On certain Series of Sections of the Regular Four-dimensional Hypersolids,

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1. In making series of sections of the regular four-dimensional figures by the method given in this paper it is only necessary to know the number of solids meeting at each vertex. The total number in each figure can be found by counting the number of solids cut in the sections.

Taking the figures bounded by tetrahedra it is evident that a section by a space cutting the edges meeting in a vertex at equal distances from that vertex will give an equilateral triangular section of each tetrahedron. Hence the complete section will be a three-dimensional regular figure bounded by equilateral triangles.

There are only three such figures, the tetrahedron, the octahedron and the icosahedron; so there will be no other four-dimensional figure bounded by tetrahedra except those which have 4, 8 or 20 at each vertex.

If groups of tetrahedra arranged so that there are 4, 8 or 20 round a point be cut by parallel spaces close enough together to pass at least once through each edge, then the number of tetrahedra cut in the three groups respectively will be 5, 16 or 600.

Next taking cubes. The section of a cube by a space cutting the edges meeting at a vertex at equal distances from that vertex is an equilateral triangle. So there will be no other figure bounded by cubes except those having 4, 8 or 20 at each vertex. But 8

cubes exactly fill three-dimensional space and cannot therefore form a four-dimensional angle. Hence there cannot be a figure whose angles are formed by eight cubes. Still less can there be one whose angles are made by 20. Similar reasoning applies to the dodecahedron; as with the cubes, there is only room round a point for four. Now the figure corresponding to the first case is the 8-cell, that corresponding to the second is the 120-cell.

Taking an octahedron. A section by a space cutting the edges meeting at a vertex equally, is a square, and, as a cube is the only regular three-dimensional figure bounded by squares, there will only be one regular figure bounded by octahedra and that will have six at each vertex. This is the 24-cell.

In this manner we meet successively all the regular cells of four-dimensional space.

2. In making a series of sections of a regular four-dimensional figure by three-dimensional spaces S^3_1, S^3_2, etc. parallel to a bounding solid Z, that solid itself may be considered the first element of the series. If Z be in a space S^3, it is the only part of the figure in that space, but its faces are the surfaces of contact of it with other bounding solids. So that in building up the solids of the figure about Z in their position in four-dimensional space, if such a thing were possible, there would be one on each face. In some cases there are also one or more on each edge and one or more on each vertex. Whether this be so or not, can be determined by means of the number of cells meeting at each vertex in the particular figure under consideration. The solids on the faces of Z may be supposed to turn about those faces until they lie wholly in S^3 and if there be any on the edges and vertices they may be supposed to turn about those edges and vertices until they too lie in S^3. We represent in fig. 1 the result of such an operation on the 8-cell. The cube HA is the solid originally in the space S^3; NA has been turned about its surface of contact with HA, namely the square CA, into S^3. The cubes PA and SA have been turned about the squares GA and EA respectively into the same space S^3.

The result of this is that the square LA, which is common to the two cubes NA and PA, has assumed two positions in S^3. It is e. g. horizontal in PA and vertical in NA.

Similarly MA and OA each appear in two positions in S^3.

If the 8-cell were cut by a space S_1^3 parallel to S^3 and passing through some point of the edge AK, each of the cubes NA, PA, SA would be cut by a plane parallel to its surface of contact with HA. The positions of these planes in the cubes could easily be determined after they have been brought into the space S^3. For instance, if S_1^3 bisects AK, the sections of NA, PA and SA will be squares parallel to CA, GA and EA through the midpoint of AK. Similarly there will be square sections of the three cubes at H and the complete section is a cube. Thus there will be three cubes in this series: 1° the cube HA, 2° a cube bounded by the sectional planes parallel to the faces of HA and 3° a cube bounded by the squares PK, NK, SK and the corresponding faces of the cubes about H. This last cube is itself a solid of the 8-cell namely that opposite to HA [1]).

3. Definition 1. — Let a point at a distance n times AB from A on the line AB be the point $A_n B$ or $B_{1-n} A$.

Definition 2. — Let pn be the projection, by a line parallel to the base on to the perpendicular of an equilateral triangle, of a length on the side equal to n times the side ($p = \frac{1}{2}\sqrt{3}$).

The 16-cell.

4. Let $ABCD$, a tetrahedron in S^3, be one of the bounding solids of a 16-cell.

In this figure there are 8 solids at each vertex, a condition that will be satisfied if a tetrahedron be put on to each face of $ABCD$, one to each edge and one to each vertex. Let the vertices of those on ABC, BCD, CDA, DBA be D', A', B', C' respectively (fig. 2). Then those on the edges will be $ABC'D'$, $BCD'A'$, $CDA'B'$, $DAB'C'$, $ACB'D'$, $BDA'C'$.

Four of the eight tetrahedra at A are represented in fig. 2.

5. If the 16-cell be cut by a space S_1^3 parallel to S^3 and passing through the point $A_{\frac{1}{4}} D'$, the tetrahedron $ABCD'$ will

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[1]) This way of dealing with the 8-cell is given in the „Scientific Romances", No. 1, „What is the fourth Dimension?" by C. H. Hinton, published by Swan, Sonnenschein & Co, London, 1884, and the subsequent book of the same author: „The New Era of Thought" London, 1888. Also in considering the 600-cell I received some valuable suggestions from Mr. H. W. Curjel.

be cut by a plane passing through that point and parallel to $A\,B\,C$; hence its section will be an equilateral triangle with a side equal to $\frac{3}{4}\,A\,D'$. In fig. 2 its vertices $A_{\frac{1}{4}}D'$, $B_{\frac{1}{4}}D'$, $C_{\frac{1}{4}}D'$ are indicated as a, b, c. Similarly the sections of $B\,C\,D\,A'$, $C\,D\,A\,B'$, $D\,A\,B\,C'$ are equilateral triangles with a side equal to $\frac{1}{4}\,A\,D'$. Their vertices are the points $(B_{\frac{1}{4}}A',\ C_{\frac{1}{4}}A',\ D_{\frac{1}{4}}A')$, $(C_{\frac{1}{4}}B',\ D_{\frac{1}{4}}B',\ A_{\frac{1}{4}}B')$, $(D_{\frac{1}{4}}C',\ A_{\frac{1}{4}}C',\ B_{\frac{1}{4}}C')$.

It has been shewn by means of the tetrahedron $ABCD'$ that S^3_1 passes through the points $A_{\frac{1}{4}}D'$, $C_{\frac{1}{4}}D'$, and by the tetrahedron $ACDB'$ that it passes through the points $A_{\frac{1}{4}}B'$, $C_{\frac{1}{4}}B'$. But the edges AD', CD' are common to $ABCD'$ and $ACB'D'$ and the edges AB', CB' are common to $ACDB'$ and $ACB'D'$. Hence we have four points on $ACB'D'$ through which S^3_1 passes, giving as section a rectangle with sides $\frac{1}{4}$ and $\frac{3}{4}$ of $A\,D'$ indicated in fig. 2 by *acde*.

There are also rectangular sections of the tetrahedra on the other edges of $A\,B\,C\,D$.

Again $A\,D'$, $A\,B'$, $A\,C'$ are edges of the tetrahedron $A\,B'\,C'\,D'$ and it has been shewn that S^3_1 cuts them in the points $A_{\frac{1}{4}}D'$, $A_{\frac{1}{4}}B'$, $A_{\frac{1}{4}}C'$, giving as section of this tetrahedron an equilateral triangle with side equal to $\frac{1}{4}\,A\,D'$. Similarly the section (fig. 3) of each of the tetrahedra on the vertices B, C, and D will be an equilateral triangle with sides equal to $\frac{1}{4}\,A\,D'$.

In figure 2 are also indicated the sections of the 16-cell by spaces S^3_2, S^3_3 parallel S^3 and cutting $A\,D'$ in the points $A_{\frac{1}{2}}D'$, $A_{\frac{3}{4}}D'$. The shapes are shewn in the figures 4 and 5.

A space S^3_4 parallel to S^3 and passing through D' would also pass through $A'\,B'$ and C', giving as the last shape of the series a tetrahedron equal to $A\,B\,C\,D$ oppositely placed, this being the bounding tetrahedron of the 16-cell opposite to $A\,B\,C\,D$.

The 24-cell.

6. Let $A\,B\,C\,D\,E\,F$ (fig. 6), an octahedron in a space S^3, be one of the bounding solids of a 24-cell. In this figure there are 6 solids at each vertex, a conditon that will be satisfied if one be

put on to each face and one to each vertex of $ABCDEF$. By an inspection of figure 6 it is clear that

on the face	ABC	is the	octahedron	ABC	(AB)	(BC) (CA),
,, ,, ,,	ACE	,, ,,	,,	ACE	(AC)	(CE) (EA),
,, ,, ,,	AEF	,, ,,	,,	AEF	(AE)	(EF) (FA),
,, ,, ,,	AFB	,, ,,	,,	AFB	(AF)	(FB) (BA),
,, ,, ,,	DBC	,, ,,	,,	DBC	(DB)	(BC) (CD),
,, ,, ,,	DCE	,, ,,	,,	DCE	(DC)	(CE) (ED),
,, ,, ,,	DEF	,, ,,	,,	DEF	(DE)	(EF) (FD),
,, ,, ,,	DFB	,, ,,	,,	DFB	(DF)	(FB) (BD).

In the same manner the octahedron

on	A	is	A	(AB)	(AC)	(AE)	(AF) A',
,,	B	,,	B	(AB)	(BC)	(DB)	(BF) B',
,,	C	,,	C	(AC)	(BC)	(CD)	(CE) C',
,,	D	,,	D	(BD)	(CD)	(DE)	(DF) D',
,,	E	,,	E	(AE)	(CE)	(DE)	(EF) E',
,,	F	,,	F	(AE)	(BF)	(DF)	(EF) F'.

7. If the 24-cell be cut by a space S^3_1 parallel to S^3 and passing through the point of bisection of $A(AC)$ the octahedron $A(CE)$ will be cut in a plane parallel to ACE, and passing through the points of bisection of $C(AC)$, $C(CE)$, $E(AE)$, $A(AE)$, $E(CE)$; the section will be a regular hexagon $(abcdef)$.

Similarly the sections by S^3_1 of the remaining seven octahedra on the faces of $ABCDEF$ are regular hexagons.

Now in the octahedron $A(CE)$ we see that S^3_1 passes through the points $A_{\frac{1}{2}}(AC)$, $A_{\frac{1}{2}}(AE)$ and in the octahedron $A(FB)$ we see that it passes through $A_{\frac{1}{2}}(AB)$ and $A_{\frac{1}{2}}(AF)$. But the lines $A(AC)$, $A(AE)$ and $A(AF)$, $A(AB)$ are also edges of the octahedron AA', whence we find that the section by S^3_1 of this octahedron is a square with side equal to half the edge of the 24-cell. There will be similar square sections of the octahedra on B, C, D, E, F.

The shape is shewn in fig. 7; it is a combination of octahedron and cube, in the crystallographic sense, the octahedron predominating.

Let S^3_2 be a space parallel to S^3 and passing through (AC), it will also pass through (CE), (AE). So that it coincides with that face of the octahedron $A(CE)$.

It also coincides with the face $(A\,B)$ $(A\,F)$ $(B\,F)$ of the octahedron $A\,(B\,F)$. The points $(A\,B)$, $(A\,F)$, $(A\,E)$, $(A\,C)$ are the vertices of a square section of the octahedron $A\,A'$. This section of the 24-cell, then, is bounded by 8 equilateral triangles, and 6 squares, the sides being equal to the edges of the 24-cell. The shape (fig. 8) is that of the combination of octahedron and cube in equilibrium.

The octahedra already grouped about the octahedron $A\,D$ give 4 solids at $(A\,C)$, namely $A\,A'$, $A\,(C\,E)$, $C\,C'$, $A\,(B\,C)$; of these the first two only are shewn in figure 6.

Putting one on the face $(A\,C)$ $(C\,E)$ $(A\,E)$ and one on the face $(A\,B)$ $(A\,C)$ $(B\,C)$ we have the required number 6 at $(A\,C)$. There will be similarly placed octahedra at $(A\,B)$, $(A\,F)$, etc.

Another parallel space S^3_3 passing through the middle point of $(A\,C)\,A'$ will give a square section of $A\,A'$ and square sections of the octahedra $C\,C'$ and $E\,E'$. These three squares determine the section of $(A\,C)\,E'$; it is a regular hexagon. This shape then is like the section by S^3_1. A space S^3_4 parallel to S^3 and passing through A' will coincide with the octahedron $A'\,B'\,C'\,D'\,E'\,F'$.

