	I.	II.	III.
Fe	69,94	69,13	69,50
O	29,97	29,60	30,46
accompanying mineral		1,2	
	99,91	99,93	99,96 [1])

Reckoned for:

	Hematite	Magnetite
Fe	70	72,41
O	30	27,59.

So that my conclusion is that we have not to do with Magnetite but with Hematite.

The results of my researches are in consequence the following:

1st. That I have had to do with Hematite with very obvious magnetism and a black streak, which in rubbing along the outlines shows a brown tint (which generally every black streak does) and not with a pseudomorphosis from Magnetite to Hematite.

2nd. That where in literature of this occurrence of Hematite has been spoken, no analysis has been added, though the magnetism and the black streak have been observed more than once.

3rd. That it is desirable to convince oneself of the chemical composition with every "Eisenrose", which shows these characteristics.

Physics. — "*Contributions to the theory of electrons.*" I. By Prof. H. A. LORENTZ.

Simplification of the fundamental equations by the introduction of new units.

§ 1. If all quantities are expressed in electromagnetic units, as I have done in former papers, the relations between the volume-density ρ of the charge of an electron, the velocity $\mathfrak{v}$ of its points, the

1) I here by have to mention that first the figure for the oxygen was determined by reduction in a hydrogen-current und weighing of the water absorbed by $CaCl_2$; that after that the figure for the iron was determined by dissolving the reduced mineral in dilute H_2SO_4 and making a titra'ion ot this solution (after reduction in a H_2S-current and after removing the H_2S by boiling in a CO_2 atmosphere) with a $KMnO_4$-solution, of which 1 cM^3 corresponded with 8,9 m.G. Fe.

The presence of Ti was shown as follows: the mineral was melted together with $KHSO_4$, the fused mass dissolved in *cold* water. This solution together with H_2O_2 gave the well-known orange colour of TiO_3. Moreover after adding a little HNO_3, the Ti after having been boiled precipitated as white TO_2. The accompanying mineral, which in microscopic examination proved to be adularia, was removed as much as possible.

dielectric displacement $\mathfrak{d}$ in the aether, the current $\mathfrak{l}$ and the magnetic force $\mathfrak{h}$ are as follows [1]:

$$div\, \mathfrak{d} = \varrho\,,$$
$$\frac{\partial \varrho}{\partial t} + div\,(\varrho\, \mathfrak{v}) = 0,$$
$$\mathfrak{l} = \dot{\mathfrak{d}} + \varrho\, \mathfrak{v}\,,$$
$$div\, \mathfrak{h} = 0\,,$$
$$rot\, \mathfrak{h} = 4\,\pi\, \mathfrak{l} = 4\,\pi\,(\dot{\mathfrak{d}} + \varrho\, \mathfrak{v})\,,$$
$$4\,\pi\, c^2\, rot\, \mathfrak{d} = -\,\dot{\mathfrak{h}}\,,$$

where c is the velocity of light in the aether. To these equations we must add the formula

$$\mathfrak{f} = 4\,\pi\, c^2\, \mathfrak{d} + [\mathfrak{v}\,.\,\mathfrak{h}]$$

for the electric force, i. e. the force, reckoned per unit charge, which the aether exerts on a charged element of volume.

The equations take a somewhat more regular form if we express ϱ, $\mathfrak{d}$, $\mathfrak{l}$ and $\mathfrak{f}$ in electrostatic units (preserving the electromagnetic unit for $\mathfrak{h}$) and a further simplification is obtained, if, instead of the units for charge and magnetic pole that are usually taken as the basis of the electrostatic and electromagnetic systems, we choose new ones, $\sqrt{4\pi}$ times smaller [2]. Introducing both modifications, we have to replace ϱ, $\mathfrak{d}$, $\mathfrak{l}$ by $\frac{\varrho}{c\sqrt{4\pi}}$, $\frac{\mathfrak{d}}{c\sqrt{4\pi}}$, $\frac{\mathfrak{l}}{c\sqrt{4\pi}}$, $\mathfrak{f}$ by $c\sqrt{4\pi}\,.\,\mathfrak{f}$, because this letter must now represent the force acting on the new unit of charge, and likewise $\mathfrak{h}$ by $\sqrt{4\pi}\,.\,\mathfrak{h}$.

This leads to the equations

$$div\, \mathfrak{d} = \varrho\,, \qquad \text{(I)}$$
$$\frac{\partial \varrho}{\partial t} + div\,(\varrho\, \mathfrak{v}) = 0\,, \qquad \text{(II)}$$
$$\mathfrak{l} = \dot{\mathfrak{d}} + \varrho\, \mathfrak{v}\,, \qquad \text{(III)}$$
$$div\, \mathfrak{h} = 0\,, \qquad \text{(IV)}$$
$$rot\, \mathfrak{h} = \frac{1}{c}\, \mathfrak{l} = \frac{1}{c}\,(\dot{\mathfrak{d}} + \varrho\, \mathfrak{v})\,, \qquad \text{(V)}$$

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[1]) See my *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern*. I shall again suppose that all quantities are continuous functions of the coordinates, so that e.g. the density ρ will be regarded as passing gradually to the value 0, which it has outside an electron. With the exception of the letters, the notations are the same as in the just mentioned treatise. The scalar product of two vectors $\mathfrak{a}$ and $\mathfrak{b}$ will be denoted by $(\mathfrak{a}\,.\,\mathfrak{b})$, the vector product by $[\mathfrak{a}\,.\,\mathfrak{b}]$. The axes of coordinates are supposed to remain at rest, relatively to the aether.

[2]) This change has been warmly advocated by Heaviside. The units I shall now use are those that have been adopted for the *Mathematische Encyclopädie*.

$$rot\, \mathfrak{d} = -\frac{1}{c}\dot{\mathfrak{h}}, \quad \ldots \quad (VI)$$

$$\mathfrak{f} = \mathfrak{d} + \frac{1}{c}[\mathfrak{v}\,.\,\mathfrak{h}]. \quad \ldots \quad (VII)$$

In connexion with the last formula it may be remarked that $\mathfrak{d}$ is the electric force that would act on an immovable charge.

The electric energy per unit-volume is given by

$$W_e = \frac{1}{2}\mathfrak{d}^2, \quad \ldots \quad (VIII)$$

the magnetic energy per unit-volume by

$$W_m = \frac{1}{2}\mathfrak{h}^2, \quad \ldots \quad (IX)$$

and POYNTING's flux of energy by

$$\mathfrak{S} = c\,[\mathfrak{d}\,.\,\mathfrak{h}]\,. \quad \ldots \quad (X)$$

We shall further write U for the total electric and T for the total magnetic energy of a system.

