$$\cos^2 \varphi = \frac{3(a-x)}{2a}$$

and from this finally for the locus

$$27\, y^2 (a - x) = (3\, x - a)^2\,;$$

so this is a cissoid, whose cusp lies at a distance $x = \frac{1}{3}\, a$ from point O and whose asymptote passes through β.

OBSERVATION. The systems of conics treated in these two cases are simply infinite systems, where more than one conic pass through a point and more than one conic touches a right line; so they are distinguished from the ordinary pencils and the tangential ones.

Thus for the first mentioned system six conics pass through a point and twelve conics touch a right line.

Astronomy. — "*On the masses and elements of Jupiter's Satellites, and the mass of the system*", by Dr. W. DE SITTER. (Communicated by Prof. J. C. KAPTEYN).

(Communicated in the meeting of February 29, 1908).

The determination of the elements and masses of the satellites of Jupiter and of the mass and the dynamical compression of the planet, which is communicated in the following pages, is based almost exclusively on heliometric and photographic observations made at the observatories at the Cape of Good Hope, Pulkowa and Helsingfors, in the years 1891 to 1904.

In addition to these I have also included in the discussion the observations made by BESSEL with the heliometer at Königsberg in 1832 to 1839, and the values of the node of the second and the perijove of the fourth satellite in 1750.0 (for which DELAMBRE's values were adopted). These latter have however, as will appear later on, only a very slight effect on the final results. The determination of all masses and elements is thus practically independent of observations of eclipses.

Previous to 1891 no series of observations of the satellites except of the eclipses had been made with the purpose of determining the elements of the orbits. Such series of observations as had been executed in the first half of the nineteenth century by BESSEL, AIRY and others, were avowedly intended only to determine the mass of Jupiter. Accordingly the satellites were by these astronomers, so far as possible, observed only near elongation. The series of observations made by GILL at the Cape in 1891, and the series of photographs taken in the same year by DONNER (on the suggestion of BACKLUND) and continued by him and by KOSTINSKY to 1898, are the first series of observations of the satellites in every point of their orbits. The discussion of the Cape observations of 1891 by me then suggested the desirability of further observations, which were executed by COOKSON with the Cape heliometer in 1901 and 1902, while photographic observations were made at the Cape in 1902, 1903 and 1904.

It will be conducive to a good understanding of what follows if I collect here at once all the notations used.

The theory, which was compared with the observations, is SOUILLART'S [1]). This theory gives the longitudes and latitudes of the satellites, referred to the plane of Jupiter's orbit of 1850·0. As I have explained in *Cape XII. 3* [2]) page 96, the orbit of 1900·0 has been used in its place.

The radii-vectores and the longitudes of the satellites in their orbits are given by the formulas:

$$r_i = a_i \varrho_i$$
$$v_i = l_i + \vartheta_i + \text{inequalities}$$
$$l_i = n_i t + \varepsilon_i .$$

We have the rigorous equations:

$$\varepsilon_1 - 3\ \varepsilon_2 + 2\ \varepsilon_3 = 180^\circ$$
$$n_1 - 3\ n_2 + 2\ n_3 = 0.$$

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1) *Théorie analytique des mouvements des satellites de Jupiter*, par M. SOUILLART, Memoirs R. A. S. XLV, 1880.

Théorie analytique des mouvements des satellites de Jupiter, seconde partie, par M. SOUILLART, Mémoires des savants étrangers, XXX, 1887.

2) *Annals of the Royal Observatory at the Cape of Good Hope*, under the direction of Sir DAVID GILL, K. C. B. Volume XII:

Part I. (Not published).

Part II. *Determination of the mass of Jupiter and orbits of the satellites*, by BRYAN COOKSON M. A. (1906).

Part III. *A determination of the inclinations and nodes of the orbits of Jupiter's satellites*, by Dr. W. DE SITTER. (1906).

Part IV. *Determination of the elements of the orbits of Jupiter's satellites*, by BRYAN COOKSON. (1907).

The titles of these papers, which I shall often have to quote, are referred to by the abbreviations used in the text above. I shall also often quote:

Publications of the Astronomical Laboratory at Groningen, N°. 17. *On the Libration of the three inner satellites of Jupiter*, by W. DE SITTER, Sc. D. (1907), which is referred to as: *Gron. Publ. 17.*

The quantities ϑ_i are the libration, which is determined by the formulas:

$$\vartheta = l_1 - 3l_2 + 2l_3 - 180^\circ = k \sin \frac{t-t_0}{T} = k \sin \beta (t-t_0)$$

$$\vartheta_i = \frac{Q_i}{\beta^2} \vartheta \ , \quad \beta^2 = Q_1 - 3Q_2 + 2Q_3.$$

The quantities Q_i (and therefore β^2 and T) depend on the masses, and have been given in *Gron. Publ.* 17, Art. 18, up to terms of the third order.

The inequalities can be divided into three groups, according to their periods, of which the first group may be divided into three subdivisions. These are:

Ia. Equations of the centre. The osculating excentricities and perijoves — excluding their periodic perturbations (which are taken into account separately as inequalities of the longitudes and radii-vectores) — are determined by the formulas:

$$h_i = 2E_i \sin \Omega_i = 2\Sigma_j \tau_{ij} e_j \sin \tilde{\omega}_j$$
$$k_i = 2E_i \cos \Omega_i = 2\Sigma_j \tau_{ij} e_j \cos \tilde{\omega}_j.$$

Here e_i and $\tilde{\omega}_i$ are the *own* excentricities and perijoves of the four satellites. The angles $\tilde{\omega}_i$ vary proportionally to the time, and the coefficients τ_{ij} depend on the masses, τ_{ii} being unity. We have then

$$\delta v_i = - \cos l_i \, h_i + \sin l_i \, k_i$$
$$\delta \varrho_i = - \tfrac{1}{2} \sin 1^\circ (\sin l_i \, h_i + \cos l_i \, k_i).$$

The squares of E are negligible, except for the fourth satellite. The corresponding term is mentioned under *Ic*.