The 120-cell.

8. The sections of this cell can be deduced in a similar manner to that in which those of the 16-cell and 24-cell were obtained.

The plans are given in the diagrams VIII—XIV.

The 600-cell.

9. Let $A\,B\,C\,D$, a tetrahedron in S^3, be one of the bounding solids of a 600-cell. In this figure there are 20 solids at each vertex and this condition is satisfied at the vertices of $A\,B\,C\,D$ if a tetrahedron be put on to each of its faces, two to each edge and 10 to each vertex. The bases of the tetrahedra in each of these groups of 20 are the faces of an icosahedron [1]). Let $I\,A$ be that one which is bounded by the bases of the tetrahedra meeting at A. The vertex of the tetrahedron on $A\,B\,C$ as base is D' and the vertex of the one on $A\,C\,D'$ is $(A\,C)$; the tetrahedron on $A\,D'$ $(A\,C)$ is $A\,D'$ $(A\,C)$ (A_D), likewise that on A $(A\,C)$ (A_D) is $A\,(A\,C)$ (A_D) (A_B) and that on A (A_D) (A_B) is $A\,(A_D)$ (A_B) (A_C).

[1]) The sections of such icosahedra appear as zones on the sections of the 600-cell.

Thus $(A_B)\ (A_C)\ (A_D)$ is the face opposite to $B\,C\,D$ on $I\,A$ (fig. 9 and 10), etc.

The five tetrahedra just given are all differently related to $A\,B\,C\,D$ and, if they be taken as types, the vertices of the remaining tetrahedra about $A\,B\,C\,D$ may at once be written down and can then easily be placed in space. For instance there are four of the form $A\,B\,C\,D'$. The form $A\,C\,B'\,(A\,C)$ gives two tetrahedra on $A\,C$ namely $A\,C\,B'\,(A\,C)$ and $A\,C\,D'\,(A\,C)$ and also two on each of the other edges as $B\,C\,D'\,(B\,C)$ and $B\,C\,A'\,(B\,C)$ on $B\,C$, and so on. The form $A\,D'\,(A\,C)\,(A_D)$ gives six tetrahedra touching A arranged in pairs:

$$\begin{array}{lcl} A\,D'\ (A\,C)\ (A_D) & \text{and} & A\,D'\ (A\,B)\,(A_D), \\ A\,C'\ (A\,B)\ (A_C) & ,, & A\,C'\ (A\,D)\ (A_C), \\ A\,B'\ (A\,C)\ (A_B) & ,, & A\,B'\ (A\,D)\ (A_B) \end{array}$$

and also those about B, C, and D. So $A\ (A\,C)\ (A_D)\ (A_B)$ gives three tetrahedra about A namely $A\,(A\,C)\,(A_D)\,(A_B)$, $A\,(A\,B)\,(A_C)\,(A_D)$ and $A\ (A\,D)\ (A_B)\ (A_C)$; about B are $B\ (B\,C)\ (B_A)\ (B_D)$ and so on.

There are also four of the form $A\,(A_B)\,(A_C)\,(A_D)$. The remaining vertices of the 600-cell are named as follows: A_1 is the vertex of the tetrahedron on $(A_B)\,(A_C)\,(A_D)$, (A^2B) is the vertex of the tetrahedron on $(A\,B)\ (A_C)\ (A_D)$ and $A'\ (A'_B)\ (A'_C)\ (A'_D)$ is related to $A'\,B\,C\,D$ as $A\,(A_B)\,(A_C)\,(A_D)$ is to $A\,B\,C\,D$ (see list of vertices).

The tetrahedron on the side of the 600-cell opposite to $A\,B\,C\,D$ is $\alpha\,\beta\,\gamma\,\delta$, the lines drawn from A to α, from B to β and so on being diameters of the figure.

The arrangement of tetrahedra about $\alpha\,\beta\,\gamma\,\delta$ is similar to that about $A\,B\,C\,D$, so that the vertices opposite to those already given may be written down by simply changing A into α, B into β, C into γ, D into δ.

10. Let the 600-cell be cut by a space S_1^3 passing through the points $A'\ B'\ C'\ D'$. It will be parallel to S^3. Now S_1^3 and $I\,A$ intersect in a plane passing through the points B', C', D', fig. 11. To find where this plane cuts the lines $B\,(A\,B)$, $C\,(A\,C)$, $D\,(A\,D)$, let $I\,A$ be projected on a plane passing through C', (A_C), $(A\,C)$, C.

This projection is shewn in fig. 18, where R, M, O, P are the points C', (A_C), $(A\,C)$, C, and N is the middle point of $(A_D)\,(A_B)$, $R\,V$ is drawn parallel to $M\,N$ and $V\,O$ is the distance from $A\,C$ at which the plane cuts $C(A\,C)$. Let $V\,O$ be a times $P\,O$ and we have the points $(A\,B)_a\,B$, $(A\,C)_a\,C$, $(A\,D)_a\,D$; for the remaining vertices of this form see list of vertices and diagram I.

Again S^3_1 and $I\,D'$ intersect in a plane passing through the points $(C\,A)_a\,C$, $(CA)_a\,A$, $(A\,B)_a\,A$, $(A\,B)_a\,B$, $(B\,C)_a\,B$, $(B\,C)_a\,C$, of fig. 12.

Let $R\,M\,O\,P$, fig. 18, be a projection of $I\,D'$ on a plane through $(A\,B)\,(D'_C)\,(C_D)\,C$. Then $R\,q = p\,a$ and $q\,V_1$ is parallel to $M\,N$. It will be found that $V_1P = a$ times $O\,P$, whence we have the points $C_a\,(C_D)$, $A_a\,(A_D)$, $B_a\,(B_D)$ of fig. 12.

From this it will be seen that the tetrahedron $A\,(A_B)\,(A_C)\,(A_D)$ is cut in the points $A_a\,(A_B)$, $A_a\,(A_C)$, $A_a\,(A_D)$ (fig. 13, $p\,q\,r$); likewise $A\,(A\,C)\,(A_B)\,(A_D)$ is cut in the points $A_a\,(A_B)$, $A_a\,(A_D)$, $(A\,C)_a\,A$ (fig. 13, $p\,r\,s$) and $A\,D'\,(A\,C)\,(A_D)$ is cut in the points $(A\,C)_a\,A$, $A_a\,(A_D)$, D' (fig. 13, $s\,r\,D'$).

The tetrahedron $A\,C\,D'\,(A\,C)$ is cut in the points $(A\,C)_a\,A$ $(A\,C)_a\,C$, D' (fig. 13, $s\,t\,D'$).

Here we have a section of a tetrahedron of each type given in Fig. 10 except $A\,B\,C\,D$ which is not cut by S^3_1, and as the tetrahedra of each type are all similarly related to $A\,B\,C\,D$ their sections will be equal.

In constructing the model of this section of the 600-cell there will be four equilateral triangles $(p\,q\,r)$ related to each other as the vertices of a tetrahedron.

The sides of the equilateral triangles are the bases of isosceles triangles ($p\,r\,s$, fig. 13) and the sides of these isosceles triangles are the bases of other isosceles triangles ($s\,r\,D'$). The last isosceles triangles are arranged in pairs with a common side, and the other sides are the sides of isosceles triangles ($s\,t\,D'$).

One of the four similar regions of this model is given in diagram I (N, M', L'); these regions are connected by triangles (M).

11. Let the 600-cell be cut by a space S^3_2 passing through $(A\,B)$, $(A\,C)$, $(A\,D)$, $(B\,C)$, $(B\,D)$, $(C\,D)$. This is parallel to S^3_1. Here

$I(A_D)$ and S^3_1 intersect in a plane passing through the points D' $A_a(AB)$, $A_a(A_C)$.

A projection of this on a plane through AD' $(A\delta)$ A_1 is given in fig. 19. There $D'r$ is the projection of the plane of intersection of IA_D and S^3_1; Ar is made equal to pa and the line $D'r$ divides (AB) A in the ratio a: $1-a$; Vr_1 is drawn through (AB) and is parallel to $D'r$. Then we have $r_1(A_C) = pa$.

Also let the length of which $D'V$ is the projection be equal to c times the edge, from which we see that S^3_2 passes through the points $(A_C)_a A$, $(A_B)_a A$, $D'_c(D'_C)$, $D'_c(D'_B)$. In diagram II these points are 2, 1, 14, 11; 6 and 5 are (AB) and (AC). A plane through IA parallel to BCD and passing through the points (AB), (AC), (AD) gives the points $D'_a(A_D)$, $B'_a(A_B)$, $C'_a(A_C)$, the first of these being point 8 of diagram II, fig. 11, (see list of vertices).

This model is constructed in the same way as the first. The points where the sectional space cuts the edges of the 600-cell are determined by means of the icosahedra [1]) and are then found on the individual tetrahedra.

12. Let the 600-cell be cut by a space S^3_3 through (A_B), (A_C), (A_D), (B_A), (B_C), (B_D), (C_A), (C_B), (C_D), (D_A), (D_C), (D_B); this is parallel to S^3_2 and its intersection with $I(A_D)$ is given in projection by the line $V_1(A_C)$ parallel to Vr_1 (fig. 19). Now V_1 bisects $D'(D'_C)$, and let $(AB)S$ be equal to f times the edge. Again $V_1(A_C)$ divides $(AB)(D'_C)$ in the ratio a: $1-a$. We have then the points $(AB)_f$ (A^2B), $(AC)_f(A^2C)$, $(AB)_a(D'_C)$, $(AC)_a$ (D'_B), $D'_{\frac{1}{2}}$ (D'_C), $D'_{\frac{1}{2}}$ (D'_B); see on diagram III the points 6, 5, 13, 14, 18, 8; 1 and 2 are (A_B) and (A_C) (consult list of vertices).

13. Let the 600-cell be cut by a space S^3_4 through (A'_B), (A'_C), (A'_D), (B'_A), (B'_C), (B'_D), (C'_A), (C'_B), (C'_D), (D'_A), (D'_B), (D'_C); this is parallel to S^3. The intersection of S^3_4 and $I(A_D)$ is shewn in projection by the line $(D'_C)r_2$ parallel to $V_1(A_C)$, fig, 19.

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[1]) See foot note on page 8.

Here $T\ (A^2 B) = f$ times the edge; r_2 bisects $(A_C) A_1$ and $(D'_C)\, r_2$ divides $(A^2B)(A_C)$ in the ratio $a : 1-a$.

This gives the points $(A_C)_{\frac{1}{2}} A_1$, $(A_B)_{\frac{1}{2}} A_1$, $(A^2B)_a (A_C)$, $(A^2C)_a (A_B)$, $(A^2B)_f (AB)$, $(A^2C)_f (AC)$. See on diagram IV the points (5, 4, 6, 3, 7, 2); 8 and 1 are the points (D'_C), (D'_B).

Again $I A_1$ and S^3_4 intersect in a plane through the point $(A^2C)_a (A_B)$ and parallel to (A_D), (A_B), (A_C). This is shewn in projection in fig. 22. It gives the points $(A_B)_a\ (A\beta)$, $(A^2C)_a (A_D)$, $(A_D)_a\ (A\delta)$, $(A^2B)_a (A_D)$.

14. Let the 600-cell be cut by a space S^3_5 in points (A^2B), (AB^2), (A^2C), (AC^2) (A^2D), (AD^2), (B^2C), (BC^2), (C^2D), (CD^2), (B^2D), (BD^2); it will be parallel to S^3. The line $V_2 r_3$, parallel to $(D'_C)\, r_2$ and passing through (A^2B), (fig. 19), is the projection of the intersection of $I(A_D)$ and S^3_5. Comparing this with the line $V r_1$, we have $A_1 r_3 = p\,c$ and $(D'_C) V_2 = p\,a$.

This gives the points $(A_1)_c (A_C)$, $(A_1)_c (A_B)$, $(D'_C)_a (A\delta)$, $(D'_B)_a (A\delta)$ (on plate V the points 4, 3, 6, 1; 5 and 2 are (A^2B) and (A^2C), etc.).

The solid $I A_1$ and S^3_5 intersect in a plane passing through (A^2B), (A^2C), (A^2D) and this plane passes through $(A\delta)_a (A_D)$, $(A\beta)_a (A_B)$, $(A\gamma)_a (A_C)$, fig. 22.

