of the ontogenetic development of these structures laid down here, entirely right and founded on facts.

A direct continuity between myofibrillae and tendon fibrillae in the sense of SCHULTZE does not exist.

Leiden, December 1914.

Physics. — *"On the field of a single centre in* EINSTEIN'*s theory of gravitation."* By J. DROSTE. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of December 30, 1914).

1. The equations which determine the field of gravitation in EINSTEIN and GROSSMANN's theory [1]), are not linear, hence the field corresponding to the tensor $\mathfrak{T}^{(1)}_{\sigma\nu} + \mathfrak{T}^{(2)}_{\sigma\nu}$ $(\sigma,\nu = 1, 2, 3, 4)$ is not the sum of the fields corresponding to the tensors $\mathfrak{T}^{(1)}_{\sigma\nu}$ and $\mathfrak{T}^{(2)}_{\sigma\nu}$. The equations, indeed, present a certain homogeneity; when all the g's are multiplied by the constant factor λ and the $\mathfrak{T}$'s also, then the equations

$$\sum_{\nu} \frac{\partial \mathfrak{T}_{\sigma\nu}}{\partial x_\nu} = \tfrac{1}{2} \sum_{\mu\nu\rho} \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \gamma_{\mu\rho} \mathfrak{T}_{\rho\nu} \quad (\varrho = 1, 2, 3, 4) \quad . \quad . \quad . \quad (1)$$

and

$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha} \left(\sqrt{-g}\, \gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) = \varkappa \left(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu} \right) \quad (\sigma,\nu = 1, 2, 3, 4) \quad . \quad (2)$$

remain valid, if they were so before the multiplication. But yet it follows by no means from this that a field would be possible, whose g's and $\mathfrak{T}$'s would be the λ-fold of a given field. Rather the contrary may be said to be the case, and this finds its cause in the accessory condition that for infinitely increasing distance to the places where $\mathfrak{T}_{\sigma\nu}$ differs from zero, g_{11}, g_{22}, and g_{33} must converge towards -1, g_{44} towards c^2.

These remarks suffice to make us see that the calculation of fields of gravitation is incomparably more difficult in the new theory than in the old. (NEWTON's theory). In the latter the field may be found by an integration; in the former theory this is impossible as appears from the above. Now equations (2) are, however, intended to pass

1) I. Entwurf einer verallgemeinerten Relativitatstheorie und einer Theorie der Gravitation, Leipzig bij B. G. TEUBNER. This treatise has been reprinted in 'Zeitschrift fur Mathematik und Physik', Vol 62.

II. Kovarianzeigenschaften der Feldgleichungen der auf die verallgemeinerte Relativitatstheorie gegründeten Gravitationstheorie. Zeitschr. für Math. u. Phys., Vol. 63.

into POISSON equations for infinitely weak fields, and so the solution of these equations may be reduced to the solution of POISSON equations, if we content ourselves with successive approximations. We start namely with supposing that the g's and γ's differ little from the values that they must have at infinity; which comes to this that the squares and the products of the differences with those 'values at infinity' are neglected. Then we have to solve ten POISSON equations, and we find the differences multiplied by the factor $\varkappa$. Then a new correction is introduced, multiplied by the factor $\varkappa^2$; this new correction is likewise the solution of a POISSON equation, the second member of which has now, however, been calculated by the aid of the first correction. Going on thus indefinitely, the whole solution is obtained in the form of a power series in $\varkappa$. For the case of a spherical body, that can be considered as an incompressible fluid, H. A. LORENTZ has calculated the field, neglecting terms which are multiplied by $\varkappa^3$ and higher powers of $\varkappa$. I have tried to follow the method used in this calculation, as I have understood it from oral communications of Prof. LORENTZ, in calculating the field of two spherical bodies at rest with respect to each other, which I hope to publish in a later communication.

2. The calculation of the field of a single centre requires only that of three functions of the distance to the centre, which may be seen in the following way, given by Prof. LORENTZ.

Let the origin be chosen in the centre of the attracting sphere. It is clear that the g's and γ's can only be functions of the distance r to the centre. Let $g_{11}=u$, $g_{22}=g_{33}=v$ and $g_{44}=w$ in a point P, lying on the x-axis. The field being supposed stationary, $g_{14}=g_{24}=g_{34}=g_{41}=g_{42}=g_{43}=0$, and as reversion of one of the three coordinate axes can have no influence on ds^2, also g_{12}, g_{13}, g_{23}, g_{21}, g_{31} and g_{32} are zero. Hence

$$ds^2 = u dx^2 + v\,(dy^2+dz^2) + w dt^2$$
$$= v\,(dx^2+dy^2+dz^2) + (u-v)\,dx^2 + w dt^2.$$

