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Physics. — "*Considerations on Gravitation*". By Prof. H. A. LORENTZ.

§ 1. After all we have learned in the last twenty or thirty years about the mechanism of electric and magnetic phenomena, it is natural to examine in how far it is possible to account for the force of gravitation by ascribing it to a certain state of the aether. A theory of universal attraction, founded on such an assumption, would take the simplest form if new hypotheses about the aether could be avoided, i. e. if the two states which exist in an electric and a magnetic field, and whose mutual connection is expressed by the well known electromagnetic equations were found sufficient for the purpose.

If further it be taken for granted that only electrically *charged* particles or ions, are directly acted on by the aether, one is led to the idea that every particle of ponderable matter might consist of two ions with equal opposite charges — or at least might contain two such ions — and that gravitation might be the result of the forces experienced by these ions. Now that so many phenomena have been explained by a theory of ions, this idea seems to be more admissible than it was ever before.

As to the electromagnetic disturbances in the aether which migh possibly be the cause of gravitation, they must at all events be of such a nature, that they are capable of penetrating all ponderable bodies without appreciably diminishing in intensity. Now, electric vibrations of extremely small wave-length possess this property; hence the question arises what action there would be between two ions if the aether were traversed in all directions by trains of electric waves of small wave-length.

The above ideas are not new. Every physicist knows LE SAGE's theory in which innumerable small corpuscula are supposed to move with great velocities, producing gravitation by their impact against the coarser particles of ordinary ponderable matter. I shall not here discuss this theory which is not in harmony with modern physical views. But, when it had been found that a pressure against a body may be produced as well by trains of electric waves, by rays of light e.g., as by moving projectiles and when the RÖNTGEN-rays with their remarkable penetrating power had been discovered, it was natural to replace LE SAGE's corpuscula by vibratory motions. Why should there not exist radiations, far more penetrating than even the X-rays, and which might therefore serve to account for a force which as far as we know, is independent of all intervening ponderable matter?

I have deemed it worth while to put this idea to the test. In what follows, before passing to considerations of a different order (§ 5), I shall explain the reasons for which this theory of rapid vibrations as a cause of gravitation can *not* be accepted.

§ 2. Let an ion carrying a charge e, and having a certain mass, be situated at the point $P(x, y, z)$; it may be subject or not to an elastic force, proportional to the displacement and driving it back to P, as soon as it has left this position. Next, let the aether be traversed by electromagnetic vibrations, the dielectric displacement being denoted by $\mathfrak{d}$, and the magnetic force by $\mathfrak{H}$, then the ion will be acted on by a force

$$4\pi V^2 e \mathfrak{d},$$

whose direction changes continually, and whose components are

$$X = 4\pi V^2 e \mathfrak{d}_x, \quad Y = 4\pi V^2 e \mathfrak{d}_y, \quad Z = 4\pi V^2 e \mathfrak{d}_z. \quad . \quad (1)$$

In these formulae V means the velocity of light.

By the action of the force (1) the ion will be made to vibrate about its original position P, the displacement (x, y, z) being determined by well known differential equations.

For the sake of simplicity we shall confine ourselves to simple harmonic vibrations with frequency n. All our formulae will then contain the factor $\cos nt$ or $\sin nt$, and the forced vibrations of the ion may be represented by expressions of the form

$$\left.\begin{aligned} x &= a e \mathfrak{d}_x - b e \dot{\mathfrak{d}}_x, \\ y &= a e \mathfrak{d}_y - b e \dot{\mathfrak{d}}_y, \\ z &= a e \mathfrak{d}_z - b e \dot{\mathfrak{d}}_z, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

with certain constant coefficients a and b. The terms with $\dot{\mathfrak{d}}_x$, $\dot{\mathfrak{d}}_y$ and $\dot{\mathfrak{d}}_z$ have been introduced in order to indicate that the phase of the forced vibration differs from that of the force (X, Y, Z); this will be the case as soon as there is a resistance, proportional to the velocity, and the coefficient b may then be shown to be positive. One cause of a resistance lies in the reaction of the aether, called forth by the radiation of which the vibrating ion itself becomes the centre, a reaction which determines at the same time an apparent increase of the mass of the particle. We shall suppose however that we have kept in view this reaction in establishing the equations of motion, and in assigning their values to the coefficients a and b.

Then, in what follows, we need only consider the forces due to the state of the aether, in so far as it not directly produced by the ion itself.

