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Physics. — *"In what way does it become manifest in the fundamental laws of physics that space has three dimensions?"* By Prof. Dr. P. EHRENFEST. (Communicated by Prof. Dr. H. A. LORENTZ).

(Communicated in the meeting of May 26, 1917).

Introduction.

"Why has our space just three dimensions?" or in other words: "By which singular characteristics do geometrics and physics in R_3 distinguish themselves from those in the other R_n's?" When put in this way the questions have perhaps no sense. Surely they are exposed to justified criticism. For does space "exist"? Is it three-dimensional? And then the question "why"! What is meant by "physics" of R_4 or R_7?

I will not try to find a better form for these questions. Perhaps others will succeed in indicating some more singular properties of R_3 and then it will become clear to what are the "justified" questions to which our considerations are fit answers.

§ 1. **Gravitation and planetary motion.**

As to the planetary motion, we shall see, that there is a difference between R_3 and R_2 as well as between R_3 and the higher R_n's with respect to the stability of the circular trajectories. In R_3 a small disturbance leaves the trajectory finite if the energy is not too great; in R_2 on the contrary this is the case for all values of the energy. In R_n for $n > 3$ the planet falls on the attracting centre or flies away infinitely. In R_n for $n > 3$ there do not exist motions comparable with the elliptic motion in R_3, — all trajectories have the character of spirals.

For the attraction under the influence of which a planet circulates in the space R_n, we put $\varkappa \frac{Mm}{r^{n-1}}$; to this corresponds for $n > 2$ a potential energy:

$$V(r) = -\varkappa \frac{Mm}{(n-2)r^{n-2}} \quad \ldots \ldots \quad (1)$$

We *deduce* this law of attraction from the differential equation of LAPLACE—POISSON The means: we assume the force to be directed towards the centre and to be a function of r only, so that it can be derived from a potential and we shall apply GAUSS' theorem for the integral of the normal component of the force over a closed surface (force-current).

The equations of motion thus have the form

$$m\frac{d^2x_h}{dt^2} = -\varkappa\frac{Mm}{r^{n-1}}\frac{x_h}{r} = -\frac{\partial V}{\partial x_h}. \qquad (h = 1,\dots n)$$

The motion takes place in a plane. In this plane we introduce polar coordinates. Then the two first integrals can be written down at once

$$\frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) = E,$$

$$m r^2 \dot{\varphi} = \Theta .$$

By elimination of $\dot{\varphi}$ we find for $\dot{r}$

$$\dot{r} = \sqrt{\frac{2E}{m} - \frac{2V}{m} - \frac{\Theta^2}{m^2r^2}},$$

$$\dot{r} = \frac{1}{r}\sqrt{Ar^2 + Br^{4-n} - C^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That r may oscillate along the trajectory between positive values, $\dot{r}$ must have real and alternately positive and negative values. Therefore the quantity from which the root has to be taken must always be positive, between two values of r for which it is zero. The discussion of the latter cases is to be found in appendix (I). There we shall also consider the case $n = 2$ for which (1) has to be replaced by

$$V = \varkappa Mm \log r$$

and (2) therefore by

$$\dot{r} = \frac{1}{r}\sqrt{\alpha r^2 - \beta r^2 \log r - \gamma^2}, \quad . \quad . \quad . \quad . \quad . \quad (2^*)$$

where

$$\alpha = \frac{2E}{m} \quad , \quad \beta = 2\varkappa M \quad , \quad \gamma^2 = \frac{\Theta^2}{m^2}.$$

The result of this discussion is

n	Circular trajectories	Motions between two positive values of r	Motion to the infinite
4 5...	instable	impossible!	possible
3	stable	possible (moreover closed)	possible
2	stable	possible (not closed)	impossible!

Remarks:

1st. In this connexion we may remind of the following theorem of BERTRAND [1]): The trajectories of a material point described under the influence of a force which is directed towards a fixed centre and a function of the distance to that centre are only then closed when the force is proportional to that distance or inversely proportional to its square.

2nd. It is remarkable that also in a non-euclidic three-dimensional space the planetary trajectories corresponding to the elliptic ones prove to be closed, if the changes in the gravitation law and in the equations of mechanics corresponding to the curvature of the space are introduced. (Comp. LIEBMANN) [2]).

3rd. We may put the question: what does of BOHR's deduction of the series in the spectra in R_n become, if $n \neq 3$. Let us change in this deduction the law of electric attraction in the same way as that of gravitation, and like BOHR quantisize the moment of momentum. From the preceding it it clear that for $n > 3$ only circular trajectories can occur. For $n > 4$ we find infinite series and for $n = 4$ a singular case which is particularly remarkable with respect to the theory of quanta. (See appendix II).