The space S^3_5 and $I\delta_1$ intersect in a plane through $(D'_B)_a (A\delta)$ that is parallel to $(D'_B)(D'_A)(D'_C)$. This plane is projected in the line $r\,v$, fig. 20.

Let $V(D'_A)$ be called h times the edge and we have the points $(D'_A)_h (\delta^2\alpha)$, $(D'_B)_h (\delta^2\beta)$, $(D'_C)_h (\delta^2\gamma)$.

Again S^3_4 and $I(A\delta)$ intersect in a plane through (D'_C), (D'_B), $(A_D)_{\frac{1}{2}} A_1$, fig. 21. Here $(D'_B) V$ and a parallel plane through (A^2B) (A^2C) is projected in the line $r\,v_1$ giving $A_1 V_1 = c$ times the edge and $(D'_B) r = p f$. Whence we have the points $(A_1)_c (A_D)$, $(D'_B)_f\ \delta_1$, $(D'_C)_f\ \delta_1$ see list.

In diagram V the zone on $I A_1$ is 23, 5, 12, 2, 15, 13 and that on $I\delta_1$ is 2, 7, 5, 8, 9, 10, 11.

15. Let the 600-cell be cut by a space S^3_6 through $A_1 B_1 C_1 D_1$ parallel to S^3; $I(A_D)$ intersects S^3_6 in a plane through A_1 parallel to its intersection with S^3_5.

In fig. 19 this plane is projected in the line $A_1 V_3$ and comparing this with $D' r$ we have the points $(A\delta)_a (D'_C)$, $(A\delta)_a (D'_B)$, $(A^2B)_a (A\delta)$, $(A^2C)_a (A\delta)$ (diagram VI points 1, 3, 4, 2, 5). Here $I\delta_1$ and S^3_6 intersect in a plane parallel to $(D'_A)(D'_B)(D'_C)$ through $(A\delta)_a (D'_B)$. This plane is projected in the line $r_1 v_1$, fig. 20, where $r_1 (A\delta) = pa$. Then $(D'_A) V_1 = a$ times the edge, whence we have the points $(D'_A)_a (\delta^2 \alpha)$, $(D'_B)_a (\delta^2\beta)$, $(D'_C)_a (\delta^2\gamma)$ (diagram VI points 12, 11, 13).

Now S^3_6 and $I A_1$ intersect in a plane parallel to $(A_B)(A_C)(A_D)$ and passing through the points $(A^2B)_a (A\delta)$.

In fig. 22 the line $(A^2B)(A\delta)$ is divided in the ratio $a : 1-a$ and through the point of division a line $r V_1$ is drawn parallel to $(A_D)(A_B)$.

Let $(A^2B)r = m$ times the edge, and it will be found that $(A\delta)(A_D)$ is divided in the ratio $h : 1-h$, $(A^2C)(A\delta)$ is divided in the ratio $a : 1-a$, $(A^2C)(\alpha'_\gamma)$ is divided in the ratio $m : 1-m$, $(A^2C)(A\beta)$ is divided in the ratio $a : 1-a$ and finally $V_1 (A\beta) = h$ times the edge; whence we have the points

$$(A^2B)_m (\alpha'_\beta), \quad (A^2B)_a (A\delta), \quad (A^2B)_a (A\gamma), \quad (A\delta)_h (A_D),$$
$$(A\gamma)_h (A_C), \quad (A^2C)_a (A\delta), \quad (A^2D)_a (A\gamma), \quad (A^2D)_m (\alpha'_\delta),$$
$$(A^2C)_m (\alpha'_\gamma), \quad (A^2C)_a (A\beta), \quad (A^2D)_a (A\beta), \quad (A\beta)_h (A_B).$$

Here S^3_6 and $I(A\delta)$ intersect in a plane through A_1 parallel to the intersection of S^3_5 with $I(A\delta)$; $A_1 r_2$, fig. 21, is a projection of this plane. Let $(D'_B) r_2 = pn$. It will be found that $(A^2C)(\alpha'_\gamma)$ is cut in the ratio $m : 1-m$, $(A^2C)(\delta^2\beta)$ is cut in the ratio $c : 1-c$, $(D'_B)(\delta^2\beta)$ is cut in the ratio $a : 1-a$, whence we have the points $(A^2C)_m (\alpha'_\gamma)$, $(A^2B)_m (\alpha'_\beta)$, $(A^2C)_c (\delta^2\beta)$, $(A^2B)_c (\delta^2\gamma)$, $(D'_B)_a (\delta^2\beta)$, $(D'_C)_a (\delta^2\gamma)$, $(D'_B)_n \delta_1$, $(D'_C)_n \delta_1$. These are, diagram VI, the points 15, 22, 24, 23, 11, 13, 26, 25, see list.

16. Let the 600 cell be cut by a space S^3_7 through $(A\beta)$, $(A\gamma)$, $(A\delta)$, $(B\alpha)$, $(B\gamma)$, $(B\delta)$, $(C\alpha)$, $(C\beta)$, $(C\delta)$, $(D\alpha)$, $(D\beta)$, $(D\gamma)$. Then I A_1 and S^3_7 intersect in a plane through $(A\beta)$, $(A\gamma)$, $(A\delta)$, fig. 22, $(A\beta)\, r_1$. It also passes through the points $(A^2B)_a\,(\alpha'_\beta)$, $(A^2C)_a\,(\alpha'_\gamma)$ $(A^2D)_a\,(\alpha'_\delta)$ (diagram VII points 3, 1 and 2). And S^3_7 and $I\delta_1$ intersect in a plane through $(A\,\delta)$, $(B\,\delta)$, $(C\,\delta)$, shewn in projection by the line $(A\,\delta)\,V_2$, fig. 20. This gives the points $(\delta^2\alpha)_a\,(D'_A)$, $(\delta^2\beta)_a\,(D'_B)$, $(\delta^2\gamma)_a\,(D'_C)$, i. e. the points 6, 5 and 4 of diagram VII. Likewise S^3_7 and $I\,(A\,\delta)$ intersect in a plane through $(A^2\,C)_a\,(\alpha'_\gamma)$ and parallel to the intersection of $I\,(A\,\delta)$ and S^3_6. This plane is shewn in projection by the line $V_2\,r_3$, fig 21; $A_1\,V_2$ and $\delta_1\,r_3$ are equal to $p\,f$; $(A^2C)\,(\delta^2\beta)$ is bisected, so that we have the points $(A_1)_f\,(\alpha'_\gamma)$, $(A_1)_f\,(\alpha'_\beta)$, $(\delta_1)_f\,(D'_B)$, $(\delta_1)_f\,(D'_C)$, $(A^2C)_a\,(\alpha'_\gamma)$, $(A^2B)_a\,(\alpha'_\beta)$, $(\delta^2\beta)_a\,(D'_B)$, $(\delta^2\gamma)_a\,(D'_C)$, $(A^2C)_{\frac{1}{2}}\,(\delta^2\beta)$, $(A^2B)_{\frac{1}{2}}\,(\delta^2\gamma)$. In diagram VII these points correspond to 7, 8, 12, 11, 1, 3, 5, 4, 10, 9. The symmetry shews that this is the central section; the other sections are now repeated in reverse order.

In practically constructing the sections I have found that their symmetry is made more obvious by colouring the faces. The letters on the faces of diagrams I—XIV denote colours, the plane and accented letters denote corresponding colours and the remaining sections are the same as those given with the plane and accented letters interchanged.

17. Numbers of tetrahedra of each colour in the 600-cell as shewn by the sections.

L and	L',	21 each,	.....	42
M ,,	M',	60 each,	.....	120
N ,,	N',	72 ,, ,	.....	144
P ,,	P'	24 ,, ,	.....	96
R ,,	R'			
S ,,	S'	12 ,, ,	.....	48
T ,,	T'			
	X	48	.....	48
	Y	96	.....	96
	Z	6	.....	6
			Total..	600

List of the 120 vertices of the 600-cell.

$$A,\ B,\ C,\ D,$$
$$A',\ B',\ C',\ D',$$

$$(AB),\ (AC),\ (AD),\ (BC),\ (BD),\ (CD),$$

$$(A_B),\ (A_C),\ (A_D),\ (B_A),\ (B_C),\ (B_D),\ (C_A),\ (C_B),\ (C_D),\ (D_A),\ (D_B),\ (D_C),$$
$$(A'_B),\ (A'_C),\ (A'_D),\ (B'_A),\ (B'_C),\ (B'_D),\ (C'_A),\ (C'_B),\ (C'_D),\ (D'_A),\ (D'_B),\ (D'_C),$$
$$(A^2B),\ (AB^2),\ (A^2C),\ (AC^2),\ (A^2D),\ (AD^2),\ (B^2C),\ (BC^2),\ (B^2D),\ (BD^2),\ (C^2D),\ (CD^2),$$

$$A_1,\ B_1,\ C_1,\ D_1,$$

$$(A\beta),\ (A\gamma),\ (A\delta),\ (B\alpha),\ (B\gamma),\ (B\delta),\ (C\alpha),\ (C\beta),\ (C\delta),\ (D\alpha),\ (D\beta),\ (D\gamma),$$

$$\alpha_1,\ \beta_1,\ \gamma_1,\ \delta_1,$$

$$(\alpha^2\beta),\ (\alpha\beta^2),\ (\alpha^2\gamma),\ (\alpha\gamma^2),\ (\alpha^2\delta),\ (\alpha\delta^2),\ (\beta^2\gamma),\ (\beta\gamma^2),\ (\beta^2\delta),\ (\beta\delta^2),\ (\gamma^2\delta),\ (\gamma\delta^2),$$
$$(\alpha'_\beta),\ (\alpha'_\gamma),\ (\alpha'_\delta),\ (\beta'_\alpha),\ (\beta'_\gamma),\ (\beta'_\delta),\ (\gamma'_\alpha),\ (\gamma'_\beta),\ (\gamma'_\delta),\ (\delta'_\alpha),\ (\delta'_\beta),\ (\delta'_\gamma),$$
$$(\alpha_\beta),\ (\alpha_\gamma),\ (\alpha_\delta),\ (\beta_\alpha),\ (\beta_\gamma),\ (\beta_\delta),\ (\gamma_\alpha),\ (\gamma_\beta),\ (\gamma_\delta),\ (\delta_\alpha),\ (\delta_\beta),\ (\delta_\gamma),$$

$$(\alpha\beta),\ (\alpha\gamma),\ (\alpha\delta),\ (\beta\gamma),\ (\beta\delta),\ (\gamma\delta),$$

$$\alpha',\ \beta',\ \gamma',\ \delta',$$
$$\alpha,\ \beta,\ \gamma,\ \delta.$$

List of vertical points of sections. [1])

Diagram I.

$$A',\ B'_{(9)},\ C'_{(7)},\ D'_{(8)},$$

$$A_a(A_B)_{(1)},\ B_a(B_A),\ C_a(C_A),\ D_a(D_A),$$
$$A_a(A_C)_{(2)},\ B_a(B_C),\ C_a(C_B),\ D_a(D_B),$$
$$A_a(A_D)_{(3)},\ B_a(B_D),\ C_a(C_D),\ D_a(D_C),$$

$$(AB)_aA_{(6)},\ (AC)_aA_{(5)},\ (AD)_aA_{(4)},\ (BC)_aB,\ (BD)_aB,\ (CD)_aC,$$
$$(AB)_aB_{(11)},\ (AC)_aC_{(12)},\ (AD)_aD_{(10)},\ (BC)_aC,\ (BD)_aD,\ (CD)_aD.$$

Diagram II.

$$(AB)_{(6)},\ (AC)_{(5)},\ (AD)_{(4)},\ (BC),\ (BD),\ (CD),$$

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[1]) The small numbers between brackets refer to the numbered points in the diagrams.

$$(A_B)_a A_{(1)},\ (B_A)_a B,\ (C_A)_a C,\ (D_A)_a D,$$
$$(A_C)_a A_{(2)},\ (B_C)_a B,\ (C_B)_a C,\ (D_B)_a D,$$
$$(A_D)_a A_{(3)},\ (B_D)_a B,\ (C_D)_a C,\ (D_C)_a D,$$

$$A'_a(B_A),\ B'_a(A_B)_{(9)}\ \ C'_a(A_C)_{(7)},\ D'_a(A_D)_{(8)},$$
$$A'_a(C_A),\ B'_a(C_B),\ \ C'_a(B_C),\ \ D'_a(B_D)_{(16)},$$
$$A'_a(D_A),\ B'_a(D_B),\ \ C'_a(D_C),\ \ D'_a(C_D)_{(17)},$$

$$A'_c(A'_B),\ B'_c(B'_A),\ \ C'_c(C'_A,\ \ D'_c(D'_A)_{(15)},$$
$$A'_c(A'_C),\ B'_c(B'_C)_{(12)},\ C'_c(C'_B)_{(10)},\ D'_c(D'_B)_{(11)},$$
$$A'_c(A'_D),\ B'_c(B'_D)_{(19)},\ C'_c(C'_D)_{(18)},\ D'_c(D'_C)_{(14)}.$$

