Analytical treatment of the polytopes regularly derived from the regular polytopes.

BY

P. H. SCHOUTE.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.

(EERSTE SECTIE).

DEEL XI. N°. 3.

AMSTERDAM,
JOHANNES MÜLLER.
1911.

Analytical treatment of the polytopes regularly derived from the regular polytopes.

BY

P. H. SCHOUTE.

INTRODUCTION.

In a memoir recently published by this Academy (*Verhandelingen*, vol. XI, n°. 1) Mrs. A. BOOLE STOTT has given the geometrical treatment of the polytopes regularly derived from the regular polytopes of polydimensional space. In these pages I wish to complete her beautiful considerations by giving the analytical counterpart [1]).

The basis of this analytical counterpart is the fact that the coordinates of the vertices of the tetrahedron may be represented by one of the symbols $(1, 0, 0, 0)$ and $\frac{1}{2}[1, 1, 1]$, while those of the vertices of cube, octahedron and icosahedron can be put in the forms $[1, 1, 1]$, $[1, 0, 0]$ and $[1+\sqrt{5}, 2, 0] : 2$ respectively. The meaning of these symbols will be explained later on.

This paper is divided into five sections. In the first, concerned with the offspring of the regular simplex, we will meet chiefly the amplifications of the symbol $(1, 0, 0, 0)$ of the tetrahedron. The second and the third, dealing in the same manner with the measure polytope and the cross polytope, will bring us chiefly amplifications of the symbols $[1, 1, 1]$ and $[1, 0, 0]$ of cube and octahedron. The fourth will deal with the half measure polytopes and allied forms represented by amplifications of the symbol $\frac{1}{2}[1, 1, 1]$. Finally in the fifth section about the extra regular polyhedra and polytopes we will have to use the symbol of the icosahedron.

---

[1]) I had the great advantage of reading the original manuscript to Mrs. STOTT; the ensuing discussion — I acknowledge this with thankfulness — has led to a simplification of the proofs of several of the theorems.

Section I: Polytopes deduced from the simplex.

A. *The symbol of coordinates.*

1. In a preceding paper (*Nieuw Archief voor Wiskunde*, vol. IX, p. 133) I found that the distance r between two points P, P', the barycentric coordinates of which — with respect to a regular simplex $S(n+1)$ of space S_n — are $\mu_1, \mu_2, \ldots, \mu_{n+1}$ and $\mu'_1, \mu'_2, \ldots, \mu'_{n+1}$, is represented, with the length of the edge of the simplex of coordinates as unit, by the simple formula

$$r^2 = \tfrac{1}{2}\sum_{i=1}^{n+1}(\mu_i - \mu'_i)^2 \ldots\ldots\ldots\ldots\ldots \quad 1).$$

We insert here a much simpler deduction of this formula; of this deduction fig. 1 gives a geometrical represensation for the particular case $n = 2$ of the plane.

Let $\sum\limits_{i=1}^{n+1} x_i = 1$ represent the space S_n determining in a space of operation S_{n+1}, on the axes of a given system of rectangular coordinates with origin O, the points $A_1, A_2, \ldots, A_{n+1}$ at positive distances unity from O.

Let P and P' with the orthogonal coordinates $m_1, m_2, \ldots, m_{n+1}$ and $m'_1, m'_2, \ldots, m'_{n+1}$ be any two points of this S_n; then, according to the expression for the distance in rectangular coordinates, we have $\overline{PP'}^2 = \sum\limits_{i=1}^{n+1} (m_i - m'_i)^2$ and, as the points lie in S_n, the conditions $\sum\limits_{i=1}^{n+1} m_i = 1$, $\sum\limits_{i=1}^{n+1} m'_i = 1$ hold.

Now let us consider the normal distance coordinates $\bar{\mu}_i$ and $\bar{\mu}'_i$ $(i = 1, 2, \ldots, n+1)$ of P and P' with respect to the regular simplex $S(n+1)$ with the vertices $A_1, A_2, \ldots, A_{n+1}$ in S_n; then from similar rectangular triangles we deduce immediately the relation

$$\frac{\bar{\mu}_i}{m_i} = \frac{\bar{\mu}'_i}{m'_i} = \frac{h}{1},$$

where h is the height of the regular simplex. But, as the barycentric coordinates are normal distance coordinates measured by the corresponding height of the simplex and all these heights are equal in the regular simplex, we find for the barycentric coordinates μ_i and μ'_i

$$\mu_i = \frac{\bar{\mu}_i}{h} = m_i, \quad \mu'_i = \frac{\bar{\mu}'_i}{h} = m'_i,$$

and therefore

$$\overline{PP'}^2 = \sum_{i=1}^{n+1} (\mu_i - \mu'_i)^2.$$

But here PP' is expressed in OA_i as unit. By taking the edge $A_i A_k = OA_i \sqrt{2}$ as unit we find as above

$$r^2 = \tfrac{1}{2} \sum_{i=1}^{n+1} (\mu_i - \mu'_i)^2.$$

The formula 1) enables us to find an answer to the following question, now forming our starting point:

"Under what circumstances will the series of points obtained by giving to the set of barycentric coordinates $x_1, x_2, \ldots, x_{n+1}$ a determinate set of values taken in all possible permutations form the vertices of a polytope all the edges of which have the same length, say the length of the edge of the simplex of coordinates?"

The very simple answer is given by the theorem:

THEOREM I. "If the $n + 1$ values $a_1, a_2, \ldots, a_{n+1}$ satisfying the relation $\sum_{i=1}^{n+1} a_i = 1$ are arranged in decreasing order, so that we have

$$a_1 \geqq a_2 \geqq \ldots \geqq a_k \geqq a_{k+1} \geqq \ldots \geqq a_{n+1},$$

the difference $a_k - a_{k+1}$ of any two adjacent values must be either one or zero."

Proof. According to 1) the square of the distance PQ between any two vertices P, Q of the set is a sum of squares; from this it is evident that in order to make the distance a minimum we have to select two points P, Q which are transformed into each other by interchanging only one pair of coordinate values, say a_k and $a_{k'}$. But then the square of PQ is $\frac{1}{2} [(a_k - a_{k'})^2 + (a_{k'} - a_k)^2] = (a_k - a_{k'})^2$, and therefore PQ itself is $a_k - a_{k'}$. Now this difference becomes a minimum, if a_k and $a_{k'}$ are unequal adjacent values. As this minimum distance must be an edge, the condition that all the edges are to have the length unity implies that the difference between any two different adjacent values must be one.

2. As the condition stated in theorem I depends upon the *differences* of the corresponding barycentric coordinates x_i and x'_i we may drop the conditions $\Sigma x_i = 1$, $\Sigma x'_i = 1$ by allowing these coordinates either to increase or to diminish *all* of them by the *same* amount, so as to make e. g. either the smallest or the greatest of the $n + 1$ values equal to zero. So in order to avoid fractions

we will indicate the system of points $(2\frac{1}{5}, 1\frac{1}{5}, \frac{1}{5}, -\frac{4}{5}, -1\frac{4}{5})$ of S_4 for which the difference of any pair adjacent values is unity by $(4, 3, 2, 1, 0)$. Indeed, it is easy to return from $(4, 3, 2, 1, 0)$, where the sum of the values is 10 and therefore too large by 9, to the real coordinate values by subtracting $\frac{9}{5}$ from each; so this important simplification — of a temporary character — can do no harm.

We indicate the entire system of points obtained by taking the values $4, 3, 2, 1, 0$ in all possible orders of succession by putting these values in decreasing order between round brackets; this symbol $(4\ 3\ 2\ 1\ 0)$ will be called the "zero symbol" in distinction with the symbol of true coordinates.

3. The simplification alluded to above allows us to indicate in a very transparent manner the sets of values furnishing the polytopes with one kind of vertex and one length of edge (equal to the edge of the simplex of coordinates) found by M[rs]. STOTT. So we have for $n = 2, 3, 4, 5$ successivily in the symbols explained in the memoir of M[rs]. STOTT:

$$n = 2$$

$$(100) = p_3\ , \qquad (110) = -p_3\ , \qquad {}^*(210) = p_6$$

$$n = 3$$

$$(1000) = T\ , \qquad (1110) = -T\ , \qquad {}^*(2110) = CO$$

$$^*(1100) = O\ , \qquad (2100) = tT\ , \qquad {}^*(3210) = tO$$

$$n = 4$$

$$(10000) = S(5)\ , \qquad (21000) = e_1\, S(5)\ , \qquad (32100) = e_1\, e_2\, S(5)$$

$$(11000) = c\,e_1\, S(5)\ , \qquad (21100) = e_2\, S(5)\ , \qquad (32110) = e_1\, e_3\, S(5) = -\,e_2\, e_3\, S(5)$$

$$(11100) = c\,e_2\, S(5) = -\,c\,e_1\, S(5)\ , \qquad {}^*(21110) = e_3\, S(5)\ , \qquad (33210) = c\,e_1\, e_2\, e_3\, S(5) = -\,e_1\, e_2\, S(5)$$

$$(11110) = c\,e_3\, S(5) = -\,S(5)\ , \qquad {}^*(22100) = c\,e_1\, e_2\, S(5)\ , \qquad {}^*(43210) = e_1\, e_2\, e_3\, S(5)$$

$$(22110) = c\,e_1\, e_3\, S(5)$$

$$= c\,e_2\, e_3\, S(5) = -\,e_2\, S(5)\ ,$$

$$n = 5$$

$$(100000) = S(6)\ , \qquad (221000) = c\,e_1\, e_2\, S(6)\ , \qquad {}^*(332100) = c\,e_1\, e_2\, e_3\, S(6)$$

$$(110000) = c\,e_1\, S(6)\ , \qquad {}^*(221100) = c\,e_1\, e_3\, S(6)\ , \qquad (432100) = e_1\, e_2\, e_3\, S(6)$$

$$^*(111000) = c\,e_2\, S(6)\ , \qquad (321000) = e_1\, e_2\, S(6)\ , \qquad (432110) = e_1\, e_2\, e_4\, S(6)$$

$$(210000) = e_1\, S(6)\ , \qquad (321100) = e_1\, e_3\, S(6)\ , \qquad {}^*(432210) = e_1\, e_3\, e_4\, S(6)$$

$$(211000) = e_2\, S(6)\ , \qquad (321110) = e_1\, e_4\, S(6) = -\,e_3\, e_4\, S(6)\ , \qquad (433210) = e_2\, e_3\, e_4\, S(6)$$

$$(211100) = e_3\, S(6)\ , \qquad (322100) = e_2\, e_3\, S(6)\ , \qquad {}^*(543210) = e_1\, e_2\, e_3\, e_4\, S(6)$$

$$^*(211110) = e_4\, S(6)\ , \qquad {}^*(322110) = e_2\, e_4\, S(6)\ ,$$

In general a form obtained in this way may present itself in two different positions with respect to the simplex $S(n+1)$ of coordinates. So we may write e. g. for (21100) also (0—1—1—2—2), or, if we invert the sign of all the coordinates and indicate that we have done so by putting the sign minus before the brackets, —(01122), i.e. —(22110); so (21100) = —(22110) and likewise (22110) = —(21100). Really the symbols with the values satisfying the condition $\Sigma x_i = 1$ corresponding to (21100) and (01122) are $(\frac{7}{5}\,\frac{2}{5}\,\frac{2}{5} - \frac{3}{5} - \frac{3}{5})$ and (— 10011) representing, if we omit the brackets, two points P, P' situated symmetrically to each other with respect to the centre of gravity $(\frac{1}{5}\,\frac{1}{5}\,\frac{1}{5}\,\frac{1}{5}\,\frac{1}{5})$ of the simplex $S(5)$ of coordinates. So (22110) is the form (21100) in opposite orientation; in the equation (22110) = —(21100) the difference in orientation is indicated by the sign minus.

The forms the symbols of which are not affected by the "inversion" mentioned are marked by an asterisk; as they do not alter as a whole when they are put into the opposite position they possess central symmetry [1]).

4. The results obtained show that the geometrical method followed by M^rs^. Stott and the analytical method developed here cover exactly the same ground, i. e. that they lead up to the same system of forms. Nevertheless we should jump to a wrong conclusion, if we deduced from this coincidence of results that by either of the methods all the possible forms with one kind of vertex and one length of edge have been found. We show this by remarking that the combination of the two zero symbols (1000) and (1110), or of the proper values (1000) and $(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2} - \frac{1}{2})$, of T and $-T$ gives us the vertices of the cube, which implies that all the forms deduced from the cube by M^rs^. Stott can be represented by couples of symbols in barycentric coordinates as derived from the tetrahedron [2]).

[1]) For the deduction of the e and c symbols from the symbols of coordinates and reversely compare the part D of this section.

The first of the tables added at the end of this memoir is destined to put on record for $n = 3, 4, 5$ the different polyhedra and polytopes deduced from the simplex with their principal properties. Of this table the first column contains M^rs^. Stott's symbol, the second the symbol of coordinates and the third the value by which the coordinates have to be diminished in order to find the true coordinate values for which $\Sigma x_i = 1$. The following columns will be explained farther on.

[2]) So the most complicated form $e_1 e_2 C = tCO$ can be represented by

$$\left(\frac{7+3'}{4},\ \frac{3-1'}{4},\ \frac{-1-1'}{4},\ \frac{-5-1'}{4}\right)\Big|$$
$$\left(\frac{7-3'}{4},\ \frac{3+1'}{4},\ \frac{-1+1'}{4},\ \frac{-5+1'}{4}\right)\Big|,$$

if 1' and 3' stand for $\sqrt{2}$ and $3\sqrt{2}$.

5. We finish the first part of this section by mentioning a theorem already proved in the paper quoted in art. 1, as this theorem will be very useful in future. It is:

"Any two spaces S_{k-1}, S_{n-k} containing together the vertices of a regular simplex $S(n+1)$ of S_n are perfectly normal to each other."

This theorem is an immediate consequence of the property that any two edges without common end point determine a regular tetrahedron and are therefore at right angles to each other. For this implies that, if S_{k-1} contains the vertices $A_1, A_2, \ldots, A_k$ and S_{n-k} the vertices $A_{k+1}, A_{k+2}, \ldots, A_{n+1}$, each of the $k-1$ independant lines $A_1 A_l$ $(l = 2, 3, \ldots, k)$ of S_{k-1} is normal to each of the $n-k$ independent lines $A_{k+1} A_{k+m}$ $(m = 2, 3, \ldots, n-k+1)$ of S_{n-k} [1]).

B. *The characteristic numbers.*

6. We will now explain how the characteristic numbers of the vertices, edges, faces, limiting bodies, etc. can be deduced from the symbol of coordinates.

The larger n is, the more elaborate the process becomes. So, in order to divide the difficulties, we will begin by treating the cases $n = 4$ and $n = 5$ at full length by means of an easy method, working from two different sides, the vertex side and the limiting polyhedron $(n = 4)$ or limiting polytope $(n = 5)$ side, of the series vertex, edge, face, etc. Afterwards we will show a more direct way leading immediately to the knowledge of the forms and the numbers of the different kinds of limits $(l)_p$ of p dimensions.

In the cases $n = 4$ and $n = 5$ of the four and the five characteristic numbers we determine for itself the first two and the last two, using the law of Euler for $n = 4$ as a check and for $n = 5$ as a means of finding the lacking middle number of the faces.

The number of vertices is easily found in all possible cases. If all the $n+1$ digits of the symbol of coordinates of the polytope in S_n are different it is $(n+1)!$ This number $(n+1)!$ must be divided by $2!$ for any two, by $3!$ for any three, by $4!$ for any four digits being equal, etc.

The number of edges can be calculated as soon as we know how many edges pass through each vertex. For the product of this latter number by the number of the vertices indicates how many times

1) Compare the theorem of art. 30 (on p. 42 of the first volume of my textbook, "Mehrdimensionale Geometrie") where the "two groups of lines through O" may be replaced by "one group of lines through O (here A_1) and an other group of lines through O' (here A_{k+1})."

an edge passes through a vertex; so the total number of edges is half this product. Now the number of edges passing through a vertex is equal to the number of vertices lying at distance unity from the chosen vertex, and this number is easily determined, as will be shown by examples for $n = 4$ and $n = 5$ separately.

In order to be able to find the number of the limiting bodies ($n = 4$) and that of the limiting polytopes ($n = 5$) we prove in general the following theorem:

THEOREM II. „The non vanishing coefficients c_i of the coordinates x_i in the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ of a limiting space S_{n-1} of the polytope deduced from the regular simplex $S(n+1)$ of S_n must all be equal to each other".

Proof. The linear equation $c_1 x_1 + c_2 x_2 + \ldots = p$ represents, as far as the vertices of the polytope are concerned, more than one equation, if the coefficients c_i are different. We show this by a simple example. If in the case (32110) of S_4 we start from the equation $2x_1 + x_2 = p$ and we try to determine the vertices of the polytope for which the expression $2x_1 + x_2$ becomes either a maximum or a minimum we find the maximum 8 for $x_1 = 3$, $x_2 = 2$ and the minimum 1 for $x_1 = 0$, $x_2 = 1$. So for values of p situated *between* 8 and 1 the space $2x_1 + x_2 = p$ *intersects* the polytope, while it contains a limiting face only — and not a limiting body — for the extreme values 8 and 1 of p, as each of the *couples* of equations $x_1 = 3$, $x_2 = 2$ and $x_1 = 0$, $x_2 = 1$ determines a plane; of these planes the first contains the triangle $x_1 = 3$, $x_2 = 2$ and $x_3, x_4, x_5 =$ (110), the second the hexagon $x_1 = 0$, $x_2 = 1$ and $x_3, x_4, x_5 = (321)$.

From this example can be deduced generally that the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ represents k different equations, as far as the vertices of the polytope are concerned, if the non vanishing coefficients c_i admit together k different values.

The theorem is not reversible, i. e. not every linear equation with equal coefficients c_i represents for the maximum or the minimum value of p a limiting space S_{n-1} of the polytope in S_n. So in the case of the simplex (10000) of S_4 only the five spaces $x_i = 0$ bear limiting tetrahedra of $S(5)$, while the ten spaces $x_i + x_k = 0$ bear faces (100), the ten spaces $x_i + x_k + x_l = 0$ bear edges (10), etc.

In order to find the number of the faces ($n = 4$) and that of the limiting bodies ($n = 5$) we determine the *form* of the limiting bodies ($n = 4$) and that of the limiting polytopes ($n = 5$). For the number of faces ($n = 4$) is half the sum of the numbers of the faces of all the limiting bodies, and the number of limiting bodies

($n = 5$) is half the sum of the numbers of the limiting bodies of the limiting polytopes.

7. We now treat at full length the example **(32110)** mentioned above.

The number of vertices is $5!$ divided by $2!$ i. e. $120:2 = 60$. The number of edges passing through each vertex is five, for in

$$3\ 2\ 1\ 1\ 0$$

each of the five brackets indicates two coordinates with difference unity the interchanging of which furnishes a new vertex joined by an edge to the "pattern vertex", i. e. to the point with the coordinates 3, 2, 1, 1, 0. So the number of edges is $\frac{60 \times 5}{2} = 150$.

In the case of this polytope we have to consider successively the equations.

$$\begin{array}{llll} a) & \ldots\ x_1 + x_2 + x_3 + x_4 = 7 & \text{or} & x_5 = 0, \\ b) & \ldots\ x_1 + x_2 + x_3 = 6 & \text{or} & x_4 + x_5 = 1, \\ c) & \ldots\ x_1 + x_2 = 5 & \text{or} & x_3 + x_4 + x_5 = 2, \\ d) & \ldots\ x_1 = 3 & \text{or} & x_2 + x_3 + x_4 + x_5 = 4. \end{array}$$

Each of the two equations placed on the same line is a consequence of the other; for in treating the polytope (32110) we have to suppose that the true coordinate values of any point have been increased by such a common amount as to make the sum of the coordinates equal to seven.

a). Both the equations $x_1 + x_2 + x_3 + x_4 = 7$ and $x_5 = 0$ give $x_1, x_2, x_3, x_4 = (3211)$, i. e. for the coordinates x_1, x_2, x_3, x_4 we can take any permutation of 3, 2, 1, 1. But if we subtract a unit from all the coordinates and write $x_5 = -1$ and $x_1, x_2, x_3, x_4 = (2100)$, whereby the constant sum 7 of the coordinates is changed into 2, we see (compare the table in art. 3) that the obtained form is a tT[1]). This tT presents itself five times, as the subscript 5 in $x_5 = 0$ may be any of the five numbers 1, 2, 3, 4, 5.

b). In the case $x_1 + x_2 + x_3 = 6$ which implies $x_4 + x_5 = 1$ we have to combine the two systems $x_1, x_2, x_3 = (321)$ and $x_4, x_5 = (10)$; i. e. we have to combine the system $x_1, x_2, x_3 = (321)$ with each of the two possibilities $x_4 = 1, x_5 = 0$ and $x_4 = 0, x_5 = 1$ giving

---

[1]) That (2100) is a polyhedron with 12 vertices, 18 edges, 8 faces limited by 4 regular hexagons and 4 equilateral triangles is immediately found by treating the symbol (2100) in the manner taught here.

two regular hexagons in parallel planes, or we have to combine the system $x_4, x_5 = (10)$ with each of the six possibilities 321, 312, 231, 213, 132, 123 for x_1, x_2, x_3 contained in $x_1, x_2, x_3 = (321)$ giving six parallel edges of the same length. So in order to prove that the result is a hexagonal prism P_6 we have only to show yet that the planes of the two hexagons are normal to the six edges. But this follows from the theorem in art. 5, as the planes of the hexagons are parallel to the plane $x_4 = 0, x_5 = 0$, i. e. to the face $A_1 A_2 A_3$ of the simplex of coordinates, while the six edges are parallel to the line $x_1 = 0, x_2 = 0, x_3 = 0$, i. e. to the opposite edge $A_4 A_5$ of that simplex.

The hexagonal prism P_6 obtained in this manner occurs ten times, as for the subscripts 4 and 5 in the equation $x_4 + x_5 = 1$ we can take any combination of the five numbers 1, 2, 3, 4, 5 by two.

c). For $x_1 + x_2 = 5$ we have either $x_1 = 3, x_2 = 2$ or $x_1 = 2, x_2 = 3$ and in both cases $x_3, x_4, x_5 = (110)$. So we find here ten prisms P_3.

d). Finally $x_1 = 3$ gives $x_2, x_3, x_4, x_5 = (2110)$; so we find here five *CO*.

All in all we have got the limiting polyhedra

$$5\ tT,\quad 10\ P_6,\quad 10\ P_3,\quad 5\ CO;$$

so their number is 30.

The number of the faces is easily found. As the numbers of faces of tT, P_6, P_3, CO are respectively 8, 8, 5, 14 we get

$$\tfrac{1}{2}(5 \times 8 + 10 \times 8 + 10 \times 5 + 5 \times 14) = 120.$$

So the final result (e, k, f, r) [1] is (60, 150, 120, 30), in accordance with the law of Euler.

Remark. In the case of the simplex (1000) of S_4 we would have to consider the equations

$$\begin{aligned}
a) &\ \ldots\ x_1 + x_2 + x_3 + x_4 = 1 \quad \text{or} \quad x_5 = 0,\\
b) &\ \ldots\ x_1 + x_2 + x_3 = 1 \quad \text{or} \quad x_4 + x_5 = 0,\\
c) &\ \ldots\ x_1 + x_2 = 1 \quad \text{or} \quad x_3 + x_4 + x_5 = 0,\\
d) &\ \ldots\ x_1 = 1 \quad \text{or} \quad x_2 + x_3 + x_4 + x_5 = 0,
\end{aligned}$$

containing — as we remarked already in art. 6 — respectively a limiting tetrahedron, a face, an edge, a vertex of the simplex $S(5)$. Therefore in the expressive language of M^rs^. STOTT the limiting polyhedra of (32110) are distinguished, as to their orientation, as

[1] This is the general symbol I used for S_4 in my textbook; here e, k, f, r stand for "ECKE, KANTE, FLÄCHE, RAUM," i. e. for vertex, edge, face, limiting body.

5 tT	of	*body*	*import*,
10 P_6	„	*face*	„ ,
10 P_3	„	*edge*	„ ,
5 CO	„	*vertex*	„ .

8. We add an other example, this time of a polytope in S_5, and choose **(432110)**, showing all possible particularities. [1])

The number of vertices is 6! divided by 2! i. e. $720 : 2 = 360$. The number of edges passing through each vertex is six, for in

$$4\ 3\ 2\ 1\ 1\ 0$$

each of the six brackets indicates two coordinates with difference unity. So the number of edges is $\dfrac{360 \times 6}{2} = 1080$.

Here we have to consider the equations

$$\begin{array}{llll}
a) & \ldots\ x_1 + x_2 + x_3 + x_4 + x_5 = 11 & \text{or} & x_6 = 0, \\
b) & \ldots\ x_1 + x_2 + x_3 + x_4 = 10 & \text{or} & x_5 + x_6 = 1, \\
c) & \ldots\ x_1 + x_2 + x_3 = 9 & \text{or} & x_4 + x_5 + x_6 = 2, \\
d) & \ldots\ x_1 + x_2 = 7 & \text{or} & x_3 + x_4 + x_5 + x_6 = 4, \\
e) & \ldots\ x_1 = 4 & \text{or} & x_2 + x_3 + x_4 + x_5 + x_6 = 7.
\end{array}$$

a). The equation $x_6 = 0$ gives $x_1, x_2, x_3, x_4, x_5 = (43211)$, or — if we diminish all the coordinates by one — we find in $x_6 = -1$ the polytope $x_1, x_2, x_3, x_4, x_5 = (32100)$, i. e. an $e_1\, e_2\, S(5)$, occurring six times [2]).

b). For $x_5 + x_6 = 1$ we have the two possibilities $x_5 = 1, x_6 = 0$ and $x_5 = 0, x_6 = 1$, combined with $x_1, x_2, x_3, x_4 = (4321)$, which may be reduced to $x_1, x_2, x_3, x_4 = (3210)$ by subtracting unity from all the coordinates. So we find a rectangular fourdimensional prism P_{tO} with tO as base, occurring fifteen times.

c. Here we have to combine the two systems $x_1, x_2, x_3 = (432)$, and $x_4, x_5, x_6 = (110)$. So we get a polytope with $6 \times 3 = 18$ vertices arranged in six equilateral triangles in planes parallel to the plane $x_1 = 0, x_2 = 0, x_3 = 0$ containing the face $A_4\, A_5\, A_6$ of

---

[1]) The fourth and the sixth columns of Table I contain the characteristic numbers and the limiting elements of the highest number of dimensions of the new polytopes. The meaning of the small subscripts in column four and of the fractions in column five will be explained in part *G* of this section.

[2]) The characteristic numbers of this form — compare Table I — can be deduced in the manner indicated in art. 7; see farther under *a'*).

In the memoir quoted of Mrs. Stott the regular simplex of space S_4 is indicated by the symbol C_5; we prefer to use here $S(5)$, as this allows us to discriminate between the regular simplex $S(8)$ of space S_7 and the measure polytope C_8 of S_4, etc.

the simplex of coordinates and in three regular hexagons in planes parallel to the plane $x_4 = 0, x_5 = 0, x_6 = 0$ containing the opposite face $A_1 A_2 A_3$ of that simplex. As these faces are perpendicular to each other according to the theorem of art 5, we find a regular prismotope (3 ; 6), occurring twenty times.

d). For $x_1 + x_2 = 7$ we have to combine each of the possibilities $x_1 = 4, x_2 = 3$ and $x_1 = 3, x_2 = 4$ with $x_3, x_4, x_5, x_6 = (2110)$. So we find fifteen prisms P_{CO}.

e). The equation $x_1 = 4$ gives $x_2, x_3, x_4, x_5, x_6 = (32110)$. So here we find six $e_1 e_3 S(5)$.

So the limiting polytopes are

$$6\, e_1 e_2 S(5),\ 15\, P_{tO},\ 20\,(3;6),\ 15\, P_{CO},\ 6\, e_1 e_3 S(5);$$

their number is 62. Of these the $6\, e_1 e_2 S(5)$ are of polytope import, the $15\, P_{tO}$ of polyhedron import, the 20 (3 ; 6) of face import, the $15\, P_{CO}$ of edge import and the $6\, e_1 e_3 S(5)$ of vertex import.

In order to find the number of the limiting polyhedra we enumerate the limiting polyhedra of the limiting polytopes.

a'). The determination of the limiting polyhedra of the six polytopes (32100) runs parallel to the investigation of (32110) in the preceding article, as we remarked already in the last footnote. So we find in the same order of succession and with the import with respect to the simplex $S(5)$ of its space: $5 tO$ of polyhedron import, $10 P_3$ of edge import and $5 tT$ of vertex import, i. e. *twenty* limiting polyhedra.

b'). The prism P_{tO} is limited by two tO and fourteen prisms, viz. six P_4 (or cubes) and eight P_6, i. e. by *sixteen* polyhedra.

c'). The prismotope (3 ; 6) is limited by *nine* prisms, six P_3 and three P_6.

d'). The prism P_{CO} is limited by two CO and fourteen prisms, viz. six P_4 (or cubes) and eight P_3, i. e. by *sixteen* polyhedra.

e'). The polytope (32110) of the preceding article is limited by *thirty* polyhedra.

So the number of polyhedra is

$$\tfrac{1}{2}(6 \times 20 + 15 \times 16 + 20 \times 9 + 15 \times 16 + 6 \times 30) = 480.$$

According to the law of Euler the number of faces is

$$1080 + 480 + 2 - (360 + 62) = 1140.$$

So the resulting symbol of characteristic numbers is (360, 1080, 1140, 480, 62).

9. The direct method alluded to in the beginning of art. 6 rests

on the division of the limiting elements $(l)_p$ of p dimensions according to their symbol into different groups; as introduction we explain what this means by considering the edges of the polytope **(432110)** treated in the preceding article.