The equations (IV) and (V) suffice for the determination of the magnetic force $\mathfrak{h}$, as soon as the current $\mathfrak{C}$ is given in every point. W_m is then known by (IX) and T follows by integration. In this sense, every motion of electricity may be said to be accompanied by a definite amount of magnetic energy.

Scalar potential and vector-potential.

§ 2. The equations of § 1 apply to every system in which charged matter moves through the aether, whether the charge be confined to certain extremely small parts of space (electrons) or otherwise distributed. Moreover, the motions may be of any kind; the electrons may have a pure translatory motion, or a rotation at the same time, and we may even suppose their form to change in the course of time. For the validity of the formulae it is however required that each element of volume whose points move with the charged matter should preserve its charge, though its form and dimensions may change. This is expressed by the equation (II) and it is on this ground that the electric current $\mathfrak{C}$, as defined by (III), (the resultant of the displacement-current $\dot{\mathfrak{d}}$ and the convection-current $\rho\mathfrak{v}$) may always be said to be solenoidally distributed, so that

$$div\, \mathfrak{C} = 0.$$

If now the motion of the charged matter is given, the electromagnetic field in the aether, within and without that matter, has to be determined by means of (I)—(VI), a problem that may be reduced to equations of the form

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which α is a known, and ψ an unknown function of x, y, z, t.

Let σ be any closed surface and n the normal to it, drawn outwards.

Then, if the equation (1) holds in the whole space S, enclosed by σ, we shall have for the value of ψ in a point P of this space, at the time t,

$$\psi = \frac{1}{4\pi}\int\frac{1}{r}[\alpha]\,dS + \frac{1}{4\pi}\int\left\{\frac{1}{r}\left[\frac{\partial\psi}{\partial n}\right] - [\psi]\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right\}d\sigma. \quad . \quad . \quad (2)$$

Here the first integral extends over the space S and the second over the boundary surface σ; r is the distance to P, and the square brackets serve to indicate the values of the enclosed quantities for the time $t - \frac{r}{c}$.

Let us now conceive the surface σ to recede on all sides to infinite distance and let the circumstances be such that the surface-integral in (2) has the limit 0. Then, ultimately:

$$\psi = \frac{1}{4\pi}\int\frac{1}{r}[\alpha]\,dS, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the integration must be extended over infinite space.

§ 3. Equations of the form (1) may be deduced from the formulae (I)—(VI) in many different ways; they may e.g. be established for each of the components of $\mathfrak{d}$ and $\mathfrak{h}$. [1]) The solution is however obtained in a simpler form [2]), if one introduces four auxiliary quantities, a scalar potential φ and the three components $\mathfrak{a}_x$, $\mathfrak{a}_y$, $\mathfrak{a}_z$ of a vector-potential $\mathfrak{a}$. These quantities satisfy the equations

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\varrho,$$

$$\Delta\mathfrak{a}_x - \frac{1}{c^2}\frac{\partial^2\mathfrak{a}_x}{\partial t^2} = -\frac{1}{c}\varrho\,\mathfrak{v}_x, \qquad \Delta\mathfrak{a}_y - \frac{1}{c^2}\frac{\partial^2\mathfrak{a}_y}{\partial t^2} = -\frac{1}{c}\varrho\,\mathfrak{v}_y, \text{ etc.,}$$

so that, with the restrictions that are required if (3) is to be true, we may write

$$\varphi = \frac{1}{4\pi}\int\frac{1}{r}[\varrho]\,dS,$$

1) LORENTZ, La théorie électromagnétique de MAXWELL et son application aux corps mouvants, Arch. néerl. T. 25, p. 476 1892.

2) See LEVI CIVITA, Nuovo Cimento, (4), vol. 6, p. 93, 1897; WIECHERT, Arch. néerl., (2), T. 5, p. 549, 1900.

$$\mathfrak{a}_x = \frac{1}{4\pi c}\int \frac{1}{r}\,[\varrho\,\mathfrak{v}_x]\,dS \quad , \quad \mathfrak{a}_y = \frac{1}{4\pi c}\int \frac{1}{r}\,[\varrho\,\mathfrak{v}_y]\,dS, \text{ etc.}$$

After having found φ and $\mathfrak{a}$, we may determine the dielectric displacement $\mathfrak{d}$ and the magnetic force $\mathfrak{h}$ by means of the relations [1])

$$\mathfrak{d} = -\frac{1}{c}\dot{\mathfrak{a}} - grad\,\varphi\,, \qquad (4)$$

$$\mathfrak{h} = rot\,\mathfrak{a}. \qquad (5)$$

It is to be remarked that the two potentials are not mutually independent; they are connected by the equation

$$div\,\mathfrak{a} = -\frac{1}{c}\dot{\varphi}. \qquad (6)$$

Theorems corresponding to the principle of D'ALEMBERT *and that of least action.*

§ 4. The physicists who have endeavoured, by means of certain hypotheses on the mecanism of electromagnetic phenomena, to deduce the fundamental equations from the principles of dynamics, have encountered considerable difficulties, and it is best, perhaps, to leave this course, and to adopt the equations (I)—(VII) — or others, equivalent to them — as the simplest expression we may find for the laws of electromagnetism. Nevertheless, even if we prefer this point of view, it deserves notice that the fundamental equations may be transformed in such a way that we arrive at theorems of the same mathematical form as the general principles of dynamics. This has been done especially by ABRAHAM in his important paper "Principien der Dynamik des Elektrons" [2]). The considerations in this and the two next paragraphs agree with those of ABRAHAM, though presented in a form differing from his.

We shall consider a system of electrons moving in the infinitely extended aether, and we shall fix our attention on the different states of this system, the aether included, that succeed each other in the course of time in any electromagnetic phenomenon. From every one of these states we shall pass to another, differing infinitely little from it, and which we shall call the *varied* state. The variation or "virtual change" will consist in infinitely small displacements $\mathfrak{q}$ of

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[1]) I shall write *grad* φ („gradient of φ") for the vector whose components are $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \varphi}{\partial y}$, $\frac{\partial \varphi}{\partial z}$.

[2]) DRUDE'S Annalen, 10, p. 105, 1903.

the points of the electrons, accompanied by infinitesimal changes in the dielectric displacement.

We shall write $\delta \mathfrak{d}$ for the difference, in a fixed point of the aether, between the dielectric displacement before and after the virtual change, the sign of variation δ having a similar meaning when it precedes other symbols representing the value of some quantity in a definite point. If it is affixed to a letter representing a quantity belonging to the system as a whole, such as the total electric energy U, it will simply serve to indicate the difference between these values in the original or real and the varied states.