Since the formulae (2) contain e as a factor, the coefficients a and b will be independent of the charge; their sign will be the same for a negative ion and for a positive one.

Now, as soon as the ion has shifted from its position of equilibrium, new forces come into play. In the first place, the force $4\pi V^2 e \mathfrak{d}$ will have changed a little, because, for the new position, $\mathfrak{d}$ will be somewhat different from what it was at the point P. We may express this by saying that, in addition to the force (1), there will be a new one with the components

$$4\pi V^2 e\left(\mathrm{x}\frac{\partial \mathfrak{d}_x}{\partial x}+\mathrm{y}\frac{\partial \mathfrak{d}_x}{\partial y}+\mathrm{z}\frac{\partial \mathfrak{d}_x}{\partial z}\right), \text{ etc.} \quad . \quad . \quad . \quad (3)$$

In the second place, in consequence of the velocity of vibration, there will be an electromagnetic force with the components

$$e(\dot{\mathrm{y}}\,\mathfrak{H}_z-\dot{\mathrm{z}}\,\mathfrak{H}_y), \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If, as we shall suppose, the displacement of the ion be very small, compared with the wave-length, the forces (3) and (4) are much smaller than the force (1); since they are periodic — with the frequency $2n$, — they will give rise to new vibrations of the particle. We shall however omit the consideration of these slight vibrations, and examine only the mean values of the forces (3) and (4), calculated for a rather long lapse of time, or, what amounts to the same thing, for a full period $\frac{2\pi}{n}$.

§ 3. It is immediately clear that this mean force will be 0 if the ion is *alone* in a field in which the propagation of waves takes place equally in all directions. It will be otherwise, as soon as a second ion Q has been placed in the neighbourhood of P; then, in consequence of the vibrations emitted by Q after it has been itself put in motion, there may be a force on P, of course in the direction of the line QP. In computing the value of this force, one finds a great number of terms, which depend in different ways on the distance r. We shall retain those which are inversely proportional to r or r^2, but we shall neglect all terms varying inversely as the higher powers of r; indeed, the influence of these, compared with that of the first mentioned terms will be of the order $\frac{\lambda}{r}$, if λ is the wave-length, and we shall suppose this to be a very small fraction.

We shall also omit all terms containing such factors as $\cos 2\pi k \frac{r}{\lambda}$ or $\sin 2\pi k \frac{r}{\lambda}$ (k a moderate number). These reverse their signs by a very small change in r; they will therefore disappear from the resultant force, as soon as, instead of *single* particles P and Q, we come to consider systems of particles with dimensions many times greater than the wave-length.

From what has been said, we may deduce in the first place that, in applying the above formulae to the ion P, it is sufficient, to take for $\mathfrak{d}$ and $\mathfrak{H}$ the vectors that would exist if P were removed from the field. In each of these vectors two parts are to be distinguished. We shall denote by $\mathfrak{d}_1$ and $\mathfrak{H}_1$ the parts existing independently of Q, and by $\mathfrak{d}_2$ and $\mathfrak{H}_2$ the parts due to the vibrations of this ion.

Let Q be taken as origin of coordinates, QP as axis of x, and let us begin with the terms in (2) having the coefficient a.

To these corresponds a force on P, whose first component is

$$4\pi V^2 e^2 a \left(\mathfrak{d}_x \frac{\partial \mathfrak{d}_x}{\partial x} + \mathfrak{d}_y \frac{\partial \mathfrak{d}_x}{\partial y} + \mathfrak{d}_z \frac{\partial \mathfrak{d}_x}{\partial z}\right) + e^2 a (\dot{\mathfrak{d}}_y \mathfrak{H}_z - \dot{\mathfrak{d}}_z \mathfrak{H}_y) \quad . \quad . \quad (5)$$

Since we have only to deal with the mean values for a full period, we may write for the last term

$$- e^2 a (\mathfrak{d}_y \dot{\mathfrak{H}}_z - \mathfrak{d}_z \dot{\mathfrak{H}}_y),$$

and if, in this expression, $\dot{\mathfrak{H}}_y$ and $\dot{\mathfrak{H}}_z$ be replaced by

$$4\pi V^2 \left(\frac{\partial \mathfrak{d}_z}{\partial x} - \frac{\partial \mathfrak{d}_x}{\partial z}\right) \quad \text{and} \quad 4\pi V^2 \left(\frac{\partial \mathfrak{d}_x}{\partial y} - \frac{\partial \mathfrak{d}_y}{\partial x}\right),$$

(5) becomes

$$2\pi V^2 e^2 a \frac{\partial (\mathfrak{d}^2)}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $\mathfrak{d}$ is the numerical value of the dielectric displacement.