The edges are obtained by joining two points which pass into each other by interchanging in the symbol of coordinates two digits with difference unity. So we have edges with four different *symbols*, viz. (43), (32), (21), (10), if by the symbol (p, q) we indicate any edge the end points of which pass into each other by interchanging two coordinates with the numerical values p and q.

It is easy to find their numbers. To that end we calculate first the numbers of edges of different symbol passing through a determinate vertex, e.g. through the pattern vertex 4, 3, 2, 1, 1, 0, which point will be indicated by P. Through P passes only one edge (43) and one edge (32), as the symbol of coordinates contains only one 4, one 3 and one 2; but two edges (21) and two edges (10) concur in P, as the symbol of coordinates contains two digits 1. So we have

$$1 \text{ edge } (43) + 1 \text{ edge } (32) + 2 \text{ edges } (21) + 2 \text{ edges } (10)$$

through P. Now the number of vertices of the polytope being 360, the numbers indicating how many times each of the four edges of different symbol passes through any vertex would be 360, 360, 720, 720; as each edge bears two vertices we find

$$180 \text{ edges } (43) + 180 \text{ edges } (32) + 360 \text{ edges } (21) + 360 \text{ edes } (10),$$

i. e. once more altogether 1080 edges.

In order to be able to extend the method of deduction of edges with different symbols to limits $(l)_p$ of p dimensions we are obliged to introduce some new terms. So we call (43), (32), (21), (10) the *unextended* symbols of the four groups of edges found above and designate as the *extended* symbols of these groups respectively

$$(43)(2)(1)(1)(0), \ (4)(32)(1)(1)(0), \ (4)(3)(21)(1)(0),$$
$$(4)(3)(2)(1)(10),$$

containing also the remaining four digits, each placed in brackets. Moreover we call the constituents (43), (2), (1), (1), (0) of the extended symbol (43) (2) (1) (1) (0) the *syllables* of that symbol and exclude from our considerations the "petrified" syllables with two or more equal digits as (22), (111), etc., where the influence of the interchange of the digits is nihil.

10. We now prove the general theorem:

THEOREM III. "We obtain all the groups of d-dimensional limiting polytopes $(P)_d$ with different symbols of any given n-dimensional polytope $(P)_n$ derived from the simplex $S(n+1)$ of S_n, if we split up the $n+1$ digits of the pattern vertex in all possible ways into $n-d+1$ groups of adjacent digits and consider these groups, each of them placed in brackets, as the syllables of the extended symbol."

We remark that the extended symbols of the four groups of edges of the preceding article satisfy the conditions of the theorem.

We represent the $n-d+1$ different syllables by $(\,.\,.\,)^{k1}, (\,.\,.\,)^{k2}, \ldots, (\,.\,.\,)^{k_{n-d+1}}$, where $k_1, k_2, \ldots, k_{n-d+1}$ indicate the numbers of the digits, so that we have $k_1 + k_2 + \ldots + k_{n-d+1} = n+1$; moreover in order to fix the ideas we suppose that the coordinate values of $(\,.\,.\,)^{k1}$ correspond to the coordinates $x_1, x_2, \ldots x_{k}$, those of $(\,.\,.\,)^{k2}$ to the coordinates $x_{k_1+1}, x_{k_1+2}, \ldots x_{k_1+k_2}$, etc.

Proof. If we put it short the three moments of the proof are:

a) As petrified syllables have been excluded we obtain by proceeding according to the indications of the theorem a d-dimensional polytope P_d, the vertices of which are vertices of P_n.

b) As the digits of the syllables are adjacent digits of the symbol of $(P)_n$, the $(P)_d$ obtained is a limiting polytope of $(P)_n$.

c) As the system of equations representing any limiting polytope $(P)_d$ of $(P)_n$ occurs under all the systems of equations corresponding to the limiting $(P)_d$ of $(P)_n$ furnished by the theorem, we obtain by means of the theorem all the limiting polytopes $(P)_d$ of $(P)_n$.

We consider each of these three parts for itself.

a) *The polytope obtained is a* $(P)_d$.

By the exclusion of petrified syllables we are sure that any syllable $(\,.\,.\,)^k$ with k digits allows the vertex, the coordinates of which are the $n+1$ digits of the symbol of $(P)_n$, to coincide successively with all the vertices of a determinate $k-1$-dimensional polytope $(P)_{k-1}$ situated in a space S_{k-1} parallel to a limiting space S_{k-1} of the simplex $S(n+1)$ of coordinates. So in the case of the $n-d+1$ syllables $(\,.\,.\,)^{k1}, (\,.\,.\,)^{k2}, \ldots, (\,.\,.\,)^{k_{n-d+1}}$ under consideration the polytope obtained will be a prismotope, the constituents of which are polytopes $(P)_{k-1}$, where k is successively $k_1, k_2, \ldots, k_{n-d+1}$, situated in spaces parallel to the limiting spaces $S_{k_1-1} = (A_1 A_2 \ldots, A_{k_1})$, $S_{k_2-1} = (A_{k_1+1} A_{k_1+2} \ldots, A_{k_1+k_2})$, etc. of the simplex of coordinates, which spaces are by two normal to one another according to art. 5. This prismotope which may be represented by the symbol $(P_{k1-1}; P_{k2-1}; \ldots; P_{k_{n-d+1}-1})$ is a polytope $(P)_d$. For its number of dimensions is the sum of the numbers $k_1 - 1, k_2 - 1, \ldots, k_{n-d+1} - 1$ of dimensions of the constituents, i. e. the sum of the numbers $k_1, k_2, \ldots, k_{n-d+1}$ diminished by the number of the constituents, i. e. $n+1$ diminished by $n-d+1$, i. e. d.

We pass from the extended symbol of a $(P)_d$ formed according to the prescriptions of the theorem to the unextended symbol by omitting the syllables containing only one digit. So the unextended symbol contains only syllables with two and more than two digits. If all the syllables of the unextended symbol bear two digits, the polytope $(P)_d$ is a measure polytope; if this symbol contains only one syllable with more than two digits, the polytope $(P)_d$ is a prism, may be of higher rank; if the symbol contains at least two syllables of more than two digits the polytope $(P)_d$ is a prismotope, may be of higher rank, in the restricted sense of the word. This explains how we have to interprete the result found above that all the limits of $(P)_n$ are prismotopes. In the particular case of the limits $(l)_{n-1}$ of the highest number of dimensions, where we have to split up the digits of the pattern vertex of $(P)_n$ into two groups, we find, if $n+1$ is split up into k and $n-k+1$, the result $(P_{k-1}; P_{n-k})$, which is a non specialized [1]) polytope $(P)_{n-1}$ for $k=1$ and $k=n$, a prism on a non specialized polytope $(P)_{n-2}$ as base for $k=2$ and $k=n-1$, and a prismotope in the narrower sense for all the intermediate values; i. o. w. of the limits $(l)_{n-1}$ represented elsewhere (*Proceedings* of the Academy of Amsterdam, vol. XIII, p. 484) in connexion with the notion of *import* by $g_0, g_1, \ldots, g_{n-1}$ the forms g_0 and g_{n-1} are non specialized polytopes, the forms g_1 and g_{n-2} are prisms and the forms $g_2, g_3, \ldots g_{n-3}$ are prismotopes.

b). *The* $(P)_d$ *obtained is a limiting body of* $(P)_n$.

A polytope $(P)_d$, the vertices of which are vertices of $(P)_n$, is a limiting body of $(P)_n$ — and not a *section* of it —, if we can indicate $n-d$ limiting spaces S_{n-1} of $(P)_n$ containing it. Now, according to the manner in which $(P)_d$ is obtained, the coordinates of its vertices satisfy the $n-d+1$ equations

$$x_1 + x_2 + \ldots + x_{k_1} = p_1,\ x_{k_1+1} + x_{k_1+2} + \ldots + x_{k_1+k_2} = p_2,$$
$$x_{k_1+k_2+1} + x_{k_1+k_2+2} + \ldots + x_{k_1+k_2+k_3} = p_3, \text{ etc.},$$

if p_1 is the sum of the first k_1 digits of the pattern vertex, p_2 the sum of the next k_2 digits, p_3 of the then next k_3 digits, etc. These $n-d+1$ equations, only connected by the relation holding for

[1]) Here "non specialized" means: according to the mode of generation neither prism nor prismotope. About this last form art. 13 will give more particulars.

all points of S_n, that the expression $\sum_{i=1}^{n+1} x_i$ is equal to the sum of all the digits of $(P)_n$, form a system of $n - d$ mutually independent equations, representing therefore, in accordance with the result of the first part of the proof, a space S_d, bearing the $(P)_d$ found above. If we write this system of equations in the form:

$$\sum_{i=1}^{k_1} x_i = p_1, \sum_{i=1}^{k_1+k_2} x_i = p_1 + p_2, \ldots, \sum_{i=1}^{k_1+k_2+\ldots+k_{n-d}} x_i = p_1 + p_2 + \ldots + p_{n-d},$$

it is evident that each of the equations represents an $n-1$-dimensional limiting space of $(P)_n$, the constant of the right hand member being a maximum.

As we remarked already the crux of the proof of this part lies in the true interpretation of the expression "adjacent digits". It cannot be replaced by the condition that all the syllables should be formed according to the first theorem. We show this by means of two simple cases concerned with the determination of faces of threedimensional polyhedra. In the case of the $(2110) = CO$ the hexagon (210) (1) is no face but a section, likewise in the case of the $(1100) = O$ the square (10) (10) is no face but a section. In both cases the syllables satisfy the conditions of the first theorem; but the impossibility of putting the syllables behind each other so as to obtain the order of succession of the digits of the pattern vertex implies the impossibility of finding an equation where the constant that is equal to the sum of some of the coordinates is either a maximum or a minimum. Under the five polyhedra T, O, tT, CO, tO which can be represented by a symbol with four digits (compare the small table of art. 3 for $n = 3$) the O and CO are the only ones with sections p_4 and p_6 with sides equal to the edges; at the same time they are the only ones with four edges through each vertex.

c). By means of the theorem we obtain all the limits $(P)_d$ of $(P)_n$.

It is always possible to represent any limit $(P)_d$ of $(P)_n$ by $n - d$ equations of spaces S_{n-1} containing $n-1$-dimensional limits $(l)_{n-1}$ of $(P)_n$; as the vertices of this $(P)_d$ are also vertices of $(P)_n$, this system of equations will be in accordance with the symbol of $(P)_n$, i. e. this system must be included into the set of systems of equations provided by the theorem.

11. We apply the theorem III to an other fivedimensional form **(321100)**, showing at the same time how we can determine the numbers of *all* the different limits.

Vertices. There is only one kind of vertex $(3)(2)(\dot{1})(1)(0)(0)$. According to the rule given in art. 6 the number of vertices is $6!$ divided by 2^2, i. e. 180.

Edges. There are three groups of edges, represented in extended and in unextended [1]) symbols by

$$(32)(1)(\dot{1})(0)(0) = (32),\ (3)(21)(1)(0)(0) = (21),$$
$$(3)(2)(1)(10)(0) = (10).$$

We indicate a new method of determining the numbers of these edge groups. In the case of (10) the coordinates corresponding to the two digits between the same brackets can be x_i, x_k where i, k is any combination of the subscripts 1, 2, 3, 4, 5, 6 by two, giving $(6)_2 = \frac{6.5}{1.2} = 15$ possibilities; these two coordinates having been chosen the four remaining ones can be assigned anyhow to the four digits (3), (2), (1), (0), giving $4! = 24$ possibilities. So the number of edges (10) is $(6)_2 . 4! = 360$. In the case of (21) the number 360 must be divided by 2 on account of the two equal syllables (0), (0), in the case of (32) this number must be divided by 2^2 on account of the two pairs of equal syllables (1), (1) and (0), (0). So we have

$$90 \text{ edges } (32) + 180 \text{ edges } (21) + 360 \text{ edges } (10)$$

i. e. altogether 630 edges.

Faces. There are six groups of faces, represented in extended and in unextended symbols by

$$(321)(1)(0)(0) = (321) = p_6,\ (32)(1)(10)(0) = (32)(10) = p_4,$$
$$(3)(211)(0)(0) = (211) = p_3,\ (3)(21)(10)(0) = (21)(10) = p_4,$$
$$(3)(2)(110)(0) = (110) = p_3,\ (3)(2)(1)(100) = (100) = p_3.$$

Taken in the order of succession of the rows the numbers of these polygons are

$$(6)_3 . 3!:2 = 60,\quad (6)_2(4)_2 . 2! = 180,\quad (6)_3 . 3!:2 = 60,$$
$$(6)_2(4)_2 . 2! = 180,\quad (6)_3 . 3! = 120,\quad (6)_3 . 3! = 120,$$

i. e. we find

$$300\,p_3 + 360\,p_4 + 60\,p_6 = 720 \text{ faces}.$$

Limiting bodies. There are seven groups of limiting bodies, viz.:

[1]) As we have seen in the preceding article the unextended symbols are deduced from the extended ones by omitting the syllables of one digit.

$$(3211)(0)(0) = (3211) = tT,\ (321)(10)(0) = (321)(10) = P_6,$$
$$(32)(110)(0) = (32)(110) = P_3,\ (32)(1)(100) = (32)(100) = P_3,$$
$$(3)(2110)(0) = (2110) = CO,\ (3)(21)(100) = (21)(100) = P_3,$$
$$(3)(2)(1100) = (1100) = O,$$

the numbers of which are respectively

$$(6)_4 \,.\, 2!:2 = 15\ ,\ (6)_3 . (3)_2 = 60\ ,\ (6)_3 . (3)_2 = 60\ ,$$
$$(6)_3 . (3)_2 = 60\ ,\ (6)_4 \,.\, 2! = 30\ ,\ (6)_3 . (3)_2 = 60\ ,$$
$$(6)_4 \,.\, 2! = 30\ ,$$

i. e.

$$15\, tT + 30\,(O + CO) + 60\, P_6 + 180\, P_3 = 315 \text{ limiting bodies.}$$

Limiting polytopes. There are four groups of limiting polytopes, viz.:

$$(32110)(0) = (32110) = e_1\, e_3\, S_5 \text{ [1]},\ (321)(100) = (6\,;\,3),$$
$$(32)(1100) = P_O,\ (3)(21100) = (21100) = e_2\, S_5,$$

the numbers of which are.

$$(6)_5 = 6\ ,\ (6)_3 = 20\ ,\ (6)_4 = 15\ ,\ (6)_5 = 6.$$

So we find

$$6\, e_1\, e_3\, S_5 + 20\,(6\,;\,3) + 15\, P_O + 6\, e_2\, S_5 = 47 \text{ limiting polytopes}$$

and the characteristic numbers are

$$(180,\ 630,\ 720,\ 315,\ 47),$$

in accordance with the law of Euler.

12. Though the introduction of the extended symbols has enabled us to simplify the theoretical considerations it cannot be denied that the unextended symbols are better fit for practical use. Therefore we insert here a corresponding version of theorem III, but to that end we have to enter first into a distinction of the digits of the syllables of the unextended symbols. We will distinguish the digits contained in any of these syllables into *end digits* and *middle digits*, the first and the last digits and the digits equal to these being the end digits, the remaining ones — if there are some — the middle digits. So in (3210) there are two middle digits 2 and 1, in (2110) there are two equal middle digits 1, while in (2210), (2100) there is only one middle digit and in (1000), (1100) none. Now we can repeat theorem III in the new form:

THEOREM III′. "We obtain a $(P)_d$ the vertices of which are vertices

[1]) Compare the small table unter art. 3.

of the given polytope $(P)_n$, if we fix either the values of $n-d$ coordinates and allow the remaining $d+1$ to interchange their values, or the values of $n-d-1$ coordinates and split up the remaining $d+2$ into two groups of interchangeable ones, or the values of $n-d-2$ coordinates and split up the remaining $d+3$ into three groups of interchangeable ones, etc., this process winding up for $n < 2d$ in a symbol with $n-d+1$ and for $n > 2(d-1)$ in a symbol with d groups."

"This $(P)_d$ will be limiting polytope of $(P)_n$, if:

1°. each syllable of the unextended symbol with middle digits exhausts these digits of the symbol of $(P)_n$,

2°. no two syllables without middle digits have the same end digits."

Proof. The first part of the new theorem is a consequence of this that in the different cases communicated the corresponding extended symbol is always consisting of $n-d+1$ syllables, i. e. of k syllables with more than one digit and $n-d-k+1$ syllables with only one digit for $k=1, 2, \ldots, d$; so it is equivalent to part *a*) of the proof of theorem III. The second part of the new theorem is equivalent to part *b*) of the proof of theorem III; for the only cases in which it is impossible to put the syllables of the extended symbol behind one another so as to obtain the order of succession of the pattern vertex are the two excluded by the two items 1° and 2°, i. e. 1° that a syllable with middle digits does not exhausts these digits and 2° that two syllables without middle digits do have the same end digits. Finally the part *c*) of the proof of theorem III can be repeated here.

By means of theorem III′ we find e. g. in the case of the $(P)_1$ represented by **(5443322210)** the following 58 different kinds of limiting $(P)_6$:

(5443322), — (544332) (21), (544332) (10), — (54433) (221), (54433) (210), — (5443) (3222), (5443) (322) (21), (5443) (322) (10), (5443) (32) (221), (5443) (32) (210), (5443) (2221), (5443) (2210), — (544) (33222), (544) (3322) (21), (544) (3322) (10), (544) (332) (221), (544) (332) (210), (544) (32221), (544) (3222) (10), (544) (322) (210), (544) (32) (2210), (544) (22210), — (54) (433222), (54) (43322) (21), (54) (43322) (10), (54) (4332) (221), (54) (4332) (210), (54) (433) (2221), (54) (433) (2210), (54) (43) (32221), (54) (43) (3222) (10), (54) (43) (322) (210), (54) (43) (32) (2210), (54) (43) (22210), (54) (332221), (54) (33222) (10), (54) (3322) (210), (54) (332) (2210), (54) (322210) — (4433222), — (443322) (21), (443322) (10), — (44332) (221), (44332) (210), — (4433) (2221), (4433) (2210), — (443) (32221), (443) (3222) (10), (443) (322) (210), (443) (32) (2210), (443) (22210), — (4332221), — (433222) (10), — (43322) (210), — (4332) (2210), — (433) (22210), — (43) (322210), — (3322210).

13. We will insert a few remarks about the character of the limiting $(P)_6$ obtained.

In the case (5443322) of one syllable we find a non specialized form (3221100) which will prove to be an $e_2 e_4 S(7)$ in M[rs]. Stott's language.

In the cases (544332) (21) and (544332) (10) we find right prisms on $(322110) = e_2 e_4 S(6)$ as base.

In the case (54433) (221) we find a prismotope the constituents of which are a $(21100) = e_2 S(5)$ and a $(110) = p_3$. So this $(P)_6$ can be generated in the following way. Consider a space S_4 and a plane S_2 perfectly normal to each other. Take in S_4 an $e_2 S(5)$, in S_2 a p_3, and let P be a definite vertex of the former, Q a definite vertex of the latter. Now move either $e_2 S(5)$ parallel to itself in such a way that P coincides successively with all the points inside p_3, or p_3 parallel to itself in such a way that Q coincides successively with all the points inside $e_2 S(5)$. Then the $(P)_6$ can be considered as the locus either of the $e_2 S(6)$ in the first case or of the P_3 in the second; its vertices are given in the first case by the three positions of $e_2 S(5)$ in which P coincides with one of the vertices of p_3, in the second by the thirty positions of p_3 in which Q coincides with one of the vertices of $e_2 S(5)$. We represent it by the symbol $\{e_2 S(5); 3\}$.

In the case (54433) (210) we find an $\{e_2 S(5); 6\}$.

In the three cases (5443) (3222), (5443) (2221), (5443) (2210) we find successively $(CO; T)$, $(CO; T)$, $(CO; tT)$.

In the cases (5443)(322)(21), (5443)(322)(10), (5443)(32)(221) we have to deal with right prisms on a $(CO; 3)$ as base, whilst (5443) (32) (210) is a right prism on a $(CO; 6)$ as base. These prisms may also be represented by the symbols $(CO; 3; 2)$ and $(CO; 6; 2)$ as prismotopes of the second rank. But a prismotope proper of the second rank is the $(P)_6$ represented by (544)(322)(210), which may be represented as such by the symbol $(3; 3; 6)$. To generate it we have to start from three planes α_1, α_2, α_3 two by two perfectly normal to one another, and to place in α_1 and α_2 equilateral triangles and in α_3 a regular hexagon; then the $(P)_6$ is obtained by the parallel motion of the hexagon in such a way that a definite vertex of that hexagon coincides successively with all the points inside the fourdimensional prismotope $(3; 3)$ determined by the two triangles.

The (54) (43) (32) (2210) is a prism of the third rank on a tT as base; it may also be considered as a prismotope $(C; tT)$.

If in the case of $(P)_n$ we deduce the limits $(P)_{n-1}$ we find them in the order of succession $g_{n-1}, g_{n-2}, \ldots, g_0$ of polytope import to vertex import, when, in proceeding from left to right we take in the first syllable as many digits as possible and keep in it the first digit as long as possible. This principle has been followed throughout in the enumeration of the limiting $(P)_6$ of the given $(P)_9$, as well as in the sixth columm of Table I.

In the notation of art. 10 (page 17 in the middle) a limit $(P)_{n-1}$, represented as to its import by g_k, is a prismotope $(P_k; P_{n-k-1})$.

14. It is worth noticing that in space S_n the series of limiting elements may include the series of the measure polytope M, for n even up to the polytope $M_{\frac{n}{2}}$ of $S_{\frac{n}{2}}$, for n odd up to the polytope $M_{\frac{n+1}{2}}$ of $S_{\frac{n+1}{2}}$. So, for $n = 2m + 1$ the $(P)_{2m+1}$ represented by

$$e_1\, e_2\, e_3 \ldots e_m\, S(2m + 2) = (2m + 1, 2m, 2m - 1, \ldots, 3, 2, 1, 0)$$

admits as limiting element $(P)_{m+1}$ the M_{m+1} with the symbol

$$(2m + 1, 2m)\,(2m - 1, 2m - 2) \ldots (3, 2)\,(1, 0).$$

On the other hand, amongst the polytopes *themselves*, no measure polytope occurs and of the cross polytopes only the octahedron presents itself. We prove that this must be so, for each of the two series separately.

Measure polytopes. The number of vertices of (5443322210) is $\frac{10!}{2!\,2!\,3!}$, which can be written in the form $\frac{10!}{3!\,(2!)^2\,(1!)^3}$ so as to be able to generalize it for any $(P)_n$ as

$$\frac{(n + 1)!}{a!\,b!\,c! \ldots k!},$$

where $a, b, c, \ldots k$ are arranged in decreasing order and their sum is $n + 1$. Now this form is a product of binomial coefficients

$$(n + 1)_a\,(n - a + 1)_b\,(n - a - b + 1)_c \ldots .$$

and there is only one possibility under which this product contains no factors different from two and is therefore a power of two, i. e. in the case $n + 1 = 2^p$, $a = 2^p - 1$, $b=1$, giving

$$\frac{2^p!}{(2^p - 1)!\;1!} = 2^p.$$

But this case corresponds to the *simplex* $S(2^p)$ of space S_{2^p-1}.

Cross polytopes. The cross polytope is characterized by the property of having all its diagonals of the same length ($=\sqrt{2}$ times an edge) and passing through the same point. So in order to represent a cross polytope the symbol of coordinates of $(P)_n$ can contain end digits only, for the supposition of *three* different digits as in (210) leads inevitably to three different distances. Let us suppose the two end digits are 1 and 0. Then we have to take in at least two of each in order to create the possibility of interchanging two pairs of digits; this gives us the octahedron, the diagonals of which are the joins of the pairs of vertices represented by

$$\left.\begin{matrix}1100\\0011\end{matrix}\right\},\quad \left.\begin{matrix}1010\\0101\end{matrix}\right\},\quad \left.\begin{matrix}1001\\0110\end{matrix}\right\},$$

and pass therefore through the point $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$. Finally in the cases $(11\overbrace{00\ldots0}^{n-1})$ and $(\overbrace{11\ldots1}^{n-1}00)$ we have to deal also with polytopes admitting only diagonals $=\sqrt{2}$ times an edge, but here these diagonals do not pass through the same point (centre). For in the case of (11000) the centre is the point all the coordinates of which are $\frac{2}{5}$ and this point lies not on the diagonal joining the points 11000 and 00110, etc.

C. *Extension number and truncation integers and fractions.*

15. "How can all the new polytopes $(a\,b\,c\ldots)$ found analytically be deduced geometrically from the regular simplex?"

As we remarked in the introduction the new polytopes *have been discovered* geometrically by M^rs. A. Boole Stott; we will consider her method thoroughly under *D.* Here we wish to indicate first that the answer to this question can also be given by the theorem:

Theorem IV. "The new polytopes, all with edges of length unity, can be found by means of a regular extension of the regular simplex of coordinates followed by a regular truncation, either at the vertices alone, or at the vertices and the edges, or at the vertices, edges and faces, etc."

Proof. This theorem is an immediate consequence of that given in art. 6 (theorem II) about the equality of the non vanishing coefficients c_i of the coordinates x_i in the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ of a limiting space S_{n-1} of the new polytope deduced from the simplex $S(n+1)$ of S_n. So in treating in art. 7 the example (32110) we found that the limiting spaces

$$\begin{array}{llll}
x_1 & = 3 & \text{containing a limiting} & CO, \\
x_1 + x_2 & = 5 & \quad ,, \quad ,, \quad ,, & P_3, \\
x_1 + x_2 + x_3 & = 6 & \quad ,, \quad ,, \quad ,, & P_6, \\
x_1 + x_2 + x_3 + x_4 & = 7 & \quad ,, \quad ,, \quad ,, & tT
\end{array}$$

are respectively parallel to the spaces

$$\begin{array}{lll}
x_1 & = 1 & \text{containing a vertex,} \\
x_1 + x_2 & = 1 & \quad ,, \quad \text{an edge,} \\
x_1 + x_2 + x_3 & = 1 & \quad ,, \quad \text{a face,} \\
x_1 + x_2 + x_3 + x_4 & = 1 & \quad ,, \quad ,, \text{ limiting body}
\end{array}$$

of the regular simplex, while they are normal to the line joining the centre O of the simplex to the centre of that limiting element. Moreover it is evident that all the spaces of the same group, say $x_k + x_l + x_m = 6$, have the same distance from the corresponding spaces $x_k + x_l + x_m = 1$, etc.

16. The meaning of the expression "extension number" is clear by itself: an extension to an amount ε transforms the simplex $S^{(1)}(n+1)$ with edge unity of S_n into a simplex $S^{(\varepsilon)}(n+1)$ of edge ε. But we have to define beforehand what we will understand e. g. by a truncation $\frac{1}{2}$. If we split up the $n+1$ vertices of the extended simplex $S^{(\varepsilon)}(n+1)$ into two groups of $k+1$ and $n-k$ points (see fig. 2, where the case $n = 6$, $k = 3$ is represented), forming the vertices of regular simplexes $S^{(\varepsilon)}(k+1)$, $S^{(\varepsilon)}(n-k)$ lying in spaces S_k, S_{n-k-1}, and we cut $S^{(\varepsilon)}(n+1)$ by any space S_{n-1} at the same time parallel to these spaces S_k, S_{n-k-1}, i. e. normal to the line joining the centres M, M' of $S^{(\varepsilon)}(k+1)$, $S^{(\varepsilon)}(n-k)$ in a certain point O, any edge PQ joining a vertex P of $S^{(\varepsilon)}(k+1)$ to a vertex Q of $S^{(\varepsilon)}(n-k)$ will be cut in a certain point R for which the ratio $\frac{PR}{PQ}$ is equal to $\frac{MO}{MM'}$, and therefore independent from the choice of the vertices P, Q. This ratio is the "truncation fraction" of $S^{(\varepsilon)}(n+1)$ at the limiting $S^{(\varepsilon)}(k+1)$ by the truncating space and its complement $\frac{QR}{QP}$ to unity is the "truncation fraction" of $S^{(\varepsilon)}(n+1)$ at the limiting $S^{(\varepsilon)}(n-k)$ opposite to $S^{(\varepsilon)}(k+1)$ by the same space.

But, if we like, we can use the term "truncation number" for the number of units contained in the segment PR or QR, according to the truncation being performed at the side of P or of Q. As the number of units of the denominator of the truncation fraction,

i. e. the number of units of the edge of the extended simplex, is the extension number ε, the truncation numbers τ, which — as ε itself — will prove to be always integer, are simply the numerators of the truncation fractions with the extension number ε as common denominator.

17. If we indicate the truncation numbers corresponding successively to a truncation at a vertex, an edge, a face,... by $\tau_0, \tau_1, \tau_2, \ldots$, the theorem holds:

THEOREM V. "Let $(m_0, m_1, m_2, \ldots, 0)$ be the zero symbol of the polytope; then the sum $m = \Sigma m_i$ of the digits is the extension number ε, and the truncation numbers $\tau_0, \tau_1, \tau_2, \ldots$ are represented by the forms

$$\tau_0 = m - m_0,\ \tau_1 = m - m_0 - m_1,\ \tau_2 = m - m_0 - m_1 - m_2, \ldots\text{"}$$

Proof. By the extension ε the simplex of coordinates $S^{(1)}(n+1) = (1\,\overline{00\ldots0}^{\,n})$ of S_n is changed into the concentric simplex $S^{(\varepsilon)}(n+1) = (\varepsilon\,\overline{00\ldots0}^{\,n})$ with edges ε. Then the space S_{n-1} represented in the latter case by $x_1 = 0$ contains a limit $(l)_{n-1}$ of the considered polytope $(P)_n$ forming a part of the limiting simplex $S^{(\varepsilon)}(n) = (\varepsilon\,\overline{00\ldots0}^{\,n-1})$ of $(\varepsilon\,\overline{00\ldots0}^{\,n})$, at which limit of the *highest* order of dimensions this $S^{(\varepsilon)}(n+1)$ is not sliced off. If now we go back to true coordinate values the last digits in the two symbols of $(P)_n$ and $S^{(\varepsilon)}(n+1)$ must still be the same, which will be the case, if we have to subtract from nought in both cases the same amount, i. e. if $\frac{\Sigma m_i - 1}{n+1}$ and $\frac{\varepsilon - 1}{n+1}$ are equal, i. e. if we have $\varepsilon = \Sigma m_i = m$.