Ib. The great inequalities. These arise (as perturbations in h_i and k_i) through the commensurability of the mean motions of the three inner satellites. They are:

$$\delta v_1 = x_1 \sin 2(l_1 - l_2) \qquad \delta \varrho_1 = - \tfrac{1}{2} \sin 1^\circ \, x_1 \cos 2(l_1 - l_2)$$
$$\delta v_2 = - x_2 \sin (l_1 - l_2) \qquad \delta \varrho_2 = \tfrac{1}{2} \sin 1^\circ \, x_2 \cos (l_1 - l_2)$$
$$\delta v_3 = - x_3 \sin (l_2 - l_3) \qquad \delta \varrho_3 = \tfrac{1}{2} \sin 1^\circ \, x_3 \cos (l_2 - l_3)$$

Ic. Minor inequalities of short periods. The periods of all the inequalities of group *I* are short (not exceeding 17 days).

II. Inequalities arising through the commensurability of the mean motions, and having periods between 400 and 500 days. These only exist for the satellites *I*, *II* and *III*. In the radii-vectores they are negligible. Their expressions are:

$$\delta v_i = \Sigma_j \varkappa_{ij} \sin \varphi_j \quad,$$
$$\varphi_i = l_2 - 2l_3 + \tilde{\omega}_i .$$

The coefficients $\varkappa_{ij}$ are proportional to e_j, and also depend on the masses.

III. Inequalities with very long periods (exceeding 12 years). These also are negligible in the radii-vectores.

The latitudes of the satellites over the plane of Jupiter's orbit are given by the formula:

$$s_i = I_i \sin (v_i - N_i).$$

The inclinations and nodes[1]) are in SOUILLART's theory determined by the formulas:

$$I_i \sin N_i = \Sigma_j \sigma_{ij} \gamma_j \sin \theta_j + \mu_i \omega \sin \theta + \text{ periodic terms}$$
$$I_i \cos N_i = \Sigma_j \sigma_{ij} \gamma_j \cos \theta_j + \mu_i \omega \cos \theta + \text{ periodic terms}$$

Mr. COOKSON and I have in all our work on the satellites referred the latitudes to a fundamental plane, which is defined by its inclination and node referred to the ecliptic and mean equinox of date. For these MARTH's values have been adopted, which are for 1900.0:

$$\mathcal{I} = 2^\circ\, 9'\, 3''.94 \quad \mathcal{N} = 336^\circ\, 21'\, 28'',4$$

The longitude of the node of this plane on LEVERRIER's orbit of Jupiter, counted *in the orbit*, and the inclination on that orbit are:

$$\theta_0 = 315^\circ\, 25'\, 48''4. \quad \omega_0 = 3^\circ\, 4'\, 4''.75.$$

The longitude of the node of the orbital plane on the fundamental plane, counted *in the fundamental plane*, is therefore

$$\theta'_0 = 135^\circ\, 24'\, 34''.3.$$

The longitudes in the fundamental plane have been counted from the point O, of which the longitude is[2])

$$O = 135^\circ\, 27'\, 2''.5.$$

If the inclination and *descending* node of the fundamental plane on the orbit of 1850.0 are represented by ω_0 and ψ_0, (thus $\psi_0 = \theta_0 + 180^\circ$) and if the longitudes of the nodes Ω_i are reckoned from this descending node, we have:

$$p_i = -\, i_i \sin \Omega_i = -\, I_i \sin (N_i - \psi_0)$$
$$q_i = \quad i_i \cos \Omega_i = \quad I_i \cos (N_i - \psi_0) + \omega_0$$

from which

$$p_i = \Sigma_j \sigma_{ij} \gamma_j \sin (\psi_0 - \theta_j) + \mu_i \omega \sin (\psi_0 - \theta) + \text{ periodic terms}$$
$$q_i = \Sigma_j \sigma_{ij} \gamma_j \cos (\psi_0 - \theta_j) + \mu_i \omega \cos (\psi_0 - \theta) + \omega_0 + \text{periodic terms.}$$

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[1]) By node I always mean ascending node, unless otherwise stated.

[2]) MARTH evidently intended to adopt $O = \theta'_0$. Probably he has applied the correction, needed to derive $'$ from $\theta + 180^\circ$, with the wrong sign.

Here γ_i and θ_i are the *own* inclinations and nodes. The angles θ_i vary proportionally with the time and the coefficients σ_{ij} depend on the masses. We have again $\sigma_{ii} = 1$. ω and θ are the inclination and node of the mean equator of the planet on the orbital plane for 1900.0. In the discussions we have used the abbreviations:

$$x_0 = -\,\omega \sin(\psi_0 - \theta)$$
$$y_0 = -\,\omega \cos(\psi_0 - \theta) - \omega_0 .$$

x_0 and y_0 thus determine the position of the equator. The adopted fundamental plane nearly co-incides with the equator, and the node ψ_0 has nearly the theoretical motion of the node θ. [1]) The angle $\psi_0 - \theta$ is therefore constant and very nearly equal to 180°.

In *Gron, Publ.* 17, Chapter IV, I have given the quantities Q_i, β^2, τ_{ij} $\frac{d\tilde{\omega}_i}{dt}$, x_i, $\varkappa_{ij}$, σ_{ij}, μ_i, $\frac{d\theta_i}{dt}$ as functions of the masses, or rather, as functions of the small quantities $\varkappa'$ and ν_i, which are defined by

$$Jb^2 = 0.0219087 \qquad (1+\varkappa) \qquad (b = 1 \text{ for } d = 39''.0)$$
$$= 0.0000\,0000\,530042\,(1+\varkappa) \qquad \text{(astronomical units).}$$
$$\varkappa' = \varkappa + 0.055$$
$$m_1 = 0.0000\,40\,(1+\nu_1) \qquad m_3 = 0.0000\,80 \qquad (1+\nu_3)$$
$$m_2 = 0.0000\,22\,(1+\nu_2) \qquad m_4 = 0.0000\,424\,751\,(1+\nu_4).$$

Of $\varkappa'$, ν_2 and ν_3 only the first power was kept, of ν_1 and ν_4 all powers, which could be derived from SOUILLART's formulas were taken into account.

For the reciprocal of the mass of the system the value

$$\mathfrak{M} = 1047.40.$$

was adopted.