Now, $\mathfrak{d}^2$ will consist of three parts, the first being $\mathfrak{d}_1^2$, the second $\mathfrak{d}_2^2$ and the third depending on the combination of $\mathfrak{d}_1$ and $\mathfrak{d}_2$.

Evidently, the value of (6), corresponding to the first part, will be 0.

As to the second part, it is to be remarked that the dielectric displacement, produced by Q, is a periodic function of the time. At distant points the amplitude takes the form $\frac{c}{r}$, where c is independent of r. The mean value of $\mathfrak{d}^2$ for a full period is $\frac{1}{2}\frac{c^2}{r^2}$ and by differentiating this with regard to x or to r, we should get r^3 in the denominator.

The terms in (6) which correspond to the part

$$2\,(\mathfrak{d}_{1x}\,\mathfrak{d}_{2x} + \mathfrak{d}_{1y}\,\mathfrak{d}_{2y} + \mathfrak{d}_{1z}\,\mathfrak{d}_{2z})$$

in $\mathfrak{d}^2$, may likewise be neglected. Indeed, if these terms are to contain no factors such as $\cos 2\pi k\frac{r}{\lambda}$ or $\sin 2\pi k\frac{r}{\lambda}$, there must be between $\mathfrak{d}_1$ and $\mathfrak{d}_2$, either no phase-difference at all, or a difference which is independent of r. This condition can only be fulfilled, if a system of waves, proceeding in the direction of QP, is combined with the vibrations excited by Q, in so far as this ion is put in motion by that system itself. Then, the two vectors $\mathfrak{d}_1$ and $\mathfrak{d}_2$ will have a common direction perpendicular to QP, say that of the axis of y, and they will be of the form

$$\mathfrak{d}_{1y} = q\cos n\left(t - \frac{x}{V} + \varepsilon_1\right)$$

$$\mathfrak{d}_{2y} = \frac{c}{r}\cos n\left(t - \frac{x}{V} + \varepsilon_2\right).$$

The mean value of $\mathfrak{d}_{1y}\,\mathfrak{d}_{2y}$ is

$$\frac{1}{2}\frac{qc}{r}\cos n(\varepsilon_1 - \varepsilon_2),$$

and its differential coefficient with regard to x has r^2 in the denominator. It ought therefore to be retained, were it not for the extremely small intensity of the systems of waves which give rise to such a result. In fact, by the restriction imposed on them as to their direction, these waves form no more than a very minute part of the whole motion.

§ 4. So, it is only the terms in (2), with the coefficient b, with which we are concerned. The corresponding forces are

$$-4\pi V^2 e^2 b\left(\dot{\mathfrak{d}}_x\frac{\partial \mathfrak{d}_x}{\partial x} + \dot{\mathfrak{d}}_y\frac{\partial \mathfrak{d}_x}{\partial y} + \mathfrak{d}_z\frac{\partial \mathfrak{d}_x}{\partial z}\right) \quad . \quad . \quad . \quad (7)$$

and

$$-e^2 b\,(\ddot{\mathfrak{d}}_y\,\mathfrak{H}_z - \ddot{\mathfrak{d}}_z\,\mathfrak{H}_y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If Q were removed, these forces together would be 0, as has already been remarked. On the other hand, the force (8), taken by itself, would then likewise be 0. Indeed, its value is

$$n^2 e^2 b\,(\mathfrak{d}_y\,\mathfrak{H}_z - \mathfrak{d}_z\,\mathfrak{H}_y)\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or, by POYNTING's theorem $\frac{n^2 e^2 b}{V^2} S_x$, if S_x be the flow of energy in a direction parallel to the axis of x. Now, it is clear that, in the absence of Q, any plane must be traversed in the two directions by equal amounts of energy.

In this way we come to the conclusion that the force (7), in so far as it depends on the part $(\mathfrak{d}_1)$, is 0, and from this it follows that the total value of (7) will vanish, because the part arising from the combination of $(\mathfrak{d}_1)$ and $(\mathfrak{d}_2)$, as well as that which is solely due to the vibrations of Q, are 0. As to the first part, this may be shown by a reasoning similar to that used at the end of the preceding §. For the second part, the proof is as follows.

