated with ore, is very abundant. The non-garnetiferous quartz-rich plagioclase-amphibolite 1523 is a closely related variety.

The garnet-epidote-amphibolites 18—41 \times and 18—42 form a well-distinguished type. They mainly consist of a deep greyish brown amphibole and epidote, while garnet is present in varying quantities. The rock 18—42 appears to be especially rich in garnet, developed in irregularly rounded crystals, averaging several millimetres in diameter. Sometimes amphibole is enclosed. The garnet appears to be much altered to green chlorite. The amphibole, which shows a strong pleochroism from greyish brown (n_7) to brown (n_8) and yellow-brown (n_a), occurs in large crystals; with the aid of the universal rotation stage the optic axial angle was measured as $2V_a = 72^\circ$; the extinction angle $n_7/c = 18^\circ$. This amphibole is considered to be a relict of igneous origin. Towards the exterior changes to a more fibrous, bluish green variety are observed; sometimes also the brown amphibole is changed to chlorite, in which case strings of minute titanite grains are observed along the cleavage lines. The younger, bluish green amphibole also occurs independent of the brown variety. Almost the entire interstitial mass consists of an aggregate of epidote, often associated with muscovite. In the rock 18—42 one or two relict crystals of monoclinic pyroxene are observed, almost changed to chlorite. Titanite and leucoxene are very abundant; titaniferous iron ore occurs, sometimes in pseudomorphs

after titanite; sometimes these minerals form peculiar vermicular patches. Some rutile and apatite are also present. The *garnet-amphibole* 2647 is a closely related type in which a small amount of felspar has been spared.

Lastly the *quartz-rich garnet-amphibole-schist* 2654 should be mentioned as a considerably divergent variety. Amphibole is again the principal constituent, but here the mineral is developed in smaller, more fibrous crystals, of an olive green colour. Quartz is very abundant. Originally the rock must have contained a considerable amount of plagioclase, but now this mineral is mostly changed to colourless mica. Some pyroxene occurs, in course of alteration to chloritic matter.

There is little doubt that many of the amphibolites and amphibole-schists described above are of igneous origin. Besides from characteristic structural features, this may in some cases be concluded from the occurrence and general character of relict amphiboles (see e.g. the garnetiferous amphibolites 18—41X, 18—42 and 2647). Other types again show some resemblance to the sausurite-gabbros from Kabaëna (see e.g. the plagioclase-rich amphibolites 18—46 and 1523). The amphibole-schists 298, 299 and 2636 may represent rather melanocratic equivalents of these sausurite-gabbros which have been subjected to considerable stress. In some cases, (e.g. the quartz-rich plagioclase-amphibole-schists 18—40X, 246 and 626), the origin is uncertain. The highly quartziferous garnet-amphibole-schist 2654 is considered to be of sedimentary origin.

Crystalline limestones

- 18—3 *Crystalline limestone*. In the Oe. Emeloro, W of Dongkala.
- 18—4 *Crystalline limestone*. Same locality.
- 18—40 *Muscovite-rich crystalline limestone*. Occurring as a band in amphibole-schists, in the upper course of the Oe. Lakambola, upstream of the point where the road from Tangkeno crosses the river.
- 18—51 *Crystalline limestone*. Along the path mounting from the Oe. Lakambola to Poë.
- 263 *Crystalline limestone*. Along the road from Dongkala to Sikele.
- 1463 *Crystalline limestone*. In the Oe. Emeloro.
- 1497 *Crystalline limestone*. Along the horse-path from Langkema to Timoekolek.
- 1516 *Crystalline limestone*. Along the path from Tangkeno to Sambara Kambola.
- 2632 *Epidote-rich crystalline limestone*. Along the beach at Dongkala.

In the *muscovite-rich crystalline limestone* 18—40 there is a well-marked parallel elongation of the calcite crystals, which show an intensive polysynthetic twinning. The rock contains a considerable quantity of colourless mica and further quartz and a subordinate amount of clinozoisite. A gradual transition of the limestone into a calc-epidote-chlorite-muscovite-schist is observed.

The *epidote-rich crystalline limestone* 2632 is a more massive type. It is a patchy greenish and brownish rock, which under the microscope besides the main constituent calcite, appears to contain a considerable amount of epidote and quartz and more subordinate amounts of muscovite, an almost

colourless chlorite-like mineral, titanite, leucoxene, apatite and haematite. The carbonate-mass is specked with small quartz grains; sometimes also lenticular quartz concentrations are found. The epidote occurs in prismatic crystals, often showing an ideal development. An interesting feature is the partial alteration of some of the epidote crystals to sericite, a change which generally preceeds from cracks in the epidote; replacement of epidote by calcite is also observed. It is a well-known fact that the epidote minerals are but seldom subject to alteration, though some instances have been described, e.a. from Celebes (Lit. 8, p. 149).

The other *crystalline limestones* investigated show but few features of interest, so no separate descriptions need be given. They are for the greater part considered to be of a lower grade than the varieties described above. The grain-size varies considerably. Carbonate is always the chief constituent, while quartz is generally present in a subordinate amount. Some types, however, appear to be fairly rich in quartz and often this mineral is concentrated in irregular dark patches together with i.a. sericite, chlorite, ore and a considerable quantity of carbonaceous matter.

Comparison with Celebes.

Up till now very little was actually known about the petrology of Kabaëna. A small number of eruptive and metamorphic rocks was described by WUNDERLIN, the metamorphic group comprising only amphibolitic varieties (Lit. 12). It is clear, however, that the island, with its schist-formation and its peridotitic intrusions, forms the continuation of the southeastern peninsula of Celebes. Here more petrological data are available, thanks to descriptions given by WUNDERLIN (Lit. 12) and GISOLF (Lit. 5) and thanks to the results of the Celebes expedition of 1929 (BROUWER, Lit. 2, 3, and DE ROEVER, Lit. 8).

The regional distribution on the southeastern peninsula, of rocks metamorphosed in the glaucophane-schist facies, shows that here there is a close relationship to the metamorphism in the eastern part of Central Celebes. DE ROEVER (Lit. 8), in his study on the igneous and metamorphic rocks in eastern Central Celebes, distinguishes between a metamorphism in the epidote-amphibolite facies, which is older than the radiolarites and the ophiolitic and spilitic igneous rocks, and a younger, presumably alpine, glaucophanitic metamorphism. He proves that many varieties, such as e.g. those intermediate between amphibolitic and glaucophanitic rocks, have been subjected to polymetamorphism. In an appendix the same author gives the names of a number of rocks from southeastern Celebes, together with some brief remarks. It appears that here too both amphibolitic and glaucophanitic rocks are represented and special attention is drawn to the occurrence of a plagioclase-amphibolite in the W. Sesoh, SE of Lake Towoeti. Another interesting fact is the occurrence in the W. Aloehoeno

⁴⁾ See also SCHMIDT (Lit. 10).

(NW of Teetedopi) of a rock intermediate between a glaucophanite and an amphibolite, and therefore polymetamorphic. Indeed it seems likely that the polymetamorphism which has played such an important part in the geological history of Central Celebes, can also be traced in the southeastern part of the island, and that here the glaucophanitic metamorphism was also preceded by an amphibolitic metamorphism. The occurrence, on the peninsula, of amphibolites containing lime-bearing plagioclase, may point to the fact that the older metamorphic phase has sometimes been of a somewhat higher grade (amphibolite facies) than in eastern Central Celebes, where the epidote-amphibolite facies was not exceeded.

For a comparison between Celebes and Kabaëna it is especially important to know the nature of the metamorphism in the more southern part of the southeastern peninsula. It appears that in Roembia both amphibolites and glaucophanitic rocks occur, the latter i.a. at Liano, not far from the coast directly opposite to Kabaëna (Lit. 12)⁵). No polymetamorphic phenomena are described from this area, but a further study of the crystalline schists, with the probability of polymetamorphism in mind, might lead to interesting results.

The detailed study, published in this paper, of metamorphic rocks from Kabaëna, shows that here the regional metamorphism was mainly in the amphibolite facies, as proved by the frequent occurrence of amphibole in association with lime-bearing plagioclase (varying from albite-oligoclase to andesine⁶). To these medium to higher grade schists are reckoned the amphibolites and amphibole-schists, the gneissic rocks, the piedmontite-bearing garnet-mica-schists and the higher grade crystalline limestones.

In contrast with eastern Celebes no indications of a glaucophanitic metamorphism are found.

Further a few thermally metamorphic rocks (lime-silicate-rocks) are described, which have been formed out of dark coloured limestones and other calcareous sediments where these are intruded by albite-diabases in the upper course of the Oe. Lakambola. This flysch-like series is but feebly metamorphosed and therefore considered younger than the crystalline schist-formation occurring close by (somewhat more upstream). Though the age of the sediments is not known the occurrence of the diabasic intrusions is of some help in solving the problem of the age-

⁵) On the neighbouring island of Boeton the schist-formation — found only in the upper course of the Oe. Moekito in the South — mainly consists of plagioclase-amphibolites and epidote-chlorite-schists (Lit. 6 p. 4). No glaucophanitic rocks are found. The schists, according to HETZEL, are older than the upper-triassic formation occurring on the island.

⁶) The paragenesis observed in many of these rocks of amphibole and lime-bearing plagioclase with minerals of the epidote-group, is attributed to the general physical conditions and in the first place to the stress which prevailed during the metamorphism and which made a still higher lime-content in the felspar impossible (see e.g. TURNER, Lit. 11, p. 81).

relations?). BROUWER considers the albite-diabases in eastern Central Celebes and on the southeastern peninsula to belong to the same differentiation-series as the gabbro-peridotitic rocks and consequently to be of about the same age (Lit. 3). It seems highly probable that the same is the case on Kabaëna, where basic and ultrabasic rocks are widely distributed. The difficulty remains that the actual age of the gabbro-peridotitic rocks in the east arc of Celebes and on Kabaëna is insufficiently known; an upper-mesozoic to lower-tertiary age of many of these rocks seems most probable⁸⁾.

Taking these various facts into consideration it seems reasonable to assume that the regional amphibolitic metamorphism on Kabaëna, which is older than the ophiolitic rocks, may be correlated with the older (pre-radiolarite and pre-ophiolite and -spilite) metamorphism distinguished on Celebes. Further it seems likely that the intrusion of the diabases, causing the contact phenomena in the flysch-like series, may be roughly correlated with that of the gabbro-peridotitic rocks which on Celebes appear to have preceded the younger phase of metamorphism responsible for the glaucophanitic rocks. The question why no traces of this glaucophanitic metamorphism have been found in the rocks investigated from Kabaëna, remains unanswered.

Summary.

The collection investigated comprises both thermally and regionally metamorphosed types.

Most of the thermally metamorphic rocks, viz. the lime-silicate-hornfelses rich in grossularite, are formed by contact-metamorphism of calcareous sediments, caused by albite-diabases. Post-metamorphic pumpellyitization has given rise to peculiar structures in some of these hornfelses.

The regionally metamorphic rocks can be divided into a number of medium to higher grade crystalline schists — e.g. epidote-amphibole-schists, plagioclase-amphibolites and amphibole-schists, garnetiferous amphibolites, garnetiferous gneisses, piedmontite-bearing garnet-mica-schists with porphyroblastic plagioclase and crystalline limestones — and a number of lower grade types, e.g. phyllitic mica-schists and crystalline limestones.

A comparison is made with eastern Celebes where two different phases of metamorphism have been distinguished. It is suggested that the higher grade crystalline schist-series on Kabaëna may represent the older phase of metamorphism on Celebes, which is considered to be older than the

⁷⁾ On Boeton contact phenomena caused by gabbroid and diabasic intrusions in the upper-triassic Winto-series have been recorded by BOTHÉ (Lit. 1), while HETZEL states that in this flysch-like series dikes and sills of diabase are a common phenomenon (Lit. 6, p. 21).

⁸⁾ See i.a. BROUWER (Lit. 3, p. 24), BOTHÉ (Lit. 1, p. 100) and HETZEL (Lit. 6, p. 21).

ophiolitic rocks. No traces are found, in the rocks investigated from Kaba-
ëna, of the younger, glaucophanitic metamorphism occurring on Celebes.
It appears that the thermal metamorphism is younger than the regional
metamorphism causing the higher grade crystalline schists. It is pointed
out that the diabasic rocks responsible for the thermal metamorphism may
be related to the gabbro-peridotitic intrusions occurring in the east arc of
Celebes and consequently approximately of the same age.

REFERENCES.

1. BOTHE, A. CH. D., Voorloopige mededeeling betreffende de Geologie van Zuid-Oost-Celebes. De Mijningenieur, 8, 97—103 (1927).
2. BROUWER, H. A., Geologische onderzoeken op het eiland Celebes. Verh. Geol. Mijnb. Gen. voor Ned. en Kol., Geol. Ser. 10, 39—217 (1934).
3. ———, Geological explorations in Celebes. Summary of the results. In: Geological explorations in the island of Celebes under the leadership of H. A. BROUWER. Amsterdam, 1—64 (1947).
4. EGELER, C. G., Contribution to the petrology of the metamorphic rocks of western Celebes. In: Geological explorations in the island of Celebes under the leadership of H. A. BROUWER, Amsterdam, 175—346 (1947). Also published as thesis, Amsterdam (1946).
5. GISOLF, W. F., Mikroskopisch onderzoek van gesteenten uit Z.O. Selébés. Jaarb. Mijnw. Ned. Indië. Verh., 66—113 (1924).
6. HETZEL, W. H., Verslag van het onderzoek naar het voorkomen van asfalt-gesteenten op het eiland Boeton. Versl. en Med. Betr. Ind. Delfstoffen, 21, Batavia, (1936).
7. PALACHE, C. and H. E. VASSAR, Some minerals of the Keweenawan copper deposits: pumpellyite, a new mineral; sericite; saponite. Amer. Min. 10, 412—418 (1925).
8. ROEVER, W. P. DE, Igneous and metamorphic rocks in eastern Central Celebes. In: Geological explorations in the island of Celebes under the leadership of H. A. BROUWER, Amsterdam, 65—174 (1947).
9. ———, Occurrences of the mineral pumpellyite in Eastern Borneo. Bull. of the Bur. of Mines and the Geol. Surv. in Indonesia, I, 1, 16—17 (1947).
10. SARASIN, P. and F., Entwurf einer geographisch-geologischen Beschreibung der Insel Celebes. (1901), with petrographical descriptions by C. SCHMIDT.
11. TURNER, F. J., Mineralogical and structural evolution of the metamorphic rocks. Geol. Soc. Amer., Mem. 30, (1948).
12. WUNDERLIN, W., Beiträge zur Kenntnis der Gesteine von Südost-Celebes. Samml. Geol. Reichsmus. Leiden, 9, 244—280 (1913).