The variations to be considered are not wholly arbitrary. We shall limit our choice by supposing in the first place that each element of volume of an electron preserves its charge during the displacements $\mathfrak{q}$; this is expressed by the relation

$$\delta \varrho + div\,(\varrho\, \mathfrak{q}) = 0\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which may be compared to (II).

In the second place we shall suppose the variations of $\mathfrak{d}$ not to violate the condition (I).

In virtue of these restrictions the vector

$$\delta \mathfrak{d} + \varrho\, \mathfrak{q}$$

will present a solenoidal distribution. Indeed, we see from (I) that

$$div\, \delta\, \mathfrak{d} = \delta \varrho,$$

and here we may, according to (7), replace the right-hand member by $-div\,(\varrho\, \mathfrak{q})$.

Let us now conceive $\mathfrak{q}$ and $\delta \mathfrak{d}$ to be chosen for every instant t, so that they vary continuously with the time. Then, in order completely to define the succession of varied states, or what we may call the „varied motion" of the system, we shall suppose the varied positions of the points of each electron to be reached at the same instants at which these points occupy the corresponding original positions in the real motion; we assume likewise that, in every point of space, the varied dielectric displacement exists at the same moments as the original one in the succession of real states.

By this the varied motion of electricity is entirely determined; indeed, since we know the velocity of matter and the rate at which $\mathfrak{d}$ changes, we are able to state what has become of the convection-current, the displacement-current, and also of the total current $\mathfrak{l}$. The first thing we have to do will be to express $\delta \mathfrak{l}$ in $\mathfrak{q}$ and $\delta \mathfrak{d}$. Of course we may be sure beforehand that the distribution of both the new $\mathfrak{l}$ and the variation $\delta \mathfrak{l}$ will be solenoidal. This must necessarily be the case, because we know 1st. that, in the states that succeed one another in the varied motion, each volume-element of an electron retains its charge, and 2nd. that the condition (I) is continually fulfilled.

§ 5. Let us begin by considering $\delta\mathfrak{v}_x$. This is the variation in a fixed point of space. Therefore, if $(\delta\mathfrak{v}_x)$ is the variation for a definite point of an electron, we shall have

$$(\delta\mathfrak{v}_x) = \delta\mathfrak{v}_x + \mathfrak{q}_x \frac{\partial \mathfrak{v}_x}{\partial x} + \mathfrak{q}_y \frac{\partial \mathfrak{v}_x}{\partial y} + \mathfrak{q}_z \frac{\partial \mathfrak{v}_x}{\partial z}.$$

As to $(\delta\mathfrak{v}_x)$, it is easily shown to have the value

$$(\delta\mathfrak{v}_x) = \frac{d\mathfrak{q}_x}{dt},$$

if we understand by $\frac{d\mathfrak{q}_x}{dt}$ the rate at which $\mathfrak{q}_x$ changes for a definite point of an electron. Comparing this to $\frac{\partial \mathfrak{q}_x}{\partial t}$ or $\dot{\mathfrak{q}}_x$, the velocity of change in a fixed point of space, we get

$$(\delta\mathfrak{v}_x) = \dot{\mathfrak{q}}_x + \mathfrak{v}_x \frac{\partial \mathfrak{q}_x}{\partial x} + \mathfrak{v}_y \frac{\partial \mathfrak{q}_x}{\partial y} + \mathfrak{v}_z \frac{\partial \mathfrak{q}_x}{\partial z}.$$

These equations, combined with (7), lead us to

$$\delta\mathfrak{l}_x = \delta(\mathfrak{d}_x + \rho\,\mathfrak{v}_x) = \delta\mathfrak{d}_x + \rho\,\delta\mathfrak{v}_x + \mathfrak{v}_x\,\delta\rho =$$

$$= \delta\dot{\mathfrak{d}}_x + \rho\,\dot{\mathfrak{q}}_x + \rho\,\mathfrak{v}_x \frac{\partial \mathfrak{q}_x}{\partial x} + \rho\,\mathfrak{v}_y \frac{\partial \mathfrak{q}_x}{\partial y} + \rho\,\mathfrak{v}_z \frac{\partial \mathfrak{q}_x}{\partial z} -$$

$$- \rho\,\mathfrak{q}_x \frac{\partial \mathfrak{v}_x}{\partial x} - \rho\,\mathfrak{q}_y \frac{\partial \mathfrak{v}_x}{\partial y} - \rho\,\mathfrak{q}_z \frac{\partial \mathfrak{v}_x}{\partial z} - \mathfrak{v}_x\, div\,(\rho\,\mathfrak{q}),$$

or, if we add to the second member the first member of (II), multiplied by $\mathfrak{q}_x$, after some further transformation,

$$\delta\mathfrak{l}_x = \frac{\partial}{\partial t}(\delta\mathfrak{d}_x + \rho\,\mathfrak{q}_x) + \rho\,\mathfrak{v}_x \frac{\partial \mathfrak{q}_x}{\partial x} + \rho\,\mathfrak{v}_y \frac{\partial \mathfrak{q}_x}{\partial y} + \rho\,\mathfrak{v}_z \frac{\partial \mathfrak{q}_x}{\partial z} - \mathfrak{v}_x\, div\,(\rho\,\mathfrak{q}) -$$

$$- \rho\,\mathfrak{q}_x \frac{\partial \mathfrak{v}_x}{\partial x} - \rho\,\mathfrak{q}_y \frac{\partial \mathfrak{v}_x}{\partial y} - \rho\,\mathfrak{q}_z \frac{\partial \mathfrak{v}_x}{\partial z} + \mathfrak{q}_x\, div\,(\rho\,\mathfrak{v}) =$$

$$= \frac{\partial}{\partial t}(\delta\mathfrak{d}_x + \rho\,\mathfrak{q}_x) + \frac{\partial}{\partial y}[\rho\,(\mathfrak{q}_x\mathfrak{v}_y - \mathfrak{q}_y\mathfrak{v}_x)] - \frac{\partial}{\partial z}[\rho\,(\mathfrak{q}_z\mathfrak{v}_x - \mathfrak{q}_x\mathfrak{v}_z)].$$

Here we may remark that the two last terms taken together represent the first component of the "rotation" of the vector whose components are

$$\rho\,(\mathfrak{q}_y\mathfrak{v}_z - \mathfrak{q}_z\mathfrak{v}_y),\quad \rho\,(\mathfrak{q}_z\mathfrak{v}_x - \mathfrak{q}_x\mathfrak{v}_z),\quad \rho\,(\mathfrak{q}_x\mathfrak{v}_y - \mathfrak{q}_y\mathfrak{v}_x),$$

and that this vector is precisely the vector-product, multiplied by ρ, of $\mathfrak{q}$ and $\mathfrak{v}$. After having calculated $\delta\mathfrak{l}_y$ and $\delta\mathfrak{l}_z$ in the same way as $\delta\mathfrak{l}_x$, we may combine the results in the formula

$$\delta \mathfrak{l} = \frac{\partial}{\partial t}(\delta \mathfrak{d} + \varrho\, \mathfrak{q}) + rot\, \{\varrho\, [\mathfrak{q} \cdot \mathfrak{v}]\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

What has already been said about the solenoidal distribution of $\delta \mathfrak{l}$ is confirmed by this equation. The two vectors represented on the right hand side both have this property, the first by what we know of the vector $\delta \mathfrak{d} + \varrho\, \mathfrak{q}$, and the second on account of the mathematical form in which it appears.