In this expression $dx^2+dy^2+dz^2=dl^2$ represents the square of an element of length in the space (x, y, z); dx^2 is nothing but dr^2. We can, therefore, also write

$$ds^2 = v dl^2 + (u-v)\,dr^2 + w dt^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and this does not contain anything that refers to the particular situation of the point P. If we had, therefore, taken P on an auxiliary axis x', i.e. if we had taken P arbitrary, ds^2 still would have been given by (3). If x, y, z are the coordinates of P, then

$$dl^2 = dx^2 + dy^2 + dz^2, \quad dr = \frac{x}{r}\,dx + \frac{y}{r}\,dy + \frac{z}{r}\,dz,$$

hence we get

$$ds^2 = v\,(dx^2+dy^2+dz^2) + (u-v)\left(\frac{x}{r}\,dx + \frac{y}{r}\,dy + \frac{z}{r}\,dz\right)^2 + w dt^2,$$

in which u, v, and w are functions of r. From the form of ds^2 we find immediately for the values of the g's the scheme

$$\begin{matrix} v + \frac{x^2}{r^2}(u-v) & \frac{xy}{r^2}(u-v) & \frac{xz}{r^2}(u-v) & 0 \\ \frac{xy}{r^2}(u-v) & v + \frac{y^2}{r^2}(u-v) & \frac{yz}{r^2}(u-v) & 0 \\ \frac{xz}{r^2}(u-v) & \frac{yz}{r^2}(u-v) & v + \frac{z^2}{r^2}(u-v) & 0 \\ 0 & 0 & 0 & w \end{matrix}$$

A similar scheme holds for the γ's, viz.

$$\begin{matrix} q + \frac{x^2}{r^2}(p-q) & \frac{xy}{r^2}(p-q) & \frac{xz}{r^2}(p-q) & 0 \\ \frac{xy}{r^2}(p-q) & q + \frac{y^2}{r^2}(p-q) & \frac{yz}{r^2}(p-q) & 0 \\ \frac{xz}{r^2}(p-q) & \frac{yz}{r^2}(p-q) & q + \frac{z^2}{r^2}(p-q) & 0 \\ 0 & 0 & 0 & s \end{matrix}$$

In this p, q, and s are functions of r satisfying the relations

$$up = vq = ws = 1, \quad \ldots \ldots \quad (4)$$

which is seen in the simplest way by choosing P on one of the axes of coordinates.

3. In order to find the differential equations, which u, v, and w or, what comes to the same thing, p, q, and s satisfy, we make use of the thesis of the calculus of variations, which occurs in the second paper of EINSTEIN and GROSSMANN cited above, and which states that the first variation of $\int H d\tau$ is equal to

$$\varkappa \int \left(\sum_{\mu,\nu} \sqrt{-g}\, T_{\mu\nu}\, \delta\gamma_{\mu\nu} \right) d\tau .$$

In this

$$H = \tfrac{1}{2} \sqrt{-g} \sum_{\alpha\beta\tau\rho} \gamma_{\alpha\beta} \frac{\partial g_{\tau\rho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} ;$$

the integration is to be performed over a region of the manifold (x, y, z, t), $d\tau$ is an element of that region, and the variations must be taken starting from the real (sought) values of the g's and the γ's, and so that they are zero at the boundary of the region.

Let us first calculate H. We must then differentiate the g's and the γ's with respect to the coordinates, then we can take all the quantities as they are in a point of the x-axis at a distance $r = x$ from the origin, and thus we find

$$g_{11} = u \ , \ g_{22} = g_{33} = v \ , \ g_{44} = w \ , \ \gamma_{11} = p \ , \ \gamma_{22} = \gamma_{33} = q \ , \ \gamma_{44} = s \, ,$$

$$\frac{\partial g_{11}}{\partial x} = u' \, , \ \frac{\partial g_{22}}{\partial x} = \frac{\partial g_{33}}{\partial x} = v' \, , \ \frac{\partial g_{44}}{\partial x} = w' \, , \ \frac{\partial g_{12}}{\partial y} = \frac{\partial g_{21}}{\partial y} = \frac{\partial g_{13}}{\partial z} = \frac{\partial g_{31}}{\partial z} = \frac{u-v}{r} \, ,$$

$$\frac{\partial \gamma_{11}}{\partial x} = p' \, , \ \frac{\partial \gamma_{22}}{\partial x} = \frac{\partial \gamma_{33}}{\partial x} = q' \, , \ \frac{\partial \gamma_{44}}{\partial x} = s' \, , \ \frac{\partial \gamma_{12}}{\partial y} = \frac{\partial \gamma_{21}}{\partial y} = \frac{\partial \gamma_{13}}{\partial x} = \frac{\partial \gamma_{31}}{\partial z} = \frac{p-q}{r} \, ,$$

$$g = uv^2 w \, .$$

In this the accents denote differentiations with respect to r, the values that have not been given are zero.