Amsterdam, Geological Institute of the University.

Crystallography. — Transformation of gnomograms and its application to the microchemical identification of crystals. II. By D. W. DIJKSTRA. (Memorandum of the Crystallographic Institute of the Rijks-Universiteit at Groningen.) (Communicated by Prof. J. M. BIJVOET.)

(Communicated at the meeting of February 26, 1949.)

§ 11. Two applications of the method dealt with in the previous part will be discussed here.

a. TERPSTRA describes in his book⁶⁾ how to make a pasteboard model of a crystal of cane sugar. The gnomogram is shown in fig. 228 (p. 275), and represented in our fig. 14 by full lines, the poles being indicated by dots. In order to make it possible to reproduce the face (001) in its true form another gnomogram has to be constructed with the pole (001) in its centre; then an orthogonal parallel projection onto the plane of drawing will produce the face with the angles between the edges having the correct values. This modified gnomogram is shown in TERPSTRA's fig. 230 (p. 280). In our fig. 14 the required transformation has been performed as described in the first part of this article (§ 8a; cf. figg. 6 and 12) without any numerical values of the angles having been used. The ——·——·——·— lines represent the modified gnomogram, the circlets its poles.

(The small distance between (001) and the centre, $13\frac{1}{2}^\circ$, causes the old zone line $h' = 0$ to coincide in the drawing practically with the new one $h = 0$, and (001) likewise with P_n . In the original drawing the radius of the gnomon circle being 6 cm the distance between O' and (001) is equal to $6 \tan 13\frac{1}{2}^\circ = 1,44$ cm and that between O' and P_n to $2 \times 6 \tan 6\frac{3}{4}^\circ = 1,42$ cm).

b. The method can be useful in microchemistry, when microscopical crystals have to be identified⁷⁾; the measuring of the angles by means of a goniometer then being impossible or at least very difficult, an ordinary microscope provided with a revolving stage with graduation is used. The investigator will see the crystal, resting with one of its faces upon the object glass, projected orthogonally onto that plane and thus will be able to measure the angles by which the edges intersect in this image. When he has some notion or other of the chemical identity of the crystal, he can look up the data on the crystallographic properties of this substance, surmise on which of its faces the crystal may rest and then construct its orthogonal projection onto this plane (if desirable onto several). If the constructed angles and the measured ones agree, the crystal will be thus identified. Moreover the orientation of the crystal on the object glass

⁶⁾ P. TERPSTRA, "Kristallometrie" (Groningen, 1946) p. 279 seqq.

⁷⁾ J. D. H. DONNAY and W. A. O'BRIEN, Anal. Ed. Ind. & Eng. Chem. 17, 593 (1945).

will be known and the question will be answered as to which of its edges are seen as lines in the image formed by the microscope.

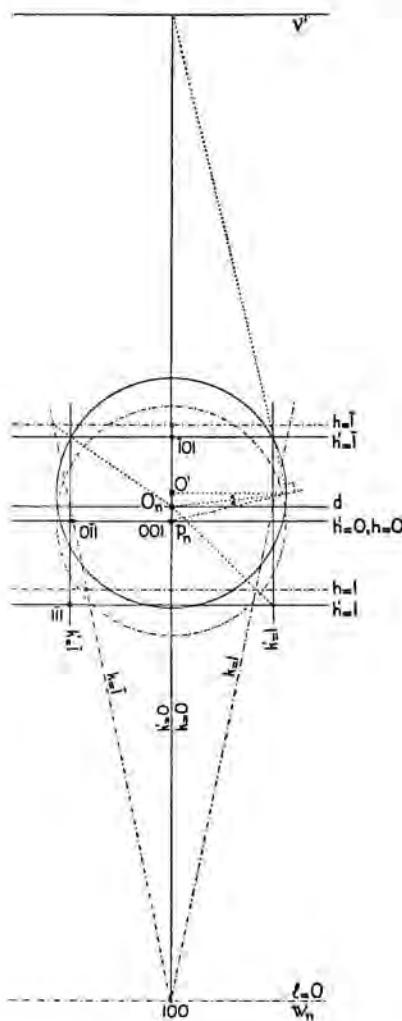


Fig. 14

In order to elucidate this we will take silverdichromate as an example, the formation of this substance being used in microchemistry to identify both silver ion and chromate ion⁸⁾. A correct performance of the reaction will result in tolerably large, well shaped crystals, coloured yellow to reddish-brown, in most cases plate-shaped and sometimes so thin that no colour is to be seen. If the crystal shown by the microscope is expected to be silverdichromate, this conjecture can be verified as follows:

⁸⁾ BEHRENS-KLEY, "Mikrochemische Analyse" (1915), vol. 1, pp. 87 and 121.
F. EMICH, "Mikrochemisches Praktikum" (1924) p. 88.

The image of the crystal is copied as seen in the microscope (fig. 15 *B, C, D*). The angles between the lines in this image can be measured, if the crystal is a suitable one, with an exactitude of about $\frac{1}{2}^\circ$, or measurements can be taken on a microphotograph of the crystal.

GROTH⁹⁾ describes the crystal of silverdicromate, measured by SCHABUS. The crystals are apt to be plate-shaped, the face (100) predominating, and elongated in the direction of the *c*-axis. The table of angular values is from TERPSTRA⁶⁾ p. 190.

Face	φ	ρ
<i>b</i>	0°	$90^\circ 0'$
<i>a</i>	$70^\circ 35'$	$90^\circ 0'$
μ	$136^\circ 11'$	$90^\circ 0'$
ω	$113^\circ 2'$	$56^\circ 41'$
σ	$40^\circ 29'$	$65^\circ 8'$
ζ	$68^\circ 9'$	$32^\circ 51'$
ϱ	$258^\circ 7'$	$11^\circ 39'$

From these angular values the gnomogram shown in fig. 15 is constructed as usual, the angles φ being located on the gnomon circle and the distances $R \tan \rho$ (R being the gnomon distance) being laid off from the centre on the radii through the corresponding points.

From the description of its habit the crystal may be expected to rest upon the object glass with face *a* {100}; obviously the gnomogram has to be changed into another one, having the pole (100) in its centre. The polar distance of *a* being 90° the bisector of $\angle O'OA$ (fig. 5) forms with the line OO' an angle of 45° , so *d* (fig. 15A) can be drawn tangent to the circle in its point of intersection with $O'a$. Now O_n will be represented by this point of intersection and v' by the line through $O' \parallel d$ (see § 5 and § 8 (a)).

It will, however, not be necessary to perform the transformation completely, only the directions of the new zone lines — and not the zone lines themselves — being wanted, since in the construction of an orthogonal projection only this simple proposition is used: the orthogonal projection e' of the edge *e* between the faces *A* and *B* onto the plane of drawing will be perpendicular to the zone line through the poles *a* and *b* of these faces¹⁰⁾.

Now after transformation (§ 6, fig. 6b) the zone lines will be parallel

⁹⁾ P. GROTH, "Chemische Kristallographie", vol. 2, p. 591.

¹⁰⁾ The demonstration of this proposition runs as follows: $Oa \perp A$, so $Oa \perp$ any line in face *A*, including the edge *e*; $Ob \perp B$ and so $\perp e$ too. This means $e \perp$ the plane through Oa and Ob and — the zone line through the poles *a* and *b* lying in this plane — $e \perp$ this zone line. The perpendicular OO' is dropped onto the plane of drawing, and this line OO' will be \perp the zone line too. The zone line, being $\perp e$ and $\perp OO'$, thus will be \perp the plane through *e* and OO' , which is the plane projecting *e*. The line of intersection e' of the plane of drawing and this plane now will be \perp the zone line.

to the lines connecting O_n with the points of intersection of the original zone lines with v' . So the directions of the new zone lines (represented

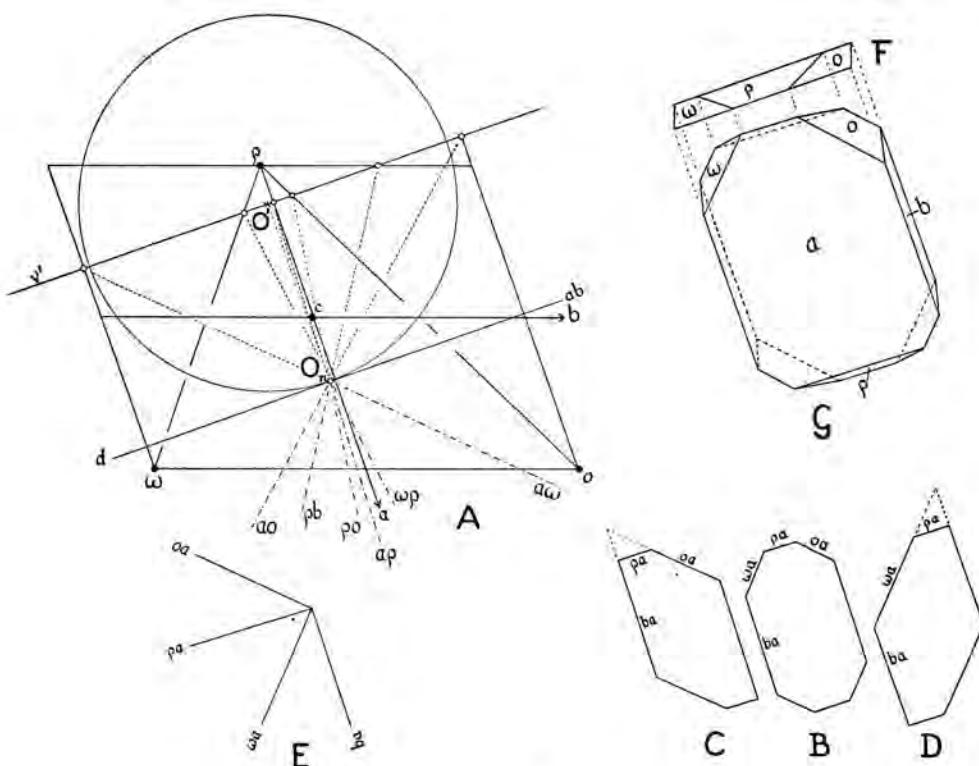


Fig. 15

by — . — . — . — lines) can be constructed as a pencil through O_n that will result, when the points of intersection of v' with the original zone lines are connected with the point O_n .

The projections of the edges, being perpendicular to these zone lines, will form a pencil similar to the one through O_n but rotated through 90° . The crystal being likely to rest on a {100}, in the first place the directions of the new zone lines through a will be constructed.

On a piece of transparent paper the directions obtained from the crystal itself are laid off from one point (fig. 15 E); this is laid on top of the pencil constructed through O_n and now the pencils are tested as to their similarity by rotation, which then should make them coincide. If this should not happen, then turn the paper upside down and try again, as the crystal may as well lie on the face (100) as on ($\bar{1}00$). In the case dealt with here the pencils turn out to coincide exactly, which justifies the conclusion that the crystal will be almost certainly one of silver-dichromate lying with face (100) uppermost.

As of each zone line it is known through which pole it passes including a , the lines in the image formed by the microscope which can be made to coincide with the zone lines, can be determined as the edges between these faces and face a .

Some elaboration in this case being worth while, the top view of the crystal is designed (fig. 15F), its lines being drawn perpendicular to the zone lines of the original gnomogram.

Then the projection of the crystal onto plane a is designed (fig. 15G), which will result when from the points of intersection in the top view perpendiculars are dropped onto the line d and the edges between the faces are added, each of them being perpendicular to the corresponding zone line in the transformed gnomogram. The outline of this fig. 15G agrees with the image of the crystal as seen by microscope (In fig. 15G the faces μ and c have not been reproduced because none of the crystals measured showed their edges.)

If the crystal is a very thin plate, the observed outline will be that of its upper face (fig. 15B), the absence of face ω resulting in fig. 15C and that of face σ in fig. 15D. All of these shapes have been observed.

These figg. C and D are almost similar to each other, especially if one of them should lie upside down; both figures can be defined as rhombs with the acute corners cut off. Moreover the angular values turn out to differ only slightly. The methods of analytic geometry have been followed in the calculation of the under mentioned angular values from the gnomogram. This calculation is, as the construction described above, very simple. The construction being restricted to intersections of lines and connections of points the calculation runs as follows: the coordinates of the given poles are determined and the equations of the zone lines, then the coordinates of their points of intersection with v' and last of all the equations of the lines connecting these points of intersection with O_n , their direction coefficients being the tangents of the angles they form with the X -axis. Only the results of this calculation will be mentioned here.

The obtuse angle of the rhomb, formed by ab and aw in fig. D, is equal to $135^\circ 45'$, that between ab and ao in fig. C to $137^\circ 16'$.

The line aq , which cuts off the acute corner of the rhomb, forms in fig. D angles of $132^\circ 4'$ and $92^\circ 11'$ and in fig. C of $134^\circ 55'$ and $87^\circ 49'$.