The data to be derived from the observations can be divided into three groups.

A. The inclinations and nodes, represented by the quantities p and q_i, i.e. the quantities determining the latitudes.

B. The data determining the longitudes and radii-vectores. These are the mean longitudes, the equations of the centre and the coefficients of the great inequalities of the three inner satellites (l_i, h_i, k_i, x_i)

C. The mean distances a_i.

A. In determining the elements from eclipse-observations, the elements of group *B* are derived from the observed epochs of the middle of the eclipse, those of group *A* from the duration of the

[1]) The motion of ψ_0 actually used by MARTH is not exactly the theoretical motion of θ. The difference is however negligible.

eclipse. This duration depends not only on those elements, but as well on the form of the shadow-cone, i.e. on the geometrical compression of the planet. This latter not being known with sufficient accuracy, it is impossible to determine the latitudes from observations of eclipses. The elements of the first group must therefore be derived exclusively from heliometric or photographic extra-eclipse-observations of the satellites.

B. For the determination of the elements of group *B*, however, the eclipses are very valuable. One eclipse-observation, which is a determination of *time*, provides a much more accurate determination of the longitude of the satellite than one extra-eclipse-observation. On the other hand the latter can be repeated as often as the weather and the available time of the observer permit, while eclipses only occur in a limited number. Another advantage of eclipse-observations is that their accuracy is independent of the distance of Jupiter from the earth, while the accuracy of extra-eclipse-observations is inversely proportional to that distance. Extra-eclipse-observations are thus generally combined in series extending over a few months on both sides of the epoch of opposition. It must not be forgotten, however, that away from the opposition the time during which Jupiter is above the horizon, and therefore the number of observable eclipses, diminishes rapidly.

For the first satellite, where eclipses are numerous, and micrometrical observations least accurate, the advantage is very probably on the side of the eclipse-observations; for the fourth, of which eclipses are rare and extra-eclipse-observations are most accurate,[1]) this ratio is reversed. So long as nothing is known about the results derived from the series of photometric eclipse-observations made at the observatory of Harvard College in the years 1878 to 1903, it is not possible to form a definite judgment regarding the relative value of the two kinds of observations. Anyhow the attempt is justified to derive also the elements of group *B* exclusively from extra-eclipse-observations.

C. The four mean distances represent only one unknown quantity, since the determination of their ratios from the mean motions (also taking into consideration the uncertainty of the perturbations which must be applied) is very much more accurate than the direct deter-

[1]) My meaning is, of course, that the determination of the jovicentric place of the satellite from extra-eclipse-observations is most accurate for IV. This is due only to the large mean distance, not to the observation itself. This latter, i.e. the determination of the relative geocentric place, seems to be slightly more accurate for II and III than for I and IV.

mination of these ratios from the observations. This one unknown — the scale-value of the system — from which the mass of the planet is derived, can naturally only be determined from extra-eclipse-observations. It has already been remarked that all series of such observations made before 1891, were made with a view to this determination.

The number of unknowns of the problem is thus 32, viz:

A.	the "own" inclinations and nodes	γ_i, θ_i . . .	8	unknowns
	the position of the equator	ω, θ . . .	2	„
	the dynamical compression	Jb^2 . . .	1	„
B.	mean longitudes with one rigorous condition	ε_i . . .	3	„
	„ motions „ „ „ „	n_i . . .	3	„
	the amplitude and phase of the libration	k, t_0 . . .	2	„
	the own excentricities and perijoves	$e_i, \tilde{\omega}_i$ . . .	8	„
	the mass of each satellite	m_i . . .	4	„
C.	the reciprocal of the mass of the system	$\mathfrak{M}$ . . .	1	„
			32	

The observations which have been used in the derivation of the results to be communicated below are the following:

1. Heliometer-observations made in 1891 at the Royal Observatory, Cape of Good Hope, by GILL and FINLAY. These have been reduced by me and were published in my inaugural dissertation [1]). After the publication some mistakes and errors of computation have been found, which have been already corrected in the results used here. The corrected results will soon be published in Cape Annals, Vol XII, Part. I.

The high order of accuracy of this series is due to the principle, introduced by GILL, to measure only distances and position-angles of the satellites relatively to each other, and not relatively to the planet [2]). Thus large systematic errors are avoided.

2. Heliometer-observations made in 1901 and 1902 at the Cape Observatory by BRYAN COOKSON, M. A., reduced by himself, and published in Cape XII.2. Corrections to the values of the unknowns as published there were afterwards given in Cape XII.4, *Appendix*. In these series

1) Reduction of Heliometer-observations of Jupiter's satellites, made by Sir DAVID GILL, K. C. B. and W. H. FINLAY, M. A., by W. DE SITTER. Groningen, J. B. WOLTERS 1901.

2) HERMANN STRUVE had previously used the same method in his observations of the satellites of Saturn.

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also only relative positions of the satellites *inter se* were determined.

From these three series all elements were derived, and all have been used in the final discussion, the values being taken unaltered from the definitive publications already quoted. The only exception is the position of the fundamental plane for 1902, the inclination of which on the ecliptic is 2°8′38″, instead of 2°11′38″, as printed in Cape XII.2 page 191 [1]).

3. Photographic plates, taken at the Cape Observatory in 1891 and 1903, measured and reduced by me, and published in Cape XII.3. The quantities p_t and q_t alone were derived for each epoch. These have been taken unaltered from Cape XII.3.

4. Photographic plates, taken at the Cape in 1904, measured and reduced by me. From these plates were derived p_t and q_t, which are published in Cape XII.3, and l_1, l_2, l_3, which are published in *Gron. Publ.* 17. The published results have been adopted unaltered.

In these last three series also only coordinates of the satellites relatively to each other were used. The planet was not measured by me.

5. Photographic plates, taken at the Cape in 1902, measured and reduced by Cookson, and published in Cape XII.4. This series requires a closer consideration.

Mr. Cookson has measured on the plates differences of *RA* and decl. of the four satellites *and Jupiter*. The pointing on the planet is, according to his own statement, "not very accurate" (Cape XII.4, p. 24). But, according to the author, high accuracy is not required, since it is eliminated in the reductions. This elimination, however, is very incomplete.