Let us call $\sqrt{-g}$ F for brevity. Then on account of (4)

$$F^2 p q^2 s = -1 \, . \qquad (5)$$

We find for H

$$H = \tfrac{1}{2} F \left\{ p \left(u'p' + 2v'q' + w's'\right) + 4 \frac{q}{r^2} (u-v)(p-q) \right\} ;$$

as in virtue of (4)

$$q(u-v)(p-q) = q\left(\frac{1}{p} - \frac{1}{q}\right)(p-q) = -p\left(1 - \frac{q}{p}\right)^2 ,$$

$$u' = -\frac{p'}{p^2} \ , \ v' = -\frac{q'}{q^2} \ \text{ and } \ w' = -\frac{s'}{s^2} ,$$

this becomes

$$H = -\tfrac{1}{2} F p \left\{ \frac{p'^2}{p^2} + 2\frac{q'^2}{q^2} + \frac{s'^2}{s^2} + \frac{4}{r^2}\left(1 - \frac{q}{p}\right)^2 \right\} .$$

We now apply the thesis of the calculus of variations to the region

$$t_1 \leqq t \leqq t_2 \quad , \quad r_1 \leqq r \leqq r_2 \ ;$$

then the first variation of $\int H d\tau$ becomes

$$\delta \int H dt = \delta \int_{t_1}^{t_2} dt \int_{r_1}^{r_2} 4\pi r^2 dr \, . \, H = -4\pi (t_2 - t_1) \, \delta \int_{r_1}^{r_2} L dr \, ,$$

if we put

$$L = -Hr^2 = \tfrac{1}{2} F p \left\{ \left(\frac{p'^2}{p^2} + 2\frac{q'^2}{q^2} + \frac{s'^2}{s^2} \right) r^2 + 4\left(1 - \frac{q}{p}\right)^2 \right\} . \quad (6)$$

For

$$\varkappa \int d\tau \sum_{\mu\nu} \sqrt{-g} \, T_{\mu\nu} \, \delta\gamma_{\mu\nu}$$

we find

$$\varkappa\,(t_2-t_1)\,.\,4\pi\int_{r_1}^{r_2} r^2 dr\,.\,\digamma\,(T_{11}\delta p + T_{22}\delta q + T_{33}\delta q + T_{44}\delta s)\,,$$

so that we get

$$\int_{r_1}^{r_2} dr\,[\delta\,(-L) - \varkappa r^2 \digamma\,(T_{12}\delta p + T_{22}\delta q + T_{33}\delta q + T_{44}\delta s)] = 0.$$

Now

$$\mathfrak{T}_{\sigma\nu} = \sum_{\mu} \sqrt{-g}\,\gamma_{\sigma\mu}\,T_{\mu\nu}$$

and therefore in our case

$$\mathfrak{T}_{\sigma\nu} = \sqrt{-g}\,\gamma_{\sigma\sigma}\,T_{\sigma\nu}\,,$$

from which follows

$$\digamma T_{11} = \frac{1}{p}\,\mathfrak{T}_{11}\ ,\ \ \digamma T_{22} = \frac{1}{q}\,\digamma_{22}\ ,\ \ \digamma T_{33} = \frac{1}{q}\,\mathfrak{T}_{33}\ \text{ and }\ \digamma T_{44} = \frac{1}{s}\,\mathfrak{T}_{44}\,.$$

By substituting this and replacing

$$\int_{r_1}^{r_2}\delta\,(-L)\,dr \ \text{ by}$$

$$\int_{r_1}^{r_2}\left[\left\{\frac{d}{dr}\left(\frac{\partial L}{\partial p'}\right) - \frac{\partial L}{\partial p}\right\}\delta p + \left\{\frac{d}{dr}\left(\frac{\partial L}{\partial q'}\right) - \frac{\partial L}{\partial q}\right\}\delta q + \left\{\frac{d}{dr}\left(\frac{\partial L}{\partial s'}\right) - \frac{\partial L}{\partial s}\right\}\delta s\right] dr,$$

we get, as the coefficients of δp, δq, and δs must be separately zero,

$$\left.\begin{array}{c}\dfrac{d}{dr}\left(\dfrac{\partial L}{\partial p'}\right) - \dfrac{\partial L}{\partial p} = \varkappa\,\dfrac{r^2}{p}\,\mathfrak{T}_{11}\ ,\ \ \dfrac{d}{dr}\left(\dfrac{\partial L}{\partial q'}\right) - \dfrac{\partial L}{\partial q} = \varkappa\,\dfrac{r^2}{q}\,(\mathfrak{T}_{22} + \mathfrak{T}_{33}),\\ \dfrac{d}{dr}\left(\dfrac{\partial L}{\partial s'}\right) - \dfrac{\partial L}{\partial s} = \varkappa\,\dfrac{r^2}{s}\,\mathfrak{T}_{44}\,.\end{array}\right\} \quad (7)$$

In this $\digamma$ must be looked upon as a known function of p, q, and s, given by (5).

The tensor $\mathfrak{T}_{\sigma\nu} : \sqrt{-g}$ possesses the same symmetry properties as $g_{\sigma\nu}$. Of the equations (1) only the first does not pass into an identity, but into

$$\frac{dP}{dr} + \frac{2}{r}\,(P-Q) + \tfrac{1}{2}\left(\frac{p'}{p}\,P + 2\,\frac{q'}{q}\,Q + \frac{s'}{s}\,S\right) = 0, \quad . \quad . \quad (8)$$

if we put $\mathfrak{T}_{11} = P$, $\mathfrak{T}_{22} = \mathfrak{T}_{33} = Q$ and $\mathfrak{T}_{44} = S$.