The vibrations excited by Q in any point A of the surrounding aether are represented by expressions of the form

$$\frac{1}{r}\,\vartheta \cos n\left(t - \frac{r}{V} + \varepsilon\right),$$

where ϑ depends on the direction of the line QA, and r denotes the length of this line. If, in differentiating such expressions, we wish to avoid in the denominator powers of r, higher than the first — and this is necessary, in order that (7) may remain free from powers higher than the second — $\frac{1}{r}$ and ϑ have to be treated as constants. Moreover, the factors ϑ are such, that the vibrations are perpendicular to the line QA. If, now, A coincides with P, and QA with the axis of x, in the expression for $\mathfrak{d}_x$ we shall have $\vartheta = 0$, and since this factor is not to be differentiated, all terms in (7) will vanish.

Thus, the question reduces itself to (8) or (9). If, in this last expression, we take for $\mathfrak{d}$ and $\mathfrak{H}$ their real values, modified as they are by the motion of Q, we may again write for the force

$$\frac{n^2 e^2 b}{V^2} S_x\,;$$

this time, however, we have to understand by S_x the flow of energy as it is in the actual case.

Now, it is clear that, by our assumptions, the flow of energy must be symmetrical all around Q; hence, if an amount E of energy traverses, in the outward direction, a spherical surface described around Q as centre with radius r, we shall have

$$S_x = \frac{E}{4 \pi r^2},$$

and the force on P will be

$$K = \frac{n^2 e^2 b E}{4 \pi V^2 r^2}.$$

It will have the direction of QP prolonged.

In the space surrounding Q the state of the aether will be stationary; hence, two spherical surfaces enclosing this particle must be traversed by equal quantities of energy. The quantity E will be independent of r, and the force K inversely proportional to the square of the distance.

If the vibrations of Q were opposed by no other resistance but that which results from radiation, the total amount of electro-magnetic energy enclosed by a surface surrounding Q would remain constant; E and K would then both be 0. If, on the contrary, in addition to the just mentioned resistance, there were a resistance of a different kind, the vibrations of Q would be accompanied by a continual loss of electro-magnetic energy; less energy would leave the space within one of the spherical surfaces than would enter that space. E would be negative, and, since b is positive, there would be attraction. It would be independent of the signs of the charges of P and Q.

The circumstance however, that this attraction could only exist, if in some way or other electromagnetic energy were continually disappearing, is so serious a difficulty, that what has been said cannot be considered as furnishing an explanation of gravitation. Nor is this the only objection that can be raised. If the mechanism of gravitation consisted in vibrations which cross the aether with the velocity of light, the attraction ought to be modified by the motion of the celestial bodies to a much larger extent than astronomical observations make it possible to admit.

§ 5. Though the states of the aether, the existence and the laws of which have been deduced from electromagnetic phenomena, are found insufficient to account for universal attraction, yet one may tiy to establish a theory which is not wholly different from that of electricity, but has some features in common with it. In order to obtain a theory of this kind, I shall start from an idea that has been suggested long ago by MOSSOTTI and has been afterwards accepted by WILHELM WEBER and ZÖLLNER.

According to these physicists, every particle of ponderable matter consists of two oppositely electrified particles. Thus, between two particles of matter, there will be four electric forces, two attractions between the charges of different, and two repulsions between those of equal signs. MOSSOTTI supposes the attractions to be somewhat greater than the repulsions, the difference between the two being precisely what we call gravitation. It is easily seen that such a difference might exist in cases where an action of a specific electric nature is not exerted.

Now, if the form of this theory is to be brought into harmony with the present state of electrical science, we must regard the four forces of MOSSOTTI as the effect of certain states in the aether which are called forth by the positive and negative ions.

A positive ion, as well as a negative one, is the centre of a dielectric displacement, and, in treating of electrical phenomena, these two displacements are considered as being of the same nature, so that, if in opposite directions and of equal magnitude, they wholly destroy each other.

If gravitation is to be included in the theory, this view must be modified. Indeed, if the actions exerted by positive and negative ions depended on vector-quantities of the same kind, in such a way that all phenomena in the neighbourhood of a pair of ions with opposite charges were determined by the resulting vector, then electric actions could only be absent, if this resulting vector were 0, but, if such were the case, no other actions could exist; a gravitation, i. e. a force in the absence of an electric field, would be impossible.

I shall therefore suppose that the two disturbances in the aether, produced by positive and negative ions, are of a somewhat different nature, so that, even if they are represented in a diagram by equal and opposite vectors, the state of the aether is not the natural one. This corresponds in a sense to MOSSOTTI's idea that positive and negative charges differ from each other to a larger extent, than may be expressed by the signs $+$ and $-$.