From what we remarked in the preceding article it follows that the truncation fraction of the extended simplex $S^{(m)}(n+1)$ at any limiting $S^{(m)}(k+1)$ can be derived from the mutual position of three parallel spaces S_{n-1} normal to the line joining the centre of $S^{(m)}(k+1)$ to the centre of the opposite $S^{(m)}(n-k)$; of these three spaces one passes through $S^{(m)}(k+1)$, an other through the opposite $S^{(m)}(n-k)$, whilst the third is the truncating space lying between these two. For, if as in the preceding article P is any vertex of $S^{(m)}(k+1)$, Q any vertex of $S^{(m)}(n-k)$ and R the point of intersection of PQ with the third of these three parallel spaces, which may be represented by $S^{(1)}{}_{n-1}, S^{(2)}{}_{n-1}, S^{(3)}{}_{n-1}$, according to definition the truncation fraction τ_k is the ratio $\frac{PR}{PQ}$ and now we have the theorem:

"If a line cuts any three parallel spaces $S^{(1)}{}_{n-1}$, $S^{(2)}{}_{n-1}$, $S^{(3)}{}_{n-1}$ of S_n represented by the three equations $b_1 x_1 + b_2 x_2 + \ldots\ldots + b_q x_q = c_t$, $(t = 1, 2, 3)$ in the points P, Q, R, we have

$$\frac{PR}{PQ} = \frac{c_1 - c_3}{c_1 - c_2}\text{''}$$

For this is obviously true for the edge $A_1 A_{q+1}$ of the simplex of coordinates, the values of x_1 for the three points of intersection with this line being determined by the relations $b_1 x_1 = c_t$, $(t = 1, 2, 3)$; therefore it is true for any transversal, according to a well known theorem, already used implicitly in the preceding article, the ratio in question being the same for all possible transversals.

Now in the case under consideration the spaces $S^{(1)}{}_{n-1}$, $S^{(2)}{}_{n-1}$, $S^{(3)}{}_{n-1}$ may be represented by the equations $x_1 + x_2 + \ldots + x_{k+1} = c_t$, $(t = 1, 2, 3)$, where c_1 and c_2 are the maximum and minimum values of the left hand member with respect to the vertices of the extended simplex $(m\,\overbrace{00\ldots\ldots0}^{n})$, while c_3 is the maximum value of the same expression with respect to the vertices of $(m_1, m_2, m_3, \ldots, 0)$. So we have

$$c_1 = m,\ c_2 = 0,\ c_3 = m_0 + m_1 + \ldots + m_k$$

giving for the truncation fraction the result $\frac{m - (m_0 + m_1 + \ldots + m_k)}{m}$ and therefore $\tau_k = m - (m_0 + m_1 + \ldots + m_k)$.

So we find in the case of the $(P)_9$ of art. 12 represented by (5443322210) $m = 26$, $\tau_0 = 21$, $\tau_1 = 17$, $\tau_2 = 13$, $\tau_3 = 10$, $\tau_4 = 7$, $\tau_5 = 5$, $\tau_6 = 3$, $\tau_7 = 1$.

For $n = 3, 4, 5$ the extension number and the truncation numbers are indicated in Table I, the seventh column containing the extension number ε, the eighth column giving what may be called the "truncation symbol". So in the case of $e_1 e_2 e_4 S(6)$ the extension number is 11, the truncation symbol is $7, 4, 2, 1$ where these numbers represent successively the values of $\tau_0, \tau_1, \tau_2, \tau_3$; so we find mentioned here a truncation $\frac{7}{11}$ at the vertices, $\frac{4}{11}$ at the edges, $\frac{2}{11}$ at the faces and $\frac{1}{11}$ at the limiting bodies.

D. *Expansion and contraction symbols.*

18. If we compare the symbols containing the operators e_i and c of expansion and contraction introduced by M[rs]. STOTT for the offspring of the $T = S(4)$ in S_3 and the $S(5)$ in S_4 with the zero symbol of these polyhedra and polytopes, we remark that all these cases underlie certain general laws, up to now of an empirical character. By proving these laws we will promote them to theorems, the first of which can be stated as follows:

THEOREM VI. "The expansion e_k, $(k = 1, 2, 3, \ldots, n-1)$ applied to the $S^{(1)}(n+1)$ of S_n changes the symbol of coordinates $(1\,\overline{00\ldots0}^{\,n})$ of that simplex into an other zero symbol which can be obtained by adding a unit to the first $k+1$ digits."

Indeed this gives (compare the small table $n = 4$ under art. 3):

$$e_1\, S(5) = (21000),\quad e_2\, S(5) = (21100),\quad e_3\, S(5) = (21110).$$

Proof. The operation of expansion e_k consists in moving the limiting $S(k+1)$ of $S^{(1)}(n+1)$ to equal distances away from the centre O of $S^{(1)}(n+1)$, each $S(k+1)$ moving in the direction of the line OM joining O to its centre M, these $S(k+1)$ "remaining parallel to their original position, retaining their original size and being moved over such a distance that the two new positions of any vertex which was common to two adjacent limits $(l)_k$ in the original $S^{(1)}(n+1)$ shall be separated by the length of an edge". [1])

Now let us consider (fig. 3) the plane through OM and any vertex A of the $S(k+1)$ of which M is the centre; then, on account of the regularity of $S^{(1)}(n+1)$, the angle AMO is a right one. This plane will also contain the new position $A'M'$ of AM. What we have to do now is this: We select from the symbol of coordinates $(1\,\overline{00\ldots0}^{\,n})$ the vertices of any limiting $S^{(1)}(k+1)$, calculate the coordinates of M, deduce from the coordinates of O and M those of M' on the supposition that $OM' : OM = \lambda$ is known. Then we have to determine the coordinates of A' by adding to the coordinates of the vertex A chosen arbitrarily among the $k+1$ vertices of the $S^{(1)}(k+1)$ the differences of the corresponding coordinates of M' and M. Finally we have to determine λ by the stated condition that two new positions of the same vertex A of $S^{(1)}(n+1)$ shall be separated by the length of an edge,

[1]) Compare p. 5 of the memoir quoted of M[rs]. STOTT.

or — which comes to the same — by the condition that the coordinates of A' satisfy the law stated in theorem I, that the difference of any two different adjacent values must be unity.

We now set to work and select for the $k+1$ vertices the vertices $A_1, A_2, \ldots, A_{k+1}$ of $S^{(1)}(n+1)$ and for A of fig. 3 the vertex A_1. According to this choice the coordinates of M are

$$x_1 = x_2 = \ldots = x_{k+1} = \frac{1}{k+1}, \quad x_{k+2} = x_{k+3} = \ldots = x_{n+1} = 0.$$

So the coordinates of the three points O, A, M satisfy the equations

$$x_2 = x_3 = \ldots = x_{k+1}, \quad x_{k+2} = x_{k+3} = \ldots = x_{n+1};$$

but then these relations hold for any point of the plane OAM, as the $n-2$ equations represent a plane. As moreover

$$AA' = MM' = (\lambda - 1)\, OM,$$

or in the notation of vector analysis

$$A' - A \equiv M' - M \equiv (\lambda - 1)(M - O),$$

we have successively for the mentioned coordinates in true values and for the mentioned differences of coordinates:

	x_1	$x_2 = x_3 = \ldots = x_{k+1}$	$x_{k+2} = x_{k+3} = \ldots = x_{n+1}$
O	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$
M	$\frac{1}{k+1}$	$\frac{1}{k+1}$	0
$M - O$	$\frac{1}{k+1} - \frac{1}{n+1}$	$\frac{1}{k+1} - \frac{1}{n+1}$	$-\frac{1}{n+1}$
$A' - A$	$(\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$(\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$-(\lambda-1)\frac{1}{n+1}$
A	1	0	0
A'	$1 + (\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$(\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$-(\lambda-1)\frac{1}{n+1}$

i. e. the difference $\dfrac{\lambda - 1}{k+1}$ of the coordinates x_{k+1} and x_{k+2} of A' is either unity or zero. But if we make it zero we get $\lambda = 1$, i. e. we find back the *original* $S^{(1)}(n+1)$. So for the *expanded* polytope we have to take $\dfrac{\lambda - 1}{k+1} = 1$ or $\lambda = k+2$, giving for A' the coordinates

$$x_1 = 2\,,\ x_2 = \ldots = x_{k+1} = 1\,,\ x_{k+2} = \ldots = x_{n+1} = 0$$

i. e. the symbol $(2\,\overbrace{11\ldots1}^{k}\,\overbrace{00\ldots0}^{n-k})$, what was to be proved.

If by the operation e_k the limits $S^{(1)}(k+1)$ of $S^{(1)}(n+1)$ are moved away from the centre O to a distance equal to λ times the original distance, the extended simplex $S^{(x)}(n+1)$, the limits $S^{(x)}(k+1)$ of which will contain these $S^{(1)}(k+1)$ in their new positions, will be a simplex $S^{(\lambda)}(n+1)$, i. o. w. $\lambda = k+2$ is the extension number of the new polytope. This comes true, for according to theorem V the sum $k+2$ of the digits of the zero symbol $(2\,\overbrace{11\ldots1}^{k}\,\overbrace{00\ldots0}^{n-k})$ *is* the extension number. So we find by the way:

THEOREM VII. "In the expansion e_k the limits $S^{(1)}(k+1)$ are moved away from the centre to a distance equal to $k+2$ times the original distance."

Remark. We may express the influence of the operation e_k on the symbol $(1\,\overbrace{00\ldots0}^{n})$ of the simplex $S^{(1)}(n+1)$ presenting only one unit interval between the first and the second digit by saying that it creates a second unit interval between the $k+1^{st}$ and the $k+2^{nd}$ digit. This remark which holds also with respect to the symbol of true coordinates will be of use in the following articles.

19. THEOREM VIII. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $S^{(1)}(n+1)$ on its zero symbol $(1\,\overbrace{00\ldots0}^{n})$ is found by adding the influences of each of the expansions taken separately".

Indeed this gives (compare the small table $n=4$ under art. 3):

$$e_1\,e_2\,S(5) = (32100),\ e_1\,e_3\,S(5) = (32110),\ e_2\,e_3\,S_5 = (32210),$$
$$e_1\,e_2\,e_3\,S(5) = (43210).$$

Proof. We begin to prove the theorem for the case of two operations of expansion only.

It is stipulated expressly by Mrs. STOTT that in the succession of two operations of expansion the subject of the second is to be what its original subject has become under the influence of the first. So in the case $e_2\,e_1\,T$ of the tetrahedron (fig. 4^a) the original triangular subject of e_2 is transformed by e_1 into a hexagon (fig. 4^b) and now the hexagon is moved out, in the case $e_1\,e_2\,T$ the linear subject of e_1 is transformed by e_2 into a square (fig. 4^c) and now the square is moved out; in both cases the result (fig. 4^d) is the same, tO. In general, for $k > l$, in the case $e_k e_l S^{(1)}(n+1)$ the subject $S^{(1)}(k+1)$ of e_k is transformed by e_l into an $e_l S^{(1)}(k+1)$ and now this $e_l S^{(1)}(k+1)$ is moved away from the centre, while in the case $e_l e_k S^{(1)}(n+1)$ the subject $S^{(1)}(l+1)$ of e_l is transformed by e_k into an $n-1$-dimensional polytope of the import l corresponding to $S^{(1)}(l+1)$ which polytope is moved away from O as a whole. Now it is evident that the geometrical condition "that the two new positions of a vertex shall be separated by the length of an edge" makes the distance over which the second motion of any of these two pairs has to take place equal to the distance described in the first motion of the other pair; i. e. if $S^{(1)}(l+1)$ is a limiting element of $S^{(1)}(k+1)$ and A is a vertex of that $S^{(1)}(l+1)$, the segments described by A in transforming $S^{(1)}(n+1)$ into the two polytopes $e_k e_l S(n+1)$ and $e_l e_k S(n+1)$ are the two pairs of sides of a parallelogram leading from A to the opposite vertex A'. In other words: we find the true coordinates of A' by adding to the coordinates of A the variations corresponding to the motions due to each of the operations e_k and e_l taken separately.

Taking for $S^{(1)}(k+1)$ the simplex $A_1 A_2 \ldots A_{k+1}$, for $S^{(1)}(l+1)$ the simplex $A_1 A_2 \ldots A_{l+1}$ and for A the point A_1 we have to vary the coordinates $\overline{1, 0, 0, \ldots 0}^{\,n}$ of A so as to admit two more unit intervals, one between the $k+1^{st}$ and the $k+2^{nd}$, an other between the $l+1^{st}$ and the $l+2^{nd}$ digit. If then afterwards we pass to the zero symbol we get $(3\ \overline{22\ldots2}^{\,l}\ \overline{11\ldots1}^{\,k-l}\ \overline{00\ldots0}^{\,n-k})$, what was to be proved.

Now we have still to add that the proof for the composition of three and more operations of expansion runs entirely on the same lines. In the case of three operations we will have to compose three displacements according to the rule of the diagonal of the parallelopipedon, in the case of more we will have to use the extension of this rule to parallelotopes. To this geometrical composition of motions always corresponds the arithmetical addition of the symbol influences, where the order of succession is irrelevant; this arithmetical addition leads to the creation of new unit intervals independently. So the general rule is proved.

The preceding developments lead to a new theorem, viz:

THEOREM IX. "The operation e_k can still be applied to any polytope deduced from the simplex in the zero symbol of which the $k+1^{st}$ and the $k+2^{nd}$ digit are equal."

This theorem enables us to find immediately the expansion symbol of a polytope with given zero symbol. We show this by the example **(5443322210)** of art. 12.

In (5443322210) five unit intervals occur, viz. if we represent the p^{th} digit by d_p between (d_1, d_2), (d_3, d_4), (d_5, d_6), (d_8, d_9), (d_9, d_{10}). Of these the first corresponds to the original unit interval of the simplex $(1\,\overline{00\ldots0}^{\,9})$, whilst the others are introduced by the expansion operations e_2, e_4, e_7, e_8. So we find $e_2\, e_4\, e_7\, e_8\, S(10)$.

Reversely it is quite as easy to find back the zero symbol of $e_2\, e_4\, e_7\, e_8\, S(10)$. As there are to be four unit intervals more than the original one the zero symbol begins by 5 and 4, and has to show a unit interval behind the third, the fifth, the eighth, the ninth digit, etc.

20. It is obvious that the system of expansion operations cannot lead to a zero symbol with two or more equal largest digits. So the system of the expansion forms is not complete as to the total number of possible forms. But the scope of this incompleteness is not so large as we might think at first. For, if the zero symbol winds up in two or more zeros, the inversion indicated in art. 3 will bring about a new zero symbol with more than one largest digit. Nevertheless, after this extension of the system of expansion forms, still the forms with a zero symbol containing two or more largest digits and two or more zeros are lacking.

So it was desirable to have at hand a new geometrical operation leading to forms with a zero symbol containing more than one largest digit. This now is given us by M^{rs}. STOTT in the operation of *contraction*; but before we show this we may devote a single word to the introduction of different kinds of contraction.

The *subject* of the operation c of contraction of an expansion form in S_n is always a group of limiting elements of the same import and of the highest order of dimensions available; so we designate the contraction c as a c_0, a c_1, a c_2, etc. according to the subject elements being of vertex import, of edge import, of face import, etc. Moreover these limits of the same import can be subject of contraction, when and only when all their vertices form together exactly all the vertices of the expansion form, each vertex taken once; in this case any two of these limits are still separated from each other by the distance of an edge at least and now the operation of contraction consists merely in this that all these limits undergo a parallel displacement, of the same amount, towards the centre O of the expansion form, by which any of these limits gets a vertex or some vertices in common with some of the other ones.

We illustrate this by the example of fig. 4. Here the results can be tabulated as follows:

$$\left.\begin{array}{l} c_0 e_1 T = \quad O \\ c_0 e_2 T = -T \\ c_0 e_1 e_2 T = -e_1 T \end{array}\right\}, \quad \left.\begin{array}{l} c_1 e_1 T = T \\ c_1 e_2 T \text{(impossible)} \\ c_1 e_1 e_2 T = e_2 T \end{array}\right\}, \quad \left.\begin{array}{l} c_2 e_1 T \text{(impossible)} \\ c_2 e_2 T = T \\ c_2 e_1 e_2 T = e_1 T \end{array}\right\}.$$

In this small table the negative sign indicates the inverse orientation; the impossibility of $c_1 e_2 T$ and $c_2 e_1 T$ is caused by the fact that the polygons, in the first case of edge and in the second of face import, forming the subject of contraction, have already a vertex or an edge in common.

But we can also account for the impossibility of $c_1 e_2 T$ and $c_2 e_1 T$ — and for other similar results — by remarking that the contraction c_k undoes the expansion e_k and that it can be applied, when and only when the expansion form has been obtained by applying amongst the different expansions the operation e_k. So c_0 is the only contraction operation which we have to introduce in order to be able to deduce all the forms with a symbol satisfying the law of theorem I.

As we will use henceforth exclusively the operation c_0, the subscript of the c can be omitted.

21. We now prove the general theorem:

Theorem X. "By applying the contraction c to any expansion form the largest digit of the zero symbol of this form is diminished by one".

Proof. The groups of polytopes of vertex import of the expansion form represented by the zero symbol $(a+1, a, b, c, .., 0)$, where $a \geq b \geq c \ldots$, is found by putting $x_i = a+1$, leaving $(a, b, c, .., 0)$ for the other coordinates. By diminishing $a+1$ by one we get an other form with the zero symbol $(a, a, b, c, .., 0)$ possessing also polytopes of vertex import represented by $(a, b, c, .., 0)$. So the polytopes g_0 of vertex import of the second form are congruent to and equally orientated with the corresponding polytopes of the first, but they lie in spaces $x_i = a$ nearer to the centre than $x_i = a+1$. For, if p is the extension number of the original form $(a+1, a, b, c, .., 0)$ of S_n, and therefore $p-1$ that of the new form $(a, a, b, c, .., 0)$, the true coordinate values of x_i corresponding to the values $a+1$ of the first and a of the second zero symbol are $a+1-\dfrac{p-1}{n+1}$

and $a - \frac{p-2}{n+1}$; as the true coordinates of O are $\frac{1}{n+1}$ in both cases, the distance to O is diminished by

$$a + 1 - \frac{p-1}{n+1} - \frac{1}{n+1} - \left(a - \frac{p-2}{n+1} - \frac{1}{n+1}\right) =$$
$$= 1 - \frac{p-1}{n+1} + \frac{p-2}{n+1} = \frac{n}{n+1}.$$

Moreover it is evident that any two of these polytopes g_0 of the first form, e. g. those lying in the spaces $x_1 = a + 1$, $x_2 = a + 1$ are separated by the right prism with the base polytopes

$$x_1 = a + 1, \quad x_2 = a \quad , \quad x_3, x_4, \ldots \quad = (b, c, \ldots, 0),$$
$$x_1 = a \quad , \quad x_2 = a + 1, \quad x_3, x_4, \ldots \quad = (b, c, \ldots, 0),$$

while the corresponding two g_0 of the second form are in contact with each other by the $n - 2$-dimensional polytope

$$x_1 = a, \quad x_2 = a, \quad x_3, x_4, \ldots \quad = (b, c, \ldots, 0).$$

By combining the theorems IX and X we can find the symbol in operators c and e_k of any contraction form, i. e. of any form the zero symbol of which contains two or more largest digits. To that end we have.

1°. to pass to the corresponding expansion form by adding one to the first digit,

2°. to treat the zero symbol of this expansion from according to the rule deduced from theorem IX,

3°. to put c before the obtained result.

In the following we give some examples of the deduction of c and e_k symbols from zero symbols, in connexion with the three possibilities which may present themselves, if we consider the two different zero symbols of a form without central symmetry, according to the appearance of the contraction symbol; they are

$(4332210) = -(4322110)$, i. e. $\quad e_2 e_4 e_5 S^{(1)}(7) = - \quad e_1 e_3 e_5 S^{(1)}(7)$,
$(4432210) = -(4322100)$, „ $\quad c e_1 e_2 e_4 e_5 S^{(1)}(7) = - \quad e_1 e_3 e_4 S^{(1)}(7)$,
$(3332100) = -(3321000)$, „ $\quad c e_2 e_3 e_4 S^{(1)}(7) = - c e_1 e_2 e_3 S^{(1)}(7)$.

Remark. According to the developments of the preceding article the contraction c_k always cancels the expansion e_k; so we can deduce from the theorems VI and IX that the operation c_k can only be applied to expansion forms in the zero symbol of which the $k + 1^{st}$ and the $k + 2^{nd}$ digit are unequal and that the zero symbol of the new form is found by subtraction of a unit from the first

$k+1$ digits of the zero symbol of the given form. Of this general result theorem X considers the special case k zero.

Now if we apply the contraction $c_0 = c$ to the simplex of coordinates $(1\overbrace{00\ldots0}^{n})$ itself we find the point with the zero symbol $(\overbrace{000\ldots0}^{n+1})$ i. e. the centre O. This result is geometrically evident: if we bring the vertices nearer to the centre so as to annihilate the separating edges the result is a single point. In this point of view the inverse operation e_0 can be considered as corresponding to the generation of the simplex starting from a point.

Remark. By introducing the operation e_0 the contraction symbol c can be shunted out. So, if $S_0^{(1)}(n+1)$ represents the point which is to become the $S^{(1)}(n+1)$ by applying the operation e_0, we can replace $c\,e_k\,e_l\,S^{(1)}(n+1)$ by $e_k\,e_l\,S_0^{(1)}(n+1)$, but this implies that we write $e_0\,e_k\,e_l\,S_0^{(1)}(n+1)$ for $e_k\,e_l\,S^{(1)}(n+1)$.

This new notation will prove to be preferable in the case of the nets (see under E the art. 30 at the end of page 57).

E. *Nets of polytopes.*

22. As to recent literature about space fillings or nets we may mention A. ANDREINI's "Sulle reti di poliedri regolari e semiregolari e sulle corrispondenti reti correlative" (Roma, 1905), two papers of mine ("Fourdimensional nets and their sections by spaces" and "The sections of the net of measure polytopes M_n of space Sp_n with a space Sp_{n-1} normal to a diagonal", *Proceedings* of Amsterdam, vol. X, pp. 536, 688) and the memoir of M^rs^. STOTT quoted several times.

We exclude what may be called a prismatic net, i. e. a net in S_n obtained by prismatizing a net of S_{n-1} in a new direction, and divide the remaining *uniform* nets derived from the simplex into two groups [1]): 1°. *pure* nets with only one (central symmetric) constituent and 2°. *mixed* nets either with one non central symmetric constituent in two opposite positions or with constituents of different kind. If we restrict ourselves to the plane the first group consists of the hexagon net only, while the second is represented e. g. by the triangle net and the net of hexagons and triangles; if we proceed to ordinary space the first group contains the tO net only, while the second is represented e. g. by the net of T and O.

[1]) This division — of no fundamental importance in itself — is introduced here, merely in order to smooth the way leading to the analytical representation of the nets.

It is our aim to unearth in the following articles *all* the nets of simplex extraction possible in space S_n from $n = 2$ to $n = 5$ included. This task, concerned with new material, breaks up into several parts. First we will have to deduce general characteristic properties of the analytical symbols which are to represent the nets. Secondly we will derive a simple rule solving the question under what circumstances the symbols obtained *do* represents possible nets. Thirdly application of this simple rule will lead to the knowledge of all the possible nets and to a tabularization of them. Finally we will pass in review the tabulated nets and devote some words to an other method by which at least a part of these results can be obtained.

23. *Theoretically* speaking a net can be determined analytically in two different ways, either as a whole or decomposed into its constituent polytopes. So we will try to find either *one* symbol of coordinates, representing all the vertices of the net at a time, or in the case of pure nets *one pair*, in the case of mixed nets *several pairs* of symbols, each pair consisting of a symbol representing all the vertices of any constituent and an other symbol from which can be deduced all the centres of the repetitions of that constituent in the same orientation occurring in the net.

In order to blow life into this theoretical skeleton — forming as it were a kind of working hypothesis — we consider the generally known and simple case of the *net of triangles* in the plane.

If we start (fig. 5) from a triangle $A_1 A_2 A_3 = p_3^{(1)}$, i. e. with sides unity, and complete the three sides produced to three systems of equidistant parallel lines, the distance of any two adjacent parallel lines being the height of triangle $p_3^{(1)}$, we get the net $N(p_3)$.

From this generation it is at once evident that with respect to the original $p^{(1)}_3$ as triangle of coordinates all the vertices of the net can be represented by the coordinate symbol (a_1, a_2, a_3), where a_i, $(i = 1, 2, 3)$ are any three integers for which $\Sigma a_i = 1$. So (a_1, a_2, a_3), $\Sigma a_i = 1$ is the *net symbol* of $N(p_3)$, under the condition stated that a_i are three integers. In this ever so simple case the round brackets may be omitted, for the faculty of taking for a_1, a_2, a_3 any set of three integers with a sum unity includes that of interchanging the three digits.

The net $N(p_3)$ consists of two sets of triangles, triangles $p^{(1)}_3$ corresponding in orientation with $A_1 A_2 A_3$ and triangles $p^{(1)}_3$ of opposite orientation. If we consider only one of these two sets of triangles and of these triangles only one of the three sets of homologous vertices we get all the vertices of the net and each vertex once. In other words: the system of the centres of either of the two sets of triangles is equipollent to the system of vertices of the net, i. e. if we move all the vertices of the net in the direction A_1O over that distance it passes into the system of the centres of the triangles corresponding in orientation with $A_1 A_2 A_3$, whilst we get the system of the centres of the other set of triangles by a motion over the same distance in opposite direction. So, as the three coordinates of any vertex of the net are found by adding to the coordinates 1, 0, 0 of A_1 three integers with a sum zero, and the true coordinates of the centres O and O_1 are $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ and $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ the symbols $(b_1 + \frac{1}{3}, b_2 + \frac{1}{3}, b_3 + \frac{1}{3})$ and $(b_1 - \frac{1}{3}, b_2 + \frac{2}{3}, b_3 + \frac{2}{3})$ represent the centres of the two sets of triangles under the condition that the three b_i are integers with sum zero. In both cases the three integers b_i with sum zero indicate what is to be added to the coordinates of any centre of each set in order to obtain the whole set; as we call the two sets of centres the "frames" of the two kinds of triangles, we call the system of *differences* b_1, b_2, b_3 the "frame coordinates" and $(b_1, b_2, b_3), \Sigma b_i = 0$ the "frame symbol" of both sets of triangles.

Recapitulating we find the following result for $N(p_3)$:

	Net symbol	$(a_1, a_2, a_3), \Sigma a = 1.$
Set of triangles	Symbol of constituent $A_1 A_2 A_3$	$(1, 0, 0),$
$A_1 A_2 A_3$	Frame	$(b_1 + \frac{1}{3}, b_2 + \frac{1}{3}, b_3 + \frac{1}{3}), \Sigma b = 0.$
Other set of	Symbol of $O_1 O_2 O_3$	$(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}),$
triangles	Frame	$(b_1 - \frac{1}{3}, b_2 + \frac{2}{3}, b_3 + \frac{2}{3}), \Sigma b = 0.$
	Frame symbol	$(b_1, b_2, b_3), \Sigma b = 0.$

Here $O_1 O_2 O_3$ represents a central triangle oppositely orientated to $A_1 A_2 A_3$. We may still remark that the second frame may be written in the more symmetrical form $(b_1 + \frac{2}{3}, b_2 + \frac{2}{3}, b_3 + \frac{2}{3})$, $\Sigma b_i = -1$, or if one likes $(b_1 - \frac{1}{3}, b_2 - \frac{1}{3}, b_3 - \frac{1}{3})$, $\Sigma b_i = 2$.

But it is much simpler here to decompose the net into the repetitions of the two triangular constituents by introducing a new symbol still, the symbol $\boldsymbol{(}b_1 + \mathbf{1}, b_2 + \mathbf{0}, b_3 + \mathbf{0}\boldsymbol{)}$, obtained by addition of the corresponding digits of the frame symbol and the symbol of the constituent $A_1 A_2 A_3$, the heavy round brackets meaning that only the parts of the digits written in heavy type are to interchange places, whilst the arbitrary integers b_i satisfy the condition $\Sigma b_i = 0$. For each system of values of the b_i satisfying the condition stated the symbol represents a definite triangular constituent of the set to which $A_1 A_2 A_3$ belongs; so by this symbol the net $N(p_3)$ is decomposed into the different constituents of the set of triangle $A_1\ A_2\ A_3$. In the same way the symbol $(b_1+1,\ b_2+1,\ b_3+0)$ characterizes the other set of triangles under the condition $\Sigma\, b_i = -1$. [1])

24. In the first place we remark that the net of triangles admits a net symbol with only integer digits and we examine now to what extent this property is a general one.

If we choose for simplex of coordinates $S^{(1)}(n+1)$ a simplex with respect to which a definite polytope of the net — let us call it the central polytope $(P)^\circ$ of the net — can be represented by its zero symbol and we restrict ourselves to the cases in which the constituents of the net are exclusively forms derived from the simplex, we can easily prove that the coordinates of all the vertices of the net must be integer. To that end we call any polytope of the net "orientated" with respect to the simplex of coordinates, if a translational motion of the polytope which brings its centre into coincidence with the centre of that simplex gives it a position in which it is represented by its zero symbol or by the reverse; this definition enables us to state the following lemma:

"If two polytopes of the net in S_n have a limiting $n-1$-dimensional polytope in common, they are both orientated with respect to the simplex of coordinates, as soon as this is the case with one of them".