The fact that the angles, in connection with the attainable accuracy of the measurements are practically equal, is very confusing, if the above interpretation should be wanting. In BEHRENS-KLEY the acute angle of the rhomb is recorded to be equal to 43° . In fig. C this angle is equal to $42^\circ 44'$, in fig. D to $44^\circ 15'$, the value 43° probably being the average of the values found for edges that were held to be the same, actually being different ones.

In this case C and D cannot be distinguished by the orientation of the extinction directions, the plates frequently being so thin, that in spite of the strong reddish-brown colour of silverdichromate they will show in white

light only a faint greyish colour, which makes the interference colour between crossed NICOLS of so low an order that the extinction direction cannot be determined with a reasonable degree of exactitude.

In general an exact calculation of the angular values will not be necessary, since the results of the construction will have about the same degree of exactitude as those obtained from the image formed by the microscope. If however, as in the case of silverdichromate, a slight difference between some of them might bring about some confusion, the required certainty can be easily obtained by means of calculation.

Finally I wish to express my gratitude to Prof. Dr P. TERPSTRA, Groningen, and to Dr W. F. DE JONG, Delft, for the interest shown to me, when this article was being written.

Rotterdam, October 1948.

Astronomy. — *Magnitude effects in G-type stars.* By P. J. GATHIER.
(Communicated by Prof. M. G. J. MINNAERT.)

(Communicated at the meeting of March 26, 1949.)

Summary.

Equivalent widths have been measured for Ca I λ 4227 and Sr II λ 4077 in the spectra of 14 stars — supergiants, giants, dwarfs — with all about the same spectral type as the sun. The observed values are satisfactorily explained by the theory of stellar atmospheres. The damping constant is found to be five to ten times larger for dwarfs than for giants and supergiants.

Introduction.

The material consists of 14 stellar spectra, taken by Professor MINNAERT at the McDONALD Observatory with the Cassegrain spectrograph, equipped with two quartz prisms. The focal length of the camera was 50 cm, the dispersion was 40 Å/mm near H_{γ} ; the plates cover the region $\lambda\lambda$ 3500—4700. The stars investigated are dwarfs, giants and supergiants of the spectral types F5—G5. Nine of the plates carried a calibration, obtained with a tube photometer. Since the exposure times did not differ by more than a factor two and since all plates were developed in the same way, it was possible to construct one mean density curve for the whole material. The spectra have been recorded by means of the Utrecht microphtometer on a 60-fold scale.

We proposed to study for these stars the relation between the observable quantities: the absolute magnitude M and the spectral type, and the theoretical parameters: temperature and effective gravitational acceleration. Just because the atmosphere of the sun has been studied so well, these stars of a similar type are well fitted for a comparative theoretical investigation.

A search for luminosity characteristics.

The stars which have been selected are but little different in spectral type; since their absolute magnitudes vary over a considerable range, videlicet from $M = +6$ to $M = -6$, it seemed interesting to look in the first place for spectral lines which are very sensitive to variations of the absolute magnitude. With the purpose to find such characteristic lines, the galvanometer deflections on the registrograms were compared for giants and for dwarfs, without converting them into intensities. In figure 1 these values are compared for the dwarf star ζ Her GO IV and for the supergiant β Aqu GO Ib. A hundred points were obtained, scattered about

a main line; this is not a straight line under an angle of 45° , because the plates have not the same density. Some points are lying well away from

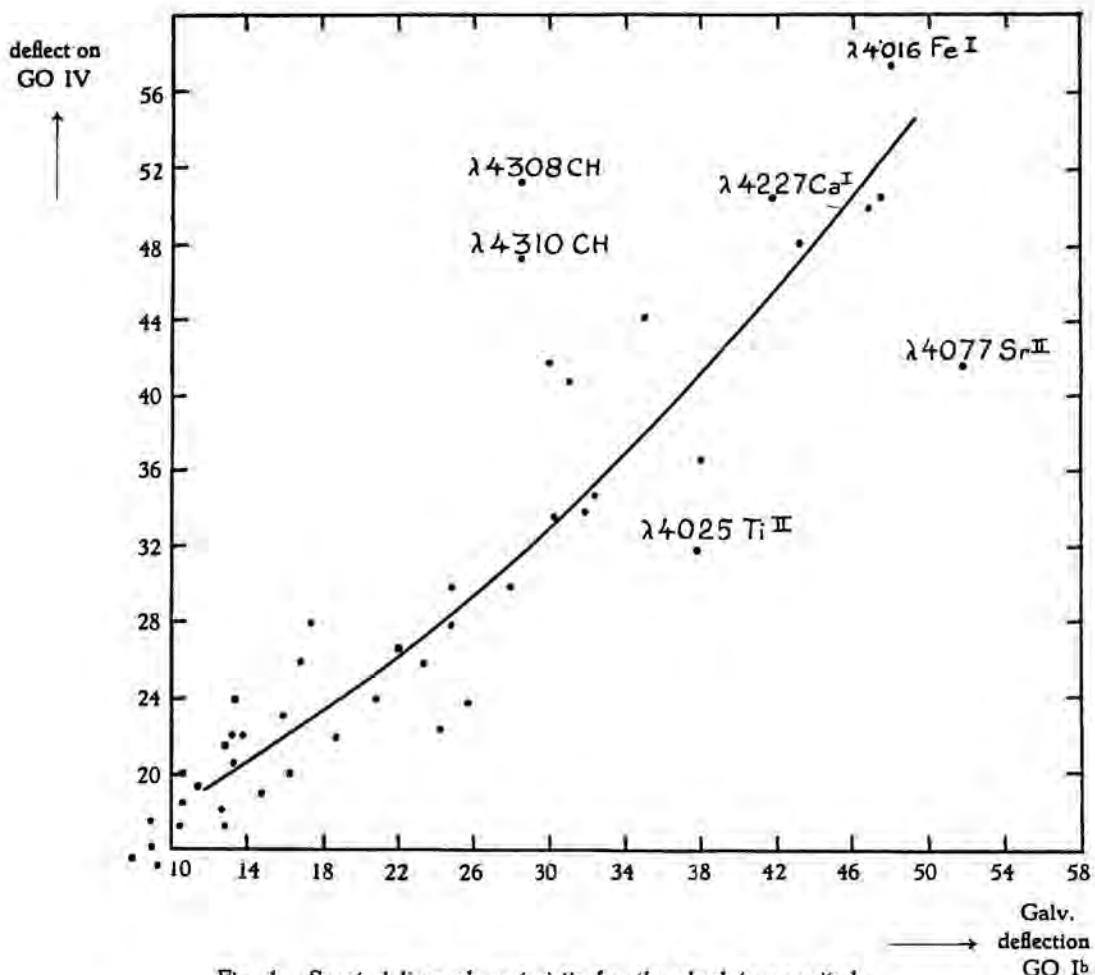


Fig. 1. Spectral lines characteristic for the absolute magnitude.
Comparison between the galvanometer deflections for several spectral lines in a dwarf
and in a supergiant.

it: these represent lines which are much stronger or much weaker in giants than in dwarfs. Strikingly stronger in giants are for example Sr II λ 4077 and Ti II λ 4025, while the CN-bands and Ca I λ 4227 are conspicuously weaker. These are just the lines, generally used as criteria of the luminosity for G-type stars. For some of these lines the variations of the galvanometer deflection with the spectral type have been determined and the ratio between the deflections for two spectral lines has been investigated as a function of the absolute magnitude (figure 2). Here for the determination of the spectral type, MORGAN's classification¹⁾ has been

¹⁾ MORGAN, KEENAN, EDITH KELLMAN, An atlas of stellar spectra.

used (table I). Two interesting lines, namely Ca I $\lambda 4227$ and Sr II $\lambda 4077$ have been studied into more details, with the purpose to connect the theory of the stellar atmospheres with the observations.

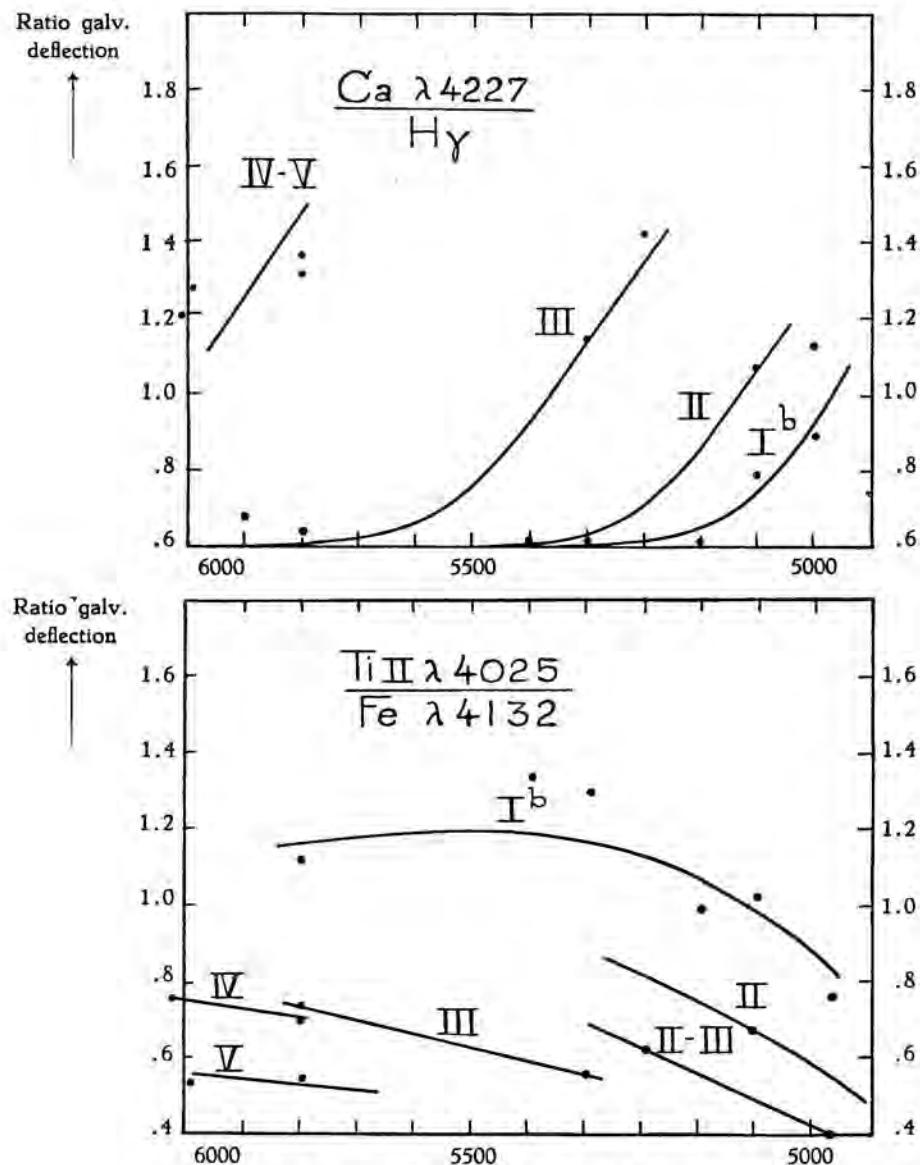


Fig. 2. The ratio of the galvanometer deflections:

- 1) for Ca I $\lambda 4227$ and $H\gamma$;
- 2) for Ti II $\lambda 4025$ and Fe I $\lambda 4132$.

Variation of these ratio's in stars of different absolute magnitudes and effective temperatures.

Ca I $\lambda 4227$.

The equivalent width W_1 was measured on the available 14 stellar spectra.

TABLE I. Stars observed for spectral line intensities.

Star	Spectrum	M	W_{4077}	W_{4227}	$T_{\text{eff}}^{\circ} \text{ K}$
β Aqu	G0 Ib	-3	2.0	1.4	4950
ζ Her	G0 IV	+3.2	1.0	1.4	5800
α Per	F5 Ib	-3	1.55	0.95	5800
π Cet	G2 V	+5.0	0.77	1.5	5900
δ CMa	F8 Ia	-6	2.3	1.5	5000
9 Peg	G2 Ib	-2.3	1.65	2.0	4700
π Cet	G2 V	+5.0	0.8	1.5	5900
β Lep	G2 II	-2.0	1.0	1.4	5000
η Cas A	G0 V	+4.9	0.6	0.9	6050
v Peg	F8 III	+0.7	0.9	0.9	5700
ϱ Pup	F6 II	-0.3	1.3	1.1	5800
γ Cyg	F8 Ib	-4.3	1.8	1.1	5300
α Sag	G1 II	-1.9	1.35	1.35	5100
η Peg	G2 II-III	-1.0	1.25	1.5	5100
9 Peg	G2 Ib	-2.3	1.7	1.9	4700
v And	F8 IV	+4.9	0.75	1.2	6000

Because $\lambda 4227$ is a resonance line and all Ca-atoms are in the fundamental level, the number of absorbing atoms is determined by the ionisation temperature T_{ion} . According to investigations on spectra taken with a considerable dispersion T_{ion} will be very much the same as the effective temperature T_{eff} .

With the help of the data of BECKER²⁾ and KUIPER³⁾ the relation between the effective temperature and the spectral type has been determined for each of MORGAN's luminosity classes (table I).

In figure 3 the measured W_{λ} -values are plotted as a function of T_{eff} , values of W_{λ} determined by other writers being also included. Two different sequences are clearly shown, one for the dwarfs and another one for the giants and supergiants.

The following simple theory was used for the calculation of the equivalent widths of Ca I $\lambda 4227$ in stellar atmospheres, characterized by different values of T and the effective gravitational acceleration g . The total number of calcium-atoms in a stellar atmosphere is determined by:

1. The total number of all atoms or ions in that part of the atmosphere which can be considered to be transparent;
2. the Ca-abundance, A_{Ca} , defined as the fraction of the atoms or ions in the atmosphere which are either Ca I or Ca II;
3. the equilibrium between the number of Ca-atoms and Ca-ions, as given by SAHA's law for every value of T and p_e , the electron pressure.