After having attributed to each of the two states an independent and separate existence, we may assume that, though both able to act on positive and negative ions, the one has more power over the positive particles and the other over the negative ones. This difference will lead us to the same result that MOSSOTTI attained by means of the supposed inequality of the attractive and the repulsive forces.

§ 6. I shall suppose that each of the two disturbances of the aether is propagated with the velocity of light, and, taken by itself, obeys the ordinary laws of the electromagnetic field. These laws are expressed in the simplest form if, besides the dielectric displacement $\mathfrak{d}$, we consider the magnetic force $\mathfrak{H}$, both together determining, as we shall now say, *one* state of the aether or one field. In accordance with this, I shall introduce two pairs of vectors, the one $\mathfrak{d}$, $\mathfrak{H}$ belonging to the field that is produced by the positive ions, whereas the other pair $\mathfrak{d}'$, $\mathfrak{H}'$ serve to indicate the state of the aether which is called into existence by the negative ions. I shall write down two sets of equations, one for $\mathfrak{d}$, $\mathfrak{H}$, the other for $\mathfrak{d}'$, $\mathfrak{H}'$, and having the form which I have used in former papers [1]) for the equations of the electromagnetic field, and which is founded on the assumption that the ions are perfectly permeable to the aether and that they can be displaced without dragging the aether along with them.

I shall immediately take this general case of moving particles.

Let us further suppose the charges to be distributed with finite volume-density, and let the units in which these are expressed be chosen in such a way that, in a body which exerts no electrical actions, the total amount of the positive charges has the same numerical value as that of the negative charges.

Let ϱ be the density of the positive, and ϱ' that of the negative charges, the first number being positive and the second negative.

Let $\mathfrak{v}$ (or $\mathfrak{v}'$) be the velocity of an ion.

Then the equations for the state ($\mathfrak{d}$, $\mathfrak{H}$) are [2])

$$\left.\begin{aligned} Div\ \mathfrak{d} &= \varrho \\ Div\ \mathfrak{H} &= 0 \\ Rot\ \mathfrak{H} &= 4\pi\varrho\mathfrak{v} + 4\pi\dot{\mathfrak{d}} \\ 4\pi V^2\, Rot\ \mathfrak{d} &= -\dot{\mathfrak{H}}; \end{aligned}\right\} \quad \ldots \ldots \text{(I)}$$

1) LORENTZ. La théorie électromagnétique de MAXWELL et son application aux corps mouvants, Arch. Néerl. XXV, p. 363; Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Korpern.

2) $Div\ \mathfrak{d} = \frac{\partial \mathfrak{d}_x}{\partial x} + \frac{\partial \mathfrak{d}_y}{\partial y} + \frac{\partial \mathfrak{d}_z}{\partial z}$.

$Rot\ \mathfrak{d}$ is a vector, whose components are $\frac{\partial \mathfrak{d}_z}{\partial y} - \frac{\partial \mathfrak{d}_y}{\partial z}$, etc.

and those for the state ($\mathfrak{d}'$, $\mathfrak{H}'$)

$$\left.\begin{aligned} Div\ \mathfrak{d}' &= \varrho' \\ Div\ \mathfrak{H}' &= 0 \\ Rot\ \mathfrak{H}' &= 4\pi\varrho'\mathfrak{v}' + 4\pi\dot{\mathfrak{d}}' \\ 4\pi V^2 Rot\ \mathfrak{d}' &= -\dot{\mathfrak{H}}'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (II)$$

In the ordinary theory of electromagnetism, the force acting on a particle, moving with velocity $\mathfrak{v}$, is

$$4\pi V^2\mathfrak{d} + [\mathfrak{v} \,.\, \mathfrak{H}],$$

per unit charge [1]).

In the modified theory, we shall suppose that a positively electrified particle with charge e experiences a force

$$k_1 = \alpha\{4\pi V^2\mathfrak{d} + [\mathfrak{v} \,.\, \mathfrak{H}]\}e \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

on account of the field ($\mathfrak{d}$, $\mathfrak{H}$), and a force

$$k_2 = \beta\{4\pi V^2\mathfrak{d}' + [\mathfrak{v} \,.\, \mathfrak{H}']\}e \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

on account of the field ($\mathfrak{d}'$ $\mathfrak{H}'$), the positive coefficients α and β having slightly different values.