We remark — in order to prove this lemma first — that, if two polytopes derived from $S^{(1)}(n+1)$ have a limiting $n-1$-dimensional polytope in common, this limit has *either* with respect to both the *same* import *or* its import with respect to the one is *complementary* to that with respect to the other. For, according to the last two lines of art. 13, any two limits $(l)_{n-1}$, represente das to their imports by g_k and $g_{k'}$, are prismotopes $(P_k;\ P_{n-k-1})$ and $(P_{k'};\ P_{n-k'-1})$, and these prismotopes cannot coincide, unless we have either $k'=k$ or $k'=n-k-1$.

This remark leads to a proof of the lemma in the following way. Let $(P)_n^a$ and $(P)_n^b$ be the two given polytopes and $(P)_{n-1}$ their common $n-1$-dimensional limit lying in the space $S_{n-1}^{a,\,b}$. Let the $S^{(1)}(n+1)$, from which $(P)_n^a$ can be derived by means of the operations $e_{\cdot\cdot}$ and c, be our simplex of coordinates; then $P_n^{(a)}$ is not only orientated with respect to that simplex but also concentric with it.

---

[1]) As soon as the idea of splitting up the digits of the symbol into two parts, an unmovable part and a permutable one, had presented itself, the analytical deduction of the nets of simplex extraction was within grasp.

Let $(P')^b_n$ and $(P'')^b_n$ represent farthermore the two [1]) polytopes congruent to $(P)^b_n$ and concentric to $(P)^a_n$ which admit of a zero symbol with respect to the simplex of coordinates. Then we have only to prove that either $(P')^b_n$ or $(P'')^b_n$ is equipollent to $(P)^b_n$. Now from the fact that $(P)^a_n$ and $(P)^b_n$ have $(P)_{n-1}$ in common it follows that $(P')^b_n$ and $(P'')^b_n$ must admit a set of limits congruent and therefore of the same or of complementary import with $(P)_{n-1}$ of $(P)^a_n$; so one of these limits of $(P')^b_n$ — say $(P')_{n-1}$ — and one of these limits of $(P'')^b_n$ — say $(P'')_{n-1}$ — must lie in spaces S'_{n-1} and S''_{n-1} parallel to $S^{a,b}_{n-1}$ and on both sides at the same distance from the centre O of $(P)^a_n$. Of these spaces $S'_{n-.}$ and S''_{n-1} let S'_{n-1} be that one which lies on opposite sides with respect to O with $S^{a,b}_{n-1}$. Then it will be possible to bring $(P')^b_n$ into coincidence with $(P)^b_n$ by means of a translational motion; for, if by such a motion the limit $(P')_{n-1}$ is brought into coincidence with $(P)_{n-1}$, the polytopes will coincide, as this is the case not only with the limits mentioned but also with their centres. So $(P)^b_n$ is orientated with respect to the simplex of coordinates.

From the lemma to the theorem in view we have only to take one step more. The lemma immediately shows that, if the net in S_n consists exclusively of polytopes derived from the simplex, all the polytopes are orientated with respect to the simplex of coordinates, as soon as this is the case with one of them; for we can always consider any two polytopes of the net as the first and the last of a series of polytopes any two adjacent ones of which are in n — 1-dimensional contact. So with respect to the simplex from which the central polytope $(P)^\circ$ has been derived all the polytopes of the net are orientated. But this includes that by passing from any vertex of the net to an adjacent one the coordinates change by integers and as we can reach any vertex of the net by means of a set of these motions — starting from a determinate vertex of $(P)^\circ$ — the coordinates of any vertex of the net must be integers.

So we have shown now that the property of admitting vertices with integer coordinates only belongs to all the nets, the polytopes of which are exclusively of simplex extraction. This very general result brings us in contact with the two following questions:

a). Can the result be expressed by saying that any net with the assigned property of its constituents admits a *net symbol* with integer coordinates only?

This question must be answered negatively. We cannot pass to

---

[1]) In the particular case of a central symmetric $(P)^b_n$ these two positions coincide, etc.

this new version, unless we prove that each net of the assigned kind *does* admit a net symbol; as soon as such a net admits a net symbol we can choose our system of coordinates in such a manner that this net symbol contains integer coordinates only. We take position with respect to this point by supposing beforehand that each net of the assigned kind admits a net symbol, which brings us under the obligation to prove afterwards that this is so.

b). Are there simplex nets not satisfying the condition that all the constituents are of simplex extraction?

We dispose of this question by pointing to three plane nets, viz.

1°. $N(p_3, p_4, p_6)$ of triangles, squares and hexagons (fig. 6),

2°. $N(p_4, p_6, p_{12})$ of squares, hexagons and dodecagons (fig. 7),

3°. $N(p_3, p_{12})$ of triangles and dodecagons (fig. 8),

which must undeniably be considered as simplex nets, as they can be derived from the three generally known plane nets $N(p_3)$, $N(p_6)$, $N(p_3, p_6)$ by means of the *e*-operations. If $N(p_x; p_y; p_z)$ represents a net with the polygonic constituents p_x, p_y, p_z of which p_x is of face, p_y of edge, p_z of vertex import, these deductions are indicated by the equations

$$e_2 N(p_3) = N(p_3; p_4; p_6); \qquad e_2 N(p_6) = N(p_6; p_4; p_3);$$
$$e_1 e_2 N(p_3) = N(p_6; p_4; p_{12}); \quad e_1 e_2 N(p_6) = N(p_{12}; p_4; p_6);$$
$$c\, e_1 e_2 N(p_3) = N(p_3; —; p_{12}).$$

As these three nets contain constituents not deducable from the simplex of the plane, the triangle, by means of the operations e_k and c, they *must* form exception to the general rule about the net symbol with integer coordinates only; for, in the coordinates with respect to the simplex, only the polytopes derived from the simplex can be represented by a symbol with integer coordinates only.

On account of the property of the three plane nets mentioned — to admit at the same time constituents derivable and constituents not derivable from the simplex — we call them "hybridous". In order to be able to deduce general results from the simple law of integers found above we discard provisionally the three hybribous plane nets and all the hybridous nets that space and hyperspace may contain, considering only the nets we call simplex nets "proper"; meanwhile we promise to come back to these exceptional cases, after having secured the general rule alluded to in art. 22 and the main results to which it leads (see art. 34).

25. In the second place we remark that the two sets of triangles of the net $N(p_3)$ admit the same frame symbol with integer coordinate

values only. We show that this property too is a general property i. e. that all the different sets of constituents of any simplex net proper admit the same frame symbol with integer coordinates.

In discussing the number of the regular polyhedra in ordinary space the plane nets $N(p_3)$, $N(p_4)$, $N(p_6)$ appear as polyhedra with an infinite number of faces, unyielding as to this that their faces remain in the same plane instead of bending round in three dimensions. Of these regular polyhedra with an infinite number of faces the centre is at infinity in the common direction of the normals to their plane in the space of three dimensions which is supposed to contain them and the anallagmatic [1]) rotations and reflections of the regular polyhedra proper pass into translations and reflections in the case of $N(p_3)$, $N(p_4)$, $N(p_6)$. In the same way each net of S_n may be considered as an $n+1$-dimensional polytope with an infinite number of limits $(l)_n$ which instead of bending round in S_{n+1} fill a space S_n. On account of this each net must be transformable in itself by a translational motion which brings a constituent polytope of the net into coincidence with any repetition of that constituent in the same orientation. By means of this property we prove now the following general theorem:

THEOREM XI. "Any possible simplex net proper admits a net symbol and for all the different sets of constituents the same frame symbol. Moreover the frames of all the possible nets of S_n are similar to each other."

a) We show first that *all the different frames of a net are equipollent.*

Let (P) and (Q) with the centres C_p and C_q be any two polytopes of different kinds of a simplex net proper having at least one vertex V in common. Let (P') be any polytope of this net equipollent to (P) and let (Q'), V', C'_p, C'_q represent the new positions of (Q), V, C_p, C_q after a translational motion which brings (P) into coincidence with (P') and therefore the net with itself. Then (Q'), V', C'_p, C'_q are respectively a polytope of the net equipollent to (Q), a vertex of the net homologous to V, the centre of (P'), the centre of (Q'). From this we derive the equipollency of the three lines VV', $C_pC'_p$, $C_qC'_q$, i. e. $C_pC'_p$ and $C_qC'_q$ are mutually equipollent as they are both equipollent to VV'. So all the different frames of a net are equipollent, i. e. each of these frames can be brought into coincidence with any other of them by means of a translational motion.

---

[1]) These rotations and reflections which transform a polytope in itself will be studied in part *G*.

b) In the second place we prove that *each net admits a frame symbol*, and that *the frames of all possible nets are similar*.

If a rectilinear translational motion of the net over a distance d brings any polytope (P) of it into coincidence with its repetition $(P)^{(1)}$ and therefore the net with itself, a rectilinear translational motion of the net over a p times larger distance pd in the same direction, p being any integer, will bring (P) into coincidence with an other of its repetitions $(P)^{(p)}$ and therefore also the net with itself. This is self evident if we consider the motion over pd as the result of p motions d in the same direction executed one after another. In other terms: the frame of any set of constituents of any simplex net proper must be characterized by the property of containing the point $C^{(p)}$ determined by the vector equation $CC^{(p)} \equiv p \,.\, CC^{(1)}$ as soon as it contains the centres C and $C^{(1)}$ and p is any integer. Now let $d_1, d_2, \ldots, d_{n+1}$ with the condition $\Sigma d_i = 0$ represent the frame coordinates of C with respect to any centre C_0 of the frame, and let us consider $d_1, d_2, \ldots, d_n$, — i. e. all these integers, d_{n+1} alone excepted —, as the rectangular coordinates of a point V lying in an other space S'_n bearing the system of coordinates $O\,(X_1, X_2, \ldots, X_n)$. Then to each point C, $C^{(1)}$, $C^{(p)}$ of the frame correspond points V, $V^{(1)}$, $V^{(p)}$ of S'_n and the vector equation $CC^{(p)} \equiv p \,.\, CC^{(1)}$ includes the vector equation $VV^{(p)} \equiv p \,.\, VV^{(1)}$, i. e. there is a correspondence one to one between the centres C of the frame and the images V in S'_n, the points V in S'_n being characterized by the property of having integer coordinates $x_1, x_2, \ldots, x_n$. But *all* the points V with integer coordinates form evidently the vertices of a net of measure polytopes with edge unity; so the system of images V is *either* the total system of vertices of this net of measure polytopes *or* a portion of it, containing always the origin O corresponding to the centre C_0 and partaking of the geometrical property of containing the point $V^{(p)}$, determined by the vector equation $VV^{(p)} \equiv p \,.\, VV'$ if it contains V, V' and p is integer. In this form it is immediately evident — in connection with the equivalence of the different coordinates — that the portion can only be a system of vertices, the coordinates of which are integers admitting a common factor r, i. e. the set of vertices of a net of measure polytopes with edge r. So we have shown now that the system of points V of S'_n must be $(rb_1, rb_2, \ldots, rb_n)$, where the n quantities b_i are arbitrary integers, whilst r is a definite integer. From this result it follows immediately that the frame of the centres C admits the frame symbol

$$(rb_1, rb_2, rb_3, \ldots, rb_n, rb_{n+1}), \ldots\ldots\ldots\ldots \quad F)$$

the arbitrary integers b_i satisfying the condition $\Sigma b_i = 0$. The quantity r, which is the same for the different frames of the same net, may vary from net to net. We call it the *period* of the net.

All the simplex nets proper have similar frames, as their images of points V are similar. [1]) If, as in art. 1, our space S_n is the space $\sum_{i=1}^{n+1} x_i = 1$ lying in a space of operation S_{n+1} and determining on the axes of a given system $O(X_1, X_2, \ldots, X_{n+1})$ of coordinates equal intersepts OA_i, we can say that the nets of S_n, the vertices of which admit with respect to the simplex of coordinates $A_1 A_2 \ldots A_{n+1}$ integer coordinates only, always admit frames *projecting* themselves normally on any of the n- dimensional spaces S'_n of coordinates of S_{n+1} as sets of vertices of systems of measure polytopes of that S'_n.

c) In the third place we prove that *each net of S_n admits a net symbol.*

By combining the zero symbol $(q_1, q_2, \ldots, q_n, 0)$ of the central polytope with the frame symbol $(rb_1, rb_2, \ldots, rb_n, rb_{n+1})$, $\Sigma\, b_i = 0$ of the net we obtain the symbol

$$(rb_1 + q_1, rb_2 + q_2, \ldots\ldots, rb_n + q_n, rb_{n+1}), \ldots\ldots N)$$

where the q_i and r are given integers, whilst for the b_i we can take any system of integers with sum zero. As this symbol contains the coordinates of *all* [2]) the vertices of the net, it is the *net symbol.*

If we write this symbol in the form

$$(rb_1 + q_1, rb_2 + q_2, \ldots . rb_n + q_n, rb_{n+1} + 0)$$

we have got a symbol representing the net decomposed into the repetitions of the central polytope.

---

[1]) We remark already here that later on cases will present themselves which are at variance with this simple result. We will treat these cases — and explain why they appear as exceptions — as soon as they turn up.

[2]) This is only true, if *each* vertex of the net is also vertex of a repetition of the central polytope in the same orientation. So, if the net contains a non central symmetric constituent in two opposite orientations and in each vertex only one of these two differently orientated constituents concurs, the net symbol corresponding to one of these constituents as central polytope would only contain half the number of vertices of the net and would have to be completed by a second symbol giving the other half. In that particular case the system of vertices breaks up into two equivalent parts P and Q with the property that the net is equipollent to itself for any two points of the same half as homologous but congruent with opposite orientation to itself for any two vertices of different halves as homologous. This particularity presents itself in the plane in the case of the net of triangles and dodecagons (fig. 8), already discarded above for an other reason. Here we exclude, also provisionally, all the eventually possible nets where this particularity of the division of the system of vertices into two equivalent systems might present itself.

At first sight it may seem that the introduction of the common factor r, by means of which the frame is only enlarged but not changed in form, is of no avail, as the *scale* of the diagrams is of no importance whatever. But then one overlooks the fact that the frame is enlarged, while the central polytope $(P)^\circ$ remains unaltered. So in the case of the central triangle $A_1 A_2 A_3$ in the plane: if we take $r=1$ we have to deal with the net $N(p_3)$ of fig. 5, whilst the supposition $r=2$ gives the vertices of the net $N(p_3, p_6)$ by means of the triangle $A_1 A_2 A_3$ and its equally orientated repetitions (fig. 9).

This simple example shows in the first place the influence of the period r. But on the other hand it gives a glimpse of the fact that with a given central polytope not all integer values of r lead to existing nets. So the supposition $r=3$ brings already the central triangle $A_1 A_2 A_3$ and its repetitions too far apart. [1])

26. We pursue our investigation in the direction of the last sentence of the preceding article, entering into details about the relationship between the period r and the largest digit q_1 of the zero symbol $(q_1, q_2, \ldots, q_n, 0)$ of the central polytope $(P)^\circ$.

If we call any repetition (P) of the central polytope $(P)^\circ$ corresponding with it in orientation *adjacent* to it, if the distance between their centres C_0 and C is as small as possible [2]), i. e. if the coordinates of C can be deduced from the equal coordinates $x_i = \frac{\Sigma q_i}{n+1}$ of C_0 by altering only *one* pair of coordinates by addition and subtraction of only *one* time r, we find:

"The central polytope and one of its adjacent repetitions *overlap* for $r < q_1$, whilst for $r = q_1$ they are *in contact* and for $r > q_1$ *free* from each other".

Of these three cases of relationship between r and q_1 we consider first the case $r = q_1$, then the two cases $r \neq q_1$ at a time.

Case $r = q_1$. The two adjacent polytopes represented by

$$(q_1, q_2, q_3, \ldots, q_n, 0), \; (r+q_1, -r+q_2, q_3, \ldots, q_n, 0)$$

have all the vertices

---

[1]) Application of the case $r=3$ to the triangle $A_1 A_2 A_3$ gives one of the two sets of triangles of the net of fig. 8, already discarded for two different reasons.

[2]) This is the case if the image V of C (compare the preceding article under b) lies on an axis OX_i at distance r from O.

$$x_1 = q_1, x_2 = 0, \ (x_3, x_4, \ldots, x_{n+1}) = (q_2, q_3, \ldots, q_n)$$

in common; this is immediately evident, if for x_1 and x_2 we take in the first symbol the digits q_1 and 0, in the second $r + 0$ and $-r + q_1 = -r + r$. In general these common vertices define a polytope of $n - 2$ dimensions situated in the space S_{n-2} for which $x_1 = q_1$, $x_2 = 0$, i. e. the two polytopes are *in contact* with each other by a limit $(l)_{n-2}$; as any common vertex of the two polytopes lies at equal distances (radius of the circumscribed spherical space) from the centres C_0 and C, this common $(l)_{n-2}$ lies in the space S_{n-1} normally bisecting $C_0\,C$, i. e. this $(l)_{n-2}$ of contact has the midpoint M of $C_0\,C$ for centre, i. e. the contact by the $(l)_{n-2}$ is *external*. But in one exceptional case, in the case $q_1 = 2$, $q_2 = q_3 = \ldots = q_n = 1$ of the central symmetric polytope $(2\,\overbrace{11\ldots1}^{n-1}0)$, the common limit $(l)_{n-2}$ shrinks together into a single point, the midpoint of $C_0\,C$, as in that case $(q_2, q_3, \ldots, q_n)$ becomes a petrified syllable. At any rate, for $r = q_1$ the net is *mixed*, as the central polytope and one of it adjacent repetitions are *not* in contact by a limit $(l)_{n-1}$.

If for brevity we represent $\dfrac{\Sigma q_i}{n+1}$ by q the coordinates of C_0 and C are

$$\left.\begin{array}{llll} C_0 \ldots\ldots & q & q, q, q, & \ldots\ldots q \\ C \ldots\ldots & r + q, & -r + q, q, q, & \ldots\ldots q \end{array}\right\}.$$

So, according to formula 1) of art. 1, the distance $C_0\,C$ is equal to the period r; this result will be useful in the treatment of the next case.

Case $r \neq q_1$. Let us start from the case $r = q_1$ treated above and vary r. As the relation $C_0\,C = r$ holds always, this variation of r implies a variation of $C_0\,C$, the effect of a translational motion of the repetition (P) of the central polytope $(P)^0$ in the direction $C_0\,C$ if r increases, in the opposite direction $C\,C_0$ if r decreases. In the first case when $C_0\,C$ is enlarged, the polytopes which were either in $(l)_{n-2}$-contact or in point contact, will become *free* from each other. In the second case when $C_0\,C$ diminishes the midpoint M of the new $C_0\,C$ will lie inside both polytopes, i. e. the polytopes will *overlap*. So the theorem is proved.

As we cannot use overlapping polytopes we have to discard all the cases $r < q_1$, i. e. we have to consider q_1 as an inferior limit of r. But if the net symbol — as we suppose — contains *all* the vertices, there is also a superior limit. For in the case $r = q_1 + k$

the distance of the two limits $(l)_{n-2}$ or of the two vertices, which coincided for $r=q$, has now become k and this distance may not surpass unity. So we have also to discard all the cases $r > q_1 + 1$. So the result is that we can only use the values $r = q_1$ and $r = q_1 + 1$, or inversely: the only values of the largest digit q_1 of the zero symbol of the central polytopes are r and $r - 1$. Now as any polytope of the net can be promoted to central polytope we have in general:

THEOREM XII. "Any possible net with period r contains only constituents with zero symbols having for largest digit q_1 either $r-1$ or r. Two adjacent repetitions of a constituent for which $q_1 = r - 1$ are free from each other, whilst two adjacent repetitions of a constituent for which $q_1 = r$ are in contact, in general by a limit $(l)_{n-2}$, but in the particular case $(2\overline{11\ldots1}^{\,n-1}0)$ by a point."

27. But now unexpectedly a difficulty presents itself. In the case $q_1 = r$ any two adjacent repetitions of a definitely orientated constituent are in $(l)_{n-2}$-contact or in point contact, in the case $q_1 = r - 1$ these two repetitions are free from each other. In both cases we need other constituents to fill up gaps, in other words all the nets are *mixed*. But this result is at variance with the existence of the net $N(p_6)$ in the plane, of the net $N(tO)$ in space. So we have to look out for a way out of this difficulty. This way will present itself immediately, if we examine how to find the other constituents of a net, the central polytope and the period of which are given.

Let the zero symbol of the central polytope $(P)^0$ of a net with period r be represented once more by $(q_1, q_2, \ldots, q_n, 0)$, where we have either $q_1 = r$ or $q_1 = r - 1$. Then we can ask by what processes we can deduce from the symbol

$$(rb_1 + q_1, rb_2 + q_2, \ldots, rb_n + q_n, rb_{n+1} + 0),$$

representing the net decomposed into the repetitions of the central polytope, other constituents. There are two of these processes completing each other in this sense that the first can be used in the case $q_n = 0$, the second in the case $q_n = 1$.

1°. In the case of the zero symbol $(q_1, q_2, \ldots, q_{n-1}, 0, 0)$ containing more than one zero we can write the decomposing symbol

$$(rb_1 + q_1, rb_2 + q_2, \ldots, rb_{n-1}, + q_{n-1}, rb_n + 0, rb_{n+1} + 0), \Sigma b_i = 0$$

in the form

$$(rb_1+q_1, rb_2+q_2, \ldots, rb_{n-1}+q_{n-1}, rb_n+0, r(b_{n+1}-1)+r), \Sigma b_i=0$$

by allowing r units to pass from the unmovable part rb_{n+1} of the digit $rb_{n+1}+0$ to the permutable part; for by that variation we alter only the *grouping* of the vertices of the net to vertices of polytopes but not the total system of vertices of the net. If now we write b_0 for $b_{n+1}-1$ and put the permutable digit r foremost we get

$$(rb_0+r, rb_1+q_1, rb_2+q_2, \ldots, rb_{n-1}+q_{n-1}, rb_n+0), \Sigma b_i=-1,$$

bringing to the fore the constituent with the zero symbol $(r, q_1, q_2, \ldots q_{n-1}, 0)$.

2°. In the case of the zero symbol $(q_1, q_2, \ldots q_{n-1}, 1, 0)$ containing only one zero an application of the same process leads from

$$(rb_1+q_1, rb_2+q_2, \ldots, rb_{n-1}+q_{n-1}, rb_n+1, rb_{n+1}+0), \Sigma b_i=0$$

to

$$(rb_0+r, rb_1+q_1, rb_2+q_2, \ldots, rb_{n-1}+q_{n-1}, rb_n+1), \Sigma b_i=-1$$

and therefore to the constituent $(r, q_1, q_2, \ldots, q_{n-1}, 1)$, the zero symbol of which is $(r-1, q_1-1, q_2-1, \ldots, q_{n-1}-1, 0)$. In order to obtain this zero symbol we can write the decomposing symbol in the form

$$\begin{gathered}(rb_0+1+r-1, rb_1+1+q_1-1, \ldots, rb_{n-1}+1 \\ +q_{n-1}-1, rb_n+1+0), \Sigma b_i=-1\end{gathered}$$

and pass to an other sum $\Sigma q_i-(n+1)$ of all the digits by omitting the unit of the unmovable part of the digits.

So, if we take notice only of the *zero symbols* of the constituents deduced by means of the two processes, we can word these processes as follows:

1°. "If the zero symbol of the given constituent contains more than one zero, we can replace one of these zeros by r".

2°. "If the zero symbol contains only one zero, we can replace this zero by r and diminish all the digits by unity afterwards".

We now come back to the difficulty about the pure nets stated above. To that end we have to ask under what circumstances one of the two processes leads back to the original constituent; therefore we repeat that:

the first deduces	$(r, q_1, q_2, \ldots, q_{n-1}, 0)$	from	$(q_1, q_2, \ldots q_{n-1}, 0, 0)$,
„ second „	$(r-1, q_1-1, q_2-1, \ldots, q_{n-1}-1, 0)$	„	$(q_1, q_2, \ldots q_{n-1}, 1, 0)$.

As a polytope with a zero symbol with $k-1$ zeros cannot be a repetition of a polytope with a zero symbol with k zeros, the first process does not suit our aim; but the second may do so under the conditions

$$r-1=q_1, q_1-1=q_2, q_2-1=q_3, \ldots, q_{n-2}-1=q_{n-1}, q_{n-1}-1=1,$$

i. e. in the case of the central polytope $(r-1, r-2, \ldots, 2, 1, 0)$, i. e. if we have in S_n the case $(n, n-1, \ldots, 2, 1, 0)$ with $r=n+1$. It is indeed easy to prove that the particular case of the reappearance of the original constituent presents itself, when and only when we have $r=n+1$ and $q_1=n$. For, according to the law of theorem I, $q_1=n$ exacts that the zero symbol contains no two equal digits and under this circumstance the substitution of $n+1$ for zero followed by the diminution of all the digits by unity reproduces the original zero symbol. In art. 30 (page 57 at the top) it will be shown that the suppositions $r=n+1$, $q_1=n$ lead to the unique self space filler of S_n.

But now that the manner in which we have to account for the existing pure simplex nets is secured we have to revise our notion of "constituent of the same kind", if we will keep the analytical theory developed just now in touch with the geometrical facts. According to that theory we are obliged to say that the plane net $N(p_6)$ contains three different *groups* of hexagons, though geometrically all the hexagons are equipollent to each other and therefore of the same *kind*. For in the case $n=2$ the suppositions $r=n+1$, $q_1=n$ give rise to the net with the decomposing symbol $(3b_1+2, 3b_2+1, 3b_3+0)$, $\Sigma b_i=0$, corresponding (fig. 10) to the set of hexagons a with a heavy lined circuit, whilst the net contains two other groups of hexagons which admit alternately thick and thin sides, one group b where the horizontal thick side is below, an other group c where the horizontal thick side is above. So, though we keep saying that the hexagon is a self plane filler, we will consider $N(p_6)$ from an analytical point of view as admitting three different groups of hexagons, using here henceforward the more precise term of "group of constituents" in order to indicate a "set of equipollent polytopes, the vertices of which form together all the vertices of the net, *each vertex taken once*". Only under this extension of our former *kind* of constituents by our now introduced *group* of constituents the theorems XI and XII are *generally true*. If we follow the interpretation of the net $N(tO)$ as a net with one kind of constituent only, we get a frame [1]) dissimilar to that of other nets, with any two adjacent repetitions of the unique constituent in contact by a limit $(l)_{n-1}$, i. e. a face here; these "exceptions" disappear, if we adhere to the analytical idea, according to which $N(tO)$ admits *four groups* of constituents. [2])

28. If we indicate by ρ_p the number of the digits of the zero symbol of the central polytope $(P)^\circ$ leaving the remainder p when divided by r, i. e. if ρ_p represents in general the number of the digits p of the zero symbol, but in the particular case of ρ_o the sum of the numbers of the digits zero and the digits r (the latter being absent in the case $q_1 = r - 1$ of the original zero symbol), we have the:

THEOREM XIII "The two operations stated above which may lead to new constituents of the same net do not affect the *circular* order of succession of the terms of the series $\rho_{r-1}, \rho_{r-2}, \ldots, \rho_1, \rho_0$. This series with the sum $n+1$ will be called "*partition cycle* of $n+1$, mod. r" of the net and be represented by $_r(\rho_{r-1}, \rho_{r-2}, \ldots, \rho_1, \rho_0)_n$".

This theorem is self evident. For the first of the two processes does not affect the series at all, whilst the second transforms it into $\rho_0, \rho_{r-1}, \rho_{r-2}, \ldots, \rho_1$.

We apply the two processes to an example in order to show the circular permutation of the partition and suppose to that end that in space S_6 there *is* a net with the period 4 admitting the constituent **(3222100)**. Then application of the two processes gives successively

			Partition cycle
(3222100)			1312
(4322210)			1312
(4432221)	=	(3321110)	2131
(4332111)	=	(3221000)	1213
(4322100)			1213
(4432210)			1213
(4443221)	=	(3332110)	3121
(4333211)	=	(3222100)	1312

Here every new symbol in the first column is derived by the first process from the one in the line immediately above it which con-

[1]) In art. 39 the system of the centres of *all* the tO of $N(tO)$ will prove to form the vertices of a net of rhombic dodecahedra, which latter net is not of simplex extraction.

[2]) In order to avoid misunderstanding we stipulate expressly that it is not our intention to *replace* the notion of kind of constituent by that of group, but that we wish to stick to the notion of kind of constituent, *complemented* by that of group as soon as the partition cycle (see the next article) is a power cycle (see page 57).

tains a zero, whilst by the second process the symbols of the second column are deduced from those of the first column placed on the same line. In the third column we find the partition cycle 1312 proceeding one step to the right at every application of the second process.

So, *if* the case considered is that of an existing net — which question does not yet interest us here —, this net must admit seven different constituents, i. e. three pairs of oppositely orientated ones

$$\begin{array}{ccc} (3222100) & , & (3321110) \\ (3221000) & , & (3332110) \\ (4322100) & , & (4432210) \end{array}$$

and the central symmetric (4322210).

This example leads us to a general rule about the number of groups of constituents any net is to have; we state it in the form of:

THEOREM XIV. "In general the number of kinds of constituents of a net of S_n is $n+1$; in the case $r=1$ it is n."

The proof of the general case runs as follows. The zero symbol $(q_1, q_2, \ldots, q_{n-1}, q_n, 0)$ of the central polytope we start with passes either into $(r, q_1, q_2, \ldots, q_{n-1}, 0)$ or into $(r-1, q_1-1, q_2-1, \ldots, q_{n-1}-1, 0)$ according to q_n being either zero or one. So, in continuing the application of the two processes of art. 27, at *each* step the digit q_1 moves one place to the right and comes back to its original place after $n+1$ moves. Moreover it reappears there with its *original* value q_1. For the increase by r at the jump from the rear to the front is exactly counterbalanced by the loss of a unit every time when of two unequal adjacent digits the right hand one jumps to the fore, this loss occurring exactly r times; indeed, in the circular permutation of the digits from the left to the right — executed for simplicity for a moment without increasing or decreasing — the zero at the end has to be replaced successively by 1, by 2, ... by $r-1$ and finally in the case $q_1 = r-1$ by zero, in the case $q_1 = r$ by r. So after $n+1$ moves the original zero symbol recurs and the total process has come to a close. [1])

In the exceptional case $r=1$ we find only $q_1 = 1$, i. e. the zero symbol of any constituent can only contain units and zeros. So we can start with the simplex $(1\,\overline{00\ldots0}^{\,n})$ and find successively $(11\,\overline{00\ldots0}^{\,n-1})$, $(111\,\overline{00\ldots0}^{\,n-2})$, etc. But when we have to pass from $(\overline{11\ldots.1}^{\,n}\,0)$

[1]) The process may come to a close sooner. Compare for this exception page 57.

to the next symbol we find by means of the second process $(\overline{00\ldots0}^{\,n+1})$, which falls out. So we only get n kinds of constituents for $r = 1$.