It is effected as follows. From the measured differences of *R. A.* $x_t - x_p$ a preliminary solution is made, the resulting values of the unknowns are substituted in the equations of condition, and residuals are formed, which may be called δx. The mean of these residuals for any one plate, say δx_0, is then considered to be the correction δx_p to be applied to x_p, i. e. the error in the pointing on the planet with reversed sign. This mean is therefore subtracted from the observed co-ordinates $x_t - x_p$. Now this method only eliminates the *accidental* part of the correction δx_p. The systematic part is already in the first approximation included in the values of the

[1]) The inclination and node referred to the equator are correct as printed, in the reduction to the ecliptic some mistake must have occurred. The consequence of this is that the inclination ω_0 of the fundamental plane on the orbit of Jupiter requires a correction of $-0°.0092$, instead of $+0°.0375$, as would appear from the printed data of Cape XII.2.

unknowns Δh_i, Δk_i, Δx_i, and is not removed from them by the subsequent approximations. The coefficients of these unknowns consist of a constant and a periodic part, of which the former amounts on, an average to three times the latter. (See e.g. COOKSON, Cape XII.4, p. 102). If this periodic part is neglected, the three unknowns cannot be separated, and they represent together only one unknown, which I have called F_i (see my dissertation, p. 69), for each satellite. Thus, if the systematic part of δx_p had been introduced as an unknown the equations of condition would have been:

$$\frac{dx}{dF} F - \delta x_p + \; \ldots = O - C.$$

Thus it would not be possible to separate F and δx_p. Whether the unknown δx_p is actually written down in the equations or not, does not affect the result; in any case the value which is found for F is not F itself, but $F - \delta x_p \Big/ \frac{dx}{dF}$, and the residuals δx_i, and therefore also their mean δx_0, do *not* contain the systematic part of the error of pointing on the disc of the planet.

If we assume that the values of F found from the simultaneous heliometer observations (see above, sub 2), are the true ones, then the differences P—H, which are given by COOKSON in Cape XII. 4, page 102 (where for F_3 — 0.0295 should be read instead of —0.0395) are proportional to this systematic error, and we have $\delta x_p = \frac{dx}{dF}(P - H)$. We thus find for the four satellites:

$$\left.\begin{aligned} \delta x_p = -\,0''.19 &\pm 0''.04 \\ -\,0\;.15 &\pm\;\; .06 \\ -\,0\;.17 &\pm\;\; .05 \\ -\,0\;.33 &\pm\;\; .04 \end{aligned}\right\} \text{mean } -0''.21 = -0'.0035.$$

The agreement of the four values is remarkable. The probable errors, of course, would only be a true measure of the accuracy, if it could be assumed that the periodic parts of the coefficients of Δh_i etc. have been entirely without any influence on the final results, which is very far from being true, especially for the fourth satellite, of which only a small number of revolutions is included in the period of observations. The mean systematic error of pointing on the disc is of the same order of magnitude as the errors which I found to exist in the measures by RENZ (see below, sub 6). So there can be little doubt that this is the true explanation of the large and systematic differences between the results from the photographs and those from the heliometer in 1902. Accordingly I have rejected all the results from the photographic series, with the exception of p_i and q_i, which depend almost exclusively on differences of declination, in which the unknowns Δh_i, Δk_i, Δx_i have small and not constant coefficients, and the elimination of dx_p is therefore much more complete. I have adopted the values derived from the solution in which the orientation was determined from the *trails*. The reason why this is to be preferred to the orientation derived from the standard stars has been explained by me in Cape XII. 3, *Appendix*. The values of Δq_i and Δp_i have been adopted unaltered from Cape XII. 4.

6. Photographic plates taken at the observatories of Helsingsfors by Prof. DONNER and of Pulkowa by Dr. KOSTINSKY, measured by RENZ, and published in the Mémoires de St. Petersbourg, VIII^th series, Vol. VII, N°. 4 and Vol. XIII, N°. 1.

From the measures by RENZ I have derived corrections Δl_1, Δl_2, Δl_3 to the mean longitudes, which have been published in Gron. Publ. 17. The values found there have been adopted unaltered.

RENZ measured the positions of the satellites relatively to Jupiter. I have commenced my discussion of these measures by rigorously eliminating the pointing on the planet. It appears that these pointings are indeed subject to very large systematic errors (Gron. Publ. 17, art. 9*b*).

7. Heliometer observations made by BESSEL in Königsberg from 1832 to 1839, published by himself in "Astronomische Untersuchungen, Band II"; re-reduced by SCHUR and published in Nova Acta Acad. Leop. Carol., Vol. 44, pages 101—180. Only the values of h_3, k_3, h_4, k_4 are included in the discussion, and only h_4 and k_4 have contributed to the final result.

BESSEL has referred the satellites to the planet. His observations are affected by large systematic errors, as has been pointed out by SCHUR, in consequence of which their real accuracy cannot be assumed to be in accordance with the probable errors.

8. The values of the "own" nodes and perijoves in 1750. These have been determined by DELAMBRE and by DAMOISEAU. Regarding the accuracy of these determinations nothing definite is known. The agreement between the two results, which is very good, cannot be taken as a measure of the accuracy, since we do not know in how far DAMOISEAU is independent of his predecessor. It will be seen below that the *rôle* played by these data in the derivation of the final results is a very subordinate one.

If from a combination of the values found on different epochs for the osculating elements we wish to derive not only the values of these elements, but also of the masses, it is necessary that the expression of the perturbations as functions of the masses be known. The masses which form the basis of SOUILLART's theory probably require considerable corrections. In consequence of the mutual commensurability of the mean motions the perturbations of higher orders are very large —: in some cases larger than those of the first order. For these reasons the perturbations cannot be assumed to be linear functions of the masses. The formulas needed to compute the corrections to the perturbations corresponding to given corrections to the masses have been developed by me, on the basis of SOUILLART's numerical theory. They have been published in *Gron. Publ.* 17, *art.* 17.