For the forces, exerted on a negatively charged particle I shall write

$$k_3 = \beta\{4\pi V^2\mathfrak{d} + [\mathfrak{v}' . \mathfrak{H}]\}e' \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and

$$k_4 = \alpha\{4\pi V^2\mathfrak{d}' + [\mathfrak{v}' \,.\, \mathfrak{H}']\}e', \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

expressing by these formulae that e is acted on by ($\mathfrak{d}$, $\mathfrak{H}$) in the same way as e' by ($\mathfrak{d}'$, $\mathfrak{H}'$), and vice versa.

§ 7. Let us next consider the actions exerted by a *pair* of oppositely charged ions, placed close to each other, and remaining so during their motion. For convenience of mathematical treatment, we may even reason as if the two charges penetrated each other, so that, if they are equal, $\varrho' = -\varrho$.

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[1]) $[\mathfrak{v} \,.\, \mathfrak{H}]$ is the vector-product of $\mathfrak{v}$ and $\mathfrak{H}$.

On the other hand $\mathfrak{v}' = \mathfrak{v}$; hence, by (I) and (II),

$$\mathfrak{d}' = -\mathfrak{d} \quad \text{and} \quad \mathfrak{H}' = -\mathfrak{H}.$$

Now let us put in the field, produced by the pair of ions, a similar pair with charges e and $e' = -e$, and moving with the common velocity $\mathfrak{v}$. Then, by (10)—(13),

$$k_2 = -\frac{\beta}{\alpha} k_1 \quad , \quad k_3 = -\frac{\beta}{\alpha} k_1 \quad , \quad k_4 = k_1.$$

The total force on the positive particle will be

$$k_1 + k_2 = k_1 \left(1 - \frac{\beta}{\alpha}\right)$$

and that on the negative ion

$$k_3 + k_4 = k_1 \left(1 - \frac{\beta}{\alpha}\right).$$

These forces being equal and having the same direction, there is no force tending to *separate* the two ions, as would be the case in an *electric field*. Nevertheless, the pair is acted on by a resultant force

$$2\, k_1 \left(1 - \frac{\beta}{\alpha}\right).$$

If now β be somewhat larger than α, the factor $2\left(1 - \frac{\beta}{\alpha}\right)$ will have a certain negative value $-\varepsilon$, and our result may be expressed as follows:

If we wish to determine the action between two ponderable bodies, we may first consider the forces existing between the positive ions in the one and the positive ions in the other. We then have to reverse the direction of these forces, and to multiply them by the factor ε. Of course, we are led in this way to NEWTON's law of gravitation.

The assumption that all ponderable matter is composed of positive and negative ions is no essential part of the above theory. We might have confined ourselves to the supposition that the state of the aether which is the cause of gravitation is propagated in a similar way as that which exists in the electromagnetic field.

Instead of introducing two pairs of vectors ($\mathfrak{d}$, $\mathfrak{H}$) and ($\mathfrak{d}'$, $\mathfrak{H}'$), both of which come into play in the electromagnetic actions, as well as in the phenomenon of gravitation, we might have assumed one pair for the electromagnetic field and one for universal attraction.

For these latter vectors, say $\mathfrak{d}$, $\mathfrak{H}$, we should then have established the equations (I), ϱ being the density of ponderable matter, and for the force acting on unit mass, we should have put

$$-\eta\{4\pi V^2 \mathfrak{d} + [\mathfrak{v} \,.\, \mathfrak{H}]\},$$

where η is a certain positive coefficient.

§ 8. Every theory of gravitation has to deal with the problem of the influence, exerted on this force by the motion of the heavenly bodies. The solution is easily deduced from our equations; it takes the same form as the corresponding solution for the electromagnetic actions between charged particles [1]).

I shall only treat the case of a body A, revolving around a central body M, this latter having a given constant velocity p. Let r be the line MA, taken in the direction from M towards A, x, y, z the relative coordinates of A with respect to M, w the velocity of A's motion relatively to M, ϑ the angle between w and p, finally p_r the component of p in the direction of r.

Then, besides the attraction

$$\frac{k}{r^2}, \quad \ldots\ldots\ldots \quad (14)$$

which would exist if the bodies were both at rest, A will be subject to the following actions.

1st. A force

$$k \,.\, \frac{p^2}{2V^2} \,.\, \frac{1}{r^2} \quad \ldots\ldots\ldots \quad (15)$$

in the direction of r.