29. We come now to the general rule about simplex nets proper; it can be stated in the following form:

THEOREM XV. "To every possible cyclical partition of $n+1$ corresponds a definite simplex net proper of S_n."

In order to prove the theorem for the general case with $n+1$ and the particular case with n groups of constituents we first of all determine a list containing these different groups of constituents, to be derived from the partition cycle. Then we select from this list a *definite* polytope $(P)_a$ of a definite group and show that this $(P)_a$ is in contact by *any* of its limits $(l)_n^{(a,b)}$ with one and only one other polytope $(P)_b$ of the list, whilst the list contains no polytope overlapping $(P)_a$.

Case $r > 1$. We start from the partition cycle ${}_r(\rho_{r-1}, \rho_{r-2}, \ldots \rho_1, \rho_0)_n$ and deduce from it the net symbol

$$(\overline{ra_i + r - 1}^{\,\rho_{r-1}},\ \overline{ra_i + r - 2}^{\,\rho_{r-2}},\ \ldots\ldots,\ \overline{ra_i + 1}^{\,\rho_1},\ \overline{ra_i + 0}^{\,\rho_0})_{(a_1, a_2, \ldots, a_{n+1})}$$

i. e. the symbol with ρ_{r-1} digits congruent to $r-1$ mod. r, ρ_{r-2} digits congruent to $r-2$ mod. r, etc., the different quotients $a_1, a_2, \ldots, a_{n+1}$ of the division of these digits by r having a sum $\Sigma a_i = 0$, whilst the sum of the remainders $r-1, r-2, \ldots, 0$, i. e. $\sum\limits_{i=1}^{r-1} i\rho_i$ may be represented by k_0.

If we write this symbol in the form

$$(\overline{ra_i + \mathbf{r} - \mathbf{1}}^{\,\rho_{r-1}},\ \overline{ra_2 + \mathbf{r} - \mathbf{2}}^{\,\rho_{r-2}},\ \ldots,\ \overline{ra_i + \mathbf{1}}^{\,\rho_1},\ \overline{ra_i + \mathbf{0}}^{\,\rho_0})_{(a_1, a_2, \ldots\ldots a_{n+1})}$$

and permutate only the remainders $r-1$, $r-2, \ldots, 0$, the net is decomposed into the group of constituents to which the central polytope belongs; but we can have this rather complicated symbol in our mind quite as well if we simplify it by omission of the unmovable parts of the digits. So the first line of the following list repeats the group of constituents to which the central polytope belongs, while the other lines give all the other groups of constituents, deduced from the "central group" in the manner and order of succession of the preceding article.

	Σa_i	k
$(\quad \overset{\rho_{r-1}}{\overline{r-1}}, \overset{\rho_{r-2}}{\overline{r-2}}, \dots, \overset{\rho_1}{\overline{1}}, \overset{\rho_0}{\overline{0}}\ )$	0	k_0
$(\overset{1}{\overline{r}}, \overset{\rho_{r-1}}{\overline{r-1}}, \overset{\rho_{r-2}}{\overline{r-2}}, \dots, \overset{\rho_1}{\overline{1}}, \overset{\rho_0-1}{\overline{0}})$	-1	k_0+r
. . . .	. . . .	. . . . . .
$(\quad \overset{\rho_0}{\overline{r}}, \overset{\rho_{r-1}}{\overline{r-1}}, \dots, \overset{\rho_2}{\overline{2}}, \overset{\rho_1}{\overline{1}})$	$-\rho_0$	$k_0+\rho_0 r$
$(\overset{1}{\overline{r+1}}, \overset{\rho_0}{\overline{r}}, \overset{\rho_{r-1}}{\overline{r-1}}, \dots, \overset{\rho_2}{\overline{2}}, \overset{\rho_1-1}{\overline{1}})$	$-(\rho_0+1)$	$k_0+(\rho_0+1)r$
. . . .	. . . .	. . . . . .
. . . .	. . . .	. . . . . .
$(\quad \overset{\rho_{r-2}}{\overline{2r-2}}, \overset{\rho_{r-3}}{\overline{2r-3}}, \dots, \overset{\rho_0}{\overline{r}}, \overset{\rho_{r-1}}{\overline{r-1}})$	$\rho_{r-1}-(n+1)$	$k_0+(n+1-\rho_{r-1})r$
$(\overset{1}{\overline{2r-1}}, \overset{\rho_{r-2}}{\overline{2r-2}}, \overset{\rho_{r-3}}{\overline{2r-3}}, \dots, \overset{\rho_0}{\overline{r}}, \overset{\rho_{r-1}-1}{\overline{r-1}})$	$\rho_{r-1}-(n+2)$	$k_0+(n+2-\rho_{r-1})r$
. . . .	. . . .	. . . . . .
$(\overset{\rho_{r-1}-1}{\overline{2r-1}}, \overset{\rho_{r-2}}{\overline{2r-2}}, \overset{\rho_{r-3}}{\overline{2r-3}}, \dots, \overset{\rho_0}{\overline{r}}, \overset{1}{\overline{r-1}})$	$-n$	k_0+nr

For each of these groups of constituents have been indicated in a second column the value of Σa_i, in a third column the value of $k = \sum_{i=1}^{r-1} i\rho_i$. Moreover, in order to point out the regularity of the process by means of the variation of these two sums, the use of the zero symbol has been sacrificed for a moment, i. e. the diminution of the digits by unity every time as the last zero is replaced by r (exacted by the second of the two processes of the preceding article) is not executed here, which implies that a digit h jumping to the fore becomes $h+r$. Here at each step Σa diminishes by a unit and k increases by r. [1])

But in the selection of a definite polytope $(P)_a$ of the list we return to the zero symbol and suppose

1) that the cyclical permutation of the partition cycle from which $(P)_a$ has been derived begins by ρ_{l-1} and winds up in ρ_l,

2) that ρ_r of the ρ_l zeros have been replaced by r,

3) that the equation of the space S_{n-1} containing $(l)^{a,b}_{n-1}$ is

[1]) This relation also holds when we pass from the last group of constituents to the first, when we diminish all the a_i by unity at the transition. From this point of view we can introduce the notion of "cycle of constituents".

determined by making the sum $x_1 + x_2 + \ldots + x_\nu$ maximum, the ν digits which are to make that sum maximum consisting of ρ_r times r, ρ_{l-1} times $r-1$, etc. and ρ_μ of the ρ_m digits $r-l+m$.

Under these circumstances the polytope $(P)_a$ is represented by the symbol

$$\Big(\overset{\rho_r}{\overline{r}}\,,\ \overset{\rho_{l-1}}{\overline{r-1}}\,,\ldots,\overset{\rho_m}{\overline{r-l+m}}\,,\ \overset{\rho_{m-1}}{\overline{r-l+m-1}}\,,\ldots,\ \overset{\rho_0}{\overline{r-l}}\,,\ \overset{\rho_{r-1}}{\overline{r-l-1}}\,,\ldots,\ \overset{\rho_{l+1}}{\overline{1}}\,,\ \overset{\rho_l-\rho_r}{\overline{0}}\Big)_{(a_1,\,a_2\,\ldots\ldots\,a_{n+1})},$$

while this symbol passes into that of $(l)^{a,b}_{n-1}$ by the introduction of intermediate brackets between the digits $a_\nu r + r - l + m$ and $a_{\nu+1} r + r - l + m$, i. e. $(l)^{a,b}_{n-1}$ is represented by

$$\Big(\overset{\rho_r}{\overline{r}}, \overset{\rho_{l-1}}{\overline{r-1}},..,\overset{\rho_\mu}{\overline{r-l+m}}\Big)_{(a_1,a_2,\ldots a_\nu)}\Big(\overset{\rho_m-\rho_\mu}{\overline{r-l+m}},\overset{\rho_{m-1}}{\overline{r-l+m-1}},..,\overset{\rho_0}{\overline{r-l}},\overset{\rho_{r-1}}{\overline{r-l-1}},\ldots,\overset{\rho_{l+1}}{\overline{1}},\ \overset{\rho_l-\rho_r}{\overline{0}}\Big)_{(a_{\nu+1},\,a_{\nu+2},\,\ldots\,a_{n+1})}$$

under the condition that of the two parts of this symbol the first refers to the coordinates $x_1, x_2, \ldots, x_\nu$ and the second to $x_{\nu+1}, x_{\nu+2}, \ldots, x_{n+1}$.

The determination of a second polytope $(P)_b$ of the list containing $(l)^{a,b}_{n-1}$ as limit must be guided by the remark that in *each* of the two parts of the symbol of $(l)^{a,b}_{n-1}$ considered for itself we may transfer the *same* amount from the unmovable parts of the digits to the permutable ones. But in order to obtain a symbol satisfying the law of theorem I, when the intermediate brackets are omitted, we have moreover to select these two amounts in such a way as to obtain a set of permutable parts containing $n+1$ integers distributed over r different ones, succeeding one another with differences unity. So we can *either* diminish all the digits $r, r-1, \ldots, r-l+m$ included between the first pair of brackets by r, counterbalancing this by increasing $a_1, a_2, \ldots, a_\nu$ by unity, *or* — which comes to the same — increase all the digits $r-l+m$, $r-l+m-1, \ldots, 0$ included between the second pair of brackets by r, counterbalancing this by diminishing $a_{\nu+1}, a_{\nu+2}, \ldots, a_{n+1}$ by unity; that these two results differ in form only can be shown by remarking that the first passes into the second if we increase all the digits $0, -1, .., -l+m, r-l+m, r-l+m-1, .., 0$ by r, counterbalancing it by diminishing $a_1+1, a_2+1, \ldots, a_\nu+1, a_{\nu+1}, a_{\nu+2}, \ldots, a_{n+1}$ by unity. So we find one and always one second polytope $(P)_b$ with $(l)^{a,b}_{n-1}$ as limit, represented by the symbol:

$$\Big(\overset{\rho_r}{\overline{0}}, \overset{\rho_{l-1}}{\overline{-1}},..,\overset{\rho_\mu}{\overline{-l+m}},\overset{\rho_m-\rho_\mu}{\overline{r-l+m}},\overset{\rho_{m-1}}{\overline{r-l+m-1}},..,\overset{\rho_0}{\overline{r-l}},\overset{\rho_{r-1}}{\overline{r-l-1}},..,\overset{\rho_{l+1}}{\overline{1}},\ \overset{\rho_l-\rho_r}{\overline{0}}\Big)_{(a_1+1,\,a_2+1,\ldots,\,a_\nu+1,\ a_\nu+1,\,a_\nu+2,\,\ldots\,a_n+1)},$$

We now pass to the determination of the centres G_a, G_b, G_l of $(P)_a$, $(P)_b$, $(l)_{n-1}^{a,b}$ which are collinear, as $G_l G_a$ and $G_l G_b$ are normal in the centre G_l to the space S_{n-1} bearing $(l)_{n-1}^{a,b}$, in order to prove that G_l lies between G_a and G_b, i. e. that $(P)_a$ and $(P)_b$ lie on different sides of that space S_{n-1}. If we consider the x_{n+1} of these three points we find for

$$G_a \ldots ra_{n+1} + \frac{1}{n+1} \Big\{ r\rho_r + (r-1)\rho_{l-1} + \ldots + \rho_m(r-l+m)$$
$$+ \ldots + \rho_0(r-l) + \rho_{r-1}(r-l-1) + \ldots + \rho_{l+1} \Big\},$$

$$G_b \ldots ra_{n+1} + \frac{1}{n+1} \Big\{ -\rho_{l-1} - 2\rho_{l-2} - \ldots - \rho_\mu(l-m)$$
$$+ (\rho_m - \rho_\mu)(r-l+m) + \ldots + \rho_0(r-l) + \rho_{r-1}(r-l-1) + \ldots + \rho_{l+1} \Big\},$$

$$G_l \ldots ra_{n+1} + \frac{1}{n+1} \Big\{$$
$$(\rho_m - \rho_\mu)(r-l+m) + \ldots + \rho_0(r-l) + \rho_{r-1}(r-l-1) + \ldots + \rho_{l+1} \Big\}.$$

Now in comparing the three values $x^{(a)}$, $x^{(b)}$, $x^{(a,b)}$ of x_{n+1} we can omit the common part ra_{n+1}. But then if we write y_{n+1} for $x_{n+1} - ra_{n+1}$ it is evident that we have $y^{(b)} < y^{(a,b)} < y^{(a)}$. For $y^{(a,b)}$, $y^{(a)}$, $y^{(b)}$ are arithmetic means, $y^{(a,b)}$ of a series S of positive integers $1, 2, \ldots, r-l+m$, each of them taken a certain number of times, $y^{(a)}$ of an other series of integers consisting of S and of positive numbers $r-l+m$, $r-l+m+1$, ..., $r-1$, r equal to or larger than the largest of S, $y^{(b)}$ of a third series of integers consisting of S and negative numbers. So G_a and G_b lie on different sides of the space S_{n-1} bearing $(l)_{n-1}^{(a,b)}$, i. e. the system of polytopes contained in the list admits no holes, every limit $(l)_{n-1}$ of an arbitrarily chosen polytope P_a being covered by an other polytope P_b.

We have still to show that no two polytopes of the net can overlap. We do so by simply remarking that the vertices of the polytopes of *any* group of constituents form together the total system of vertices of the symbol derived from the partition cycle [1]) (see above at the beginning of the treatment of the case $r > 1$ under consideration), as each of the groups of constituents of the list has been deduced from that symbol according to the processes of art. 27. For — while overlapping of polytopes of the *same* group

[1]) This fact can also be put on duty in the proof about the position of two polytopes with common $(l)_{n-1}$ on different sides of that limit.

is already excluded (art. 26) — this remark excludes overlapping of any two polytopes, as we can derive from it that not a single vertex can lie inside any polytope of any group of constituents.

Case $r = 1$. In this case the enumeration of the zero symbols

$$(1\,\overbrace{00\,..\,0}^{n}),\ (11\,\overbrace{00\,..\,0}^{n-1}),\ \ldots,\ (\overbrace{11\,..\,1}^{p}\,\overbrace{00\,..\,0}^{n-p+1}),\ \ldots,\ (\overbrace{11\,..\,1}^{n}\,0)$$

of the n groups of constituents is much simpler. Moreover the polytopes of the first group and those of the last admit only one kind of limits $(l)_{n-1}$ viz. simplexes, those of any other group only two, limits $(l)_{n-1}$ with respect to $(1\,\overbrace{00\,..\,0}^{n})$ of the lowest and of the highest import.

Here overlapping is also excluded, as can be shown by means of the same remark used above. Here the polytope $(P)_a$ can be represented by

$$(a_1 + \mathbf{1},\ a_2 + \mathbf{1},\ \ldots\ldots,\ a_p + \mathbf{1},\ a_{p+1} + \mathbf{0},\ \ldots\ldots,\ a_{n+1} + \mathbf{0}),$$

its limit $(l)_{n-1}^{(a,b)}$ lying in the space S_{n-1} with the equation $x_1 = a_1 + 1$ by

$$(a_1 + 1)\ (a_2 + \mathbf{1},\ \ldots\ldots,\ a_p + \mathbf{1},\ a_{p+1} + \mathbf{0},\ \ldots\ldots,\ a_{n+1} + \mathbf{0}),$$

the second polytope $(P)_b$ of the list containing also this limit by

$$(a_1 + 1 + \mathbf{0},\ a_2 + \mathbf{1},\ \ldots\ldots,\ a_p + \mathbf{1},\ a_{p+1} + \mathbf{0},\ \ldots\ldots,\ a_{n+1} + \mathbf{0}).$$

For the rest the proof can be copied from that given above.

Now that theorem XV has been proved we go back to the polytopes $(P)_a$ and $(P)_b$ in contact with each other by a common limit $(l)_{n-1}$ in order to indicate a relation between the *import* of that common limit with respect to $(P)_a$ and $(P)_b$ on one hand and the *places* of the groups of constituents, to which $(P)_a$ and $(P)_b$ belong, in the list of polytopes of the general case $r > 1$ on the other. To that end we indicate by $G_1, G_2, \ldots, G_n, G_{n+1}$ successively the kinds of polytopes represented by the first, the second, ... the last but one, the last line of the list of polytopes and — as on page 17 — by the symbols $g_0, g_1, g_2, \ldots, g_{n-1}$ in relation to any n-dimensional polytope limits $(l)_{n-1}$ of vertex import, edge import, face import, ..., the highest import of that polytope. Then we find:

THEOREM XVI. "If two polytopes of the net, $(P)_a$ of group G_k and $(P)_b$ of group $G_{k-\nu}$, are in $(l)_{n-1}$ contact, the common limit is a $g_{\nu-1}$ for $(P)_a$ and a $g_{n-\nu}$ for $(P)_b$".

The proof of this theorem lies in the remark that $(P)_b$, according to the subscript $(a_1 + 1, a_2 + 1, \ldots, a_\nu + 1, a_{\nu+1}, a_{\nu+2}, \ldots, a_{n+1})$

of the symbol representing it, gives for Σa a value surpassing the corresponding sum for $(P)_a$ by ν, on account of the units added to the digits $a_1, a_2, \ldots a_\nu$. As Σa diminishes by a unit if we go down one line in the list, our $(P)_b$ belongs to group $G_{k-\nu}$ if $(P)_a$ belongs to group G_k. Now we know that the sum of the ν definite coordinates is maximum for $(P)_a$ and minimum for $(P)_b$ in the space bearing the common limit, which proves that this limit is a $g_{\nu-1}$ for $(P)_a$ and a $g_{n-\nu}$ for $(P)_b$.

By means of this theorem we can indicate the group to which belong the polytopes touching a given polytope along its limits of a given import; if $(P)_a$ belongs to group G_k and it has limits g_h, it is touched along these limits by polytopes $(P)_b$ belonging to group G_{n-h-1}.

The theorem also holds for the case $r = 1$, where the zero symbols of the successive groups $G_1, G_2, \ldots, G_p, \ldots, G_n$ are

$$(1\overbrace{00\,..\,0}^{n}),\ (11\overbrace{00\,..\,0}^{n-1}),\ \ldots,\ (\overbrace{11\,..\,1}^{p}\overbrace{00\,..\,0}^{n-p+1}),\ \ldots,\ (\overbrace{11\,..\,1}^{n}0).$$

There we can state it in this form: "Any polytope $(P)_a$ of a net for which $r = 1$ is touched along its limits of vertex import by polytopes of the preceding, along its limits of highest import by polytopes of the following group".

30. We now apply the theorem XV to the cases $n = 2, 3, 4, 5$ and put the results on record in the second table added at the end of this memoir.

First one word about the general plan of this table. Horizontally it is divided into four parts, corresponding successively to the cases $n = 2, 3, 4, 5$. Vertically it breaks up into seven columns with the first five of which we are concerned here. The first column, indicating the rank number of the net, enables us to individualize each net by a very short symbol, consisting of the value of n in italian figures, bearing at the right a roman rank index, 2_{III} indicating the net of hexagons in the plane. The second column gives the value of the period r from 1 to $n + 1$ upward. The third column contains the partition cycle, represented by that permutation in which the first digit is as small as possible. The fourth column brings the net symbol corresponding to that cyclical permutation of the partition cycle, whilst the fifth is concerned with the zero symbols of the different groups of constituents. With respect to these columns — the others will be explained in part G — we have to insert a few remarks.

In the cases 2_{III}, 3_V, 4_{VII}, 5_{XII}, where the partition cycle consists of $n+1$ units, we find back the self space fillers of simplex extraction, $p_6 = (210)$, $tO = (3210)$, etc. The net symbol of these self space fillers is characterized by the property that its $n+1$ digits, when divided by $n+1$, leave all possible remainders $n, n-1, \ldots, 1, 0$, each remainder once.

In the case of the partition 2,2 of the net 3_{III} an other particularity presents itself: in the process of formation of new zero symbols we fall back at the second step on the original symbol

$$(1100), (2110), (2211) = (1100).$$

This is due to the fact that the partition cycle consists of (two) equal parts. So this particularity repeats itself in the cases 5_{IV}, 5_{VII} and 5_X with the partition cycles (3, 3), (2, 2, 2) and (1, 2, 1, 2), in general if we have $n+1 = uv$ and the partition cycle consists of the v digits $a_1, a_2, \ldots a_v$, this set of v digits being repeated in the same order of succession so as to have u sets. In the latter case where the partition cycle is said to be "a cycle of power v", we find only $\frac{n+1}{v} = u$ constituents of different form; it includes the self space fillers, which present themselves for $v = n+1$, $u = 1$ [1]).

We point out two other particularities occurring for the first time in S_5. The two partition cycles 1, 2, 3 and 1, 3, 2 of which the second written in the form 3, 2, 1 is the inversion of the first, have been inscribed both as 5_{VI}, as these two nets, differing only in orientation with respect to the simplex of coordinates, are essentially the same. On the other hand the two nets 5_{IX} and 5_X are essentially different, though the four digits of the partition cycle are two times 2 and two times 1 for both.

The fifth column forms the principal part of the table. As to the number of different constituents of a net in S_n this column is subdivided into $n+1$ small ones. In the first of these $n+1$ small columns is placed the central polytope; on each horizontal line the polytope mentioned in a following small column is deduced by the two processes of art. 27 from that in the immediately preceding one. For brevity we have only inscribed the *geometrically* different forms, using from $n = 4$ upward the symbol e_0 explained at the end of art. 21 and indicating the orientation by means of the signs.

[1]) If we wish to indicate the number of constituents of different form and orientation we can complete theorem XIV by saying that this number is n for $r = 1$ and $\frac{n+1}{v}$ if the partition cycle is a cycle of power v.

We now come back to the particularity of the nets $5_{VI}{}^{a}$, $5_{VI}{}^{b}$ hinted at above. We see now at a glance that these two nets are one and the same, the polytope of the p^{th} small column of the one being equal but oppositely orientated to the polytope of the $7 - p^{th}$ small column of the other ($p = 1, 2, \ldots, 6$). So in each the six constituents present themselves in only one of the two possible orientations, which implies that none of them can be central symmetric, as in the range of the $n + 1$ different constituents of a net of S_n in the order of succession obtained by a regular application of the processes of art. 27 *adjacent* polytopes of a central symmetric one differ in orientation only. Or otherwise: two opposite limits $(l)_{n-1}$ of a central symmetric constituent are covered by two congruent but oppositely orientated polytopes, i. e. if we project on the line CC' joining the centre C of any polytope of the net to the centre C' of any limit $(l)_{n-1}$ of this polytope all the polytopes of the net the centres of which lie on that line, the projection of any central symmetric polytope with its centre on CC' acts as a "turn table" with respect to that projection.

31. The simple rule of theorem XV enables us to extend the list of nets to any value of n we like. So we would find for $n = 6$ and $n = 7$ respectively the 17 and the 29 cases represented as to their roman rank index, their partition cycle and the character of their constituents in the following small tables, where the three subdivisions of each last column give successively the number of central symmetric constituents, the number of the asymmetric constituents occurring in pairs and the number of asymmetric constituents occurring in one orientation only.

$n = 6$.

I	7		6		VII	133	1	6		XIII	11113	1	6	
II	16	1	6		VIII	223	1	6		XIV	11122	1	6	
III	25	1	6		IX	1114	1	6		XV	11212	1	6	
IV	34	1	6		X	1123			7	XVI	111112	1	6	
V	115	1	6		XI	1213	1	6		XVII	1111111	1		
VI	124			7	XII	1222	1	6						

$n = 7$.

I	8	1	6		XI	1115		8		XXI	11213			8
II	17		8		XII	1124			8	XXII	11222	2	6	
III	26	2	6		XIII	1214	2	6		XXIII	12122	2	6	
IV	35		8		XIV	1133	2	6		XXIV	111113		8	
V	44	2	2		XV	1313		4		XXV	111122	2	6	
VI	116	2	6		XVI	1223		8		XXVI	111212		8	
VII	125			8	XVII	1232		8		XXVII	112112	2	2	
VIII	134			8	XVIII	2222	2			XXVIII	1111112	2	6	
IX	224	2	6		XIX	11114	2	6		XXIX	11111111	1		
X	233	2	6		XX	11123			8					

Under $n = 6$ no cases of a power partition cycle (except 6_{XVII}, the self space filler) present themselves, as $n+1$ is prime here. For $n = 7$ we find besides 7_{XXIX} still 7_V, 7_{XV}, 7_{XXVII} with $v = 2$ and 7_{XVIII} with $v = 4$.

Instead of pushing this general investigation any further we will give here the generalizations of the three nets of the plane to space S_n.

THEOREM XVII. "In space S_n the central symmetric polytope with the zero symbol $(n, n-1, n-2, \ldots, 1, 0)$, represented also by the expansion symbol $e_1 e_2 e_3 \ldots e_{n-2} e_{n-1} S(n+1)$, is the only self space filler of simplex extraction. This unique geometric constituent of the net presents itself in $n+1$ different groups with the property that the vertices of the constituents of each group form the vertices of the net, each vertex taken once, in other words: that no two constituents of the same group have a vertex in common. In this "cycle of constituents" (compare the footnote of art. 29) formed by these groups $G_0, G_1, \ldots, G_n$ any polytope of the group G_k is touched along its limits g_0 of vertex import by $(n+1)_1$ polytopes of group G_{k-1}, along its limits g_1 of edge import by $(n+1)_2$ polytopes of group G_{k-2}, etc. So, in order to perform the task of colouring the polytopes of this net in such a way that any two polytopes bearing the same colour are free from each other (a polydimensional bud of the renowned shrub "map colouring") it will be necessary and sufficient to have at hand $n+1$ different paints, one for the polytopes of each group".

"The n-dimensional angle of the self space filler of S_n is $\dfrac{2^n}{n+1}$ right ones".

What is said above about contact of constituents of different groups by a limit $(l)_{n-1}$ is a mere application [1]) of theorem XVI.

As the net of measure polytopes M_n of S_n shows, the n-dimensional space round any point contains 2^n right angles, if the n-dimensional angle of M_n is called a right one. As all the n-dimensional angles of the self space filler are equal and $n+1$ of these polytopes concur in a vertex of the net, the n-dimensional angle of the self space filler is $\frac{2^n}{n+1}$ right angles.

THEOREM XVIII. "The net of S_n with the period unity admits the n constituents $(\overbrace{11\,..\,1}^{p}\ \overbrace{00\,..\,0}^{n-p+1})$, $(p = 1, 2, \ldots, n)$ consisting of $\frac{n}{2}$ constituents in both orientations for n even and of $\frac{n-1}{2}$ constituents in both orientations and one central symmetric constituent for n odd".

THEOREM XIX. "The net of S_n with the partition cycle $1, n$ admits the $n+1$ constituents $(1\ \overbrace{00\,..\,0}^{n})$, $(\overbrace{22\,..\,2}^{p}\ 1\ \overbrace{00\,..\,0}^{n-p})$ for $p = 1, 2, .., n-1$ and $(\overbrace{11\,..\,1}^{n}0)$ consisting of $\frac{n}{2}$ constituents in both orientations and one central symmetric constituent for n even and of $\frac{n+1}{2}$ constituents in both orientations for n odd".

These theorems immediately follow by specializing the general results. We give them here expressis verbis as we will indicate later on an other deduction of them. [2])

32. A survey of the results for $n = 2, 3, \ldots, 7$ suggests one or two general remarks.

The first can be stated in the form of:

THEOREM XX. "Every simplex polytope partakes in the formation of two nets. This is true without any reserve for the central symmetric constituents, it is also true for *each* of the two different positions of an asymmetric constituent."

---

[1]) It is an easy task to demonstrate theorem XVII by itself by showing that the image points of the centre of the central polytope with respect to the spaces S_{n-1} bearing the limits $(l)_{n-1}$ as mirrors form the centres of the polytopes in $n-1$-dimensional contact with the central polytope. We consider this verification as a useful exercise, even in the special case $n = 3$ of ordinary space.

[2]) Though we do not wish to push the general investigation any further we still mention the following theorem:

"The net of S_{2^n-1} with the power partition cycle 2^n is built up of two central symmetic constituents only".

Let us take the polytope (221000) of S_5. This polytope can belong to a net the period r of which is either 2 or 3. In the first case we find ${}_2(221000)$ which can be reduced to ${}_2(100000)$ by going two steps backward; in the second case we have ${}_3(221000)$ which passes into ${}_3(211100)$ also by going two steps backward.