The data required for the determination of the masses are:

I. The motions of the nodes, especially of θ_2. The inclination of satellite I is too small to allow the motion of its node to be determined with accuracy, and the motions of θ_3 and θ_4 are too slow to be of any importance for the determination of the masses, compared with θ_2. The motion of θ_2 is the datum from which the constant of compression Jb^2 must be derived.

II. The *motions of the perijoves, especially of* $\tilde{\omega}_4$. *The excentricities* of I and II are again too small to allow a determination of the motion of the perijove to be made. The motion of $\tilde{\omega}_3$ on the other hand, if it could be accurately determined, would be of little value for the determination of the masses on account of the small coefficients of these masses. The motion of $\tilde{\omega}_4$, which owing to the large excentricity of this satellite can be very accurately determined, is used for the derivation of the value of m_3.

III*a*. The great inequalities in the longitudes and radii-vectores of the first and third satellites. These depend chiefly on m_2, and serve to determine this mass.

III*b*. The great inequality of the second satellite. This furnishes an equation involving m_1 and m_3.

These data are those used by LAPLACE. To these I have added:

IV. The period of the libration. This depends on m_1, m_2 and m_3. Of these m_2 only has a small coefficient, consequently the observed period practically gives an equation between m_1 and m_3, from which combined with III*b* these two masses can be found [1]).

[1]) See "*Over de libratie der drie binnenste satellieten van Jupiter, en eene nieuwe methode ter bepaling van de massa van satelliet I,*" door Dr. W. DE SITTER. Handelingen van het 10e Ned. Nat. en Geneesk. Congres, (Arnhem 1905), pages 125—128.

Finally I add for the sake of completeness:

V. The ratio of the two excentricities of III, from which m_4 must be determined. It has not been possible to determine this ratio from the data at my disposal, and I have therefore been compelled to leave m_4 uncorrected.

The investigation can thus be divided into the following parts, or subordinate investigations:

I. The determination of the inclinations and nodes on the different epochs, and of the motions of the nodes. This discussion must at the same time give the position of the mean equator, since the major part of the motions of the nodes is due to the compression of the planet, and consequently the plane of the equator is the one to which the theoretical motions are referred, and on which the own inclinations are constant. This discussion has been made with preliminary values of p_i and q_i in *Cape XII.* 3, *Chapters XV—XXI.*

II. The determination of the equations of the centre and of their secular variations. This was done in *Gron. Publ.* 17, Art. 19.

III. The determination of the great inequalities. These have been adopted unaltered from the heliometer observations of 1891, 1901 and 1902.

IV. The determination of the libration. This was carried out in *Gron. Publ.* 17.

The determination of the masses from the equations of condition furnished by these various subordinate investigations was effected in *Gron. Publ.* 17, so far as it was possible with the data which were then at my disposal. I there found the masses:

$$\left.\begin{array}{ll} \varkappa' = +\,0.025 & \nu_2 = +\,0.050 \\ \nu_1 = -\,0.360 & \nu_3 = +\,0.025 \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (A)$$

The equations of condition from which corrections to these values were derived, will be communicated below. I will now first describe the various subordinate investigations I to IV, to which I add V: the determination of the mean motions, and VI: the determination of the mass of the system.

I. *Inclinations and Nodes.*

The data are the values of p_i and q_i for the five epochs 1891.75 1901.61, 1902.60, 1903.72, 1904.89. The unknowns are γ_i, Γ_{i0}, x_0, y_0 and the motions of the nodes [1]). These latter depend on $\varkappa'$ and ν_i, of which only $\varkappa'$ has been introduced as unknown. The

[1]) In this investigation we put for abbreviation $\Gamma_i = \psi_0 - \theta_i$.

discussion is carried out in Cape XII. 3, based on the masses of SOUILLART's theory. It must now be repeated with the masses (A). Further the following corrections must be applied.

a. The observed values of p_i and q_i must be reduced to one and the same fundamental plane for all epochs. At the time when the discussion of Cape XII. 3 was made, I had not at my disposal the data for carrying out this reduction for the epochs 1901 and 1902.

b. In the discussion of Cape XII. 3 I was compelled to reject the observations of the satellites III and IV in 1901 and 1902. COOKSON had found in the latitude of IV an empirical term, which had also influenced the results for III, and which could be demonstrated not to exist in the observations of 1891, 1903 and 1904. Mr. COOKSON has since then found the true explanation of this apparent periodic term, and has corrected his results accordingly. The corrected results must now be introduced into the discussion. It appears that now not only nothing must be rejected, but that also the representation of the observations generally is much improved.

c. The results of the photographs of 1902, which were not yet known when the discussion of Cape XII. 3 was made, must be taken into account.

It seems unnecessary to mention here all the details of the discussion. It will be published in Cape XII. 1, *Appendix*, and it will suffice here to state the results.

It may be remembered that in Cape XII. 3 two final solutions were made, of which Sol. VI was based exclusively on modern observations, while in Sol. VII the motion of θ_2 was derived from a comparison with DELAMBRE (1750), and the motions of the other nodes theoretically corresponding with this were adopted [1]). Thus $\varkappa'$ was not introduced as an unknown in this solution. The agreement of the solutions VI and VII was very good, with the exception of $\varkappa'$ and y_0. The values (A) of $\varkappa$ and ν_i are chosen so that the corresponding motions of the nodes are about the means of those found in Sol. VI and Sol. VII.

The corrections (a), (b) and (c) were now applied, the quantities σ_{ij} and μ_i, which are used in the solution were altered so as to correspond with the masses (A), and a new solution was made (Sol. VIII) in which, similarly to Sol. VI, the unknowns were γ_i, Γ_{i0}, x_0, y_0 and $d\varkappa'$. The method by which the solution was effected is the same as in Cape XII. 3, and has also been described

[1]) The correspondence was only approximate, the expressions of the motions of the nodes as functions of the masses (*Gron. Publ.* 17, art. 17), not yet being computed at that time.

in detail in these Proceedings (March 1906). The values found for γ_i and Γ_{i0} were very nearly equal to those found previously. The correction to $\varkappa'$ was very small, viz.:

$$d\varkappa' = +\, 0.0026 \pm .0058.$$

The masses now become

$$\left.\begin{array}{ll} \varkappa' = +\, 0.0276 & \nu_2 = +\, 0.050 \\ \nu_1 = -\, 0.360 & \nu_3 = +\, 0.025 \end{array}\right\} \quad . \; . \; . \; . \; . \quad (B)$$

The motions of the nodes were now made to correspond with these masses, the values found for γ_i, Γ_{i0}, x_0 and y_0 were introduced into the equations of condition, and residuals $\Delta\gamma_i$ and $\sin\gamma_i\, \Delta\Gamma_i$ were formed. From these latter I then derived for each satellite separately a correction to the motion of the node. These corrections are given below sub I. The values of the nodes in 1750.0 were next computed and compared with those determined by DELAMBRE. This comparison gave the corrections II to the motions of the nodes.