2nd. A force whose components are

$$-\frac{k}{2V^2}\frac{\partial}{\partial x}\left(\frac{p_r^2}{r}\right), \quad -\frac{k}{2V^2}\frac{\partial}{\partial y}\left(\frac{p_r^2}{r}\right), \quad -\frac{k}{2V^2}\frac{\partial}{\partial z}\left(\frac{p_r^2}{r}\right) \,. \quad (16)$$

1) See the second of the above cited papers.

3rd. A force

$$-\frac{k}{V^2}p \cdot \frac{1}{r^2}\frac{dr}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

parallel to the velocity p.

4th. A force

$$\frac{k}{V^2}\frac{1}{r^2} p\, w \cos \vartheta\, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in the direction of r.

Of these, (15) and (16) depend only on the common velocity p, (17) and (18) on the contrary, on p and w conjointly.

It is further to be remarked that the additional forces (15)—(18) are all of the second order with respect to the small quantities

$$\frac{p}{V} \text{ and } \frac{w}{V}.$$

In so far, the law expressed by the above formulae presents a certain analogy with the laws proposed by WEBER, RIEMANN and CLAUSIUS for the electromagnetic actions, and applied by some astronomers to the motions of the planets. Like the formulae of CLAUSIUS, our equations contain the absolute velocities, i. e. the velocities, relatively to the aether.

There is no doubt but that, in the present state of science, if we wish to try for gravitation a similar law as for electromagnetic forces, the law contained in (15)—(18) is to be preferred to the three other just mentioned laws.

§ 9. The forces (15)—(18) will give rise to small inequalities in the elements of a planetary orbit; in computing these, we have to take for p the velocity of the Sun's motion through space. I have calculated the *secular* variations, using the formulae communicated by TISSERAND in his Mécanique céleste.

Let a be the mean distance to the sun,

e the eccentricity,

φ the inclination to the ecliptic,

θ the longitude of the ascending node,

$\tilde{\omega}$ the longitude of perihelion,

$\varkappa'$ the mean anomaly at time $t = 0$, in this sense that, if n

be the mean motion, as determined by a, the mean anomaly at time t is given by

$$\varkappa' + \int_0^t n\, dt.$$

Further, let λ, μ and ν be the direction-cosines of the velocity p with respect to: 1st. the radius vector of the perihelion, 2nd. a direction which is got by giving to that radius vector a rotation of 90°, in the direction of the planet's revolution, 3rd. the normal to the plane of the orbit, drawn towards the side whence the planet is seen to revolve in the same direction as the hands of a watch.

Put $\omega = \tilde{\omega} - \Theta$, $\frac{p}{V} = \delta$ and $\frac{n\,a}{V} = \delta'$ ($n\,a$ is the velocity in a circular orbit of radius a).

Then I find for the variations *during one revolution*

$$\Delta a = 0$$

$$\Delta e = 2\pi \sqrt{(1-e^2)} \left\{ \lambda \mu \delta^2 \frac{(2-e^2) - 2\sqrt{(1-e^2)}}{e^3} - \lambda \delta \delta' \frac{1-\sqrt{(1-e^2)}}{e^2} \right\}$$

$$\Delta \varphi = \frac{2\pi}{\sqrt{(1-e^2)}}\, \nu \left\{ [-\lambda \delta^2 \cos \omega + \delta (e\delta' - \mu\delta) \sin \omega] \frac{1-\sqrt{(1-e^2)}}{e^2} + \mu \delta^2 \sin \omega \right\}$$

$$\Delta \Theta = -\frac{2\pi}{\sqrt{(1-e^2)} \sin \varphi}\, \nu \left\{ [\lambda \delta^2 \sin \omega + \delta (e\delta' - \mu\delta) \cos \omega] \frac{1-\sqrt{(1-e^2)}}{e^2} + \right.$$
$$\left. + \mu \delta^2 \cos \omega \right\}$$

$$\Delta \tilde{\omega} = \pi (\mu^2 - \lambda^2) \delta^2 \frac{(2-e^2) - 2\sqrt{(1-e^2)}}{e^4} + 2\pi \mu \delta \delta' \frac{\sqrt{(1-e^2)} - 1}{e^3} -$$
$$- \frac{2\pi\, tg\, \frac{1}{2}\varphi}{\sqrt{(1-e^2)}}\, \nu \left\{ [\lambda \delta^2 \sin \omega + \delta (e\delta' - \mu\delta) \cos \omega] \frac{1-\sqrt{(1-e^2)}}{e^2} + \mu \delta^2 \cos \omega \right\}$$

$$\Delta \varkappa' = \pi (\lambda^2 - \mu^2) \delta^2 \frac{(2+e^2)\sqrt{(1-e^2)} - 2}{e^4} - 2\pi \delta^2 - 2\pi \mu^2 \delta^2 -$$
$$- 2\pi \mu \delta \delta' \frac{(1-e^2) - \sqrt{(1-e^2)}}{e^3}.$$