§ 2. **Translation—rotation, force—pair of forces, electric field—magnetic field.**

In R_3 there is dualism between rotation and translation, in so far as both are defined by three characterizing numbers. This is closely connected to the fact that the number of planes through the pairs of axes of coordinates equals the number of axes themselves.

In every other R_n these two numbers are not equal. The number of axes of coordinates is n. Taking two of these at a time we can draw through them $\frac{n(n-1)}{2}$ planes. Evidently $\frac{n(n-1)}{2} > n$ for $n > 3$, while $n > \frac{n(n-1)}{2}$ for $n < 3$ e.g.:

for $n = 2$ we have only one rotation and two translations,
for $n = 4$ we have 6 rotations and 4 translations.

This corresponds to the dualism which exists only for $n = 3$ between the three components of the force and the three components of a pair of forces which together can replace an arbitrary system of forces.

1) J. BERTRAND, Comptes Rendus. T. 77, 1873, p. 849.
2) H. LIEBMANN, Nicht-euklidische Geometrie. 2e Aufl. 1912, p. 207.

Starting from the formulae of the theory of relativity we easily see that also the dualism between the electric and magnetic quantities is restricted to R_3.

In R_n the electric field is determined by n components, the magnetic one by $\frac{n(n-1)}{2}$ numbers.

The space-coordinates in the $(n+1)$-dimensional "world" will be denoted by $x_1 \ldots . x_n$ and t will be replaced by $x_0 = ict$. The electric and magnetic forces can be deduced from an $(n+1)$-fold potential (corresponding to the four-fold retarded potential in R_3) : $\varphi_0, \varphi_1, \ldots \varphi_n$.

The $\frac{n(n-1)}{2}$ components of its rotation

$$\frac{\partial \varphi_h}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_h} \qquad \begin{pmatrix} h \text{ and } k = 1, \ldots n \\ \neq 0 \end{pmatrix}$$

give the magnetic field and the n components of the rotation:

$$\frac{\partial \varphi_h}{\partial x_0} - \frac{\partial \varphi_0}{\partial x_h} \qquad (h = 1, \ldots n)$$

the electric field.

§ 3. Integrals of the equation of vibration in $\mathbf{R}_n$.

(Generalization of the retarded potentials).

The integrals of the equation :

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 0,$$

have the following properties in R_3 : If at the time $t=0$ we have everywhere $\varphi = 0$ and $\frac{\partial \varphi}{\partial t} = 0$ except in a small domain γ, then we have at an arbitrary later moment t (if only t is taken large enough) still everywhere $\varphi = 0$, $\frac{\partial \varphi}{\partial t} = 0$, except in a thin layer between two surfaces (fig. A), which in the limit, when γ becomes small enough, become spherical surfaces with the centre at γ.

In R_2 we have something else : here we have except a disturbance of equilibrium between two concentric lines round γ still an asymptotically diminishing disturbance of equilibrium in the whole extension (III) enclosed by the inner line.

In this respect all R_{2n+1}'s behave like R_3, all R_{2n}'s like R_2 (see appendix III).

But among the R_{2n+1}'s R_3 is characterized by a particularity

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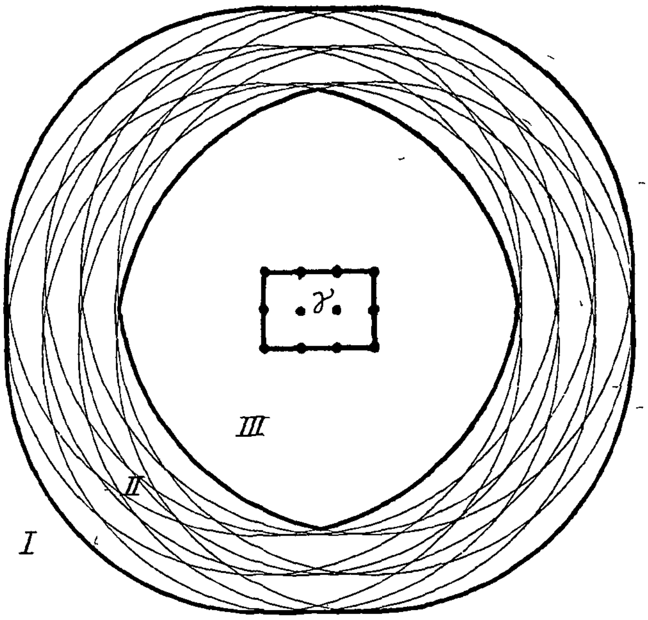

Fig. A.

which becomes evident when the retarded potentials i.e. the integrals of the differential equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = \varrho$$

for R_3 are compared with those for the higher R_{2n+1}'s.