Correction to the motion of	I (*modern*)	II (DELAMBRE)	*Adopted*
θ_1	$+\,0^\circ.0094 \pm .0029$		
θ_2	$-\,0\,.00001 \pm 00009$	$-\,0^\circ.00042 \pm {}^\circ.00020$	$-\,0^\circ.00010 \pm 00008$
θ_3	$-\,0\,.00048 \pm \quad 23$	$-\,0\,.00034 \pm \quad 20$	$-\,0\,.00041 \pm \quad 15$
θ_4	$-\,0\,.00013 \pm \quad 11$	$+\,0\;00008 \pm \quad 50$	$-\,0\,.00010 \pm \quad 10$

These corrections have been used as the right-hand members of equations of condition, from which, together with those derived from the other subordinate investigations, corrections to the values (B) of the masses have been determined. These equations will be given further on. It will be seen that the adopted values agree within the probable errors with those derived from the modern observations alone. If thus these latter were adopted, the final results could only be altered within their probable errors. The finally adopted masses are:

$$\left.\begin{array}{l} \varkappa' = +\, 0.0326 \pm .0075 \\ \nu_1 = -\, 0.350 \pm .030 \\ \nu_2 = +\, 0.050 \pm .050 \\ \nu_3 = +\, 0.005 \pm .020 \\ \nu_4 = \quad 0 \quad \pm 0.25 \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (C)$$

These were now introduced into the quantities σ_{ij} and μ_i and a new solution was made (Sol. IX), in which the motions of the nodes corresponding to the masses (C) were adopted, and accordingly $d\varkappa'$ was not introduced as unknown. The result is:

TABLE I INCLINATIONS AND NODES

Series	Observed correction	probable error	Residual	Observed correction	probable error	Residual	Observed correction	probable error	Residual	Observed correction	probable error	Residual
	p_1			p_2			p_3			p_4		
1891 *H*	+0°.0360	± °.0045	+ °.0023	+0°.0752	± °.0031	+ °.0025	−0°.0029	± °.0020	+ °.0032	+0°.0630	± °.0010	+ °.0022
„ *P*	+ .0372	± 50	+ 35	+ 0733	± 36	+ 6	− .0024	± 19	+ 37	+ 0638	± 12	+ 30
1901	+ .0338	± 71	+ 20	+ .1119	± 52	− 64	− 0105	± 33	+ 18	+ .0564	± 17	− 54
1902 *H*	+ .0091	± 65	+ 1	+ .0923	± 40	+ 19	− 0097	± 25	+ 18	+ .0636	± 13	+ 16
„ *P*	+ .0026	± 80	− 64	+ .0941	± 42	+ 37	− .0095	± 27	+ 21	+ .0612	± 14	− 8
1903	+ .0021	± 60	+ 75	+ .0526	± 33	− 33	− .0199	± 22	− 97	+ 0583	± 12	− 20
1904	− .0028	± 78	− 58	+ .0158	± 44	+ 8	− 0104	± 28	− 11	+ .0648	± 13	+ 27
	q_1			q_2			q_3			q_4		
1891 *H*	−0°.0273	± °0049	− °.0063	+0°.0867	± °0029	+ °0015	−0°.0681	± °.0017	+ °.0016	−0°.0137	± °.0010	− °.0005
„ *P*	− .0258	± 61	− 48	+ .0813	± 35	− 39	− 0748	± 23	− 51	− .0147	± 11	+ 15
1901	− .0793	± 79	− 57	− .1658	± 45	+ 76	− .0369	± 30	+ 103	− .0191	± 16	0
1902 *H*	− .0769	± 59	− 37	− 1896	± 36	+ 41	− .0431	± 21	− 4	− .0172	± 13	+ 22
„ *P*	− .0820	± 62	− 88	− 1904	± 37	+ 33	− 0384	± 22	+ 43	− .0170	± 13	+ 24
1903	− .0597	± 48	− 58	− .2120	± 32	− 29	− .0442	± 20	− 4	− .0209	± 11	− 6
1904	− .0336	± 77	− 34	− .2253	± 48	− 90	− .0477	± 26	− 54	− .0210	± 17	0

Solution IX.

$\gamma_1 = 0^\circ.0272 \pm {}^\circ.0028$ $\quad \theta_1 = 60^\circ.2 \pm 7^\circ.0 - \{0^\circ.13614 \pm {}^\circ.00100\}\, t$

$\gamma_2 = 0.4683 \pm 16$ $\quad \theta_2 = 293.18 \pm 0.19 - \{0.032335 \pm .000240\}\, t$

$\gamma_3 = 0.1839 \pm 26$ $\quad \theta_3 = 319.73 \pm 0.52 - \{0.006854 \pm .000180\}\, t$

$\gamma_4 = 0.2536 \pm 23$ $\quad \theta_4 = 11.98 \pm 0.67 - \{0.001772 \pm .000030\}\, t$

The time t is counted in days from 1900 Jan. 0, mean Greenwich noon. The nodes are reckoned from the first point of Aries. The motions contain the precession, for which NEWCOMB's value was adopted. The probable errors of the motions of the nodes were computed from those of the masses (C). For the position of the mean equator referred to LEVERRIER's plane of Jupiter's orbit for 1900.0 I find

$$\omega = 3^\circ.1153 \pm 0^\circ.0014$$

$$\theta = 315.800 \pm 0.025 \quad (1900 \text{ Jan. } 0.0)$$

Table I contains the observed corrections to SOUILLART's theory, their probable errors derived from the discussion of each series separately, and the residuals which remain after the subtitution of the final values of γ_ι, θ_ι, ω and θ.