§ 10. I have worked out the case of the planet Mercury, taking 276° and + 34° for the right ascension and declination of the apex of the Sun's motion. I have got the following results:

$$\Delta a = 0$$

$$\Delta e = 0{,}018\,\delta^2 + 1{,}38\,\delta\,\delta'$$

$$\Delta \varphi = 0{,}95\,\delta^2 + 0{,}28\,\delta\,\delta'$$

$$\Delta \theta = 7{,}60\,\delta^2 - 4{,}26\,\delta\,\delta'$$

$$\Delta \tilde{\omega} = -\,0{,}09\,\delta^2 + 1{,}95\,\delta\,\delta'$$

$$\Delta \varkappa' = -\,6{,}82\,\delta^2 - 1{,}93\,\delta\,\delta'$$

Now, $\delta' = 1{,}6 \times 10^{-4}$ and, if we put $\delta = 5{,}3 \times 10^{-5}$, we get

$$\Delta e = 117 \times 10^{-10}\ ,\quad \Delta\varphi = 51 \times 10^{-10},$$

$$\Delta\theta = -\,137 \times 10^{-10},\quad \Delta\tilde{\omega} = 162 \times 10^{-10},\quad \Delta\varkappa' = -\,355 \times 10^{-10}.$$

The changes that take place in a century are found from these numbers, if we multiply them by 415, and, if the variations of φ, θ, $\tilde{\omega}$ and $\varkappa'$ are to be expressed in seconds, we have to introduce the factor $2{,}06 \times 10^5$. The result is, that the changes in φ, θ, $\tilde{\omega}$ and $\varkappa'$ amount to a few seconds, and that in e to 0,000005.

Hence we conclude that our modification of NEWTON's law cannot account for the observed inequality in the longitude of the perihelion — as WEBER's law can to some extent do — but that, if we do not pretend to explain this inequality by an alteration of the law of attraction, there is nothing against the proposed formulae. Of course it will be necessary to apply them to other heavenly bodies, though it seems scarcely probable that there will be found any case in which the additional terms have an appreciable influence.

The special form of these terms may perhaps be modified. Yet, what has been said is sufficient to show that gravitation may be attributed to actions which are propagated with no greater velocity than that of light.

As is well known, LAPLACE has been the first to discuss this question of the velocity of propagation of universal attraction, and later astronomers have often treated the same problem. Let a body B be attracted by a body A, moving with the velocity p. Then, if the action is propagated with a finite velocity V, the influence which reaches B at time t, will have been emitted by A at an anterior moment, say $t - \tau$. Let A_1 be the position of the acting body at this moment, A_2 that at time t. It is an easy matter to calculate the distance between these positions. Now, if the action at time

t is calculated, as if A had continued to occupy the position A_1, one is led to an influence on the astronomical motions of the order $\frac{p}{V}$; if V were equal to the velocity of light, this influence would be much greater than observations permit us to suppose. If, on the contrary, the terms with $\frac{p}{V}$ are to have admissible values, V ought to be many millions of times as great as the velocity of light.

From the considerations in this paper, it appears that this conclusion can be avoided. Changes of state in the aether, satisfying equations of the form (I), are propagated with the velocity V; yet, no quantities of the first order $\frac{p}{V}$ or $\frac{w}{V}$ (§ 8), but only terms containing $\frac{p^2}{V^2}$ and $\frac{pw}{V^2}$ appear in the results. This is brought about by the peculiar way — determined by the equations — in which moving matter changes the state of the aether; in the above mentioned case the condition of the aether will *not* be what it would have been, if the acting body were at rest in the position A_1.

Physiology. — "*On the power of resistance of the red blood corpuscles*". By Dr. H. J. HAMBURGER.

(Will be published in the Proceedings of the next meeting.)

Physics. — "*On the critical isotherm and the densities of saturated vapour and liquid in the case of isopentane and carbonic acid*". By Dr. J. E. VERSCHAFFELT (Communicated by Prof. H. KAMERLINGH ONNES).

(Will be published in the Proceedings of the next meeting).

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