Diagram III.

$$(A_B)_{(1)},\ (B_A),\ (C_A),\ (D_A),$$
$$(A_C)_{(2)},\ (B_C),\ (C_B),\ (D_B),$$
$$(A_D)_{(3)},\ (B_D),\ (C_D),\ (D_C),$$

$$(AB)_f(A^2B)_{(6)},\ (AB)_f(AB^2)_{(22)},$$
$$(AC)_f(A^2C)_{(5)},\ (AC)_f(AC^2)_{(20)},$$
$$(AD)_f(A^2D)_{(4)},\ (AD)_f(AD^2)_{(21)},$$
$$(BC)_f(B^2C),\ \ (BC)_f(BC^2),$$
$$(BD)_f(B^2D),\ \ (BD)_f(BD^2),$$
$$(CD)_f(C^2D),\ \ (CD)_f(CD^2),$$

$$(AB)_a(C'_D)_{(12)},\ (AC)_a(B'_D)_{(15)},\ (AD)_a(B'_C)_{(10)},\ (BC)_a(A'_D),\ (BD)_a(A'_C),\ (CD)_a(A'_B),$$
$$(AB)_a(D'_C)_{(13)},\ (AC)_a(D'_B)_{(14)},\ (AD)_a(C'_B)_{(11)},\ (BC)_a(D'_A),\ (BD)_a(C'_A),\ (CD)_a(B'_A),$$

$$A'\tfrac{1}{2}(A'_B),\ B'\tfrac{1}{2}(B'_A),\ \ C'\tfrac{1}{2}(C'_A),\ \ D'\tfrac{1}{2}(D'_A)_{(25)},$$
$$A'\tfrac{1}{2}(A'_C),\ B'\tfrac{1}{2}(B'_C)_{(9)},\ C'\tfrac{1}{2}(C'_B)_{(17)},\ D'\tfrac{1}{2}(D'_B)_{(8)},$$
$$A'\tfrac{1}{2}(A'_D),\ B'\tfrac{1}{2}(B'_D)_{(16)},\ C'\tfrac{1}{2}(C'_D)_{(7)},\ D'\tfrac{1}{2}(D'_C)_{(18)}.$$

Diagram IV.

$$(A'_B),\ (B'_A),\ \ (C'_A),\ \ (D'_A)_{(24)},$$
$$(A'_C),\ (B'_C)_{(19)},\ (C'_B)_{(21)},\ (D'_B)_{(1)},$$
$$(A'_D),\ (B'_D)_{(18)},\ (C'_D)_{(22)},\ (D'_C)_{(8)},$$

$$(A_B)\tfrac{1}{2}A_{1(4)},\ (B_A)\tfrac{1}{2}B_1,\ (C_A)\tfrac{1}{2}C_1,\ (D_A)\tfrac{1}{2}D_1,$$
$$(A_C)\tfrac{1}{2}A_{1(5)},\ (B_C)\tfrac{1}{2}B_1,\ (C_B)\tfrac{1}{2}C_1,\ (D_B)\tfrac{1}{2}D_1,$$
$$(A_D)\tfrac{1}{2}A_{1(27)},\ (B_D)\tfrac{1}{2}B_1,\ (C_D)\tfrac{1}{2}C_1,\ (D_C)\tfrac{1}{2}D_1,$$

$$(A_B)_a(A\beta)_{(10)},\ (B_A)_a(B\alpha),\quad (C_A)_a(C\alpha),\quad (D_A)_a(D\alpha),$$
$$(A_C)_a(A\gamma)_{(11)},\ (B_C)_a(B\gamma),\quad (C_B)_a(C\beta),\quad (D_B)_a(D\beta),$$
$$(A_D)_a(A\delta)_{(9)},\ (B_D)_a(B\delta)_{(25)},\ (C_D)_a(C\delta)_{(26)},\ (D_C)_a(D\gamma),$$

$$(A^2B)_f(AB)_{(7)},\ (AB^2)_f(AB)_{(23)},$$
$$(A^2C)_f(AC)_{(2)},\ (AC^2)_f(AC)_{(17)},$$
$$(A^2D)_f(AD)_{(4)},\ (AD^2)_f(AD)_{(20)},$$
$$(B^2C)_f(BC),\quad (BC^2)_f(BC),$$
$$(B^2D)_f(BD),\quad (BD^2)_f(BD),$$
$$(C^2D)_f(CD),\quad (CD^2)_f(CD),$$

$$(A^2C)_a(A_B)_{(3)},\ (A^2D)_a(A_B)_{(13)},$$
$$(A^2D)_a(A_C)_{(14)},\ (A^2B)_a(A_C)_{(6)},$$
$$(A^2B)_a(A_D)_{(16)},\ (A^2C)_a(A_D)_{(12)},$$

$$(B^2C)_a(B_A),\quad (B^2D)_a(B_A),$$
$$(B^2D)_a(B_C),\quad (B^2A)_a(B_C),$$
$$(B^2A)_a(B_D),\quad (B^2C)_a(B_D),$$

$$(C^2B)_a(C_A),\quad (C^2D)_a(C_A),$$
$$(C^2D)_a(C_B),\quad (C^2A)_a(C_B),$$
$$(C^2A)_a(C_D),\quad (C^2B)_a(C_D),$$

$$(D^2B)_a(D_A),\quad (D^2C)_a(D_A),$$
$$(D^2C)_a(D_B),\quad (D^2A)_a(D_B),$$
$$(D^2A)_a(D_C),\quad (D^2B)_a(D_C).$$

Diagram V.

$$(A^2B)_{(5)},\ (A^2C)_{(2)},\ (A^2D)_{(13)},\ (B^2C),\ (B^2D),\ (C^2D),$$
$$(AB^2)_{(22)},\ (AC^2)_{(30)},\ (AD^2)_{(17)},\ (BC^2),\ (BD^2),\ (CD^2),$$

$$(A_1)_c(A_B)_{(3)},\ (B_1)_c(B_A),\ (C_1)_c(C_A),\ (D_1)_c(D_A),$$
$$(A_1)_c(A_C)_{(4)},\ (B_1)_c(B_C),\ (C_1)_c(C_C),\ (D_1)_c(D_B),$$
$$(A_1)_c(A_D)_{(7)},\ (B_1)_c(B_D),\ (C_1)_c(C_D),\ (D_1)_c(D_C),$$

$$(A\beta)_a(A_B)_{(23)},\ (B\alpha)_a(B_A),\ (C\alpha)_a(C_A),\ (D\alpha)_a(D_A),$$
$$(A\gamma)_a(A_C)_{(15)},\ (B\gamma)_a(B_C),\ (C\beta)_a(C_B),\ (D\beta)_a(D_B),$$
$$(A\delta)_a(A_D)_{(12)},\ (B\delta)_a(B_D),\ (C\delta)_a(C_D),\ (D\gamma)_a(D_C),$$

$$(A'_B)_h(\alpha^2\beta),\ (B'_A)_h(\alpha\beta^2),\quad (C'_A)_h(\alpha\gamma^2),\quad (D'_A)_h(\alpha\delta^2)_{(27)},$$
$$(A'_C)_h(\alpha^2\gamma),\ (B'_C)_h(\beta^2\gamma)_{(16)},\ (C'_B)_h(\beta\gamma^2)_{(18)},\ (D'_B)_h(\beta\delta^2)_{(11)},$$
$$(A'_D)_h(\alpha^2\delta),\ (B'_D)_h(\beta^2\delta)_{(13)},\ (C'_D)_h(\delta\gamma^2)_{(21)},\ (D'_C)_h(\gamma\delta^2)_{(8)},$$

$$(A'_B)_a(C\alpha),\quad (A'_B)_a(D\alpha),$$
$$(A'_C)_a(D\alpha),\quad (A'_C)_a(B\alpha),$$
$$(A'_D)_a(B\alpha),\quad (A'_D)_a(C\alpha),$$

$$(B'_A)_a(C\beta),\quad (B'_A)_a(D\beta),$$
$$(B'_C)_a(D\beta),\quad (B'_C)_a(A\beta)_{(15)},$$
$$(B'_D)_a(A\beta)_{(14)},\ (B'_D)_a(C\beta),$$

$$(C'_A)_a(B\gamma),\quad (C'_A)_a(D\gamma),$$
$$(C'_B)_a(D\gamma),\quad (C'_B)_a(A\gamma)_{(19)},$$
$$(C'_D)_a(A\gamma)_{(20)},\ (C'_D)_a(B\gamma),$$

$$(D'_A)_a(B\delta)_{(28)},\ (D'_A)_a(C\delta)_{(26)},$$
$$(D'_B)_a(C\delta)_{(25)},\ (D'_B)_a(A\delta)_{(1)},$$
$$(D_C)_a(A\delta)_{(6)},\ (D'_C)_a(B\delta)_{(29)},$$

$$(A'_B)_f\alpha_1,\ (B'_A)_f\beta_1,\ (C'_A)_f\gamma_1,\ (D'_A)_f\delta_{1\,(24)},$$
$$(A'_C)_f\alpha_1,\ (B'_C)_f\beta_1,\ (C'_B)_f\gamma_1,\ (D'_B)_f\delta_{1\,(10)},$$
$$(A'_D)_f\alpha_1,\ (B'_D)_f\beta_1,\ (C'_D)_f\gamma_1,\ (D'_C)_f\delta_{1\,(9)}.$$

Diagram VI.

$$A_{1(1)},\ B_1,\ C_1,\ D_1,$$

$$(A^2B)_a(A\delta)_{(2)},\ (A^2B)_a(A\gamma)_{(21)},$$
$$(A^2C)_a(A\beta)_{(16)},\ (A^2C)_a(A\delta)_{(5)},$$
$$(A^2D)_a(A\gamma)_{(19)},\ (A^2D)_a(A\beta)_{(18)},$$

$$(AB^2)_a(B\gamma),\quad (AB^2)_a(B\delta),$$
$$(B^2C)_a(B\delta),\quad (B^2C)_a(B\alpha),$$
$$(B^2D)_a(B\alpha),\quad (B^2D)_a(B\gamma),$$

$$(AC^2)_a(C\beta),\quad (AC^2)_a(C\delta),$$
$$(BC^2)_a(C\delta),\quad (BC^2)_a(C\alpha),$$
$$(C^2D)_a(C\alpha),\quad (C^2D)_a(C\beta),$$

$$(AD^2)_a(D\beta),\quad (AD^2)_a(D\gamma),$$
$$(BD^2)_a(D\gamma),\quad (BD^2)_a(D\alpha),$$
$$(CD^2)_a(D\alpha),\quad (CD^2)_a(D\beta),$$

$$(A^2B)_m(\alpha'_\beta)_{(22)},\ (AB^2)_m(\beta'_\alpha)_{(47)},\ (AC^2)_m(\gamma'_\alpha)_{(29)},\ (AD^2)_m(\delta'_\alpha)_{(37)},$$
$$(A^2C)_m(\alpha'_\gamma)_{(15)},\ (B^2C)_m(\beta'_\gamma),\quad (BC^2)_m(\gamma'_\beta),\quad (BD^2)_m(\delta'_\beta),$$
$$(A^2D)_m(\alpha'_\delta)_{(50)},\ (B^2D)_m(\beta'_\delta),\quad (C^2D)_m(\gamma'_\delta),\quad (CD^2)_m(\delta'_\gamma),$$

$$(A\beta)_h\ (A_B)_{(17)},\ (B\alpha)_h\ (B_A),\quad (C\gamma)_h\ (C_A),\quad (D\alpha)_h\ (D_A),$$
$$(A\gamma)_h\ (A_C)_{(20)},\ (B\gamma)_h\ (B_C),\quad (C\beta)_h\ (C_B),\quad (D\beta)_h\ (D_B),$$
$$(A\delta)_h\ (A_D)_{(14)},\ (B\delta)_h\ (B_D),\quad (C\delta)_h\ (C_D),\quad (D\gamma)_h\ (D_C),$$