Then this equation with the three equations (7) form a system of four differential equations for the determination of p, q, and s, and say P, if, in connection with the nature of the substance, we know two more relations between P, Q, and S. If e.g. $Q = P$ and $S = const.$, we have the case of an incompressible fluid; $Q = P$, $S = f(P)$ represents the case of a compressible liquid or gas.

4. In some cases it is possible to derive another relation from (7) and (8), in which only first derivatives occur, a socalled first integral. For this purpose we multiply the equations (7) successively by p', q', and s', and we then add them. The result may be written in the form

$$\frac{d}{dr}\left[p'\frac{\partial L}{\partial p'}+q'\frac{\partial L}{\partial q'}+s'\frac{\partial L}{\partial s'}\right]+\frac{\partial L}{\partial r}-\frac{dL}{dr}=\varkappa r^2\left(\frac{p'}{p}P+2\frac{q'}{q}Q+\frac{s'}{s}S\right).$$

From (6) we find that

$$p'\frac{\partial L}{\partial p'}+q'\frac{\partial L}{\partial q'}+s'\frac{\partial L}{\partial s'}-L=\tfrac{1}{2}Fp\left\{\left(\frac{p'^2}{p^2}+2\frac{q'^2}{q^2}+\frac{s'^2}{s^2}\right)r^2-4\left(1-\frac{q}{p}\right)^2\right\}$$

and

$$\frac{\partial L}{\partial r}=Fpr\left(\frac{p'^2}{p^2}+2\frac{q'^2}{q^2}+\frac{s'^2}{s^2}\right)$$

so that we get in connection with (8)

$$\left.\begin{array}{l}\dfrac{d}{dr}\left[\tfrac{1}{2}Fp\left\{\left(\dfrac{p'^2}{p^2}+2\dfrac{q'^2}{q^2}+\dfrac{s'^2}{s^2}\right)r^2-4\left(1-\dfrac{q}{p}\right)^2\right\}\right]+\\ +Fpr\left(\dfrac{p'^2}{p^2}+2\dfrac{q'^2}{q^2}+\dfrac{s'^2}{s^2}\right)=-2\varkappa r^2\dfrac{dP}{dr}-4\varkappa r(P-Q)\end{array}\right\}\quad . \qquad (9)$$

For the equations (7), written in full, we find, after having multiplied them successively by p, $\frac{1}{2}q$, and s,

$$\left.\begin{array}{l}\dfrac{d}{dr}\left(r^2Fp\dfrac{p'}{p}\right)-\tfrac{1}{2}L-4Fp\left(1-\dfrac{q}{p}\right)\dfrac{q}{p}=\varkappa r^2P\\ \dfrac{d}{dr}\left(r^2Fp\dfrac{q'}{q}\right)-\tfrac{1}{2}L+2Fp\left(1-\dfrac{q}{p}\right)\dfrac{q}{p}=\varkappa r^2Q\\ \dfrac{d}{dr}\left(r^2Fp\dfrac{s'}{s}\right)+\tfrac{1}{2}L=\varkappa r^2S.\end{array}\right\}\quad . \;.\;.\qquad (10)$$

We now add twice the second equation to the first, and get in this way

$$\frac{d}{dr}\left[r^2Fp\left(\frac{p'}{p}+2\frac{q'}{q}\right)\right]+\tfrac{1}{2}L=\varkappa r^2(P+2Q).\quad .\;.\qquad (11)$$

When we subtract twice (11) from r times (9) we find

$$r\frac{d}{dr}\left[\tfrac{1}{2}Fp\left\{\left(\frac{p'^2}{p^2}+2\frac{q'^2}{q^2}+\frac{s'^2}{s^2}\right)r^2-4\left(1-\frac{q}{p}\right)^2\right\}\right]-$$

$$-2\frac{d}{dr}\left[Fpr^2\left(\frac{p'}{p}+2\frac{q'}{q}\right)\right]+\tfrac{1}{2}Fp\left\{\left(\frac{p'^2}{p^2}+2\frac{q'^2}{q^2}+\frac{s'^2}{s^2}\right)r^2-4\left(1-\frac{q}{p}\right)^2\right\}=$$

$$=-2\varkappa\left\{r^3\frac{dP}{dr}+r^2(2P+Q)\right\}$$

or

$$\frac{d}{dr}\left[\tfrac{1}{2}\, F p r \left\{\left(\frac{p'^2}{p^2}+2\frac{q'^2}{q^2}+\frac{s'^2}{s^2}\right) r^2 - 4\left(1-\frac{q}{p}\right)^2\right\} - 2\, F p r^2 \left(\frac{p'}{p}+2\frac{q'}{q}\right)\right] =$$

$$= -2\varkappa \left\{ r^3 \frac{dP}{dr} + r^2 (2P+Q) \right\}.$$

For a fluid $Q = P$, and

$$\left.\begin{array}{c} \tfrac{1}{2}\, F p r \left\{\left(\frac{p'^2}{p^2}+2\frac{q'^2}{q^2}+\frac{s'^2}{s^2}\right) r^2 - 4\left(1-\frac{q}{p}\right)^2\right\} - 2 F p r^2 \left(\frac{p'}{p}+2\frac{q'}{q}\right) + \\ + 2\varkappa r^3 P = const. \end{array}\right\} \quad (12)$$