The probable error of weight unity, determined from these residuals is

$$\pm 0^\circ.0097.$$

Weights had originally been assigned, corresponding to a probable error of weight unity of

$$\pm 0^\circ.0100.$$

Comparing each residual with its probable error, we find the following distribution

	actual	*theoretical*
smaller than ϱ	$30\frac{1}{2}$	28.0
between ϱ and 2ϱ	$16\frac{1}{2}$	18.1
,, 2ϱ ,, 3ϱ	6	7.5
exceeding 3ϱ	3	2.4

Remembering that the corrections Δp_ι and Δq_ι are for each epoch the results of a series of observations, made for the different epochs by different observers and different instruments, and reduced absolutely independently of each other, we must consider this excellent agreement of the actual distribution with the ideal one according to the law of errors as a strong proof of the freedom of the observations from systematic errors. Accordingly the probable errors of the resulting inclinations and nodes can confidently be regarded as a correct measure of the accuracy. How much better the observations are represented by these values than by those adopted in SOUILLART's theory is evident at once by comparing the residuals with the observed corrections.

For 1750.0 I now find:

	Sol. IX.	DELAMBRE	DAMOISEAU
θ_2	$264^\circ.7 \pm 13^\circ.2$	$283^\circ.3$	$282^\circ.0$
θ_3	335.2 ± 10.0	352.5	353.5
θ_4	109.1 ± 1.8	105.0	98.3

The agreement with the values found by DELAMBRE and DAMOISEAU is now satisfactory. If the probable errors of θ_2 and θ_3 in 1750 are estimated at $\pm 5^\circ$ (see Cape XII. 3 page 111), the difference in both cases hardly exceeds the sum of the probable errors. As has been already said, I consider the probable errors of Solution IX as a true measure of the accuracy. This however they only remain for 1750 on the assumption that the theory, by means of which the elements have been carried back from 1900 to 1750, be correct. This, however, cannot be assumed without some qualification. It is well known that SOUILLART has integrated the equations of motion by two different methods. The difference between the motion of the node of II in 150 years according to the two methods is nearly $1^\circ.4$. It is thus quite possible that the terms of higher order in the masses, which are neglected in *both* methods, may also amount to a very appreciable quantity. In the interval of 150 years θ_2 has completed nearly five revolutions, while its motion is practically derived from the interval 1891—1904, during which the node has moved about 155° degrees. Remembering this, the agreement between the values carried back from 1900 to 1750, and those directly determined, is as good as could reasonably be expected.

In Cape XII. 3. I pointed out that the solutions VI (modern observations alone) and VII (motion of θ_2 from comparison with DELAMBRE) were in perfect agreement except for the motions of the nodes and for y_0. I then stated as my opinion, that the substitution of better masses for those of SOUILLART could be expected to reconcile the two solutions. This expectation has been entirely fulfilled. With regard to the motions of the nodes, (which are practically the same in Sol. VIII and Sol. IX) we have just seen that the agreement with 1750 is satisfactory. With regard to y_0 the following comparison of the different solutions shows that indeed the difference between Sol. VIII and IX is much smaller than between VI and VII, and now leaves nothing to be desired.

Values of y_0.

Sol. VI	$+ 0^\circ.0388 \pm {}^\circ.0044$	Sol. VIII	$+ 0^\circ.0454 \pm {}^\circ.0029$
Sol. VII	$+ \quad .0490 \pm \quad 24$	Sol. IX	$+ \quad .0473 \pm \quad 14.$

For the other unknowns the differences between the solutions VIII and IX are entirely negligible. In addition to the improvement of the masses, also the reduction to one and the same fundamental plane, and the corrections applied by COOKSON to the values for 1901 and 1902 are largely responsible for this improvement in the agreement of the two solutions.

II. *Equations of the centre.* The values of the own excentricities and perijoves were derived by me from the heliometer observations of 1891, 1901 and 1902, in *Gron. Publ.* 17, *Art.* 19. (See also these Proceedings, June 1907). The discussion was there carried out for two sets of coefficients τ_{ij}, the results agreeing within their probable errors. It is therefore unnecessary to repeat it here with the coefficients corresponding to the masses (C), which are intermediate between the two sets there used. The reasons why the photographic results of 1902 must be rejected, have already been given above. The finally adopted values are thus the same as in *Gron. Publ.* 17, with only a few unimportant alternations in the last decimal places, viz:

$$
\begin{array}{llll}
e_1 = 0^\circ.0031 \pm {}^\circ.0080 & \tilde{\omega}_1 = 155^\circ.5 \pm \infty & + \{0^\circ.14703 & \pm {}^\circ.00144\}\, t \\
e_2 = 0\ .0172 \pm \quad 40 & \tilde{\omega}_2 = \ 62\ .7 \pm 10^\circ.0 & + \{0\ .038955 & \pm .000455\}\, t \\
e_3 = 0\ \ 0868 \pm \quad 65 & \tilde{\omega}_3 = 338\ .3 \pm \ 3\ .0 & + \{0.\ 007032 & \pm .000180\}\, t \\
e_4 = 0\ .4264 \pm \quad 20 & \tilde{\omega}_4 = 283\ .15 \pm \ 0\ .30 & + \{0\ .001896 & \pm .000021\}\, t
\end{array}
$$

The probable errors depend on judgment, and are probably estimated rather too large. The values of e_1 and $\tilde{\omega}_1$ were not derived from the observed values of h_1 and k_1, but from the inequalities of group II, as will appear below when we treat of the libration. The adopted p. e. of e_1 is the largest value which can still be considered to be not improbable having regard to the observed values of h_1 and k_1. This p. e. being larger than the value of e_1 itself, the p. e. of $\tilde{\omega}_1$ cannot be stated.

The motions have been computed by the masses (C) and their probable errors correspond to the probable errors of these masses.

These values of e_i and $\tilde{\omega}_i$, and the values of τ_{ij} corresponding to the masses (C) give the residuals contained in Table II, together

TABLE II. EQUATIONS OF THE CENTRE.