§ 6. We may next proceed to determine the variation δT of the magnetic energy. In doing so we shall start from the assumption that the varied motion of electricity involves a definite magnetic energy [1]), to be determined as stated at the end of § 1.

The formula

$$T = \frac{1}{2}\int \mathfrak{h}^2\, dS$$

leads immediately to

$$\delta T = \int (\mathfrak{h}_x\, \delta\mathfrak{h}_x + \mathfrak{h}_y\, \delta\mathfrak{h}_y + \mathfrak{h}_z\, \delta\mathfrak{h}_z)\, dS = \int (\mathfrak{h} \cdot \delta\mathfrak{h})\, dS,$$

where the integration covers all space. The same will be the case with the other volume-integrals appearing in the following transformations. If an integration is performed, or if the process of integration by parts is applied, one obtains integrals over the infinite surface which we may conceive as the boundary of the field of integration. These surface-integrals however will be supposed to vanish.

We begin by writing $rot\, \mathfrak{a}$ instead of $\mathfrak{h}$, as may be done in virtue of (5); and we shall next integrate by parts, keeping in mind that, on account of (V),

$$rot\, \delta\mathfrak{h} = \frac{1}{c}\, \delta\mathfrak{l}.$$

The result is

$$\delta T = \int (rot\, \mathfrak{a} \cdot \delta\mathfrak{h})\, dS = \int (\mathfrak{a} \cdot rot\, \delta\mathfrak{h})\, dS = \frac{1}{c}\int (\mathfrak{a} \cdot \delta\mathfrak{l})\, dS\,, \quad . \qquad (9)$$

or, if we substitute for $\delta\mathfrak{l}$ its value (8),

$$\delta T = \frac{1}{c}\int \left(\mathfrak{a} \cdot \frac{\partial}{\partial t}\left\{\delta\mathfrak{d} + \varrho\mathfrak{q}\right\}\right) dS + \frac{1}{c}\int \left(\mathfrak{a} \cdot rot \left\{\varrho\,[\mathfrak{q} \cdot \mathfrak{v}]\right\}\right) dS\,. \quad (10)$$

Using (4), we may put for the first term

[1]) This assumption only means to define the value of T we shall assign to the wholly fictitious varied state.

$$\frac{1}{c}\frac{d}{dt}\int(\mathfrak{a}\cdot\{\delta\mathfrak{d}+\varrho\mathfrak{q}\})\,dS-\frac{1}{c}\int(\dot{\mathfrak{a}}\cdot\{\delta\mathfrak{d}+\varrho\mathfrak{q}\})\,dS=$$

$$=\frac{1}{c}\frac{d}{dt}\int(\mathfrak{a}\cdot\{\delta\mathfrak{d}+\varrho\mathfrak{q}\})\,dS+\int(\mathfrak{d}\cdot\{\delta\mathfrak{d}+\varrho\mathfrak{q}\})\,dS+\int(grad\,\varphi\cdot\{\delta\mathfrak{d}+\varrho\mathfrak{q}\})\,dS. \quad (11)$$

Now, it appears from (9) that

$$\frac{1}{c}\int(\mathfrak{a}\cdot\{\delta\mathfrak{d}+\varrho\mathfrak{q}\})\,dS \quad \ldots\ldots \quad (12)$$

is the change the magnetic energy of the system would undergo, if we gave to the current the change $\delta\mathfrak{d}+\varrho\mathfrak{q}$. We shall write $\delta'\mathfrak{i}$ for *this* variation of the current, and $\delta'\mathfrak{h}$, $\delta' T$ for the corresponding variations of $\mathfrak{h}$ and T. As to $\delta'\mathfrak{i}$, it may be defined as the current that would exist if the changes represented by $\mathfrak{q}$ and $\delta\mathfrak{d}$ were accomplished in unit of time.

On the other hand, $\int(\mathfrak{d}\cdot\delta\mathfrak{d})\,dS$ is the variation of the electric energy U and the last integral in (11) is 0, because the vector $\delta\mathfrak{d}+\varrho\mathfrak{q}$ is solenoidally distributed. Thus, the first term in (10) becomes

$$\frac{d\delta' T}{dt}+\delta U+\int(\mathfrak{d}\cdot\varrho\mathfrak{q})\,dS.$$

For the last term in that equation we find, integrating by parts,

$$\frac{1}{c}\int(rot\,\mathfrak{a}\cdot\{\varrho[\mathfrak{q}\cdot\mathfrak{v}]\})\,dS=\frac{1}{c}\int\varrho\,(\mathfrak{h}\cdot[\mathfrak{q}\cdot\mathfrak{v}])\,dS=\frac{1}{c}\int\varrho\,(\mathfrak{q}\cdot[\mathfrak{v}\cdot\mathfrak{h}])\,dS,$$

so that finally

$$\delta T=\frac{d\delta' T}{dt}+\delta U+\int\varrho\left(\mathfrak{q}\cdot\left\{\mathfrak{d}+\frac{1}{c}[\mathfrak{v}\cdot\mathfrak{h}]\right\}\right)dS.$$

Now, the equation (VII) shows that the last term is precisely the work done, during the displacements $\mathfrak{q}$, by the electric forces exerted by the aether on the electrons.

Writing δE for this work, we have

$$\delta E=\delta(T-U)-\frac{d\delta' T}{dt} \quad \ldots\ldots \quad (13)$$

an equation closely corresponding to D'ALEMBERT's principle in common dynamics.

§ 7. The motion of the electrons themselves may be determined by ordinary methods; it will be governed by the electric forces whose work has been denoted by δE, together with forces of any other kind that may come into play. We shall confine ourselves to those cases in which these latter forces depend on a potential energy U_1; then the total virtual work of all forces acting on the electrons will be $\delta E - \delta U_1$. Moreover we shall ascribe to the electrons a certain kinetic energy T_1, which they have by virtue of their mass in the ordinary sense of the word. Should there be no such "true" mass, we have only to put $T_1 = 0$.