$$(A^2B)_c\,(\gamma^2\delta)_{(45)},\ (AB^2)_c\ (\gamma^2\delta)_{(26)},\ (A^2B)_c\ (\gamma\delta^2)_{(23)},\ (AB^2)_c(\gamma\delta^2)_{(48)},$$
$$(A^2C)_c\ (\beta^2\delta)_{(31)},\ (AC^2)_c\ (\beta^2\delta)_{(30)},\ (A^2C)_c\ (\beta\delta^2)_{(24)},\ (AC^2)_c\ (\beta\delta^2)_{(28)},$$
$$(A^2D)_c\ (\beta^2\gamma)_{(39)},\ (AD^2)_c\ (\beta^2\gamma)_{(36)},\ (A^2D)_c\ (\beta\gamma^2)_{(40)},\ (AD^2)_c\ (\beta\gamma^2)_{(38)},$$

$$(B^2C)_c\ (\alpha^2\delta),\quad (BC^2)_c\ (\alpha^2\delta),\quad (B^2C)_c\ (\alpha\delta^2),\quad (BC^2)_c\ (\alpha\delta^2),$$
$$(B^2D)_c\ (\alpha^2\gamma),\quad (BD^2)_c\ (\alpha^2\gamma),\quad (B^2D)_c\ (\alpha\gamma^2),\quad (BD^2)_c\ (\alpha\gamma^2),$$
$$(C^2D)_c\ (\alpha^2\beta),\quad (CD^2)_c\ (\alpha^2\beta),\quad (C^2D)_c\ (\alpha\beta^2),\quad (CD^2)_c\ (\alpha\beta^2),$$

$$(A'_B)_a\ (\alpha^2\beta),\quad (B'_A)_a\ (\alpha\beta^2),\quad (C'_A)_a\ (\alpha\gamma^2),\quad (D'_A)_a\ (\alpha\delta^2)_{(12)},$$
$$(A'_C)_a\ (\alpha^2\gamma),\quad (B'_C)_a\ (\beta^2\gamma)_{(35)},\ (C'_B)_a\ (\beta\gamma^2),\quad (D'_B)_a\ (\beta\delta^2)_{(11)},$$
$$(A'_D)_a\ (\alpha^2\delta),\quad (B'_D)_a\ (\beta^2\delta)_{(32)},\ (C'_D)_a\ (\gamma^2\delta)_{(44)},\ (D'_C)_a\ (\gamma\delta^2)_{(13)},$$

$$(A'_B)_n\,\alpha_1,\ (B'_A)_n\,\beta_1,\ (C'_A)_n\,\gamma_1,\ (D'_A)_n\,\delta_{1(27)},$$
$$(A'_C)_n\,\alpha_1,\ (B'_C)_n\,\beta_1,\ (C'_B)_n\,\gamma_1,\ (D'_B)_n\,\delta_{1(26)},$$
$$(A'_D)_n\,\alpha_1,\ (B'_D)_n\,\beta_1,\ (C'_D)_n\,\gamma_1,\ (D'_C)_n\,\delta_{1(25)},$$

$$(C\alpha)_a(A'_B),\quad (D\alpha)_a(A'_B),$$
$$(D\alpha)_a(A'_C),\quad (B\alpha)_a(A'_C),$$
$$(B\alpha)_a(A'_D),\quad (C\alpha)_a(A'_D),$$

$$(C\beta)_a(B'_A),\quad (D\beta)_a(B'_A),$$
$$(D\beta)_a(B'_C),\quad (A\beta)_a(B'_C)_{(34)},$$
$$(A\beta)_a(B'_D)_{(33)},\ (C\beta)_a(B'_D),$$

$$(B\gamma)_a(C'_A),\quad (D\gamma)_a(C'_A),$$
$$(D\gamma)_a(C'_B),\quad (A\gamma)_a(C'_B)_{(42)},$$
$$(A\gamma)_a(C'_D)_{(43)},\ (B\gamma)_a(C'_D),$$

$$(B\delta)_a(D'_A),\quad (C\delta)_a(D'_A),$$
$$(C\delta)_a(D'_B),\quad (A\delta)_a(D'_B)_{(4)},$$
$$(A\delta)_a(D'_C)_{(3)},\ (B\delta)_a(D'_C).$$

Diagram VII.

$$(A_1)_f(\alpha'_\beta)_{(8)},\ (A_1)_f(\alpha'_\gamma)_{(7)},\ (A_1)_f(\alpha'_\delta)_{(37)},$$
$$(B_1)_f(\beta'_\alpha),\quad (B_1)_f(\beta'_\gamma),\quad (B_1)_f(\beta'_\delta),$$
$$(C_1)_f(\gamma'_\alpha),\quad (C_1)_f(\gamma'_\beta),\quad (C_1)_f(\gamma'_\delta),$$
$$(D_1)_f(\delta'_\alpha),\quad (D_1)_f(\delta'_\beta),\quad (D_1)_f(\delta'_\gamma),$$
$$(\alpha_1)_f(A'_B),\quad (\alpha_1)_f(A'_C),\quad (\alpha_1)_f(A'_D),$$
$$(\beta_1)_f(B'_A),\quad (\beta_1)_f(B'_C),\quad (\beta_1)_f(B'_D),$$
$$(\gamma_1)_f(C'_A),\quad (\gamma_1)_f(C'_B),\quad (\gamma_1)_f(C'_D),$$
$$(\delta_1)_f(D'_A)_{(35)},\ (\delta_1)_f(D'_B)_{(12)},\ (\delta_1)_f(D'_C)_{(11)},$$

$$(A^2B)_a(\alpha'_\beta)_{(3)},\ (A^2C)_a(\alpha'_\gamma),\quad (A^2D)_a(\alpha'_\delta)_{(2)},$$
$$(AB^2)_a(\beta'_\alpha)_{(23)},\ (B^2C)_a(\beta'_\gamma),\quad (B^2D)_a(\beta'_\delta),$$
$$(AC^2)_a(\gamma'_\alpha)_{(16)},\ (BC^2)_a(\gamma'_\beta),\quad (C^2D)_a(\gamma'_\delta),$$
$$(AD^2)_a(\delta'_\alpha)_{(22)},\ (BD^2)_a(\delta'_\beta),\quad (CD^2)_a(\delta'_\gamma),$$
$$(\alpha^2\beta)_a(A'_B),\quad (\alpha^2\gamma)_a(A'_C),\quad (\alpha^2\delta)_a(A'_D),$$
$$(\alpha\beta^2)_a(B'_A),\quad (\beta^2\gamma)_a(B'_C)_{(20)},\ (\beta^2\delta)_a(B'_D)_{(18)},$$
$$(\alpha\gamma^2)_a(C'_A),\quad (\beta\gamma^2)_a(C'_B)_{(24)},\ (\gamma^2\delta)_a(C'_D)_{(26)},$$
$$(\alpha\delta^2)_a(D'_A),\quad (\beta\delta^2)_a(D'_B)_{(5)},\ (\gamma\delta^2)_a(D'_C)_{(4)},$$

$$(A\beta)_{(19)},\ (A\gamma)_{(25)},\ (A\delta)_{(36)},\ (B\alpha),\ (B\gamma),\ (B\delta)_{(13)},$$
$$(C\alpha),\quad (C\beta),\quad (C\delta)_{(14)},\ (D\alpha),\ (D\beta),\ (D\gamma).$$

The following lines are all bisected

$$(A^2B)(\gamma^2\delta)_{(33)},\ (A^2C)(\beta^2\delta)_{(30)},\ (A^2D)(\beta^2\gamma)_{(31)},$$
$$(A^2B)(\gamma\delta^2)_{(34)},\ (A^2C)(\beta\delta^2)_{(10)},\ (A^2D)(\beta\gamma^2)_{(32)},$$
$$(B^2C)(\alpha^2\delta),\quad (B^2D)(\alpha^2\gamma),\quad (C^2D)(\alpha^2\beta),$$
$$(B^2C)(\alpha\delta^2),\quad (B^2D)(\alpha\gamma^2),\quad (C^2D)(\alpha\beta^2),$$
$$(AB^2)(\gamma^2\delta)_{(27)},\ (AC^2)(\beta^2\delta)_{(17)},\ (AD^2)(\beta^2\gamma)_{(21)},$$
$$(AB^2)(\gamma\delta^2)_{(29)},\ (AC^2)(\beta\delta^2)_{(15)},\ (AD^2)(\beta\gamma^2)_{(23)},$$
$$(BC^2)(\alpha^2\delta),\quad (BD^2)(\alpha^2\gamma),\quad (CD^2)(\alpha^2\beta),$$
$$(BC^2)(\alpha\delta^2),\quad (BD^2)(\alpha\gamma^2),\quad (CD^2)(\alpha\beta^2).$$

(6 February 1900).

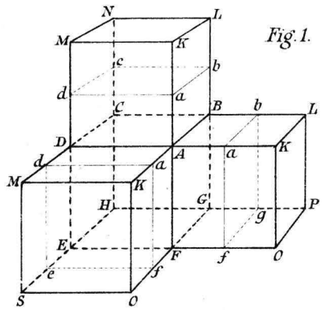


Fig. 1.

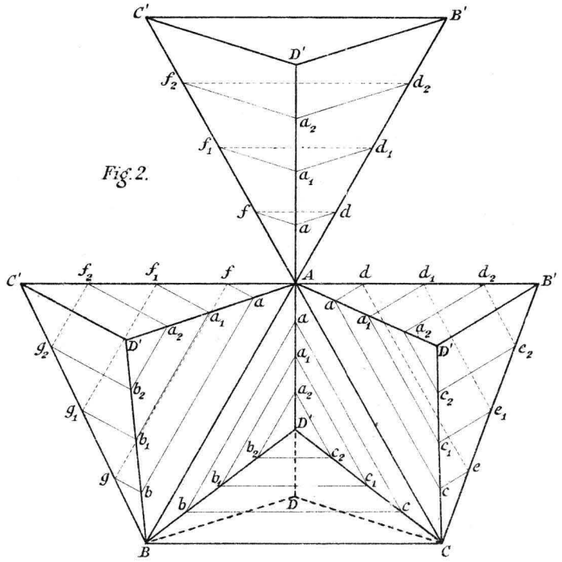
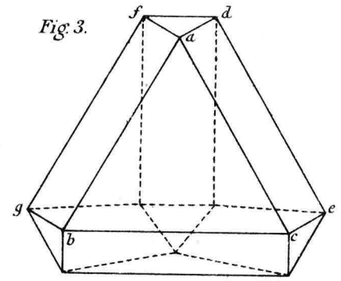


Fig. 2.



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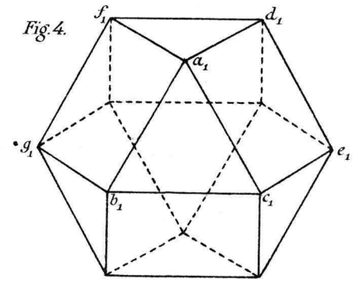

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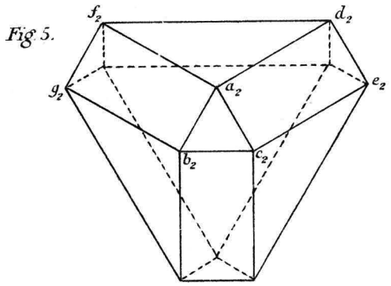

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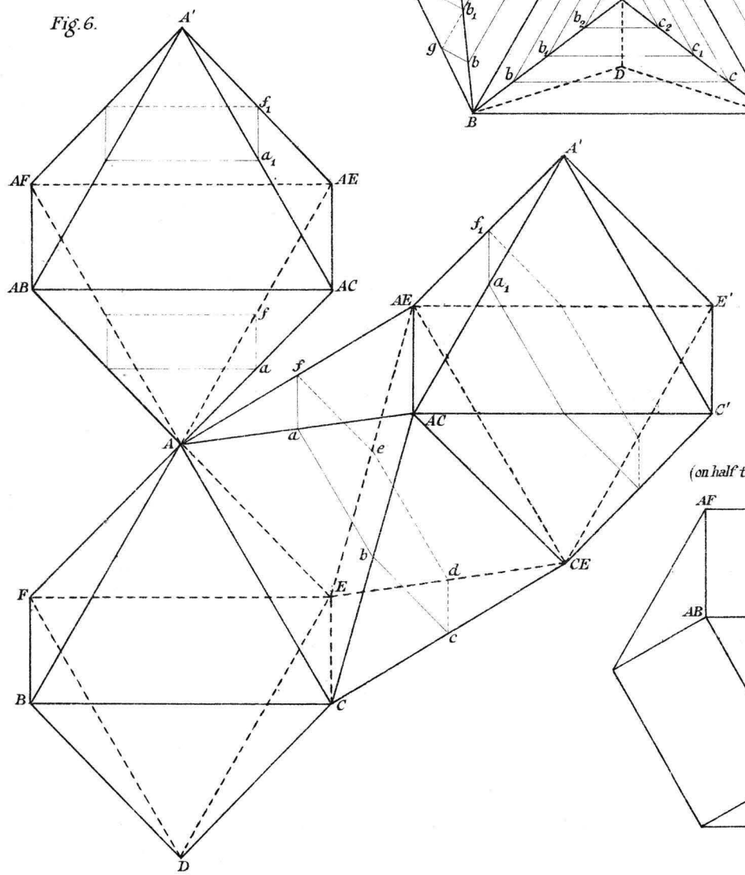