For the calculation in these three steps, the numerical values of the different quantities will be chosen as well as possible in accordance with

²⁾ BECKER, Zs. f. Ap. 25, 145 (1948).

³⁾ KUIPER, Ap. J. 88, 429 (1938).

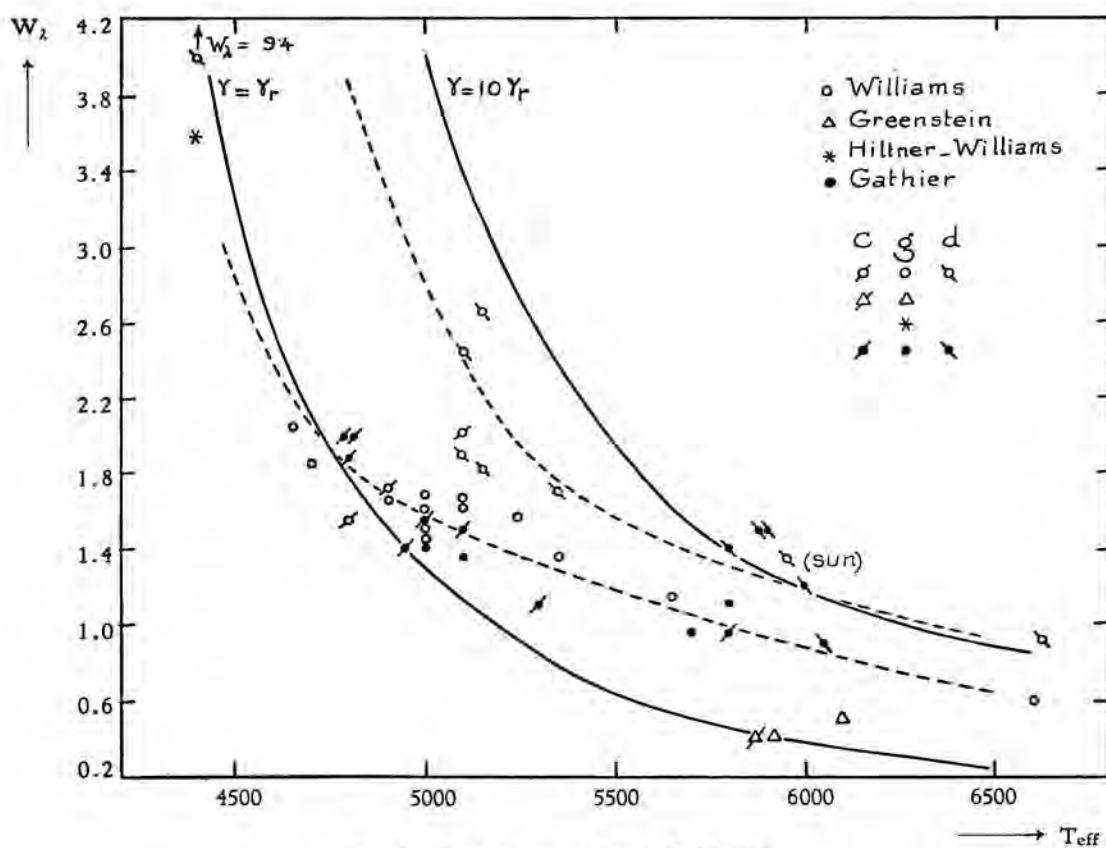


Fig. 3. Equivalent widths of $\text{Ca I} \lambda 4227$.

Fulldrawn lines: theoretical equivalent widths for $\gamma = \gamma_r$, and for $\gamma = 10\gamma_r$.

Dotted lines: observed values of W_{4227} as a function of temperature, for giants-supergiants (left line) and for dwarfs (right line).

The dispersions used are (near H_γ):

WILLIAMS⁴⁾ 17 Å/mm; GREENSTEIN⁵⁾ 3 Å/mm;

HILTNER-WILLIAMS⁶⁾ 3 Å/mm; GATHIER 40 Å/mm.

earlier results of other authors. If N is the average number of atoms or ions per cm^3 and H is the effective height of the atmosphere, we have:

$$NH = \frac{\tau_0}{\mu m_H \times}$$

where τ_0 is the effective optical depth, μ the mean atomic weight, m_H the mass of the H -atom and \times the continuous absorption coefficient⁷⁾.

We assume⁸⁾ $\mu = 1.5$ and provisionally $\tau_0 = 0.7$, which is a convenient value for the sun⁹⁾.

⁴⁾ WILLIAMS, P. A. S. P. **48**, 113 (1936).

⁵⁾ GREENSTEIN, Ap. J. **107**, 151 (1948).

⁶⁾ HILTNER-WILLIAMS, A Photometric Atlas of Stellar spectra.

⁷⁾ UNSÖLD, Physik der Sternatmosphären 1938, Chapter XII.

⁸⁾ Miss ROSA, Zs. f. Ap. **25**, 9 (1948).

⁹⁾ TEN BRUGGENGATE, Veröff. Univ. Sternw. Göttingen **76** (1947).

At present it seems well established that the continuous absorption for stars of the spectral type investigated is due to the H^- -ions; the absorption coefficient κ is known from theory¹⁰⁾.

For A_{Ca} a value 1.45×10^{-6} has been taken. The electron pressure p_e can be expressed as a function of T and the gas pressure p . With the aid of the relation $p = \frac{r}{\kappa} g$ we introduce g as parameter instead of p_e ¹¹⁾.

So, if N_a is the number of active Ca-atoms, it is possible to calculate $N_a H$ for every value of T and g (fig. 4). From this figure one can see that in

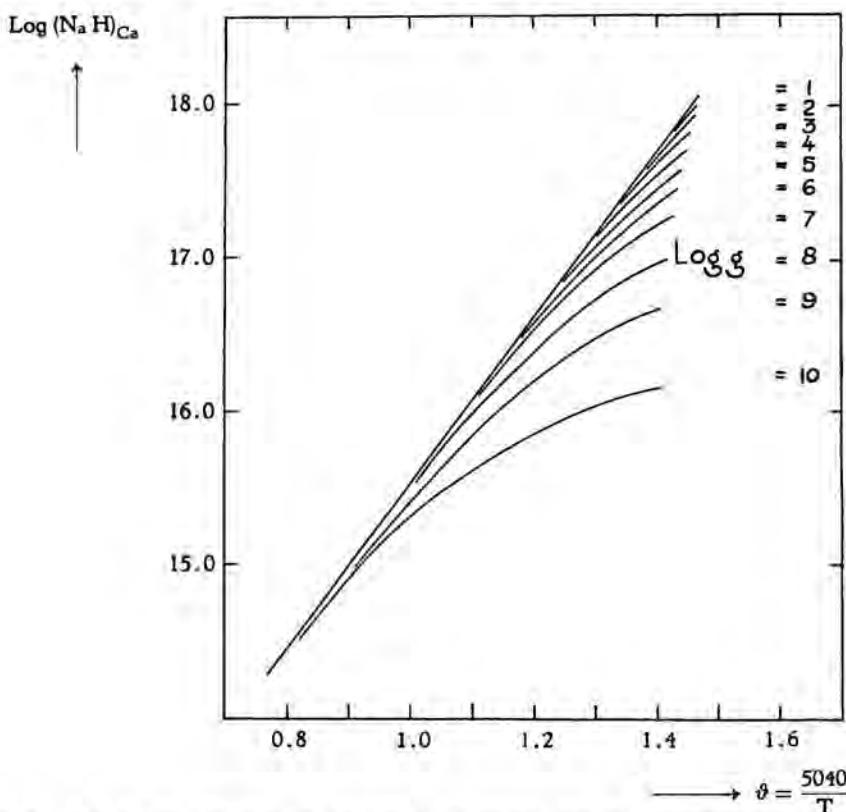


Fig. 4. The theoretical number of active Ca-atoms, for different temperatures and effective gravitational accelerations.

the temperature range here investigated, since $\log g$ is always < 5 , the values of $N_a H$ cannot be expected to be different for dwarfs and for giants.

The Ca-line at $\lambda 4227$ is a strong line and so its equivalent width can be calculated by the relation:

$$W_\lambda = \frac{\lambda^2}{c} \sqrt{\frac{e^2}{mc} N_a H \gamma f}.$$

¹⁰⁾ CHANDRASEKHAR and MÜNCH, Ap. J. 104, 446 (1946).

¹¹⁾ Compare also Miss ROSA, Zs. f. Ap. 25, 4 (1948).

The best value for f is 1.2. About the damping there is always the uncertainty how much of it is due to the radiation and how much to collisional damping. Therefore W_1 has been calculated for two different cases:

1. for pure radiation damping, $\gamma = \gamma_r = 1.6 \times 10^8$; and
2. for the damping as derived from the solar spectrum, which is ten times the damping caused by radiation. In figure 3 both theoretical cases have been considered.

In comparing the theoretical curves with the observations, it is found that the E.W., measured on small dispersion plates, are in general too great; GREENSTEIN, working with the Coudé-spectrograph, finds lower values which are much closer to the theoretical ones. This effect has been found many times before; it may be due to blends, which are not well discerned if the dispersion is small; or perhaps to photographic effects. We conclude that probably some correction of the order of -0.4 \AA will have to be applied to our measurements, but that the main differences between the luminosity classes will subsist and that the values of the damping constants will not materially be altered.

The giants and supergiants are pretty well in agreement with the theoretical curve for $\gamma = \gamma_r$. Evidently γ is greater for dwarfs, because the pressures in their atmospheres and the collisional damping are much higher than in giants or even supergiants. To the change from giants to supergiants corresponds no further decrease of γ ; this seems to show that we have already reached a minimum value for γ in the giants, which must evidently be γ_r . Thus for stars with $M < 3$ or $\log g < 2$, we have $\gamma = \gamma_r$, while for larger values of M the damping increases rapidly. For the sun ($M = 5$, $\log g = 4.4$) usually $\gamma = 10 \gamma_r$ is adopted; WRIGHT supposes that this applies to all dwarfs¹²⁾.

From figure 3 a somewhat smaller value for γ is deduced but this may be due to our assumption, that the effective optical depth is the same in dwarfs and in giants. Moreover, the material is still too small to derive more definitive values.

Sr II $\lambda 4077$.

Also for this spectral line the equivalent widths have been measured on the available plates (figure 5). The theoretical values of $\lambda 4077$ have been calculated in the same way as for $\lambda 4227$. For the Sr-abundance we assumed 2.35×10^{-9} ¹³⁾, while for all other quantities the same numerical values as for $\lambda 4227$ have been used. In figure 5 on the left side the theoretical values of W_1 have been plotted for $\gamma = \gamma_r$, and at the right for $\gamma = 10 \gamma_r$. The assertion that $\gamma = \gamma_r$ for giants and supergiants and $\gamma = 10 \gamma_r$ for dwarfs seems well-confirmed; the g -values from figure 5

¹²⁾ WRIGHT, Dom. Ap. Obs. 8, 1 (1948).

¹³⁾ UNSÖLD, Zs. f. Ap. 24, 306 (1948).

agree in a satisfactory way with those, mentioned in the literature for supergiants, giants and dwarfs^{14), 15), 16).}

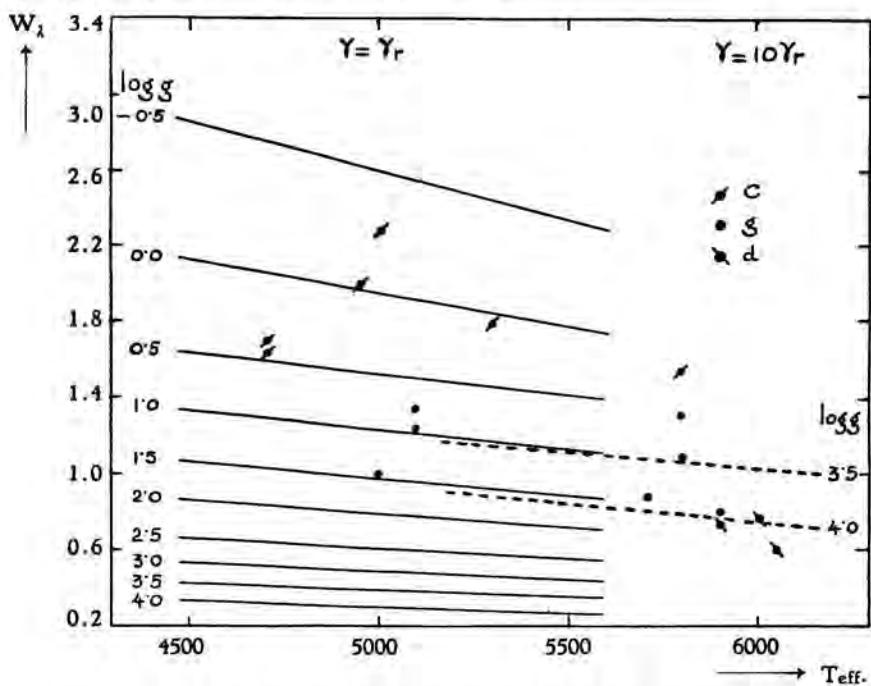


Fig. 5. Equivalent widths of Sr II λ 4077.

Fulldrawn lines: the theoretical values of W_{4077} for different values of $\log g$, computed for $\gamma = \gamma_r$.

Dotted lines: the same for $\gamma = 10\gamma_r$. The points represent the measured W_{4077} .

I am much indebted to Professor M. MINNAERT, who suggested this investigation and gave me his continuous help.