For R_3:

$$\varphi = \frac{1}{C_3}\iiint d\omega\,\frac{[\varrho]}{r},$$

For R_5:

$$\varphi = \frac{1}{3C_5}\iiiint\!\!\int d\omega\left\{\frac{[\varrho]}{r^3} + \frac{\frac{1}{c}\left[\frac{\partial\varrho}{\partial t}\right]}{r^2}\right\},$$

For R_7:

$$\varphi = \frac{1}{5C_7}\int\cdots\int d\omega\left\{\frac{[\varrho]}{r^5} + \frac{1}{c}\frac{\left[\frac{\partial\varrho}{\partial t}\right]}{r^4} + \frac{1}{c^2}\frac{\left[\frac{\partial^2\varrho}{\partial t^2}\right]}{r^3}\right\}.$$

(see appendix IV).

Here $C_3 = 4\pi$, $C_5 = \frac{8}{3}\pi^2$, $C_7 = \frac{11}{15}\pi^3$, are the areas of spheres with a radius equal to unity in R_3, R_5, R_7 respectively. The symbol $[\varrho]$, $\left[\frac{\partial\varrho}{\partial t}\right]$, $\left[\frac{\partial^2\varrho}{\partial t^2}\right]$ expresses that the values must be taken at the time $t - \frac{r}{c}$ (the "retarded values"). While in R_3 the retarded potentials depend on ϱ only, we see that in R_5, R_7 etc. they are functions of the differential coefficients of ϱ with respect to the time too.

It must be remarked here that for high values of r (which in radiation problems are the only ones we are concerned with) the highest differential coefficient is the most important because here the lowest power of r occurs in the denominator. An electron with sharply bounded charge causes therefore by its motion high singularities.

Appendix.

I. The discussion mentioned in § 1 may be illustrated by fig. 1, where the dotted lines give the terms Ar^2 and Br^{4-n} as functions of r, while the full curve represents their sum and the horizontal line the part C^2 to be subtracted. In this graphical representation the condition is that the horizontal line cuts the full curve in two points between which the line lies below the curve so that the difference $(Ar^2 + Br^{4-n}) - C^2$ is here positive.

For $n = 2$ we have added fig. 2 of analogous structure; the lines represent: $\alpha r^2 - \beta r^2 \log r$, [1]) their sum and γ^2. Then the condition is satisfied.

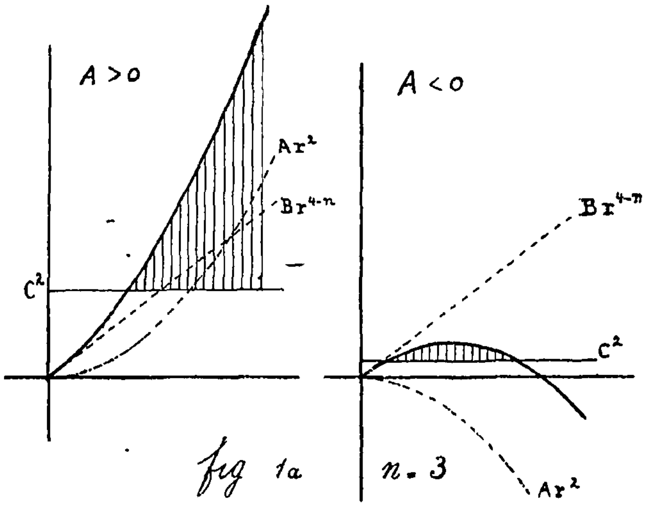

fig 1a

[1]) $-A$ divided by $\frac{2}{m}$ is the energy a planet must have in order to be brought