In this case therefore we have a first integral. If S is only different from zero, when $r \leq R$, the same thing is the case with P and Q, whether P be equal to Q or not. For $r > R$ (12) then becomes always a first integral, if we put $P = 0$. In this case we can get another first integral for $r > R$, by subtracting the third equation (10) from (11), viz.:

$$r^2 F p \left(\frac{p'}{p} + 2\frac{q'}{q} - \frac{s'}{s}\right) = const. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

5. I have not succeeded in finding other first integrals of the system (10); in what follows we shall therefore content ourselves with the calculation of the approximation already found by LORENTZ; but we shall for this purpose start from the equations (10), and besides we shall not suppose $\mathfrak{T}_{44}$ to be constant. However intricate the way may be in which the different quantities $\mathfrak{T}_{\sigma\nu}$ depend on each other and on the field, $\mathfrak{T}_{44}$ can only depend on r; hence we put

$$\mathfrak{T}_{44} = \varrho(r).$$

We suppose the values of the other $\mathfrak{T}$'s only different from zero in consequence of the gravitation and therefore we may suppose these values to be zero in first approximation. We now think $p, q,$ and s expanded in a series of powers of $\varkappa$, and the expansion broken off after the term of the first degree in $\varkappa$. We then find from (10), neglecting terms with $\varkappa^2$ etc.,

$$\frac{d}{dr}(r^2 p') = 4(p-q), \quad \frac{d}{dr}(r^2 q') = -2(p-q), \quad \frac{d}{dr}(r^2 s') = -\frac{\varkappa \varrho}{c^3} r^2.$$

From the first two equations it follows, that

$$r^2 (p'+2q') = const. \quad \text{and} \quad r^2 (p'-q') = const.$$

As p' and q' must be infinite for $r = 0$, the two constants appear to be zero, hence $p' = q' = 0$ and $p = q = -1$. No terms of the first order will occur, therefore, in p and q. Further

$$r^2 s' = -\frac{\varkappa}{c^3}\int_0^r \varrho r^2 dr = -\frac{\varkappa}{c^3}\alpha(r),$$

if we put

$$\int_0^r \varrho r^2 dr = \alpha(r).$$

Hence

$$s = \frac{1}{c^2} + \frac{\varkappa}{c^3}\int_r^\infty \frac{\alpha}{r^2}\, dr.$$

Let this approximation of s be called s_1. We now go a step further, by retaining the terms with $\varkappa^2$ in p, q, s, and in the equations (10). We may put

$$L = -\tfrac{1}{2} c^6 r^2 s_1'^2 = -\tfrac{1}{2}\frac{\varkappa^2 \alpha^2}{c r^2}.$$

We now put

$$s = s_1 + \zeta,$$

which makes the third equation (10) pass into

$$-\frac{d}{dr}\left[r^2 F \frac{s_1'}{s_1} + c^3 r^2 \zeta'\right] - \frac{\varkappa^2 \alpha^2}{4 c r^2} = \varkappa r^2 \varrho$$

Now, up to the terms of the first order,

$$\frac{F}{s_1} = s_1^{-\frac{3}{2}} = c^3\left(1 + \frac{\varkappa}{c}\int_r^\infty \frac{\alpha}{r^2}\, dr\right)^{-\frac{3}{2}} = c^3\left(1 - \frac{3\varkappa}{2c}\int_r^\infty \frac{\alpha}{r^2}\, dr\right),$$

so that we find

$$-\frac{d}{dr}(c^3 r^2 \zeta') - \frac{d}{dr}(r^2 c^3 s_1') + \frac{3\varkappa}{2c}\frac{d}{dr}\left(r^2 c^3 s_1' \int_r^\infty \frac{\alpha}{r^2}\, dr\right) - \frac{\varkappa^2 \alpha^2}{4 c r^2} = \varkappa r^2 \varrho,$$

which in consequence of

$$\frac{d}{dr}(r^2 s_1') = -\frac{\varkappa \varrho}{c^3} r^2$$

passes into

$$\frac{d}{dr}(r^2 \zeta') = -\frac{3\varkappa^2}{2c^4}\frac{d}{dr}\left(\alpha\int_r^\infty \frac{\alpha}{r^2}\, dr\right) - \frac{\varkappa^2 \alpha^2}{4 c^4 r^2}.$$