¹⁴⁾ GREENSTEIN, Ap. J. 107, 151 (1948).

¹⁵⁾ WRIGHT, Dom. Ap. Obs. 8, 1 (1948).

¹⁶⁾ PANNEKOEK and VAN ALBADA, Publ. Astr. Inst. Univ. Amsterdam, 6 (1946).

Botany. — *De hypothese voor de erfelijkheidsformules van de twee zuivere lijnen I en II van Phaseolus vulgaris op grond van kruisingssproeven. II.* By G. P. FRETS. (Communicated by Prof. J. BOEKEL.)

(Communicated at the meeting of January 29, 1949.)

We gaan nu over tot het onderzoek van de erfelijkheid van de bonen met de grootste breedten en resp. met een zeer grote en een niet zeer grote lengte van de F_3 -zaadgeneratie van 1935. Daarvoor gaan we de erfelijke samenstelling van enige bonenopbrengsten van tab. 5 na.

TABLE 5. The beanyields with the greatest mean breadths of F_3 -1935 and comparison beanyields of the I- and the II-line of 1935.

N	Pl.	n	B	Gr. var.	Sm. var.	L	Gr. var.	Sm. var.
F ₃ -1935. 1—6. The mean length is great; 7—12 the mean length is small.								
1	73	35	95.4	112	73	159	183	120
2	71	29	96.3	104	85	155.6	182	138
3	95	52	98	105	89	155	177	139
4	63	50	97.6	111	78	153.5	174	127
5	84	52	91.5	98	83	149.7	167	133
6	60	64	92.7	104	81	149.5	166	119
7	93	56	91.7	102	80	136.3	155	115
8	47	75	91.8	113	80	140.6	160	119
9	58	47	96.7	113	83	142.3	157	117
10	80	100	92.5	105	78	143.9	171	120
11	68	51	96.9	115	78	144.2	170	126
12	87	58	95.7	106	75	146.7	164	107
I-line 1935								
	1	36	96.4	117	73	158.2	193	128
	16	45	94.7	130	73	160.2	216	125
	10	49	90.6	106	66	149.3	178	112
II-line 1935								
	22	48	90.4	97	76	120.8	130	98

Dimensions in 0.1 mm. The standarddeviations and variation coefficients are omitted. The mean error (m) of the mean dimensions is for the lengths, $m_l = 1.5—2.5$, for the breadths $m_b = 1—2$.

Pl. 73. De bonenopbrengst van pl. 73 heeft de grootste gemiddelde lengte van de bonenopbrengsten van de F_3 -generatie van 1935. De uitgangsboon voor pl. 73 is van pl. 81, F_2 -1934 en heeft eveneens een zeer grote lengte (de grenswaarde) en grote breedte (tab. 6 en fig. 1, blz. 427). De formule van de gemiddelden van de bonenopbrengst van pl. 81 is $L_1 l_2 B Th$, cl. 1b. De bonenopbrengst van pl. 81 bevat, naast bonen met een grote breedte en een zeer grote lengte, zoals de uitgangsboon voor pl. 73, ook

TABLE 6. The heredity of beans of some beanyields of F_3 -1935 with great mean breadths. Four succeeding generations.

N.	Pl.	Bean	L	B	Th	W	LB	LTh	BTh	N.	Pl.	n	L	B	Th	W	LB	LTh	BTh
Initial beans are crosses (F_1) I \times II and II \times I										F_2 — 1934. Averages									
1	II \times I		124	79	67	43	64	54	85	1	81	90	142	87	70	62	61	49	80
2	II \times I		128	84	68	50	66	53	81	2	82	91	141	85	69	59	61	49	81
3	I \times II		152	86	68	58	57	45	79	3	70	51	139	92	74	69	67	53	80
4	I \times II		154	90	67	61	58	44	75	4	76	64	134	88	69	59	66	52	79
Initial beans of F_2 — 1934 for beanyields of F_3 — 1935										F_3 — 1935. Averages									
1	81	4p 1b	155	96	67	73	61	43	69	1	73	35	159	95	63	66	60	40	66
2	82	13p 3b	158	90	71	71	57	45	79	2	93	56	136	92	67	59	67	49	74
3	70	6p 3b	132	94	77	70	71	58	82	3	58	47	142	97	64	63	69	45	66
4	76	12p 2b	149	93	64	62	62	43	69	4	68	51	144	97	66	67	67	46	69
Initial beans of F_3 — 1935 for beanyields of F_4 — 1936										F_4 — 1936. Averages									
1	73	2p 1b	183	112	73	101	61	40	65	1	309	38	154	100	65	66	66	42	64
2a	93	1p 2b	134	94	71	61	70	53	76	2a	352	23	140	88	63	56	63	45	72
2b	93	6p 2b	155	102	75	74	66	48	74	2b	353	24	131	85	60	48	65	46	70
2c	93	10p 1b	116	83	56	39	72	48	68	2c	354	25	143	91	65	61	64	45	71
3a	58	1p 2b	152	113	60	75	74	39	53	3a	259	21	119	85	53	41	71	44	62
3b	58	7p 1b	167	110	64	83	66	38	58	3b	260	23	126	86	58	45	68	46	67
4a	68	1p 1b	170	109	65	98	64	38	60	4a	291	15	140	96	55	55	69	39	57
4b	68	4p 4b	143	96	72	68	67	50	75	4b	292	25	132	90	55	46	68	42	61
4c	68	9p 3b	156	103	75	87	66	48	73	4c	293	23	129	88	59	47	68	46	67
4d	68	13p 2b	137	100	68	68	73	50	68	4d	294	14	134	93	63	55	69	47	68
Initial beans of F_4 — 1936 for beanyields of F_5 — 1937										F_5 — 1937. Averages									
1a	309	7p 1b	178	113	57	84	63	32	50	1a	239	25	162	101	66	81	62	41	65
1b	309	3p 1b	168	107	71	77	64	42	66	1b	151	13	149	89	61	59	60	41	68
1c	309	3p 3b	142	103	69	66	73	49	67	1c	152	21	149	90	68	68	61	46	76
1d	309	3p 5b	150	99	68	69	64	45	69	1d	153	29	137	85	61	54	62	45	72
2	—									2	—								
3aa	259	6p 1b	135	95	56	50	70	42	59	3aa	206	29	131	87	61	54	66	47	70
3ab	259	6p 2b	128	95	55	49	74	43	58	3ab	207	28	125	89	55	45	71	44	62
3b	260	3p 1b	139	89	59	52	64	42	66	3b	208	30	131	86	62	53	66	47	72
4a	291	—								4a	—								
4ba	292	3p 1b	147	101	63	64	69	43	62	4ba	221	32	144	92	66	67	64	46	72
4bb	292	6p 1b	161	103	62	74	64	39	60	4bb	222	30	157	95	64	74	61	41	67
4bc	292	6p 2b	158	104	68	76	66	43	65	4bc	223	29	152	94	70	71	62	46	74
4bd	292	6p 3b	153	100	67	71	65	44	67	4bd	224	26	141	91	64	62	65	45	70
4ca	293	5p 1b	144	96	71	68	67	49	74	4ca	225	29	135	88	63	57	65	47	72
4cb	293	5p 2b	138	94	74	65	68	54	79	4cb	226	29	134	90	66	60	67	49	73
4d	294	2p 1b	140	102	68	67	73	49	67	4d	227	30	127	86	57	47	68	45	66

bonen met een grote breedte en een niet zeer grote lengte e.a. Bij de bonenopbrengst van 35 bonen van pl. 73, F_3 -1935 (tab. 7) zijn 9 bonen met de grote breedte, $b = 10.1$ — 11.2 mm. Al deze 9 bonen hebben een zeer grote lengte $l = 16.6$ — 18.3 mm. Er zijn 11 bonen met de grote breedte 9.6—10.0 mm. Ook deze 11 bonen hebben alle een zeer grote lengte $l = 15.7$ — 17.5 mm. Van 7 bonen met de breedte, $b = 9.1$ — 9.5 mm, is de lengte $l = 14.5$ — 15.3 mm. Van de overige 8 bonen is de breedte, $b = 7.3$ (dan volgt

TABLE 7. The breadths and lengths of the beans of 4 beanyields of F₃-1935 with great mean breadths of which the first (pl. 73) has a very great mean length whereas the mean length of the 3 other beanyields is not very great and comparison beanyields of the I- and the II-line of 1935.

Pl.	I	I	I	II	F ₃	F ₃	F ₃	F ₃	Pl.	I	I	I	II	F ₃	F ₃	F ₃	F ₃
	1	16	10	22	73	93	58	68		1	16	10	22	73	93	58	68
Br.	Length of beans								Br.	Length of beans							
66			112						90				120		137	135	
72			115						90				127			137	
73	128	132	124		98	120			90				124				
76									90				123				
77	132	132	134						90				118				
77		135	127						90				117				
78		131							90				127				
78		135							91	162			144	122	152	136	139
78		134							91	148			126		126	134	136
78		125							91				121				
79			127						91				127		148		
80	146	128				121			91				128				
80	138				118				92				127	149	136	138	143
81		137			128	115			92				166	130	145	141	140
81					120				92				163	127	146	131	
82		140			131				92				120		136		
82			134						92				123		145		
83			142	116	141	129	118		92				144				
83						116			93				152	155	128	153	141
83						118			93				156	120	153	138	142
84			134	111		133			93				150	123		138	129
84			142						93				150		130	138	146
84			139						93				152		133	145	
85		140	145			140			93						142		
85		131	147						93						137		
86	142	149	157	120	136	138	135	133	94				127	149	134		141
86	148		151	114		141			94					150	135		135
86	145		143			130			94						132		
87	145		140	115		125			94						139		
87	148		142	116		134			94						139		
87	152					136			94						128		
87	141								94						143		
87	149								95	159			160	118		146	140
88	155	151	149	116	146		136	134	95				166	127			140
88	148	147		110	152				95				154	119			
88				105					95				157	127			
88				123					95				155	123			
89	152	171	153	120		134	137		95				145				
89				120			138		96				159	150	124	161	145
89				121			137		96				157	158	122	157	146
89				128					96						158	142	140
89				114					96						141	133	145
89				116					96								141
90	154	157	123	156	142	130	137		97	158	152	160	130	158	142	152	149
90	157	153	118	154	142	139	144		97	165	169	164		169	139	142	145

TABLE 7 (Continued).

Pl.	I 1	I 16	I 10	II 22	F ₃ 73	F ₃ 93	F ₃ 58	F ₃ 68	Pl.	I 1	I 16	I 10	II 22	F ₃ 73	F ₃ 93	F ₃ 58	F ₃ 68
Br.	Length of beans								Br.	Length of beans							
97	157	177			136	153	148	103	161		159				157	152	
97		165				146	142	103	168		153				144	148	
97						144		103							171	150	145
98	164	168	149		174	141	156	103							176	148	156
98		192	163		175		145	104		181						145	
98		182			159		140	105	177								146
99			163		168		145	105									156
99							146	106			175						
99							133	107	178	158					176	155	
99							146	107									151
99							154	108		173							160
99							149	108		181							
99							139	109		180							170
99							136	110	162	160							167
100		167	148		166		150	143	110	168	183						
100					161		141	140	110	188	182						
100							152	145	111	148							
100								137	111	167							
100								150	112		195				183		
101					166			150	112		196						
101					166			146	113								152
101					178			153	115	183							
102	175	162	178			155	152	153	115	186							
102		171	160			152	147		117	193							
102			157					139		121		179					
102								147		130		216					

8.0)—9.0 mm, en de lengte, $l = 12.0$ (dan volgt 12.5)—15.6 mm. De bonenopbrengst van pl. 73 bevat dus veel bonen met een grote breedte (9.6—11.2 mm) en een zeer grote lengte (formule $L_1 L_2 B_1 b_2$), daarbij is er geen met een grote breedte en niet zeer grote lengte (formule $L_1 l_2 B_1 b_2$). Er is overeenstemming met bonenopbrengsten van de I-lijn van 1935 (tab. 7 en blz. 425).

Van pl. 73 is in 1936 een boon voortgekweekt. Ze leverde pl. 309, F₄-generatie 1936, fig. 2, blz. 430. Van de 38 bonen van deze bonenopbrengst, hebben 22 bonen de grote breedte $b = 10.1$ —11.3 mm. Twee bonen ($b = 10.2$ en $= 10.3$ mm) hebben de lengte $l = 14.2$ mm, één boon ($b = 10.3$ mm) heeft de lengte $l = 14.5$ mm. Vijf bonen ($b = 10.1$, 10.4—10.8 mm) hebben de lengte $l = 15.0$ —15.5 mm. De overige 14 bonen hebben alle een lengte groter dan 15.5 mm ($l = 15.6$ —17.8 mm). De bonenopbrengst van pl. 309 bevat brede en minder brede peulen. De brede peulen bevatten zeer brede bonen. Deze bonen hebben een hoge LB-index ($LB = 68$ —73). Daardoor is ook de gemiddelde LB-index hoog, ($LB_{\text{gem}} = 66$, tab. 6) en ligt in het characterogram LB hoog (fig. 2b). De bonenopbrengst is ten opzichte van bonen met de form. $L_1 L_2 B_1 b_2$ niet geheel eenvormig. Ze bevat veel bonen met een grote breedte en een zeer grote lengte, zodat de samenstelling van de bonenopbrengst wel wijst op de erfelijkheid van bonen met de form. $L_1 L_2 B_1 b_2$ voor de lengte en de breedte.

Van pl. 309 zijn in 1937 4 bonen voortgekweekt. Ze leverden pl. 239 en 251—253, F₅-1937. De uitgangsboon van pl. 309 voor pl. 239 is de boon met de grootste breedte en de grootste lengte van deze bonenopbrengst. De breedte is zeer groot, daardoor is de L B-index vrij hoog (tab. 6) en ligt ook in het characterogram (fig. 3a) LB vrij hoog.