Or, let us go back to the constituent

$$(\overset{\rho_r}{\overline{r}},\ \overset{\rho_{l-1}}{\overline{r-1}},\ \ldots\ldots,\ \overset{\rho_0}{\overline{r-l}},\ \overset{\rho_{r-1}}{\overline{r-l-1}},\ \ldots\ldots,\ \overset{\rho_{l+1}}{\overline{1}},\ \overset{\rho_l-\rho_r}{\overline{0}}\ )$$

used in art. 29 in order to make the proof as general as possible. This constituent can belong to two nets, one with the period r, an other with the period $r+1$; the two partition cycles of these nets are

$$\left.\begin{array}{r} {}_r(\rho_{l-1}, \rho_{l-2}, \ldots, \rho_0, \rho_{r-1}, \ldots, \rho_{l+1}, \quad \rho_l \quad) \\ {}_{r+1}(\rho_r, \rho_{l-1}, \rho_{l-2}, \ldots, \rho_0, \rho_{r-1}, \ldots, \rho_{l+1}, \rho_l - \rho_r) \end{array}\right\}$$

and may be reduced to

$$\left.\begin{array}{r} {}_r(\rho_{r-1}, \rho_{r-2}, \ldots, \rho_{l+1}, \quad \rho_l \quad, \rho_{l-1} \ldots, \rho_1, \rho_0) \\ {}_{r+1}(\rho_r, \rho_{r-1}, \rho_{r-2}, \ldots, \rho_{l+1}, \rho_l - \rho_r, \rho_{l-1} \ldots, \rho_1, \rho_0) \end{array}\right\}.$$

In the case of an asymmetric constituent it may happen as we have seen that a definitely orientated one occurs in two different nets, if we consider as different two nets as 5_{VI}^{a}, 5_{VI}^{b} which are each others reversions. So under this point of view the two positions of ${}_3(221000)$ occur together in three different nets. But the statement of the theorem about *each* of the two positions of an asymmetric constituent holds under any point of view.

A second remark refers to the expansion symbols used in the table. In order to bring the two different orientations of the asymmetric constituents into evidence we have introduced the expansion symbols provided with the negative sign. But the law of succession of the different constituents of each net proceeding in the list from column to column would have been much more evident if we had stuck to expansion symbols without sign. Then the order of succession in the case of net 5_I would have been e_0, e_1, e_2, e_3, e_4 leading to the supposition that in general at each step the index of each e increases by unity, an illusion which is already destroyed by the series $e_0, e_0e_1, e_1e_2, e_2e_3, e_3e_4, e_4$. At any rate this second remark places us before the question by which rule the expansion symbols of the constituents of a net can be deduced from the partition cycle. The answer may be given in the form of:

THEOREM XXI. "The constituents of the net of S_n corresponding to the partition cycle ${}_r(\rho_{r-1}, \rho_{r-2}, \ldots, \rho_0)$ are found by applying to the system of digits consisting of the series

$$-1, \rho_{r-1}-1, \rho_{r-1}+\rho_{r+2}-1, \ldots\ldots, n-\rho_0$$

preceded by the repetition of all its terms after having subtracted $n+1$ from each (of which repetition the negative terms with an absolute value surpassing n may be omitted) a number of n times the process of increasing all the digits by unity and throwing out (as soon as they appear) digits surpassing $n-1$ and (afterwards when the process is finished) all the negative digits. Then the $n+1$ rows obtained represent the indices of the e operations to be applied to $(\overline{00\ldots0}^{\,n+1})$ in order to obtain the expansion symbols of the constituents".

Before proving this general rule we elucidate its meaning by applying it to an example, for which we choose the case $5_{VI}{}^{b}$. Here the series is $-1, 0, 3$ which has to be proceeded by -3. So, if we indicate in heavy type the figures which are to be kept, the operation is

$$\begin{array}{rrrr}
-3 & -1 & \mathbf{0} & \mathbf{3} \\
-2 & \mathbf{0} & \mathbf{1} & \mathbf{4} \\
-1 & \mathbf{1} & \mathbf{2} & \\
\mathbf{0} & \mathbf{2} & \mathbf{3} & \\
\mathbf{1} & \mathbf{3} & \mathbf{4} & \\
\mathbf{2} & \mathbf{4} & &
\end{array}$$

giving $e_0 e_3, e_0 e_1 e_4, e_1 e_2, e_0 e_2 e_3, e_1 e_3 e_4, e_2 e_4$ for the six expansion symbols of $5_{VI}{}^{b}$. As we have $e_1 e_3 e_4 = -e_0 e_1 e_3$ and $e_2 e_4 = -e_0 e_2$ this series is the same as that inscribed in the table.

The proof of this general theorem splits up into three parts. In the first we show that the top row corresponds to the constituent for which ${}_r(\overline{r-1}^{\,\rho_{r-1}}, \overline{r-2}^{\,\rho_{r-2}}, \ldots, \overline{1}^{\,\rho_1}, \overline{0}^{\,\rho_0})$ is the zero symbol. In the second we explain that the addition of a unit to all the digits corresponds to what happens to the digits in the processes of art. 27 but for the transplantation of the digit at the end to the beginning. In the third we will be concerned with the influence of that transplantation.

The first and the second parts are mere consequences of theorem IX. In the case of the zero symbol ${}_r(\overline{r-1}^{\,\rho_{r-1}}, \overline{r-2}^{\,\rho_{r-2}}, \ldots, \overline{1}^{\,\rho_1}, \overline{0}^{\,\rho_0})$ the unit intervals present themselves behind the digits of rank

$$\rho_{r-1}, \rho_{r-1}+\rho_{r-2}, \ldots\ldots, n+1-\rho_0$$

and this proves in connection with theorem IX the first part. Moreover the circular permutation over one digit to the right happening at each step of the two processes of art. 27 (see the example following theorem XIII) changes the ranks $k+1$ and $k+2$ of two adjacent digits into $k+2$ and $k+3$, i. e. — according to theorem IX — the operation e_{k+1} is still to be applied or has already been performed on the new constituent according to the operation e_k being still to be applied or having already been performed on the original constituent; i. e. if e_k occurs in the e-symbol of the original constituent, e_{k+1} must occur in the e-symbol of the new one, what proves the second part.

In the third part we have to consider all the possible cases of the transplantation of a digit from the end to the beginning; these cases, four in number, are the following:

$(r-1, .., 1, 0)$ becomes $(r-1, r-2, .., 0)$ .. loss of e_{n-1}, gain of e_0,
$(r-1, .., 0, 0)$ „ $(r, r-1, \ldots\ldots, 0)$ .. gain of e_0,
$(r, \ldots\ldots, 1, 0)$ „ $(r-1, r-2, .., 0)$ .. loss of e_{n-1},
$(r, \ldots\ldots, 0, 0)$ „ $(r, r, \ldots\ldots\ldots, 0)$ .. neither loss nor gain.

So we find the two rules:

1°. If e_{n-1} appears in the symbol of the original constituent it falls out in the next one, though an other e_{n-1} may be introduced (if e_{n-2} was contained also in the original symbol).

2°. If the number of the operation factors e_k is $r-1$ the symbol e_0 appears in the next constituent.

But this is also the effect of the operation indicated in the theorem, the first rule being a consequence of the omission of the digits surpassing $n-1$, the second being deducable from the repetition of the series -1, $\rho_{r-1}-1$, $\rho_{r-1}+\rho_{r-2}-1$, etc. If, in order to add still one word about the second rule, G_{p-1}, G_p, G_{p+1} indicate three constituents, consecutive in the sense of the theorem, and the e-symbol of G_p bears only $r-1$ expansion factors, then the e-symbol of G_{p-1} contains $n-1$ and therefore also $-(n+1)+(n-1)=-2$, before the negative digits have been omitted; this -2 becomes -1 for G_p and 0 for G_{p+1}.

33. The theorem XXI enables us to show how the "principal" net of S_n, i. e. the net with the period $r=1$ always inscribed first, can be transformed successively into all the other ones.

The result for S_3 is given in the following table, in the left half in the symbols to be applied to the different constituents of $N(T, O)$, in the right half by the results of this application.

	Constituents			Vertex	Constituents			Vertex
I	e_0	e_1	e_2	gap	T	O	$-T$	gap
II	—	0	1	e_2	T	tT	$-tT$	$-T$
	1	2	—	e_1	tT	$-tT$	$-T$	T
III	2	—	0	e_1	CO	O	CO	O
IV	1	02	1	$e_0 e_2$	tT	tO	$-tT$	CO
	2	0	01	$e_1 e_2$	CO	tT	tO	$-tT$
	12	2	0	$e_0 e_1$	tO	$-tT$	CO	tT
V	12	02	01	—	tO	tO	tO	—

According to this table the principal net $N(T, O)$ can be transformed into the net 3_{II} either by applying to O and $-T$ the operations e_0 and e_1 or by applying to T and O the operations e_1 and e_2; these two transformations are of the same kind, as they pass into each other by interchanging the two sets of tetrahedra and at the same time the two sets of four non adjacent faces of each octahedron in contact with them. Whilst each of the two nets 3_{III} and 3_V can be deduced in one way only, there are three manners of deduction of net 3_{IV}; of these the first stands by itself and the second and the third pass into each other by the indicated interchange of the two sets of tetrahedra, etc.

The table for S_4 is the following

	Constituents				Vertex	Constituents				Vertex
I	e_0	e_1	e_2	e_3	gap	e_0	e_1	e_2	e_3	gap
II	—	0	1	2	e_3	e_0	$e_0 e_1$	$e_1 e_2$	$-e_0 e_1$	$-e_0$
	1	2	3	—	e_0	$e_0 e_1$	$e_1 e_2$	$-e_0 e_1$	$-e_0$	e_0
III	2	3	—	0	e_1	$e_0 e_2$	$-e_0 e_2$	$-e_1$	$e_0 e_3$	e_1
	3	—	0	1	e_2	$e_0 e_3$	e_1	$e_0 e_2$	$-e_0 e_2$	$-e_1$
IV	1	02	13	2	$e_0 e_3$	$e_0 e_1$	$e_0 e_1 e_2$	$-e_0 e_1 e_2$	$-e_0 e_1$	$e_0 e_3$
	3	0	01	12	$e_2 e_3$	$e_0 e_3$	$e_0 e_1$	$e_0 e_1 e_2$	$-e_0 e_1 e_2$	$-e_0 e_1$
	12	23	3	0	$e_0 e_1$	$e_0 e_1 e_2$	$-e_0 e_1 e_2$	$-e_0 e_1$	$e_0 e_3$	$e_0 e_1$
V	2	03	1	02	$e_1 e_3$	$e_0 e_2$	$e_0 e_1 e_3$	$e_1 e_2$	$-e_0 e_1 e_3$	$-e_0 e_2$
	13	2	03	1	$e_0 e_2$	$e_0 e_1 e_3$	$e_1 e_2$	$-e_0 e_1 e_3$	$-e_0 e_2$	$e_0 e_2$
	23	3	0	01	$e_1 e_2$	$-e_0 e_1 e_3$	$-e_0 e_1$	$e_0 e_2$	$e_0 e_1 e_3$	$e_1 e_2$
VI	12	023	13	02	$e_0 e_1 e_3$	$e_0 e_1 e_2$	$e_0 e_1 e_2 e_3$	$-e_0 e_1 e_2$	$-e_0 e_1 e_3$	$e_0 e_1 e_3$
	13	02	013	12	$e_0 e_2 e_3$	$e_0 e_1 e_3$	$e_0 e_1 e_2$	$e_0 e_1 e_2 e_3$	$-e_0 e_1 e_2$	$-e_0 e_1 e_3$
	23	03	01	012	$e_1 e_2 e_3$	$-e_0 e_1 e_3$	$e_0 e_1 e_3$	$e_0 e_1 e_2$	$e_0 e_1 e_2 e_3$	$-e_0 e_1 e_2$
	123	23	03	01	$e_0 e_1 e_2$	$e_0 e_1 e_2 e_3$	$-e_0 e_1 e_2$	$-e_0 e_1 e_3$	$e_0 e_1 e_3$	$e_0 e_1 e_2$
VII	123	023	013	012	—	$e_0 e_1 e_2 e_3$	$e_0 e_1 e_2 e_3$	$e_0 e_1 e_2 e_3$	$e_0 e_1 e_2 e_3$	—

In the following table for S_5 we give only the indices of the e-symbols which are to be applied to the constituent of the principal net in order to obtain all the other ones.

	Constituents					Vertex gap
I	**0**	**1**	**2**	**3**	**4**	
II	—	0	1	2	3	4
	1	2	3	4	—	0
III	2	3	4	—	0	1
	4	—	0	1	2	3
IV	3	4	—	0	1	2
V	1	02	13	24	3	04
	4	0	01	12	23	34
	12	23	34	4	0	01
VI^a	2	03	14	2	03	14
	13	24	3	04	1	02
	34	4	0	01	12	23
VI^b	3	04	1	02	13	24
	14	2	03	14	2	03
	23	34	4	0	01	12
VII	24	3	04	1	02	13

	Constituents					Vertex gap
I	**0**	**1**	**2**	**3**	**4**	
VIII	12	023	134	24	03	014
	14	02	013	124	23	034
	34	04	01	012	123	234
	123	234	34	04	01	012
IX	13	024	13	024	13	024
	24	03	014	12	023	134
	124	23	034	14	02	013
	234	34	04	01	012	123
X	23	34	14	02	013	124
	134	24	03	014	12	023
XI	123	0234	134	024	013	0124
	124	023	0134	124	023	0134
	134	024	013	0124	123	0234
	234	034	014	012	0123	1234
	1234	234	034	014	012	0123
XII	1234	0234	0134	0124	0123	

We only remark here that the number of ways in which the principal net can be transformed into any other one is equal to the number of different cyclical permutations of the partition symbol of the latter, if we make allowance for the fact that two of these ways may be essentially the same as they pass into each other by interchanging the different positions of the constituents without central symmetry, etc.

34. In the outset of this paragraph (art. 22) we have excluded prismatic nets, restricting ourselves to uniform ones; moreover we have disregarded 1°. all cases in which not all the constituents are of simplex extraction (the hybridous nets of art. 24, *b*) and 2°. the nets with two systems of vertices (art. 25, *c*). Now that our general considerations about simplex nets are come to a close we wish to add a few words about these two exceptional groups of nets.

Hybridous nets. In order not to become too circumstantial we only *mention* the decomposing symbols of the three plane hybridous nets indicated in art. 24. They are (in the notation of art. 24):

$$N(p_3;p_4;p_6)\ldots\left\{\begin{array}{l}(a_1(1+\sqrt{3})+\mathbf{1},\ a_2(1+\sqrt{3})+\mathbf{0},\ a_3(1+\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0,\\ (a_1(1+\sqrt{3})+\tfrac{1}{3}\sqrt{3}+\mathbf{0},\ a_2(1+\sqrt{3})+\tfrac{1}{3}\sqrt{3}+\mathbf{1},\\ \qquad a_3(1+\sqrt{3})+\tfrac{1}{3}\sqrt{3}+\mathbf{1}),\ \Sigma a_i=-1.\end{array}\right.$$

$$N(p_6;p_4;p_3)\ldots\quad (a_1(1+\tfrac{1}{3}\sqrt{3})+\mathbf{2},\ a_2(1+\tfrac{1}{3}\sqrt{3})+\mathbf{1},\ a_3(1+\tfrac{1}{3}\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0.$$

$$N(p_6;p_4;p_{12})\ldots\quad (a_1(1+\sqrt{3})+\mathbf{2},\ a_2(1+\sqrt{3})+\mathbf{1},\ a_3(1+\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0,$$

the a_i different from each other with respect to mod. 3.

$$N(p_3;-;p_{12})\ldots\left\{\begin{array}{l}(a_1(2+\sqrt{3})+\mathbf{1},\ a_2(2+\sqrt{3})+\mathbf{0},\ a_3(2+\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0,\\ ((a_1+\tfrac{1}{3}\sqrt{3})(2+\sqrt{3})+\mathbf{0},\ (a_2+\tfrac{1}{3}\sqrt{3})(2+\sqrt{3})+\mathbf{1},\\ \qquad (a_3+\tfrac{1}{3}\sqrt{3})(2+\sqrt{3})+\mathbf{1}),\ \Sigma a_i=-2.\end{array}\right.$$

In space we find two hybridous nets. If $N(A;B)$ represents a net with the polyhedric constituents A, B, the first being of body, the second of vertex import, these two nets and their generation are indicated by the equations

$$e_2N(T,\underline{O})=N(T,RCO;C),\ e_1e_2N(T,\underline{O})=N(tT,tCO;tC),$$

the stroke under O referring to this that the expansions are to be applied to O. Here we even abstain from mentioning decomposing symbols.

Which prospect opens hyperspace for the hunting up of hybridous simplex nets? Very probably none at all. For the most powerful instrument in the plane, the operation e_n, is quite ineffective in ordinary space already, whilst the two hybridous nets of that space are due to the special character of the octahedron as simplex polyhedron.

Nets with two kind of vertices. Neither is it probable that hyperspace contains nets with a constituent occurring in such a manner in two different orientations that any vertex of the net only belongs to one polytope of one of the two sets; for in S_2 and in S_3 the only nets admitting this particularity are precisely hybridous nets, the net $N(p_3,p_{12})$ with respect to p_3, the net $e_2N(T,\underline{O})$ with respect to T and the net $e_1e_2N(T,\underline{O})$ with respect to tT.

35. We finish this paragraph by mentioning other generations of the nets n_I and n_{II} of the theorems XVIII and XIX.

"If we start from a simplex $S(n+1)^{(1)}$ of S_n and complete the $n+1$ spaces S_{n-1} bearing the $n-1$-dimensional limits $S(n)^{(1)}$ to $n+1$ systems of equidistant parallel spaces S_{n-1}, the distance between any two adjacent parallel spaces S_{n-1} being either the height of $S(n+1)^{(1)}$ or twice that height, we get either the net n_I or the net n_{II}."

"If we intersect a net of measure polytopes M_{n+1} of space S_{n+1} by a space S_n normal to a diagonal of a measure polytope and we make that S_n to pass either through a vertex or through the centre of an edge of that polytope we generate either a net n_I or a net n_{II}. In order to obtain nets n_I and n_{II} with length of edge unity we must start in the first case from a net $N(M_{n+1}^{\frac{1}{2}\sqrt{2}})$, in the second case from a net $N(M_{n+1}^{\sqrt{2}})$."

The first generation is easily proved, if we consider the cases of the triangle net $N(p_3)$ and the triangle and hexagon net $N(p_3, p_6)$ of the plane and the cases of the net $N(T, O)$ and the net $N(T, tT)$ of threedimensional space first.

But the second generation, used already in two different papers, [1]) has this great advantage that it furnishes at the same time an easy method of deducing the character of the different constituents. We only trace this method here, as the different constituents have been found otherwise already.

The generation itself shows that all the constituents are sections of the measure polytope M_{n+1} by a space S_n normal to a diagonal. In the first of the two papers quoted just now is demonstrated that "the section of M_{n+1} by a space S_n normal to a diagonal can always be regarded as a part of that space S_n enclosed by two definite, concentric, oppositely orientated, regular simplexes $S(n+1)$ of that space", i. e. that this section is a "regularly truncated regular simplex". Moreover the second of the two papers indicates how to find the amount of these truncations, whilst finally the theorem V, or rather its inversion, teaches how to deduce the zero symbol from the truncation numbers.

F. *Polarity.*

36. If we polarize one of the regular or one of the Archimedian semiregular polyhedra with respect to any concentric sphere, i. e. if we replace that polyhedron characterized by its vertices by the polyhedron included by the polar planes of these vertices with respect to that sphere, we pass from a body with one kind of vertex and edges of the same length to a body with one kind of face and equal dihedral angles. We suppose the simple laws of this "inversion" to be known; so we state only that the lines bearing the edges of the

---

[1]) *Proceedings* of the Academy of Amsterdam, vol. X, pp. 485 and 688.

new body are the reciprocal polars of the lines bearing the edges of the original one, that the vertices of the new body are the poles of the planes bearing the faces of the original one, etc.

This definition of reciprocal polyhedra can be extended immediately to space S_n, where we have to use a concentric spherical space (with ∞^{n-1} points) as polarisator. If this concentric spherical space is the circumscribed one, the limiting spaces S_{n-1} of the new polytope pass through the corresponding vertices of the original one and are normal in these points to the lines joining these points to the centre. We use this most simple disposition in order to show that the property of having one length of edge is transformed into that of the equality of the dispatial angles. To that end we consider (fig. 11) the plane determined by any edge AB and the centre O of the original polytope and remark that the polar spaces S_{n-1} of A and B project themselves onto that plane in the lines a and b, in A and B normal to OA and OB respectively; so the space of intersection S_{n-2} of these two spaces S_{n-1} projects itself in the point C common to a and b and the angle ACB is the dispatial angle between the two spaces S_{n-1}; but this angle is the supplement of the angle AOB which is constant, $OA = OB$ and AB being constant.

By applying this inversion to any semiregular polytope of simplex extraction the characteristic number symbol of it is inverted too. So the symbol (15, 60, 80, 45, 12) of $ce_1\ S(6)$ — see the table — passes into (12, 45, 80, 60, 15). [1])

If, in inverting a definite polytope of simplex descent in S_n, we assume as polarisator the imaginary spherical space for which the vertices of the simplex from which the polytope was derived admit as polar spaces S_{n-1} the opposite limiting spaces S_{n-1} of that simplex, and $(a_1, a_2 \dots a_{n+1})$ is the coordinate symbol of the

---

[1]) It is a very good exercise to deduce the limiting bodies of the reciprocal polytopes of S_4 by polarizing the properties of the edges passing through the vertices of the original simplex polytopes. So, if $Le_1 e_2 e_3$ stands for "the limiting bodies of the reciprocal polytope of $e_1 e_2 e_3 S$ (5)", if $T\,(1_3, 1_{2+1}, 2_{1+1+1})$ indicates a tetrahedron of which one vertex bears three equal edges, one two equal and one unequal edges, two three different edges, if $P^1{}_{\text{deltoid}}$ means pyramid on a deltoid base, $P^2{}_{2+1}$ double pyramid on an isosceles triangle as base, Rh rhombohedron, the results to be obtained are represented by the equations

$$
\begin{array}{rl|rl}
Le_1 = & 20\ T\,(1_2, 3_{2+1}), & Le_1\, e_3 = & L\,(-e_2\, e_3) = 60\ P^1{}_{\text{deltoid}}, \\
Le_2 = & 30\ P^2{}_{2+1}, & Le_1\, e_2\, e_3 = & 120\ T\,(2_{2+1}, 2_{2+1}), \\
Le_3 = & 20\ Rh, & Lce_1 = & 10\ P^2{}_3, \\
Le_1\, e_2 = & 60\ T\,(1_3, 1_{2+1}, 2_{1+1+1}), & Lce_1\, e_2 = & 30\ T\,(4_{2+1}).
\end{array}
$$

polytope in true value coordinates, this symbol also represents all the limiting spaces S_{n-1} of the new polytope in space coordinates, i. e. that these spaces S_{n-1} are represented by the equations $b_1 x_1 + b_2 n_2 + \ldots + b_{n+1} x_{n+1} = 0$, where $b_0, b_1, \ldots, b_{n+1}$ stands for any permutation of the $n + 1$ digits a_i. [1])

Finally it is easy to see in what manner the process of truncation is transformed by inversion. As we have no intention of studying the new system of semiregular polytopes for itself, it may suffice here to remark that truncation at a limit $(l)_p$, which implies the determination of the intersection of a definite space S_{n-1} with the limits $(l)_{p+1}$ passing through that $(l)_p$, is transformed into the assumption of a point in the line joining the centre of a limit $(l)_{n-p-1}$ of the new polytope to the centre O of that polytope, which implies that this point is joined to all the limits $(l)_{n-p-2}$ of that $(l)_{n-p-1}$ by new limits $(l)_{n-p-1}$ replacing the chosen one, etc.

37. We now prove the theorem:

Theorem XXII. "Any polytope $(P)_n$ of simplex descent in S_n has the proporty that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining that vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices V of $(P)_n$ include an other polytope $(P)'_n$, the reciprocal polar of $(P)_n$ with respect to a certain spherical space with O as centre".

In order to prove the theorem we consider the polytope with the zero symbol $(a_1, a_2, \ldots a_{n+1})$ and in connection with it the linear expression

$$a_1 x_1 + a_2 x_2 + \ldots \qquad + a_{n+1} x_{n+1}.$$

This expression assumes the value

$$a^2_1 + a^2_2 + \ldots \qquad + a^2_{n+1}$$

for the pattern vertex V and the same value diminished by unity for each of the points V_i adjacent to V. For we pass from the pattern vertex V to any vertex V_i adjacent to it by making two digits p and $p-1$ interchange places and by this process the sum $p^2 + (p-1)^2$ contained in $\sum\limits_1^{n+1} a^2_i$ is replaced by $p(p-1) + (p-1)p = 2p^2 - 2p$. So the coordinates of the points V_i adjacent to the

[1]) Compare "*Nieuw Archief voor Wiskunde*", vol IX, p. 138—141.

pattern vertex satisfy the equation $\sum_1^{n+1} a_i x_i = \sum_1^{n+1} a^2_i - 1$; as this equation represents a space S_{n-1} normal to the line from the centre of the coordinate simplex to the pattern vertex [1]), the first part of the theorem is proved.

From the regularity of the considered polytope it can be deduced that the distance OP' from the centre O to the space S_{n-1}c ontaining the vertices V_i adjacent to any vertex V does not change with that vertex. So all the vertices of the considered polytope are transformed into the spaces S_{n+1} containing their adjacent vertices by means of an inversion with respect to the spherical space with O as centre and $\sqrt{OP.\,OP'}$ as radius.

38. If we use the symbol $S_0(n+1)^{(1)}$ introduced in art. 21 we have:

Theorem XXIII. "The two polytopes

$$e_a\, e_b\, e_c \ldots e_r\, e_s\, e_t\, S_0\,(n+1)^{(1)},\quad e_{a'}\, e_{b'}\, e_{c'} \ldots e_{r'}\, e_{s'}\, e_{t'}\, S_0\,(n+1)^{(1)}$$

are equal and concentric, but of opposite orientation, if and only if we have generally

$$a+t'=b+s'=c+r'= \ldots. = r+c'=s+b'=t+a'=n-1\text{''}$$

"For $a=a', b=b', c=c', \ldots, r=r', s=s', t=t'$ the polytope in which the two given ones coincide is central symmetric, if and only if we have

$$a+t=b+s=c+r= \ldots = n-1$$

under which conditions there may be an unpaired middle expansion $e_{\frac{n-1}{2}}$ for n odd".

This theorem gives in analytical form the results published in a joint paper of M^rs Stott and myself [2]), already quoted on page 17, as far as the simplex offspring is concerned; for the supposition that the reciprocal polytopes A and A' mentioned in art. 3 of that paper are $e_0\, S_0\,(n+1)^{(1)}$ and $e_{n-1}\, S_0\,(n+1)^{(1)}$, i. e two concentric and equal simplexes $S\,(n+1)^{(1)}$ of opposite orientation, specializes the general results found there to the simplex theorem stated just now here. To prove the latter analytically we have only to write out the result of the operations $e_a e_b e_c \ldots e_r e_s e_t$ and $e_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'}$

---

[1]) Compare "*Nieuw Archief voor Wiskunde*", vol. IX, p. 140, remark I.

[2]) Reciprocity in connection with semiregular polytopes and nets, "*Proceedings* of the Academy of Amsterdam", September, 1910.

on $S_0(n+1)^{(1)}$ and to investigate under what circumstances the zero symbol of the one is the inversion of that of the other. If each of the two products

$$e_a\, e_b\, e_c \ldots e_r\, e_s\, e_t \quad , \quad e_{a'}\, e_{b'}\, e_{c'} \ldots e_{r'}\, e_{s'}\, e_{t'} \;,$$

where we have

$$a < b < c < \ldots < r < s < t \quad , \quad a' < b' < c' < \ldots < r' < s' < t'$$

bears k factors, the two results are represented by

$$(\overbrace{k, k, \ldots k}^{a+1}, \overbrace{k-1, k-1, \ldots k-1}^{b-a}, \overbrace{k-2, k-2, \ldots k-2}^{c-b}, \ldots \overbrace{22 \ldots 2}^{s-r}, \overbrace{11 \ldots 1}^{t-s}, \overbrace{00 \ldots 0}^{n-t})$$

and the same expression in which the $a, b, c, \ldots r, s, t$ are dashed.

So the conditions are

$$a+1 = n-t',\; b-a = t'-s',\; c-b = s'-r', \ldots$$
$$\ldots, s-r = c'-b',\; t-s = b'-a,\; n-t = a'+1,$$

giving immediately

$$a+t' = b+s' = c+r' = \ldots \quad = r+c' = s+b' = t+a' = n-1.$$

So the first part is proved and the second is deduced from this by suppression of the dashes. In this second part the unpaired middle expansion $e_{\frac{n-1}{2}}$ occurs, if and only if both n and k are odd.

It is an easy task to return to the e and c symbols referring to the simplex $(1\,\overbrace{00\,\ldots\,0}^{n})$; to that end we have to omit the e_0 symbol and to add c to any expansion form, where e_0 is lacking.

In doing so we arrive for $n = 3, 4, 5$ by means of the first part of the theorem to all the cases, as $e_2\, e_3\, S(5) = -\, e_1\, e_3\, S(5)$, of equal and concentric polytopes of opposite orientation mentioned in the table, and by means of the second part to all the cases, as $ce_2\; S(6)$, of central symmetry.

39. In the joint paper of M^rs^ STOTT and myself quoted in the preceding article, the notion of reciprocal polytopes has been extended to that of reciprocal nets by considering a net of S_n as a polytope with an infinite number of limits $(l)_n$ in S_{n+1}. In this case the centre of the circumscribed spherical space of the polytope lies at infinity in the direction of the normal to the space S_n bearing the space filling, from which it ensues that the poles of the limits $(l)_n$ coincide with the centres of these polytopes. So one obtains a net reciprocal to a given one by considering the centres

of the polytopes of the given net as vertices (see art. 51 of ANDREINI's memoir, quoted in art. 22).

Under what circumstances polarization of a simplex net leads to an other simplex net? The answer to this question is: "this only happens in the plane with the nets $N(p_3)$, $N(p_6)$, $N(p_3, p_6)$ and the for two reasons discarded net $N(p_3, p_{12})$". For, if otherwise the net contains two or more different constituents the reciprocal net will contain two or more differents kinds of vertices, and if the net is formed by one constituent only and this self space filler is partially regular the vertices of the new net will be partially regular. So the only possible case of *two reciprocal simplex nets* is that of the pair $N(p_3)$ and $N(p_6)$ in the plane, the centres of the two sets of triangles of $N(p_3)$ being the vertices of an $N(p_6)$, the centres of the hexagons of $N(p_6)$ being the vertices of an $N(p_3)$.