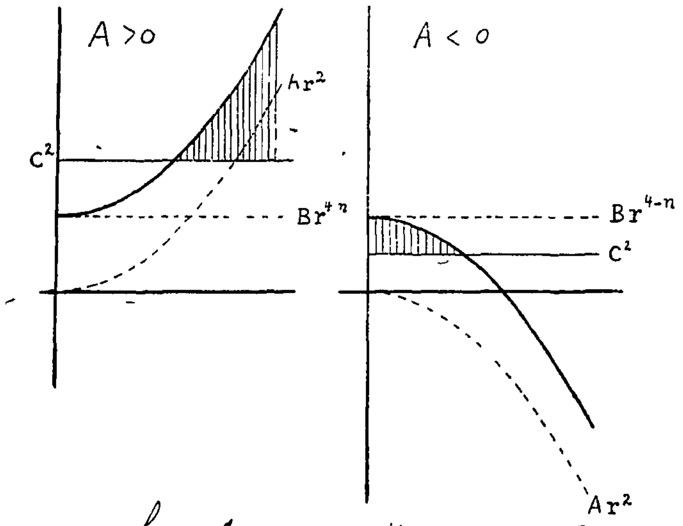

fig 1b $n = 4$

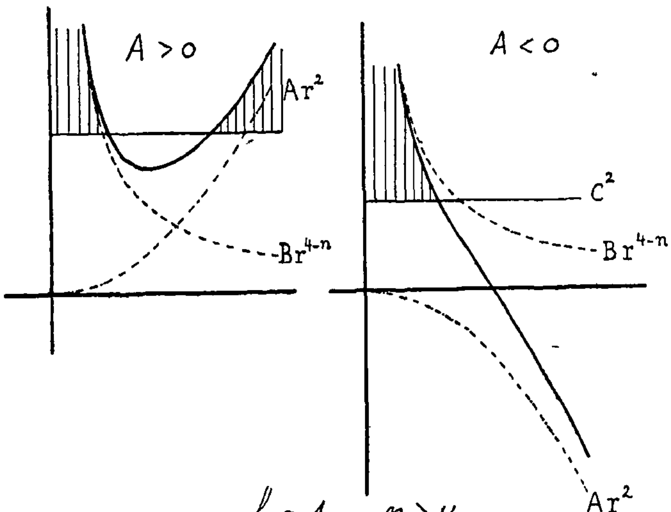

fig 1c $n > 4$

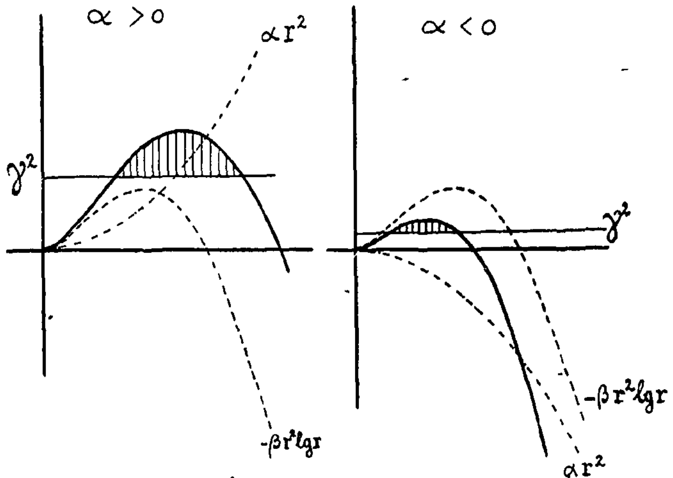

fig 2 $n = 2$

II. That the electric attraction gives the centripetal force for the circular motion is expressed by the equation

$$mr\dot{\varphi}^2 = \frac{e^2}{r^{n-1}}. \qquad (A)$$

BOHR's condition for the stationary circular paths gives

$$mr^2\dot{\varphi} = \frac{\tau h}{2\pi}$$

where τ is a whole number.

For the τ^{th} circle the energy is therefore

$$E_\tau = \tau^{-\frac{2(n-2)}{4-n}} \left(\frac{4\pi^2 m}{h^2}\right)^{\frac{(n-2)}{4-n}} e^{\frac{4}{4-n}} \frac{n-4}{2(n-2)},$$

where $n > 2$.

For R_n too we suppose the radiated frequencies to be calculable from

$$\nu_{\sigma,\tau} = \frac{E_\sigma - E_\tau}{h}$$

For $n = 4$ we have a singular case. Equation (A) becomes then

$$r^4\dot{\varphi}^2 = \frac{e^2}{m}$$

so that

$$mr^2\dot{\varphi} = e\sqrt{m}.$$

The moment of momentum can thus have only one perfectly defined value: $e\sqrt{m}$, so that the coefficient of attraction must be connected with h if the quantum condition (necessarily with only one value of τ) remains. For $n > 4$ we find

$$\nu_{\sigma,\tau} = \nu_0(\sigma^\varkappa - \tau^\varkappa),$$

where $\varkappa$ is a positive fraction in general. Thus we obtain series in the spectrum which for constant τ and increasing σ contain lines in the ultraviolet which become more and more distant from one another.