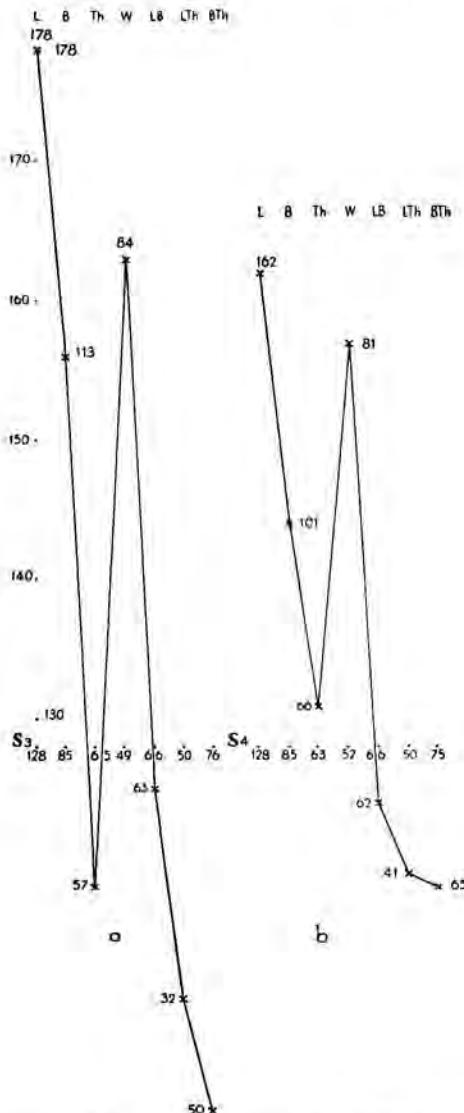


Fig. 3. a. Characterogram of 7 p 1 b, initial bean of pl. 309, F₄-1936 for pl. 239, F₅-1937.
b. Idem of the average of the beanyield of pl. 239, F₅-1937. S₃ = Standardcharacterogram of 1936; S₄ = id. of 1937.

De bonenopbrengst van 25 bonen van pl. 239, F₅-1937, fig. 3b, bevat veel bonen met een grote breedte en een zeer grote lengte. Van 13 bonen met de grote breedte, $b = 10.1 - 10.9$ mm, is de lengte 15.8—17.5 (van één boon, $b = 10.3$ is $l = 15.2$ mm). Van 7 bonen met de grote breedte $b = 9.6 - 10.0$ mm, is de lengte, $l = 15.7 - 17.0$ mm. Van de overige 5 bonen is $b = 8.9 - 9.5$ mm en de lengte, $l = 14.8 - 15.4$ mm. De samenstelling van de bonen-

opbrengst van pl. 239 beantwoordt voor de lengte en de breedte aan de form. $L_1 L_2 B_1 b_2$. De uitgangsboon 7 p i b van pl. 309, F₄-1936 voor pl. 239 heeft voor de lengte en de breedte de form, $L_1 L_2 B_1 b_2$ in homozygote of bijna homozygote vorm. De uitgangsbonen van pl. 309 voor de pl. 151—153 F₅-1937 zijn de 1ste, 3de en 5de (laatste) boon van dezelfde peul. De lengten van deze 3 bonen verschillen zeer (tab. 6).

Pl. 151. De 13 bonen van de kleine bonenopbrengst van pl. 151 bevat 9 bonen met de grote breedte 9.8—8.6 mm. Van deze 9 bonen hebben er 4 de zeer grote lengte 16.6—15.6 mm. De LB-index is, LB = 57—60. Van de 5 overige van deze 9 bonen is de lengte, 1 = 13.8—15.1 mm; tweemaal is de LB-index, LB = 59; van 3 bonen is zij 64—67. Een boon heeft dus een hoge LB-index, bovendien een grote breedte ($b = 9.2$ mm) en een niet zeer grote lengte ($1 = 13.8$ mm). Ze beantwoordt aan de form, $L_1 l_2 B_1 b_2$. Er zijn twee grens gevallen. Vier bonen hebben de breedte 8.0 (dan 8.4)—8.5 mm en de lengten 13.6—15.6 mm. De LB-index is, LB = 54—62 (dan 58). De bonenopbrengst van pl. 151 bevat vooral bonen met de form. $L_1 L_2 B_1 b_2$ en enkele met de form. $L_1 l_2 B_1 b_2$. De uitgangsboon van pl. 309 voor pl. 151 is niet homozygoot voor de form. $L_1 L_2 B_1 b_2$.

Pl. 152. De uitgangsboon van pl. 309 voor pl. 152 is een zeer goed voorbeeld van een boon met de form. $L_1 l_2 B_1 b_2$ (tab. 6). Van de 21 bonen van de bonenopbrengst van pl. 152 hebben er 19 de grote breedte, $b = 8.6—9.8$ mm. Acht er van hebben de zeer grote lengte 15.6—17.2 (dan 16.2) mm. De index van deze 8 bonen is laag, LB = 57—59. Van 4 bonen is de lengte 1 = 15.2—15.4 mm; van deze bonen is LB = 56—64 (dan 63). De 7 bonen met de niet zeer grote lengte 12.9—14.9 (dan 14.4) mm, hebben alle een hoge index, LB = 63—69. De bonenopbrengst van pl. 152 heeft een gemengde samenstelling. Ze bestaat bijna uitsluitend uit bonen met een grote breedte (van één boon is $b = 8.2$ mm en er is een onvolgroeide boon). Daarvan zijn er met een zeer grote, doch ook met een niet zeer grote lengte. Er zijn veel bonen met een lage en met een hoge LB-index. Er zijn 10 bonen met de LB-index 55—60 (7 met LB = 55—57) en 9 met de LB-index 63—69. De uitgangsboon, ofschoon phaenotypisch duidelijk $L_1 l_2 B_1 b_2$ is genotypisch heterozygoot.

Pl. 153. De uitgangsboon van pl. 309 voor pl. 153 heeft een grotere lengte dan die voor pl. 152 en een kleinere lengte en breedte dan die voor pl. 151 (tab. 6). Van de bonenopbrengst van 29 bonen zijn er 12 met de grote breedte 8.7—9.4 mm. Er zijn er enige met een zeer grote lengte en vele met een niet zeer grote lengte. Er zijn 8 bonen met een hoge LB-index, LB = 63 (dan 65)—69. Er zijn zeer veel bonen met een kleine breedte, $b = 7.5—8.5$ mm. Daarbij zijn er 7 met een kleine lengte. Er zijn hoge en lage LB-indices. Van 7 bonen is LB = 63 (dan 65)—67, van 10 bonen is LB = 54—61. Ook de bonenopbrengst van pl. 153 is samengesteld. Naast bonen met de formules $L_1 L_2 L_1 l_2$ zijn er met de form. $L_1 l_2 B_1 b_2$. Er zijn ook veel bonen met de form. $L_1 l_2 b_1 b_2$ en $l_1 l_2 b_1 b_2$.

Uit de samenstelling van de bonenopbrengsten van de pl. 151—153 blijkt, dat de uitgangsbonen van pl. 309 voor deze planten de form. $L_1 L_2 B_1 b_2$ niet in de homozygote vorm hebben. Ze blijven hierin achter bij de uitgangsboon van pl. 309 voor pl. 239. Wel zijn de gemiddelde LB-indices van de pl. 151—153 laag. Deze bonenopbrengsten, besluiten we, steunen wel de opvatting, dat de uitgangsboon van pl. 73 voor pl. 309 in belangrijke mate homozygoot is voor de form. $L_1 L_2 B_1 b_2$.

De gegevens van de descendantie en de ascendantie van de bonenopbrengst van pl. 73, F₃-1935, die betrekking hebben op een onderzoek over 4 generaties, bevestigen, dat van deze bonenopbrengst de formule voor de lengte en de breedte $L_1 L_2 B_1 b_2$ is. Er blijkt de erfelijkheid uit van bonen met een grote breedte en een zeer grote lengte. Het gaat er steeds om, om de erfelijkheid te herkennen te midden van een grote niet-erfelijke variabiliteit.

Pl. 93. De bonenopbrengst van pl. 93 heeft de kleinste gemiddelde lengte van de bonenopbrengsten met een grote gemiddelde breedte van de F_3 -generatie van 1935 (tab. 6). De bonenopbrengst van 56 bonen van pl. 93 (tab. 7) bevat slechts 2 bonen met de grote breedte, $b = 10.2$ mm. Deze bonen hebben niet een zeer grote lengte ($l = 15.5$ en $= 15.2$ mm). Er zijn 8 bonen met een breedte van $b = 9.6$ — 10.0 mm (de grootste breedte is $b = 9.8$ mm). De lengte van deze bonen, is $l = 13.6$ — 14.6 mm. Er zijn dus geen bonen met een grote breedte en een zeer grote lengte. Maar de grootste breedten treffen we ook niet aan in de bonenopbrengst van pl. 93. Er zijn 25 bonen met de breedte, $b = 9.1$ — 9.5 mm; de lengte is $l = 12.8$ (dan volgen $l = 13.0$ en 13.2)— 14.8 , (dan volgt $l = 14.6$) mm. Het is een zeer gelijkmatig samengestelde groep bonen met een grote breedte, waarbij er geen enkele is met een zeer grote lengte. De uitgangsboon voor pl. 93 is van pl. 82, F_2 -1934. Deze boon heeft een matig grote breedte en een zeer grote lengte. De overige 5 bonen van de peul van de uitgangsboon hebben alle een kleinere lengte ($l = 13.2$ — 15.2 mm).

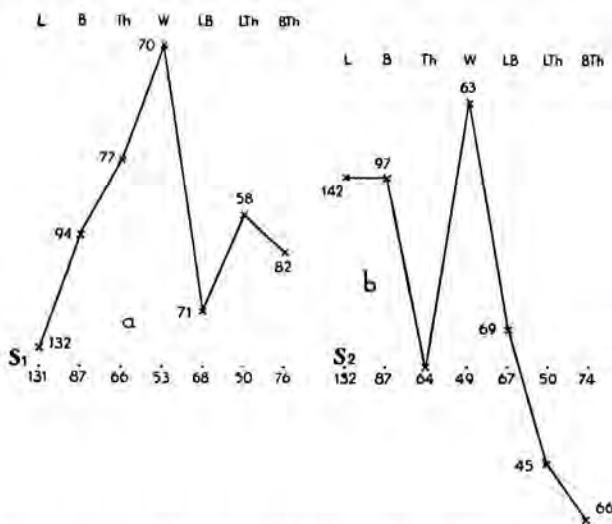


Fig. 4. a. Characterogram of 6 p 3 b, initial bean of plant 70, F_2 -1934 for pl. 58, F_3 -1935.
b. Idem of the averages of the beanyield of pl. 58, F_3 -1935.

Van pl. 93 zijn in 1936 3 bonen voortgekweekt. Ze leverden in 1936 de pl. 352, 353 en 354, F_4 -1936. Van de uitgangsboon van pl. 93 voor pl. 352 (tab. 6) behoort de breedte tot de breedte-groep 9.1—9.6 mm, de lengte is niet zeer groot. Alle 6 bonen van de peul van de uitgangsboon hebben een overeenkomstige breedte, $b = 9.3$ — 9.4 mm, ook een overeenkomstige lengte, $l = 13.0$ (dan 13.4)—14.1 mm. De LB-index is hoog; LB = 66—72. Van de bonenopbrengst van pl. 352 van 23 bonen, is de grootste breedte, $b = 10.7$ mm. Van deze boon is de lengte $l = 14.3$ mm. Dan volgen 3 bonen met de breedten, $b = 9.9 = 9.4$ en $= 9.3$ mm en de lengten $l = 14.6, = 14.3$ en $= 13.5$ mm. Deze bonen hebben dus als phaenotype de form, $L_1 l_2 B_1 b_2$; ze zijn alle van eenzelfde peul. Van de overige bonen is de breedte 8.2—8.9 mm en de lengte 13.5—14.3 mm. Dit zijn geen kenmerkende bonen voor de form, $L_1 l_2 B_1 b_2$. De uitgangs-

boon van pl. 93 voor pl. 352 heeft de form. $L_1 l_2 B_1 b_2$ niet in de homozygote of bijna homozygote vorm.

Pl. 353. De uitgangsboon van pl. 93 heeft een grote breedte $b = 10.2$ mm, de lengte heeft de grenswaarde voor $L_1 L_2$ ($l = 15.5$ mm). Van de bonenopbrengst van 27 bonen van pl. 353 heeft één boon de grote breedte, $b = 9.8$ mm, de lengte is $l = 15.0$ mm, dit is dus geen kenmerkende boon voor de form. $L_1 l_2 B_1 b_2$. Van de overige bonen is de breedte, $b = 9.3$ — 7.5 mm en de lengte $l = 11.5$ — 13.7 mm. Deze afmetingen van de bonen van de bonenopbrengst van pl. 353 passen weinig bij die van de uitgangsboon van pl. 93, die zelf, vergeleken met de bonenopbrengst van pl. 93, een niet-erfelijke plus-variant is. Ook moeten we er mee rekening houden, dat bonenopbrengsten van op elkaar volgende jaren, slechts ten dele vergelijkbaar zijn. Bij de bonenopbrengst van pl. 353 staat aangetekend „matig, enkele peulen groen” (dus misschien onrijp). Dergelijke aantekeningen zijn er bij veel bonenopbrengsten. Het gewas van 1936 was waarschijnlijk iets minder goed dan dat van 1935. Tab. 2 geeft de gemiddelde lengte en het gemiddelde gewicht van het materiaal van de I- en de II-lijn van 1934, '35, '36 en '37 ter vergelijking. De LB-indices van pl. 353 (en van pl. 352) zijn nog al hoog, ook die van de uitgangsboon van pl. 93 is vrij hoog. Hier blijkt dus verwantschap met de form. $L_1 l_2 B_1 b_2$. Alles tezamen geeft de bonenopbrengst van pl. 353 geen duidelijke aanwijzingen voor de formule van de lengte en de breedte van de uitgangsboon.