In the treatise "Sulle reti, ecc." quoted once more above M^r. ANDREINI has indicated how to draw up a complete list of all the reciprocal nets of threedimensional space; in this research he comes to the remarkable result (art. 59) that the rhombic dodecahedron and some other less regular polyhedra into which this semiregular polyhedron of the second kind can be decomposed form the constituents of the different reciprocal nets. If we restrict ourselves to the cases concerned with nets of simplex extraction this result is that the constituent of the reciprocal net of

$N(T, O)$	is the rhombic dodecahedron RD,
$N(T, tT)$	„ the rhombohedron ($\frac{1}{4}$ RD),
$N(O, CO)$	„ a double pyramid on a square ($\frac{1}{6}$ RD),
$N(tT, tO, CO)$	„ a pyramid on a lozenge ($\frac{1}{12}$ RD),
$N(tO)$	„ a tetrahedron limited by four equal isosceles triangles ($\frac{1}{24}$ RD).

What corresponds to this remarkable result in space S_n? It goes without saying that this question deserves an answer. But that answer can only be fragmentary, unless we surpass the limits between which we wish to confine ourselves in this paper. So all we can do now is to express the hope that we may be able to give a complete answer to that question in a new paper of its own. Only we cannot retain the remark that the constituent of the reciprocal net of the net corresponding to the undivided partition $n+1$ of $n+1$ and that of the reciprocal net of the net corresponding to the partition of $n+1$ consisting of units only are very interesting polytopes, worthy of study for their own sake.

G. *Symmetry, considerations of the theory of groups, regularity.*

40. We begin by determining the spaces S_{n-1} of symmetry which may be indicated by Sy_{n-1} and we consider to that end successively the case of the simplex $S(n+1)$ of S_n and that of any polytope $(P)_s$ deduced from that simplex $S(n+1)$ by the operations of expansion and contraction.

Case of the simplex. The vertices of $S(n+1)$ lying outside a space of symmetry Sy_{n-1} of this $S(n+1)$ occur in couples. Now there must be at least one of these couples, as Sy_{n-1} cannot contain all the vertices of $S(n+1)$, and on the other hand there cannot be more than one of these couples, as $S(n+1)$ does not admit parallel edges. So any space Sy_{n-1} must bisect orthogonally one edge of $S(n+1)$, i. e. the number of spaces Sy_{n-1} is $\frac{1}{2}n(n+1)$.

It is not at all difficult to indicate the equations of the $\frac{1}{2}n(n+1)$ spaces Sy_{n-1} of $(1\,\overbrace{00\ldots0}^{n})$. For the space S_{n-1} bisecting normally the edge $A_k A_l$ joining the points A_k and A_l with the coordinates $(x_k = 1,\ x_{not\,k} = 0)$ and $(x_l = 1,\ x_{not\,l} = 0)$ is represented by the equation $x_k = x_l$.

Case of the polytope $(P)_s$ deduced from the simplex. It goes without saying that the $\frac{1}{2}n(n+1)$ spaces of symmetry $x_k = x_l$ of $S(n+1)$ are at the same time spaces Sy_{n-1} for any polytope $(P)_s$ derived from that $S(n+1)$ by the operations e and c, and that any two limits of that $(P)_s$ which are each others mirror images with respect to any of these Sy_{n-1} are of the same import. So the only question is, if the polytope $(P)_s$ can possess a space of symmetry which is no Sy_{n-1} for the $S(n+1)$ from which the $(P)_s$ has been derived. To answer this question we suppose there *is* such a space Sy_{n-1} and we examine the consequences to which this supposition leads. According to this supposition $(P)_s$ is its own mirror image with respect to that definite Sy_{n-1}, which may be represented by the symbol $\overline{Sy}_{n-1}$, whilst the mirror image of the simplex $S(n+1)$ from which $(P)_s$ has been deduced is an other simplex $S'(n+1)$ concentric to $S(n+1)$. But then the figure consisting of $(P)_s$ on one hand and the two simplexes $S(n+1)$, $S'(n+1)$ on the other is symmetric with respect to $\overline{Sy}_{n-1}$; so it must be possible to deduce $(P)_s$ by the same set of expansion operations from the new simplex $S'(n+1)$. From this we can draw two conclusions, one with respect to the two simplexes, an other with respect to $(P)_s$. If we can deduce the same polytope $(P)_s$ from two different simplexes, these simplexes must be concentric and *oppositely orientated*; if we can do so by means of the *same* set of expansions, $(P)_8$ must be *central symmetric.* So we have to solve first the new question if the simplex $S(n+1)$ admits a space $\overline{S}_{n-1}$ reflecting it into a concentric simplex $S'(n+1)$ oppositely orientated to $S(n+1)$. We once more suppose that there *is* such a space $\overline{S}_{n-1}$ and we examine the consequences of this supposition. Let $A_1 A_2 \ldots . A_n$ be the given simplex and $A_1' A_2' \ldots . A'_{n+1}$ the concentric simplex of opposite orientation, the common centre O being at the same time centre of the $n+1$ segments $A_i A_i'$, $(i = 1, 2, \ldots, n+1)$ Then the image of A_1 wit respect to $\overline{S}_{n-1}$ must be either A'_1 or one of the other vertices of $S'(n+1)$, say A'_2. We consider these two different cases each for itself.

If A'_1 is the mirror image of A_1, the space $\overline{S}_{n-1}$ is normal to the line $A_1 O$ joining in $S(n+1)$, if produced, the vertex A_1 with the centre M_1 of the opposite limit $S(n)$, which line $A_1 M_1$ may be called a "first transversal" of $S(n+1)$; this $S(n)$ with the vertices $A_2 A_3 \ldots A_{n+1}$ is contained in a space S_{n-1} parallel to $\overline{S}_{n-1}$, whilst the mirror image of it is the limit $S'(n)$ of $S'(n+1)$ opposite to A'_1, with the vertices $A'_2, A'_3 \ldots, A'_{n+1}$. So, as a whole, these $S(n)$ and $S'(n)$ have to be at the same time equipollent and oppositely orientated to each other, equipollent as reflections of figures lying in spaces S_{n-1} and S'_{n-1} parallel to the mirror $\overline{S}_{n-1}$, oppositely orientated as corresponding parts of the oppositely orientated simplexes $S(n+1)$ and $S'(n+1)$. This is impossible for $n > 2$, e. g. two triangles lying in parallel planes cannot be equipollent and oppositely orientated at the same time. Now the case $n = 1$, meaningless in itself, leads to two conciding simplexes, i. e. to a point of symmetry of the simplex of the linear domain, the line segment. So the case $n = 2$ of the triangle $A_1 A_2 A_3$ with the lines through O parallel to the sides is the only remaining one.

If A'_2 (fig. 12) is the mirror image of A_1, we consider the triangle $A_1 A_2 A'_2$ with a right angle in A_1, as the centre O of $A_2 A'_2$ is at equal distance from the three vertices; if M_{12} is the centre of $A_1 A_2$, the line $M_{12} O$, parallel to $A_1 A'_2$, passes, if produced, through the centre of the limit $S(n-1)$ of $S(n+1)$ containing $A_3, A_4, \ldots, A_{n+1}$ as vertices and may therefore be called a "second transversal" of $S(n+1)$. Now the mirror $\overline{S}_{n-1}$ bisects $A_1 A_2$ orthogonally and is therefore normal to the second transversal $M_{12} O$, while the limits $S(n-1)$ with the vertices $A_3 A_4, \ldots, A_{n+1}$ and $S'(n-1)$ with the vertices $A'_3 A'_4 \ldots A'_{n+1}$ lie in parallel spaces S_{n-2} and S'_{n-2}. Here too these limits have to be at the same time equipollent and oppositely orientated to each other, which is impossible for $n-1 > 2$. So we find here two cases, the case $n = 2$ found above and the case $n = 3$ of the tetrahedron $A_1 A_2 A_3 A_4$ with the planes through O parallel to a pair of opposite edges. So we have proved the general theorem:

THEOREM XXIV. "The simplex $(1\,\overbrace{00\,..\,0}^{n})$ of S_n and the polytopes deduced from it by expansion and contraction admit $\frac{1}{2}n(n+1)$ spaces Sy_{n-1} of symmetry, the spaces $x_i = x_k$. Moreover in the plane the $e_1(p_3)$ admits the three new axes of symmetry $x_1 = y$ of the hexagon, whilst in space the $ce_1 T = O$, $e_2 T = CO$, $e_1 e_2 T = tO$ admit the new planes of symmetry $x_i + x_j = x_k + x_l$ of the octahedron".

41. We now prove the following theorem [1]:

THEOREM XXV. "The order of the group of anallagmatic displacements of the simplex $S(n+1)$ of S_n and of the polytopes deduced from it by expansion and contraction is $\frac{1}{2}(n+1)!$"

"The order of the extended group of anallagmatic displacements of these polytopes, reflexions with respect to spaces Sy_{n-1} of symmetry included, is $(n+1)!$ In this extended group the first group of order $\frac{1}{2}(n+1)!$ forms a perfect subgroup".

"For $n = 2$ and $n = 3$ these general results have to be completed in the generally known way".

The simplest proof of this theorem is connected with the remark that reflexion of the polytopes with respect to any space Sy_{n-1} corresponds to the interchanging of any pair of vertices of the simplex. So the order of the group of reflexions (and anallagmatic displacements) is equal to the number of permutations of the $n+1$ vertices of $S(n+1)$, i. e. $(n+1)!$, and the group of the anallagmatic displacements is of an order half as large, i. e. of order $\frac{1}{2}(n+1)!$

For the cases $n = 2$ and $n = 3$ we refer to F. KLEIN's "Vorlesungen über das Ikosaeder" (Leipsic, Teubner, 1884).

42. The manner in which the polytopes considered here have been derived from the simplex is a guarantee that all the vertices are of the same kind and all the edges have the same length. But this is all that can be asserted; so e. g. the polyhedron tT has *two* kinds of edges, edges common to two hexagons lying in planes including a definite acute angle and edges common to a hexagon and a triangle lying in planes including the obtuse supplementary angle. So in judging of the regularity we have to look at the edges from two different points of view; we must not only take into account the length but also consider angles on or faces through the edges, etc.

[1]) Compare *Report of the British Association*, 1894, p. 563.

In his dissertation — which is about to appear — M^r^. E. L. ELTE has created an artificial system [1]) according to which it is possible to count the degree of regularity of the partially regular polytopes deduced from the regular polytopes by regular truncation. In this system the regularity of such a polytope is expressed by a fraction, the denominator of which is equal to the number of dimensions, while each group of limiting elements as vertices, edges, faces, etc. may contribute a unit to the numerator. With the exception of the group of vertices [2]) every group of limiting elements has this unit subdivided into two halves, one half for equality of form, the other half for equality of position with respect to the surroundings; moreover only *successive* contributions count, beginning at the vertices. So in the case of tT the contributions of vertices and edges are 1, $\frac{1}{2}$ and the degree of regularity is $\frac{1+\frac{1}{2}}{3} = \frac{1}{2}$ and this is the case with all the Archimedian semiregular polyhedra, except CO and ID, where the dihedral angles on the edges are equal and the degree of regularity is $\frac{1+1}{3} = \frac{2}{3}$.

Of the two halves corresponding to equality of form and to equality of position with respect to the surroundings the first needs no explanation, while the second may seem rather difficult to grasp. But this second half also will become clear, if we indicate it as follows. Equality of vertices means that the figures formed by the systems of edges concurring in the different vertices (vertex polyangles) are congruent, equality of edges means that the edges have the same length (first $\frac{1}{2}$) and that the figures formed by the systems of intersecting lines of the faces passing through the different edges with spaces S_{n-1} normal to the edges (edge polyangles) are congruent (second $\frac{1}{2}$), equality of faces means that the faces are congruent (first $\frac{1}{2}$) and that the figures formed by the systems of intersecting lines of the limiting threedimensional spaces passing through the faces with spaces S_{n-2} normal to the faces (face polyangles) are congruent (second $\frac{1}{2}$), etc.

So we will be able to determine the regularity fraction of a given polytope derived from the simplex in the scale of M^r^. ELTE, if we have found the *different subgroups* of each of the limiting

[1]) We can only give a glimpse of the system here. For more particulars we must refer to the dissertation written in English.

[2]) If we count from the other side (see the next page) we must say: "with the exception of the group of limits $(l)_{n-1}$", etc.

elements $(l)_1, (l)_2, \ldots, (l)_{n-1}$. So this research is closely related with theorem III of art. 10 which enables us to find the subgroups of the same system of limiting elements l_d characterized by *different symbols*, the more so as we have the theorem:

THEOREM XXVI. "Any two limiting elements of the same group $(l)_d$ belong to the same subgroup or to different subgroups, in the sense of the scale of regularity, according to their zero symbols being equal or different, if we consider two different zero symbols of a central symmetric polytope as being equal when they pass into each other by inversion."

This theorem is nearly self evident. A rigid proof of it can be based on the consideration of the limits $(l)_{n-1}$ passing through the $(l)_d$. So in the case of the form (321100) treated in art. 11 the different unextended edge symbols (32), (21), (10) correspond to subgroups of edges with different positions in relation to the surroundings. For, if we consider the four groups (32110), (321)(100), (32)(1100), (21100) of limiting polytopes it is immediately evident that the second group distinguishes (10) from the others, that the fourth group distinguishes (32) from the others, whilst the third group alone shows already that no two of the three subgroups of edges can be equal.

We remarked above that we count the contributions to the numerator of the regularity fraction beginning at the vertices and taking in only successive contributions. But the case may present itself that a polytope derived from the simplex shows also some regularity at the side of the limiting elements $(l)_{n-1}$ of the highest number of dimensions. We then indicate two fractions of regularity, one for each side, as will be shown in an example in the next article.

The fifth column of Table I contains the regularity fraction of the different forms obtained in the cases $n = 3, 4, 5$, only counted from the vertex side. In the fourth column the subscripts indicate the numbers of the different subgroups of each limiting element $(l)_d$.

43. We elucidate the theory by applying it to several examples:

a). *Example* **(321100)**. Here we find three different groups of edges. So the vertices contribute 1, the edges contribute $\frac{1}{2}$ to the numerator and the fraction is $\dfrac{1+\frac{1}{2}}{5} = \dfrac{3}{10}$.

b). *Example* **(110000)**. This form has only one kind of edge (10) but two subgroups (110) and (100) of triangular faces. So we find $\dfrac{1+1+\frac{1}{2}}{5} = \dfrac{1}{2}$.

c). *Example* **(111000)**, This central symmetric form has one kind of edge (10), one kind of face (110) = — (100), but two subgroups (1110) = — (1000) = T and (1100) = O of limiting bodies and once more one kind of limiting polytopes (11100) = — (11000). So we find $\left(\frac{3}{5}; \frac{1}{5}\right)$.

Remark. The degree of regularity of the polytopes of S_n found here is at least $\frac{1+\frac{1}{2}}{n} = \frac{3}{2n}$ and therefore for $n = 3$ at least $\frac{1}{2}$. So the Archimedian polyhedra of the stereometry are *semi*regular in the right sense of the word, if we take semiregular to mean that the degree of regularity is $\frac{1}{2}$ at least but less than unity.

44. As the scale used for the determination of the regularity is independent from the number of vertices, edges, faces, etc. of the polytope, the same method may be applied to nets of polytopes, by considering a net in S_n as as polytope limited by an infinite number of limits $(l)_n$ in S_{n+1}. This new application depends only on the problem how to determine the different kinds of vertices, edges, faces, etc. of the net.

All the nets considered here have vertices of the same kind and edges of the same length. So for a net in S_n the fraction of regularity is at least $\frac{1+\frac{1}{2}}{n+1}$, i. e. $\frac{3}{2(n+1)}$. So in the most frequent number of cases in which a constituent of the nets admits two or more differently shaped faces we have only the choice between $\frac{2}{n+1}$ and $\frac{3}{2(n+1)}$ of which the first value corresponds to the case of only one kind of edge, the second to that of two or more differents kinds of edges.

In order to make the determination of the fraction of regularity of the nets in S_4 and S_5 as easy as possible we enumerate in Table III the different limits $(l)_4$, $(l)_3$, $(l)_2$ of the nets in S_4 and the different limits $(l)_5$, $(l)_4$, $(l)_3$, $(l)_2$ of the nets in S_5. In the part corresponding to $n = 4$ we find under the seven headings I, II, . . , VII the subdivisions 4, 3, 2 standing for $(l)_4$, $(l)_3$, $(l)_2$, in the part corresponding to $n = 5$ likewise under I, II, . . , XII the subdivisions 5, 4, 3, 2 standing for $(l)_5$, $(l)_4$, $(l)_3$, $(l)_2$. These limits are indicated in abridged notation: under 5 the symbols 1, ce_1, ce_2, etc. denote $S(6)$, $ce_1S(6)$, $ce_2S(6)$, etc.; under 4 the symbols 1, ce_1, etc. signify $S(5)$, $ce_1S(5)$, etc.

The results of Table III are inscribed in Table II in the sixth column under the headings $(l)_0, (l)_1, \ldots, (l)_5$; so the number 11 on the lines of the nets $5_{VI}{}^a$, $5_{VI}{}^b$ under $(l)_4$ indicates that in these equal nets of space S_5 the constituents admit together eleven *differently shaped* limits $(l)_4$. What is taken from Table III — and what is self evident — is incribed in small type. The other numbers — inscribed in heavy type —, of which only two correspond to faces, have been found separately. We treat here two of these cases in detail.

Case 4_{III}. Here the constituent (11000) has only one kind of edge. Does this *imply* that the *net* has only one kind of edge? The example of the net 3_{IV} where the CO admits also only one kind of edge, whilst ANDREINI rightly mentions the fact (see his treatise, p. 32 under n°. 21) that of the five edges concurring in a vertex one is common to 2 tT and 2 tO and each of the four others to tT, tO, CO, must prevent us from jumping too rashly to this conclusion. So we investigate this point and examine if, e. g. in the case of the constituent (21100) with two kinds of edges, (21)100 and 21(10)0 these two edges are different with respect to the net or not. So we enumerate first the different limits $(l)_4$ to which the vertex 21100 is common. They are

$$\begin{array}{rlllllll}
e_2\ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & \mathbf{0}, & \mathbf{0}) & & \\
-e_2\ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & -2+\mathbf{2}, & \mathbf{0}) & & \\
-e_2\ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & \mathbf{0}, & -2+\mathbf{2}) & & 1 \\
-ce_1\ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & -2+\mathbf{2}, & -2+\mathbf{2}) & & 2 \\
e_3\ldots & (\mathbf{2}, & -2+\mathbf{3}, & \mathbf{1}, & -2+\mathbf{2}, & -2+\mathbf{2}) & & \\
e_3\ldots & (\mathbf{2}, & \mathbf{1}, & -2+\mathbf{3}, & -2+\mathbf{2}, & -2+\mathbf{2}) & & 3 \\
ce_1\ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -2+\mathbf{2}, & -2+\mathbf{2}) & & \\
e_2\ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -2+\mathbf{2}, & -4+\mathbf{4}) & & \\
e_2\ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -4+\mathbf{4}, & -2+\mathbf{2}) & & 4 \\
-e_2\ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -4+\mathbf{4}, & -4+\mathbf{4}) & &
\end{array}$$

Starting from (2 11 00) we have indicated in this list of ten polytopes first the two polytopes deduced from (21100) by varying the form of one of the digits 1, 1, 0, 0, then the only polytope obtained by varying two of the digits, etc., see the curved brackets and the numbers 1, 2, 3, 4 at the right. As we can augment all the interchangeable parts by the same integer provided that we diminish all the unmovable parts by the same amount, we find in this manner all the polytopes to which the chosen vertex 21100 is common, though we leave the first digit 2 alone.

If we denote the ten polytopes of the list by $(P)_1, (P)_2, \ldots, (P)_{10}$

we find that the edge (21) 100 is common to $(P)_1, (P)_2, (P)_3, (P)_4, (P)_6$ and the edge 21(10)0 to $(P)_1, (P)_3, (P)_6, (P)_7, (P)_8$. So both edges are common to $3\,e_2$, ce_1, e_3.

But this fact is not yet decisive, as the possibility exists that the grouping of the sets of five polytopes around the edges (21) and (10) is different. In order to decide this point we draw up the following table of threedimensional contact, where 1, 2, ..., 10 stand for $(P)_1, (P)_2, \dots, (P)_{10}$ and contact by a prism is indicated by a small asterisk.

1	2	3	4	5 & 6	7	8	9	10
2	1	1	2	1*	1	3	2	4
3	4	4	3	2*	5	5*	5*	5*
5*	5*	5*	5	3*	6	6*	6*	6*
6*	6*	6*	6	4	8	7	7	8
7	9	8	10	7	9	10	10	9
				8*				
				9*				
				10*				

This table shows that if we arrange each of the two sets of five polytopes as follows in three groups

$$\begin{array}{l} P_1, P_4 \;-\; P_2, P_3 \;-\; P_6 \\ P_3, P_7 \;-\; P_1, P_8 \;-\; P_6 \end{array}$$

each polytope is in bodily contact with the polytopes of the other groups of its horizontal row, whilst two polytopes in the same column are equal. So there is no difference whatever in the threedimensional contact, i. e. there is only one kind of edge

Case 5_V. Here the point 321000 is common to the 17 polytopes

$$\begin{array}{rl|r}
e_1 e_2 \dots & (\mathbf{3}, \quad \mathbf{2}, \quad \mathbf{1}, \quad \mathbf{0}, \quad \mathbf{0}, \quad \mathbf{0}) & 1 \\
ce_1 e_2 e_3 . & (\mathbf{3}, \quad \mathbf{2}, \quad \mathbf{1}, -3+\mathbf{3}, \quad \mathbf{0}, \quad \mathbf{0}) & 3 \\
-e_1 e_2 \dots & (\mathbf{3}, \quad \mathbf{2}, \quad \mathbf{1}, -3+\mathbf{3}, -3+\mathbf{3}, \quad \mathbf{0}) & 3 \\
-e_1 \dots . & (\mathbf{3}, \quad \mathbf{2}, \quad \mathbf{1}, -3+\mathbf{3}, -3+\mathbf{3}, -3+\mathbf{3}) & 1 \\
e_4 \dots . & (\mathbf{3}, \quad \mathbf{2}, -3+\mathbf{4}, -3+\mathbf{3}, -3+\mathbf{3}, -3+\mathbf{3}) & 1 \\
e_1 \dots . & (\mathbf{3}, -3+\mathbf{5}, -3+\mathbf{4}, -3+\mathbf{3}, -3+\mathbf{3}, -3+\mathbf{3}) & 1 \\
e_1 e_2 \dots & (\mathbf{3}, -3+\mathbf{5}, -3+\mathbf{4}, -3+\mathbf{3}, -3+\mathbf{3}, -6+\mathbf{6}) & 3 \\
ce_1 e_2 e_3 . & (\mathbf{3}, -3+\mathbf{5}, -3+\mathbf{4}, -3+\mathbf{3}, -6+\mathbf{6}, -6+\mathbf{6}) & 3 \\
-e_1 e_2 \dots & (\mathbf{3}, -3+\mathbf{5}, -3+\mathbf{4}, -6+\mathbf{6}, -6+\mathbf{6}, -6+\mathbf{6}) & 1
\end{array}$$

In this list we have availed ourselves of the occurrence of the three zeros in order to represent the 17 polytopes by nine symbols. So the second line represents three different polytopes which can be obtained by putting seccessively $-3+3$ under each of the three zeros. For each line the number at the right indicates how many different polytopes through the point 321000 correspond to that line.

This list shows that we may find here two kinds of edges though we have three groups, edges (32) 1000 common to 9 polytopes, edges 3(21)000 common to 16 polytopes, edges 32(10)00 common to 9 polytopes, as in the first and the last group the sets of 9 polytopes are both $4\, e_1\, e_2$, $3\, ce_1\, e_2\, e_3$, e_1, e_4. By investigating the fourdimensional contact between the nine polytopes of each set can be found whether the edges (32) and (10) belong to the same kind or not.

From the numbers of differently shaped limits the fraction of regularity has been deduced; it is given in the last column of Table II.

45. We finish this part of our memoir concerned with the offspring of the simplex by a remark about what may be called the "circumpolytope" of a net. This polytope, which has for vertices the vertices of the net joined by edges to any arbitrarily chosen vertex of the net, is by its form a criterion for the regularity of the net. If the net admits one kind of edge the circumpolytope must admit one kind of vertex, etc. This circumpolytope is in the cases of the threedimensional nets:

3_I ... a CO,
3_{II} ... a prismoid limited by two equilateral and six equal isosceles triangles,
3_{III} ... a prismoid limited by two squares and eight equal isosceles triangles,
3_{IV} ... a pyramid on a rectangular base,
3_V ... a tetrahedron limited by four equal isosceles triangles; of these five polyhedra only the fourth has vertices of two different kinds.

The theories developed here enable us to find the circumpolytopes corresponding to the different nets of simplex extraction in S_4 and S_5. But instead of deducing these polytopes here we conclude by the following general problem, for the proof of which we refer to the dissertation of Mr. ELTE:

THEOREM XXVII. "If the regularity of the circumpolytope of a simplex net of S_n is expressed by the fraction $\frac{p}{n}$, the regularity of the net itself will be represented in the general case by $\frac{p+1}{n+1}$ and in the special case $p=0$ by $\frac{3}{2(n+1)}$".

(To be continued).

Groningen, September 1911.

Errata.

Page 7, last line of middle column under $n=4$ omit "$= c\, e_2\, e_3\, S(5)$",
„ 27, line 18 from top replace "$(m_1, m_2, m_3, \ldots, 0)$", by "$(m_0, m_1, m_2, \ldots, 0)$".

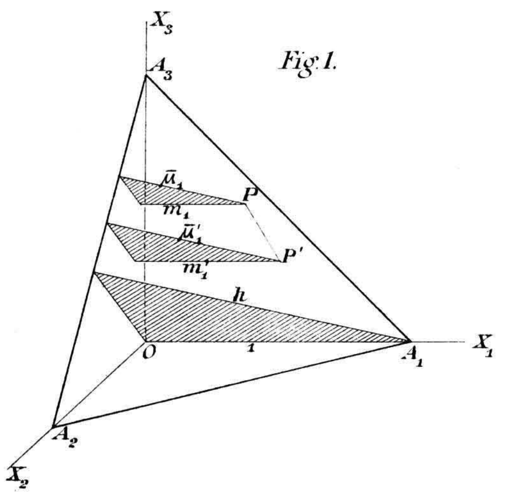

Fig. 1.

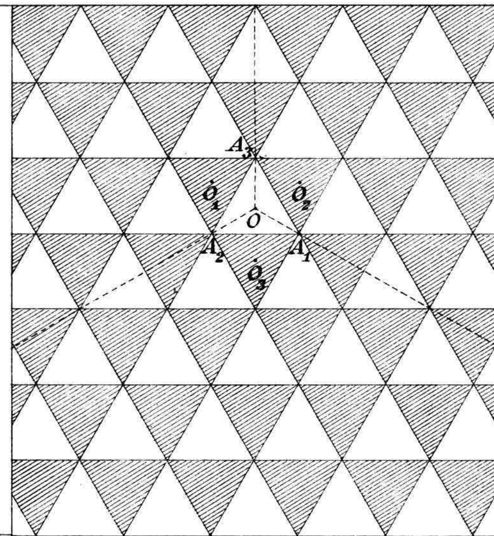

Fig. 5.

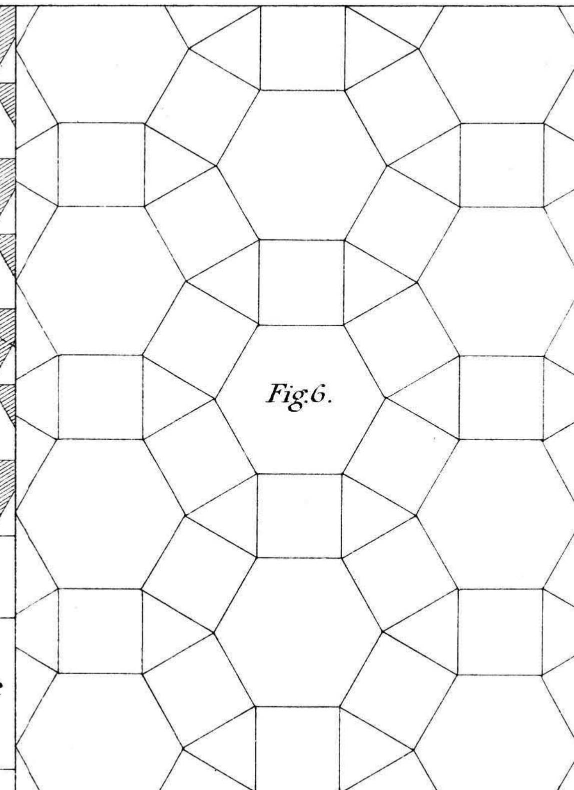
Fig. 6.

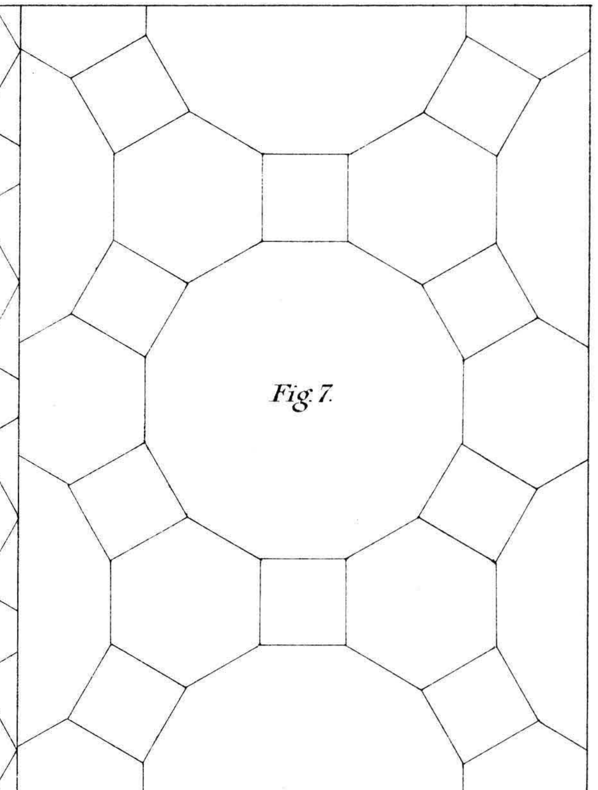
Fig. 7.