From this we find

$$\zeta = \frac{3\varkappa^2}{2c^4}\int_r^\infty \frac{\alpha}{r^2}\, dr \int_r^\infty \frac{\alpha}{r^2}\, dr + \frac{\varkappa^2}{4c^4}\int_r^\infty \frac{dr}{r^2}\int_0^r \frac{\alpha^2}{r^2}\, dr$$

and therefore

$$s=\frac{1}{c^2}+\frac{\varkappa}{c^3}\int_r^\infty \frac{\alpha}{r^2}\,dr+\frac{\varkappa^2}{c^4}\left\{\tfrac{3}{2}\int_r^\infty \frac{\alpha}{r^2}\,dr\int_r^\infty \frac{\alpha}{r^2}\,dr+\tfrac{1}{4}\int_r^\infty \frac{dr}{r^2}\int_0^r \frac{\alpha^2}{r^2}\,dr\right\}.$$

At a great distance from the attracting centre we may put $\alpha=\alpha_\infty$ (constant) and $\varrho=0$. In this way we get

$$s=\frac{1}{c^2}+\frac{\varkappa\alpha_\infty}{c^3 r}+\frac{\varkappa^2}{4c^4 r}\int_0^\infty \frac{\alpha^2}{r^2}\,dr+\frac{5\varkappa^2\alpha_\infty^2}{8c^4 r^2}.$$

If we now put

$$\frac{1}{2}\varkappa c\alpha_\infty+\frac{1}{4}\varkappa^2\int_0^\infty \frac{\alpha^2}{r^2}\,dr=k,$$

we may write $4k^2/c^2\alpha_\infty^2$ for $\varkappa^2$ in the last term of s, and so we find

$$s=\frac{1}{c^2}\left(1+\frac{2k}{c^2 r}+\frac{5k^2}{2c^4 r^2}\right) \quad \ldots\ldots \quad (14)$$

We further put

$$p=-1+\xi\,,\,q=-1+\eta.$$

The first and second of the equations (10) then become

$$\frac{d}{dr}(r^2\xi')-4(\xi-\eta)+\frac{\varkappa^2\alpha^2}{4c^2r^2}=\frac{\varkappa}{c}Pr^2,$$

$$\frac{d}{dr}(r^2\eta')+2(\xi-\eta)-\frac{\varkappa^2\alpha^2}{4c^2r^2}=\frac{\varkappa}{c}Qr^2,$$

from which it follows that

$$\left.\begin{aligned}\frac{d}{dr}[r^2(\xi'+2\eta')]&=\frac{\varkappa}{c}r^2(P+2Q)+\frac{\varkappa^2\alpha^2}{4c^2r^2},\\ \frac{d}{dr}[r^2(\xi'-\eta')]-6(\xi-\eta)&=\frac{\varkappa}{c}r^2(P-Q)-\frac{\varkappa^2\alpha^2}{2c^2r^2}.\end{aligned}\right\} \quad \ldots \quad (15)$$

In this P and Q must be calculated up to the terms of the first order, which can take place by the aid of an equation, that follows from (8) viz.

$$\frac{dP}{dr}+\frac{2}{r}(P-Q)=\frac{\varkappa\varrho\alpha}{2cr^2},$$

if one more relation is given between P and Q. If e.g. $P=Q$, then

$$\xi+2\eta=\frac{\varkappa^2\alpha^3}{8c^2r^2}+\frac{\varkappa^2}{2c^2}\int_r^\infty \alpha\varrho\,dr-\frac{\varkappa^2 r^2}{4c^2}\int_r^\infty \frac{\alpha\varrho}{r^2}\,dr,$$

$$\xi - \eta = \frac{\varkappa^2 r^2}{10c^2}\int_r^\infty \frac{\alpha^2}{r^5}\,dr + \frac{\varkappa^2}{10c^2 r^3}\int_0^r \alpha^2 dr.$$

But whatever may be the particular properties of the central body, we can put $P = Q = \varrho = 0$ and $\alpha = \alpha_\infty$ at a large distance, in consequence of which we find from (15)

$$\xi + 2\eta = \frac{\varkappa^2 \alpha^2_\infty}{8\,c^2\,r^2} \quad , \quad \xi - \eta = \frac{\varkappa^2 \alpha^2_\infty}{8\,c^2\,r^2} + \frac{3B}{r^3},$$

in which B is a constant of the second order.

From this it follows that

$$p = -1 + \frac{\varkappa^2 \alpha^2_\infty}{8\,c^2\,r^2} + \frac{2B}{r^3} \quad , \quad q = -1 - \frac{B}{r^3}.$$

6. We shall now examine how a particle moves in the field of a single centre.

The motion is determined by a principle corresponding to that of HAMILTON, viz.

$$\delta\int_{t_1}^{t_2} L\,dt = \delta\int_{t_1}^{t_2} dt\,\sqrt{g_{11}\dot{x}_1^{\,2} + \ldots + g_{44}} = 0.$$

In the case under consideration, we have

$$ds^2 = v\,(dx^2 + dy^2 + dz^2) + (u - v)\,dr^2 + w dt^2.$$

If we introduce polar coordinates r, ϑ, φ, we get

$$ds^2 = w dt^2 + u dr^2 + v r^2\,d\vartheta^2 + v r^2 \sin^2\vartheta\,d\varphi^2,$$

hence

$$L = \sqrt{w + u\dot{r}^2 + v r^2\,\dot{\vartheta}^2 + v r^2 \sin^2\vartheta\,\dot{\varphi}^2}$$