Pl. 354. De uitgangsboon van pl. 93 is zeer klein (tab. 6). De 3 andere bonen van de peul komen onderling overeen en zijn veel groter. Er zijn 4 bonen met een grote breedte, $b = 9.6$ — 10.1 mm en de lengte 14.9—16.1 mm. De overige bonen hebben de breedte, $b = 8.5$ — 9.4 mm en de lengte 13.5—15.1 mm. Er zijn slechts 2 lage LB-indices ($LB = 59$ en $= 60$). Ook de bonenopbrengst van pl. 354 is niet kenmerkend voor bonenopbrengsten met de formule $L_1 L_2 B_1 b_2$ of $L_1 l_2 B_1 b_2$ van de uitgangsboon.

Pl. 58 is de 3de bonenopbrengst van F_3 -1935, die we hier bespreken, de 2de met een niet zeer grote gemiddelde lengte. Ze heeft een grote gemiddelde breedte en een niet zeer grote gemiddelde lengte; de gemiddelde LB-index is hoog (tab. 5 en 6).

Pl. 58. De uitgangsboon is van pl. 70, F_2 -1934 (tab. 6 en fig. 4) en heeft fraai de formule $L_1 l_2 B_1 b_2$ voor de lengte en de breedte. Ook de 2de en de 4de boon van de peul van de uitgangsboon hebben deze formule. In de bonenopbrengst van 51 bonen van pl. 70 zijn 11 bonen met een grote breedte en een niet zeer grote lengte ($b = 9.6$ — 10.0 ; $l = 13.3$ (dan 13.9)— 15.2 mm; $LB = 65$ — 74). De bonenopbrengst van 46 bonen van pl. 58, F_3 -1935 (tab. 7) heeft 5 bonen met de grote breedten, $b = 10.7$ — 11.3 mm. Van 3 van deze bonen is de lengte niet zeer groot, $l = 15.1$ — 15.5 mm, van 2 zeer groot, $l = 16.0$ en $= 16.7$ mm; $LB = 66$ en $= 68$). Van 9 bonen is de breedte, $b = 10.1$ — 10.4 mm en de lengte, $l = 13.9$ — 15.7 (dan volgt 15.2) mm. De LB-index van de boon met $l = 15.7$ mm, is $LB = 66$. Er zijn 12 bonen met de breedte, $b = 9.6$ — 10.0 mm; de lengte is $l = 13.3$ (dan 14.0)— 15.2 mm. Al deze bonen hebben dus een niet zeer grote lengte. Er zijn ten slotte 11 bonen met de breedte, $b = 9.1$ — 9.6 mm en $l = 11.7$ (deze is van een boon, die de laatste is in de peul; dan volgt $l = 12.8$)— 14.5 (dan volgt 14.0) mm. Er zijn in de bonenopbrengst van 47 bonen van pl. 58, dus 26 bonen met een grote breedte en deze hebben bijna alle een niet zeer grote lengte. Bij de bonenopbrengst van pl. 58 staat aangetekend: „Alle bizonder brede peulen”. Deze bonenopbrengst voldoet zeer goed

aan de eisen van een bonenopbrengst, waarvan de formule van de lengte en de breedte van de uitgangsboon $L_1 l_2 B_1 b_2$ is. Wij zagen, dat ook het phaenotype van de uitgangsboon deze formule heeft.

Van pl. 58 zijn in 1936 2 bonen voortgekweekt. Ze leverden de pl. 259 en 260, F₄-1936 (tab. 6). De uitgangsboon voor pl. 259 heeft de grote breedte, $b = 11.3$ mm en de niet zeer grote lengte, $l = 15.2$ mm, L B = 74, (fig. 5). Nog 3 bonen van deze peul hebben eveneens een grote breedte (10.2—10.3 mm); de lengte van deze 3 bonen is $l = 14.7$, 15.2 en 15.7 mm. Bij de bonenopbrengst van pl. 259 van F₄-1936, staat aangegetekend: „slecht, veel schimmel, bonen meest iets aan de rimpelige kant” (blz. 584 en tab. 3). De bonen zijn klein, hebben een klein gewicht; gew. = 30—50 cg (dan volgen 49 en 44 cg). Er zijn slechts 2 bonen met een breedte groter dan 9 mm, nl. $b = 9.5$ mm. Deze hebben een niet zeer grote en een kleine lengte, $l = 13.5$ en $= 12.8$ mm. De L B-index is L B = 70 en = 74. Er zijn slechts 3 bonen met de breedte 8.6—9.0 mm, nl. $b = 9.0$, 8.9 en 8.7 mm. De kleinste breedte is $b = 7.7$ mm ($l = 11.4$ mm). De L B-indices zijn alle hoog, (L B = 68—74). Het kan zijn, — het is waarschijnlijk —, dat we hier met kleine bonen als gevolg van slechte milieu-verhoudingen te doen hebben, dus met een achterblijven in de groei. De bonen zijn dan phaenotypisch klein. De hoge L B-indices wijzen daarbij op de mogelijkheid, dat de formule voor de lengte en de breedte genotypisch van deze bonen $L_1 l_2 B_1 b_2$ is.

De 2 bonen met de grootste breedten, beide van dezelfde peul (het is een peul met 2 bonen) zijn in 1937 voortgekweekt en leverden de pl. 206 en 207, F₅-1937 (tab. 6).

Pl. 206. F₅-1937. Er zijn in de bonenopbrengst van 29 bonen van pl. 206 8 bonen met de breedte, $b = 9.1$ —9.5 mm; de lengte is $l = 13.0$ —14.4 (dan nog eens 14.4, dan 13.9) mm. Van één boon is de L B-index = 65; van de overige bonen is L B = 66—71. Er zijn 10 bonen met de grote breedte, $b = 8.6$ —9.0 mm; de lengte is $l = 13.0$ —13.8 mm. De L B-index is 64—68. Van 11 bonen ten slotte is $b = 7.5$ (dan volgt 8.0)—8.5 mm; de lengte is 10.7 (dan volgt $l = 11.2$, dan 12.0)—13.6 (dan 13.2) mm. L B-index = 62—72. Ook van pl. 206 is het gewicht der meeste bonen niet groot; w = 29 (dan 38)—70 (dan 65) cg. De bonenopbrengst van pl. 206 voldoet goed aan het phaenotype $L_1 l_2 B_1 b_2$ en de uitgangsboon daarvan aan het genotype $L_1 l_2 B_1 b_2$.

Pl. 207. Het is van belang, om met pl. 206, pl. 207 te vergelijken, waarvan de uitgangsboon van dezelfde peul is. Deze uitgangsboon is de laatste boon van de peul met 2 bonen en is een goed voorbeeld van een boon met de form. $L_1 l_2 B_1 b_2$ (tab. 6 en in Proc. 52, no. 6, fig. 6). In de bonenopbrengst van pl. 207 zijn 9 bonen met de grote breedte $b = 9.2$ —9.8 mm; de lengte is $l = 12.6$ —14.4 (dan volgen $l = 14.2$, dan $l = 13.7$) mm. De L B-index is L B = 68—76. Er zijn 12 bonen met de grote breedte, $b = 8.6$ —8.9 mm; de lengte is $l = 11.6$ —12.8 (dan 12.5) mm. De L B-index is L B = 68—74. Ten slotte zijn er 7 bonen met de breedte, $b = 8.1$ —8.5 mm en de lengte $l = 11.3$ (dan 11.8)—13.2 (dan 12.7, dan 12.1) mm. De L B-index is L B = 64—72. De bonenopbrengst van pl. 207 heeft een zeer gelijkmatige samenstelling. Ze is een goed voorbeeld van het phaenotype $L_1 l_2 B_1 b_2$. Met het phaenotype van de bonenopbrengst van pl. 207 stemt het phaenotype $L_1 l_2 B_1 b_2$ van de uitgangsboon nog beter overeen, dan dit bij de bonenopbrengst van pl. 206 het geval is. We nemen aan, dat het genotype van de uitgangsboon van pl. 259, F₄-1936 voor pl. 207, F₅-1937, $L_1 l_2 B_1 b_2$ is.

Pl. 260. F₄-1936. De uitgangsboon van pl. 58, F₃-1935 voor pl. 260 heeft, evenals die voor pl. 259, een zeer grote breedte ($b = 11.0$ mm), doch een zeer grote lengte ($l = 16.7$ mm). De L B-index is L B = 66. De 2de en laatste boon van de peul met 2 bonen van de uitgangsboon heeft een grote breedte ($b = 10.2$ mm) en een niet zeer grote lengte ($l = 14.7$ mm). De L B-index is L B = 69. Van de 23 bonen van de bonenopbrengst van pl. 260 hebben 17 bonen de breedte $b = 8.6$ —9.1 (dan volgt 8.9) mm en de lengte 12.3—13.9 (dan volgt 13.5) mm. Deze 17 bonen hebben een niet zeer grote lengte (L B = 64 (dan 66)—71). Alle bonen van de bonenopbrengst hebben een hoge L B-index.

Daaronder zijn er 6 met een niet grote breedte ($b = 7.7-8.5$ mm). De bonenopbrengst voldoet goed aan het phaenotype $L_1 l_2 B_1 b_2$ en de uitgangsboon kan het genotype $L_1 l_2 B_1 b_2$ hebben. We moeten daarbij aannemen, dat de zeer grote lengte van de uitgangsboon ($l = 16.7$ mm) een niet-erfelijke variant is van de formule $L_1 l_2$.

Pl. 208. F₅-1937. Van pl. 260, F₄-1936 is in 1937 één boon voortgekweekt; ze leverde pl. 208 (tab. 6). De uitgangsboon van pl. 260 voor pl. 208 is de boon met de grootste lengte ($l = 13.0$ mm, tab. 6 en Proc. 52, no. 6). Van de overige 4 bonen van de peul met 5 bonen is de lengte $l = 12.9-13.5$ (dan 13.2) mm. De LB-indices zijn LB = 66-67. Er zijn in de bonenopbrengst van 30 bonen van pl. 208, F₅-1937 18 bonen met de grote breedte 8.6-9.2 mm; de lengte is 12.7-14.0 (dan 13.7) mm. De LB-index is 64-69. Van de overige 12 bonen is $b = 8.0-8.5$ mm, $l = 12.0-13.4$ (dan 13.0) mm; LB = 62-69. Er zijn dus 11 bonen met een kleine lengte (d.w.z. $l = 13.0$ mm en kleiner). Het phaenotype van de bonenopbrengst, voor de lengte en de breedte, is $L_1 l_2 B_1 b_2$.

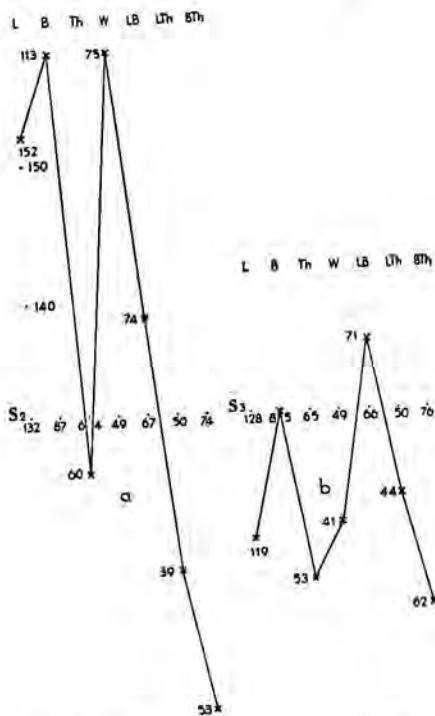


Fig. 5. a. Characterogram of 1 p 2 b, initial bean of pl. 58, F₅-1935 for pl. 259, F₄-1936.
b. Idem of the averages of the beanyield of pl. 259, F₄-1936.

De uitgangsbonen voor de pl. 208 en 207, van pl. 260 en 259 verschillen voor de lengte en de breedte beide zeer, daardoor ook de LB-indices. De bonenopbrengsten van de pl. 260 en 259 verschillen in mindere mate (tab. 6). De bonenopbrengst van pl. 260 en haar uitgangsboon beantwoorden in minder raszuivere vorm aan de formule $L_1 l_2 B_1 b_2$ van de lengte en de breedte dan de bonenopbrengst van pl. 259 en haar uitgangsboon. Hiermee is in overeenstemming, dat ook de uitgangsboon van pl. 58 voor pl. 260 in dezelfde richting verschilt van de uitgangsboon van pl. 58 voor pl. 259 (tab. 6 en Proc. 52, no. 6).

Anatomy. — *The digital formula in relation to age, sex and constitutional type.* II. By J. HUIZINGA. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of March 26, 1949.)

7. Considerations on the causation of differences in prominence.

ECKER (1875) believed the problem of prominence differences to be connected with the mobility of the metacarpals. As a measure of this mobility he uses the extent to which the capituli can be passively displaced in a volo-dorsal direction. The most laterally situated metacarpals (I and V) are movable to the greatest extent and correspond to the least prominent fingers. Metacarpal III is almost immovable and corresponds to the most prominent finger. In the human hand metacarpal IV is movable to a greater extent than II and should, therefore, correspond to a finger shorter than that belonging to II. If mobility and relative length are really correlated in this way, this suggests that Rd. prominence is typical of man, a conclusion which — as we have already pointed out — WOOD JONES (1944) reached by another route. In the Uln. type of the anthropoids, metacarpal II should, according to this reasoning, be more mobile than metacarpal IV, although ECKER admits 'I do not know whether the index finger in apes is more mobile than the ring finger'.