One of the forms that may be given to the variational equation of motion for a system of material particles is

$$\delta A = \frac{d\delta' T_1}{dt} - \delta T_1 ,$$

δT_1 being the change of T_1, if we pass from the real motion to some varied motion in which the varied positions are reached at the same moments as the original positions in the real motion, δA the virtual work of the forces, and $\delta' T_1$ the increment that would be acquired by the kinetic energy T_1, if variations, equal to the virtual changes of the coordinates, were imparted to the corresponding velocities (the coordinates themselves being kept constant). For our system of electrons

$$\delta A = \delta E - \delta U_1 ;$$

hence, if we use for δE the formula (13),

$$\delta \{(T + T_1) - (U + U_1)\} - \frac{d\delta'(T+T_1)}{dt} = 0.$$

We shall finally multiply this by dt and integrate from t_1 to t_2. In case both the displacements $\mathfrak{q}$ and the variations $\delta\mathfrak{d}$ vanish at the limits, we find

$$\delta \int_{t_1}^{t_2} \{(T + T_1) - (U + U_1)\} \, dt = 0$$

This is analogous to the principle of least action.

§ 8. In what precedes there has been question of the variations of the energies T and U, taken for the system of electrons together with the surrounding aether, which extends to infinite distance. Similar though somewhat less simple results are obtained, if one understands by T and U the magnetic and the electric energies, in so far only as they belong to the space within an immovable closed surface σ. In what follows it is to be understood that this surface may have, relatively to the system of electrons, any position we like; for simplicity's sake however we shall suppose that it cuts none of them, so that, in every point of σ, the density $\rho = 0$. As to the virtual variations, determined by $\mathfrak{q}$ and $\delta\mathfrak{d}$, they need not at all be confined to the part of the system within the surface. We shall denote by n the normal to the surface, drawn towards the outside, and by λ, μ, ν the angles between this normal and the positive axes of coordinates.

If now we repeat the above calculations, we have to do with volume-integrals confined to the space within σ, and every integration by parts will give rise to a surface-integral.

Thus, to the last member of (9) we shall have to add the term

$$\int \begin{vmatrix} \cos\lambda, & \cos\mu, & \cos\nu \\ \mathfrak{a}_x, & \mathfrak{a}_y, & \mathfrak{a}_z \\ \delta\mathfrak{h}_x, & \delta\mathfrak{h}_y, & \delta\mathfrak{h}_z \end{vmatrix} d\sigma = \int [\mathfrak{a} \cdot \delta\mathfrak{h}]_n \, d\sigma$$

and the value of (12) will no longer be $\delta' T$, but

$$\delta' T - \int \begin{vmatrix} \cos\lambda, & \cos\mu, & \cos\nu \\ \mathfrak{a}_x, & \mathfrak{a}_y, & \mathfrak{a}_z \\ \delta'\mathfrak{h}_x, & \delta'\mathfrak{h}_y, & \delta'\mathfrak{h}_z \end{vmatrix} d\sigma = \delta' T - \int [\mathfrak{a} \cdot \delta'\mathfrak{h}]_n \, d\sigma. \quad (14)$$

The last integral of (11) becomes

$$c\int (grad\,\varphi \cdot \{rot\,\delta'\mathfrak{h}\})\, dS = c\int (rot\,grad\,\varphi \cdot \delta'\mathfrak{h})\, dS - c\int [grad\,\varphi \cdot \delta'\mathfrak{h}]_n \, d\sigma \quad (15)$$

Here the first term on the right-hand side is 0, since $rot\,grad\,\varphi = 0$. The transformation of the last part of (10) remaining as it was, as we have supposed $\varrho = 0$ in all points of the surface, we finally find for the second member of (13) the additional term

$$\int \left\{ -[\mathfrak{a} \cdot \delta\mathfrak{h}]_n + \frac{\partial}{\partial t}[\mathfrak{a} \cdot \delta'\mathfrak{h}]_n + c\,[grad\,\varphi \cdot \delta'\mathfrak{h}]_n \right\} d\sigma.$$

But, on account of (4),

$$\frac{\partial}{dt}[\mathfrak{a} \cdot \delta'\mathfrak{h}]_n + c\,[grad\,\varphi \cdot \delta'\mathfrak{h}]_n =$$

$$= \left[\mathfrak{a} \cdot \left\{\frac{\partial \delta'\mathfrak{h}}{\partial t}\right\}\right]_n + [\dot{\mathfrak{a}} \cdot \delta'\mathfrak{h}]_n + c\,[grad\,\varphi \cdot \delta'\mathfrak{h}]_n =$$

$$= \left[\mathfrak{a} \cdot \left\{\frac{\partial \delta'\mathfrak{h}}{\partial t}\right\}\right]_n - c\,[\mathfrak{d} \cdot \delta'\mathfrak{h}]_n,$$

We get therefore, instead of (13),

$$\delta E = \delta(T - U) - \frac{d\delta' T}{dt} + \int \left\{ \left[\mathfrak{a} \cdot \left\{\frac{\partial \delta'\mathfrak{h}}{\partial t} - \delta\mathfrak{h}\right\}\right]_n - c\,[\mathfrak{d} \cdot \delta'\mathfrak{h}]_n \right\} d\sigma \quad (16)$$

§ 9. The following are some examples of the applications that may be made of the formulae (13) and (16).

a. Let the virtual changes in the position of the electrons and in the dielectric displacement be proportional to the rates of change in the real motion, i. e. let

$$\mathfrak{q} = \varepsilon\mathfrak{v}, \qquad \delta\mathfrak{d} = \varepsilon\dot{\mathfrak{d}},$$

ε being a constant infinitely small factor. From these assumptions it follows at once that

$$\delta' \mathfrak{l} = \varepsilon \mathfrak{l}, \qquad \delta' \mathfrak{h} = \varepsilon \mathfrak{h}.$$

Now the magnetic energy may be considered as a homogeneous quadratic function of the components of the current; it will therefore change in ratio of 1 to $1 + 2\varepsilon$, if the current becomes $(1 + \varepsilon)\mathfrak{l}$. Thus: $\delta' T = 2 \varepsilon T$.

We may also infer from our assumptions that the position of the electrons and the values of $\mathfrak{d}$ are, in the varied motion at the time t, what they are in the real motion at the time $t + \varepsilon$, so that the only difference between the two motions is that the one is in advance of the other by an interval ε.