One of the three equations of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vartheta}}\right) = 0,$$

which shows that if $\dot{\varphi}$ once is zero, it remains so; we see from this that the motion takes place in a plane, and, knowing this, we can choose the coordinates so that this plane becomes the plane $\vartheta = \frac{\pi}{2}$. Accordingly

$$L = \sqrt{w + u\dot{r}^2 + v r^2\,\dot{\varphi}^2}$$

and the equations of motion become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 0. \quad \ldots \quad (16)$$

The equation of energy

$$L - \dot{r}\frac{\partial L}{\partial \dot{r}} - \dot{\varphi}\frac{\partial L}{\partial \dot{\varphi}} = constant$$

and the equation

$$\frac{\partial L}{\partial \dot{\varphi}} = constant$$

are first integrals, which together can replace the equations of motion. If we call the first constant h and the second Ah, then

$$\frac{w}{\sqrt{w + u\dot{r}^2 + vr^2\dot{\varphi}^2}} = h \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$\frac{-v}{w} r^2\dot{\varphi} = A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

By these two equations φ and r are given as functions of t; (18) presents close resemblance to KEPLER's second law.

Eliminating $\dot{\varphi}$ from (17) and (18), we find

$$u\left(\frac{dr}{dt}\right)^2 = w^2\left(\frac{1}{h^2} - \frac{A^2}{vr^2}\right) - w,$$

by which r is defined as a function of t; (18) then gives φ as a function of t.

In the case that the orbit just extends into infinity, $\dot{r}^2 + r^2\dot{\varphi}^2$, and also $u\dot{r}^2 + vr^2\dot{\varphi}^2$ must be zero for $r = \infty$, hence $h = c$ according to (17). If $h < c$, then r remains finite, and if $h > c$, the velocity is different from zero also for infinitely increasing r.

The orbit may also be circular; as in virtue of (18) $\dot{\varphi}$ is constant in this case, $\partial L/\partial \dot{r}$ will be constant, and the first equation (16) shows that

$$\frac{\partial L}{\partial r} = 0,$$

i. e.

$$\frac{dw}{dr} + \dot{\varphi}^2 \frac{d}{dr}(vr^2) = 0,$$

by which the angular velocity is determined as a function of r.

7. In order to examine closer the motion of a particle we make use of the approximations for u, v, and w, found above. If we put in (17)

$$u = v = -1, \qquad w = c^2\left(1 - \frac{2k}{c^2 r}\right),$$

we get, expanding the root,

$$\frac{k}{c^2 r} - \frac{\dot{r}^2 + r^2 \dot{\varphi}^2}{2c^2} = 1 - \frac{h}{c}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (17a)$$

and from (18) we find, by putting $v = -1$ and $w = c^2$,

$$r^2 \dot{\varphi} = A c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18a)$$

The formulae (17*a*) and (18*a*) lead to the ordinary planetary motion as described by KEPLER's laws. We now shall go a step further with the approximation. Equation (17*a*) shows that k/cr^2 and $\dot{r}^2 + r^2 \dot{\varphi}^2/c^2$ are of the same order of magnitude; both quantities are small, as the second represents the square of the ratio of the planetary velocity to the velocity of light. We shall call these quantities (also $1-h/c$) of the first order of magnitude, and we wish to retain in (17) also the quantities of the second order of magnitude. For this purpose we still need not go further in u and v than to terms without $\varkappa$, as ξ and η contain the factor $\varkappa^2$, and are of the second order of magnitude, but would give terms of the third order of magnitude in (17), because they occur there multiplied by $\dot{r}^2$ and $r^2 \dot{\varphi}^2$. The motion of the material point will, accordingly, not depend on the special properties of the substance of the attracting body.

Let us now put for brevity

$$1 - \frac{h}{c} = l, \quad w = c^2(1-\delta+\varepsilon),$$

in which l and δ are of the first order, ε of the second order in $\varkappa$. We now expand the root in (17), and omit terms of higher order than the second; this implies that in the terms of the second order we may apply equation (17^a), i.e.:

$$\frac{\dot{r}^2+r^2\dot{\varphi}^2}{2c^2} = \tfrac{1}{2}\delta - l,$$

in order to eliminate $\dot{r}^2 + r^2\dot{\varphi}^2$ from the terms of the second order. The result is

$$\dot{r}^2 + r^2\dot{\varphi}^2 = -2c^2 l(1+\tfrac{3}{2}l) + c^2\delta(1+4l) - c^2(\varepsilon+\delta^2). \quad . \quad (17^b)$$

To proceed a step further with the approximation in (18), we need only put $v = -1$ and $w = c^2(1-\delta)$; this gives

$$r^2\dot{\varphi} = Ac^2(1-\delta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18^b)$$