WEISSENBERG (1895) states that BRAUNE was of the opinion that the Uln. type develops as a result of the action of the considerably stronger flexors, whereby the hand undergoes ulnar reflexion so that the apparent length of the ring finger is increased. WEISSENBERG criticises this view remarking, that Rd. types would then be expected to be the rule in cases where 'the hand position in question has not yet had the opportunity of reaching its full development', e.g. in children 'which is, however, not the case'.

But we have already seen that WOLOTZKOV (1924) states that the Rd. type does actually occur with greater frequency in children, a finding that is confirmed by our own investigations (see below).

WECHSLER (1939) also attempted a functional explanation: as a result of the relatively greater use of the radially-situated fingers, a corresponding growth in breadth of the muscles in this region is to be expected. The stronger are these muscles the more the second metacarpo-phalangeal joint will be pushed thumb-wards, producing a bend in the course of II. Through this bending II becomes shortened as regards prominence and the Uln. type develops. 'It would be of interest in this connection to study the degree of correlation between the L/Br index of the hand with the force of pressure.' WOOD JONES (1944) remarks casually, in connection with the development of the elongated index finger:

'Many have dwelt upon some of the separate outcomes of the human dominance of the index finger, and in a rather topsyturkey way the human ability to point out objects has received certain attention, but there is a grave danger of mixing up physical and psychical specialisations, if the act of pointing is made the mainspring of the development'.

III. Investigation.

We had the opportunity of studying sexual dimorphism, the variation with age and the connection with constitutional type.

A. Subjects.

In 1454 male and 858 female individuals from the age of four years upwards we recorded the age and the mode of prominence of the fingers; in 121 boys and 74 girls aged 7 to 11 years we also determined the constitutional type. (By a child aged 10 years, for instance, we understand one older than 9.6 years and younger than 10.6 etc.).

The groups are shown, together with other data in a table which will be discussed.

In 56 male and 44 females foetuses we made an estimate of the age and determined the prominence type of the right hand.

B. Method.

As stated above (see under 6) we used the method described by WOLOTZKOI (1924) for determination of the prominence type.

The observations on foetuses were all checked independently by a medical colleague.

C. Results.

Arranged according to age (in months) and prominence types, the (right) hands *before birth* show the following picture:

Muths.	♂ ♂			♀ ♀		
	Rd.	Uln.	T.	Rd.	Uln.	T.
3	1	—	—	2	—	—
4	9	—	1	6	—	3
5	6	2	2	6	—	2
6	5	—	1	4	1	—
7	5	3	2	2	1	—
8	6	3	4	5	5	1
9	4	1	1	4	2	—
Total	36	9	11	29	9	6

Although the *ulnar* prominence type occurs in the first six months of gestation, it does not appear in any considerable proportion until the *last three months*. *Increasing age* is accompanied — at any rate during intrauterine life — by *loss of the radial prominence type* in favour of the

ulnar type. The *mean* frequency of the Rd. type before birth for the two sexes is 65 %.

Our results, thus, differ appreciably from those of MIERZECKI (see remarks above); no other data were available for purposes of comparison. Like WOLOTZKOI, we arranged our (post-natal) data in the age-groups proposed by STRATZ.

Our group aged 1—4 years was too small to be of much significance in itself. The percentages calculated for these children (in brackets) fit reasonably well into the general picture when taken in connection with the other age groups (including the foetal group).

Males.

Age (yrs)	n	Rd.	Uln.	T.	%/0/Rd.	%/0/Uln.	%/0/T.
1 to 4 incl.	13	9	4	—	(69)	(31)	—
5 " 7 "	218	111	90	17	51	41	8
8 " 10 "	405	191	186	28	47	46	7
11 " 14 "	362	148	192	22	41	53	6
15 " 20 "	284	108	164	12	38	58	4
21 etc.	172	60	97	15	35	56	9
Total	1454	627	733	94			

Females.

Age (yrs)	n	Rd.	Uln.	T.	%/0/Rd.	%/0/Uln.	%/0/T.
1 to 4 incl.	7	6	1	—	(86)	(14)	—
5 " 7 "	178	102	59	17	57	33	10
8 " 10 "	331	206	98	27	62	30	8
11 " 14 "	236	139	78	19	59	33	8
15 " 20 "	56	23	26	7	41	46	13
21 etc.	50	24	20	6	48	40	12
Total	858	500	281	76			

The frequency of occurrence of the prominence types in the different extrauterine age-groups shows (see also figs. 1 and 2) that our observation on foetuses is not specific for this group. After birth also it appears that increasing age is accompanied by a loss of the Rd. type, but without this type disappearing altogether.

The figures and tables enable us to draw certain conclusions as to sexual dimorphism.

a. *Males.* In the majority of males the Rd. prominence type is present at birth. The number of such types undergoes a (probably gradual) decrease and corresponds to about 50 % in the first extension period (5—7 years).

The radial types, and probably also the T. types, change into ulnar types.

About the 21st. year the phenomenon referred to WOLOTZKOI is found: The number of ulnar forms shows a slight decrease although not, as stated

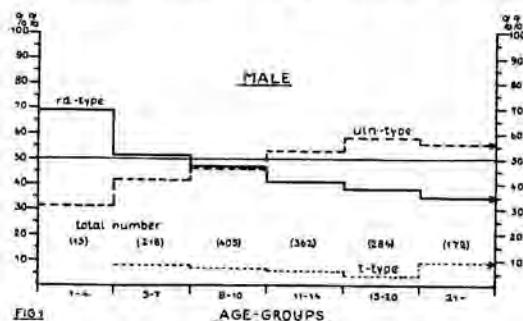


Fig. 1.

by this investigator, in favour of radial types but in favour of the T. types.

If we estimate roughly the percentage of radial forms at birth to be almost 70 %, we see that this falls to 35 % for men over 21 years of age.

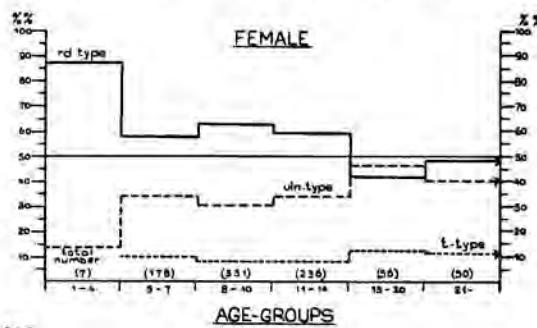


Fig. 2.

b. Females.

In the majority of females also, the radial prominence type is present at birth (probably rather more than 70 %).

This proportion remains high for much longer than is found to be the case in boys, passing the 50 % about the middle of the period of puberty.

This longer stay at the level corresponding to childhood in females than in males has also been repeatedly found — at least for physical characters — in our other investigations (1947, 1948).

We have, of course, no intention of asserting that this is necessarily true of all characters (exceptions are known to exist), nor do we wish at this stage to make any statement other than that given above.

The decrease in the number of radial types is less gradual in females.

In the period of maturation (15 to 20) years, we found in our subjects a large difference in frequency of the radial type between males and females (see also fig. 3), after this period the radial type appears to be more frequent in women than in men.

In women too, we see the phenomenon already described in men to appear about the 21st. year: a decrease in the number of ulnar types which, however, in this case does proceed in favour of the radial type.

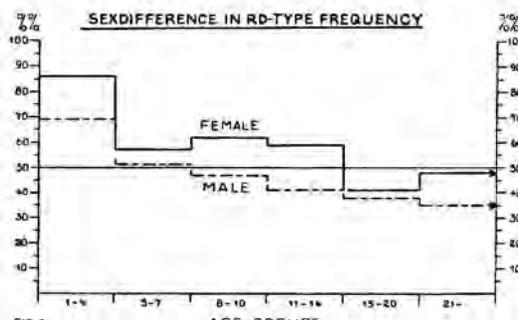


Fig. 3.

If the figures analysed provide us with reliable data, it is presumably the case that the falling-off in the number of radial forms in girls during puberty constitutes a minimum, which is then followed by a rise. In males we have seen that this decrease continues.

Some of the contradictions we have encountered in the literature are undoubtedly to be attributed to age differences in the groups compared.

The possible connection between prominence type and type of physical build has not been sufficiently investigated. Apart from the rather peculiar typology of ROMICH already discussed, we have seen than ECKER was under the impression that he had seen rather more radial types in leptosome individuals.

Using KRETSCHMER's typology with the modern terminology: eurysome (pynic), leptosome and (the intermediate)mesosome, we determined both prominence and physical type in a number of *children*. The results are shown in the following tables.

Boys.				Girls.			
Type	Rd.	Uln.	T.	Type	Rd.	Uln.	T.
Leptos.	31	21	4	56	15	5	—
Mesos.	28	21	—	49	29	16	1
Eurys.	10	5	1	16	7	1	—
Total	69	47	5	121	51	22	1
							74

We find, thus, that in the whole group of *boys* ($n = 121$) the Rd. type appears in 57% ($n = 69$). In the leptosome types among these boys ($n = 56$) the Rd. type is present in 55% ($n = 31$), in the mesosome types ($n = 49$) in 57% ($n = 28$) and in the eurysome types ($n = 16$) in 62% ($n = 10$).

The Uln. type showed an equal absence of preference for a given bodily build.

For the Rd. type in *girls* the percentages are: all girls, 69 %; leptosome, mesosome and eurysome types, 75 %, 63 % and 7 of 8 respectively.

Our study of children, thus, gives no indications of any linking of a given type of physical build to a given type of prominence. We believe, however, that this by no means excludes the possibility that an affinity of this kind may be found to exist when larger groups and older persons with fixed physical type are studied (see below).

We have little to add to our remarks on the causation of prominence differences (II, 7). We should, however, like to draw attention to the possibility that a chronological difference in the formation of the bone centres in the carpus may be *partly* responsible for difference in prominence type in men and women. The available data, however, point only vaguely in this direction.

Another problem is that of prominence type in the light of evolution. We have already seen that the literature does not permit any conclusions as to the occurrence or non-occurrence of Rd. forms in the anthropoids. We ourselves were unable to obtain any suitable anthropoid specimens, still less anthropoid foetuses.

A study of the latter would offer possibilities of examining the problem of prominence from the point of view of L. BOLK's retardation and foetalization theory.

This assumes that the rate of development is retarded in man. This retarded development is believed to have the result that the foetal character of man remains more clearly preserved than that of the other primates. BOLK enumerated a number of characters in which this foetal character is indeed shown (see DE FROE, 1948, p. 306).

It is possible that other workers are in possession of sufficient pre- and postnatal data on anthropoids to (1) establish the occurrence or non-occurrence of Rd. forms and, if these forms are found (2) to ascertain to what extent their occurrence is connected with age.

It would, in our opinion, be well worth while to be able to add further characters to the list given by BOLK or — in the case of a 'negative' result — to study the problem of prominence from different points of view.

We should not omit to remark here that PORTMANN (1944) draws attention to the increasing frequency of leptosomia as a progressive type. In the foregoing we left open the possibility that a connection might exist between, e.g., leptosomia and prominence type. In view of BOLK's theory we might perhaps expect to find a low frequency of the Rd. type among leptosome individuals. It is our opinion that considerations of this kind are sufficient reason for including prominence in the morphological part of the anthropological investigation programme.

REFERENCES.

1. BAKER, F., Anthropological notes on the human hand, *American Anthropologist*, vol. I, 70 (1888).

2. CASANOVA, Mémoires. T. 6, p. 252, Bruxelles, 1871.
3. ECKER, A., Einige Bemerkungen über einen schwankenden Charakter in der Hand des Menschen. Archiv f. Anthropol., Bnd. 8, 67—75 (1875).
4. FROE, A. DE, Inleiding tot de studie en de beoefening der anthropologie. Amsterdam, 1948.
5. HUIZINGA, J., Structural alterations indicated in the development of the human cranium. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 51, 76—87 (1948).
6. ———, Cephalometrische verwantschap tusschen verwanten van den eersten graad. Amsterdam (1947).
7. JONES, F. WOOD, Principles of anatomy as seen in the hand, London (1944).
8. MANTEGAZZA, P., Della lunghezza relativa dell'indice e dell'anulare nella mano umana. Arch. per l'Anthrop., p. 22 (1877).
9. MIERZECKI, H., Over de morphologie der hand. Ciba-Tijdschrift, 18, 573—574 (1946).
10. PFITZNER, W., Anthropologische Beziehungen der Hand- und Fuszmaasse. Morphologische Arbeiten (Schwalbe), Bnd. II, 93—206 (1893).
11. PORTMANN, A., Biologische Fragmente zu einer Lehre vom Menschen. Basel (1944).
12. ROMICH, S., Fingerlängen bei verschiedenen Konstitutionstypen. Anthropol. Anzeiger., IX, 264—267 (1932).
13. RUGGLES, G., Human Finger Types. The Anat. Record. Vol. 46, 199—204 (1930).
14. SCHAAFFHAUSEN, Einige Eigentümlichkeiten der Hand. Corr. Blatt d. dtsch. Gesellsch. f. Anthropol., Ethnol., u. Urgesch. XV Jhrg., p. 94 (1884).
15. SCHULTZ, A. H., Growthstudies on primates bearing upon man's evolution. Am. J. of phys. Anthropol., vol. VII, 149—165 (1924).
16. STRATZ, C. H., Der Körper des Kindes, 1903.
17. WECHSLER, W., Anthropologische Untersuchung der Handform mit einem familienkundlichen Beitrag. Zürich, 1939.
18. WEISSENBERG, S., Ueber die Form der Hand und des Fuszes. Zschr. f. Ethnol., 82—111 (1895).
19. WOLOTZKOI, M., Ueber die zwei Formen der menschlichen Hand, J. Russe d'Anthrop., XIII, 3/4, p. 70—81, (German summary), 1924.