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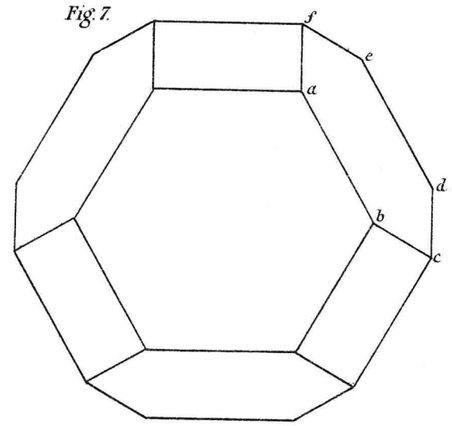

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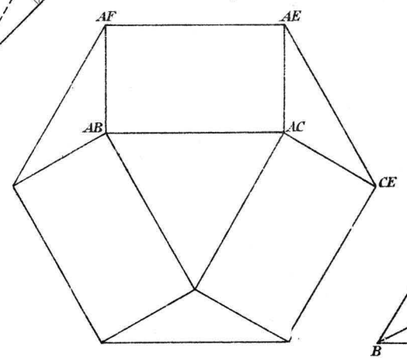
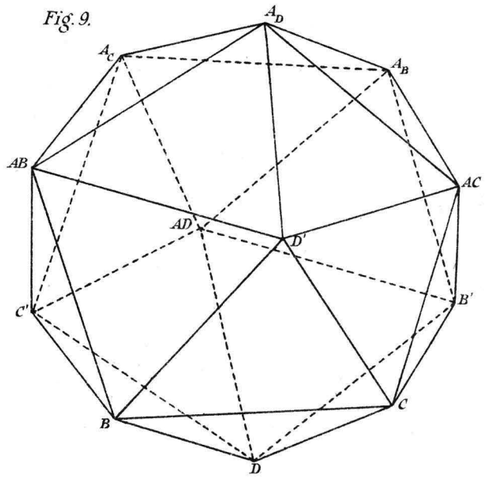


Fig. 8.
(on half the scale of Figs. 6 and 7)



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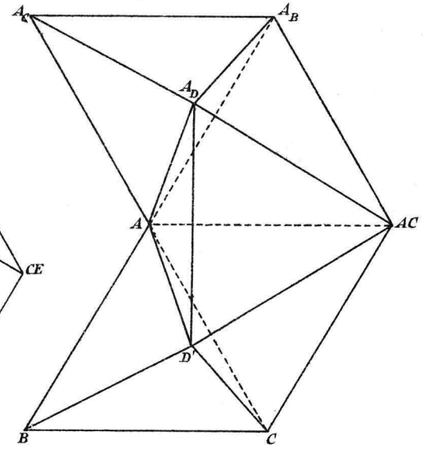

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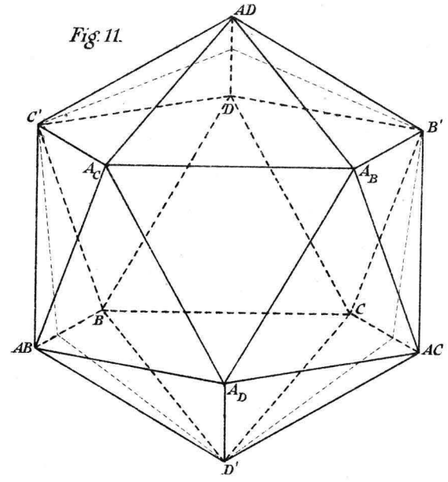

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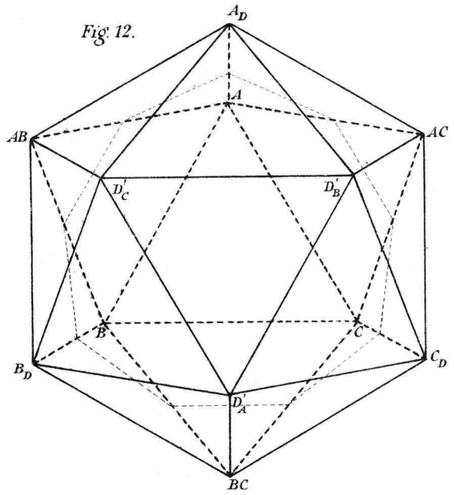

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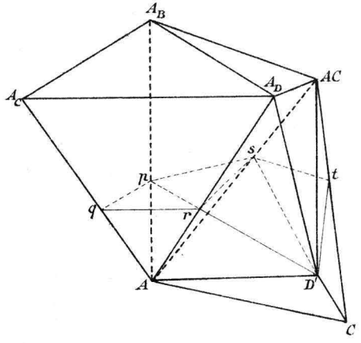

Fig. 13.

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Fig. 14 (IA_D)

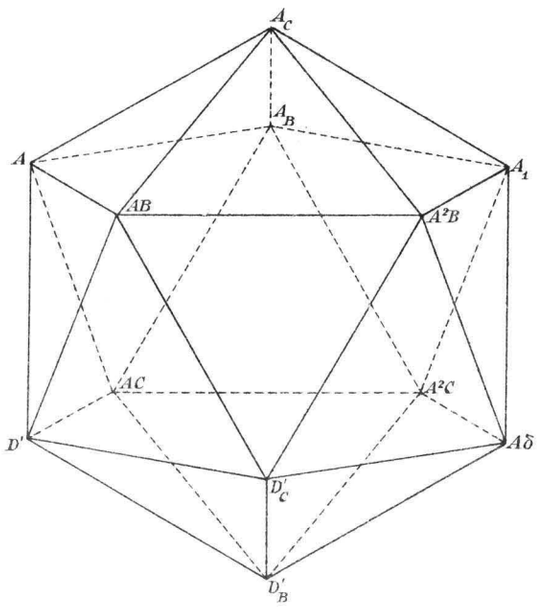


Fig. 17 ($I\delta_1$)

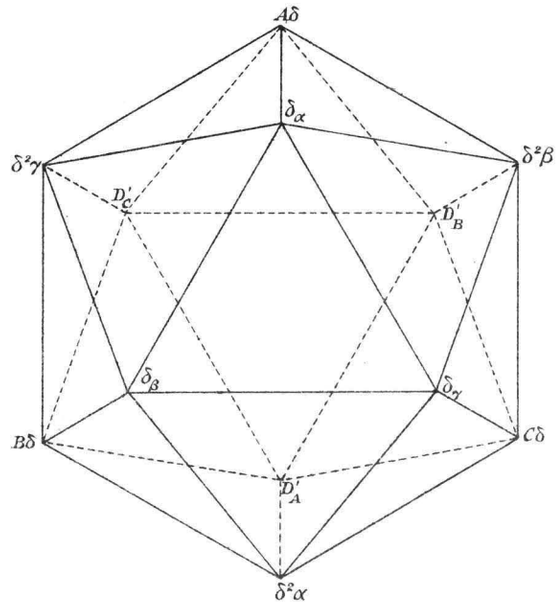


Fig. 20 ($I\delta_1$)

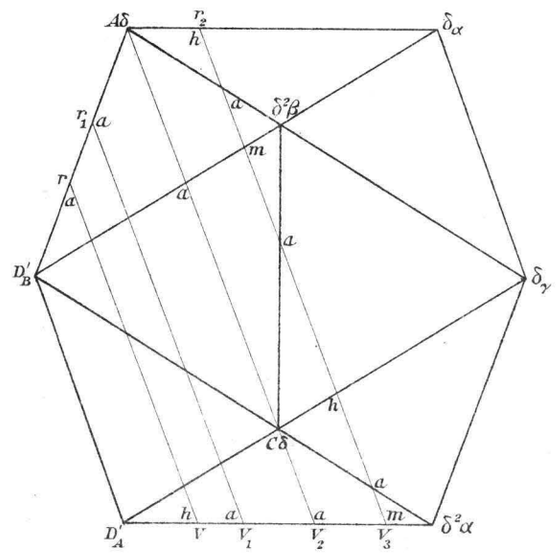


Fig. 15 (IA_1)

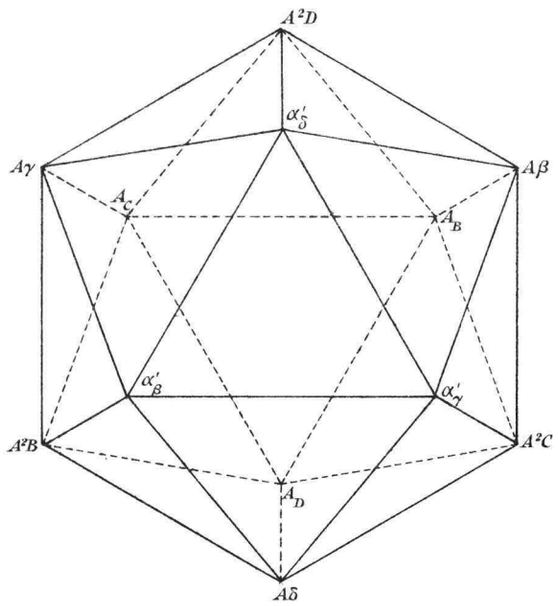


Fig. 18.

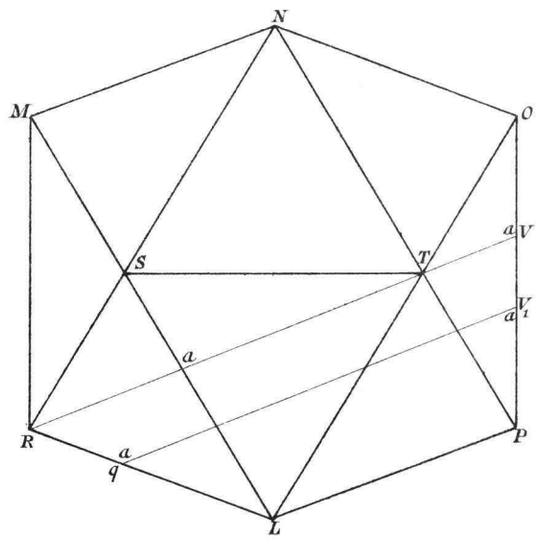


Fig. 21 ($I1\delta$)

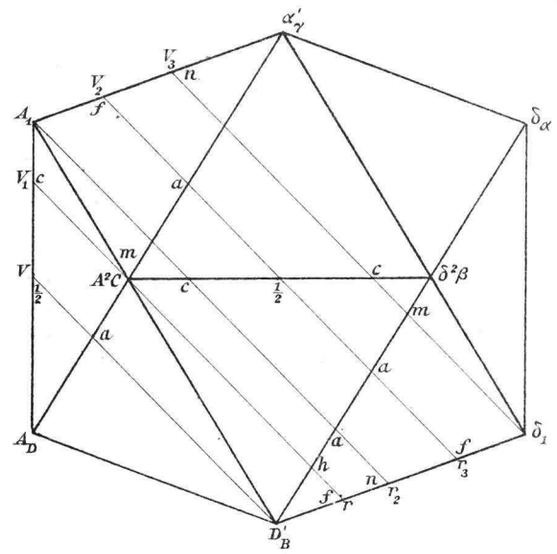


Fig. 15 (IA_1)

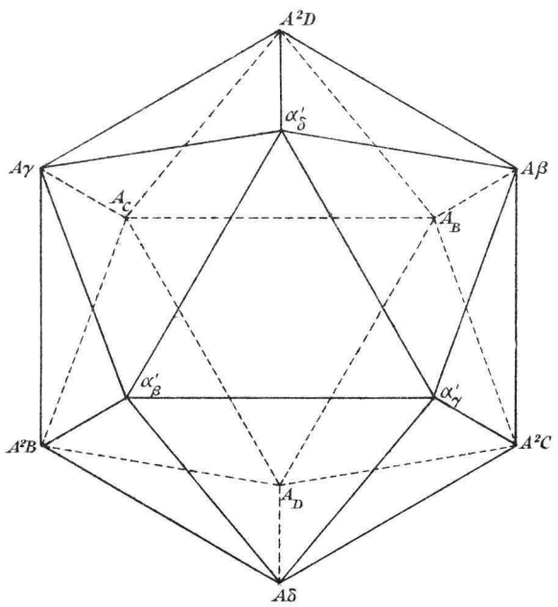


Fig. 18.