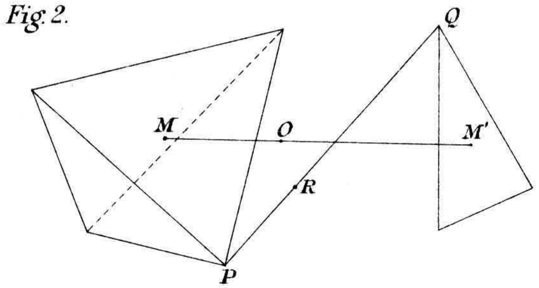

Fig. 2.

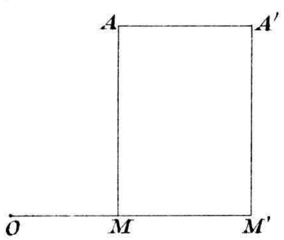

Fig. 3.

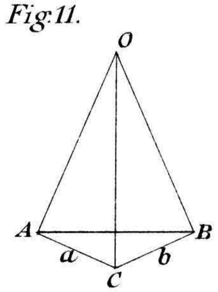

Fig. 11.

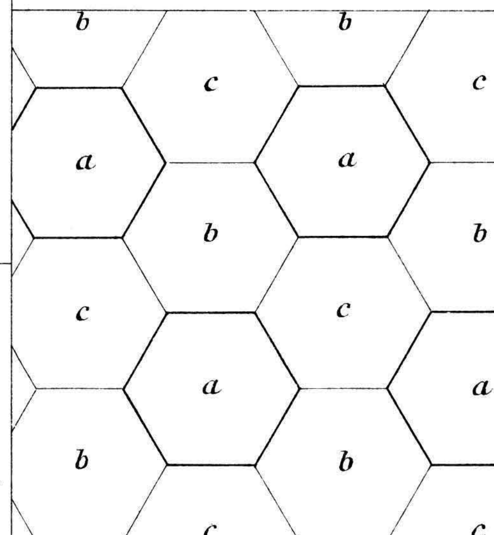

Fig. 10.

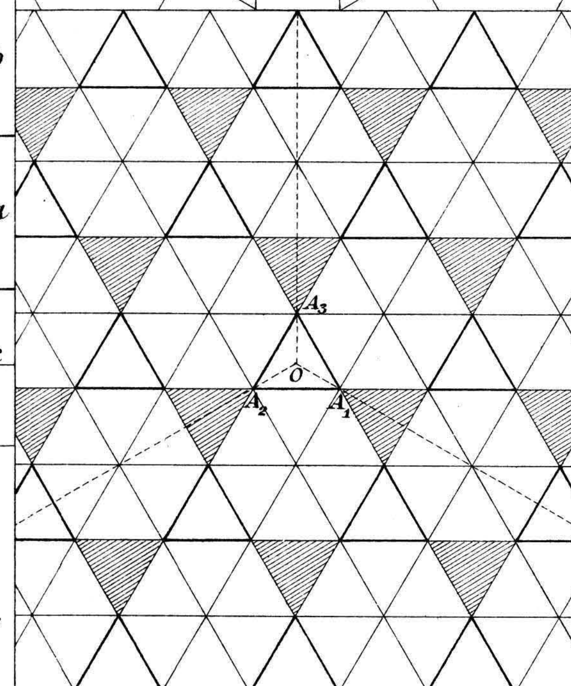

Fig. 9.

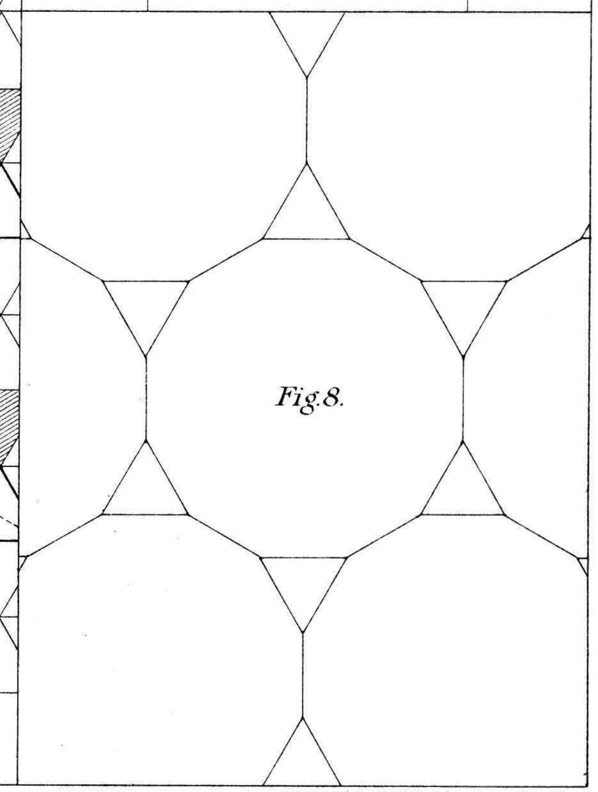
Fig. 8.

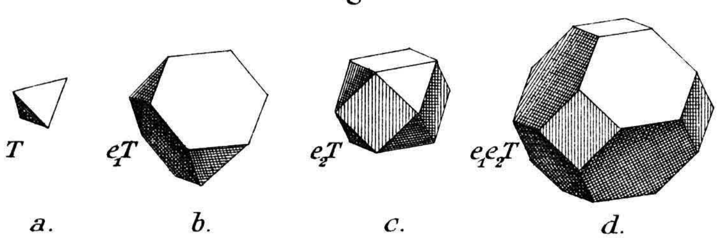

Fig. 4.

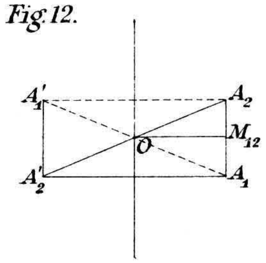

Fig. 12.

J. Bijtel lith. P. J. Mulder impr. Leiden.

$n = 3$

					4	6	4			
$S(4) = T$	(1000)	0	$(4_1\ \ 6_1\ \ 4_1)$	1	p_3	—	—	1		
$e_1\, S(4) = tT$	(2100)	$\frac{2}{4}$	$(12_1\ \ 18_2\ \ 8_2)$	$\frac{1}{2}$	p_6	—	p_3	3	1	
$e_2\, S(4) = CO$	*(2110)	$\frac{3}{4}$	$(12_1\ \ 24_1\ \ 14_2)$	$\frac{2}{3}$	p_3	p_4	p_3	4	2, 1	c. s.
$e_1\, e_2\, S(4) = tO$	*(3210)	$\frac{5}{4}$	$(24_1\ \ 36_2\ \ 14_2)$	$\frac{1}{2}$	p_6	p_4	p_6	6	3, 1	c. s.
$c\, e_1\, S(4) = O$	*(1100)	$\frac{1}{4}$	$(6_1\ \ 12_1\ \ 8_1)$	1	p_3	—	p_3	2	1	c. s.

$n = 4$

					5	10	10	5			
$S(5)$	(10000)	0	$(5_1\ \ 10_1\ \ 10_1\ \ 5_1)$	1	T	—	—	—	1		
$e_1\, S(5)$	(21000)	$\frac{2}{5}$	$(20_1\ \ 40_2\ \ 30_2\ \ 10_2)$	$\frac{3}{8}$	tT	—	—	T	3	1	
$e_2\, S(5)$	(21100)	$\frac{3}{5}$	$(30_1\ \ 90_2\ \ 80_4\ \ 20_3)$	$\frac{3}{8}$	CO	—	P_3	O	4	2, 1	
$e_3\, S(5)$	*(21110)	$\frac{4}{5}$	$(20_1\ \ 60_1\ \ 70_2\ \ 30_2)$	$\frac{1}{2}$	T	P_3	P_3	T	5	3, 2, 1	c. s.
$e_1\, e_2\, S(5)$	(32100)	1	$(60_1\ \ 120_3\ \ 80_4\ \ 20_3)$	$\frac{3}{8}$	tO	—	P_3	tT	6	3, 1	
$e_1\, e_3\, S(5)$	(32110)	$\frac{6}{5}$	$(60_1\ \ 150_3\ \ 120_5\ \ 30_4)$	$\frac{3}{8}$	tT	P_6	P_3	CO	7	4, 2, 1	$= e_2\, e_3\, S(5)$
$e_2\, e_3\, S(5)$	(32210)	$\frac{7}{5}$	$(60_1\ \ 150_3\ \ 120_5\ \ 30_4)$	$\frac{3}{8}$	CO	P_3	P_6	tT	8	5, 3, 1	$= e_1\, e_3\, S(5)$
$e_1\, e_2\, e_3\, S(5)$	*(43210)	$\frac{9}{5}$	$(120_1\ \ 240_2\ \ 150_4\ \ 30_2)$	$\frac{3}{8}$	tO	P_6	P_6	tO	10	6, 3, 1	c. s.
$c\, e_1\, S(5)$	(11000)	$\frac{1}{5}$	$(10_1\ \ 30_1\ \ 30_2\ \ 10_2)$	$\frac{5}{8}$	O	—	—	T	2	1	
$c\, e_1\, e_2\, S(5)$	*(22100)	$\frac{4}{5}$	$(30_1\ \ 60_1\ \ 40_2\ \ 10_1)$	$\frac{1}{2}$	tT	—	—	tT	5	3, 1	

$n = 5$

					6	15	20	15	6			
$S(6)$	(100000)	0	$(6_1\ \ 15_1\ \ 20_1\ \ 15_1\ \ 6_1)$	1	$S(5)$	—	—	—	—	1		
$e_1\, S(6)$	(210000)	$\frac{2}{6}$	$(30_1\ \ 75_2\ \ 80_2\ \ 45_2\ \ 12_2)$	$\frac{3}{10}$	$e_1\, S(5)$	—	—	—	$S(5)$	3	1	
$e_2\, S(6)$	(211000)	$\frac{3}{6}$	$(60_1\ \ 240_2\ \ 290_4\ \ 135_4\ \ 27_3)$	$\frac{3}{10}$	$e_2\, S(5)$	—	—	P_T	$c\, e_1\, S(5)$	4	2, 1	
$e_3\, S(6)$	(211100)	$\frac{4}{6}$	$(60_1\ \ 270_2\ \ 420_4\ \ 255_5\ \ 47_4)$	$\frac{3}{10}$	$e_3\, S(5)$	—	(3 ; 3)	P_O	$-\, c\, e_1\, S(5)$	5	3, 2, 1	
$e_4\, S(6)$	*(211110)	$\frac{5}{6}$	$(30_1\ \ 120_1\ \ 210_2\ \ 180_2\ \ 62_3)$	$\frac{2}{5}$	$S(5)$	P_T	(3 ; 3)	P_T	$S(5)$	6	4, 3, 2, 1	c. s.
$e_1\, e_2\, S(6)$	(321000)	$\frac{5}{6}$	$(120_1\ \ 300_3\ \ 290_4\ \ 135_4\ \ 27_3)$	$\frac{3}{10}$	$e_1\, e_2\, S(5)$	—	—	P_T	$e_1\, S(5)$	6	3, 1	
$e_1\, e_3\, S(6)$	(321100)	1	$(180_1\ \ 630_3\ \ 720_6\ \ 315_7\ \ 47_4)$	$\frac{3}{10}$	$e_1\, e_3\, S(5)$	—	(6 ; 3)	P_O	$e_2\, S(5)$	7	4, 2, 1	
$e_1\, e_4\, S(6)$	(321110)	$\frac{7}{6}$	$(120_1\ \ 420_3\ \ 570_5\ \ 330_7\ \ 62_5)$	$\frac{3}{10}$	$e_1\, S(5)$	P_T	(6 ; 3)	P_T	$e_3\, S(5)$	8	5, 3, 2, 1	$= e_3\, e_4\, S(6)$
$e_2\, e_3\, S(6)$	(322100)	$\frac{7}{6}$	$(180_1\ \ 540_3\ \ 570_6\ \ 255_6\ \ 47_4)$	$\frac{3}{10}$	$e_2\, e_3\, S(5)$	—	(3 ; 3)	P_{tT}	$c\, e_1\, e_2\, S(5)$	8	5, 3, 1	
$e_2\, e_4\, S(6)$	*(322110)	$\frac{8}{6}$	$(180_1\ \ 720_2\ \ 900_4\ \ 420_5\ \ 62_3)$	$\frac{3}{10}$	$e_2\, S(5)$	P_{CO}	(3 ; 3)	P_{CO}	$e_2\, S(5)$	9	6, 4, 2, 1	c. s.
$e_3\, e_4\, S(6)$	(322210)	$\frac{9}{6}$	$(120_1\ \ 420_3\ \ 570_5\ \ 330_7\ \ 62_5)$	$\frac{3}{10}$	$e_3\, S(5)$	P_T	(3 ; 6)	P_{tT}	$e_1\, S(5)$	10	7, 5, 3, 1	$= e_1\, e_4\, S(6)$
$e_1\, e_2\, e_3\, S(6)$	(432100)	$\frac{9}{6}$	$(360_1\ \ 900_4\ \ 810_7\ \ 315_7\ \ 47_4)$	$\frac{3}{10}$	$e_1\, e_2\, e_3\, S(5)$	—	(3 ; 6)	P_{tT}	$e_1\, e_2\, S(5)$	10	6, 3, 1	
$e_1\, e_2\, e_4\, S(6)$	(432110)	$\frac{10}{6}$	$(360_1\ \ 1080_4\ \ 1140_8\ \ 480_9\ \ 62_5)$	$\frac{3}{10}$	$e_1\, e_2\, S(5)$	P_{tO}	(6 ; 3)	P_{CO}	$e_1\, e_3\, S(5)$	11	7, 4, 2, 1	$= e_2\, e_3\, e_4\, S(6)$
$e_1\, e_3\, e_4\, S(6)$	*(432210)	$\frac{11}{6}$	$(360_1\ \ 1080_2\ \ 1080_5\ \ 420_5\ \ 62_3)$	$\frac{3}{10}$	$e_1\, e_3\, S(5)$	P_{tT}	(6 ; 6)	P_{tT}	$e_1\, e_3\, S(5)$	12	8, 5, 3, 1	c. s.
$e_2\, e_3\, e_4\, S(6)$	(433210)	2	$(360_1\ \ 1080_4\ \ 1140_8\ \ 480_9\ \ 62_5)$	$\frac{3}{10}$	$e_2\, e_3\, S(5)$	P_{CO}	(6 ; 3)	P_{tO}	$e_1\, e_2\, S(5)$	13	9, 6, 3, 1	$= e_1\, e_2\, e_4\, S(6)$
$e_1\, e_2\, e_3\, e_4\, S(6)$	*(543210)	$\frac{14}{6}$	$(720_1\ \ 1800_3\ \ 1560_6\ \ 540_6\ \ 62_3)$	$\frac{3}{10}$	$e_1\, e_2\, e_3\, S(5)$	P_{tO}	(6 ; 6)	P_{tO}	$e_1\, e_2\, e_3\, S(5)$	15	10, 6, 3, 1	c. s.
$c\, e_1\, S(6)$	(110000)	$\frac{1}{6}$	$(15_1\ \ 60_1\ \ 80_2\ \ 45_2\ \ 12_2)$	$\frac{1}{2}$	$c\, e_1\, S(5)$	—	—	—	$S(5)$	2	1	
$c\, e_2\, S(6)$	*(111000)	$\frac{2}{6}$	$(20_1\ \ 90_1\ \ 120_1\ \ 60_2\ \ 12_1)$	$\frac{3}{5}$	$-\, c\, e_1\, S(5)$	—	—	—	$c\, e_1\, S(5)$	3	2, 1	c. s.
$c\, e_1\, e_2\, S(6)$	(221000)	$\frac{4}{6}$	$(60_1\ \ 150_2\ \ 140_3\ \ 60_3\ \ 12_2)$	$\frac{3}{10}$	$c\, e_1\, e_2\, S(5)$	—	—	—	$e_1\, S(5)$	5	3, 1	
$c\, e_1\, e_3\, S(6)$	*(221100)	$\frac{5}{6}$	$(90_1\ \ 360_1\ \ 420_3\ \ 180_3\ \ 32_2)$	$\frac{2}{5}$	$e_2\, S(5)$	—	—	—	$e_2\, S(5)$	6	4, 2, 1	c. s.
$c\, e_1\, e_2\, e_3\, S(6)$	*(332100)	$\frac{8}{6}$	$(180_1\ \ 450_2\ \ 420_3\ \ 180_3\ \ 32_2)$	$\frac{3}{10}$	$e_1\, e_2\, S(5)$	—	—	—	$e_1\, e_2\, S(5)$	9	6, 3, 1	c. s.

LIST OF NETS DEDUCED FROM THE SIMPLEX.

Table II.

$n = 2$

							$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	3	(a_1, a_2, a_3)	$(100) = p_3$	$(110) = -p_3$		1	1	1				1
II	2	1, 2	$(2a_1+1, 2a_2, 2a_3)$	$(100) = p_3$	$(210) = {}^*p_6$	$(110) = -p_3$	1	1	2				$\frac{2}{3}$
III	3	1, 1, 1	$(3a_1+2, 3a_2+1, 3a_3)$	$(210) = {}^*p_6$			1	1	1				1

$n = 3$

								$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	4	(a_1, a_2, a_3, a_4)	$(1000) = T$	$(1100) = {}^*O$	$(1110) = -T$		1	1	1	2			$\frac{3}{4}$
II	2	1, 3	$(2a_1+1, 2a_2, 2a_3, 2a_4)$	$(1000) = T$	$(2100) = tT$	$(2210) = -tT$	$(1110) = -T$	1	1	2	2			$\frac{1}{2}$
III	2	2, 2	$(2a_1+1, 2a_2+1, 2a_3, 2a_4)$	$(1100) = {}^*O$	$(2110) = {}^*CO$			1	1	2	2			$\frac{1}{2}$
IV	3	1, 1, 2	$(3a_1+2, 3a_2+1, 3a_3, 3a_4)$	$(2100) = tT$	$(3210) = {}^*tO$	$(2210) = -tT$	$(2110) = {}^*CO$	1	2	3	3			$\frac{3}{8}$
V	4	1, 1, 1, 1	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4)$	$(3210) = {}^*tO$				1	1	2	1			$\frac{1}{2}$

$n = 4$

									$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	5	$(a_1, a_2, a_3, a_4, a_5)$	$(10000) = e_0$	$(11000) = e_1$	$(11100) = -e_1$	$(11110) = -e_0$		1	1	**1**	2	2		$\frac{3}{5}$
II	2	1, 4	$(2a_1+1, 2a_2, 2a_3, 2a_4, 2a_5)$	$(10000) = e_0$	$(21000) = e_0 e_1$	$(22100) = {}^* e_1 e_2$	$(22210) = -e_0 e_1$	$(11110) = -e_0$	1	**1**	2	2	3		$\frac{2}{5}$
III	2	2, 3	$(2a_1+1, 2a_2+1, 2a_3, 2a_4, 2a_5)$	$(11000) = e_1$	$(21100) = e_0 e_2$	$(22110) = -e_0 e_2$	$(11106) = -e_1$	$(21110) = {}^* e_0 e_3$	1	**1**	2	4	3		$\frac{2}{5}$
IV	3	1, 1, 3	$(3a_1+2, 3a_2+1, 3a_3, 3a_4, 3a_5)$	$(21000) = e_0 e_1$	$(32100) = e_0 e_1 e_2$	$(33210) = -e_0 e_1 e_2$	$(22210) = -e_0 e_1$	$(21110) = {}^* e_0 e_3$	1	**2**	3	4	3		$\frac{3}{10}$
V	3	1, 2, 2	$(3a_1+2, 3a_2+1, 3a_3+1, 3a_4, 3a_5)$	$(21100) = e_0 e_2$	$(32110) = e_0 e_1 e_3$	$(22100) = {}^* e_1 e_2$	$(32210) = -e_0 e_1 e_3$	$(22110) = -e_0 e_2$	1	**3**	3	5	3		$\frac{3}{10}$
VI	4	1, 1, 1, 2	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4, 4a_5)$	$(32100) = e_0 e_1 e_2$	$(43210) = {}^* e_0 e_1 e_2 e_3$	$(33210) = -e_0 e_1 e_2$	$(32210) = -e_0 e_1 e_3$	$(32110) = e_0 e_1 e_3$	1	**2**	3	5	3		$\frac{3}{10}$
VII	5	1, 1, 1, 1, 1	$(5a_1+4, 5a_2+3, 5a_3+2, 5a_4+1, 5a_5)$	$(43210) = {}^* e_0 e_1 e_2 e_3$					1	1	2	2	1		$\frac{2}{5}$

$n = 5$

										$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	6	$(a_1, a_2, a_3, a_4, a_5, a_6)$	$(100000) = e_0$	$(110000) = e_1$	$(111000) = {}^* e_2$	$(111100) = -e_1$	$(111110) = -e_0$		1	1	**1**	2	2	3	$\frac{1}{2}$
II	2	1, 5	$(2a_1+1, 2a_2, 2a_3, 2a_4, 2a_5, 2a_6)$	$(100000) = e_0$	$(210000) = e_0 e_1$	$(221000) = e_1 e_2$	$(222100) = -e_1 e_2$	$(222210) = -e_0 e_1$	$(111110) = -e_0$	1	**1**	2	2	3	3	$\frac{1}{3}$
III	2	2, 4	$(2a_1+1, 2a_2+1, 2a_3, 2a_4, 2a_5, 2a_6)$	$(110000) = e_1$	$(211600) = e_0 e_2$	$(221100) = {}^* e_1 e_3$	$(222110) = -e_0 e_2$	$(111100) = -e_1$	$(211110) = {}^* e_0 e_4$	1	**1**	2	4	5	4	$\frac{1}{3}$
IV	2	3, 3	$(2a_1+1, 2a_2+1, 2a_3+1, 2a_4, 2a_5, 2a_6)$	$(111000) = {}^* e_2$	$(211100) = e_0 e_3$	$(221110) = -e_0 e_3$				1	**1**	2	3	4	2	$\frac{1}{3}$
V	3	1, 1, 4	$(3a_1+2, 3a_2+1, 3a_3, 3a_4, 3a_5, 3a_6)$	$(210000) = e_0 e_1$	$(321000) = e_0 e_1 e_2$	$(332100) = {}^* e_1 e_2 e_3$	$(333210) = -e_0 e_1 e_2$	$(222210) = -e_0 e_1$	$(211110) = {}^* e_0 e_4$	1	**2**	3	4	5	4	$\frac{1}{4}$
VI[a]	3	1, 2, 3	$(3a_1+2, 3a_2+1, 3a_3+1, 3a_4, 3a_5, 3a_6)$	$(211000) = e_0 e_2$	$(321100) = e_0 e_1 e_3$	$(332110) = -e_0 e_2 e_3$	$(222100) = -e_1 e_2$	$(322210) = -e_0 e_1 e_4$	$(221110) = -e_0 e_3$	1	**3**	3	6	11	6	$\frac{1}{4}$
VI[b]	3	1, 3, 2	$(3a_1+2, 3a_2+1, 3a_3+1, 3a_4+1, 3a_5, 3a_6)$	$(211100) = e_0 e_3$	$(321110) = e_0 e_1 e_4$	$(221000) = e_1 e_2$	$(322100) = e_0 e_2 e_3$	$(332210) = -e_0 e_1 e_3$	$(222110) = -e_0 e_2$	1	**3**	3	6	11	6	$\frac{1}{4}$
VII	3	2, 2, 2	$(3a_1+2, 3a_2+2, 3a_3+1, 3a_4+1, 3a_5, 3a_6)$	$(221100) = {}^* e_1 e_3$	$(322110) = {}^* e_0 e_2 e_4$					1	**1**	2	4	3	2	$\frac{1}{3}$
VIII	4	1, 1, 1, 3	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4, 4a_5, 4a_6)$	$(321000) = e_0 e_1 e_2$	$(432100) = e_0 e_1 e_2 e_3$	$(443210) = -e_0 e_1 e_2 e_3$	$(333210) = -e_0 e_1 e_2$	$(322210) = -e_0 e_1 e_4$	$(321110) = e_0 e_1 e_4$	1	**2**	3	5	7	3	$\frac{1}{4}$
IX	4	1, 1, 2, 2	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4+1, 4a_5, 4a_6)$	$(321100) = e_0 e_1 e_3$	$(432110) = e_0 e_1 e_2 e_4$	$(332100) = {}^* e_1 e_2 e_3$	$(433210) = -e_0 e_1 e_2 e_4$	$(332210) = -e_0 e_1 e_3$	$(322110) = {}^* e_0 e_2 e_4$	1	**3**	3	7	8	4	$\frac{1}{4}$
X	4	1, 2, 1, 2	$(4a_1+3, 4a_2+2, 4a_3+2, 4a_4+1, 4a_5, 4a_6)$	$(322100) = e_0 e_2 e_3$	$(432210) = {}^* e_0 e_1 e_3 e_4$	$(332110) = -e_0 e_2 e_3$				1	**2**	3	4	5	2	$\frac{1}{4}$
XI	5	1, 1, 1, 1, 2	$(5a_1+4, 5a_2+3, 5a_3+2, 5a_4+1, 5a_5, 5a_6)$	$(432100) = e_0 e_1 e_2 e_3$	$(543210) = {}^* e_0 e_1 e_2 e_3 e_4$	$(443210) = -e_0 e_1 e_2 e_3$	$(433210) = -e_0 e_1 e_2 e_4$	$(432210) = {}^* e_0 e_1 e_3 e_4$	$(332110) = e_0 e_1 e_2 e_4$	1	**3**	3	6	8	4	$\frac{1}{4}$
XII	6	1, 1, 1, 1, 1, 1	$(6a_1+5, 6a_2+4, 6a_3+3, 6a_4+2, 6a_5+1, 6a_6)$	$(543210) = {}^* e_0 e_1 e_2 e_3 e_4$						1	1	2	3	3	1	$\frac{1}{3}$

$n = 4$

I			II			III			IV			V			VI			VII		
4	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**
1	T	p_3	1	T	p_3	ce_1	O	p_3	e_1	tT	p_3	e_2	CO	p_3	$e_1 e_2$	tO	p_3	$e_1 e_2 e_3$	tO	p_4
ce_1	O		e_1	tT	p_6	e_2	T	p_4	$e_1 e_2$	T	p_4	$e_1 e_3$	P_3	p_4	$e_1 e_2 e_3$	P_3	p_4		P_6	p_6
			$ce_1 e_2$			e_3	CO		e_3	tO	p_6	$ce_1 e_2$	O	p_6	$e_1 e_3$	tT	p_6			
							P_3			P_3			tT			P_6				
													P_6			CO				

$n = 5$

I				II				III				IV				V				VI			
5	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**
1	1	T	p_3	1	1	T	p_3	ce_1	ce_1	O	p_3	ce_2	ce_1	O	p_3	e_1	e_1	tT	p_3	e_2	e_2	CO	p_3
ce_1	ce_1	O		e_1	e_1	tT	p_6	e_2	1	T	p_4	e_3	e_3	T	p_4	$e_1 e_2$	1	T	p_4	$e_1 e_3$	P_T	P_3	p_4
ce_2				$ce_1 e_2$	$ce_1 e_2$			$ce_1 e_3$	e_2	CO			(3 ; 3)	P_3		$ce_1 e_2 e_3$	$e_1 e_2$	tO	p_6	$e_2 e_3$	ce_1	O	p_6
								e_4	P_T	P_3			P_O			e_4	P_T	P_3		$ce_1 e_2$	$e_1 e_3$	T	
									(3 ; 3)								(3 ; 3)			$e_1 e_4$	(6 ; 3)	tT	
																				e_3	P_O	P_6	
																					(3 ; 3)		
																					P_{tT}		
																					$ce_1 e_2$		
																					e_1		
																					e_3		

VII				VIII				IX				X				XI				XII			
5	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**
$ce_1 e_3$	e_2	CO	p_3	$e_1 e_2$	$e_1 e_2$	tO	p_3	$e_1 e_3$	$e_1 e_3$	tT	p_3	$e_2 e_3$	$e_2 e_3$	CO	p_3	$e_1 e_2 e_3$	$e_1 e_2 e_3$	tO	p_3	$e_1 e_2 e_3 e_4$	$e_1 e_2 e_3$	tO	p_4
$e_2 e_4$	P_{CO}	P_3	p_4	$e_1 e_2 e_3$	P_T	P_3	p_4	$e_1 e_2 e_4$	(6 ; 3)	P_6	p_4	$e_1 e_3 e_4$	(3 ; 3)	P_3	p_4	$e_1 e_2 e_3 e_4$	(3 ; 6)	P_6	p_4		P_{tO}	P_6	p_6
	(3 ; 3)	O		$e_1 e_4$	e_1	tT	p_6	$ce_1 e_2 e_3$	P_O	P_3	p_6		P_{tT}	P_6	p_6	$e_1 e_2 e_4$	P_{tT}	P_3	p_6		(6 ; 6)	P_4	
		P_4			$e_1 e_2 e_3$	T		$e_2 e_4$	e_2	CO			$ce_1 e_2$	tT		$e_1 e_3 e_4$	$e_1 e_2$	tT					
					(3 ; 6)	P_6			$e_1 e_2$	O			(6 ; 6)				P_{tO}	P_4					
					P_{tT}				P_{tO}	tO							(6 ; 6)	CO					
					e_3				P_{CO}	P_4							P_{CO}						
									(3 ; 3)								$e_1 e_3$						