In this way it is seen that

$$\delta T = \varepsilon \frac{dT}{dt}, \quad \delta U = \varepsilon \frac{dU}{dt}, \quad \delta \mathfrak{h} = \varepsilon \frac{\partial \mathfrak{h}}{\partial t}, \quad \frac{\partial\, \delta' \mathfrak{h}}{\partial t} - \delta \mathfrak{h} = 0.$$

Substituting these values in the equation (16), we get, after division by ε and multiplication by dt, denoting by dE the work done by the electric forces in the real motion, during the time dt,

$$dE = - d(T + U) - c\, dt \int [\mathfrak{d} . \mathfrak{h}]_n \, d\sigma. \quad . \quad . \quad . \quad (17)$$

This is the equation of energy. The last term represents the flow of energy through the surface.

b. Applying (17) to a single electron, whose motion is a translation with variable velocity along a straight line, one may calculate the force with which it is acted on by the aether, and which, under certain simplifying assumptions, is found to be proportional to the acceleration and directed oppositely to it. The quotient of this force, divided by the acceleration, may appropriately be called the *electromagnetic mass* of the electron.

c. There will likewise be a force proportional and opposed to the acceleration, if the latter is perpendicular to the direction of motion. In this case however, of which the uniform motion of an electron in a circle furnishes the simplest example, we must recur to the equation (16), in order to determine the force. The surface σ may be supposed to lie at infinite distance and the virtual displacement must be taken in the direction of the acceleration. The ratio of the force and the acceleration may again be called the *electromagnetic mass*, though, except for small velocities, its value is not equal to that of the corresponding ratio in the case *b*.

In both cases the result agrees with what has been found by Abraham.

Ponderomotive action on a system of electrons.

§ 10. A virtual change of a very simple kind is an infinitely small translation of all the electrons, combined with what we may call an equal translation in the same direction of the whole electric field. Applying to these variations — which we give as well to the part of the system outside the surface σ as to the part enclosed by it — the equation (16), one may calculate the resulting force exerted by the aether on the electrons within the surface. This force may be shown to consist of two parts, the first of which is the force with which we should have to do, if the surface σ were subjected to the stresses in the aether, whose components have been already determined by MAXWELL, whereas the second part is determined by the rate of change of a certain integral, relating to the space S within σ. The latter part will therefore vanish if the state is stationary, and may be left out of account if, for periodic states, we wish only to know the mean value of the resulting force, taken for a full period. I need not here work out the formulae, having formerly deduced the result in a more direct way. The components of MAXWELL's stress are

$$\left.\begin{aligned} X_x &= \frac{1}{2}(\mathfrak{d}_x^2 - \mathfrak{d}_y^2 - \mathfrak{d}_z^2) + \frac{1}{2}(\mathfrak{h}_x^2 - \mathfrak{h}_y^2 - \mathfrak{h}_z^2), \text{ etc.} \\ X_y &= Y_x = \mathfrak{d}_x \mathfrak{d}_y + \mathfrak{h}_x \mathfrak{h}_y, \text{ etc.} \end{aligned}\right\} \quad (18)$$

and the just mentioned volume-integral is

$$-\frac{1}{c^2}\int \mathfrak{S}_h \, d\, S,$$

$\mathfrak{S}_h$ being the flux of energy in the direction h, for which we seek the resulting force.

Thus, the resulting force in the direction of x is given by

$$\Xi = \int X_n \, d\,\sigma - \frac{1}{c^2}\frac{d}{dt}\int \mathfrak{S}_x \, d\, S. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The vector $\frac{1}{c^2}\int \mathfrak{S} \, d\, S$ is called by ABRAHAM the *electromagnetic momentum.*

§ 11. Similar results would be obtained if we chose for the virtual variation, instead of a translation, an infinitely small rotation about an axis passing through the origin of coordinates; the equation (16) would then serve to determine the resulting couple, arising from all the forces exerted by the aether on the electrons within the surface σ. The moment of this couple may however be calculated in a shorter way, if we start from what we know already about the forces.

Indeed, in virtue of the formula (19) and the two corresponding to it, the components of the force acting on an element of volume $d\,S$ may be represented as follows:

$$\left.\begin{aligned} X\,dS &= \left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right)dS-\frac{1}{c^2}\,\dot{\mathfrak{S}}_x\,dS,\\ Y\,dS &= \left(\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right)dS-\frac{1}{c^2}\,\dot{\mathfrak{S}}_y\,dS,\\ Z\,dS &= \left(\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right)dS-\frac{1}{c^2}\,\dot{\mathfrak{S}}_z\,dS \end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

and these formulae give immediately for the components of the couple

$$\int (yZ-zY)\,dS=\int (yZ_n-z\,Y_n)\,d\sigma-\frac{1}{c^2}\int (y\dot{\mathfrak{S}}_z-z\dot{\mathfrak{S}}_y)\,dS. \quad . \quad (21)$$

§ 12. Another consequence of the equations (20), analogous to the well known virial-theorem in ordinary kinetic theory, will perhaps be thought of some interest. In order to find it, we have only to add the three equations, multiplied by x, y, z, and to integrate the result over the space S, within the surface σ. Transforming such terms as $\int x\frac{\partial X_x}{\partial x}\,dS$ by means of partial integration, we find

$$\int (Xx+Yy+Zz)\,dS=\int (X_nx+Y_ny+Z_nz)\,d\sigma-$$

$$-\int (X_x+Y_y+Z_z)\,dS-\frac{1}{c^2}\frac{d}{dt}\int (\mathfrak{S}_xx+\mathfrak{S}_yy+\mathfrak{S}_zz)\,dS. \quad . \quad (22)$$

For stationary states the last term will vanish, so that, if we substitute in the term preceding it the values (18),

$$\int (Xx+Yy+Zz)\,dS=\int (X_n\,x+Y_n\,y+Z_n\,z)\,d\sigma+T+U.$$

Particular cases of ponderomotive action.

§ 13. In a large variety of cases, in which the system of electrons is confined to a space of finite dimensions, the electric and magnetic intensities in the surrounding field become so feeble at great distances that the surface-integrals in 19) and (21) approach the limit 0, if the surface σ moves to infinite distance. Moreover, the volume-integrals will vanish if the state is stationary. We then come to the conclusion that the resulting force and the resulting couple are 0 for the whole system. If the system consists of two parts A and B, we may express the same thing by saying that the total ponderomotive action on one of these is equal and opposite to the total action on the other.

Of course this will be equally true if, for a system whose state changes periodically, we have only in view the mean ponderomotive action during a full period.

These theorems are useful whenever the phenomena in one of the parts, say in A, are not well enough known to permit a direct calculation of the force acting on this part of the system. If the phenomena in B are less complicated, so that we encounter no difficulty in determining the force or the couple acting on this part, the action on A will be found at the same time.

We may apply this in the first place to well-known experiments on electromagnetic rotations.

Let us consider a cylindrical magnet, touched in two points of its surface by the ends of a conducting wire W. Let this wire be the seat of an electromotive force, producing a current that flows through W and through part of the magnet. The ponderomotive forces acting on the wire are known with certainty and may easily be deduced from the formula (VII); they produce a couple, tending to turn the wire about the axis of the magnet. Without entering into any speculations concerning the motion of the electrons in its interior, we may infer that the magnet will be acted on by an equal couple in the opposite direction.