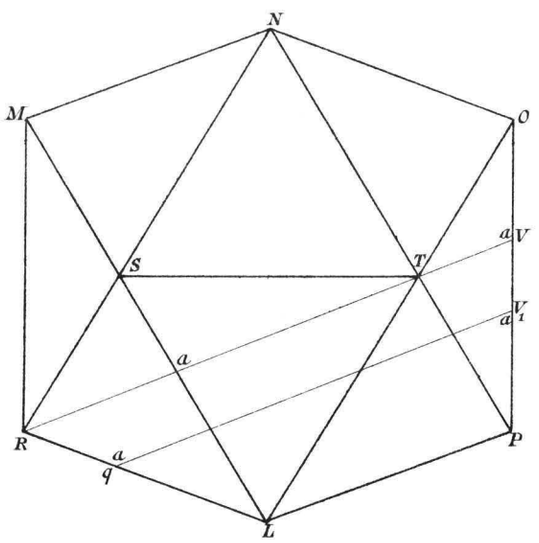
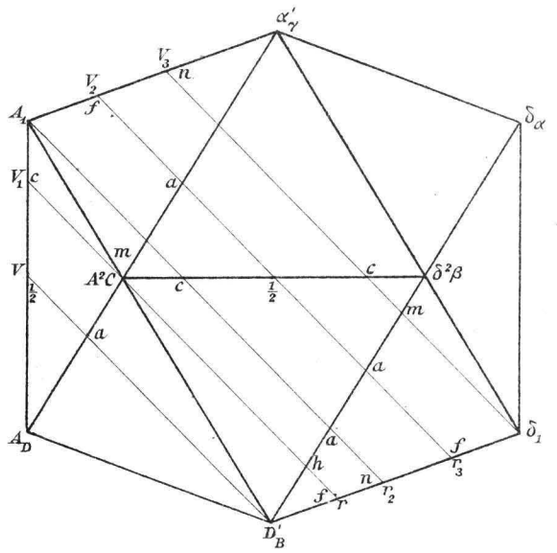
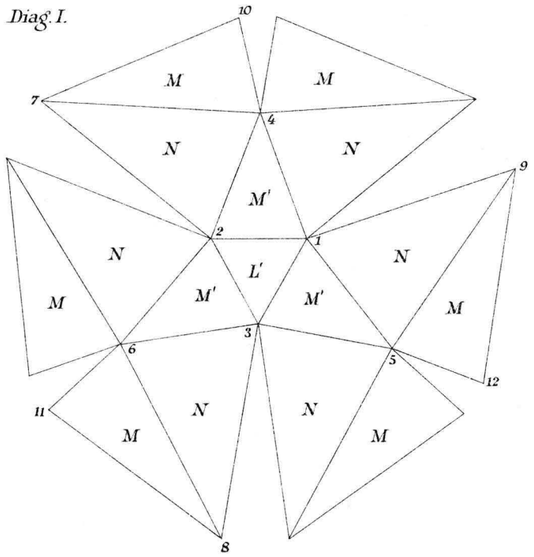


Fig. 21 ($I1\delta$)



Diag. I.



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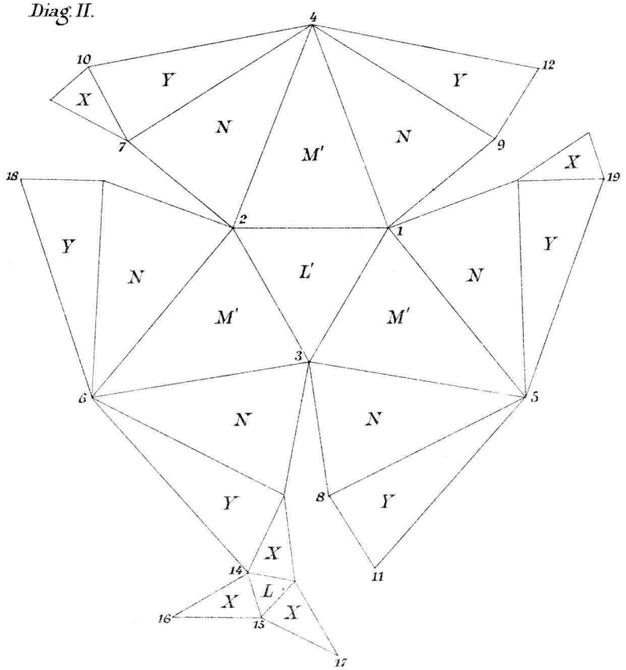

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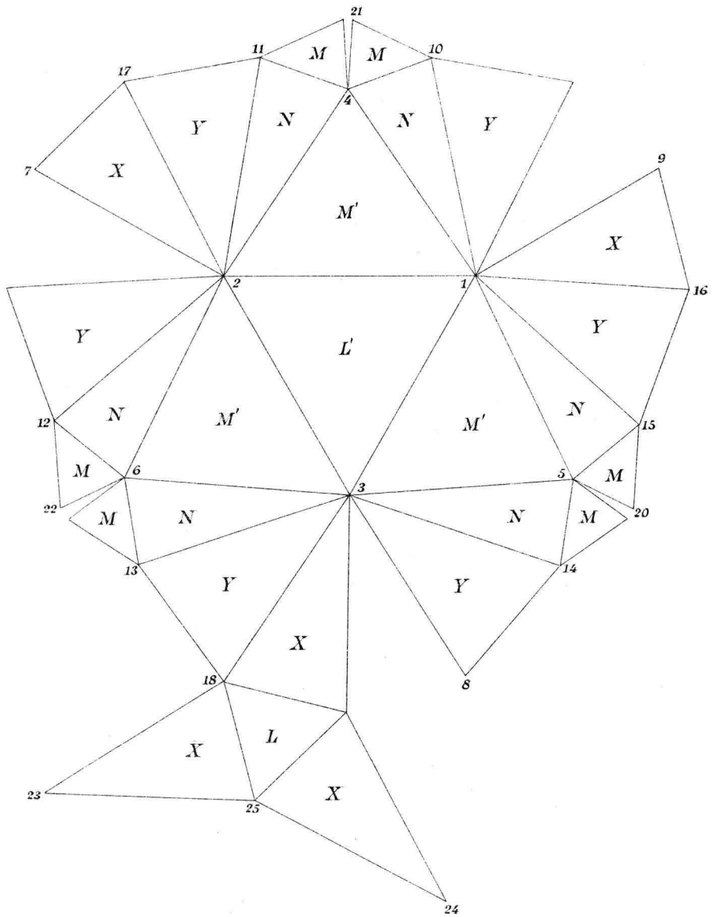

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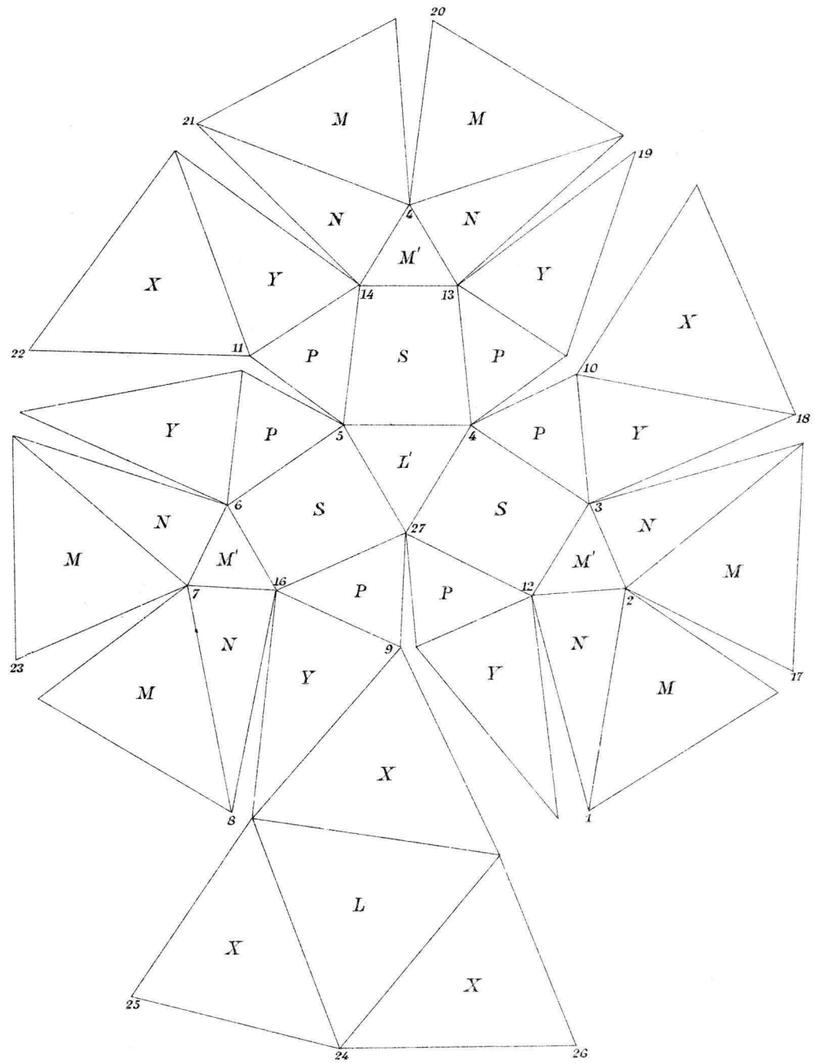

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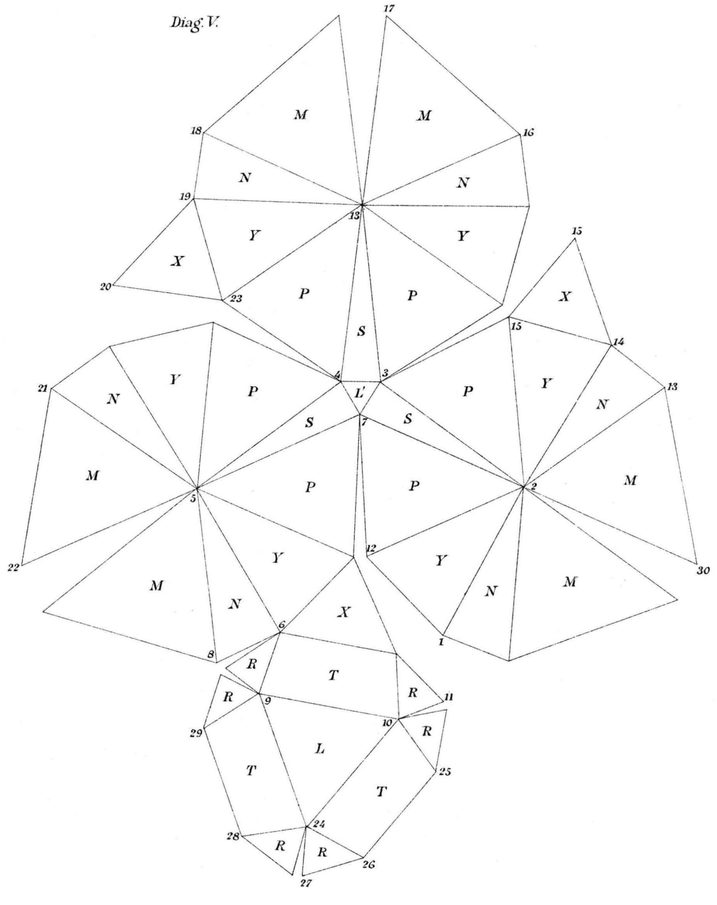