In connection with this we may write for (17^b)

$$\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} = -\frac{2l}{A^2c^2}(1+\tfrac{3}{2}l) + \frac{\delta}{A^2c^2} + \frac{\delta^2-\varepsilon}{A^2c^2}.$$

As $ws = 1$ we get

$$s = \frac{1}{c^2}\{1+\delta+(\delta^2-\varepsilon)\}.$$

If we compare this with (14), and moreover put $r\xi = 1$, then

$$\left(\frac{d\xi}{d\varphi}\right)^2 + \xi^2 = -\frac{2l}{A^2c^2}(1+\tfrac{3}{2}l) + \frac{2k}{A^2c^4}\xi + \frac{5k^2}{2A^2c^6}\xi^2.$$

The function

$$\xi = \alpha + \beta \cos \gamma(\varphi + C)$$

solves this differential equation by suitable choice of the values α, β, and γ; we can take the integration constant C to be zero, as this choice only determines from where we measure φ. The function

$$\xi = \alpha + \beta \cos \gamma\varphi$$

satisfies the differential equation, if

$$\frac{2}{A^2c^2}l(1+\tfrac{3}{2}l) = (\alpha^2 - \beta^2)\gamma^2, \quad \frac{k}{A^2c^4} = \alpha\gamma^2, \quad 1 - \frac{5k^2}{2A^2c^6} = \gamma^2.$$

Instead of the integration constants l and A, which we introduced before, we can now consider α and β as such γ differs from 1 only in terms of the second order, and therefore the equation

$$\alpha = \frac{k}{A^2c^4}$$

is accurate up to terms of the second order.

We may, therefore, use the value of A^2c^4, which follows from this, for the calculation of γ, and so we find

$$\gamma^2 = 1 - \frac{5k}{2c^2}\alpha,$$

and from this

$$\frac{1}{\gamma} - 1 = \frac{5k}{4c^2}\alpha.$$

If we now put $\gamma\varphi = \psi$, then

$$\varphi = \psi + \frac{5k}{4c^2}\alpha\psi$$

and

$$\xi = \frac{1}{r} = \alpha + \beta \cos \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This is the equation of a conic section in polar coordinates.

The angle $5\,k\alpha\psi/4\,c^2$, between the major axis and the fixed line $\varphi = 0$, is proportional to the angle ψ, between the radius vector and the major axis. For one revolution the 'motion of the perihelium' is $450\,k\alpha/c^2$ degrees; it depends only on the parameter $1/\alpha$ of the orbit. As Prof. DE SITTER has calculated from equations of motion determined by Prof. LORENTZ, it amounts for Mercurius to 18″ per century, the observed motion being 44″. It is worthy of note that the motion of the perihelium does not depend on the particular properties of the substance of which the centre of attraction consists.

The time of revolution T (the time in which φ increases by 2π) can easily be calculated. It follows namely from (18^b) and (19) that

$$\dot{\psi} = Ac^2\gamma (\alpha+\beta \cos\psi)^2 \left\{1 - \frac{2k}{c^2} (\alpha+\beta \cos\psi)\right\},$$

and from this, accurate up to quantities of the second order of magnitude,

$$Ac^2\gamma dt = \frac{d\psi}{(\alpha+\beta \cos\psi)^2} + \frac{2k}{c^2} \frac{d\psi}{\alpha+\beta \cos\psi}.$$

From this it easily follows, that

$$Ac^2 T = \frac{2\pi}{\sqrt{\alpha^2-\beta^2}} \left(\frac{\alpha}{\alpha^2-\beta^2} + \frac{k}{c^2}\right).$$

Let us call a half the major axis of the ellipse, then

$$a = \frac{\alpha}{\alpha^2-\beta^2}$$

and we get

$$a\left(a + \frac{k}{c^2}\right)^2 : T^2 = \frac{k}{4\pi^2}.$$

T depends therefore still exclusively on the major axis of the orbit; this is, however, not the case with the time of revolution in the ellipse. In the first member we may substitute $4\pi^2 a^3/c^2 T^2$ for k/c^2, and thus we get

$$a^3\left[1 + \tfrac{2}{3}\left(\frac{2\pi a}{cT}\right)^2\right]^3 : T^2 = \frac{k}{4\pi^2}$$

instead of the third law of Kepler.

Chemistry. — "*On gas equilibria, and a test of* Prof. J. D. van der Waals Jr.'s *formula*". II. By Dr. F. E. C. Scheffer. (Communicated by Prof. J. D. van der Waals).

(Communicated in the meeting of Dec. 30, 1914.)

7. *The equilibrium* $I_2 \rightleftarrows 2I$. (Continued).

In my preceding paper [1]) I have shown that from the determinations of the iodine equilibrium the value 0.41 10^{-8} cm. follows for the radius of inertia of the iodine molecule; the iodine dissociation can therefore be represented by equation 8, when $\lambda = 15\mu$ and $\log M = -38.20$ are there substituted. That this equation sufficiently represents the experimentally found values, appears from

[1]) These Proc. 17, 695 (1914/15).