Of course this reasoning must be justified by showing that the surface-integral in (21) is really 0, if it is taken for a surface at infinite distance. This is readily seen to be the case, if we keep in mind that, at great distances, the magnetic force produced by the system varies inversely as the third power of the distance, and that the intensity of the electric field, if it exist at all, will certainly contain no terms diminishing more slowly than the square of the distance.

§ 14. I shall choose as a second example some experiments, lately made by WHITEHEAD [1]) for the purpose of testing a consequence of MAXWELL's theory that has been admitted by many physicists and is unavoidable in the theory of electrons, viz. that a ponderable dielectric, which is the seat of a variable dielectric displacement, and therefore of a displacement-current, when placed in a magnetic

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[1]) WHITEHEAD, Ueber die magnetische Wirkung elektrischer Verschiebung, Physikalische Zeitschr., 4, p. 229, 1903.

field, will be acted on by a similar force as a body carrying a conduction-current. In WHITEHEAD's apparatus two cylindric metallic plates, having the same vertical axis $P\,Q$, formed a condenser, in which a rapidly alternating electric field was maintained; at the same time alternating currents were passed through the horizontal windings of a circular coil, surrounding the condenser; the axis of the coil, which is at the same time the axis of its magnetic field, coincided with $P\,Q$. A sensitive torsion-balance was suspended by a wire passing along the axis of the instrument; the ends of the beam carried each a piece of some solid dielectric, so that these two equal pieces hung, diametrically opposite each other, in the air-space between the condenser-plates. The two fields, the electric and the magnetic, had exactly the same period, being produced by the same alternate current-machine; besides, the arrangements were such that there was a phase-difference of a quarter period between the two fields. Thus, at the instants at which the magnetic force had its maximum values, the rate of change of the electric field and consequently the intensity of the displacement-current was likewise at its maximum. Under these circumstances a sensible couple acting on the dielectric was expected, but no deviation of the beam, attributable to such a couple, could with certainty be observed.

We may remark in the first place that in WHITEHEAD's formula for the expected effect, the specific inductive capacity K appears in the numerator. If this were right, a couple would act on the aether between the plates itself. According to the theory of electrons, as here presented, ponderomotive force acts only on the electrons contained in ponderable bodies, but in no case on the aether. The theory therefore regards every ponderomotive action as due to the *difference* between the properties of the body acted upon and the aether; it can lead to a formula containing in the numerator $K—1$, but never to one, containing, instead of this factor, the coefficient K itself.

In the second place WHITEHEAD has overlooked a circumstance by which the effect he sought for must have been, at least for the greater part, compensated. The compensation may be shown to be complete if the properties of the dielectric used differ from those of the aether to so small extent, that quantities which are in this respect of the second order of magnitude, i. e. of the order $(K—1)^2$, may be neglected.

If this may be done, the ponderomotive action on a ponderable dielectric, placed between the condenser-plates, may be considered not to be altered by the presence in the field of a second or third piece of the same dielectric. Now, the two bodies suspended at the ends of WHITEHEAD's torsion-balance may be taken to have been parts of a complete dielectric ring, bounded by a surface of revolution with the axis PQ. Moreover it will be safe to assume that the action on the two bodies which it was sought to observe, did not depend on their relative positions with respect to the wires leading to the condenser-plates, and remained therefore the same, in whatever position the torsion-balance was turned. If this was the case, the action on a body that is the n^{th} part of the ring (being cut out of it by two planes passing through the axis) must have been the n^{th} part of the couple, acting on the complete ring. Consequently, it will suffice to show that the effect is 0, if the experiment is made with a complete dielectric ring.

§ 15. For simplicity's sake we shall suppose the condenser-plates to be united by a wire W and their alternating electric charges to be produced by a periodic electromotive force in this wire. As to the currents in the coil, they may be regarded as due to electromotive forces of the same period, acting in the windings themselves; indeed, the action on the dielectrics can only depend on the magnetic field and not on the way in which it is produced. For this same reason it is allowable to ascribe to the windings so small a resistance that they do not carry any appreciable charges.

Then no other but electromagnetic forces will act on the windings of the coil and these cannot give rise to any couple about the axis PQ, because such forces are perpendicular to the elements of the windings. By the theorem of § 13 the couple acting on the torsion-balance must therefore have been equal and opposite to the moment of rotation, acting on the condenser-plates and the wire W. It remains to show that this last moment has been 0.

I shall denote by I the electromotive forces acting in the connecting wire W, by II those existing in the windings of the coil, and I shall distinguish by the suffixes 1 and 2 the states arising from these two causes. Let us indicate by A_1 the charges of the plates and the currents in these and the wire W, in so far as they are due to I, and let A_2 have the same meaning with respect to II; also, let F_1 and F_2 be the electromagnetic fields excited by the two causes. In each of these fields there will be an electric force $\mathfrak{d}$ (acting on charges that are in rest), as well as a magnetic force $\mathfrak{h}$; in virtue of the first, the field will exert a ponderomotive force on the charges of the plates and in virtue of the second on the currents, one of these actions being determined by the first, and the other by the last term in the general equation (VII). If we denote by the symbol (F, A) the couple acting on the plates and the wire, in so far as it is due to a field F and a state A of these bodies, the two actions we shall have to consider may be represented by

$$(F_1, A_2) \text{ and } (F_2, A_1).$$

The first of these is readily seen to be 0. Indeed, the magnetic field, produced by the forces II, though modified by the presence of the dielectric ring, is symmetrical around the axis PQ. Therefore, if the periphery of the condenser-plates is nowhere interrupted, the state A_2 will consist in circular currents in these plates, without any electric charge. It is impossible that the field F_1 should, by its action on these currents, give rise to a couple, since, whatever be the nature of this field, each element of the stream-tubes will only be acted on by a force perpendicular to its length.

In reality the case was somewhat different, each condenser-plate being cut by a vertical slit. There must have been equal and opposite charges at the edges of each slit and the field F_1 must have acted on these charges, in virtue of the electric force existing in it. These forces may however be supposed to have annulled each other, because the distance between the charges on the two edges was very small.

§ 16. The action (F_2, A_1) is therefore the only one that remains to be considered. Now, in the state A_1, the plates of the condenser were the seat of charges, whose amount was modified by the influence of the dielectric ring, and whose alternations were accompanied by currents in the wire W and in part of the plates themselves. In so far as they are currents of conduction, i. e. in so far as they consist in a motion of electrons, these currents are evidently unclosed. We may decompose the whole system of them into infinitely thin stream-tubes, the tubes being all thronged together in the connecting wire, and widening out in the plates, at whose surfaces each stream-tube ends in two elements of surface.