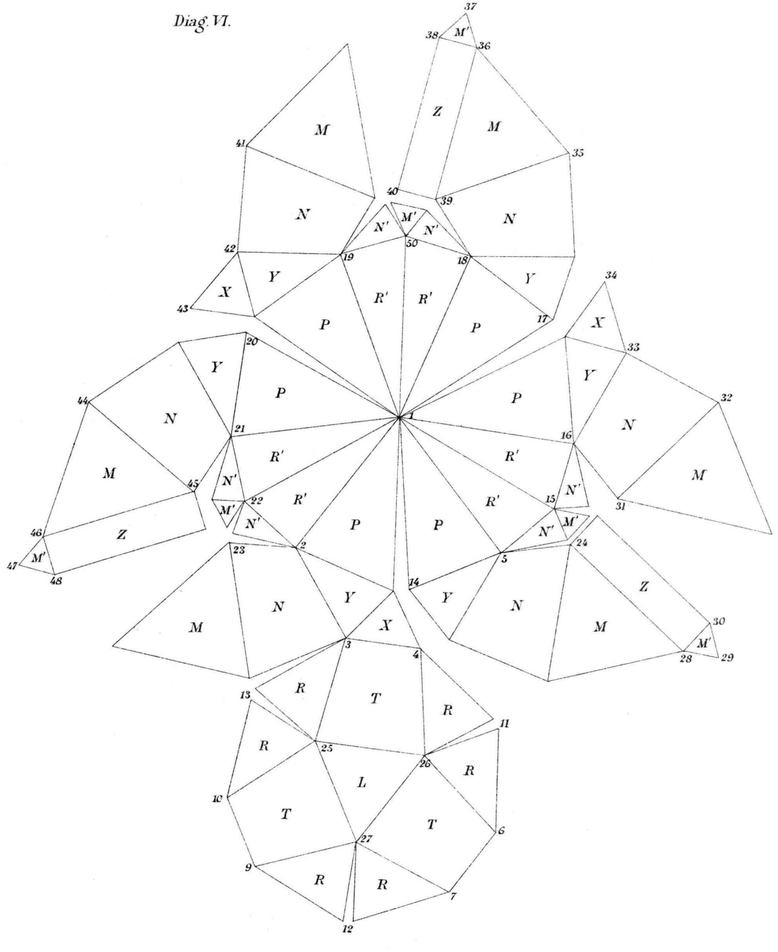

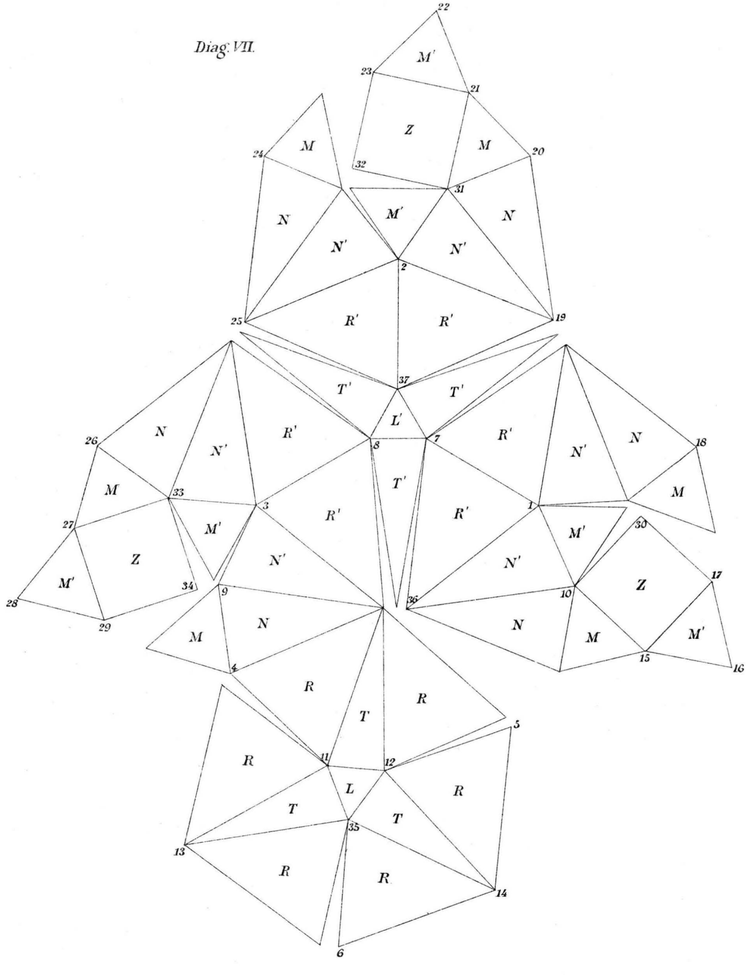

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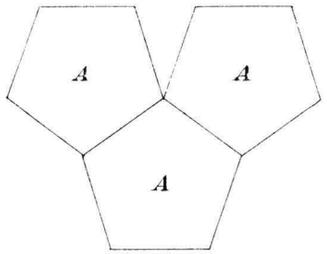
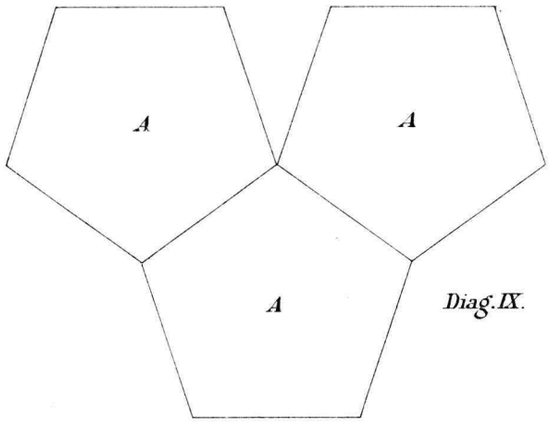
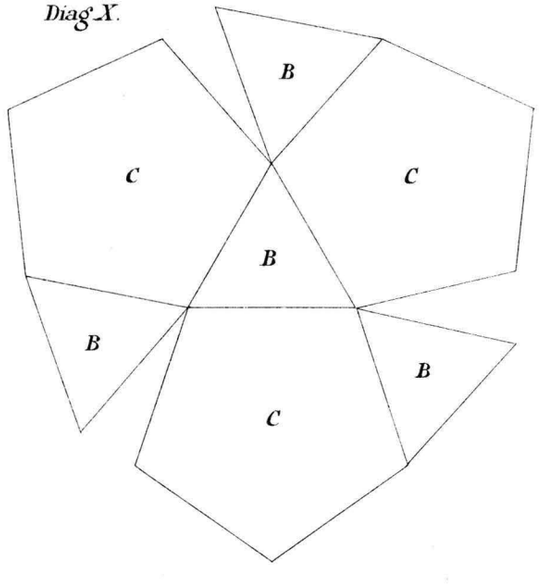
Diag. VIII.





Diag. IX.

Diag. X.



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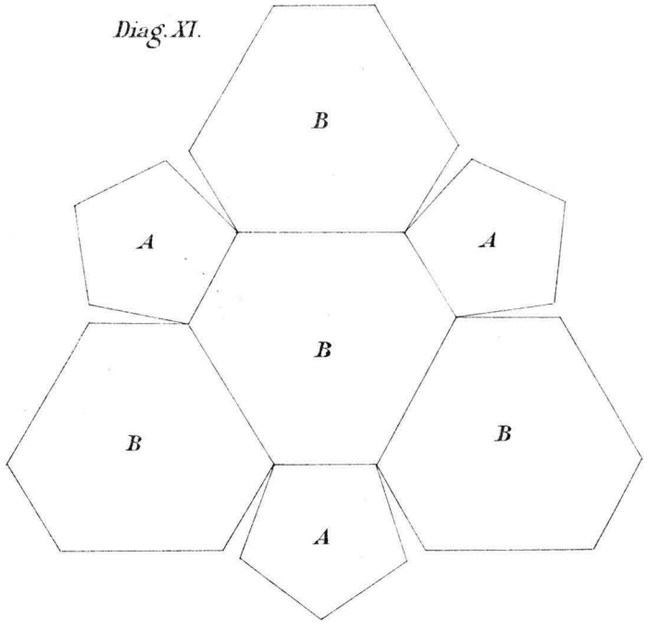

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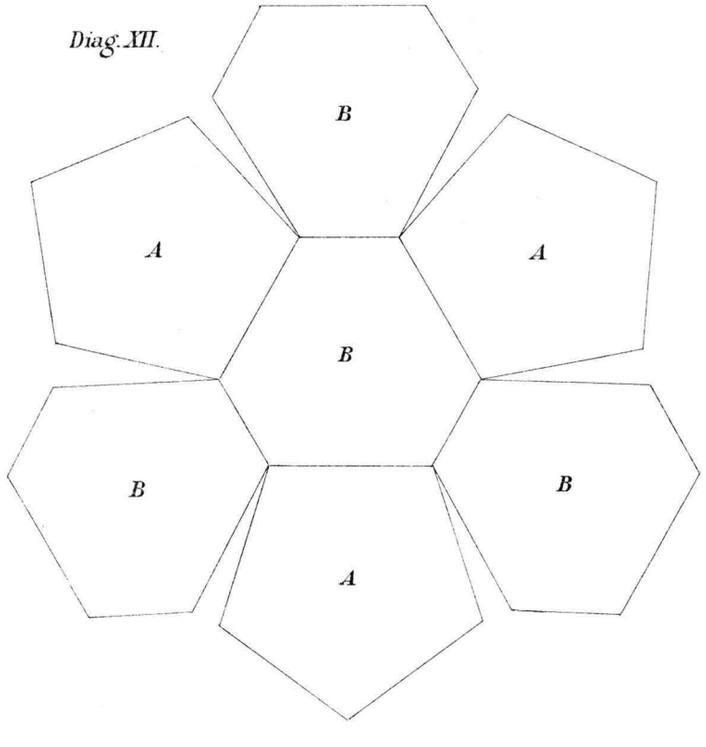

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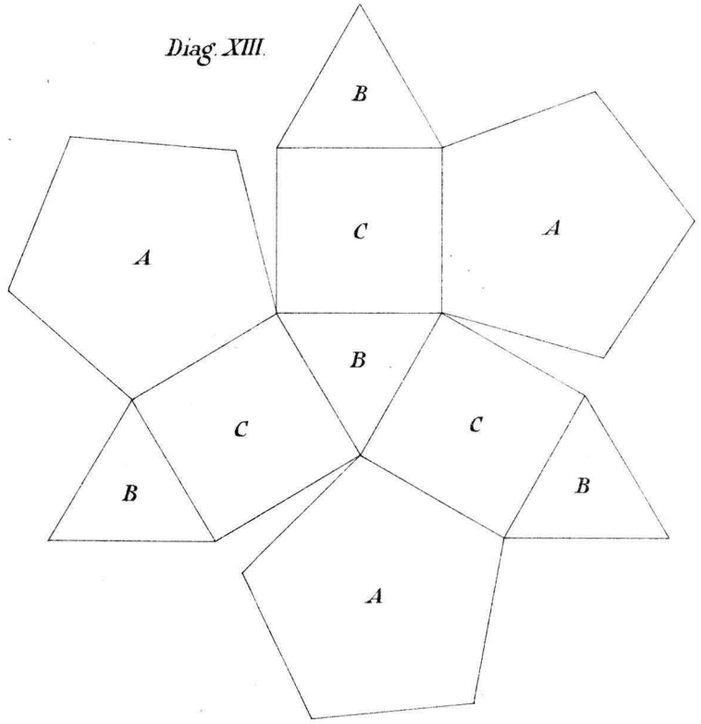

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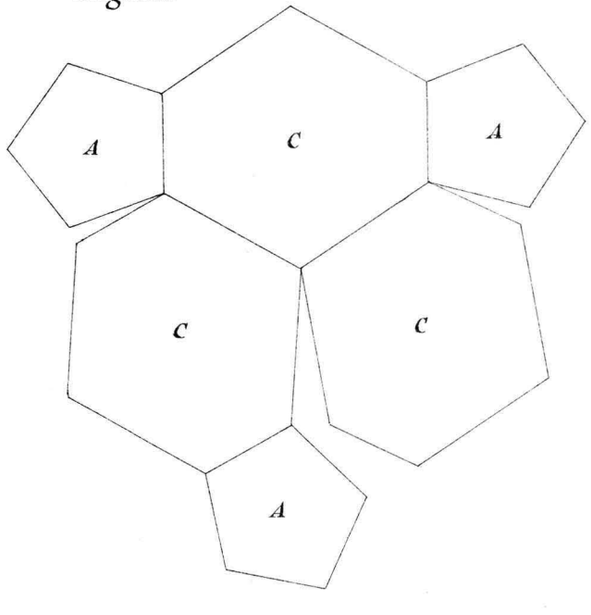

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