III. The solution of the equation of vibration for a membrane can be derived from that for a three-dimensional body by supposing in the latter case the disturbances of equilibrium to be in the beginning independent of one of the rectangular coordinates e.g. of

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to an infinite distance without velocity, $-\varkappa$ divided by $\frac{2}{m}$ on the other hand the energy required to carry it without velocity to the distance 1 from the centre,

z. Then spheres with a radius $r = ct$ are continually cutting the domain of the original disturbance of equilibrium. Working out the calculation we find that the number of integrations to be executed if one of the coordinates does not occur is still the same as when it occurs.[1]) That is the reason why in R_2 a disturbance of equilibrium never vanishes there where it once appeared. In an analogous way we can pass from a solution for R_{2n+1} to one for R_{2n}. In this way it becomes clear that the continuation of a disturbance of equilibrium is a common property of all R_{2n}'s.

IV. The easiest way to find these solutions is by means of KIRCHHOFF's method.[2]) A special solution χ of the equation without right-hand side is then used. This χ is a function of t and of the distance r to a fixed centre P only so that the equation which is satisfied by χ, becomes in R_n

$$\frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} - \frac{n-1}{r}\frac{\partial\chi}{\partial r} - \frac{\partial^2\chi}{\partial r^2} = 0 .$$

Applying the operation $D = \frac{1}{r}\frac{\partial}{\partial r}$ to a special solution of this equation we find a solution of the same equation for $n+2$ instead of for n. For odd values of n the special solution is

$$\chi = D^{\frac{n-1}{2}} \left\{ G\left(t + \frac{r}{c}\right) \right\},$$

viz. for $n = 1$:

$$G\left(t + \frac{r}{c}\right),$$

where G is an arbitrary function;

for $n = 3$:

$$\frac{1}{r}\frac{\partial G}{\partial r} \text{ or also } = \frac{1}{r} F\left(t + \frac{r}{c}\right)$$

(F an arbitrary function);

for $n = 5$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r} F\left(t + \frac{r}{c}\right)\right) = -\frac{1}{r^3} F\left(t + \frac{r}{c}\right) + \frac{1}{r^2 c} F'\left(t + \frac{r}{c}\right)$$

etc.

Applying GREEN's identity to the required solution ψ and this χ (e.g. for $n = 5$) in the whole space ω outside a small sphere with radius R round P we find

[1]) Comp. e.g. H. A. LORENTZ: The theory of electrons. Note 4, p. 233.

[2]) See e.g. RAYLEIGH, Theory of Sound, Ch. XIV, § 275.

$$-\iiiint d\Sigma\left(\psi\frac{\partial\chi}{\partial N}-\chi\frac{\partial\psi}{\partial N}\right)=$$

$$=\frac{d}{dt}\iiiint d\omega\left(\chi\frac{\partial\psi}{\partial t}-\psi\frac{\partial\chi}{\partial t}\right)+\iiiint d\omega\,\chi\varrho.$$

where Σ represents the area of the sphere and N its normal drawn towards ω.

Now we must integrate with respect to t from a high negative value t_1 to a high positive one t_2 [1]). For the arbitrary function F occurring in χ we take a function which is zero for all values of the argument except for those very near zero (there we must pass to the limit) in this way however that for zero the integral of F over that small domain has just the value 1. By interchanging the passage to the limit and the integration [2]) and by contraction of the sphere the identity becomes

$$-3C_5\psi_{P,(t=0)}=-\int_{t_1}^{t_2}dt\iiiint d\omega\,\varrho\left\{-\frac{1}{r^3}F\left(t+\frac{r}{c}\right)+\frac{1}{r^2c}F'\left(t+\frac{r}{c}\right)\right\},$$

or after a partial integration

$$\psi_{P,(t=0)}=\frac{1}{3C_5}\left\{\int d\omega\frac{(\varrho)_{t=-\frac{r}{c}}}{r^3}+\int d\omega\frac{\left(\frac{\partial\varrho}{\partial t}\right)_{t=-\frac{r}{c}}}{r^2c}\right\}.$$

A translation of the point $t=0$ then gives the result we want.

1) If we want to be accurate the extension must also be delimited at the outside. For the largest value of r which occurs $t_1+\frac{r}{c}$ must still be negative. Only afterwards we pass to the limit of an infinite extension.

2) This interchange which is not further justified will be known to be characteristic of KIRCHHOFF's method. Here we shall simply borrow it from KIRCHHOFF. If we want to execute the integration rigorously, we shall have to avail ourselves of a method given by J. HADAMARD: Acta Math. **31** (1908) p. 333; especially § 22 Comp. for further literature J. HADAMARD, Journ. de Phys. 1906.