Let S be one of the stream-tubes, G the end of it on the outer, and H that on the inner plate, e the charge in G, $-e$ that in H,

$$i = \frac{de}{dt} \quad \ldots \ldots \ldots \quad (23)$$

the current in the tube in the direction from H towards G, and let us consider the action (F_2, A_1) only in so far as it depends on this current i and on the charges e and $-e$.

In the first place there will be an electromagnetic force on the tube S, owing to the current i. The couple arising from it depends on the course of the magnetic lines of force in the field F_2; it is most easily found by remarking that its work during a complete revolution of S about the axis PQ is numerically equal to the product of $\frac{i}{c}$ by the number of lines of force that are cut by S. These lines are precisely those that are intersected by the surface described by S in its revolution, a surface which may have different forms, according to the form of the wire W, but has at all events for its boundaries the circles described by the points G and H. Let N be the number of these lines, taken positive if the middle one of them passes *upwards* along PQ, and let us take as positive directions for the rotation and for the couple the direction corresponding to the upward direction. Then, for a full revolution in the positive direction, the work of the couple will be $-\frac{1}{c} iN$, whence we find for the couple itself

$$-\frac{1}{2\pi c} iN \qquad \qquad (24)$$

If this were all, we should indeed come to an effect such as was expected by WHITEHEAD. We must however keep in mind that there can never be a variable magnetic field without *electric* forces. Such forces, represented in direction and intensity by the vector $\mathfrak{d}$, will exist in the field F_2, the lines of electric force being circles around the axis PQ.

We must therefore add to (24) the couple arising from the action of the field on the charges e and $-e$; its moment may again be found by considering the work done in a complete revolution in the positive direction.

The force on the charge e being $e\,\mathfrak{d}$, its work is equal to the product of e by the line-integral of $\mathfrak{d}$ along the circle described by G. Similarly, the work of the force acting on the charge $-e$ in H is the product of $-e$ by the line-integral of $\mathfrak{d}$ along the circle described by H, or, what amounts to the same thing, the product of $+e$ by the line-integral for this circle, if it is taken in the negative direction. Now, if we follow the circle G in the positive and the circle H in the negative direction, we shall have gone along the whole contour of the surface described by the stream-tube S, in a direction corresponding to the positive direction of the magnetic force. Hence, by a well known theorem, of which the fundamental equation (VI) is the expression, the sum of the two line-integrals by which e must be multiplied, will be

$$-\frac{e}{c}\frac{dN}{dt},$$

and the couple to be added to (24) will be given by

$$-\frac{1}{2\pi c}\, e\, \frac{dN}{dt}.$$

Taking into account (23), we find for the total couple

$$-\frac{1}{2\pi c}\left(i\,N + e\frac{dN}{dt}\right) = -\frac{1}{2\pi c}\,\frac{d(eN)}{dt}.$$

Since this is the rate of change of a periodic quantity, the mean value will be 0, as above asserted.

The above somewhat complicated reasoning has been used in order to avoid the difficulties arising in a closer examination of the phenomena going on in the ponderable dielectrics. The result may however be verified by making suitable assumptions concerning these phenomena. It will suffice for our purpose to replace one of the dielectric bodies by a single pair of electrons A and B, the first of which is immovable, whereas the second may be displaced over an infinitely small distance, in a radial direction, by the electric forces of the field F_1. We shall denote by $-e$ and $+e$ the charges of A and B, by r the distance of A to the axis, by s the infinitely small distance AB, and we shall write $\mathfrak{h}_z$ for the vertical component of the magnetic force in the field F_2 and D for the value of the delectric displacement in this field at a distance r from the axis. We shall take the positive directions as follows: for s outwards, for $\mathfrak{h}_z$ upwards, and for D along the circular line of electric force in a direction corresponding to the positive direction of $\mathfrak{h}_z$, i. e. in the direction of a positive rotation about the axis.

Now, owing to the velocity $\frac{ds}{dt}$ of the electron B, there will be, according to the formula (VII), a force

$$-\frac{e}{c}\mathfrak{h}_z\frac{ds}{dt}$$

acting on this electron along a circle about the axis, and producing a moment

$$-\frac{e}{c}\,r\,\mathfrak{h}_z\frac{ds}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24')$$

This is the couple of which WHITEHEAD has sought to prove the existence. It is however annulled by the moment arising from the action of the field F_2 in virtue of its electric force D. For the particle A this moment is

$$-\,e\,r\,\mathrm{D}$$

and for the particle B it is obtained if we replace $-e$ by $+e$, taking at the same time the value of r D at the distance $r+s$ from the axis.

The algebraic sum of the two moments will therefore be

$$e\,s\,\frac{\partial}{\partial r}(r\,\mathrm{D})$$

and for this we may write

$$-\frac{e}{c}\,s\,r\,\frac{\partial \mathfrak{h}_z}{\partial t}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24'')$$

since, by the equation (VI)

$$\frac{\partial}{\partial r}(r\,\mathrm{D}) = -\frac{1}{c}\,r\,\frac{\partial \mathfrak{h}_z}{\partial t}.$$

For the sum of (24′) and (24″) we may write

$$-\frac{e}{c}\,r\,\frac{d\,(s\,\mathfrak{h}_z)}{dt},$$

whence it is immediately seen that its mean value is 0 for a full period.

Physics. — *Methods and apparatus used in the cryogenic laboratory. III. Baths of very uniform and constant temperature in the cryostat (continued). A cryostat of modified form for apparatus of small dimensions. IV. A permanent bath of liquid nitrogen at ordinary and at reduced pressure. V. Arrangement of a* Burckhardt-Weiss *vacuum-pump for use in the circulations for low temperatures.* Communication N°. 83 (continued) from the Laboratory at Leiden. By Prof. H. Kamerlingh Onnes. (Read February 28, 1903).

III. § 6. *A cryostat of modified form for apparatus of small dimensions.* If the cross sections of the apparatus that is to be immersed into the bath are small, vacuum glasses may be profitably used in the construction of the cryostat. For, vacuum glasses of comparatively small diameter can then accommodate the stirrer and the temperature indicator in addition to the measuring apparatus. Plate IV shows a cryostat of the kind, viz. the one used in the determinations by Hyndman and myself on the critical state of oxygen.

Obviously the arrangement could be much simpler, as it was not necessary to watch the liquefied gas streaming from the jet or to use the generated cold vapour for the cooling and as no particles of dust from the leads had to be feared, a filter was not required. (Comp. Comm. 51, Sept. '99 § 2. Y_1 p. 12). The principles for obtaining a uniform con-