Series	Observed correction	Probable error	Residual	Observed correction	Probable error	Residual	Observed correction	Probable error	Residual	Observed correction	Probable error	Residual
	h_1			h_2			h_3			h_4		
1891	+ 0°·031	± °·013	+ °·034	+ 0°·008	± °·008	+ °·001	+ 0°·060	± °·005	+ °·003	+ 0°·0617	± °·0026	+ °·0001
1901	+ ·091	± 28	+ 86	— ·015	± 19	+ 6	+ 007	± 14	+ 13	+ ·0653	± 51	— 108
1902	— ·008	± 34	— 41	— ·040	± 18	— 27	— ·022	± 11	— 13	+ ·0847	± 34	— 16
	k_1			k_2			k_3			k_4		
1891	— 0°·064	± °·021	— °·067	— 0°·055	± °·016	— °·005	— 0°·053	± °·008	+ °·003	+ 0°·0261	± °·0038	— °·0026
1901	— ·102	± 54	— 96	+ ·030	± 31	+ 1	— ·069	± 17	— 23	+ ·0390	± 59	+ 175
1902	— ·060	± 25	0	+ ·037	± 15	+ 4	— ·017	± 10	+ 28	+ ·0237	± 56	+ 35

with the observed corrections to SOUILLART's theory and their probable errors.[1]) The residuals are very satisfactory, especially so if satellite I is left out of account. (See also *Gron. Publ.* 17, pages 92 and 115).

From the values of $\tilde{\omega}_4$ in 1900, 1836 and 1750 I have already in Gron. Publ. 17 derived the motion of $\tilde{\omega}_4$. The value found there requires however a small correction. The values which BESSEL, and following his example SCHUR also, gives for E_4 *sin* Ω_4 and E_4 *cos* Ω_4, i. e. for h_4 and k_4, are in reality the values of $e_4 \sin \tilde{\omega}_4$ and $e_4 \cos \tilde{\omega}_4$. This was not noticed at first, and must now be corrected.

I now find for 1836

$$h_4 = -0^\circ.704 \qquad k_4 = -0^\circ.395.$$

Using, as before, the most probable values of e_3, $\tilde{\omega}_3$ and τ_{43}, we find from this:

$$e_4 \sin \tilde{\omega}_4 = -0^\circ.351 \quad e_4 \cos \tilde{\omega}_4 = -0^\circ.208$$
$$\tilde{\omega}_4 = 239^\circ.4 \pm 0^\circ.8.$$

We have now:

	$\tilde{\omega}_4$	*Residual*
1750.0	180°.4	+ 0°.1
1836.0	239. 4	0 .0
1900.0	283. 1	0 .0

from which:

$$\frac{d\tilde{\omega}_4}{dt} = 0^\circ.001872 \pm 0^\circ.000020 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

If the probable error were derived from the residuals, or from the probable errors for the separate epochs, we should find a much smaller value. The larger value has been adopted chiefly on account of the possibility of systematic errors of BESSEL, which will be mentioned below.

COOKSON has already (Cape XII. 2. page 197) derived the motion of $\tilde{\omega}_4$ from the observations of 1836, 1879 (SCHUR) 1891, 1901 and 1902. He finds:

$$\frac{d\tilde{\omega}_4}{dt} = 0^\circ.001892 \pm 0^\circ\ 000024, \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

The values (α) and (β) agree within their probable errors. So, if (β) were adopted instead of (α), the final results could only be

[1]) In deriving these residuals the longitudes of the perijoves are, of course, counted from the point O, as was done in the tabular places.

altered within their probable errors. They would then be entirely independent of eclipse observations.

With the finally adopted elements we find for BESSEL the following residuals.

BESSEL 1836.0

	Observed	Residual		Observed	Residual
h_3	$-0^\circ.033 \pm {}^\circ.010$	$+{}^\circ.008$	h_4	$-0^\circ.704 \pm {}^\circ.007$	$+{}^\circ.028$
k_3	$-0\ .188 \pm\ \ 14$	$+\ .020$	k_4	$-\ \ .395 \pm\ \ 9$	$+\ .026$

It thus appears that, although $\tilde{\omega}_4$ is well represented, h_4 and k_4 leave large residuals. It is remarkable that all four residuals are positive. This must probably be ascribed to systematic errors in the observations, which have already been proved to exist by SCHUR's discussion, and which probably are not entirely eliminated by the empirical corrections applied by SCHUR.

The theoretical values of h_3 and k_3 are:

$$\tfrac{1}{2}\, h_3 = \tau_{31}\, e_1 \sin \tilde{\omega}_1 + \tau_{32}\, e_2 \sin \tilde{\omega}_2 + e_3 \sin \tilde{\omega}_3 + \tau_{34}\, e_4 \sin \tilde{\omega}_4$$

$$\tfrac{1}{2}\, k_3 = \tau_{31}\, e_1 \cos \tilde{\omega}_1 + \tau_{32}\, e_2 \cos \tilde{\omega}_2 + e_3 \cos \tilde{\omega}_3 + \tau_{34}\, e_4 \cos \tilde{\omega}_4 .$$

The two first terms are exceedingly small, but $\tau_{34}\ e_4$ is large, and this term has been used by LAPLACE to determine m_4, with which the coefficient τ_{34} is roughly proportional. An attempt to derive τ_{34} from a comparison of the equations of the centre in 1836 and 1900 had to be given up, as will be easily understood by considering the residuals and probable errors stated above. Also a comparison with 1750 is not possible, for DELAMBRE and DAMOISEAU both state nothing but the values of the coefficients and the arguments, and it is not possible to derive from these the values of h_i and k_i as found directly from the observations. I have thus been compelled to leave m_4 uncorrected.

The values of $\tilde{\omega}_3$ and $\tilde{\omega}_4$ for 1750, computed from the final values for 1900.0 and the final motions, are:

		DELAMBRE	DAMOISEAU
$\tilde{\omega}_3$	$313^\circ.0 \pm 10^\circ.3$	$309^\circ.4$	$315^\circ.0$
$\tilde{\omega}_4$	$179.3 \pm\ \ 1.2$	180.3	180.4

The agreement is excellent, in fact better than could have been expected.

(To be continued).